SOME TOPICS CONCERNING GRAPHS, SIGNED GRAPHS AND MATROIDS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

By

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2012

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ABSTRACT

We discuss well-quasi-ordering in graphs and signed graphs, giving two short proofs of the bounded case of S. B. Rao’s conjecture. We give a characterization of graphs whose bicircular matroids are signed-graphic, thus generalizing a theorem of Matthews from the 1970s. We prove a recent conjecture of Zaslavsky on the equality of frustration number and frustration index in a certain class of signed graphs. We prove that there are exactly seven signed Heawood graphs, up to switching isomorphism. We present a computational approach to an interesting conjecture of D. J. A. Welsh on the number of bases of matroids. We then move on to study the frame matroids of signed graphs, giving explicit signed-graphic representations of certain families of matroids. We also discuss the cycle, bicircular and even-cycle matroid of a graph and characterize matroids arising as two different such structures. We study graphs in which any two vertices have the same number of common neighbors, giving a quick proof of Shrikhande’s theorem. We provide a solution to a problem of E. W. Dijkstra. Also, we discuss the flexibility of graphs on the projective plane. We conclude by mentioning partial progress towards characterizing signed graphs whose frame matroids are transversal, and some miscellaneous results.
To my family
I would like to thank Prof. Neil Robertson for sharing his deep understanding of structural aspects of graphs. I have learned a lot about graphs from the numerous weekly meetings I have enjoyed with him. His penetrating questions and constructive comments have streamlined many of the arguments presented in this dissertation. His personal accounts of the Fantastic Four - Graph Minors, Hadwiger’s Conjecture for $k = 6$, Four Color Theorem, and Strong Perfect Graph Theorem, were very interesting, and gave me a very clear picture of how research progresses in phases. Daniel Slilaty gave a series of talks here at Ohio State during November 2007, which is the basis for many of the topics in this thesis. He first defined matroids for me and pointed to Oxley’s book for further study. I have immensely benefitted from several discussions with Thomas Zaslavsky. My first encounter with serious combinatorics was in a course by Prof. Ákos Seress. He is a wonderful teacher and his spectacular treatment of matching theory eventually led me to topics related to transversal matroids. I have learned several probabilistic techniques from Prof. Boris Pittel and Matthew Kahle. I am also grateful to the following students of Prof. Robertson with whom I had the privilege to interact: John Maharry, Xiangqian Zhou, Christopher McClain, Christian Altomare and Oguz Kurt. John Maharry suggested several changes in the original version of this dissertation, and this has improved the readability of this document.
I am grateful to Prof. Bert Gerards who hosted me for two weeks in Maastricht and also for inviting me to participate and give a talk in the third workshop on graphs and matroids that he organized. It was there that Luis Goddyn gave an interesting talk on the excluded minors for the class of bicircular matroids, which inspired me to look at the intersection of bicircular matroids and signed-graphic matroids. I thank Stefan van Zwam for explaining several matroid concepts to me, and for installing the computer software SAGE in my laptop. I have used SAGE to verify a conjecture of Welsh for small values (Chapter 5).
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FIELDS OF STUDY

Major Field: Mathematics

Specialization: Graph Theory, Matroid Theory and Signed Graphs
# TABLE OF CONTENTS

Abstract ......................................................... ii
Dedication ......................................................... iii
Acknowledgments ................................................... iv
Vita ................................................................. vi
List of Figures ..................................................... xi
List of Tables ...................................................... xii

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction ........................................... 1</td>
</tr>
<tr>
<td></td>
<td>1.1 Graphs .............................................. 1</td>
</tr>
<tr>
<td></td>
<td>1.2 Signed graphs ....................................... 2</td>
</tr>
<tr>
<td></td>
<td>1.3 Matroids ............................................. 3</td>
</tr>
<tr>
<td></td>
<td>1.4 Background material ................................ 4</td>
</tr>
<tr>
<td></td>
<td>1.5 Organization of the dissertation .................. 5</td>
</tr>
<tr>
<td>2</td>
<td>Well-quasi-ordering in graphs ......................... 7</td>
</tr>
<tr>
<td></td>
<td>2.1 Two short proofs of the bounded case of S. B. Rao’s degree sequence conjecture ..................... 10</td>
</tr>
<tr>
<td></td>
<td>2.2 Well-quasi-ordering series-parallel graphs ........ 14</td>
</tr>
<tr>
<td></td>
<td>2.3 Well-quasi-ordering signed graphs ................. 17</td>
</tr>
<tr>
<td></td>
<td>2.4 An infinite antichain of series-parallel graphs ... 18</td>
</tr>
<tr>
<td>3</td>
<td>A signed-graphic analogue of Matthews’ theorem ........ 19</td>
</tr>
<tr>
<td></td>
<td>3.1 Terminology and notation ........................... 19</td>
</tr>
<tr>
<td></td>
<td>3.2 Statement of Matthews’ theorem ..................... 20</td>
</tr>
<tr>
<td></td>
<td>3.3 Statement of the main theorem ....................... 20</td>
</tr>
<tr>
<td></td>
<td>3.4 Bicircular matroids and their properties ........... 22</td>
</tr>
<tr>
<td></td>
<td>3.5 Signed-graphic matroids and their properties ....... 22</td>
</tr>
<tr>
<td>Section</td>
<td>Page</td>
</tr>
<tr>
<td>----------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>3.6 Proof of the main theorem</td>
<td>23</td>
</tr>
<tr>
<td>3.7 Questions and concluding remarks</td>
<td>28</td>
</tr>
<tr>
<td>4 Seven signings of the Heawood graph</td>
<td>30</td>
</tr>
<tr>
<td>4.1 Introduction</td>
<td>32</td>
</tr>
<tr>
<td>4.2 Structure of the Heawood graph</td>
<td>33</td>
</tr>
<tr>
<td>4.3 Signings</td>
<td>35</td>
</tr>
<tr>
<td>4.4 Frustration</td>
<td>40</td>
</tr>
<tr>
<td>4.5 Number of negative circles</td>
<td>43</td>
</tr>
<tr>
<td>4.6 Clusterability</td>
<td>45</td>
</tr>
<tr>
<td>4.7 Coloring</td>
<td>48</td>
</tr>
<tr>
<td>4.8 Signings of the Petersen graph</td>
<td>49</td>
</tr>
<tr>
<td>4.9 Concluding Remarks</td>
<td>50</td>
</tr>
<tr>
<td>5 On a conjecture of Welsh</td>
<td>52</td>
</tr>
<tr>
<td>5.1 Introduction</td>
<td>52</td>
</tr>
<tr>
<td>5.2 A computational approach to Welsh’s conjecture</td>
<td>52</td>
</tr>
<tr>
<td>5.3 SAGE results</td>
<td>53</td>
</tr>
<tr>
<td>5.4 Four related conjectures</td>
<td>53</td>
</tr>
<tr>
<td>5.5 A proof of de Mier’s observation</td>
<td>54</td>
</tr>
<tr>
<td>6 Signed-graphic matroids</td>
<td>56</td>
</tr>
<tr>
<td>6.1 Properties of signed-graphic matroids</td>
<td>56</td>
</tr>
<tr>
<td>6.2 Signed-graphic representations of ternary swirls</td>
<td>58</td>
</tr>
<tr>
<td>6.3 Signed-graphic representations of $R_{10}$</td>
<td>61</td>
</tr>
<tr>
<td>6.4 Graphic vs. Signed-graphic matroids</td>
<td>64</td>
</tr>
<tr>
<td>6.5 Operations preserving the frame matroid structure</td>
<td>64</td>
</tr>
<tr>
<td>6.6 A Venn diagram</td>
<td>65</td>
</tr>
<tr>
<td>7 Three matroids on the edge set of a graph</td>
<td>67</td>
</tr>
<tr>
<td>7.1 Graphic matroids</td>
<td>67</td>
</tr>
<tr>
<td>7.2 Bicircular matroids</td>
<td>68</td>
</tr>
<tr>
<td>7.3 Even-cycle matroids</td>
<td>69</td>
</tr>
<tr>
<td>7.4 When do these matroids coincide?</td>
<td>69</td>
</tr>
<tr>
<td>7.5 Questions</td>
<td>69</td>
</tr>
<tr>
<td>7.6 When is $B(G)$ an even-cycle matroid?</td>
<td>70</td>
</tr>
<tr>
<td>8 Strongly regular graphs with $\lambda = \mu$</td>
<td>73</td>
</tr>
<tr>
<td>8.1 Graphs in which any two vertices have the same number of common neighbors</td>
<td>73</td>
</tr>
<tr>
<td>8.2 Graphs in which any two vertices have exactly two common neighbors</td>
<td>80</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Breaking into pieces depending on where the parallel addition operation is used first.</td>
<td>15</td>
</tr>
<tr>
<td>3.1 Forbidden graphs in Matthews’ characterization of graphs whose bicircular matroids are graphic.</td>
<td>21</td>
</tr>
<tr>
<td>3.2 Forbidden graphs in the characterization of graphs whose bicircular matroids are signed-graphic.</td>
<td>22</td>
</tr>
<tr>
<td>4.1 The Heawood graph</td>
<td>30</td>
</tr>
<tr>
<td>4.2 The seven switching isomorphism types of signed Heawood graphs.</td>
<td>31</td>
</tr>
<tr>
<td>6.1 Base case: $\pm C_5$ uniquely represents its frame matroid.</td>
<td>59</td>
</tr>
<tr>
<td>6.2 Induction step: uniqueness of extension and coextension.</td>
<td>60</td>
</tr>
<tr>
<td>6.3 The six signed-graphic representations of $R_{10}$</td>
<td>62</td>
</tr>
<tr>
<td>6.4 Shuffling a $-K_4$</td>
<td>64</td>
</tr>
<tr>
<td>6.5 A Venn diagram of the class of matroids</td>
<td>66</td>
</tr>
<tr>
<td>8.1 (a) The Shrikhande graph (b) The line graph of $K_{4,4}$</td>
<td>80</td>
</tr>
<tr>
<td>8.2 The Shrikhande Graph</td>
<td>81</td>
</tr>
<tr>
<td>8.3 Constructing the Shrikhande Graph</td>
<td>81</td>
</tr>
<tr>
<td>10.1 Forbidden minors for projective-planarity</td>
<td>90</td>
</tr>
<tr>
<td>10.2 W-twist</td>
<td>92</td>
</tr>
<tr>
<td>10.3 P-twist and Q-twist</td>
<td>92</td>
</tr>
</tbody>
</table>
## LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Frustration index of signed Heawood graphs.</td>
</tr>
<tr>
<td>4.2</td>
<td>Number of negative hexagons in signed Heawood graphs.</td>
</tr>
<tr>
<td>4.3</td>
<td>Number of negative Hamilton circles in signed Heawood graphs.</td>
</tr>
<tr>
<td>4.4</td>
<td>Cluster number of signed Heawood graphs.</td>
</tr>
<tr>
<td>4.5</td>
<td>Inclusterability index of the seven minimal signed Heawood graphs.</td>
</tr>
<tr>
<td>4.6</td>
<td>The chromatic numbers of signed Heawood graphs.</td>
</tr>
<tr>
<td>6.1</td>
<td>A comparison of graphic and signed-graphic matroids</td>
</tr>
</tbody>
</table>
CHAPTER 1
INTRODUCTION

1.1 Graphs

Graph theory is one of the most colorful parts of combinatorics. Any collection of objects with a binary relation can be faithfully represented by a graph, and the theory of graphs guides one to analyze the situation. Formally, a graph is a triple \((V, E, \psi)\), where \(V\) and \(E\) are sets and \(\psi\) is a function from \(E\) to the set of two-element multisets of \(V\). Elements of \(V\) are called vertices, elements of \(E\) are called edges, and the endpoints (or endvertices) of an edge \(e\) are the elements of \(\psi(e)\). Although the definition of a graph looks extremely simple, deep and intricate techniques have been developed to analyze graphs. Hundreds, if not thousands, of graph invariants have been introduced and carefully studied. Several classes of “nice” graphs have been culled out and properties tabulated. Graphs have been viewed from different angles and perspectives. Structural questions typically ask for the existence of substructures when the value of an invariant is known. Extremal questions usually involve understanding the effect of one invariant on another. Complexity questions ask whether a particular property can be tested in polynomial time. Topological questions deal with understanding the connection between discrete (graphs) and the continuous (surfaces). Algebraic questions reduce the problem of understanding a graph to that of
its matrices and vector spaces. Probabilistic questions ask how “most” graphs look like.

1.2 Signed graphs

The concept of a graph has been very useful, both in mathematics and outside. But there are situations where an extension or a variation of this notion is necessary. For example, suppose we are representing roads by edges, and junctions by vertices. It might be necessary to record the length of each road. In this case, we associate with each edge the corresponding length, thus giving rise to what is called a weighted graph. One can also find several other kinds of objects, like directed graphs, infinite graphs, etc. The concept of a group is extremely important in modern mathematics, and hence it makes sense to associate each edge of the graph with an element of a given group. Such an object is called a group-labeled graph and has been useful in analyzing flows and in matroid theory.

A graph can be thought of as a group-labeled graph with edges labeled from the trivial group (group with one element). The next case, in which the group has two elements, is interesting, and the objects thus obtained are called signed graphs. Thus a signed graph is a graph with each edge receiving either a positive sign or a negative sign, and the signs are thought of as the elements of the group of order two. Signed graphs actually came to prominence as a tool to study interpersonal relationships. Suppose we have a group of people and we would like to understand how people mingle with each other. This can be represented by a signed graph, where vertices are the people, and a positive (negative) edge signifies that the two corresponding people like (hate) each other, while the absence of an edge indicates indifference. In mathematics, signed graphs are closely related to two-graphs, a topic pioneered by J. J. Seidel. Zaslavsky showed that one can associate several different matroids on
the edge set of a signed graph and that the resulting matroids are genuine extensions of those obtained from graphs.

1.3 Matroids

In 1736, Euler published a paper giving a solution of the Königsberg bridge problem. This is believed to be the first paper in the theory of graphs. Almost 200 years after Euler’s paper, something spectacular happened. Hassler Whitney was thinking about graphs and noticed an uncanny similarity between matrices and graphs. For example, all trees on \( n \) vertices have the same number of edges, much like the well-known fact that all bases of a finite-dimensional vector space have the same number of elements. Whitney’s key insight was to extract some key properties satisfied by the edges of a graph and the columns of a matrix, and define an abstract set system satisfying these properties. He called these objects matroids, a term that signifies its kinship to matrices (see Appendix A for six ways to define a matroid). Since then matroids have served to unify several theories, most notably, graph theory, matrix theory, transversal theory, lattice theory, geometry, block designs, and combinatorial optimization.

Three quarters of a century later, spectacular advances have been made in the theory of matroids. Several classes of matroids have been studied carefully. The most important of these is the class of graphic matroids. Graph theory provides not only source and motivation for matroid theory, but also a fantastic testing ground for conjectures. As Tutte points out, whenever a theorem about graphs can be stated only in terms of edges and cycles, it probably has a counterpart in matroids. This symbiotic relationship between graphs and matroids has been useful in understanding both these structures and we will see several examples of this in this work.
1.4 Background material

We will define terms as and when needed. Most of the terminology is standard. There are several graph theory texts in the literature. Diestel’s book [20] is a very good reference. Matroids can be defined in several different ways. Each one of them gives a new perspective on these objects and a particular viewpoint taken depends on one’s taste and intended applications. The reader can find six of the most common ways to define a matroid in Appendix A of this dissertation. Oxley’s book [39] is a fantastic resource for matroid theory and if the reader needs clarification on a definition or a piece of notation, this is the book to consult.

A graph $H$ is said to be an induced subgraph of $G$ if (an isomorphic copy of) $H$ can be obtained from $G$ by deleting some vertices. A graph $H$ is said to be a subgraph of $G$ if (an isomorphic copy of) $H$ can be obtained from $G$ by deleting vertices and deleting edges. A graph $H$ is said to be a topological minor of $G$ if (an isomorphic copy of) $H$ can be obtained from $G$ by deleting vertices, deleting edges, and contracting series edges. A graph $H$ is said to be a minor of $G$ if (an isomorphic copy of) $H$ can be obtained from $G$ by deleting vertices, deleting edges and contracting edges.

A signed graph $\Sigma$ is a pair $(G, \sigma)$ where $\sigma$ is a function from $E(G)$ to $\{\pm 1\}$. A circle or path in a signed graph $\Sigma$ is called positive if the product of signs on its edges is positive, otherwise the circle or path is called negative. A subgraph $\Sigma_1$ of $\Sigma$ is called balanced if all circles in $\Sigma_1$ are positive. A vertex $v$ in $\Sigma$ is called a balancing vertex if $\Sigma \setminus v$ is balanced.

A switching function on a signed graph $\Sigma = (G, \sigma)$ is a function $\eta : V(\Sigma) \to \{\pm\}$. The signed graph $\Sigma^\eta = (G, \sigma^\eta)$ has sign function $\sigma^\eta$ on $E(G)$ defined by $\sigma^\eta(e) = \eta(v)\sigma(e)\eta(w)$ where $v$ and $w$ are the endvertices of $e$. Two signed graphs $\Sigma_1$ and $\Sigma_2$ are said to be switching equivalent if there exists a switching function $\eta$ such that $\Sigma_1^\eta = \Sigma_2$. We say $\Sigma_1$ and $\Sigma_2$ are isomorphic if there is an underlying graph isomorphism that
preserves edge signs. They are switching isomorphic if $\Sigma_2$ is isomorphic to a switching of $\Sigma_1$.

In a signed graph $\Sigma = (G, \sigma)$, the deletion of an edge $e$ from $\Sigma$ is defined as $\Sigma \setminus e = (G \setminus e, \sigma)$ where $\sigma$ is understood to be restricted to the domain $E(\Sigma) \setminus e$. The contraction of an edge $e$ is defined depending on whether $e$ is a link, positive loop, or a negative loop. If $e$ is a link, then $\Sigma/e = (G/e, \sigma^\eta)$ where $\eta$ is a switching function satisfying $\sigma^\eta(e) = 1$. If $e$ is a positive loop, then $\Sigma/e = \Sigma \setminus e$. If $e$ is a negative loop with endpoint $v$, then $\Sigma/e$ is the signed graph on $V(\Sigma)$ obtained from $\Sigma$ as follows: delete $e$, links incident to $v$ become negative loops incident to their other endpoint, loops incident to $v$ become positive loops at $v$, and edges not incident to $v$ remain unchanged.

The complements of the bases of a matroid $M$ also from the bases of another matroid, called the dual of $M$ and denoted $M^\ast$. For an element $e$ of a matroid $M$, the deletion of $e$ from $M$, denoted $M \setminus e$, is the matroid whose ground set is $E(M) \setminus e$, and whose bases are exactly the bases of $M$ contained in $E(M) \setminus e$. The contraction of $e$ from $M$, denoted $M/e$, is defined to be $(M^\ast \setminus e)^\ast$.

1.5 Organization of the dissertation

In chapter 2 we discuss well-quasi-ordering in graphs and signed graphs. We give two short proofs of the bounded case of S. B. Rao’s conjecture. We give a direct proof of the graph minor theorem for graphs with tree-width at most two using Nash-Williams’ minimal bad sequence argument. Then we prove that signed graphs with bounded frustration number are well-quasi-ordered by the minor relation. Finally, we mention a new infinite antichain of graphs with respect to the topological minor relation. In chapter 3 we give a characterization of graphs whose bicircular matroids are signed-graphic, thus generalizing a theorem of Matthews from the 1970s. In chapter 4 we
prove a recent conjecture of Zaslavsky on the equality of frustration number and frustration index in a certain class of signed graphs. Also, we show that there are exactly seven signed Heawood graphs, up to switching isomorphism. In chapter 5 we present a computational approach to an interesting conjecture of D. J. A. Welsh on the number of bases of matroids. In chapter 6 we study the frame matroids of signed graphs, giving explicit signed-graphic representations of certain families of matroids. In chapter 7 we discuss the cycle, bicircular and even-cycle matroid of a graph and characterize matroids arising as two different such structures. In chapter 8 we study graphs in which any two vertices have the same number of common neighbors, giving a quick proof of Shrikhande’s theorem. In chapter 9 we provide a solution to a problem of E. W. Dijkstra. In chapter 10 we discuss the flexibility of graphs on the projective plane. This chapter is joint work with John Maharry, Neil Robertson, and Daniel Slilaty. Here we give three operations, P-twist, Q-twist, and W-twist, and prove that any two embeddings of a nonplanar graph on the projective plane can be explained by a sequence of these three operations. In chapter 11 we mention partial progress towards characterizing signed graphs whose frame matroids are transversal. Bondy proved a very nice theorem where he precisely finds out the reason why the cycle matroid of a graph fails to be transversal. We attempt to mimic his success in the signed-graphic case. In chapter 12 we give a construction of a family of 4-regular graphs with girth 5 on 19k vertices. Then we give a unified proof of coloring theorems of Brooks and Catlin.
A quasi-order is a pair \((Q, \leq)\), where \(Q\) is a set and \(\leq\) is a reflexive and transitive relation on \(Q\). An infinite sequence \(q_1, q_2, \ldots\) in \((Q, \leq)\) is good if there exist \(i < j\) such that \(q_i \leq q_j\). An infinite sequence \(q_1, q_2, \ldots\) in \((Q, \leq)\) is perfect if there exist \(i_1 < i_2 < \ldots\) such that \(q_{i_1} \leq q_{i_2} \leq \ldots\). A subset \(A\) of \(Q\) is called an antichain if for any two distinct elements \(a_1, a_2 \in A\), we have \(a_1 \not\leq a_2\). A lower ideal of \((Q, \leq)\) is a subset \(I \subseteq Q\) such that \(q_1 \leq q_2 \in I\) implies \(q_1 \in I\). A quasi-order \((Q, \leq)\) is finite if the underlying set \(Q\) is finite. If \((Q_1, \leq_1)\) and \((Q_2, \leq_2)\) are quasi-orders, then their Cartesian product is \((Q_1 \times Q_2, \leq)\), where \((q_1, q_2) \leq (q_1', q_2')\) if and only if \(q_1 \leq_1 q_1'\) and \(q_2 \leq_2 q_2'\).

A quasi-order \((Q, \leq)\) is a well-quasi-order (WQO) if one (and hence all) of the following equivalent conditions holds:

1. Every sequence is good.

2. Every sequence is perfect.

3. There is no infinite antichain and no infinite strictly descending chain.

4. Every sequence of strictly descending lower ideals is finite.

The proof of the above equivalence is routine and uses one application of Ramsey’s theorem [20].
Some examples follow.

1. If \((Q, \leq)\) is well-ordered, then \((Q, \leq)\) is a WQO. Canonical example is the set of natural numbers under the usual size relation.

2. If a quasi-order \((Q, \leq)\) is finite, then it is a WQO.

3. If \((Q, \leq)\) is a WQO and \(H \subseteq Q\), then \((H, \leq)\) is a WQO.

4. A finite union of WQOs is a WQO.

5. If \((Q_1, \leq_1)\) and \((Q_2, \leq_2)\) are both WQOs, then so is their Cartesian product \((Q_1 \times Q_2, \leq)\).

6. If \((Q, \leq)\) is a WQO, so is \((Q^{<\omega}, \leq_H)\). This is known as Higman’s Lemma or Higman’s Finite Sequences Theorem (see [20]).

7. If \((Q, \leq)\) is a WQO, then \((Q^{\omega}, \leq)\) need not be a WQO (Rado’s counterexample).
   Let \(R = \{(i, j) | i, j \in \mathbb{N}, i < j\}\). Define \((i, j) \leq (i', j')\) if \((i = i'\) and \(j \leq j')\) or \(j < i'\). It is easy to see that \((R, \leq)\) is a WQO. Let \(A_i = (i, i + 1), (i, i + 2), \ldots\). It is easy to see that \(A_1, A_2, \ldots\) is an infinite antichain.

8. If \((Q, \leq)\) is a well-order, then \((Q^{\omega}, \leq)\) is a WQO.

There are many ways to define a relation on graphs. Depending on the relation the set of all graphs might or might not be a WQO. Even if the set of all graphs is not a WQO under a particular notion of containment, we can recover well-quasi-ordering by restricting to a subclass. We consider the following notions of containment in graphs, starting with the strongest one:

1. Induced subgraph relation

2. Subgraph relation
3. Topological minor relation

4. Minor relation

1. The class of finite graphs is not a WQO under the induced subgraph relation. The set of circuits is an antichain. But the class of graphs not containing a path of length $k$ ($k$ fixed) as an induced subgraph is a WQO under the induced subgraph relation. The class of trees with at most one vertex of degree greater than 2 is a WQO under the induced subgraph relation. The class of trees with at most two vertices of degree greater than 2 is not a WQO under subgraph relation.

2. The class of finite graphs is not a WQO under the subgraph relation. The set of circuits is again an antichain. Although the class of split graphs is not a WQO under the induced subgraph relation, it is a WQO under the subgraph relation.

3. The class of finite graphs is not a WQO under the topological minor relation. The set of doubled circuits (each edge of a circuit replaced by a digon) is an infinite antichain. Although the class of trees is not a WQO under subgraph relation, it is a WQO under the topological minor relation. (Kruskal’s tree theorem [30])

4. The class of finite graphs is a WQO under the minor relation. (Robertson-Seymour theorem or Graph minor theorem [43])

5. The class of finite graphs is a WQO under the immersion relation. (conjectured by Nash-Williams and proved by Robertson and Seymour)

6. The class of graphic sequences is a WQO under Rao containment. (conjectured by S. B. Rao [41] and proved by Chudnovsky and Seymour [16]; Rao containment is defined later)
7. The class of binary matroids is a WQO under the minor relation. (Geelen, Gerards and Whittle)

Since the class of all graphs is not a WQO under topological minor relation, it is of great interest to know which topological ideals are well-quasi-ordered under the topological minor relation.

We give two short proofs of the bounded case of a conjecture of S. B. Rao. Then we prove a special case of the celebrated Graph Minor Theorem of Robertson and Seymour. More precisely, we prove the graph minor theorem for graphs with treewidth at most 2. Then we prove that signed graphs with bounded frustration number are well-quasi-ordered under the minor relation. Finally, we give an infinite antichain of series-parallel graphs with respect to topological inclusion.

2.1 Two short proofs of the bounded case of S. B. Rao’s degree sequence conjecture

S. B. Rao conjectured that graphic sequences are well-quasi-ordered under an inclusion based on induced subgraphs. This conjecture has now been settled completely by M. Chudnovsky and P. Seymour [16]. One part of the proof establishes the result for the bounded case, a result proved independently by C. J. Altomare [3]. We give two short proofs of the bounded case of S. B. Rao’s conjecture. This section is based on [47].

All our graphs are finite and simple, i.e., we allow neither loops nor multiple edges. An integer sequence \( D = (d_1, d_2, \ldots, d_n) \) with \( d_1 \geq d_2 \geq \ldots \geq d_n \geq 1 \) is called graphic if there exists a graph \( G \) whose degree sequence is \( D \). A graph \( G \) is said to be a realization of an integer sequence \( D \) if the degree sequence of \( G \) is \( D \).

**Definition 2.1.1.** Let \( D_1 \) and \( D_2 \) be graphic sequences. Then we write \( D_1 \leq D_2 \) to
mean there exist graphs $G_1$ and $G_2$ such that $G_i$ is a realization of $D_i$ for $i = 1, 2$, and $G_1$ is an induced subgraph of $G_2$.

The above order (also known as Rao containment) is obviously reflexive, and easily seen to be antisymmetric and transitive. We will prove Theorem 2.1.3, the bounded-degree case of S. B. Rao’s conjecture, in two different ways. The first proof is based on an idea of C. J. Altomare that, it is sometimes advantageous to use the regularity sequence (defined later) instead of the degree sequence of a graph. The second proof is based on an observation of N. Robertson that, by virtue of Higman’s finite sequences theorem, it suffices to prove that bounded graphic sequences can be realized by graphs with bounded component size. Both the proofs use the fact that if the number of entries in an integer sequence (with even sum) is much larger than its highest term, then it is necessarily graphic [65]. We will prove this using the Erdős-Gallai condition for an integer sequence to be graphic.

**Theorem 2.1.2** (Erdős-Gallai [24]). Let $D = (d_1, d_2, \ldots, d_n)$ be an integer sequence with $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ and $\sum_{i=1}^{n} d_i$ even. Then $D$ is graphic if and only if for every $k \in \{1, 2, \ldots, n\}$, the following holds:

$$
\sum_{i=1}^{k} d_i \leq k(k - 1) + \sum_{i=k+1}^{n} \min(d_i, k).
$$

**Theorem 2.1.3.** Let $N$ be a fixed positive integer. Let $D_1, D_2, \ldots$ be an infinite sequence of graphic sequences with no entry in any sequence exceeding $N$. Then there exist indices $i < j$ such that $D_i \leq D_j$.

We will need the following easy result, a special case of a theorem of I. E. Zverovich and V. E. Zverovich [65].

**Proposition 2.1.4.** Let $D = (d_1, d_2, \ldots, d_n)$ be an integer sequence with $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ and $\sum_{i=1}^{n} d_i$ even. If $n \geq (d_1)^2$, then $D$ is graphic.
Proof. If \( d_1 = 1 \), then clearly \( D \) is graphic. Hence we will assume that \( d_1 > 1 \).

Suppose \( k > d_1 \). Then

\[
\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} d_1 \\
\leq k(k - 1).
\]

Suppose \( 1 < k \leq d_1 \). Then

\[
\sum_{i=1}^{k} d_i \leq kd_1 \\
\leq (d_1)^2 \\
\leq n,
\]

and

\[
k(k - 1) + \sum_{i=k+1}^{n} \min(d_i, k) \geq k(k - 1) + n - k \\
= n + k(k - 2) \\
\geq n.
\]

Suppose \( k = 1 \). Then

\[
\sum_{i=1}^{k} d_i = d_1,
\]

and

\[
k(k - 1) + \sum_{i=k+1}^{n} \min(d_i, k) \geq n - 1 \\
\geq d_1^2 - 1 \\
\geq d_1.
\]

Hence the Erdős-Gallai condition is satisfied. By Theorem 2.1.2, \( D \) is graphic. ☐
For the first proof, we will need the notion of regularity sequence, first used by C. J. Altomare [3]. We associate a vector to every graphic sequence as follows: If $D$ is the graphic sequence in which $i$ occurs $a_i$ times, i.e., $D = (N^{a_N}, \ldots, 2^{a_2}, 1^{a_1})$, then the regularity sequence of $D$ is $(a_N, \ldots, a_2, a_1)$.

We will also need the following easy result. Let $k$ be a positive integer. Let $\mathbb{N}$ denote the set of non-negative integers. Consider the quasi-order $(\mathbb{N}^k, \leq_H)$, where $(a_1, \ldots, a_k) \leq_H (b_1, \ldots, b_k)$ if $a_i \leq b_i$ for all $1 \leq i \leq k$. Since a Cartesian product of WQOs is a WQO, we have the following.

**Proposition 2.1.5.** $(\mathbb{N}^k, \leq_H)$ is a WQO.

**First proof of Theorem 2.1.3.** We look at the corresponding sequence of regularity sequences $V_1, V_2, \ldots$. Note that $V_i \in \mathbb{N}^N$ for all $i$. By Proposition 2.1.5, we have an infinite ascending subsequence with respect to $\leq_H$. Restrict to that subsequence, whose elements we now denote $V_1, V_2, \ldots$. If the sum of the entries in the vectors $\{V_k\}$ is bounded, then there exist indices $i < j$ such that $V_i = V_j$, and hence $D_i \leq D_j$. If not, let $j$ be such that the sum of entries in $V_j$ is at least $N^2$ plus the sum of entries in $V_1$. Let $H$ be a graph realizing $D_1$ and let $K$ be a graph whose regularity sequence is $V_j - V_1$ (such a graph exists by Proposition 2.1.4). The disjoint union of $H$ and $K$ realizes $D_j$, and hence $D_1 \leq D_j$. □

N. Robertson asked whether bounded graphic sequences can be realized by graphs with bounded component size? The following proposition answers this question.

**Proposition 2.1.6.** If $D = (d_1, d_2, \ldots, d_n)$ is a graphic sequence with $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$, then there exists a graph $G$ with degree sequence $D$ none of whose components have more than $3(d_1)^2$ vertices.

**Proof.** Let $L = (d_1)^2$. Let $q$ and $r$ be integers such that $n = qL + r$ such that
0 \leq r < L. If q = 0, the result is obvious. If not, divide D into q integer sequences \( D_1, D_2, \ldots, D_q \) as follows: For \( i = 1, \ldots, q - 1 \), the \( i \)th integer sequence is \( (d_{(i-1)L+1}, \ldots, d_{iL}) \), and the \( q \)th integer sequence is \( (d_{(q-1)L+1}, \ldots, d_n) \). Arbitrarily pair and combine integer sequences in the collection \{D_1, D_2, \ldots, D_q\} that have odd sum to get integer sequences \( P_1, P_2, \ldots, P_k \), each of which has even sum and length between \( L \) and \( 3L \). By Proposition 2.1.4, the \( P_i \)'s are graphic; let \( G_i \) be a realization of \( P_i \). Then the disjoint union of the \( G_i \)'s is a realization of \( D \) with each component having at most \( 3L \) vertices. □

**Second proof of Theorem 2.1.3.** Let \( \mathcal{G} \) be the set of all (non-isomorphic) graphs with at most \( 3N^2 \) vertices and maximum degree at most \( N \). Clearly \( \mathcal{G} \) is a finite set. Since a finite quasi-order is a WQO, \( \mathcal{G} \) is a WQO under the isomorphism relation. Higman’s finite sequences theorem tells us that the set of graphs, each of whose components is in \( \mathcal{G} \) is also a WQO under the induced subgraph relation. By Proposition 2.1.6, each \( D_i \) can be realized by a graph \( G_i \) that is a disjoint union of graphs in \( \mathcal{G} \). Hence there exist indices \( i < j \) such that \( G_i \) is an induced subgraph of \( G_j \). This implies \( D_i \leq D_j \), and completes the second proof of Theorem 2.1.3. □

### 2.2 Well-quasi-ordering series-parallel graphs

We prove that subcubic series-parallel graphs are well-quasi-ordered by topological inclusion. The proof is by rooting the graphs, and using the minimal bad sequence argument. The same proof can be used, with minor changes, to prove that series-parallel graphs are well-quasi-ordered by minor relation.

A graph is said to be *series-parallel* if it can be obtained from the complete graph on 2 vertices \( (K_2) \) by a sequence of the following two operations: (1) subdividing an existing edge (2) adding an edge in parallel to an existing edge. A *rooted series-parallel*
(SP) graph is a triple \((S, a, b)\) where \(S\) is a series-parallel graph, \(a, b \in V(S)\), and \(S\) can be obtained from \(K_2\) with endvertices \(a\) and \(b\) by a sequence of subdividing edges and adding edges in parallel. A rooted SP graph is subcubic if every vertex has degree at most 3. For notational convenience we will not write the two roots explicitly. We write \(S \leq S'\) if \(S\) can be obtained from \(S'\) by suppressing vertices of degree two, by deleting edges and deleting vertices. We work up to isomorphism, but the isomorphism must respect the roots.

![Diagram](image)

Figure 2.1: Breaking into pieces depending on where the parallel addition operation is used first.

**Proposition 2.2.1.** The class of subcubic rooted SP graphs is a WQO under \(\leq\).
Proof. We use the classic minimal bad sequence argument developed by Nash-Williams. Suppose that the class of subcubic rooted SP graphs contains a bad sequence. Given \( n \in \mathbb{N} \), assume inductively that we have chosen a sequence \( S_0, \ldots, S_{n-1} \) of rooted subcubic SP graphs such that some bad sequence of rooted subcubic SP graphs starts with this sequence. Choose a rooted subcubic SP graph \( S_n \) with minimum number of edges such that some bad sequence starts with \( S_0, \ldots, S_n \). For each \( n \in \mathbb{N} \), denote the roots of \( S_n \) by \( a_n \) and \( b_n \). Note that \((S_n)_{n \in \mathbb{N}}\) is a bad sequence.

We break each \( S_i \) into 2, 3, or 4 rooted subcubic SP graphs as shown in Figure 2.1. (For each rooted SP graph, we fix a way to obtain it using the two operations. A rooted SP graph will look like one of the five diagrams on the left, but it is important to note that the two operations can be applied in each of the edges and that some of the edges shown could be paths.) In each of the five cases shown, since the graphs are subcubic, the rooted graphs on the right side are \( \leq \) the corresponding graph on the left side. Let \( A_i \) denote the set of rooted graphs obtained from \( S_i \) and let \( A := \bigcup A_i \). We show that \( A \) is a WQO under \( \leq \) and then we will be done by the fact that Cartesian product of WQOs is a WQO and the observation that the operations for topological minor can be done independently in each of the pieces shown in Figure 2.1.

Let \((S^n)_{n \in \mathbb{N}}\) be a sequence of elements from \( A \). For each \( k \in \mathbb{N} \), let \( n(k) \) be an integer such that \( S^k \in A_{n(k)} \). Choose a \( k \) such that \( n(k) \) is minimum. Then \( S_0, \ldots, S_{n(k)-1}, S^k, S^{k+1}, \ldots \) is a good sequence, by the minimal choice of \( S_{n(k)} \) and \( |E(S^k)| < |E(S_{n(k)})| \).

Let \( S, S' \) be a good pair of this sequence. Since \((S_n)_{n \in \mathbb{N}}\) is bad, \( S \) cannot be among the first \( n(k) \) members \( S_0, \ldots, S_{n(k)-1} \) of our sequence: then \( S' \) would be some \( S^i \) with \( i \geq k \), i.e. \( S \leq S' \leq S_{n(i)} \); since \( n(k) \leq n(i) \) by the choice of \( k \), this would make \((S, S_{n(i)})\) a good pair in the bad sequence \((S_n)_{n \in \mathbb{N}}\). Hence \((S, S')\) is a good pair also in \((S^k)_{k \in \mathbb{N}}\), completing the proof that \( A \) is well-quasi-ordered. \( \square \)
The above proposition clearly implies that subcubic SP graphs are well-quasi-ordered under topological containment. Note that we need to impose the condition that the graphs are subcubic because of the existence of the infinite antichain consisting of doubled circuits (which are subquartic). Suppose we are working with ordinary minors instead of topological minors. Then we don’t have to worry about our graphs being subcubic, because we can contract any edge. Thus, the above proof can be used, with minor changes, to prove the following:

**Proposition 2.2.2.** SP graphs are well-quasi-ordered under the minor relation.

Note that a graph has tree-width \( \leq 2 \) if and only if every block of it is a series-parallel graph.

### 2.3 Well-quasi-ordering signed graphs

We prove that signed graphs that can be balanced by deleting a bounded number of vertices are well-quasi-ordered under minor inclusion.

A signed graph is said to be *balanced* if all its circles are positive. Balanced signed graphs tend to behave like graphs. An important invariant of a signed graph is its frustration number. The *frustration number* of a signed graph is defined to be the minimum number of vertices to be deleted so that the resulting signed graph is balanced.

**Theorem 2.3.1.** Let \( k \in \mathbb{N} \). The set of signed graphs with frustration number at most \( k \) is a well-quasi-order under the minor relation.

**Proof.** Let \( \Sigma_1, \Sigma_2, \ldots \) be a sequence of signed graphs, each with frustration number at most \( k \). We may assume that each \( \Sigma_i \) has a set \( X_i \) of vertices with \( |X_i| = k \) such that \( \Sigma_i \setminus X_i \) is balanced. We may also assume that all \( X_i = \{v_1, \ldots, v_k\} \) and that
$\Sigma_i : X_i$ are all isomorphic. Now label each vertex $v$ of $\Sigma_i \setminus X_i$ by a vector of length $2k$ as follows: The $(2i - 1)$-coordinate is the number of positive edges joining $v$ and $v_i$ and the $(2i)$-coordinate is the number of negative edges joining $v$ and $v_i$. By the graph minor theorem we know that there exist $i < j$ such that $\Sigma_i \setminus X_i$ is a minor of $\Sigma_j \setminus X_j$, where the minor respects labels. This clearly implies that $\Sigma_i$ is a minor of $\Sigma_j$, proving the theorem. $\square$

**Conjecture 2.3.2.** Let $k \in \mathbb{N}$. The set of signed graphs with no $k$ vertex-disjoint negative circles is a well-quasi-order under minor relation.

A signed graph is said to be *tangled* if it is unbalanced, does not contain a balancing vertex, and does not contain two vertex-disjoint negative circles. Can a tangled signed graph be balanced by deleting at most a constant number of vertices? By considering the one-sided projective grids (or the Escher walls of Bruce Reed) with every edge negative, it is easy to see that the number of vertices to be removed from a tangled signed graph to make it balanced is not bounded by a constant.

### 2.4 An infinite antichain of series-parallel graphs

In this section, we give an infinite antichain of series-parallel with respect to the topological minor relation.

Let $G_i$ be the graph obtained from a doubled path of length $i$ by adding four pendant edges, two at each end. Then \{ $G_i$ \} is an infinite antichain.
CHAPTER 3
A SIGNED-GRAPHIC ANALOGUE OF MATTHEWS’ THEOREM

A theorem of Matthews [36] gives a characterization of graphs whose bicircular matroids are graphic. We give a characterization of graphs whose bicircular matroids are signed-graphic. This chapter is based on [48]. All graphs mentioned here are assumed to be finite.

3.1 Terminology and notation

Standard reference for matroid theory is Oxley’s treatise [39]. $U_{k,n}$ denotes the uniform matroid with $n$ elements and rank $k$. $K_n$ denotes the complete graph on $n$ vertices. A subdivision of a graph $G$ is obtained by replacing some edges of $G$ by paths, where the internal vertices of the paths are disjoint from the vertices of $G$. Alternatively, a subdivision of $G$ is a graph obtained by repeatedly applying the operation of inserting a vertex of degree two in an edge. Two graphs $G$ and $H$ are said to be homeomorphic if there exists a graph $K$ such that both $G$ and $H$ are subdivisions of $K$. A matroid is said to be binary if it is the vector matroid of a matrix with entries in $GF(2)$. A matroid is said to be ternary if it is the vector matroid of a matrix with entries in $GF(3)$. A matroid is graphic if it is the cycle matroid of a graph. Definitions not found here can be found in Oxley [39].
Recall that a signed graph $\Sigma$ is a pair $(G, \sigma)$, where $G$ is a graph and $\sigma : E(G) \rightarrow \{-1, 1\}$ is a function.

### 3.2 Statement of Matthews’ theorem

Let $G$ be graph. The *bicircular matroid* of $G$, denoted $B(G)$, is the matroid on the edge set of $G$, where a set of edges in $G$ is independent if the graph spanned by it has at most one cycle in each component.

**Theorem 3.2.1** (Matthews, [36]). Let $G$ be a graph. Then the following conditions are equivalent:

1. $B(G)$ is graphic.
2. $B(G)$ is binary.
3. $B(G)$ is regular (that is, representable over every field).
4. Each component of $G$ can be obtained, by (repeated) addition of pendant edges, from either a theta graph or a graph homeomorphic to a tree with loops at some vertices.
5. $G$ has no subgraph homeomorphic to any of the graphs shown in Figure 1 (where a graph with a dotted edge represents either the graph itself or the graph obtained when the dotted edge is contracted).

### 3.3 Statement of the main theorem

Let $k$ be a positive integer. $k$-skein is the graph with two vertices and $k$ edges, none of which is a loop. A $k$-theta graph is a subdivision of the $k$-skein. A subdivision of
the graph with two vertices and four edges, one of which is a loop is called a theta-handcuff. We denote the graphs in Figure 3.2 (from left to right) by \( G_1, G_2, G_3, G_4, G_5 \) and \( G_6 \). We are now ready to state the signed-graphic version of Matthews’ theorem.

**Theorem 3.3.1.** Let \( G \) be a graph. Then the following conditions are equivalent:

1. \( B(G) \) is signed-graphic.

2. \( B(G) \) is ternary.

3. \( B(G) \) is near-regular (that is, representable over every field with at least three elements).

4. Each component of \( G \) can be obtained, by (repeated) addition of pendant edges, from a subdivision of a tree with loops at some vertices and some edges doubled (an edge can be tripled (quadrupled) if one (both) of its endpoints is (are) pendant and loopless).

5. \( G \) does not contain a subgraph that is a subdivision of any of the graphs shown in Figure 3.2 (where a graph with a dotted edge represents either the graph itself or the graph obtained when the dotted edge is contracted).
3.4 Bicircular matroids and their properties

Bicircular matroids were introduced by Simões-Pereira [45], and studied in detail by Matthews [36].
The circuits of the bicircular matroid of a graph are of three types: theta graphs, tight handcuff, and loose handcuff, because these are the only connected subgraphs which contain more than one cycle, but the deletion of any edge destroys all but one cycle.

Bicircular matroids are transversal, and hence inherit all nice properties of transversal matroids. In particular, they are representable over all sufficiently large fields and over all infinite fields (Ingleton and Piff [29]). They are base-orderable, and can be written as a union of rank-1 matroids. But their behavior is much better than that of transversal matroids: while the former is minor-closed, the latter is not. In fact, a recent project of DeVos et al. [19] is aimed at getting a complete list of excluded minors for the class of bicircular matroids.

3.5 Signed-graphic matroids and their properties

Signed-graphic matroids were introduced by Zaslavsky [60], and studied in detail by Zaslavsky and Slilaty. A list of work done on topics related to signed graphs has been meticulously collected and maintained by Zaslavsky [63].

Figure 3.2: Forbidden graphs in the characterization of graphs whose bicircular matroids are signed-graphic.
The frame matroid of a signed graph Σ, denoted \( M(Σ) \), is the matroid on the edge set of Σ, where a set of edges in \( G \) is independent if the signed graph spanned by it has no positive circle and at most one negative circle in each component. The circuits of the matroid are of three types: positive circle, tight handcuff with both circles negative, and loose handcuff with both circles negative.

The class of signed-graphic matroids properly contains the class of graphic matroids. Signed-graphic matroids need not be binary because the four-point line is signed-graphic. If an element of a signed-graphic matroid is deleted or contracted, the resulting matroid is still signed-graphic, and hence the class of signed-graphic matroids is closed under taking minors.

Signed-graphic matroids are dyadic, i.e., representable over all fields whose characteristic is not 2 (Dowling-Zaslavsky [60]). The complete list of excluded minors for the class of signed-graphic matroids is not known, although all such regular matroids are known (Qin, Slilaty, Zhou [40]).

### 3.6 Proof of the main theorem

**Proof of Theorem 3.3.1.** We will prove the following chain of five implications:

\[ (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5) \Rightarrow (4) \Rightarrow (1). \]

- \( (1) \Rightarrow (2) \) It is well known that signed-graphic matroids are ternary (Dowling-Zaslavsky, [60]). A suitably chosen incidence matrix is a ternary representation for the frame matroid.

- \( (2) \Rightarrow (3) \) Bicircular matroids are representable over all sufficiently large finite fields and over all infinite fields (in fact, this is true for the bigger class of transversal matroids) (Ingleton and Piff, [29]). In particular, a bicircular matroid is representable over \( \mathbb{Q} \) and a field of characteristic 2. A theorem of
Whittle [57] says that if a matroid $M$ is representable over $GF(3), \mathbb{Q}$ and over a field of characteristic two, then $M$ is near-regular. Suppose $B(G)$ is ternary. We know that $B(G)$ is representable over $\mathbb{Q}$ and over a field of characteristic 2. Invoking Whittle’s theorem, we see that $B(G)$ is near-regular.

- $(3) \Rightarrow (5)$ The following statements are easy to check:
  
  - $B(G_1) \cong U_{4,6}$.
  - $U_{3,5}$ is a minor of $B(G_2)$ and $B(G_3)$.
  - $U_{2,5}$ is a minor of $B(G_4)$ and $B(G_5)$.
  - $B(G_6) \cong U_{2,5}$.

Suppose $B(G)$ is near-regular. In particular, $B(G)$ is ternary. But none of $U_{4,6}, U_{2,5}, U_{3,5}$ is ternary; in fact $U_{2,5}$ and $U_{3,5}$ are excluded minors for the class of ternary matroids. This together with the fact that the class of bicircular matroids is closed under taking minors shows that $G$ has no subgraph homeomorphic to any of the graphs $G_i$.

- $(5) \Rightarrow (4)$

We need the following well-known characterization of graphs none of whose subgraphs is a subdivision of $K_4$.

**Lemma 3.6.1.** Let $G$ be a graph. The following statements are equivalent:

- No subgraph of $G$ is a subdivision of $K_4$.
- $G$ can be iteratively constructed from $K_1$ by the following operations:
  
  * Adding a loop.
  * Adding a pendant edge.
  * Adding an edge in parallel to an existing edge.
Subdividing an edge.

A graph none of whose subgraphs is a subdivision of $K_4$ is called a series-parallel graph. A proof of the above lemma can be easily obtained from the following easy result, whose proof can be found in many standard graph theory texts:

**Proposition 3.6.2.** Let $G$ be a (non-null) simple graph with minimum degree at least 3. Then $G$ has a subgraph that is a subdivision of $K_4$.

We may assume that $G$ is connected. Since $G$ contains no subdivision of $K_4$ as a subgraph, by Lemma 3.6.1, it can be obtained from $K_1$ by a sequence of four operations mentioned above.

**Lemma 3.6.3.** Let $G$ be obtained by adding an edge $f$ in parallel to an edge $e$ in a 3-theta graph $H$. Then $G$ is either a subdivision of $G_2$ or a 4-theta graph.

**Proof.** Let $u, v$ be the trivalent vertices of $H$. If the endpoints of $f$ are different from $u, v$, then $G$ is a subdivision of $G_2$ where the dotted edge is an edge. If $f$ has exactly one endpoint in $u, v$, then $G$ is a subdivision of $G_2$ where the dotted edge is contracted. If $u, v$ are the endpoints of $f$, then $G$ is a 4-theta graph. □

**Lemma 3.6.4.** Let $G$ be a series-parallel graph with minimum degree at least 3. Suppose that no subgraph of $G$ is a subdivision of either $G_2$ or $G_3$. Let $B$ be a block in which the base edge is duplicated exactly once. Then $B$ is isomorphic to the 2-skein.

**Proof.** Let the endpoints of the base edge be $u$ and $v$. Suppose that the base edge has an internal vertex $w$. Up to symmetry, there are three possibilities: (a) Two edges with endpoints $u$ and $w$, and two edges with endpoints $v$ and $w$. Together with the duplicated edge $f$, we have $G_2$. (b) Two edges with endpoints
$u$ and $w$, and two edges with endpoints $v$ and $x$ and an edge with endpoints $w$ and $x$. Together with the duplicated edge $f$, we have $G_2$. (c) There is another block containing $v$. Hence there is an endblock containing a cycle and we have a subdivision of $G_3$. □

**Lemma 3.6.5.** Let $G$ be a series-parallel graph with minimum degree at least 3. Suppose that no subgraph of $G$ is a subdivision of either $G_2$ or $G_3$. Then every block of $G$ is either a loop or a $k$-skein for some positive integer $k$.

*Proof.* Let $B$ be a block of $G$. We may assume that $B$ is not a loop. If $G$ is a $k$-theta graph, then $G$ is isomorphic to $k$-skein. By Lemma 3.6.3, we know that if the base edge is duplicated at least twice, it has to be a theta-graph. If the base edge is duplicated only once, by Lemma 3.6.4, $B$ is isomorphic to the 2-skein. □

**Lemma 3.6.6.** Let $G$ be a series-parallel graph with minimum degree at least 3. Suppose that no subgraph of $G$ is a subdivision of $G_4$. If $B$ is a block of $G$ isomorphic to the 3-skein, then $B$ is an endblock.

Armed with the above two lemmas, we are now ready to prove the implication $(5) \Rightarrow (4)$. Suppose no subgraph of $G$ is a subdivision of any of $G_i$, $1 \leq i \leq 6$. Let $G'$ be obtained from $G$ by a sequence of operations, the operations being deleting pendant vertices and contracting edges in series. It is routine to check that $G'$ is well-defined.

It is important to note that no subgraph of $G'$ is a subdivision of any of $G_i$, $1 \leq i \leq 6$. In particular, $G'$ is series-parallel and none of its subgraph is a subdivision of $G_2$ or $G_3$. By Lemma 3.6.5, each block of $G'$ is either a loop or a $k$-skein. Note that $G_6$ is the 5-skein, hence each non-loop block of $G'$ is
a $k$-skein for $k = 1, 2, 3$, or 4. Suppose $G'$ contains a block that is isomorphic to the 4-skein. Since $G'$ does not contain a subdivision of $G_3$, we can easily see that $G'$ is isomorphic to the 4-skein. Now, suppose that no block of $G'$ is isomorphic to the 4-skein. Let $G'$ have a block $B$ that is isomorphic to the 3-skein. Then, by Lemma 3.6.6, $B$ is an endblock of $G'$. Hence $G'$ is isomorphic either to the 4-skein or to a tree with some edges doubled and loops added, and tripling of edges allowed only when one of its endpoints is a leaf of the tree and has no loops on it. By reversing the operations done to get $G'$ from $G$, we see that $G$ can be obtained, by (repeated) addition of pendant edges, from a subdivision of a tree with loops at some vertices and some edges doubled (an edge can be tripled (quadrupled) if one (both) of its endpoints is (are) pendant and loopless);

- $(4) \Rightarrow (1)$: Since the class of signed-graphic matroids is closed under taking direct sums, it suffices to prove the result for connected graphs. Henceforth we will assume that $G$ is connected. Note that the bicircular matroid of the 3-skein is isomorphic to the frame matroid of a positive triangle, and the bicircular matroid of the 4-skein or the 3-skein with a loop is isomorphic to the frame matroid of the signed expansion of $K_2$ (four-point line).

Suppose $G$ is a graph obtained from a tree by doubling some edges and adding loops. We will construct a signed graph $\Sigma$ such that $B(G) = M(\Sigma)$, proving that $B(G)$ is signed-graphic. Let the underlying graph of $\Sigma$ be $G$. Also, make all loops and exactly one edge of each digon negative. Note that a cycle in $G$ corresponds to a negative circle in $\Sigma$. Also, there are no positive circles in $\Sigma$. Hence $B(G) = M(\Sigma)$.

Suppose $G$ is a graph obtained from a tree by doubling some edges and adding
loops, and some pendant edges tripled when the corresponding pendant vertex has no loops. Then we can construct a graph $G'$ by replacing an edge in each triple (three edges in parallel) by a loop at the corresponding pendant endpoint. It is easy to check that $B(G) = B(G')$. Now we repeat the procedure above to get a signed graph $\Sigma$ with $B(G') = M(\Sigma)$. This implies $B(G) = M(\Sigma)$ and hence $B(G)$ is signed-graphic.

We conclude, by using the following two easy lemmas (whose trivial proofs we omit) to show, that if $G$ is of the form mentioned in (4), then $B(G)$ is signed-graphic.

**Lemma 3.6.7.** Let $H$ be obtained from a graph $G$ by subdividing an edge. If $B(G)$ is signed-graphic, so is $B(H)$.

**Lemma 3.6.8.** Let $H$ be obtained from a graph $G$ by adding a pendant edge. If $B(G)$ is signed-graphic, so is $B(H)$.

This concludes all the five implications, and hence concludes the proof of the theorem.

\[ \square \]

### 3.7 Questions and concluding remarks

Also, using the above theorem, we are able to answer the following question (see Chapter 7):

**Question 3.7.1.** Characterize graphs whose bicircular matroids are even-cycle matroids.

We conclude with some open-ended questions:

**Question 3.7.2.** Characterize graphs whose bicircular matroids are cosigned-graphic.
Question 3.7.3. Characterize signed graphs whose frame matroids are bicircular.

Question 3.7.4. Characterize signed graphs whose frame matroids are transversal.

Question 3.7.5. Characterize set systems whose transversal matroids are signed-graphic.

Question 3.7.6. When is the dual of a signed-graphic matroid signed-graphic?

Question 3.7.7 (Dominic Welsh). When is the dual of a transversal matroid transversal?

Question 3.7.8. It is known that we cannot determine in polynomial time whether a matroid, given in terms of independence oracle, is signed-graphic (Geelen-Mayhew). Can we determine in polynomial time whether a matroid, given in terms of independence oracle, is bicircular? Can we determine in polynomial time whether a matroid, given in terms of independence oracle, is signed-graphic and bicircular?

A graph can be thought of as a signed graph with all edges positive. Since a graph with all edges positive has no negative circles, the frame matroid of an all-positive graph is equal to the cycle matroid of the graph, and hence graphic matroids are signed-graphic. Using the fact that $U_{2,4}$ is not graphic, Matthews’ theorem is an immediate corollary of the main result of this chapter.
CHAPTER 4
SEVEN SIGNINGS OF THE HEAWOOD GRAPH

The Heawood graph (shown in Figure 4.1) has many incarnations, most notable being the Levi point-line incidence graph of the Fano plane. It is a highly symmetric graph and illustrates several aspects of signed graph theory. In this chapter, we study the different ways in which signs can be assigned to the edges of the Heawood graph and analyze the resulting signed graphs. Along the way, we will prove and use several interesting properties of the Heawood graph. We also prove a conjecture of Zaslavsky on frustration in signed subcubic graphs. This chapter is based on [49].

Figure 4.1: The Heawood graph
Figure 4.2: The seven switching isomorphism types of signed Heawood graphs.
4.1 Introduction

Signed graphs appear often in mathematics, computer science, sociology, psychology and various other fields ([5], [14], [17]). It is an extremely useful device to model interaction in social groups. It is a very good source of matroids ([50], [60]). Recently, Zaslavsky made a detailed study of signed Petersen graphs [63]. There he discusses several important notions: balance, frustration, coloring, automorphism groups and clusterability. In this chapter, we deal with signed cubic graphs with particular emphasis on the Heawood graph. We write $[14]$ to denote the set $\{1, 2, \ldots, 14\}$.

Definition 4.1.1. The Heawood graph $H$ is defined as $H = (V, E)$, where $V = \{v_i : i \in [14]\}$, and $E = \{v_i v_{i+1} : i \in [14]\} \cup \{v_i v_{i+5} : i \in [14] \text{ and } i \text{ odd}\}$, with addition carried out in modulo 14.

The Heawood graph is the unique $(3, 6)$-cage, meaning there is no other graph (up to isomorphism) $G$ having number of vertices less than or equal to that of the Heawood graph such that $G$ is both cubic and has girth 6. It is also the point-line incidence graph of the Fano plane. A fundamental difference between the Heawood graph and the Petersen graph is that the former is both Hamiltonian and bipartite, while the latter has neither of those properties.

The two main results of this chapter are stated below. The first one is about the number of signed Heawood graphs, up to switching isomorphism.

Theorem 4.1.2. There are exactly seven signed Heawood graphs up to switching isomorphism. They are $+H$, $H_1$, $H_{2,1}$, $H_{2,2}$, $H_{3,1}$, $H_{3,2}$ and $H_4$.

The seven signed graphs are shown in Figure 4.2. (Black lines represent positive edges; red lines represent negative edges.) The second result is a proof of a conjecture of Zaslavasky [63]. For a signed graph $\Sigma$, $l_0(\Sigma)$ denotes the minimum number of vertices
to be deleted to restore balance, and $l(\Sigma)$ denotes the minimum number of edges to be deleted to restore balance.

**Theorem 4.1.3.** For every signed subcubic graph $\Sigma$, $l_0(\Sigma) = l(\Sigma)$.

Neil Robertson suggested the problem of looking at the signings of the Heawood graph. Thomas Zaslavsky (private communication) has an independent proof of Theorem 4.1.2.

### 4.2 Structure of the Heawood graph

We count the number of cycles in the Heawood graph. Some of the counts will be used later to count the number of negative circles of a particular length in signed Heawood graphs that will be used to prove that two signed graphs are indeed not equivalent under switching and automorphism. Since the Heawood graph is bipartite, it has no odd cycles.

- $H$ has 14 vertices, 21 edges, independence number 7, chromatic index 3, and is bipartite and toroidal. Its dual in the torus is the complete graph on 7 vertices. It is both vertex- and edge-transitive.

- $H$ has 28 hexagons.

**Proof.** The number of hexagons containing one chord is 7. The number of hexagons containing two chords is 14. The number of hexagons containing three chords is 7. The total number of hexagons is 28. □

- $H$ has 21 octagons.

**Proof.** Let $C$ be an octagon in $H$. Then $H - C$ is a tree on 6 vertices. There are three possibilities. But only one can extend to a Heawood graph (we are
using the fact that the girth is 6 and the graph is bipartite). This graph is just an edge with the neighbors of its endpoints. Hence we can establish a bijection between the set of octagons and the set of edges of $H$. □

- $H$ has 84 decagons.

Proof. One can establish a bijection between the set of sets of four vertices that induce a path and the set of decagons. Also, the number of sets of four vertices inducing a path is $\frac{14 \times 3 \times 2 \times 2}{2} = 84$. Hence the number of 10-cycles in the Heawood graph is 84. □

- $H$ has 56 dodecagons.

Proof. For each pair of non-adjacent vertices $x, y$ ($x$ and $y$ in different color classes of $H$), $H - \{x, y\}$ has exactly 2 dodecagons (we are using the fact that $H$ is edge-transitive). And, for each pair of adjacent vertices $\{x, y\}$, or $x$ and $y$ in the same color class of $H$, $H - \{x, y\}$ has no dodecagon. Hence the number of dodecagons is equal to twice the number of non-adjacent pairs of vertices in $H$ with the two vertices in different color classes of $H$, which is precisely 56. □

- $H$ has 24 perfect matchings and 24 Hamilton cycles.

Proof. We can easily count the number of perfect matchings in which each edge participates. It is 8. Hence there are $\frac{8 \times 21}{7} = 24$ perfect matchings. There is a natural bijection between the set of perfect matchings in $H$ and the set of Hamilton cycles in $H$, proving that the number of Hamilton cycles in $H$ is 24 as well. □

It is interesting to note that, if we define the disjointness relation on the set of 28 hexagons, we get the remarkable Coxeter graph [7].
4.3 Signings

How many signings are there of the Heawood graph? In other words, how many different ways can we sign the edges of the Heawood graph, where two signings are different if they are not switching isomorphic. The obvious ones are the all-positive and all-negative Heawood graphs. But, since $H$ is bipartite, the two are switching equivalent. It is of interest to find out how many there are. We claim that there are exactly 7 of them. We will use the following lemma.

**Lemma 4.3.1.** Let $\Sigma = (H, \sigma)$ be a signed Heawood graph. Then $\Sigma$ contains a positive Hamilton circle.

**Proof.** Let $M_1 = \{v_i v_{i+1} : i \in [14] \text{ and } i \text{ odd}\}$. Let $M_2 = \{v_i v_{i+1} : i \in [14] \text{ and } i \text{ even}\}$. Let $M_3 = \{v_i v_{i+5} : i \in [14] \text{ and } i \text{ odd}\}$. By the pigeon-hole principle, the product of the signs of the edges in two of the $M_i$’s is the same. The union of those two $M_i$’s is a positive Hamilton circle. \[ \square \]

The proof of Proposition 4.3.1 actually tells us more: it shows that the number of positive Hamilton circles in a signed Heawood graph is even and at least 8. Note that the above proof shows that any graph that has three perfect matchings, the union of any two being a Hamilton cycle, has a positive Hamilton circle in any signing. But it is not clear whether a characterization of such graphs exists or not.

**Proof of Theorem 4.1.2.** By switching and Lemma 4.3.1, we may assume that we have an all-positive Hamilton circle. We have to figure out how many different ways there are to put signs on the 7 chords. Note that we are looking only for signed Heawood graphs up to switching isomorphism. So, if we have a signed graph that is switching isomorphic to some signed graph in the list that we have already constructed, we just ignore it. The symmetry of the Heawood graph is of great help. For odd $i$, $1 \leq i \leq 14$, let us denote the edge $v_i v_{i+5}$ by $e_i$. 

35
• No chord is negative. This gives rise to the all-positive Heawood graph ($\Sigma_1$)

• Exactly one chord is negative. Negative edge $e_1$ ($\Sigma_2$)

• Exactly two chords are negative. There are 3 ways to do this. Negative edges $e_1, e_3$ ($\Sigma_3$) Negative edges $e_1, e_5$ ($\Sigma_4$) Negative edges $e_1, e_7$ ($\Sigma_5$)

• Exactly three chords are negative. There are 5 ways to do this. Negative edges $e_1, e_5, e_9$ ($\Sigma_6$) Negative edges $e_1, e_3, e_5$ ($\Sigma_7$) Negative edges $e_1, e_3, e_7$ ($\Sigma_8$) Negative edges $e_1, e_3, e_9$ ($\Sigma_9$) Negative edges $e_1, e_3, e_{11}$ ($\Sigma_{10}$)

• Exactly four chords are negative. There are 5 ways to do this. Negative edges $e_3, e_7, e_{11}, e_{13}$ ($\Sigma_{11}$) Negative edges $e_5, e_7, e_{11}, e_{13}$ ($\Sigma_{12}$) Negative edges $e_3, e_9, e_{11}, e_{13}$ ($\Sigma_{13}$) Negative edges $e_5, e_9, e_{11}, e_{13}$ ($\Sigma_{14}$) Negative edges $e_7, e_9, e_{11}, e_{13}$ ($\Sigma_{15}$)

• Exactly five chords are negative. There are 3 ways to do this. Negative edges $e_5, e_7, e_9, e_{11}, e_{13}$ ($\Sigma_{16}$) Negative edges $e_3, e_7, e_9, e_{11}, e_{13}$ ($\Sigma_{17}$) Negative edges $e_3, e_5, e_9, e_{11}, e_{13}$ ($\Sigma_{18}$)

• Exactly six chords are negative. 1 way to do this. Negative edges $e_3, e_5, e_7, e_9, e_{11}, e_{13}$ ($\Sigma_{19}$)

• All chords are negative. Negative edges $e_1, e_3, e_5, e_7, e_9, e_{11}, e_{13}$ ($\Sigma_{20}$)

Hence we get 20 signed Heawood graphs. We still need to eliminate duplicates (switching isomorphic). This is what we do next.

We use the cycle notation to write a permutation. For example, $(v_1 v_2 v_3)$ refers to the map taking $v_1$ to $v_2$, $v_2$ to $v_3$ and $v_3$ to $v_1$. Also, we write $\Sigma_1 \sim \Sigma_2$ to denote that $\Sigma_1$ is switching isomorphic to $\Sigma_2$. 
1. $\Sigma_5 \sim \Sigma_4$:

The map $(v_2v_{14})(v_3v_9v_{11}v_{13})(v_4v_8v_{10}v_{12})(v_5v_7)$ from $V(\Sigma_4)$ to $V(\Sigma_5)$ is an automorphism.

2. $\Sigma_6 \sim \Sigma_7$:

The map $(v_1v_{13}v_7v_9v_{11}v_3v_5)(v_2v_4v_6v_{14}v_{12}v_8v_{10})$ from $V(\Sigma_7)$ to $V(\Sigma'_6)$ is an automorphism, where $\Sigma'_6$ is obtained from $\Sigma_6$ by switching at $\{v_1, v_2\}$.

3. $\Sigma_8 \sim \Sigma_3$:

The map $(v_1v_6v_5v_4v_3v_9v_{14})(v_2v_7v_{10}v_{13})(v_{11}v_{12})$ from $V(\Sigma_3)$ to $V(\Sigma'_6)$ is an automorphism, where $\Sigma'_8$ is obtained from $\Sigma_8$ by switching at $\{v_6, v_7, v_8\}$.

4. $\Sigma_{10} \sim \Sigma_5$:

The map $(v_1v_{14}v_{13}v_4v_5v_6)(v_2v_9v_{12}v_3v_{10}v_7)(v_8v_{11})$ from $V(\Sigma_5)$ to $V(\Sigma_{10}')$ is an automorphism, where $\Sigma'_{10}$ is obtained from $\Sigma_{10}$ by switching at $\{v_1, v_2, v_3\}$.

5. $\Sigma_{12} \sim \Sigma_4$:

The map $(v_1v_{10}v_{13}v_2v_5v_{14}v_{11}v_4)(v_3v_6v_9v_{12})(v_7v_8)$ from $V(\Sigma_4)$ to $V(\Sigma_{12}')$ is an automorphism, where $\Sigma'_{12}$ is obtained from $\Sigma_{12}$ by switching at $\{v_{10}, v_{11}, v_{12}, v_{13}\}$.

6. $\Sigma_{13} \sim \Sigma_7$:

The map $(v_1v_3)(v_4v_{14}v_8v_6)(v_5v_{13}v_9v_7)(v_{10}v_{12})$ from $V(\Sigma_7)$ to $V(\Sigma'_{13})$ is an automorphism, where $\Sigma'_{13}$ is obtained from $\Sigma_{13}$ by switching at $\{v_1, v_2, v_3, v_{13}, v_{14}\}$.

7. $\Sigma_{14} \sim \Sigma_7$:

The map $(v_2v_5v_{14})(v_3v_7v_{13})(v_4v_8v_{12})(v_5v_9v_{11})$ from $V(\Sigma_7)$ to $V(\Sigma'_{14})$ is an automorphism, where $\Sigma'_{14}$ is obtained from $\Sigma_{14}$ by switching at $\{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$.

8. $\Sigma_{15} \sim \Sigma_3$:
The map \((v_1v_{14}v_{13}v_4v_5v_6)(v_2v_9v_{12}v_3v_{10}v_7)(v_8v_{11})\) from \(V(\Sigma_3)\) to \(V(\Sigma_{15})\) is an automorphism, where \(\Sigma_{15}'\) is obtained from \(\Sigma_{15}\) by switching at \(\{v_{11}, v_{12}, v_{13}, v_{14}\}\).

9. \(\Sigma_{16} \sim \Sigma_4:\)

The map \((v_1v_{10}v_3v_{12})(v_2v_{11})(v_4v_7v_{14}v_5v_8v_{13}v_6v_9)\) from \(V(\Sigma_5)\) to \(V(\Sigma_{16})\) is an automorphism, where \(\Sigma_{16}'\) is obtained from \(\Sigma_{16}\) by switching at \(\{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}\).

Since \(\Sigma_5 \sim \Sigma_4\), we get \(\Sigma_{16} \sim \Sigma_4\).

10. \(\Sigma_{17} \sim \Sigma_5:\)

The map \((v_1v_{14}v_{13}v_4v_5v_6)(v_2v_9v_{12}v_3v_{10}v_7)(v_8v_{11})\) from \(V(\Sigma_{10})\) to \(V(\Sigma_{17}')\) is an automorphism, where \(\Sigma_{17}'\) is obtained from \(\Sigma_{17}\) by switching at \(\{v_{11}, v_{12}, v_{13}, v_{14}\}\).

Since \(\Sigma_{10} \sim \Sigma_5\), we get \(\Sigma_{17} \sim \Sigma_5\).

11. \(\Sigma_{18} \sim \Sigma_5:\)

The map \((v_1v_{13})(v_2v_{12})(v_3v_7)(v_4v_6)\) from \(V(\Sigma_{10})\) to \(V(\Sigma_{18}')\) is an automorphism, where \(\Sigma_{18}'\) is obtained from \(\Sigma_{18}\) by switching at \(\{v_8, v_9, v_{10}, v_{11}\}\). Since \(\Sigma_{10} \sim \Sigma_5\), we get \(\Sigma_{18} \sim \Sigma_5\).

12. \(\Sigma_{19} \sim \Sigma_3:\)

The map \((v_5v_{13})(v_6v_{14})(v_7v_9)(v_{10}v_{12})\) from \(V(\Sigma_8)\) to \(V(\Sigma_{19}')\) is an automorphism, where \(\Sigma_{19}'\) is obtained from \(\Sigma_{19}\) by switching at \(\{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}\). Since \(\Sigma_8 \sim \Sigma_3\), we get \(\Sigma_{19} \sim \Sigma_3\).

13. \(\Sigma_{20} \sim \Sigma_9:\)

The map \((v_1v_9v_3)(v_2v_{14}v_8)(v_4v_6v_{10})(v_7v_{11}v_{13})\) from \(V(\Sigma_9)\) to \(V(\Sigma_{20}')\) is an automorphism, where \(\Sigma_{20}'\) is obtained from \(\Sigma_{20}\) by switching at \(\{v_1, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}\).
Our list of 20 has reduced to 7. We list the remaining seven signed Heawood graphs, together with the corresponding frustration index and number of negative hexagons.

- \( +H \): All positive Heawood graph (Frustration index 0, number of negative hexagons 0)
- \( H_1 \): Heawood graph with edge \( e_1 \) negative (Frustration index 1, number of negative hexagons 8)
- \( H_{2,2} \): Heawood graph with edges \( \{e_1, e_3\} \) negative (Frustration index 2, number of negative hexagons 14)
- \( H_{2,1} \): Heawood graph with edges \( \{e_1, e_5\} \) negative (Frustration index 2, number of negative hexagons 12)
- \( H_{3,2} \): Heawood graph with edges \( \{e_1, e_3, e_5\} \) negative (Frustration index 3, number of negative hexagons 16)
- \( H_{3,1} \): Heawood graph with edges \( \{e_1, e_3, e_9\} \) negative (Frustration index 3, number of negative hexagons 14)
- \( H_4 \): Heawood graph with edges \( \{e_3, e_7, e_{11}, e_{13}\} \) negative (Frustration index 3, number of negative hexagons 18)

We conclude, by looking at the frustration index and the number of negative hexagons, that the seven signed graphs above (see Figure 4.2) are pairwise non-switching-isomorphic. This concludes the proof of Theorem 4.1.2. □


## 4.4 Frustration

In this section, we prove two results. The first result is a proof of a conjecture of Zaslavsky, and the second one is an upper bound for the frustration index of a signed graph whose underlying graph is simple, triangle-free and cubic.

A signed graph may become balanced when some of its vertices or edges are deleted. This is captured by two parameters: frustration number and frustration index. Let $\Sigma$ be a signed graph. The **frustration index** of $\Sigma$, denoted $l(\Sigma)$, is the smallest number of edges whose deletion from $\Sigma$ leaves a balanced signed graph. The **frustration number** of $\Sigma$, denoted $l_0(\Sigma)$, is the smallest number of vertices whose deletion from $\Sigma$ leaves a balanced signed graph. We prove the following conjecture of Zaslavsky [63, Conjecture 7.1] that these numbers are the same for any signed subcubic graph.

**Proof of Theorem 4.1.3.** Let $\Sigma$ be a signed subcubic graph. Since $\Sigma$ is subcubic, it can have at most one negative loop incident to a vertex, and hence the presence of a negative loop increases both the frustration index and number by one (frustration number increases by one because if a negative loop is present, the vertex incident to it must be deleted, whereas if the negative loop is deleted, the deletion of the vertex (now isolated or pendant) does not affect the balance of the signed graph). Hence we may assume that $\Sigma$ is loopless. It is trivial to see that $l_0(\Sigma) \leq l(\Sigma)$. To see the other direction, suppose $l_0(\Sigma) = k$. Let $v_1, \ldots, v_k \in V(\Sigma)$ such that $\Sigma \setminus \{v_1, \ldots, v_k\}$ is balanced. Let $X = \{v_1, \ldots, v_k\}$ and $Y = V(\Sigma) - X$. Switch so that $\Sigma : Y$ consists of only positive edges. If in the resulting signed graph, $v_i$ is incident to more than one negative edge for some $i$, $1 \leq i \leq k$, switch at $v_i$ to reduce the number of negative edges (since $\Sigma$ is cubic). We keep doing this until we have a signed graph $\Sigma'$, switching equivalent to $\Sigma$, with at most one negative edge incident to each $v_i$, and since switching at a vertex in $X$ does not change the sign of an edge with both
Table 4.1: Frustration index of signed Heawood graphs.

<table>
<thead>
<tr>
<th>$(H, \sigma)$</th>
<th>$+H$</th>
<th>$H_1$</th>
<th>$H_{2,1}$</th>
<th>$H_{2,2}$</th>
<th>$H_{3,1}$</th>
<th>$H_{3,2}$</th>
<th>$H_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l(H, \sigma)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

endpoints in $Y$, $\Sigma'$ : $Y$ consists of only positive edges. Let $S$ be the set of negative edges in $\Sigma'$. Since every edge in $S$ has an endpoint in $X$, and no two of them share an endpoint in $X$, $|S| \leq |X| = k$. Now, $\Sigma' \setminus S$ consists only of positive edges, and is therefore balanced. This implies $l(\Sigma) = l(\Sigma') \leq |S| \leq k = l_0(\Sigma)$. \( \Box \)

Note that $l$ can be arbitrarily large compared to $l_0$. For example, consider the Dutch windmill graph $DW_n$ (graph obtained from $n$ triangles by identifying a vertex in each triangle) with one edge in each triangle negative. Call it $\Sigma_n$. Then $l_0(\Sigma_n) = 1$ and $l(\Sigma_n) = n$. But having a bound on the degree or knowing the degree sequence of the underlying graph can be useful. As Zaslavsky (private communication) points out, the above proof can be used to prove the following:

**Proposition 4.4.1.** Let $G$ be a graph with degree sequence $(d_1, \ldots, d_n)$, where $d_1 \geq \ldots \geq d_n$. Let $\Sigma$ be a signed graph with underlying graph $G$. Then

$$l(\Sigma) \leq \sum_{i=1}^{l_0(\Sigma)} \left\lfloor \frac{d_i}{2} \right\rfloor .$$

What can we say about the frustration number of a signed subcubic loopless graph? Suppose that $\Sigma$ is a signed subcubic loopless graph. Then we can switch $\Sigma$ to a signed graph $\Sigma'$ such that the negative edges in $\Sigma'$ form a matching (cf. Corollary 7.4 [63]). Hence we have $l_0(\Sigma) \leq \frac{|V(\Sigma)|}{2}$. Can this bound be improved? We give two families of signed cubic graphs to show that without any additional restrictions on the underlying graph, the bound cannot be improved. First family: $-kK_4$ (disjoint
union of $k$ copies of $K_4$, with every edge negative). Second family: Disjoint union of $k$ copies of the signed graph consisting of two vertices and three links, exactly one of them negative.

The following result tells us that the frustration index of a signed graph whose underlying graph is simple, triangle-free, cubic, and has $n$ vertices, is at most $\frac{3n}{8}$.

**Theorem 4.4.2.** Let $\Sigma$ be a signed graph whose underlying graph is simple, triangle-free and cubic. Then $l(\Sigma) \leq \frac{3}{8}|V(\Sigma)|$.

**Proof.** We will “reduce” $\Sigma$ by using the following two operations: 1) If the number of negative edges at a vertex $v$ is more than 1, switch at $v$. 2) If there exist vertices $u, v, w$ such that $uv$ and $vw$ are positive edges, and each of the three vertices $u, v, w$ is incident to a negative edge, then switch at $\{u, v, w\}$. We keep doing these operations (operation 2 should be used only if operation 1 cannot be done) on $\Sigma$. Let the resulting signed graph be $\Sigma'$ (since both the operations reduce the number of negative edges, the process terminates after a finite number of steps). The fact that operation 1 cannot be done in $\Sigma'$ tells us that the negative edges form a matching. Let us denote the set of endpoints of the edges in the matching by $X$. The fact that operation 2 cannot be done in $\Sigma'$, together with the absence of loops, multiple edges, and triangles in the underlying graph of $\Sigma'$, tells us that the positive edges in $\Sigma'$ : $X$ form a matching. Since $\Sigma'$ is cubic, there is at least one edge from every vertex in $X$ to $V(\Sigma') - X$. Hence the number of edges between $X$ and $V(\Sigma')$ is at least $|X|$ and at most $3|V(\Sigma') - X|$:

$$|X| \leq 3|V(\Sigma') - X|.$$ 

This gives

$$|X| \leq \frac{3}{4}|V(\Sigma')|.$$ 

42
Deleting exactly one of the endpoints of each of the negative edges in $\Sigma'$ : $X$ yields a balanced signed graph. Hence

$$l(\Sigma) = l(\Sigma') \leq \frac{3}{8}|V(\Sigma')| = \frac{3}{8}|V(\Sigma)|,$$

and this completes the proof. $\square$

Theorem 4.4.2, together with Lemma 4.6.3, implies a result of Zaslavsky ([63, Theorem 10.5]) on the largest inclusterability index of a signed Petersen graph.

### 4.5 Number of negative circles

It is not hard to find out the number of negative hexagons in each of the seven switching isomorphism types of signed Heawood graph. Since the number of negative (positive) circles of a particular length is invariant under switching and automorphism, it gives us a way to see that there are at least six switching isomorphism types of signed Heawood graph. Clearly, $+H$ has no negative hexagons. Each edge in the Heawood graph is in exactly 12 hexagons, and hence $H_1$ has 12 negative hexagons. For $H_{2,1}$ and $H_{2,2}$, the number of hexagons with both the negative edges is 2 and 1, respectively, and hence we get the numbers in the table.

Proposition 4.3.1 tells us that the number of negative Hamilton circles in a signed Heawood graph is even and at most 16. But the following result tells us that it is more restricted.
Proposition 4.5.1. The number of negative Hamilton circles in each of the seven signed Heawood graphs is as shown in Table 4.3. The number of negative Hamilton circles in a signed graph based on Heawood graph is either 0, 8, 12, or 16.

Proof. Clearly the number of negative Hamilton circles in $+H$ is 0. The number of negative Hamilton circles in $H_1$ is precisely the number of Hamilton cycles containing a given edge. Since $H$ is edge-transitive, this number is $\frac{24 \cdot 14}{2!} = 16$. Consider $H_{2,1}$. The number of Hamilton cycles containing $e_1$ but not containing $e_5$ is 4. The number of Hamilton cycles containing $e_5$ but not containing $e_1$ is 4. Hence the number of negative Hamilton circles in $H_{2,1}$ is exactly 8. Consider $H_{2,2}$. The number of Hamilton cycles containing $e_1$ but not containing $e_3$ is 6. The number of Hamilton cycles containing $e_3$ but not containing $e_1$ is 6. Hence the number of negative Hamilton circles in $H_{2,2}$ is exactly 12. Consider $H_{3,2}$. The number of Hamilton cycles containing $e_1, e_3$ and $e_5$ is 7. The number of Hamilton cycles containing $e_1$ but not containing $e_3$ or $e_5$ is 1. The number of Hamilton cycles containing $e_3$ but not containing $e_1$ or $e_5$ is 3. The number of Hamilton cycles containing $e_5$ but not containing $e_1$ or $e_3$ is 1. Hence the number of negative Hamilton circles in $H_{3,2}$ is exactly 12. Consider $H_{3,1}$. The number of Hamilton cycles containing none of $e_1, e_3, e_9$ is 1. The number of Hamilton cycles containing $e_1$ and $e_3$ but not containing $e_9$ is 1. The number of Hamilton cycles containing $e_1$ and $e_9$ but not containing $e_3$ is 3. The number of Hamilton cycles containing $e_3$ and $e_9$ but not containing $e_1$ is 3. Hence the number of positive

<table>
<thead>
<tr>
<th>$(H, \sigma)$</th>
<th>$+H$</th>
<th>$H_1$</th>
<th>$H_{2,1}$</th>
<th>$H_{2,2}$</th>
<th>$H_{3,1}$</th>
<th>$H_{3,2}$</th>
<th>$H_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c^{-}_{14}(H, \sigma)$</td>
<td>0</td>
<td>16</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 4.3: Number of negative Hamilton circles in signed Heawood graphs.
Hamilton circles in $H_{3,1}$ is exactly 8. Since $H$ has 24 Hamilton cycles, $H_{3,1}$ has exactly 16 negative Hamilton circles. Consider $H_4$. The number of Hamilton cycles containing $e_3$ but none of $e_7, e_{11}$ and $e_{13}$ is 0. The number of Hamilton cycles containing $e_7$ but none of $e_3, e_{11}$ and $e_{13}$ is 0. The number of Hamilton cycles containing $e_{11}$ but none of $e_3, e_7$ and $e_{13}$ is 1. The number of Hamilton cycles containing $e_{13}$ but none of $e_3, e_7$ and $e_{11}$ is 1. The number of Hamilton cycles containing $e_3, e_7$ and $e_{11}$ but not containing $e_{13}$ is 3. The number of Hamilton cycles containing $e_3, e_7$ and $e_{13}$ but not containing $e_{11}$ is 3. The number of Hamilton cycles containing $e_3, e_{11}$ and $e_{13}$ but not containing $e_7$ is 2. The number of Hamilton cycles containing $e_7, e_{11}$ and $e_{13}$ but not containing $e_3$ is 2. Hence the number of negative Hamilton circles in $H_4$ is exactly 12.

From Table 4.3 and the fact that switching preserves the number of cycles of given length and sign, it is clear that the number of negative Hamilton circles in a signed graph based on Heawood graph is either 0, 8, 12, or 16. □

4.6 Clusterability

A signed graph $\Sigma$ is called clusterable if its vertex set can be partitioned into sets, called clusters, so that each edge within a cluster is positive and each edge between two clusters is negative. The cluster number of a clusterable graph $\Sigma$ is the smallest number of clusters into which its vertex set can be partitioned. This is denoted by $\text{clu}(\Sigma)$. The inclusterability index of $\Sigma$, denoted $Q(\Sigma)$, is the smallest number of edges whose deletion from $\Sigma$ leaves a clusterable signed graph [63].

In the following proposition, we give a bound for the smallest number of edges to be deleted to make a signed graph clusterable, and characterize signed graphs that achieve the bound.

**Proposition 4.6.1.** For every signed loopless graph $\Sigma$, $Q(\Sigma) \leq \frac{|E(\Sigma)|}{2}$, with equality
if and only if $\Sigma$ can be obtained from a loopless graph by replacing every edge by a negative digon.

Proof. Delete either all the positive edges or all the negative edges, depending on whichever is less (if equal, choose either one). The resulting signed graph has no circle with exactly one negative edge, and hence by Davis’s criterion, clusterable. Hence $Q(\Sigma) \leq \frac{|E(\Sigma)|}{2}$.

If $\Sigma$ is obtained from a loopless graph by replacing every edge by a negative digon, then clearly $Q(\Sigma) = \frac{|E(\Sigma)|}{2}$. Now, let $\Sigma$ be a signed graph that cannot be obtained from a loopless graph by replacing every edge by a negative digon. If the number of positive edges in $\Sigma$ is not equal to the number of negative edges in $\Sigma$, we can delete whichever is less to get a clusterable signed graph and hence $Q(\Sigma) < \frac{|E(\Sigma)|}{2}$.

Hence we will assume that $\Sigma$ has equal number of positive and negative edges. There exists two vertices $u$ and $v$ such that the number of positive edges between $u$ and $v$ is strictly greater than the number of negative edges between $u$ and $v$. The signed graph obtained from $\Sigma$ by deleting all negative edges between $u$ and $v$ and all positive edges except the ones between $u$ and $v$ is clusterable, and hence $Q(\Sigma) < \frac{|E(\Sigma)|}{2}$. $\square$

We now calculate the inclusterability index of the seven signed Heawood graphs given in Figure 4.2. We use the following, known as Davis’ criterion [17], to decide if a signed graph is clusterable or not.

**Proposition 4.6.2** (Davis’ criterion). A signed graph is clusterable if and only if no circle has exactly one negative edge.

Since $+H$ has no negative edge, we can group all the vertices in a single cluster and hence $clu(+H) = 1$. All the other six signed Heawood graphs have a cycle with exactly one negative edge, and hence are not clusterable (see Table 4.4; “-” denotes
that the corresponding signed graph is not clusterable). Note that the inclusterability index is not an invariant under switching automorphism.

Since \( +H \) has no edges, it is clusterable, and hence its inclusterability index is 0. \( H_1 \) contains exactly one negative edge (which is in a circle), and hence its inclusterability index is 1. Both \( H_{2,1} \) and \( H_{2,2} \) can be made clusterable by deleting the two negative edges, but neither of them can be made clusterable by deleting just one edge. Hence both of them have inclusterability index 2. Note that \( H_4 \) can be switched at \( \{v_{11}, v_{12}, v_{12}\} \) to get a signed graph with exactly three negative edges. All three of \( H_{3,1}, H_{3,2} \) and \( H_4 \) can be made clusterable by deleting three edges but it is easy to check that none of them can be made clusterable by deleting just two edges. Hence all the three have inclusterability index 3 (see Table 4.5).

Since clusterability is not switching invariant, knowing the inclusterability indices for the seven minimal signed Heawood graphs is not sufficient to understand the inclusterability indices of all signed Heawood graphs. Zaslavsky proved that the

<table>
<thead>
<tr>
<th>((H, \sigma))</th>
<th>(+H)</th>
<th>(H_1)</th>
<th>(H_{2,1})</th>
<th>(H_{2,2})</th>
<th>(H_{3,1})</th>
<th>(H_{3,2})</th>
<th>(H_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(clu(H, \sigma))</td>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.4: Cluster number of signed Heawood graphs.

<table>
<thead>
<tr>
<th>((H, \sigma))</th>
<th>(+H)</th>
<th>(H_{1})</th>
<th>(H_{2,1})</th>
<th>(H_{2,2})</th>
<th>(H_{3,1})</th>
<th>(H_{3,2})</th>
<th>(H_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q(H, \sigma))</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 4.5: Inclusterability index of the seven minimal signed Heawood graphs.
largest inclusterability index of any signed Petersen graph is 3. We show that this is true for any signed Heawood graph as well. We first need an easy lemma.

**Lemma 4.6.3.** If $\Sigma_1$ is switching isomorphic to $\Sigma_2$, then $Q(\Sigma_1) \leq l(\Sigma_2)$.

**Proof.** $\Sigma_1$ has a set $X$ of $l(\Sigma_1)$ edges such that $\Sigma_1 \setminus X$ is balanced. By Davis’ criterion, a balanced signed graph is clusterable. Hence $Q(\Sigma_1) \leq l(\Sigma_1)$. Since $l(\Sigma_1) = l(\Sigma_2)$, we get $Q(\Sigma_1) \leq l(\Sigma_2)$.

**Theorem 4.6.4.** The largest inclusterability index of any signed Heawood graph is 3.

**Proof.** Clearly, there exist signed Heawood graphs with inclusterability index 3 (Table 4.5). Let $\Sigma$ be a signed Heawood graph. By Theorem 4.1.2, we know that $\Sigma$ is switching isomorphic to $\Sigma_1$, one of the seven minimal signed Heawood graphs. By Lemma 4.6.3, the inclusterability index of $\Sigma$ is at most the frustration index of $\Sigma_1$, which is at most 3 (Table 4.1).

Let $G$ be graph. Suppose that $\{\Sigma_1, \ldots, \Sigma_k\}$ is a set of signings of $G$ such that any signing of $G$ is switching isomorphic to one of $\Sigma_1, \ldots, \Sigma_k$. Lemma 4.6.3 tells us that the $\max\{l(\Sigma_1), \ldots, l(\Sigma_k)\}$ is an upper bound for the inclusterability index of any signing of $G$. This shows that, in [63], Theorem 10.5 follows as a corollary from Theorem 5.1 and Table 7.1.

### 4.7 Coloring

A proper $k$-coloring of a signed graph $\Sigma$ is a function $c : V(\Sigma) \rightarrow \{0, \pm 1, \ldots, \pm k\}$ such that, if $vw$ is an edge of $\Sigma$, then $c(w) \neq \sigma(vw)c(v)$. The chromatic number of $\Sigma$, denoted $\chi(\Sigma)$, is the smallest $k$ such that there is a proper $k$-coloring of $\Sigma$. The zero-free chromatic number of $\Sigma$, denoted $\chi^*(\Sigma)$, is the smallest $k$ such that there is
(H, σ) | +H | H₁ | H₂₁ | H₂₂ | H₃₁ | H₃₂ | H₄  
---|---|---|---|---|---|---|---
χ(H, σ) | 1 | 1 | 1 | 1 | 1 | 1 | 1  
χ*(H, σ) | 1 | 2 | 2 | 2 | 2 | 2 | 2  

Table 4.6: The chromatic numbers of signed Heawood graphs.

a proper k-coloring of Σ that does not use the color 0. Zaslavsky [63] showed that
the two chromatic numbers are preserved under switching and isomorphism.
The chromatic and zero-free chromatic number of signed Heawood graphs are given
in Table 4.6. The numbers in the table follow from the fact that H is bipartite. In
fact, if the underlying graph of a signed graph is bipartite, then its chromatic number
is 1 and zero-free chromatic number is 1 or 2.

4.8 Signings of the Petersen graph

Two signed graphs Σ₁ and Σ₂ are switching isomorphic if a switching of Σ₁ is isomor-
phic to Σ₂. It is easy to see that, up to switching isomorphism, K₃ has 2 signings,
K₄ has 3 signings, and K₅ has 7 signings. Zaslavsky [64] has calculated the number
of signings of the Petersen graph. We give a proof of this result.

Theorem 4.8.1 (Zaslavsky). The Petersen graph has exactly six signings, up to
switching isomorphism.

Proof. Consider a signed Petersen graph. We split the analysis into three cases.
Suppose there exist two vertex-disjoint positive 5-circles. Then, by switching, we see
that all the non-spoke edges are positive, and at least three of the spokes are positive.
This gives rise to three signed graphs: Petersen graph with all edges positive (+P),
Petersen graph with one negative edge ($P_1$), and Petersen graph with two negative edges where the two negative edges are in a perfect matching ($P_2$). Consider the case in which there exist two vertex-disjoint negative 5-circles. Then, by switching we see that all the non-spoke edges are negative, and at least three of the spokes are negative. This gives rise to three signed graphs: Petersen graph with all edges negative (−$P$), Petersen graph with one positive edge ($P_3$), and Petersen graph with two positive edges where the two positive edges are in a perfect matching ($P_4$). It is easy to check that $P_4$ is switching isomorphic to $P_2$. The only case that is left is where any two disjoint 5-circles have opposite parity. This means that every 8-circle is positive. Then we can find a negative 9-circle. We can easily argue that such a signed graph is switching isomorphic to the Petersen graph with three positive edges where the positive edges are not in a perfect matching ($P_5$). This completes all the cases, and we have six signed graphs: $+P, -P, P_1, P_2, P_3, P_5$. It is easy to see that no two of them are switching isomorphic. This proves the theorem. \(\square\)

4.9 Concluding Remarks

What kinds of structure should occur in any signing of a graph? Proposition 4.3.1 tells us that any signing of the Heawood graph must contain a positive Hamilton circle. An obvious question is the following.

**Problem 4.9.1.** Characterize graphs $G$ such that any signing of $G$ contains a positive Hamilton circle.

It is curious that the frustration index and the inclusterability index turn out to be the same for the seven signed Heawood graphs shown in Figure 4.2. The fact that frustration is switching-invariant whereas clusterability is not makes it hard to explain this coincidence.
Analyzing the signings of a graph helps us to understand the graph better. The number of signings of a graph (up to switching isomorphism) is a good indicator of the symmetry of the graph. Given a graph $G$, it might be useful to know bounds on the number of signings of $G$ in terms of some other invariants of $G$. Also, since finding the frustration index of a signed graph is NP-complete [2], it might be instructive to work with signed graphs whose underlying graphs have bounded tree-width.

**Problem 4.9.2.** Let $k$ be a fixed integer. Devise a polynomial-time algorithm to determine the frustration index of signed graphs whose underlying graphs have tree-width at most $k$, if one exists.
Welsh conjectured [55] that if $n, r, b$ are integers satisfying $0 \leq r \leq n$ and $1 \leq b \leq \binom{n}{r}$, then there is an $n$-element rank-$r$ matroid with $b$ bases. Anna de Mier found a small counterexample: No rank-3 6-element matroid can have exactly 11 bases. Dillon Mayhew and Gordon Royle verified [35] that this is the only counterexample up to $n = 9$, and conjectured that de Mier’s instance is the only counterexample to Welsh’s conjecture. We use SAGE (System for Algebra and Geometry Experimentation, a free and open source software) to verify the conjecture up to $n = 12$. We also mention three weakenings and one strengthening of Welsh’s conjecture.

5.1 Introduction

Dillon Mayhew and Gordon Royle [35] updated Dominic Welsh’s conjecture as follows:

**Conjecture 5.1.1.** Let $n, r, b$ be integers such that $0 \leq r \leq n$, $1 \leq b \leq \binom{n}{r}$ and $(n, r, b) \neq (6, 3, 11)$. Then there exists a rank-$r$ matroid on $n$ elements with exactly $b$ bases.

5.2 A computational approach to Welsh’s conjecture

Let us consider verifying the conjecture for $n = 10$ and $r = 5$. SAGE has a command to compute the number of bases of a matroid. But enumerating all the matroids on
10 elements having rank 5 and then checking their number of bases is not a good idea since a catalogue of 10-element rank-5 matroids is not available and probably hard to come by ([35]). But we don’t need to know all the matroids on 10-elements to discuss the validity of Welsh’s conjecture. We take the following approach. We ask SAGE to generate a random $5 \times 10$ matrix $A$ with entries in $GF(2)$. Now we ask SAGE to compute the number of bases of the vector matroid of $A$ as viewed over $GF(2)$. We record this, and repeat this step. We also change the field, and repeat the step. We halt the process if we have seen all integers between 1 and $\binom{10}{5}$. This experiment was done and the process stopped verifying that Welsh’s conjecture is true for $n = 10$ and $r = 5$.

### 5.3 SAGE results

We report the results obtained using SAGE. Experiments verified that Welsh’s conjecture is true for $n = 10, 11, 12$ (for all appropriate values of $r$), and using results of Mayhew and Royle, we can see that Welsh’s conjecture is true for all $n \leq 12$, with the exception of de Mier’s counterexample.

### 5.4 Four related conjectures

We mention three weakenings and one strengthening of Welsh’s conjecture.

**Conjecture 5.4.1.** There exists an absolute constant $c$ such that for every $n$ there are at most $c$ counterexamples to Welsh’s conjecture for matroids on $n$ elements.

**Conjecture 5.4.2.** There exists an absolute constant $c$ such that for every $n, r$ with $r \leq n$, there are at most $c$ counterexamples to Welsh’s conjecture for matroids on $n$ elements and rank $r$. 

53
Let $B_{n,r}$ denote the set $\{b(M) : M$ is a matroid on $n$ elements with rank $r\}$, and let $b_{n,r} = |B_{n,r}|$.

**Conjecture 5.4.3.** For a fixed non-negative integer $r$, $\lim_{n \to \infty} \frac{b_{n,r}}{\binom{n}{r}} = 1$.

Given the current interest in the properties of a typical matroid, and research on random matroids, Conjecture 5.4.3 is more likely to be solved with current techniques.

**Conjecture 5.4.4.** Welsh’s conjecture is true even when we restrict to representable matroids. More precisely, if $n, r, b$ are integers satisfying $0 \leq r \leq n$, $1 \leq b \leq \binom{n}{r}$, and $(n, r, b) \neq (6, 3, 11)$, then there is an $n$-element rank-$r$ representable matroid with $b$ bases.

Note the following chain of implications: Conjecture 5.4.4 $\Rightarrow$ Conjecture 5.1.1 $\Rightarrow$ Conjecture 5.4.1 $\Rightarrow$ Conjecture 5.4.2 $\Rightarrow$ Conjecture 5.4.3.

## 5.5 A proof of de Mier’s observation

In this section we focus on the following fact (Proposition 5.5.1) first observed by Anna de Mier. The following is a proof of this nice observation. We will show that no rank-$3$ matroid on six elements can have exactly nine dependent $3$-sets using the circuit elimination axiom (CEA) (Axiom C3 in Appendix A).

**Proposition 5.5.1.** Let $M$ be a rank-$3$ matroid on six elements. Then $M$ cannot have exactly eleven bases.

**Proof.** Let $E(M) = \{1, 2, 3, 4, 5, 6\}$. The result follows easily if $M$ contains a loop. Hence we will assume that $M$ is loopless. If $M$ has a non-trivial parallel class with size at least three or three non-trivial parallel classes, then $M$ has at least ten dependent $3$-subsets, and we are done. If $M$ has two non-trivial parallel classes, say $\{1, 2\}$ and $\{3, 4\}$, the eight $3$-subsets of $E(M)$ containing either $\{1, 2\}$ or $\{3, 4\}$ are dependent.
CEA guarantees that there cannot be exactly one more dependent 3-subset of $E(M)$. Hence the number of dependent 3-subsets of $E(M)$ is not nine, and we are done. Suppose that $M$ contains exactly one non-trivial parallel class, say $\{5,6\}$. We may assume that $M$ contains two 3-circuits having two elements in common, say $\{1,2,3\}$ and $\{1,2,4\}$ (the other case, where $\{1,2,3\}$ and $\{1,2,5\}$ are 3-circuits, is similar).

Using CEA we see that, $\{1,3,4\}$ and $\{2,3,4\}$ are also circuits. CEA also guarantees that there cannot be exactly one more dependent 3-subset of $E(M)$. Hence the number of dependent 3-subsets of $E(M)$ is not nine, and we are done. Hence we may assume that $M$ is simple. We may assume, by the argument outlined above, that $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$ are circuits. We may also assume that $\{1,2,5\}$ is a circuit. Using CEA, we see that $\{2,3,5\}$, $\{1,3,5\}$, $\{2,4,5\}$, $\{1,4,5\}$, and $\{3,4,5\}$ are also circuits, and hence $M$ has at least ten 3-element circuits. The proof is complete. □
CHAPTER 6
SIGNED-GRAPHIC MATROIDS

6.1 Properties of signed-graphic matroids

We list some properties of signed-graphic matroids that play an important role in their analysis.

• Signed-graphic matroids are dyadic (see [60] for a proof).

• Graphic matroids are signed-graphic.

Proof. Let $G$ be a graph. Then the cycle matroid of $G$ is identical to the frame matroid of $(G, +)$, the signed graph with underlying graph $G$ and all edges positive. □

• The class of signed-graphic matroids is closed under taking minors.

• The class of signed-graphic matroids is not closed under duality. For example, $M^*(K_7)$ is not signed-graphic. This fact can be proved either from first principles or by appealing to a theorem of Slilaty [52] stating that $M^*(G)$ is signed-graphic if and only if $G$ is projective-planar.

• A relaxation of a graphic matroid is always signed-graphic.
Proof. It is not hard to show that if a graphic matroid has a circuit-hyperplane, then it is a subgraph of a wheel graph. Hence a relaxation of a graphic matroid is a restriction of a whirl matroid, which is well-known to be signed-graphic. This completes the proof. □

- Grafts need not be signed-graphic. For example, the graft matroid of \((K_4, V(K_4))\) is isomorphic to the Fano matroid, which is not signed-graphic.

- Signed-graphic matroids need not be binary. For example, the unique obstruction for a matroid being binary, the four-point line, is signed-graphic.

- A uniform matroid \(U_{r,n}\) is signed-graphic if and only if \(r = 0, 1, n - 1, n\) or \(\{r, n\} = \{2, 4\}\).

Proof. It is well known that \(U_{r,n}\) is graphic if and only if \(r = 0, 1, n - 1\), or \(n\) (cf. [39]). Now let \(U_{r,n}\) be signed-graphic where \(r \notin \{0, 1, n - 1, n\}\). Since neither \(U_{2,5}\) nor \(U_{3,5}\) is signed-graphic and since the class of signed-graphic matroids is closed under minors, we see that \(r = n - r = 2\). Since \(U_{2,4}\) is a signed-graphic matroid, we see that \(U_{r,n}\) is signed-graphic if and only if \(r = 0, 1, n - 1\), or \(n\), or \(\{r, n\} = \{2, 4\}\). □

- A simple rank-\(r\) signed-graphic matroid has at most \(r^2\) elements, with equality occurring if and only if the matroid is a ternary Dowling geometry.

- A signed-graphic matroid can be made graphic by deleting at most half of its elements.

Proof. Let \(M\) be a signed-graphic matroid, and \(\Sigma\) be a signed graph with \(M(\Sigma) = M\). It is well known that a loopless signed graph can be made balanced by deleting at most half of its edges. Apply this result to the signed
graph obtained from $\Sigma$ by deleting all the loops, and then put back all the loops. The resulting signed graph has at least half the number of edges of $\Sigma$, and is joint-unbalanced. Since the frame matroid of a joint-unbalanced signed graph is graphic, we are done. \( \square \)

- It is not possible to determine whether or not a matroid, given in terms of an independence oracle, is signed-graphic by using only polynomial number of oracle calls (Geelen-Mayhew).

- Every simple signed-graphic matroid is a restriction of a ternary Dowling geometry

- The smallest minor-closed class of matroids containing the even-cycle matroids is the class of signed-graphic matroids.

- Out of the 70 matroids on at most 5 elements, only two are not signed-graphic: $U_{2,5}$ and $U_{3,5}$ (See Appendix B for details).

### 6.2 Signed-graphic representations of ternary swirls

We discuss signed-graphic representations of ternary swirls. $\pm G$ denotes the signed graph obtained from a graph $G$ by replacing every edge by a negative digon (Zaslavsky calls this the signed expansion of $G$). The frame matroid of $\pm C_n$ is called a ternary swirl.

**Proposition 6.2.1.** $\pm C_n$ $(n \geq 5)$ uniquely represents its frame matroid.

**Proof.** The proof is by induction on $n$. Let us prove it for the base case $n = 5$. First we check that $M(\pm C_3)$ has exactly 12 representations (see Figure 6.1). Then we can verify that $M(\pm C_4)$ has exactly 3 representations. And then it is easy to see that
Figure 6.1: Base case: $\pm C_5$ uniquely represents its frame matroid.
Figure 6.2: Induction step: uniqueness of extension and coextension.
only one of them extends to a representation of $M(\pm C_5)$. For the induction step, assume that the statement is true for $n$ ($n \geq 5$).

In the first step, we need to add the element $b_{n+1}$ in such a way that $\{a_1, b_1, b_{n+1}\}$ and $\{a_n, b_n, b_{n+1}\}$ are triangles in the matroid. The only way to do this to add it as a negative loop at vertex $v$. In the second step, we need to add the element $a_{n+1}$ in such a way that $\{a_1, b_1, a_{n+1}, b_{n+1}\}$ and $\{a_n, b_n, a_{n+1}, b_{n+1}\}$ are 4-circuits in the matroid. The only way to do this is to split vertex $v$, and out of the many possibilities of splitting, only one will preserve the matroid structure (shown in bottom right of Figure 6.2, and hence the the statement is true for $n + 1$. We are done by induction.

6.3 Signed-graphic representations of $R_{10}$

The matrix $[I_5|A]$ represents $R_{10}$ over any field, where

$$A = \begin{bmatrix}
-1 & 1 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 \\
0 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & 1 \\
1 & 0 & 0 & 1 & -1
\end{bmatrix}.$$ 

Note: $R_{10}$ is also represented over GF(2) by the $(5 \times 10)$-matrix where the columns consist of the ten 5-tuples with exactly 3 ones each.

Figure 6.3 shows the six signed-graphic representations of $R_{10}$.
Figure 6.3: The six signed-graphic representations of $R_{10}$
<table>
<thead>
<tr>
<th>Property</th>
<th>Graphic matroids</th>
<th>Signed-graphic matroids</th>
</tr>
</thead>
<tbody>
<tr>
<td>Representability</td>
<td>Regular</td>
<td>Dyadic</td>
</tr>
<tr>
<td></td>
<td>(Tutte-Rado)</td>
<td>(Dowling-Zaslavsky)</td>
</tr>
<tr>
<td>Recognition in rank oracle model</td>
<td>Polynomials many calls</td>
<td>Exponentially many calls</td>
</tr>
<tr>
<td></td>
<td>(Seymour)</td>
<td>(Geelen-Mayhew)</td>
</tr>
<tr>
<td>Bound for simple matroids</td>
<td>$</td>
<td>E</td>
</tr>
<tr>
<td>WQO under minors?</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>(Robertson-Seymour)</td>
<td>(Geelen-Gerards-Whittle)</td>
</tr>
<tr>
<td>Excluded minors</td>
<td>$U_{2,4}, F_7, F_7^*$,</td>
<td>not known</td>
</tr>
<tr>
<td></td>
<td>$M^<em>(K_5), M^</em>(K_{3,3})$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(Tutte)</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.1: A comparison of graphic and signed-graphic matroids
6.4 Graphic vs. Signed-graphic matroids

Graphic matroids possess remarkably nice properties. It is interesting to know how many of those properties are possessed by classes that properly contain graphic matroids. Lot of research has gone into this question for the classes of regular and binary matroids (see [39]). Here we discuss properties of graphic matroids that can be carried over to signed-graphic matroids. In Table 6.1, some properties of signed-graphic matroids are compared with that of graphic matroids.

6.5 Operations preserving the frame matroid structure

Refer [50] for details on operations 1 – 4.

![Diagram of switching a $-K_4$]

Figure 6.4: Shuffling a $-K_4$

1. Switching: Since switching a signed graph does not change the list of positive circles, the frame matroid remains invariant under the switching operation.
2. Balancing vertex: If all the negative edges of a signed graph \( \Sigma \) are incident on a vertex \( v \), then we can form a graph \( G \) where \( v \) is replaced by two vertices, \( v^+ \) and \( v^- \), where all positive edges incident on \( v \) are now incident on \( v^+ \), and all negative edges incident on \( v \) are now incident on \( v^- \). It is easy to check that \( M(G) = M(\Sigma) \).

3. Joint-unbalanced: Form a new graph \( G \) from \( \Sigma \) by replacing every joint by an edge from the corresponding endpoint of the joint to a newly added vertex. It is easy to check that \( M(G) = M(\Sigma) \).

4. Negative edges induce a triangle: Let \( \Sigma \) be a signed graph with exactly 3 negative edges, \( e_1, e_2 \) and \( e_3 \), where \( e_1 = \{a, b\} \), \( e_2 = \{b, c\} \) and \( e_3 = \{c, a\} \). Delete the three negative edges, and let \( e_1 = \{c, z\} \), \( e_2 = \{a, z\} \) and \( e_3 = \{b, z\} \) where \( z \) is a new vertex. Let the new signed graph be \( \Sigma' \). Then \( M(\Sigma) = M(\Sigma') \).

5. Negative edges induce a \( K_4 \): This operation is shown in Figure 6.4. Here we assume that the given signed graph is switching isomorphic to a signed graph with exactly six negative edges and that they induce a \( K_4 \). We switch the edges of the three pairs of matchings: edges 1 and 3 get swapped, 2 and 4 get swapped, and 5 and 6 get swapped and the positive edges stay put. It is not hard to verify that the frame matroids of the original signed graph and the new one coincide.

6.6 A Venn diagram

Figure 6.5 shows some important classes of matroids and how they intersect with each other.
Figure 6.5: A Venn diagram of the class of matroids
CHAPTER 7
THREE MATROIDS ON THE EDGE SET OF A GRAPH

There are three well-known matroids on the edge set of a graph: cycle matroid, bicircular matroid, and even-cycle matroid [46]. We will compare and contrast these three matroids. Also, we will discuss when a matroid from one class can be realized as a matroid from the other classes.

7.1 Graphic matroids

Let $G$ be a graph. If we let a set of edges of $G$ to be independent if the graph spanned by the edge set is a forest, we get the cycle matroid of $G$, denoted $M(G)$. The circuits of the matroid are the edge sets of the cycles of the graph, and a basis corresponds to the edge set of a spanning forest (spanning tree in each component). An abstract matroid isomorphic to the cycle matroid of a graph is called a graphic matroid. Graphic matroids have remarkably nice properties. They are regular, meaning representable over every field. This can be seen by taking the vertex-edge incidence matrix of an arbitrary orientation of the graph, viewed over $\mathbb{Q}$, and noting that it is totally unimodular. Also the class of graphic matroids is minor-closed i.e., if we delete or contract an element of a graphic matroid, the resulting matroid is also graphic. It is a natural question to ask for the complete set of excluded minors for
this class. Thanks to groundbreaking work of Tutte, this question is completely answered: A matroid is graphic if and only if it has no minor isomorphic to any of $U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3})$.

Given a matroid in terms of an independence oracle, it is possible to decide in polynomial time whether or not the matroid is graphic (Seymour). Unfortunately, the class of graphic matroids is not closed under duality. But the intersection of the class of graphic matroids and the class of cographic matroids is well understood. The cycle matroid of a graph is cographic if and only if it is planar (Tutte).

### 7.2 Bicircular matroids

Let $G$ be a graph. If we let a set of edges of $G$ to be independent if the graph spanned by the edge set contains at most one cycle in each component, we get the bicircular matroid of $G$, denoted $B(G)$. The circuits of the matroid correspond to the edge sets of a theta-graph, a tight handcuff, or a loose handcuff. An abstract matroid isomorphic to the bicircular matroid of a graph is called a bicircular matroid.

Bicircular matroids are representable over sufficiently large finite fields and all infinite fields (Piff and Ingleton). This can be seen by taking the vertex-edge incidence matrix, using algebraically independent elements in place of non-zero entries. Also, the class of bicircular matroids is minor-closed. It is a natural question to ask for the complete set of excluded minors for this class. This question is now almost answered: The set of excluded minors for the class of bicircular matroids is finite and most likely consists of 27 members (DeVos, Goddyn, Mayhew, Royle).

Like the class of graphic matroids, the class of bicircular matroids is not closed under duality. The intersection of the class of bicircular matroids with the class of cobicircular matroids has not been studied.
7.3 Even-cycle matroids

Let $G$ be a graph. If we let a set of edges of $G$ to be independent if the graph spanned by the edge set contains no even cycle and at most one odd cycle in each component, we get the even-cycle matroid of $G$, denoted $F(G)$. This matroid has some connections with factors of the graph, and hence the reason for the notation. Note that the circuits of the matroid correspond to the edge sets of either an even cycle or an odd handcuff. An abstract matroid isomorphic to the even-cycle matroid of a graph is called a even-cycle matroid. Even-cycle matroids are dyadic, meaning representable over every field whose characteristic is not 2. This can be seen by taking the vertex-edge incidence matrix viewed over the appropriate field. The class of even-cycle matroids is not minor-closed. This is a serious limitation and makes it hard to understand this class.

Note that the class of even-cycle matroids is a proper subset of the class of signed-graphic matroids. In fact, the smallest minor-closed class containing all the even-cycle matroids is the class of signed-graphic matroids. The intersection of the class of even-cycle matroid with the class of co-even-cycle matroids has not been studied.

7.4 When do these matroids coincide?

$M(G) = B(G)$ if and only if $G$ has no cycles i.e., $G$ is a forest. $B(G) = F(G)$ if and only if $G$ has no even cycles i.e., every block of $G$ is an odd cycle. $M(G) = F(G)$ if and only if $G$ has no odd cycles i.e., $G$ is bipartite.

7.5 Questions

Since we can associate three matroids to a graph, there are six natural questions to ask:
1. When is \( M(G) \) bicircular?

2. When is \( B(G) \) graphic?

3. When is \( M(G) \) an even-cycle matroid?

4. When is \( F(G) \) graphic?

5. When is \( B(G) \) an even-cycle matroid?

6. When is \( F(G) \) bicircular?

Questions 1 and 2 were answered by Matthews. We answer the fifth question.

### 7.6 When is \( B(G) \) an even-cycle matroid?

In this section we characterize graphs whose bicircular matroids are even-cycle. The following is the main result.

**Theorem 7.6.1.** Let \( G \) be a graph. Then \( B(G) \) is an even-cycle matroid if and only if each component of \( G \) can be obtained, by (repeated) addition of pendant edges, from a subdivision of a tree with loops at some vertices and some edges doubled (an edge can be tripled (quadrupled) if one (both) of its endpoints is (are) pendant and loopless), with the additional condition that all the inner digons are subdivided odd number of times.

We prove the theorem by using the following two lemmas.

**Lemma 7.6.2.** Let \( G \) be a graph such that each component of \( G \) can be obtained, by (repeated) addition of pendant edges, from a subdivision of a tree with loops at some vertices and some edges doubled (an edge can be tripled (quadrupled) if one (both) of its endpoints is (are) pendant and loopless), with the additional condition that all
the inner digons are subdivided odd number of times. Then \( B(G) \) is an even-cycle matroid.

Proof. Let \( G \) have the structure as given in the lemma. We construct a graph \( H \) such that \( B(G) = F(H) \). Let \( H \) be obtained from \( G \) by replacing an end even cycle by an edge and odd cycle and by replacing a theta graph by an odd cycle and an edge and a cycle. Since all cycles in \( H \) have odd length, we have \( B(G) = F(H) \). This shows that \( B(H) \) is an even-cycle matroid. \( \square \)

Let \( H \) be the graph on two vertices with a loop at each vertex and two non-loop edges.

Lemma 7.6.3. Let \( G \) be a graph that is a subdivision of \( H \). Suppose that the digon has been subdivided even number of edges. Then \( B(G) \) is not an even-cycle matroid.

Proof. Assume, to the contrary, that \( B(G) \) is an even-cycle matroid. Then \( B(H) \) is an even-cycle matroid. But \( B(H) \cong U_{2,4} \). It is easy to see that \( U_{2,4} \) is not an even-cycle matroid. Indeed, a negative digon with a negative loop at each vertex is the unique signed-graphic representation of \( U_{2,4} \), showing that \( U_{2,4} \) is not an even-cycle matroid. We conclude that \( B(G) \) is not an even-cycle matroid. \( \square \)

Now we are ready to prove Theorem 7.6.1.

Proof of Theorem 7.6.1. Let \( G \) be a graph having the structure specified in the theorem. Then, by Lemma 7.6.2, we know that \( B(G) \) is an even-cycle matroid. Conversely, let \( G \) be a graph such that \( B(G) \) is an even-cycle matroid. By Theorem 3.3.1, we know that each component of \( G \) can be obtained, by (repeated) addition of pendant edges, from a subdivision of a tree with loops at some vertices and some edges doubled (an edge can be tripled (quadrupled) if one (both) of its endpoints is (are) pendant and loopless). If any of the inner digons is subdivided even number
of times, then, by Lemma 7.6.3, $B(G)$ is not an even-cycle matroid, a contradiction. Hence all inner digons are subdivided odd number of times. This completes the proof of the theorem. $\square$
CHAPTER 8

STRONGLY REGULAR GRAPHS WITH $\lambda = \mu$

8.1 Graphs in which any two vertices have the same number of common neighbors

Let $\lambda$ be a positive integer greater than 1. Let $G$ be a finite graph with any two distinct vertices having exactly $\lambda$ common neighbors. It is well known that there are at most $f(\lambda)$ different possible values for $|V(G)|$, where $f(\lambda)$ denotes the number of factors of $\lambda$. We sharpen the bound by showing that if $\lambda = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3} \cdots p_l^{\alpha_l}$, then the number of possible different values of $|V(G)|$ is less than or equal to

$$\left\lceil \frac{\alpha_1 + 3}{2} \right\rceil \left\lceil \frac{\alpha_2 + 3}{2} \right\rceil \left\lceil \frac{\alpha_3 + 3}{2} \right\rceil \cdots \left\lceil \frac{\alpha_l + 3}{2} \right\rceil.$$  \hspace{1cm} (8.1)

Introduction. Suppose we have a group of people such that any two of them have exactly one friend in common. Then there is a person who is a friend of everybody. This is the friendship theorem [1]. In this note, we discuss the generalized version of the problem by requiring that any two people have exactly $\lambda$ common friends, where $\lambda$ is any positive integer greater than 1.

Suppose a graph has the property that any two distinct vertices have the same number of common neighbors. If this number is greater than 1, then such a graph has to be regular by the following lemma, due to Erdős. The proof of the lemma is due to Bose and Shrikhande [11].
Lemma 8.1.1 (Erdős). Let $\lambda$ be a positive integer greater than 1. Let $G$ be a $n$-vertex graph satisfying the following for any two distinct vertices $x$ and $y$,

$$|N(x) \cap N(y)| = \lambda.$$  \hspace{1cm} (8.2)

Then $G$ is regular.

Proof. Let the vertices of $G$ be $v_1, v_2, v_3, \ldots, v_n$. Fix $i \in [n]$. Double count $S = \{(l, m, j) \mid l, m, j \in [n] \text{ and } l, m \in N(v_i) \cap N(v_j) \text{ and } l \neq m \text{ and } i \neq j\}$. There are $\binom{d(v_i)}{2}$ ways of choosing $l$ and $m$, and for each one of these choices, there are $\lambda - 1$ ways of choosing $j$. Thus we have,

$$|S| = \binom{d(v_i)}{2} (\lambda - 1).$$  \hspace{1cm} (8.3)

Alternatively, there are $n - 1$ ways of choosing $j$ and for each such $j$, there are $\binom{\lambda}{2}$ ways of choosing $l$ and $m$. Thus we also have,

$$|S| = (n - 1) \binom{\lambda}{2}.$$  \hspace{1cm} (8.4)

Equating (8.3) and (8.4), we get

$$d(v_i)(d(v_i) - 1) = (n - 1)\lambda.$$  \hspace{1cm} (8.5)

Thus $d(v_i)$ is uniquely determined by $n$ and $\lambda$, proving that $G$ is regular. $\blacksquare$

Lemma 8.1.2. Let $\lambda$ and $\mu$ be positive integers greater than 1. Let $G$ be a $n$-vertex graph such that every pair of adjacent vertices has $\lambda$ common neighbors, and every pair of distinct non-adjacent vertices has $\mu$ common neighbors. Then $G$ is regular.

Proof. Let $x$ be a vertex in $G$. Let $m = d(x)$. We will count the number of quadrilaterals passing through $x$ in two different ways.

$$\frac{m\lambda}{2}(\lambda - 1) + \left(\binom{m}{2} - \frac{m\lambda}{2}\right)(\mu - 1)$$ \hspace{1cm} (8.6)

$$m\binom{\lambda}{2} + (n - m - 1)\binom{\mu}{2}$$ \hspace{1cm} (8.7)
From equations (8.6) and (8.7), we see that \(m\) is a solution of

\[x^2 + (\mu - \lambda - 1)x - (n - 1)\mu = 0\]  

(8.8)

Since equation (8.8) cannot have two positive roots, \(m\) is uniquely determined by \(\lambda\), \(\mu\), and \(n\). This completes the proof. \(\square\)

A regular graph is said to be strongly regular if it has the property that the number of common neighbors of two distinct vertices depends only on whether the two vertices are adjacent or not. Strongly regular graphs are often referred to as SRGs.

**Definition 8.1.3.** A graph \(G\) is said to be \(SRG(v, k, \lambda, \mu)\) if

1. \(G\) has \(v\) vertices.
2. \(G\) is \(k\)-regular.
3. every pair of adjacent vertices has \(\lambda\) common neighbors.
4. every pair of distinct non-adjacent vertices has \(\mu\) common neighbors.

Stated more formally, a graph \(G\) is \(SRG(v, k, \lambda, \mu)\) if it has \(v\) vertices and satisfies

\[|N(x) \cap N(y)| = \begin{cases} 
  k, & \text{if } x = y \\
  \lambda, & \text{if } x \neq y \text{ and } x \text{ adjacent to } y \\
  \mu, & \text{if } x \neq y \text{ and } x \text{ not adjacent to } y
\end{cases}\]

for \(x, y \in V(G)\).

We will be interested in the case where \(\lambda = \mu\).

**Proposition 8.1.4.** For every positive integer \(\lambda\), there exists a \(SRG(v, k, \lambda, \lambda)\) for some values of \(v\) and \(k\).

**Proof.** The complete graph \(K_{2+\lambda}\) is \(SRG(\lambda + 2, \lambda + 1, \lambda, \lambda)\). \(\square\)
In light of proposition (8.1.4), we will call complete graphs trivially strongly regular, and hence will not be including them in our classification.

**Theorem 8.1.5.** Let $\lambda$ be a positive integer greater than 1. Let $\lambda = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3} \cdots p_l^{\alpha_l}$ be the prime factorization of $\lambda$ and $G$ be a non-complete $v$-vertex graph in which any two distinct vertices have exactly $\lambda$ common neighbors. Then the number of possible different values $v$ can take is less than or equal to

$$\left\lceil \frac{\alpha_1 + 3}{2} \right\rceil \left\lceil \frac{\alpha_2 + 3}{2} \right\rceil \left\lceil \frac{\alpha_3 + 3}{2} \right\rceil \cdots \left\lceil \frac{\alpha_l + 3}{2} \right\rceil.$$  

(8.9)

**Proof.** By lemma (8.1.1), we see that $G$ is regular. From definition 8.1.3, we have $G$ is $SRG(v, k, \lambda, \lambda)$, where $v, k, \text{ and } \lambda$ are related by

$$k(k - 1) = \lambda(v - 1).$$  

(8.10)

Following Erdős, Rényi and Sós [23], we find the eigenvalues of $A^2(G)$, where $A(G)$ is the adjacency matrix of $G$.

$$A^2(G) = (r - k)I + kJ.$$  

(8.11)

The eigenvalues of $A^2(G)$ are $k - \lambda$ and $k^2$. The eigenvalues of $A(G)$ are $\pm \sqrt{k - \lambda}$ and $k$. Let the multiplicity of the eigenvalue $-\sqrt{k - \lambda}$ be $m$. Since $tr(A(G)) = 0$, we get

$$(v - 2m - 1)\sqrt{k - \lambda} + k = 0.$$  

(8.12)

$$\quad (v - 2m - 1)^2(k - \lambda) = k^2.$$  

(8.13)

Equations (8.10) and (8.13) severely restrict the possible values of $v$.

Equation (8.12) implies $\sqrt{k - \lambda}$ divides $k$, and since $\sqrt{k - \lambda}$ divides $k - \lambda$, $\sqrt{k - \lambda}$ divides $-\lambda$. Hence the number of different possible values of $k - \lambda$ is less than or equal to the number of factors of $\lambda$. A value for $k - \lambda$ gives unique value for $k$ and this uniquely determines $v$.  

76
Let $k - \lambda = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \cdots p_l^{\beta_l}$.

If $0 < \beta_i < \lceil \alpha_i + \frac{1}{2} \rceil$, then $p_i^{\alpha_i}$ does not divide $k$, which contradicts (8.10).

Hence we have the following bounds on $\beta_i$ ($1 \leq i \leq l$).

$$\left\lceil \frac{\alpha_i + 1}{2} \right\rceil \leq \beta_i \leq \alpha_i$$

or

$$\beta_i = 0.$$

There are $\left\lceil \frac{\alpha_i + 3}{2} \right\rceil$ possible values of $\beta_i$. This means that there are

$$\left\lceil \frac{\alpha_1 + 3}{2} \right\rceil \left\lceil \frac{\alpha_2 + 3}{2} \right\rceil \left\lceil \frac{\alpha_3 + 3}{2} \right\rceil \cdots \left\lceil \frac{\alpha_l + 3}{2} \right\rceil$$

possible values for $k - \lambda$, and hence for $k$. Given $\lambda$ and $k$, equation (8.10) forces $v$ to be unique. Hence the total number of possible values of $v$ is at most

$$\left\lceil \frac{\alpha_1 + 3}{2} \right\rceil \left\lceil \frac{\alpha_2 + 3}{2} \right\rceil \left\lceil \frac{\alpha_3 + 3}{2} \right\rceil \cdots \left\lceil \frac{\alpha_l + 3}{2} \right\rceil.$$

Hence the theorem is established. □

**Proposition 8.1.6.** Let $p$ be a prime. Let $G$ be a non-complete finite graph with any two distinct vertices having exactly $p$ common neighbors. Then $G$ is

$$SRG(p^2(p + 2), p^2 + p, p, p).$$

**Proof.** By Lemma 8.1.1 $G$ is regular. Let us denote the number of vertices and the degree of $G$ by $v$ and $k$, respectively. We see that $G$ is $SRG(v, k, p, p)$. By equation (8.12), we know that $\sqrt{k - p}$ divides $p$. Hence $\sqrt{k - p} = 1$ or $\sqrt{k - p} = 1$. Since $G$ is not complete, the former cannot happen. Therefore $\sqrt{k - p} = p$, giving $k = p^2 + p$. Using equation (8.10), we get $v = p^2(p + 2)$. □
Proposition 8.1.7. Let $p$ and $q$ be distinct primes ($p < q$) with $q^2 \not\equiv 1 \pmod{p}$.
Let $G$ be a non-complete finite graph with any two distinct vertices having exactly $pq$
common neighbors. Then $G$ is

\[ \text{SRG}(p^2q^2(pq + 2), p^2q^2 + pq, pq, pq). \]

Proof. By Lemma 8.1.1 $G$ is regular. Let us denote the number of vertices and the
degree of $G$ by $v$ and $k$, respectively. We see that $G$ is $\text{SRG}(v, k, \lambda, \lambda)$, where $\lambda = pq$.
Equation (8.13) and the note that follows immediately after that implies that $k - \lambda$
can take only four possible values: 1, $p^2$, $q^2$, or $p^2q^2$. We will show that the only
possible value for $k - \lambda$ is $p^2q^2$ by showing that the first three possibilities cannot
occur.

- $k - \lambda = 1$ implies that $G$ is complete, which is not true.

- $k - \lambda = p^2$ implies that $k = p^2 + pq$. Equation (8.10) implies that $q|(p^2 + pq - 1)$.
  This can happen only if $q|p^2 - 1$, which can happen only when $p = 2$ and $q = 3$.
  This is not possible since $3^2 \equiv 1 \pmod{2}$, contradicting the congruence condition
  in the proposition.

- $k - \lambda = q^2$ implies that $k = q^2 + pq$. Equation (8.10) implies that $p|(q^2 + pq - 1)$.
  This can happen only if $p|q^2 - 1$, which contradicts the congruence condition
  in the proposition.

We conclude that $k - \lambda = p^2q^2$, and hence $k = pq + p^2q^2$. Using this in equation
(8.10), we get $v = p^2q^2(pq + 2)$. □

Definition 8.1.8. A graph $G$ is said to be a $(v, k, \lambda)$-graph if $G$ is $\text{SRG}(v, k, \lambda, \lambda)$.

Proposition 8.1.9. The number of triangles in a $(v, k, \lambda)$-graph is $\frac{v(k\lambda)}{6}$.
Proof. Let $G$ be a $(v, k, \lambda)$-graph. Double counting $S = \{ (T, e) \mid T$ is a triangle in $G$, $e$ is an edge in $T \}$, we get

$$3t(G) = e\lambda.$$ 

This gives

$$t(G) = \frac{e\lambda}{3} = \frac{vk\lambda}{6},$$ 

proving the claim. □

**Proposition 8.1.10.** The number of quadrilaterals in a $(v, k, \lambda)$-graph is $\frac{v(v-2)}{2} \frac{\lambda(\lambda-2)}{2}$.

Proof. Let $G$ be a $(v, k, \lambda)$-graph. There are $\binom{v}{2}$ ways of choosing two vertices $a$ and $b$, and $\binom{k}{2}$ ways of choosing two vertices $c$ and $d$ such that both $a$ and $b$ are adjacent to $c$ as well as $d$. Since we are counting every quadrilateral twice, we get

$$q(G) = \frac{\binom{v}{2} \binom{\lambda}{2}}{2},$$ 

proving the claim. □

**Lemma 8.1.11.** Let $\lambda$ and $k$ be positive integers with $\lambda \mid k$ and let the multiplicity of every prime in $\lambda$ be odd. The number of distinct values of $v$ for which there exists a $(v, k, \lambda)$-graph is less than

$$f(\lambda) \frac{2}{2\pi(\lambda)}.$$ 

Proof. Following exactly the steps in proof of Theorem 8.1.5, and noting that none of the $\beta_i$ is 0, we get the possible values of $v$ for which there exists a $(v, k, \lambda)$-graph is less than

$$\left( \frac{\alpha_1 + 1}{2} \right) \left( \frac{\alpha_2 + 1}{2} \right) \left( \frac{\alpha_3 + 1}{2} \right) \ldots \left( \frac{\alpha_l + 1}{2} \right)$$
which is equal to

\[
\frac{f(\lambda)}{2\pi(\lambda)},
\]

proving the claim. □

The total number of non-isomorphic $SRG(16, 6, 2, 2)$ is known and we discuss this in the next section. By using an extensive computer search, Edward Spence has determined that the total number of non-isomorphic $SRG(45, 12, 3, 3)$ is 78 (see [53]).

8.2 Graphs in which any two vertices have exactly two common neighbors

![Figure 8.1: (a) The Shrikhande graph (b) The line graph of $K_{4,4}$](image)

The question of existence of configurations with specific parameters is a common theme in combinatorial mathematics. In graph theory, we are interested in determining whether graphs exist with certain properties, and if so, list all of them. In general
THE SHRIKHANDE GRAPH

Figure 8.2: The Shrikhande Graph

Figure 8.3: Constructing the Shrikhande Graph
this can be a very difficult problem. For example, the existence of a 57-regular Moore graph is a famous open problem in this area. Here we consider graphs where any two vertices have the same number of common adjacent vertices.

**Theorem 8.2.1.** Let $G$ be a finite simple graph on at least two vertices such that any two distinct vertices have exactly two common adjacent vertices. Then $G$ is isomorphic to $K_4$, $L(K_{4,4})$, or the Shrikhande graph.

**Lemma 8.2.2.** Let $G$ be a finite simple graph on at least two vertices such that any two distinct vertices have exactly two common adjacent vertices. Then $G$ is $r$-regular, where \( (r/2) = |V(G)| - 1 \).

*Proof.* Let $x \in V(G)$. Counting the number of quadrilaterals passing through $x$ in two different ways, we get that \( (d(x)/2) = |V(G)| - 1 \). This shows that $G$ is $r$-regular, where \( (r/2) = |V(G)| - 1 \). \[ \square \]

**Lemma 8.2.3.** Let $G$ be a finite simple $r$-regular graph on at least two vertices such that any two distinct vertices have exactly two common adjacent vertices. Then $r \in \{3, 6\}$.

*Proof.* Let $n = |V(G)|$. Note that $A^2(G) = 2J + (r - 2)I$. The eigenvalues of $A^2(G)$ are $2n + r - 2$ (multiplicity 1) and $r - 2$ (multiplicity $n - 1$). This means that the eigenvalues of $A(G)$ are $\sqrt{2n + r - 2}$ (multiplicity 1) and $\sqrt{r - 2}$ (multiplicity $m_1$) and $-\sqrt{r - 2}$ (multiplicity $m_2$), where $m_1 + m_2 = n - 1$. Since $tr(A(G)) = 0$, we get $(m_1 - m_2)\sqrt{r - 2} + \sqrt{2n + r - 2} = 0$. This leads to $(m_1 - m_2)^2(r - 2) = 2n + r - 2$, yielding that $(r - 2)|2n$. Hence we have $2n \equiv 0 \pmod{r - 2}$. By lemma 8.2.2, \( (r/2) = n - 1 \), or $r^2 - r = 2n - 2$. Taking modulo $r - 2$, we get $2 \equiv -2 \pmod{r - 2}$. Hence we conclude that $r - 2|4$. This leads to three possibilities: $r = 3, 4, \text{ or } 6$. If $r = 4$, then $n = 7$, and $(m_1 - m_2)\sqrt{2} + 4 = 0$ for some nonnegative integers $m_1$ and $m_2$, implying that $\sqrt{2}$ is rational. Hence $r = 4$ is not possible. \[ \square \]
Lemma 8.2.4 (Shrikhande [44]). Let $G$ be a simple 16-vertex 6-regular graph such that any two distinct vertices have exactly two common adjacent vertices. Then $G$ is isomorphic to $L(K_{4,4})$, or the Shrikhande graph.

Proof. Let $v$ be a vertex of $G$. Then $G[N(x)]$ is clearly a 2-regular graph on 6 vertices. There are only two possibilities for $G[N(x)]$: (i) cycle of length 6; (ii) disjoint union of two triangles. It is routine to check that each of these possibilities gives rise to a unique graph: the former leads to the Shrikhande graph (Figure 8.2) and the latter leads to the line graph of $K_{4,4}$. This completes the proof. 

Note 8.2.5. Consider the first possibility. Without loss of generality we may assume that the graph looks as shown in Figure 8.3. The following is a sequence of edges that are forced:

$12, 23, 46, 56, 78, 79, 17, 47, 25, 28, 36, 69, 34, 48, 15, 59, 38, 19$, giving rise to the Shrikhande graph (figure 8.2). The second case is easier than the first and we omit the details.

Now we can complete the proof of Theorem 8.2.1.

Proof of Theorem 8.2.1. By Lemma 8.2.2, we know that $G$ is regular. By Lemma 8.2.3, we know that $G$ is either 3-regular, or 6-regular. If $G$ is 3-regular, then it has 4 vertices, and $G \cong K_4$. If $G$ is 6-regular, then $G$ has 16 vertices, and by Lemma 8.2.4, $G$ is isomorphic to either the line graph of $K_{4,4}$, or the Shrikhande graph. This completes the proof of the theorem. 

---

1The details are given in Note 8.2.5
The following problem is listed in a set of hand-written articles that E. W. Dijkstra [21] wrote during different periods of time. He also gives a solution to the problem. Our object in this chapter is to give a different proof of the result, and we hope that this new proof can be useful for attacking problems of similar nature.

Let two positive integers $p$ and $q$ be given. We start with a bag containing a finite number of integers. When the bag contains some number, $x$ say, at least twice, a move is possible; the move consists of replacing in the bag two occurrences of $x$ by $x + p$ and $x - q$ respectively. Show that this game terminates.

In the bag are integers $a_1, a_2, \ldots, a_s$. We may arrange these in any order, so suppose

$$a_1 \leq a_2 \leq \ldots \leq a_s.$$ 

Perform some moves to get a new sequence: we do not reorder the resulting values. If equal values $\ldots c, \ldots, c, \ldots$ occur in a sequence, we may replace them by $\ldots c - q, \ldots, c + p, \ldots$ always with $c - q$ in the earlier position, $c + p$ in the latter.

There are sometimes choices involved in making moves since several pairs of equal values could occur. The claim here is that no matter how the moves are made, the game must eventually end.

To prove this we introduce some notation.
$a_1^{(t)}, a_2^{(t)}, \ldots, a_s^{(t)}$ is the sequence after $t$ moves. Then $a_i^{(0)} = a_i$ and only two positions differ between sequence $t$ and sequence $t + 1$.

Let $M_k = \max\{a_k^{(t)}\}$ be the maximum value in the $k$ position as game is played. Note that $a_1^{(t)} \leq a_1$ because any move involving the first position can only decrease that value. Hence $M_1 = a_1$.

**Lemma 9.0.6.** $M_1, M_2, M_3, \ldots, M_s$ are finite.

To see this, let’s assume $M_1, M_2, M_3, \ldots, M_k$ are known to be finite, and analyze $M_{k+1}$. At step 0, the $(k + 1)$ position is $a_{k+1}$. Suppose after $t$ moves the value $a_{k+1}^{(t)}$ is larger than $a_{k+1}$ for the first time. Then at the previous step we have $a_{k+1}^{(t-1)} = b$ for some $b \leq a_{k+1}$ and the next move bumps it up to $a_{k+1}^{(t)} = b + p > a_{k+1}$. But this can happen only if $b$ also appears earlier in the sequence: $b = a_j^{t-1}$. Then $b \leq M_j$ and $a_{k+1} < a_{k+1}^{(t)} = b + p \leq M_j + p$.

If this situation happens, then $a_{k+1} < M_j + p \leq \max\{M_1, M_2, \ldots, M_k\} + p$. Consequently, if $a_{k+1} \geq \max\{M_1, M_2, \ldots, M_k\} + p$, then this increase cannot occur, so that $a_{k+1}^{(t)} \leq a_{k+1}$ for every $t$. That is, $M_{k+1} \leq a_{k+1}$.

To continue we will assume $a_{k+1} < \max\{M_1, M_2, \ldots, M_k\} + p$. The values of $a_{k+1}^{(t)}$ could increase beyond $a_{k+1}$, but we claim the following:

**Lemma 9.0.7.**

$$a_{k+1}^{(t)} \leq \max\{M_1, M_2, \ldots, M_k\} + p$$

for all $t$.

**Proof.** If false, let $t$ be the first time it fails. Then value

$$c = a_{k+1}^{(t-1)} \leq \max\{M_1, M_2, \ldots, M_k\} + p$$

but at the next step

$$c + p = a_{k+1}^{(t)} > \max\{M_1, M_2, \ldots, M_k\} + p \quad (9.1)$$
This move can happen only if value $c$ also appears earlier in the $(t - 1)$ sequence: $\ c = a_j^{t-1}$ for some $j \leq k$. Then $c \leq M_j$, but inequality 9.1 above says $c > \max\{M_1, M_2, \ldots, M_k\}$, a contradiction. Therefore

$$M_{k+1} \leq \max\{M_1, M_2, \ldots, M_k\} + p$$

in this case. \[ \square \]

To summarize, we have proved $M_{k+1}$ is finite in all cases. In fact:

$$M_{k+1} \leq \max\{\max\{M_1, M_2, \ldots, M_k\} + p, a_{k+1}\}.$$  \hspace{1cm} (9.2)

In particular, we have

$$M_1 = a_1.$$

$$M_2 \leq \max\{a_1 + p, a_2\}.$$

$$M_3 \leq \max\{a_1, \max\{a_1 + p, a_2\} + p, a_3\} = \max\{a_1 + p, a_1 + 2p, a_2 + p, a_3\} = \max\{a_1 + 2p, a_2 + p, a_3\}.$$

It follows that

$$M_{k+1} \leq \max\{a_1 + kp, a_2 + (k - 1)p, \ldots, a_k + p, a_{k+1}\}.$$  \hspace{1cm} (9.3)

Now to get a lower bound on the entries, let

$$m_k = \min\{a_k^{(t)}\} = \text{minimum value in the } k\text{th position as game is played}.$$

To analyze this value we consider the negative of our game. Start with the initial sequence

$$b_1 = -a_s, \quad b_2 = -a_{s-1}, \ldots, b_j = -a_{s-j+1}, \ldots, b_s = -a_1$$
and allow moves

\[ c, c \rightarrow c - p, c + q. \]

Every move in this game corresponds exactly to a move of the original game, with all signs reversed:

\[ b_j^{(t)} = -a_{s-j+1}^{(t)}. \]

In particular, \( M'_k = \) maximum for the negative game becomes

\[ M'_k = -m_k. \]

The earlier bound says

\[ m_{k+1} \geq \min\{a_s - kq, a_{s-1} - (k - 1)q, \ldots, a_{s-k+1} - q, a_{s-k}\}. \]

1. **Case I** \( p \neq q \) The work above proves that for a given initial sequence \( a_1 \leq a_2 \leq \ldots \leq a_s \), there are absolute bounds \( B_1 < B_2 \) (depending on \( a_1, a_2, \ldots, a_s \)) such that

\[ B_1 < a_k^{(t)} < B_2 \]

for all \( 1 \leq k \leq s \) and for all \( t = 0, 1, 2, \ldots \).

Now to prove the game must terminate, let us track how the sum of all values changes as moves are made. As we make the \( (t + 1) \) st move, we go from \( \sum_{k=1}^{s} a_k^{(t)} \) to \( \sum_{k=1}^{s} a_k^{(t+1)} \). This change is equal to \( p - q \), since some \( c, c \rightarrow c - q, c + p \). If \( p - q \neq 0 \) there cannot be an infinitely long game because \( \sum_{k=1}^{s} a_k^{(t)} \) is bounded.

2. **Case II** \( p = q \) Since all the entries are bounded below, let’s add an appropriate constant to all entries to ensure that every \( a_k^{(t)} > 0 \) for all \( k \) and for all
Now we examine the net change of the product $P^{(t)} = \prod_{k=1}^{s} a_k^{(t)}$. From one step to the next the product changes by altering some terms $c, \ldots, c$ to terms $c - p, \ldots, c + p$ (with all other terms unchanged). Then $P^{(t)}$ becomes

$$P^{(t+1)} = \frac{(c - p)(c + p)}{c,c} P^{(t)} = \left(1 - \frac{p}{c}\right) \left(1 + \frac{p}{c}\right) P^{(t)} = \left(1 - \left(\frac{p}{c}\right)^2\right) P^{(t)} < P^{(t)}.$$ 

Since these products are positive integers they cannot decrease forever. The process must stop. Hence the claim.

Note that the above proof gives an upper bound for the number of moves for a given finite multiset of integers and positive integers $p$ and $q$. 

88
CHAPTER 10
FLEXIBILITY OF PROJECTIVE-PLANAR EMBEDDINGS

This chapter is based on joint work with John Maharry, Neil Robertson, and Daniel Slilaty [34]. The following is the main result.

Theorem 10.0.8 (Maharry, Robertson, Sivaraman, Slilaty). Let $G$ be a connected nonplanar graph. If $\sigma_1$ and $\sigma_2$ are two inequivalent embeddings of $G$ on the projective plane, then there exists a sequence of $Q$-twists, $P$-twists, and $W$-twists taking $\sigma_1$ to $\sigma_2$.

Let $n$ be an integer greater than 2. The Möbius ladder of order $n$, denoted $V_{2n}$, is the cubic graph on $2n$ vertices obtained from a cycle on $2n$ vertices by connecting every pair of antipodal vertices by an edge.

10.1 Projective-planar graphs

The projective plane can be thought of as a disc with antipodal points identified. This surface is non-orientable with genus 1 (sphere with one cross-cap added). A graph is projective-planar if it can be embedded in the projective plane without edges crossing. The projective plane has no two disjoint essential (non-contractible) cycles. The Petersen graph, $K_6$, $V_8$, and $K_{3,4}$ are nonplanar but projective-planar. Which graphs are projective-planar? Archdeacon-Glover-Huneke-Wang [4] proved that the
Figure 10.1: Forbidden minors for projective-planarity
set of forbidden minors for projective-planarity has exactly 35 members (see Figure 10.1). $K_n$ is projective-planar if and only if $n \leq 6$. $K_{m,n}(m \leq n)$ is projective-planar if and only if $m \leq 2$ or $(m = 3$ and $3 \leq n \leq 4$).

Some important questions are as follows:

- In how many essentially different ways does a graph embed in the projective plane?
- How can we get from any one embedding to any other?
- What can we say if our graph is highly connected?
- How does representativity\(^1\) affect embedding flexibility?

10.2 The three operations

The three operations featuring in the main theorem are shown in Figures 10.2 and 10.3. More details can be found in [34].

10.3 Sketch of proof of the main theorem

For the reader’s convenience, we list the main theorem once again.

**Theorem 10.3.1.** Let $G$ be a connected nonplanar graph. If $\sigma_1$ and $\sigma_2$ are two inequivalent embeddings of $G$ on the projective plane, then there exists a sequence of Q-twists, P-twists, and W-twists taking $\sigma_1$ to $\sigma_2$.

We give the overall structure of the proof. Details can be found in [34].

---

\(^1\)The *representativity* (face-width) of a graph embedded in a surface $S$ is the smallest number $k$ such that $S$ contains an essential closed curve that intersects the graph in $k$ points. It is a measure of how densely a graph is embedded on a surface.
Figure 10.2: W-twist

Figure 10.3: P-twist and Q-twist
Connectivity reduction: It suffices to prove the following: Suppose a 3-connected nonplanar graph $G$ has two inequivalent embeddings on the projective plane, say $\sigma_1$ and $\sigma_2$. Then $\sigma_2$ can be obtained from $\sigma_1$ by a sequence of Q-twists and P-twists.

Start with a subgraph with known flexibility and see how the bridges behave with respect to the embedding. $V_8$ was a natural choice considering the fact that the octagon has to embed as a contractible cycle in any embedding of $V_8$, and the fact that the structure of graphs not containing $V_8$ is known. Hence, the proof is in two parts, depending on whether the graph contains $V_8$ as a topological subgraph or not. If the given graph does not contain $V_8$, we use the following theorem of Robertson.

**Theorem 10.3.2** (Robertson). An internally 4-connected graph\(^2\) $G$ has no $V_8$-minor if and only if one of the following holds:

- $G$ is planar
- $G$ has at most 7 vertices
- $G$ has two vertices $u, v$ such that $G - \{u, v\}$ is a cycle
- $G$ has four vertices $u, v, w, x$ such that $G - \{u, v, w, x\}$ is edgeless
- $G \cong L(K_{3,3})$

**Case 1:** Graphs containing $V_8$

Let $G$ be a 3-connected graph with two inequivalent embeddings on the projective plane, say $\sigma_1$ and $\sigma_2$. Let $H$ be a $V_{2n}$-subdivision contained in $G$ with $n$ maximum.

\(^2\)A graph $G$ is said to be **internally 4-connected** if it is 3-connected and, for every separation $(A, B)$ of $G$ of order 3, one of $A - B$, $B - A$ contains at most one vertex.
• Identify the reembeddable $H$-bridges.

• Define a sequence of $P$-twists and $Q$-twists taking $\sigma_1$ to $\sigma_2$.

• Make sure that there are no bridges that obstruct the twists.

The analysis splits into 5 subcases, depending on how the outer cycle of the Möbius ladder embeds. Note that $V_{2n}$ has exactly $n + 1$ projective embeddings.

**Case 2:** Graphs not containing $V_8$

This case is again split into three subcases:

• 3-sum of two planar graphs

• 3-sum of two non-planar graphs

• Reduction to I-4-C frame: A 3-connected nonplanar graph $G$ admits a patch decomposition with an internally 4-connected frame $F_G$ and patches $P_i$ when either $G$ is internally 4-connected (and is its own frame with no patches) or

$$G = F_G \bigoplus_3 P_1 \bigoplus_3 P_2 \bigoplus_3 \ldots \bigoplus_3 P_k$$

where

− $F_G$ is an internally 4-connected non-planar graph

− Each $P_i$ is planar and summed into a triangle of $F_G$

− The triangle of summation is peripheral in $P_i$

− No three $P_i$ are summed into the same triangle of $F_G$

Having certain patches will force the graph to contain one of the 35 forbidden graphs, and hence the possibility of what can occur as patches is severely restricted.

We use the theorem of Robertson mentioned above to analyze the third subcase. The analysis is split into the following cases.
• Frames on five vertices: $K_5$

• Frames on six vertices: $K_6, K_6 \setminus e, DW_4, K_{3,3}$

• Frames on seven vertices: Out of the 28 I-4-C graphs on 7 vertices, one is planar and seven are non-projective-planar

• Frames that are subgraphs of double wheels

• Frames that contain $K_{3,4}$

• Frames that are 4-vertex coverable graphs

• Frames isomorphic to $L(K_{3,3})$

This concludes all the cases and establishes the theorem.

10.4 Some corollaries

As corollaries of our main theorem, we get the following results, proved earlier by Vitray, Negami and Kitakubo.

• If $G$ is 3-connected and has a 4-representative embedding in the projective plane, then the embedding is unique.

• If $G$ is 3-connected and has a 3-representative embedding in the projective plane, then the number of inequivalent embeddings of $G$ in the projective plane is a divisor of 12.

• If $G \not\cong K_6$, is 5-connected, and has a 3-representative embedding in the projective plane, then this embedding is the unique embedding of $G$ in the projective plane.
10.5 Connection to the signed-graphic matroid representation problem

The signed-graphic matroid representation problem can be stated as follows: Given a matroid $M$, find all signed graphs that represent $M$ (whose frame matroid is $M$). Also, we would like to know if there exists a finite set of operations that can explain any two different signed-graphic representations of a matroid. If the matroid is regular, the problem becomes more tractable.

A graph embedded on the projective plane gives rise to a signed graph as follows: Make edges that use the crosscap negative and all the other edges positive. Dan Slilaty [50] proved the following important theorem as an analogue of a theorem of Tutte characterizing cycle matroids of planar graphs:

**Theorem 10.5.1** (Slilaty [50]). Let $G$ be a 2-connected graph. If $M^*(G)$ is signed-graphic, then $G$ is projective-planar.

Slilaty also noticed an important connection between the flexibility of graphs on the projective plane and the different signed-graphic representations of regular matroids. In order to understand the different signed-graphic representations of a regular matroid, it suffices to understand the different projective-planar embeddings of a suitable graph. Each of the operations mentioned in this chapter should correspond to an operation on signed graphs that will preserve the frame matroid structure. One disadvantage with this approach is that this works only for regular matroids. For more details we refer the interested reader to [52] and [34].
CHAPTER 11
TOWARDS A SIGNED-GRAPHIC ANALOGUE OF BONDY’S THEOREM

11.1 Introduction

A theorem of Bondy [8] gives a characterization of graphs whose cycle matroids are transversal. We give partial results towards a characterization of signed graphs whose frame matroids are transversal.

All graphs and matroids mentioned here are assumed to be finite. A subdivision of a signed graph Σ is obtained by replacing some edges of Σ by paths (with the same sign as that of the corresponding edge), where the internal vertices of the paths are disjoint from the vertices of G.

11.2 Transversal matroids and their properties

Transversal matroids were introduced by Edmonds and Fulkerson [22]. We will take a bipartite view of them. Let $G(A \cup B, E)$ be a bipartite graph. The transversal matroid of $G$ is the matroid on $A$ whose independent sets are the subsets of $A$ that can be covered by a matching in $G$. An isolated vertex in $A$ corresponds to a loop in the transversal matroid. Two pendant vertices in $A$ with the same neighbor correspond to a pair of parallel elements in the transversal matroid.
Transversal matroids are representable over all sufficiently large fields and over all infinite fields (Ingleton and Piff [29]). They are base-orderable, and can be written as the union of rank-1 matroids. But the class of transversal matroids is not minor-closed, although a restriction of a transversal matroid is transversal. Transversal matroids are gammoids. In fact, the smallest minor-closed class of matroids containing the transversal matroids is the class of gammoids. Also, all uniform matroids are transversal. A recent wonderful survey [10] by J. Bonin on transversal matroids is a good pointer for the interested reader.

11.3 Statement of Bondy’s theorem

Theorem 11.3.1 (Bondy, [8]). Let $G$ be a graph. Then $M(G)$ is transversal if and only if no subgraph of $G$ is a subdivision of $K_4$ or $C_k^2$, for $k \geq 3$.

An important lemma that Bondy used to prove his result is the following:

Lemma 11.3.2. Let $M$ be a matroid. Let $e$ and $f$ be distinct elements of $M$. Suppose that $e$ and $f$ are in series. If $M$ is transversal, so is $M/f$.

11.4 When is $M(\Sigma)$ transversal?

The reader can find the four signed graphs whose frame matroids are isomorphic to $M(K_4)$ in Figure 6.1.

Lemma 11.4.1. Let $\Sigma$ be a signed graph. If a subdivision of any of the four signed graphs representing $M(K_4)$ is a subgraph of $\Sigma$, then $M(\Sigma)$ is not transversal.

Proof. Note that $M(K_4)$ is not transversal. We are done by Lemma 11.3.2. □

Lemma 11.4.2. Let $\Sigma$ be a signed graph. If $\pm C_4 \setminus e$ is a minor of $\Sigma$, then $M(\Sigma)$ is not transversal.
Proof. Note that $M(\pm C_4 \setminus e) \cong (F_7^-)^*$, the dual of the non-Fano matroid. It is well known that $(F_7^-)^*$ is not representable over any field of characteristic two (cf. [39]). If $M(\Sigma)$ was transversal, it would be representable over some field of characteristic two, and $(F_7^-)^*$, being a minor of it, will be representable over some field of characteristic two, a contradiction. Hence $M(\Sigma)$ is not transversal. □

**Lemma 11.4.3.** Let $M$ be a matroid. If $M$ has a circuit $C$ with at least 3 elements, and each element of $C$ is in a non-trivial parallel class, then $M$ is not transversal.

**Proof.** Suppose that $M$ is transversal. Let $B$ be a bipartite graph representing $M$ ($E(M)$ is a partite set of $B$). If an element $e \in E(M)$ is in a non-trivial parallel class, then it has degree 1 in $B$. Since any pair of elements in $C$ is independent, no two elements in $C$ have a common neighbor, and hence $C$ is independent, a contradiction. □

**Lemma 11.4.4.** Let $\Sigma$ be a signed graph. If a subdivision of a doubled positive circle, or a negative handcuff is a subgraph of $\Sigma$, then $M(\Sigma)$ is not transversal.

**Proof.** Follows from Lemma 11.4.3 and Lemma 11.3.2. □

Recall that $W_n$ denotes the $n$-wheel, the graph obtained from the cycle graph on $n$ vertices by adding a vertex adjacent to every vertex in the cycle.

**Lemma 11.4.5.** Let $n \geq 4$. Then $M(-W_n)$ is not transversal.

**Proof.** By virtue of Lemma 11.3.2, it suffices to prove the result for $n = 4$ and $n = 5$. Suppose $n = 4$. It is easy to see that the frame matroid of $-W_4$ is isomorphic to the cycle matroid of a subdivision of $K_4$, and hence not transversal. Suppose $n = 5$. It is well known and easy to check that $M(-W_5)$ is isomorphic to $M^*(K_5)$. A theorem of de Sousa and Welsh [18] says that a binary transversal matroid is graphic. Since $M^*(K_5)$ is not graphic, we conclude that $M(-W_5)$ is not transversal. This completes the proof. □
Lemma 11.4.6. Let $\Sigma$ be a signed graph. If a subdivision of the negative $n$-wheel ($n \geq 4$) is a subgraph of $\Sigma$, then $M(\Sigma)$ is not transversal.

Proof. Follows from Lemma 11.4.5 and Lemma 11.3.2. \qed

Having figured out some of the obstructions, we offer the following conjecture:

Conjecture 11.4.7. Let $\Sigma$ be a signed graph not containing the following signed graphs as a subdivision: The four signed graphs representing $M(K_4)$, doubled positive circles, doubled negative loose handcuffs, doubled negative tight handcuffs, negative wheels of order at least 5. Furthermore, suppose that $\pm C_4 \setminus e$ is not a minor of $\Sigma$. Then $M(\Sigma)$ is transversal.

Bondy proved his theorem with lemma stating that if a graph $G$ has neither $K_4$ nor any of $C_4^2$ ($k > 2$) as a subdivision, then $G$ has a vertex with exactly two neighbors. We need something similar for signed graphs not containing the above mentioned objects.

We conclude with the following two open problems. A matroid $M$ is said to be base-orderable if for any two bases $B$ and $B'$, there exists a bijective map $\sigma : B \to B'$ such that, for all $x \in B$, both $B - x + \sigma(x)$ and $B' - \sigma(x) + x$ are bases of $M$. A matroid $M$ is said to be strongly base-orderable if for any two bases $B$ and $B'$, there exists a bijective map $\sigma : B \to B'$ such that, for all $X \subseteq B$, both $B - X + \sigma(X)$ and $B' - \sigma(X) + X$ are bases of $M$.

**Question 11.4.8.** Characterize signed graphs whose frame matroids are base-orderable.

**Question 11.4.9.** Characterize signed graphs whose frame matroids are strongly base-orderable.
12.1 A construction of 4-regular graphs of girth 5 on 19\(k\) vertices

We construct a family of 4-regular graphs of girth 5 using elementary modulo arithmetic. In particular, an easy algebraic construction is given for the Robertson graph, the unique (4, 5)-cage.

Let \(k\) be a positive integer. Let \(G = (V, E)\), where \(V\) and \(E\) are defined as follows:

\[
V = A \cup B \cup C, \quad \text{where}
\]
\[
A = \{a_1, a_2, \ldots, a_{4k}\}
\]
\[
B = \{b_1, b_2, \ldots, b_{3k}\}
\]
\[
C = \{c_1, c_2, \ldots, c_{12k}\}
\]

and \(E = E_1 \cup E_2 \cup E_3 \cup E_4\), where

\[
E_1 = \{\{a_i, a_{i+2k}\} \mid i = 1, 2, \ldots, 2k\}
\]
\[
E_2 = \{\{c_i, c_{i+1}\} \mid i = 1, 2, \ldots, 12k - 1\} \cup \{c_{12k}, c_1\}
\]
\[
E_3 = \{\{b_i, c_j\} \mid j \equiv i \mod 3k\}
\]
\[
E_4 = \{\{a_i, c_j\} \mid j \equiv i \mod 4k\}
\]

**Proposition 12.1.1.** \(G\) is a connected 4-regular graph with girth 5 on 19\(k\) vertices.

**Proof.** Number of vertices in \(G = |A| + |B| + |C| = 4k + 3k + 12k = 19k\). Clearly \(G\) is
connected and is 4-regular. It is easy to see, by using elementary modulo arithmetic, that $G$ has neither triangles (3-cycles) nor quadrilaterals (4-cycles). The five cycle $a_1c_1b_1c_{6k+1}a_{2k+1}$ shows that $G$ has girth 5. □

The special case $k = 1$ gives the Robertson graph [42], the unique (4, 5)-cage. In fact, the modulo argument presented here was obtained by analyzing the Robertson graph. It is easy to see from this definition that the automorphism group of the Robertson graph is the dihedral group of order 24. Indeed, there is a 1-1 correspondence between $\text{Aut}(G[C])$ and $\text{Aut}(G)$, implying that the automorphism group of $G$ is the dihedral group of order $24k$.

12.2 A proof of Brooks’ theorem and Catlin’s theorem

All graphs in this section are simple and finite. Let $G$ be a graph. $\alpha(G)$ denotes the independence (stability) number of $G$, the maximum number of pairwise non-adjacent vertices in $G$. If $X$ is a set of vertices in $G$, then $G \setminus X$ denotes the graph obtained from $G$ by deleting the vertices in $X$. An $n$-coloring of $G$ is a partition of $V(G)$ into $n$ independent sets. The chromatic number of $G$ is the smallest $k$ such that $G$ has a $k$-coloring, and is denoted $\chi(G)$.

Several proofs of Brooks’ theorem appear in the literature, the most famous one being [32]. We begin by proving the result for triangle-free cubic graphs, and the general result follows from that by induction. Our method has the advantage of implying a result of Catlin [15], which is a strengthening of Brooks’ theorem stating that every graph $G$, with $\Delta(G) \geq 3$ and no clique of size $\Delta(G) + 1$, has a $\Delta(G)$-coloring in which one of the color classes has size $\alpha(G)$.

Lemma 12.2.1. Let $G$ be a triangle-free cubic graph. Then $\chi(G) \leq 3$.

Proof. Choose an independent set $I$ of size $\alpha(G)$ such that the number of odd cycles
in $G\setminus I$ is minimum. Suppose $G\setminus I$ contains an odd cycle $C$. Let $x \in V(C)$. Consider
the set $S$ of all paths $P$ starting at $x$ and alternating between non-isolated vertices of $G\setminus I$ and elements of $I$ subject to $V(G\setminus I) \cap V(P)$ being independent. Let $P_0$ be a member of $S$ of maximum length. Let $I'$ be the symmetric difference of $I$ and $V(P_0)$. It is easy to check that $I'$ is an independent set of size $\alpha(G)$. Since no cycle in $G\setminus I'$ contains a vertex from $I \cap V(P_0)$, every odd cycle in $G\setminus I'$ is an odd cycle in $G\setminus I$. Hence the number of odd cycles in $G\setminus I'$ is strictly less than that of $G\setminus I$, a contradiction. Hence $G\setminus I$ contains no odd cycle, therefore bipartite, and can be colored with two colors. Giving all the vertices of $I$ a new (third) color, we conclude that $\chi(G) \leq 3$. $\square$

**Theorem 12.2.2** (Brooks’ theorem [12]). Let $d$ be an integer greater than 2. Let $G$ be a graph with maximum degree $d$. Suppose that $K_{d+1}$ is not a subgraph of $G$. Then $\chi(G) \leq d$.

*Proof.* Induction on $|V(G)|$. In the base case $G$ is a non-complete graph on $d + 1$ vertices, and $\chi(G) \leq d$ in this case. Note that we may assume that $G$ is regular. We consider two cases depending on whether or not $K_d$ is a subgraph of $G$. Suppose that $K_d$ is a subgraph of $G$. Let $v_1, v_2, \ldots, v_d$ be pairwise adjacent vertices of $G$. Let $a_1, a_2, \ldots, a_d$ be the respective neighbors of vertices $v_1, v_2, \ldots, v_d$ outside $\{v_1, v_2, \ldots, v_d\}$. Using the induction hypothesis, we can find a $d$-coloring of $G\setminus\{v_1, v_2, \ldots, v_d\}$ such that not all of $a_1, a_2, \ldots, a_d$ receive the same color. (If $\{a_1, a_2, \ldots, a_d\}$ is independent, we have to add a suitable edge with endpoints in $\{a_1, a_2, \ldots, a_d\}$ to force the induction hypothesis to provide a coloring where at least two vertices of $\{a_1, a_2, \ldots, a_d\}$ have different colors.) It is easy to extend this to a $d$-coloring of $G$.

Suppose that $K_d$ is not a subgraph of $G$. If $d > 3$, then consider a maximum independent set $I$ in $G$. Since $I$ is a maximum independent set, the maximum degree
of a vertex in $G \setminus I$ is at most $d - 1$. By the induction hypothesis, $G \setminus I$ can be colored with at most $d - 1$ colors. By giving a new color to all the vertices in $I$, we see that $\chi(G) \leq d$. The case $d = 3$ follows from Lemma 12.2.1. In all cases, $\chi(G) \leq d$, proving the theorem. □

Let us call the above argument of extending a $d$-coloring of a graph with a clique of size $d$ deleted to a $d$-coloring of the whole graph an extension argument. We now show that the above proof idea easily implies the following theorem of Catlin.

**Theorem 12.2.3** (Catlin’s theorem [15]). Let $d$ be an integer greater than 2. Let $G$ be a graph with maximum degree $d$. Suppose that $K_{d+1}$ is not a subgraph of $G$. Then there is a $d$-coloring of $G$ in which one of the color classes has size $\alpha(G)$.

Note that it suffices to prove the result for graphs $G$ with $\Delta(G) = \chi(G)$. Indeed, for other graphs $H$, simply give a color to all the vertices in an independent set of maximum size, and color the rest of the graph with $\chi(H)$ colors.

Now we explain why our method of proof of Brooks’ theorem implies Catlin’s theorem. We will use the obvious fact that no independent set contains more than one vertex of any clique. Let $G$ be a graph with $\Delta(G) = \chi(G)$. If $G$ does not contain a $K_{\Delta(G)}$-subgraph, we appeal to Lemma 12.2.1 if $\Delta(G) = 3$, and if $\Delta(G) \neq 3$, we can give a color to the vertices of an independent set of maximum size, and color the remaining graph with $\chi(G) - 1$ colors (such a coloring exists by Brooks’ theorem). Suppose $G$ contains a $K_{\Delta(G)}$-subgraph. We apply induction on the number of vertices. Let $d = \Delta(G)$.

Let $v_1, v_2, \ldots, v_d$ be pairwise adjacent vertices of $G$. If one of the $v_i$’s has degree less than $d$, we are done by induction and the extension argument. Let $a_1, a_2, \ldots, a_d$ be the respective neighbors of vertices $v_1, v_2, \ldots, v_d$ outside $\{v_1, v_2, \ldots, v_d\}$. Note that the $v_i$’s need not be distinct but neither can they all be equal. If $\{a_1, a_2, \ldots, a_d\}$ is
not independent, we are done by induction and the extension argument. Suppose \( \{a_1, a_2, \ldots, a_d\} \) is independent. If \( \{a_1, a_2, \ldots, a_d\} \) occurs in every independent set of maximum size in \( G \setminus \{v_1, \ldots, v_d\} \), then \( \alpha(G) = \alpha(G \setminus \{v_1, \ldots, v_d\}) \), and we apply induction after adding a suitable edge with endpoints in \( \{a_1, a_2, \ldots, a_d\} \). If \( \{a_1, a_2, \ldots, a_d\} \) does not occur in every independent set of maximum size in \( G \setminus \{v_1, \ldots, v_d\} \), we can add a suitable edge with endpoints in \( \{a_1, a_2, \ldots, a_d\} \) to force the induction hypothesis to provide a \( d \)-coloring of \( G \setminus \{v_1, \ldots, v_d\} \) where at least two vertices of \( \{a_1, a_2, \ldots, a_d\} \) have different colors and one of the color classes has size \( \alpha(G) - 1 \). This easily extends to a \( d \)-coloring of \( G \), in which one of the color classes has size \( \alpha(G) \).
Appendix A

MATROID CRYPTOMORPHISMS

There are several equivalent ways to define a matroid. We list below six of the most important ones.

Let $E$ be a finite set. Let $\mathcal{I}, \mathcal{B}, \mathcal{C} \subseteq 2^E$. Let $r : 2^E \to \mathbb{N}$, $cl : 2^E \to 2^E$, and $g : 2^E \to \mathbb{N}^+ \cup \{\infty\}$ be functions.

1. **INDEPENDENT SETS:** $\mathcal{I}$ is the set of *independent sets* of a matroid on $E$ if

   I1) $\emptyset \in \mathcal{I}$.

   I2) If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.

   I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists $x \in I_2 - I_1$ such that $I_1 \cup x \in \mathcal{I}$.

2. **CIRCUITS:** $\mathcal{C}$ is the set of *circuits* of a matroid on $E$ if

   C1) $\emptyset \notin \mathcal{C}$.

   C2) If $C_1, C_2 \in \mathcal{C}$ and $C_2 \subseteq C_1$, then $C_1 = C_2$.

   C3) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.

3. **BASES:** $\mathcal{B}$ is the set of *bases* of a matroid on $E$ if

   B1) $\mathcal{B} \neq \emptyset$. 

106
B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$, then there exists $y \in B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

4. **RANK FUNCTION**: $r$ is the rank function of a matroid on $E$ if

   R1) If $X \subseteq E$, then $r(X) \leq |X|$.

   R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

   R3) If $X, Y \subseteq E$, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

5. **CLOSURE FUNCTION**: $cl$ is the closure operator of a matroid on $E$ if

   Cl1) If $X \subseteq E$, then $X \subseteq cl(X)$.

   Cl2) If $X \subseteq Y \subseteq E$, then $cl(X) \subseteq cl(Y)$.

   Cl3) If $X \subseteq E$ and $y \in E$ such that $y \in cl(X \cup x) - cl(X)$, then $x \in cl(X \cup y) - cl(X)$.

   Cl4) If $X \subseteq E$, then $cl(cl(X)) = cl(X)$.

6. **GIRTH FUNCTION**: $g$ is the girth function of a matroid on $E$ if

   G1) If $X \subseteq E$ and $g(X) < \infty$, then $X$ has a subset $Y$ such that $g(X) = g(Y) = |Y|$.

   G2) If $X \subseteq Y \subseteq E$ and $g(X) < \infty$, then $g(Y) \leq g(X)$.

   G3) If $X, Y \subseteq E$ with $X \neq Y$, $g(X) = |X|$, $g(Y) < |Y|$, and $e \in X \cap Y$, then $g((X \cup Y) - e) < \infty$. 
Appendix B

MATROIDS ON AT MOST FIVE ELEMENTS

It is well known that the number of non-isomorphic matroids on 0, 1, 2, 3, 4 elements is respectively, 1, 2, 4, 8, 17, and with the exception of $U_{2,4}$, all are graphic. Since $U_{2,4}$ is the frame matroid of a negative digon with a negative loop at each end, we conclude that all matroids on at most 4 elements are signed-graphic.

Up to isomorphism, there are exactly 38 matroids on 5 elements. 36 of them are signed-graphic, of which 32 are graphic. This tells us that out of the 70 matroids on at most 5 elements, two are not signed-graphic, and 7 are not graphic. The two 5-element matroids that are not signed-graphic are $U_{2,5}$ and $U_{3,5}$.

Of the 38 matroids on 5 elements, 17 have loops, 12 are loopless having a 2-circuit, and 9 of them are simple. For brevity and style, we will write 123 for $\{1, 2, 3\}$, and so on. We list the 38 matroids on $\{1, 2, 3, 4, 5\}$ below by specifying the set of circuits.

- $M_1$: 1 2 3 4 5
- $M_2$: 1 2 3 4
- $M_3$: 1 2 3
- $M_4$: 1 2 3 45
- $M_5$: 1 2
- $M_6$: 1 2 3 45
• $M_7$: 1 2 34
• $M_8$: 1 2 34 35 45
• $M_9$: 1
• $M_{10}$: 1 2345
• $M_{11}$: 1 234
• $M_{12}$: 1 234 235 245 345
• $M_{13}$: 1 45 234 235
• $M_{14}$: 1 23 24 25 34 35 45
• $M_{15}$: 1 23 45
• $M_{16}$: 1 23 24 34
• $M_{17}$: 1 23
• $M_{18}$: 12 13 14 15 23 24 25 34 35 45
• $M_{19}$: 12 13 14 23 24 34
• $M_{20}$: 12 13 23 24 34 2345
• $M_{21}$: 12
• $M_{22}$: 12 345
• $M_{23}$: 12 34
• $M_{24}$: 12 34 135 235 145 245
• $M_{25}$: 12 13 23
• \( M_{26} \): 12 13 23 145 245 345
• \( M_{27} \): 12 13 23 45
• \( M_{28} \): 12 345 134 234 145 135 245 235
• \( M_{29} \): 12 134 234
• \( M_{30} \): 12345
• \( M_{31} \): 1234 1235 1245 1345 2345
• \( M_{32} \): 1234 1235 1245 345
• \( M_{33} \): 1234
• \( M_{34} \): 1234 125 345
• \( M_{35} \): \( \emptyset \)
• \( M_{36} \): 123
• \( M_{37} \): 123 124 234 134
• \( M_{38} \): 123 124 125 134 135 145 234 235 245 345

Note that \( M_{31} \cong U_{3,5} \) and \( M_{38} \cong U_{2,5} \), and are the only two matroids in the list that are not signed-graphic. Also, none of \( M_{12}, M_{28}, M_{32}, \) and \( M_{37} \) is graphic.

**Problem B.0.4.** Find how many matroids on 6 elements are signed-graphic.

Let \( m(n), g(n), s(n) \) denote respectively, the number of non-isomorphic matroids, graphic matroids, signed-graphic matroids on \( n \) elements.

**Problem B.0.5.** It is known that most matroids are non-graphic i.e., \( \lim_{n \to \infty} \frac{g(n)}{m(n)} = 0 \).

Are most matroids non-signed-graphic i.e., is \( \lim_{n \to \infty} \frac{s(n)}{m(n)} = 0 \)?
Problem B.0.6. Are most signed-graphic matroids non-graphic? Does $\lim_{n \to \infty} \frac{g(n)}{s(n)}$ exist? If so, what is the limit?
Appendix C

NOTES

Takeo Nakasawa, a Japanese mathematician is also believed to have discovered matroids independently. He wrote three papers “On axiomatics of linear dependence I, II, and III” in mid 1930s. A recent book [38] contains his life story and his three papers.

Transversal matroids were discovered by Edmonds and Fulkerson in 1965 [22]. Bondy [8] characterized graphs whose cycle matroids are transversal (base-orderable) in 1972. Matthews characterized graphs whose cycle matroids are bicircular as well as graphs whose bicircular matroids are graphic. Signed-graphic matroids were first explicitly defined by Zaslavsky in 1982. The history of the concept of well-quasi-ordering before 1972 is well documented in a paper of Kruskal [30].

Welsh [55] posed a list of 36 problems in a conference in 1969, one of which is the conjecture discussed here. Some of them have been solved, and partial results are known for some others.

Robertson and Seymour [43] proved their graph minor theorem in a series of 23 papers published between 1983 and 2012. In the language of matroids, their result states that graphic matroids are well-quasi-ordered under the minor relation. It is well known that well-quasi-ordering fails for arbitrary matroids. The situation with matroids representable over finite fields is dramatically different. Recently, Geelen, Gerards and Whittle have extended the graph minor theorem by proving that binary
matroids are well-quasi-ordered under the minor relation. They also see no major conceptual difficulty in extending the proof to matroids representable over a fixed finite field.

The exact number of (non-isomorphic) matroids on \( n \) elements is known only for \( n \) up to 9, thanks to recent work of Gordon Royle and Dillon Mayhew [35]. Nevertheless, our understanding of the asymptotic behavior of the number of matroids on \( n \) elements is much better, due to various authors, Knuth, Piff, Welsh, recent work of Bansal, Pendavingh, and van der Pol.


Brooks [12] published his famous coloring result in 1941. It has been generalized, extended and strengthened in several different ways.

The well-quasi-ordering of signed graphs under the minor relation does not seem to follow from the theory of graph minors. However, it follows easily from the well-quasi-ordering of binary matroids by considering the lift matroid of signed graphs.

Dijkstra’s problem on multisets of integers is listed in his collective writings, known as EWDs and maintained by the University of Texas at Austin. Dijkstra achieved ever-lasting fame after he discovered a very simple and elegant greedy algorithm to solve the shortest path problem in a weighted graph. The analysis of this algorithm was one of the reasons for the discovery of greedoids, a significant generalization of matroids.

Bondy’s theorem was also proved independently by Las Vergnas [31].

R. Rado proved that matroids are precisely the structures for which the greedy algorithm always works. This important result gave matroids a centre stage in combinatorial optimization. Tutte proved that a matroid is binary if and only if it has
no $U_{2,4}$-minor. Such a characterization, known as excluded minor characterization is known for several other classes.

There are several excellent graph theory texts. We mention a few: Diestel, Bondy-Murty, West, Harary, Bollobás. There are two standard matroid theory texts: Oxley and Welsh. There is no text on signed graphs, although papers of Zaslavsky and Slilaty can partially rectify that problem.
Appendix D

SOME OPEN PROBLEMS

1. Give a characterization of matroids that are both signed-graphic and cosigned-graphic\(^1\).

2. Given a ternary matrix \(A\), determine whether \(M[A]\) is signed-graphic or not.

3. Suppose that \(M(\Sigma_1) = M(\Sigma_2)\) for two signed graphs \(\Sigma_1\) and \(\Sigma_2\). How can we obtain \(\Sigma_2\) from \(\Sigma_1\)?

4. Given a signed graph \(\Sigma\), design a polynomial time algorithm to determine 
   \(b(M(\Sigma))\), the number of bases in the frame matroid of \(\Sigma\).

5. Characterize matroids that are both signed-graphic and gammoid\(^2\).

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\(^{1}\)A matroid \(M\) is said to be cosigned-graphic if \(M^*\) is signed-graphic.

\(^{2}\)Any minor of a transversal matroid is called a gammoid.


[59] T. Zaslavsky, Characterizations of signed graphs, J. Graph Theory 5 (1981),
no. 4, 401-406.


215-228.


[63] T. Zaslavsky, A mathematical bibliography of signed and gain graphs and
allied areas, Electronic Journal of Combinatorics, Dynamic Surveys in Com-
binatorics, #DS8.

[64] T. Zaslavsky, Six signed Petersen graphs, and their automorphisms, Discrete
Math. 312 (2012), no. 9, 1558-1583.

[65] I. E. Zverovich and V. E. Zverovich, Contributions to the theory of graphic