A Study of Student Understanding of the Sine Function through Representations and the Process and Object Perspectives

THESIS

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Abstract

A major topic that is addressed during the second half of the high school curriculum is the topic of trigonometric functions. In order to analyze how students represent a function like sine and how they connect those representations together, the framework developed by Moschkovich, Schoenfeld, and Arcavi, (1993) will be utilized. Their framework contained two dimensions. The first dealt with the means in which students represent functions and the second dealt with the perspective from which a function is seen and operated on. 6 students of varying mathematical ability were interviewed to see how they understood sine, represented sine, and connected representations together. Students’ responses were analyzed qualitatively to see depth of understanding by looking at the representations, perspective, and the connections between representations that were used for answering questions. The representations that students utilized were algebraic in the form of an equation, graphical in the form of a sine wave, and geometric in the form of right triangles and the unit circle. Particular attention was paid to if and when a student utilized the Cartesian Connection (Moschkovich et al., 1993). The analysis also attempted to discover where student knowledge is lacking and how this relates to the ability of students to use multiple representation and perspectives. Results indicate that students had some difficulty seeing sine as a function with inputs and outputs. The analysis also revealed that students’ use of the Cartesian Connection
was dictated by the context of the question being asked. In addition, students under-utilized the graphical representation during the interview and mostly used the object perspective for this representation. Finally, some implications for teaching are discussed, including getting students to go beyond memorization of facts and more opportunities to use the graphical representation for problem solving and building connections.
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Table of Contents

Abstract .......................................................................................................................... ii

Acknowledgments ........................................................................................................ iv

Vita ............................................................................................................................... v

Introduction .................................................................................................................... 1

Research Questions ....................................................................................................... 1

Theoretical Framework ................................................................................................. 2

Literature Review .......................................................................................................... 3

Multiple Representations ............................................................................................ 3

Process and Object Perspectives .................................................................................. 5

Context .......................................................................................................................... 6

Learning Difficulties with Multiple Representations and Perspectives ...................... 7

Multiple Representations in Trigonometry .................................................................. 13

Learning Difficulties with Trigonometric Representations ......................................... 15

Other Learning Difficulties with Trigonometry ........................................................... 16

Influence of Technology on Student Learning .............................................................. 19

Procedural versus Conceptual Knowledge ................................................................... 21
Reasons for this Study............................................................................................................23

Methods ......................................................................................................................................24

Participants .............................................................................................................................24

Description of Courses..............................................................................................................25

Trigonometric Topics Covered During Instruction ..................................................................26

Data Collection ..........................................................................................................................27

Interview Protocol .....................................................................................................................28

Question 1 ...................................................................................................................................29

Question 2 ...................................................................................................................................29

Question 3 ...................................................................................................................................30

Question 4 ...................................................................................................................................31

Questions 5 and 6 .......................................................................................................................31

Questions 7 and 8 .......................................................................................................................32

Question 9 ...................................................................................................................................33

Question 10 ..............................................................................................................................33

Question 11 ..............................................................................................................................34

Questions 12 and 13 ..................................................................................................................34

Question 14 ..............................................................................................................................35

Question 15 ..................................................................................................................................35
Question 15 .................................................................................................................. 153
Second Interview Analysis .......................................................................................... 165
Questions 1 .................................................................................................................. 165
Questions 2 and 3. ....................................................................................................... 176
Question 4 .................................................................................................................. 182
Question 5 .................................................................................................................. 201
Question 6 .................................................................................................................. 202
Conclusion ................................................................................................................... 209
Implications for Teaching ......................................................................................... 218
Future Research .......................................................................................................... 222
References .................................................................................................................. 225
Appendix A: Verbal Recruitment Script ...................................................................... 227
Appendix B: Consent Form ......................................................................................... 228
Appendix C: Assent Form ............................................................................................ 232
Appendix D: Parental Permission Form ....................................................................... 236
Appendix E: First Interview Questions ....................................................................... 240
Appendix F: Second Interview Questions ................................................................... 243
Appendix G: First Interview Transcript, Student A .................................................... 247
Appendix H: First Interview Transcript, Student B ..................................................... 257
Appendix I: First Interview Transcript, Student C .......................................................... 264
Appendix J: First Interview Transcript, Student D ............................................................... 272
Appendix K: First Interview Transcript, Student E ............................................................... 283
Appendix L: First Interview Transcript, Student F ............................................................... 295
Appendix M: Second Interview Transcript, Student B ......................................................... 307
Appendix N: Second Interview Transcript, Student C ......................................................... 312
Appendix O: Second Interview Transcript, Student D ......................................................... 318
List of Figures

Figure 1 .....................................................................................................................49
Figure 2 .....................................................................................................................63
Figure 3 .....................................................................................................................69
Figure 4 .....................................................................................................................70
Figure 5 .....................................................................................................................245
Figure 6 .....................................................................................................................245
Figure 7 .....................................................................................................................246
Introduction

A major topic that is addressed during the second half of the high school curriculum is that of trigonometric functions. Most high school curricula devote time to learning about linear functions, quadratic and polynomial functions, exponential and logarithmic functions, and trigonometric functions at some point. During this process, students are also introduced to several representations for the functions including algebraic representations, such as their equations, and graphical representations.

Trigonometric functions share additional unique representations. In addition to equations and graphs, students are introduced to right triangles and the unit circle to make sense of sine and cosine functions. Making connections between these different representations of functions and within the representation itself are essential to mathematical learning at the high school level and in preparation for a college education. This paper will attempt to address the importance of learning trigonometric functions through the connections students make between and within their representations.

Research Questions

1. How do students understand sin(x)? How do they justify their understandings?
2. In what ways do students represent \( \sin(x) \) and is the representation based on the context of the question being asked? Are they able to move to other representations and use them to make sense of representations where their knowledge lacks?

3. When looking at the graph of a trigonometric function, do students utilize the Cartesian Connection? Do they see the graph as a series of points or as a whole object, or both? Can they use a graph to solve an equation and other tasks? What other connections are they making between representations?

Theoretical Framework

In order to analyze how students represent a function like sine and how they connect those representations together, the framework developed by Moschkovich, Schoenfeld, and Arcavi, (1993) was utilized. Their framework contains two dimensions. The first deals with the means with which students represent functions and the second deals with the perspective from which a function is seen and operated on. The representations they focus on for linear functions were algebraic, graphical, and tabular representations. They specifically consider two different perspectives: the process perspective and the object perspective. The process perspective for a function is viewing an \( x \)-value as being linked to a \( y \)-value by some relationship or rule. This relationship can be seen as a transformation of quantities by some repeatable means (p. 72). The focus is on the relationship between \( x \) and \( y \), or the input and output (p. 79), via the process or procedure that must be performed on \( x \) values to obtain \( y \) values. However what is
important to the process perspective is that the procedure is carried out on individual members of a set.

The object perspective is viewing the function as an entity to be operated on such as a graph as a whole instead of a collection of individual ordered pairs. If students treat the representation as an object as a whole instead of a collection of individual entities to find properties or connections then the students are utilizing the object perspective (p. 72-73). For example, if the student sees the graph of a line as a set or ordered pairs from a formula then the student is using the process perspective, but if the student sees a line as an object with properties like slope that can be manipulated then the student is using the object perspective (p. 84).

One important way students can make a connection between representations of functions is the Cartesian Connection. The authors defined this connection as: “A point is on the graph of line L if and only if its coordinates satisfy the equation of L” (p. 73). In other words, a graphical representation of a function can be connected to the algebraic representation of a function or vice versa by looking at all the (x, y) coordinates that satisfy a given equation and realizing that these are the same points that make up the graph.

Literature Review

*Multiple Representations*

It is important to take a closer look at what the research has to say about how students represent functions in mathematics, the connections they make between those
representations, and how students learn trigonometry. The three most common representation used in secondary mathematics are algebraic, graphical, and tabular representations. The form that student choose can determine the degree of difficulty or easiness of a task (Knuth, 2000b). The NCTM standards also emphasize that it is important for students to be able to use these three types of representations and move from one to another (Knuth, 2000a). A proper representation is a tool to engage in mathematical thinking (Santos-Trigo, 2002). As a result, from the time students begin learning about functions, they are exposed to all three representations and expected to use all three forms to guide their understanding.

Research has shown that students are aware of multiple representations and the advantages of using them. Santos-Trigo (2002) found that students are able to realize the importance of viewing tasks from multiple perspectives. Herman (2007) found that students understand that for a given situation, one representation may be more beneficial than another. Students’ posttest scores were even improved through the use of multiple representations (p. 44). One reason for this was that students who were able to use multiple representations were able to solve the problem multiple ways and often used a second way to check the first (p. 47). Using multiple representations is also a good motivational tool for students. Byers (2010) stated that using multiple representations can prepare students for activities in professional work environments.

While being able to use representations is important, connecting them together and moving from one to another is vital to student learning. The ability to use representations and flexibly move from one representation to another allows for
institution of rich relationships, deeper conceptual understanding, and better problem solving skills (Even, 1998). Pesek and Kirshner (2000) found that understanding is influenced by the strength of connections among representations. Having the ability to move from one representation to another when need arises is obviously beneficial to students. However, to do so requires that students understand the connection between two representations (Van Dyke & White, 2004). Previous knowledge can also be improved through new connections of representations (Santos-Trigo, 2002). By connecting new knowledge to that which was previously learned, students can develop a richer understanding of mathematical topics.

Even (1998) found that using a representation could be specific to the type of function, demonstrating the importance of context, or could be more general for all functions. The kind and nature of a function can also determine if a student will switch representations when solving a problem (p. 115). Even also stated that there are times when a function must be looked at in a point-by-point approach, which advocates the process perspective and other times when a global approach was necessary, leading to students use of the object perspective (p. 112). As a result, students must be able to not only use multiple representations but multiple perspectives with functions during problem solving.

Process and Object Perspectives

While using multiple representations play an important part in learning, knowing and understanding the conceptual perspective a student takes with the representation is
equally as important. Hiebert and Lefevre (1986) discuss how some procedures in mathematics can operate on symbols while others operate on objects. While they are not discussing the process and object perspectives, this supports the idea that a procedure can be in the process perspective or the object perspective depending on how the students are using them. Moschkovich et al. (1993) discuss the importance of not only being able to change representations, but also perspectives. Utilizing both the process perspective and the object perspective and knowing when to use each is essential to learning. They found that for linear equations it may appear to be a natural progression for the process to the object perspective because students learn to link x and y values together first and then look at features like slope. However, they also found examples that showed students moving from the object perspective to the process perspective and concluded that one perspective is not always mastered or used before the other (p. 87). Even (1998) found that students using a global approach to graphical representations, which would support the object perspective, were much more successful at solving problems than those who used a point-by-point perspective, which would be the process perspective. Whether students prefer one perspective over another or utilize both, real competency deals with being able to get the job done easily, regardless of perspective, not doing it the same way each time or using the same perspective each time (Moschkovich et al., 1993).

**Context**

The context of a problem or question also plays a key role in the representations and the perspectives that students adopt. Among other things, context can be the wording
of tasks, the ordering of tasks, or what resources students have available for completing the tasks. Even (1998) and Moschkovich et al. (1993) both discuss the importance the context plays in problems solving. Even found that the kind and nature of a function played an important role in what representation students chose (p. 115). Moschkovich et al. stated that the context influences which perspective a student chooses to utilize because students may find one perspective beneficial for a given context (p. 72). Hiebert and Lefevre (1986) found the knowledge students acquire is usually context based. As a result, it is important to keep in mind that the context of a question, not just the knowledge and ability of a particular student, will play a key role in the representations and perspectives used to solve a problem.

**Learning Difficulties with Multiple Representations and Perspectives**

Multiple representations and perspectives may offer students several ways to problem solve in mathematics, but they also create some learning difficulties as well. High school mathematics teachers often operate under the assumption that students can easily understand connections among representations (Knuth, 2000b). However, this may not be the case. Many students show difficulty linking representations or lack the ability to move flexibly across representations and perspectives (Even, 1998; Moschkovich et al., 1993). In fact, many students leave high school without an understanding of the connections among the many representations they use in class (Knuth, 2000a). Research has also found that students may not be troubled when arriving
at different answers using different representation (Silver, 1986). This would imply that students view representations as different entities and not variations of the same thing.

Other learning difficulties may arise from the students’ inability to create an accurate representation. Obviously, a student must represent a topic like a function first to be able to begin to understand its properties. However, sometimes students develop inadequate constructions of representations. As a result, the errors students make are a direct result of those misconceptions (Gur, 2009). Problems also can arise not from a lack of knowledge, but difficulty putting together relevant information (Moschkovich et al., 1993). Tall and Bakar (1991) looked at errors in representation caused by positive and negative resonance for students’ mental prototypes of the function concept. They classified positive resonance errors as using inappropriate properties to create a representation. As a result, the students incorrectly label something to be a function that is not. Negative resonance errors occur when the original representation is too narrow in scope to include all possibilities. Negative resonance will cause students to label a function as a non-function (p. 104). Both errors led to incorrect or incomplete representations that can seriously hinder the learning process.

The way students chose to represent functions can also led to some difficulty with representations and perspectives. Van Dyke and White (2004) found that student knowledge may be lacking in the concept of a function. This would lead to incorrect or inadequate representations of functions and difficulty with perspectives as well. Tall and Bakar (1991) found that while students may understand the process aspect of a function, they often fail to mention that it applies to a certain input domain or that it takes a certain
range of values. This lack of knowledge could lead to problems connecting algebraic and graphical representations. Herman (2007) found that students tend to identify a function with only one representation, most often algebraic by way of symbolic manipulation, and will stick to that representation when solving problems. If students are hesitant to use graphical representations to solve problems or do not readily recall them, they will miss out on a wealth of important information regarding functions. Such over-reliance on algebraic representations is a hurdle many students face. School instruction places a tremendous emphasis on symbolic manipulation in the algebraic form which causes students to think algebraically more often than visually and graphically (Knuth 2000b). Knuth elaborates on how students are far more comfortable with algebraic solutions and that they rely on them even when graphical solutions seem easier and more efficient (p. 505). Once again, the context can play a key role. Knuth states that many of the problems that the students are asked to solve lead them to use algebraic approaches by nature (p. 505). Herman (2007) reported similar results as she stated that students in her study chose symbolic manipulation as the primary solution method to solve problems even when they could use the calculator to create tables and graphs (p. 27). However, by the end of the study, students saw the value of multiple representations and moved to using a combination of representations to solve problems (p. 44). Knuth (2000a) found that when students attempted to match a graph with an equation they tended to rely on algebraic approaches. The same students, when asked to perform a task by plugging in values to an equation or finding points on a graph still chose the algebraic approach (p. 49). Even when students had to approximate a y-value on a graph they plugged in values
into the equation instead of using the graph (p. 50). It is important to examine how students use tables and graphs to organize and interpret data (Santos-Trigo, 2002).

However, students are not always given enough opportunities to do so. In fact, many students perceive graphs as unnecessary and use them only as a means to support algebraic methods (Knuth, 2000a). Santos-Trigo (2002) found that students relied on the quadratic formula and symbolic manipulation to solve quadratic equations more often than graphing them, even when they had access to a calculator. It would appear that many students fail to fully understand all the information they could extract from a graph.

Other problems with the graphical representation deal with a failure on the part of students to use the process or object perspective. Graphs present visual representations of relationships among points (Santos-Trigo, 2002). However, if students have difficulty viewing the graph from a process perspective, then several challenges arise. Knuth (2000b) found that students were unable to see the graph as a series of points and only as an object. In essence, they were unable to make the Cartesian Connection. If this connection is not made possible, there is less potential for vital information to be extracted from the graphical representation of a function. This leaves students unable to grasp most of the information that is in a graph (p. 504). Van Dyke and White (2004) found that students did not associate the points on a curve as being the coordinates \((a, f(a))\) for a \(f(x)\) function. In this case, students were able to see the points but could not connect those points back to the equation of the function. Finally, Even (1998) found that students were able to make conclusions about graphs from a point-by-point perspective but were not able to understand why their conclusions worked. The research
cited above indicates that the Cartesian Connection does not appear to be an all or nothing understanding but one of varying levels.

Difficulties also arise from using the objet perspective with a graph. To correctly use all the information in a graph, students must know what to focus their attention on when solving a problem (Knuth, 2000b). When students view a graph as an object, they may leave out important components that hinder understanding, demonstrating that using a object approach to a graph does not guarantee that students may understand it (Even, 1998).

Regardless of perspective, students tend to be less comfortable with graphs in general than they are with algebraic representations (Van Dyke & White, 2004). Many students even fail to recognize a graphing representation might be suitable for solving problems involving linear functions (Knuth, 2000b). Choi-Koh (2003) found that while graphical feedback was convincing to a student, it was not as convincing as expected. When non-integral values are involved, students appear to be much more hesitant to use graphs possibly because of a decreased level of precession that comes from estimating coordinates (Knuth, 2000a). This would again point students to using another representation, most likely the algebraic representation, to solve a problem.

Several student difficulties with utilizing the Cartesian Connection have also been documented. Teachers assume that once a student has been exposed to the Cartesian Connection no review of it is necessary (Knuth, 2000b). However, there is no guarantee that students maintain the understanding of this connection as more functions or representations are introduced. Even those who do possess an understanding of the
Cartesian Connection can still miss other important connections between algebraic and graphical representations and may go as far as to treat algebraic and graphical representations as being independent entities (Moschkovich et al., 1993; Van Dyke & White, 2004).

One reason why the Cartesian Connections may give students so much difficulty is the absence of opportunities for them to connect graphs to equations. Students spend a great deal of time learning how to graph an equation, but not how to get an equation from a graph. Most of their understanding is limited to translations of graphs (Knuth, 2000b). Hirsch, Weindhold, and Nichols (1991) stated that writing the equation from a graph is a significant step is developing a student’s graphical sense. Students that are unable to do this, lack a major connection between representations. It is not enough to make a connection between representations moving from one to another. Instead students must be given opportunities to shift in both directions in order to fully grasp the connections among them.

Research has also documented students’ lack of precise understanding of solutions of functions, which in turn impacts how they solve problems or integrate them. With graphical representations, students are often unsure if the solutions are the x-intercepts or any point on the curve (Van Dyke & White, 2004). This misunderstanding leaves many students reluctant to use graphical representations and may be one of the reasons for their overreliance on algebraic representations. Likewise, Knuth (2000a) found that when solving quadratic equations students confused the roots, which would only be the x-intercepts, with the solutions to the equation, which would be any point on
the line. It is important to understand what a student is trying to represent before one is able to determine if a student actually has an understanding of a mathematical topic like finding solutions.

**Multiple Representations in Trigonometry**

Although there is much research on representations and perspectives in the area of algebra learning and teaching, less attention has been paid to how students specifically use multiple representations to learn trigonometry (Byers, 2010). Weber (2005) also found little research on how students learn about trigonometric functions. The trigonometric functions and their representations constitute an essential part of the high school and college curriculum. The study of trigonometry helps deepen students’ understanding of functions and their properties (Hirsh, et al., 1991). Trigonometric functions usually offer the first instance of oscillating functions offering examples of functions that pass the vertical line test but fail the horizontal line test. Trigonometric functions can also lead to a better understanding of parent graphs and transformations. Hirsh et al. (1991) found that students who studied families of functions were able to extend this concept to include the trigonometric graphs thus deepening their understanding of functions in general. The authors further discussed the importance of using sine and cosine graphs to introduce periodicity in order to allow students an opportunity to visualize not only the periodic nature of graphs but also their oscillation. By analyzing the parent graphs of sine and cosine, the participants in their study were able to visualize the oscillation between 1 and -1 easily (p. 99).
The study of trigonometry also exposes the learner to the use of multiple representations, some of which are unique to the topic. Since it is one of the earliest topics that link algebraic, geometric, and graphical representations together by using ratios, triangles, and circles, to name a few, there are a number of ways a student can begin to represent sine and cosine (Weber, 2005). Calzada and Scariano (2006) found that a smooth transition can be made into trigonometry simply by bringing back the concepts of area of triangles and rectangles and the Pythagorean Theorem and using these to building connections between algebra, geometry and trigonometry.

Despite this, there appears to be no major consensus within the community regarding effective ways to represent trigonometric functions. Byers (2008) found that the six most common ways to represent trigonometric functions are by using right triangles, ratios, input and output functions, the unit circle, the sinusoidal wave, and vectors. In contrast, Weber (2008) claimed that there were only two major ways to represent the operations involved with trigonometry: ratios and functions. Still a third study conducted by Brown (2006) stated that sine and cosine are most often represented as coordinates on the unit circle, as horizontal and vertical distances on the unit circle, and as ratios of sides of right triangles with given reference angles. However, despite the three different classifications, similarities exist among them. All three involve right triangles, the unit circle, graphs, and equations in some form. While it is debatable which representations could be included in the study of trigonometry, problems students experience in learning trigonometry usually arise more from how it is taught rather than how it is represented.
Learning Difficulties with Trigonometric Representations

The unit circle, as one of the representations mentioned, appears to be vital to the understanding of trigonometry. Weber (2005) stated that understanding the process that creates the unit circle is crucial to understanding trigonometric functions. Brown (2006) studied students’ understanding of the connection between right triangles and the unit circle and reported a fragile concept of a rotation angle and unit among the participants in her study. She found that students fail to connect a rotation on the unit circle with a point on the graph of sine or cosine. Even when not connecting the unit circle to another representation, students still had difficulty with the rotation angle. Few realized that the coordinate of sine or cosine connects with the vertical and horizontal distance for the axes (228). If a student cannot correctly visualize the unit circle then using it to solve problems or connect to other representations will not be possible.

Additional student problems with representing trigonometric functions have been reported by various researches. Brown (2006) found that many of her participants had an incomplete or fragmented understanding of the ways to represent sine and cosine. Weber (2005) reported that students that had difficulty answering questions lacked a way to even approach the problem solving process. According to him, to successfully understand trigonometric functions students must be able to relate triangles and numerical relationships as well as to manipulate symbols. This can only be achieved if multiple representations are introduced and used along with the discussion of their connections when teaching trigonometry.
It is likely that the problems that students experience when working with representations are caused by the emphasis placed on memorizing the information, like the unit circle, without establishing a conceptual context for what they mean. Gur (2009) found that most instruction and textbooks emphasize this, concluding that many students’ errors are simply a mechanical error in the application of a rule. Weber (2008) also found that much of trigonometry instruction focused on procedures and paper-and-pencil computations without an emphasis on applying the process. Such experiences fail to assist students to form connections between representations or see trigonometry as unified construct. Hence, students end up seeing sine and cosine as a step-by-step prescription and not as a goal-oriented process (Weber, 2005). To address problems like this Hirsch et al. (1991) stated that trigonometry programs needed to shift from memorization and paper-and-pencil skills to conceptual understanding, multiple representations, modeling, and problem solving. The fact that the problems still seem to exist decades later would imply that instruction has not been responsive to this need.

Other Learning Difficulties with Trigonometry

Other problems with learning trigonometry that are due to how the subject is taught revolve around whether or not students view sine and cosine as actual functions. Weber (2005) found that many students were not able to explain why sin(x) is a function and that students who had trouble with trigonometric operations also had trouble viewing them as functions. He proposed teaching students a geometric model to help them overcome this dilemma. By teaching it as a geometric process, students were better able
to interpret sine and cosine as functions (Weber, 2008). Brown (2006) found that students had difficulty understanding that sine and cosine can be both ratios of triangles’ sides and numbers. If students only see sine and cosine as a ratio, it may be difficult for them to think of their values in terms of input and outputs thus making it difficult to classify them as functions. Similar results were reported by Weber (2005). He found that when students could reason with numerical values and connect that with geometric processes, like creating right triangles, they developed a better understanding of trigonometric functions.

There have also been studies on how textbooks treat trigonometry and ways in which they supply students with the necessary knowledge needed to understand trigonometry at the college level. Byers (2010) argued that when instruction starts with definitions and then time is spent on concept development students experience greater difficulty when trying to construct representations for trigonometry. Despite this, the author also found that much of the classroom instructional patterns and the majority of the textbooks followed such a routine to teaching trigonometry. Her study compared the instructional practices of textbooks for trigonometry in secondary schools and college. The study examined textbook’s treatments of trigonometric representations to identify sources of learning difficulties. Byers reported gaps in the textbook pathways as students moved from secondary school to college. The secondary school textbooks focused on ratios, right triangles, and functions whereas college textbooks used vectors, right triangles, tables, and sinusoidal waves. Such a discrepancy leads to difficulties for students, especially for those who may not have experienced a vector approach.
Pesek and Kirshner (2000) also discussed how teaching for meaning after instruction on procedures tended to negatively affect learners. However, they were not focusing solely on students learning trigonometry. Van Dyke and White (2004) discussed how students may have difficulty remembering mathematical definitions that have strong associations outside of the mathematics classroom. They found words like increasing, decreasing, and constant gave students a hard time when reviewing graphs (p. 114). Gur (2009) argued that most trigonometry instruction is teacher-centered so students have difficulty constructing new knowledge about concepts, processes, and procepts dealing with trigonometry. This is most likely a result of students lacking connections and strong conceptual knowledge. According to Gur, a concept produces a mathematical object, a process involves the ability to use operations, and a procept is the ability to think of operations and objects by using both concepts and processes (p. 68).

Simply memorizing information without grounding them conceptually also hinders students learning trigonometry. Gur (2009) found that many students only memorized definitions for trigonometry and had no understanding of what those definitions actually meant and the knowledge was not retained long term. Weber (2005) also found that students tended to describe sine in terms of the procedures they were shown in class. If students are unable to put definitions or concepts into their own words or formulate their own understanding, then they will encounter difficulties remembering and understanding them later.
Influence of Technology on Student Learning

Some studies examined how technology affected student learning and their representations of trigonometry. As students become more proficient with graphing calculators, dynamic geometric software, and other forms of technology, the ways to represent and connect mathematical topics increase. Santos-Trigo (2002) looked at how technology can be a powerful tool for building understanding by allowing students to use different representations. Byers (2010) discussed how dynamic software can help students develop representations for right triangles. Zengin, Furkan, and Kutluca (2012) looked specifically at how GeoGebra, one such example of dynamic geometric software, affected student achievement in learning trigonometry. They concluded that GeoGebra was a good supplement to the constructivist instruction and improved scores on a posttest (p. 187).

The calculator is also shown to have an influence on both how students learn mathematics and their attitudes towards the subject. Herman (2007) outlined the impact of calculators on students’ abilities to produce multiple representations arguing that they enhance students’ understanding of topics (p. 28) and develop a better understanding of algebra and calculus concepts (p. 30). Trigonometry is a topic that is essential to calculus and involves algebra so the calculator could have a large impact on the development of trigonometry skills as well. Hirsch et al. (1991) found that calculator use specific to trigonometry can reduce the time devoted to finding and operating on values of trigonometric functions and their inverses. They also found that technology enabled students to solve all types of trigonometric functions, not just ones from the unit circle,
presenting students with media to see why multiple solutions to an equation might exist. By viewing and manipulating a graph on the calculator, it can become obvious to a student that there are infinite solutions to an equation due the graph’s periodic nature (p. 102). This is not always evident to the student when solving an equation algebraically.

Van Dyke and White (2004) discussed how graphing calculators have made graphical representations more accessible to students even when they did not understand a graph’s connection to its equation. Students with a strong graphical sense are able to check graphs produced by technology and use them to problem solve (Hirsch et. al, 1991). Choi-Koh (2003) looked at the effect a graphing calculator had on the ability of a tenth grade student to learn the rules for graphing trigonometric functions. The graphing calculator gave the student the ability to manipulate the graphs as objects. This allowed the student a way to test conjectures about how changing the equation of trigonometric function affected its graph. By the end of the study, the student was able to explain all the transformation properties simply by observing graphs and manipulating the equations.

Calculator use in the classroom has also revealed knowledge about students’ attitudes about problems solving. Herman (2007) found that students believed symbolic manipulation was the correct way to solve problem and perceived graphs and tables on a calculator as only means to check an answer already obtained through symbolic manipulation. Some students went as far as to say that calculator use was a form of cheating and not a part or true mathematics. The end result was that the participants in the study found the calculator to be a computational device instead of a tool for understanding (p. 50).
Procedural versus Conceptual Knowledge

Finally, the representations and perspectives students utilize during learning and problem solving are influenced by their procedural and conceptual knowledge they possess. Therefore, it is important to take a closer look at their influence on understanding. Silver (1986) stated that procedural and conceptual knowledge are what gives students the power for application. The relationship between conceptual knowledge and procedural knowledge is far more important than the distinction between the two (p. 181). Hiebert and Lefevre (1986) stated that conceptual knowledge is rich in relationships and how pieces of information relate together. Therefore, for students to make connections between representations, they must possess conceptual knowledge to make those connections. Procedural knowledge, on the other hand, is made up of formal language and symbol representation (p. 6). Therefore, it is from a student’s procedural knowledge that he or she begins to form representations. However, symbols must be connected to conceptual knowledge for them to have meaning (p. 10) so conceptual knowledge also influences how students form representations. Likewise, making connections between representations also involves procedural knowledge. Students tend to use procedural knowledge as a basis for determining if two problems are mathematically related (Silver, 1986). Similarities in procedures could also then be used as a basis for connecting different representations together.

Just as learning difficulties occur when representations are not linked, procedural knowledge that is not linked with conceptual knowledge can negatively affect understanding. Silver (1986) stated that procedural knowledge that is not connected to
conceptual knowledge is very limited for students. They may lack the ability to transfer their procedural knowledge to new situations or solve novel problems (p. 185).

Emphasizing procedural knowledge during instruction is also not enough to achieve conceptual knowledge because students can possess strong procedural knowledge without having conceptual knowledge (p. 185). Pesek and Kirshner (2000) stated that students who focus only on procedural knowledge focus on symbolic manipulation and do not consider the validity of answers. Initial rote learning can lead to interference with meaningful learning later. Students who also only memorize procedures see finding connections a distraction (p. 537). Teachers may also influence students’ lack of conceptual knowledge for a particular topic by focusing on instrumental instruction only. Due to time constraints, they tend to avoid the long application problems that force students to make connections and develop conceptual knowledge. Teachers view relational activities like these as extra problems that can be cut out to save time (p. 527). Hiebert and Lefevre (1986) found that mathematics instruction does a better job of teaching procedures than the relationship between them that develop concepts. Weber (2005) found that most approaches to trigonometry teach only procedural skills and do not allow students to fully understand trigonometric functions. This could greatly hinder the ability of students to make connections between representations.

Moschkovich et al. (1993) emphasized competence instead of procedural mastery. Students still must understand concepts even if they understand a procedure; otherwise the students do not gain much from the procedure (Van Dyke & White, 2004). Students must understand both what to do and why to be fully competent mathematicians (Pesek &
Kirshner, 2000). If procedural and conceptual knowledge are not linked together then students may generate an answer but not understand what they are doing (Hiebert & Lefevre, 1986). The effects of this are seen during the learning of trigonometry.

Reasons for this Study

Research has looked at the representations students and instruction use for the teaching and learning of trigonometry, but not how those representations are connected together. This study attempted to find how students connect the representations they made for sin(x). While many studies looked at the Cartesian Connection to link algebraic and graphical representations, none looked at this connection in the context of trigonometry. This study aimed to address this issue. The process-object theoretical framework developed by Moschkovich et al. has never been applied to students’ use of trigonometric functions. This study attempted to analyze how students use the object and process perspectives for their representations of sin(x).
Methods

Participants

Six student participants were taken from classes taught by the author at a private, Catholic high school. The school is predominantly Caucasian and almost all students move on to some level of college after graduation. The students were enrolled in one of three courses at the school: Algebra II and Trigonometry, Algebra III and Trigonometry, or Calculus. All three courses included the instruction of trigonometry at some point during the school year. There were a total of 121 students in the classes from which the students were chosen. The pool of potential subjects included 25 students enrolled in Algebra II and Trigonometry, 72 students enrolled in Algebra III and Trigonometry, and 24 students enrolled in Calculus. Six students were chosen based on mathematical ability, as measured by in-class performance, and their availability and interest to take part in the study. Participation was completely voluntary for the students. A script was read to them during class, and if they were interested, appropriate consent and assent forms were sent home for the students and parents/guardians (if they were under the age of 18) to review and sign. Copies of the verbal script, consent form, and assent form are in the appendix.
Description of Courses

The Algebra II and Trigonometry course is a junior level course in the middle tier of a three tier system of math classes the school offers. Students in this class usually do not include accelerated students or students who tend to struggle with mathematics. Topics covered are consistent with the major topics that most Algebra II level courses cover and the course spends approximately 10 weeks devoted to the study of trigonometry. The course is the first detailed study of trigonometry for the students outside of simple right triangle ratios. The goal of the course is to prepare students for a pre-calculus or calculus level class as high school seniors.

The Algebra III and Trigonometry course is a senior level course in the lowest tier of a three tier system of math classes the school offers. Students in this class include students who tend to struggle with mathematics during high school as well as students who do not plan to major in a field requiring a great deal of mathematics in college. The course is designed for students who took an Algebra II level class as juniors that did not include the study of trigonometry (as opposed to Algebra II and Trigonometry). Students spend time reviewing many of the prior Algebra II topics and after that the course will spend approximately 15 weeks on the study of trigonometry. The course is the first detailed study of trigonometry for the students outside of simple right triangle ratios. The goal of the course is to prepare students for a college algebra or college pre-calculus level class as they enter college.

Calculus is a senior level class in the upper tier of a three tier system of math classes the school offers. The class does include some junior students. It is not an
Advance Placement level class that offers college credit. The students enrolled in the class have taken Algebra II and Trigonometry, Accelerated Algebra II and Trigonometry (honors level course) or Pre-Calculus as juniors. The class is designed for both accelerated students and students on a normal track who wish to be challenged. The major focus of the course is differential calculus, but the first 9 weeks of the course review Pre-Calculus concepts and approximately 2 weeks of that time is devoted to the study of trigonometry. The goal of the course is to prepare students for a first of second year Calculus class at the college level.

Trigonometric Topics Covered During Instruction

The trigonometry units in Algebra II and Trigonometry and Algebra III and Trigonometry introduce students to trigonometric functions through the study of angles in radians and degrees, ratios of right triangles, and coordinates in the Cartesian plane. Students spend time working with special right triangles and the unit circle as well. The unit also includes lessons on graphing sine, cosine, and tangent as well as transformations of those graphs. Students spend times solving trigonometric equations and looking at the inverse and reciprocal functions for sine, cosine and tangent. Additional topics include the law of sines, law of cosines, and trigonometric identities.

The major representations used during the unit include ratios of right triangles, the unit circle, graphs, and algebraic representations like equations. Students are taught to solve equations algebraically and graphically, both during the unit and throughout the year. In addition, the graphs of the sine and cosine and the unit circle were presented
with GeoGebra at various points during class and students were able to utilize graphing
calculators during the unit. Students did not do any constructions by hand or with
computer software.

Calculus covers many of the same topics during the review in the beginning of the
year but moves through the material rapidly with the goal of preparing students for
differential calculus. Students in Calculus also work with sine, cosine, and tangent
functions, as well as their reciprocals, during the study of limits and derivatives.

Data Collection

<table>
<thead>
<tr>
<th>Timeframe</th>
<th>Procedure</th>
<th>Students Involved</th>
</tr>
</thead>
<tbody>
<tr>
<td>End of School Year</td>
<td>Participant Selection</td>
<td>121 students</td>
</tr>
<tr>
<td>Early June</td>
<td>First Interview</td>
<td>6 students</td>
</tr>
<tr>
<td>Mid June</td>
<td>Second Interview</td>
<td>3 students</td>
</tr>
</tbody>
</table>

The data collection was in the form of 2 one-on-one interviews with students.
The interviews occurred after the end of the school year when students had finished all
exams. Since the trigonometry units did not occur at the very end of the year, this would
ensure that their answers were a reflection of what they were able to remember and recall
and not just simply memorized for the sake of the interview. 4 of the students chosen
were seniors enrolled in Algebra III and Trigonometry, 1 student was a junior enrolled in
Algebra II and Trigonometry, and 1 student was a senior enrolled in Calculus. The
student enrolled in Calculus also took the Algebra II and Trigonometry course the
previous year taught by author as well. All 6 students participated in the first round of
interviews. After the first round was completed, the responses were analyzed for the purpose of creating a second round or questions to be used in a follow-up interview. 3 of the original participants were called back based on their ability level and availability.

During both interviews, the interviewer asked guided questions and follow-up questions to gain a better understanding of students’ responses and understanding. The questions were meant to allow students opportunities to demonstrate how they represent sine. Questions were similar to the types of tasks and problems students encountered during the school year. Some of the questions were similar to the questions used in other research studies (Weber, 2005). Students responded both verbally and using pencil and paper during the interview. They had access to a graphing calculator, although one student preferred to use a personal calculator which did not have graphing capabilities.

Interview Protocol

The author took the role of interviewer for the student interviews and asked the following scripted questions as well as follow-up questions to the original scripted questions as needed in order to gain better insight into student thinking. The author did offer hints and clarifications to students when they asked for them. The complete list of questions for both interviews, as well as the transcripts, can be found in the appendix.
**Question 1: How are the equation of a function and the graph of a function related?**

*Explain your thinking.*

The purpose of this question was to see if the students had formed any connections between the equation of a function and its graph and what perspectives they used for each of these representations. Because the question did not mention anything with regards to trigonometry, it would also give some insight into how students used these representations for functions in general. One possible connection the students could make was the Cartesian Connection. If the students answered that the points on the graph were also solutions to the given equation, then it would be apparent that the students understood the Cartesian Connection and had used it to connect the algebraic representation with the graphical representation. It was also important to see which perspective, process or object, students used during their explanations and if it had any effect on their answers. Did students see equations and graphs as a process that links x and y together, or did they see the equation and graph as objects on which operations are performed and properties noticed? It was also entirely possible that students could utilize both perspectives at some point in their responses.

**Question 2: The following is the graph of y = x² - 2x – 3. How many points are on this graph?**

The main purpose of this question was to see if the students could utilize a graph from the process perspective. While it was possible for the students to connect the equation and graph together by citing properties of both, such as the y-intercept, a
stronger connection could be made if the students discussed how the graph, made up of individual x-values and y-values, connected to the solutions of the given equation. The question was also meant to show how many and what specific points the students saw on the graph. If the students had a strong concept of a graph, they should be able to see more points than just the coordinates with integer values. Finally, since the function the students saw in the question was quadratic, it also allowed for an analysis of the students’ reasoning about graphical representations without involving trigonometry.

*Question 3: Can you give examples of functions and examples of non-functions? How can you tell when you have one? Are the multiple ways to tell?*

The purpose of this question was to assess if the students understood the concept of a function, how they might represent it, and what procedures were known to them for determining if a given representation was a function. It could also give some insight in the perspectives students took when representing a function. Research has shown that students readily rely on algebraic representations first (Knuth, 2000). However, a common way to determine if a relation is a function is using the vertical line test. Using this test would require students to change representations. Observing what connections students made when changing representations is a very important component of this study. Also, this question purposefully did not incorporate trigonometry in order to see how students represented functions in general. Later, students were given opportunities to represent trigonometric functions.
Question 4: What does $\sin(x)$ mean? Is it a function? How can you tell?

The purpose of this question was to see exactly how students represent $\sin(x)$. Research has shown that many students do not see sine or cosine as a function (Weber, 2005). In the previous question, students represented functions as equations, graphs, and tables. They had the most success while working with the graphical representation. It was interesting to see if students would stick to these representations to make sense of $\sin(x)$ or began to use new representations. Students’ earliest forms of representing sine and cosine dealt with ratios of sides of right triangles. SOH CAH TOA is the mnemonic device just about every high school student has either learned or memorized to represent the trigonometric functions as a ratio of the sides of a right triangle. Later they are introduced to the unit circle and graphs, as well as equations. This question was meant to see if students can connect the representations for sine and cosine with the representations they had for functions in general.

Questions 5: How could you find or estimate the $\sin(30^\circ)$? How could you find or estimate the $\sin(20^\circ)$?

Question 6: Can you think of other ways to do #5?

The purpose of these questions was to see how students will use their representations for $\sin(x)$ to solve tasks. Students had represented $\sin(x)$ as a ratio from a right triangle, a coordinate on the unit circle, and a graph. While $\sin(30^\circ)$ could easily be recalled from a special right triangle or the unit circle, $\sin(20^\circ)$ could not, so it would be of interest to see if students used the same representation and perspective to answer both
questions. It was also important to observe if the students arrived at the answer using the
calculator if there was any conceptual knowledge to reinforce the calculator value or if
the student simply accepted the calculator answer as accurate. If a student used multiple
representations to solve, it was be important to see if they used the Cartesian Connection
or some other connection to tie the representations together.

Question 7: Explain how you would solve the following equations: \( \sin(x) = \frac{1}{3} \) and
\( 2\sin(x) + 1 = \frac{2}{3} \).

Questions 8: Can you think of any other way to solve the equations in #7?

Just like the previous question grouping, this set of questions was used to see
what representations and perspectives students used in their responses. Unlike, the
previous questions, coordinates from the unit circle could not be used. To find the
answer on the calculator, student also had to utilize the inverse sine function. The
context of the second equation also led many of the students to use algebra to isolate a
variable. It attempted to show what effect such an equation had on the students’
representations. Once again the students were asked to solve the equations in multiple
ways to see if they were able to use multiple representations. This would demonstrate if
they had any connections for those representations. Unlike the previous question set, the
answers here were angle measures and there were an infinite number of solutions. It was
interesting to see which students realized either of these two factors and what
representations they used.
Question 9: If \( \sin(x) = 2 \), what is the value of \( x \)? How do you know?

The purpose of this question was to see if students had a conceptual understanding of why there is no answer. The question was purposefully worded so the students would expect to find a value. If a student simply relied on an algebraic representation and used the calculator to find the inverse sine of 2, then the calculator would give the students an error message about the domain. This question forced the students to draw on their understanding of sine to figure out what the error message meant or to connect that error message to another representation to answer the second part of the question. If the question was worded so that the students knew for a fact the there was no solution, then they would most likely stop once the calculator gave the error message and use the error as their justification.

Question 10: Is there a connection between the unit circle and the graph of \( \sin(x) \)?

The purpose of this question was to specifically see what connections students make with the unit circle and graphs. Both were essential representations for understanding trigonometry (Byers, 2008; Weber 2008, Brown, 2006). However, whether by the context of the question or the students’ unfamiliarity or discomfort with graphs, students had not really used graphical representations throughout their responses to the trigonometry questions. This question was meant to see if they were able to use a graphical representation and what connections they could make to the unit circle. In all three courses students were shown, using GeoGebra, how the unit circle coordinates

33
connect with the graphs for \( \cos(x) \) and \( \sin(x) \). If the students were able to conceptualize that connection, they should have been able to demonstrate it in their response.

**Question 11: Explain how you can graph \( \sin(x) \).**

The purpose of this question was to get the students to work with the graphical representation. Students could have approached it from the process perspective and used the Cartesian Connection to link the coordinates of the points to solutions to the equation \( y = \sin(x) \), or students could have approached it from the object perspective and discussed the period, amplitude, and overall shape of the sinusoidal wave. Students could have also connected the graph to the unit circle, much like they did in the previous questions and attempted to use the coordinates from the unit circle to create the graph. Students saw all of these methods and perspectives used at some point during instruction in all three courses. The perspectives they took may affect the connections they made.

**Question 12: Using the graph, can you find or estimate \( \sin(20^\circ) \)?**

**Questions 13: Using the graph, if \( \sin(x) = -1 \), what is the value of \( x \)?**

The purpose of this question set was to see if the students were able to use the graphical representation to complete tasks. Since the students did not use a graphical representation to answer questions similar to this earlier, it was interesting to see how well they could do it when a graph was provided for them. It was not possible to tell if the students choose not to use a graph because they felt it was not possible or just did not think of using a graph due to the context of the question. Now the context specifically
asked them to use a graph. The second question would also help to determine if the graphical representation allowed the students to see that there were multiple solutions to the equation, which none of them originally realized when solving the equations algebraically, with the unit circle, or with a triangle.

*Question 14: Look at the following graph. Is this graph a function? How can you tell?*

The graph for this question was a sinusoid graph that has been transformed from an original parent graph. The purpose was to see if the student could incorporate such a graph into their understanding of functions. Most of the students used the vertical line test to determine if a graph was a function earlier, but no student cited a trigonometric graph at that point. It was necessary to see if students still made the connection for trigonometric graphs or if they treated trigonometric graphs as a separate entity.

*Question 15: Can you tell if the graph is of a sin(x) or cos(x) function? Can it be both? How do you know or what information do you need to make a decision? How would you write the equation for this graph?*

The purpose of this equation was to see what connections students had with a graph and equation. Research has shown the moving from a graph to an equation is something that gives students a great deal of difficulty (Knuth 2000b). Due to the complexity of the equation and difficult nature of trigonometric functions, it was not expected that the student would be able to produce an equation. Rather, it was important to see the connections the students tried to make, what perspective they took, and what
understanding they had in general about the graphical representation and algebraic representation of $\sin(x)$ and $\cos(x)$. Since most of the students utilized an object perspective both when moving from a graph to an equation and when creating the graph for $\sin(x)$, it was probable that most students would focus on features like amplitude, period, and shape of the graph and how these features connected to an equation.

The second interview only involved three students. The questions were created to give the students more opportunities to use the graphical representation and to see if the students’ representation for sine as a function included an understanding of the inverse sine operation.

*Questions 1: Graph $y = \sin(x)$. Explain how you created your graph. Using your graph find the following: a) $y = \sin(80^\circ)$, b) $y = \sin(-150^\circ)$, c) $0.75 = \sin(x)$, d) $-1/4 = \sin(x)$*

The purpose of this question was to make the students actually create the graph for $\sin(x)$. All the students explained how to make the graph in the previous interview and relied on the object perspective to do so. However, because no student had to make a graph, it was possible that some did not utilize the process perspective or possibly the Cartesian Connection because they did not need to produce the actual graph or plot specific points. This question would see if students saw the need to change to the process perspective to physically plot points to create an accurate graph for $\sin(x)$ or if their knowledge from the object perspective was enough for an accurate graph.
To answer the second part of the question the students would almost have to view the graph from the process perspective. If the students created the graph from the object perspective, it would determine if they are able to change perspectives with the graphical representation that they had created. Also, some of the students had difficulty both with an understanding of negative numbers with the sine function and multiple solutions. Using the graph to find these four values would determine if the graphical representation helped students develop a better understanding for both. The tasks for which the students used the graphical representation in the first interview were instances where they had previously found the answer with another representation, \((\sin(20^\circ))\) or could have easily determined the answer using the unit circle representation \((\sin(x) = -1))\). Some of the students even stated that they did use this prior knowledge to answer the questions instead of using the graph. These questions were meant to see if the students could use the graphical representation as the primary means to get the answer and not switching or connecting with another representation to produce an answer. However, if the students did change representations to check or to help them understand the graph, that would be important as well.
Question 2: Which of the equations best describes the graph? Explain how you know.

a) \( y = 2\sin(x) - 1 \), b) \( y = \sin(2x - 1) \), c) \( y = -2\sin(x) - 1 \), d) \( y = -\sin(2x) - 1 \)

Question 3: How could you check your answer to number 2? Are there multiple ways to check?

The purpose of this question set was to see what procedure the students would use to get an answer. In the previous interview, they were given a similar question but were not given four possible choices. The students focused on the graph as an object and mentioned things like amplitude, an x-axis reflection, and a vertical translation with the graph and attempted to describe how each affected the equation. Now that the students had four choices for equations, the students also had the option of guessing and checking by either graphing the equations on the graphing calculator, or checking random points on the graph with the 4 equations to see which equations worked. If the students chose this option, it would be interesting to see how many points they used before they were convinced an equation was the correct one. Also, if the students chose to plug each equation into the calculator, it would be important to see if the students tried all the equations or could make a decision after graphing one or two of them on the calculator. It was also be interesting to see if the students connected an equation with a graph, or the graph with an equation. The direction the connection was made appeared to affect the responses students gave both in perspective and strength of connection during the first interview.
Question 4: Given the unit circle, special right triangles, and the graph of \( y = \sin(x) \), answer the following questions.  

a) What is the \( \sin(45^\circ) \),  
b) What is the \( \sin(65^\circ) \),  
c) If \(-1/2 = \sin(x)\), what is the value of \( x \)?,  
d) If \( 0 = \sin(x) \) what is the value of \( x \)?,  
e) If \( 3/2 = \sin(x) \), what is the value of \( x \)?,  
f) solve the equation \( \sin(x) + 1 = 1/2 \),  
g) solve the equation \( 3\sin(x) = 2 \)

Note: The students were given a graph of \( y = \sin(x) \) with an x-axis in degrees, a unit circle labeled with positive degree and radian measures with the 16 most used coordinates, and two special right triangles with the sides and angles labeled (30\(^\circ\)-60\(^\circ\)-90\(^\circ\) and 45\(^\circ\)-45\(^\circ\)-90\(^\circ\)). (see figures 5, 6, and 7 in Appendix F)

The purpose of this question was to see what representations the students would choose to complete each task. In the previous interview, when the students were provided with a graphical representation and asked to complete similar tasks, they were able to utilize the graphical representation for their answers. However, before that point, the use of the graphical representation for \( \sin(x) \) was limited to use for the vertical line test to prove it was a function. Now that the students had 3 representations and a graphing calculator, it would be interesting to see how it affected their responses. The unit circle representation has been utilized throughout both interviews, especially as a set of memorized points. Problems occurred when students incorrectly recalled points for this representation. By giving the students the unit circle, it insured that any mistake is a result in the understanding of the unit circle representation and not a matter of poor memory. Finally, it was important to see if students attempted to complete all the tasks with the same representation or choose the representation based on the task itself. If the
students have mastered the content, they should be able to choose the best representation for a task, not do it the same way each time (Moschkovich et al, 1993). The tasks were meant to highlight some of the strengths and weaknesses of each representation, so while it was possible to use the same representation to solve each task, a student with a strong understanding of multiple representations should have been able to distinguish which representation worked best for a given task.

Question 5: How do your answers to a and b differ from c through e?

The purpose of this question was to see if the students realized the difference between the regular sine operation and the inverse sine operation. Students were able to use both operations in different representations, but it was not clear if they were able to connect the process of inputting an angle and outputting a ratio with the sine operation and inputting a ratio and outputting an angle as the inverse sine operation. Students could look at their answers and see that parts a) and b) were ratios and c) through e) were angles, but it would be interesting to see if they knew why.

Question 6: Discuss sine and inverse sine in terms of inputs and outputs. How can you represent them in a triangle? On the unit circle? On a graph?

The purpose of this question was to see if the students could discuss sine and inverse sine in terms of inputs and outputs. This would help demonstrate if the students could connect it to the definition of an inverse function. Students spent time studying inverse functions during the year, so it was important to see if they have a way to
represent them and if they can incorporate sine and inverse sine into that understanding. The questions attempted to show if students knew that the inputs and outputs switch in an inverse operation. In this case, did the students understand that sine inputted an angle and outputted a ratio and inverse sine inputted a ratio and outputted an angle? It was important to see if the students realized this change with inputs and outputs was because they were inverse operations or if they realized this change by observing the procedures. In addition, this question aimed to see if the students were able to represent the procedure for sine and the procedure for inverse sine in the different representations and if they were able to make connections between those representations.

Data Analysis

Student responses were analyzed qualitatively to investigate students’ depth of understanding of representations and connections between those representations as they answered questions. An item analysis was done for each question looking at what representations were used as well as whether the students used the process or object perspective. The three main representations that students used during the trigonometric units were algebraic, graphical, and geometric, in the form of right triangles and the unit circle. Responses were also analyzed to see if students utilized the process perspective, the object perspective, or both. Particular attention was paid to identifying if and when a student utilized the Cartesian Connection. The analysis also attempted to discover where students’ knowledge is lacking and how this relates to the ability of the students to use multiple representation and perspectives. The patterns of thinking that were most visible
were documented as well as where the students were challenged. The analysis also found unique patterns of thinking. Finally, a cross examination of all responses allowed the author to answer the research questions.
Analysis and Results

The purpose of the study was to see how students understood and represented sine and what connections students had among their representations. The study also considered if the students used the process perspective, the object perspective, or both with their representations. Another focus of the study was to see if the students utilized the Cartesian Connection to connect the equation and graph together. A summary of the results is given for each of the questions and then evidence is provided in the form of an analysis from the responses of all the students who participated in the interview.

First Interview Analysis

Question 1: How are the equation of a function and the graph of a function related?
Explain your thinking.

Results

While all the students were able to connect the representations together, five of the six attempted to do so with the Cartesian Connection to some degree. However, Student A’s connection was not strong and eventually switched to the object perspective in an attempt to make a more convincing connection. Student A and Student D both used the object perspective during their answers to make a connection and neither one used the Cartesian Connection at that point. It is also interesting to note that changing the context of the connection had some impact on the responses in terms of the perspective being
used. Connecting an equation to a graph did not always bring about the same connections or perspectives as connecting a graph to an equation.

Evidence

Student A right away connects the equation and the graph from the process perspective by discussing how both are made up of points.

TEACHER: OK, so first question. How are the equation of a function and the graph of a function related?
STUDENT A: Um, both can provide points to the function.
TEACHER: OK. What do you mean by points?
STUDENT A: Both can plot points on the graph like if you’re given an equation, you can like, just by the equation can’t you get like points on a graph? Is that right?

The student views the equation as a process that yields points to plot on a graph but does not specifically state how that process yields the coordinates. Therefore, from this explanation there is not enough detail to conclude that Student A is using the Cartesian Connection. When asked further about how the points on the graph relate to the equation, there is some evidence of a fragile connection because Student A does not continue to use the process perspective or the Cartesian Connection but switches to the object perspective in an attempt to find a connection.

TEACHER: So I guess what I’m trying to ask is what makes a particular graph go with one particular equation? You have a graph there and that equation (motioning to question #2). How do you know that the two go together? What tells you that they are related?
STUDENT A: The type of function it is. Like there are different types of functions, like quadratic and cubic.
TEACHER: And how can you tell that?
STUDENT A: By the exponent right there (points to equation). It think it’s quadratic.
TEACHER: Anything else that relates the equation to its graph?
STUDENT A: Um, it depends on what is in the equation, like when it, like, for this it would start at (0, 0) (pause). No, I’m wrong (pause). Well, I thought in the equation it will tell you when to move the graph, like the original graph, um, depending on what’s in it, you can move it down, up, left, or right.

Student A is now viewing the equation as an object that has properties, like exponents, that affect what the graph will look like as an object. This allows the student to make a connection between the properties of the equation and the properties of the graph. Student A understands how to transform the equation in a way that will translate the graph, as an object, in the coordinate plane in certain directions. The student has once again connected the two representations together, but from an object perspective and not utilizing the Cartesian Connection.

What is interesting to note is the effect that the context of the interview questions may have had on his/her answer. The original question was presented in such a way that the student started with an equation, an algebraic representation, and then connected it to a graph, a graphical representation. During the follow-up interaction the question asked him/her to start with a graph and connect it back to an equation. It is possible that the change in context led to a change in perspective by the student. However, including the evidence for the second response, it is still not possible to determine if he/she truly understands the Cartesian Connection because the student has not been able to specifically state how a point on the graph is directly linked to an equation. The student appears to have some procedural knowledge in understanding that an equation will yield points on a graph or that making changes to an equation will translate the graph up and down or left and right. However, the conceptual knowledge of the significance of those
points to the equation and graph appears to be absent from the explanation. This is another possible explanation for why Student A switched perspectives in an attempt to find a connection between the representations.

The response for Student B is very similar to Student A in that it begins in the process perspective.

STUDENT B: Ok, well, the equation of the function is always going to equal y so, um, for every... on the other side of the equals there’s an x so... for every number there you put in for x you will get an answer for y and then on the graph for every, um, on this one for example (looking at equation in question two), if you put in 3 for x you got 0 for y and that’s how they relate.

He/she has connected the equation to the graph, but unlike Student A, specifically states this happens by the input/output process of the equation. That process yields an x-value and y-value that translate to a coordinate on a graph. The student appears to be utilizing the Cartesian Connection if only for one point in the function. The follow-up question this time does not cause the him/her to change perspectives, but rather the student gives solid evidence that there is an understanding of the Cartesian Connection.

TEACHER: Ok, very good. So the (3, 0) again relates to the graph how?
STUDENT B: Um, that it’s a point on the graph and you found that point with the function of the graph because you put in 3 for x and once you solved the equation you got 0 for y.

Student B expresses the understanding that the pair of coordinates for a point on the graph is a solution to the equation. The context of the question though is still phrased in a way that the student started with an equation and then connected it to a graph. Student B’s conceptual knowledge of functions seems to be richer than that of Student A because
he/she articulates a connection between the equation and graph. This may have also made a change in perspectives unnecessary.

With Student C comes the first instance of using two different perspectives to initially connect an equation and graph. Student C’s response seems to indicate the he/she is viewing the equation as an object with properties, but because Student C sees that the point (3, 0) is on the graph via the process of relating an x-value to produce a y-value, he/she is using the process perspective for the graphical representation. However, the connection between the two representations does not seem to be very strong, quite possibly because they are from two different perspectives.

STUDENT C: The equation of a function is more mathematical, kind of like specific numbers like 2x and 3², things like that. And a graph of a function is more about the visual in showing you more points like show you specific points on a graph.

When the student is asked to provide more detail with the explanation of what points are, Student C has to change perspectives for the equation and discusses the equation from the process perspective to attempt to better connect the two representations.

TEACHER: And what connects them together?
STUDENT C: Um, the solutions.
TEACHER: Ok, could you expand on that? What do you mean by solutions?
STUDENT C: The, um, answers. Um, I don’t know how to expand on solutions.
TEACHER: Well, what do they look like?
STUDENT C: Points on a graph or getting x by itself or any variable.

Student C does appear to end up utilizing the Cartesian Connection with the answer to the follow-up by tying the solutions to the equation with the points on the graph, but Student C had to view both representations from the same perspective before doing so. However,
the answer does not specifically show that the student understands a point is made up of an x and y value that can also be found from the equation. If this is the case, then the student is not using the Cartesian Connection. The student cites solving for x as a means for generating solutions. The student appears to be using procedural knowledge for solving equations with one variable. It is unclear if the student has the conceptual knowledge for how functions with two variables (independent and dependent) relate.

It would appear that based on the responses of the three students so far, a student must be using the process perspective for both the equation and the graph to cite the Cartesian Connection as a means of connecting the representations together. The process perspective for a function involves linking the x-values and y-value together via a process of producing outputs from inputs (Moschkovich et al., 1993). So when students use the process perspective, they have the opportunity to make the Cartesian Connection. When the students view the equation or graph as an object, their focus is on the properties of equation and the shape of the graph and not on how the x-values and y-values relate.

Student D is able to connect the equation and graph together from both perspectives but does not use the Cartesian Connection at all. The initial response shows the student citing the definition of a function to explain how both representations are connected and viewing both representations for the process perspective.

STUDENT D: Um, the equation of a function for every, can I write this down? For every x there is only one y so then for a graph of a function, um, every x-coordinate can have only one y output. Do you want me to write down my responses?
TEACHER: No. So, if you have an equation that’s a function and you have a graph that’s a function, how can you tell that the two go together?
STUDENT D: Um, because if this was like, I think about it was like if there was something like this (see figure 1) and there was like $x_2$ then it’s $y_2$. It can’t be like a straight up and down line.

Figure 1

The student is using the graph to show that two points cannot have the same y-value, but in the explanation, he/she has incorrectly stated this to mean that every x-value has to have a unique y-value. However, the student does understand that the equation of the function and graph of the function have x-values and y-values that are linked together and no x-value can have multiple y-values. This connection the student has made is based on the definition of a function and not the Cartesian Connection, but it does not explain why a specific equation and graph go together. When the student again is asked exactly how the equation and graph are linked together, he/she approaches the second answer from the object perspective and ties the two representations together by the features of the equation and graph as objects. However, since Student D does not discuss the x and y points, there is still no evidence of the Cartesian Connection being used. Also, it is not clear if the student would be able to make this same connection for functions that are not linear.

TEACHER: I guess what I’m trying to ask is: You see a graph, you draw a graph and you know the equation of the graph. How do you know that’s the equation of the graph?
STUDENT D: Um, I put it in \( y = mx + b \) form and then I look at \( b \) and usually find out what the, um, the y-intercept is and then I’ll match it up and see if it’s the same. And then otherwise I think that’s about it.

The student is directly linking the equation to the graph by characteristics common to both. Even though the student cites the y-intercept he/she is still using the object perspective because Student D is using the point as part of the graph as an object and not as an x-value that is linked with a y-value. Student D sees both the equation and graph as objects with a matching property, in this case the y-intercept. The context of the follow-up questions again goes from graph to equation and the change in context could have caused the student to change from the process perspective to the object perspective, just as Student A did. Based on students responses so far, it appears that starting with an equation, an algebraic representation, causes students to take on a process perspective for the representations whereas viewing the graphical representation initially cause the students to take on an object perspective. This means they most likely view an equation as a procedure whereas a graph, being more visual, is more likely to be seen as an object with properties.

Student E is able to quickly utilize the Cartesian Connection as a means of connecting the two representations together, and once again, views both representations from the process perspective.

TEACHER:Alright, so first question, how are the equation of a function and the graph of a function related? Explain your thinking.
STUDENT E: OK, um, well, like the equation of a function is like the input and like the output, right? I mean, like, what you put into the function and what you get is the output. The graph is like, like the visual interpretation of that. Like, you know, the x’s and the y’s, those are your inputs and outputs.

TEACHER: Ok. What is special about the x’s and y’s on the graph?
STUDENT E: Um, I mean, like each x is what you input into the function and then each y is what you get out of it, and the equation is obviously essential to that.

It is clear that Student E understands how the solutions to an equation translate into points on a graph. When the context is changed during the follow-up question, Student E does not change perspectives as Student A did, but that is most likely because the follow-up specifically asks about the x-values and the y-values, which could lead students to continue to use the process perspective. However, Student E’s answer is very detailed in stating that an equation inputs an x-value and outputs a y-value and the graph is made up of the same x-value input and y-value output as a point. These are details no previous student has explicitly provided yet so it appears that Student E has the strongest conceptual knowledge to go with the procedural knowledge. As a result, even with the change of context, he/she feels no need to change perspectives because of the strength of the connection between representations in the process perspective. It does not appear to be a coincidence that both students that showed a strong understanding of the Cartesian Connection did not change perspectives in their answers.

Finally, with Student F there is one more example of a student connecting the two representations together by using the Cartesian Connection in the context of going from an equation to a graph.

STUDENT F: Um (talking indistinctly), the function, well, isn’t it the function, the function is the equation technically, and um, the equation is like how you get the inputs and outputs of the function.
TEACHER: Ok (pause). And the graph?
STUDENT F: The… Oh, the graph. I forgot that part and then the graph is the visual demonstration of the function and the equation.
TEACHER: So can you explain how the equation and the graph are related?
STUDENT F: The, um…
TEACHER: You said it was a demonstration. What do you mean by that?
STUDENT F: Well, I mean you plot the points, like plug in x and y, you find the values and then you plot them onto a graph. Then you draw the graph.

Student F is also viewing the equation and graph from the process perspective by citing how an equation has x-values and y-values and those values are the points that are on a graph. Student F understands that the inputs and outputs of an equation make up the individual points on a graph. From the answer it is not possible to see if the student would be able to move from a graph to an equation. It is also interesting to note that he/she cites the equation as the function itself. It would be interesting to see how this student approaches working with functions from the graphical or tabular standpoint only.

**Question 2:** The following is the graph of \( y = x^2 - 2x - 3 \). **How many points are on this graph?**

**Results**

All the students were able to see the graph as a series of points and many utilized the graph from both perspectives during their responses. Students showed a tendency to label the intercepts first which means they view those points as being the most important or easiest to see, most likely due to the emphasis intercepts received during instruction. Some students seemed to be bothered by the lack of accuracy an estimated coordinate from the graph gives as an answer and realized that a change in representation back to the equation in the algebraic representation will be more accurate. This was consistent with
the research (Knuth, 2000a). Only Student D and Student F were comfortable giving estimated coordinates from the graph as an answer.

Evidence

When Student A initially hears the question, he/she creates a table, most likely to produce points, which would mean Student A has focused on the equation from a process perspective. However, the student soon realizes that the graph from a process perspective can be used to answer the question and begins to state the coordinates of points directly from the graph.

STUDENT A: Um, (long pause). I can write on this, right? (makes x/y t-table) Um, (long pause). Oh, uh, isn’t it wherever… isn’t it, can’t you tell one way by if it crosses the x-axis? Right here, (motions to x-axis on graph), or is that wrong?
TEACHER: For what?
STUDENT A: Points
TEACHER: You’re just looking for points. Any point on the graph.
STUDENT A: Do I just say it point by point? Um, (3, 0), (-1, 0).
TEACHER: Any others?
STUDENT A: Um, I don’t know what it’s called but where it meets (marks vertex of graph at (1, -4)), like where it like hits… (makes parabolic motion with hands for emphasis) Um, (pause) Um, I want to say (0, -3) because there’s a -3 there (points to equation).
TEACHER: Ok
STUDENT A: So four?

The coordinates that the student marks are very interesting. The student marks the points that are usually emphasized most during instruction, the vertex and the intercepts, but no others. During the school year when creating graphs of quadratic equations, students were taught to include the vertex and the y-intercept, and the student also seems to be trying to make some connection with the x-intercepts. Although it is not possible to determine why Student A views the x-intercepts as important, students were taught to
find the roots of equations, including quadratic equations, by finding the x-intercepts on a graph. It is possible he/she is recalling the procedural knowledge associated with that but lacks the conceptual knowledge as to why the x-intercepts are important. It is only when the student is asked if there are more points that the student states that there are an infinite amount of points on the graph.

TEACHER: So that graph has four points?
STUDENT A: Yes. It has more (pause). Infinite. I just don’t know where they are. I don’t know. That’s all I got.
TEACHER: Alright. How many points are between here and here (points to (0, -3) and (3, 0)). How many points are on that graph?
STUDENT A: Um, in between these two? Four, just because they’re on the line.

This response again is very interesting for a few reasons. First, even though the student says there are an infinite number of points, the student is only able to identify four additional points, most likely because there are only four coordinates ((0, -3), (1, -4), (2, -3), and (3, 0)) that have integer values for x and y between (0, -3) and (3, 0). The student appears unable to realize that an x-coordinate could be rational or even irrational. The student most likely has memorized the fact that a graph is made up of an infinite number of points but lacks the procedural or conceptual knowledge for finding the infinite number of points aside from the ones with integer values. Second, the student made an x-y table in the beginning before abandoning it when he/she realized the question could be answered by looking at the graph. The fact that the student made a table could imply that Student A understands the connection between the equation and its graph, which would mean Student A was about to utilize the Cartesian Connection. However, when using the graphical representation, this idea did not seem to be present. It is possible Student A
only understands the Cartesian Connection for the equation to graph standpoint and cannot connect the points on the graph back to the equation.

In contrast to Student A’s response, Student B immediately states that there are an infinite number of points and quickly points out all the coordinates on the graph that have integer values for x and y. Student B also realizes there are more but finding them is not “easy” and that they fall between the lines that make up the grid of the graph.

STUDENT B: Well, there’s infinite points.
TEACHER: Could you mark some on the paper for me?
STUDENT B: Well, there’s (3, 0) and there is (1, -4). There is (-1, 0). There’s the (0, -3). Um, (2, -3). That’s all the easy ones.
TEACHER: Ok, what do you mean by easy?
STUDENT B: Well, those are the only ones that I see. There’s other ones that have y on the cross between the x and y axis. There’s a bunch of other ones but they’re not… I mean you’ll have to do a bunch of math to find them.

The student does not want to estimate those points from the graph but instead appears to imply using the equation to find them when Student B says, “do a bunch of math to find them.” This does not indicate that Student B can actually find those additional points, only that Student B knows they exist and are possible to find from the equation. It is also interesting to note that from the response, is appears that a graph from a process perspective should only be utilized to find the integer coordinates. To find others, it is necessary to switch representations and use the equation from a process perspective. So once again, just like Student A, Student B seems to understand that utilizing the Cartesian Connection from the equation to the graph will generate points on the graph.

With Student C, there is some confusion with what a solution to the equation really means. While the student knows that the equation will yield points on the graph, it
appears that he/she might assume the solutions refer to only the x-intercepts. Student C has confused the definition of solution to the equation with root or zero of the equation. This was consistent with findings of previous research (Van Dyke & White, 2004; Knuth, 2000a).

TEACHER: Alright, so if you have a solution to the equation, where does it appear on the graph?
STUDENT C: Um, on the x-axis.
TEACHER: Ok. So, next question, the following is the graph of $y = x^2 - 2x - 3$. How many points are on this graph?
STUDENT C: Do you mean points by solutions or answers?
TEACHER: Just, whatever you think.
STUDENT C: Um, one… two…
TEACHER: Mark them too.
STUDENT C: I’ve got five so far.
TEACHER: Ok, any others?
STUDENT C: Uh, well, all the lines are points. It’s just the main points are, um, I wouldn’t know how to explain it. I think that’s it, yeah.
TEACHER: Ok, and what’s special about the points on that graph.
STUDENT C: Um, I don’t know.

Student C is using the word answer to mean a point on the graph, but solution to mean root or zero of the function. Student C clearly understands that the graph is made up of multiple points, but the five the student marked on the paper were the five points with integer coordinates. Student C is able to state that there are other points but has no way of finding or labeling them. Unlike the response to the first question, Student C is not able to utilize the Cartesian Connection because of the confusion with definitions. The student earlier stated that the solutions, which imply an understanding of the x-intercepts on the graph, are also the solutions to the equation. However, he/she could not explain how the other points on the graph, which Student C referred to as answers, relate to the equation.
Student D is able to answer the entire question by using the graph from the process perspective. Unlike the three previous students’ answers, Student D has no trouble finding a non-integer value for a coordinate.

TEACHER: The following is the graph of \( y = x^2 - 2x - 3 \). How many points are on this graph?

STUDENT D: Um infinite.

TEACHER: Can you show me some? Mark them on your paper.

STUDENT D: Um, (-1, 0), (3, 0), (0, -3)

TEACHER: Any others?

STUDENT D: Um, (2, -3), (-1 1/2, 2)

However, the student does start with the intercepts, again seeming to demonstrate the importance students perceived the intercepts were given during instruction. It is interesting to note that the student did not use the vertex as one of the points mentioned. This would imply that the student is only using the graph from the process perspective. If he/she would have shown the vertex as a point, that could have meant that the student had also realized that graph was quadratic and an important property of a quadratic graph is its vertex. This would imply the student would have been using the object perspective simultaneously. The student is not using the equation during the response either. From the non-integer point Student D proposed, it is possible to see the student is not utilizing the equation from a process perspective because if \( x = -1 1/2 \), then \( y \) will equal 2.25. Therefore, the student used the graph to estimate the point and is comfortable with using the estimate as a point on the graph. No other student used this approach during the response without being directly prompted to do so.
Student E is also able realize the graph contains an infinite number of points but while labeling the points, it appears he/she is using both the process and object perspective simultaneously.

STUDENT E: So the second one is about a graph. How many points are on this graph? An infinite amount.
TEACHER: Ok, can you mark some for me?
STUDENT E: Yes, here, here, here (laughter). Here, here. Do you want me to tell you what they are?
TEACHER: Sure
STUDENT E: Um, so like (1, -4), and like (2, -3), and (0, -3). (-1, 0), (3, 0)

The student understands the graph is a parabola, and therefore an object. By starting with the vertex of (-1, 4) and then moving up both sides of the parabola to name additional ordered pairs, the student is also clearly viewing the graph as a parabola. Student E moves up each side of the parabola when naming additional points. This is a pattern that students learned during the unit on quadratic equations and is dependent on the vertex, not the equation, for finding order pairs.

Student E is then prompted to find some additional points to see if he/she will find some that have non-integer values. Although the student knows that the points exist, he/she shows the tendency to want to be as accurate as possible and realizes a change in representations will allow for such. The student is now using the equation as an algebraic representation in the process perspective to generate points on the graph.

TEACHER: Ok, without extending the graph anymore, can you give me some more coordinates?
STUDENT E: Um, yes, I mean, yes, like if I plugged in like 5, 5² - 2(5) – 3. 12, right?
TEACHER: Ok, what about another point between 0 and 3? Are there any more points between 0 and 3 that you could tell me?
STUDENT E: Yes, $1^2 - 2(1) = -3$. Oh wait, I just told you (1, -4). Sorry, um, like $.5^2 - 2(.5) = -3$. -3.75 so (1/2, -3 3/4).

The Cartesian Connection is used to change representations and the need to change representations is brought on by a desire for more accuracy. When asked to estimate a point on the graph without using the equation, the student has no trouble estimating a point with non-integer coordinates. However, he/she goes back to the object perspective and reflects the point (1/2, -3 3/4) over the line of symmetry to find another point. Once again the student is back to using the graph as an object. It is only when he/she is specifically asked to estimate an additional point that cannot be found through reflection that Student E gives an approximate, non-integer coordinate for an answer.

TEACHER: Ok, now without using the calculator, just looking at the graph, can you find another point? Estimate another point.
STUDENT E: Estimate one? Uh, well, if I just did that one, then 1 1/2 would be (1 1/2, -3 3/4).
TEACHER: Ok, go ahead and mark that one too. Alright, now go ahead and estimate one more for me. Any other point.
STUDENT E: Any other point? Um, just estimating it on the graph? Hmm, like, um, maybe like (3 1/2, 2).
TEACHER: Ok
STUDENT E: ish (laughter)

The final statement by Student E shows how uncomfortable the student is with giving an answer that is just an estimate. The “ish” implies the student wants to add emphasis to the fact that the answer is only an estimate. Since the student has the ability to change representations to give accurate answers and understands how the change will affect answers, Student E has a deep understanding of the connections between equations and graphs.
Similar to the previous students, Student F immediately states there are an infinite number of points on the graph. The student then labels the intercepts first but not the vertex. If the student is using the object perspective, he/she does not see the object as a parabola, merely a graph with x-intercepts. However, it is not necessary to view the graph as a parabola to answer the question. Perhaps the student is only using a representation and perspective with enough detail to answer the question asked.

TEACHER: The following is the graph of $y = x^2 - 2x - 3$. How many points are on this graph?
STUDENT F: Infinite
TEACHER: Ok, can you label a couple?
STUDENT F: Um, yeah. We have (0, -1), no wait, (-1, 0).
TEACHER: Go ahead and mark it on your paper.
STUDENT F: And you have (0, -3), and (3, 0).
TEACHER: Ok, can you find more points between here and here? (pointing to (-1, 0) and (0, -3))
STUDENT F: Um, like (-.75, -1)

When prompted to find a point between (-1, 0) and (0, -3) the student easily estimates a point with a non-integer value for x. Unlike student E, Student F is not bothered by the accuracy of the answer and never changes from the process perspective or the graphical representation during the response. Since the point is only an estimate (.75 does not yield output of -1 in the equation), the student did not use the equation for the answer. Since the student did not change representations, it is not possible to tell what connections the student is able to make for this question.
Question 3: Can you give examples of functions and examples of non-functions? How can you tell when you have one? Are the multiple ways to tell?

Results

The students relied on graphs to represent functions. Only two students were successfully able to represent function as equations in addition to graphs. This is most likely due to their ability to connect graphs to equations and really speaks to the strength of understanding for those individuals. It was also interesting to see that when students began to speak about inputs and outputs of a function, it was most often accompanied by a tabular representation. Many of the students also had a great deal of difficulty working with functions in the algebraic representation and most ended up switching to a graphical representation to utilize the vertical line test. Nevertheless, all the students did have at least a basic understanding of a function and at least one working representation of a function and non-function, even if the idea of a function was misconceived.

Evidence

Consistent with what previous research has established, Student A starts with an algebraic representation in the form of an equation. Using the object perspective and focusing on the appearance of the equation in terms of \( f(x) = \) and \( y = \), Student A then realizes that this is not going to help determine what a function is in this representation.

STUDENT A: Um, ok. I can just write any examples down? (writes two equations: \( f(x) = 2x^2 \) and \( y = 2x + 1 \)) Wait, (pause). Isn’t if it has \( f \) of \( x \) equals something then it’s a function and if it’s just like \( y \) (pause), then it’s not a function? Maybe (pause), I don’t know. Wait, (pause). That looks like it might be one too (referring to \( y = 2x +1 \) equation on paper). I want to say that they are both functions. I don’t really know what a non-function is. A non-function would be something that doesn’t pass the vertical line test? I’m trying to think of
a non-function. Oh, um, hmm (pause). Well, to answer that question, there are multiple ways you can tell.

The student does realize that both equations, f(x) = 2x² and y = 2x + 1, were functions, so Student A does have an algebraic representation of what a function is but is unable to think of what a function is not. As he/she attempts to find a non-example of an equation, Student A recalls that the vertical line test can be used to determine if something is a function. However, it appears that this is most likely a memorized response because the connection does not help the student to represent a non-function or give any insight into determining what makes an equation a function. The student does realize that the vertical line test will not help with an equation but still cannot offer an alternative.

Only after the teacher prompts a change in representation is the student able to represent a non-function. Research has shown that students tend to avoid graphical representations until there is an error or misconception with algebraic representations (Even, 1998).

TEACHER: You’ve given me two equations (f(x) = 2x² and y = 2x + 1). Can you show me a function any other way besides an equation?
STUDENT A: Oh, can’t you have a graph? (draws parabola, see figure 2) Function.
TEACHER: Can you give me a graph that’s not a function?
STUDENT A: (draws x-y axis, and stops to think, and then draw vertical line, see figure 2) That’s not a function.
TEACHER: Ok, so that’s not a function?
STUDENT A: Well, isn’t it if you have something over zero it’s undefined. Ok, that’s wrong. (draws square root graph, see figure 2) Ok, not a function.
TEACHER: Why?
STUDENT A: Because it doesn’t pass the horizontal line test.
The change to graphical representations is enough to finally allow the student to represent non-functions, but the student still is not able to explain what a function is in either the algebraic or graphical representation. Student A’s concept of a function is underdeveloped because he/she appears to experience some confusions with the vertical line test and horizontal line test as well as why a vertical line is not a function. This could be attributed to the fact that the student is approaching the graphs from an object standpoint. Student A knows the slope of a vertical line is undefined which he/she has connected with division by zero. The student seems to realize that this is not what is causing it to be a non-function but is unable to determine anything else. He/she has some procedural knowledge in using the vertical and horizontal line tests, but lacks the conceptual knowledge as to why they work or why they are important.
The teacher then prompts the student to change back to an algebraic representation to see if he/she can connect the vertical line test with a rule for algebraic representations. Even though the student can represent a non-function in graphical form, he/she is unable to represent it in the form of an equation. The difficulty the student has in moving from the graphical representation to the algebraic is consistent with Knuth’s (2000a) findings.

TEACHER: Ok, so you gave me some graphs that are not functions, can you give me any equations?
STUDENT A: Yes, I’m trying to think of an equation.
TEACHER: That first one you made there (parabola), what’s the equation of that one?
STUDENT A: Isn’t that a quadratic equation?
TEACHER: Is that a function?
STUDENT A: Yes
TEACHER: What about the second one (vertical line)?
STUDENT A: I don’t know. I kind of just did it. Um (pause), could that not be a function?
TEACHER: Why?
STUDENT A: Because it doesn’t pass the vertical line test.

The problem may be a result of the perspective the student is using. Student A is viewing the graph from the object perspective and is only able to link features of the graph and equation together, i.e. the fact a parabolic graph is a quadratic equation. The student did not attempt to change perspectives or utilize the Cartesian Connection when changing representations. As a result, he/she is unable to generate the equations to match the graphs.

However, the student appears to eventually change to the process perspective with an equation to look at inputs and outputs in an attempt to make a connection.
Representing the function in this manner takes Student A to the definition of a function and causes another change of representations, this time into a tabular representation.

STUDENT A: Oh my gosh! Isn’t it, that like, ok, I can’t remember which one is which, um, the input can have only one output?
TEACHER: Ok, so what does that mean?
STUDENT A: (Draws t-table with values and graph). Ok, that’s just another way to see if it passes the vertical line test.
TEACHER: Ok, but what are you doing?
STUDENT A: Um, why am I doing that? I’m determining if it’s a function or not.
TEACHER: And how are you doing that? Is that a function right there?
STUDENT A: No because there’s no equation. So if I was given an equation, I would try to figure out the points and if there’s more than one output for an input then it’s not a function.
TEACHER: Ok, but does that satisfy the definition you gave me?
STUDENT A: No
TEACHER: Why not?
STUDENT A: Because if it’s a function there’s no output (changes table). That’s not a function.

The student is able to connect the tabular form to the graphical form and the vertical line test and realizes that the method for tabular representations will also work for an equation. However, his/her tabular representation is underdeveloped because Student A believes it must be connected to an equation for it to be a function. This implies the student does not view a table as a standalone representation. The student also has some confusion with what “the input can have only one output” really means. He/she creates the following table on the left and then modifies it to the table on the right during the response to be an example of a non-function.
The student states that no input can have two outputs but the table he/she creates of a non-function has an output with two different inputs. The student has incorrectly represented this idea. The student is able to represent a function in algebraic, graphical, and tabular form, and a non-function is graphical and tabular form (although incorrect), but Student A has a great deal of difficulty connecting these representations together and verbalizing the procedure for determining a function for a given representation.

Unlike Student A, Student B immediately starts with a graphical representation of a function, most likely because the student had just used a graph to answer the previous question. From a graphical standpoint he/she can confidently state how to determine what a function is and is not and even states the definition of a function as Student B understands it.

STUDENT B: Oh, um… That is a function (motioning to graph in questions #2). I know because if you graph an equation, um, you can do the vertical line test and so if you draw a bunch of lines and they never… they only intersect the graph once then it’s a function. But if you turn it like that (rotates papers 90° so parabola opens sideways) and this was the x-axis and that was the y-axis and you did (motions vertically on the graph with hands), it wouldn’t work because it crosses twice. So that’s one way you can tell. Um, there are multiple ways you can tell but I can’t remember them. Um, or wait. Oh yeah, that was, wasn’t there, uh… I think if you can’t… every… every x has to have its own y I think. I think so. Um, so like if you make a t-chart you can’t have, um, you couldn’t have, wait, uh, you couldn’t have like two x’s and one of them have like a different, like 1 and 3 (meaning the point (1, 3)). You couldn’t do that.
It is interesting that once again that by stating the definition of a function with inputs and outputs the student switches to a tabular representation. This time though, Student B is able to correctly represent one input with two outputs in a table. Student B offers the following example of a non-function in tabular form.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Even though Student B is able to represent functions and non-functions in graphical and tabular form, he/she does not appear able to represent them as equations. Student B knows that the equation that goes with the graph in question two is an example of a function, but is unable to generate anything else.

TEACHER: What about an equation?
STUDENT B: Hmm (talking indistinctly). I don’t know. I would have to graph it or like find out some points, like put it into a t-chart. Um, I don’t know if there’s a way to find it out just by looking at the equation.
TEACHER: Can you write an equation that’s a function and an equation that’s not a function from your recollection?
STUDENT B: Well, that one is a function (motioning to equation in questions #2).
TEACHER: What about one that is not?
STUDENT B: Um (pause). I don’t think I can.

Student B states that a change of representations is necessary to determine if an equation is a function. As a result, the student is unable to generate an equation that is a non-function. This also implies that Student B is unable to represent a graph or table as an equation; much like Student A was unable.
From Student C’s response it can be concluded that although the student does have a working representation for a function, it is a misconceived one.

TEACHER: Let’s say, give me an example of a function, whatever comes to mind.
STUDENT C: Um, like y equations, y equals.
TEACHER: Go ahead and write it down.
STUDENT C: Y equals, like that would be a function (writes $y = 2x^2 + 3x - 1$).
TEACHER: Ok, do you know how you can tell, or is it just your intuition telling you?
STUDENT C: Um, intuition.
TEACHER: Ok. What about something that’s not a function?
STUDENT C: Like it looks like that but it’s not a function? Um, oh, if they’re, um, like if you graph it, and it’s not, none of the points cross the x-axis. So if they’re all imaginary. That’s one way. Oh, there’s another way (pause). I can’t remember the other one.

The student begins with an algebraic representation but appears to believe that any equation in the form of $y =$ will be a function. To generate a non-function, Student C must change representations, just as Student A did, but in doing so reveals a misconception Student C has about functions. Rather than cite the vertical line test as the two previous students did, Student C’s understanding of a function is a graph that crosses the x-axis. Both graphs he/she draws are parabolas but the non-example graph is a parabola that is strictly above the x-axis. The example of a function is a parabola that passes through the x-axis. Student C also had confusion with what solutions are to a function, so it appears that the definition of a solution may be causing this misconception. This misconception is also a major reason why when asked to generate an equation of a non-function Student C is unable.

Student D chooses to start with a tabular representation in the process perspective during the response. Once again it appears that the use of tabular representations seems
to be tied to the use of inputs and outputs. Using the same accurate definition the student used in the response to the first question, he/she generates two tables, one to illustrate a function and one to illustrate a non-function. The non-function table does include one x-value with two different y-values. It appears that most students chose to represent inputs and outputs of a function in tabular form. As soon as the students began to verbally representing a function as inputs and outputs, they also physically begin to represent a function in a table of values.

STUDENT D: I think of a table. (makes t-table, see figure 3) I think like 1, 2, 3, 4 (x-values in the table) and then it’s like 8, 9, 10, 11 (y-values in the table), and then that’s a function. And not-a-function would be like (makes second table), 1, 2, 3, 4 (x-values in the table), and if there were two outputs for one input, 8, 9, 10, 11, 12 (y-values, but adds a second 1 on the x side to match up with 12). Then that one gives you two answers.

![Figure 3](image)

What is interesting with Student D is that the student understands that “for every x there is only one y” and is able to use this definition regardless of representation. The other students cited the vertical line test for graphical representation and some even connected the vertical line test to this definition. However, Student D never mentions the vertical line test because the understanding of the definition alone for Student D is strong enough,
making the vertical lines test unnecessary. Using this definition he/she is able to generate graphs and equations that are and are not functions.

TEACHER: Ok, you mentioned a graph. How would do it with a graph?
STUDENT D: Um, this would be a function (draws a graph that is approximately y = x in the first quadrant only, see figure 4) because for every x you can see that there is only going to be one y and then like if you had a graph like this (draws a second graph of a parabola opening sideways, see figure 4). This would not be.
TEACHER: Ok
STUDENT D: Because for one you could have like whatever this y is and this y. So you’d have all those points for just one x.
TEACHER: Ok, so you’re saying the graph is not a function because…
STUDENT D: Because for every x, like it goes this way, there’s more than one y output.
TEACHER: Ok, what about an equation. Can you give me an equation that’s a function and an equation that’s not? And how can you tell?
STUDENT D: Um, I forget how to write the equation of this (referring to graph on right in figure 4). It’s… You can write the equation of this and then restrict it to be a function, but I forget how to write the equation. I know, um, like you want to do y = 1/2 + 2. That would be a function.
TEACHER: Ok, what about an equation that’s not a function?
STUDENT D: Um, (pause). Maybe x = y², something like that.

Figure 4

Student D is the first student that is able to give examples of function and non-functions in equation form and appears to have done so simply by using the definition. Just as the student did in the response to the first question, the definition was enough to move from
one representation to another and connect the representations together. The student is clearly viewing the representations from a process perspective because of the linking of inputs and outputs, but there is no evidence to support or contradict the use of the Cartesian Connection during the response.

Student E begins using algebraic representations, much like most of the other students have. The student uses the equation from Question 2 as an example of a function, but much like most of the previous students, is unable to create an equation that is a non-function. However, he/she is able to switch representation and think of a graph that is not a function and then change that representation back to an equation to accomplish the task.

STUDENT E: um, ok, so an example of a function would be like f(x) equals, I don’t know, well, that (pointing to equation in number 2), x² - 2x – 3, right? And then an example of a non-function, er, well this is a parabola, so then an example of a non-function would be like, oh gosh, I can’t think of an equation, but like, like the semi-circle one, radical whatever.

It appears that the student have connected both equations to graphical representation from the object perspective because the student states the first equation is a parabola and the second is a semi-circle. The student shows no evidence for determining if an equation is a function in the algebraic representation. It is only by changing representations to a graph that the student is able to determine if an algebraic representation is a function. However, the student has incorrectly verbally labeled a semi-circle as a non-function. When the student is prompted to draw the picture, it is enough for Student E to correct the mistake and explain why the correction was necessary.

TEACHER: Can you draw a picture for me?
STUDENT E: Oh, wait, no, that’s not one. This one, a full circle.
TEACHER: Ok and how can you tell it’s not a function?
STUDENT E: Um, well, like just doing the equation algebraically you can tell it’s not a function because there is more than output for every input and graphically there’s the vertical line test and if passes through more than one point at any, like, given one.
TEACHER: Ok, so draw me a graph that is a function.
STUDENT E: That is a function? Like that one is a function (graph in question #2), or like that is a function (draws graph of a linear equation). That’s a function (draws graph of a cubic equation).

Much like Student D, Student E understands the input/output definition of a function, but Student E views this definition in the graphical representation as the equivalent of the vertical line test. Unlike Student D, Student E sees the vertical line test as necessary for determining if a graph is a function. It is also interesting that student E never uses a table during the response. The only examples he/she gives of equations are directly tied to the graphs the student uses. This means he/she is using the object perspective. The student is using the vertical line test on the graph and uses the graph as an object, and then connects a graph with its given equation by the features of the graph and equation. It appears this is done from memory because there are not details to support the use of the process perspective and the student is mostly likely connecting a graph with its parent equation.

Student F also starts with the input and output definition to represent functions and much like the other students, uses this definition to represent functions in graphical and tabular form.

STUDENT F: Um, well, a non-function is when the, um, the non-function is when there’s more that one of the same input? And the function is when there’s one input for every output, and, um, you can tell if, there’s like the vertical line test, making a table of values, and that’s all I remember.
The student has correctly defined a function and has stated the procedure for determining if something is a function (the vertical line test) but Student F lacks the conceptual knowledge and the ability to really connect the representations together. When asked to present concrete examples, the student resorts to guessing.

TEACHER: Ok, can you give me some examples?
STUDENT F: Of a function?
TEACHER: Or a non-function
STUDENT F: Like $y = 5x + 4$
TEACHER: Is that a function or not?
STUDENT F: I’m going to say that’s a function.
TEACHER: Because?
STUDENT F: Because I don’t know (laughter). Um, (pause). I don’t know. I made it up.
TEACHER: Ok, what about an equation that’s not a function?
STUDENT F: Um, (pause). $y = x^3 - 2$
TEACHER: And why do you think that’s not a function?
STUDENT F: I just think it is. I don’t feel like it would pass the vertical line test when I picture it in my mind.

The student finally does connect an equation with its graph and then uses the vertical line test as a determination. It is interesting that even though the student mentions inputs and outputs and representing functions as a table of values earlier, the student still uses $y = x^3 - 2$ as an example of a non-function because the student believes the graph of that equation would not pass the vertical line test. If the student has used inputs and outputs or a table of values, then it is possible the student would realize that this equation is a function. However, it appears he/she is much more comfortable representing functions and non-functions in graphical form in the object perspective. When asked to draw some graphs, the student quickly draws an example of a function that passes the vertical line
test and two examples of non-functions that fail pass the vertical line test. The student is unable to represent these functions in any other representation.

**Question 4: What does sin(x) mean? Is it a function? How can you tell?**

**Results**

It was very interesting that when asked about determining if something is a function, most of the student relied on a graphical representation, but when asked about sin(x) being a function only a few students utilized the same representation. Despite the fact that all the students had been introduced to the sin(x) graph, the stronger representation for the students was a right triangle ratio and no student was able to use that representation to determine if sin(x) is a function without changing to another representations first. It is also worthy to note that no student used the Cartesian Connection at any point during the response to the question. The students that did use a graph to represent sin(x) never attempted to explain how it connects with the equation y = sin(x) so there is no evidence to support the idea that students understand the graph of sin(x) is made up of points that are solutions to the equation y = sin(x). The students that used the graph did so from the object perspective and only stated that the graph passed the vertical line test or that the graph made a wave.

**Evidence**

Right away Student A uses the SOH CAH TOA approach to represent sin(x) as the opposite side over the hypotenuse. However, this does not seem compatible with how
the student represents functions because according to Student A, functions have to have points therefore $\sin(x)$ is not a function.

STUDENT A: Not a function because you are not given any points. But doesn’t it mean (pause) hypotenuse over (pause) opposite? Wait, (pause). Wait, yeah, isn’t it hypotenuse over opposite side, right? Oh my gosh! No! Yeah! Wait (pause), um, it’s opposite over hypotenuse.

The student is recalling the ratio from memory and does not have any procedural or conceptual knowledge to help understand why it is opposite over hypotenuse. As a result, the student has some difficulty recalling the correct ratio. Because he/she is relying on what appears to be a memorized fact, it is unclear if the student’s representation is from the process or object perspective. When asked to explain further, it becomes apparent that he/she is viewing the right triangle representation as a process that links sine with the length of the opposite divided by the length of the hypotenuse of a triangle and eventually a process that links $\sin(x)$ with the $y$-coordinate of the unit circle

STUDENT A: (drawing a triangle in a coordinate plane) OK, so that’s just a graph and that’s an example of the triangle if you were trying to find a point. That’s the point we’re given and, uh, opposite over adjacent. So it’s a circle on a graph.

TEACHER: Ok

STUDENT A: Um, if that’s a circle, wait (pause), hold on, I’m trying to, uh, (pause), I have to wait for my mind to catch up. I know what I’m trying to say. Ok, so just imagine it as a circle like you gave us and it had all the points, like that was $0^\circ$, that was $90^\circ$, I’m just trying to explain sine. Um, ok, so this is how I see it. You have a point here and you have a triangle right here and so imagine that this is really touching this (referring to triangle inside the circle student drew). Um, so to describe the triangle when we go over (horizontally), this is cos. Ok, so we have the points. The points are always cos and then the sine for $x$ and $y$. That’d be cos and that’d be sine and in that triangle I said it would be corresponding to that point would be opposite over adjacent.

TEACHER: Opposite over adjacent?

STUDENT A: No, wait. Opposite over hypotenuse and then cosine is adjacent over hypotenuse or $x$ over $r$, and $r$ is the hypotenuse.
The student has created a strong geometrical representation for \( \sin(x) \). Student A has demonstrated some strong conceptual knowledge because the student has taken a simple ratio from the sides of a right triangle and used it to justify why a coordinate on the unit circle will yield \((\cos(x), \sin(x))\). The student is clearly viewing \( \sin(x) \) from the process perspective. Now that the representation includes coordinates, the teacher asks the student again if \( \sin(x) \) is a function.

TEACHER: So you said sine of \( x \) is not a function.
STUDENT A: Hmm (pause). Wait (pause). Yes. No. Yes. We’ll, you’re not given any numbers.
TEACHER: \( x \) is your number.
STUDENT A: So variables, they could be anything. It’s a function because you can replace \( x \) with any number because it’s a variable and \( x \) is only there to fill in the space.

Although the student now believes \( \sin(x) \) is a function, the support of this position is not very strong. He/she does not offer any evidence as viewing \( \sin(x) \) as an equation that links \( x \) and \( y \) together or even an input with an output so it is unclear why the student believes it is a function. From this answer it appears that the student has not connected this representation to any representation the student used for functions in the previous question. It is interesting that the student does not consider the graph of \( \sin(x) \). Student A used graphs to represent functions and non-functions in the previous response but makes no attempt to do the same thing with \( \sin(x) \). It appears the student’s representation for \( \sin(x) \) and \( \cos(x) \) are not compatible with the representations he/she has for functions. This could be an effect of using a geometric representation to understand \( \sin(x) \) instead of an algebraic representation.
In contrast to Student A, Student B is able to see the input and output for \( \sin(x) \) and viewing it from the process perspective, determines that \( y = \sin(x) \) must be a function.

STUDENT B: … Um, yes, yeah it would be a function because an \( x \), a number for \( x \), you get a \( y \) and you can graph it. What does it mean? It’s a (pause), something that is used for finding sides of triangles. Um, opposite over hypotenuse. Um, and I can tell it’s a function because if you put a number in for \( x \) you get a \( y \) out of the equation. As far as I know, that makes it a function. I don’t know what the graph looks like.

Student B does see \( \sin(x) \) as an equation and realizes that the equation can be represented as a graph. However, the student gives no indication that the graph passes the vertical line test, which is the student’s basis for determining if something is a function. It appears the simple fact that it can be represented in graphical form is now enough to convince Student B that \( \sin(x) \) is a function. Even though the student is able to represent \( \sin(x) \) as an equation with inputs and outputs and knows it can be graphed, Student B uses the ratios of a right triangle, just as Student A did, to give meaning to \( \sin(x) \). Student B is able to state a purpose for \( \sin(x) \) so it can be concluded that Student B does have some conceptual knowledge.

Student C is the first to use a graph to represent \( \sin(x) \) and what is more interesting is that Student C is viewing that graph from the object perspective. The student had a misconception about what makes a graph a function and the student applies this same misconception to the graph for \( \sin(x) \) and concludes that it must be a function because the graph crosses the \( x \)-axis.

TEACHER: Ok, does anything come to mind when you hear sine of \( x \)? What are you picturing when I say sine of \( x \)?
STUDENT C: The wave. Middle, up, middle, down, middle.
TEACHER: So is that a function?
STUDENT C: Yes, it is because it crosses the x-axis.

Student C is focused on the shape of the wave, not the actually points that make up the graph, therefore the student views the graph as an object. It is also interesting that he/she never refers to the ratio from the right triangle as the previous students did. It appears that the student is unable to use sin(x) in the process perspective because when asked if there are other ways sin(x) can be represented, Student C gives two alternative representations in the object perspective. Student C mentions the reciprocal of sin(x) and the y-value on the unit circle. Because these two responses are treated as objects with no linkage to a process, they appear to be memorized facts and of little help for establishing connections.

TEACHER: Anything else come to mind when you say sine of x? Anything special?
STUDENT C: Um…
TEACHER: Doesn’t have to be a graph, whatever you think of.
STUDENT C: Opposite, um, I don’t know what it’s called. I want to say cosecant. Is that correct?
TEACHER: Cosecant is the reciprocal of sine.
STUDENT C: Yes! Got one! And the unit circle.
TEACHER: What about the unit circle?
STUDENT C: Um, the sine is the y part of the points.

Unlike Student B, Student C’s representations do not indicate there is any understanding of a purpose or use for sin(x), only simple facts or definitions. Even though the student does have some representations for sin(x) and the student does understand it is a function, the representations do not appear to be very detailed or contain a great deal of conceptual knowledge.
During the previous question, Student D was able to use the definition of a function and represent that idea as an equation, graph, and table. It is very intriguing then that even though he/she is able to create representations for sin(x), the student is unable to conclude if those representations are also functions or even that they may not be functions. It is also interesting that the student does not try to use the same algebraic, graphical, and tabular representations that were utilized in the previous question.

TEACHER: What does sine of x mean? Is it a function? How can you tell?
STUDENT D: Um, sine of x to me means opposite over hypotenuse.
TEACHER: Ok
STUDENT D: Of x though (adds (x) after the word sine in the equation), and I do not know if that’s a function.
TEACHER: Ok, is there any other way you can represent sine of x besides opposite over hypotenuse and what do you mean by opposite over hypotenuse too?
STUDENT D: I mean if you have a triangle and this was x, 60°, then, well, that wouldn’t work. Then you would know that, well, yeah, x is supposed to be an angle? Yes? Then I would do opposite over, opposite one, hypotenuse.
TEACHER: Ok, very good. So the previous question, you looked at a table, you looked at an equation. You looked at a graph to see if it's a function. Can you do the same thing with sine of x?
STUDENT D: Possibly (pause). Um, (pause), um, no. I can't recall.

Student D is using the same geometric representation that Student A and Student B used and views the representation from the process perspective. The student also is the first to realize that the input into sin(x) must be an angle so there is strong conceptual knowledge to go with the procedural knowledge for generating the opposite over hypotenuse ratio. However, he/she never uses the word input with sin(x). The lack of the ability to see the input and output of sin(x) is probably why the student is unable to tie sin(x) back the definition of a function he/she gave and used in the previous two questions. In addition, even when directly asked to represent sin(x) in the same way the student represented
functions previously, he/she is unable to do so. Apparently Student D is unable to connect trigonometry to the algebra the student already knows and understands.

In the previous question, Student E used the graphical representation as the major way to represent functions and built an understanding by changing other representations to a graph and then using the vertical line test. It is no surprise then that Student E also uses a graph to initially represent $\sin(x)$. With this representation, the student can easily state that $\sin(x)$ is a function. It appears that he/she is using the object perspective when creating the graph because the student is focused on the overall pattern of the graph and not the points of which the graph consists.

TEACHER: So, when you hear sine of x. what does sine of x mean? Is it a function? How can you tell?
STUDENT E: Um, uh, well, I mean it’s a function, right? Like this thing (draws wavy graph), or well not that thing. This thing (correctly sketches sine wave).
TEACHER: Ok
STUDENT E: Um, so yeah, it’s a function and you can tell because it passes the vertical line test at every point, right? (laughter), and…
TEACHER: So, when you hear sine of x, you think of a graph?
STUDENT E: Yeah
TEACHER: Anything else?
STUDENT E: Um, I think SOH CAH TOA
TEACHER: What about SOH CAH TOA? Explain SOH CAH TOA.
STUDENT E: Um, like the right triangle thing. So like sine would be like the length of the opposite side over the hypotenuse, right? Of like any given angle, like this one (draws picture of a right triangle and labels one angle). And then same thing for cosine and this angle.

In addition to the graphical representation, Student E is also able to represent $\sin(x)$ using the ratio from the right triangle just as most of the previous students have done. The student has now changed to the process perspective to create the ratio and links the angle with the ratio. Like Student D, Student E also understands the input should be an angle.
However, Student E does not attempt to make any connections between the two representations created for sin(x). He/she never mentions how the graph and the ratio are connected so it is unclear if the student understands the two representations are connected.

Student F also initially represents sin(x) as a graph and uses that to determine that sin(x) is a function because it passes the vertical line test. However, just like the other students’ graph use, Student F is viewing the graph only from the object perspective because Student F mentions the waves of the graph. Because of this, it is not possible to tell how strong the conceptual understanding of sin(x) is with respect to the graphical representation. The student never mentions how the points on the graph relate to the equation y = sin(x) so there is also no evidence to see if he/she understands the Cartesian Connection.

TEACHER: Now we’re into the trig questions, so what does sine of x mean? Is it a function? How can you tell?
STUDENT F: Um, sine of x I think is a function. I think it does pass the vertical line test even with all the sine waves and, (pause).

When the student is asked to represent the function in a second way, the student gives some more insight into the understanding of sin(x). The student uses a right triangle and the opposite over hypotenuse ratio to represent sin(x) and shows the conceptual understanding that it can be used to find missing sides and angles in a right triangle. He/she is also using the process perspective for this representation by attempting to link the angle in a triangle with a ratio of the sides.

TEACHER: Ok, very good. So when you hear sine of x, what do you think of?
STUDENT F: Um, sin(x)? Triangles.
TEACHER: OK, can you explain it a little bit more?
STUDENT F: (laughs) Like finding, you know, missing sides of triangles and angles and all that fun stuff.

However, when asked to explain exactly how to find a missing side or angle, the student may understand how the problems are set up, but does not realize that the explanation is missing an unknown value to find.

TEACHER: Ok, so what did you do?
STUDENT F: Um, well, I drew a right triangle and then I wrote down some random numbers and then I put a random angle in there and since sine is opposite over hypotenuse and, the SOH CAH TOA would be 50/12.

The student has the procedural knowledge but lacks the conceptual knowledge of seeing the big picture. The ratio he/she has given is also greater than one. The student has done enough drill and practice with this representation to be able to explain what is involved in the set-up of a task of this kind, but does not quite have the understanding that the triangle has to have an unknown angle or side. Even when asked further about this the student does not realize the problem with the representation he/she has just created.

Finally there is also evidence that Student F represents sin(x) as a point on the unit circle.

TEACHER: Ok, excellent. So anything else come to mind when you hear sine of x?
STUDENT F: The circle stuff
TEACHER: What do you mean by circle stuff?
STUDENT F: Like, the unit circle, you know, sine of… I’ve forgotten all of this (laughter).

It is not possible to determine what perspective the student is using or how the student connects the unit circle to the other representation. It is most likely being recalled as
something that is synonymous with $\sin(x)$ that he/she used throughout the unit on trigonometry.

**Questions 5: How could you find or estimate the $\sin(30^\circ)$? How could you find or estimate the $\sin(20^\circ)$?**

**Question 6: Can you think of other ways to do #5?**

**Results**

Overall students seemed most comfortable representing $\sin(30^\circ)$ as a coordinate on the unit circle. However, the students had varying levels of understanding in terms of the conceptual and procedural knowledge for generating the coordinate in that representation and most students relied on memory. Almost all relied on the calculator for verification and few were able to use the unit circle representation to find points that were not already memorized. It was interesting once again to see that no students attempted to use a graph of $y = \sin(x)$ for either task even though they had access to a graphing calculator. It is quite possible that this could be the due to the context of the question. Students did not use a graph to solve questions of this nature during class very often. It is also possible that since the answer to $\sin(20^\circ)$ is irrational, students are not comfortable using coordinates from a graph to answer the question. Most of the students avoided using graphical estimates for a quadratic function, so it may be that students did not choose this representation to avoid giving an approximate answer. It should be noted that the students never mentioned graphing as a means to get to an answer, which was consistent with the research (Knuth, 2000a).
Evidence

To find \( \sin(30^\circ) \) Student A initially recalls the unit circle representation. The student is using the unit circle from the process perspective because the student connects the actual coordinates on the unit circle with the pattern \((\cos(x), \sin(x))\). It is not immediately apparent if the student has connected the unit circle to any other representation to generate the coordinates. Instead the student is relying on memory for the coordinates of 30°.

TEACHER: Ok, next question, how could you find or estimate the sine of 30°?
STUDENT A: Um, sine of 30°? Um, so this has to be 60, 30 (draws first quadrant of unit circle). I mean I know in my head but try and find it.
TEACHER: What is it?
STUDENT A: \( \sqrt{3}/2 \)
TEACHER: Ok, convince me it’s \( \sqrt{3}/2 \) or at least show me what you’ve done.
STUDENT A: See, that how good I have the unit circle down, Mr. Marchi. I remembered it. Oh, well, that’s a triangle, so (draws triangle the unit circle for 30°). How do I explain that? I don’t know how to explain how I remember.
TEACHER: In your drawing where’s sine?
STUDENT A: Sine is (pause). Wait, wait (pause). You asked for sine, right?
TEACHER: Yes
STUDENT A: Oh, it’s 1/2
TEACHER: 1/2?
STUDENT A: Yes
TEACHER: Ok. Because? What changed your mind?
STUDENT A: Because I just remembered that it’s 1/2

When asked to explain how the student arrived at the answer, the student shows some understanding that the unit circle representation is connected to a right triangle representation because the student draws a right triangle with 30° at the origin and labels the length of the adjacent side along the x-axis as \( \sqrt{3}/2 \) and the length of the side opposite 30° as 1/2. Even though the student is unable to explain why the triangle is labeled this way, the student does now realize the answer to \( \sin(30^\circ) \) should be 1/2. It appears he/she
is switching to this combination of representations to realize that the correct coordinates for 30° should have 1/2 as the y-coordinate. From the drawing, the student sees that the y-coordinate on the unit circle connects with the vertical distance in the triangle, which he/she labels as 1/2. It is less likely that the student uses the process of making a ratio of sides in the triangle to connect the y-coordinate of 30° with 1/2 because there is no evidence of Student A using the hypotenuse of the triangle. When the student is asked to use the picture to prove the sin(30°) is 1/2, he/she is unable to verbalize an answer. The student did have an understanding to create the drawing, but lacks the ability to communicate that understanding.

When asked to find sin(20°) Student A stays with the unit circle representation and cites that it would be possible using a sum or difference identity.

STUDENT A: Um, you could take (pause), knowing what the coordinates or something are like the coordinates for 30° is (√3/2, 1/2) so…
TEACHER: Ok, so write that down.
STUDENT A: Oh, and then I don’t know how to explain it but I know that forty-five degrees is (√2/2, √2/2) and that 60° is (1/2, √3/2) so and then 90° is (0, 1/2). What you could do is find two points on the circle, er, ok, how I was going to say I was going to find points on the graph that you know what the coordinates are and then you just subtract it and there’s a certain formula you have to subtract it and then you find the sine of 20°.

The student has recalled a procedure (difference identity) for using the unit circle representation to find additional points where the coordinates on the unit circle are not easily known by the student. However, when learning this identity during class, the student was never faced with a situation where the angle measure could not be easily found from the sum or difference of known angles, so he/she does not realize the limitations of using the identities in this manner.
Since the student did not actually generate an answer, Student A is then asked if .7 would be a reasonable answer. The student is able to use the combination of the unit circle and right triangle representation to conclude it should be less than .5 because 20° is smaller than 30°.

TEACHER: I just want a quick estimate on what the sine of 20° should be. Based on what you have right there, can you tell me a way to estimate the sine of 20°?
(Long pause). So, I say the sine of 20° is .7.
STUDENT A: Are you asking me if that’s right?
TEACHER: Yes
STUDENT A: Um, hmm…
TEACHER: Or 7/10 or I say the sine of 20° is 7/10. Do you agree or disagree?
STUDENT A: Um, ok, I’m going to answer your question, I’m just trying to figure it out (talking indistinctly) 1/4, 1/8, wait, what did you say it was? .7? 20°? Um…
TEACHER: I just made up the value too.
STUDENT A: Wait, what’s the .7?
TEACHER: Sine of 20°. I’m just making up a value. The sine of 20° is 7/10. Yes or no? Does that sound reasonable or is it way off?
STUDENT A: Oh, if that’s .5, it sounds reasonable because the sine of 30° is .5 so obviously 20° would be higher.
TEACHER: Why would it be higher?
STUDENT A: Because (pause), wait, wait, I take back my answer. Ok, it’s not reasonable so the sine of 20° would be smaller because 30° is .5 so then 20° is below it so the sine would have to be below .5.

The student once again is using both the unit circle representation and the right triangle representation together. He/she looks at the same drawing that was used to figure out sin(30°) and reasons that since the 20° is smaller, the side opposite the angle would also be smaller. The student emphasized this by purposefully tracing over the opposite side in the triangle during the explanation to show the connection the angle and that side has.

It is interesting to note that the student never used the calculator during the explanation. It was only after being directly prompted by the teacher that he/she used the
calculator to check both of the answers. The representation the student has was strong enough to convince the student of the accuracy of the answers and the student felt some vindication when the calculator reinforced the answers he/she gave. The only other way the student had to try and evaluate the two angle measure was to switch the angle measure to radians but he/she stopped short of creating any representation to go with the radian measures and as a result was unable to do anything with this change from degrees to radians.

Much like Student A, Student B utilizes the unit circle representation to evaluate sin(30°). Unlike Student A though, Student B’s understanding of the unit circle is simply memorized information with no conceptual knowledge to tie it all together.

TEACHER: What’s the easiest way to find the sine of 30° that you know?
STUDENT B: Um, you have to use the unit circle, right? No, yes. Yeah, yeah, you would, so… Um (draws triangle), and then (1/2, 2/3) maybe, I think.
TEACHER: Ok
STUDENT B: I know it’s one of those.
TEACHER: Do you know which one?
STUDENT B: Um, the first one, maybe?

The fact that the student drew a right triangle shows that he/she understands there is a connection between the unit circle and a right triangle, but it is not a strong enough connection to allow the student to accurately recall any of the relevant information. Although he/she does understand that sin(x) can be represented with the unit circle, the student lacks the procedural and conceptual knowledge to use that representation for any kind of problem solving. Student B is unsure is sin(x) is the x-value or y-value and the student does not know the correct coordinates for 30°. While Student A’s answer was originally incorrect, the student was able to use conceptual knowledge and the connection
between a right triangle and the unit circle to arrive at the correct answer. Student B does not possess the same level on understanding with those representations and resorts to simply guessing for an answer.

When asked if the problem can be solved another way, the student cannot think of one. Only after the teacher suggest using the calculator to plug in values is he/she able to solve the problem another way. Although this is changing representations to an algebraic form, there again is no conceptual knowledge to go with this procedure or any evidence that the student have a connection between the answer from the unit circle and the answer the calculator provides. It is also interesting to note that when he/she gets a different answer than the unit circle guess, there is no concern on the part of the student until the teacher points it out.

TEACHER: Ok, you can use the unit circle to find the sine of 30°. Is there any other way you can find that ratio?
STUDENT B: Um you could just have it memorized (pause). I don’t know. I give up.
TEACHER: Can you find it with a calculator?
STUDENT B: Hmm (typing on calculator). Oh yeah, you can. (calculator gives answer of -.9880316241)
TEACHER: What did you type in?
STUDENT B: Sine of 30°
TEACHER: Does that answer make sense to you? You told me here it’s either 1/2 or 2/3. Is that (calculator answer) 1/2 or 2/3?
STUDENT B: (laughing) No, um…
TEACHER: Check your mode
STUDENT B: There you go. .5 after I changed it to degree mode. So it was 1/2.

It appears that even though Student B is able to represent sin(30°) and give a correct answer, the understanding of it is very limited and a great deal of it depends on luck.
When Student B is asked to find \( \sin(20^\circ) \) the only way he/she has to find it is using the calculator. The student is unable to make sense of it on the unit circle or in a triangle representation. When asked to compare it with the answer for \( \sin(30^\circ) \) the student does realize that \( \sin(20^\circ) \) should be smaller than \( \sin(30^\circ) \) but offers no representation or evidence of conceptual knowledge as to why this should be. The student appears to simply base this on the idea that since 20 is less than 30 the answer for 20 should also be less than the answer for 30. However, this will not always be the case in trigonometry, so the student is using conceptual knowledge from algebra.

Student C used the graph as an object to represent \( \sin(x) \) but when asked to find \( \sin(30^\circ) \) the student does not attempt to use the graph. This is further evidence that the student is only able to use the graph from the object perspective and unable to utilize the Cartesian Connection. To find an answer, Student C instead represents \( \sin(30^\circ) \) as a point on the unit circle and knows the answer is 1/2 because it is the y-coordinate on the unit circle.

**TEACHER:** Alright. Ok, so the sine of 30°. Can you find that value any way you want? The sine of 30°.
**STUDENT C:** Um, it’s 1/2.
**TEACHER:** OK, how do you know it’s 1/2?
**STUDENT C:** Because it’s on the unit circle.
**TEACHER:** Ok, so show me on the page there or anywhere how you got that from the unit circle.
**STUDENT C:** So it’s 60°, 45°, 30°. 30° is \( \sqrt{3}/2 \) and 1/2. 45° is \( (\sqrt{2}/2, \sqrt{2}/2) \), and 60° is \( (1/2, \sqrt{3}/2) \).

The student is using the unit circle representation from the process perspective but it does not appear that the representation is connected to a triangle, but rather it appears that Student C is referencing a set of memorized coordinates. The student never references a
triangle at any point during the response or when drawing the unit circle on paper. This also may be the reason why the only other way the student is able to find sin(30°) is by plugging it into the calculator.

When given sin(20°) the student is able to realize that the unit circle representation is not going to work because 20° is not linked with a coordinate that he/she knows. Since the student lacks the conceptual knowledge that goes with generating the unit circle, Student C is not able to use it for an estimate as Student A did, even when the teacher prompts the student to try. He/she, however, is able to correctly mark where 20° is on the unit circle but it does not help to form an estimate.

TEACHER: Ok, let’s go to the unit circle for a second. You told me that the sine of 30° is on the unit circle. Where is the sine of 20° at?
STUDENT C: Right… there (marking drawing).
TEACHER: And what do you think the coordinates there are?
STUDENT C: Um, I have no clue. We talked about this once. I just can’t remember.

Even though the student is utilizing a process perspective, he/she is unable to use the process to find or estimate a value that is not already memorized. The only way the student is able to produce an answer is by plugging the expression into the calculator.

Student D was unable to see sin(x) as a function most likely because the lack of an ability to see inputs and outputs to tie it to the definition Student D had for functions. When Student D is presented with sin(30°) the student immediately uses the unit circle representation and is able to generate an answer. However there appears to some confusion between representing the unit circle with radians and degrees.
STUDENT D: It would be \( \pi/4 \), wait, no that would be 1/2 because of the unit circle. It’s in here, 30°, 45, 60, and then I believe it’s \((1/2, \sqrt{3}/2)\). Oh wait, and sine equals y so then it would be \( \sqrt{3}/2 \).

Much like Student C, Student D is recalling the representation from memory but has incorrectly stated the coordinates for 30°. This seems to show that the student simply has the coordinates memorized with no conceptual knowledge tied to it for understanding.

When asked to represent it another way, the student chooses to use the calculator. Unlike Student B, this time when the calculator yields a different answer, the student does realize there is a mistake somewhere and concludes that the memorized point is actually backwards. Unlike Student B, Student D realizes that the calculator and the unit circle should give the same answer.

TEACHER: Alright, the calculator says it’s \( \sqrt{3}/2 \), or sorry, the calculator says it’s 1/2 and the unit circle says is \( \sqrt{3}/2 \). So which one do you think is correct?
STUDENT D: Um, I think they’re… (pause)
TEACHER: Well, can they both…
STUDENT D: Oh, is, maybe I got this mixed up.
TEACHER: Ok
STUDENT D: Maybe this is \((1/2, \sqrt{3}/2)\)

The student is able to reconcile the different answers by switching the coordinates and appears convinced that the answer is 1/2. Finally, unlike the previous students, Student D is able to represent \( \sin(30^\circ) \) as a triangle also. The student creates a right triangle where the measure of one of the angles is 30° and the side opposite that angle is 5 and the hypotenuse is 10. The student does not explain why the use of 5 and 10 are warranted. It appears that is mostly likely because the student understands the ratio of the opposite side to hypotenuse must be 1/2 and not because for any 30°-60°-90° right triangle the ratio of
the sides is set. Nevertheless, the student was able to use a representation the other students were unable to utilize during their responses.

When presented with $\sin(20^\circ)$ Student D immediately goes back to the unit circle representation from the process perspective and is able to estimate $\sin(20^\circ)$ to approximately equal .3. It is clear that Student D is able to use the unit circle representation for more than just recalling coordinates of the most common angles measures. Student D is able to connect a coordinate on the unit circle to the solution to $\cos(x)$ and $\sin(x)$.

TEACHER: Can you find the sine of 20° or estimate that?
STUDENT D: Um, you could estimate it being, um, sine of 20° would be less than, um, the sine of 30°, which would be 1/2 and greater than 0, er, greater than 0.
TEACHER: Why greater than 0?
STUDENT D: Because I’m thinking in terms of the unit circle. It’s greater than 0°, er, will be greater than 0 and then less than that because it’s between those two.
TEACHER: Alright. Would it be closer to the sine of 30° or the sine of 0?
STUDENT D: The sine of 30°
TEACHER: Ok. So can you give me a decimal approximation?
STUDENT D: Um, .3?

Student D was one of the few students comfortable using a graph to estimate a solution earlier. The ability to produce as estimate in the coordinate plane may also be the reason the student is able to use the unit circle representation to produce an estimate for $\sin(20^\circ)$ while other students were forced to change representations. Student D also attempts to change to the triangle representation that the student used with $\sin(30^\circ)$ but realizes that it will not be possible to determine the lengths of the sides.

TEACHER: Ok, is there any other way, and that’s a great way, you could estimate the sine of 20°? You used the unit circle, what else could you do?
STUDENT D: You could use a right triangle. 20°, well, you’d have to have your, your sides and do something like that.

It is interesting that the student does not try to label the sides with values that create a ratio of .3 as he/she did when using the representation for \(\sin(30^\circ)\). It could be that the student does not realize that .3 is 3/10 or he/she realizes that since .3 is only an estimate, Student D does not know how to create an estimate in this representation.

It appears that most of the students readily understand that \(\sin(30^\circ)\) can easily be answered by recalling a coordinate on the unit circle and that is also the representation that Student E chooses first. Like the other students, Student E is not totally convinced with an answer that is drawn from memory. To check, the student changes representations to evaluate the expression in the calculator in the algebraic representation.

TEACHER: Alright, how can you find or estimate the sine of 30°?
STUDENT E: Wait, sine of 30°, so like the unit circle, right? Um 30° and then it’s \((\sqrt{3}/2, 1/2)\), right? Or is that, that’s not what we’re talking about. We’re talking about radians, no, no we’re not. Can I use my calculator? (types in calculator) Yeah, that’s the answer I was talking about. So like 1/2.
TEACHER: Ok, and where did the \((\sqrt{3}/2, 1/2)\) come from?
STUDENT E: The unit circle

It is not possible to determine what conceptual knowledge the student has with the unit circle representation from this response, only that the calculator verification is enough to convince Student E that this representation was valid. It interesting to note that the student did not mention the unit circle representation with \(\sin(x)\) in the previous question but used a graph and SOH CAH TOA to describe \(\sin(x)\) as a function. When a concrete value is given for \(x\), the student uses an entirely different representation, most likely
because of the procedural knowledge the student has connected with a the task of evaluating \( \sin(30°) \).

The student also believes \( \sin(30°) \) can be represented with a 30°-60°-90° right triangle, but is unable to produce a working representation. The fact that the student’s intuition suggested it is not surprising given that he/she used right triangles and SOH CAH TOA to represent the \( \sin(x) \) in the previous question.

TEACHER: Ok, can you think of any other way to do the sine of 30°? Show me the sine of 30° is 1/2.
STUDENT E: Um…
TEACHER: Besides what you already showed me with the calculator and unit circle. Those are two great ways. Can you give me a third one?
STUDENT E: Um, I feel you can do it with like 30-60-90 triangle thing. I was never very good at that.

Student E, like the other students so far, is relying on memorized procedures in an attempt to create other representations for the trigonometric functions. However, each student appears to lack enough conceptual knowledge to create a working representation or connect it to other known representations. This is most likely a result of having less familiarity with trigonometry than they do with other functions. When presented with the task of evaluating \( \sin(20°) \) he/she is unable to find a way to represent it other than evaluating it on the calculator. The student’s knowledge of the unit circle representation only includes the coordinates of well known angles and there is not a connection to triangles for an angle like 20°. However, when prompted to use the unit circle to support the answer he/she got on the calculator, the student does produce an argument that shows some conceptual knowledge of the unit circle representation.

TEACHER: Ok, so on the unit circle, which you said the sine of 30° is 1/2…
STUDENT E: Yes
TEACHER: How would the sine of 20° compare to that?
STUDENT E: Well, I mean, it has to be less than that because 30°, right? And it has to be less than that, so it’s like down here, right?
TEACHER: So, what does that mean? What’s less than that besides the angle measure?
STUDENT E: Um, like the length here, right?
TEACHER: Ok
STUDENT E: Is that what you’re talking about?
TEACHER: Sure. Is that what you’re talking about?
STUDENT E: (laughter) (pause)
TEACHER: Ok
STUDENT E: (laughter) Well, I mean like this angle, right?
TEACHER: Alright, that angle is 20°.
STUDENT E: Yes, this angle is 20° and then like the little arc thingy is less than that, right?

For the argument, Student E is connecting the right triangle representation to the unit circle, much like Student A did, to argue that the length representing $\sin(20°)$ is going to be less than the length representing $\sin(30°)$. However, the student is using the length of the arc on the unit circle to represent $\sin(x)$ not the length on the opposite side in the right triangle. As a result, the student is using the representation to fit a given answer the student wants instead of creating a representation that truly generates the answer to the task at hand. This means the representation is based on convenience and not conceptual understanding.

Just as all the other students have done, Student F represents $\sin(30°)$ as a coordinate on the unit circle and is not totally confident that the answer the student is recalling from memory is accurate. Like most of the other students, Student F’s only other way to evaluate $\sin(30°)$ is by using the calculator to change to an algebraic
representation. When the calculator gives the same output as the unit circle, the student is convinced the answer is right.

STUDENT F: Um, well, if you know the unit circle you could just remember that 30° is, the sine is the y-coordinate so that would be, that would be 1/2.
TEACHER: Ok, is that a guess or do you know for sure it’s 1/2?
STUDENT F: It’s a 95% I know for sure and 5% because it could also be √3/2.
TEACHER: Ok, so how could you figure that out right now if you had to?
STUDENT F: Um, (pause). I do not know.
TEACHER: Ok, do you know of a way?
STUDENT F: My friend the calculator.

The student’s representation of the unit circle from the process perspective includes a connection of the y-coordinate with sin(x) so the student does have the correct procedural knowledge, but there is no evidence to suggest there is conceptual knowledge of where the coordinates come from. Like most of the students, Student F simply has the major coordinates memorized and has no other representation to connect it. However, the student does mention that if cos(30°) was known then it would be possible to find sin(30°). The argument the student makes show that the student has changed to a right triangle ratio representation but that it is not connected to the unit circle.

STUDENT F: Um, hold on. I’m just going off of if you didn’t happen to know the unit circle. If you’d have to know like where, what quadrant it would be in. Um, (pause). Maybe if you knew the cosine or something.
TEACHER: Ok, if you knew the cosine, how would that help you?
STUDENT F: Well, because if you knew the cosine you might also know the hypotenuse and then you could like cross multiply or something like that.
TEACHER: Ok. So you’re thinking of using a triangle and using ratios?
STUDENT F: Yes

This right triangle representation from the process perspective shows that the student understands that cosine is also a ratio for a right triangle that would give him/her two
sides of the triangle and that there is a way then to find the third side. However, from the response the student gave, there is not enough detail to determine if he/she is convinced that there is enough information to make the sine ratio for 30° from the cosine ratio. The student suggests using only ratios and never mentions the Pythagorean Theorem or even an identity to find the value for sin(30°).

When presented with sin(20°) the student makes the same arguments that previous students have made. Student F is unable to represent it any other way than as a value the calculator outputs. When asked to justify that value with a unit circle representation, the student states that since 20° is less than 30°, sin(20°) should be less than sin(30°). As mentioned before, it is not possible to tell exactly what level of understanding the student has for the outputs of sin(x). Student F could draw the same conclusion simply by looking at the inputs alone.

Question 7: Explain how you would solve the following equations: sin(x) = 1/3 and 2sin(x) + 1 = 2/3.

Questions 8: Can you think of any other way to solve the equations in #7?

Results

Most of the students approached solving the equations from an algebraic standpoint, which is consistent with what the research found (Knuth, 2000a; Herman, 2007). Students also had difficulty linking the representations together, which is another point the research made (Even, 1998). However, the students were able to produce an answer, even if they lacked the conceptual understanding to what the answer actually
meant. The only student that was able to fully conceptualize the answer was Student E, but he/she was only able to realize that the answer was an infinite number of angles after being presented with a graphical representation later in the interview. Due to the context of the question suggesting an algebraic representation, no student utilized a graphical representation during the response, so this may have been a major reason students failed to fully conceptualize the answer.

_Evidence_

Student A begins using the unit circle representation from the process perspective just as the student did in the previous question. However, the student realizes that 1/3 is not a value among those he/she has memorized and decides to change representations. This adds further evidence that the unit circle representation for this student is only made up of memorized angles and coordinates and does not include the conceptual knowledge to find additional angles or coordinates. The student changes to an algebraic representation and uses the calculator to find answers to both equations.

STUDENT A: Oh, this was one my placement test. I don’t think I did it right. Well, for this, wouldn’t you do the inverse of 1/3? But that’s not on the unit circle.
TEACHER: So how would you do it?
STUDENT A: You’d have to type it in the calculator. Um, so you do, mine is second and then you do the sine and then I type in 1/3 and that equals 19.741. And then for this one you have to subtract 1 so you have 2sin(x) =, hold on I need to use the calculator, -1/3 and then you divide by 2 or times it by the reciprocal which gives you 1/6. Hold on, I’m going to check that. And then you do the inverse of that. It equals -9.594.

The student has strong procedural knowledge from the algebraic representation and can use it to generate an answer. However, when he/she is asked to explain what the answer
means, it becomes apparent that the student lacks the conceptual knowledge to add
meaning to the answer. Student A does not realize that the answer is an angle measure or
that because it is an angle measure, there are more than one angle measures that will
work in the equation.

TEACHER: So, what is x? Can you tell me what x is? You said x is 19.741. What does that mean?
STUDENT A: X is the y coordinate on the graph

The student does have some understanding of what the answers are. The student has
connected the answers with the y-coordinate in the unit circle representation but does not
realize that it should be the angle measure not the y-coordinate. He/she understands in
the algebraic representation to use the inverse of sine to find x, but does not connect the
inverse to mean an angle instead of a coordinate on the unit circle. The student is also
not able to represent the problems in any other representations when asked to do so,
including using a graph.

Student B also approaches the problem in the unit circle representation. However,
Student B incorrectly uses 1/3 as a coordinate for one of the memorized angles.
Although the student understands the correct procedure that it is necessary is finding an
x-value or y-value on the unit circle of 1/3 and giving the angle measure as an answer,
he/she is unable to give an accurate answer.

STUDENT B: Alright… Um, oh, um the x would be a degree and the 1/3 is one
of the x or y coordinates of one of the angles on the unit circle and I think it’s like
60° maybe.
The student is unable to connect the representation to anything else or approach the problem for an algebraic representation and use the calculator to solve for the angle so it is not possible for the student to check the answer. Even though Student B does understand the answer will be an angle measure and demonstrates a representation that could yield an accurate answer, he/she lacks the procedural and conceptual knowledge to understand that there are multiple answers. This shows that the student’s understanding of the unit circle representation does not include an understanding that there should be another quadrant that has the same ratio for the sine value as well as the idea of all coterminal angles having the same sine value.

The student is also unable to use the representation to answer the second equation and is unable to use any other representation. Instead he/she deems it unsolvable. This is most likely because some algebra must be done in the equation to change the equation into a form that matches the first equation. It is interesting that the student did not attempt to use an algebraic representation and solve it using equation-solving procedures. The student does not appear to connect an equation with sin(x) in it to other equations of non-trigonometric functions.

Unlike the previous students, Student C immediately starts in the algebraic representation. The context of the question appears to be the reason the student starts in this representation. The student does not see the question as something unique to trigonometry but understands the questions are asking the student to solve equations. The representation he/she uses for this process is an algebraic one. The student appears to understand that to isolate x is this representation, it is necessary to use the inverse of sine.
Student C does not seem to understand the answer is an angle, but understands that if you input it into the equation, it will output 1/3.

STUDENT C: Uh, or sine of 1/3, I take the inverse of, uh, sine of 1/3. Which is... Then you type it in the calculator, (talking indistinctly), 19.47.
TEACHER: Ok, what does 19.47 mean?
STUDENT C: Um, that it’s the sine of 1/3.

In addition to not realizing the answer is an angle measure, the student also lacks the conceptual knowledge in this representation to understand that there are multiple inputs that can give an output of 1/3. Because the student is not able to connect it to any other representation or even create another representation independently, he/she is unable to expand on the answer.

For the second equation the student again uses the algebraic representation and uses the calculator to arrive at an answer, but there is evidence that the student is using more than just procedural knowledge to generate the answer.

STUDENT C: Um, alright for the second one, minus 1 from both sides. Divided by 2. Equals -1/6. Then you take the inverse of both sides. Which is -9.59.
TEACHER: Ok, and again, what does that answer mean?
STUDENT C: It is the sine of -1/6. That feels wrong.
TEACHER: Why?
STUDENT C: Because this is like your starting equation so -9 is an x so you are trying to get all of that to 2/3. So technically, it’s the sine of 2/3.

The student shows an awareness of what the procedure is attempting to do when he/she states that the answer means the input that should yield a desired output. However, Student C is confused with which output the answer should be linked. Since the student is unable to connect it to another representation or expand this representation any further,
the student has no way to clear up the misunderstanding and instead elects to move on to
the next question.

While the first three students were most successful with an algebraic
representation, Student D utilizes the geometric representations of right triangles to get to
a solution. The student does begin with an algebraic representation for the second
equation to isolate \( \sin(x) \). However, once both equations have \( \sin(x) \) isolated the student
changes to a right triangle representation to find the value of \( x \). It is through this
representation that the student realizes that the inverse of sine is needed to find the
answer, which is different from what the three previous students did

STUDENT D: Alright. I haven’t done this in awhile. Let’s see, I would take, do
I take the inverse sine? I know for this one I would subtract 1 so then it would be
-1/3. So then 2sin(x) = -1/3 and then I divide by 2 so then…
TEACHER: Ok, so you have the \( \sin(x) = 1/3 \) there and the \( \sin(x) = -1/6 \). How
would you find the solutions? What would you do next?
STUDENT D: Um, I always go to a triangle.
TEACHER: Ok
STUDENT D: And I think of 1 over 3 and then I use the Pythagorean, uh, the
theorem, and I do 3², which is 9 and minus 1², which equals 8, so \( \sqrt{8} \) and then this
I do a triangle also, and then this has to be -1 and 6, 36 minus 1 is \( \sqrt{34} \), and then
this has to put it in the coordinate plane would be this way.
TEACHER: Alright, so how do those triangles help you find the value of \( x \)?
STUDENT D: Um because I just think of SOH CAH TOA and then when,
because if this is \( x \), this is what you’re looking for, and the sine of \( x \) equals
opposite over hypotenuse. So then you take opposite over hypotenuse of \( x \) equals
1/3 and then, uh, (pause). Then I would take the inverse sine, wouldn’t I?
Because I’m trying to find the angle. So this would help me find a side. OK, so
then if it’s the angle, then I take inverse sine 1/3 and I get 19.5.
TEACHER: So what is 19.5?
STUDENT D: Degrees

It is through this representation that Student D realizes that the answers are angle
measures. The student does feel it is necessary to find the missing side in the triangle
even though it is unnecessary for finding the answer to the equation, but the student does
demonstrate the procedural knowledge for constructing a complete triangle representation
from the given information. It is interesting to see how the student handles a negative
ratio with the triangle representation.

STUDENT D: And for this I would do inverse sine -1/6. Oh, wait it wouldn’t be -
1 because this is just showing where it is. Inverse sine, 1 divided by 6. 9.6…
degrees.
TEACHER: Alright, so you changed from -1/6 to positive 1/6. Why?
STUDENT D: Because you can’t have negative degrees, and I was thinking if this
was to be placed on the coordinate plane then the -1 would be there because this is
always your x, your θ.
TEACHER: Ok
STUDENT D: So then (talking indistinctly) and draw. Your opposite is negative
and it would have to be there or there (Quadrants III and IV). And the hypotenuse
is… So then both have to be negative. I’m got to say it’s there (Q III).
TEACHER: So what’s your final answer then for x?
STUDENT D: 9.6°
TEACHER: And 9.6° takes you to the third quadrant?
STUDENT D: No, I’m doing something incredibly wrong (laughter). Um I don’t
know. Maybe I bet it’s… (pause). So if it’s -9.6°, then I would be thinking it’s
going this way. Maybe it’s over here (Q IV). I don’t know.
TEACHER: Ok, so you’re going to stick with 9.6 as your answer? 9.6°?
STUDENT D: Yes

The student has modified the triangle representation to include the coordinate plane,
mostly likely drawing a connection with the unit circle. This is how the student attempts
to justify a negative measure for a length of a side in a triangle. He/she understands that
a length for a triangle side cannot be negative, but if the triangle is in the coordinate
plane, it can have negative coordinates corresponding to the sides of a triangle. As a
result, he/she is able to realize that the triangle is not in the first quadrant, but because the
student’s understanding does not include a representation for negative angles, Student F
is unclear in what quadrant the triangle should be placed. Although he/she demonstrates
a deep level of conceptual knowledge to create this representation, it is not deep enough to allow the student to clear up the confusion with the negative ratio. However, at the end of the interview after he/she has seen a graphical representation, Student D is able to incorporate negative angles into the student’s understanding by seeing that the graph has negative angles and that the point (-9.6, -1/6) is on the graph. Student D has utilized the Cartesian Connection, but only after being shown the graphical representation.

When asked to think of another way to solve the equations, Student D attempts to use the unit circle representation from the object perspective. Using this representation as an object, the student is able to state that the angle for \( \sin(x) = 1/3 \) should be between 0° and 30° by using the coordinates of 0° and 30° on the unit circle instead of trying to locate where the actual point would be first and then determine its coordinates. Student D was the only student that was able to successfully use the unit circle representation to estimate a solution for this equation.

Student E immediately uses the algebraic representation with the inverse to solve both equations and gets an answer for both but is not able to say what the answer means, just like Students A and C. The student has strong procedural knowledge and the task of solving both equations is carried out quickly and effectively, but there appears to be a lack of conceptual knowledge in knowing what that answer means. As with the previous students, since Student E does not see that the answers as angles, he/she is unable to reason that there would be multiple angles that connect with the given ratio. However, at the end of the interview, when asked if the student would like to modify any answers, he/she comes back to these two equations and realizing that they are angles, also realizes
that there are multiple solutions. As a result, the student modifies the answer to incorporate this understanding.

TEACHER: Do you want to modify your answer to those based on the graph?
STUDENT E: Based on my graphs?
TEACHER: Look at your answer to number thirteen.
STUDENT E: Oh, I need to do ±360°k.
TEACHER: Why?
STUDENT E: Because it’s the same thing I mentioned with that. There is more than one input that can give you that number.
TEACHER: Do you want to modify your answer to those based on the graph?
STUDENT E: Based on my graphs?
TEACHER: Look at your answer to number thirteen.
STUDENT E: Oh, I need to do ±360°k.
TEACHER: Why?
STUDENT E: Because it’s the same thing I mentioned with that. There is more than one input that can give you that number.

This change was done because later in the interview, when the student is looking at a graphical representation, Student E realizes that there are multiple outputs for an equation in the form of sin(x) = y. Student E is able to make the connection with the graphical representation to this algebraic representation, but only after physically seeing the graphical representation. The connection from the graphical to algebraic representation appears to utilize the Cartesian Connection because he/she realizes that every point on the graph where sin(x) equals a desired ratio was a solution to the equation and those solutions appear to occur in a recognizable pattern that the student translates into the “±360°k.” The final result is still missing the concept of having two angles between 0° and 360° where a given ratio occurs, but if the student had been viewing the graph with the intent of answering these equations, then it is entirely possible that the student, using the graphical representation, would again modify the final answer.
Student F also approaches the first equation from an algebraic representation in the process perspective but rather than use the calculator to find the inverse, decides to change representations to the unit circle in an attempt to find an angle measure.

STUDENT F: Um, I think for \( \sin(x) = 1/3 \) you would just do the inverse so it would be inverse sine. Wait, no that’s wrong. You’re trying to get \( x \) by itself, right?
TEACHER: Um hmm
STUDENT F: Ok, so, (pause). Would you have to find what equals… what sine, what value of sine equals 1/3? Something like that?
TEACHER: Um hmm
STUDENT F: Ok, um, so (pause)
TEACHER: So what are you thinking right now?
STUDENT F: Um, (pause). I want to say like something with the unit circle, but I don’t really know where to go from there.
TEACHER: What are you trying to do with the unit circle?
STUDENT F: I am trying to see if there is any point that is on, (interruption from PA). I am trying to see if there is any, like points, like sine, cosine, or tangent, preferably sine that equal 1/3.
TEACHER: Ok, do you know of any on the unit circle?
STUDENT F: Not off the top of my head.

The student has connected the inverse sine operation with the process of finding an angle on the unit circle with a given ratio. As a result, rather than use the operation on the calculator, the student changed representations to the unit circle. When Student F was unable to find a desired angle with a ratio of 1/3, the student is unable to continue solving the problem. What is interesting is the student is able to explain what needs to be accomplished in the algebraic representation because he/she states, “you’re trying to get \( x \) by itself,” but fails to realize that the inverse will accomplish this. Instead, the student’s line of thinking causes a change in representations.

Since the student is unsure how to proceed from this point, the interviewer attempts to get him/her to back track and go back to the algebraic representation. The
student has realized that the unit circle representation, as Student F is using it, is not going to help produce an answer because the student’s representation only includes memorized coordinates for special angles. When asked what it means to isolate x, Student F begins recalling procedural knowledge for the process but is unable to connect anything in the algebraic representation with finding an angle on the unit circle.

TEACHER: Ok, so you mention you had to get x by itself.
STUDENT F: Um hmm
TEACHER: How do you get x by itself?
STUDENT F: I normally would divide but there’s nothing really to divide by.
TEACHER: Ok, so what other options do you have?
STUDENT F: Multiply?
TEACHER: Will that help?
STUDENT F: Probably not
TEACHER: Ok, so to get x by itself, what do you have to do?
STUDENT F: Um, you have to get rid of sine.
TEACHER: And how do you get rid of sine?
STUDENT F: Inverse. That doesn’t work either because there’s no inverse to the inverse. I mean, there’s no… I mean would you be able to do the sine inverse equals 1/3 inverse?

The student is able to reason back to using the inverse sine operation but it appears the student has a misconception with the procedure for that operation. The student wrote the following equation on the paper:

$$\sin^{-1}(x) = \sin^{-1}(1/3)$$

The student appears to understand that the inverse sine procedure will cancel sine in an equation, but the student has failed to realize that that canceling process will also eliminate the inverse sine operation in the equation and isolate x. Instead, the student now sees the equation of having the inverse sine of x equals a value instead of x equals a
value. As a result, he/she believes the value obtained from this process is the value of the inverse sine of x and not x by itself.

The teacher then asks the student to find \( \sin^{-1}(1/3) \) on the calculator to see if that will help the student to overcome the error in reasoning. The student states the answer but still does not connect the process to same thing as finding an angle on the unit circle.

STUDENT F: Uh, well, the inverse sine of 1/3 is 19.47
TEACHER: So what does that mean?
STUDENT F: Um (pause). It means that the sine inverse of 1/3 is 19.47.
TEACHER: Ok, so what does 19.47 mean?
STUDENT F: Um, I don’t know.
TEACHER: Is that the solution to your equation?
STUDENT F: I’m going to say no.
TEACHER: Ok, so the sine of 19.47 doesn’t equal 1/3?
STUDENT F: Yes (checks on calculator). Yes
TEACHER: It does?
STUDENT F: Just about.
TEACHER: Ok, so what does that mean?
STUDENT F: That the sine of 19.47 equals 1/3 so x would equal 19.47.
TEACHER: Ok, and x is what type of thing?
STUDENT F: It’s an angle.

It is only when the teacher asks the student to plug the value into the equation that Student F is finally able to see that 19.47 is the angle the student is looking for. The student is able to directly state that \( \sin(19.47) = 1/3 \) but it is not apparent from the student’s reaction if he/she really understands or believes it. The student does not appear to have reconciled the problem with the inverse procedure but accepts what is done only because it gave a correct answer.

With the next problem, the student uses the algebraic representation again but becomes confused when the ratio in the problem becomes negative. The student does not believe negative numbers are possible for trigonometric functions. This appears to be an
example of incorrectly associating a property with a function (Tall & Bakar, 1991). It appears the student is incorrectly recalling a restriction on the domain. In the algebraic representation, students learned about several instances of restrictions on domain and range during the school year. Some of the instances where the domain and range have restrictions with negative values include absolute value functions, square root functions, exponential functions, and logarithmic functions. Students also learn that the range of trigonometric functions is a restricted set of values. It appears that the student understands that there is a restriction associated with trigonometric functions, but has incorrectly used it for this particular representation.

TEACHER: Ok. Alright next one, 2\sin(x) + 1 = 2/3
STUDENT F: Um, you would subtract 1 first
TEACHER: Ok
STUDENT F: So that would be 2/3 minus 1 would be -1/3 and then… Are you allowed to have a negative answer? I don’t think you’re allowed to have a negative answer.
TEACHER: Because?
STUDENT F: I just remember learning in your class that a negative answer is not allowed somewhere in the unit on trigonometry.
TEACHER: Ok. So you have 2\sin(x) = -1/3. What does that mean literally?
STUDENT F: That the angle would be less than 1.
TEACHER: Why?
STUDENT F: Because it’s a negative.
TEACHER: Ok
STUDENT F: Or less than 0
TEACHER: Can an angle be less than 0?
STUDENT F: Not really, no.
TEACHER: We didn’t have any angles less than 0?
STUDENT F: I don’t remember.

The student has connected the negative ratio in the algebraic representation with a negative angle measure and is unable to visualize what this would mean. Even though the students were presented with negative angles measures during the trigonometry unit,
including negative angles in the unit circle representation, Student F has no representation for a negative angle and uses that to justify why a negative ratio is not allowed in the algebraic representation.

However, it appears that the student is not completely convinced that negative angles measures are not possible because when the teacher asks the student to continue the problem anyway, the student continues solving the problem in the algebraic representation even though he/she believes that an answer is not possible.

TEACHER: Ok, what if we kept solving the problem and forgot that point. What do you get from this?
STUDENT F: You would have to divide by 2. That would be -.166 and then you would pretty much do… I guess you could change that to a fraction, which is -1/6. And then you would pretty much do what you did in the first one. Find the inverse.
TEACHER: Ok, so try that.
STUDENT F: So the x would be -9.59.
TEACHER: You said it would be a negative number, right?
STUDENT F: Yes
TEACHER: Can x be -9.59?
STUDENT F: Um, I guess so.
TEACHER: Ok, how could you check?
STUDENT F: By plugging in the sine of -9.59 and see if I get -1/6.
TEACHER: So does it work?
STUDENT F: Yes

The student is able to execute the problem in the algebraic representation without a connection to another representation and this time takes no issue with using the inverse. It is not apparent if the student has the conceptual knowledge to understand what is happening or if the student is simply following a procedure to solve for x because the teacher requested it. The student does appear to accept the final answer as being correct but only after the calculator verifies it. Student F offers no argument or representation for
what the answer means. While the student does have procedural knowledge for solving equations and can utilize the algebraic representation for trigonometric equations, it appears that he/she has no way of connecting this representation to the unit circle or to a graphical representation to add conceptual meaning to the process or determine that there are multiple solutions.

*Question 9: If \( \sin(x) = 2 \), what is the value of \( x \)? How do you know?*

*Results*

It was not a real surprise that most of the students relied on the calculator to initially solve the problem. However, when the calculator did not present the full picture to the solution, few students could make sense of it in another representation. It is interesting that the only student who used a graphical representation, Student E, most likely did so because the graph was in front of the student. As a result, it was a case of the student being resourceful instead of intuitive. Again, most of the students readily used graphs to illustrate their understanding of functions, but when it came to representing trigonometric functions, the few that did still utilize the graph of \( \sin(x) \) did so from an object perspective or just for the sake of citing the vertical line test. It is unclear why students are not using this representation more often. It is most likely a result of not using graphical representations very often in class to answer these types of questions. Students did find some success with the unit circle, but because most of the students are using a representation that is memorized with no conceptual knowledge behind it, mistakes in memory have lead to larger mistakes with representations.
Evidence

Despite using an algebraic approach to solve the previous equations, Student A begins by using the unit circle representation and realizes that 2 is not a value for any of the coordinates. However, this is not enough to convince the student that there is not a solution, instead Student A tries to change representations to attempt to find an answer.

STUDENT A: Saw that on my placement test. Um (talking to self indistinctly), well, I want to use the unit circle but I know that there’s nothing there that equals straight up two. Um…

TEACHER: What does that mean?

STUDENT A: You have to figure it out another way.

TEACHER: What other ways do you have?

STUDENT A: You could fill in x with like a number or, no, can’t think of how.

Student A goes back to an algebraic representation from the process perspective, but because the student suggests plugging in values for x instead of using the inverse sine function, Student A does not see it as an equation to be solved. The student wants to guess and check but seems to realize that this is not going to be productive. It is probably the context of the question that is hindering the student’s thought process, because as soon as the teacher asks him/her to solve for x, the student begins to use the same process that was used in the previous question and uses the inverse sine function on the calculator.

TEACHER: What if I said solve for x?

STUDENT A: Oh, well, yeah, just type it in your calculator (types in calculator twice). It says domain error.

TEACHER: What does that mean?

STUDENT A: That means there’s no solution

TEACHER: Why?

STUDENT A: For whatever equation it was a solution to it could have gone out of the domain and range and that domain and range specify limits that the equation has. Or it’s not a rational number on the graph so it’s irrational.
TEACHER: So there’s a solution?
STUDENT A: No, I said no solution. Ok, then it means it could possibly not be a solution for whatever equation it’s given for. I mean there could be other solutions for it. I don’t know. That’s what I got.

The student is able to state that there is no solution but is not able to connect it to the unit circle or triangle representation that he/she used previously. Because of the lack of connection, there is limited conceptual understanding of what the calculator error means. The student lists several reason as to what he/she believes is causing the domain error but the student is unable to represent those reasons with anything concrete that would connect it to a trigonometric representation that the student understands. As a result he/she cannot confidently say what the final outcome is.

Student B was not able to use an algebraic representation to solve the equations and attempted to use the unit circle representation instead in the previous question. He/she uses the same representation again to answer this question. However, because the student’s unit circle representation is made up of memorized values with no conceptual knowledge as to where those values come from, the student incorrectly states that 2 is a value for one of the coordinates on the unit circle.

STUDENT B: The 2, 2 is on the unit circle, right? Yeah, I don’t know what angle it is for but it’s on there somewhere. The first quadrant I think. Um, it’s for one of the angles. 2 is one of the coordinates for one of the x or y angles. I know that. So x would be one of the angles and I don’t know the x but… How do I know that it’s one of the angles on the unit circle? Because you taught me that (laughter).

The student does understand that an angle measure is necessary for the answer and it is the angle where 2 is one of the coordinates. Because he/she believes that this process will give the correct answer and 2 is a coordinate on the unit circle, there is no reason to
change representations. However, the fact that the student is not able to actually state or even estimate the angle measure, other than to say it is in the first quadrant, most likely means that he/she has no connections for the unit circle to other representations. This would also explain why the student does not attempt to use the calculator to find an answer.

Student C uses the same algebraic approach that he/she utilized in the previous question and uses the inverse sine operation on the calculator. However, when the calculator displays an error message and mentions the domain, the student attempts to create a graphical representation to express what this means.

STUDENT C: The value of x is... It’s going to be... (typing on calculator). Oh no! Hmm (pause). Um, I know a way to do this (pause).
TEACHER: What’s the calculator say?
STUDENT C: It says error.
TEACHER: Why do you think that is?
STUDENT C: Probably the domain which is... If you were to graph it, um, it would not have, um, oh, (drawing a graph). If you go this way not all the x points are used. Because the domain is going to be on one side of the... So it would probably look like... look like that where it doesn’t cross the x-axis or where there’s no... it’s empty down here (pointing to quadrants III and IV on the graph).

The graphical representation the student makes is not consistent with the representations he/she used for defining a function and determining if \( \sin(x) \) was a function because it appears to contradict those. The student creates a graph of \( \sin(x) \) that is completely above the x-axis. Student C believes that the domain error means that there are no coordinates on the graph that fall in the third or fourth quadrant. The student earlier used a graphical representation to determine if something is a function by stating that any graph that crossed the x-axis is going to be a function. He/she had also used that
argument to conclude that \( \sin(x) \) was a function. However, with this representation, the student has created a graph that by his/her definition is not a function. The student makes this representation in an attempt to explain the error with the domain. The student does not incorporate any of the earlier understanding of functions or \( \sin(x) \) into this explanation. It is another example of a representation created for convenience.

It is also interesting to consider the perspective the student is using for the graphical representation that was created. It appears that because the student is discussing the points on the graph that he/she is utilizing the process perspective. However, because the student has taken the graph of \( \sin(x) \) and translated the entire graph above the x-axis, the student has acted on the entire graph as an object and is now using the object perspective. The student states that the domain error is caused by not using the third or fourth quadrant which shows that the student is not able to visualize the graph from a process perspective at this point. If the student was able to do this, then he/she should realize that all the x-values are being used, they are just all paired with positive y-values.

Just like Student C, Student D begins by using the inverse sine operation on the calculator because the student understands that in an algebraic representation, inverse sine will cancel sine and help solve for \( x \). The error message that the calculator outputs also causes him/her to change representations, but Student D elects to change to the right triangle representation that the student used to solve the equations. The error message on the calculator is also not enough information alone to allow him/her to conclude that there is no answer.
STUDENT D: On the calculator, it didn’t work.
TEACHER: Do you know why?
STUDENT D: I’m trying to figure that out (pause). Um, I always think of it when it’s a sine and things like that I always think of a triangle because it’s the easiest way for me to think about it, and then opposite over hypotenuse that would be 4 over 2. Oh, because, um, this wouldn’t make sense on a triangle because the hypotenuse of a triangle always has to be the longest side and for here you’d have this side being the longest which wouldn’t make sense.
TEACHER: So what does that mean?
STUDENT D: That means that it’s not a triangle.
TEACHER: Ok, what does that mean your answer is to the questions of the \( \sin(x) = 2 \), what is the value of \( x \)?
STUDENT D: Um, I don’t know. I don’t remember how to do that.

Student D is able to accurately represent the situation with a triangle but is still unable to conclude that the equation has no solution. The student understands that the calculator is not going to give a solution and the triangle representation is not going to work, but he/she is not able to connect those two with no solution. It is possible the misleading nature of the questions is causing this. From the context of the question, the student is expecting to find a value and may not be aware that no solution is a possibility. When the student is pressed further, the student concludes that the angle must be larger than 90° to fit with the student’s understanding for triangles because the biggest angle is always opposite the biggest side. He/she does not explain why the calculator cannot find this angle measure.

TEACHER: What did the calculator say the answer was?
STUDENT D: It said it didn’t work.
TEACHER: Ok, so what does that mean in terms of the triangle?
STUDENT D: Um, I don’t know.
TEACHER: Well, tell me about the triangle you just drew.
STUDENT D: I drew a triangle with a base of 4 and a hypotenuse of 2.
TEACHER: Ok
STUDENT D: A 90° triangle.
TEACHER: Ok, and what’s wrong with that triangle?
STUDENT D: It can’t work.
TEACHER: Why not?
STUDENT D: Because the hypotenuse has to be the longest side.
TEACHER: Ok, so what does that mean in terms, well, what are you trying to find in the triangle to answer the question?
STUDENT D: I’m trying to find this angle.
TEACHER: Ok, so what does that mean?
STUDENT D: It means that, it’s an obtuse angle maybe?
TEACHER: Ok
STUDENT D: So, it must be bigger, well, it means this angle is bigger than this angle.

It is interesting that the student does not attempt to connect it with the unit circle. Student D used the unit circle to estimate values in the previous two questions. However, it appears that the student was able make the connection to the unit circle in that case because Student D already knew the answers. With this latest task, the student does not know the answer to the question and is therefore unable to represent it with the unit circle. In the previous question, Student D, like Student C did, used an answer believed to be correct to create a representation, rather than use a representation to create an answer. As a result, when the answer is not known, as was the case with this explanation, the representation cannot be used as a tool for understanding.

It is also interesting to see how different students interpret the information the calculator gives them. Student E begins the task in the same way most of the others did. The student types in the inverse sine of 2 on the calculator and when he/she receives an error message, Student E is fairly convinced that there is no answer because it is not possible. This belief clearly impacts how the student chooses to represent the situation.

TEACHER: Ok, alright, so if the sine of x equals 2, what is the value of x? How do you know?
STUDENT E: Um, can I use the calculator again?
TEACHER: Sure
STUDENT E: I feel like it’s not
TEACHER: Do you think…
STUDENT E: I feel like it can’t. I mean it can’t really be 2, can it?
TEACHER: Why not?
STUDENT E: I mean think of the unit circle, right? I mean it only, like the highest number is 1. Well, except, no, yeah, yes.

The student quickly moves to a unit circle representation in an effort to offer validation for the claim that there is no answer instead of using it to discover an answer. Student D changed representations in an effort to find an answer while Student E changes representations to verify an answer. However, this could also be another case of using an answer to create a representation. There is not enough evidence yet to determine if the student really knows that 2 is not a value on the unit circle. Student E wants there to not be a place where 2 is on the unit circle. However, when asked to draw a picture to illustrate it, Student E changes representations again and is able to create another representation to verify the fact that there is no solution.

TEACHER: Ok, can you draw a picture to convince me?
STUDENT E: Well, like I mean, like can I use this one (graph of sin(x) on the bottom of the page)?
TEACHER: Sure
STUDENT E: I mean, like for the graph of y = sin(x) like it never goes up, like y is never equal to 2. The highest it ever gets is 1.
TEACHER: OK, what about with a triangle?
STUDENT E: What?
TEACHER: What about a triangle? Can you represent it with a triangle?
STUDENT E: With a triangle?
TEACHER: Sin(x) = 2, try to draw a triangle.
STUDENT E: (laughter) Ok, so we’re going back to SOH CAH TOA, right? Um, alright, let’s see. So, I mean, um, I feel like the hypotenuse has to be the longest one…
TEACHER: Yes
STUDENT E: So, I’m just going to use A. I guess I should use H for hypotenuse and the opposite, right?
TEACHER: Ok
STUDENT E: So, I mean like the hypotenuse always has to the greater than the opposite and like it’s on the bottom (of the ratio), like it can never be, like if it, just it can’t be 2, you know, like because this one (denominator of the ratio) is always going to be greater than that one (numerator of ratio).

This change in representations the student makes does not involve creating a representation but using one that is in front of the student. He/she locates the graph of sin(x) on the bottom of the sheet that is going to be used in one of the upcoming questions and uses it to illustrate that there is no solution. It now appears that the student truly understands what each representation is saying and has connected them together. The calculator said there was an error, which the student understands to mean that there is no place on the unit circle where 2 is a value of one of the coordinates, and the graph of sin(x) never reaches 2 for a y-value. The student has connected three different representations all to say that there is no solution. It is interesting to note that Student E is the first to use the graph of sin(x) in the response, but that is more likely a result of it being on the paper in front of the student than the student readily recalling it to prove a point.

Nevertheless, Student E is still able to produce one more representation at the request of the teacher by using a right triangle representation. Student E does the same thing that student D did, except Student E focuses on the ratio in the representation instead of angle. The student concluded that it is not possible because the ratio of opposite side to hypotenuse can never equal 2. This fact, aided with the understanding that there is no solution, allowed Student E to utilize this representation whereas Student D simply was able to conclude that there was something wrong with the right triangle.
Student F used a very unique approach to answer the question. The student began by using the calculator, as most of the other students had. However when the calculator gave the error message that caused the other students to change representation to try and make sense of the situation, Student F remains in the algebraic representation and finds a connection between the error message for this equation and the solutions to the two equations found in the previous question.

STUDENT F: Um, sine of x is 2? Um, you could technically just see the inverse sine of 2 and then that would be x. Because then you’d know that the sine of that value is x. Sine inverse, 2 (on calculator), which is an error so apparently there is no value x according to the calculator.
TEACHER: Ok, do you know why?
STUDENT F: Because 2 is a whole number?
TEACHER: What’s wrong with whole numbers?
STUDENT F: I don’t know, just the other two I tried that method with were fractions and they worked fine.
TEACHER: Ok, what if I said \(\sin(x) = \frac{3}{2}\)?
STUDENT F: Then that’s technically a fraction. It’s an improper fraction. I mean you could try the same method and you still get an error.

Although the student offers no explanation for why \(\sin(x)\) works this way, the student has noticed that if \(\sin(x)\) equals a whole number or an improper fraction then the calculator will output an error message. When the teacher asks the student to check what happens when \(\sin(x) = 1\), the student is then able to connect this observation to the unit circle.

TEACHER: What about 1?
STUDENT F: (types on calculator) You get 90.
TEACHER: So what does that mean?
STUDENT F: It means that 1 is a y-value on the unit circle for 90°.
TEACHER: So 1 works?
STUDENT F: Yeah
TEACHER: But no improper fractions work?
STUDENT F: Or 2.
TEACHER: Ok, do you know why that would be?
STUDENT F: No
Unfortunately, it does not look like the connection is strong enough to allow the student to make a connection between the calculator and the unit circle that will lead to an understanding of why improper fractions and 2 do not work. The connection the student has made to the unit circle appears to be based solely on memory of certain angles and coordinates. While the observation is remarkable and unique, due to the lack of conceptual knowledge it probably does not have many problem solving applications for the student and probably will not connect with any other representation the student makes for trigonometry. It appears to be another case of a student creating a representation to fit an answer instead of a student using a representation to gain a better conceptual understanding.

**Question 10: Is there a connection between the unit circle and the graph of sin(x)?**

**Results**

All of the students did realize that there was some connection between the unit circle and the graph. However, when the students took an in-depth look at the graphical representation for sin(x) it appeared that many students lacked a deep enough conceptual understanding of the representation to fully utilize it. In fact, it appears that many students are lacking the Cartesian Connection for the non-integer values on the graph, and maybe even the integer values too. This may explain why the graphical representation has not been used in most of the responses so far. While the students were able to identify, for example, the coordinate (90°, 1), it was unclear from some of the
explanations that the student understood this to mean that \( \sin(90^\circ) = 1 \). Instead, they seemed to draw the connection that the graph had a \( y \)-coordinate of 1 because the unit circle had a \( y \)-coordinate of 1 for \( 90^\circ \). While an understanding like this was helpful to establishing a connection for the sake of a response to the question, it brings up the issue of if students truly have a deep conceptual understanding of \( \sin(x) \) or are simply relying on procedural knowledge and memorized facts to generate their representations.

**Evidence**

Student A is not able to voice any connection but just the similarities in the two representations. The student understands that both can described in radians and degrees, both are plotted in the coordinate plane, and both contain important coordinates. However, the student does not make an attempt to mention how the coordinates of the two representations may connect. It appears that the student is using the object perspective for both representations and finding similarities in their properties, but the student does not represent either from a process perspective and does not see any similarity in how the coordinates are generated.

Likewise Student B appears to be viewing the graph and unit circle from the object perspective and finding their similarities because the student is focused on features like the angles for both and the amplitude for the graph. The student’s connections are similar to Students A’s in that it appears to be common features and no conceptual or procedural knowledge involved.

**STUDENT B:** The exact connection I do not know exactly what it is but there is I think because the angles are just like the angles on the unit circle. Um, every \( 90^\circ \) it hits. It goes from like 0 to the amplitude. The amplitude on this one is 1. Any sine of \( x \) graph will have 1. Then it goes back down to 0 and then it goes back
down to -1 and 0 and that would be a full 360. So, 0, 1, 0, 1, 0 would be a full 360. And how that exactly relates to the unit circle… (laughter).

Once again, because the student is not focused on the process that generates the unit circle and the process that generates the graph, he/she is unable to find any conceptual connection.

Student C is able to utilize the process perspective for both representations and as a result is able to make a meaningful connection. Student C realizes that both share common angles and coordinates. However, what is different about Student C’s understanding that separates it from the previous responses is that Student C sees that a particular angle that is common to both representations also shares a particular coordinate. This specific connection was missing from the previous two students’ responses. The student is therefore able to use the unit circle representation to generate the graphical representation.

STUDENT C: Yes, there is a connection. It, um, it helps you, uh, I know… I can’t… I know this part, I just can’t think of it. It, it helps you find the points on the, um, where to like start and end and things of that sort on the graph.
TEACHER: Can you give me an example?
STUDENT C: Um, like, it would help you find, like, where to start so, I’d start on 0, right here, starting point, and ending point would be 360. So I would be right here, here, here, here. (drawing wave on a graph). It helps you find these points.
TEACHER: The unit circle does?
STUDENT C: Yes.
TEACHER: Why?
STUDENT C: (Pause), It… I can’t think of it. I had something but I can’t remember.
TEACHER: Ok. Do you know how that point is connected to the unit circle? How that point is? What are the points right here? (pointing to graph).
STUDENT C: This is 90. This is 180. This is 2… No, I lied (changes x-axis scale). They’re the… These points are these four points on the circle and you use the circle to find these points to plug them in correctly.

123
However, even thought he/she is able to see both representation as a process, the representations only contain coordinates for $0^\circ$, $90^\circ$, $180^\circ$, $270^\circ$, and $360^\circ$. The student is not able to state the specific $x$-coordinates and $y$-coordinates for these angle measures but understands that the $y$-coordinate for an angle on the graph somehow comes from the coordinates for that angle on the unit circle. It does not appear if Student C knows that the same holds true for all the angle and points in between these five angles or if the student even knows there are points there to compare. Nevertheless, this connection is strong enough for him/her to generate a graph for $\sin(x)$ simply by looking at the unit circle representation.

Student D also looks at both the unit circle and the graph from a process perspective. However, Student D does not know what the connection between the two representations is when asked originally, but is actually able to use the two representations to build a connection during the response. Whereas Student C used the unit circle to create the graph, Student D looks at both representations point by point to find a connection.

TEACHER: Ok, so, can you explain to me what the connection is in what you’re doing right there?
STUDENT D: Um, I was just looking at the degrees given here on the unit circle, which is 30, 45, and 60, and then I kind of approximate it on the graph.

By doing this, the student is able to realize that the $y$-coordinate on the graph for a given angle matches the $y$-coordinate on the unit circle for that same angle measure. However, the student is not able to say why it is the $y$-coordinate on the unit circle and not the $x$-
coordinate. Even though he/she has found a connection, it is only one based on similarity and not on a mathematical procedure. The student’s understanding is lacking the Cartesian Connection because the student never mentions that the y-value on the graph is the sine of that given angle and likewise the student does not seem to realize that the x-value on the unit circle is the cosine of the angle and the y-value on the unit circle is the sine of the angle. The teacher continues to question the student about the coordinates on the unit circle to attempt to get the student to realize how they are generated, but the student is not able to make this connection.

However, once the student is ask to create the graph for cos(x) and connect it to the unit circle, the student is able to realize that the x-coordinate on the unit circle matches the y-coordinate on the graph of cos(x). Even though the student has connected the graphs to the x-coordinate and y-coordinate on the unit circle, Student F still does not know why this connection exists.

TEACHER: Ok, so what if, for example, the graph of cosine, would you still use the degree and y-value to get the graph of cosine from the unit circle?
STUDENT D: The graph of cosine would be like (draws graph) this. I believe like that and cosine starts up.
TEACHER: Ok, so at 0°, what is the y-coordinate?
STUDENT D: 1
TEACHER: On the unit circle, what is the coordinate at 0°?
STUDENT D: (1, 0)
TEACHER: Ok
STUDENT D: No, wait, yeah (1, 0)
TEACHER: Ok, go ahead and write that down up here. So on cosine, 0° is what on the graph? The y-coordinate at 0° is…
STUDENT D: 1
TEACHER: And at, on the sine graph at 0°, what’s the y-coordinate?
STUDENT D: 0. Oh, so maybe it’s opposite for sine and cosine. You use the x-coordinate for cosine and the y-coordinate for sine.
It is interesting that even though the student has the procedural and conceptual knowledge to develop a deep understanding of both representations, the student is not able to make what seems like the most obvious connection: \( \cos(x) \) and \( \sin(x) \) are the \( x \)-coordinate and \( y \)-coordinate on the unit circle and the \( y \)-coordinate on their respective graphs. What is blocking the understanding appears to just be a failure to realize it. The student has not trouble generating all three representations and does not appear to be the slightest bit confused by the change in meaning of the \( (x, y) \) coordinates on the unit circle and on the graph.

Finally, the teacher asks the student to use the unit circle to evaluate the cosine of a given angle. By using the procedural knowledge of this simple task, the student is able to see the connection.

TEACHER: Now, if we’re talking just about the unit circle, and I want to find the cosine of 135°, what are you going to do?
STUDENT D: Um, I going to go to 135°, which would be right here ish. Quizzing me on my unit circle. It’s like…
TEACHER: The coordinate I’ll tell you is \((-\sqrt{2}/2, \sqrt{2}/2)\).
STUDENT D: Oh, ok, \((-\sqrt{2}/2, \sqrt{2}/2)\) so then…
TEACHER: I’m asking what’s the cosine of 135°
STUDENT D: The cosine? The cosine is \( x \).
TEACHER: So cosine is \( x \)?
STUDENT D: Yes, cosine is \( x \).
TEACHER: So is there a connection to why you chose the \( x \)-coordinate for the graph?
STUDENT D: Oh! Yes, because cosine is \( x \)!

It is interesting that within the unit circle representation, the student knew that the \( x \)-coordinate was the cosine of the angle and the \( y \)-value was the sine of the angle but never used this knowledge until the teacher specifically asked for it. Once the student recalled the information, the connection is quickly made. It again supports the idea that much of
the information about the representation appears to come from memorized information and memorized information can lose the conceptual meaning.

Just like Student D, student E approaches both representations from the process perspective, but appears to have already made the connection between the representations. Student E begins with the unit circle and uses it to make sense of the graph. For this reason, the student avoids the confusion of why they y-values on the graph of \( \sin(x) \) match with the y-values on the unit circle. The student is implying that the y-value on the unit circle is the sine of the angle during the explanation and circles the y-coordinates on the unit circle to emphasize this point.

STUDENT E: Um, well, ok, so you have your unit circle, right? Um, ok, for \( \sin(x) \), we’re just talking about sine. Um, so like at 30°, we’re using degrees, um, so (\( \sqrt{3}/2 \), 1/2), so 1/2 is our sine value, right?
TEACHER: Why 1/2?
STUDENT E: We talked about this (laughter). Because it is, because the unit circle says so.
TEACHER: Ok
STUDENT E: Can I go with that? Um, or like 90°, the unit circle at 90°, like the sine of 90° is 1.
TEACHER: Ok
STUDENT E: So at 90° on here the y-value is going to have to equal 1, like the output.
TEACHER: Ok
STUDENT E: So like for 30°, it’s going to be 1/2.

When the student states, “The unit circle says so,” the student is implying that the sine of the angle is the y-value from the unit circle. He/she then connects it with the y-value on the graph and also appears to understand that it is the output to sine of that angle measure too. However, it is not clear if the student understands the y-value on the graph is also the output for the equation \( y = \sin(x) \) or if it is the y-value because the unit circle says it
is. It would be interesting to see if Student E is able to produce a graph without utilizing the unit circle.

Student F also uses both representations from the process perspective to find similarities in the two representations. However, Student F is not able to correctly label any of the coordinates on the graph so it appears the connection based on similarity is done by recognizing that the angle measures on the unit circle and x-axis (in the graph on the page in front of the student) contain the same angle measures and then assuming that the coordinates on the unit circle must be on the graph also.

STUDENT F: Um, well some of the points for the sine of x could be on the unit circle. Like I see … I see like 180 on there. It’s a point of the unit circle.
TEACHER: Ok, what do you mean by given the degrees?
STUDENT F: I mean like 30°, 45, 60, 90…
TEACHER: Ok.
STUDENT F: Some others would be 180, these points are on the unit circle.
TEACHER: How are they points? They’re just random degrees. Points have an x and a y.
STUDENT F: Well, it’d be like (\sqrt{3}/2, 1/2) and (\sqrt{2}/2, \sqrt{2}/2).
TEACHER: Ok, and how does that connect to this graph in front of you?
STUDENT F: Um, it connects because (pause). I don’t know.

The student is unable to see a connection with the coordinates from the unit circle and the y-coordinates on the graph of sin(x). The connection he/she sees is based solely on the common angle measures, much like the connection Student A made. However, Student F does have a little more conceptual knowledge with the connection because the student believes that the coordinates from the unit circle do connect. The student is unable to voice that connection, but that does not mean that the student does not realize that a connection exists.
As the teacher begins to ask the student to take a closer look at the coordinates on the unit circle and try and connect them to the graph of \( \sin(x) \), it becomes apparent that the ability to read the graph of \( \sin(x) \) may be hindering the connection the student is making. Whereas Student D could find coordinates on the graph for \( \sin(x) \), Student F is unable to fully grasp the graph from the process perspective.

**TEACHER:** Ok, what is the sine of 30°?
**STUDENT F:** 1/2
**TEACHER:** Ok, on the graph, find 30°
**STUDENT F:** (points to 30° on the x-axis) Right here.
**TEACHER:** Ok, what’s the y-value there?
**STUDENT F:** 1 and 0. There’s no y. I don’t know. It could be any y-value.
**TEACHER:** On the graph?
**STUDENT F:** Like infinite because you could go up or down for y-values as long (talking indistinctly)
**TEACHER:** Alright find 30° on the y-axis.
**STUDENT F:** Right here
**TEACHER:** Now find 30° on the graph. (pause)
**STUDENT F:** I don’t know what you mean by that though.

The student can use the unit circle from the process perspective to match an angle measure with the correct coordinate to find \( \sin(x) \), but when the student attempts to use the graph to do the same thing, the student finds the angle on the x-axis and does not see that x-value angle is linked to a y-coordinate on the actual sine wave at 30°. However the student can correctly make this connection for the x-intercepts as well as the maxima and minima of the graph.

**TEACHER:** Ok, let’s try another one. Find 90° on your scale. Now find 90° on your graph.
**STUDENT F:** Would it be right here? (points to (90°, 1))
**TEACHER:** Ok, so what is that point?
**STUDENT F:** That is 90° corresponds to -1.
**TEACHER:** Ok, so mark that point. Label the coordinates again. So what is this right here?
STUDENT F: (90°, -1)
TEACHER: Why is it -1?
STUDENT F: Because there’s -1 right there. (points to positive 1 on the y-axis)
TEACHER: So what’s that value right there (points to -1 on the y-axis)
STUDENT F: That is also a negative. So that would be 1 (changes coordinate to (90°, 1)).

While the student is not able to find the coordinates of the points for angles like 30° on the graph, the student is able to find the coordinates of angles like 0°, 90°, and 180°. In the dialogue above, the student simply misreads the scale on the y-axis but was able to realize the mistake and correct it. Even though the student is using the process perspective to link x and y together, the process perspective is lacking full understanding because the student is unable to estimate y-values that are not 0, 1, or -1. It appears the connection the student is making between these representations then is based on the fact that they both have similar angles and coordinates that involve 0, 1, and -1. There is no understanding how coordinates including 1/2, √3/2, or √2/2 are connected because the student is unable to represent these values in the graphical representation.

However, it appears that only using the points the student can read on the graph is enough for the student to voice a connection between the unit circle and the graph, even if it is only voiced as a response to direct questioning by the teacher.

TEACHER: Ok so that’s (90°, 1). What’s the coordinate on the unit circle at 90°?
STUDENT F: (1, 0)
TEACHER: What’s the coordinate at 0?
STUDENT F: Oh wait, no, no, no. 90° is (0, 1)
TEACHER: and 180°?
STUDENT F: (-1, 0)
TEACHER: Ok. What are the coordinates at 0° on the graph?
STUDENT F: 0 is (0, 0).
TEACHER: Ok, good. What are the coordinates of 180° on the graph?
STUDENT F: Um, (180°, 0)
TEACHER: Alright, write that down. Now look at 0° on the graph and on the unit circle, look at 90° on the graph and on the unit circle, look at 180° on the graph and on the unit circle. What do you notice?
STUDENT F: Um, the sine of x is the same. The y-value is the same.

The student is able to state the connection between the unit circle and the graph, but does not use the connection then to include the understanding for the coordinates at 30° or 45° or wherever the y-coordinate incorporates non-integer values. As a result, while a connection does exist, it is not very strong and would probably be of limited use in problem solving or another context.

Question 11: Explain how you can graph sin(x).

Results

It was interesting that all of the students saw the graphical representation in the same perspective. It should be noted that the object perspective was being utilized for the purpose of creating the graph. Other students have been able to use the process perspective for the graph in a different context. It also appears to be the direct result of the instruction they received for why they chose this perspective. What is still not apparent is if it has an effect on the ability to use the Cartesian Connection with \( y = \sin(x) \) or if it has hurt some students ability to see more than just certain points on the graph. It could be possible that by only seeing the graph in the object perspective, some of the students were unable to see the point (30°, 1/2) was on the graph or that other points with irrational coordinates were also possible.
Evidence

Student A uses the object perspective to discuss properties of the graph as an object, like its amplitude and period. The student does not attempt to discuss any specific points on the graph, so it is unclear if the student can view the graph from a process perspective.

STUDENT A: Um, well, I forget how to get to these, but there’s like period, um, starting point, ending point, um (talking indistinctly). Depending on where it starts you can determine that there’s no reflection to it, that there’s no transition, it’s just a normal sine wave. Um, there’s (pause), I want to say the period would be $\pi$ or 180° and then…

TEACHER: Let’s say create the graph of sine of $x$.

STUDENT A: Um since there’s no transition or anything, um, well sine wave looks like that and then cosine wave looks like that. I want to say the amp is one, which is how high it goes (talking to herself indistinctly). So it starts at (0, 0) because that’s the starting point for sine. Um, amp is one. Um, I think this is how you get the period (pause). Or $\pi/2b$. I don’t know. I don’t remember which is which or how you get it.

TEACHER: What’s the period of sine?

STUDENT A: $\pi$

TEACHER: Ok, what does that mean?

STUDENT A: I forget how we got that. Um, I don’t know. I’m not sure. I forget how you get how far you plot the points from each other, but that’s what I got.

Once again, the student is using memorized information to create the representation.

There is evidence of some conceptual knowledge tied to this information because the student discusses how to determine if the graph has been reflected or translated, although the student refers to the latter as a “transition.” Other memorized knowledge appears to have no conceptual understanding tied to it like the overall pattern for a sine or cosine wave or the length of the period. The student never looks at the graph from the process perspective so he/she does not attempt to connect it to the equation for $\sin(x)$ or the unit
circle. This does not mean the student does not have this connection, only that he/she chose not to use it.

Student B connects the graphical representation with the algebraic representation of the equation \( y = \sin(x) \). Both representations are viewed from the object perspective and the student is focused on the features of the equation and graph much like Student A was.

STUDENT B: Well, you know that the sine graph starts at 0, and the cosine graph starts at 1. So it starts at 0, it starts at the origin and, um, um, so its, er, depending on, well, just a normal sine graph, like \( y = \sin(x) \), um, it will always go up first because there is no other numbers to or negative signs to change it, like go down first or something. Um, and the amplitude of any, um, sine or cosine graph is always 1, so it will always go up to 1 and likewise down to -1. So that’s where it starts and how high it goes and how low it goes and the, um, on a standard like \( y = \sin(x) \) it’s, that it’s always, um, in degrees, it’s always 0, 90°, 180, 270, 360 and, um, in radians it’s something else like 0… or no π first… 2?

TEACHER: And the numbers you are giving me, why are they significant? 0°, 90°, 180°, 0, π?
STUDENT B: Why are they significant? (Pause) Because they are (laughter). Uh (pause), oh, because they’re the, uh, each like the main, uh, the main angles of the unit circle.

The student talks about the overall shape of the sine graph and represents that as an object, not one that is made up of coordinates. Even though he/she mentions specific y-values and then specific angle measures later, those values are treated as objects on the graph and not necessarily linked values. This student also explains how the graph of \( y = \sin(x) \) will create a “normal” sine wave meaning that there are no transformations. He/she never uses the process perspective or the Cartesian Connection to link any points on the graph with the equation. The student included those specific angle measures
because of the connection the student has made with the unit circle in the previous question, not because they have a connection to the equation $y = \sin(x)$.

Student C also approaches the graph from the object perspective by looking at the features of the graph and not the points on the graph. Like Student B, Student C has also linked the equation, as an object, to the graph and can use the features of the equation to determine the features of the graph. He/she specifically mentions a procedure learned in class for producing a graph, but because the student does not see the graph as a collection of points, does not use the process perspective to execute the procedure.

STUDENT C: You use that trick, those steps. The starting, no, if your starting point is… Oh my God! Amplitude. I’m thinking slope, point-slope something like that. It’s for the slope, starting point, ending point. We use like the amplitude and all that of an equation to find where you start your plotting so if there’s no amplitude, if the amplitude is just one, then we leave it the same so that would look like… It would go above this line or below it here. This would be 1 and this would be -1.

TEACHER: Ok.
STUDENT C: Scale, scale… Scale shows, um, how far it stretches, um… I find all that out using the equation…

The student is recalling a memorized procedure that once again at certain steps has no real conceptual understanding. The student says, “point-slope,” because the step where students look for a phase shift was always abbreviated P.S. so Student C wrote the letters on the paper and is trying to make sense of what P.S. stands for. He/she does understand that all of this information in this procedure comes from information from the graphical representation and the student is able to explain how changing an equation can change the amplitude of the graph. There is still no evidence yet that any of the students can view the graph as a series of points that are solutions to the equation $y = \sin(x)$. 

134
It is interesting to see that Student D also uses this same approach to graphing $\sin(x)$. The student looked at both the unit circle and the graphs of $y = \sin(x)$ and $y = \cos(x)$ in the process perspective in the previous question in order to establish a connection between those representations. However, the student changes to the object perspective for this response and begins to discuss the graph of $y = \sin(x)$ in terms of the amplitude and period of the graph. This seems to really emphasize that this is how students perceive the graphical representation during instruction and may be a reason why so many of the students do not utilize the graph from a process perspective.

Student D makes the same connection to the equation to determine the features of the graph that Student C did and produces a graph of $y = \sin(x)$ that is based on the pattern of the wave and not the points that make up the wave.

STUDENT D: $\sin(x)$? Ok, so then first you have to find the amplitude.
TEACHER: Ok
STUDENT D: Which is 1 because there’s nothing there.
TEACHER: So what does that mean?
STUDENT D: That means the amplitude is 1. Then you go on your graph, so that means since sine starts here, the maximum and, like, minimum that you’re going to go up is 1.
TEACHER: So why does sine start there again?
STUDENT D: Because sine is y and at 0°, your y is 0.
TEACHER: Ok
STUDENT D: Um, so then, so then you start at zero and sine goes up, er, middle, up, middle, down, middle.
TEACHER: Ok, what does that mean?
STUDENT D: Like middle, up, middle, down, middle (draws graph).
TEACHER: Ok
STUDENT D: So then, sine, that’s your amplitude, your period would be 1 or $\pi$.
TEACHER: What about degrees?
STUDENT D: Um, 90°
TEACHER: What’s the period?
STUDENT D: The period is like, um, maybe it would be $2\pi$. The period is how long, like within what length it takes to complete the graph of…
However, unlike Student C, Student D does make the connection of what the point (0, 0) on the graph actually means. The means the student is able to see the graph from the process perspective but is choosing to use the object perspective. This was something that could not be determined with the responses from the previous students. He/she is also using the connection made to the unit circle in the previous question to get the coordinates (0, 0). The student is also able to use the process perspective to justify the graph.

TEACHER: Ok, and what are these marks right here? (marks on x-axis)
STUDENT D: Those are degrees so then if this was, if the period was 360, then this would be… This would be 90 no that would be 180.
TEACHER: How did you know it couldn’t be 90?
STUDENT D: Because, um, it wouldn’t make sense because if this was 90, then this would have to be 180 and this would have to be, oh wait, that’d make sense. No wait, it wouldn’t because if your graph ends there and if this was 90 that would have to be 180, that would be 270 and that wouldn’t equal 360.

The student is using the unit circle coordinates from the process perspective to check the x-intercepts on the graph and is able to determine why the x-intercepts are at 0°, 180°, and 360° and why the maximum of the graph is at 90° and the minimum is at 270°. This also means that the student would have to be viewing the graph from both the object perspective and the process perspective simultaneously.

Student E also uses graphical representation from the objective perspective but does so to describe what the graph of anything of the form \( y = A \sin(Bx \pm C) \pm D \) will look like. The student is connecting the conceptual meaning of what all the parameters in the algebraic form will do to the sine graph. The student also states that all graphs for sine will follow a set pattern that the student has learned directly from instruction.
STUDENT E: OK, well, you taught us that when you graph sine of x, it’s going to be, like going to look like this thing, or like some variation of it, right? You have to like move it over, so you’re going to have like the, oh my gosh, I forgot the up and down thing, the phase shift (laughter). Um, so this is like what you’re going to have. Like y equals, I forget what you call them. It’s like Asin(x) like Bx, ±C or something like that, and ±D, right? Or something like that (laughs) and so A here is going to be your amplitude and that’s going to like tell you, um, like up, like not like your up or down because that’s what D is, but it’s going to like tell you how, you know, like how far up and down it goes.

TEACHER: Ok

STUDENT E: And then (talking indistinct). I forget how to do phase shift. There’s like phase shift, right? And the period and then your up or down shift, right?

TEACHER: Ok

STUDENT E: And it’s like, I don’t know, like your up and down, ±D here, that’s going to tell you how far up or down it’s going to go. Like how you’re going to move the whole graph, not like stretch it, like amplitude does.

TEACHER: Ok

STUDENT E: And then, I forget how to get the phase shift.

TEACHER: What is the phase shift?

STUDENT E: Phase shift is like moving it left and right. Like the same way that you know like D moves it up and down.

TEACHER: Ok

STUDENT E: Uh, like moving the whole graph left and right and then the period is how long it’s like, how it stretches and shrinks sort of like amplitude does.

The student does have some difficulty recalling the procedures for the phase shift and period, but he/she does know how each impact the graph. The connection appears strong enough that the student does not need to discuss the actual points on the graph so the process perspective is not necessary for the sake of discussing how to create a graph for sine. The student draws the pattern for sine but does not tie it to the unit circle as in the previous question so he/she is basing the pattern on a memorized sequence. However, it is probable that Student E does have the conceptual understanding of the pattern from the connection the student was able to make to the unit circle earlier.
Approaching the graphical representation from the object perspective to create a graph has worked for the students. However, it is dependent on the students remembering the procedures and patterns associated with the graph. Student F is able to mention the pattern, but does not have a detailed memory of the process, so the student is unable to explain how to create the graph aside from stating that it follows a set pattern for a sine wave.

**Question 12:** Using the graph, can you find or estimate $\sin(20^\circ)$?

**Question 13:** Using the graph, if $\sin(x) = -1$, what is the value of $x$?

**Results**

It was interesting to see the ease with which most student were able to produce answers to the questions and how most of them, using a graphical representation that was already provided, were able to determine that there were multiple solutions to the second question when earlier using other representations did not lead to that conclusion. It was surprising that most students easily moved from object to process perspective when the opportunity called for it. It again demonstrates the importance the context of the question plays in representations. When creating a graph, students used the objet perspective, but when utilizing the graph to answer questions, student switched to the process perspective. While some students connected the graph with the unit circle, others were unable to see the graph as anything more that a collection of points that were not similar to other representations they had for trigonometry.
Evidence

Student A does not believe it is possible to answer the first question using a graph. However, once the teacher tells the student it is possible, the student easily estimates \( \sin(20^\circ) \).

STUDENT A: I don’t think it’s the right graph for this type of situation.
TEACHER: What if I told you it was?
STUDENT A: Well, then you count the points (talking indistinctly trying to decipher x-axis scale) Then the sine of 20° would be somewhere, well, the way I did it was find the middle, between 0 and 90, and the middle between 45 and 0°, which is 22.5 and then go down slightly from that and bring it up to wherever the line is and that would be the solution.
TEACHER: Ok, and what is that solution?
STUDENT A: Um, 1, 2, 3. It would be... Would it be like 3? No, .35 or something.

It appears that the student’s reluctance to use the graph to find the answer was not because the student did not think it was possible, but rather it seems he/she is reluctant to estimate the solution on the graph. Earlier with quadratic equations, most of the students avoided using the graph for estimates but instead only used it to find points where both coordinates were integers. It is possible that Student A views this graphical representation for \( \sin(x) \) the same way. The only points the student would be comfortable using on the graph are the ones with integer coordinates. It is not because the student does not realize there are additional points between those integer coordinates. The student is using the process perspective for a graph, even though the student created a graph of sine using the object perspective in the previous question. The student also appears to be utilizing the Cartesian Connection to find an estimate for \( \sin(20^\circ) \) from the graph.
Student A then states that the second question is not solvable but for a different reason. The student understands that the answer is now an angle measure on the graph, which indicates the student understands how to use the representation to get an answer. The student also is able to say the answer is 270° because the coordinate on the unit circle at 270° is (0, -1) so the student has used a unit circle to represent the problem and solution. However, the student is unable to find the coordinate (270°, -1).

STUDENT A: Um, wouldn’t it not be a solution for this?
TEACHER: Why?
STUDENT A: Because it’s not on the… Well, I know that from the unit circle that the sine of -1 equals 270° or is the y-coordinate for 270°, but this line doesn’t hit that so that wouldn’t be a solution.
TEACHER: What point is?
STUDENT A: 180° and then 360°.
TEACHER: Ok, on the graph there, on the unit circle you have (0, -1), but on the graph what is the point for 270°?
STUDENT A: Well, it would be (270°, 0)
TEACHER: Is that on the graph?
STUDENT A: No
TEACHER: Should it be?
STUDENT A: No. Wait, yes.
TEACHER: Why?
STUDENT A: Because you’re trying to find the solution for sine of x
TEACHER: Ok, so the sine of x never equals -1?
STUDENT A: Not in this case. If you’re asking me to look at this graph then no, it doesn’t. If it were another graph that was hitting the point 270° then I would be like yes, -1 is a solution for that graph but it’s not a solution for this one.

It appears that the student believes solutions in the graphical representation should be x-intercepts. This might be another reason why the student was reluctant to use the graph to find \( \sin(20°) \). He/she states that the graph is saying the answers are 180° and 360° but the student knows the answer should be 270°. The understanding of Student A is that a solution in a graphical representation means the point (270°, 0) should be on the graph.
When the student is asked to point out the solutions on the graph, the student points to (180°, 0) and then (270°, -1). When the student sees the coordinates of the second point, it is enough for the student to realize that it is the solution he/she is looking for.

TEACHER: What makes it a solution?
STUDENT A: The points that are on the line that hit the line.
TEACHER: Point to one
STUDENT A: 180° (pointing to (180°, 0). This one down here (pointing to (270°, -1)).
TEACHER: What’s that point?
STUDENT A: Oh my goodness! What!? No! That’s right. Well, it misses it by, no wait, yep it’s a solution.
TEACHER: What is?
STUDENT A: Isn’t it? Yeah, it is. (270°, -1) is a solution because you go over and then you go down and it hits the line.

It is not clear why the student identifies (270°, -1) as a solution. It appears that the definition of the word “solution” has changed meaning possibly because the student sees the follow-up question in a different context. For the follow-up question, the student switches to labeling the integer-value coordinates on the graph. It is possible that the student is now viewing the graph without connecting it to the equation so solutions now mean points on the graph instead of x-intercepts. When the student realizes that the point is (270°, -1), it triggers the connection the student has with the unit circle and the student states that it is actually a solution.

The student is then asked if there are more solutions and he/she is able to point out that -90° is also a solution and there are more if the graph is extended. While it is not possible to tell if the student would have made this connection without being asked for additional solutions, Student A was able to use the representation to find multiple answers whereas in the algebraic representation the student was not.
Student B is first asked to find $\sin(30^\circ)$ on the graph. The student had previously used the unit circle and calculator to determine that the answer is $\frac{1}{2}$. The student uses the graph to estimate the answer to be $0.4$. He/she realizes that the answer is different than the answer given previously and points out that the graph is only an estimate. The student then looks at the graph a second time to conclude $\sin(30^\circ)$ does in fact equal $\frac{1}{2}$. The last time he/she tried to find $\sin(30^\circ)$, the calculator gave a different answer than the one the student got from the unit circle, but Student B did not see a need to reconcile this difference. It is interesting that when the difference is pointed out the student immediately offers an explanation. It appears that the student is confident the answer is $\frac{1}{2}$ and realizes the limitation of a graph.

STUDENT B: $30^\circ$ would be, uh, (tracing on graph), it would be like $0.4$. It would be point four. That doesn’t sound right but…
TEACHER: What did you say it was earlier?
STUDENT B: I said it was $\frac{1}{2}$, (laughter). Um…
TEACHER: So which one is more accurate, the $\frac{1}{2}$ or your $0.4$?
STUDENT B: The $\frac{1}{2}$ because I did it on the calculator and graphs are sometimes not exactly precise if you’re just like looking at the lines between them.
TEACHER: Can you show me with the graph how the sine could be $\frac{1}{2}$? I mean what would it have to be to be on the graph, for it to be $\frac{1}{2}$ on the graph?
STUDENT B: On, no, it is $\frac{1}{2}$ (laughter).
TEACHER: Because?
STUDENT B: Because the line goes over and hits. Because if you, um, because $30^\circ$ is right here. That’s $60$ (marks $x$-axis) and that’s $90$, so if you take the $30$ and take the line straight up (drawing vertical line) it’s this one right here. It intersects the graph of this. The line of the graph’s right there, and if you take that over (draws horizontal line) it goes through $0.5$.

Just like Student A, Student B has no trouble using the graph from the process perspective. It appears the reason that students were not using the process perspective when creating the graph was because of the lack of emphasis the instruction gave that
perspective at that time. It is still not clear if it has affected the students’ ability to connect it with their understandings of functions.

When the student is asked to find $\sin(20°)$ he/she once again has no trouble finding the exact point on the graph to give an accurate estimate. The student gives an estimate of .32 and recalls that the answer the calculator gave was .34. Because he/she is also able to point out the exact coordinate on the graph that is producing the estimate, the student is not just making the answer match the one the calculator gave. The student is able to utilize the Cartesian Connection going from the equation back to the graph.

When asked about $\sin(x) = -1$, the student is able to make sense of this in the graphical representation. Student B understands that the answer is now going to be an angle that has $-1$ as the $y$-value.

STUDENT B: On the graph, sine of $x$ is $-1$. So earlier we were finding, we’re finding if, like, the sine of $20°$ was like $x$. That would be $y$ so it’s here and here (pointing to the points $(270°, -1)$ and $(-90°, -1)$). It would equal, um, 90 or $-90$ or $270$ or $-270$ or any angle that would be if you took 90 and added $180°$ to it. And you could just keep on finding a bunch of angles that equal, uh, $\sin(x) = -1$. So there’s a bunch of different values for $x$.

The student was able to use a unit circle representation to answer questions similar to this one before but the student made no mention of multiple answers. Now that the student is given a graphical representation, the student, again using the process perspective, quickly points out that there are 2 solutions on the graph and many more if you extend the graph in both directions. The change in representations from the previous questions allows the student to develop a better understanding of the answers.
Since the student was able to answer this question in the graphical representation, the teacher also asked the student if it is now possible to find where \( \sin(x) = 2 \). The student earlier was using the unit circle representation to answer this question and stated that somewhere in the first quadrant there was a \( y \)-value that equaled 2 and that angle measure was the answer. The student now realizes that since the graph never reaches 2, there is no solution and that meant that his/her unit circle representation was incorrect.

STUDENT B: Ok, um, but the graph does not intersect a point with 2 or -2 and… so I was wrong before. I think (pause). I’m pretty sure because of the graph. The graph would have to like go like all the way up here and come back down and then go down to -2 (drawing wave up to 2 and down to -2 on the graph of \( y = \sin(x) \)).

TEACHER: Does that mean the unit circle doesn’t hit 2 either?
STUDENT B: Um…
TEACHER: You said the unit circle did. Did you just remember incorrectly or it doesn’t hit 2 at all?
STUDENT B: I’m going to say that it doesn’t now. So I mean that’s some pretty good proof here that it doesn’t so I think that I was remembering incorrectly.

It is interesting that both students have been very successful in using the graphical representation. However, they were only using the graphs because they were asked to do so. Even though the students had access to a calculator, they did not choose to use this representation earlier. It could be the context of the question or it could also be the difficulty of generating an accurate representation in graphical form. The students may find it easier to use something like the unit circle because they have so many of the points memorized. For the graphical representation, the students understanding of it is mostly from an object perspective. That understanding might not be strong enough to produce a graph that can be utilized in the process perspective so they avoid using it altogether.
Just like Student A and Student B, Student C has no trouble using a graphical representation from the process perspective. The student easily uses the graph to find the solutions to both \( \sin(30^\circ) \) and \( \sin(20^\circ) \) and does so by utilizing the Cartesian Connection.

**STUDENT C:** You would go over 1, 2, since these are degrees. You would go 1, 2, 3, for \( 30^\circ \) and then find where it crosses at.

**TEACHER:** Good, Ok, so what’s the answer?

**STUDENT C:** .5. .5 y.

**TEACHER:** Ok, excellent. Now doing the exact same thing, can you find the sine of \( 20^\circ \)? Remember earlier you said there’s no way you could do it other than the calculator. Can you use the graph now to find the sine of \( 20^\circ \)?

**STUDENT C:** Yes, go over 2 and up 1, 2, 3, where it crosses the, uh, that line and it’s about .34. Like right there.

However, unlike Student B, Student C runs into some trouble with \( \sin(x) = -1 \). The student understands that the answer is not an angle measure and as a result believes that it may not be possible to use the graph to answer the question. As the student looks at the graph, the student is able to make a connection with the \( x \)-coordinate angles and \( y \)-coordinates to the unit circle.

**STUDENT C:** It’s not a degree. Wait a minute. Yes, you could because all you do is go to… I figured out how to use the unit circle! Ah ha! Because you can use it to find, um, like sine equals -1 because that’s what’s 1. I know that’s 0 for sure. 1, or is it the other way? Never mind, I won’t worry about that one, but I know you use all these points. If this is like -1, it’s 180 so for finding the sine of -1 you go to 180 and see where it’s at, or wait I’m getting somewhere. I’m collaborating.

**TEACHER:** It’s very good.

**STUDENT C:** I’m close. I feel I’m very close. You use the graph. You go to where -1 is on the unit circle. I know that for sure, but I don’t know which one is -1.

**TEACHER:** What do you mean which one is -1?

**STUDENT C:** Like I know this is 0, 90, 180, 270, 360. I don’t know which one of these is -1. I think it’s 180. I’m so sure it’s 180.
The problem the student appears to have at this moment is a failure to recall from memory the coordinates of the unit circle. He/she has made the connection that the coordinates for the angles on the unit circle should match the y-coordinate on the graph, but the student has not figured out if it is supposed to be the x-coordinate or the y-coordinate from the unit circle. The student continues to compare the y-values on the unit circle with the coordinates on the graph. The student draws a unit circle and only uses the sine value instead of the full coordinate for 0°, 90°, 180° and 270°. However, he/she has difficulty correctly recalling what the sine values are for those angles. The student is able to use the graphical representation to finally correctly create the unit circle representation. Once this is accomplished, Student C is able to make the connection between the unit circle and graph.

Now that the student has made the connection, Student C is not only able to say that the answer is 270° but also -90°. Without being prompted by the teacher, the student’s understanding of the representation is strong enough to realize there are multiple solutions to the original question.

Student D can also use the graph from a process perspective and easily finds an estimate for \( \sin(20°) \). Based on the understanding that the student has displayed in the previous questions this was not surprising. However, Student D does have some trouble with the second question. The student is able to realize that for the graphical representation that the question is asking for an angle measure that has a y-value of -1. The student gives the answer in radians and states it is \( 3\pi/4 \). It appears the student thinks that this radian measure is equal to 270° because the student is pointing to 270° on the
The student does not realize that two answers are possible even though the graphical representation is showing this. The teacher asks the student to plug both answers into the calculator for verification, but that still does not convince Student D. Finally, since Student D was not able to decide if $\sin(x)$ was a function, the teacher asks him/her if having two answers means $\sin(x)$ is not a function.

STUDENT D: There’s more than one answer when $\sin(x) = -1$
TEACHER: Does that mean it’s still a function? Was it a function first of all?
STUDENT D: Um, yes.
TEACHER: Ok, but can an answer have, can there be two answers and it still be a function?
STUDENT D: Yes, because every $x$ can have only one $y$, but um, but then because this is $-90^\circ$ and this is $270^\circ$ and they both equal $-1$, but that’s ok because they’re not the same $x$.

The student is finally representing $\sin(x)$ in a representation that is compatible with the student’s understanding of functions. The student did not use the vertical line test earlier and still does not. Instead the student relies on the definition to decide that it is a function. Because the student understands the inputs to be angles, the student realizes
that no input has two outputs. This also helps the student accept the fact that there are two solutions. However, unlike Student B and Student C, the student does not mention that there are more than the two solutions on the graph. It is possible that the student is unaware of the additional answers in this representation.

Student E is also easily able to answer both questions. This is not surprising based on the level of understanding demonstrated during most of the responses. The student is also able to state there are multiple answers to $\sin(x) = -1$ and they would all fit the pattern of $270° \pm 360°k$. Also, after seeing this representation, the student goes back to the previous question that was similar and modifies that answer. Seeing the graphical representation allowed the student to realize that there were multiple solutions. This was consistent with the research (Hirsch et al., 1991). He/she is also then able to connect it as being the same process that was carried out in an algebraic representation, but in that representation the student had failed to see that there were multiple answers.

Student F had a great deal of trouble reading the graph of $\sin(x)$ in the process perspective earlier when trying to connect it to the unit circle. So it could be expected that the student would have difficulty using a graphical representation to find $\sin(20°)$. The student showed an understanding that the graph was made up of individual points, but he/she was only able to see points like $(0°, 0)$, $(90°, 1)$, and $(180°, 0)$. The student was unable to recognize the points in between the maxima, minima, and x-intercepts. The student does understand that the solution to this equation lies with finding where $\sin(20°)$ is on the graph, but to find that point, the student approaches the graph from an object perspective instead of a process perspective.
TEACHER: Alright, so using that graph, can you estimate the sine of 20°?
STUDENT F: Um, I could.
TEACHER: Alright, what would you need?
STUDENT F: I mean, if you wanted to go and find like where 20° is on the graph.
TEACHER: Can you do that?
STUDENT F: It would be like around here.
TEACHER: Why?
STUDENT F: Because 90° is up here.
TEACHER: Ok. Do you know what your scale is on the graph? What each mark is?
STUDENT F: Um, the scale is 180°. I think.
TEACHER: So each mark is 180°?
STUDENT F: Well, from like here, each like hill thing is 180.

The student sees the sine graph as a “hill” and each hill is 180° long. Student F then uses the length of 180° to estimate where 20° would be to form an estimate. The student is unable to use the actual scale of the graph and is confused when the teacher suggests trying this approach. He/she was able to easily read the scale of the quadratic graph to estimate a point earlier in the interview, so it is either the fact that the scale is now in degrees or the fact that it a trigonometric graph that appears to be hindering the ability of the student to connect the scale to the his/her previous representation for graphs. When the teacher suggests the scale is 10°, Student F does not know how to check if it is reasonable.

TEACHER: Ok, but the actual graph not the… the graph paper itself, do you know what each mark’s increment is?
STUDENT F: No
TEACHER: How could you find that out?
STUDENT F: Um, I don’t know.
TEACHER: Ok, what if I told you that each mark on the x-axis was 10°? Does that seem reasonable?
STUDENT F: Sure
TEACHER: Because?
STUDENT F: Because I don’t have any other reason.
TEACHER: Ok, how many marks between 0 and 180?
STUDENT F: 0 and 180? Like here to here? There’s not technically any.
TEACHER: Each of these marks (point to grid marks on graph).
STUDENT F: Ok, I got you. 7 not including 0 and 90.
TEACHER: OK, so is it 10° then? If you count by 10’s does it work?
STUDENT F: Um, let’s see… 30, 40, 50, 60, 70, 80. Yeah.

It appears that the degree measure for a unit on the graph is what the student is having difficulty understanding. When given a way to verify that the scale is 10° the student is able to count using the scale so it does not appear the trouble is not using a graph, but connecting a meaning of a unit measure that is in degrees with an actual increment on a graph.

When the student does locate 20° on the x-axis, the student does not automatically use it to locate the point on the graph where the x-coordinate is 20°. Instead the student states that the answer to sin(20°) is probably 0.

TEACHER: Ok, so where’s 20° at?
STUDENT F: Right here
TEACHER: Ok, now use that to estimate the sine of 20°, or can you?
STUDENT F: Um, you mean the sine of 20°? Well, would it still be 0 because it’s close enough to the 0 coordinate?
TEACHER: Show me what coordinate you’re looking at.
STUDENT F: Right here (points to (20°, 0)).
TEACHER: Ok, is that on the graph?
STUDENT F: It is not on the graph.
TEACHER: Ok, so what would be?
STUDENT F: This would be on the graph (points to (30°, .5)).
TEACHER: Ok, so what’s the point there?
STUDENT F: Um (20°, .5).

When the student realizes that point is not on the graph, the student identifies a point, although accidentally marks 30° on the graph instead of 20°. Because of this mistake the student uses the graph to estimate the value to be .5. What is interesting is that at this
moment, the student in not connecting the graphical representation to any other representation. The student earlier used the unit circle and the calculator to determine \( \sin(30^\circ) \) was \( .5 \). Now in a different representation, the student is looking at \( \sin(20^\circ) \) and saying that equals \( .5 \). The student either does not have a strong enough connection between the representations, or does not see the relevance of the previous answer to this one. When asked to compare it to the answer the student got for \( \sin(20^\circ) \) early, the student knows that it is different and concludes that \( .5 \) is only an estimate and not a very accurate one. However, he/she does not state if the inaccuracy is caused by the student misreading the point or the graph not being precise enough to make accurate estimates.

The second question does not give Student F nearly as much difficulty because the student has demonstrated the ability to use the process perspective for intercepts, maxima, and minima and connect those points to the unit circle. The student knows the answer is \( 270^\circ \) without using the graph, because of his/her understanding of the unit circle. By connecting the unit circle to the graph, the student is able to locate the point \((270^\circ, -1)\) on that graph. However, because he/she is using the unit circle representation and connecting it to the graph to find the solution, Student F does not mention there is a second answer on the graph or there are multiple answers to the question. The graphical representation was only used to verify the information the student obtained from the unit circle so the student is not actually using the graphical representation to answer the question.
Question 14: Look at the following graph. Is this graph a function? How can you tell?

Results

Almost all of the students used the exact same process for this graph as they did earlier. Student A, Student B, Student E, and Student F immediately said it was a function because it passed the vertical line test. Student D said it was because every x has only one y. This was consistent with the approach the Student D used earlier. The student never mentioned the vertical line test, but based a test for functions on the actual definition. Student C initial response was inconsistent with his/her earlier response but when questioned further about it, changed it.

Evidence

Student C’s response was very interesting. Student C had early stated that any graph that crosses the x-axis is a function and used this definition not only with quadratic graphs, but also the graph for sin(x). According to the student’s definition, this graph should be a function because it does cross the x-axis. However, Student C states the graph is not a function. The reasoning though has nothing to do with the graph crossing the x-axis. The student is using the object perspective again and saying that because more of the graph is above the x-axis than below, the graph is not a function.

STUDENT C: It is not because the bottom of the graph is at -1 like the wave but the top is at 7.
TEACHER: And that means it’s not a function because?
STUDENT C: Because it’s not like fully, um, like the bottom should be either at -7 or the top should be at one.
TEACHER: You told me earlier though that, um, if the graph crosses the x-axis then it’s a function. That was your definition.
STUDENT C: Oh, hmm. Well, then it is a function because it does cross the x-axis and it does use, uh, more than like all the points on both sides.
TEACHER: What do you mean uses all the points on both sides?

152
STUDENT C: If goes on both sides instead of just one side. Like two, where it had no… one domain was not used. It’s just not a very good graph.

The student’s graphical representation of a trigonometric function is one that is symmetrical with respect to the x-axis. As a result, this graph does not fit that representation and he/she says it is not a function. When the teacher reminds the student of the definition the student used early, the student now sees the question in a different context and states that it is. The student saw the graph as not being a function not based on the student’s understanding of functions, but on the graph not fitting the student’s representation for trigonometric graphs.

*Question 15: Can you tell if the graph is of a sin(x) or cos(x) function? Can it be both? How do you know or what information do you need to make a decision? How would you write the equation for this graph?*

*Results*

All of the students were able to utilize the object perspective to make connections between the graph and the equation. However, only Student E was able to correctly connect the period of the graph with the period of the equation. The same student was the only one that elected to check the final answer and as a result changed perspectives to do so. As mentioned earlier, it was expected, based on how the material was taught, that most students would use the object perspective but some students still had difficulty connecting the graph to their representations for sine and cosine as objects. The connections between the graph and equation were also mostly procedural and based off
of memorized procedures and not necessarily on any conceptual understanding between the two representations.

Evidence

Student A uses the graph from an object standpoint and understands that the amplitude of the graph has been changed. The student also knows that to find the equation it is important to know other features like the period and any transformations that have occurred with the parent graph. This also implies that he/she is connecting this graph to the parent graphs for sin(x) and cos(x) in an effort to compare those graphs as objects.

TEACHER: To determine sine or cosine what do you have to know from the graph? Can you figure it out?
STUDENT A: Well, if you’re given parts of it like the period and the amp and the starting point or ending point.
TEACHER: Ok, you said the period and amplitude. Can you find the period and amplitude from the graph?
STUDENT A: It would be hard to tell because it’s stretched. You also need like (pause) like in the function you’d be able to tell, it will tell you, like if it shifts right or left, if it shifts up or down, and like if it expands, like if there’s a number before the equation. I don’t think you can tell from this.

The student also shows an understanding of how the graph connects to the equation from an object perspective. However, he/she is unable to distinguish if the pattern of the graph fits the sine pattern or cosine pattern the student uses for a representation. It is unclear from the response if Student A understands that a vertical translation has occurred. It appears that this may be the reason why the student is having difficulty determining if the pattern fits sin(x) or cos(x). When the teacher begins to give a starting point for the wave, it is enough for the student to incorporate the graph into his/her representations of
the parent graphs. By moving the starting point of the wave, the teacher is able to allow
the student to see the graph as both a sine wave and a cosine wave. However, the student
is unable to use this information to connect with anything else in the equation.

Student B also uses the object perspective to analyze the graph and connects it
back to the parent graph representations he/she has for sin(x) and cos(x). The student
realizes that it is a sine graph translated up because of where the maxima and minima of
the graph occur.

STUDENT B: (Talking indistinctly) It is, (pause). That is a sine graph. Yes, that
is a sine graph because the sine graph you can move the origin up or down on the
y, on the y-axis. Um, the cosine graph too, but the cosine graph, like the peak is
on the y-axis, er, yeah, the peak or the lowest point. Like this or that part (points
to graph) would be here (y-axis) but it’s not. The middle point in between, like,
the line segment is on the origin.

The student realizes that the “peak” to the graph is not on the y-axis so it cannot be a
cosine graph. The student also is able to make the same connections between the graph
and equation in the object perspective that Student A did. However, while Student B is
looking at the features of the graph, he/she becomes confused when the scale for the
graph does not match up with the students representation for sin(x).

STUDENT B: You need to know what kind of graph it is. It’s a sine graph. So
sine, and you need to know the amplitude.
TEACHER: Do you know the amplitude of this graph?
STUDENT B: Which is... um, (counting), 8. Um, (pause), this is confusing.
TEACHER: What is confusing?
STUDENT B: Um, because the, the valleys and peaks are not on the, like every
90°, they’re not on, like this one’s not on 90°. This one’s not on 180°. This one’s
not on 270°. So it’s like shifted over so now I’m thinking it could be a cosine
graph because maybe... I don’t know. I can’t exactly remember if you can like
shift a graph like left or right. Other graphs you can so I think you can.
To clear up this confusion, the student remembers that graphs can be shifted left and right as well up and down. This also causes the student to realize that the graph could be a cosine graph too. Even though he/she is using the height of the entire graph as the amplitude, the student shows an understanding of this feature of the graph from an object perspective. The student states that if it is a cosine graph, then the graph has been translated to the right. He/she is also able to realize that the period has been changed, although the student does not use the word period.

STUDENT B: You need to know the, this number, (motioning horizontally with hands). I don’t know what that’s called because I forget but the… because the amplitude is how far and how low it goes and there’s the something number which is, like, how squished or how wide it gets. Um and how you figure that out or how you get that I do not know. Um…

TEACHER: So, how can you tell how squished or how wide? How do you know? Can you tell me that from the graph or do you have to look at the equation to figure it out?

STUDENT B: You can tell from the graph because it is squished some. It’s condensed because the peaks and valleys are not on like 90°, 180°, 270 and 360. Because if you opened up a little bit, like desquished it, this one would move over to 90, this one would move over to 180, and this one to 270, and this one to 360. Same thing for over here, so… (pause). Yeah, um, if it’s one that’s like normal, every peak and valley like on a 90, like it’s on a 90°. Um, so this looks like its 1/2 because the valley, like peak or valley, is like halfway between 0 and 90. So I think, I’m thinking like .5 maybe because that number, what that number is called I don’t remember, is .5.

The student has also made a connection with this properties and the equation for the graph. However, Student B does not know where this number goes in the equation. The student does know that the new period is 1/2 the length of the old one.

STUDENT B: It’s been shifted up, so there has to be like a plus in the equations somewhere. And then I don’t think it’s been shifted left or right anymore because the… I mean, the .5 number would make it fit correctly and then, oh, there’s also like, inverted. Normally, it goes up first, right? Yeah, normally it goes like this but it’s not so it has to be a negative in there somewhere, maybe like in front of, is
where the negative would go, but there is a negative in there somewhere. So there’s a negative eight, .5, plus three and I think that it’s a sine.

The student also connects a translation up with adding a number after the equation, a negative sign in front of the equation with a reflection, and a constant in front of the equation with amplitude of the graph. However, he/she does not make an attempt to check the solution by switching to a process perspective and checking certain points on the graph with the equation the student has created. It is not clear if Student B is choosing not do this because the student believes it is not necessary or because the student is unable to change perspectives in this context. Nevertheless, the student was able to make several connections between the graph and equation.

Earlier, Student C used a learned procedure from instruction to explain how to create a graph for sin(x). The student discussed the graph from objet perspective for that response as well. He/she uses the same approach when discussing the new graph. The student has a memorized pattern for a sine wave and a cosine wave and is able to see that this graph could fit either pattern so the graph could be both a sine graph and a cosine graph. Student A’s representations for sin(x) and cos(x) used the x-axis as a reference and this caused trouble with this graph because the x-axis is not in the center of this graph. Student C’s representation does not appear to include the x-axis for the pattern because vertical translation does not confuse Student C. It is interesting that the student is able to do this because the student earlier stated the graph is not a function because more of the graph was above the axis than below it. The change in context in this
question has caused the student to look at the graph as a set pattern and not as a function so the student is not using the x-axis in the representation at the moment.

When asked to determine what information is needed to write the equation, the student cites the same procedure that was used to graph sin(x). He/she is able to connect most of these procedural steps with the equation, again in the object perspective.

STUDENT C: You need to know the amplitude, starting point, ending point, scale. I know it’s P.S. but I can’t remember what it stands for. Point, no, point-slope? No, I know it’s P.S. but I can’t think of what it’s called.

TEACHER: What if I told you P.S. stands for phase shift. What does that mean?
STUDENT C: Phase Shift. Uh, whether it goes up or down. The middle moves up or down.

TEACHER: Anything else?
STUDENT C: So that’s phase shift, starting point, ending point. No, I think that’s it. Oh, if there’s a negative in front of the equation or not.

The student has listed several features of the graph that are tied to features of the equation. Student C then begins to connect the two representations together.

TEACHER: What do you know about the equation?
STUDENT C: It probably starts out as negative 1, 2, 3, 4. 1, 2, 3, 4. 4, um, parentheses, x. I know there’s something in the middle to turn it into something and then out here would be 3, plus 3. Ah (erases 4 and moves it outside parentheses).

TEACHER: Alright so why did you move the 4?
STUDENT C: Because like 4 is how far it’s stretching. Wait, no, the amplitude is in front. There, that’s correct. So it’s -4x + 3.

What is interesting is that the student does not include the sine function in the equation.

Although the student has correctly identified how the reflection, amplitude, and vertical translation affect the equation of the graph, the student makes a linear function with the same features. The student still does not seem to notice, even when the teacher tries to point it out. Instead of writing sin( ) in the equation, Student C sets the entire equation
equal to \( \sin(x) \) so he/she now has \( \sin(x) = -4x + 3 \) on the paper. The student’s object perspective representation of an equation does not differentiate based on the function. The most likely explanation is the emphasis the instruction has put on equations of the form \( y = \pm A(Bx \pm C) \pm D \). This is the representation that students saw for discussing transformations. The student understands how \( A \) and \( D \) connect to the graph, but the student’s representation does not include a way to express the individual sine or cosine function. He/she also is not able to recognize that the period has been changed and does not alter the equation any further.

Student D is able to make the same connections between the graph and equation from an object perspective that the other students were able to make. He/she understands from the representations for the parent graphs that it could be a sine or cosine graph. Student D is able to create the most detailed equation yet. The only feature that Student D is unable to accurately state is the change in period. Just like Student B, Student D understands that a change has taken place, and that the period is now \( 1/2 \) of the original parent graph, but is unclear how to connect that to the equation.

TEACHER: Ok, so let’s say I wanted an equation for sine, so it’s a sine equation.
STUDENT D: So then I would start with \( y = -\sin(x) \). I know it’s negative because sine goes middle, up, middle, down, middle.
TEACHER: Ok
STUDENT D: And it is flipped over the x-axis.
TEACHER: Ok
STUDENT D: And moved up 3, so it would be \( y = -3 \), wait, yeah, no that’s not right, \( \sin(x) \). The amplitude is 4 so it would be \( -4\sin(x) \). Let’s see if I recall, so \( b \), the period, would be \( \pi/2 \).
TEACHER: Why is it \( \pi/2 \)?
STUDENT D: Er, (pause). Um, it would be \( 180^\circ \), which would be \( \pi \).
TEACHER: Ok, so why is the period \( 180^\circ \)?
STUDENT D: Because if starts here and ends on the graph at \( 180^\circ \).
TEACHER: Ok
STUDENT D: Um, so I forget how to write that in. Maybe, I think I do y. I did this before. No, + 1/2? Maybe. I forget how to do that part, but then I know it’s moved up 3 so we add 3.

Unlike the previous students, there was very little hesitation or need to think about many of the connections. This implies that Student D has a stronger memory of the material possibly because he/she has stronger conceptual knowledge and stronger connections between representations.

Student E is also able to identify that the graph looks like a reflected sine graph so the student is using the objective perspective and has representations for the parent graphs of sine and cosine much like the other students do. The student also states that it could be a cosine function but there would be a horizontal translation. He/she has no trouble with recognizing both of these even with the orientation of the center of the graph above the x-axis. The student understands that knowing the phase shift (horizontal translation) would be the information needed to correctly identify the graph as a sine or cosine function. This implies that Student E knows that the defining features of the graphs would all be identical other than the phase shift. No other student was able to voice this distinction, although that does not mean others did not realize it. The student is easily able to identify both a reflection and amplitude in the graphical representation by using the object perspective and correctly connects those features to the equation.

STUDENT E: So, alright, this would be my amplitude here. If it was, I guess you don’t (talking indistinctly). We’re just going to go with sine for right now. Um, this would be my amplitude right here, right? So, -1, 2, 3, 4, so – 4, right?
TEACHER: Why use the negative again?
STUDENT E: Well, because like a normal sine function is like this, like a positive one and it’s like x-axis reflection, that’s what I’m looking for.
TEACHER: Ok, excellent
STUDENT E: Alright, so -4sin and then, um, I’m going to use sin(x), right? Because this is my no phase shift one and then plus that moves up 3, maybe. I don’t know. Wait, can I check?

However, unlike the other students, Student E wants to check the equation and changes perspectives to do so. The student types the equation into the graphing calculator to check its graph. Originally he/she only looks at the graph as an object on the calculator but realizes that because of the viewing window the calculator may not be precise enough. Instead the student checks specific points that the student knows should be common to both graphs.

TEACHER: So what are you doing now?
STUDENT E: Um, I’m just putting in the thing into the little y= thing.
TEACHER: Ok
STUDENT E: And then I changed my window to zoom trig
TEACHER: Ok
STUDENT E: And I don’t know. I don’t think it looks right.
TEACHER: How can you check?
STUDENT E: Because, like, alright, so at what is this? Like 45°? Probably ish. Um, so if I put in… Um, let’s just go with 90.
TEACHER: What did you get?
STUDENT E: I got -1
TEACHER: Ok, is that good or bad?
STUDENT E: Bad
TEACHER: Because?
STUDENT E: Because it’s definitely not negative. It’s like 3, so that’s telling me I did something wrong.

The graph on the calculator has a negative y-value when x is 90° and the graph on the page has the coordinates (90°, 3) so that tells the student that the graphs are not identical. By looking at individual points though, instead of the features of both graphs, the student is utilizing the process perspective. However, this mistake in the equation causes the
student to analyze the graph from the object perspective again and then realize that that he/she has not accounted for a change in period in the algebraic representation.

The student is unable to recall the rule that connects the change in period in the graph with a change in period in the equation but does know what part of the equation is affected. The student then uses the calculator to guess and check. Because he/she understands that the period is half of the original period for sine, the student tries the equation \( y = -4\sin(x/2) + 3 \). Now that the student is checking the answer, the student switches back to the process perspective and by checking only one point realizes that the equation is incorrect.

STUDENT E: Um, oh, it’s the wrong period. Um, because this one, the period is from, it stops over here, 180 instead of all the way to 360, so it’s like (pause). Is it like sine of x/2 then? I forgot how to do period. Nope, that’s not it.
TEACHER: why not?
STUDENT E: Because for this I get .17 and that still not right.
TEACHER: OK
STUDENT E: So I went the wrong way. 2x. Ok that’s better. That gave me what I wanted. Alright.

The student understands because the points do not match, the equation is still not accurate. However, the student has made a connection previously with the period of the graph and period of an equation because he/she knows only to modify one part of the equation and simply changes the process form multiplication to division. Now the student checks three points and that is enough to convince the student that \( y = -4\sin(2x) + 3 \) is the correct equation. The fact that the student checked more than one points seems to show that Student E has enough conceptual knowledge to know that two different trigonometric equations can contain the same coordinates for some points on the graph,
but not all. Having three that match though, gives him/her enough confidence in the final answer, but the student does state that it could still be a coincidence.

Student E is then asked to try and find the equation of the graph in terms of cosine. Again, the student knows that the only feature in the graph that will change is the phase shift. This means that the only alteration he/she needs to make to the equation is to add or subtract something after the 2x to get the graph to move left or right, respectively. The student has a very strong understanding of the connection between the graph and the equation and the graph of sine and cosine. The student is unable to remember the procedure for producing a phase shift in the algebraic form, so the student switches to the process perspective to use the calculator to guess and check. Although it takes some time and several hints from the teacher, Student E is able to produce a cosine equation for the graph as well.

Student F is able to make some connections between the graph and the equation utilizing the object perspective, but they do not appear to be as strong as the other students. Student F does not think the equation can be both sine and cosine. This would imply that the student’s representations are mostly independent and the student may not realize what features the two graphs have in common. The student sees both graphs as being made of different patterns and does not realize that it is the same pattern for the wave, just starting at a different phase.

The student understands how the amplitude of the graph connects with the equation but not how a vertical translation connects with the equation. It appears that the
students representations of sine and cosine as objects are not detailed enough to see any of the other important features of this graph.

TEACHER: Ok, so I want you to write an equation for sine that describes the function.
STUDENT F: Ok
TEACHER: So it’s a sine graph.
STUDENT F: Sine graph? Ok, um, (pause)
TEACHER: So you said 6x, why is it 6x or 6sin, sorry.
STUDENT F: Um, because it’s a pretty big amplitude and 6 seemed like a good number.
TEACHER: Ok, and the +2?
STUDENT F: +2? Um, well, it’s small in between down here so I forget where the +2 comes in.
TEACHER: Ok, anything else?
STUDENT F: No

The student does realize that the graph is different than the parent graph based on its orientation to the x-axis, but has resorted to guessing how this affects the equation.

When the student is asked to write the equation in the form of cosine, the student keeps all the information the same but changes the sine function to the cosine function in the equation.

TEACHER: How about if I said it was an equation for cosine?
STUDENT F: I mean, don’t you figure out how to graph sine and cosine using like the same information? Like you could use the same formula.
TEACHER: Ok, so what would be different?
STUDENT F: Um, the pattern of how you draw the waves.

Student F seems to understand that information connecting it to the equation is identical, but that sine and cosine are two separate entities that are not connected. The student sees the procedures for creating the graphs as identical except for the pattern used for the
graph. This adds further evidence as why he/she did not think the graph could be both sine and cosine.

Second Interview Analysis

The second interview only involved three students, Student B, Student C, and Student D. The questions were created to give the students more opportunities to use the graphical representation and to see if the students’ representation for sine as a function includes an understanding of the inverse sine operation.

Questions 1: Graph \( y = \sin(x) \). Explain how you created your graph. Using your graph find the following: a) \( y = \sin(80^\circ) \), b) \( y = \sin(-150^\circ) \), c) \( .75 = \sin(x) \), d) \(-1/4 = \sin(x)\)

Results

It appears that all of the students’ knowledge of the graph as an object is strong enough to produce an accurate graph. However, two of the students also connected it to coordinates on the unit circle. In addition to seeing the graph for \( \sin(x) \) as an object, the students were able to view the graph as a set of points when the context called for it. The students understood that the coordinates on the graph were linked to the equation \( y = \sin(x) \) for a given \( x \)-values or \( y \)-values. As a result, the students used the Cartesian Connection to answer the questions in this context but did not use this connection for others. It also appears that the graph’s connection to the unit circle is limited to certain angles because when asked about an angle of \(-150^\circ\), no student connected that angle with an angle measures on the unit circle.
Evidence

Student B is able to produce an accurate representation for sin(x) and it appears that the student utilizes the object perspective to accomplish this.

STUDENT B: Sine starts… here (points to the origin). It’s between 1 and -1. Right there and there. It starts, um, (pause). It starts at 0. Then it goes up at 90. (draws sine wave)
TEACHER: Alright, those points you plotted, how did you know to plot those there?
STUDENT B: Because that’s just the standard position and stuff.

Even though the student states, “it starts at 0.” and “it goes up at 90.” he/she is not using these as coordinates to form points but understands the sine graph as a pattern that starts there and moves every 90°. There is no evidence to suggest that Student B uses additional points between 0° and 90° or that the angles are connected with specific y-values. The student appears to be recalling a memorized object. However, it appears that the student does change perspectives to answer the next part of the question and uses the process perspective to adjust the original graph.

TEACHER: Ok, very good. So using that graph you just made, uh, can you find those four values? Y = sin(80°) first.
STUDENT B: (long pause) It would be, this graph is not that great, but, it would be just below 1.
TEACHER: Ok
STUDENT B: It would be something like that (adjusting curve of graph)
TEACHER: Ok, so estimate that.
STUDENT B: Let’s say like .95

By looking for the y-value at 80°, the student realizes that the graph he/she has drawn is giving a y-value that is greater than 1, and according to the graph as the student understands it, this is incorrect. The student has incorrectly drawn a curve that had a
maximum before (90°, 1) so the student redraws the curve between 0° and 90° to make the maximum occur at (90°, 1). Although it appeared that the student simply recalled a memorized object, when the student used the graph to find specific points, it was necessary to change to the process perspective. This perspective allowed the student to see the need to modify the graph, now seen as a collection of linked points, based on the student understanding. The modified graph, just like the original graph, was based on the fact that the graph, as an object, stays between 1 and -1, not because the range of sin(x) is -1 to 1, so the student is utilizing both the process perspective and the object perspective at this moment.

Student B is not bothered by a negative angle measure and quickly finds the value of sin(-150°) to be -.45. Because the answer is -.45 and not -1/2 or -.5 this indicates that he/she is not connecting the graph to the unit circle for this answer. Either the student does not need to make the connection in order to get an answer or he/she does not have a representation of negative angles with the unit circle. Since the student showed a tendency to only use integer values when discussing coordinates on the quadratic equation graph during the first interview, it is possible that the student still has not connected the graph to the equation for non-integer coordinates. In the first interview Student B stated that, “you’ll have to do a bunch of math” to find other points on the quadratic graph but makes no attempt to do that here for the graph of sin(x). It appears that any connection he/she has between the trigonometric graph and its equation is not as strong as the connection between a quadratic graph and its equation. It is also not
apparent if the student realizes that \(-150^\circ\) is a point that can be found using the unit circle representation.

Student B also shows no difficulty using the graph to find an angle measure because he/she is easily able to estimate answers for \(0.75 = \sin(x)\) and \(-1/4 = \sin(x)\). However, when answering these questions, the student makes no mention of multiple answers as he/she did for \(\sin(x) = -1\) during the first interview.

STUDENT B: Yeah, yeah. .75, so that would be like right here (marks .75 on the y-axis). So that would be like there ish. So around like 60, 60° about. Ok, -1/4 (long pause). So like -20° ish.

This is interesting because in the previous interview the student was able to point out that the answer to \(\sin(x) = -1\) was \(270^\circ\) and \(-90^\circ\) and there were others if the graph was extended. He/she is unable to make the same conclusion even though there are 3 ordered pairs for both equations where the y-value is the one the student is looking for on the graph. It could be the context of the tasks because Student B found one answer for the first two in this problem set and then could assume that one answer was sufficient for the next two. The student may have just stopped looking at the graph once an answer was obtained and the thought process for obtaining answers stopped too. As a result, the student failed to notice the other points on the graph that have the same y-value. The fact that these answers do not occur at a maxima, minima, or x-intercept of the graph could also affect why the student does not see multiple answers.

It is also interesting that Student B gave a positive angle measure for the positive ratio and a negative angle for the negative ratio. Again, since it appeared the student was only looking for one answer, it appears that he/she found the solution that was closest to
the y-axis and then stopped. Nevertheless, because the student was linking y-value ratios with x-value angles, the student is using the graph from a process perspective, and because he/she is making a connection between the linked coordinates on the graph and solutions to the equations, the student is using the Cartesian Connection.

To create the graph for sin(x), Student C uses a connection with the unit circle. In the previous interview, the student discussed the graph as an object with features like amplitude, but now Student C plots points on the graph that come from the student’s unit circle representation. Because he/she plots specific ordered pairs, the student now sees the graph as a collection of points and is therefore using the process perspective. These points are connected to the unit circle though, not the equation y = sin(x).

STUDENT C: Since it’s just sine of x, there’s nothing to, uh, change it or disform it or whatever, so you would just start on (0, 0), (1, 90), 180, (-1, 270), and (0, 360), is one wave. Well, here (moving last point)
TEACHER: And how did you know to plot those points?
STUDENT C: Um, Because of the unit circle, that’s the points it has.
TEACHER: Ok
STUDENT C: The four points are like that, there.
TEACHER: Ok, very good. And is that the entire graph or…
STUDENT C: Oh, uh, you can continue, going backwards you just put what they are so, (-1, 90), (0, 180), (1, 270) and (0, 360). (actually plotting the points on the negative side of the x-axis even though stating positive angle measures)

The student still recognizes the connection between the graph and equation in the object perspective when the student states that “there’s nothing to… change it of disform it,” meaning the student sees that the graph has no transformations. The student is also utilizing the process perspective simultaneously to find distinct points on the graph but at a level of limited understanding. It is not apparent from the explanation if the student understands how the x-coordinates and y-coordinate are linked on the graph only that
they are linked because of the unit circle. The ordered pairs also are part of a memorized pattern. Because of this, although the x-coordinates and y-coordinates are switched in the explanation above, Student C is able to create an accurate graph. The switch is a mistake that does not affect the student’s understanding of the representations, but the student does not appear to be using the Cartesian Connection because there is nothing in the response to support the idea that he/she knows that (0, 0) is on the graph because the sin(0°) = 0. The student is simply using a memorized set of linked x-values and y-values that has meaning because the student connects them with the unit circle.

Further support that Student C can see the graph from the process perspective is demonstrated by the fact that the student can find additional points on the graph aside from his/her memorized set. The student finds both sin(80°) and sin(-150°) on the graph without much difficulty. This demonstrates he/she can find a specific ordered pair on the graph that is a solution to a given equation. Just like Student B, it appears the student is only using the graph and not the unit circle because the student answers that sin(-150°) is -.48. For this estimated answer, the student is not connecting it to the unit circle which would produce an exact value answer. Just like Student B, Student C is able to see the graph of sin(x) from both perspectives and change perspectives when necessary.

It is interesting that the student did not use the unit circle for sin(-150°). The student has been able to use the unit circle to find values like sin(30°). However, the student relied on memorized coordinates from the unit circle so it is possible the student does not have the point (-150°, -1/2) memorized. Student C also demonstrates how to plot the coordinates for the graph of sin(x) in the negative direction to produce the
negative angle measure for the graph so it may also be a case of not understanding the negative angles measures’ connection to the unit circle. The student’s use of memorized values on the unit circle appear to be for angles like 0°, 90°, 180°, and 270° (axes coordinates) and their coterminal negative angle measures. For other questions in the first interview, the student was able to identify 30°, 45°, and 60° (first quadrant coordinates), but was never asked to offer the coordinates for an angle like 150° or -150°. This could mean the student is not able to recall an angle in another quadrant like -150° from memory without be specifically told to do so.

In the previous interview, Student C had some difficulty using the graph to answer \( \sin(x) = -1 \). However, while using the graphical representation to answer that question, the student found a connection between the unit circle and the graph. This appears to be the connection the student used to create the graph for this response. It appears though that he/she is still unable to use the graphical representation to answer the next two questions because the student uses the calculator to find the inverse instead of using the graph to do so. The student does not understand that it is possible to find an x-value for a given y-value by only using the graph.

STUDENT C: So, that would be… Try to get y by itself, so do we inverse or, inverse sine of .75 so we want x, no, wait. Yeah, equals sine inverse 3 over 4. (picks up calculator) And it equals 48.5, 48.6. So y equals sine of 48.6°. Let’s see, is that in here? (looking at graph). There’s 40, so you go up to about 48.6. We’ll say it’s right there. Then go up that’s probably about .5 or 6.1, .61.

The student does have an understanding of what the questions are asking, but has no way to represent the inverse operation in a graphical representation and therefore switches to an algebraic representation that utilizes the calculator. It appears the context of the
question has also led to some more confusion for the student. Once he/she gets an angle measure on the calculator, the student believes that since the question was asking to use the graph, it is necessary to find the y-value on the graph for that angle. In the previous interview, the student correctly used the inverse function on the calculator to find the correct answer to an equation and stopped there. The change in context for this question has led to a misunderstanding that the graph has to be used to produce an answer. The student believes that by taking the inverse of both sides in the equation for \( .75 = \sin(x) \), the equation is turned into \( y = \sin(48.6^\circ) \) and the student understands this to mean finding the y-value on the graph for that angle. As a result, in this context there are not multiple solutions to be found either. The student then repeats the exact same process for the second equation and gets an answer in the same way.

Asking the students to make the graph has not forced the students to change to the process perspective and connect the graph to the equation. It is possible to make a connection with the graph as an object and the unit circle as a process. Student D uses this connection to begin to produce the graph for \( y = \sin(x) \). Student D understands the unit circle as a process that links an angle with a y-value, not as a collection of memorized points. Even though the student has incorrectly recalled a coordinate from the unit circle, when the teacher questions Student D, the student is able to correct the mistake because he/she understands this relationship.

STUDENT D: So sine starts up because sine equals x. At least that’s what I remember from last time. Um, sine is x.
TEACHER: Does sine equal x?
STUDENT D: No! Sine is y, yeah, sine is y. So then at 0° y equals 0. So it starts there and goes up, middle, down, and middle. \( Y = \sin(x) \).
To produce the rest of the graph, the student uses a memorized pattern as an object, much like Student B and Student C did. Because the pattern is seen as an object, he/she incorrectly draws a graph that has a period of $180^\circ$. When the teacher questions the student to see if he/she is able to see this mistake, Student D does not, which means the student does not see the mistake with the period from an object perspective.

TEACHER: Ok, what is the period of that graph?
STUDENT D: Um, oh, well that would be $\pi/8$, right? No, $\pi/4$.
TEACHER: So the period is $\pi/4$?
STUDENT D: No, it would be $\pi$. This would $\pi$ and this would be $2\pi$. Since it goes to 180, it would be $\pi$.
TEACHER: Ok, so what is the sine of $90^\circ$?
STUDENT D: Sine of $90^\circ$ would be… Sine equals y so that’s supposed to be 1. (erasing graph). So then it should be here at one. Oh my God that was right, because sine equals y. So then it would be… You said the sine of $90^\circ$?
TEACHER: What’s the coordinate at $0^\circ$?
STUDENT D: Right here? (student referring to graph)
TEACHER: On the unit circle
STUDENT D: (1, 0)
TEACHER: So what’s the coordinate at $90^\circ$? (referring to the unit circle)
STUDENT D: Are you asking me on… (student looking at the graph)
TEACHER: No, you have (1, 0) and (1, 0). (on unit circle)
STUDENT D: Oh, (0, 1), no, (0, 1), and (1, 0) (corrects unit circle).

However, because the student has connected the graph with points from the unit circle as a process, the student is able to use the unit circle to eventually correct the graph. The student is showing the ability to change perspectives. The problem is that the student is incorrectly recalling the point for $90^\circ$ from memory. To correct the problem it does not appear that the student is using the fact the $\cos(90^\circ) = 0$ and $\sin(90^\circ) = 1$ to get the point (0, 1). Instead, it appears the student realizes that based on the location of the ordered pair in the coordinate plane for the unit circle representation he/she drew, the point is (0, 1). The student knows the horizontal distance in the coordinate plane is the x-value and
the vertical distance is the y-value. Therefore, instead of using the unit circle as a memorized object, he/she is using the unit circle in the coordinate plane as process to figure out the point on the unit circle is (0, 1). Therefore sin(90°) on the unit circle and the graph of the sin(x) should both have a y-coordinate of 1. The student is now able to correctly label the unit circle’s 4 axes points and use these to generate an accurate graph for sin(x) with a period of 360°.

To find the sin(80°) the student is able to see the graph as a series of x-coordinates paired with y-coordinates which means he/she is using the process perspective with the graph. Because the student is offering the coordinate as a solution to the equations y = sin(80°), the student is utilizing the Cartesian Connection.

STUDENT D: Well, sine of 80°. So it would be a little less than 1.
TEACHER: Ok, so give me an estimate.
STUDENT D: 9.5, er, .95

The student knows that the point (90°, 1) is on the graph and the y-coordinate for 80° is to the left of that point, so the y-value would have to be less than 1 based on where the coordinate would fall on the graph. The student is able to generate an estimate based on this understanding and also point out the coordinates on the sine curve where the student believes (80°, .95) lies. The student also generates an estimate for -150° to be .6. The student is looking at the point (-150°, -.6) but says .6. A few moments later, the student realizes that it should be a negative coordinate because it is below the x-axis and corrects it to be -.6.

Again, it is interesting that Student D did not connect this point with the unit circle. It is most likely due to the context of the question that he/she does not use the unit circle.
circle to find the answer. The student was one of the few that gave estimates for coordinates on the quadratic graph when it was possible to find a more accurate answer with the equation. Just like the question on the quadratic graph, Student D uses the graph to give an estimate because it is what the question asked and there is no requirement to use the unit circle or a representation that could give an exact answer.

For the next two questions the student realizes that the inverse sine operation is needed to get the answer, and he/she is able to represent this operation by starting at the y-value on the graph and finding an x-value angle.

STUDENT D: And then, .75 = sin(x). So wouldn’t I have to do negative sine to get that? So .75, I would look... I don’t understand if it means like... Uh 65. Next -1/4. Um, 15, 30°
TEACHER: Pardon?
STUDENT D: 30°, -30°, sorry.

The student says “negative sine” to mean the inverse of sine and understands that both answers are angles. He/she also notices that there are multiple places on the graph that have the desired y-value. This creates some confusion for the student.

STUDENT D: The only thing that confuses me is for these, that, I’m looking for it on this axis now I think.
TEACHER: Ok
STUDENT D: But I could find that in like numerous places
TEACHER: So what does that mean?
STUDENT D: It means there are multiple solutions
TEACHER: Ok, so what would the other ones be?
STUDENT D: Um, what did I say for c the first time?
TEACHER: Um, 65°
STUDENT D: Well it could be -250, and then it could go on off the graph. And like for this one it could also be 190.
TEACHER: Ok
STUDENT D: And also negative…
TEACHER: You said -30
STUDENT D: Yeah, I did. Oh then also -170, 160.
Using the graphical representation, Student D appears to come to the understanding that there can be multiple solutions. The graph also appeared to convince the student in the previous interview as well. He/she then shows an understanding that there are more answers if the graph is extended. It appears for this part, the student is back to using the object perspective. Knowing the graph is an object that follows a set pattern, the student knows it will reach the desired y-value again. Because the student is only looking at the graph as an object, the student does not give additional coordinates. However, this does not mean the student would be unable to if asked.

**Question 2:** Which of the equations best describes the graph? Explain how you know.

a) $y = 2\sin(x) - 1$, b) $y = \sin(2x - 1)$, c) $y = -2\sin(x) - 1$, d) $y = -\sin(2x) - 1$

**Question 3:** How could you check your answer to number 2? Are there multiple ways to check?

**Results**

It was interesting that even though each student earlier demonstrated the ability to see the graph as a collection of points, no student checked points on the graph with the equation to check an answer. Student C and Student D relied on the features of the graph and equation as objects to determine the correct answer. Student B graphed all four equations on the calculator to see which graph matched the one on the page. While the students appeared to have a strong connection with the graph and equation as objects, there does not appear to be as strong of a connection between the graph and equation as a
process. The students demonstrated an understanding that a specific x-value is linked to a specific y-value in the previous question, but the context of the previous questions gave them a specific x-value or y-value to start with. Therefore, for a question like this one, the Cartesian Connection is not utilized possibly because the students are not able to narrow their focus to a specific coordinate.

Evidence

Student B chooses to guess and check by graphing each equation on the calculator and comparing the graphs to the graph on the paper. To determine if the graph on the calculator matches the graph on the paper, the student is using the object perspective to look at both graphs as objects and is not checking individual ordered pairs.

STUDENT B: I’m going to plug them all in so I’m going to use the calculator for that (laughs) (works on calculator and then crosses off choice a)
TEACHER: Why is it not that one?
STUDENT B: Because it doesn’t look like it at all (laughter).
TEACHER: Alright
STUDENT B: Because this line (motioning with hands) was backwards so it was like so it was more like right there. (types in second equation into calculator).
It’s not that one either because it’s all… This one’s below the line and this one’s like normal.
TEACHER: Ok
STUDENT B: -2 sine x parentheses, minus 1. This one is the same. It’s close but it’s still above it. So it’s most likely d, but (types in last equation). There it is. So it’s d.

The student plugs in all four equations and uses features of the graphs like where the curve is with respect to the x-axis. Student B notices that the graph of the first equation is increasing at 0° and the graph on the paper is decreasing. This is what the student means when the student says, “this line was backwards.” However, the student is not able to connect that feature of the graph with an x-axis reflection because that would have
also eliminated option b). Instead, Student B types this equation in next and eliminates option b) because the graph on the paper has a vertical translation and the graph on the calculator does not. Once again, he/she does not voice how the translation connects to the equation. Instead, the student types in the third equation without making an educated guess on whether or not it will work. He/she sees that option c) also has a graph that is partially above the x-axis whereas the graph on the paper is not. Since the student has made no attempt to connect any of the features of the graph to the equations, the student then types in the final equation to verify that it is in the correct answer. When it appears to be a match, the student is convinced.

When asked if there was another way to check the answer, the student does make the connection between the features of the graph and the features of the equation. However, the student is using the answer to discuss these features so it is not apparent if the student would be able to do it without knowing the correct answer first.

TEACHER: How could you check your answer other than graphing on the calculator?
STUDENT B: You could actually, like, know it (laugh). You could actually know like what – 1 will do and –sin and the 2x would do because the normal sine equation is just sine of x. But like the negative sign and the minus -1 all does stuff. So the -1 I’m guessing, well yeah, it like flips it, like makes this one instead of going this way, go this way.

The student also does not appear to know exactly how each part of the equation connects with the graph. Student B does not mention that the period has changed or that the graph has been translated down 1 unit during the response. He/she is starting with the equation and connecting it to the graph. This may explain why the student is unable to mention all the transformations the graph has undergone. In the previous interview, looking at the
graph, the student was able to state that adding a number in the equation will translate the graph up. However, looking at the equations this time, the student is not able to discuss what causes a vertical translation.

Student C uses the connection between the graph and the equation from the object perspective to figure out which one is correct. The student realizes that the graph has been translated down one unit and that connects with the $-1$ outside of the parentheses in the equation and he/she eliminates $y = \sin(2x - 1)$. The student is also able to recognize a reflection and connects this with a negative in front of the equation and eliminates $y = 2\sin(x) - 1$. The student then eliminates $y = -2\sin(x) - 1$ because he/she knows the 2 in front of $\sin(x)$ stretches the graph and changes the amplitude, and the graph on the paper has no change in amplitude from the parent $\sin(x)$ graph. This means that in addition to an understanding of the connections between an equation and graph, Student C has a representation for the parent graph of $\sin(x)$ and its features.

STUDENT C: So, let’s see. I think it is… Oh, that’s right (pause)
TEACHER: So what are you trying to do?
STUDENT C: I’m trying to figure out like… I know that it’s either a or… It’s not b.
TEACHER: Ok
STUDENT C: Because the number has to be outside (the parentheses) for it to go up or down, in this case it’s down 1, so it’s a, c, or d. I’m just trying to figure out how far it moves over left or right to, um, figure what is right so… So it’s not a.
TEACHER: Why is it not a?
STUDENT C: Because it’s the, um, the way the wave is reversed so there’s a negative in front of the equation. Because it goes middle, down, middle, up, middle instead of up… middle, up, middle, down, middle.
TEACHER: Ok
STUDENT C: So it’s not stretched any so I’m going to say it’s d because there’s a negative in front, it’s not stretched this way, but it is moved down.
Even though the student does not know what the 2 in the equations $y = -\sin(2x) - 1$ does to the graph, he/she knows that it has to be this equation because there was something to eliminate the other three choices. When asked if there is a way to check this answer, the student is unable to think of any other way. When asked to figure out what the 2 does to the graph, the student is able to recognize that the graph on the paper has a shortened period so this must be connected to the 2 in the equation. The student was able to figure out all the information using the graph and equations as objects and never attempted to look at any specific points.

Student D, just like Student C, connects the graph to the equations from an object perspective to answer the question. However, it appears that Student D is using the characteristics of the graph and matching them to the equations. Student C seemed to look at the individual features that were unique to an equation and use that to narrow the choices. Student D started by looking at the amplitude of the graph and noticing that the amplitude was 1. This eliminates $y = 2\sin(x) - 1$ and $y = -2\sin(x) - 1$ because both equations had an amplitude of 2. He/she eliminates $y = \sin(2x - 1)$ next. However, it is not clear if the student looked at the two remaining equations to determine the next step, or if the student just decided to look for a reflection next instead of a vertical translation. It appears that the student looks at the two remaining equations and notices that one has an x-axis reflection, option d), and the other does not, option b). Student D then realizes the graph has a reflection so the correct answer is option d). A vertical translation seems to be easier to distinguish from the graph, especially since a horizontal translation can take the place of a reflection. This means the student was probably looking at the
equations at that point and not the graph because when looking at the equations from left to right, the reflection happens in the equation before any translation.

STUDENT D: Well first I look at the amplitude because that’s the easiest to find. And the amplitude is 1. So that eliminates these two (a and c).
TEACHER: Ok
STUDENT D: And, since sine starts at 0 and goes up then we know it’s not reflected over the, wait, then you know it is reflected over the x-axis because it goes down. So it would be d.

To check the answer, just like Student B and Student C, Student D does not look at individual points. The student discusses how the remaining features of the equation connect to the graph. The student knows the – 1 is a vertical translation and the 2x shortens the period. The only other method that Student D suggests to check the answer is using the learned procedure for graphing trigonometric functions to graph option d) to see if it does in fact produce the same graph as the one on the paper.
Question 4: Given the unit circle, special right triangles, and the graph of \( y = \sin(x) \), answer the following questions. a) What is the \( \sin(45^\circ) \), b) What is the \( \sin(65^\circ) \), c) If \(-1/2 = \sin(x)\), what is the value of \( x \)?, d) If \(0 = \sin(x)\) what is the value of \( x \)?, e) if \(3/2 = \sin(x)\), what is the value of \( x \)?, f) solve the equation \( \sin(x) + 1 = 1/2 \), g) solve the equation \( 3\sin(x) = 2 \)

Note: The students were given a graph of \( y = \sin(x) \) with an x-axis in degrees, a unit circle labeled with positive degrees and radian measures with the 16 most used coordinates, and two special right triangles with the sides and angles labeled (30°-60°-90° and 45°-45°-90°). (see figures 5, 6, and 7 in Appendix F)

Results

No student attempted to use the same representation for all the questions. While sometimes it was out of a lack of understanding with how to use the representation to answer a given question, the students did show an understanding that in certain situations one representation had a benefit over another. This was most often the case with the graph being better for generating an estimate. Student D realized that the unit circle can give an exact value whereas a graph can only give an estimate for something like \( \sin(45^\circ) \). It was interesting that the triangle representation was not really used to solve any of the tasks. While Student B and Student C both demonstrated that they recalled the activity in class that connected the triangles to the unit circle, no student was able to use a triangle representation to solve any of the questions unless it had already been solved with another representation first. The tasks in the questions were in a form that the
students did not rely on triangles to solve during class, and unlike the graph, they were not able to figure out how to use them while they answered the questions.

**Evidence**

Student B starts by using the 45°-45°-90° right triangle and places it inside the unit circle in an attempt to find sin(45°). The student is doing this because when creating the unit circle during class students looked at the two special right triangles and placed them inside the unit circle to find the coordinates for specific angles measures. Because he/she has been given both of these representations, this is the connection the student remembers. However, during the process, Student B realizes that because the coordinates are already given, the triangle is not necessary for finding the answer.

STUDENT B: Um, sine of 45° I’m going to use the 45-45 triangle, and place it like that and put this little point there in the center and then it will be, I didn’t even need to use this triangle because (talking indistinctly). So 45 degree mark, obviously and sine that’s the (pause) either the first one or the second one. I think it’s the first one. In this case it would matter because they’re both the same thing. So it’s √2/2 for the first one.

The student uses the unit circle in the process perspective to get the answer. He/she knows that the angle measure is linked to a point and either the x-coordinate or the y-coordinate at that point is the answer. However, the student does not show a full understanding or this process because Student B guesses it’s the first coordinate and offers no way to check the answer. In this case, Student B shows some knowledge that the point is generated by the sides of the right triangle, but the process that the student uses links the x-coordinate and y-coordinate to either sin(x) or cos(x).
For the next task, the student knows that 65° is not an angle on the unit circle so the student changes to the graphical representation to answer that question. Student B has been unable to use the unit circle to estimate angles or coordinates during both interviews so a change in representations is necessary.

STUDENT B: Ok, the sine of 65. 65 is, well none of these will help me (motioning to triangles and unit circle) because it’s not any of those. So I’ll use the graph to find the sine of 65°. 65° is… 90, 80, 65 is right here. That’s .9

The student uses the graph to produce an accurate estimate, much like he/she did in the first question. What is important here is not that the student produced an accurate representation, but that the student saw the need to change representations to do so. The student’s understanding of the unit circle from the process perspective is one that links an angle with a point. However, this process can only be used for a set number of angles and an angle like 65° is not in that set. Student B also sees the special right triangles as being used only to generate the points on the unit circle and not a process for making estimates for other angles. Because the student does not have a process for using either representation to form an estimate, he/she must use the graph.

It is interesting that to answer the next two tasks, the student continues with the graphical representation. \(-1/2 = \sin(x)\) and \(0 = \sin(x)\) can both easily be solved with the unit circle, but it appears that Student B’s understanding of the inverse operation is strongest in the graphical representation and for this reason the student uses the graph as a process to complete the next two tasks.

STUDENT B: If \(-1/2 = \sin(x)\), what is the value of x? OK, I’ll use the graph again. Alright so -1/2 is here. Yeah. The value of x would be right her, which is there, which is -30°.
TEACHER: Ok
STUDENT B: If $0 = \sin(x)$, what is the value of $x$? Zero. (pointing to origin)
Zero. (pointing to origin) Zero. (laughs)

The student understands that to solve this equation, a process of starting with a $y$-value and linking it to an $x$-value must be used. Just as in the first question, the student fails to realize that there are multiple solutions using this process. It again appears the student is only looking for one answer and never checks the graph for a second one.

When the student looks at $\frac{3}{2} = \sin(x)$, the student stares at it for a few moments and is unsure how to proceed. When the teacher asks what the student is thinking, Student B responds that it is probably necessary to use the unit circle for this question. The question is misleading because it leads the students to assume that a value of $x$ does exist for an answer. This appears to be what is confusing Student B and the student suggests using the unit circle because $\sqrt{3}/2$ is a value on the unit circle and the student believes it may be equal to $3/2$.

STUDENT B: Alright, if $3/2 = \sin(x)$ what is the value of $x$? (long pause)
TEACHER: So what are you trying to use to answer?
STUDENT B: Well, I’m thinking unit circle because… but I don’t know if that’s right because there’s $\sqrt{3}/2$ in some spots but I don’t think it’s the same, but that wouldn’t make sense. I could look at the graph. It would be there (pointing to $3/2$ on y-axis). But there’s nothing, so it’s not the same, I’m pretty sure.
TEACHER: Ok, so what’s your answer then?
STUDENT B: Nothing. It’s impossible. No solution.

The student is using both the graph and the unit circle to answer this question. The student makes the connection that if $\sqrt{3}/2$ and $3/2$ are the same, then the graph should have a $y$-value of $3/2$ somewhere. Since the graph does not reach $3/2$, this convinces the student that $3/2$ is also not on the unit circle and there must not be a solution.
Since the next tasks ask the student to “solve the equation” the student utilizes an algebraic representation first to get $\sin(x)$ by itself in the equation $\sin(x) - 1 = 1/2$.

However, the student does not seem to understand that in the algebraic representation that the inverse sine operation must be used to get the final answer. Instead, the student goes back to the graph to find the angle where the y-value is -1/2 to get an answer of -30°.

During the first interview, the student also showed a lack of understanding with inverse sine in the algebraic representation.

STUDENT B: Alright. (long pause) Well, $x$ has to be -1/2. (typing on calculator). It has to be -1/2 for that to work out. So $\sin$ of $x$ has to equal -1/2. So that’d be there (points to graph). So it’d be like -30°.

TEACHER: Ok
STUDENT B: That makes sense to me. But it might not make mathematical sense.

TEACHER: Why not?
STUDENT B: I don’t know (laughter). I’m second guessing myself here. Um, so I said negative… -30°.

It is interesting that the student is second guessing the answer. It is not apparent why he/she decides to say this. It could be that the student noticed that there are multiple solutions on the graph. However he/she quickly moves on to the next question and seems to be satisfied with -30° as an answer.

For the equation $\sin(x) + 1 = 1/2$, the student knows to subtract 1 from both sides to get $\sin(x) = -1/2$. However, with the equation $3\sin(x) = 2$, he/she is not able to understand that to get $\sin(x)$ by itself, both sides must be divided by three. Instead the student approaches the problem for a more literal process. The sine of an angle multiplied by 3 must equal 2. Student B also knows that to use the graph to find the answer, it is necessary to find a y-value that when multiplied by 3 equals 2.
STUDENT B: Ok, next one, so 3\sin(x) = 2. So, (long pause)
TEACHER: So what are you thinking?
STUDENT B: Um, I’m trying to figure out, I’m trying to think of what would sine of x equal, have to equal for that equation to be 2. But sine of x would have to be, have to equal something that would multiply by 3 to equal 2. Right?
TEACHER: Ok
STUDENT B: Um, so, uh, then I can’t think of what that would be. What is 3 times something that would be 2? (types on calculator)
TEACHER: What did you just type in?
STUDENT B: I typed in 3 times .75 and that equals 2.25.
TEACHER: Ok, why did you choose .75?
STUDENT B: I thought it would possibly be… Well, because point…, 1/2 wouldn’t equal it because that would be 1.5 and 1 wouldn’t because that would be 3. So it has to be somewhere in between. So .75.
TEACHER: Ok
STUDENT B: And that didn’t work, so I’m kind of, kind of stumped here.
TEACHER: Ok, so if you know it’s between 1/2 and 1 what does that tell you?
STUDENT B: Um, well, it’s, well, it’s, it’s within this, this area (pointing to region on graph).
TEACHER: Ok, can you give me a range of angles?
STUDENT B: A range of angles would be… (counting on graph) So that would be 30 to 90°. So I don’t know what that accomplished.
TEACHER: Ok (student laughs). OK, so you can say it’s between 30° and 90°. That’s all that you can say?
STUDENT B: Um, I don’t know. Should it be more or should there be more? There should be more.
TEACHER: Ok
STUDENT B: I think, unless you’re just trying to trick me again. Um, I’m trying to think what I should do. 3 sine of 70. (types on calculator) No. Um, that’s too big. 3 sine of 60. (on calculator) No. That’s closer. 45. (on calculator) Ok. 44. (on calculator) 43. (on calculator) 42. (on calculator) Well it’s between 41 and 42.
TEACHER: Ok
STUDENT B: So 41.5? (on calculator ) No. I’m not doing this right. 41.7?
TEACHER: Well, it might not be an exact value but just approximate.
STUDENT B: Ok, well let’s try… we’ll go with 41.8 or 41.9.

Even though the student is only able to narrow the possible y-value to something between 1/2 and 1, he/she is able to use the graphical representation to produce a range of angle measures that the estimate would correspond with. The student then uses the calculator
to guess and check values in this range and narrows it down to produce an estimate for
the answer. Even though the student has no algebraic representation for the inverse
operation, the student does have a graphical representation and was able to use this
understanding to guess and check to arrive at an answer.

Student C also uses the unit circle to find an answer for \( \sin(45^\circ) \). However,
Student C’s understanding of the process that links \( \sin(x) \) with a coordinate includes the
understanding that \( \cos(x) \) is the \( x \)-coordinate and \( \sin(x) \) is the \( y \)-coordinate for a given
angle. For \( 65^\circ \), the student is unable to use any of three representations to produce an
answer. The student tries to put the \( 30^\circ-60^\circ-90^\circ \) triangle inside the unit circle, just like
Student B did earlier, but Student C does not know how that will help with \( 65^\circ \). This
does show that both students do see a connection between the triangles and the unit circle
as a result of the classroom activity, but the connection is not strong enough to use for
any conceptual understanding. Instead Student C uses the calculator to evaluate the
expression.

TEACHER: Ok. What is the sine of \( 65^\circ \)?
STUDENT C: I don’t know how to use this (triangle).
TEACHER: What were you trying to do?
STUDENT C: I was trying to like see if I could put in on there and move it to go
to 65.
TEACHER: Ok
STUDENT C: Um, well, I guess I could just, sine, calculator, sine 65 close
parentheses equals… But that does not give me a fraction. It gives me a point. Or
is that the answer? Would that be the answer? Or is there no decimals?
TEACHER: It can be a decimal.
STUDENT C: Ah, then the sine of 65 is .91.

The student seems to be expecting an exact value for an answer. Much like Student B,
Student C has no way of using the unit circle or triangles to produce an estimated value.
The student did use the graph to estimate values in the first question, so the fact that he/she did not use the graphical representation first seems to mean that the student was expecting an exact value answer. If the student is expecting an exact value, then the graph, which produces estimates, is not going to be helpful. When the student is then asked if any of the three given representations can be used to find that value, the student realizes that the graph can be used.

TEACHER: Alright, can you find that using the unit circle, your graph, or the triangles? Any of the three?
STUDENT C: I can use the graph.
TEACHER: Ok
STUDENT C: Just go to 60 on the graph, which is about…
TEACHER: 65
STUDENT C: 65. Go to 65, man these are (talking indistinctly) and go to that point and it looks like it’s .91.

However, the student is using the answer that was already known and verifying the answer with the graph.

For \(-1/2 = \sin(x)\) the student goes back to the calculator and types in the inverse sine operation to get \(-30^\circ\) as the answer. Unlike Student B, Student C is able to understand how to solve an equation like this in algebraic form. When the student is asked to use one of the other representations to get the same answer, the student demonstrates a misconception with negative angles on the unit circle.

TEACHER: Alright can you use the unit circle, the graph, or the triangles to show me that?
STUDENT C: Um, since it’s negative, since it’s a \(-1/2\), you go to sine on the… You go to 1/2 on the… 1/2 sine on the unit circle and since it’s negative you put a negative in front of that degrees so it’s -30.
It appears that even though he/she has all the coordinates labeled on the unit circle, Student C is only familiar with the first quadrant and axes points. Because the student is unfamiliar with the points in the other quadrants, he/she finds a y-value of 1/2 in the first quadrant and to account for the negative sign for the angle, simply says the final answer should be negative. This implies that if \( \sin(x) = a \) then \( \sin(-x) = -a \). The student does not seem to be making this assumption based on any conceptual understanding but one made for convenience of verifying a known answer. It is very strange that the student does not look at the other three quadrants during the response. This must be due to a lack of familiarity with those three quadrants.

It is interesting that the student knew the answer was \(-30^\circ\) but did not simply find \(-30^\circ\) on the unit circle to verify the answer. It appears one of the reasons the student chose to look at \((30^\circ, 1/2)\) is because the student has a very limited understanding of negative angles in the unit circle representation. This may be the case because when the student switched to a graphical representation to explain the answer, the student first appears looks at \(30^\circ\) but then switches to \(-30^\circ\) on the graph and finds the y-value there. He/she figures out how to use a negative angle measure on the graph, but not the unit circle. However, what makes this more interesting is that during the first question, the student, citing points from the unit circle, produced the graph of \(\sin(x)\) in the negative direction. The coordinates that Student B used had x-values and y-values reversed and the student did not make any of the angle measures negative, but the student plotted each point in the intended negative spot in the coordinate plane. (student said, “\((-1, 90), (0, 180), (1, 270)\) and \((0, 360)\)” but actually plotted \((-90^\circ, -1), (-180^\circ, 0), (-270^\circ, 1)\) and \(-190\).
360°, -1)). This again shows the student is recalling them from memory but this does not mean that he/she can actually find these points on the unit circle. It was the understanding of the pattern of the graph that allowed him/her to correctly plot them. The student has a weak understanding of a negative angle with the unit circle representation.

For 0 = \sin(x) the student goes back to the calculator and uses the inverse to get an answer of 0. However, because the student is familiar with where the coordinates contain a 0 on the unit circle, the student sees the answer and points out that the unit circle can be used to find this answer as well. It is interesting that when using the unit circle, he/she first looks at 180° and then at 0° before saying the answer is 0°. The student realizes that the answer could also be 180° based on the understanding of the unit circle from the process perspective, but because the calculator only gives one answer, the student appears to disregard the second answer.

STUDENT C: Hmm, (pause). Ok, you can do it on the calculator. Do inverse sine of 0 it’s 0. You could also go on the unit circle (pause). So it’s at (1, 0) for sine because sine is y and the graph goes to zero and that’s 0.
TEACHER: Ok, but on the unit circle you also looked at 180°. Why?
STUDENT C: Because it’s also at… 0 at the sine.
TEACHER: So what does that mean?
STUDENT C: (pause) Um, it means it’s sine of 0. It’s… I don’t know.

It is only when the teacher questions the student again about the final answer that Student C realizes that there are multiple solutions. The student states that there are now two answers and that there are no more. But when the teacher presents the student with the graphical representation and asks the same question, Student C realizes that there are
more, but only the ones on the actual graph between -360° and 360°, not the additional solutions that can be found by extending the graph.

TEACHER: So is your answer 0°…
STUDENT C: No! It’s 0 and 180. Aha! I got two answers. Because they’re both, both of their sines, or y’s, are 0.
TEACHER: Ok. Are there just two answers?
STUDENT C: Yes
TEACHER: Alright, look at your graph, where it the sine 0? You said 0 and 180°. Anywhere else?
STUDENT C: Also 360 because it’s at 0.
TEACHER: Anywhere else?
STUDENT C: No, just 360, 180, and 0
TEACHER: Ok
STUDENT C: Do the negatives count too?
TEACHER: Yes
STUDENT C: Ah, so -180 and -360.

By using the unit circle and the graph from the process perspective the student is able to find multiple answers. However, the student’s understanding of both representations is limited to what is on the paper in front of the student. He/she is still unable to view the unit circle in terms of negative angles or for angles greater than 360° and the student is unable to see the that continuing the graph in either direction will add more solutions.

Again, it is important to point out that the student’s familiarity with the axes points and first quadrant points allowed him/her to make the multiple solution connection with the unit circle. The student was unable to come to the same realization when the value was -1/2.

With 3/2 = sin(x), it appears that the misleading nature of the question confuses the student. Because the question implies there is a value of x, Student C is unable to realize that there are no solutions. He/she utilizes the calculator first again, and when that
outputs an error message the student tries to change representations. Just like Student B, Student C wants to say that 3/2 is the same as $\sqrt{3}/2$. The student though is able to realize that the calculator would have given 3/2 as the answer if they were the same. When the teacher asks the student to consider the graph, Student C is finally able to say that there is no answer because the graph never reaches 3/2. However, it was only after the teacher suggested the graph that the student is able to do this.

For the equation $\sin(x) + 1 = 1/2$ the student begins in the algebraic representation and solves for $x$. Just like before, he/she is about to type the inverse sine operation into the calculator, but the teacher stops the student and asks Student B to use one of the three representations to find the answer first. The student decides to use the unit circle. When faced with the exact same situation earlier ($x = \sin^{-1}(-1/2)$), the student looked at 1/2 in the first quadrant, saw that the angle that matched was 30°, and then said the answer was -30° since the ratio was negative. This time the student actually finds the two places on the unit circle where -1/2 is the y-coordinate and says the answer is 210° and 330°. It appears that answering the previous two questions has reminded the student of the existence of multiple solutions. So now instead of using the unit circle to find first quadrant and axes coordinates only, he/she now knows that is important to check all four quadrants to get answers.

The student is still not able to use the unit circle to get anymore than two answers. The teacher now tells the student to check the answer on the calculator. When the calculator gives an answer of -30° the student is surprised it is not one of the two answers the student got from the unit circle. To deal with this discrepancy, Student C then
changes to the graphical representation to verify the answer. This is most likely because the student has no way to represent -30° on the unit circle.

STUDENT C: Yes, um, I’m going to use the unit circle. So I’m going to go anywhere where it’s the y-coordinate -1/2, which is 210, um, 330, and that’s it.

TEACHER: Ok, now try the calculator and see what you get.

STUDENT C: And that is… (talking indistinctly). The inverse sine of -1/2 equals -30. What? Um, (points to -30 on the graph) so it’s at (-30, 1/2), -150, and it’s also at 210 and I’m guessing that 330. Yeah, that’s 330. So we have the first two which we find on the unit circle, one we find on the calculator, which is -30. On the unit circle it’s 210, 330. And on the graph we’re going to use it to find -150.

The student is able to use the graphical representation to find all the answers but makes no attempt to explain why the same cannot be done on the calculator or unit circle.

However, when asked if there are still additional answers, unlike in part d) the student now understands the graph can be extended to find additional answers past -360° and 360°.

TEACHER: Ok, very good. Do you think there’s any more?
STUDENT C: Yes, but it’s on the graph, if it were stretched further than 360.

TEACHER: Ok. Is there a way you could find it on the unit circle?
STUDENT C: Um, yes. If you keep going around, let’s say this is 360, so this would be 390, um 405, um, 420, and this would be 450 and so forth and so on.

TEACHER: Ok, what about -30°? Where would -30° be?
STUDENT C: There’s 30. Well, if you started here and go backwards.

TEACHER: Ok, so where’s -30° at?
STUDENT B: It would be here. (points to 330°)

TEACHER: Ok, what about -150°?
STUDENT C: (points to 210°) It’s close.

TEACHER: Which one?
STUDENT C: Because it was flipped if you’re going backwards.

It appears that by using the graphical representation, the student is finally able to connect the multiple answers in the graphical representation with multiple answers on the unit circle and is eventually able to incorporate negative angles measures into the
understanding in the unit circle representation. He/she realizes that making a second revolution around the unit circle is same thing as adding 360° to each angle measure. To make the connection with the negative angles, the student is able to see that the angle measures are “flipped” over the x-axis in the negative direction. The student realized that moving clockwise will produce a negative angle measure, so the student is using the unit circle from the process perspective and not from the object perspective which would be reflecting the angles over the x-axis.

For the final equation, 3sin(x) = 2, the student uses an algebraic representation to manipulate it into sin(x) = 2/3. Once again the student is asked not to use the calculator. The student becomes confused because 2/3 is not a coordinate on the unit circle. The teacher then suggests using the graph and the student is able to use the graphical representation to estimate the answer to be approximately 50°.

TEACHER: Ok, what about the graph?
STUDENT C: 2/3 equals .66 repeating which is about .67 and on the graph look at .67. It’s right there. So it’s probably about, 1, 2, 3, 4, 5. So 50°
TEACHER: Ok. You just have one answer this time?
STUDENT C: Yeah
TEACHER: Ok, now try it on the calculator and see if it’s about the same thing.
STUDENT C: It would be 42.

It was also interesting that he/she did not see that there are multiple solutions even though the student did for the previous equation. The confusion caused by not being able to represent the solution with the unit circle must be enough that the student is not connecting this solution with the same procedure in the previous question.
Student D uses the unit circle just like Student C did to find the y-coordinate on the unit circle at 45° and states the answer is $\sqrt{2}/2$. However, Student D also points out that the graph will give the same answer in decimal form if he/she finds the y-value on the graph for an angle of 45°. Not only has Student D used the process perspective to get both answers, but the student also points out that the answers, although different forms are the same value.

To find $\sin(65°)$ the student decides that the unit circle is not going to help since 65° is “not on there.” This is interesting because Student D had used the unit circle to estimate value in the first interview, but it is possible that Student D realizes that other representations, like the graph, would give a more accurate estimate. However, rather than use the graph, he/she attempts to use the triangles. In the first interview, Student D explained that when confused, he/she prefers to think in terms of triangles so it is possible tist is why the student tries this representation. The student picks up the 30°-60°-90° triangle and looking at the 60° angle, states the answer is $\sqrt{3}/2$ because that ratio is opposite over hypotenuse.

STUDENT D: That’d be this one (talking indistinctly). Um the sine of 65°since it’s not on there (unit circle) I should probably use this. Or I could use these here (triangles) for the sine of 65° since it’s just opposite over hypotenuse. It’s $\sqrt{3}/2$. TEACHER: Why, why did you use that triangle over there? The 60-30-90 right triangle? STUDENT D: Oh, wait, that’s 65°. So that would not equal. I would use the graph. So that’s probably about 45° there and it would be more than that. I would say a little less than 1, so .9.

It is not apparent why the student chose to use this triangle for 65°. However, when the teacher asks why, it is enough for the student to realize that the answer is not the one the
student wants for $\sin(65^\circ)$. Instead the student changes representations again to the graph to find a y-coordinate for $65^\circ$ and states the answer is .9.

For the $-1/2 = \sin(x)$ the student goes straight to the unit circle and understands this to mean the places on the unit circle where the y-value is $-1/2$. Unlike Student C, Student D checks the entire circle and finds two answers, $210^\circ$ and $330^\circ$. Just like before, the student demonstrates that the same answers can be found on the graph. However, by looking at the graphical representation, he/she realizes that negative angles are possible as well and finds two additional answers.

STUDENT D: Um, what is the value of $x$, so $-1/2$, so I have to find… So I go here (unit circle). I would just look for when $y$ equals $-1/2$.

TEACHER: Ok

STUDENT D: Which it does it two places. So there’s more than one solution, just like there would be on the graph. Um, so the value of $x$ could either be $210^\circ$ or $330^\circ$. And then we could check that here (graph). It would equal $-1/2$, $210^\circ$, and that would match up.

TEACHER: Where’s $210^\circ$ at?

STUDENT D: Oh, it’s right here. So it would be that there. Which is in line with that. And then $330$. And it could also be negative too.

TEACHER: Ok

STUDENT D: So then that would probably be like $-30$, oh yeah, that would be $-30^\circ$ (points to $330^\circ$ on unit circle) and negative, (counting on unit circle) $-150$. And so that was what you’re looking for, for that one?

The student has used the graphical representation to remember about negative angles measures and then connects that understanding to the unit circle to show why $-30^\circ$ and $-150^\circ$ would be solutions in that representation. In first interview, negative angle measures also gave Student D some trouble until he/she saw the graphical representation.

It appears that the understanding the student has for negative angle measures is best recalled in the graphical representation.
Now that the student has a way to represent multiple solutions on the unit circle, the student does not need the graph to find all the answers to $0 = \sin(x)$. The student understands the solutions on the unit circle are $0^\circ$ and $180^\circ$ but that $-180^\circ$ is also a solution and there are more if you continue to add $360^\circ$ to the answers because adding $360^\circ$ to an angle will “get to the same place” on the unit circle and therefore have the same y-value. The student has been able to create a procedure for finding all the solutions on the unit circle.

For $3/2 = \sin(x)$ the student uses the graph to realize that there is no solution. When asked why he/she used the graph instead of the unit circle, the response indicates the student chooses the graph because Student D knows that $3/2$ was not a coordinate on the unit circle, so it is not able to produce an answer, just like $\sin(65^\circ)$ could not be answered with the unit circle. Because Student D said this and knows that the unit circle gives an exact answer for $\sin(45^\circ)$ while the graph gave a decimal approximation, the student is showing an understanding of when to use the unit circle representation and when to use the graphical representation.

STUDENT D: Um, $3/2 = \sin(x)$. If you look for $1 1/2$, $1 1/2$ equals… Um, it’s one of the trick ones. Because you’re looking for this, right? (pointing to 1.5 on y-axis) on this graph.
TEACHER: So what does that mean?
STUDENT D: That means (pause) there is, the graph doesn’t go there. Um, (pause) the amplitude would have to be bigger to be up there.
TEACHER: Ok. So what do you want to say your answer is?
STUDENT D: Well, right now since I don’t see it, I would say there’s no solution.
TEACHER: Ok
STUDENT D: Um, (pause), yea because this is $\sqrt{3}/2$ (pointing to unit circle). I guess there is no solution because I don’t see it.
TEACHER: Ok, why did you look at the graph first that time? You’ve been looking at the unit circle first. Why did you look at the graph?
STUDENT D: Because I know it wasn’t $\sqrt{3}/2$ and just $3/2$ isn’t anywhere on here.

It is interesting that the student has not tried to use the triangle representation at any other point since trying to find $\sin(65^\circ)$. The student has shown a tendency to use triangle representation throughout the first interview. Even though the student has connected the meaning of the unit circle answer with the meaning of the answer on the graph, the student has left out the triangle representation altogether. This may be because the student has difficulty with the triangles in multiple quadrants. The student realized that is possible in the first interview, but was confused by the negative angles and negative side measures. The student never connected the negative angles or sides with the graph or unit circle during the first interview, so it is possible a strong connection does not exist for the student. The student is able to represent $3/2 = \sin(x)$ with a triangle and concludes it will not work because if the ratio is $3/2$, it implies the hypotenuse is not the longest side in the right triangle.

For the next task, the student realizes that $\sin(x) + 1 = 1/2$ is the same thing as $\sin(x) = -1/2$ and so the answer will match the answer the student gave in part c). He/she uses an algebraic representation to change this equation into the same equation the student saw earlier, and then is able to go straight to the answers without needing to change from the algebraic representation.

For $3\sin(x) = 2$, the student immediately sees this as $\sin(x) = 2/3$. Again, the student uses the algebraic representation to do this. The student also appears to realize that 2 and 3 were the same numbers he/she used to show that there was not a triangle in
part e) so the student again draws a triangle to show that since the ratio is reversed, the hypotenuse is now the longer side and the triangle is possible.

STUDENT D: Ok. Um, 2/3, so then (drawing triangle). That would make sense this time. So (opening calculator), it will be easier to find the decimal on here (pointing to graph). So then that would almost be .7 (plugged is 2/3 on the calculator), it would probably be 45°.

TEACHER: Ok
STUDENT D: But it could also be 135 and also would be 225. I just want to see something.

TEACHER: What are you typing in?
STUDENT D: Um, √2/2. I just want to see… Yeah because that’s really close to .6. So it wouldn’t be exactly 225, it would be a little less than that.

TEACHER: Ok, so 45°, 135°. What were your other two answers? Or other answer?
STUDENT D: What did I say?
TEACHER: You said 45°, 135°, and …
STUDENT D: Um, it’s 220 and, er -220 and -300.

The student does not have a way to use a triangle to find a decimal that is not 30°, 45°, or 60°, so the student must change representation to continue. Student D changes to a graphical representation. He/she most likely chooses not to use the unit circle because 2/3 is not a memorized ratio on the unit circle. Using the graphical representation, the student finds answers of 45°, 135°, and 225°, although the student seems to mean -225°.

The teacher asks the student to repeat the solution to see if Student D can catch this mistake. The student does but also realizes that the answers are just estimates because 2/3 is about √2/2.
Question 5: How do your answers to a and b differ from c through e?

Results

It was possible that the question was too vague and the students confused how to differentiate the answers because they appear to see a different procedure for the two types of operations, but there is no connection with the differences and an operation and its inverse. It appears that the students were not connecting the representations with the algebraic representation. The students that solved the equations on the calculator knew to use inverse sine to find an angle, but it did not appear the students understood what the inverse sine process is on the unit circle or graph. The inability to see this difference in operations may be the reason there was such inconsistency with being able to determine the situations when multiple solutions were possible or even when no solution was a possible response.

Evidence

Student B looks at the answers and the way each problem was actually set up. While the student does not mention sine and inverse sine, the student does realize that a) and b) were asking for the sine of an angle and c) through e) had a number equal to sin(x).

STUDENT B: Um, well, these are angles and no solution, c through, oh wait, c through e. Um, the questions are different. A and b are what is the sine of blank, and c through e is if blank equals sine of x, what is the value of x. Um, (long pause) these are… these are y values and these are x-values.

While the student does see the difference in the way the problems are set up, he/she does not state the procedures involved are inverses of one another.
Student C does not see any noticeable difference. However, the student realizes that part c) should have multiple answers because the problem is just like part f) where Student C found multiple solutions. The student does not look at the way the problems are set up, as Student B did. Student C is basing the answer solely on the answers for each but makes no mention of how the answers are different.

Student D, much like Student B is able to see that a) and b) start with an angle and end with a y-value and that c) through e) start with a y-value and ask for an angle. Once again though, the student is not able to make a connection to sine and inverse sine.

Question 6: Discuss sine and inverse sine in terms of inputs and outputs. How can you represent them in a triangle? On the unit circle? On a graph?

Results

While the students were not able to a connection sine and inverse sine with inverse operations, it did not hinder their ability to use or represent each during the interview. However, the inability to represent the output for sine in more than one way did. If students lack the understanding that the output can be a fraction, ratio, coordinate etc., it affects their ability to connect the representations together. It was especially interesting that although most students understood sine as opposite over hypotenuse, only one of the three students was able to connect this ratio with a coordinate or y-value on the unit circle or graph.
Evidence

Student B is able to state the differences in the inputs and outputs, but it appears that the statement is made by observing the procedure for the different representations and not because he/she has made a connection to inverse operations. The student understands that when an angle is inputted, a coordinate is outputted and when a coordinate is inputted, an angle is outputted.

STUDENT B: Um, um, sine is to be used if you’re trying to find a, a uh… Inverse sine you use to try find, if you’re trying to find an angle. Yeah, so you’re trying to find an angle you use the second sine button to find it, and then you would plug in like, uh… (pause) What do you plug in if you’re trying to find and angle? You plug in a coordinate, so like 1/2 or $\sqrt{2}/2$ or $\sqrt{3}/2$. Plug that in and find the angle using inverse sine.
TEACHER: Ok
STUDENT B: Now sine, you use to plug in an angle and find a coordinate so 30 would give you 1/2 or 45 would give you $\sqrt{3}/2$, 30 would give you 1/2, right?

The student is using the representations to figure out what the inputs and outputs are for sine and inverse sine. Student B does not make any mention of switching the input and outputs for inverse functions so it is not possible to tell if he/she understands this relationship.

For representing sine and inverse sine in a triangle, the student understands that inverse sine is used to find an angle and inputs a ratio of the sides, but it appears that he/she does not fully understand what sine does in a triangle. The student’s understanding of sine is inputting an angel and outputting a coordinate. Using that understanding the student is unable to figure out what sin(45°) means in the triangle.

STUDENT B: Ok. Um, right, so if you were trying to find this one (using 45°-45°-90° right triangle). This angle up here, um it would be, so it would be opposite over adjacent so it’s be 1/1 and you would plug that into the sine, the
inverse sine, you would, right? 2nd, sine, 1 over 1, which would be 45. No! Um, oh, opposite over hypotenuse, so 1/√2. 45, there you go.

TEACHER: OK
STUDENT B: So that’s inverse sine and then sine you’re trying to find one of these, that would just be sine of 45 which give you .707 which would not be √2 Um (pause). Sine 90, which equals 1.

TEACHER: So what are you trying to do?
STUDENT B: Well, sine of 90 equals 1 so it’s one of those, so yeah.

The student does not realize that sin(45°) is a ratio of two sides in the triangle, even though the student used the ratio during the explanation of inverse sine. The student thinks that the answer is the length of one of the sides in the triangle. When the student types in sin(90°) and gets 1, then he/she believes it demonstrates sine in a triangle representation because 1 is the length of one of the sides in the triangle.

The student says the unit circle is the same process because the unit circle is all the special right triangles together. Again, he/she is referencing the activity done during class using special right triangles to create the unit circle. However, it does appear that the student understands how to represent both sine and inverse sine on the unit circle because Student B discussed inputting and angle and outputting a coordinate and inputting a coordinate and outputting an angle. The coordinates and angles were all ones from the unit circle.

The connection to the graph for sine and inverse sine is weak. The student is not able to state the procedure for sine and inverse sine in a graphical representation even though he/she used both earlier.

STUDENT B: Um, well, for sine, it’s just like a normal sine graph so it’s related to the unit circle part. You get the unit circle from the graph because at 90° it’s 1 and 180 it’s 0, at 90 it’s 1, 180, 0, Oh crap. Cosine is the first one. Cosine is x, oh
well, too late. Um, so this is just the unit circle without all the points in between. This is just the four angles.

TEACHER: Alright
STUDENT B: The inverse sine I never used with a graph.

It is interesting that the student is unable to realize that he/she is utilizing the inverse sine operation when starting with a y-value and finding an x-value. The student clearly sees a difference in the inverse operation on the unit circle and the triangle, but cannot make a similar connection with the graph.

Student C also understands the input and outputs for sine and inverse sine but there is no connection to how it fits in relation to an operation and its inverse. It is interesting that Student B saw sine as inputting an angle and outputting a coordinate and Student C sees sine as inputting an angle and outputting a fraction. This representation of inputs and outputs may be one reason why Student B has some difficulty making sense of the graph. Student C is able to use “fractions” on the unit circle and on the graph.

STUDENT C: Sine, that’s the (pause). Sine of x is like what you use to find the fractions or like 1/2 or √3/2 if you have the degrees, and if you have a fraction you use that for the inverse sine to find the degrees. If you have 1/2 sine, inverse sine of 1/2, the degrees would be 30.

The student is able to use this understanding to represent sine and inverse sine in both the unit circle representation and graphical representation, but not the triangle representation. It is interesting that Student C also never referred to the opposite over hypotenuse ratio during the first interview when asked how to represent sine. This may explain why the student is unable to understand sine and inverse sine in a triangle representation. Even though he/she sees sine as pairing an angle with a fraction, the student may not see this
fraction as a ratio or the sides in a triangle and therefore cannot use it in this representation.

Student D is able to represent sine and inverse sine in all three representations. However, where Student B referred to answers for sine as coordinates and Student C referred to them as fractions, Student D changes what the output is depending on the representation.

STUDENT D: Ok, um, sine would be equal to, um, opposite over hypotenuse, and is also equal to y, the y-value on the unit circle. Um, how can I represent that in a triangle? In a triangle it would be opposite over hypotenuse, on the unit circle it would be y, and then on a graph it would be on the y-axis, that is your sine for your degree.

It is interesting that Student D only mentions the input with the last representation. It is difficult to tell if Student D understands that sine has to have an angle input with it to be linked with an output. From the explanation, the student is using the process perspective, but not necessarily to link an angle with an output. In the triangle representation, the student links sine with opposite over hypotenuse and in the unit circle the student links sine with the y-value. The student appears to be differentiating sine from cosine with the linking process in the first two representations. In the graph of sine representation, which cannot be used to represent cosine, the student sees that the angle is linked to the output. This may explain why Student D earlier had trouble seeing sin(x) as a function. Going back to the explanation in the first interview, Student D originally said that sine is opposite over hypotenuse. It was only after the student wrote it on the paper that the student changed it to say sine of x is opposite over hypotenuse. The student was unable to

206
represent unique inputs in a representation until a graphical representation was used. It was at this point the student was able to determine that sine was a function.

It is interesting that all the representations for Student D that involve inverse sine do involve a definite input and output. This means that while the student may not see the need for a specific input for sine, the student does need a specific input for inverse sine.

Student B: Um, inverse sine, let’s see, on a triangle, that’s when, this is your x (labeling angle) and that’s when you’re given that (labels opposite and hypotenuse in triangle) so then it would be 4/5, inverse sine, and that’s when you’re trying to find the degree.

TEACHER: Ok

STUDENT D: On the unit circle that’d be like these (pointing to c is question 4), so then that would be when you’re given, um, like your y-value, so then you’d know the sine of x equals that so then you’d take inverse sine to figure out what sine is.

TEACHER: Ok

STUDENT D: So I guess I could have done that for here too (letter c in question 4). And then on a graph, inverse sine, that would be, that would be, where your y value’s lying (tracing y-axis). So then I guess if you’re given the y-value, you can figure out what the degree is, the x-value by using inverse sine.

From this explanation, each time the student realizes that the answer is an angle. It is interesting also that the student says that each output for the inverse is an x-value.

However, it does not appear that the student makes the connection that this has with an inverse function operation. It is clear that he/she knows the purpose of the inverse function in all three representations.

Finally, the student is asked to specifically discuss inputs and outputs. Again, the student’s inability to represent sine as inputs and outputs is demonstrated because the student finds the question confusing.

TEACHER: Um, in terms of input and outputs, how do sine and inverse sine differ?
STUDENT D: The input and outputs, x and y. Let’s see, like what are you saying? If like…
TEACHER: For sine, what is your input and what is your output? For inverse sine, what is your input and what is your output?
STUDENT D: I don’t think I understand the question because when I think of sine I think of like the y-value and when I think of input and output I think of x and y.

Student D does not see an input for the sine function in all of the representations. If the student is given sin(x), then he/she has no trouble working with it, but in the mental representation for the student, it is only sine with no input. In the student’s understanding, there is a distinction between sine and sin(x). If the student talks about sin(x), then some of the representations are changed to incorporate an input.

TEACHER: Ok, um, sin(x) is a function, right? So what do you input into sin(x) and what do you get out of sin(x)?
STUDENT D: Oh, ok, so then if I had like the sine of 90 it would be 1. So then are you asking like if I have inverse sine of 90, something like that? Is that what you’re trying to ask? You would have degrees, you’d have inverse sine of 1. So they’re opposite, I guess.
TEACHER: Ok. So what do you input into sine?
STUDENT D: Into sine you input degrees.
TEACHER: Ok, what do you input into inverse sine?
STUDENT D: Um, y-values.

It Student D thinks sine, the student is using a representation that makes it unique from cosine or tangent, but when the student hears sin(x), the student is using a representation to actually find an output. The student is also able to state that sine and inverse sine are “opposites” but he/she does not say inverse operations have opposite input and outputs.
Conclusion

The research questions this study attempted to address were:

1. How do students understand sin(x)? How do they justifying their understandings?
2. In what ways do students represent sin(x) and is the representation based on the context of the question being asked? Are they able to move to other representations and use them to make sense of representations where their knowledge lacks?
3. When looking at the graph of a trigonometric function, do students utilize the Cartesian Connection? Do they see the graph as a series of points or as a whole object, or both? Can they use a graph to solve an equation and other tasks? What other connections are they making between representations?

The first research question dealt with how participants understood the sine function. Their understanding of sine not only affected their ability to see it as a function, but also how they chose to represent it and connect the representations together. The influence of the instruction they received was also apparent from how they represented sine and what understanding they demonstrated. Studies had found that students had trouble identifying trigonometric functions as functions (Weber, 2005). Though many students in the current study seemed to understand sin(x) as a function, most were only able to draw that conclusion after citing the vertical line test in the graphical representation. Since not all the students immediately connected sin(x) with a graphical
representation, some struggled to connect their representations for \( \sin(x) \) with their understanding of functions. For example, some students who relied on a right triangle representation and the opposite over hypotenuse ratio for sine had trouble connecting it to the definition of a function they had used earlier in the interview. The problem appeared to stem from the inability to see inputs and outputs in the triangle representation. This problem was only alleviated after a change in representations. When students switched to a graph or the unit circle they were able to see angles and coordinates for inputs and outputs or simply apply the vertical line test to the graph.

Research has found that even when students know that \( \sin(x) \) is a function, they may have difficulty explaining why (Weber, 2005). This appeared to be the case with a few of the students in the current study. The only explanation that most students offered for why \( \sin(x) \) was a function was that the graph passed the vertical line test. Explanations for other representations fell short. One student believed that the ability to input values into \( \sin(x) \) verified that it was a function, and the fact the equation could be graphed was also enough to verify it was a function. There was no supporting evidence that the students had the ability to use the triangle representation to state why sine is a function.

The goal of the second research question was to see what representations students used for \( \sin(x) \) and if they were able to connect those representations together. The three most common representations in secondary mathematics are algebraic, graphical, and tabular (Moschkovich et al., 1993; Knuth, 2000b). However, for a topic like trigonometry, new representations specific to the topic are also utilized. Up to six
different representations were found for sine in some research studies (Byers, 2008). The representations the students utilized during the interviews were algebraic in the form of an equation, a right triangle, a graph in form of a sine wave, and the unit circle. Some of the use of representations was directly tied to the way the topic was presented during class and therefore was tied to the context of the question being asked (Weber, 2005). If students were presented with a task like finding the value of \( \sin(30°) \), they were most likely to use the unit circle representation because of the instructional emphasis put on the unit circle for these types of problems. For solving an equation like \( \sin(x) + 1 = 1/2 \), many students used an algebraic approach, at least at the point of entering into the problem, because that was what they had seen being utilized in class. Students also tended to avoid using a graphical representation unless they were specifically told to do so. This likely stems from a lack of the use of graphical representation being used to problem solve during instruction.

The study also investigated the connections students made between the representations. Researchers had found that students experienced difficulty seeing sine and cosine as ratios and numbers (Brown, 2006). That is, students have trouble seeing \( \sqrt{3}/2 \) as both a ratio of the sides in a triangle and a coordinate on a graph, especially if the coordinate is in decimal form. Understanding such a connection is essential to connecting the triangle representation to the unit circle or graph. Consistent with the findings of the previous research, many students in this study exhibited the same difficulty. In some of their responses students only referred to the outputs for \( \sin(x) \) as coordinates or fractions and struggled with using the triangle representation in the context.
of connecting it to the graph or unit circle. They were unable to use the triangle to estimate values because if they were using a decimal value, they did not know how to make this a ratio of the sides in a triangle.

A major problem in this study that affected participants’ understanding and their ability to make connections was that much of the information for representations, like the unit circle, was drawn from memorized coordinates or patterns. As a result of relying on memory, some students lacked a conceptual understanding of how the representations could be created and therefore they were unable to find similarities in the process for creating different representations. This pattern was consistent with findings previously reported (Gur, 2009). Knowledge students acquire through rote learning or memorization is usually context based (Hiebert & Lefevre, 1986). This was definitely the case for students who had memorized a coordinate and could say that the \( \sin(45^\circ) = \sqrt{2}/2 \) using the unit circle, but was unable to make the same conclusion on the graph for \( \sin(x) \).

Other students incorrectly recalled information and made false connections, particularly when trying to connect the graph for \( \sin(x) \) with the unit circle.

Research had found that students could have a great deal of knowledge of a representation but lack the ability to move flexibly to other representations or to change perspectives (Moschkovich et al., 1993). Similar patterns of thinking were observed among the participants in this study. For example, some students were able to use the graphical representation to find the sine of negative angles and the existence of multiple solutions to an equation that involved using the inverse sine operation. However, the same students were not able to find those solutions using a unit circle representation or
algebraic representation solved with a calculator. Their explanations appeared to show that their solutions to these types of questions were subject to the representations being used. The reluctance to change perspectives also affected students’ ability to make connections among the representations. Some students were able to connect a graph of a sine wave with an equation but only when operating from an object perspective. As a result, they were constrained when checking answers. They could not correctly account for a change in period in the equation, even when they realized it occurred in the graph.

Even (1998) identified three critical factors influencing the process of linking representations: different ways of approaching functions, context of presentation, and underlying notions. All three factors appeared to play a role in the connections students made among representations. Although there may be other ways of representing sine besides an algebraic representation, graph, unit circle, or triangle, these were the ways the instruction approached \( \sin(x) \). As a result, students similarly chose to approach \( \sin(x) \) with the same representations and it affected connections being made. Most students’ approach to representing \( \sin(x) \) in the unit circle was through memorization of coordinates for certain angles. This made estimating coordinates for an angle like 65° and connecting it the graphical representation difficult. Likewise, when representing \( \sin(x) \) in a triangle, it usually involved using a triangle that already had all of the information present. When students had to create a right triangle, for example a right triangle to illustrate 65°, they were unable. Given a triangle that had the dimensions necessary, they would probably have no trouble using the inverse operation to verify an angle measure of 65° or create a ratio that would equal \( \sin(65°) \).
Context played a key role in how the participants solved a problem. The ordering of the questions played a role in the representations the student used. As previously mentioned, students did not choose to use a graphical representation until they were told to do so or were given one to work with. The context of the questions led students to use representations other than graphs to answer questions. Their belief about whether or not an answer should be exact or an estimate also affected what representation they used. If the context of the question led students to believe an exact value was needed, then students were reluctant to use a graphical representation. The context of some of the questions also seemed misleading and confusing to students. For example, the students were asked to find a solution to \( \sin(x) = 2 \). Because the students were asked to find an answer, some continued to change representations in an attempt to find an answer even when a representation supported the idea that there was no solution.

Finally, students’ underlying notions about \( \sin(x) \) played a role because the way they understood a representation affected how they understood \( \sin(x) \) and connected it to other representations. The belief that the unit circle only contained memorized points affected how they connected it to the graph and when they choose to change representations to find an answer. One student even referenced an incorrect notion of what makes a graph a function; in this case any graph that crosses the x-axis, as a basis for determining that \( \sin(x) \) was a function. Another student could not identify sine as a function because sine was a ratio of opposite over hypotenuse and therefore had no inputs or outputs.
The third research question in the study focused on whether students could connect the equation and graph of sine using the Cartesian Connection. However, the most common connection the students appeared to make was between the graph and the unit circle. No evidence was obtained about whether students were able to use the Cartesian Connection to connect the graph to the equation, and specifically connect a point \((x, y)\) on the graph with the equation \(y = \sin(x)\). The students conveyed the understanding that using the graph for \(\sin(65^\circ)\) meant finding the \(y\)-value on the graph for an angle of \(65^\circ\), and some were able to figure out that \(2/3 = \sin(x)\) meant starting with a \(y\)-value of \(2/3\) on the graph and then looking for the \(x\)-value angle measure for the answer. Some of the students also went as far as to say that the \(x\)-value answer meant that the place on the graph where \(\sin(x) = 2/3\) or the \(y\)-value answer is the place that equaled \(\sin(65^\circ)\). Those students were able to use and understand the Cartesian Connection in this instance only that connected the equation to the graph for a specific value.

The use of the Cartesian Connection appeared to be limited to the context used. When students were asked to create a graph for \(\sin(x)\), they did not use the equation \(y = \sin(x)\) directly during the explanation. Instead, they approached the graph from an object perspective to find a pattern for the wave and supplemented it with memorized coordinates for the axes angles on the unit circle. As a result, it is not possible to determine if the students knew that \((90^\circ, 1)\) or \((180^\circ, 0)\) were points on the graph because they were also solutions to the equation \(y = \sin(x)\). Instead, the students understood them as points on the graph because the pattern for a sine wave starting at 0 was middle, up,
middle, down, middle and the up was at (90°, 1) because 1 is the coordinate on the unit circle for 90°. This was not the same thing as connecting the point to the equation \( y = \sin(x) \) because if the unit circle representation was removed, then so was the ability to locate the coordinates. Proof of this can be found when students incorrectly recalled a point on the unit circle to say that \( \sin(30°) = \sqrt{3}/2 \) or the sine graph starts at 1 because the unit circle has a coordinate of (1, 0) and sine is the x-value. To fix these errors, students had to be convinced that their unit circle representation was incorrect first. The students had a much stronger understanding of the connection between the graph and the unit circle than they did of the graph and the equation.

When attempting to find the equation of a graph most students focused on the object perspective for both the equation and the graph and were successful in finding the features to create an equation. However, only one student actually used the Cartesian Connection to check specific points on the graph to check the accuracy of an equation. Other students had no other way to check the graph except to graph the equation from the object perspective, which utilized features like amplitude, period, reflections, and translations and not finding specific \((x, y)\) coordinates on the graph.

This study also aimed to take a closer look at how and when the students used graphical representations. These results may also explain why the students did not use the Cartesian Connection more extensively when making connections. Research has shown that students rely on an object perspective for easier examples of sine graphs and switch to a combination of object and process perspectives when the graphs are more complicated (Even, 1998). This appeared to be the case with the students’ use of the
object perspective for the graphs for which they were trying to find the equation. In the first interview, the students used the object perspective to find all the major components that they could in the equation like amplitude and translations, and then stopped because they could not figure out how to proceed from there. The one student who continued with the problem switched to the process perspective to finish writing the equation. It appears that the inability to use the process perspective to find an equation from a graph may play a role in the inability to use Cartesian Connection in that context.

Research has also shown that students do not always associate a point on a curve to its coordinate \((a, f(a))\) (Van Dyke & White, 2004). Because of the strong connection the students had formed between the graph and the unit circle, this appears to have been the case in the current study. Students in this study readily connected a point on the graph to being an angle and the \(y\)-coordinate on the unit circle, but there was little evidence that they also understood it to mean \((x, \sin(x))\), other than in the context where the student is asked to find the sine of a specific angles measure.

Knuth (2000a) found that students rely on algebraic representations and symbolic manipulation first before trying graphical representations, especially when estimating a solution is required. Most students in the current study showed this tendency by using the calculator to solve equations or produce estimates for tasks such as \(\sin(20^\circ)\) or \(2\sin(x) + 1 = 2/3\). However, the lack of understanding of the inverse operation in the algebraic representation forced some students to change representations and others to change representations to verify an answer. During the first interview, no student used a graphical representation to find the answer unless it was suggested to do so. It would
appear that they did not know that using a graph was a method for finding a solution. This was also consistent with prior research (Knuth, 2000b). Later in the interview, when the students were told to change to a graphical representation, they were able to use the Cartesian Connection, but this was a similar context to the tasks mentioned earlier.

In some instances the Cartesian Connection was not used because students had other connections between the representations that were stronger. The concept of a parent function is very important to the understanding of trigonometry (Hirsch et al., 1991). The representation of the parent graph for \( \sin(x) \) is one that all the students appeared to have used based on their responses. They were able to use the parent graph representation to build connections to the unit circle and the algebraic representation of an equation. The strength of these connections lessened the need to use the Cartesian Connection.

Implications for Teaching

Research has shown that too much of trigonometry is focused on procedural knowledge and paper-and-pencil skills and lacks opportunities for students to apply the procedures they have learned (Weber, 2008). Research has also recommended that instruction should focus on conceptual understanding, the use of multiple representations, modeling, and problem solving (Hirsch et al., 1991). While the instruction participating students received did include some of these elements, it could benefit from spending more time developing the conceptual knowledge and applying it. Much of what the students were asked to do during class was accomplished by executing a procedure or
recalling memorized information. Asking the students to synthesize information or develop a better conceptual framework for a representation will help the students to not only recall the information from memory more accurately but also to apply it as necessary. This will lead to a better conceptual understanding. Procedural knowledge is limited unless it is connected to conceptual understanding (Silver, 1986). This was definitely the case with many of the responses from the students.

The use of graphs during instruction could also be more extensively and appropriately utilized so to guide student understanding. While the students in this study had success using the sine graph from an object perspective and even connected the graph with an equation, there were limited connections that the students made from the process perspective. It seems that too much emphasis was put on the graph as an end result instead of establishing it as a means to end or as a tool for problem solving. While students showed an ability to memorize a pattern and connect features like amplitude, reflections, and translations to and equation, the students did not readily use a graph for completing tasks or connecting representations. Instead, once the students were able to create the graph their understanding of it stopped. They did not think to use this graph as a tool unless they were told to do so. Students need the ability to know which representations and perspectives are likely to be useful in a particular context and to switch flexibly among representations and perspectives (Moschkovich et. al, 1993). It does not appear the students in the current study have the ability to fully incorporate a graphical representation into such an understanding.
Students could also benefit from an emphasis on the equation for sine in the form of \( \sin(\text{angle}) = \text{ratio} \). By highlighting that a ratio can also be represented as a coordinate on the unit circle and as a y-coordinate on the graph, students will be in a better position to rely on this interpretation when needed. Placing more emphasis on this form of the equation should help students see \( \sin(x) \) as a function. If they are able to connect the angle with an input of \( x \) or even \( \theta \), it should help them represent \( \sin(x) \) and \( \cos(x) \) with multiple representations. The confusion students had with connecting the triangle representation with a SOH CAH TOA ratio to another representation like the unit circle appeared to be caused by the students linking sine or cosine with the appropriate ratio without an angle input. For example, sine is opposite over hypotenuse. When working within the unit circle, there was always an angle linked with a coordinate. Because their representations for right triangles did not involve specific angle measures, students did not appear to realize that the individual coordinate for the angle measure was also the SOH CAH TOA ratio for the angle in the right triangle. Similar difficulty existed with connecting a right triangle to the specific graph where the ratio was just the y-value. Emphasizing \( \sin(\text{angle}) = \text{opposite over hypotenuse} \) could help students make this necessary connection.

Problems with representations do not necessarily emerge from a lack of knowledge but from difficulty putting relevant information together to form connections (Moschkovich et al., 1993). From the responses students provided, it became apparent that they did indeed possess a great deal of knowledge for each individual representation, but the knowledge of connections among those representations was not strong enough to
allow the students to develop a deep understanding of trigonometric functions. Emphasis on this form of the equation in those representations could be a key to motivating a deeper understanding.

Emphasis on \( \sin(\text{angle}) = \text{ratio} \) would also help students with the inverse sine operation. Since students spend time learning about inverse operations, they should be able to make the connection that the inverse operation for sine switches the input and output. This should lead then to the understanding that \( \sin^{-1}(\text{ratio}) = \text{angle} \). An understanding such as this could be used to remind students that there are multiple solutions by connecting it with the unit circle or graphical representation where there are multiple angles that have the same ratio for sine.

Another way that could help students see \( \sin(x) \) as a function is to work more with the tabular representations. When students were discussing inputs and outputs with functions, most automatically generated a table for part of the explanation. However, when the students discussed trigonometry, there were only a few instances that students tried to employ a tabular representation. The instruction they received could be improved if it included more use of trigonometric functions in tabular form. This may also help the students to approach a graph from the process perspective and allow students to utilize the Cartesian Connection by expressing the points from the graph in a table and then connecting the table to the equation. The students appeared to understand that a table is usually connected to an equation. If students saw these representations more in class, then it is possible they would eventually be able to make the same connection between the graph and equation in the process perspective without relying on a table.
Finally, students could benefit from using representations to make estimates instead of finding exact-value solutions all the time. Research has found that for many students knowledge of trigonometry consists of memorized rules that are used with the unit circle (Gur, 2009). For the students who took part in the interviews, this knowledge was apparent, and for the most part, they were only able to use it to find values for the 16 common angles on the unit circle. If students were given more opportunities to use the unit circle and right triangles to estimate the sine value for other angles, students could develop not only a better understanding of sine as a function, but also find new connections for representations like the right triangle, unit circle, and graph.

Weber (2008) stated that to understand a trigonometric function a student should be able to anticipate the approximate result and reason about it without actually performing the steps of the procedure for calculating it. From the results of the interview in the current study, it would appear that not all the students were quite at this stage in their understanding. However, based on their knowledge of individual representations, if they were able to make better connections, then the students could eventually reach this level. Instruction more focused on the connections of representations, including the Cartesian Connection, would greatly aid the students on the path to a better understanding of trigonometric functions.

Future Research

The instruction the students received appeared to limit their ability to use graphs to problem solve. The emphasis was placed on creating graphs, not using the information
within the graphs to complete tasks. It would be interesting to see how students who received instruction that included graphing as a tool for understanding would approach the tasks in the interview. It is possible that the students would use different connections among their representations and utilize the process perspective more often for \( \sin(x) \). This may also cause students to place more value on the graphical representation and Cartesian Connection and use them more often.

Another important aspect to consider is the role radians and degrees play in the students’ understanding or trigonometry. For simplicity purposes, the interviews only involved tasks that used angles measured in degrees. It is not known what role expressing angles in radians could have on students’ representations for \( \sin(x) \) as well as what connections and perspectives they would use. Since many radian measures are expressed in terms of \( \pi \), it would be important to investigate how students make sense of this with their representations. Along the same lines it is important to take a more detailed look at how obtuse angles measures and negative angles measures affect student understanding. While the study did include some tasks that involved such angles measures, more research on how students represent these types of angles is still necessary.

Finally, it would be important to consider how students’ representations for tangent affect their understanding of trigonometry. While it could be expected that the students would use similar representations and connections for the cosine function, many of the same representations for tangent would not be as straightforward. For example, while \( \sin(x) \) or \( \cos(x) \) can be represented on the unit circle as a coordinate, tangent would
have to be represented as a ratio of the coordinates. It would be of interest to see how
c students incorporate this into their representation of the unit circle and the connections
they make to other representations. Similarly, the graph of tangent with vertical
asymptotes and an unrestricted range could present more difficulty to students. How
would this affect their connections and perspective for the graph? While this study did
shed some light on students’ understanding of trigonometry, there is still more that is not
known.
References


Appendix A: Verbal Recruitment Script

I am currently doing a study for my master’s thesis on the relationship between trigonometric equations and their graphs. For my study I am in need of some student input to give me insight into how students in my classes use the knowledge they have learned in the course to represent mathematics and solve problems. I am in need of six volunteers to take part in a one-on-one interview. Participation is completely voluntary and all necessary precautions will be taken to make sure your identity is not revealed in any part of the study. There will be two separate interviews. Each one will last between forty-five and sixty minutes and will be recorded. However, no identifiable information, such as your name, will be include in any recordings or transcripts and no one else other than me and my advisor will have access to the interview results. If you are interested in helping my study, please see me for a consent form for you and your parents to sign. I thank you in advance for your consideration.
Appendix B: Consent Form
The Ohio State University Consent to Participate in Research

Study Title: A Study of the Cartesian Connection with Respect to Trigonometric Functions

Researcher: Dominic Marchi

This is a consent form for research participation. It contains important information about this study and what to expect if you decide to participate.

Your participation is voluntary.

Please consider the information carefully. Feel free to ask questions before making your decision whether or not to participate. If you decide to participate, you will be asked to sign this form and will receive a copy of the form.

Purpose: To obtain knowledge and insight into students’ understandings of trigonometry

Procedures/Tasks: Two one-on-one interviews that will be video recorded. The interviewer will ask questions related to trigonometry to gauge student understanding.

Duration: 45-60 minutes each

You may leave the study at any time. If you decide to stop participating in the study, there will be no penalty to you, and you will not lose any benefits to which you are otherwise entitled. Your decision will not affect your future relationship with The Ohio State University.

Risks and Benefits: No risk to participants. A better knowledge of the subject matter may result from the interactions during the interview.

Confidentiality:

Efforts will be made to keep your study-related information confidential. However, there may be circumstances where this information must be released. For example, personal information regarding your participation in this study may be disclosed if required by state law. Also, your records may be reviewed by the following groups (as applicable to the research):

- Office for Human Research Protections or other federal, state, or international regulatory agencies;
- The Ohio State University Institutional Review Board or Office of Responsible Research Practices;
- The sponsor, if any, or agency (including the Food and Drug Administration for FDA-regulated research) supporting the study.

Incentives: None
Participant Rights:

You may refuse to participate in this study without penalty or loss of benefits to which you are otherwise entitled. If you are a student or employee at Ohio State, your decision will not affect your grades or employment status.

If you choose to participate in the study, you may discontinue participation at any time without penalty or loss of benefits. By signing this form, you do not give up any personal legal rights you may have as a participant in this study.

An Institutional Review Board responsible for human subjects research at The Ohio State University reviewed this research project and found it to be acceptable, according to applicable state and federal regulations and University policies designed to protect the rights and welfare of participants in research.

Contacts and Questions:

For questions, concerns, or complaints about the study, or if you feel you have been harmed by participation, you may contact Dominic Marchi at marchi.9@osu.edu or dmarchi@ceducation.org

For questions about your rights as a participant in this study or to discuss other study-related concerns or complaints with someone who is not part of the research team, you may contact Ms. Sandra Meadows in the Office of Responsible Research Practices at 1-800-678-6251.
Signing the consent form

I have read (or someone has read to me) this form and I am aware that I am being asked to participate in a research study. I have had the opportunity to ask questions and have had them answered to my satisfaction. I voluntarily agree to participate in this study.

I am not giving up any legal rights by signing this form. I will be given a copy of this form.

Printed name of subject  Signature of subject

AM/PM

Date and time

Printed name of person authorized to consent for subject  Signature of person authorized to consent for subject
(when applicable)  (when applicable)

AM/PM

Date and time

Investigator/Research Staff
I have explained the research to the participant or his/her representative before requesting the signature(s) above. There are no blanks in this document. A copy of this form has been given to the participant or his/her representative.

Printed name of person obtaining consent  Signature of person obtaining consent

AM/PM

Date and time
Appendix C: Assent Form
The Ohio State University Assent to Participate in Research

**Study Title:** A Study of the Cartesian Connection with Respect to Trigonometric Functions

**Researcher:** Dominic Marchi

- You are being asked to be in a research study. Studies are done to find better ways to treat people or to understand things better.
- This form will tell you about the study to help you decide whether or not you want to participate.
- You should ask any questions you have before making up your mind. You can think about it and discuss it with your family or friends before you decide.
- It is okay to say “No” if you don’t want to be in the study. If you say “Yes” you can change your mind and quit being in the study at any time without getting in trouble.
- If you decide you want to be in the study, an adult (usually a parent) will also need to give permission for you to be in the study.

1. **What is this study about?**
   
   The study is about your understanding of trigonometry and how you use it to solve problems as well as what representations you use.

2. **What will I need to do if I am in this study?**
   
   You will take part in one-on-one interviews with the researcher where you will discuss questions that attempt to gauge how you understand, represent, and use trigonometry.
3. How long will I be in the study?

You will be asked to participate in at most two one-on-one interviews lasting between 45 and 60 minutes each.

4. Can I stop being in the study?

You may stop being in the study at any time.

5. What bad things might happen to me if I am in the study?

There are no risks. All the questions do not have right or wrong answers but are meant to help the researcher understand how you learn and use trigonometry. Your grade is in no way affected by how you answer the questions or if you chose to stop participating.

6. What good things might happen to me if I am in the study?

You may develop a better understanding of the trigonometric material discussed during the one-on-one interviews.

7. Will I be given anything for being in this study?

There are no incentives for participating.

8. Who can I talk to about the study?

For questions about the study you may contact Dominic Marchi at marchi.9@osu.edu or dmarchi@cdeducation.org.

To discuss other study-related questions with someone who is not part of the research team, you may contact Ms. Sandra Meadows in the Office of Responsible Research Practices at 1-800-678-6251.
Signing the assent form

I have read (or someone has read to me) this form. I have had a chance to ask questions before making up my mind. I want to be in this research study.

Signature or printed name of subject

Date and time

AM/PM

Investigator/Research Staff

I have explained the research to the participant before requesting the signature above. There are no blanks in this document. A copy of this form has been given to the participant or his/her representative.

Printed name of person obtaining assent

Signature of person obtaining assent

Date and time

AM/PM

This form must be accompanied by an IRB approved parental permission form signed by a parent/guardian.
Appendix D: Parental Permission Form
The Ohio State University Parental Permission
For Child’s Participation in Research

Study Title: A Study of the Cartesian Connection with Respect to Trigonometric Functions

Researcher: Dominic Marchi

This is a parental permission form for research participation. It contains important information about this study and what to expect if you permit your child to participate.

Your child’s participation is voluntary.

Please consider the information carefully. Feel free to discuss the study with your friends and family and to ask questions before making your decision whether or not to permit your child to participate. If you permit your child to participate, you will be asked to sign this form and will receive a copy of the form.

Purpose: To obtain knowledge and insight into students’ understandings of trigonometry

Procedures/Tasks: Two one-on-one interviews that will be video recorded. The interviewer will ask questions related to trigonometry to gauge student understanding.

Duration: 45-60 minutes each

Your child may leave the study at any time. If you or your child decides to stop participation in the study, there will be no penalty and neither you nor your child will lose any benefits to which you are otherwise entitled. Your decision will not affect your future relationship with The Ohio State University.

Risks and Benefits: No risk to participants. A better knowledge of the subject matter may result from the interactions during the interview
Confidentiality:

Efforts will be made to keep your child’s study-related information confidential. However, there may be circumstances where this information must be released. For example, personal information regarding your child’s participation in this study may be disclosed if required by state law. Also, your child’s records may be reviewed by the following groups (as applicable to the research):

- Office for Human Research Protections or other federal, state, or international regulatory agencies;
- The Ohio State University Institutional Review Board or Office of Responsible Research Practices;
- The sponsor, if any, or agency (including the Food and Drug Administration for FDA-regulated research) supporting the study.

Incentives: None

Participant Rights:

You or your child may refuse to participate in this study without penalty or loss of benefits to which you are otherwise entitled. If you or your child is a student or employee at Ohio State, your decision will not affect your grades or employment status.

If you and your child choose to participate in the study, you may discontinue participation at any time without penalty or loss of benefits. By signing this form, you do not give up any personal legal rights your child may have as a participant in this study.

An Institutional Review Board responsible for human subjects research at The Ohio State University reviewed this research project and found it to be acceptable, according to applicable state and federal regulations and University policies designed to protect the rights and welfare of participants in research.

Contacts and Questions:

For questions, concerns, or complaints about the study, or if you feel your child has been harmed by participation, you may contact Dominic Marchi at marchi.9@osu.edu or dmarchi@cdeducation.org.

For questions about your child’s rights as a participant in this study or to discuss other study-related concerns or complaints with someone who is not part of the research team, you may contact Ms. Sandra Meadows in the Office of Responsible Research Practices at 1-800-678-6251.
Signing the parental permission form

I have read (or someone has read to me) this form and I am aware that I am being asked to provide permission for my child to participate in a research study. I have had the opportunity to ask questions and have had them answered to my satisfaction. I voluntarily agree to permit my child to participate in this study.

I am not giving up any legal rights by signing this form. I will be given a copy of this form.

__________________________________________________________________________

Printed name of subject

__________________________________________________________________________

Printed name of person authorized to provide permission for subject Signature of person authorized to provide permission for subject

__________________________________________________________________________

Relationship to the subject Date and time

__________________________________________________________________________

Investigator/Research Staff

I have explained the research to the participant or his/her representative before requesting the signature(s) above. There are no blanks in this document. A copy of this form has been given to the participant or his/her representative.

__________________________________________________________________________

Printed name of person obtaining consent Signature of person obtaining consent

__________________________________________________________________________

Date and time AM/PM
1) How are the equation of a function and the graph of a function related? Explain your thinking.

2) The following is the graph of $y = x^2 - 2x - 3$. How many points are on this graph?

3) Can you give examples of functions and examples of non-functions? How can you tell when you have one? Are the multiple ways to tell?

4) What does $\sin(x)$ mean? Is it a function? How can you tell?

5) How could you find or estimate the $\sin(30°)$? How would find or estimate $\sin(20°)$?

6) Can you think of any other ways to do #5?
7) Explain how you would solve the following equations:
\[
\begin{align*}
\sin(x) &= \frac{1}{3} \\
2\sin(x) + 1 &= \frac{2}{3}
\end{align*}
\]

8) Can you think of any other way to solve the equations in #8?

9) If \( \sin(x) = 2 \), what is the value of \( x \)? How do you know?

10) Is there a connection between the unit circle and the graph of \( \sin(x) \)?

11) Explain how you can graph \( \sin(x) \)

Look at the following graph of \( y = \sin(x) \)

12) Using the graph, can you find or estimate \( \sin(20^\circ) \)?

13) Using the graph, if the \( \sin(x) = -1 \), what is the value of \( x \)?
Look at the following graph

14) Is this graph a function? How can you tell?

15) Can you tell if the graph is of a sin(x) function or a cos(x) function? Can it be both? How do you know or what information do you need to make a decision? How would you write the equation for this graph?
Appendix F: Second Interview Questions

1) Graph $y = \sin(x)$. Explain how you created your graph.

Using your graph, find the following:

a) $y = \sin(80^\circ)$

b) $y = \sin(-150^\circ)$

c) $0.75 = \sin(x)$

d) $-\frac{1}{4} = \sin(x)$
2) Which of the following equations best describes the graph? Explain how you know.

a) $y = 2\sin(x) - 1$

b) $y = \sin(2x - 1)$

c) $y = -2\sin(x) - 1$

d) $y = -\sin(2x) - 1$

3) How could you check your answer to number 2? Are there multiples ways to check?

4) Given the unit circle, special right triangles, and the graph of $y = \sin(x)$, (figures 5, 6, and 7) answer the following questions.
   a) What is the $\sin(45^\circ)$
   b) What is the $\sin(65^\circ)$
   c) If $-1/2 = \sin(x)$, what is the value of $x$?
   d) If $0 = \sin(x)$, what is the value of $x$?
   e) If $3/2 = \sin(x)$ what is the value of $x$?
   f) Solve the equation $\sin(x) + 1 = 1/2$
   g) Solve the equation $3\sin(x) = 2$

5) How do your answers to a and b differ from c through e?

6) Discuss sine and inverse sine in terms of inputs and outputs. How can you represent them in a triangle? On the unit circle? On a graph?
Figure 5

Figure 6

245
Appendix G: First Interview Transcript, Student A

TEACHER: OK, so first question. How are the equation of a function and the graph of a function related?
STUDENT A: Um, Do I have to write this down or...
TEACHER: Whatever you want. You can just tell me.
STUDENT A: Um, both can provide points to the function.
TEACHER: OK. What do you mean by points?
STUDENT A: Both can plot points on the graph like if you we’re given an equation, you can like, just by the equation can’t you get like points on a graph? Is that right?
TEACHER: So, if a point is on the graph, what does that mean?
STUDENT A: Um, (pause). If it’s a function it can form a line?
TEACHER: So I guess what I’m trying to ask is what makes a particular graph go with one particular equation? You have a graph there and that equation (motioning to question #2). How do you know that the two go together? What tells you that they are related?
STUDENT A: The type of function it is. Like there are different types of functions, like quadratic and cubic.
TEACHER: And how can you tell that?
STUDENT A: By the exponent right there (points to equation). It think it’s quadratic.
TEACHER: Anything else that relates the equation to its graph?
STUDENT A: Um, it depends on what is in the equation, like when it, like, for this it would start at (0, 0) (pause). No, I’m wrong (pause). Well, I thought in the equation it will tell you when to move the graph, like the original graph, um, depending on what’s in it, you can move it down, up, left, or right.
TEACHER: Alright. Very good. Next question, the following is the graph of y = x² - 2x – 3. How many points are on this graph?
STUDENT A: Um, (pause). I can write on this, right? (makes x/y t-table) Um, (pause). Oh, uh, isn’t it wherever… isn’t it, can’t you tell one way by if it crosses the x-axis? Right here, (points to x-axis on graph), or is that wrong?
TEACHER: For what?
STUDENT A: Points
TEACHER: You’re just looking for points. Any point on the graph.
STUDENT A: Do I just say it point by point? Um, (3, 0), (-1, 0).
TEACHER: Any others?
STUDENT A: Um, I don’t know what it’s called but where it meets (marks vertex of graph at (1, -4)), like where it like hits… (makes parabolic motion with hands for emphasis) Um, (pause) Um, I want to say (0, -3) because there’s a -3 there (points to equation)
TEACHER: Ok
STUDENT A: So four?
TEACHER: So that graph has four points?
STUDENT A: Yes. It has more (pause). Infinite. I just don’t know where they are. I
don’t know. That’s all I got.
TEACHER: Alright. How many points are between here and here (points to (0, -3) and
(3, 0)). How many points are on that graph?
STUDENT A: Um, in between these two? Four, just because they’re on the line.
TEACHER: Ok, very good. Next question, can you give examples of functions and
examples of non-functions? How can you tell when you have one? Are the multiple ways
to tell?
STUDENT A: Um, ok. I can just write any examples down? (writing equation) Wait,
(pause). Isn’t if it has f of x equals something then it’s a function and if it’s just like y
(pause), then it’s not a function? Maybe (pause), I don’t know. Wait, (pause). That
looks like it might be one too (referring to y = equation on paper). I want to say that they
are both functions. I don’t really know what a non-function is. A non-function would be
something that doesn’t pass the vertical line test? I’m trying to think of a non-function.
Oh, um, hmm (pause). Well, to answer that question, there are multiple ways you can
tell.
TEACHER: Besides the vertical line test, what are the other ways?
STUDENT A: There’s the horizontal line test for linear equation. I could be so wrong.
Um (pause). I can’t think of a non-function
TEACHER: You’ve given me two equations. Can you show me a function any other
way besides an equation?
STUDENT A: Oh, can’t you have a graph? (draws a graph) Function.
TEACHER: Can you give me a graph that’s not a function?
STUDENT A: (draws a graph) That’s a function.
TEACHER: Ok, so that’s not a function?
STUDENT A: Well, isn’t it if you have something over zero it’s undefined. Ok, that’s
wrong. Ok, not a function.
TEACHER: Why?
STUDENT A: Because it doesn’t pass the horizontal line test.
TEACHER: Ok, so you gave me some graphs that are not functions, can you give me any
equations?
STUDENT A: Yes, I’m trying to think of an equation.
TEACHER: That first one you made there, what’s the equation of that one?
STUDENT A: Isn’t that a quadratic equation?
TEACHER: Is that a function?
STUDENT A: Yes
TEACHER: What about the second one?
STUDENT A: I don’t know. I kind of just did it. Um (pause), could that not be a
function?
TEACHER: Why?
STUDENT A: Because it doesn’t pass the vertical line test.
TEACHER: Ok, so it has to pass both the vertical line test and the horizontal line test?
STUDENT A: No, um (pause), I don’t know!
TEACHER: Ok, that’s fine. So besides the vertical line test and horizontal line test, is there any other way you can tell a function is a function or not?
STUDENT A: Oh my gosh! Isn’t it, that like, ok, I can’t remember which one is which, um, the input can have only one output?
TEACHER: Ok, so what does that mean?
STUDENT A: (Draws table with values and graph). Ok, that’s just another way to see if it passes the vertical line test.
TEACHER: Ok, but what are you doing?
STUDENT A: Um, why am I doing that? I’m determining if it’s a function or not.
TEACHER: And how are you doing that? Is that a function right there?
STUDENT A: No because there’s no equation. So if I was given an equation, I would try to figure out the points and if there’s more than one output for an input then it’s not a function.
TEACHER: Ok, but does that satisfy the definition you gave me?
STUDENT A: No
TEACHER: Why not?
STUDENT A: Because if it’s a function there’s no output (changes table). That’s not a function.
TEACHER: Alright, moving on. Now we’re moving on to the trig stuff. So that was a review of algebra concepts to give me some background about how you use algebra. Now we’re looking at trigonometry. What does sine of x mean? Is it a function? How can you tell?
STUDENT A: Not a function because you are not given any points. But doesn’t it mean (pause) hypotenuse over (pause) opposite? Wait, (pause). Wait, yeah, isn’t it hypotenuse over opposite side, right? Oh my gosh! No! Yeah! Wait (pause), um, it’s opposite over hypotenuse.
TEACHER: Because?
STUDENT A: (drawing a triangle in a coordinate plane) OK, so that’s just a graph and that’s an example of the triangle if you were trying to find a point. That’s the point we’re given and, uh, opposite over adjacent. So it’s a circle on a graph.
TEACHER: Ok
STUDENT A: Um, if that’s a circle, wait (pause), hold on, I’m trying to, uh, (pause), I have to wait for my mind to catch up. I know what I’m trying to say. Ok, so just imagine it as a circle like you gave us and it had all the points, like that was 0°, that was 90°, I’m just trying to explain sine. Um, ok, so this is how I see it. You have a point here and you have a triangle right here and so imagine that this is really touching this (referring to triangle inside the circle student drew). Um, so to describe the triangle when we go over, this is cos. Ok, so we have the points. The points are always cos and then the sine for x and y. That’d be cos and that’d be sine and in that triangle I said it would be corresponding to that point would be opposite over adjacent.
TEACHER: Opposite over adjacent?
STUDENT A: No, wait. Opposite over hypotenuse and then cosine is adjacent over hypotenuse or $x$ over $r$, and $r$ is the hypotenuse.
TEACHER: Ok, very good. So you said sine of $x$ is not a function.
STUDENT A: Hmm (pause). Wait (pause). Yes. No. Yes. We’ll, you’re not given any numbers.
TEACHER: $x$ is your number.
STUDENT A: So variables, they could be anything. It’s a function because you can replace $x$ with any number because it’s a variable and $x$ is only there to fill in the space.
TEACHER: Ok, next question, how could you find or estimate the sine of $30^\circ$?
STUDENT A: Um, sine of $30^\circ$? Um, so this has to be 60, 30. I mean I know in my head but try and find it.
TEACHER: What is it?
STUDENT A: $\sqrt{3}/2$
TEACHER: Ok, convince me it’s $\sqrt{3}/2$ or at least show me what you’ve done.
STUDENT A: See, that how good I have the unit circle down, Mr. Marchi. I remembered it. Oh, well, that’s a triangle, so (pause). How do I explain that? I don’t know how to explain how I remember.
TEACHER: In your drawing where’s sine?
STUDENT A: Sine is (pause). Wait, wait (pause). You asked for sine, right?
TEACHER: Yes
STUDENT A: Oh, it’s 1/2
TEACHER: 1/2?
STUDENT A: Yes
TEACHER: Ok. Because? What changed your mind?
STUDENT A: Because I just remembered that it’s 1/2
TEACHER: In your drawing there, can you use it to show me why it’s 1/2?
STUDENT A: Um (long pause).
TEACHER: What are you trying to do?
STUDENT A: I don’t know. I thought that would help me, more numbers.
TEACHER: More numbers? What were you going to do? I see $2\pi$, and $\pi$. What are you trying to do?
STUDENT A: Well, that’s $\pi/2$.
TEACHER: Why are you switching to radians?
STUDENT A: I thought it would help me because that’s in fractions too.
TEACHER: Did it?
STUDENT A: No.
TEACHER: So, the sine of $30^\circ$ is 1/2, from that picture. Ok, how does that picture prove the sine of $30^\circ$ equals 1/2? How does it convince you?
STUDENT A: Well, on a perfect circle it’s the same distance from the center
TEACHER: Your picture is fine. What are you trying to do with it?
STUDENT A: I don’t know. Um, (pause), well, (pause), I don’t know (pause). How do I (pause), wait, (pause), I’m lost for words. I do not know how to explain.
TEACHER: That’s fine. So your intuition said it was 1/2. You try to draw a picture which is good, but you can’t use it to explain. Is there any other way you can explain why sine of 30° is 1/2?

STUDENT A: Oh! I think so. No, I don’t want to change to radians.

TEACHER: What about a calculator?

STUDENT A: What about a calculator? The calculator will obviously tell you. Um, you hit the sine button. Wait; make sure it’s in radians, or no degrees. Then you input sine of 30°.

TEACHER: Ok, very good. Now, if the sine of 30° is 1/2, how could you estimate the sine of 20° or if you know the exact value?

STUDENT A: Um, you could take (pause), knowing what the coordinates or something are like the coordinates for 30° is (√3/2, 1/2) so…

TEACHER: OK, so write that down

STUDENT A: Oh, and then I don’t know how to explain it but I know that forty-five degrees is √2/2, √2/2) and that 60° is (1/2, √3/2) so and then 90° is (0, 1/2). What you could do is find two points on the circle, er, ok, how I was going to say I was going to find points on the graph that you know what the coordinates are and then you just subtract it and there’s a certain formula you have to subtract it and then you find the sine of 20°.

TEACHER: Ok, that works. You’re talking about identities?

STUDENT A: Yes

TEACHER: So, we don’t have two angles to subtract to get 20° on the unit circle that you know, right?

STUDENT A: No

TEACHER: So if I want an estimate not the exact value, you just described how to find the exact value, I just want a quick estimate on what the sine of 20° should be. Based on what you have right there, can you tell me a way to estimate the sine of 20°? (Long pause). So, I say the sine of 20° is .7.

STUDENT A: Are you asking me if that’s right?

TEACHER: Yes

STUDENT A: Um, hmm

TEACHER: Or 7/10 or I say the sine of 20° is 7/10. Do you agree or disagree?

STUDENT A: Um, ok. I’m going to answer your question, I’m just trying to figure it out (talking indistinctly) 1/4, 1/8, wait, what did you say it was? .7? 20°? Um…

TEACHER: I just made up the value too.

STUDENT A: Wait, what’s the .7?

TEACHER: Sine of 20°. I’m just making up a value. The sine of 20° is 7/10. Yes or no? Does that sound reasonable or is it way off?

STUDENT A: Oh, if that’s .5, it sounds reasonable because the sine of 30° is .5 so obviously 20° would be higher.

TEACHER: Why would it be higher?

STUDENT A: Because (pause), wait, wait, I take back my answer. Ok, it’s not reasonable so the sine of 20° would be smaller because 30° is .5 so then 20° is below it so the sine would have to be below .5.
TEACHER: Ok, very good. Is there any other way you can give me an argument besides the unit circle?
STUDENT A: Hmm (attempts to change angle to radians). I thought radians would help. Um, well, I don’t know if this is right. I did it by converting to radians so π/9 and π is roughly 3.14 and so I would just round that to 3 and so that would really equal 1/3 which is .3 repeating so that’s smaller than .5, which is what the sine of 30° is.
TEACHER: Ok, so type it in the calculator to see if you’re right.
STUDENT A: Oh! Look at me go! Yea!
TEACHER: Ok, that answers number six. Question number seven. Explain how you would solve the following equations. The \( \sin(x) = \frac{1}{3} \) and \( 2\sin(x) + 1 = \frac{2}{3} \).
STUDENT A: Oh, this was one my placement test. I don’t think I did it right. Well, for this, wouldn’t you do the inverse of 1/3? But that’s not on the unit circle.
TEACHER: So how would you do it?
STUDENT A: You’d have to type it in the calculator. Um, so you do, mine is second and then you do the sine and then I type in 1/3 and that equals 19.741. And then for this one you have to subtract 1 so you have \( 2\sin(x) = \), hold on I need to use the calculator, -1/3 and then you divide by 2 or times it by the reciprocal which gives you 1/6. Hold on, I’m going to check that. And then you do the inverse of that. It equals -9.594.
TEACHER: So, what is x? Can you tell me what x is? You said x is 19.741. What does that mean?
STUDENT A: X is the y coordinate on the graph
TEACHER: Now you used algebra there to solve the equations by isolating x. Is there any other way you can solve the equation without algebra?
STUDENT A: I can’t think of anything.
TEACHER: Ok, that’s fine. OK, questions number nine, if the \( \sin(x) = 2 \), what is the value of x? How do you know?
STUDENT A: Saw that on my placement test. Um (talking to self indistinctly), well, I want to use the unit circle but I know that there’s nothing there that equals straight up two. Um…
TEACHER: What does that mean?
STUDENT A: You have to figure it out another way.
TEACHER: What other ways do you have?
STUDENT A: You could fill in x with like a number or, no, can’t think of how.
TEACHER: Can’t use a calculator?
STUDENT A: Um, I don’t know
TEACHER: What if I said solve for x?
STUDENT A: Oh, well, yeah, just type it in your calculator (types in calculator twice). It says domain error.
TEACHER: What does that mean?
STUDENT A: That means there’s no solution
TEACHER: Why?
STUDENT A: For whatever equation it was a solution to it could have gone out of the domain and range and that domain and range specify limits that the equation has. Or it’s not a rational number on the graph so it’s irrational.
TEACHER: So there’s a solution?
STUDENT A: No, I said no solution. Ok, then it means it could possibly not be a solution for whatever equation it’s given for. I mean there could be other solutions for it. I don’t know. That’s what I got.
TEACHER: Ok, next question, is there a connection between the unit circle and the graph of sine of x?
STUDENT A: Do you want me to name the similarities between them?
TEACHER: Yes. How are the graph of sine of x and the unit circle related?
STUDENT A: Well, it can be described in degrees and radians. Um, hmm (pause), since both can be plotted on the graph both can have coordinates or points.
TEACHER: Ok, how do you get those points? That goes along with question eleven. How do you graph sin of x? How do you go about getting those points?
STUDENT A: Um, well, I forget how to get to these, but there’s like period, um, starting point, ending point, um (talking indistinctly). Depending on where it starts you can determine that there’s no reflection to it, that there’s no transition, it’s just a normal sine wave. Um, there’s (pause), I want to say the period would be π or 180° and then…
TEACHER: Let’s say create the graph of sine of x.
STUDENT A: Um since there’s no transition or anything, um, well sine wave looks like that and then cosine wave looks like that. I want to say the amp is one, which is how high it goes (talking to herself indistinctly). So it starts at (0, 0) because that’s the starting point for sine. Um, amp is one. Um, I think this is how you get the period (pause). Or π/2b. I don’t know. I don’t remember which is which or how you get it.
TEACHER: What’s the period of sine?
STUDENT A: π
TEACHER: Ok, what does that mean?
STUDENT A: I forget how we got that. Um, I don’t know. I’m not sure. I forget how you get how far you plot the points from each other, but that’s what I got.
TEACHER: Alright. Four more questions. Looking at the graph you just had, can you use it to find the sine of 20°?
STUDENT A: I want to say no because it’s not a normal graph. I don’t think it’s the right graph for this type of situation.
TEACHER: What if I told you it was?
STUDENT A: Well, then you count the points (talking indistinctly trying to decipher x-axis scale) Then the sine of 20° would be somewhere, well, the way I did it was find the middle, between 0 and 90, and the middle between 45 and 0°, which is 22.5 and then go down slightly from that and bring it up to wherever the line is and that would be the solution.
TEACHER: Ok, and what is that solution?
STUDENT A: Um, 1, 2, 3. It would be... Would it be like 3? No, .35 or something.
TEACHER: Ok, what did you tell me the answer was earlier?
STUDENT A: .34717. Oh, the one I came up with? Point .3 repeating.
TEACHER: Alright. Now, if the sine of x equals -1, using your graph, what is the value of x?
STUDENT A: Um, wouldn’t it not be a solution for this?
TEACHER: Why?
STUDENT A: Because it’s not on the… Well, I know that from the unit circle that the sine of -1 equals 270° or is the y-coordinate for 270°, but this line doesn’t hit that so that wouldn’t be a solution.
TEACHER: What do you mean?
STUDENT A: The line doesn’t… the point isn’t on the line
TEACHER: What point is?
STUDENT A: 180° and then 360°.
TEACHER: What point? There should be two parts to a point.
STUDENT A: What?
TEACHER: A point has two parts, an x and a y so what is the point for 270°?
STUDENT A: Um, (0, -1)
TEACHER: Ok, on the graph there, on the unit circle you have (0, -1), but on the graph what is the point for 270°?
STUDENT A: Well, it would be (270°, 0)
TEACHER: Is that on the graph?
STUDENT A: No
TEACHER: Should it be?
STUDENT A: No. Wait, yes.
TEACHER: Why?
STUDENT A: Because you’re trying to find the solution for sine of x
TEACHER: Ok, so the sine of x never equals -1?
STUDENT A: Not in this case. If you’re asking me to look at this graph then no, it doesn’t. If it were another graph that was hitting the point 270° then I would be like yes, -1 is a solution for that graph but it’s not a solution for this one.
TEACHER: What makes it a solution?
STUDENT A: The points that are on the line that hit the line.
TEACHER: Point to one
STUDENT A: 180°. This one down here (pointing to (270°, -1)).
TEACHER: What’s that point?
STUDENT A: Oh my goodness! What!? No! That’s right. Well, it misses it by, no wait, yep it’s a solution.
TEACHER: What is?
STUDENT A: Isn’t it? Yeah, it is. (270°, -1) is a solution because you go over and then you go down and it hits the line.
TEACHER: Are there any other solutions to the sin(x) = -1?
STUDENT A: Yeah, It’s -90°.
TEACHER: Ok, anymore if you continue the graph?
STUDENT A: Yes
TEACHER: Where would they be at?
STUDENT A: Well, they’d be both directions. If you’re asking me, they’d be both directions but they will constantly hit negative one because that’s how far the amp extends above and below the x-axis.
TEACHER: Why is it a solution?
STUDENT A: Because when you look at the graph and then you graph the points that are easy to tell on the line, you go over to 270° and go down you see that it’s the point -1 on the y-axis.
TEACHER: Ok, label the x, y coordinate
STUDENT A: That doesn’t look right. Um, I could put it in radians.
TEACHER: Well, why doesn’t it look right?
STUDENT A: It looks strange. One’s in degrees and one’s not
TEACHER: Well, look at your scale.
STUDENT A: True. Ok, then it looks right.
TEACHER: How are 270° and -1 related?
STUDENT A: Because the point where the graph stops hits negative one directly and you can go straight up and the point where it meets it is 270°.
TEACHER: Ok, two more questions. Look at the following graph. Is it a function? How can you tell?
STUDENT A: Yes
TEACHER: Because?
STUDENT A: Passes the vertical line test
TEACHER: Is that all you need to say?
STUDENT A: Well, I know it stretches because I know it’s bigger
TEACHER: Is that going to affect the fact that it’s a function or not?
STUDENT A: No
TEACHER: Ok, do you need to say more than it passes the vertical line test?
STUDENT A: No
TEACHER: Ok, so given that, Can you tell if the graph is of a sine of x function or a cosine of x function? Can it be both? How do you know or what information do you need to make a decision? How would you write the equation for this graph?
STUDENT A: Um, I don’t know if you can tell.
TEACHER: Ok, what information would you need to make a decision?
STUDENT A: You would need a function first of all.
TEACHER: Can you figure out what function it is here?
STUDENT A: Uh, well (pause).
TEACHER: To determine sine or cosine what do you have to know from the graph? Can you figure it out?
STUDENT A: Well, if you’re given parts of it like the period and the amp and the starting point or ending point.
TEACHER: Ok, you said the period and amplitude. Can you find the period and amplitude from the graph?
STUDENT A: It would be hard to tell because it’s stretched. You also need like (pause) like in the function you’d be able to tell, it will tell you, like if it shifts right or left, if it shifts up or down, and like if it expands, like if there’s a number before the equation. I don’t think you can tell from this.
TEACHER: What if I said it starts right there?
STUDENT A: Original starting point?
TEACHER: Yes. It ends right there.
STUDENT A: Then that’s a cos
TEACHER: Because?
STUDENT A: Because sine always starts at (0, 0) and cosine always starts at (1, 0), no (0, 1)
TEACHER: Ok, so what if I said it starts right here and ends right here?
STUDENT A: Oh, now I see what you’re saying. Oh well, I didn’t see it, oh well, ok, so are you trying to isolate all this? Or that’s what all this is?
TEACHER: No, you said you need a starting point and ending point.
STUDENT A: That doesn’t help
TEACHER: What if I said here’s your starting point and here’s your ending point?
STUDENT A: Still wouldn’t help. I mean, I was wrong on the starting point and ending point. I don’t think you can tell just by looking at what it is.
TEACHER: Ok, well, thanks a lot.
Appendix H: First Interview Transcript, Student B

TEACHER: First question, how are the equation of a function and the graph of a function related? Explain your thinking.

STUDENT B: Ok, well, the equation of the function is always going to equal y so, um, for every… on the other side of the equals there’s an x so… for every number there you put in for x you will get an answer for y and then on the graph for every, um, on this one for example, if you put in 3 for x you got 0 for y and that’s how they relate.

TEACHER: Ok, very good. So the (3, 0) again relates to the graph how?

STUDENT B: Um, that it’s a point on the graph and you found that point with the function of the graph because you put in 3 for x and once you solved the equation you got 0 for y.

TEACHER: So, looking at the graph, the following is the graph of $y = x^2 - 2x - 3$. How many points are on this graph?

STUDENT B: Well, there’s infinite points.

TEACHER: Could you mark some on the paper for me?

STUDENT B: Well, there’s (3, 0) and there is (1, -4). There is (-1, 0). There’s the (0, -3). Um, (2, -3). That’s all the easy ones.

TEACHER: Ok, what do you mean by easy?

STUDENT B: Well, those are the only ones that I see. There’s other ones that have y on the cross between the x and y axis. There’s a bunch of other ones but they’re not… I mean you’ll have to do a bunch of math to find them.

TEACHER: Ok, very good. Now talk about functions again. Can you give examples of functions and examples of non-functions? How can you tell when you have one? Are the multiple ways to tell?

STUDENT B: Oh, um… That is a function (motioning to graph in questions #2). I know because if you graph an equation, um, you can do the vertical line test and so if you draw a bunch of lines and they never… they only intersect the graph once then it’s a function. But if you turn it like that (rots papers 90° so parabola opens sideways) and this was the x-axis and that was the y-axis and you did (motions vertically on the graph with hands), it wouldn’t work because it crosses twice. So that’s one way you can tell. Um, there are multiple ways you can tell but I can’t remember them. Um, or wait. Oh yeah, that was, wasn’t there, uh… I think if you can’t… every… every x has to have its own y I think. I think so. Um, so like if you make a t-chart you can’t have, um, you couldn’t have, wait, uh, you couldn’t have like two x’s and one of them have like a different, like 1 and 3. You couldn’t do that.

TEACHER: Ok, very good.

STUDENT B: Um, other examples would be like a graph like I said. Um…
TEACHER: What about an equation?
STUDENT B: Hmm (talking indistinctly). I don’t know. I would have to graph it or like find out some points, like put it into a t-chart. Um, I don’t know if there’s a way to find it out just by looking at the equation.
TEACHER: Can you write an equation that’s a function and an equation that’s not a function from your recollection?
STUDENT B: Well, that one is a function (motioning to equation in questions #2).
TEACHER: What about one that is not?
STUDENT B: Um (pause). I don’t think I can.
TEACHER: Ok, now moving into the trig questions. What does sine of x mean? Is it a function? How can you tell?
STUDENT B: Um (long pause). Sine of x is not a function because there is no y equals.
A function can be a function if it has y equals.
TEACHER: Ok, if it’s y equals sine of x, is it a function?
STUDENT B: Yes, maybe. Um, yes, yeah it would be a function because an x, a number for x, you get a y and you can graph it. What does it mean? It’s a (pause), something that is used for finding sides of triangles. Um, opposite over hypotenuse. Um, and I can tell it’s a function because if you put a number in for x you get a y out of the equation. As far as I know, that makes it a function. I don’t know what the graph looks like.
TEACHER: So using any means you want how could you find or estimate the sine of 30°?
STUDENT B: Um…
TEACHER: What’s the easiest way to find the sine of 30° that you know?
STUDENT B: Um, you have to use the unit circle, right? No, yes. Yeah, yeah, you would, so… Um (draws triangle), and then (1/2, 2/3) maybe, I think.
TEACHER: Ok
STUDENT B: I know it’s one of those.
TEACHER: Do you know which one?
STUDENT B: Um, the first one, maybe?
TEACHER: Ok, you can use the unit circle to find the sine of 30°. Is there any other way you can find that ratio?
STUDENT B: Um you could just have it memorized (pause). I don’t know. I give up.
TEACHER: Can you find it with a calculator?
STUDENT B: Hmm (typing on calculator). Oh yeah, you can.
TEACHER: What did you type in?
STUDENT B: Sine of 30°
TEACHER: Does that answer make sense to you? You told me here it’s either 1/2 or 2/3. Is that (calculator answer) 1/2 or 2/3?
STUDENT B: (laughing) No, um…
TEACHER: Check your mode
STUDENT B: There you go. .5 after I changed it to degree mode. So it was 1/2.
TEACHER: Ok, very good. Sine of 30° is 1/2 because you told me from the unit circle and the calculator. Can you find or estimate the sine of 20°, since that’s not one of the points on the unit circle?
STUDENT B: You could use the calculator, which gives you .3420201433.
TEACHER: Does that answer sound reasonable based on the fact that the sine of 30° is 1/2?
STUDENT B: Yes
TEACHER: Why?
STUDENT B: Because 20° is less than 30°, so it would. If this is 30, 20ish would be here so that angle would be smaller.
TEACHER: Is there any other way you could find the sine of 20°?
STUDENT B: I guess you could have it memorized again, but that’s kind of a big stretch. You already have to have the unit circle memorized (talking indistinctly).
TEACHER: Questions number seven; explain how you would solve the following equations.
STUDENT B: Alright… Um, oh, um the x would be a degree and the 1/3 is one of the x or y coordinates of one of the angles on the unit circle and I think it’s like 60° maybe.
TEACHER: Ok, you’re getting that from the unit circle. You’re looking for a ratio on the unit circle?
STUDENT B: Yes
TEACHER: Ok. What if the ratio is not on the unit circle?
STUDENT B: Then um, it’s probably not solvable. Hmm… (long pause). I don’t know how to do that one.
TEACHER: Question number nine. If sin(x) = 2, what is the value of x? How do you know?
STUDENT B: (Long pause) The 2, 2 is on the unit circle right? Yeah, I don’t know what angle it is for but it’s on there somewhere. The first quadrant I think. Um, it’s for one of the angles. 2 is one of the coordinates for one of the x or y angles. I know that. So x would be one of the angles and I don’t know the x but… How do I know that it’s one of the angles on the unit circle? Because you taught me that (laughter). Um…
TEACHER: Ok, very good. Ok, you keep referring to the unit circle. Is there a connection between the unit circle and the graph of sine of x?
STUDENT B: Yes there is. The exact connection I do not know exactly what it is but there is I think because the angles are just like the angles on the unit circle. Um, every 90° it hits. It goes from like 0 to the amplitude. The amplitude on this one is 1. Any sine of x graph will have 1. Then it goes back down to 0 and then it goes back down to -1 and 0 and that would be a full 360. So, 0, 1, 0, 1, 0 would be a full 360. And how that exactly relates to the unit circle… (laughter).
TEACHER: So, when you had to graph the sine of x during class, what did you do? Explain the process. It doesn’t have to be the way I taught you, but what comes to mind?
STUDENT B: Well, you know that the sine graph starts at 0, and the cosine graph starts at 1. So it starts at 0, it starts at the origin and, um, um, so its, er, depending on, well, just a normal sine graph, like y equals sine of x, um, it will always go up first because there is no other numbers to or negative signs to change it, like go down first or something. Um, and the amplitude of any, um, sine or cosine graph is always 1, so it will always goes up to 1 and likewise down to -1. So that’s where it starts and how high it goes and how low it goes and the, um, on a standard like y equals sine of x it’s, that it’s always, um, in
degrees, it’s always 0, 90°, 180, 270, 360 and, um, in radians it’s something else like 0… or no π first… 2?
TEACHER: And the numbers you are giving me, why are they significant? 0°, 90°, 180°, 0, π?
STUDENT B: Why are they significant? (Pause) Because they are (laughter). Uh (pause), oh, because they’re the, uh, each like the main, uh, the main angles of the unit circle.
TEACHER: Ok, very good. Alright, looking at that same graph, can you find of estimate the sine of 30°?
STUDENT B: 30° would be, uh, (tracing on graph), it would be like .4. It would be point four. That doesn’t sound right but…
TEACHER: What did you say it was earlier?
STUDENT B: I said it was 1/2, (laughter). Um…
TEACHER: So which one is more accurate, the 1/2 or your .4?
STUDENT B: The 1/2 because I did it on the calculator and graphs are sometimes not exactly precise if you’re just like looking at the lines between them.
TEACHER: Can you show me with the graph how the sine could be 1/2? I mean what would it have to be to be on the graph, for it to be 1/2 on the graph?
STUDENT B: On, no, it is 1/2 (laughter).
TEACHER: Because?
STUDENT B: Because the line goes over and hits. Because if you, um, because 30° is right here. That’s 60 (marks x-axis) and that’s 90, so if you take the 30 and take the line straight up (drawing vertical line) it’s this one right here. It intersects the graph of this. The line of the graph’s right there, and if you take that over (draws horizontal line) it goes through .5.
TEACHER: Alright. Now do the same thing with 20°. Try to estimate 20°. Earlier you used the calculator. Can you use the graph to find an estimate for 20°?
STUDENT B: Let’s see, 10°. 10° is right here (marks x-axis), so it’s right here ish (marks x-axis). So it’s like .32ish.
TEACHER: Ok
STUDENT B: .34 from the calculator.
TEACHER: So are you convinced that it’s correct?
STUDENT B: Yes, I am.
TEACHER: Alright. Using the graph, if the sin(x) = -1, what is the value of x?
STUDENT B: If the sine of x…
TEACHER: It’s question thirteen if you need to see it.
STUDENT B: On the graph, sine of x is -1. So earlier we were finding, we’re finding if, like, the sine of 20° was like x. That would be y so it’s here and here (pointing to the points (270°, -1) and (-90°, -1)). It would equal, um, 90 or -90 or 270 or -270 or any angle that would be if you took 90 and added 180° to it. And you could just keep on finding a bunch of angles that equal, uh, sin(x) = -1. So there’s a bunch of different values for x.
TEACHER: Good, ok, now earlier I asked you if the \( \sin(x) = 2 \), what did \( x \) equal? So look at your graph. Can you answer that question now? You didn’t know what angle on the unit circle had that so where is the \( \sin(x) = 2 \) at?

STUDENT B: Nowhere

TEACHER: Because?

STUDENT B: There’s… it’s not here

TEACHER: What’s not?

STUDENT B: The, uh, 2, -2. It’s not on the y-axis.

TEACHER: Right there (pointing to y-axis). Right there is 2 and there’s -2 down there.

STUDENT B: Ok, um, but the graph does not intersect a point with 2 or -2 and… so I was wrong before. I think (pause). I’m pretty sure because of the graph. The graph would have to like go like all the way up here and come back down and then go down to -2 (drawing wave up to 2 and down to -2 on the graph of \( y = \sin(x) \)).

TEACHER: Does that mean the unit circle doesn’t hit 2 either?

STUDENT B: Um…

TEACHER: You said the unit circle did. Did you just remember incorrectly or it doesn’t hit 2 at all?

STUDENT B: I’m going to say that it doesn’t now. So I mean that’s some pretty good proof here that it doesn’t so I think that I was remembering incorrectly.

TEACHER: Alright, good. Two more questions. So there’s a new trig graph. First of all, is this graph a function? How can you tell?

STUDENT B: Yes, it is a function because no matter how many lines I put down, it will never, the line through it would only cross the graph of the line of the graph once.

TEACHER: How many points are on that graph?

STUDENT B: An infinite amount of points because the x-axis could just keep on expanding as long as you wanted it to.

TEACHER: Is that graph there a graph of sine of \( x \) or cosine of \( x \) or both? Can it be both?

STUDENT B: It cannot be both.

TEACHER: So which one is it?

STUDENT B: (Talking indistinctly) It is, (pause). That is a sine graph. Yes, that is a sine graph because the sine graph you can move the origin up or down on the y, on the y-axis. Um, the cosine graph too, but the cosine graph, like the peak is on the y-axis, er, yeah, the peak or the lowest point. Like this or that part (points to graph) would be here (y-axis) but it’s not. The middle point in between, like, the line segment is on the origin.

TEACHER: So no cosine graph can have a peak that doesn’t start on the y-axis, is that what you’re saying?

STUDENT B: Yes, a cosine graph has the origin, has to go on a peak or valley. Sine graphs have to, it’s the other way around.

TEACHER: So if there’s a peak on the y-axis then it’s cosine, if there’s not a peak then it’s going to be sine.

STUDENT B: Yes.

TEACHER: So, if I want to write the equation of that graph, what information do you need to write the equation for this graph?
STUDENT B: You need to know what kind of graph it is. It’s a sine graph. So sine, and you need to know the amplitude.
TEACHER: Do you know the amplitude of this graph?
STUDENT B: Which is… um, (counting), 8. Um, (pause), this is confusing.
TEACHER: What is confusing?
STUDENT B: Um, because the, the valleys and peaks are not on the, like every 90°, they’re not on, like this one’s not on 90°. This one’s not on 180°. This one’s not on 270°. So it’s like shifted over so now I’m thinking it could be a cosine graph because maybe... I don’t know. I can’t exactly remember if you can like shift a graph like left or right. Other graphs you can so I think you can.
TEACHER: Ok
STUDENT B: But, then I don’t know how you can tell if it’s a sine or cosine graph. It could be either. It could be shifted over to the right like… 1, 2, 3, 4. 4ish or 5ish points and it would look like a cosine graph.
TEACHER: Ok. So what question could you ask me to answer that for you? What do you have to know to determine if it’s sine or cosine?
STUDENT B: Um, I would ask you, um, can you shift a sine or cosine graph left or right on the x-axis. Then I could figure out then. If you couldn’t then it would be a sine graph. Then if you could, I will still be kind of stuck because I still really wouldn’t know.
TEACHER: Ok, I’ll tell you yes, you can shift it left and right which would imply that it could be a sine or cosine. So that could be a sine graph or cosine graph. So let’s say I wanted you to write an equation for sine. It’s a graph of sine. I want you to write the equation. You said you knew the amplitude. What else do you need to know to get the equation of the graph?
STUDENT B: You need to know the, this number, (motioning horizontally with hands). I don’t know what that’s called because I forget but the… because the amplitude is how far and how low it goes and there’s the something number which is, like, how squished or how wide it gets. Um and how you figure that out or how you get that I do not know. Um…
TEACHER: So, how can you tell how squished or how wide? How do you know? Can you tell me that from the graph or do you have to look at the equation to figure it out?
STUDENT B: You can tell from the graph because it is squished some. It’s condensed because the peaks and valleys are not on like 90°, 180°, 270 and 360. Because if you opened up a little bit, like desquished it, this one would move over to 90, this one would move over to 180, and this one to 270, and this one to 360. Same thing for over here, so… (pause). Yeah, um, if it’s one that’s like normal, every peak and valley like on a 90, like it’s on a 90°. Um, so this looks like its 1/2 because the valley, like peak or valley, is like halfway between 0 and 90. So I think, I’m thinking like .5 maybe because that number, what that number is called I don’t remember, is .5.
TEACHER: So the distance vertically, the distance horizontally is important for writing the equation. Is there any other information you need to know?
STUDENT B: Any other information I need to know? Oh, yeah, um, if this is a sine graph the origin should be at (0,0). Right now it’s at, uh, (0, 3).
TEACHER: What does that tell you?
STUDENT B: It’s been shifted up, so there has to be like a plus in the equations somewhere. And then I don’t think it’s been shifted left or right anymore because the... I mean, the .5 number would make it fit correctly and then, oh, there’s also like, inverted. Normally, it goes up first, right? Yeah, normally it goes like this but it’s not so it has to be a negative in there somewhere, maybe like in front of, is where the negative would go, but there is a negative in there somewhere. So there’s a negative eight, .5, plus three and I think that it’s a sine.

TEACHER: Ok, very good. Well thanks a lot.
Appendix I: First Interview Transcript, Student C

TEACHER: Ok, first question, how are the equation of a function and the graph of a function related? Explain your thinking.
STUDENT C: Do you want me to write it down or just talk?
TEACHER: If you need to write something down then write something down.
STUDENT C: The equation of a function is more of mathematical, kind of like specific numbers like 2x and 3², things like that. And a graph of a function is more about the visual in showing you more points like show you specific points on a graph.
TEACHER: And what connects them together?
STUDENT C: Um, the solutions.
TEACHER: Ok, could you expand on that? What do you mean by solutions?
STUDENT C: The, um, answers. Um, I don’t know how to expand on solutions.
TEACHER: Well, what do they look like?
STUDENT C: Points on a graph or getting x by itself or any variable.
TEACHER: Alright, so if you have a solution to the equation, where does it appear on the graph?
STUDENT C: Um, on the x-axis.
TEACHER: Ok. So, next question, the following is the graph of y = x² - 2x - 3. How many points are on this graph?
STUDENT C: Do you mean points by solutions or answers?
TEACHER: Just, whatever you think.
STUDENT C: Um, one… two…
TEACHER: Mark them too.
STUDENT C: I’ve got five so far.
TEACHER: Ok, any others?
STUDENT C: Uh, well, all the lines are points. It’s just the main points are, um, I wouldn’t know how to explain it. I think that’s it, yeah.
TEACHER: Ok, and what’s special about the points on that graph.
STUDENT C: Um, I don’t know.
TEACHER: Next question, can you give examples of functions and non-functions? How can you tell when you have one? Are there multiple ways to tell?
STUDENT C: Um, an example of functions? Why do you ask me such hard questions, Mr. Marchi? (laughter) Um…
TEACHER: Let’s say, give me an example of a function, whatever comes to mind.
STUDENT C: Um, like y equations, y equals.
TEACHER: Go ahead and write it down.
STUDENT C: Y equals, like that would be a function (writes \( y = 2x^2 + 3x - 1 \)).
TEACHER: Ok, do you know how you can tell, or is it just your intuition telling you?
STUDENT C: Um, intuition.
TEACHER: Ok. What about something that’s not a function?
STUDENT C: Like it looks like that but it’s not a function? Um, oh, if they’re, um, like if you graph it, and it’s not, none of the points cross the x-axis. So if they’re all imaginary. That’s one way. Oh, there’s another way (pause). I can’t remember the other one.
TEACHER: Ok, so draw a graph that’s not a function then.
STUDENT C: (draws graph) That would not be a function.
TEACHER: Now draw a graph that would be a function.
STUDENT C: (draw another graph)
TEACHER: So, other than crossing the x-axis, can you think of any other way to tell if a graph if a function?
STUDENT C: I can’t really think of any.
TEACHER: Ok, that’s ok. Can you give think of an equation that’s not a function?
STUDENT C: Nothing is coming to mind.
TEACHER: Ok, fair enough. Alright, now we’re into the trig questions.
STUDENT C: Oh great.
TEACHER: So, again, just tell me what you’re thinking. What does sine of x mean? Is it a function? How can you tell?
STUDENT C: Sine is a function. I know there’s a bigger word for it. I just can’t remember what sine is. I can’t remember. I forgot… (talking indistinctly).
TEACHER: Ok, does anything come to mind when you hear sine of x? What are you picturing when I say sine of x?
STUDENT C: The wave. Middle, up, middle, down, middle.
TEACHER: So is that a function?
STUDENT C: Yes, it is because it crosses the x-axis.
TEACHER: Anything else come to mind when you say sine of x? Anything special?
STUDENT C: Um…
TEACHER: Doesn’t have to be a graph, whatever you think of.
STUDENT C: Opposite, um, I don’t know what it’s called. I want to say cosecant. Is that correct?
TEACHER: Cosecant is the reciprocal of sine.
STUDENT C: Yes! Got one! And the unit circle.
TEACHER: What about the unit circle?
STUDENT C: Um, the sine is the y part of the points.
TEACHER: Ok, anything else?
STUDENT C: No, I think that’s it.
TEACHER: Alright. Ok, so the sine of 30°. Can you find that value any way you want? The sine of 30°.
STUDENT C: Um, it’s 1/2.
TEACHER: OK, how do you know it’s 1/2?
STUDENT C: Because it’s on the unit circle.
TEACHER: Ok, so show me on the page there or anywhere how you got that from the unit circle.
STUDENT C: So it’s 60°, 45°, 30°. 30° is $\sqrt{3}/2$ and 1/2. 45° is ($\sqrt{2}/2$, $\sqrt{2}/2$), and 60° is (1/2, $\sqrt{3}/2$).
TEACHER: Besides the unit circle, how else could you find the sine of 30°? It’s correct. It’s excellent.
STUDENT C: Type it in the calculator (laughs).
TEACHER: Ok, show me.
STUDENT C: Uh, on, sine, 30, parentheses, equals, .5.
TEACHER: Excellent. So we have the unit circle, we have the calculator, any other way else you can find the sine of 30°?
STUDENT C: I know there’s one other way but I can’t think of it.
TEACHER: Ok. Alright, now what about the sine of 20°?
STUDENT C: That one’s not on the unit circle, but I can still use the calculator. Sine, 20°, parentheses, equals. It’s a big decimal.
TEACHER: Ok, round to the nearest hundredth.
STUDENT C: .342.
TEACHER: Ok, so without just saying it’s on the calculator, can you convince me that the sine of 20° is .342? Can you give me an argument to convince me that the sine of 20° should be .342?
STUDENT C: I could not. It’s probably because the other way I can’t remember.
TEACHER: Ok, let’s go to the unit circle for a second. You told me that the sine of 30° is on the unit circle. Where is the sine of 20° at?
STUDENT C: Right… there (marking drawing).
TEACHER: And what do you think the coordinates there are?
STUDENT C: Um, I have no clue. We talked about this once. I just can’t remember.
TEACHER: That’s fine. Is there anything else that you want to talk about with the sine of 30° or the sine of 20° that coming to your mind right now?
STUDENT C: Hmm. That’s it right now.
TEACHER: Ok, let’s look at some equations. You have sin(x) = 1/3 and 2sin(x) + 1 = 2/3. Explain how you would solve them.
STUDENT C: Uh, or sine of 1/3, I take the inverse of, uh, sine of 1/3. Which is… Then you type it in the calculator, (talking indistinctly), 19.47.
TEACHER: Ok, what does 19.47 mean?
STUDENT C: Um, that it’s the sine of 1/3.
TEACHER: Ok.
STUDENT C: Um, alright for the second one, minus 1 from both sides. Divided by 2. Equals -1/6. Then you take the inverse of both sides. Which is -9.59.
TEACHER: Ok, and again, what does that answer mean?
STUDENT C: It is the sine of -1/6. That feels wrong.
TEACHER: Why?
STUDENT C: Because this is like your starting equation so -9.59 is an x so you are trying to get all of that to 2/3. So technically, it’s the sine of 2/3.
TEACHER: Ok
STUDENT C: I want to say (laughter).
TEACHER: Alright, you used algebra and the calculator to solve both of those equations. Is there any other way you can think of to help find the answers to these questions?
STUDENT C: Not that I can think of.
TEACHER: Ok.
STUDENT C: Yep, I’m pretty sure.
TEACHER: Ok, if \( \sin(x) = 2 \), what is the value of \( x \)? How can you tell?
STUDENT C: The value of \( x \) is… It’s going to be… (typing on calculator). Oh no! Hmm (pause). Um, I know a way to do this (pause).
TEACHER: What’s the calculator say?
STUDENT C: It says error.
TEACHER: Why do you think that is?
STUDENT C: Probably the domain which is… If you were to graph it, um, it would not have, um, oh, (drawing a graph). If you go this way not all the \( x \) points are used. Because the domain is going to be on one side of the… So it would probably look like… look like that where it doesn’t cross the \( x \)-axis or where there’s no… it’s empty down here (pointing to quadrants III and IV on the graph).
TEACHER: Ok, very good. Is there a connection between the unit circle and the graph of sine of \( x \)?
STUDENT C: Yes, there is a connection. It, um, it helps you, uh, I know… I can’t… I know this part, I just can’t think of it. It, it helps you find the points on the, um, where to like start and end and things of that sort on the graph.
TEACHER: Can you give me and example?
STUDENT C: Um, like, it would help you find, like, where to start so, I’d start on 0, right here, starting point, and ending point would be 360. So I would be right here, here, here, here. (drawing wave on a graph). It helps you find these points.
TEACHER: The unit circle does?
STUDENT C: Yes.
TEACHER: Why?
STUDENT C: (Pause), It… I can’t think of it. I had something but I can’t remember.
TEACHER: Ok. Do you know how that point is connected to the unit circle? How that point is? What are the points right here? (pointing to graph).
STUDENT C: This is 90. This is 180. This is 2… No, I lied (changes \( x \)-axis scale). They’re the… These points are these four points on the circle and you use the circle to find these points to plug them in correctly. But there’s no…
TEACHER: What’s the point here that gives you that one?
STUDENT C: (0, 1). No, that would be (0, 0), right? No, (0, 1)
TEACHER: How does that translate to that point?
STUDENT C: (Pause) Gives me this point here in the middle. This one is 1, this one, oh my God, it’s something like that. I can’t think of it.
TEACHER: Ok.
STUDENT C: It’s in my head!
TEACHER: It’s ok. It’s the week after school’s out.
STUDENT C: I’m doomed!
TEACHER: Um, so you already started to answer the next question. Explain how you can graph sine of x. So walk through again how you know to start at (0, 0) and everything. So explain how you graph sine of x.
STUDENT C: You use that trick, those steps. The starting, no, if your starting point is… Oh my God! Amplitude. I’m thinking slope, point-slope something like that. It’s for the slope, starting point, ending point. We use like the amplitude and all that of an equation to find where you start your plotting so if there’s no amplitude, if the amplitude is just one, then we leave it the same so that would look like… It would go above this line or below it here. This would be 1 and this would be -1.
TEACHER: Ok.
STUDENT C: Scale, scale… Scale shows, um, how far it stretches, um… I find all that out using the equation, like say the equation is y = 2x -1, I think would be a good one (laughs). And use it to find these.
TEACHER: Where’s your trig function?
STUDENT C: That’s not a trig function?
TEACHER: is it?
STUDENT C: I was hoping…
TEACHER: Alright, continue.
STUDENT C: Then you do that to find these and you get these to plot.
TEACHER: Alright, so what would the amplitude be?
STUDENT C: 2
TEACHER: Ok.
STUDENT C: So, instead of right here being 1 (point to y-axis of graph), it would be 2and -2. So the wave would go woo, like that.
TEACHER: Ok, excellent. Number twelve. Look at that graph there. Can you use it to find the sine of 30°?
STUDENT C: (Counting) Yes, you can.
TEACHER: Alright, so show me.
STUDENT C: You would go over 1, 2, since these are degrees. You would go 1, 2, 3, for 30° and then find where it crosses at.
TEACHER: Good, Ok, so what’s the answer?
STUDENT C: .5. .5 y.
TEACHER: Ok, excellent. Now doing the exact same thing, can you find the sine of 20°? Remember earlier you said there’s no way you could do it other than the calculator. Can you use the graph now to find the sine of 20°?
STUDENT C: Yes, go over 2 and up 1, 2, 3, where it crosses the, uh, that line and it’s about .34. Like right there.
TEACHER: How does that compare with the answer you gave me earlier? Do you know what it was? Did you write it down?
STUDENT C: Yeah. That’s close because it’s almost… Where’d it go?
TEACHER: On page two, questions number five.
STUDENT C: I did not write it down.
TEACHER: Here you go: .342. So does that sound reasonable?
STUDENT C: I said it was about .34, so it’s right there. Because it passed there, but not all the way up to 4 yet.

TEACHER: Alright, now earlier we talked about if the sin(x) = 2, what does x equal? Now using the graph, can you answer that question?

STUDENT C: Um.

TEACHER: The sin(x) =2. Let’s back up. Questions number thirteen. Using the graph, if the sin(x) = -1, what is the value of x?

STUDENT C: Sine of… -1? Um, I could. I believe you could. Could I tell you? Most likely not.

TEACHER: Why not? You found the sine of 20°, which you said was about .3, which is correct. The sin(x) = -1. Why can’t you do that one? What’s different about it?

STUDENT C: It’s not a degree. Wait a minute. Yes, you could because all you do is go to… I figured out how to use the unit circle! Ah ha! Because you can use it to find, um, like sine equals -1 because that’s what’s 1. I know that’s 0 for sure. 1, or is it the other way? Never mind, I won’t worry about that one, but I know you use all these points. If this is like -1, it’s 180 so for finding the sine of -1 you go to 180 and see where it’s at, or wait I’m getting somewhere. I’m collaborating.

TEACHER: It’s very good.

STUDENT C: I’m close. I feel I’m very close. You use the graph. You go to where -1 is on the unit circle. I know that for sure, but I don’t know which one is -1.

TEACHER: What do you mean which one is -1?

STUDENT C: Like I know this is 0, 90, 180, 270, 360. I don’t know which one of these is -1. I think it’s 180. I’m so sure it’s 180.

TEACHER: Ok, so you’re saying 180° should be -1.

STUDENT C: Yes.

TEACHER: Alright, look at the graph. Is the graph… back it up?

STUDENT C: Uh, it does not.

TEACHER: What does the graph say it should be?

STUDENT C: Uh, 9, 18. It’s 18. It’s at (0, 18),

TEACHER: Pardon?

STUDENT C: It’s at (0, 18)

TEACHER: (0, 18)?

STUDENT C: or 180, it’s at 180.

TEACHER: So 180° is what? According to the graph?

STUDENT C: 180° is 0.

TEACHER: Ok, so…

STUDENT C: Ah, I think I found out. Yes I did! Haha! I was wrong, it’s (1, 0). Wait. I keep losing it right there! I got it and I’m like, I lose it, but…

TEACHER: So what are you doing right now? How did you know that was 1 and not 0?

STUDENT C: Because on the graph 90° is one.

TEACHER: Oh.

STUDENT C: 180 is 0, 270 is -1, and 360 is also 0. So to find the sine of -1, it’s 270 because it’s where it’s down at -1.

TEACHER: So the answer to the questions in number thirteen is 270°?
STUDENT C: And also -90.
TEACHER: Why -90?
STUDENT C: Because it is also at -1.
TEACHER: Ok, that’s very good.
STUDENT C: See, I have good math skills. I just don’t know how to put it all together.
TEACHER: You just did though. Alright, we’ve got two more questions left, more difficult. That was an excellent job using the graph there. Ok, so we have that new graph there, Ok, so we have that new graph there, right there, is that graph a function and how can you tell?
STUDENT C: It is not because the bottom of the graph is at -1 like the wave but the top is at 7.
TEACHER: And that means it’s not a function because?
STUDENT C: Because it’s not like fully, um, like the bottom should be either at -7 or the top should be at one.
TEACHER: You told me earlier though that, um, if the graph crosses the x-axis then it’s a function. That was your definition.
STUDENT C: Oh, hmm. Well, then it is a function because it does cross the x-axis and it does use, uh, more than like all the points on both sides.
TEACHER: What do you mean uses all the points on both sides?
STUDENT C: If goes on both sides instead of just one side. Like two, where it had no… one domain was not used. It’s just not a very good graph.
TEACHER: Ok. So is that graph, you think, sine of x or cosine of x graph?
STUDENT C: I think cosine.
TEACHER: Because?
STUDENT C: Sine is middle, up, middle, down, middle, up. Cosine is up, middle, down, middle, up, and it starts probably on the seven. Middle, up, middle… middle, down, middle, up. Instead of middle, up, middle, down, middle, up. Yeah, I’m going with cosine.
TEACHER: Can it be both sine and cosine? Could it be a cosine curve and a sine curve at the same time?
STUDENT C: It could.
TEACHER: Ok, if it’s sine, what would change?
STUDENT C: Um, Oh, this would be… it would be shifted probably to where it’s even like the middle is right here. Up is… wait that’s the middle but, uh, no… It is sine.
TEACHER: Ok
STUDENT C: Well, it could be both.
TEACHER: Why could it be both?
STUDENT C: Because sine is... it could start right here at 3 and go down, er, down, middle, up, middle, down and sine could be negative in front of it .
TEACHER: Ok
STUDENT C: To where it flips. Or it could be cosine where it starts up or starts here at seven. Up, middle, down, middle, up.
TEACHER: Ok
STUDENT C: And that would be a normal cosine.
TEACHER: So if I wanted you to write the equation, if you can’t remember that’s fine, what information do you need to know to write that equation?
STUDENT C: You need to know the amplitude, starting point, ending point, scale. I know it’s P.S. but I can’t remember what it stands for. Point, no, point-slope? No, I know it’s P.S. but I can’t think of what it’s called.
TEACHER: What if I told you P.S. stands for phase shift. What does that mean?
STUDENT C: Phase Shift. Uh, whether it goes up or down. The middle moves up or down.
TEACHER: Anything else?
STUDENT C: So that’s phase shift, starting point, ending point. No, I think that’s it. Oh, if there’s a negative in front of the equation or not.
TEACHER: So can you attempt to write the equation for sine?
STUDENT C: Let’s see. It’s sine.
TEACHER: What do you know about the equation?
STUDENT C: It probably starts out as negative 1, 2, 3, 4, 1, 2, 3, 4. 4, um, parentheses, x. I know there’s something in the middle to turn it into something and then out here would be 3, plus 3. Ah (erases 4 and moves it outside parentheses).
TEACHER: Alright so why did you move the 4?
STUDENT C: Because like 4 is how far it’s stretching. Wait, no, the amplitude is in front. There, that’s correct. So it’s -4x + 3.
TEACHER: Ok, and how can you tell from that equation that it’s sine?
STUDENT C: Just like that (adds sin(x) in place of y).
TEACHER: Ok, very good. Anything else you want to go back and change now that you had your little epiphany with the graph? Any answers you can go back and add to?
STUDENT C: Um, (looking back through questions). I know the unit circle one I can’t remember the question. How the unit circle can be used…
TEACHER: Um, number ten.
STUDENT C: Yes, there is because you can use it if you have easier, not easy, but, um, nothing extended or stretching to find your points on the x-axis. Like say 360… 0 is 0.
TEACHER: OK
STUDENT C: 90 is 1, 180 is 0, 270 is -1, 360 is 0. That’s if there’s no amplitude so that’d be if it’s like sine of x equals (pause). Ok, let’s say there’s no amplitude so x is probably stretched a little bit so (talking indistinctly). I never do good with equations so I’m going to say sin(x) + 1.
TEACHER: Ok
STUDENT C: I’m guessing on that. I don’t know if it’s kind of like uh, there’s nothing like making it different.
TEACHER: Ok, anything else?
STUDENT C: I think that’s it.
TEACHER: Alright, excellent. Thank you very much for your time. I really appreciate it.
STUDENT C: Oh, it was fun. Using math skills for once.
Appendix J: First Interview Transcript, Student D

TEACHER: So, first question, how are the equation of a function and the graph of a function related? Explain your thinking.
STUDENT D: Um, the equation of a function for every, can I write this down? For every x there is only one y so then for a graph of a function, um, every x-coordinate can have only one y output. Do you want me to write down my responses?
TEACHER: No. So, if you have an equation that’s a function and you have a graph that’s a function, how can you tell that the two go together?
STUDENT D: Um, because if this was like, I think about it was like if there was something like this and there was like x² then it’s y². It can’t be like a straight up and down line.
TEACHER: Ok
STUDENT D: Because every x can have only one y.
TEACHER: I guess what I’m trying to ask is you see a graph, you draw a graph and you know the equation of the graph. How do you know that’s the equation of the graph?
STUDENT D: Um, I put it in y = mx + b form and then I look at b and usually find out what the, um, the y-intercept is and then I’ll match it up and see if it’s the same. And then otherwise I think that’s about it.
TEACHER: Alright. That’s very good. Excellent. Next question. Uh, the following is the graph of y = x² - 2x – 3. How many points are one this graph?
STUDENT D: Um infinite.
TEACHER: Can you show me some? Mark them on your paper.
STUDENT D: Um, (-1, 0), (3, 0), (0, -3)
TEACHER: Any others?
STUDENT D: Um, (-1 1/2, 2)
TEACHER: Ok, very good. Alright, next question. Can you give examples of functions and examples of non functions? How can you tell when you have one? Are there multiple ways to tell?
STUDENT D: A graph or just a…
TEACHER: Whatever comes to mind.
STUDENT D: I think of a table. (makes t-table) I think like 1, 2, 3, 4 (x-values in the table) and then it’s like 8, 9, 10, 11 (y-values in the table), and then that’s a function. And not-a-function would be like (makes second table), 1, 2, 3, 4 (x-values in the table), and if there were two outputs for one input, 8, 9, 10, 11, 12 (y-values, but adds a second 1 on the x side to match up with 12). Then that one gives you two answers.
TEACHER: Ok, you mentioned a graph. How would do it with a graph?
STUDENT D: Um, this would be a function because for every x you can see that there is only going to be one y and then like if you had a graph like this. This would not be.

TEACHER: Ok

STUDENT D: Because for one you could have like whatever this y is and this y. So you’d have all those points for just one x.

TEACHER: Ok, so you’re saying the graph is not a function because…

STUDENT D: Because for every x, like it goes this way, there’s more than one y output.

TEACHER: Ok, what about an equation. Can you give me an equation that’s a function and an equation that’s not? And how can you tell?

STUDENT D: Um, I forget how to write the equation of this. It’s… You can write the equation of this and then restrict it to be a function, but I forget how to write the equation. I know, um, like you want to do y = 1/2 + 2. That would be a function.

TEACHER: Ok, what about an equation that’s not a function?

STUDENT D: Um, (pause). Maybe x = y², something like that.

TEACHER: Ok, very good. Alright, so that’s algebra. Now we’re into the actual trig questions. So, first off, what does sine of x mean? Is it a function? How can you tell?

STUDENT D: Um, sine of x to me means opposite over hypotenuse.

TEACHER: Ok

STUDENT D: Of x though (adds (x) after the word sine in the equation), and I do not know if that’s a function.

TEACHER: Ok, is there any other way you can represent sine of x besides opposite over hypotenuse and what do you mean by opposite over hypotenuse too?

STUDENT D: I mean if you have a triangle and this was x, 60°, then, well, that wouldn’t work. Then you would know that, well, yeah, x is supposed to be an angle? Yes? Then I would do opposite over, opposite one, hypotenuse.

TEACHER: Ok, very good. So the previous question, you looked at a table, you looked at an equation. You looked at a graph to see if it’s a function. Can you do the same thing with sine of x?

STUDENT D: Possibly (pause). Um, (pause), um, no. I can’t recall.

TEACHER: Alright, so sine means opposite over hypotenuse. You can’t tell it it’s a function. Is there any other way you can represent sine of x using anything else?

STUDENT D: Yes, if it was on, um, a coordinate plane or something like that and this was like, um, (-5, 5) and this was like (-5, 0). And then I could figure out that this was 5 so then that would be 5 over… Then I’d do √50

TEACHER: Alright, next question. How can you find or estimate the sine of 30°?

STUDENT D: It would be π/4, wait, no that would be 1/2 because of the unit circle. It’s in here, 30°, 45, 60, and then I believe it’s (1/2, √3/2). Oh wait, and sine equals y so then it would be √3/2

TEACHER: Ok, you used the unit circle to get 1/2. How else could you get or check that answer?

STUDENT D: Um, you could…

TEACHER: or √3/2, sorry.

STUDENT D: Um…

TEACHER: Besides the unit circle.
STUDENT D: Besides the unit circle? Um, I would do 30°, multiply by $\pi/180$ and change it into radians and then I don’t know how you would get the sine of that. You could do it in the calculator.

TEACHER: Alright, show me.

STUDENT D: (laughter) In the calculator? Sine, 30, .5

TEACHER: Oh, so is it .5 or $\sqrt{3}/2$?

STUDENT D: Both

TEACHER: Both? The sine of 30° is both?

STUDENT D: (laughter) Well, yes (pause). Um, is it… (types in calculator) No, it doesn’t equal the same thing when you do it in the calculator.

TEACHER: What did you just do now?

STUDENT D: I did $\sqrt{3}$ divided by 2 to figure out if it equals .5 but it doesn’t.

TEACHER: Alright, the calculator says it’s $\sqrt{3}/2$, or sorry, the calculator says it’s 1/2 and the unit circle says is $\sqrt{3}/2$. So which one do you think is correct?

STUDENT D: Um, I think they’re… (pause)

TEACHER: Well, can they both…

STUDENT D: Oh, is, maybe I got this mixed up.

TEACHER: Ok

STUDENT D: Maybe this is (1/2, $\sqrt{3}/2$)

TEACHER: Ok, so you used the unit circle, you used the calculator. Anything else to find the sine of 30°?

STUDENT D: Um, triangle

TEACHER: Ok

STUDENT D: This is 30°. Opposite over hypotenuse

TEACHER: Ok, how did that equal 1/2?

STUDENT D: If this was 5 and this was 10

TEACHER: Ok, very good. Alright, now what about the sine of 20°? Can you find the sine of 20° or estimate that?

STUDENT D: Um, you could estimate it being, um, sine of 20° would be less than, um, the sine of 30°, which would be 1/2 and greater than 0, er, greater than 0.

TEACHER: Why greater than 0?

STUDENT D: Because I’m thinking in terms of the unit circle. It’s greater than 0°, er, will be greater than 0 and then less than that because it’s between those two.

TEACHER: Alright. Would it be closer to the sine of 30° or the sine of 0?

STUDENT D: The sine of 30°

TEACHER: Ok. So can you give me a decimal approximation?

STUDENT D: Um, .3?

TEACHER: Ok, is there any other way, and that’s a great way, you could estimate the sine of 20°? You used the unit circle, what else could you do?

STUDENT D: You could use a right triangle. 20°, well, you’d have to have your, your sides and do something like that.

TEACHER: Ok. Anything else you can do to find the sine of 20°?

STUDENT D: Um, (pause). That’s all I can think of.
TEACHER: What about all the ways you tried for the sine of 30°? You tried the triangle, unit circle, what else did you do?
STUDENT D: I did this, changed it into radians, but I’m not sure how that would help you get the sine.
TEACHER: And what about the calculator?
STUDENT D: Oh yeah, sine of 20. Hey! I was pretty close.
TEACHER: What is it?
STUDENT D: .34
TEACHER: That’s pretty good. OK, next question. Number seven, explain how you would solve the following equations. Just walk me through what you would do to find the answers to those equations. The first one you have sin(x) = 1/3 and the second you have 2sin(x) + 1 = 2/3.
STUDENT D: Alright. I haven’t done this in awhile. Let’s see, I would take, do I take the inverse sine? I know for this one I would subtract 1 so then it would be -1/3. So then 2sin(x) = -1/3 and then I divide by 2 so then…
TEACHER: Ok, so you have the sin(x) = 1/3 there and the sin(x) = -1/6. How would you find the solutions? What would you do next?
STUDENT D: Um, I always go to a triangle.
TEACHER: Ok
STUDENT D: And I think of 1 over 3 and then I use the Pythagorean, uh, the theorem, and I do 3², which is 9 and minus 1², which equals 8, so √8 and then this I do a triangle also, and then this has to be -1 and 6, 36 minus 1 is √34, and then this has to put it in the coordinate plane would be this way.
TEACHER: Alright, so how do those triangles help you find the value of x?
STUDENT D: Um because I just think of SOH CAH TOA and then when, because if this is x, this is what you’re looking for, and the sine of x equals opposite over hypotenuse. So then you take opposite over hypotenuse of x equals 1/3 and then, uh, (pause). Then I would take the inverse sine, wouldn’t I? Because I’m trying to find the angle. So this would help me find a side. OK, so then if it’s the angle, then I take inverse sine 1/3 and I get 19.5.
TEACHER: So what is 19.5?
STUDENT D: Degrees
TEACHER: Ok
STUDENT D: And for this I would do inverse sine -1/6. Oh, wait it wouldn’t be -1 because this is just showing where it is. Inverse sine, I divided by 6. 9.6… degrees.
TEACHER: Alright, so you changed from -1/6 to positive 1/6. Why?
STUDENT D: Because you can’t have negative degrees, and I was thinking if this was to be placed on the coordinate plane then the -1 would be there because this is always your x, your θ.
TEACHER: Ok
STUDENT D: So then (talking indistinctly) and draw. Your opposite is negative and it would have to be there or there (Quadrants III and IV). And the hypotenuse is… So then both have to be negative. I’m got to say it’s there (Q III).
TEACHER: So what’s your final answer then for x?
STUDENT D: 9.6°
TEACHER: And 9.6° takes you to the third quadrant?
STUDENT D: No, I’m doing something incredibly wrong (laughter). Um I don’t know. Maybe I bet it’s… (pause). So if it’s -9.6°, then I would be thinking it’s going this way. Maybe it’s over here (Q IV). I don’t know.
TEACHER: Ok, so you’re going to stick with 9.6 as your answer? 9.6°?
STUDENT D: Yes
TEACHER: Ok, alright so you used triangles and some trigonometry and inverse trig functions to get your answers. Can you think of any other way you could solve those two equations? If doesn’t have to be some way you learned in class, just in general, can you think of any other way?
STUDENT D: Um, maybe try using the unit circle.
TEACHER: Ok, so the first one you have sin(x) = 1/3. How would you use the unit circle to find that?
STUDENT D: Um, sine equals y, so then I’d think (talking to self as student draws unit circle).
TEACHER: So what are you trying to do right now?
STUDENT D: I’m trying to figure out what this is (30°) on the unit circle because I think I got it wrong.
TEACHER: Ok, and what did you do on the calculator just now?
STUDENT D: Um, I did, oh, I did 20. I’m doing 30 divided by 180 to change it into radians. Um, was this 1/9?
TEACHER: You said it was 1/2
STUDENT D: So (√3/2, 1/2). Where was I going again?
TEACHER: You were trying to tell me how you find sin(x) = 1/3 on the unit circle.
STUDENT D: So, since sine equals y, then I looked at y so it must be between 30° and 45°, no that can’t be right because I’m looking at 1/3.
TEACHER: Ok
STUDENT D: Oh, then it’d be between 30° and 0
TEACHER: Ok
STUDENT D: Because… yeah.
TEACHER: Ok, that’s very clever. Can you think of any other way, I mean that’s great. It’s awesome. You’re ahead of everybody else so far. Can you think of a third way?
STUDENT D: Um, to figure out x?
TEACHER: Yes
STUDENT D: Hmm, (pause). I cannot.
TEACHER: Ok, that’s fine. Alright, moving on, next question. If sin(x) = 2, what is the value of x? How can do you know?
STUDENT D: Um, I would take inverse sine of 2 because, oh, that doesn’t work (plugged it in calculator). I would do that because inverses are supposed to cancel.
TEACHER: Ok
STUDENT D: Um…
TEACHER: So, it didn’t work, why?
STUDENT D: On the calculator, it didn’t work.
TEACHER: Do you know why?
STUDENT D: I’m trying to figure that out (pause). Um, I always think of it when it’s a sine and things like that I always think of a triangle because it’s the easiest way for me to think about it, and then opposite over hypotenuse that would be 4 over 2. Oh, because, um, this wouldn’t make sense on a triangle because the hypotenuse of a triangle always has to be the longest side and for here you’d have this side being the longest which wouldn’t make sense.
TEACHER: So what does that mean?
STUDENT D: That means that it’s not a triangle.
TEACHER: Ok, what does that mean your answer is to the questions of the sin(x) = 2, what is the value of x?
STUDENT D: Um, I don’t know. I don’t remember how to do that.
TEACHER: What did the calculator say the answer was?
STUDENT D: It said it didn’t work.
TEACHER: Ok, so what does that mean in terms of the triangle?
STUDENT D: Um, I don’t know.
TEACHER: Well, tell me about the triangle you just drew.
STUDENT D: I drew a triangle with a base of 4 and a hypotenuse of 2.
TEACHER: Ok
STUDENT D: A 90° triangle.
TEACHER: Ok, and what’s wrong with that triangle?
STUDENT D: It can’t work.
TEACHER: Why not?
STUDENT D: Because the hypotenuse has to be the longest side.
TEACHER: Ok, so what does that mean in terms, well, what are you trying to find in the triangle to answer the question?
STUDENT D: I’m trying to find this angle.
TEACHER: Ok, so what does that mean?
STUDENT D: It means that, it’s an obtuse angle maybe?
TEACHER: Ok
STUDENT D: So, it must be bigger, well, it means this angle is bigger than this angle.
TEACHER: Ok, alright, anything else you want to add? So you’re saying x would be an obtuse angle, is what you’re saying?
STUDENT D: Well, not necessarily obtuse, just bigger than this angle (right angle),
TEACHER: Ok
STUDENT D: Because since this is 2 and this is 4 then that has to be bigger.
TEACHER: Ok, can you estimate what it would be?
STUDENT D: Uh, double this angle maybe.
TEACHER: Ok, is that just a guess?
STUDENT D: Yeah, that’s just a guess.
TEACHER: Ok, next question, getting back to the unit circle. Is there a connection between the unit circle and the graph of sin(x)?
STUDENT D: Let’s see. I have to draw the unit circle. Ok. Then let’s see, x would be √3/2 so I’m just trying to see (pause) and then 30° is… Somewhere in between probably
back here, um, um, I’m trying to make a relationship between 30° or π/4 or π/6 and um x and y. So as I’m looking here and between here I see 30° and it’s at .5 so that would make sense.

TEACHER: Because
STUDENT D: Because it goes up to there. Because if you have an x of 30° here (graph) and your y is .5, it would go up to the there which makes sense.

TEACHER: Ok
STUDENT D: And then up here, \( \sqrt{2}/2 \) is a little less than 1 so if I had 45° and it’s here, it would be like a little less than 1. I guess that makes sense. Because that’s between 1 and 1/2.

TEACHER: Ok, so, can you explain to me what the connection is in what you’re doing right there?
STUDENT D: Um, I was just looking at the degrees given here on the unit circle, which is 30, 45, and 60, and then I kind of approximate it on the graph.

TEACHER: Ok
STUDENT D: And I looked at, I used degree because degrees is used here on the x-axis. I used 30° over and I looked at the y on the unit circle.

TEACHER: Ok, why did you look at the y and not the x?
STUDENT D: Because I looked at the normal, like, um, like when you plot a normal point, y is up or down.

TEACHER: Ok
STUDENT D: But then I don’t know why we didn’t use the x.

TEACHER: So, you don’t know why you didn’t use the \( \sqrt{3}/2 \)? What would the point be at 90°?
STUDENT D: 1
TEACHER: What’s the point on the unit circle at 90°?
STUDENT D: 1, right?

TEACHER: What is the point? The x and y coordinate at 90°
STUDENT D: (0, 1)

TEACHER: So do you know why you used the y again? Other than it was the vertical distance moved on the graph?

STUDENT D: Um, what are you trying to ask me?

TEACHER: I’m trying to figure out why you know not to use the x-coordinate. Why are you using only the y coordinate? Is it just because you are moving vertically on the graph?
STUDENT D: Um, I used only the y-coordinate because this was given in degrees.

TEACHER: Ok

STUDENT D: So then it wouldn’t make sense for me to use this if it’s not in degrees. At least it wouldn’t make sense to me.

TEACHER: Ok, so what if, for example, the graph of cosine, would you still use the degree and y-value to get the graph of cosine from the unit circle?

STUDENT D: The graph of cosine would be like (draws graph) this. I believe like that and cosine starts up.

TEACHER: Ok, so at 0°, what is the y-coordinate?
STUDENT D: 1
TEACHER: On the unit circle, what is the coordinate at 0°?
STUDENT D: (1, 0)
TEACHER: Ok
STUDENT D: No, wait, yeah (1, 0)
TEACHER: Ok, go ahead and write that down up here. So on cosine, 0° is what on the graph? The y-coordinate at 0° is...
STUDENT D: 1
TEACHER: And at, on the sine graph at 0°, what’s the y-coordinate?
STUDENT D: 0. Oh, so maybe it’s opposite for sine and cosine. You use the x-coordinate for cosine and the y-coordinate for sine.
TEACHER: Ok. Do you know why that is?
STUDENT D: I don’t know why but I can see the relationship.
TEACHER: Ok, so explain the relationship one last time.
STUDENT D: Um, for when the, when the x-axis is given in degrees, for a graph of sine you are going to take the y-coordinate when graphing, you are going to use the degrees and y-coordinate.
TEACHER: Ok
STUDENT D: That explains it, and for cosine you’re going to use the degrees and the x-coordinate.
TEACHER: Ok, very good. Now, if we’re talking just about the unit circle, and I want to find the cosine of 135°, what are you going to do?
STUDENT D: Um, I going to go to 135°, which would be right here ish. Quizzing me on my unit circle. It’s like...
TEACHER: The coordinate I’ll tell you is (-√2/2, √2/2).
STUDENT D: Oh, ok, (-√2/2, √2/2). so then...
TEACHER: I’m asking what’s the cosine of 135°
STUDENT D: The cosine? The cosine is x.
TEACHER: So cosine is x?
STUDENT D: Yes, cosine is x.
TEACHER: So is there a connection to why you chose the x-coordinate for the graph?
STUDENT D: Oh! Yes, because cosine is x!
TEACHER: Ok
STUDENT D: So then you’d use the x-coordinate which would make sense.
TEACHER: Ok, excellent. Ok now let’s back up again. That’s a wonderful job. Next question, explain how you can graph sin(x). So pretend the graph is not there and you wanted to teach someone who’s never seen the graph of sin(x). Go through step by step how you would produce a graph of sin(x).
STUDENT D: One of my favorite things. Ok, then if it’s this I graph, it’s that way because it’s easier to draw. I’d first look at, um, I’d have an equation.
TEACHER: Just do sin(x)
STUDENT D: Sin(x)? Ok, so then first you have to find the amplitude.
TEACHER: Ok
STUDENT D: Which is 1 because there’s nothing there.
TEACHER: So what does that mean?
STUDENT D: That means the amplitude is 1. Then you go on your graph, so that means since sine starts here, the maximum and, like, minimum that you’re going to go up is 1.
TEACHER: So why does sine start there again?
STUDENT D: Because sine is y and at 0°, your y is 0.
TEACHER: Ok
STUDENT D: Um, so then, so then you start at zero and sine goes up, er, middle, up, middle, down, middle.
TEACHER: Ok, what does that mean?
STUDENT D: Like middle, up, middle, down, middle (draws graph).
TEACHER: Ok
STUDENT D: So then, sine, that’s your amplitude, your period would be 1 or π.
TEACHER: What about degrees?
STUDENT D: Um, 90°
TEACHER: What’s the period?
STUDENT D: The period is like, um, maybe it would be 2π. The period is how long, like within what length it takes to complete the graph of...
TEACHER: Ok, so what’s the period of the graph there?
STUDENT D: The period of the graph here would be 2π.
TEACHER: Ok, and then in degrees?
STUDENT D: 360
TEACHER: Ok
STUDENT D: No, yeah 2π.
TEACHER: Ok, so given your graph, you started at (0, 0), amplitude is 1 and period is 360.
STUDENT D: Something like that.
TEACHER: Ok, and what are these marks right here? (marks of x-axis)
STUDENT D: Those are degrees so then if this was, if the period was 360, then this would be… This would be 90 no that would be 180.
TEACHER: How did you know it couldn’t be 90?
STUDENT D: Because, um, it wouldn’t make sense because if this was 90, then this would have to be 180 and this would have to be, oh wait, that’d make sense. No wait, it wouldn’t because if your graph ends there and if this was 90 that would have to be 180, that would be 270 and that wouldn’t equal 360.
TEACHER: Ok
STUDENT D: So then that would have to be 180. Then that would be 270. That would be 360 or π/4, π/2, 3π/4.
TEACHER: Ok, very good. Anything else?
STUDENT D: Nope
TEACHER: Ok, so looking at the that graph still, can you use it to find, you already found the sine of 30°, which was 1/2, can you find the sine of 20°?
STUDENT D: On this graph? Um, hmm, it would be around there, like .4.
TEACHER: Ok. How does that compare with what you gave me earlier?
STUDENT D: I said it was .33.
TEACHER: Pretty close. Alright, now earlier we talked about the sin(x) = 2. Let’s back up, so question number thirteen. Using the graph, if sin(x) = -1, what is the value of x?
So looking at that same graph, if the sin(x) = -1, what is the value of x?
STUDENT D: (pause) So then y = -1. What is the value of x? Is that what it’s asking?
(talking indistinctly) Like 3π/4?
TEACHER: Ok
STUDENT D: Is that what you’re looking for?
TEACHER: Yes. So if sine of x = -1, then the value of x is 270° in degrees?
STUDENT D: Oh wait, if it’s -1, yes because it’s y on, wait, that doesn’t… because it’s there too. Hmm (pause). This doesn’t make sense. The way I did that, there’s two places where y equals -1.
TEACHER: So what does that mean?
STUDENT D: That means if sine equals y, I don’t know what it means.
TEACHER: Ok, so what do you want to say your final answer for number thirteen is now?
STUDENT D: Um (pause). I would just go with 3π/4.
TEACHER: Ok, so you’re going to disregard -90° altogether?
STUDENT D: I mean if I had to put an answer and since it was my first one I’d use that, but it doesn’t make sense.
TEACHER: Ok, type in the sine of -90° on your calculator and see what it says.
STUDENT D: Of what? -90°? -1
TEACHER: Ok, what about the sine of 270°?
STUDENT D: -1
TEACHER: So what does that mean?
STUDENT D: There’s more than one answer when sin(x) = -1
TEACHER: Does that mean it’s still a function? Was is a function first of all?
STUDENT D: Um, yes.
TEACHER: Ok, but can an answer have, can there be two answers and it still be a function?
STUDENT D: Yes, because every x can have only one y, but um, but then because this is -90° and this is 270° and they both equal -1, but that’s ok because they’re not the same x.
TEACHER: Ok, good.
STUDENT D: That’s the only way I can explain it.
TEACHER: Alright, you’re got two questions left. They both have to do with that last graph right there. So there’s some random trig graph. Is that graph a function? How can you tell?
STUDENT D: This? Yes.
TEACHER: Because?
STUDENT D: Because every x has only one y.
TEACHER: Ok. Alright. Can you tell if the graph is of a sin(x) function or a cos(x) function? Can it be both? How do you know or what information do you need to make a decision?
STUDENT D: Um, do I have an equation?
TEACHER: No
STUDENT D: No, I don’t? Ok, then I, well, I would need the equation.
TEACHER: Ok, so you couldn’t tell me right now it’s sine or cosine or both?
STUDENT D: Well, it could be either.
TEACHER: Because?
STUDENT D: Because, well, it depends if this is where it’s saying it starts, but it could be either because you could start a cosine function up here or you could start a sine function in the middle.
TEACHER: Ok, so let’s say I wanted an equation for sine, so it’s a sine equation.
STUDENT D: So then I would start with $y = -\sin(x)$. I know it’s negative because sine goes middle, up, middle, down, middle.
TEACHER: Ok
STUDENT D: And it is flipped over the x-axis.
TEACHER: Ok
STUDENT D: And moved up 3, so it would be $y = -3$, wait, yeah, no that’s not right, $\sin(x)$. The amplitude is 4 so it would be $-4\sin(x)$. Let’s see if I recall, so $b$, the period, would be $\pi/2$.
TEACHER: Why is it $\pi/2$?
STUDENT D: Er, (pause). Um, it would be $180^\circ$, which would be $\pi$.
TEACHER: Ok, so why is the period $180^\circ$?
STUDENT D: Because if starts here and ends on the graph at $180^\circ$.
TEACHER: Ok
STUDENT D: Um, so I forget how to write that in. Maybe, I think I do $y$. I did this before. No, + 1/2? Maybe. I forget how to do that part, but then I know it’s moved up 3 so we add 3.
TEACHER: Ok, excellent. Um, is there anything you want to go back to and add to your answers now that you’ve been through the entire thing?
STUDENT D: Um, it was these, I could figure those out now because sine equals $y$, so then you put 1/3 and go back to the graph you gave me.
TEACHER: Ok
STUDENT D: SO then that would be $y = 1/3$, which would be…
TEACHER: You said the answer was $19.5^\circ$
STUDENT D: So it would be when this was .33 so then it would be in between here so then that would make sense. I’m going to stick with that. Then for this one, um -1/6, so then look… I’m looking for when it’s -1/6. Well, that makes sense then, well, -9.6.
TEACHER: Ok, anything else?
STUDENT D: Is there anything else I need to answer?
TEACHER: Nope
STUDENT D: Alright
TEACHER: Well, thank you. Excellent job. I have a lot to analyze. Excellent job.
Appendix K: First Interview Transcript, Student E

TEACHER: Alright, so first question, how are the equation of a function and the graph of a function related? Explain your thinking.
STUDENT E: OK, um, well, like the equation of a function is like the input and like the output, right? I mean, like, what you put into the function and what you get is the output. The graph is like, like the visual interpretation of that. Like, you know, the x’s and the y’s, those are your inputs and outputs.

TEACHER: Ok. What is special about the x’s and y’s on the graph?
STUDENT E: Um, I mean, like each x is what you input into the function and then each y is what you get out of it, and the equation is obviously essential to that.

TEACHER: How so?
STUDENT E: What?
TEACHER: How so?
STUDENT E: Oh, um, because it determines what the output is (laughter).

TEACHER: Very good.
STUDENT E: Is that… Am I good?

TEACHER: Um hmm.

STUDENT E: So the second one is about a graph. How many points are on this graph? An infinite amount.

TEACHER: Ok, can you mark some for me?

STUDENT E: Yes, here, here, here (laughter). Here, here. Do you want me to tell you what they are?

TEACHER: Sure

STUDENT E: Um, so like (1, -4), and like (2, -3), and (0, -3). (-1, 0), (3, 0).

TEACHER: Ok, without extending the graph anymore, can you give me some more coordinates?

STUDENT E: Um, yes, I mean, yes, like if I plugged in like 5, 5² - 2(5) – 3. 12, right?

TEACHER: Ok, what about another point between 0 and 3? Are there any more points between 0 and 3 that you could tell me?


TEACHER: Ok, now without using the calculator, just looking at the graph, can you find another point? Estimate another point.

STUDENT E: Estimate one? Uh, well, if I just did that one, then 1 1/2 would be (1 1/2, -3 3/4).

TEACHER: Ok, go ahead and mark that one too. Alright, now go ahead and estimate one more for me. Any other point.

STUDENT E: Any other point? Um, just estimating it on the graph? Hmm, like, um, maybe like (3 1/2, 2).

TEACHER: Ok

STUDENT E: ish (laughter)
TEACHER: Excellent. Very good, next, question number three, can you give examples of functions and examples of non-functions? How can you tell when you have one? Are there multiple ways to tell?
STUDENT E: um, ok, so an example of a function would be like f(x) equals, I don’t know, well, that (pointing to equation in number 2), $x^2 - 2x - 3$, right? And then an example of a non-function, er, well this is a parabola, so then an example of a non-function would be like, oh gosh, I can’t think of an equation, but like, like the semi-circle one, radical whatever.
TEACHER: Can you draw a picture for me?
STUDENT E: Oh, wait, no, that’s not one. This one, a full circle.
TEACHER: Ok, and how can you tell it’s not a function?
STUDENT E: Um, well, like just doing the equation algebraically you can tell it’s not a function because there is more than output for every input and graphically there’s the vertical line test and if passes through more than one point at any, like, given one.
TEACHER: Ok, so draw me a graph that is a function.
STUDENT E: That is a function? Like that one is a function (graph in question #2), or like that is a function (draws graph). That’s a function (draws second graph).
TEACHER: Ok, very good. (student laughs). Are there any more ways you can represent functions and non-functions? Besides graphs and equations?
STUDENT E: Um, not that I can think of right now (laughs).
TEACHER: Ok, that’s fine. So that was algebra, now we’re into the trig stuff.
STUDENT E: Exciting
TEACHER: So, when you hear sine of x. what does sine of x mean? Is it a function? How can you tell?
STUDENT E: Um, uh, well, I mean it’s a function, right? Like this thing (draws wavy graph), or well not that thing. This thing (correctly sketches sine wave).
TEACHER: Ok
STUDENT E: Um, so yeah, it’s a function and you can tell because it passes the vertical line test at every point, right? (laughter), and…
TEACHER: So, when you hear sine of x, you think of a graph?
STUDENT E: Yeah
TEACHER: Anything else?
STUDENT E: Um, I think SOH CAH TOA
TEACHER: What about SOH CAH TOA? Explain SOH CAH TOA.
STUDENT E: Um, like the right triangle thing. So like sine would be like the length of the opposite side over the hypotenuse, right? Of like any given angle, like this one (draws picture). And then same thing for cosine and this angle.
TEACHER: Anything else come to mind for sine? Other ways you can represent it for me?
STUDENT E: Um, other than those two, nothing else comes to mind right now.
TEACHER: Alright, how can you find or estimate the sine of 30°?
STUDENT E: Wait, sine of 30°, so like the unit circle, right? Um 30° and then it’s ($\sqrt{3}/2, 1/2$), right? Or is that, that’s not what we’re talking about. We’re talking about
radians, no, no we’re not. Can I use my calculator? (types in calculator) Yeah, that’s the answer I was talking about. So like 1/2.

TEACHER: Ok, and where did the (√3/2, 1/2) come from?
STUDENT E: The unit circle
TEACHER: The unit circle? (student laughs) So the sine of 30° is 1/2 because it’s on the unit circle and you used your calculator?
STUDENT E: Yes
TEACHER: Ok, can you think of any other way to do the sine of 30°? Show me the sine of 30° is 1/2.
STUDENT E: Um…
TEACHER: Besides what you already showed me with the calculator and unit circle. Those are two great ways. Can you give me a third one?
STUDENT E: Um, I feel you can do it with like 30-60-90 triangle thing. I was never very good at that.
TEACHER: Could you try it for me?
STUDENT E: (laughs) Um, I forget how to do it. Um, it’s like the hypotenuse is like, one, right? If it’s on the unit circle.
TEACHER: Alright. Fair enough. Ok, now what about the sine of 20°? Can you find or estimate the sine of 20°?
STUDENT E: I can with my calculator.
TEACHER: Ok, go ahead.
STUDENT E: I can estimate it with my calculator
TEACHER: Ok
STUDENT E: .3420201453
TEACHER: Ok, now without the calculator, convince me that the sine of 20° is .342
STUDENT E: (laughter) Um…
TEACHER: Use triangles, the unit circle, whatever you want. Convince me that the sine of 20° is .342.
STUDENT E: Um, give me a second.
TEACHER: Ok, take all the time you need.
STUDENT E: (long pause) Nope, I forget.
TEACHER: Ok, so on the unit circle, which you said the sine of 30° is 1/2…
STUDENT E: Yes
TEACHER: How would the sine of 20° compare to that?
STUDENT E: Well, I mean, it has to be less than that because 30°, right? And it has to be less than that, so it’s like down here, right?
TEACHER: So, what does that mean? What’s less than that besides the angle measure?
STUDENT E: Um, like the length here, right?
TEACHER: Ok
STUDENT E: Is that what you’re talking about?
TEACHER: Sure. Is that what you’re talking about?
STUDENT E: (laughter) (pause)
TEACHER: OK
STUDENT E: (laughter) Well, I mean like this angle, right?
TEACHER: Alright, that angle is 20°.
STUDENT E: Yes, this angle is 20° and then like the little arc thingy is less than that, right?
TEACHER: Sure, ok, fair enough. So the sine of 20° is .342. You used the calculator and there’s no other way you could think of to show me?
STUDENT E: No
TEACHER: Alright, so next explain how to solve the following equations. \( \sin(x) = \frac{1}{3} \) and \( 2\sin(x) + 1 = \frac{2}{3} \).
STUDENT E: Um, I mean you could take the inverse sine of x for the first one, right?
TEACHER: Ok
STUDENT E: (talking to self) Right?
TEACHER: Ok, what do you get?
STUDENT E: I got 19.47.
TEACHER: So what does that mean?
STUDENT E: What do you mean?
TEACHER: 19.47, what does it mean?
STUDENT E: It’s my answer.
TEACHER: Ok
STUDENT E: Ok, second one, um, like isolating x, right? So it’s like negative one, \( \frac{1}{3} \), right? (talking to self) (laughter)Alright, so -1/6. So for that one, -9.59.
TEACHER: Ok, very good. So if you couldn’t use the calculator to get those two numbers there, 19.47 and -9.59, can you think of another way to find those?
STUDENT E: (sighs) I haven’t done this in awhile. Um, (pause), alright, I should be talking about what I’m thinking. Um, I guess I’m trying to think of what \( \sin(x) \) actually means. Um, (pause), no, I don’t know.
TEACHER: Ok, alright, so if the sine of x equals 2, what is the value of x? How do you know?
STUDENT E: Um, can I use the calculator again?
TEACHER: Sure
STUDENT E: I feel like it’s not
TEACHER: Do you think…
STUDENT E: I feel like it can’t. I mean it can’t really be 2, can it?
TEACHER: Why not?
STUDENT E: I mean think of the unit circle, right? I mean it only, like the highest number is 1. Well, except, no, yeah, yes.
TEACHER: So what does that mean?
STUDENT E: The sine of x can’t be 2.
TEACHER: Ok, can you draw a picture to convince me?
STUDENT E: Well, like I mean, like can I use this one (graph of \( \sin(x) \))?
TEACHER: Sure
STUDENT E: I mean, like for the graph of \( y = \sin(x) \) like it never goes up, like y is never equal to 2. The highest it ever gets is 1.
TEACHER: OK, what about with a triangle?
STUDENT E: What?
TEACHER: What about a triangle? Can you represent it with a triangle?
STUDENT E: With a triangle?
TEACHER: Sin(x) = 2, try to draw a triangle.
STUDENT E: (laughter) Ok, so we’re going back to SOH CAH TOA, right? Um, alright, let’s see. So, I mean, um, I feel like the hypotenuse has to be the longest one…
TEACHER: Yes
STUDENT E: So, I’m just going to use A. I guess I should use H for hypotenuse and the opposite, right?
TEACHER: Ok
STUDENT E: So, I mean like the hypotenuse always has to the greater than the opposite and like it’s on the bottom, like it can never be, like if it, just it can’t be 2, you know, like because this one is always going to be greater than that one.
TEACHER: Ok, excellent. Alright, is there a connection between the unit circle and the graph of sine of x?
STUDENT E: I’m going to go with yes.
TEACHER: Ok, can you explain the connection?
STUDENT E: Um, well, ok, so you have your unit circle, right? Um, ok, for sin(x), we’re just talking about sine. Um, so like at 30°, we’re using degrees, um, so (√3/2, 1/2), so 1/2 is our sine value, right?
TEACHER: Why 1/2?
STUDENT E: We talked about this (laughter). Because it is, because the unit circle says so.
TEACHER: Ok
STUDENT E: Can I go with that? Um, or like 90°, the unit circle at 90°, like the sine of 90° is 1.
TEACHER: Ok
STUDENT E: So at 90° on here the y-value is going to have to equal 1, like the output.
TEACHER: Ok
STUDENT E: So like for 30°, it’s going to be 1/2 (talking indistinctly).
TEACHER: Ok, very good. OK, can you walk me through how to get a graph of sine of x? How would you go about graphing sine of x?
STUDENT E: OK, well, you taught us that when you graph sine of x, it’s going to be, like going to look like this thing, or like some variation of it, right? You have to like move it over, so you’re going to have like the, oh my gosh, I forgot the up and down thing, the phase shift (laughter). Um, so this is like what you’re going to have. Like y equals, I forget what you call them. It’s like Asin(x) like Bx, ±C or something like that, and ±D, right? Or something like that (laughs) and so A here is going to be your amplitude and that’s going to like tell you, um, like up, like not like your up or down because that’s what D is, but it’s going to like tell you how, you know, like how far up and down it goes.
TEACHER: Ok
STUDENT E: And then (talking indistinct). I forget how to do phase shift. There’s like phase shift, right? And the period and then your up or down shift, right?
TEACHER: Ok

287
STUDENT E: And it’s like, I don’t know, like your up and down, ±D here, that’s going to tell you how far up or down it’s going to go. Like how you’re going to move the whole graph, not like stretch it, like amplitude does.

TEACHER: Ok

STUDENT E: And then, I forget how to get the phase shift.

TEACHER: What is the phase shift?

STUDENT E: Phase shift is like moving it left and right. Like the same way that you know like D moves it up and down.

TEACHER: Ok

STUDENT E: Uh, like moving the whole graph left and right and then the period is how long it’s like, how it stretches and shrinks sort of like amplitude does.

TEACHER: Ok

STUDENT E: Um, I forget how to get that. I don’t know. I haven’t done this in awhile.

TEACHER: Ok, so if you knew all that information, what would you do to get your graph?

STUDENT E: Well...

TEACHER: So, you know the period, the amplitude, and the phase shift and the up and down shift.

STUDENT E: Um, then I guess I would just find out where it starts with like the phase shift, right?

TEACHER: Ok

STUDENT E: I know how long it goes with the period. So then I would like make my scale, find like my, kind of like origin point, take like my phase shift, and my up and down, and like find out where it’s going to like start, you know? Where I need to start this thingy and then I would use the period to decide where it’s going to kind of end and then the amplitude to figure out how far up and down I needed to go.

TEACHER: Ok

STUDENT E: If that makes sense. Like for my max.

TEACHER: Ok, very good. Almost done. So, using the graph there for sin(x)...

STUDENT E: (laughter) Bring back sine.

TEACHER: Well, first off, the sine of 30°. Point to it on the graph.

STUDENT E: Right there, 1/2

TEACHER: Ok, now can you use the graph to estimate the sine of 20°?

STUDENT E: Yes, sort of. Alright, so if this is 1/2, you’re going by like .1, right? So then like .1, .2,.3, and that’s kind of like halfway between them. I’m going to guess 3.4202 (laughter). Or .342 or whatever.

TEACHER: Ok, so are you convinced it’s .342 then?

STUDENT E: Yes

TEACHER: Did the calculator convince you enough or did you have to look at the graph to know?

STUDENT E: Uh, the calculator convinced me pretty well.

TEACHER: ok

STUDENT E: But the graph definitely reassured me.
TEACHER: Ok, fair enough. Alright, so using the graph, if the sine of x equals -1, what is the value of x?
STUDENT E: Well, the sine of x = -1, then it’s like, it’s oh my gosh, it’s like 270° if we’re going to stay in degrees, ±… I can’t use the $2\pi k$ so like 360°k. Can I use that?
TEACHER: Sure. And how is this answer different from your answer in 12? Or are they not different?
STUDENT E: Well, sine of 20° is like one exact input for the function.
TEACHER: Ok
STUDENT E: And like sine of x equals -1 is like it can equal -1 and any of those points but at sine of 20° it only has one output.
TEACHER: Ok
STUDENT E: There are various inputs that make -1 but not various outputs for sine of 20°.
TEACHER: Ok, and using that graph, can you explain again why the sine of x can’t be 2?
STUDENT E: Because it doesn’t, it doesn’t go like, sine of x equals y is your equation, it can’t equal, it can’t go above 1. It oscillates between 1 and -1.
TEACHER: Ok, excellent. Alright, two more questions.
STUDENT E: Ok
TEACHER: If there’s a graph, is it a function, how can you tell?
STUDENT E: Well, this graph is a function because it passes the vertical line test.
TEACHER: Ok, excellent. Can you tell if the graph is of a sine of x function or a cosine of x function? Can it be both?
STUDENT E: Um, (pause) my first instinct was sine just because it looks like, you know just looking at it right away because it’s like kind of looks like a negative sine function, you know? Like that one flipped, but I mean I guess I feel like it can’t be a cosine function for some reason, but I mean I guess it could be because you know it can do this thing too, right?
TEACHER: Ok
STUDENT E: It’s what a cosine function does, right? Um, I don’t know, I’ll just be diplomatic and say both.
TEACHER: Ok, so what information do you need to decide? Or do you want to stick with both?
STUDENT E: I mean I guess I would want to know, like, I would want to know the phase shift and stuff to help. Like that would be important. It would tell you if it’s moved over or not.
TEACHER: Ok, can you write the equation of that trig function? All or part of it?
STUDENT E: Um, maybe part of it. We’ll go with part of it. Give me a second.
TEACHER: Ok
STUDENT E: So, alright, this would be my amplitude here. If it was well, I guess you don’t (talking indistinctly). We’re just going to go with sine for right now. Um, this would be my amplitude right here, right? So, -1, 2, 3, 4, so – 4, right?
TEACHER: Why use the negative again?
STUDENT E: Well, because like a normal sine function is like this, like a positive one
and it’s like x-axis reflection, that’s what I’m looking for.
TEACHER: Ok, excellent
STUDENT E: Alright, so -4sin and then, um, I’m going to use sin(x), right? Because this
is my no phase shift one and then plus that moves up 3, maybe. I don’t know. Wait, can
I check?
TEACHER: Um hmm
STUDENT E: No, wait
TEACHER: So what are you doing now?
STUDENT E: Um, I’m just putting in the thing into the little y= thing.
TEACHER: Ok
STUDENT E: And then I changed my window to zoom trig
TEACHER: Ok
STUDENT E: And I don’t know. I don’t think it looks right.
TEACHER: How can you check?
STUDENT E: Because, like, alright, so at what is this? Like 45°? Probably ish. Um, so
if I put in… Um, let’s just go with 90.
TEACHER: What did you get?
STUDENT E: I got -1
TEACHER: Ok, is that good or bad?
STUDENT E: Bad
TEACHER: Because?
STUDENT E: Because it’s definitely not negative. It’s like 3, so that’s telling me I did
something wrong.
TEACHER: Ok, so can you tell what’s wrong with the graph?
STUDENT E: Um, oh, it’s the wrong period. Um, because this one, the period is from, it
stops over here, 180 instead of all the way to 360, so it’s like (pause). Is it like sine of x/2
then? I forgot how to do period. Nope, that’s not it.
TEACHER: why not?
STUDENT E: Because for this I get .17 and that still not right.
TEACHER: OK
STUDENT E: So I went the wrong way. 2x. Ok that’s better. That gave me what I
wanted. Alright.
TEACHER: So what are you doing right now?
STUDENT E: I’m just checking points to see.
TEACHER: Ok, how many points are you going to check before you’re convinced?
STUDENT E: Um, just the ones that I can find.
TEACHER: How many points?
STUDENT E: I checked 3 points.
TEACHER: Ok, they all worked?
STUDENT E: Yeah, well, it might have been a coincidental 3 points but…
TEACHER: OK, now do you know how to change it so it’s a graph of cosine? That’s the
sine graph.
STUDENT E: I forgot how to do phase shift. Um, it has some to do with like B and C being related to each other, but I needed to like, and like if you subtract it goes to the right and if you add it goes to the left.

TEACHER: Ok

STUDENT E: I don’t remember how it all relates to each other.

TEACHER: Ok, what if I told you the graph was shifted 45° to the left?

STUDENT E: Oh, that one. I still don’t know how to do it. Um, I mean…

TEACHER: Could you figure it out?

STUDENT E: Maybe. I can try. Um, ok, so… This one… Cosine graph looks like the little bowl right? So this one wouldn’t be negative. This one would just still be 4 though. 4cos and then, this, (pause). It’s like, um, oh, I’m going to leave that part blank. And, um, ok, I’m going to play for a second, alright?

TEACHER: Ok, just tell me what you’re plugging in.

STUDENT E: Alright, so I’m plugging in 4cos just x +3 this time, or, well I guess I want 2x, right? Because it would certainly be the same. Well, maybe not since I’m going to be adding something.

TEACHER: So what are you typing in?

STUDENT E: Um, right now I have 4cos(x) + 3 in. Well, x, parentheses around the x. And then, now I’m putting in 4cos(x), er cos(2x) in parentheses, + 3. So that still doesn’t give me my phase shift, but it kind of looks similar. So I want it to move left.

TEACHER: Ok

STUDENT E: So, I’m going to add something. I know that. Um, I’m just going to do +1 and see what it does.

TEACHER: So 4cos(2x + 1) + 3?

STUDENT E: Yes, and now I’m going to plug in a point and see what it gives me. Alright, so that didn’t give me what I wanted. That gave me like almost -1. So that can’t be right. Why don’t I just do -45?

TEACHER: So you changed the plus one to a plus 45?

STUDENT E: No, I just changed the point I was trying to -45. Alright, it’s weird for me to think in degrees too. Alright, so that didn’t work. Alright (pause). Alright let’s see. So, (pause) um, (pause), I don’t know. I’m just trying 4.

TEACHER: What are you trying now?

STUDENT E: I’m just trying 4, but it’s not moving?

TEACHER: + 4?

STUDENT E: Yeah, it’s not doing what I want it to.

TEACHER: Ok, the + 4 moves it how far?

STUDENT E: I don’t know. That’s what I’m trying to figure out.

TEACHER: Ok. What is your x-scale?

STUDENT E: What do you mean?

TEACHER: Your x-scale on your graph?

STUDENT E: Oh, 90. Oh wait, so…

TEACHER: So each mark on your x-axis stands for what?

STUDENT E: Well, 90°.

TEACHER: Each of these, go ahead and hit your graph button.
STUDENT E: There we go.
TEACHER: Each movement along there is what?
STUDENT E: 90, well, 90.
TEACHER: So that 90°, that’s -180°, so on and so forth?
STUDENT E: Right
TEACHER: Ok, so if you add one, it’s going to move it how far?
STUDENT E: Is it going to move it one?
TEACHER: One what?
STUDENT E: Unit? Degree?
TEACHER: Degree
STUDENT E: Is it just one degree? Oh, that’s stupid. So do I try + 45? I’m going to try + 45.
TEACHER: Ok
STUDENT E: Ok, this looks better.
TEACHER: So you tried 4cos(2x + 45)?
STUDENT E: Yeah, it’s still not what I want. I got, no wait, hang on, I got (-45, 5.828),
TEACHER: What should it be?
STUDENT E: Well, you said it was shifted 45°, so it should be 7.
TEACHER: Ok, so where is the point that you want at? Like that point, where’s it at on your graph that you have in front of you on the calculator?
STUDENT E: I don’t know
TEACHER: So what are you doing now?
STUDENT E: I’m moving up my y-max. Um, it’s not -30. It’s like close to -15, -20ish.
TEACHER: What if I told you it was -22.5? See if that works.
STUDENT E: Then I would say I need to double mine. 45, right? Oh! Oh! Is it because it C/B? Is that what the phase shift is?
TEACHER: Try it.
STUDENT E: Like 90/2. It is! (laughter)
TEACHER: Ok, so what is your final equation?
STUDENT E: Uh, y = 4cos(2x + 90) + 3
TEACHER: Ok, thanks (student laughs). Alright, now that you’ve been through the entire questionnaire, would you like to go back and modify any of your answers? Add to them? Take away anything?
STUDENT E: Um…
TEACHER: Or are you content with what you have?
STUDENT E: Um, hang on, I’m pretty content with page one. I really think I got that one. Um, so where was that thing we did when we were talking about the graphs? Do I need to change this one?
TEACHER: Do you?
STUDENT E: Well, we figured out that phase shift is C/B, right? And then minus is right and plus is left.
TEACHER: Ok
STUDENT E: Figured that one out.
TEACHER: What about the period?
STUDENT E: The period is something over, like, 4 or something like that.
TEACHER: Ok
STUDENT E: Like (laughter), I don’t know if that’s off a little. Um, the period is like, ok, so sine of x, and the period is like, like 360°, right? And we figured out that the period is cut in half when I put 2x for b right?
TEACHER: Ok
STUDENT E: So is it like… It’s not just like B/4, is it?
TEACHER: Well, what was B?
STUDENT E: B is this thing.
TEACHER: In the equation over there B was 2, right?
STUDENT E: 2/4 is 1/2. Oh, and the period is not 1/2.
TEACHER: That’s the period?
STUDENT E: The period is 180°
TEACHER: Ok
STUDENT E: So is it like 4? No it’s not.
TEACHER: A, B, C, D. Which of those affect the period?
S; B
TEACHER: Ok, so it’s just B, nothing else is affecting the period?
STUDENT E: Does C?
TEACHER: Does it?
STUDENT E: No, it didn’t affect my period at all for my cosine equation.
TEACHER: So, B is the only one that affects the period. When B was 2, the period was 180°.
STUDENT E: Yes
TEACHER: So what does that mean?
STUDENT E: That means that it was… so if B (talking indistinctly), so that’s 1, that’s 360°. So it’s like, give me a second. Um, (pause). It’s like, hang on, I’m going back to my graph.
TEACHER: Ok, what are you trying?
STUDENT E: Um, I’m just, I don’t know. I’m just trying to think about it. Like if…
TEACHER: Without graphing it, if B gets bigger, does the period get smaller or longer?
STUDENT E: Small
TEACHER: So it implies what is happening?
STUDENT E: (laughter) If B gets bigger then my answer gets smaller, right?
TEACHER: Ok, and if it gets smaller, what happens to the period?
STUDENT E: It gets bigger.
TEACHER: Ok, so what mathematical operation would support that?
STUDENT E: Wait, if B gets smaller, it gets bigger so… division? Division, right? I think it’s something over 4.
TEACHER: Since we’re running out of time, it’s just period over B.
STUDENT E: Oh. I couldn’t remember the formula.
TEACHER: One final thing. Look at number seven.
STUDENT E: Oh, that’s the scale I was thinking about. Ok, looking at number seven. Where is number seven?
TEACHER: Seven?
STUDENT E: Got it.
TEACHER: Do you want to modify your answer to those based on the graph?
STUDENT E: Based on my graphs?
TEACHER: Look at your answer to number thirteen.
STUDENT E: Oh, I need to do ±360°k.
TEACHER: Why?
STUDENT E: Because it’s the same thing I mentioned with that. There is more than one input that can give you that number.
TEACHER: Ok
STUDENT E: So like ±360°k, right? Ok.
TEACHER: Excellent. Awesome job.
Appendix L: First Interview Transcript, Student F

TEACHER: Alright, so first question, how are the equation of a function and the graph of a function related? Explain your thinking.
STUDENT F: Um (talking indistinctly), the function, well, isn’t it the function, the function is the equation technically, and um, the equation is like how you get the inputs and outputs of the function.
TEACHER: Ok (pause). And the graph?
STUDENT F: The… Oh, the graph. I forgot that part and then the graph is the visual demonstration of the function and the equation.
TEACHER: So can you explain how the equation and the graph are related?
STUDENT F: The, um…
TEACHER: You said it was a demonstration. What do you mean by that?
STUDENT F: Well, I mean you plot the points, like plug in x and y, you find the values and then you plot them onto a graph. Then you draw the graph.
TEACHER: Ok, very good. Next question, the following is the graph of \( y = x^2 - 2x - 3 \). How many points are on this graph?
STUDENT F: Infinite
TEACHER: Ok, can you label a couple?
STUDENT F: Um, yeah. We have (0, -1), no wait, (-1, 0).
TEACHER: Go ahead and mark it on your paper.
STUDENT F: And you have (0, -3), and (3, 0).
TEACHER: Ok, can you find more points between here and here? (pointing to (-1, 0) and (0, -3))
STUDENT F: Um, like (.75, -1)
TEACHER: Ok, very good. Can you give examples of functions and non-functions? How can you tell when you have one? Are there multiple ways to tell?
STUDENT F: Um, well, a non-function is when the, um, the non-function is when there’s more that one of the same input? And the function is when there’s one input for every output, and, um, you can tell if, there’s like the vertical line test, making a table of values, and that’s all I remember.
TEACHER: Ok, can you give me some examples?
STUDENT F: Of a function?
TEACHER: Or a non-function
STUDENT F: Like \( y = 5x + 4 \)
TEACHER: Is that a function or not?
STUDENT F: I’m going to say that’s a function.
TEACHER: Because?
STUDENT F: Because I don’t know (laughter). Um, (pause). I don’t know. I made it up.

TEACHER: Ok, what about an equation that’s not a function?

STUDENT F: Um, (pause). \( y = x^3 - 2 \)

TEACHER: And why do you think that’s not a function?

STUDENT F: I just think it is. I don’t feel like it would pass the vertical line test when I picture it in my mind.

TEACHER: Ok, can you draw me a couple graphs that are functions and not functions, since you’re using the vertical line test? You can just sketch it on the page there anywhere, it doesn’t matter. It can be real quick.

STUDENT F: (drawing) Um, (pause), not, (continues drawing)

TEACHER: Alright, is that one a function?

STUDENT F: Yes, that is a function.

TEACHER: Is that one a function?

STUDENT F: No

TEACHER: Is that one a function?

STUDENT F: No

TEACHER: Ok, very good. Alright, so that was a review of some algebra. Now we’re into the trig questions, so what does sine of x mean? Is it a function? How can you tell?

STUDENT F: Um, sine of x I think is a function. I think it does pass the vertical line test even with all the sine waves and, (pause).

TEACHER: Ok, very good. So when you hear sine of x, what do you think of?

STUDENT F: Um, \( \sin(x) \)? Triangles.

TEACHER: OK, can you explain it a little bit more?

STUDENT F: (laughs) Like finding, you know, missing sides of triangles and angles and all that fun stuff.

TEACHER: Can you give me an example?

STUDENT F: Am I supposed to write on this?

TEACHER: Um hmm

STUDENT F: Ok

TEACHER: Ok, so what did you do?

STUDENT F: Um, well, I drew a right triangle and then I wrote down some random numbers and then I put a random angle in there and since sine is opposite over hypotenuse and, the SOH CAH TOA would be 50/12.

TEACHER: Ok, so what would the purpose of doing that be? What’s the ultimate goal?

STUDENT F: Um, to find this missing angle because you already have a 90° angle. This is incorrect but…

TEACHER: Why is it incorrect?

STUDENT F: Because 90 and then 45 and 45 (continues indistinctly)

TEACHER: So that equation you wrote would help you find the measure of that angle?

STUDENT F: Yeah, eventually.

TEACHER: How so?

STUDENT F: Um, well, let me think. You would take the sine of B over 6, or it might be flipped. Equals 50 over sine of 40°. I think it’s flipped actually (rewrites proportion)
TEACHER: Ok, excellent. So anything else come to mind when you hear sine of x?
STUDENT F: The circle stuff
TEACHER: What do you mean by circle stuff?
STUDENT F: Like, the unit circle, you know, sine of... I’ve forgotten all of this (laughter).
TEACHER: It’s ok. Alright, so next question, how could you find or estimate the sine of 30°?
STUDENT F: Um, well, if you know the unit circle you could just remember that 30° is, the sine is the y-coordinate so that would be, that would be 1/2.
TEACHER: Ok, is that a guess or do you know for sure it’s 1/2?
STUDENT F: It’s a 95% I know for sure and 5% because it could also be \( \sqrt{3}/2 \).
TEACHER: Ok, so how could you figure that out right now if you had to?
STUDENT F: Um, (pause). I do not know.
TEACHER: Ok, do you know of a way?
STUDENT F: My friend the calculator.
TEACHER: Ok, try it. (student picks up calculator) What are you doing right now?
STUDENT F: Checking my mode.
TEACHER: Ok
STUDENT F: 1/2
TEACHER: Alright, what did you type in?
STUDENT F: sine 30°
TEACHER: And you got?
STUDENT F: .5
TEACHER: Alright, any other way that you could convince me the sine of 30° is 1/2 besides the unit circle and besides the calculator? Which are both correct by the way.
STUDENT F: I... I feel like there is, but I can’t think of it. I would need more information.
TEACHER: What type of information?
STUDENT F: Um, hold on. I’m just going off of if you didn’t happen to know the unit circle. If you’d have to know like where, what quadrant it would be in. Um, (pause). Maybe if you knew the cosine or something.
TEACHER: Ok, if you knew the cosine, how would that help you?
STUDENT F: Well, because if you knew the cosine you might also know the hypotenuse and then you could like cross multiply or something like that.
TEACHER: Ok. So you’re thinking of using a triangle and using ratios?
STUDENT F: Yes
TEACHER: Ok. Alright, very good. Now what about the sine of 20°? Because that’s not one of the points on the unit circle.
STUDENT F: Um
TEACHER: Either estimate it or tell me how I could find the value of 20°.
STUDENT F: I mean you could use your calculator.
TEACHER: Ok
STUDENT F: And that’s .34202021433
TEACHER: Ok, now convince me that that’s the actual answer without saying the calculator told you so.
STUDENT F: (laughter) Um, (pause). I don’t remember.
TEACHER: Ok. What if we look at the unit circle? So can you make an argument with the unit circle that the sine of 20° should be .342?
STUDENT F: Um
TEACHER: Based on what you know from the unit circle
STUDENT F: Well, you technically could because 20, if 30°, if the sine of 30° is 1/2, then 20 is less than 30, then it would be less than 1/2 and not significantly.
TEACHER: Ok, what about with triangles?
STUDENT F: Um, (pause). I mean you would have to be given more information than just that. Then you could do the ratio.
TEACHER: Ok, that’s very good. So, explain how to solve the following equations.
You have sin(x) = 1/3 and you have 2sin(x) + 1 = 2/3.
STUDENT F: Um, I think for sin(x) = 1/3 you would just do the inverse so it would be inverse sine. Wait, no that’s wrong. You’re trying to get x by itself, right?
TEACHER: Um hmm
STUDENT F: Ok, so, (pause). Would you have to find what equals… what sine, what value of sine equals 1/3? Something like that?
TEACHER: Um hmm
STUDENT F: Ok, um, so (pause)
TEACHER: So what are you thinking right now?
STUDENT F: Um, (pause). I want to say like something with the unit circle, but I don’t really know where to go from there.
TEACHER: What are you trying to do with the unit circle?
STUDENT F: I am trying to see if there is any point that is on, (interruption from PA). I am trying to see if there is any, like points, like sine, cosine, or tangent, preferably sine that equal 1/3.
TEACHER: Ok, do you know of any on the unit circle?
STUDENT F: Not off the top of my head.
TEACHER: Ok, so how else could you go about it if the unit circle is not going to help?
STUDENT F: Um, I don’t know.
TEACHER: Ok, so you mention you had to get x by itself.
STUDENT F: Um hmm
TEACHER: How do you get x by itself?
STUDENT F: I normally would divide but there’s nothing really to divide by.
TEACHER: Ok, so what other options do you have?
STUDENT F: Multiply?
TEACHER: Will that help?
STUDENT F: Probably not
TEACHER: Ok, so to get x by itself, what do you have to do?
STUDENT F: Um, you have to get rid of sine.
TEACHER: And how do you get rid of sine?
STUDENT F: Inverse. That doesn’t work either because there’s no inverse to the inverse. I mean, there’s no… I mean would you be able to do the sine inverse equals 1/3 inverse?

TEACHER: Write it down. (student writes equation) Alright, what happened to you sine of x?

STUDENT F: I inversed it.

TEACHER: Ok, so what happens if you inverse sine? What do you get?

STUDENT F: (typing on calculator) You get error on the calculator.

TEACHER: Ok, what did you type in?

STUDENT F: Sine, inverse

TEACHER: Ok. So you have to put something in there obviously. (student types on calculator) So what did you do?

STUDENT F: I haven’t done anything yet.

TEACHER: What are you trying to do?

STUDENT F: Does it have to be in radians for this particular one?

TEACHER: Why?

STUDENT F: Because it’s a fraction and I don’t like to associate fractions with degrees.

TEACHER: You did right here (question number five).

STUDENT F: I did? Oh, cool. Nevermind. Uh, well, the inverse sine of 1/3 is 19.47

TEACHER: So what does that mean?

STUDENT F: Um (pause). It means that the sine inverse of 1/3 is 19.47.

TEACHER: Ok, so what does 19.47 mean?

STUDENT F: Um, I don’t know.

TEACHER: Is that the solution to your equation?

STUDENT F: I’m going to say no.

TEACHER: Ok, so the sine of 19.47 doesn’t equal 1/3?

STUDENT F: Yes (checks on calculator). Yes

TEACHER: It does?

STUDENT F: Just about.

TEACHER: Ok, so what does that mean?

STUDENT F: That the sine of 19.47 equals 1/3 so x would equal 19.47.

TEACHER: Ok and x is what type of thing?

STUDENT F: It’s an angle.

TEACHER: Ok, in radians or degrees?

STUDENT F: Degrees

TEACHER: Ok. Alright next one, 2sin(x) + 1 = 2/3

STUDENT F: Um, you would subtract 1 first

TEACHER: Ok

STUDENT F: So that would be 2/3 minus 1 would be -1/3 and then… Are you allowed to have a negative answer? I don’t think you’re allowed to have a negative answer.

TEACHER: Because?

STUDENT F: I just remember learning in your class that a negative answer is not allowed somewhere in the unit on trigonometry.

TEACHER: Ok. So you have 2sin(x) = -1/3. What does that mean literally?
STUDENT F: That the angle would be less than 1.
TEACHER: Why?
STUDENT F: Because it’s a negative.
TEACHER: Ok
STUDENT F: Or less than 0
(PA interruption)
TEACHER: You were saying?
STUDENT F: Huh?
TEACHER: What were you saying?
STUDENT F: That angle would be less than 0 because it’s negative.
TEACHER: Can an angle be less than 0?
STUDENT F: Not really, no.
TEACHER: We didn’t have any angles less than 0?
STUDENT F: I don’t remember.
TEACHER: Ok, can an angle be -33°?
STUDENT F: Um,
TEACHER: Can you draw an angle that is -33°?
STUDENT F: No
TEACHER: Ok, what if we kept solving the problem and forgot that point. What do you get from this?
STUDENT F: You would have to divide by 2. That would be -.166 and then you would pretty much do… I guess you could change that to a fraction, which is -1/6. And then you would pretty much do what you did in the first one. Find the inverse.
TEACHER: Ok, so try that.
STUDENT F: So the x would be -9.59.
TEACHER: You said it would be a negative number, right?
STUDENT F: Yes
TEACHER: Can x be -9.59?
STUDENT F: Um, I guess so.
TEACHER: Ok, how could you check?
STUDENT F: By plugging in the sine of -9.59 and see if I get -1/6.
TEACHER: So does it work?
STUDENT F: Yes
TEACHER: Ok, very good. So you used algebra there to solve those two equations. Can you think of any other way to solve those two equations? Can you think of any other way to solve those two equations without using the algebra you just did?
STUDENT F: Um (pause).
TEACHER: No? Ok, that’s fine. Alright, if \( \sin(x) = 2 \), what is the value of x? How do you know?
STUDENT F: Um, sine of x is 2? Um, you could technically just see the inverse sine of 2 and then that would be x. Because then you’d know that the sine of that value is x. Sine inverse, 2 (on calculator), which is an error so apparently there is no value x according to the calculator.
TEACHER: Ok, do you know why?
STUDENT F: Because 2 is a whole number?
TEACHER: What’s wrong with whole numbers?
STUDENT F: I don’t know, just the other two I tried that method with were fractions and
they worked fine.
TEACHER: Ok, what if I said sin(x) was 3/2?
STUDENT F: Then that’s technically a fraction. It’s an improper fraction. I mean you
could try the same method and you still get an error.
TEACHER: Ok, so if it’s an improper fraction, you’re going to get an error.
STUDENT F: Yeah
TEACHER: What about 1?
STUDENT F: (types on calculator) You get 90.
TEACHER: So what does that mean?
STUDENT F: It means that 1 is a y-value on the unit circle for 90°.
TEACHER: So 1 works?
STUDENT F: Yeah
TEACHER: But no improper fractions work?
STUDENT F: Or 2.
TEACHER: Ok, do you know why that would be?
STUDENT F: No
TEACHER: Ok. Is there a connection between the unit circle and the graph of sine of x?
STUDENT F: Um, well some of the points for the sine of x could be on the unit circle.
Like I see … I see like 180 on there. It’s a point of the unit circle.
TEACHER: Ok, what do you mean by given the degrees?
STUDENT F: I mean like 30°, 45, 60, 90…
TEACHER: Ok.
STUDENT F: Some others would be 180, these points are on the unit circle.
TEACHER: How are they points? They’re just random degrees. Points have an x and a y.
STUDENT F: Well, it’d be like (√3/2, 1/2) and (√2/2, √2/2).
TEACHER: Ok, and how does that connect to this graph in front of you?
STUDENT F: Um, it connects because (pause). I don’t know.
TEACHER: Ok, so what is the coordinate on the unit circle at 0°?
STUDENT F: The coordinate on the unit circle at 0°? (pause) (1, 0)
TEACHER: Ok, so find 0° on the graph of sine of x and what do you see.
STUDENT F: I see that it meets at (0, 0).
TEACHER: Ok, now look at 180°. What is the coordinate at 180°?
STUDENT F: (180°, 0)
TEACHER: Ok, what is the coordinate on the unit circle at 180°?
STUDENT F: (0, 1)
TEACHER: Are you sure?
STUDENT F: Well…
TEACHER: Draw the circle.
STUDENT F: Oh! This would be 90°. OK, so then it would be (-1, 0).
TEACHER: Ok, on the graph you have (180°, 0), on the unit circle you have (-1, 0) Do you see a connection?
STUDENT F: Yes
TEACHER: What’s the connection?
STUDENT F: The coordinates are the same.
TEACHER: How so? Like show me a couple on the graph of sine of x. (long pause)
TEACHER: So we have 0°, 180°... Do 0°, 90°, and 180° for me. Show me how they connect.
STUDENT F: Um 0, um, 0 would be (0, 0) on the unit circle. Wait, I guess they don’t.
TEACHER: On the unit circle, what does (1, 0) mean?
STUDENT F: (1, 0)? I forget.
TEACHER: Alright, 30°. What is the point at 30°?
STUDENT F: The coordinate at 30° is (√3/2, 1/2).
TEACHER: Ok, so what does that mean about 30°?
STUDENT F: The x value is √3/2 and the y is 1/2
TEACHER: Ok. Why is that significant? Why do we bother to remember that?
STUDENT F: Because so that you can plug in values easier and get more values.
TEACHER: Ok. So at 30°, you have (√3/2, 1/2). What is special about (√3/2, 1/2)?
Other than it’s the answer.
STUDENT F: Um, I forget. The only thing I remember is using the triangles to find those values on the unit circle.
TEACHER: Ok. What is the cosine of 30°?
STUDENT F: Cosine 30°? √3/2
TEACHER: Why?
STUDENT F: Because it’s the x-value.
TEACHER: Ok, what is the sine of 30°?
STUDENT F: 1/2
TEACHER: Ok, on the graph, find 30°
STUDENT F: (points to 30° on the x-axis) Right here.
TEACHER: Ok, what’s the y-value there?
STUDENT F: 1 and 0. There’s no y. I don’t know. It could be any y-value.
TEACHER: On the graph?
STUDENT F: Like infinite because you could go up or down for y-values as long (talking indistinctly)
TEACHER: Alright find 30° on the y-axis.
STUDENT F: Right here
TEACHER: Now find 30° on the graph.
(pause)
STUDENT F: I don’t know what you mean by that though.
TEACHER: Ok, let’s try another one. Find 90° on your scale. Now find 90° on your graph.
STUDENT F: Would it be right here?
TEACHER: Ok, so what is that point?
STUDENT F: That is 90° corresponds to -1.
TEACHER: Ok, so mark that point. Label the coordinates again. So what is this right here?
STUDENT F: (90°, -1)
TEACHER: Why is it -1?
STUDENT F: Because there’s -1 right there. (points to positive 1 on the y-axis)
TEACHER: So what’s that value right there (points to -1 on the y-axis)
STUDENT F: That is also a negative. So that would be 1 (changes coordinate to (90°, 1)).
TEACHER: Ok so that’s (90°, 1). What’s the coordinate on the unit circle at 90°?
STUDENT F: (1, 0)
TEACHER: Alright label it. What’s the coordinate at 0?
STUDENT F: Oh wait, no, no, no. 90° is (0, 1)
TEACHER: and 180°?
STUDENT F: (-1, 0)
TEACHER: Ok. What are the coordinates at 0° on the graph?
STUDENT F: 0 is (0, 0).
TEACHER: Ok, good. What are the coordinates of 180° on the graph?
STUDENT F: Um, (180°, 0)
TEACHER: Alright, write that down. Now look at 0° on the graph and on the unit circle, look at 90° on the graph and on the unit circle, look at 180° on the graph and on the unit circle. What do you notice?
STUDENT F: Um, the sine of x is the same. The y-value is the same.
TEACHER: Ok. Do you think there’s a reason to that?
STUDENT F: I’m sure there is.
TEACHER: Do you know what it could be?
STUDENT F: No.
TEACHER: Ok. Alright, next question. Explain how to graph sine of x. So how would you go about getting that graph if you had to create the graph for nothing on a piece of paper? How would you go about graphing sine of x?
STUDENT F: Um, well, like if I had an equation of just sine of x?
TEACHER: Your equation is y = sin(x).
STUDENT F: Y = sin(x)? Ok, um, well, hopefully I would remember the pattern that you taught me and like up, down, middle up, but that could be cosine though.
TEACHER: Ok
STUDENT F: And then I would draw the scale and amplitude and starting and ending point.
TEACHER: Ok, anything else?
STUDENT F: No
TEACHER: Alright, so using that graph, can you estimate the sine of 20°?
STUDENT F: Um, I could.
TEACHER: Alright, what would you need?
STUDENT F: I mean, if you wanted to go and find like where 20° is on the graph.
TEACHER: Can you do that?
STUDENT F: It would be like around here.
TEACHER: Why?
STUDENT F: Because 90° is up here.
TEACHER: Ok. Do you know what your scale is on the graph? What each mark is?
STUDENT F: Um, the scale is 180°. I think.
TEACHER: So each mark is 180°?
STUDENT F: Well, from like here, each like hill thing is 180.
TEACHER: Ok, but the actual graph not the… the graph paper itself, do you know what each mark’s increment is?
STUDENT F: No
TEACHER: How could you find that out?
STUDENT F: Um, I don’t know.
TEACHER: Ok, what if I told you that each mark on the x-axis was 10°? Does that seem reasonable?
STUDENT F: Sure
TEACHER: Because?
STUDENT F: Because I don’t have any other reason.
TEACHER: Ok, how many marks between 0 and 180?
STUDENT F: 0 and 180? Like here to here? There’s not technically any.
TEACHER: Each of these marks.
STUDENT F: Ok, I got you. 7 not including 0 and 90. (counts from 0° to 90°)
TEACHER: OK, so is it 10° then? If you count by 10’s does it work?
STUDENT F: Um, let’s see… 30, 40, 50, 60, 70, 80. Yeah.
TEACHER: Ok, so where’s 20° at?
STUDENT F: Right here
TEACHER: Ok, now use that to estimate the sine of 20°, or can you?
STUDENT F: Um, you mean the sine of 20°? Well, would it still be 0 because it’s close enough to the 0 coordinate?
TEACHER: Show me what coordinate you’re looking at.
STUDENT F: Right here (points to (20°, 0)).
TEACHER: Ok, is that on the graph?
STUDENT F: It is not on the graph.
TEACHER: Ok, so what would be?
STUDENT F: This would be on the graph (points to (30°, .5)).
TEACHER: Ok, so what’s the point there?
STUDENT F: Um (20°, .5).
TEACHER: What did you say it was earlier?
STUDENT F: .34
TEACHER: Ok, do you think it’s a very accurate estimate?
STUDENT F: No
TEACHER: Ok, next question. Using the graph, if the sine of x equals -1, what is the value of x?
STUDENT F: It would be I think 270, something like that.
TEACHER: Ok, why 270?
STUDENT F: Because on the unit circle the sine of 270 is (-1, 0), no (0, -1).
TEACHER: Ok, and on the graph there?
STUDENT F: Well, like 270… -270 would be right here.
TEACHER: Ok
STUDENT F: And that would be (-270, 1) but then for positive 270 the y-value would be -1 over here.
TEACHER: So if the sine of x equals -1, the answer is 270°?
STUDENT F: Yes
TEACHER: Ok, two more questions. Number fourteen, you have that graph right there in front of you, is this graph a function? How can you tell?
STUDENT F: Uh, it is because it passes the vertical line test.
TEACHER: Ok, now, can you tell if the graph is of a sin(x) or cos(x) function? Can it be both? How do you know or what information do you need to make a decision?
STUDENT F: Um, I don’t know. Um, I think I would need more information to figure it out.
TEACHER: Ok, what information would you need?
STUDENT F: Um, probably points from the graph.
TEACHER: Ok. Can you find any points on the graph?
STUDENT F: There is, (pause). I can but I forget how to.
TEACHER: Ok, so you can’t decide if it’s sine or cosine?
STUDENT F: Not visually
TEACHER: Ok, let’s say it’s… Well, can it be both sine and cosine or is it one or the other?
STUDENT F: Um, I think it has to be one or the other.
TEACHER: Why?
STUDENT F: I don’t think you can just switch from sine to cosine all of a sudden.
TEACHER: Ok, so I want you to write an equation for sine that describes the function.
STUDENT F: Ok
TEACHER: So it’s a sine graph.
STUDENT F: Sine graph? Ok, um, (pause)
TEACHER: So you said 6x, why is it 6x or 6sin, sorry.
STUDENT F: Um, because it’s a pretty big amplitude and 6 seemed like a good number.
TEACHER: Ok, and the +2?
STUDENT F: +2? Um, well, it’s small in between down here so I forget where the +2 comes in.
TEACHER: Ok, anything else?
STUDENT F: No
TEACHER: How about if I said it was an equation for cosine?
STUDENT F: I mean, don’t you figure out how to graph sine and cosine using like the same information? Like you could use the same formula.
TEACHER: Ok, so what would be different?
STUDENT F: Um, the pattern of how you draw the waves.
TEACHER: Ok, so which one do you think is more accurate?
STUDENT F: cosine

305
TEACHER: Because?
STUDENT F: Um, (pause), I don’t know
TEACHER: Ok, that’s alright. Now that you’ve been through the entire interview, are there any questions you’d like to go back to and add to or change?
STUDENT F: To the interview?
TEACHER: Like any questions that now that you’ve been thinking about it long enough, you want to add something to it or want to change your answer.
STUDENT F: Um, no.
TEACHER: Ok. Thank you for your time. You’ve been a huge help. Good job.
Appendix M: Second Interview Transcript, Student B

TEACHER: Ok, so first question, graph y = sin(x). Explain how you created your graph.
STUDENT B: It’s getting harder and harder as time goes on. Um, (talking indistinctly). Sine starts... here (points to the origin). It’s between 1 and -1. Right there and there. It starts, um, (pause). It starts at 0. Then it goes up at 90. (draws sine wave)
TEACHER: Alright, those points you plotted, how did you know to plot those there?
STUDENT B: Because that’s just the standard position and stuff.
TEACHER: Ok, very good. So using that graph you just made, uh, can you find those four values? Y = sin(80°) first.
STUDENT B: (long pause) It would be, this graph is not that great, but, it would be just below 1.
TEACHER: Ok
STUDENT B: It would be something like that (adjusting curve of graph)
TEACHER: Ok, so estimate that.
STUDENT B: Let’s say like .95
TEACHER: Ok
STUDENT B: That’s that
TEACHER: Ok, what about the sine of -150°?
STUDENT B: So like negative, -.45.
TEACHER: -.45?
STUDENT B: Yeah, yeah. .75, so that would be like right here (marks .75 on the y-axis). So that would be like there ish. So around like 60, 60° about. Ok, -1/4 (long pause). So like -20° ish.
TEACHER: Ok. Alright, so you are you happy with all those answers?
STUDENT B: Um, yeah (laughter).
TEACHER: Alright. Ok, questions number two. There’s a graph there. Which of the following equations best describes the graph? Explain how you know. So there are four choices. You can use whatever you want to figure out the graph so, um, just decide which one. (student picks up calculator) So what are you trying to do right now?
STUDENT B: I’m going to plug them all in so I’m going to use the calculator for that (laughs) (works on calculator and then crosses off choice a)
TEACHER: Why is it not that one?
STUDENT B: Because it doesn’t look like it at all (laughter).
TEACHER: Alright
STUDENT B: Because this line (motioning with hands) was backwards so it was like so it was more like right there. (types in second equation into calculator). It’s not that one either because it’s all… This one’s below the line and this one’s like normal.
TEACHER: Ok
STUDENT B: -2 sine x parentheses, minus 1. This one is the same. It’s close but it’s still above it. So it’s most likely d, but (types in last equation). There it is. So it’s d.
TEACHER: Ok, so besides plugging each of them into the calculator to see which one is the closest, what are other ways to check your answer? So you’re saying it’s d?
STUDENT B: Yes
TEACHER: How could you check your answer other than graphing on the calculator?
STUDENT B: You could actually, like, know it (laugh). You could actually know like what – 1 will do and –sine and the 2x would do because the normal sine equation is just sine of x. But like the negative sign and the minus -1 all does stuff. So the -1 I’m guessing, well yeah, it like flips it, like makes this one instead of going this way, go this way.
TEACHER: Ok
STUDENT B: Or so, yeah that’s the negative. I don’t know. But, if you actually know it and you can actually remember it, then you can check it that way. It would probably be faster just to do it that way then actually check it with this.
TEACHER: Ok, so calculator, knowing the features of the equation, is there a third way?
STUDENT B: Huh? Um, (pause, tapping fingers) I don’t know. Those are the two simplest ways to me.
TEACHER: Ok
STUDENT B: I don’t know if there’s a third. There might be. Probably
TEACHER: Ok. Alright so now, I’m going to give you a unit circle, uh, a graph of sine of x, just like the one you just made, and then two special right triangles. We have a 45-45-90 and the 30-60-90. So what I did, I put them on both sides in case you want to look at the triangle however you want. So you have those three, those three things to help you.
STUDENT B: Ok
TEACHER: So I have, uh, seven questions for you there to answer, and, uh, if you can answer it in your head that great, otherwise tell me, you know, which of the three, things, you’re going to use to get the question, answer the question.
STUDENT B: Ok
TEACHER: Ok, so number one, what is the sine of 45°?
STUDENT B: Um, sine of 45° I’m going to use the 45-45 triangle, and place it like that and put this little point there in the center and then it will be, I didn’t even need to use this triangle because (talking indistinctly). So 45 degree mark, obviously and sine that’s the (pause) either the first one or the second one. I think it’s the first one. In this case it would matter because they’re both the same thing. So it’s √2/2 for the first one.
TEACHER: Ok
STUDENT B: Ok, the sine of 65. 65 is, well none of these will help me (motioning to triangles and unit circle) because it’s not any of those. So I’ll use the graph to find the sine of 65°. 65° is... 90, 80, 65 is right here. That’s .9
TEACHER: .9? Ok.
STUDENT B: Yes. If -1/2 = sin(x), what is the value of x? OK, I’ll use the graph again. Alright so -1/2 is here. Yeah. The value of x would be right her, which is there, which is -30°.
TEACHER: Ok
STUDENT B: If $0 = \sin(x)$, what is the value of $x$? Zero. (pointing to origin)
Zero. (pointing to origin) Zero. (laughs) Alright, if $3/2 = \sin(x)$ what is the value of $x$?
(long pause)
TEACHER: So what are you trying to use to answer?
STUDENT B: Well, I’m thinking unit circle because… but I don’t know if that’s right because there’s $\sqrt{3}/2$ in some spots but I don’t think it’s the same, but that wouldn’t make sense. I could look at the graph. It would be there. But there’s nothing, so it’s not the same, I’m pretty sure.
TEACHER: Ok, so what’s your answer then?
STUDENT B: Nothing. It’s impossible. No solution.
TEACHER: Ok. Alright, next solve the equation $\sin(x) + 1 = 1/2$.
STUDENT B: Alright. (long pause) Well, $x$ has to be -1/2. (typing on calculator). It has to be -1/2 for that to work out. So sine of $x$ has to equal -1/2. So that’d be there (points to graph). So it’d be like -30°.
TEACHER: Ok
STUDENT B: That makes sense to me. But it might not make mathematical sense.
TEACHER: Why not?
STUDENT B: I don’t know (laughter). I’m second guessing myself here. Um, so I said negative… -30°. Ok, next one, so $3\sin(x) = 2$. So, (long pause)
TEACHER: So what are you thinking?
STUDENT B: Um, I’m trying to figure out, I’m trying to think of what would sine of $x$ equal, have to equal for that equation to be 2. But sine of $x$ would have to be, have to equal something that would multiply by 3 to equal 2. Right?
TEACHER: Ok
STUDENT B: Um, so, uh, then I can’t think of what that would be. What is 3 times something that would be 2?
TEACHER: What did you just type in?
STUDENT B: I typed in 3 times .75 and that equals 2.25.
TEACHER: Ok, why did you choose .75?
STUDENT B: I thought it would possibly be… Well, because point, 1/2 wouldn’t equal it because that would be 1.5 and 1 wouldn’t because that would be 3. So it has to be somewhere in between. So .75.
TEACHER: Ok
STUDENT B: And that didn’t work, so I’m kind of, kind of stumped here.
TEACHER: Ok, so if you know it’s between 1/2 and 1 what does that tell you?
STUDENT B: Um, well, it’s, well, it’s, it’s within this, this area (pointing to region on graph).
TEACHER: Ok, can you give me a range of angles?
STUDENT B: A range of angles would be… (counting on graph). So that would be 30° to 90°. So I don’t know what that accomplished.
TEACHER: Ok (student laughs). OK, so you can say it’s between 30° and 90°. That’s all that you can say?
STUDENT B: Um, I don’t know. Should it be more or should there be more? There should be more.
TEACHER: Ok
STUDENT B: I think, unless you’re just trying to trick me again. Um, I’m trying to think what I should do. 3 sine of 70. No. Um, that’s too big. 3 sine of 60. No. 50? No.
That’s closer. 45. Ok. 44. 43. 42. Well it’s between 41 and 42.
TEACHER: Ok
STUDENT B: So 41.5? No. I’m not doing this right. 41.7?
TEACHER: Well, it might not be an exact value but just approximate.
STUDENT B: Ok, well let’s try… we’ll go with 41.8 or 41.9.
TEACHER: Ok, very good. So of these three representations, which one do you feel most comfortable using?
STUDENT B: The graph.
TEACHER: The graph, ok. Which one did you feel most comfortable using when you walked in here, do you think?
STUDENT B: The graph, I like visual things.
TEACHER: Ok
STUDENT B: This is helpful (pointing to the unit circle) but I like this (graph)
TEACHER: Ok. Question number five, how do you answers to a and b differ from c, d, and e?
STUDENT B: Um, well, these are angles and no solution, c through, oh wait, c through e. Um, the questions are different. A and b are what is the sine of blank, and c through e is if blank equals sine of x, what is the value of x. Um, (long pause) these are… these are y values and these are x-values.
TEACHER: Ok. Ok, very good. Alright, last question, discuss sine and inverse sine in terms of inputs and outputs. How can you represent them in a triangle, on the unit, and on a graph? So what can you tell me about sine and inverse sine?
STUDENT B: Um, um, sine is to be used if you’re trying to find a, a uh… Inverse sine you use to try find, if you’re trying to find an angle. Yeah, so you’re trying to find an angle you use the second sine button to find it, and then you would plug in like, uh… (pause) What do you plug in if you’re trying to find and angle? You plug in a coordinate, so like 1/2 or √2/2 or √3/2. Plug that in and find the angle using inverse sine.
TEACHER: Ok
STUDENT B: Now sine, you use to plug in an angle and find a coordinate so 30 would give you 1/2 or 45 would give you √2/2, 30 would give you 1/2, right?
TEACHER: Um hmm
STUDENT B: Ok, Um, so how can you represent them in a triangle? (pause) Well, um…
TEACHER: You can make your own triangle. You don’t have to use that one.
STUDENT B: Um, just like show the things that you would use in the triangle?
TEACHER: Sure
STUDENT B: Ok. Um, right, so if you were trying to find this one. This angle up here, um it would be, so it would be opposite over adjacent so it’s be 1/1 and you would plug that into the sine, the inverse sine, you would, right? 2nd, sine, 1 over 1, which would be 45. No! Um, oh, opposite over hypotenuse, so 1/√2. 45, there you go.
TEACHER: OK
STUDENT B: So that’s inverse sine and then sine you’re trying to find one of these, that would just be sine of 45 which give you .707 which would not be $\sqrt{2}$. Um (pause). Sine 90, which equals 1.
TEACHER: So what are you trying to do?
STUDENT B: Well, sine of 90 equals 1 so it’s one of those, so yeah.
TEACHER: Alright, what about the unit circle?
STUDENT B: The unit circle. Uh, it’s basically the same as this, the same as the triangles because the unit circle is just all the triangles just in one nice little package. Do you want me to explain it again?
TEACHER: No
STUDENT B: Ok, good (laughs). Um, the graph, (pause) um, I never used inverse sine related to a graph.
TEACHER: Ok
STUDENT B: Um, well, for sine, it’s just like a normal sine graph so it’s related to the unit circle part. You get the unit circle from the graph because at 90° it’s 1 and 180 it’s 0, at 90 it’s 1, 180, 0, Oh crap. Cosine is the first one. Cosine is x, oh well, too late. Um, so this is just the unit circle without all the points in between. This is just the four angles.
TEACHER: Alright
STUDENT B: The inverse sine I never used with a graph.
TEACHER: Ok, very good. Excellent job again. Thanks for your time.
TEACHER: Alright, first question, graph y = sin(x) and explain how you created your graph.
STUDENT C: Since it’s just sine or x, it’s just a normal one?
TEACHER: Um hmm
STUDENT C: Since it’s just sine of x, there’s nothing to, uh, change it or disform it or whatever, so you would just start on (0, 0), (1, 90), 180, (-1, 270), and (0, 360), is one wave. Well, here (moving last point)
TEACHER: And how did you know to plot those points?
STUDENT C: Um, Because of the unit circle, that’s the points it has.
TEACHER: Ok
STUDENT C: The four points are like that, there.
TEACHER: Ok, very good. And is that the entire graph or…
STUDENT C: Oh, uh, you can continue, going backwards you just put what they are so, (-1, 90), (0, 180), (1, 270) and (0, 360). (actually plotting the points on the negative side of the x-axis)
TEACHER: Ok, very good. Now from your graph can you find y = sin(80°)?
STUDENT C: Yes, so it’s… 90 is right here, um, so 80 would be like here, so it’s probably, I’ll say it’s .98.
TEACHER: - .98?
STUDENT C: Yes
TEACHER: And the sine of -150°?
STUDENT C: Ok, -180 is here. You go 1, 2, 3, over and go straight down, it’s probably about -.48.
TEACHER: - .48?
STUDENT C: Yes
TEACHER: Ok, and .75 = sin(x)?
STUDENT C: So, that would be… Try to get y by itself, so do we inverse or, inverse sine of .75 so we want x, no, wait. Yeah, equals sine inverse 3 over 4. (picks up calculator) And it equals 48.5, 48.6. So y equals sine of 48.6°. Let’s see, is that in here? (looking at graph). There’s 40, so you go up to about 48.6. We’ll say it’s right there. Then go up that’s probably about .5 or 6.1, .61.
TEACHER: Ok. So you answer is .61?
STUDENT C: Yes
TEACHER: Ok, and -1/4 =sin(x)
STUDENT C: (writing equation) It is -1/4 right?
TEACHER: Yes
STUDENT C: -1/4, so y equals (typing on calculator)-14.5. So there is 0, 14, 14.5 right there (on graph). So that’s .32.
TEACHER: So the answer is .32?
STUDENT C: Yes, sounds correct.
TEACHER: Ok. So question number two, which of the following equations best describes the graph? Explain how you know. So you’re going to use that graph there and there are four choices below.
STUDENT C: So, let’s see. I think it is… Oh, that’s right (pause)
TEACHER: So what are you trying to do?
STUDENT C: I’m trying to figure out like… I know that it’s either a or… It’s not b.
TEACHER: Ok
STUDENT C: Because the number has to be outside for it to go up or down, in this case it’s down 1, so it’s a, c, or d. I’m just trying to figure out how far it moves over left or right to, um, figure what is right so… So it’s not a.
TEACHER: Why is it not a?
STUDENT C: Because it’s the, um, the way the wave is reversed so there’s a negative in front of the equation. Because it goes middle, down, middle, up, middle instead of up… middle, up, middle, down, middle.
TEACHER: Ok
STUDENT C: So it’s not stretched any so I’m going to say it’s d because there’s a negative in front, it’s not stretched this way, but it is moved down.
TEACHER: Ok. Alright, so how could you then check your answer? So you looked at the features of the equation there to answer it, which is great. Now how could you go about checking your answer to see if it’s correct?
STUDENT C: Um, (pause) I don’t know.
TEACHER: Ok, can you think of a way?
STUDENT C: Um, other than looking at the features, like it’s moved down to the middle. I know that’s -1, and then the negative in front is flipping it so…
TEACHER: Ok. Like that 2, what does the 2 do to the equation?
STUDENT C: Um, Doesn’t it move the starting point? I know this 2 stretches it, it’s the amplitude, like that (motioning with hands). This isn’t stretched so…
TEACHER: Alright, well look at you graph of sine of x and look at that graph. Besides the fact that it’s been flipped, what else do you notice?
STUDENT C: It’s shifted, it’s shifted to the… the… it’s shifted to the right.
TEACHER: Ok, so where’s it starting now?
STUDENT C: So instead of starting here, it starts here at 90.
TEACHER: Ok. Do you notice anything else about it?
STUDENT C: Um, oh it’s a shortened wave. So the 2 shortens the wave.
TEACHER: But you can’t think of a way other than what you just explained to check the graph?
STUDENT C: Not that I can think of.
TEACHER: Ok, that’s fine. Alright, so now I’m going to give you the unit circle, two special right triangles. So I have a 30-60-90 right triangle. It’s on both sides in case you want to rotate it around, and a 45-45-90 right triangle. And then the graph of y = sin(x).
STUDENT C: Ok
TEACHER: So, using those three things and whatever else you want, answer the following questions that are on the sheet there.

STUDENT C: Alright

TEACHER: So first off, what is the sine of 45°?

STUDENT C: So cosine equals x, sine equals y. \( \sqrt{2}/2 \).

TEACHER: Ok, what did you do to get that answer there?

STUDENT C: Um, I looked at the unit circle and I was at 45 and since they’re both the same and cosine equals x and sine is y, I looked at the y coordinate and saw the it was \( \sqrt{2}/2 \).

TEACHER: Ok. What is the sine of 65°?

STUDENT C: I don’t know how to use this (triangle).

TEACHER: What were you trying to do?

STUDENT C: I was trying to like see if I could put in on there and move it to go to 65.

TEACHER: Ok

STUDENT C: Um, well, I guess I could just, sine, calculator, sine 65 close parentheses equals… But that does not give me a fraction. It gives me a point. Or is that the answer? Would that be the answer? Or is there no decimals?

TEACHER: It can be a decimal.

STUDENT C: Ah, then the sine of 65 is .91.

TEACHER: Alright, can you find that using the unit circle, your graph, or the triangles? Any of the three?

STUDENT C: I can use the graph.

TEACHER: Ok

STUDENT C: Just go to 60 on the graph, which is about…

TEACHER: 65

STUDENT C: 65. Go to 65, man these are (talking indistinctly) and go to that point and it looks like it’s .91.

TEACHER: Ok. Very good, so next one, if \(-1/2 = \sin(x)\), what is the value of x?

STUDENT C: So you start of by switching or putting the inverse sine of 1/2, yeah inverse sine of 1/2, -1/2. It’s -30 (on calculator). Alright, equals -30.

TEACHER: Alright can you use the unit circle, the graph, or the triangles to show me that?

STUDENT C: Um, since it’s negative, since it’s a -1/2, you go to sine on the… You go to 1/2 on the… 1/2 sine on the unit circle and since it’s negative you put a negative in front of that degrees so it’s -30.

TEACHER: Ok

STUDENT C: It’s logical

TEACHER: What about the graph or the triangles?

STUDENT C: Um, you could do the same thing on the graph. Go about 1, 2, 30 and then… or – 30. 1, 2, 3, and go down and -1/2.

TEACHER: Ok. Alright if \( x = \sin(x) \), what is the value of \( x \)?

STUDENT C: Hmm. (pause). Ok, you can do it on the calculator. Do inverse sine of 0 it’s 0. You could also go on the unit circle (pause). So it’s at (1, 0) for sine because sine is y and the graph goes to zero and that’s 0.
TEACHER: Ok, but on the unit circle you also looked at 180° Why?
STUDENT C: Because it’s also at… 0 at the sine.
TEACHER: So what does that mean?
STUDENT C: (pause) Um, it means it’s sine of 0. It’s… I don’t know.
TEACHER: So is your answer 0°…
STUDENT C: No! It’s 0 and 180. Aha! I got two answers. Because they’re both, both of their sines, or y’s, are 0.
TEACHER: Ok. Are there just two answers?
STUDENT C: Yes
TEACHER: Alright, look at your graph, where it the sine 0? You said 0 and 180°. Anywhere else?
STUDENT C: Also 360 because it’s at 0.
TEACHER: Anywhere else?
STUDENT C: No, just 360, 180, and 0
TEACHER: Ok
STUDENT C: Do the negatives count too?
TEACHER: Yes
STUDENT C: Ah, so -180 and -360.
TEACHER: Ok, very good. If 3/2 = sin(x), what is the value of x?
STUDENT C: 3/2? So domain error if I type it in the calculator. So it would be… That would turn into \sqrt{3}/2 wouldn’t it? Because that makes no sense. (pause). Well, if it was \sqrt{3}/2 then I could get it.
TEACHER: Alright, so looking at the unit circle or the graph, can you try and figure out where the sin(x) is 3/2? What do you need to answer that question?
STUDENT C: To know what the sine of 3/2 equals. Wait a second (types on calculator). No, I don’t know
TEACHER: So, do you have a final answer?
STUDENT C: Um, I don’t think I do, no.
TEACHER: What about the graph? Look at the graph. What are you looking for on the graph to answer the question?
STUDENT C: 3/2 which is 1 1/2. Ok, so I don’t think it is anything because 3/2 equals 1 1/2 and the graph doesn’t go higher than 1.
TEACHER: Ok, so what is your final answer then?
STUDENT C: Uh, 0, none. None is my final answer.
TEACHER: Alright, now solve the equation sin(x) + 1 = 1/2.
STUDENT C: Oh, there it is. Minus 1 from both sides and you have sine of x equals -1/2. And the inverse sine of -1/2. And the answer is…
TEACHER: Now can you do it without the calculator first? Try using one of these three…
STUDENT C: Yes, um, I’m going to use the unit circle. So I’m going to go anywhere where it’s the y-coordinate -1/2, which is 210, um, 330, and that’s it.
TEACHER: Ok, now try the calculator and see what you get.
STUDENT C: And that is… (talking indistinctly). The inverse sine of -1/2 equals -30. What? Um, (points to -30 on the graph) so it’s at (-30, 1/2), -150, and it’s also at 210 and
I’m guessing that 330. Yeah, that’s 330. So we have the first two which we find on the unit circle, one we find on the calculator, which is -30. On the unit circle it’s 210, 330. And on the graph we’re going to use it to find -150.

TEACHER: Ok, very good. Do you think there’s any more?
STUDENT C: Yes, but it’s on the graph, if it were stretched further than 360.
TEACHER: Ok. Is there a way you could find it on the unit circle?
STUDENT C: Um, yes. If you keep going around, let’s say this is 360, so this would be 390, um 405, um, 420, and this would be 450 and so forth and so on.
TEACHER: Ok, what about -30°? Where would -30° be?
STUDENT C: There’s 30. Well, if you started here and go backwards.
TEACHER: Ok, so where’s -30° at?
STUDENT B: It would be here. (points to 330°)
TEACHER: Ok, what about -150?
STUDENT C: (points to 210°) It’s close.
TEACHER: Which one?
STUDENT C: Because it was flipped if you’re going backwards.
TEACHER: Ok, and last question in this set, solve the equation 3sin(x) = 2.
STUDENT C: Ok, I would divide 3 by both sides. Sin(x) = 2/3. Then take the inverse sine…
TEACHER: So again, try it without the calculator first.
STUDENT C: Ok. Um (pause). I forgot how to do it. I know there’s a way to do it I just don’t remember.
TEACHER: Ok, with what? The graph, the unit circle…
STUDENT C: The unit circle.
TEACHER: Ok, what about the graph?
STUDENT C: 2/3 equals .66 repeating which is about .67 and on the graph look at .67. It’s right there. So it’s probably about, 1, 2, 3, 4, 5. So 50°
TEACHER: Ok. You just have one answer this time?
STUDENT C: Yeah
TEACHER: Ok, now try it on the calculator and see and see if it’s about the same thing.
STUDENT C: It would be 42.
TEACHER: Ok, so fairly close given the estimation of the graph. Good job. Now, how do your answers in a and b, differ from your answers in c, d, and e?
STUDENT C: So I have a and b, c, d, e. Um, they’re different because with a it was already on the circle, it was easy because it was already on the unit circle, like right there and 65 was, just plug it straight into the calculator and it will give you one answer, but c, d, and e, um… C had more than one answer, didn’t it?
TEACHER: Ok, what are the other answers?
STUDENT C: Um, this (pointing to answers in f) 210, 330, and -150. Yeah, so they all have more than one answer. Alright, uh, so we have a and b only have one answer where as c and d and e had more than one because it wasn’t just, um, it wasn’t the fraction of it. It was the degree. Instead of 45° it was 1/2.
TEACHER: Ok
STUDENT C: Does that make sense?
TEACHER: Sure, alright and last thing discuss sine an inverse sine in terms of inputs and outputs. How can you represent them in a triangle, on the unit circle, and on a graph? So have sine of x and you have inverse sine of x.

STUDENT C: Sine, that’s the (pause). Sine of x is like what you use to find the fractions or like 1/2 or √3/2 if you have the degrees, and if you have a fraction you use that for the inverse sine to find the degrees. If you have 1/2 sine, inverse sine of 1/2, the degrees would be 30.

TEACHER: Ok, so on the unit circle, what do you do for regular sine, what do you do for inverse sine?

STUDENT C: For regular sine, so if it’s just like sine of 30 you go to 30 and go to the coordinates and you go to sine which is 1/2. And if it was the inverse, let’s say it was the inverse of √2/2 then you go to radical… No, no, no, you go to √2/2 and any other one that has a radical… If it was positive √2/2 you go to any one that has positive √2/2 and then you find, alright so it would be 45, uh, 135, and that would be it on the unit circle.

TEACHER: Ok, what about on the graph, if you have sine and inverse sine?

STUDENT C: If you have sine of 90 you would go to 90 and you would go up to 1 because that’s where it crosses. If you have inverse sine of -1, then you go to -1 and follow that up to 270 and it equals -90.

TEACHER: Ok, and what about with a triangle, can you do it with a triangle?

STUDENT C: Um, no I can’t do it with this triangle.

TEACHER: So of these three representations, which one is most helpful to you and why?

STUDENT C: Uh, I find the unit circle is more helpful because it’s more printed out as, it’s more shown like instead of looking at lines and trying to count if it’s there.

TEACHER: Ok. Alright, anything you want to go back and change?

STUDENT C: Uh, no.

TEACHER: Ok, nice job again. Thank you very much again.
Appendix O: Second Interview Transcript, Student D

TEACHER: Ok, so first off, just graph $y = \sin(x)$. Explain what you’re doing.
STUDENT D: Alright. So I don’t do these yet?
TEACHER: Not yet, just graph $y = \sin(x)$.
STUDENT D: So sine starts up because sine equals $x$. At least that’s what I remember from last time. Um, sine is $x$.
TEACHER: Does sine equal $x$?
STUDENT D: No! Sine is $y$, yeah, sine is $y$. So then at $0^\circ$ $y$ equals 0. So it starts there and goes up, middle, down, and middle. $y = \sin(x)$.
TEACHER: Ok, what is the period of that graph?
STUDENT D: Um, oh, well that would be $\pi/8$, right? No, $\pi/4$.
TEACHER: So the period is $\pi/4$?
STUDENT D: No, it would be $\pi$. This would $\pi$ and this would be $2\pi$. Since it goes to 180, it would be $\pi$.
TEACHER: Ok, so what is the sine of $90^\circ$?
STUDENT D: Sine of $90^\circ$ would be… Sine equals $y$ so that’s supposed to be 1. (erasing graph). So then it should be here at one. Oh my God that was right, because sine equals $y$. So then it would be… You said the sine of $90^\circ$?
TEACHER: What’s the coordinate at $0^\circ$?
STUDENT D: Right here?
TEACHER: On the unit circle
STUDENT D: (1, 0)
TEACHER: So what’s the coordinate at $90^\circ$?
STUDENT D: Are you asking me on…
TEACHER: No, you have (1, 0) and (1, 0).
STUDENT D: Oh, (0, 1), no, (0, 1), and (1, 0) (corrects unit circle).
TEACHER: Ok, so how does that change your graph?
STUDENT D: It was wrong. So then this would be at 1. At $0^\circ$ $y$ is 0. 180°, (-1, 0) and then there (drawing graph), -1. So then for 180° it would be 0. And then… That wouldn’t make sense if it was just…
TEACHER: Well, what’s the point at 270?
STUDENT D: (0, -1). So then I’d have to go down.
TEACHER: Just connect your three dots there.
STUDENT D: Ok. It’s very long, long graph.
TEACHER: Ok. So, now using that graph, uh, can you find those four values underneath, on your sheet there?
STUDENT D: Well, sine of $80^\circ$. So it would be a little less than 1.
TEACHER: Ok, so give me an estimate.
STUDENT D: 9.5, er, .95
TEACHER: Ok, the sine of 150°? Or -150°, sorry. The sine of -150°.
STUDENT D: -150 would be like around here. So then if I continue this is down
(drawing graph). It would be uh…
TEACHER: Now why did you curve up right there?
STUDENT D: Because here it curves down.
TEACHER: But where does it cross on the positive side?
STUDENT D: Oh, it crosses at 180. I don’t know why… It’s supposed to be over here.
So then (talking indistinctly). So then sine of 150 would be like .6. (pointing to .6 on
the graph)
TEACHER: Ok, .6?
STUDENT D: And then, .75 = sin(x). So wouldn’t I have to do negative sine to get that?
So .75, I would look… I don’t understand if it means like… Oh that would be negative
point, -6. (referring to previous answer) Uh 65. Next -1/4. Um, 15, 30°
TEACHER: Pardon?
STUDENT D: 30°, -30°, sorry.
TEACHER: Alright, anything else?
STUDENT D: The only thing that confuses me is for these, that, I’m looking for it on this
axis now I think.
TEACHER: Ok
STUDENT D: But I could find that in like numerous places
TEACHER: So what does that mean?
STUDENT D: It means there are multiple solutions
TEACHER: Ok, so what would the other ones be?
STUDENT D: Um, what did I say for c the first time?
TEACHER: Um, 65°
STUDENT D: Well it could be -250, and then it could go on off the graph. And like for
this one it could also be 190.
TEACHER: Ok
STUDENT D: And also negative…
TEACHER: You said -30
STUDENT D: Yeah, I did. Oh then also -170, 160.
TEACHER: Ok, that’s very good. So question number two, which of the following
equations best describes the graph? Explain how you know. So I give you four choices
there at the bottom. Figure out which one is the equation of the graph.
STUDENT D: Well first I look at the amplitude because that’s the easiest to find. And
the amplitude is 1. So that eliminates these two (a and c).
TEACHER: Ok
STUDENT D: And, since sine starts at 0 and goes up then we know it’s not reflected over
the, wait, then you know it is reflected over the x-axis because it goes down. So it would
be d.
TEACHER: Ok. So how could you check your answer? You just looked at the features
of the equation and matched it to the graph, which is fine, how else can you check your
answer, besides using the actual features of the equation?
STUDENT D: Besides using the equation? How would I find…
TEACHER: Besides saying the amplitude is 1, there is a reflection, it moves down one. You know the features from the equation itself. Without using amplitudes and reflections, and periods, how can you check your answer?
STUDENT D: Like actually drawing the graph? Is that what you mean? Or…
TEACHER: You said the answer is d.
STUDENT D: Yes
TEACHER: How could you check your answer?
STUDENT D: Without looking at the graph?
TEACHER: Whatever you want. How could you check your answer?
STUDENT D: Um, well I always go and double check the, um, shift that it makes because it common sense since it’s just a down one then it would start at -1 which is down.
TEACHER: Ok
STUDENT D: And um, the period would be π because, yeah it would be π, I think. Yeah that would be right, because it would be, um, 2π/2 I think, which would be π. So then here, that’s what you have.
TEACHER: Ok
STUDENT D: Um, I mean other than that, if I wanted to check I would just not look at the graph and try to sketch it on my own.
TEACHER: Any other ways you can check your answer?
STUDENT D: Um, that’s all that I can think of now that I would do off the top of my head.
TEACHER: Ok. Alright, now, I’m going to give you a unit circle, some right triangles, and I put them on both sides in case you want to rotate the triangle around as you see fit. So there’s two special right triangles, and here is the graph of $y = \sin(x)$. Ok, so I gave you seven questions there you can use any of those or multiple ones to get your answers but just let me know what you’re thinking about using and why to answer those questions.
STUDENT D: Ok, um the sine of 45°, I just go to here (unit circle) just because it’s the one I’m most comfortable with and since sine equals y I would know that is $\sqrt{2}/2$. And I could also go, since this is the graph of sine, I could also go to 45° which would be right here. And then that would probably give me the answer in decimal form.
TEACHER: Ok
STUDENT D: That’d be this one (talking indistinctly). Um the sine of 65°since it’s not on there (unit circle) I should probably use this. Or I could use these here (triangles) for the sine of 65° since it’s just opposite over hypotenuse. It’s $\sqrt{3}/2$.
TEACHER: Why, why did you use that triangle over there? The 60-30-90 right triangle?
STUDENT D: Oh, wait, that’s 65°. So that would not equal. I would use the graph. So that’s probably about 45° there and it would be more than that. I would say a little less than 1, so .9.
TEACHER: Ok
STUDENT D: Um, what is the value of x, so -1/2, so I have to find… So I go here (unit circle). I would just look for when y equals -1/2.
TEACHER: Ok
STUDENT D: Which it does it two places. So there’s more than one solution, just like there would be on the graph. Um, so the value of x could either be 210° or 330°. And then we could check that here (graph). It would equal -1/2, 210°, and that would match up.
TEACHER: Where’s 210° at?
STUDENT D: Oh, it’s right here. So it would be that there. Which is in line with that. And then 330. And it could also be negative too.
TEACHER: Ok
STUDENT D: So then that would probably be like 30, oh yeah, that would be -30° (points to 330° on unit circle) and negative, (counting on unit circle) -150. And so that was what you’re looking for, for that one?
TEACHER: Um hmm
STUDENT D: And then for sin(x) = 0 I look at y and I have 0 and 180°. It could also be -180°.
TEACHER: How do you know that?
STUDENT D: Because you get to the same place (traces -180° on unit circle). And it could really be like, um, like 180 plus 360,whatever that would be.
TEACHER: Ok
STUDENT D: Um, if you just keeping adding 360 you’ll eventually get back to 0° again for each one.
TEACHER: Ok
STUDENT D: And, is that all you’re looking for, for that one?
TEACHER: Um hmm
STUDENT D: Um, 3/2 = sin(x). If you look for 1 1/2, 1 1/2 equals… Um, it’s one of the trick ones. Because you’re looking for this, right? (pointing to 1.5 on y-axis) on this graph.
TEACHER: So what does that mean?
STUDENT D: That means (pause) there is, the graph doesn’t go there. Um, (pause) the amplitude would have to be bigger to be up there.
TEACHER: Ok. So what do you want to say your answer is?
STUDENT D: Well, right now since I don’t see it, I would say there’s no solution.
TEACHER: Ok
STUDENT D: Um, (pause), yea because this is √3/2 (pointing to unit circle). I guess there is no solution because I don’t see it.
TEACHER: Ok, why did you look at the graph first that time? You’ve been looking at the unit circle first. Why did you look at the graph?
STUDENT D: Because I know it wasn’t √3/2 and just 3/2 isn’t anywhere on here.
TEACHER: Ok
STUDENT D: Um, and then here it just makes more sense to me to be able to find 1.5, but then I was just thinking (drawing triangle) if I had a triangle and x, 2, and 3. Yea that’s what I did last time too and this wouldn’t, that would make sense that there’s no solution because the hypotenuse has to be the longest side.
TEACHER: Ok. Alright so next thing, you have two equations to solve.
STUDENT D: Ok, um, so the sine of x would be -1/2, um so then that would be the same answers for this one.
TEACHER: Why?
STUDENT D: Because it’s the same thing just written in a different way. -1/2 = sin(x), sin(x) = -1/2.
TEACHER: Ok
STUDENT D: So then it’d be 210, 330, -150, -30. Did you get the answer?
TEACHER: Um hmm
STUDENT D: Ok. Um, 2/3, so then (drawing triangle). That would make sense this time. So (opening calculator), it will be easier to find the decimal on here (pointing to graph). So then that would almost be .7 (plugged is 2/3 on the calculator), it would probably be 45°.
TEACHER: Ok
STUDENT D: But it could also be 135 and also would be 225. I just want to see something.
TEACHER: What are you typing in?
STUDENT D: Um, √2/2. I just want to see… Yeah because that’s really close to .6. So it wouldn’t be exactly 225, it would be a little less than that.
TEACHER: Ok, so 45°, 135°. What were your other two answers? Or other answer?
STUDENT D: What did I say?
TEACHER: You said 45°, 135°, and …
STUDENT D: Um, it’s 220 and, er -220 and -300.
TEACHER: Ok. Alright, with those three representations, which one do you find most useful?
STUDENT D: Uh finding 2/3?
TEACHER: Just answering those questions in general, which one is most helpful to you?
STUDENT D: Well, it depends, if it’s something like this where it’s 2/3 it’s easiest for me to find the decimal and then match it up with a degree (on the graph) but then if it’s something like, um, -1/2 then it’s easier for me to look at here (unit circle) and say, oh, -1/2 is here and here. And then it’s easier for me to look and see, well every time I get back to that point on the unit circle I can add 360 then I’ll be there again. So that’s where the next solution would be. But I’d say this would be the most useful (graph).
TEACHER: Ok. Alright so how do you answers to a and b differ from your answers from c, d, and e?
STUDENT D: A and b. Um, well for these (a and b) you’re looking at the y-value on the unit circle or you’re looking at or you’re finding the answer that would be on the y-axis here. The for c, d, and e you’re given what’s on the y-axis, so then you have to, um, find the degrees.
TEACHER: Ok
STUDENT D: Um, I guess these are a little different. They’re just two different types.
TEACHER: Ok, what do you mean by type?
STUDENT D: Like, for this you’re given the degree and this you’re given the y-value, I guess you could say. And you have to find the y-value for when you’re given the degree and the degree for when you’re given the y-value.
TEACHER: Ok. Alright so final question, discuss sine and inverse sine in terms of inputs and outputs. How can you represent them in a triangle, on the unit circle, and on a graph?

STUDENT D: Ok, um, sine would be equal to, um, opposite over hypotenuse, and is also equal to y, the y-value on the unit circle. Um, how can I represent that in a triangle? In a triangle it would be opposite over hypotenuse, on the unit circle it would be y, and then on a graph it would be on the y-axis, that is your sine for your degree. Um, inverse sine, let’s see, on a triangle, that’s when, this is your x (labeling angle) and that’s when you’re given that (labels opposite and hypotenuse in triangle) so then it would be 4/5, inverse sine, and that’s when you’re trying to find the degree.

TEACHER: Ok

STUDENT D: On the unit circle that’d be like these (pointing to c is question 4), so then that would be when you’re given, um, like your y-value, so then you’d know the sine of x equals that so then you’d take inverse sine to figure out what sine is.

TEACHER: Ok

STUDENT D: So I guess I could have done that for here too (letter c in question 4). And then on a graph, inverse sine, that would be, that would be, where your y value’s lying (tracing y-axis). So then I guess if you’re given the y-value, you can figure out what the degree is, the x-value by using inverse sine. Um, did I answer all those? Yeah.

TEACHER: Um, in terms of input and outputs, how do sine and inverse sine differ?

STUDENT D: The input and outputs, x and y. Let’s see, like what are you saying? If like...

TEACHER: For sine, what is your input and what is your output? For inverse sine, what is your input and what is your output?

STUDENT D: I don’t think I understand the question because when I think of sine I think of like the y-value and when I think of input and output I think of x and y.

TEACHER: Ok, um, sin(x) is a function, right? So what do you input into sin(x) and what do you get out of sin(x)?

STUDENT D: Oh, ok, so then if I had like the sine of 90 it would be 1. So then are you asking like if I have inverse sine of 90, something like that? Is that what you’re trying to ask? You would have degrees, you’d have inverse sine of 1. So they’re opposite, I guess.

TEACHER: Ok. So what do you input into sine?

STUDENT D: Into sine you input degrees.

TEACHER: Ok, what do you input into inverse sine?

STUDENT D: Um, y-values.

TEACHER: Ok, excellent. Anything you want to add?

STUDENT D: I don’t think so. Anything I missed?

TEACHER: No. Excellent job. Thank you very much again.