ARCHIMEDEAN DERIVATIVES AND RANKIN-SELBERG INTEGRALS

DISSertation

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In this dissertation, we first define two notions: derivatives of smooth admissible representations of moderate growth on $GL_n(\mathbb{R})$ and exceptional poles. Then we study their basic properties and relate them to the archimedean Rankin-Selberg integrals. This is part of an ongoing project to develop the archimedean theory analogous to $p$-adic case in [10].
Dedicated to my parents
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CHAPTER 1
INTRODUCTION

Let $F$ be a $p$-adic field, $\pi$ a smooth admissible representation of $GL_n(F)$, in [2], J. Bernstein and A. Zelevinsky introduced the notion of $k$-th derivative, denoted as $\pi^{(k)}$, for $\pi$, and used it to study the properties of $\pi$. More precisely, let $P_n(F)$ be the mirabolic subgroup of $GL_n(F)$ consisting of matrices like

$$\begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix}$$

where $g \in GL_{n-1}(F)$, $u \in F^{n-1}$. Then J. Bernstein and A. Zelevinsky first defined two functors: one is $\Phi^{-1}$ from the category of smooth representations of $P_n(F)$ to the category of smooth representations of $GL_{n-1}(F)$, and the other is $\Psi^{-1}$ from the category of smooth representations of $P_n(F)$ to the category of smooth representations of $P_{n-1}(F)$. Then for any smooth representation $\pi$ of $GL_n(F)$, restrict it to $P_n$, which is still a smooth representation, and define the $k$-th derivative $\pi^{(k)}$ as $\Psi^{-1} \circ (\Phi^{-1})^{k-1}(\pi)$. Using these notions, J. Bernstein and A. Zelevinsky obtained a lot of properties of $\pi$. For more details, see [2].

In [J-PS2], J. Cogdell and Piatetski-Shapiro was the first to use derivatives, together with another important notion exceptional poles, to study the poles of Rankin-Selberg integrals for a pair of irreducible smooth generic admissible representations $\pi$ and $\pi'$ of $GL_n(F)$. One main result in their paper is Theorem 3.1.1. below, based
on which they can compute the $L$-function $L(s, \pi \times \pi')$ in terms of the $L$-functions for the inducing data of $\pi$ and $\pi'$. This is Theorem 6.1. in their paper.

Now it is the purpose of this paper to set up some preliminary foundations for carrying out the same calculations in Archimedean theory. In fact, what we have done here is to introduce the notion of derivative for a large class of representations of $GL_n(\mathbb{R})$, and relate it to Rankin-Selberg integrals.

There are a couple of differences between Archimedean theory and $p$-adic theory. First of all, in $p$-adic case, we naturally worked in the category of smooth admissible representations for any $p$-adic group, reductive or not. But in Archimedean case, we have multiple choices. For any Lie group $G$, there are extensive studies on unitary representations of $G$. When $G$ is reductive, let $\mathfrak{g}$ be its complexified Lie algebra and $K$ its maximal compact subgroup, a larger class of representations than unitary ones is called $(\mathfrak{g}, K)$ modules. Classical modular forms gives us examples of this type representation.

Thus for applications to automorphic forms, it is better to work with $(\mathfrak{g}, K)$ modules, not only unitary ones. But for irreducible admissible $(\mathfrak{g}, K)$ modules, unlike smooth representations in $p$-adic case, the Whittaker functional is not unique, which is very important in the theory of Rankin-Selberg integrals.

Fortunately, by the fundamental work of Casselman and Wallach, for each finitely generated admissible $(\mathfrak{g}, K)$ module $(\pi, V)$, there exists a unique canonical completion $(\pi, V^\infty)$. This is an admissible smooth representation in a Frechet space with certain growth conditions. A nice property for such representations is that there exists at most one continuous Whittaker functional on $\pi$ provided it is irreducible. The precise definitions and properties of smooth representations of moderate growth are contained in Chapter 2.

The second difference is how to define the derivatives. In $p$-adic case, it involves the
study of smooth representations on mirabolic subgroup \( P_n \). But this group is not reductive, and so far we don’t know too much about its representations in Archimedean case. In particular, the notion of smooth representation of moderate growth doesn’t apply to this group.

The way to solve this is that we still define the derivatives formally as the composition \( \Psi^{-1} \circ (\Phi^{-1})^{k-1}(\pi) \) of functors, but in order to study the properties of derivatives, we avoid working with representations of \( P_n \). Instead, we make use of zero degree \( n \) homology \( H_0(n, V^\infty) \), where \( n \) is the complexified Lie algebra of the nilpotent radical of some maximal parabolic subalgebra. These are dealt in Chapter 3.

Another difference is the definition of exceptional poles. Different from \( p \)-adic case, we introduced two types of exceptional poles: type 1 and type 2. Type 1 exceptional pole is the Archimedean analog of \( p \)-adic case, while type 2 is certain representation theoretic invariants for a pair of irreducible admissible smooth representations \( \pi \) and \( \pi' \). Type 2 exceptional poles include type 1 poles, and in fact we expect they are the same. Under some special cases, like principal series and some cases involving discrete series, we can compute exceptional poles of type 2 for \( \pi \) and \( \pi' \), and show that they are the same as type 1 poles. But in general, we haven’t obtained this fact. For more details see Chapter 4.

The main results of this paper is summarized in Theorem 4.2.9 and Theorem 4.3.8. We can also use the techniques developed here to study the exterior square \( L \)-integrals.
CHAPTER 2
PRELIMINARIES

2.1 Smooth Representations of Moderate Growth

In this section, we will collect some concepts and results in representation theory which are useful for this paper. We prove a number of results about smooth representations of moderate growth, which are difficult to find in the existing standard literatures. For convenience, we will restrict ourselves to the special case $GL_n(\mathbb{R})$, though most of the notions and results are also applied to general real reductive group.

Let $G_n = GL_n(\mathbb{R})$ be the general linear group of invertible $n \times n$ matrices over $\mathbb{R}$, $K = K_n = O(n)$ be the orthogonal subgroup of $G_n$, which is a maximal compact subgroup of $G_n$. We use $\mathfrak{g} = \mathfrak{g}_n$, $\mathfrak{k} = \mathfrak{k}_n$ to denote the complexified Lie algebras of $G_n$ and $K_n$ respectively.

Definition. A $(\mathfrak{g}, K)$–module $(\pi, V)$ of $G_n$ is a vector space $V$ together with a structure of $\mathfrak{g}$–module and an action of $K$, such that they are compatible in the following sense:

1. Every vector $v$ is $K$ finite, that is, the subspace $W(v)$ spanned by $\{k.v : k \in K\}$ is finite dimensional, and the action of $K$ on $W(v)$ is continuous for every $v \in V$;
2. The two actions of $\mathfrak{k}$, one as subalgebra of $\mathfrak{g}$, and one induced from the action of $K$, are the same;
(3) For any \( k \in K, X \in \mathfrak{g} \), we have

\[
\pi(\text{Ad}(k)X)v = \pi(k)\pi(X)\pi(k^{-1})v
\]

Suppose \( V \) is a Fréchet space, \( f : \mathbb{R}^n \to V \) is a continuous function taking values in \( V \). We say \( f \) is differentiable at \( x_0 \) if there exists a unique linear map \( f'(x_0) : \mathbb{R}^n \to V \) such that

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)|x - x_0|}{|x - x_0|} = 0
\]

where \(| \cdot |\) denote the usual norm in \( \mathbb{R}^n \).

Use \( \text{End}(\mathbb{R}^n, V) \) to denote the space of all linear maps from \( \mathbb{R}^n \) to \( V \) with the compact open topology. If \( f \) is differentiable at \( x_0 \), then we say \( f \) is of class \( \mathcal{C}^1 \) if the map \( x_0 \to f'(x_0) \) from \( \mathbb{R}^n \) to \( \text{End}(\mathbb{R}^n, V) \) is continuous, and \( f \) is of class \( \mathcal{C}^2 \) if the map \( x_0 \to f'(x_0) \) is of class \( \mathcal{C}^1 \), and so on. We say \( f \) is smooth at \( x_0 \) if \( f \) is of class \( \mathcal{C}^k \) for any \( k \geq 1 \) at \( x_0 \). If \( f \) is smooth at every point, then we simply say \( f \) is a smooth function.

**Definition.** A smooth representation \((\pi, V)\) of \( G_n \) is a linear action of \( G_n \) on a Fréchet space \( V \) such that

1. The map \( G_n \times V \to V \), sending \((g, v)\) to \( \pi(g)v \), is jointly continuous for all \( g \in G_n, v \in V \);
2. For each \( v \in V \), the function from \( G_n \) to \( V \) given by \( g \to \pi(g)v \) is smooth.

When there is no confusion, we will write \( g.v \) for \( \pi(g)v \), and simply write \( V \) for \((\pi, V)\).

An advantage of smooth representation is that it naturally induces a representation of the Lie algebra. For any \( X \in \mathfrak{g} \), then \( \pi(X)v = \frac{d}{dt}|_{t=0}\pi(e^{tX})v \) defines a representation of \( \mathfrak{g} \).

Given a smooth representation \((\pi, V)\) of \( G_n \), if we set \( V_K \) to be the set of all \( K \)-finite vectors, then \( V_K \) is a \((\mathfrak{g}, K)\)-module, and \( V_K \) is dense in \( V \).
A smooth representation $V$ is called admissible, if any irreducible representation of $K$ occurs in $V$ with finite multiplicity. Similarly we can define admissible $(\mathfrak{g}, K)$ modules. An admissible $(\mathfrak{g}, K)$–module $V$ is also called a Harish-Chandra module.

Now we introduce the following important class of smooth representation, following [6] and [18].

**Definition.** A smooth representation $(\pi, V)$ is called a representation of moderate growth if for every seminorm $\rho$ on $V$, there exists a positive integer $N$ and a seminorm $\nu$ such that for every $g \in G_n$, $v \in V$, we have

$$|\pi(g)v|_\rho \leq ||g||^N|v|_\nu$$

where $||g|| = Tr(g^t g) + Tr(g^{-1} g^t)$ and $g^t = g^{-1}$.

Note that we always have $||g|| \geq 1$.

If in addition, $V$ is admissible, then we call it a Harish-Chandra representation.

Given a Harish-Chandra representation, its underlying $(\mathfrak{g}, K)$ module is a Harish-Chandra module, a deep result of Casselman and Wallach is the following theorem. For more details, see [6] or [18].

**Theorem 2.1.1.** (Casselman and Wallach): For any finitely generated Harish-Chandra module $W$, there exists exactly one smooth representation $(\pi, V)$ of moderate growth, up to canonical topological isomorphism, such that the underlying $(\mathfrak{g}, K)$ module $V_K$ is isomorphic to $W$. Moreover, the assignment $W \rightarrow V$ is an exact functor from the category of finitely generated Harish-Chandra modules to the category of finitely generated Harish-Chandra representations.

In this case, we will call $V$ the completion or globalization of $W$.

Now given a smooth representation $V$ of $G_n$, if $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$, we use $H_0(\mathfrak{n}, V)$ to denote the quotient of $V$ by the closure of the subspace spanned by $\{X.v : X \in \mathfrak{n}, v \in V\}$. When $V_K$ is a $(\mathfrak{g}, K)$ module, use $H_0(\mathfrak{n}, V_K)$ to denote $V_K/\mathfrak{n}V_K$. 
If $N$ is a unipotent subgroup of $G_n$, denote by $H_0(N,V)$ the quotient of $V$ by the closure of the subspace spanned by vectors $\{\pi(u)v - v : u \in N, v \in V\}$.

If $\psi$ is a smooth character of $N$, let $\mu$ denote the differential of $\psi$. That is, for any $X$ in the Lie algebra $n$ of $N$, we have

$$
\mu(X) = \lim_{t \to 0} \frac{\psi(exp(tX)) - 1}{t}
$$

which defines a linear form on $n$.

**Proposition 2.1.2.** Let $N$ be a connected and simply connected Lie subgroup of $G_n$. Let $\psi$ be a smooth character of $N$, $\mu$ is the differential of $\psi$. Let $(\pi, V)$ be a smooth representation of $G_n$. Suppose $\Lambda$ is a continuous linear functional on $V$, then $\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$ for any $u \in N$, $v \in V$ if and only $\Lambda(\pi(X)v) = \mu(X)\Lambda(v)$ for any $X \in n$, $v \in V$.

**Proof.** Suppose if we have $\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$ for any $u \in N$, $v \in V$. Now for any $X \in n$, recall that $\pi(X)v = \frac{d}{dt}|_{t=0}\pi(exp(tX))v$. Now apply $\Lambda$ to both sides and note that it is continuous, so we have

$$
\Lambda(\pi(X)v) = \Lambda\left(\frac{d}{dt}|_{t=0}\pi(exp(tX))v\right) = \Lambda\left(\lim_{t \to 0} \frac{\pi(exp(tX))v - v}{t}\right)
$$

$$
= \lim_{t \to 0} \frac{\Lambda(\pi(exp(tX))v) - \Lambda(v)}{t} = \lim_{t \to 0} \frac{(\psi(exp(tX)) - 1)\Lambda(v)}{t}
$$

$$
= \frac{d}{dt}|_{t=0}\psi(exp(tX))\Lambda(v) = \mu(X)\Lambda(v)
$$

This proves one direction.

For the other direction, $\Lambda$ satisfies $\Lambda(\pi(X)v) = \mu(X)\Lambda(v)$, which means that $X.\Lambda = \mu(X)\Lambda$. This is a one dimensional representation of $n$. Since $N$ is connected and simply connected, by Theorem 3.27 [19] for example, this can be lifted to a one dimensional representation $\psi$ of $N$, that is, $u.\Lambda = \psi(u)\Lambda$, which finishes the other direction.

\[\square\]
Corollary 2.1.3. As in the above proposition, the quotient of $V$ by the closure of the subspace $\{\pi(u)v - \psi(u)v : u \in N, v \in V\}$ has the same continuous dual as the quotient of $V$ by the closure of the subspace $\{\pi(X)v - \mu(X)v; X \in n, v \in V\}$.

Proof. Let $W_1$ be the quotient $V/\{\pi(u)v - \psi(u)v : u \in N, v \in V\}$, and $W_2 = V/\{\pi(X)v - \mu(X)v; X \in n, v \in V\}$. Then any continuous linear functional on $W_1$ gives a continuous linear functional $\Lambda$ on $V$ with $\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$, and conversely, any continuous linear functional $\Lambda$ on $V$ with the above property factors through $W_1$. Hence the continuous dual of $W_1$ is the set of continuous linear functionals $\Lambda$ on $V$ satisfying $\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$.

On the other hand, apply the above argument to $W_2$, we see that the continuous dual of $W_2$ is the set of continuous linear functionals $\Lambda$ on $V$ with $\Lambda(\pi(X)v) = \mu(X)\Lambda(v)$.

Hence by Proposition 2.1.2., the continuous dual of $W_1$ and $W_2$ are the same. This finishes the proof.

Now if $(\pi, V)$ is an irreducible Harish-Chandra representation of $G_n$, $V_K$ its $K$–finite vectors, hence a Harish-Chandra module. If $P$ is a standard parabolic subgroup of $G_n$, with Levi decomposition $P = MN$, where $N$ is the unipotent subgroup of $P$. Let $p, m, n$ be their complexified Lie algebras respectively. It is well known (for example, by Theorem 2.4 [3]) that $H_0(n, V_K)$ is a nonzero admissible $(m, K \cap M)$ module, and it is finitely generated over $U(m)$, here $U(m)$ denotes the universal enveloping algebra of $m$.

For $H_0(n, V)$, this is a quotient of a Frechet space $V$ by its closed subspace $\overline{nV}$, hence it is also a Frechet space. The seminorms on $H_0(n, V)$ are defined as follows. Let $| \cdot |_{p_i}$ be a seminorm on $V$, define

$$|v + \overline{nV}|_{p_i} = \inf_{y \in \overline{nV}}(|v + y|_{p_i})$$
then $|\cdot|_{q_i}$ is a seminorm on $V/\overline{nV}$, and the collection of all such seminorms $|\cdot|_{q_i}$ makes $H_0(n, V)$ a Frechet space. For example see [17].

Now for any $h \in M$, define $h.(v + \overline{nV}) = h.v + \overline{nV}$. Then this defines a smooth representation of $M$ on $H_0(n, V)$. Moreover, for any seminorm $|\cdot|_{q_i}$, any $v + \overline{nV} \in H_0(n, V)$, we have

$$|h.v + \overline{nV}|_{q_i} = \inf_{y \in \overline{nV}} |h.v + h^{-1}y|_{p_i}$$

$$\leq \inf_{y \in \overline{nV}} |h.v + h^{-1}y|_{p_j} = \inf_{y \in \overline{nV}} |h.v + y|_{p_j} = ||h||^N |v + \overline{nV}|_{q_j}$$

Thus this is a representation of moderate growth.

Next we will consider the natural embedding of $H_0(n, V_K)$ into $H_0(n, V)$ sending $v + nV_K$ to $v + \overline{nV}$ for any $v \in V_K$. Moreover, we have

**Proposition 2.1.4.** $H_0(n, V)$ is a finitely generated Harish-Chandra representation of $M$, and its $K \cap M$ finite vectors are exactly $H_0(n, V_K)$. So it is the completion of $H_0(n, V_K)$.

**Proof.** We have checked that $H_0(n, V)$ is a smooth representation of moderate growth.

Consider the embedding $H_0(n, V_K) \rightarrow H_0(n, V)$, the image is a $(m, K \cap M)$ module. Since $V_K$ is dense in $V$, the image is also dense in $H_0(n, V)$. Hence $H_0(n, V_K)$ can be identified with the underlying $(m, K \cap M)$ module of $H_0(n, V)$. As $H_0(n, V_K)$ is nonzero, finitely generated and admissible, so is $H_0(n, V)$.

Hence $H_0(n, V)$ is the completion of $H_0(n, V_K)$.

$\Box$

- It is an unpublished result of B. Casselman that $H_*(n, V)$ is the completion of $H_*(n, V_K)$, see for example Theorem 1.5 [5].

Given two smooth representations $(\pi, V)$ and $(\rho, W)$ of $G_n$ and $G_m$ respectively, use $V \otimes W$ to denote the algebraic tensor product of $V$ and $W$. If $|\cdot|_p$ and $|\cdot|_q$ are
seminorms on $V$ and $W$ respectively, as in Proposition 43.1 in [17], for all $x \in V \otimes W$, the formula
\[
|x|_{p \otimes q} = \inf \left( \sum_j |v_j|_p |w_j|_q \right)
\]
defines a seminorm on $V \otimes W$, where the infimum is taken over all possible finite number of pairs $(v_j, w_j)$ with $x = \sum_j v_j \otimes w_j$. These seminorms define a metric on $V \otimes W$, and we use $\overline{V \otimes W}$ to denote its completion. This is again a Frechet space.

Now $G_n \times G_m$ acts on $V \otimes W$ by $\pi \otimes \rho$, which extends to $V \otimes W$ by continuity. Take the underlying smooth vectors, we thus obtain the complete projective tensor product, denoted as $(\pi \hat{\otimes} \rho, V \hat{\otimes} W)$. If $V_{K_n}$ and $W_{K_m}$ are the underlying Harish-Chandra modules, then $V_{K_n} \otimes W_{K_m}$ is dense in $\pi \hat{\otimes} \rho$, and exactly its $K_n \times K_m$ finite vectors.

Moreover, if both $\pi$ and $\rho$ are representations of moderate growth, we want to show the representation $\pi \hat{\otimes} \rho$ is also of moderate growth. For this purpose, by Theorem in section 5 of [11], it suffices to check that the representation $\pi \otimes \rho$ on the completion $\overline{V \otimes W}$ is of moderate growth.

First given any seminorm $| \cdot |_{p \otimes q}$, for any vector $v \in V \otimes W$, write $v = \sum_i x_i \otimes y_i$, with $x_i \in V$, $y_i \in W$. For any $(g_1, g_2) \in G_n \times G_m$, $(g_1, g_2).v = \sum_i (g_1.x_i) \otimes (g_2.y_i)$. Note that both $\pi$ and $\rho$ are of moderate growth, we have
\[
|(g_1, g_2).v|_{p \otimes q} \leq \sum_i |g_1.x_i|_p |g_2.y_i|_q \leq \sum_i ||g_1||^{N_1} ||x_i|_{p'} ||g_2||^{N_2} ||y_i|_{q'}
\]
\[
\leq ||g_1||^{N_1} ||g_2||^{N_2} \sum_i |x_i|_{p'} |y_i|_{q'}
\]
Hence
\[
|(g_1, g_2).v|_{p \otimes q} \leq ||(g_1, g_2)||^{N_0} |v|_{p' \otimes q'}
\]
where $N_0 = \max N_1, N_2$. This shows that $\pi \otimes \rho$ is of moderate growth on $V \otimes W$.

Now in general, if $v \in \overline{V \otimes W}$ which is a limit of $\{v_n \in V \otimes W\}$, that is, $v = \lim_{n \to \infty} v_n$, then
with each $v_n \in V \otimes W$. Given a seminorm $|\cdot|_{p \otimes q}$, there exists a positive integer $N$, a seminorm $|\cdot|_{p' \otimes q'}$, such that for any $v_n$, any $(g_1, g_2)$, we have

$$|(g_1, g_2) \cdot v_n|_{p \otimes q} \leq ||(g_1, g_2)||^N |v_n|_{p' \otimes q'}$$

Taking the limit for the right hand side, we have

$$|(g_1, g_2) \cdot v_n|_{p \otimes q} \leq ||(g_1, g_2)||^N |v|_{p' \otimes q'}$$

which implies that

$$|(g_1, g_2) \cdot v|_{p \otimes q} \leq ||(g_1, g_2)||^N |v|_{p' \otimes q'}$$

This finishes checking $\pi \hat{\otimes} \rho$ is of moderate growth.

The following result is known to experts, we include a proof here for the sake of completeness.

**Proposition 2.1.5.** If $(\pi, V)$ is an irreducible Harish-Chandra representation of $G_n \times G_m$, then there exist irreducible Harish-Chandra representations $\rho$ and $\sigma$ on $G_n$ and $G_m$ respectively, such that $\pi$ is isomorphic to $\rho \hat{\otimes} \sigma$. Moreover, the equivalence classes of $\rho$ and $\sigma$ are uniquely determined by $\pi$.

**Proof.** First consider the underlying Harish-Chandra module $V_{K_n \times K_m}$ of $V$, by Proposition 3.4.4. and Theorem 3.4.2. of [4], $V_{K_n \times K_m}$ is a tensor product of irreducible Harish-Chandra modules $W_{K_n}$ and $U_{K_m}$ on $G_n$ and $G_m$ respectively, and the equivalence classes of $W_{K_n}$ and $U_{K_m}$ are uniquely determined by $V_{K_n \times K_m}$. Now let $(\rho, W)$ and $(\sigma, U)$ be the global completion of $W_{K_n}$ and $U_{K_m}$ respectively, then $\rho \hat{\otimes} \sigma$ is an irreducible Harish-Chandra representation of $G_n \times G_m$ with its Harish-Chandra module isomorphic to $W_{K_n} \otimes U_{K_m}$, hence isomorphic to $V_{K_n \times K_m}$. This implies that $\pi$ is isomorphic to $\rho \hat{\otimes} \sigma$ by the uniqueness of global completion of Harish-Chandra modules.

$\square$
2.2 Langlands correspondence and $L$–factors

In this section, we review some basics about Langlands correspondence, specified to $GL_n(\mathbb{R})$ in particular. We will follow [15] very closely.

Let $SL^\pm_2(\mathbb{R})$ be the subgroup of $G_2$ consisting of matrices $g$ with $|\text{det} g| = 1$. For any positive integer $l \geq 1$, use $D^+_l$ to denote the discrete series of $SL_2(\mathbb{R})$, which is defined as follows. Let $H_l$ be the space of holomorphic functions on upper half plane with

$$||f||^2 = \int \int |f(z)|^2 y^{l-1} dx dy < \infty$$

where $z = x + iy$ in the upper half plane.

The action of $D^+_l$ on $H_l$ is given by, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$,

$$D^+_l(g)f(z) = (bz + d)^{-(l+1)} f \left( \frac{az + c}{bz + d} \right)$$

It is known that $D^+_l$ is an irreducible unitary representation of $SL_2(\mathbb{R})$.

We use $D_l$ to denote the induced representation

$$D_l = \text{ind}_{SL^\pm_2(\mathbb{R})}^{SL_2(\mathbb{R})}(D^+_l)$$

For any complex number $t$, we use $[+, t]$ to denote the character $1 \otimes |\cdot|^t$ of $GL_1$, $[-, t]$ denotes the character $\text{sgn} \otimes |\cdot|^t$, and $[l, t]$ denotes the representation $D_l \otimes |\text{det} \cdot|^t$ of $GL_2$.

If we have a partition of $n = (n_1, ..., n_r)$ into 1’s and 2’s, that is, each $n_i$ is either 1 or 2 and $\sum n_i = n$, we associate the diagonal block group

$$M = G_{n_1} \times ... \times G_{n_r}$$

Then $M$ is the Levi part of the standard parabolic subgroup $P$ associated to the
partition \((n_1, \ldots, n_r)\). If \(\sigma_i\) is a representation of the above type on \(G_{n_i}\), use \(I(\sigma_1, \ldots \sigma_r)\) to denote the normalized induced representation

\[
\text{Ind}_{P}^{G_n}(\sigma_1 \otimes \sigma_2 \otimes \ldots \otimes \sigma_r)
\]

Now we are ready to state the Langlands classification theorem for \(GL_n\).

**Theorem 2.2.1.** (1). If the parameter satisfy

\[
n_1^{-1} \text{Re}(t_1) \geq \ldots \geq n_r^{-1} \text{Re}(t_r)
\]

then \(I(\sigma_1, \ldots \sigma_r)\) has a unique irreducible quotient \(J(\sigma_1, \ldots, \sigma_r)\), and if \(J(\sigma_1, \ldots, \sigma_r)\) and \(J(J(\sigma'_1, \ldots, \sigma'_r))\) are isomorphic, then \(r = r'\) and up to a permutation \(j(i)\) of \(\{1, 2, \ldots, r\}\), \(\sigma_i = \sigma'_{j(i)}\);

(2). Any irreducible admissible representation of \(G_n\) arises in this way, that is, for any irreducible admissible representation \(\pi\) of \(G_n\), there exists a partition \((n_1, \ldots, n_r)\) of \(n\), and representations \(\sigma_i\) of the type as above on \(G_{n_i}\), such that \(\pi\) is isomorphic to \(J(\sigma_1, \ldots, \sigma_r)\).

In order to describe the Langlands correspondence, we need to introduce the Weil group and its finite dimensional representations.

The Weil group \(\mathbb{W}_\mathbb{R}\) of \(\mathbb{R}\), is given by

\[
\mathbb{W}_\mathbb{R} = \mathbb{C}^\times \cup j\mathbb{C}
\]

and the group operations are \(j^2 = -1\) and \(jzj^{-1} = \overline{z}\), where \(z \in \mathbb{C}\) and \(-\) denotes the complex conjugation. \(\mathbb{W}_\mathbb{R}\) has three equivalence classes of irreducible finite dimensional representations, two of them are one dimensional and the other is two dimensional, which are listed as below. Here \(l \geq 1\).

(1). \((+, t)\): \(z.1 = |z|^t\) and \(j.1 = 1\).

(2). \((- , t)\): \(z.1 = |z|^t\) and \(j.1 = -1\).
(3), $(l, t)$: There exists a basis $u$ and $u'$ such that

$$
(l, t) = \begin{cases}
  z.u = r^{2i} e^{i\theta} u & j.u = u' \\
  z.u' = r^{2i} e^{-i\theta} u' & j.u' = (-1)^l u
  \end{cases}
$$

A basic property about finite dimensional representation of $W_R$ is the following lemma.

**Lemma 2.2.2.** Every finite dimensional semisimple representation of $W_R$ is completely reducible.

**Proof.** See Lemma in section 3, [15].

Thus for any $n$ dimensional semisimple representation $\rho$ of $W_R$, we can find a partition of $n = (n_1, ..., n_r)$, with each $n_i$ equal to either 1 or 2. Use $\rho_i$ to denote the corresponding $n_i$ dimensional representation in the decomposition of $\rho$, then

$$\rho = \rho_1 \oplus ... \oplus \rho_r$$

and we write $\rho = (\rho_1, ..., \rho_r)$.

Now if we set a map

$$[+, t] \rightarrow (+, t),$$

$$[-, t] \rightarrow (-, t),$$

$$[l, t] \rightarrow (l, t).$$

then in this way we have a correspondence between any $n$ dimensional semisimple representation of $W_R$ and any irreducible admissible representation of $G_n$, sending $\rho = (\rho_1, ..., \rho_r)$ to $J(\sigma_1, ... \sigma_r)$, where $\sigma_i$ corresponds to $\rho_i$ according to the above table. This is the local Langlands correspondence for $GL_n(\mathbb{R})$, and we state it in the following form.
Theorem 2.2.3. The map sending $\rho = (\rho_1, ..., \rho_r)$ to $J(\sigma_1, ... \sigma_r)$ is a one to one correspondence between the set of $n$ dimensional semisimple representations of $W_\mathbb{R}$ and the set of irreducible admissible representations of $GL_n(\mathbb{R})$.

For each $n$ dimensional semisimple representation $\rho$ of $W_\mathbb{R}$, Weil associates a local factor $L(s, \rho)$ by Lemma 2.2.2. and the following table for irreducible ones

$$L(s, \rho) = \begin{cases} 
\pi^{-(s+t)/2} \Gamma\left(\frac{s+t}{2}\right) & \text{if } \rho = (+, t) \\
\pi^{-(s+t+1)/2} \Gamma\left(\frac{s+t+1}{2}\right) & \text{if } \rho = (-, t) \\
\pi^{-(s+t+\frac{1}{2})} \Gamma\left(\frac{s+t+\frac{1}{2}}{2}\right) \Gamma\left(\frac{\frac{s+t+\frac{1}{2}}{2}}{2}\right) & \text{if } \rho = (l, t)
\end{cases}$$

Now if $\pi$ (or $\pi'$) is an irreducible admissible representation of $G_n$ (or $G_m$), by the Langlands correspondence, $\pi$ (or $\pi'$) corresponds to a $n$ (or $m$) dimensional semisimple representation of $W_\mathbb{R}$, denoted as $\rho$ (or $\rho'$). Now consider the tensor product $\rho \otimes \rho'$, which defines a semisimple representation of $W_\mathbb{R}$ with dimension $mn$. Denote the corresponding $L$ factor as $L(s, \rho \otimes \rho')$, or $L(s, \pi \times \pi')$, and call it as the tensor product $L$ factor for $\pi$ and $\pi'$.

2.3 Archimedean Rankin-Selberg integrals

In this section we discuss some basic facts about archimedean Rankin-Selberg integrals. We refer to the papers [Ja 2], [J-S1] for details.

Now let $N_n$ be the upper triangular unipotent subgroup of $G_n$. Let $\psi$ be the additive character of $\mathbb{R}$ given by $\psi(x) = exp(2\pi \sqrt{-1}x)$, define a character on $N_n$, still denoted as $\psi$, as follows:

$$\psi(u) = \psi(\sum_i u_{i,i+1})$$

where $u = (u_{ij}) \in N_n$. 

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**Definition.** A Whittaker functional $\Lambda$ with respect to $\psi$ on a smooth representation $(\pi, V)$ of $G_n$ is a continuous linear functional on $V$ satisfying

$$\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$$

for all $u \in N_n$, $v \in V$.

Unlike the non-archimedean case, we can also define the Whittaker functional in terms of the Lie algebra action. Let $\mu$ be the differential of $\psi$, then $\mu$ is a linear form on $n_n$, the Lie algebra of $N_n$, vanishing on $[n_n, n_n]$.

**Definition.** A Whittaker functional $\Lambda$ with respect to $\mu$ on $(\pi, V)$ is a continuous linear functional on $V$ satisfying

$$\Lambda(\pi(X)v) = \mu(X)\Lambda(v)$$

for all $X \in n_n$, $v \in V$.

- By **Proposition 2.1.3.**, these two definitions are equivalent.

By [16], if $V$ is irreducible, then there exists at most one Whittaker functional with respect to a given nontrivial $\psi$, unique up to a scalar. When $(\pi, V)$ admits a nontrivial Whittaker functional, we call it a generic representation.

Through our paper, we will assume $\pi$ to be generic. Let $\Lambda$ be the Whittaker functional on $\pi$, for any $v \in V$, define a function $W_v(g) = \Lambda(\pi(g)v)$. Then $W_v(g)$ is called the Whittaker function on $G_n$ corresponding to $v$, and the space $\mathcal{W}(\pi, \psi) = \{W_v(g) : v \in V\}$ is called the Whittaker model of $\pi$.

For any $W_v(g) \in \mathcal{W}(\pi, \psi)$, define $\tilde{W}_v(g) = W_v(\omega_ng^t)$, here

$$w_n = \begin{pmatrix} 0 & \ldots & 1 \\ \vdots \\ 1 & \ldots & 0 \end{pmatrix}$$

and $g^t = {}^t g^{-1}$. Then by [12], it is known that $\{\tilde{W}_v(g) : v \in V\}$ is a Whittaker model for $\tilde{\pi}$ with respect to $\tilde{\psi}$, the contragredient of $\pi$. 

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Let $S(\mathbb{R}^n)$ be the space of Schwartz functions on $\mathbb{R}^n$.

Now we are ready to introduce the Rankin-Selberg integrals. Suppose $(\pi, V)$ and $(\pi', V')$ are generic irreducible Harish-Chandra representations of $G_n$ and $G_m$ respectively, with Whittaker models $\mathcal{W}(\pi, \psi)$ and $\mathcal{W}(\pi', \tilde{\psi})$.

If $m = n$, set

$$I(s, W, W', \Phi) = \int_{N_n \setminus GL_n} W(g) W'(g) \Phi(\epsilon_n g) |\det(g)|^s dg$$

for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \tilde{\psi})$, $\Phi \in S(\mathbb{R}^n)$, and $\epsilon_n = (0, 0, ..., 1) \in \mathbb{R}^n$.

If $n > m$, set

$$I(s, W, W') = \int_{N_m \setminus G_m} W \begin{pmatrix} g & 0 & 0 \\ 0 & I_{n-m} & \\ & & \end{pmatrix} W'(g) |\det(g)|^{s - \frac{n-m}{2}} dg$$

In general for $0 \leq j \leq n - m - 1$, set

$$I_j(s, W, W') =$$

$$\int_{M(m \times j, \mathbb{R})} \int_{N_m \setminus G_m} W \begin{pmatrix} g & 0 & 0 \\ & X & I_j \\ & 0 & 0 & I_{n-m-j} \\ & & & \end{pmatrix} W'(g) |\det(g)|^{s - \frac{n-m}{2}} dgdX$$

Now we can state the following theorem of Jacquet and Shalika, see [13].

**Theorem 2.3.1**

(1) These integrals converge for $Re(s) > 0$;

(2) Each integral has a meromorphic continuation to all $s \in \mathbb{C}$, which is a holomorphic multiple of $L(s, \pi \times \pi')$;

(3) The following functional equations are satisfied:

$$I_j(1 - s, \tilde{W}, \tilde{W}') = \omega'(-1)^{n-1} \gamma(s, \pi \times \pi', \psi) I_{n-m-1-j}(s, W, W')$$

and

$$I(1 - s, \tilde{W}, \tilde{W}', \tilde{\Phi}) = \omega'(-1)^{n-1} I(s, W, W', \Phi)$$
where $\hat{\Phi}$ is the Fourier transform of $\Phi$, given by the following formula

$$\hat{\Phi}(X) = \int \Phi(Y) \psi(-Tr(\ ^tXY))dY$$
CHAPTER 3
ARCHIMEDEAN DERIVATIVES

3.1 p-adic Derivatives

In this section, we review the notion of p-adic derivatives. This was first defined and studied in [2].

Let $F$ be a p-adic field, and in this section $G_n$ denotes $GL_n(F)$. Let $P_n$ be the mirabolic subgroup, that is, the subgroup of $G_n$ consisting matrices with last low $(0,...,0,1)$. Then $P_n$ is the semidirect product of $G_{n-1}$ and $U_n$, where $G_{n-1}$ embeds to $G_n$ on the left upper corner, and $U_n$ is the subgroup consisting of matrices of the form

$$
\begin{pmatrix}
I_{n-1} & u \\
0 & 1
\end{pmatrix}
$$

If $\psi$ is a nontrivial additive character of $F$, define a character of $U_n$, still denoted as $\psi$, by $\psi(u) = \psi(u_{n-1,n})$ for each $u = (u_{ij}) \in U_n$.

Now define the following two functors $\Phi^{-1}$ and $\Psi^{-1}$.

$\Phi^{-1}$ is a functor from the category of smooth representations of $P_n$ to the category of smooth representations of $G_{n-1}$. Given any smooth representation $(\rho, V)$ of $P_n$, Let $V_\rho = \{\rho(u)v - v : u \in U_n, v \in V\}$, and form the quotient $V/V_\rho$. $\Phi^{-1}(\rho)$ acts on this quotient by

$$\Phi^{-1}(\rho)(g)\bar{v} = |\det g|^{-1/2}\rho(g).v + V_\rho$$
for any $g \in G_{n-1}$. Then this defines a smooth representation of $G_{n-1}$.

The functor $\Psi^{-1}$ is from the category of smooth representations of $P_n$ to the category of smooth representations of $P_{n-1}$. For any smooth representation $(\rho, V)$ of $P_n$, let $V_{\rho, \psi} = \{\rho(u)v - \psi(u)v : u \in U_n, v \in V\}$, and form the quotient $V/V_{\rho, \psi}$. Then $\Psi^{-1}(\rho)$ acts on this quotient by

$$\Psi^{-1}(p)\bar{v} = |\text{det} p|^{-1/2} \rho(p)v + V_{\rho, \psi}$$

for any $p \in P_{n-1}$. Then this defines a smooth representation of $P_{n-1}$.

Now given any smooth representation $(\pi, V)$ of $G_n$, we restrict it to $P_n$, which becomes a smooth representation of $P_n$, then define the $k$-th derivative, denoted as $\pi^{(k)}$, by the formula $\Phi^{-1} \circ (\Psi^{-1})^{k-1}(\pi)$. This defines a smooth representation of $G_{n-k}$.

- From the definition, $\pi^{(0)} = \pi$, $\pi^{(n)}$ is the linear dual of the space of Whittaker functionals.

Now define the Rankin-Selberg integrals for a pair of generic irreducible admissible smooth representations of $G_n$ and $G_m$ similarly as the archimedean case. That is, suppose $(\pi, V)$ and $(\pi', V')$ are generic irreducible admissible smooth representations of $G_n$ and $G_m$ respectively, with Whittaker models $W(\pi, \psi)$ and $W(\pi', \bar{\psi})$.

If $m = n$, set

$$I(s, W, W', \Phi) = \int_{N_n \backslash GL_n} W(g)W'(g)\Phi(\epsilon_n g)|\text{det}(g)|^s dg$$

for $W \in W(\pi, \psi), W' \in W(\pi', \bar{\psi}), \Phi \in S(F^n)$, and $\epsilon_n = (0, 0, ..., 1) \in F^n$.

If $n > m$, set

$$I(s, W, W') = \int_{N_m \backslash G_m} W \left( \begin{array}{cc} g & 0 \\ 0 & I_{n-m} \end{array} \right) W'(g)|\text{det}(g)|^{s-n-m/2} dg$$

In general for $0 \leq j \leq n - m - 1$, set

$$I_j(s, W, W') = \int_{N_m \backslash G_m} W \left( \begin{array}{cc} g & 0 \\ 0 & I_{n-m} \end{array} \right) W'(g)|\text{det}(g)|^{s-n-m-j/2} dg$$
\[
\int_{M(m \times j, \mathbb{R})} \int_{N_m \backslash G_m} W \begin{pmatrix} g & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{n-m-j} \end{pmatrix} W'(g) |\det(g)|^{s - \frac{n-m}{2}} dg dX
\]

In [10], J. Cogdell and I.I. Piatetski-Shapiro used the notion of derivatives, together with another notion exceptional poles, to analyze the poles of such Rankin-Selberg integrals \( I(s, W, W') \). One main result in their paper is the following theorem.

**Theorem 3.1.1.** *(Theorem 3.2 in [10])*. Let \( \pi \) and \( \sigma \) be generic irreducible admissible smooth representations of \( G_n \) and \( G_m \) respectively, such that all derivatives of \( \pi \) and \( \sigma \) are completely reducible. Then the poles (counting multiplicity) of the local \( L \) factor \( L(s, \pi \times \sigma) \) are exactly the same as exceptional poles of the Rankin-Selberg integrals for all possible components of derivatives \( \pi^{(n-k)} \) and \( \sigma^{(m-k)} \).

It is the main purpose of this paper to introduce the archimedean analog of derivatives and exceptional poles, and to prove a similar result for archimedean Rankin-Selberg integrals.

### 3.2 Archimedean Derivatives

In this section, we define the notion: archimedean derivatives.

First for any \( 1 \leq k \leq n \), let \( U_{n-k+1} \) be the subgroup of \( N_n \) consisting of matrices having the form

\[
\begin{pmatrix}
I_{n-k} & x \\
& u
\end{pmatrix}
\]

where \( x \) is a \( (n - k) \times k \) matrix and \( u \in N_k \) is an upper triangular matrix with 1 on the diagonal. Note that \( U_1 = N_n \). The corresponding Lie algebras are denoted by \( u_{n-k+1} \).
As before, fix a nontrivial additive character $\psi$ of $\mathbb{R}$ by setting $\psi(x) = exp(2\pi\sqrt{-1}x)$, for any $u = (u_{ij}) \in N_n$, we have defined a character on $N_n$, still denoted as $\psi$, by setting $\psi(u) = \psi(u_{12} + u_{23} + \ldots + u_{n-1,n})$.

The differential of $\psi$, denoted by $\mu$, is a linear form on $n_n$, vanishing on $[n_n, n_n]$. It has the form, for any $X = \{X_{ij}\} \in n_n$, $\mu(X) = 2\pi\sqrt{-1}(X_{12} + X_{23} + \ldots + X_{n-1,n})$.

Define a linear form $\mu_{n-k+1}$ on each $u_{n-k+1}$, by

$$\mu_{n-k+1}(X) = \mu(X_{n-k+1,n-k+2} + \ldots + X_{n-1,n})$$

Note that $\mu_n = 0$ and $\mu_1 = \mu$.

Now let $(\pi, V)$ be a Harish-Chandra representation of $G_n$, $V_K$ its underlying $(g, K)$ module. For $1 \leq k \leq n$, let $V_k$ be the closure of the subspace spanned by $\{X.v - \mu_{n-k+1}(X)v : v \in V, X \in u_{n-k+1}\}$.

**Definition.** For each integer $0 \leq k \leq n$, we define the $k$–th derivative of $\pi$, denoted by $(\pi^{(k)}, V^{(k)})$, as follows:

1. If $k = 0$, put $(\pi^{(0)}, V^{(0)}) = (\pi, V)$;
2. If $1 \leq k \leq n$, put $V^{(k)} = V/V_k$ and define the action $\pi^{(k)}$ as follows, for any $g \in GL_{n-k}$,

$$\pi^{(k)}(g).(v + V_k) = |det g|^{-k/2}\pi(g)v + V_k$$

Remark:

- Since conjugation by element in $G_{n-k}$ (embedded in $G_n$ in the left upper corner) preserves the space $V_k$, the action of derivative $\pi^{(k)}$ is well defined.
- By definition, $V_n = \{X.v - \mu(X)v : v \in V, X \in n_n\}$, hence any continuous linear functional on $V/V_n$ gives a continuous Whittaker functional on $V$, and conversely any
continuous Whittaker functional on $V$ factors through $V/V_n$. Thus the continuous dual of $V/V_n$ is the space of continuous Whittaker functionals on $V$.

- A. Aizenbud, D. Gourevitch and S. Sahi have defined another type of derivatives in their paper [1], which is different from ours. If we use $U^{n-k}$ to denote the subgroup of $GL_{n-k}$ consisting of matrices

$$
\begin{pmatrix}
I_{n-k-1} & x & 0 \\
0 & 1 & 0 \\
0 & 0 & I_k
\end{pmatrix}
$$

and $u^{n-k}$ its Lie algebra, then they define the $k$-th derivative of $\pi$, denoted as $D^k(\pi)$ by

$$
D^k(\pi) = |\text{det}|^{-k/2} \otimes \lim_{l \to \infty} V^{(k)}/\text{Span}\{Y.v : v \in V^{(k)}, Y \in (u^{n-k})^\otimes l\}
$$

For more details about their derivatives, see [1].

Define a continuous partial Whittaker functional $\Lambda_k$ to be a continuous linear functional on $V$ such that $\Lambda_k(\pi(X)v) = \mu_{n-k+1}(X)\Lambda_k(v)$ for any $X \in u_{n-k+1}$ and $v \in V$, then any $\Lambda_k$ factors through $V/V_k$, and conversely, any continuous linear functional on $V/V_k$ gives a continuous partial Whittaker functional. Thus the continuous dual of $V/V_k$ is the space of continuous Whittaker functionals.

In view of Proposition 2.1.2., we can also define the derivatives in terms of the action of group elements. Let $V'_k$ be the closure of the subspace spanned by $\{u.v - \psi_{n-k+1}(u)v : u \in U_{n-k+1}, v \in V\}$, where $\psi_{n-k+1}$ is the character on the group $U_{n-k+1}$ by $\psi_{n-k+1}(u) = \psi(u_{n-k+1,n-k+2} + \ldots + u_{n-1,n})$, then $V^{(k)} = V/V'_k$ and the action of $GL_{n-k}$ is the same as above.

Now we introduce the following notations. Use $P_{n-k,k}$ to denote the standard parabolic subgroup of $G_n$ associated to the partition $(n-k,k)$ of $n$. Let $P_{n-k,k} = M_{n-k,k}N_{n-k,k}$ be its Levi decomposition, with Levi component $M_{n-k,k}$ isomorphic to
GL_{n-k} \times GL_k$ and unipotent part $N_{n-k,k}$. Let $p_{n-k,k}$, $m_{n-k,k}$ and $n_{n-k,k}$ be their complexified Lie algebras respectively.

Note that the group $U_{n-k+1} = V_{n-k+1} N_{n-k,k}$ with the decomposition

$$
\begin{pmatrix}
I_{n-k} & x \\
v & 0
\end{pmatrix} =
\begin{pmatrix}
I_{n-k} & 0 \\
v & 0
\end{pmatrix}
\begin{pmatrix}
I_{n-k} & x \\
0 & I_k
\end{pmatrix}
$$

where $V_{n-k+1}$ is the standard unipotent subgroup $N_k$ of $G_k$ embedded in $G_n$ in the right lower corner.

Let $v_{n-k+1}$ be the complexified Lie algebra of $V_{n-k+1}$. Note that the character $\mu_{n-k+1}$ is trivial on $n_{n-k,k}$. Let $Y_k$ to be the closure of the space spanned by $\{X.\bar{v} - \mu_{n-k+1}(X)\bar{v} : X \in v_{n-k+1}, \bar{v} \in H_0(n_{n-k,k}, V)\}$. Then we have the following proposition.

**Proposition 3.2.1.** $V^{(k)} = H_0(n_{n-k,k}, V)/Y_k$.

**Proof.** By definition, $V^{(k)} = V/V_k$, and

$$
V_k = \{X.v - \mu_{n-k+1}(X)v : v \in V, X \in u_{n-k+1}\}
$$

Since $u_{n-k+1} = v_{n-k+1} + n_{n-k,k}$, then

$$
H_0(n_{n-k,k}, V)/Y_k =
(V/n_{n-k,k}V)/(\{X.v - \mu_{n-k+1}(X)v : X \in u_{n-k+1}, v \in V\}/n_{n-k,k}V) = V/V_k
$$

which completes the proof.

- Note that if we set $Y'_k$ to be the closure of the space generated by $\{u.\bar{v} - \psi_{n-k+1}(u)\bar{v} : u \in V_{n-k+1}, \bar{v} \in H_0(n_{n-k,k}, V)\}$, then by **Corollary 2.1.3.**, $H_0(n_{n-k,k}, V)/Y_k \cong H_0(n_{n-k,k}, V)/Y'_k$.
Proposition 3.2.2. For each $k$, $\pi^{(k)}$ is a Harish-Chandra representation of $G_{n-k}$.

Proof. By Proposition 2.1.4., $H_0(n_{n-k}, V)$ is a Harish-Chandra representation of $M_{n-k}$, which is isomorphic to $GL_{n-k} \times GL_k$. Suppose $H_0(n_{n-k}, V) = W_0 \supset W_1 \supset \ldots \supset W_l \supset 0$ is a composition chain of of $H_0(n_{n-k}, V)$, with each composition factor isomorphic to an irreducible Harish-Chandra representation. By Proposition 2.1.5., we may assume $W_i/W_{i+1}$ is isomorphic to $E_i \otimes F_i$, where $E_i$ and $F_i$ are irreducible Harish-Chandra representations of $GL_{n-k}$ and $G_k$ respectively.

Now consider the quotient $W_i' = W_i/\{X.\bar{v} - \mu_{n-k+1}(X)v : X \in v_{n-k+1}, \bar{v} \in W_i\}$ and we have a chain $H_0(n_{n-k}, V)/Y_k = W_0' \supset W_1' \supset \ldots \supset W_l' \supset 0$.

The same as in the proof of Proposition 3.2.1., $W_i'/W_{i+1}'$ is isomorphic to the quotient of $W_i/W_{i+1}$ by the closure of the space $\{X.\bar{w}_i - \mu_{n-k+1}(X)\bar{w}_i : X \in v_n, \bar{w}_i \in W_i/W_{i+1}\}$.

Since $W_i/W_{i+1}$ is isomorphic to $E_i \otimes F_i$, note that $v_{n-k+1}$ acts trivially on $E_i$, hence $X.(e_i \otimes f_i) = X.e_i \otimes f_i + e_i \otimes X.f_i = e_i \otimes X.f_i$, thus the closure of the space spanned by $\{X.(e_i \otimes f_i) - \mu_{n-k+1}(X)e_i \otimes f_i : X \in v_n, e_i \in E_i, f_i \in F_i\}$ equals the closure of the space spanned by $\{e_i \otimes X.f_i - e_i \otimes (\mu_{n-k+1}(X)f_i) : X \in v_n, e_i \in E_i, f_i \in F_i\} = \{e_i \otimes (X.f_i - \mu_{n-k+1}(X)f_i) : X \in v_n, e_i \in E_i, f_i \in F_i\}$.

Since $F_i$ is an irreducible Harish-Chandra representation of $G_k$, by the uniqueness of Whittaker functionals, up to a scalar multiple, there is at most one continuous linear functional $\Lambda$ on $F_i$ such that $\Lambda(X.f_i) = \mu_{n-k+1}(X)\Lambda(f_i)$. Hence continuous dual of the quotient space $F_i/\{X.f_i - \mu_{n-k+1}(X)f_i : X \in v_n, f_i \in F_i\}$ is at most one dimensional, which implies this quotient is at most one dimensional.

Thus each factor $W_i'/W_{i+1}'$ is either isomorphic to $E_i$ or $0$. This shows that $\pi^{(k)}$ is of finite length and hence admissible. So $\pi^{(k)}$ is a Harish-Chandra representation of $G_{n-k}$.
3.3 Some Basic Properties

In this section, we will study some properties of the derivatives.

Let \((\pi, V)\) be an irreducible smooth admissible generic representation of moderate growth on \(G_n\), denote by \(V_K\) its \(K\)-finite vectors, which is a Harish-Chandra module. We assume \((\pi, V)\) is in the general position, in the sense that it has a regular infinitesimal character. In this section, unless otherwise stated, we will drop the subscript for the standard upper triangular parabolic subgroup associated with the partition \((n - k, k)\) of \(n\), so write \(P = P_{n-k,k}\) with its Levi \(M\) isomorphic to \(G_{n-k} \times G_k\), and write \(N\) for the unipotent radical of \(P\). The corresponding complexified Lie algebras are denoted by \(p, m\) and \(n\).

Consider the \(n\)-homology \(V_K / nV_K\), it is known by Lemma 12.2.4 in [18] that this is nonzero and is a semisimple Harish-Chandra module for \((m, K \cap M)\). We also consider \(n\)-homology \(V / nV\) of \(V\), here \(nV\) denotes the closure of the subspace \(nV\).

By Proposition 2.1.4., \(V / \overline{nV}\) is the smooth completion of \(V_K / nV_K\).

It follows that \(V / \overline{nV}\) is also semisimple, so we can write it as

\[
V / \overline{nV} = \bigoplus_{i=1}^r A_i
\]

where each \(A_i\) is an irreducible smooth admissible representation of moderate growth on \(M\), hence isomorphic to \(\rho_i \hat{\otimes} \sigma_i\), where each \(\rho_i\) and \(\sigma_i\) are irreducible Harish-Chandra representations on \(G_{n-k}\) and \(G_k\) respectively. Note that it is possible to have \(A_i \cong A_j\) for \(i \neq j\). We use \(\rho^K_i\) and \(\sigma^K_i\) to denote the representations on the underlying Harish-Chandra modules. Let \(p_i\) be the natural projection from \(V_K / nV_K\) onto \(\rho^K_i \otimes \sigma^K_i\), and also be the projection from \(V / \overline{nV}\) onto \(\rho_i \hat{\otimes} \sigma_i\). We will also use \(p\) to denote the projections \(V \rightarrow V / \overline{nV}\). For each \(i\), use \(L_i\) to denote the subspace of \(V\) containing \(\overline{nV}\) corresponding to \(A_i\). Thus \(A_i\) can be realized as the quotient \(L_i / \overline{nV}\).

Lemma 3.3.1. Each \(i, \rho_i\) and \(\sigma_i\) are generic representations.
Proof. By Frobenious reciprocity for \( n \)-homology, for example Theorem 4.9 in [H-S], the projection \( p_i \) induces an injective map \( V_K \to Ind(|det|^{-k/2} \rho_i^K \otimes |det|^{(n-k)/2} \sigma_i^K) \).

By Theorem 2.1.1. this extends to an injective intertwining map

\[
V \to Ind(|det|^{-k/2} \rho_i \hat{\otimes} |det|^{(n-k)/2} \sigma_i)
\]

Since \( V \) is generic, and the Whittaker functor is exact on the category of finitely generated Harish-Chandra representations (see for example Theorem 8.2, [16]), it follows that \( Ind(|det|^{-k/2} \rho_i \hat{\otimes} |det|^{(n-k)/2} \sigma_i) \) is generic. Now by Theorem 15.4.1. [18], the representation \( |det|^{-k/2} \rho_i \hat{\otimes} |det|^{(n-k)/2} \sigma_i \) is generic, hence so are \( \rho_i \) and \( \sigma_i \) for each \( i \).

\[\square\]

Fix the non-trivial additive character \( \psi(x) = \exp(2\pi \sqrt{-1} x) \) of \( \mathbb{R} \), and denote by \( \mathcal{W}(\rho_i, \psi) \) the Whittaker model for \( \rho_i \).

**Proposition 3.3.2.** For every \( W_i \in \mathcal{W}(\rho_i, \psi) \) and every \( \Phi \in \mathcal{S}(\mathbb{R}^{n-k}) \), there is a Whittaker function \( W_v \in \mathcal{W}(\pi, \psi) \) such that

\[
W_v \left( \begin{pmatrix} g \\ I_k \end{pmatrix} \right) = W_i(g)\Phi(\epsilon_{n-k} g).
\]

**Proof.** First as in the proof of last lemma, we have an injection

\[
V \to Ind(|det|^{-k/2} \rho_i \hat{\otimes} |det|^{(n-k)/2} \sigma_i)
\]

Denote by \( Q \) its quotient and we have a short exact sequence of smooth representations of moderate growth

\[
0 \to V \to Ind(|det|^{-k/2} \rho_i \hat{\otimes} |det|^{(n-k)/2} \sigma_i) \to Q \to 0 \to \cdots \ (1)
\]

The underlying \((g, K)\) modules also form a short exact sequence

\[
0 \to V_K \to Ind(|det|^{-k/2} \rho_i^K \otimes |det|^{(n-k)/2} \sigma_i^K) \to Q_K \to 0 \to \cdots \ (2)
\]
By taking the dual (contragredient representation) of short sequence (2), we have

\[ 0 \longrightarrow Q^*_K \longrightarrow (\text{Ind}(|\det|^{-k/2}\rho_i^K \otimes |\det|^{(n-k)/2}\sigma_i^K))^* \longrightarrow V_K^* \longrightarrow 0 \]

By Lemma 4.5.2 in [18], we have

\[ (\text{Ind}(|\det|^{-k/2}\rho_i^K \otimes |\det|^{(n-k)/2}\sigma_i^K))^* \cong \text{Ind}((|\det|^{-k/2}\rho_i^K)^* \otimes (|\det|^{(n-k)/2}\sigma_i^K)^*)) \]

So we have an exact sequence

\[ 0 \longrightarrow Q^*_K \longrightarrow \text{Ind}((|\det|^{-k/2}\rho_i^K)^* \otimes (|\det|^{(n-k)/2}\sigma_i^K)^*)) \longrightarrow V_K^* \longrightarrow 0 \]

which induces a short exact sequence for their smooth completions

\[ 0 \longrightarrow Q^* \longrightarrow \text{Ind}((|\det|^{-k/2}\rho_i^K)^* \otimes (|\det|^{(n-k)/2}\sigma_i^K)^*)) \longrightarrow V^* \longrightarrow 0 \cdots \cdots (3) \]

Now for any representation \((\tau, U)\), define representation \((\tau^s, U)\) by \(\tau^s(g).u = \tau(t^sg^{-1}).u\) for any \(g \in G_n, u \in U\), then \(\tau^s\) is isomorphic to \(\tau^*\) when \(\tau\) is irreducible, by for example [12]. Note that we are working in the same space, but simply change the action. So if we have a short exact sequence

\[ 0 \longrightarrow (\tau_1, U_1) \longrightarrow (\tau_2, U_2) \longrightarrow (\tau_3, U_3) \longrightarrow 0 \]

Applying the operation \(s'\), we then have a new exact sequence

\[ 0 \longrightarrow (\tau_1^s, U_1) \longrightarrow (\tau_2^s, U_2) \longrightarrow (\tau_3^s, U_3) \longrightarrow 0 \]

Now apply operation \(s'\) to the sequence (3), then we have the following

\[ 0 \longrightarrow (Q^s)^* \longrightarrow (\text{Ind}((|\det|^{-k/2}\rho_i^K)^* \otimes (|\det|^{(n-k)/2}\sigma_i^K)^*))^* \longrightarrow (V^s)^* \longrightarrow 0 \cdots \cdots (4) \]

Now we need the following claim. Let \(P'\) be the parabolic subgroup of \(G_n\) consisting of matrices

\[
\begin{pmatrix}
g_{n-k} & 0 \\
X & g_k
\end{pmatrix}
\]
with \( g_{n-k} \in G_{n-k}, g_k \in G_k. \)

**Claim.** \((\text{Ind}_{P}^{G^n}(\tau))^s \cong \text{Ind}_{P'}^{G^n}(\tau^s)\)

**Proof of Claim.** For any \( f \in \text{Ind}_{P}^{G^n}(\tau), \) define \( f' : G_n \to U \) by \( f'(g) = f(t^g). \) Then for any \( p' = \begin{pmatrix} g_1 & 0 \\ X & g_2 \end{pmatrix} \in P', \) any \( g \in G_n, \) we have

\[
f'(p'g) = f(t^{p'^{-1}(t^g)}) = f(p \cdot t^g) = \nu_P(p)^{1/2}\tau(p)f(t^g)
\]

where

\[
p = (p')^{-1} = \begin{pmatrix} t^{-1} & t(-g_2^{-1}Xg_1^{-1}) \\ 0 & t^{-1}g_2^{-1} \end{pmatrix} \in P
\]

This shows that \( f' \in \text{Ind}_{P'}^{G^n}(\tau^s). \)

Moreover,

\[
((\text{Ind}(\tau))^s(g).f)'(x) = ((\text{Ind}(\tau))^s(g).f)(t^{x^{-1}}) = f(t^{x^{-1}} \cdot t^g)
\]

which shows that the map \( f \to f' \) is a homomorphism.

Conversely, if \( \phi \in \text{Ind}_{P'}^{G^n}(\tau^s), \) define \( \tilde\phi(g) = \phi(t^g), \) then as above we have \( \tilde\phi \in \text{Ind}_{P}^{G^n}(\tau), \) and \( \phi \to \tilde\phi \) is also a homomorphism.

\( \tilde f'(g) = f'(t^g) = f(g), \) hence the composition \( f \to f' \to \tilde f' \) is identity. Similarly the composition \( \phi \to \tilde\phi \to \tilde\phi' \) is also the identity. Thus we have the isomorphism in the Claim.

\( \square \)

It follows by the Claim the sequence (4) becomes

\[
0 \to (Q^*)^s \to \text{Ind}_{P'}^{G^n}((|\det|^{-k/2}\rho_i)^s \otimes (|\det|^{(n-k)/2}\sigma_i)^s) \to (V^*)^s \to 0
\]
Since $\pi$, $\rho_i$ and $\sigma_i$ are irreducible, the above is

$$0 \rightarrow (Q^*)^s \rightarrow \text{Ind}_{P'}^G(|\text{det}|^{-k/2}\rho_i \hat{\otimes} |\text{det}|^{(n-k)/2}\sigma_i) \rightarrow V \rightarrow 0$$

Since $(\pi, V)$ is generic, there is a unique up to constant continuous Whittaker functional $\Lambda$ on $V$. Composed with the projection

$$\text{Ind}_{P'}^G(|\text{det}|^{-k/2}\rho_i \hat{\otimes} |\text{det}|^{(n-k)/2}\sigma_i) \rightarrow V$$

we get a nontrivial continuous Whittaker functional $\Lambda'$ on

$$\text{Ind}_{P'}^G(|\text{det}|^{-k/2}\rho_i \hat{\otimes} |\text{det}|^{(n-k)/2}\sigma_i)$$

By Theorem 15.4.1 in [18], there is a unique up to constant continuous Whittaker functional on $\text{Ind}_{P'}^G(|\text{det}|^{-k/2}\rho_i \hat{\otimes} |\text{det}|^{(n-k)/2}\sigma_i)$, so it must be $\Lambda'$. Then we can conclude that the space of Whittaker functions $W(\pi, V)$ for $\pi$ and that for $\text{Ind}_{P'}^G(|\text{det}|^{-k/2}\rho_i \hat{\otimes} |\text{det}|^{(n-k)/2}\sigma_i)$, are the same.

So in order to prove the existence of $W_v$ in $W(\pi, V)$ as in the proposition, it suffices to find some Whittaker function for $\text{Ind}_{P'}^G(|\text{det}|^{-k/2}\rho_i \hat{\otimes} |\text{det}|^{(n-k)/2}\sigma_i)$ with the required property. Now this follows from Proposition 14.1 in [13], and the first part of the proposition is proved.

To be more precise, we give the details of the construction in [13].

Now let $\Omega$ be the subset of $G$ consisting of matrices like

$$\begin{pmatrix} g_{n-k} & 0 \\ X & g_k \end{pmatrix} \begin{pmatrix} I_{n-k} & Z \\ 0 & I_k \end{pmatrix}$$

Fix nontrivial continuous Whittaker functionals $\lambda_1$ on $\rho_i$ and $\lambda_2$ on $\sigma_i$. Use $v_1 \in \rho_i$ to denote the vector corresponding to $W_i$, and take a vector $v_2 \in \sigma_i$ such that $\lambda_2(v_2) = 1$. 

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Then the element \( f \in \text{Ind}^{G_n}_{\Pi^G}(|\det|^{-k/2} \rho_i \hat{\otimes} |\det|^{(n-k)/2} \tau_i) \) constructed in [13] has the following form: if \( g \) is not in \( \Omega \), \( f(g) = 0 \), and if 
\[
\begin{pmatrix}
g_1 & 0 \\
X & g_2
\end{pmatrix}
\begin{pmatrix}
I_{n-k} & Z \\
0 & I_k
\end{pmatrix}
\in \Omega,
\]
then
\[
f(g) = \Psi(Z) |\det g_1|^{-k/2} |\det g_2|^{(n-k)/2} (|\det|^{-k/2} \rho_i)(g_1).v_1 \otimes (|\det|^{(n-k)/2} \tau_i)(g_2).v_2
\]
where \( \Psi(Z) \) is a Schwartz function on \( Z \in \mathbb{R}^{(n-k)\times k} \) such that
\[
\Phi(\epsilon_{n-k} g) = \int \Psi(Z) \psi(-\text{Tr}(\delta g Y)) dY
\]
and here \( \delta \) is the matrix with \( k \) rows and \( n - k \) columns whose last row is \((0, ..., 0, 1)\) and all other rows are zero.

Now we can check the Whittaker function \( W_f \) corresponding to \( f \) satisfies
\[
W_f \left( \begin{array}{c} g \\ 1 \end{array} \right) = \int \lambda_1 \otimes \lambda_2 \left( f \left( \begin{array}{cc} I_{n-k} & Z \\ I_k & \end{array} \right) \right) \bar{\psi} \left( \begin{array}{cc} I_{n-k} & Z \\ I_k & \end{array} \right) dZ
\]
\[
= \int \lambda_1 \otimes \lambda_2 f \left( \begin{array}{c} \left( \begin{array}{c} g \\ I_k \end{array} \right) \left( \begin{array}{cc} I_{n-k} & g^{-1} Z \\ I_k & \end{array} \right) \bar{\psi} \left( \begin{array}{cc} I_{n-k} & Z \\ I_k & \end{array} \right) \\ dZ
\]
\[
= \int \lambda_1 \otimes \lambda_2 (\Psi(g^{-1} Z) |\det g|^{-k} \rho_i(g_1) \otimes v_2) \bar{\psi} \left( \begin{array}{c} I_{n-k} \\ I_k \end{array} \right) dZ
\]
\[
= W_{v_1}(g) \lambda_2(v_2) \int \Psi(Z) \bar{\psi} \left( \begin{array}{c} I_g Z \\ 1 \end{array} \right) dZ = W_{v_1}(g) \Phi(\epsilon_{n-1} g)
\]

Now as we discussed before, this \( W_f(g) \) represents a Whittaker function \( W_{v}(g) \in \mathcal{W}(\pi, V) \). This finishes the proposition.

\[\square\]

Now recall that the derivative \( \pi^{(k)} \) is isomorphic to \( |\det|^{-k/2} \bigoplus_i \rho_i \), thus we have the following corollary.
Corollary 3.3.3. For every Whittaker function $W_i$ in any irreducible component of $\pi^{(k)}$, and any Schwartz function $\Phi$ on $\mathbb{R}^{n-k}$, we can always find some $W_v \in W(\pi, \psi)$, such that

$$W_v \left( \begin{array}{c} g \\ I_k \end{array} \right) = W_i(g)\Phi(\epsilon_{n-k}g)|detg|^{k/2}$$

Let $\omega_i(x) = sgn(x)^{\varepsilon_i}|x|^{c_i}$ be the central character of $\rho_i$, where $\varepsilon_i = 0$ or $1$ and $c_i$ is a complex number. Now we present the following lemma.

Lemma 3.3.4. $\omega_i$ and $\omega_j$ are distinct if $i \neq j$.

Proof. The proof relies on computation of $n$ homology by results in [8].

Let $P' = M'N'$ be a parabolic subgroup of $G_n$, with $M' = G_1^p \times G_2^q$ with $p+2q = n$. Write $C_2$ for the cyclic group $\{\pm 1\}$, $G_1 \simeq \mathbb{R}_{>0} \times C_2$ and $G_2 \simeq \mathbb{R}_{>0} \times SL_2^\pm$, where $SL_2^\pm$ stands for the subgroup of $G_2$ consisting of matrices with determinant $\pm 1$. So $M' = (\mathbb{R}_{>0})^{p+q} \times C_2^p \times (SL_2^\pm)^q$.

Let $T_m$ be the discrete series of $SL_2^\pm$ with parameter $m \in \mathbb{Z}_{>0}$. We will use notation $(s_1, ..., s_p)$ to denote the character on $(\mathbb{R}_{>0})^p$ sending $(x_1, ..., x_p)$ to $\prod_{i=1}^p x_i^{s_i}$.

And let $\epsilon$ be a character on $C_2$. Then form the tensor product

$$\sigma = (s_1, ..., s_p, 2t_1, ..., 2t_q) \otimes (\epsilon_1 \otimes ... \otimes \epsilon_p \otimes T_{m_1} \otimes ... \otimes T_{m_q})$$

This is a representation on $M'$, and then we get the normalized parabolic induced representation $Ind(\sigma)$. Let $\pi = Ind(\sigma)$ be a generic irreducible Harish-Chandra representation in the general position in the sense that

$$s_i, t_j, s_i - s_j(i \neq j) \notin \mathbb{Z}, t_i - t_j(i \neq j) \notin \frac{1}{2}\mathbb{Z}, s_i - t_j \notin \frac{1}{2}\mathbb{Z}$$

Note that this means that $\pi$ has regular infinitesimal character.

In view of Theorem 4.9 in [H-S], Corollary 3.4 and Theorem 4.2 in [8] are of the following form using the notations in [8].
Corollary 3.4:

\[ H_0(n_m, \pi) = \bigoplus_{\pi(\mu; \nu) \in P_0} \pi(\mu; \nu) \otimes \delta_m^{1/2} \]

Theorem 4.2:

\[ H_0(n_Q, \pi) = \bigoplus_{\pi(\mu; \nu) \in P_0/\sim Q} J_{LQ}(\mu; \nu) \otimes \delta_Q^{1/2} \]

where \( \delta_m \) is the modular character of the Borel subgroup, \( Q \) is a standard parabolic subgroup of \( G_n \) and \( \delta_Q \) its modular character.

Next we are going to show Lemma 3.3.4.

Suppose \( Q \) is the parabolic subgroup of \( G_n \) associated to the partition \((n - 1, 1)\). Let \( N_{n-1,1} \) be the unipotent part of \( Q \), and let \( n_{n-1,1} \) be the complexified Lie algebra of \( N_{n-1,1} \). Next we want to describe the \( n_{n-1,1} \) homology of \( \pi \) according to [8].

First let \( \theta = \epsilon_1 \otimes ... \otimes \epsilon_p \otimes \delta_{p+1} \otimes ... \otimes \delta_{p+2q} \) be a character on \( C_n^2 \) with \( \epsilon_i \) given as above and \( \delta_j \) arbitrary, let

\[ \lambda = (s_1, ..., s_p, t_1 + m_1/2, t_1 - m_1/2, ..., t_q + m_q/2, t_q - m_q/2) \]

be the infinitesimal character of \( \pi \), which also represents a character on \((\mathbb{R}_{>0})^n\). We will use \( \pi(\lambda, \theta) \) to denote the character on \( G_n^m \) given by \( \lambda \otimes \theta \). Let \( W \) be the Weyl group of \( G_n \). This is the permutation group \( S_n \). Use \( \delta(m) \) to denote the parity of \( m \), that is, \( \delta(m) = 0 \) if \( m \) is even, and \( \delta(m) = 1 \) if \( m \) is odd.

Now set

\[ \mathcal{P} := W \{ \pi(\lambda, \theta) : \delta_{p+2j-1}(1)\delta_{p+2j}(-1) = \delta(m_j), j = 1, 2, ..., q \} \]

and

\[ \mathcal{P}_0 = \{ \pi(\mu, \theta) \in \mathcal{P} : t_j + m_j/2 \ precedes \ t_j - m_j/2 \ for \ all \ j \} \]

Now we can describe the components of \( H_0(n_{n-1,1}, \pi) \) according to section 4 of [8]. First let’s recall the equivalence relation defined on \( \mathcal{P}_0 \) as in [8] for the partition \( n = (n - 1, 1) \).
If \( \lambda \) appears in the first component of some element in \( P_\emptyset \), set
\[
I_\lambda = \{ l : \lambda_l = t_j + m_j/2 \text{ or } \lambda_l = t_j - m_j/2 \text{ for some } j \text{ with } \lambda_n \neq t_j - m_j/2 \}
\]
Note that for elements in \( P_\emptyset \), \( f_j + m_j/2 \) must precede \( t_j - m_j/2 \), hence \( \lambda_n \neq t_j + m_j/2 \).

Now let \( \sim \) be the following equivalence relation defined on \( P_\emptyset \) as in [8]. \( \pi(\lambda, \theta) \sim \pi(\lambda', \theta') \) if

1. If \( \lambda = \lambda' \), then \( \theta_l = \theta'_l \) for all \( l \not\in I_l \).
2. If \( \lambda \neq \lambda' \), then there exists transitions \( r_1, r_2, ..., r_r \in S_{n-1} \), permuting the first \( n - 1 \) components of \( \lambda \) and \( \theta \), such that \( r_h...r_1.\lambda = \lambda' \), and \( r_h...r_1.\pi(\lambda, \theta) \sim \pi(\lambda', \theta') \) in the sense of (1), and for all \( i = 0, ..., h - 1, r_{i+1}r_i...r_1.\pi(\lambda, \theta) \in P_\emptyset \).

For a representative \( \pi(\lambda, \theta) \) from each equivalence class of \( \sim \), view this as a character on the \( \mathbb{R}^{n-1} \times \mathbb{R} \), then the induced representation \( \text{Ind}_{G^{n-1} \times G_1}^{G_n G_1} \) has a unique submodule, denoted as \( J(\lambda, \theta) \). By Theorem 4.2 in [8], we have
\[
H_0(n_{n-1,1}, \pi) = \bigoplus_{\pi(\lambda, \theta) \in P_\emptyset / \sim} J(\lambda, \theta) \otimes \delta^{1/2}_{n-1,1}
\]
here \( \delta_{n-1,1} \) is the modular character for the parabolic subgroup associated to the partition \( n = (n - 1, 1) \).

Now we can describe \( H_0(n_{n-1,1}, \pi) \) based on the above. We first determine a set of representatives for the relation \( ' \sim ' \).

Give \( \pi(\lambda, \theta) \), if \( \lambda_n = s_i \) for some \( 1 \leq i \leq p \), then by condition (2), we may permute the first \( n - 1 \) components of \( \lambda \) and \( \theta \) to the form
\[
\lambda_i = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_p, t_1 + m_1/2, t_1 - m_1/2, ..., t_q + m_q/2, t_q - m_q/2, s_i)
\]
and
\[
\theta_i = (\epsilon_1 \otimes ... \otimes \epsilon_{i-1} \otimes \epsilon_{i+1} \otimes ... \otimes \epsilon_p \otimes \delta_{p+1} \otimes .... \otimes \delta_{p+2q} \otimes \epsilon_i)
\]
So in this case a representative is given by
\[
\pi(\lambda, \theta)
\]
with

\[ \lambda_i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p, t_1 + m_1/2, t_1 - m_1/2, \ldots, t_q + m_q/2, t_q - m_q/2, s_i) \]

and

\[ \theta_i = (\epsilon_1 \otimes \ldots \otimes \epsilon_{i-1} \otimes \epsilon_{i+1} \otimes \ldots \otimes \epsilon_p \otimes \delta_{p+1} \otimes \ldots \otimes \delta_{p+2q} \otimes \epsilon_i) \]

Now if \( \lambda_n = t_j - m_j/2 \) for some \( 1 \leq j \leq q \), then first by condition (1), if we modify \( \theta_n \) by multiplying a sign character, then a get an inequivalent element. By condition (2), we can permute the first \( n - 1 \) components of \( \lambda \) and \( \theta \) to the form

\[ \lambda^j = (s_1, \ldots, s_p, t_j + m_j/2, t_1 + m_1/2, t_1 - m_1/2, \ldots, t_q + m_q/2, t_q - m_q/2, t_j - m_j/2) \]

and

\[ \theta^j = (\epsilon_1 \otimes \ldots \otimes \epsilon_p \otimes \delta_{p+2j-1} \otimes \delta_{p+1} \otimes \ldots \otimes \delta_{p+2q} \otimes \delta_{p+2j}) \]

Here the requirement is that \( \delta_{p+2j-1}(-1)\delta_{p+2j}(-1) = \delta(m_j) \).

So we can describe the inducing character on \( G_1^{m-1} \times G_1 \) as follows.

For each \( s_i, 1 \leq i \leq p \), we have the following character on \( G_1^{m-1} \times G_1 \):

\[ \pi(\lambda_i, \theta_i) \otimes \pi(s_i, \epsilon_i) \]

where

\[ \lambda_i = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p, t_1 + m_1/2, t_1 - m_1/2, \ldots, t_q + m_q/2, t_q - m_q/2) \]

and

\[ \theta_i = (\epsilon_1 \otimes \ldots \otimes \epsilon_{i-1} \otimes \epsilon_{i+1} \otimes \ldots \otimes \epsilon_p \otimes \delta_{p+1} \otimes \ldots \otimes \delta_{p+2q}) \]

with \( \delta_{p+2j-1} = \delta_{p+2j} \) is the trivial character if \( m_j \) is even, and \( \delta_{p+2j-1} \) is the sign character while \( \delta_{p+2j} \) is trivial if \( m_j \) is odd.
For each \( t_j - m_j/2, 1 \leq j \leq q \), we have the following character on \( G_1^{n-1} \times G_1 \):

\[
\pi(\lambda^j, \theta^j) \otimes \pi(t_j - m_j/2, \delta_{p+2j})
\]

where

\[
\lambda^j = (s_1, \ldots, s_p, t_j + m_j/2, t_1 + m_1/2, t_1 - m_1/2, \ldots, t_q + m_q/2, t_q - m_q/2)
\]

and

\[
\theta^j = (\epsilon_1 \otimes \cdots \otimes \epsilon_p \otimes \delta_{p+2j-1} \otimes \delta_{p+1} \otimes \cdots \otimes \delta_{p+2q})
\]

Here the requirement is that \( \delta_{p+2j-1}(-1)\delta_{p+2j}(-1) = \delta(m_j) \).

For each \( \pi(\lambda_i, \theta_i) \) above, there is a unique irreducible subrepresentation, denoted as \( J_i \), in the representation of \( G_1^{n-1} \) induced from \( G_1^{n-1} \times G_1 \) by \( \pi(\lambda_i, \theta_i) \). Similarly, for each \( \pi(\lambda^j, \theta^j) \), there is a unique irreducible subrepresentation, denoted as \( J^j \), in the representation of \( G_1^{n-1} \times G_1 \) by \( \pi(\lambda^j, \theta^j) \). Then by Theorem 4.2 above, we have

\[
H_0(n_{n-1}, \pi) = \bigoplus_{i,j} (J_i \otimes \pi(s_i, \epsilon_i)) \otimes \delta_1^{1/2} \oplus (J^j \otimes \pi(t_j - m_j/2, \delta_{p+2j})) \otimes \delta_1^{1/2} \\
\ldots \ldots (1)
\]

Now we can check Lemma 3.3.4. First note that for principal series \( Ind_{G_1^{n-1}}^{G_1^{n-1}}(\chi_1 \otimes \cdots \otimes \chi_{n-1}) \), then its central character is given by

\[
\delta_1^{1/2} \cdot \chi_1 \cdot \cdots \cdot \chi_{n-1}
\]

where \( \delta_m \) is the modular character of the standard Borel subgroup in \( G_1^{n-1} \).

So to check the central characters are different, it suffices to check the products of \( \chi_i \) are different. According to decomposition (1), it suffices to consider the characters \( \pi(\lambda_i, \theta_i), 1 \leq i \leq p \) and \( \pi(\lambda^j, \theta^j), 1 \leq j \leq q \).
If central characters corresponding to the inducing characters \( \pi(\lambda_i, \theta_i) \) and \( \pi(\lambda_l, \theta_l) \) are the same, which means that the character sending \( a \in \mathbb{R}^x \) to
\[
(\epsilon_1 \ldots \epsilon_{i-1} \epsilon_{i+1} \ldots \epsilon_p \delta_{p+1} \ldots \delta_{p+2q})(a)|a|^\sum_{\alpha \neq i}s_\alpha + \sum_j 2t_j
\]
is equal to the character sending \( a \in \mathbb{R}^x \) to
\[
(\epsilon_1 \ldots \epsilon_{l-1} \epsilon_{l+1} \ldots \epsilon_p \delta_{p+1} \ldots \delta_{p+2q})(a)|a|^\sum_{\alpha \neq l}s_\alpha + \sum_j 2t_j
\]
Thus we have \( |a|^{s_i} = |a|^{s_l} \) for any \( a \in \mathbb{R}^x \), which forces \( s_l = s_i \), and this is a contradiction to the condition \( \pi \) being in general position.

If central characters corresponding to the inducing characters \( \pi(\lambda^j, \theta^j) \) and \( \pi(\lambda^l, \theta^l) \) are the same, which means that the character sending \( a \) to
\[
(\epsilon_1 \ldots \epsilon_p \delta_{p+1} \ldots \delta_{p+2q})(a)|a|^\sum_i s_i + t_j + m_j/2 + \sum_\alpha \neq j 2t_\alpha
\]
is equal to the character sending \( a \) to
\[
(\epsilon_1 \ldots \epsilon_p \delta_{p+1} \ldots \delta_{p+2q})(a)|a|^\sum_i s_i + t_l + m_l/2 + \sum_\alpha \neq l 2t_\alpha
\]
It follows that \( \delta_{p+2j} = \delta_{p+2l} \) and \( |a|^{t_j - m_j/2} = |a|^{t_l - m_l/2} \). Hence \( t_j - m_j/2 = t_l - m_l/2 \).
So have \( t_l - t_j \in \mathbb{Z}/2 \mathbb{Z} \), which is a contradiction.

If central characters corresponding to the inducing characters \( \pi(\lambda_i, \theta_i) \) and \( \pi(\lambda^j, \theta^j) \) are the same, which means that the character sending \( a \) to
\[
(\epsilon_1 \ldots \epsilon_{i-1} \epsilon_{i+1} \ldots \epsilon_p \delta_{p+1} \ldots \delta_{p+2q})(a)|a|^\sum_{\alpha \neq i}s_\alpha + \sum_j 2t_j
\]
is equal to the character sending \( a \) to
\[
(\epsilon_1 \ldots \epsilon_p \delta_{p+1} \ldots \delta_{p+2q})(a)|a|^\sum_i s_i + t_j + m_j/2 + \sum_\alpha \neq j 2t_\alpha
\]
So we have \( |a|^{s_i} = |a|^{t_j - m_j/2} \), which implies \( s_i = t_j - m_j/2 \), and we have a contradiction again.
Hence we have checked Lemma 3.3.4.

This implies in the decomposition of \( n \) homology

\[
V/nV \cong \bigoplus_i A_i
\]

\( A_i \cong A_j \) only when \( i = j \). This is too special when the representation has regular infinitesimal character and \( n \) is associated to \((n-1,1)\). But in general, we don’t have this multiplicity one result.

Finally, we will record the following result, though we don’t need it in later chapters.

**Proposition 3.3.5.** Let \((\pi, V)\) be an irreducible general HC representation. For any \(1 \leq k \leq n - 1\), if \(\sigma\) is an equivalence class of some direct summand of \(\pi^{(k)}\), then it also occurs as a direct summand in \(\pi^{(k-1)(1)}\) and vice versa.

**Proof.**

- Let \(n_1\) be the Lie subalgebra consisting of matrices like

\[
\begin{pmatrix}
0_{(n-k)\times(n-k)} & x_{(n-k)\times1} & 0_{(n-k)\times(k-1)} \\
0_{1\times(n-k)} & 0_{1\times1} & 0_{1\times(k-1)} \\
0_{(k-1)\times(n-k)} & 0_{(k-1)\times1} & 0_{(k-1)\times(k-1)}
\end{pmatrix}
\]

where the subscript in \(x_{m\times n}\) indicates the size of the matrix.

Let \(n_2\) be the Lie subalgebra consisting of matrices like

\[
\begin{pmatrix}
0_{(n-k)\times(n-k)} & 0_{(n-k)\times1} & 0_{(n-k)\times(k-1)} \\
0_{1\times(n-k)} & 0_{1\times1} & z_{1\times(k-1)} \\
0_{(k-1)\times(n-k)} & 0_{(k-1)\times1} & 0_{(k-1)\times(k-1)}
\end{pmatrix}
\]
Let $n_0$ be the Lie subalgebra consisting of matrices like
\[
\begin{pmatrix}
0_{(n-k)\times(n-k)} & 0_{(n-k)\times1} & y_{(n-k)\times(k-1)} \\
0_{1\times(n-k)} & 0_{1\times1} & 0_{1\times(k-1)} \\
0_{(k-1)\times(n-k)} & 0_{(k-1)\times1} & 0_{(k-1)\times(k-1)}
\end{pmatrix}
\]

Now let’s study $\pi^{(k)}$ and $\pi^{(k-1)(1)}$.

For $\pi^{(k)}$, we first take the quotient $V/(n_1 + n_0)V$. By Proposition 2.1.4, this is the smooth completion of $V_K/(n_1 + n_0)V_K$. By Theorem 3.8.3 in [18], $V_K/n_mV_K$ is nonzero. Since $n_1 + n_0$ is a Lie subalgebra of $n_m$, it follows that $V_K/(n_1 + n_0)V_K$ is also nonzero. Consequently, $V/(n_1 + n_0)V$ is nonzero.

By Lemma 12.2.4 in [18], $V_K/(n_1 + n_0)V_K$ has a direct sum decomposition, so is $V/(n_1 + n_0)V$, and we may write
\[
V/(n_1 + n_0)V = \bigoplus B_i \hat{\otimes} C_i
\]
where each $B_i$ is an irreducible general HC representation on $G_{n-k}$ and $C_i$ is a such representation on $G_k$.

Since each $C_i$ is generic, then take the Whittaker vector in each $C_i$, tensor with $B_i$, and then sum up, then we get $\pi^{(k)}$. Thus $\pi^{(k)}$ can be identified with $\bigoplus i B_i$.

- For $\pi^{(k-1)(1)}$, we first take the quotient $V/(n_2 + n_0)V$. By Proposition 2.1.4, this is the smooth completion of $V_K/(n_2 + n_0)V_K$. By Theorem 3.8.3 in [18], $V_K/n_mV_K$ is nonzero. Since $n_2 + n_0$ is a Lie subalgebra of $n_m$, it follows that $V_K/(n_2 + n_0)V_K$ is also nonzero. Consequently, $V/(n_2 + n_0)V$ is nonzero.

By Lemma 12.2.4 in [18], $V_K/(n_2 + n_0)V_K$ has a direct sum decomposition, so is $V/(n_2 + n_0)V$, and we may write
\[
V/(n_2 + n_0)V = \bigoplus E_j \hat{\otimes} F_j
\]
where each $E_j$ is an irreducible general HC representation on $G_{n-k+1}$ and $F_j$ is a such representation of $G_{k-1}$. As above, we can identify $\pi^{(k-1)}$ as $\bigoplus_j E_j$.

To get $\pi^{(k-1)(1)}$, for each $E_j$, we form the quotient $E_j/\overline{n_1 E_j}$, As in the above argument, first by Proposition 2.1.4, this is the completion of $(E_j)_{K_{n-k+1}}/n_1(E_j)_{K_{n-k+1}}$. By Theorem 3.8.3 in [18], $(E_j)_{K_{n-k+1}}/n_1(E_j)_{K_{n-k+1}}$ is nonzero, so is $E_j/\overline{n_1 E_j}$.

By Lemma 12.2.4 in [18], $(E_j)_{K_{n-k+1}}/n_1(E_j)_{K_{n-k+1}}$ has a direct sum decomposition, so is $E_j/\overline{n_1 E_j}$, and we may write

$$E_j/\overline{n_1 E_j} = \bigoplus_l H_{jl} \otimes D_{jl}$$

where each $H_{jl}$ is an irreducible general HC representation on $G_{n-k}$ and each $D_{jl}$ is a character on $G_1$. As above, $\pi^{(k-1)(1)}$ is isomorphic to $\bigoplus_{j,l} H_{jl}$.

In summary, $\pi^{(k-1)(1)}$ can be identified as follows: we first take the quotient

$$V/(n_2+n_0) = \bigoplus_j E_j \otimes F_j$$

then on each $E_j$, we have

$$E_j/\overline{n_1 E_j} = \bigoplus_l H_{jl} \otimes D_{jl}$$

and $\pi^{(k-1)(1)}$ is isomorphic to $\bigoplus_{j,l} H_{jl}$.

We can now think about this from another point of view. Consider the quotient $V/(n_1+n_2+n_0)$. Again first by Proposition 2.1.4, this is the smooth completion of $V_K/(n_1+n_2+n_0)V_K$. By Lemma 3.8.3 in [18], $V_K/(n_1+n_2+n_0)V_K$ is nonzero, thus $V/(n_1+n_2+n_0)V$ is also nonzero.

Now by Lemma 12.2.4 in [18], $V_K/(n_1+n_2+n_0)V_K$ has a direct sum decomposition, so is $V/(n_1+n_2+n_0)V$ and we may write

$$V/(n_1+n_2+n_0)V = \bigoplus H \otimes D \otimes F$$

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where each $H$ is an irreducible general HC representation on $G_{n-k}$, each $F$ is a such representation on $G_{k-1}$, and each $D$ is a character on $G_1$. We then take the Whittaker vector on each $F$, tensor with $D$ and $H$, then sum up. And $\pi^{(k-1)(1)}$ is isomorphic to $\bigoplus H$.

• Now we can get the relation between $\pi^{(k)}$ and $\pi^{(k-1)(1)}$.

We first take the quotient $V/(n_1 + n_0)\overline{V}$ and it has a decomposition

$$V/(n_1 + n_0)\overline{V} = \bigoplus_i B_i \hat{\otimes} C_i$$

as before. Then the differences between $\pi^{(k)}$ and $\pi^{(k-1)(1)}$ becomes:

For $\pi^{(k)}$, we simply take the Whittaker vector on each $C_i$, so $\pi^{(k)}$ is the sum of $B_i$.

For $\pi^{(k-1)(1)}$, we first form the quotient $C_i/\overline{n_2C_i}$. This is never zero and has a decomposition

$$C_i/\overline{n_2C_i} = \bigoplus D \hat{\otimes} F$$

where each $D$ is a character on $G_1$ and each $F$ is an irreducible general HC representation on $G_{k-1}$. Then we take the Whittaker vector on each $F$, tensor with $D$ and $B_i$, then sum up. And $\pi^{(k-1)(1)}$ is the sum of all possible copies of $B_i$.

Then the conclusion of the proposition follows immediately.
CHAPTER 4
RANKIN-SELBERG INTEGRALS

In this section, we are going to study Rankin-Selberg integrals.

Let \((\pi, V)\) and \((\pi', V')\) be generic irreducible smooth admissible representations of moderate growth on \(G_n\) and \(G_m\) with regular infinitesimal characters, respectively. As before \(\psi\) is the nontrivial additive character of \(\mathbb{R}\) given by \(\psi(x) = \exp(2\pi \sqrt{-1}x)\). We use \(\mathcal{W}(\pi, \psi)\) and \(\mathcal{W}(\pi', \psi^{-1})\) to denote the corresponding Whittaker models, and use \(V_{K_n}\) and \(V'_{K_m}\) to denote the underlying \(K\)-finite vectors. These are irreducible Harish-Chandra modules.

4.1 Exceptional Poles

In this section, we will introduce another important notion of this paper: exceptional poles.

First of all, note that the Schwartz function space \(\mathcal{S}_n = \mathcal{S}_n(\mathbb{R}^n)\) has a natural filtration as follows. Let

\[
\mathcal{S}_n^m = \{f \in \mathcal{S} : f \text{ vanishes at least of order } m \text{ at zero}\}
\]

then each \(\mathcal{S}_n^m\) is a closed subspace and we have a filtration

\[
\mathcal{S}_n = \mathcal{S}_n^0 \supset \mathcal{S}_n^1 \supset ... \supset \mathcal{S}_n^m \supset ...
\]
By Theorem 38.1 in [17], $S_n^m/S_n^{m+1}$ is isomorphic to the space of homogeneous polynomials on $\mathbb{R}^n$ of degree $m$. We will use $E_n^m$ to denote this space. The group $G_n$ acts on $S_n$ via $(g, \Phi)(x) = \Phi(xg)$, where $x \in \mathbb{R}^n, g \in G_n$ and $xg$ denotes the matrix multiplication from the right. Then this $G_n$ action preserves the above filtration and therefore induces an action on $E_n^m$.

On $G_n \times G_n$, the Rankin-Selberg integrals for $\pi$ and $\pi'$ are defined in section 2.3, that is, for $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \psi^{-1}), \Phi \in \mathcal{S}$,

$$I(s, W, W', \Phi) = \int_{N_n \backslash GL_n} W(g)W'(g)\Phi(\epsilon_n g)|\det(g)|^s dg$$

where $\epsilon_n = (0, 0, \ldots, 1) \in \mathbb{R}^n$, $s \in \mathbb{C}$ a complex number, $N_n$ stands for the standard maximal unipotent subgroup in $G_n$. By Theorem 2.3.1., these integrals converge when $s$ is in some right half plane, and have a meromorphic continuation to the whole complex plane.

For any integer $1 \leq k \leq n$, for $v \in \pi$, $v' \in \pi'$ and $\Phi \in \mathcal{S}_k$, we define the following family of integrals

$$I_k(s, W_v, W_{v'}, \Phi) = \int_{N_k \backslash G_k} W_v \begin{pmatrix} g & \epsilon_k g \\ I_{n-k} & g \end{pmatrix} W_{v'} \begin{pmatrix} g & \epsilon_k g \\ I_{n-k} & g \end{pmatrix} \Phi(\epsilon_k g)|\det g|^{s-n+k} dg$$

- When $k = n$, $I_n$ is the usual Rankin-Selberg integrals for $\pi$ and $\pi'$.

**Lemma 4.1.1.** The integrals $I_k$ belong to the space of Rankin-Selberg integrals for $\pi$ and $\pi'$.

**Proof.** First by Proposition 6.1 in [13], there exists a function $W_{v_0} \in \mathcal{W}(\pi, \psi)$ such that

$$W_{v_0} \begin{pmatrix} g & \epsilon_k g \\ I_{n-k} & g \end{pmatrix} = W_v \begin{pmatrix} g & \epsilon_k g \\ I_{n-k} & g \end{pmatrix} \Phi(\epsilon_k g)$$
Thus the integral $I_k$ now becomes the integral

$$\int_{N_k \setminus G_k} W_{v_0} \begin{pmatrix} g & \vdots \\ I_{n-k} & \vdots \end{pmatrix} W_{v'} \begin{pmatrix} g & \vdots \\ I_{n-k} & \vdots \end{pmatrix} |detg|^{s-n+k} dg$$

Now the lemma follows from Lemma 14.1 in [13].

Thus it follows that $I_k$ converges when $Re(s)$ is large and has a meromorphic continuation to the whole complex plane of $s$.

Suppose for some choices of $W, W'$ and $\Phi$, $s = s_0$ is a pole of order $d$ for the integral $I_k(s, W, W', \Phi)$, and we have an expansion

$$I_k(s, W, W', \Phi) = \frac{B_{s_0,k}(W, W', \Phi)}{(s - s_0)^d} + ...$$

where $B_{s_0,k}(W, W', \Phi)$ is a trilinear form on $V \times V' \times S_k$ satisfying the following invariance property:

$$B_{s_0,k}(gW, gW', g\Phi) = |detg|^{-s_0+n-k} B_{s_0,k}(W, W', \Phi)$$

for any $g \in G_k$, $W \in W(\pi, \psi)$, $W' \in W(\pi', \psi^{-1})$, $\Phi \in S_k$. Now we need to make the following assumption.

**Assumption 4.1.2.** The trilinear form $B_{s_0,k}$ is continuous with respect to the topologies involved.

- When $k = n$, this follows from a result of Cogdell and I.I.Piatetski-Shapiro in [9]. But for general $k$, we haven’t proved this.

**Definition 4.1.1.** We say a pole $s = s_0$ is an exceptional pole of type 1, with level $m$ and depth $n - k$, if the corresponding $B_{s_0,k}$ is zero on $S_{k}^{m+1}$, but not identically zero on $S_k^{m}$. In this case, we also say $s_0$ is an exceptional pole for the integrals $I_k(s, W_v, W'_{v'}, \Phi)$. 

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If $s_0$ is an exceptional pole of order $m$, then $B_{s_0}$ defines a continuous linear form on $V \times V' \times E_{k}^m$ such that for any $g \in G_k$

$$B_{s_0,k}(g.W, g.W', g.\Phi) = |\det g|^{-s_0 + n - k}B_{s_0,k}(W, W', \Phi)$$

We also introduce the following concepts.

**Definition 4.1.2.** We say a complex number $s = s_0$ is an exceptional pole of type 2, with level $m$, for $\pi$ and $\pi'$, if there exists a continuous trilinear form $l : V \times V' \times E_{n}^m \to \mathbb{C}$ such that for $g \in G_n$

$$l(g.W, g.W', g.\Phi) = |\det g|^{-s_0}l(W, W', \Phi)$$

**Lemma 4.1.3.** If $s = s_0$ is an exceptional pole of type 1, with level $m$ and depth 0, then $s_0$ is also an exceptional pole of type 2, with level $m$ for $\pi$ and $\pi'$.

**Proof.** The trilinear form $B_{s_0,n}$ gives the required linear form $l$ in **Definition 4.1.2**.

• It is not known if the converse is also true.

Next we want to relate the exceptional poles for the integrals $I_k$ to the exceptional poles of type 2 for the components of $\pi^{(n-k)}$ and $\pi'^{(n-k)}$.

Let $P_{k,n-k}$ be the standard parabolic subgroup of $G_n$ associated to the partition $(k, n-k)$. We will drop the subscripts in this section, and just write $P$ for $P_{k,n-k}$.

Let $P = MN$ be the Levi decomposition with $M \cong G_k \times G_{n-k}$. Use $n$ to denote the Lie algebra of $N$. 

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Lemma 4.1.4. If $X = (X_{ij}) \in \mathfrak{n}$, then there exists a linear form $P_X$ on $\mathbb{R}^k$, such that for any $v \in V$, we have

$$W_{\pi(X),v} \left( \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} \right) = P_X(\epsilon_k g) W_v \left( \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} \right)$$

Proof.

$$W_{\pi(X),v} \left( \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} \right) = \frac{d}{dt} \bigg|_{t=0} W_v \left( \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} \right) \begin{pmatrix} tX \\ I_{n-k} \end{pmatrix} = \frac{d}{dt} \bigg|_{t=0} W_v \left( \begin{pmatrix} I_k \\ tX \\ I_{n-k} \end{pmatrix} \right)$$

$$= \frac{d}{dt} \bigg|_{t=0} e^{2\pi \sqrt{-1} t} \sum_{j=1}^{k} g_{kj} X_{j,k+1} W_v \left( \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} \right)$$

$$= 2\pi \sqrt{-1} \sum_{j=1}^{k} g_{kj} X_{j,k+1} W_v \left( \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} \right)$$

So define a linear form $P_X(a_1, \ldots, a_k) = 2\pi \sqrt{-1} \sum_{j=1}^{k} X_{j,k+1}a_j$ on $\mathbb{R}^k$, then $P_X(\epsilon_k g) = 2\pi \sqrt{-1} \sum_{j=1}^{k} g_{kj} X_{j,k+1}$, which proves the lemma.

□

Proposition 4.1.5. Let $s = s_0$ be an exceptional pole of level $m$ for the integrals $I_k$, then the continuous trilinear form $B_{s_0,k}$ defines a continuous trilinear form on $V/\bar{\mathfrak{n}}V \times V'/\bar{\mathfrak{n}}V' \times E^m_k$.

Proof. It suffice to show that the form $B_{s_0,k}$ vanishes on $\bar{\mathfrak{n}}V$ and $\bar{\mathfrak{n}}V'$ when restricted to $S^m_k$.

So for any $W_{\pi(X),v}, X \in \mathfrak{n}$, any $W_v$, and any $\Phi \in S^m_k$, by Lemma 4.1.4, we have

$$W_{\pi(X),v} \left( \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} \right) = P_X(\epsilon_k g) W_v \left( \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} \right)$$
for some linear form $P_X$ on $\mathbb{R}^k$.

It follows that

$$I_k(s, W_{\pi(X).v}, W_{v'}, \Phi) =$$

$$\int W_{\pi(X).v} \begin{pmatrix} g & \Phi(\epsilon_k g)|det g|^{s-k+n} dg 
\end{pmatrix}$$

$$= \int_{N_k \setminus G_k} W \begin{pmatrix} g & \Phi(\epsilon_k g) |det g|^{s-k+n} dg 
\end{pmatrix}$$

where $\Psi_k(\epsilon_k g) = P_X(\epsilon_k g) \Phi(\epsilon_k g)$.

Since $\Phi \in S^m_\mathbf{r}$, thus $\Psi = P_X \Phi \in S^{m+1}_\mathbf{r}$. Note that $s_0$ is an exceptional pole with level $m$, so

$$B_{s_0,k}(W_{\pi(X).v}, W_{v'}, \Phi) = B_{s_0,k}(W_v, W_{v'}, \Psi_k) = 0$$

Similarly, $B_{s_0,k}$ vanishes when $v' \in nV'$. Thus the proposition follows.

Now since both $\pi$ and $\pi'$ are in general positions, we have the following direct sum decompositions as in section 3.3.

$$\frac{V}{nV} = \bigoplus (\rho_i, A_i) \hat{\otimes} (\sigma_i, B_i)$$

and

$$\frac{V'}{nV'} = \bigoplus (\rho'_i, A'_i) \hat{\otimes} (\sigma'_j, B'_j)$$

By Proposition 4.1.5, if $s = s_0$ is an exceptional pole of level $m$ for $I_k$, then $B_{s_0,k}$ defines a nontrivial continuous trilinear form on $\frac{V}{nV} \times \frac{V'}{nV'} \times E^m_k$. Thus it has to be nontrivial on some components

$$B_{s_0,k} : (\rho_i, A_i) \hat{\otimes} (\sigma_i, B_i) \times (\rho'_j, A'_j) \hat{\otimes} (\sigma'_j, B'_j) \times E^m_k \to \mathbb{C}$$

with the invariance property

$$B_{s_0,k}(g.v, g.v', \Phi) = |det g|^{-s_0+n-k} B_{s_0,k}(v, v', \Phi)$$
for all $g \in G_k$, $v \in A_i \hat{\otimes} B_i$, $v' \in A'_i \hat{\otimes} B'_i$ and $\Phi \in E^m_k$.

Since the tensor products $A_i \otimes B_i$ and $A'_i \otimes B'_i$ are dense in $A_i \hat{\otimes} B_i$ and $A'_i \hat{\otimes} B'_i$ respectively, $B_{s_0,k}$ is nontrivial on the subspace $A_i \otimes B_i \times A'_i \otimes B'_i \times E^m_k$.

Now fix $v_2 \in B_i$, $v'_2 \in B'_i$, so that $B_{s_0,k}$ is nontrivial on $A_i \otimes v_2 \times A'_i \otimes v'_2 \times E^m_k$. Then the restriction of $B_{s_0,k}$ to this subspace induces a nontrivial continuous trilinear form, still denoted as $B_{s_0,k}$, on $A_i \times A'_i \times E^m_k$, with

$$B_{s_0,k}(g.v_1, g.v'_1, g.\Phi) = |det g|^{-s_0 + n - k} B_{s_0,k}(v_1, v'_1, \Phi)$$

for any $v_1 \in A_i$, $v'_1 \in A'_i$, $\Phi \in E^m_k$ and $g \in G_k$. Note that $|det|^{(n-k)/2} \rho_i$ is a component for $\pi^{(n-k)}$, thus we have proved the following theorem.

**Theorem 4.1.6.** If $s = s_0$ is an exceptional pole of type 1 with level $m$ and depth $n - k$, then $s_0$ is an exceptional pole of type 2 with level $m$ for some components of $\pi^{(n-k)}$ and $\pi'^{(n-k)}$.

### 4.2 Case 1: $G_n \times G_n$

In the rest of this chapter, we are going to use archimedean derivatives and exceptional poles to analyze the Rankin-Selberg integrals. In this section we will first consider the case $G_n \times G_n$.

Now suppose a pole $s = s_0$ is not an exceptional pole for the integrals $I_n$. We still have the Laurent expansion around $s_0$

$$I_n(s, W, W', \Phi) = \frac{B_{s_0}(W, W', \Phi)}{(s - s_0)^d} + ...$$

and $B_{s_0}$ is continuous on $V \times V' \times E^m_k$ with the invariance property

$$B_{s_0,n}(g.W, g.W', g.\Phi) = |det g|^{-s_0} B_{s_0,n}(W, W', \Phi)$$

Since $s_0$ is not exceptional, now for any integer $m$, we can find some $\Phi \in S^m$, such that the form $B_{s_0,n}(W, W', \Phi)$ is nonzero for some choices $W$ and $W'$. Because
of the continuity of $B_{s_0,n}$, we may further assume $W$ and $W'$ are both $K_n$-finite. By Iwasawa decomposition, we have

$$I_n(s,W,W',\Phi) = \int_{K_n} \int_{N_n \backslash P_n} W(pk)W'(pk)|\text{det}p|^{s-1} \int_{\mathbb{R}^\times} \omega(a)\omega'(a)|a|^{ns} \Phi(\epsilon_n ak) d^x a \, dp \, dk$$

Take $\{W_i\}$ to be some base vectors in the $K$ span subspace of $W$, and we write $W(gk) = \sum_i f_i(k)W_i(g)$, where $f_i$ are continuous functions on $K$. Similarly, write $W'(gk) = \sum_i f'_i(k)W'_i(g)$, where $\{W'_i\}$ are some base vectors of the $K$ span subspace of $W'$, and $f'_i$ are continuous functions on $K$. Now $I(s,W,W',\Phi)$ equals

$$\sum_{i,j} \int_{N_n \backslash P_n} W_i(p)W_j(p)|\text{det}p|^{s-1} \int_{\mathbb{R}^\times} \omega(a)\omega'(a)|a|^{ns} \int_K f_i(k)f'_j(k)\Phi(\epsilon_n ak) dpd^x a \, dk$$

**Lemma 4.2.1.** For any continuous function $f(k)$ on $K$, the function $\Psi(a) = \int_K f(k)\Phi(\epsilon_n ak) dk$ belongs to $S^m(\mathbb{R})$ if $\Phi$ is in $S^m(\mathbb{R}^n)$.

**Proof.** Since we are integrating over a compact group, and the function $f(k)\Phi(\epsilon_n ak)$ is continuous on $k$ and smooth on $a$, we can interchange the differentiation and the integration. More precisely, by Fubini theorem, we have

$$\frac{d}{da} \int_0^a \int_K \frac{\partial}{\partial t} (f(k)\Phi(\epsilon_n tk)) dk \, dt = \frac{d}{da} \int_K \int_0^a \frac{\partial}{\partial t} (f(k)\Phi(\epsilon_n tk)) dt \, dk$$

$$= \frac{d}{da} \int_K (f(k)\Phi(\epsilon_n ak) - f(k)\Phi(0)) \, dk$$

$$= \frac{d}{da} \int_K f(k)\Phi(\epsilon_n ak) \, dk = \frac{d}{da} \Psi(a)$$

So $\Psi(a)$ is smooth.

To show $\Psi(a)$ is a Schwartz function, for any nonnegative integers $p$ and $q$, we have

$$|a^p \frac{d^q}{da^q}(\Psi(a))| \leq \int_K |f(k)||a^p \frac{d^q}{da^q}(\Phi(\epsilon_n ak))| \, dk$$

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\[ \leq \int_{K} |f(k)||\alpha_k|^{p/2}\left(\sum_{\alpha} |D^\alpha\Phi(\epsilon_\alpha a_k)|\right) \]

where the sum is over all \( \alpha = (n_1, \ldots, n_r) \) with \( n_i \geq 0 \) and \( \sum n_i = q \).

Now since \( \Phi \) is a Schwartz function on \( \mathbb{R}^n \), put \( z_i = ak_{ni} \), then

\[ |(ak_{n_1}, ak_{n_2}, \ldots, ak_{nn})|^{p/2}\left(\sum_{\alpha} |D^\alpha\Phi(\epsilon_\alpha a_k)|\right) = (\sum_{i} |z_i|^2)^{p/2}\left(\sum_{\alpha} |D^\alpha\Phi(z_1, \ldots, z_n)|\right) \]

\[ < C \]

for some constant \( C \) and for all \( a \).

Thus

\[ |a^p d^p/(da^q)(\Psi(a))| \leq C \int_{K} |f(k)|dk \]

which shows that \( \Psi(a) \) is a Schwartz function.

Since \( \Phi \) vanishes at 0 at least of order \( m \), by Theorem 38.1 in [17] again, this means that there exists a homogeneous polynomial \( P(x_1, \ldots, x_n) \) of degree \( m \), such that the Taylor expansion of \( \Phi \) at 0 has the form

\[ \Phi(x_1, \ldots, x_n) = P(x_1, \ldots, x_n) + \ldots \]

Then

\[ \Psi(a) = \int_{K} f(k)\Phi(\epsilon_\alpha a_k)dk = \int_{K} f(k)P(\epsilon_\alpha a_k)dk + \ldots \]

\[ = a^m \int_{K} f(k)P(\epsilon_\alpha k)dk + \ldots \]

This shows that \( \Psi(a) \) vanishes at least of order \( m \) at zero, which finishes the proof.

\[ \square \]

**Lemma 4.2.2.** If \( \Phi \in \mathcal{S}^m(\mathbb{R}) \) for some \( m > 0 \), then as a function of \( s \in \mathbb{C} \), the function

\[ \int_{0}^{\infty} a^s \Phi(a) d^x a \]

is holomorphic in the half plane \( Re(s) > -m \).
Proof. Since $\Phi$ is a Schwartz function, the integral
\[ \int_\epsilon^\infty a^s \Phi(a) d^\times a \]
is holomorphic in $s$, when $\epsilon$ is away from 0.

In a neighborhood of 0, when $Re(s) > -m$, and $\Phi \in S^m(\mathbb{R})$, the function $a^s \Phi(a)$ is continuous in this neighborhood of 0, thus the integral
\[ \int_0^\epsilon a^s \Phi(a) d^\times a \]
is also holomorphic in $s$.

Hence the lemma is proved.

□

Now let's continue our analysis of $I_n(s, W, W', \Phi)$. By Lemma 4.2.1, as a function of $a$, the integral over $K$
\[ \int_K f_i(k) f_j'(k) \Phi(\epsilon a k) dk \]
belongs to $S^m_n(\mathbb{R})$, and by Lemma 4.2.2, when we choose $m$ large enough, the function
\[ \int_{\mathbb{R}^n} \omega'(a) |a|^n s \int_K f_i(k) f'_j(k) \Phi(\epsilon a k) d^\times a dk \]
is homomorphic in the half plane containing $s = s_0$. Hence the pole $s_0$ has to occur in the sum
\[ \sum_{i,j} \int_{N_n \setminus P_n} W_i(p) W_j'(p) |det p|^{s-1} dp \]
and we may assume one of the terms
\[ \int_{N_n \setminus P_n} W_i(p) W_j'(p) |det p|^{s-1} dp \]
contains the pole $s_0$. But this integral descends to the integral
\[ \int_{N_{n-1} \setminus GL_{n-1}} W_i \left[ \begin{array}{cc} g & 0 \\ 0 & 1 \end{array} \right] W_j' \left[ \begin{array}{cc} g & 0 \\ 0 & 1 \end{array} \right] |det g|^{s-1} dg \]
on $N_{n-1} \setminus GL_{n-1}$ since any $p \in P_n$ can be written as
\[
\begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.
\]

Combining all the above, we have the following result.

**Proposition 4.2.3.** If a pole $s_0$ of order $d$ is not exceptional for $I_n$, then it occurs as a pole of order $d$ for the family of integrals
\[
\int_{N_{n-1} \setminus GL_{n-1}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W' \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} |detg|^{s-1} dg
\]
with $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$.

By Lemma 10.4 in [J-S1], each $W_v \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ can be written as a finite sum
\[
\sum_i W_i \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \Phi_i(\epsilon_{n-1}g)
\]
for some functions $W_i \in \mathcal{W}(\pi, \psi)$ and Schwartz functions $\Phi_i$ on $\mathbb{R}^{n-1}$. Thus the integral
\[
\int_{N_{n-1} \setminus GL_{n-1}} W \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W' \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} |detg|^{s-1} dg
\]
becomes
\[
\sum_i \int_{N_{n-1} \setminus GL_{n-1}} W_i \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W' \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \Phi_i(\epsilon_{n-1}g)|detg|^{s-1} dg
\]
which are integrals belonging to $I_{n-1}$. So we have the following corollary.

**Corollary 4.2.4.** If a pole $s_0$ of $I_n$ of order $d$ is not exceptional of type 1, then it occurs as a pole of order $d$ for the integrals $I_{n-1}$.

Now in general, we have the following result.
**Proposition 4.2.5.** If a pole $s_0$ of $I_k$ is not an exceptional pole for the integrals $I_k$, then it is a pole for $I_{k-1}$.

**Proof.** By Proposition 2 in [14], there exists a finite set of functions $\{\xi\}$ on $(\mathbb{R}^\times)^k$, which have the form $\xi(z_1, ..., z_k) = \prod_{j=1}^{k} \chi_j(z_j)(\log|z_j|)^{n_j}$, where $\chi_j$ is a character on $\mathbb{R}^\times$, and Schwartz functions $\phi_\xi$ on $\mathbb{R}^k \times O(n)$, such that

$$W_v(\alpha x) = \sum_{\xi} \xi(a_1, ..., a_k)\phi_\xi(a_1, ..., a_k, x)$$

where $x \in O(n)$, and

$$\alpha = \text{diag}(a_1a_k, a_2a_k, ..., a_{k-1}a_k, a_k)$$

which will be viewed as

$$\text{diag}(a_1a_k, a_2a_k, ..., a_{k-1}a_k, a_k, 1, ..., 1) \in G_n$$

Since $\phi_\xi$ is a Schwartz function, for each $x$, it has a Taylor expansion around 0,

$$\phi_\xi(a_1, ..., a_k, x) = f(x)P_\xi(a_1, ..., a_k) + ...$$

where $f(x)$ is some continuous function of $x$, and $P_\xi$ denotes the sum of leading coefficients in the Taylor expansion, which is a polynomial in $a_1, ..., a_k$.

It follows that around 0, we can write

$$W_v(\alpha x) = \sum_{\xi} \{f(x)\xi(a_1, ..., a_k)P_\xi(a_1, ..., a_k) + ...\} \cdots \cdots (13)$$

Similarly, around 0, we have

$$W'_v(\alpha x) = \sum_{\xi'} \{f'(x)\xi'(a_1, ..., a_k)P_{\xi'}(a_1, ..., a_k) + ...\} \cdots \cdots (14)$$

By Iwasawa decomposition, we have

$$I_k = 53$$
\[
\int W_v \begin{pmatrix} pax & 0 \\ 0 & I_{n-k} \end{pmatrix} W_v' \begin{pmatrix} pax & 0 \\ 0 & I_{n-k} \end{pmatrix} \Phi(\epsilon_k ax)|\text{det}p|^{s-n+k-1}|a|^{k(s-n+k)}dpdx d^\times a
\]

with \( p \in N_k \setminus P_k \), where \( P_k \) is the mirabolic subgroup in \( G_k \), \( x \in O(k) \), \( a \in \mathbb{R}^\times \).

Note that \( N_k \setminus P_k = N_{k-1} \setminus G_{k-1} \), so we can write \( pax = n_{k-1} a y x \) for some \( n_{k-1} \in N_{k-1} \),

\[ \alpha = \text{diag}(a_1 \ldots a_{k-1} a, a, 1, \ldots, 1) \]

and \( y \in O(k-1) \).

Thus by (13), around 0 we have

\[
W_v(pax) = \psi(n_k) \sum_\xi \{ f(yx) \xi(a_1, \ldots, a_{k-1}, a) P_\xi(a_1, \ldots, a_{k-1}, a) + \ldots \}
\]

and

\[
W_v'(pax) = \psi^{-1}(n_k) \sum_{\xi'} \{ f'(yx) \xi'(a_1, \ldots, a_{k-1}, a) P_{\xi'}(a_1, \ldots, a_{k-1}, a) + \ldots \}
\]

Note that the poles of \( I_k \) is caused by the integration around 0, and in a neighborhood of 0, the integral is

\[
\sum_{\xi, \xi'} \int f(yx) f'(yx) dy dx
\]

\[
\int (\xi P_\xi P_{\xi'})(a_1, \ldots, a_{k-1}, a) \Phi(\epsilon_k ax)|a|^{k(s-n+k)}|a_1|^{c_1} \ldots |a_{k-1}|^{c_{k-1}} d^\times a_1 \ldots d^\times a_{k-1} + \ldots
\]

where \( c_1, \ldots, c_{k-1} \) are some complex numbers depending on \( s \).

First since \( s_0 \) is a pole for this integral, and \( O(k), O(k-1) \) are compact, it follows that this pole occurs as a pole for the integral with respect to the variables \( a_1, \ldots, a_{k-1}, a \), and the integration with respect to \( x, y \) are nonzero.

Since \( s_0 \) is not an exceptional pole, we can choose the Schwartz function \( \Phi \) so that the integral on \( a \) in the above expression is holomorphic in a region containing \( s_0 \), by Lemma 4.2.2.
Thus the pole is caused by the integration with respect to the variables $a_1, ..., a_{k-1}$. This implies that the integral

$$\int W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k+1} \end{pmatrix} W'_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k+1} \end{pmatrix} |\text{det}g|^{s-n+k-1} dg$$

has the pole $s = s_0$. This integral belongs to the integrals $I_{k-1}$, the proposition follows.

□

**Corollary 4.2.6.** Any pole of the Rankin-Selberg integrals $I_n$ for $\pi$ and $\pi'$ is an exceptional pole of type 2 for some components of $\pi^{(k)}$ and $\pi'^{(k)}$, $0 \leq k < n$.

**Proof.** This follows from the above proposition and Theorem 4.1.6.

□

For the other direction we first need the following result. Suppose $\sigma$ and $\sigma'$ are a pair of components of $\pi^{(k)}$ and $\pi'^{(k)}$ respectively.

**Proposition 4.2.7.** The Rankin-Selberg integrals of $\sigma$ and $\sigma'$ belong to the Rankin-Selberg integrals of $\pi$ and $\pi'$.

**Proof.** For any $W_{v_1} \in \mathcal{W}(\sigma, \psi)$, $W'_{v_1} \in \mathcal{W}(\sigma', \psi^{-1})$, and $\Phi \in \mathcal{S}_{n-k}$, we have the Rankin-Selberg integral for $\sigma$ and $\sigma'$

$$I(s, W_{v_1}, W'_{v_1}, \Phi) = \int_{N_{n-k}\setminus G_{n-k}} W_{v_1}(g)W'_{v_1}(g)\Phi(\epsilon_{n-k}g)|\text{det}g|^{s} dg$$

By Corollary 3.3.3., there exists some $W_v(g) \in \mathcal{W}(\pi, \psi)$, such that

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = W_{v_1}(g)\Phi(\epsilon_{n-k}g)|\text{det}g|^{k/2}$$

Thus the above integral is

$$I(s, W_{v_1}, W'_{v_1}, \Phi) = \int_{N_{n-k}\setminus G_{n-k}} W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} W'_{v_1}(g)|\text{det}g|^{s-k/2} dg$$
By Proposition 6.1 in [13], we can write

\[ W_v \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = \sum_j W_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} \Phi_j(\epsilon_{n-k}g) \]

with \( W_j \in \mathcal{W}(\pi, \psi) \), and Schwartz functions \( \Phi_j \) on \( \mathbb{R}^{n-k} \). So we have

\[ I(s, W_v, W_{v'}, \Phi) = \sum_j \int W_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} W_{v'}(g) \Phi_j(\epsilon_{n-k}g) |\text{det}g|^s |\text{det}g|^{-k/2} dg \]

Using Corollary 3.3.3. again, we can find \( W'_{j}(g) \in \mathcal{W}(\pi', \psi^{-1}) \), such that for each \( j \), we have

\[ W'_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} = W_{v'}(g) \Phi_j(\epsilon_{n-k}g) |\text{det}g|^{k/2} \]

Hence

\[ I(s, W_v, W_{v'}, \Phi) = \sum_j \int W_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} W'_j \begin{pmatrix} g & 0 \\ 0 & I_k \end{pmatrix} |\text{det}g|^s |\text{det}g|^{-k} dg \]

Then by Lemma 14.1 in [13], the integrals on the right side belong to the Rankin-Selberg integrals for \( \pi \) and \( \pi' \). Thus the proposition follows.

\[ \square \]

**Corollary 4.2.8** Any exceptional pole of type 1 of depth 0 for Rankin-Selberg integrals of \( \sigma \) and \( \sigma' \) is a pole of the Rankin-Selberg integrals \( I_n \) for \( \pi \) and \( \pi' \).

**Proof.** This follows the above proposition.

\[ \square \]

Let us summarize the above results as follows.

**Theorem 4.2.9.** Let \( \pi \) and \( \pi' \) be irreducible generic Harish-Chandra representations of \( G_n \) in general position. Then any pole of the Rankin-Selberg integrals for \( \pi \) and \( \pi' \) is an exceptional pole of type 2 for a pair of components of \( \pi^{(k)} \) and \( \pi'^{(k)} \), \( 0 \leq k \leq n - 1 \). On the other hand, any exceptional pole of type 1 of depth 0 for a
pair of components of \( \pi^{(k)} \) and \( \pi'^{(k)} \), \( 0 \leq k \leq n - 1 \), is a pole of the Rankin-Selberg integrals of \( \pi \) and \( \pi' \).

**Remark:**
- It is expected that an exceptional pole of type 2 for \( \pi \) and \( \pi' \) is also a pole for the Rankin-Selberg integrals of \( \pi \) and \( \pi' \). Moreover we expect exceptional poles of type 2 and type 1 are in fact equivalent, at least when both representations have regular infinitesimal characters. We have obtained some progress in this direction, but in general, this problem is still open.

### 4.3 Case 2: \( G_n \times G_m \)

In this section, we consider the case \( G_n \times G_m \) with \( n > m \). The method is the same as the one in previous case.

Assume \( \pi \) and \( \pi' \) are generic irreducible Harish-Chandra representations of \( G_n \) and \( G_m \) in general position. Let \( \mathcal{W}(\pi, \psi) \) and \( \mathcal{W}(\pi', \psi^{-1}) \) be their Whittaker models.

The family of Integrals are given by

\[
I(s, W, W') = \int_{N_m \backslash G_m} W \begin{pmatrix} g & 0 \\ 0 & I_{n-m} \end{pmatrix} W'(g) |\text{det}(g)|^{s - \frac{n-m}{2}} dg
\]

and for \( 1 \leq j \leq n - m - 1 \)

\[
I_j(s, W, W') = \int_{M(m \times j, \mathbb{R})} \int_{N_m \backslash G_m} W \begin{pmatrix} g & 0 & 0 \\ X & I_j & 0 \\ 0 & 0 & I_{n-m-j} \end{pmatrix} W'(g) |\text{det}(g)|^{s - \frac{n-m}{2}} dgdX
\]

with \( W \in \mathcal{W}(\pi, \psi) \) and \( W' \in \mathcal{W}(\pi', \psi^{-1}) \). Since it is known in [13] that \( I(s, W, W') \) has the same poles with the same multiplicities with \( I_j(s, W, W') \) for each \( j \), thus it suffices to consider the poles of \( I(s, W, W') \).
Now introduce the following family of integrals for each \(1 \leq k \leq m\), let \(\Phi\) be a Schwartz function on \(\mathbb{R}^k\), and let

\[
I_k(s, W, W', \Phi) = \int_{N_k \setminus G_k} W \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} W' \begin{pmatrix} g & 0 \\ 0 & I_{m-k} \end{pmatrix} \Phi(\epsilon_k g) |\text{det}(g)|^{s - \frac{n+m}{2} + k} \, dg
\]

By Lemma 10.3 and 10.4 in [J-S1], this family of integrals is the same as integrals

\[
I_k(s, W, W') = \int_{N_k \setminus G_k} W \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} W' \begin{pmatrix} g & 0 \\ 0 & I_{m-k} \end{pmatrix} |\text{det}(g)|^{s - \frac{n+m}{2} + k} \, dg
\]

• When \(k = m\), we get the above integrals \(I(s, W, W')\).

By Lemma 14.1 in [13], the integrals \(I_k\) belong to the family \(I_m\), which implies they are convergent when \(\text{Re}(s)\) is large, and have meromorphic continuation to the whole plane.

Now suppose \(s = s_0\) is a pole for \(I_k(s, W, W', \Phi)\), then we have a Laurent expansion

\[
I_k(s, W, W', \Phi) = \frac{B_{s_0, k}(W, W', \Phi)}{(s - s_0)^d} + \ldots
\]

where \(B_{s_0, k}(W, W', \Phi)\) is a trilinear form on \(V \times V' \times S_k\) satisfying the following invariance property: for any \(g \in G_k\),

\[
B_{s_0, k}(gW, gW', g\Phi) = |\text{det}g|^{-s_0 + (n+m)/2 - k} B_{s_0, k}(W, W', \Phi)
\]

**Assumption 4.3.1.** The trilinear form \(B_{s_0, k}\) is continuous.

• Again, when \(k = n\), the continuity follows from a result of Cogdell and I.I. Piatetski-Shapiro in [9]. But for general \(k\), this is open.

Now we can introduce the following two types of exceptional poles.

**Definition 4.3.1.** We say a pole \(s = s_0\) is an exceptional pole of type 1, with level \(l\) and depth \(m - k\), if the corresponding \(B_{s_0, k}\) is zero on \(S_k^{l+1}\), but not identically
zero on $S^l_k$. In this case, we also say $s_0$ is an exceptional pole for the integrals $I_k(s, W_v, W_{v'}, \Phi)$.

- If $s_0$ is an exceptional pole of order $m$, with level $l$ and depth $m - k$, then $B_{s_0}$ defines a continuous linear form on $V \times V' \times E^l_k$ such that for any $g \in G_k$

\[ B_{s_0,k}(g.W, g.W', g.\Phi) = |\det g|^{-s_0+(n+m)/2-k} B_{s_0,k}(W, W', \Phi) \]

We also introduce the following concepts.

**Definition 4.3.2.** We say a complex number $s = s_0$ is an exceptional pole of type 2, with level $l$, for $\pi$ and $\pi'$, if there exists a continuous trilinear form $l : V \times V' \times E^l_k \to \mathbb{C}$ such that

\[ l(g.W, g.W', g.\Phi) = |\det g|^{-s_0+(n-m)/2} l(W, W', \Phi) \]

**Lemma 4.3.2.** If $s = s_0$ is an exceptional pole of type 1, with level $l$ and depth 0, then $s_0$ is also an exceptional pole of type 2, with level $l$ for $\pi$ and $\pi'$.

**Proof.** The trilinear form $B_{s_0,m}$ gives the required linear form $l$ in Definition 4.3.2.

\[ \square \]

- As before, we don’t know if the converse is also true.

Now as before let $P_{k,n-k}$ be the standard parabolic subgroup of $G_n$ associated to the partition $(k, n-k)$. Let $P_{k,n-k} = M_{k,n-k}N_{k,n-k}$ be the Levi decomposition, and $n_{k,n-k}$ be the Lie algebra of $N_{k,n-k}$.

Next we have analogs of Proposition 4.1.5. and Theorem 4.1.6.

**Proposition 4.3.3.** Let $s = s_0$ be an exceptional pole of level $l$ for the integrals
$I_k$, then the continuous trilinear form $B_{s_0,k}$ defines a continuous trilinear form on $V/\mathfrak{n}_{k,n-k}V \times V'/\mathfrak{n}_{k,m-k}V' \times E'_k$.

**Proof.** It suffice to show that the form $B_{s_0,k}$ vanishes on $\mathfrak{n}_{k,n-k}V$ and $\mathfrak{n}_{k,m-k}V'$ when restricted to $S'_k$.

So for any $W_{\pi(X),v}$, $X \in n$, any $W_{v'}$ and any $\Phi \in S'_k$, by Lemma 4.1.4, we have

$$W_{\pi(X),v} \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} = P_X(\epsilon_kg)W_v \begin{pmatrix} g \\ I_{n-k} \end{pmatrix}$$

for some linear form $P_X$ on $\mathbb{R}^k$.

It follows that

$$I_k(s, W_{\pi(X),v}, W_{v'}, \Phi) =$$

$$\int_{N_k \backslash G_k} W_{\pi(X),v} \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} W_{v'} \begin{pmatrix} g \\ I_{m-k} \end{pmatrix} \Phi(\epsilon_kg)|detg|^{s-k + \frac{n+m}{2}} dg$$

$$= \int_{N_k \backslash G_k} W_v \begin{pmatrix} g \\ I_{n-k} \end{pmatrix} W_{v'} \begin{pmatrix} g \\ I_{m-k} \end{pmatrix} \Psi(\epsilon_kg)|detg|^{s-k + \frac{n+m}{2}} dg$$

where $\Psi_k(\epsilon g) = P_X(\epsilon_kg)\Phi(\epsilon_kg)$.

Since $\Phi \in S'_k$, thus $\Psi = P_X\Phi \in S'_k^{l+1}$. Note that $s_0$ is an exceptional pole with level $l$, so

$$B_{s_0,k}(W_{\pi(X),v}, W_{v'}, \Phi) = B_{s_0,k}(W_v, W_{v'}, \Psi) = 0$$

which proves the proposition.

□

**Theorem 4.3.4.** If $s = s_0$ is an exceptional pole of type 1 with level $l$ and depth $m - k$, then $s_0$ is an exceptional pole of type 2 with level $l$ for some components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$.  

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Proof. Now since both $\pi$ and $\pi'$ are in general positions, we have the following direct sum decompositions as in section 3.3.

$$V/\mathfrak{n}_{k,n-k}V = \bigoplus (\rho_i, A_i) \otimes (\sigma_i, B_i)$$

and

$$V'/\mathfrak{n}_{k,m-k}V' = \bigoplus (\rho'_i, A'_i) \otimes (\sigma'_i, B'_i)$$

By Proposition 4.1.5, if $s = s_0$ is an exceptional pole of level $l$ for $I_k$, then $B_{s_0,k}$ defines a nontrivial continuous trilinear form on $V/\mathfrak{n}_{k,n-k}V \times V'/\mathfrak{n}_{k,m-k}V' \times E^l_k$. Thus it has to be nontrivial on some components

$$B_{s_0,k} : (\rho, A_i) \otimes (\sigma, B_i) \times (\rho'_j, A'_j) \otimes (\sigma'_j, B'_j) \times E^l_k \rightarrow \mathbb{C}$$

with the invariance property

$$B_{s_0,k}(g.v, g.v', g.\Phi) = |\text{det}g|^{-s_0 + \frac{n-m}{2}} B_{s_0,k}(v, v', \Phi)$$

for all $g \in G_k$, $v \in A_i \otimes B_i$, $v' \in A'_j \otimes B'_j$ and $\Phi \in E^l_k$.

Since the tensor products $A_i \otimes B_i$ and $A'_i \otimes B'_i$ are dense in $A_i \otimes B_i$ and $A'_j \otimes B'_j$ respectively, $B_{s_0,k}$ is nontrivial on the subspace $A_i \otimes B_i \times A'_i \otimes B'_i \times E^l_k$.

Now fix $v_2 \in B_i$, $v'_2 \in B'_i$, so that $B_{s_0,k}$ is nontrivial on $A_i \otimes v_2 \times A'_i \otimes v'_2 \times E^l_k$. Then the restriction of $B_{s_0,k}$ to this subspace induces a nontrivial continuous trilinear form, still denoted as $B_{s_0,k}$, on $A_i \times A'_i \times E^l_k$, with

$$B_{s_0,k}(g.v_1, g.v'_1, g.\Phi) = |\text{det}g|^{-s_0 + \frac{n-m}{2}} B_{s_0,k}(v_1, v'_1, \Phi)$$

for any $v_1 \in A_i$, $v'_1 \in A'_i$, $\Phi \in E^l_k$ and $g \in G_k$. Thus we have proved the theorem.

□

Use the same method to analyze the poles of integrals $I_k(s, W_v, W_{v'}, \Phi)$, the key reduction step is the following analogous of Proposition 4.2.5.
**Proposition 4.3.5.** If a pole $s_0$ of $I_k$ is not an exceptional pole for these integrals, then it is a pole of $I_{k-1}$.

**Proof.** By Proposition 2 in [14], there exists a finite set of functions $\{\xi\}$ on $(\mathbb{R}^\times)^k$, which have the form $\xi(z_1, ..., z_k) = \prod_{j=1}^{k} \chi_j(z_j)(\log|z_j|)^{n_j}$, where $\chi_j$ is a character on $\mathbb{R}^\times$, and Schwartz functions $\phi_\xi$ on $\mathbb{R}^k \times O(k + 1)$, such that

$$W_v(\alpha x) = \sum_\xi \xi(a_1, ..., a_k)\phi_\xi(a_1, ..., a_k, x)$$

where $x \in O(k + 1)$, and

$$\alpha = \text{diag}(a_1a_k, a_2a_k, ..., a_{k-1}a_k, a_k)$$

which will be viewed as

$$\text{diag}(a_1a_k, a_2a_k, ..., a_{k-1}a_k, a_k, 1, ..., 1) \in G_n$$

Since $\phi_\xi$ is a Schwartz function, for each $x$, it has a Taylor expansion around 0,

$$\phi_\xi(a_1, ..., a_k, x) = f(x)P_\xi(a_1, ..., a_k) + ...$$

where $f(x)$ is some continuous function of $x$, and $P_\xi$ denotes the sum of leading coefficients in the Taylor expansion, which is a polynomial in $a_1, ..., a_k$.

It follows that around 0, we can write

$$W_v(\alpha x) = \sum_\xi \{f(x)\xi(a_1, ..., a_k)P_\xi(a_1, ..., a_k) + ...\} \cdots \cdots \cdots (13)$$

Similarly, around 0, we have

$$W_v'(\alpha x) = \sum_{\xi'} \{f'(x)\xi'(a_1, ..., a_k)P_{\xi'}(a_1, ..., a_k) + ...\} \cdots \cdots \cdots (14)$$

where $x \in O(k + 1)$, and

$$\alpha = \text{diag}(a_1a_k, a_2a_k, ..., a_{k-1}a_k, a_k)$$

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which will be viewed as

\[ \text{diag}(a_1 \ldots a_k, a_2 \ldots a_k, \ldots, a_{k-1} a_k, a_k, 1, \ldots, 1) \in G_m \]

By Iwasawa decomposition, we have

\[ I_k = \int \]

\[ W_v \begin{pmatrix} pax & 0 \\ 0 & I_{n-k} \end{pmatrix} W_{v'} \begin{pmatrix} pax & 0 \\ 0 & I_{m-k} \end{pmatrix} \Phi(\epsilon_k ax) |det p|^{s-\frac{n+m}{2}+k-1} |a|^{k(s-\frac{n+m}{2}+k)} dpdx dx \]

with \( p \in N_k \setminus P_k \), where \( P_k \) is the mirabolic subgroup in \( G_k \), \( x \in O(k) \), \( a \in \mathbb{R}^\times \).

Note that \( N_k \setminus P_k = N_{k-1} \setminus G_{k-1} \), so we can write \( pax = n_{k-1} axy \) for some \( n_{k-1} \in N_{k-1} \),

\[ \alpha = \text{diag}(a_1 \ldots a_{k-1} a, \ldots, a_{k-1} a, a, 1, \ldots, 1) \]

and \( y \in O(k-1) \).

Thus by (13), around 0 we have

\[ W_v \begin{pmatrix} pax \\ I_{n-k} \end{pmatrix} = \psi(n_k) \sum_{\xi} \{ f(yx) \xi(a_1, \ldots, a_{k-1}, a) P_{\xi}(a_1, \ldots, a_{k-1}, a) + \ldots \} \]

and

\[ W_{v'} \begin{pmatrix} pax & 0 \\ 0 & I_{m-k} \end{pmatrix} = \psi^{-1}(n_k) \sum_{\xi'} \{ f'(yx) \xi'(a_1, \ldots, a_{k-1}, a) P_{\xi'}(a_1, \ldots, a_{k-1}, a) + \ldots \} \]

Note that the poles of \( I_k \) is caused by the integration around 0, and in a neighborhood of 0, the integral is

\[ \sum_{\xi, \xi'} \int f(yx) f'(yx) dydx \]

\[ \int (\xi P_{\xi'} P_{\xi'})(a_1, \ldots, a_{k-1}, a) \Phi(\epsilon_k ax) |a|^{k(s-n+k)} |a_1|^{c_1} \ldots |a_{k-1}|^{c_{k-1}} d^x a \ldots d^x a_{k-1} + \ldots \]
where \( c_1, \ldots, c_{k-1} \) are some complex numbers linearly depending on \( s \).

First since \( s_0 \) is a pole for this integral, and \( O(k), O(k-1) \) are compact, it follows that this pole occurs as a pole for the integral with respect to the variables \( a_1, \ldots, a_k, a \), and the integration with respect to \( x, y \) are nonzero.

Since \( s_0 \) is not an exceptional pole, we can choose the Schwartz function \( \Phi \) so that the integral on \( a \) in the above expression is holomorphic in a region containing \( s_0 \), by Lemma 4.2.2.

Thus the pole is caused by the integration with respect to the variables \( a_1, \ldots, a_{k-1} \).

This implies that the integral

\[
\int W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k+1} \end{pmatrix} W_v' \begin{pmatrix} g & 0 \\ 0 & I_{m-k+1} \end{pmatrix} |detg|^{s-\frac{n+m}{2}+k-1} dg
\]

has the pole \( s = s_0 \). This integral belongs to the integrals \( I_{k-1} \), the proposition follows.

\[ \square \]

**Corollary 4.3.6.** Any pole of the Rankin-Selberg integrals \( I_m \) for \( \pi \) and \( \pi' \) is an exceptional pole of type 2 for some components of \( \pi^{(n-k)} \) and \( \pi'^{(m-k)} \), \( 0 \leq k < m \).

Conversely, we have the following proposition.

**Proposition 4.3.7.** Any exceptional pole of type 1 of depth 0 for a pair of components of \( \pi^{(n-k)} \) and \( \pi'^{(m-k)} \) is a pole of the Rankin-Selberg integrals \( I_n \) for \( \pi \) and \( \pi' \).

**Proof.** Suppose \( \sigma \) and \( \sigma' \) are a pair of components of \( \pi^{(n-k)} \) and \( \pi'^{(m-k)} \) respectively. For any \( W_{v_1} \in \mathcal{W}(\sigma, \psi) \), \( W'_{v_1} \in \mathcal{W}(\sigma', \psi^{-1}) \), and \( \Phi \in \mathcal{S}_k \), we have the Rankin-Selberg integral for \( \sigma \) and \( \sigma' \)

\[
I(s, W_{v_1}, W'_{v_1}, \Phi) = \int_{N_k \backslash G_k} W_{v_1}(g)W'_{v_1}(g)\Phi(\epsilon_k g)|detg|^s dg
\]
By Corollary 3.3.3., there exists some $W_v(g) \in \mathcal{W}(\pi, \psi)$, such that

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} = W_{v_1}(g) \Phi(\epsilon_k g) |detg|^\frac{n-k}{2}$$

Thus the above integral is

$$I(s, W_{v_1}, W_{v'_1}, \Phi) = \int_{N_k \backslash G_k} W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} W_{v'_1}(g) |detg|^s |detg|^{s-\frac{n-k}{2}} dg$$

By Lemma 10.4 in [J-S1], we can write

$$W_v \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} = \sum_j W_j \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} \Phi_j(\epsilon_k g)$$

with $W_j \in \mathcal{W}(\pi, \psi)$, and Schwartz functions $\Phi_j$ on $\mathbb{R}^k$. So we have

$$I(s, W_{v_1}, W_{v'_1}, \Phi) = \sum_j \int W_j \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} W_{v'_1}(g) \Phi_j(\epsilon_k g) |detg|^s |detg|^{s-\frac{n-k}{2}} dg$$

Using Corollary 3.3.3. again, we can find $W'_j(g) \in \mathcal{W}(\pi', \psi^{-1})$, such that for each $j$, we have

$$W'_j \begin{pmatrix} g & 0 \\ 0 & I_{m-k} \end{pmatrix} = W_{v'_1}(g) \Phi_j(\epsilon_k g) |detg|^\frac{n-k}{2}$$

Hence

$$I(s, W_{v_1}, W_{v'_1}, \Phi) = \sum_j \int W_j \begin{pmatrix} g & 0 \\ 0 & I_{n-k} \end{pmatrix} W'_j \begin{pmatrix} g & 0 \\ 0 & I_{m-k} \end{pmatrix} |detg|^s |detg|^{s-\frac{n+m}{2}+k} dg$$

Then by Lemma 14.1 in [13], the integrals on the right side belong to the Rankin-Selberg integrals for $\pi$ and $\pi'$. Thus the proposition follows.

□

Let us summarize the above results as follows.
Theorem 4.3.8. Let $\pi$ and $\pi'$ be irreducible generic Harish-Chandra representations of $G_n$ in general position. Then any pole of the Rankin-Selberg integrals for $\pi$ and $\pi'$ is an exceptional pole of type 2 for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $1 \leq k \leq m$. On the other hand, any exceptional pole of type 1 of depth 0 for a pair of components of $\pi^{(n-k)}$ and $\pi'^{(m-k)}$, $1 \leq k \leq m$, is a pole of the Rankin-Selberg integrals of $\pi$ and $\pi'$. 
BIBLIOGRAPHY


   http://www.math.umn.edu/~garrett/m/v/moderate_growth.pdf


