COMPARING INVARIANTS OF 3-MANIFOLDS DERIVED FROM HOPF ALGEBRAS

DISSERTATION

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This paper will compare two different quantum 3-manifold invariants, both of which are given using a finite dimensional Hopf Algebra $H$. One is the Hennings invariant, given by an algorithm involving the link surgery presentation of a 3-manifold and the Drinfeld double $D(H)$; the other is the Kuperberg invariant, which is computed using a Heegaard diagram of the 3-manifold and the same $H$. We show that when $H$ is semi-simple, these two invariants are equal. The proof is entirely in Hopf algebraic terms and does not rely on the representation theory of $H$ or general results involving categorical invariants.
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# TABLE OF CONTENTS

Abstract .................................................. ii
Dedication .................................................. ii
Acknowledgments ............................................ iv
Vita ............................................................ vi
List of Figures ............................................... ix

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
</tr>
<tr>
<td>2</td>
<td>Outline and Preliminaries</td>
</tr>
<tr>
<td></td>
<td>2.1 Hopf Algebras</td>
</tr>
<tr>
<td></td>
<td>2.1.1 Vertical Diagram Notation</td>
</tr>
<tr>
<td></td>
<td>2.2 Quasi-Triangular Hopf Algebras and The Drinfeld Double</td>
</tr>
<tr>
<td></td>
<td>2.3 Integrals and Co-Integrals</td>
</tr>
<tr>
<td></td>
<td>2.4 The Category $\mathcal{TGL}$</td>
</tr>
<tr>
<td></td>
<td>2.5 The Category $\mathcal{BEAD}_H$</td>
</tr>
<tr>
<td>3</td>
<td>The Hennings and Involutory Kuperberg Constructions</td>
</tr>
<tr>
<td></td>
<td>3.1 The Hennings 3-manifold Invariant</td>
</tr>
<tr>
<td></td>
<td>3.1.1 A Hennings Invariant example: $L(p, 1)$</td>
</tr>
<tr>
<td></td>
<td>3.2 The Involutory Kuperberg Invariant</td>
</tr>
<tr>
<td></td>
<td>3.2.1 A Kuperberg invariant example: $L(p, 1)$</td>
</tr>
<tr>
<td></td>
<td>3.2.2 The Kuperberg Invariant of $L(3, 2)$</td>
</tr>
<tr>
<td>4</td>
<td>The Hennings Invariant as a Tangle Functor</td>
</tr>
<tr>
<td>5</td>
<td>The Kuperberg Invariant of a Link</td>
</tr>
<tr>
<td></td>
<td>5.1 Blackboard framed links as braid closures</td>
</tr>
<tr>
<td></td>
<td>5.2 Constructing a Heegaard Diagram from a Link</td>
</tr>
</tbody>
</table>
5.2.1 Basepoints, Orientations, and Orderings on Framed Braid Links .................................................. 57

5.3 A Kuperberg algorithm for each tangle piece .................. 63

6 Proof of the Main Theorem ................................................. 78

6.1 Defining the tensor $A$ ................................................... 78

6.2 Proof of Theorem 1 ...................................................... 86

Bibliography ....................................................................... 94

viii
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>9</td>
</tr>
<tr>
<td>2.2</td>
<td>10</td>
</tr>
<tr>
<td>2.3</td>
<td>10</td>
</tr>
<tr>
<td>2.4</td>
<td>11</td>
</tr>
<tr>
<td>2.5</td>
<td>11</td>
</tr>
<tr>
<td>2.6</td>
<td>12</td>
</tr>
<tr>
<td>2.7</td>
<td>13</td>
</tr>
<tr>
<td>2.8</td>
<td>23</td>
</tr>
<tr>
<td>2.9</td>
<td>24</td>
</tr>
<tr>
<td>2.10</td>
<td>27</td>
</tr>
<tr>
<td>2.11</td>
<td>28</td>
</tr>
<tr>
<td>2.12</td>
<td>30</td>
</tr>
<tr>
<td>2.13</td>
<td>33</td>
</tr>
<tr>
<td>3.1</td>
<td>38</td>
</tr>
<tr>
<td>3.2</td>
<td>38</td>
</tr>
<tr>
<td>3.3</td>
<td>40</td>
</tr>
<tr>
<td>3.4</td>
<td>41</td>
</tr>
<tr>
<td>3.5</td>
<td>43</td>
</tr>
</tbody>
</table>
3.6 A two-dimensional L(3,2) Heegaard diagram

4.1 A pictorial definition of $\mathcal{F}$

5.1 A closed braid example with $Height(x)$

5.2 Step 3 of the algorithm

5.3 In the diagram, the link projection lying on the 3-ball is the thick black line. The black dot represents a point where two different tangle pieces come together. The 1-handle rises over this meeting point and attaches to the 3-ball on either side.

5.4 The Heegaard diagram pieces for $\{\}$, $\cap$, and $\cup$ respectively

5.5 The Heegaard diagram pieces for the positive and negative crossings

5.6 The orientation and base point of lower circles

5.7 Oriented diagram for the positive crossing

5.8 Oriented diagram for the negative crossing

5.9 An ordered link with two components

5.10 The local picture at a lower circle and the corresponding tensor

5.11 The two tensors corresponding to the higher and lower diagram piece at a joint

5.12 The oriented positive crossing Heegaard diagram

5.13 The unsimplified tensor for a positive crossing

5.14 The simplified tensor for a positive crossing

5.15 The oriented negative crossing Heegaard diagram

5.16 The unsimplified tensor for a negative crossing

5.17 The simplified tensor for a negative crossing
Two well-known quantum invariants of 3-manifolds are the Hennings Invariant and the the Kuperberg Invariant. The Hennings Invariant, first described by Hennings in [4] and later reformulated by Kauffman and Radford in [8], is developed using a quasi-triangular Hopf algebra and a projection of a link $L$. After multiplying by a proper scalar, it can be made into an invariant of 3-manifolds by the surgery correspondence of links and manifolds. The Kuperberg Invariant was introduced by Kuperberg in [14] and [15] and is calculated using a Hopf algebra and a Heegaard diagram of the 3-manifold. The purpose of this paper is to give an algebraic proof of the fact that for a semi-simple Hopf algebra $H$, $Kup(M, H) = Henn(M, D(H))$, where $D(H)$ is the Drinfeld double of $H$.

This fact had been a long standing conjecture with many partial results. It was first suggested by Kuperberg in 1990 [14] that his invariant was closely related to the Reshetikhin-Turaev construction. Kerler and Lyubashenko ([11], [12]) later showed that the Hennings Invariant was a special case of the the Reshetikhin-Turaev TQFT. Barrett and Westbury showed [1] that the Kuperberg Invariant was a special case of the Turaev-Viro TQFT. Finally in 2010 Turaev and Virelizier [20] showed that the Reshetikhin-Turaev and Turaev-Viro constructions were equivalent when applied to spherical fusion categories. Combined with the earlier results, this proved the conjecture.
The results described above rely heavily on the semi-simplicity of categories and decompositions into simple objects. Because our proof will explicitly be described in terms of Hopf algebra calculations, the method outlined in this paper can likely be generalized to the case where $H$ is not semi-simple. Also, the proof in this paper is concise and direct, and does not rely on the representation theory of $H$ or abstract categorical constructions.

The way we begin to show the equivalence is by defining a functor $F_H$ from the category of framed tangles to the category of vector spaces. For a fixed quasi-triangular involutory Hopf algebra $H$, $F_H$ sends a framed tangle of the form $(n, m)$ to an element of $\text{Hom}(H^\otimes n, H^\otimes m)$. In particular, when $F_H$ is applied to a link $L$, (a tangle of type $(0, 0)$) we get an element of the ground field $k$. This field element will be shown to be a scalar multiple of $\text{Henn}(L, H)$, thus giving us a generalization of the Hennings Invariant to tangles.

Next we will construct a Heegaard diagram for a 3-manifold $M_L$ given by surgery on a link $L$. This diagram is built out of smaller Heegaard diagram pieces corresponding to pieces of the link $L$. This allows us to define a function $L_H$ that maps pieces of an oriented link to Hopf algebra morphisms, producing a piecewise version of the Kuperberg Invariant for $M_L$. We then will show that $L_H(L) = \text{Kup}(M_L, H)$.

The final step is to show that for all links $L$ and Hopf algebras $H$ that are involutory, unimodular, and semi-simple, $F_{D(H)}(L)$ is a scalar multiple of $L_H(L)$. The equivalence of these functions gives us our desired equivalence of invariants.

The outline for this paper will be as follows: Chapter 2 will outline the main theorems, detail the various Hopf Algebra constructions that will be used throughout the paper, and define the important categories $\mathcal{TGL}$ and $\mathcal{BEAD}_H$. Chapter 3 will describe the constructions of the Hennings Invariant and the Kuperberg Invariant. In Chapter 4 we will construct the functor $F_H$ and show that it is a generalization
of the Hennings Invariant. Chapter 5 will construct the function $\mathcal{L}_H$ and show how it is related to $Kup(M, H)$. Finally in Chapter 6 we will show that $\mathcal{L}_H$ is equal to $\mathcal{F}_{D(H)}$ (up to a scalar multiple). This will prove our main theorem.
CHAPTER 2
OUTLINE AND PRELIMINARIES

Throughout this paper, $M$ will denote a closed, orientable 3-manifold, $H$ will be a Hopf algebra, and $D(H)$ will the Drinfeld double of $H$. If $H$ is quasi-triangular, then $Henn(M, H)$ will denote the Hennings Invariant as described in [4] and [8]. When $H$ is an involutory Hopf algebra, $Kup(M, H)$ will denote the Kuperberg Invariant as described in [14] and [15]. By involutory, we mean that $S^2 = id$ in $H$. $TGL$ and $BEAD_H$ will denote the category of framed tangles and the category of beaded diagrams respectively. $\mathsf{Vect}$ will denote the category of vector spaces. Both $TGL$ and $BEAD_H$ will be defined in detail later in this section.

The main theorem that will be proven in this thesis is the following:

**Theorem 1.** Let $H$ be an involutory Hopf algebra such that the characteristic of the ground field does not divide the dimension of $H$. Then for any 3-manifold $M$, $Kup(M, H) = Henn(M, D(H))$.

The theorem will be proved in several steps. First we must detail the various Hopf algebra and geometric constructions needed to define the invariants. Next we will prove the following theorem, which describes a new way to obtain the Hennings Invariant:

**Theorem 2.** Let $H$ be an involutory Hopf Algebra such that the characteristic of the ground field does not divide the dimension of $H$ and $D(H)$ be its Drinfeld Double.
Then there exists a functor $F : TGL \to VECT$, such that if $M$ is a 3-manifold given by surgery on a link $L$, $F(L) = (\dim H)^{c(L)} \text{Henn}(M, D(H))$, where $c(L)$ is the number of components in $L$.

The next step in the process is to describe a way to formulate the Kuperberg invariant, which was originally only described for manifolds, as a link invariant. It turns out that by recording additional information about our link, we can define a function on the generators of $TGL$ that gives the Kuperberg invariant.

**Theorem 3.** Let $L$ be a framed link, and $M_L$ be the manifold given by surgery along $L$. For an involutory Hopf Algebra $H$, there is a map $\mathcal{L}_H$ that sends the generators of $TGL$ to morphisms in $VECT$ such that $\mathcal{L}_H(L) = \text{Kup}(M_L, H)$

Both of these functions preserve rank in the sense that a tangle of the form $(n, m)$ gets sent to an element of $\text{Hom}(H^\otimes n, H^\otimes m)$. In particular, a $(0,0)$ tangle (i.e. a link) gets sent to an automorphism of the ground field. When we perform $S^3$ surgery with this link, both functors give 3-manifold invariants. The final step in proving the main theorem is to show that the Hennings functor applied with $D(H)$ is the same as the Kuperberg map applied with $H$.

### 2.1 Hopf Algebras

We begin this section by recalling Hopf algebras and their relevant properties. For the purposes of this paper a *Hopf algebra* is a finite dimensional vector space $H$ over a field $k$ equipped with the following maps:

- **Multiplication:** $M : H \otimes H \to H$
- **Unit:** $\eta : k \to H$
- **Co-multiplication:** $\Delta : H \to H \otimes H$
• Co-Unit: \( \varepsilon : H \rightarrow k \)

• Antipode: \( S : H \rightarrow H \)

We will also use some more standard notation to describe calculations. For instance, \( M(x \otimes y) \) will be written simply as \( xy \). We will often use the standard Sweedler notation for co-multiplication, namely \( \Delta(x) = x' \otimes x'' \) or \( \Delta(x) = x^{(1)} \otimes x^{(2)} \), suppressing any summation signs that may occur. Also, \( \eta(1) \) will often be simply written as \( 1_H \).

The maps satisfy the following properties. Let \( x, y, z \in H \):

1. Associativity: \( (xy) \cdot z = x \cdot (yz) \)

2. Unit: \( 1_H \cdot x = x = x \cdot 1_H \)

3. Co-Associativity: \( \Delta(x') \otimes x'' = x' \otimes \Delta(x'') \)

4. Co-Unit: \( \varepsilon(x') \otimes x'' = x = x' \otimes \varepsilon(x'') \)

5. Bi-Algebra Axiom: \( \Delta(xy) = x' \cdot y' \otimes x'' \cdot y'' \)

6. Antipode Axiom: \( x' \cdot S(x'') = S(x') \cdot x'' = \varepsilon(x) \cdot 1_H \)

Often it will make calculations more convenient if we make use of dualization. To this end, the following fact is useful:

**Lemma 4** ([6]). Let \( (H, M, \eta, \Delta, \varepsilon, S) \) be a Hopf algebra. Then \( (H^*, \Delta^*, \varepsilon^*, M^*, \eta^*, S^*) \), \( (H^{op}, M \circ \tau, \eta, \Delta, \varepsilon, S^{-1}) \) and \( (H^{cop}, M, \eta, \tau \circ \Delta, \varepsilon, S^{-1}) \) are all Hopf Algebras as well and are all isomorphic to \( H \).

Here the superscript * on a linear map indicates the transpose and \( \tau \) is the switch map, i.e. \( \tau(x \otimes y) = y \otimes x \). Also, \( H^{op} \) and \( H^{cop} \) denote \( H \) with opposite multiplication and opposite co-multiplication respectively.
Given a Hopf algebra $H$, we can define left and right $H$-actions on $H^*$, denoted ‘$\triangleleft$’ and ‘$\triangleright$’, that make $H^*$ into an $H$-module. Let $h, x \in H$ and $\delta \in H^*$. Then these actions are defined by:

$$ (\delta \triangleleft h)(x) := \delta(hx) \quad (2.1.1) $$

and

$$ (h \triangleright \delta)(x) := \delta(xh) \quad (2.1.2) $$

Similarly, we can use co-multiplication to define left and right actions of $H^*$ on $H$.

$$ \delta \rightarrow h := h' \otimes \delta(h'') \quad (2.1.3) $$

and

$$ h \leftarrow \delta := \delta(h') \otimes h'' \quad (2.1.4) $$

Also we will often use the action of $H$ (or $H^*$) on itself given by left or right multiplication. These will be denoted by $L$ and $R$. Namely, for $a, x \in H$, we have:

$$ L_a : H \rightarrow H := L_a(x) = ax \quad (2.1.5) $$

and

$$ R_a : H \rightarrow H := R_a(x) = xa \quad (2.1.6) $$

Given a Hopf algebra $H$, we can define a new category $\mathcal{T}(H)$ as the full subcategory of $\mathcal{V}ECT$ whose objects are finite tensor products of the algebras $H$ and $H^*$. Four particularly important morphisms in $\mathcal{T}(H)$ are the following:
\[ ev : H \otimes H^* \to k := ev(x \otimes \delta) = \delta(x) \quad (2.1.7) \]
\[ \tilde{ev} : H^* \otimes H \to k := \tilde{ev}(\delta \otimes x) = \delta(x) \quad (2.1.8) \]
\[ coev : k \to H \otimes H^* := coev(1) = \sum_i e_i \otimes f_i \quad (2.1.9) \]
\[ \tilde{coev} : k \to H^* \otimes H := \tilde{coev}(1) = \sum_i f_i \otimes e_i \quad (2.1.10) \]

where \( x \in H, \delta \in H^* \), \( \{e_i\} \) is a basis for \( H \) and \( \{f_i\} \) is its dual basis in \( H^* \).

When these four morphisms are spliced with the standard Hopf algebra maps, we need to apply the switch map \( \tau \) in between.

**Lemma 5.** We have the following identities:

\[ ev \circ (M \otimes id^*) = ev \circ (id \otimes ev \otimes id^*) \circ (\tau \otimes \Delta^*) \quad (2.1.11) \]
\[ ev \circ (id \otimes M^*) = ev \circ (id \otimes ev \otimes id^*) \circ (\Delta \otimes \tau) \quad (2.1.12) \]
\[ (id \otimes \Delta^*) \circ coev = (M \otimes \tau) \circ (id \otimes coev \otimes id^*) \circ coev \quad (2.1.13) \]
\[ (\Delta \otimes id^*) \circ coev = (\tau \otimes M^*) \circ (id \otimes coev \otimes id^*) \circ coev \quad (2.1.14) \]

The corresponding results also hold for \( \tilde{ev} \) and \( \tilde{coev} \).

### 2.1.1 Vertical Diagram Notation

The Hopf algebra calculations described in this paper are quite complicated, and appear quite cumbersome when they are written out as formulas. In order to make the results more clear for the reader, we will use a version of a vertical diagram calculus to visualize these formulas.

First we must fix a Hopf algebra \( H \). In this calculus, some morphisms in \( \mathcal{T}(H) \) will be described by vertical lines labeled with arrows. The diagrams are read from top to bottom, with the lines at the top of the diagram encoding the initial space.
and lines at the bottom encoding the target space. Arrows pointing downward in the
diagram correspond to elements of $H$ and arrows pointing upward correspond to $H^*$.  
The order of the tensor factors is determined by reading left to right. Composition is
given by stacking and connecting two diagrams and tensor products are denoted by
juxtaposition in order from left to right. If a particular morphism has already been
described formulaically, we draw a box labeled with the morphism, as in Figure 2.1.1.

![Diagram](image)

**Figure 2.1:** This diagram denotes a linear map $f : H \otimes H^* \otimes H \rightarrow H^* \otimes H$

The morphisms on $\mathcal{H}(H)$ that will use most frequently in this paper have their
own special diagrams without boxes. For instance, if we fix a Hopf algebra element
$x \in H$, the morphism $1_x : k \rightarrow H$ given by $1_x(1) = x$ is denoted by a line pointing
downward labeled with $x$ at the top. Similarly, if $\delta \in H^*$, the morphism $1_\delta : k \rightarrow H^*$
where $1 \mapsto \delta$ is denoted by an arrow pointing up labelled with $\delta$. Generally, we will
use Roman letters for elements of $H$ and Greek letters for elements of $H^*$.

Multiplication in $H$ is denoted by two separate lines coming together to form one
line, and co-multiplication is denoted by a single vertical line forking into two lines
(thus preserving the rule for tensor juxtaposition). The antipode map will be denoted
by a large black dot on the vertical line.
Figure 2.2: This picture represents the function \( k \to H \otimes H^* \otimes H \) defined by \( 1 \mapsto x \otimes \delta \otimes y \).

Figure 2.3: These morphisms are \( 1_{xy} \), \( 1_{x' \otimes x''} \), and \( 1_{S(x)} \) respectively.

Composition in this calculus is given by stacking diagrams on top of one another. The diagram should be read from top to bottom, so any operation at the top of the diagram would be the first operation calculated in the formula.

Oftentimes diagrams will contain elements of both \( H \) and \( H^* \). The evaluation maps from \( ev \) and \( \tilde{ev} \) are denoted by lower semi-circular arcs with arrows pointing right and left respectively. The co-evaluation maps \( coev \) and \( \tilde{coev} \) are denoted by upper semi-circular arcs with arrows pointing right and left respectively. When the context is made clear with labeling, these arcs may sometimes be omitted. See Figure 2.4.

We will denote the switch map \( \tau(x \otimes y) = y \otimes x \) on \( H \) by two arrows pointing down that cross each other.

Finally, we can also denote the standard Hopf algebra calculations on \( H^* \) in a
Figure 2.4: Both of these represent $\delta(x') \otimes 1_{x''}$, for $\delta$ in $H^*$

Figure 2.5: This represents the switch map $\tau$

similar manner. However, in this case we would like the co-multiplication picture to denote \textit{opposite} co-multiplication in $H^*$. Combined with Lemma 5, this allows the diagrams containing this operation to be invariant under rotations of $180^\circ$, as demonstrated in the picture below. This seemingly odd convention will be convenient when we begin working with the Drinfeld double.
Finally, Figure 2.7 gives a vertical diagram interpretation of the four relations of Lemma 5 under these conventions.

2.2 Quasi-Triangular Hopf Algebras and The Drinfeld Double

A Hopf algebra is called quasi-cocommutative if there exists an invertible element $R \in H \otimes H$ such that for all $x \in H$,

$$\Delta^{op}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}$$  \hspace{1cm} (2.2.1)

The element $\mathcal{R}$ is called the universal $R$-matrix for $H$. If in addition $\mathcal{R}$ satisfies the two relations:

$$\left(\Delta \otimes id_H\right)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23} \quad \left(id_H \otimes \Delta\right)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$$  \hspace{1cm} (2.2.2)

we say that $H$ is a quasi-triangular Hopf algebra. (Here we are using the standard Drinfeld subscript notation, in which if $\mathcal{R} = \sum a^\nu \otimes b^\nu$, then $\mathcal{R}_{31} = \sum \overline{b^\nu} \otimes 1 \otimes a^\nu$ for example)
Figure 2.7: Lemma 5 written in vertical diagram notation
One of the most important properties of $\mathcal{R}$ implied by equations 2.2.1 and 2.2.2 is that it satisfies the Yang-Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

(2.2.3)

Other important properties of $\mathcal{R}$ are the following:

**Lemma 6 ([6]).**

\[
(\varepsilon \otimes id_H)(\mathcal{R}) = 1 = (id_H \otimes \varepsilon)(\mathcal{R}) \tag{2.2.4}
\]

\[
(S \otimes id_H)(\mathcal{R}) = \mathcal{R}^{-1} = (id_H \otimes S^{-1})(\mathcal{R}) \tag{2.2.5}
\]

\[
(S \otimes S)(\mathcal{R}) = \mathcal{R} \tag{2.2.6}
\]

If we are given a quasi-triangular Hopf algebra we can use the R-matrix to construct other elements with certain desirable and useful properties.

**Lemma 7 ([3]).** Let $H$ be a quasi-triangular Hopf algebra with universal R-matrix $\mathcal{R} = \sum_{\nu} a_{\nu} \otimes b_{\nu}$. Define $u \in H$ by $u = \sum_{\nu} S(b_{\nu})a_{\nu}$. Then $u$ is invertible and has the property that $S^2(x) = uxu^{-1}$ for all $x$ in $H$.

We generally refer to $u$ as the **Drinfeld element**. Let $H$ be a quasi-triangular Hopf algebra with special element $u$ defined above. We call $H$ a **ribbon** Hopf algebra if there exists an element $G$ such that:

1. $\Delta(G) = G \otimes G$
2. $\varepsilon(G) = 1$
3. $G^{-1}u$ is in the center of $H$.
4. $G^2 = S(u)^{-1}u$
The element \( G \) is referred to as the *special group-like element*. If \( H \) is a ribbon Hopf algebra we denote the element \( G^{-1}u \) by \( v \) and call it the *ribbon element* of \( H \). By definition, \( v \) is central and \( S(v) = v \).

Given any Hopf algebra \( H \), there is a systematic way of producing a closely related Hopf algebra, \( D(H) \), which is always a quasi-triangular Hopf algebra. The process was discovered by Drinfeld in [3] and the algebra \( D(H) \) is called the *Drinfeld double* of \( H \).

As a vector space, \( D(H) \) is isomorphic to \( H^{*\text{cop}} \otimes H \). The Hopf algebra structure is given by the following:

Let \( f, g \in H^{*\text{cop}} \) and \( a, b \in H \)

- **Multiplication:** \( (f \otimes a)(g \otimes b) = \sum_{(a)} (a') \triangleright f g \trianglelefteq (S^{-1}(a'')) \otimes a''b \)
- **Unit:** \( \varepsilon \otimes 1 \)
- **Co-multiplication:** \( \Delta_D(f \otimes a) = \sum_{(a)(f)} (f' \otimes a') \otimes (f'' \otimes a'') \)
- **Co-Unit:** \( \varepsilon_D(f \otimes a) = \varepsilon(a)f(1) \)
- **Antipode:** \( S_D(f \otimes a) = (\varepsilon \otimes S(a))(S(f) \otimes 1) \)

**Theorem 8 ([3]).** With these definitions \( D(H) \) satisfies all of the axioms of a quasi-triangular Hopf algebra.

The multiplication above is an algebraic construction known as a *bi-crossed product*. Now let \( f_i \) be a basis for \( H^{*\text{cop}} \) and \( e_i \) its dual basis in \( H \). Then the universal \( R \)-matrix of \( D(H) \) is given by:

\[
\mathcal{R} = \sum_{i \in I} (1 \otimes e_i) \otimes (f_i \otimes 1) \in D(H) \otimes D(H) \tag{2.2.7}
\]

For a proof that the above \( \mathcal{R} \) is indeed a universal \( R \)-matrix see [6].
2.3 Integrals and Co-Integrals

The definition of the Kuperberg invariant makes heavy use of co-integrals and integrals. We will briefly recall their definitions and some important results that will be used in the remainder of the paper.

A left co-integral, denoted $\Lambda_L$, is an element of $H$ such that for any $x \in H, x \cdot \Lambda_L = \varepsilon(x)\Lambda_L$. Similarly a right co-integral $\Lambda_R$ satisfies $\Lambda_R \cdot x = \varepsilon(x)\Lambda_R$.

A left integral, denoted $\mu_L$, for $H$ is an element of $H^*$ such that $x' \otimes \mu_L(x'') = 1_H \cdot \mu_L(x)$ for all $x \in H$. Similarly a right integral $\mu_R$ for $H$ satisfies $\mu_R(x') \otimes x'' = 1_H \cdot \mu_R(x)$.

It follows from the definitions and standard duality theorems that a left co-integral for $H$ is a left integral for $H^*$ and vice-versa. (The same also holds for right integrals and co-integrals.)

We have the following facts:

**Lemma 9** ([16]).

1. All finite dimensional Hopf Algebras have left (and right) co-integrals.

2. Left (and right) co-integrals form a dimension 1 subspace of $H$, i.e. $\Lambda_L$ is unique up to a scalar.

3. Left (and right) integrals form a dimension 1 subspace of $H^*$

4. It is always possible to scale $\mu_L$ so that $\mu_L(\Lambda_L) = 1$

Duality gives us the same existence and uniqueness results for $\mu_L$ and $\mu_R$. For the rest of this paper, we will assume that $\mu_L$ is scaled so that $\mu_L(\Lambda_L) = 1$.

Sometimes it is the case that the subspace of left co-integrals for $H$ is equal to the subspace of right co-integrals. If this holds we say $H$ is *unimodular*. (Notationally
we drop the subscript and simply write \( \Lambda \). Similarly, if \( \mu_L = \mu_R \) we say \( H^* \) is unimodular and we denote the integral as \( \mu \).

Finally, we will say that a Hopf algebra \( H \) is semi-simple if \( H \) is semi-simple as an algebra. If \( H \) has the property that \( S^2 = id \) we say that \( H \) is involutory. All three of these properties are closely related:

**Lemma 10** ([16]).

1. If \( H \) is semi-simple then \( H \) is unimodular.

2. Let \( S \) be the antipode for \( H \). Then \( H \) and \( H^* \) are both semi-simple if and only if \( Tr(S^2) \neq 0 \)

Combining 1) and 2) above gives us that if \( H \) is involutory and \( \text{dim}(H) \) is not a multiple of \( \text{char}(k) \) then \( H \) and \( H^* \) are both unimodular. In particular, if \( k = \mathbb{C} \) and \( H \) is involutory then \( H \) and \( H^* \) are both unimodular.

The following Hopf algebra relations are very useful:

**Lemma 11.** Let \( H \) be a Hopf algebra that is unimodular. Let \( x \in H \) and \( \mu(\Lambda) = 1 \). Then

\[
S(x) = \mu(\Lambda x) \otimes \Lambda'' = \Lambda' \otimes \mu(x\Lambda'') = \Lambda' \otimes \mu(\Lambda''x) = \mu(x\Lambda') \otimes \Lambda'' \quad \text{(2.3.1)}
\]

**Proof.** We will only show the first one, the proofs of the other 3 relations are similar. We have that

\[
x \otimes 1 = x' \otimes x'' S(x''') = \Delta(x') (1 \otimes S(x''))
\]

Thus

\[
\mu(\Lambda'x) \otimes \Lambda'' = (\mu \otimes 1)[\Delta(\Lambda)(x \otimes 1)] =
\]

\[
(\mu \otimes 1)[\Delta(\Lambda x')(1 \otimes S(x''))] =
\]

\[
(\mu \otimes 1)[\Delta(\Lambda)(1 \otimes S(\varepsilon(x')x'')] =
\]

\[
\mu(\Lambda') \otimes \Lambda'' S(x) = \mu(\Lambda)S(x) = S(x)
\]

17
The fact that co-integrals form a 1-dimensional subspace has an interesting consequence that allows us to relate $\Lambda_L$ and $\Lambda_R$. If we fix an element $x \in H$, we see that $\Lambda_L \cdot x$ is itself a left co-integral in $H$. Thus $\Lambda_L \cdot x = \lambda \cdot \Lambda_L$ for some $\lambda \in k$. If we set $\alpha(x) := \lambda$ this defines an element of $H^*$, called the co-modulus. Dualizing this process gives a special element in $H$, called the modulus, which is generally denoted by $a$.

We say that an element $g \in H$ is group-like if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. It follows quickly from the axioms for a Hopf algebra that all group like elements $g$ are invertible with $g^{-1} = S(g)$. Since $\alpha(xy) = \alpha(x) \cdot \alpha(y)$, we see that the co-modulus is group-like in $H^*$. Thus $\alpha$ has an inverse in $H^*$, namely $\alpha^{-1} = S(\alpha)$. The modulus has these same properties in $H$.

The following lemma describes some useful properties involving $a$ and $\alpha$.

**Lemma 12 ([17], [19]).** Let $H$ be a Hopf algebra with left integral $\Lambda_L$, left co-integral $\mu_L$, modulus $a$ and co-modulus $\alpha$. Then:

1. $S(\Lambda_L) = \alpha \twoheadrightarrow \Lambda_L$
2. $S^{-1}(\mu_L) = a \triangleright \mu_L$
3. $\mu_L((\alpha \twoheadrightarrow x) \cdot y) = \mu_L(yS^2(x))$
4. $\mu(xy) = \mu(S^2(y \leftarrow \alpha)x)$
5. $\Lambda'' \otimes \Lambda' = \Lambda' \otimes S^2(\Lambda'')a$
6. $a(\alpha \twoheadrightarrow x \leftarrow \alpha^{-1})a^{-1} = S^4(x)$

Now that we have defined both $\alpha$ and $a$, we can now describe the ribbon element (when it exists) in $D(H)$:
Lemma 13 ([7]).  1. $D(H)$ is a ribbon Hopf algebra if and only if both $\alpha$ and $\alpha$ have group-like square roots such that $\sqrt{a}(\sqrt{\alpha}) \rightarrow x \leftarrow \sqrt{\alpha}^{-1})\sqrt{a}^{-1} = S^2(x)$ for all $x \in H$. In this case, the special group-like element $G$ can be chosen as $G = \sqrt{\alpha} \otimes (\sqrt{a})^{-1}$

2. If $H$ is unimodular, co-unimodular, and involutory then $u$ is a ribbon element in $D(H)$ and $G$ can be chosen to be $1_{D(H)}$

Finally we will state some important results about how $D(H)$ and $H$ are related.

Lemma 14 ([18], [4]).  1. Let $\Lambda_R$ be a right co-integral in $H$ and $\mu_L$ be a left integral for $H$. Then $D(H)$ is unimodular with co-integral $\Lambda_{D(H)} = \mu_L \otimes \Lambda_R$.

2. A left integral for $D(H)$ is given by $\mu_{D(H)} = \Lambda_R \otimes \mu_L$.

3. Let $\alpha$ be the modulus of $H$ and $\alpha$ be the co-modulus in $H^\ast$. Then the modulus of $D(H)$ is $a_{D(H)} = \alpha \otimes a$. By (1), $\alpha_{D(H)} = 1_H \otimes \epsilon_H$.

4. $S_D^2 = (S^*)^2 \otimes S^2$. In particular, we have that $D(H)$ is involutory if and only if $H$ is involutory.

2.4 The Category $\mathcal{TGL}$

The categories $\mathcal{TGL}$ and $\mathcal{BEAD}_H$ are central to this paper so it is best to build them from first principles with great detail. The definitions are based on what appears in [9].

We will begin by first defining $\mathcal{TOP}$, the category of framed topological tangles. A tangle $C$ is an embedding of a 1-manifold $S = (\bigcup S^1) \cup (\bigcup [0,1])$ into $\{\mathbb{R}^2 \times [0,1]\}$ that is transverse to the boundary of $\{\mathbb{R}^2 \times [0,1]\}$. Under this embedding, we require that $\partial S$ gets mapped to $\{\mathbb{N} \times \{0\} \times \partial[0,1]\}$. Furthermore, we require that if $\partial C$
intersects \( \mathbb{R} \times \{0\} \times \{0\} \) (resp. \( \mathbb{R} \times \{0\} \times \{1\} \)) in \( n \) points, these ends be placed exactly on the first \( n \) natural numbers \( \{1, 2, 3, \ldots, n\} \).

We would like to place our tangle \( C \) in general position. In order to do this precisely, we define a projection map \( \pi : \mathbb{R}^2 \times [0, 1] \to \mathbb{R} \times [0, 1] \) where \( \pi : (x, y, z) \mapsto (x, z) \). We also define a height map \( h : \mathbb{R}^2 \times [0, 1] \to [0, 1] \) where \( h : (x, y, z) \mapsto z \). We say a tangle \( C \) is in general position if we have the following:

- \( \pi \circ C \) is an immersion everywhere
- For \( x \in \mathbb{R}^2 \times [0, 1] \), we have \(|(\pi \circ C)^{-1}(x)|\) is equal to 0 or 1 for all but finitely many \( x \), and \(|(\pi \circ C)^{-1}(x)|\) is at most 2. Any \( x \) such that \(|(\pi \circ C)^{-1}(x)| = 2 \) will be called a crossing point
- All crossings are transverse
- \( h \) takes different value on all crossing points
- \( h \circ C : S \to [0, 1] \) has a finite number of non-degenerate maxima and minima
- None of the local minima or maxima of \( h \circ C \) are also crossing points

It follows from standard differential topology arguments that:

**Lemma 15.** The set of tangles in general position is dense and open.

Mostly we will be working with framed tangles. A **framed tangle** consists of a pair \((C, \eta)\), where \( C \) is a tangle in general position and \( \eta \) is a continuous section of the unit normal bundle of \( C \) embedded in the manifold \( \mathbb{R}^2 \times [0, 1] \). A general point of a framed tangle consists of a pair \((x, \eta(x))\) where \( x \) is a point on \( C \) and \( \eta(x) \) is a vector associated to \( x \). We also will assume that \( \eta \) is a so-called right hand framing, namely that \( \eta \) restricted to \( \partial C \) is identically the vector \(<1, 0, 0>\). Also, we want to assume
that our framing vector is never \(< 0, 1, 0 >\) or \(< 0, -1, 0 >\). It is clear that the space of framed diagrams without this condition is dense in \(\mathcal{TOP}\).

Framed tangles are subject to isotopy: Two framed tangles \((C_1, \eta_1)\) and \((C_2, \eta_2)\) are isotopic if there is a continuous isotopy of the vector bundle \((C_1, \eta_1)\) onto \((C_2, \eta_2)\) such that \(\partial C\) is fixed and both the embedding and the discretedness conditions for crossing points are maintained at all times.

We say that \(C\) is a \textit{tangle of the form} \((n, m)\) if it has \(n\) open ends on top and \(m\) open ends on the bottom. We can compose a tangle \(C_1\) of the form \((p, q)\) with a tangle \(C_2\) of the form \((q, r)\) by stacking \(C_1\) on top of \(C_2\) and then scaling the height of the box, producing a new tangle called \(C_2 \circ C_1\). It is clear that this composition can be generalized to framed tangles with a right-hand framing. This operation can also be reversed. If we have a tangle \(C\) such that \(C \cap \{\mathbb{R} \times \{\frac{1}{2}\} \times [0, 1]\} \subseteq \{\mathbb{N} \times \{\frac{1}{2}\} \times \{0\}\}\) we can use an isotopy to write \(C\) as the composition of two separate tangles \(C = C_2 \circ C_1\).

Finally we can define a tensor product of topological tangles. Given a tangle \(C_1\) of the form \((n, m)\) and another tangle \(C_2\) of the form \((p, q)\) we can define a tensor product \(C_1 \otimes C_2\) by juxtaposing \(C_2\) to the right of \(C_1\). Namely, we place the tangle \(C_2\) directly to the right of \(C_1\) and then use a small isotopy to allow the \(i\)-th open end of \(C_2\) on the top to lie on the natural number \((n + i)\). We arrange the open ends on the bottom in a similar way, and then we have a topological tangle \(C_1 \otimes C_2\) of the form \((n + p, m + q)\).

**Definition 16.** The tensor category \(\mathcal{TOP}\) is the category whose objects are the set of non-negative integers \(\{0, 1, 2, \ldots\}\). The set of morphisms between two integers \(p\) and \(q\) are all the isotopy classes of framed tangles of the form \((p, q)\). Composition in this category is defined by the stacking procedure described above. The tensor product of two objects \(p\) and \(q\) is given by \(p + q\) and the tensor product of the morphisms is defined by the juxtaposition procedure defined above.
We should also point out that a framed tangle of the form \((0,0)\) is equivalent to the usual notion of a framed link. In this case we can think of the tips of the vectors in our field as tracing out the path of the framing curve. Using this framing curve we can construct 3-manifolds by surgery on \(S^3\) in the standard way.

As with knots and links, it is often more convenient to work with a two dimensional planar projection of a tangle. We will now begin to define a new category \(TGL\), a combinatorial category where the morphisms are generated by a fixed set of curves in the plane, modulo some relations. Once it is defined, it follows from standard differential topology arguments that \(TGL\) and \(TOP\) are equivalent categories.

Naively, we think of \(TGL\) as the category generated by applying \(\pi\) to a morphism in \(TOP\). Under this projection, all of the framing vectors get mapped to vectors lying in the plane. Formulaically, we write \(\pi : <a,b,c>\mapsto <a,c>\). This induces the so-called blackboard framing on the category \(TGL\).

The objects of \(TGL\) will once again the set of non-negative integers \(\{0,1,2,\ldots\}\). The morphism set of \(TGL\) will be defined with a fixed set of pictures that should be read from top to bottom. For example, a picture that has 5 open ends at the top and 3 open ends at the bottom will be considered an element of \(Hom(5,3)\). We can define a tensor product on this category by juxtaposing two different morphism pictures, and we once again define composition by stacking pictures. The set of morphisms is generated by taking tensor products and composites of five basic morphisms: \(\{\mid, \cap, \cup, \times, \times\}\). The pictures of these morphisms are shown below. Note that \(\{|\}\) is in \(Hom(1,1)\), \(\{\cap\}\) is in \(Hom(0,2)\), \(\{\cup\}\) in \(Hom(2,0)\) and finally \(\{\times\}\) and \(\{\times\}\) are in \(Hom(2,2)\).
Composition in this category is initially given by stacking pictures on top of each other. However, we will eventually want $\mathcal{TGL}$ to represent projections of framed topological tangles, and thus we must implement some relations on this composition in order to produce a well defined functor between $\mathcal{TOP}$ and $\mathcal{TGL}$. Those relations are as follows:

[1*] the modified 1st Reidemeister move

[2] the 2nd Reidemeister move

[3] the 3rd Reidemeister move
[TI] Cancelling out a consecutive minimum and maximum (or vice-versa)

[T2] pushing a strand across a maximum or a minimum.

[IND] the independence move, changing the relative heights of crossings, maxima or minima in distant sections of the diagram

Figure 2.9: The Relations in $\mathcal{TGL}$
Definition 17. The Category $\mathcal{TGL}$

The objects are the set of non-negative integers $\{0,1,2,\ldots\}$. The set of morphisms is generated by tensor products and composites of the five morphisms: $\{\mid, \cap, \cup, \times, \triangleright\}$ defined above, modulo the five relations above. The tensor product of two objects $p$ and $q$ is given by $p+q$ and the tensor product of the morphisms is defined by juxtaposition.

Lemma 18 ([21]). The categories $\mathcal{TOP}$ and $\mathcal{TGL}$ are equivalent.

The proof of Lemma 18 is similar to the well known piecewise linear proof of Reidemeister’s Theorem. We note the obvious functor $\mathcal{TGL} \rightarrow \mathcal{TOP}$ given by ‘pushing out’ a 2 dimensional diagram in $\mathcal{TGL}$ into 3-space. For the remainder of the paper, we will no longer differentiate between $\mathcal{TOP}$ and $\mathcal{TGL}$, and when making calculations we will work exclusively with the combinatorial category $\mathcal{TGL}$.

Occasionally we will wish to make use of another type of tangle category that does not have the modified first Reidemeister move as one of its relations. This will allow us to keep track of more precise framing information. We will refer to this category as $\mathcal{TGL}^*$. It is defined precisely as follows:

Definition 19. The objects are the set of non-negative integers $\{0,1,2,\ldots\}$. The morphisms are given by tensor products of the five morphisms: $\{\mid, \cap, \cup, \times, \triangleright\}$ defined above. Composition is defined by stacking pictures, modulo the relations $\{[2], [3], [T1], [T2]\}$ defined above. The tensor product of two objects $p$ and $q$ is given by $p+q$ and the tensor product of the morphisms is defined by juxtaposition.

2.5 The Category $\mathcal{BEAD}_H$

Another very important category in this paper is $\mathcal{BEAD}_H$, an immersion category closely related to $\mathcal{TGL}$ that intertwines framed tangle information with a ribbon
Hopf algebra $H$. The morphism set of this category is fairly complicated so it is best to break it down into several components and describe $Bead_H$ as a quotient of a simpler category.

The first step is to define an immersion category called $\mathcal{FLAT}$. It is a quotient category of $TGL^*$ where we do not differentiate between positive and negative crossings.

**Definition 20.** Define an equivalence relation $R$ in $TGL^*$ by $R = \{X = X\}$. Then the category $\mathcal{FLAT}$ is defined by:

$$\mathcal{FLAT} = TGL^*/R$$

We will denote the equivalence class of $X$ and $X$ in $\mathcal{FLAT}$ simply as $X$. Thus the morphisms of $\mathcal{FLAT}$ are given by tensor products of the four morphisms: $\{\mid, \cap, \cup, \times\}$. The relations of $TGL^*$ ([2],[3],[T1], [T2]) factor through as relations in $\mathcal{FLAT}$ called $[F2], [F3], [FT1], [FT2]$ respectively. (See Figure 2.10). We will occasionally refer to morphisms in $\mathcal{FLAT}$ as immersed tangles.

At this point it will be convenient to introduce some terminology that will be used for the duration. The morphism in $\mathcal{FLAT}$ given by $(\mid \otimes \cup) \circ (\times \mid) \circ (\mid \otimes \cap)$ will be referred to as a *right-hand loop*. Similarly, $(\cup \otimes \mid) \circ (\mid \otimes \times) \circ (\cap \otimes \mid)$ will be referred to as a *left-hand loop*. See Figure 2.11. Note that since we do not have any version of a type I Reidemeister move in $TGL^*$, these loops are non-trivial in $\mathcal{FLAT}$.

**Lemma 21 ([9]).** In the category $\mathcal{FLAT}$, a right-hand loop followed by immediately a left-hand loop (or vice-versa) is equivalent to the morphism $\{\mid\}$.

This is a special case of what is known as the *Whitney Trick*. The proof follows entirely from the relations in Figure 2.10. It allows us to prove the following lemma:

**Lemma 22.** Any morphism of type $(0,0)$ in $\mathcal{FLAT}$ is equivalent to a number of
Figure 2.10: The Relations on $\mathcal{FLAT}$. 

\begin{align*}
\begin{array}{c}
\text{F2} \\
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{f2} \\
\end{array}
\end{array}
&=egin{array}{c}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{f3} \\
\end{array}
\end{array} \\
\text{FT1} \\
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{ft1} \\
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
\text{FT2} \\
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{ft2} \\
\end{array}
\end{array}
\end{align*}
disjoint simple closed curves decorated with a sequence of only left-hand or only right-hand loops.

Proof. It is clear that we can separate the different components of our closed morphism using the Type II and Type III moves above, so we may assume that there is only one component. The Whitney-Graustein Theorem says that the regular homotopy type of any curve immersed in the plane depends only on its total curvature. Combined with Lemma 21, the relations on $\mathcal{FLAT}$ preserve regular homotopy of immersed curves, so we may choose the representative of our isomorphism class to be in the form of the lemma.

Now we can begin to describe $\mathcal{BEAD}$. We start by taking a morphism $C$ in $\mathcal{FLAT}$ and decorate it with a finite number of dots or ‘beads’. We do this by replacing an occurrence of $\{\|$ with $\{\bullet^k\}$, where $k$ is an integer unique to that bead. We emphasize that we only place these beads on vertical pieces of $C$, and not at crossing points, minima or maxima. If there are $n$ beads placed on a curve $C$, we label each bead with a unique integer from the set $\{1, 2, \ldots, n\}$. This gives us an ordering of the beads. This object will be called a decorated tangle.

We can define a composition of decorated tangles. Let $C_1$ be a decorated tangle with $n$ beads of the form $(p, q)$ and $C_2$ be a decorated tangle with $m$ beads of the
form \((q,r)\). Then we can define \(C_2 \circ C_1\), the composition of \(C_1\) and \(C_2\), in the following manner. We compose the underlying immersed tangles via the stacking method described for \(\mathcal{FLAT}\). We then have to change the labeling of the \(m\) beads on \(C_2\); if a bead on \(C_2\) was originally labeled with \(i\) we re-label it with \(i + n\). We do not change the labeling of the beads on \(C_1\). This gives us a new decorated tangle, \(C_2 \circ C_1\), with \(n + m\) beads of the form \((p,r)\).

Similarly we can define the tensor product of two decorated tangles. Let \(C_1\) be a decorated tangle with \(n\) beads and \(C_2\) a decorated tangle with \(m\) beads. We begin by juxtaposing \(C_2\) directly to the right of \(C_1\) as in \(\mathcal{FLAT}\). We then once again relabel the beads of \(C_2\) by adding \(n\) to the previous label. The resulting decorated tangle with \(n + m\) beads is defined to be \(C_1 \otimes C_2\).

In our category, decorated tangles are subject to a form of isotopy. First of all, if a piece of the decorated tangle is unbeaded, the five moves (Figure 2.10) from the category \(\mathcal{FLAT}\) still apply to that piece. We also have two additional moves that allow us to move the beads around the diagram:

\[
[B1] \quad (| \circ \mathbf{k}) \simeq (\mathbf{k} \circ |) \quad \text{for any } k \\
[B2] \quad (\times \circ \mathbf{m} \otimes \mathbf{k}) \simeq (\mathbf{k} \otimes \mathbf{m} \circ \times) \quad \text{for any } k \text{ and } m \quad \text{(See Figure 2.12)}
\]

Putting this all together we have the following:

**Definition 23. The category \(\mathcal{BEAD}\)**

The objects are non-negative integers \(\{0, 1, 2, \ldots\}\). The morphisms are given by tensor products of the five morphisms: \(\{|, \cap, \cup, \times, \mathbf{k}\}\). Composition is defined by the stacking and relabeling procedure, modulo the seven moves described above. The tensor product of two objects \(p\) and \(q\) is given by \(p + q\) and the tensor product of the morphisms is defined by juxtaposition and relabeling procedure above.
Finally we are ready to begin to describe the category $\mathcal{BEAD}_H$. Each morphism of $\mathcal{BEAD}_H$ has two components: The first is a decorated tangle $C$ with $n$ beads and the other is an element $a$ of the Hopf algebra $H^\otimes n$. A typical element $a$ of $H^\otimes n$ is a finite sum of elements of the form $\sum_\nu a_1^\nu \otimes a_2^\nu \otimes \cdots \otimes a_n^\nu$ where all the $a_i^\nu \in H$. However, as is typical in Hopf algebra literature, we will suppress all of the $\sum$ signs. Adopting this convention, we write out a general element $a \in H^\otimes n$ as $a = a_1^\nu \otimes a_2^\nu \otimes \cdots \otimes a_n^\nu$, then we associate the symbol $a_i^\nu \in H$ with the $i$-th bead in our decorated diagram $C$. To any decorated tangle with 0 beads, we assign the element $1 \in k$ to this tangle.

The morphism set of our category $\mathcal{BEAD}_H$ will be isotopy classes of pairs $\{C, a\}$, where $C$ is a curve decorated with $n$ beads and $a$ is an element of the Hopf algebra $H^\otimes n$ over all $n \in \mathbb{N}$.

Composition in $\mathcal{BEAD}_H$ is straightforward. If $C_1$ and $C_2$ are two decorated tangles such that $C_2 \circ C_1$ exists in $\mathcal{BEAD}$, we define $\{C_2, b\} \circ \{C_1, a\}$ to be $\{C_2 \circ C_1, a \otimes b\}$. A tensor product of morphisms is also well defined. Namely, if we have two morphisms $\{C_1, a\}$ and $\{C_2, b\}$, the tensor product is defined by $\{C_1, a\} \otimes \{C_2, b\} = \{C_1 \otimes C_2, a \otimes b\}$, with $C_1 \otimes C_2$ defined as in $\mathcal{BEAD}$. 

Figure 2.12: The crossing map in $\mathcal{BEAD}$
When defining the category we will also include a number of relations that will provide information about the interaction between the tangle and the Hopf algebra. These are outlined below:

For all of the following let \( \{C, a\} \) be a pair where \( C \) is a decorated tangle with \( n \) beads and \( a \in H^\otimes n \) with \( a = a_1^\nu \otimes a_2^\nu \otimes \cdots \otimes a_n^\nu \) with all the summation signs suppressed. For a geometric version of these relations, see Figure 2.13.

[B3] **Permutation Equivalence**: Let \( \pi \) be an element of the permutation group \( S_n \). Then \( \{C, a\} \cong \{C^\pi, \pi(a)\} \), where \( C^\pi \) is \( C \) except a bead labeled with \( i \) is relabeled with \( \pi(i) \) and \( \pi(a) = a_{\pi(1)}^\nu \otimes a_{\pi(2)}^\nu \otimes \cdots \otimes a_{\pi(n)}^\nu \)

[B4] **Bead Birth/Death**: Let \( C \) have an instance of the morphism \( \{\}\). Then \( \{C, a \otimes 1\} \cong \{C', a\} \), where \( C' \) is same decorated tangle as \( C \), except that this \( \{\}\) is replaced with \( \{\bullet^{n+1}\} \)

[B5] **Combination**: Let \( C \) be such that two beads lie on the same vertical strand. (Algebraically, this means that an instance of \( \bullet^k \otimes \bullet^l \) occurs somewhere in \( C \).) By [B3] above, we may assume that these beads are labeled with \( n \) and \( n - 1 \), and \( n - 1 \) lies directly above \( n \) with respect to the vertical direction. Then we have that \( \{C, a\} \cong \{C', a'\} \) where \( C' \) is the same decorated tangle except with beads 1 and 2 combined into a single bead, and \( a' \in H^\otimes (n-1) \) with \( a' = a_1^\nu \otimes a_2^\nu \otimes \cdots \otimes a_{n-2}^\nu \otimes a_{n-1}^\nu \otimes a_n^\nu \)

[B6] **Cap move**: Let \( C \) be such that the bead labeled with \( i \) is directly under the right hand piece of the \( \cap \). Then \( \{C, a\} \cong \{C', a'\} \) where \( C' \) is the same decorated tangle except the \( i \)-th bead will be located directly under the left side of the same \( \cap \) and \( a' = a_1^\nu \otimes \cdots a_{i-1}^\nu \otimes S(a_i^\nu) \otimes a_{i+1}^\nu \otimes \cdots \otimes a_n^\nu \)

[B7] **Cup move**: Let \( C \) be such that the bead labeled with \( i \) is directly over the left hand piece of the \( \cup \). Then \( \{C, a\} \cong \{C', a'\} \) where \( C' \) is the same decorated
tangle except the $i$-th bead will be located directly over the right side of the same $\cup$ and $a' = a_1^{\nu} \otimes \ldots a_{i-1}^{\nu} \otimes S(a_i^{\nu}) \otimes a_{i+1}^{\nu} \otimes \ldots \otimes a_n^{\nu}$

[B8] G-bead Birth: This move only applies in the special case where $H$ is quasi-triangular. Let $C$ have a right-hand loop. Then we have that $\{C, a\} \cong \{C', a \otimes G^{-1}\}$ where $C'$ is the same decorated tangle except we have removed the right hand loop and replaced it with $\{\bigotimes_n \nu \bigoplus^{n+1}\}$. For left-hand loops, we have that $\{C, a\} = \{C', a \otimes G\}$ where $C'$ is the same decorated tangle except we have removed the left hand loop and replaced it with $\{\bigotimes_n \nu \bigoplus^{n+1}\}$.

**Definition 24.** The set of objects of $\mathcal{BEAD}^*_H$ is the set of non-negative integers $\{0, 1, 2, \ldots\}$. The morphisms are pairs $\{C, a\}$, where $C$ is a morphism of $\mathcal{BEAD}$ with $n$ beads and $a$ is an element of $H^{\otimes n}$. The pair is itself an isotopy class modulo the six relations described above. Composition is given by the composing the tangles in $\mathcal{BEAD}$ and tensoring the Hopf algebra elements. Tensor product of morphisms are given by tensoring the tangles in $\mathcal{BEAD}$ and tensoring the Hopf algebra elements.

This concludes the formal definition of $\mathcal{BEAD}_H$. Since $\mathcal{BEAD}_H$ is clearly a very geometric category, it will greatly help the exposition of the paper if we describe a geometric shorthand for working with it. We can draw out a tangle diagram in $\mathcal{BEAD}$ and instead of labeling each of the beads with an integer, we label them directly with the corresponding elements of the Hopf algebra. Abstractly, these labels might represent a single element of $H$, or they may be a finite sum of such elements. In general we will indicate that the label represents a sum by writing a $\nu$ (or other appropriate summation index) on the label. We can think of the eight relations given above as a calculus for pushing the beads around the diagram. As an example, Figure 2.13 below gives geometric pictures of the relations [B4]-[B8]. Our geometric shorthand parallels the notation defined in [8].
Figure 2.13: Relations [B4]-[B8] in the geometric shorthand notation
3.1 The Hennings 3-manifold Invariant

We will briefly describe the construction of the Hennings Invariant. We will mostly follow the method described in [8] with a couple of small differences that we will point out along the way. Throughout this section, we will have a fixed quasi-triangular and unimodular Hopf algebra $H$ with universal R-matrix $R = \sum \nu a_\nu \otimes b_\nu$.

The Hennings invariant was first constructed as an invariant of framed links. However, after multiplying by a particular scalar, it factors through the Kirby moves, and is therefore a 3-manifold invariant via Dehn surgery.

We start with an abstract framed link $L$ embedded in $S^3$. Recall that a framed link is a framed topological tangle of the form $(0,0)$. Using Lemma 18, we can regard $L$ as a morphism in the category $\mathcal{TGL}$. Henceforth, whenever we refer to the link $L$ we really will be referring to this composition of morphisms in $\mathcal{TGL}$.

The first step in constructing the Hennings invariant will be a functor from $\mathcal{TGL}$ to $\mathcal{BEAD}_H$. Using Definition 17, we know that all the morphisms in $L$ are generated by the following five morphisms: $\{\|, \cup, \cap, \mathbf{x}, \mathbf{y}\}$. We will define our functor $\mathcal{G}$ in terms of these generators.

**Definition 25.** The tensor functor $\mathcal{G} : \mathcal{TGL} \to \mathcal{BEAD}_H$ is defined by:
\* \( G(n) = n \)

\* \( G(\sqcup) = \sqcup \), \( G(\sqcap) = \sqcap \)

\* \( G(\sqcap) = (\times \circ (\bullet^1 \otimes \bullet^2), b'' \otimes a'') \)

\* \( G(\sqcup) = (\times \circ (\bullet^1 \otimes \bullet^2), a'' \otimes S^{-1}(b'')) \)

The functor \( G \) sends both the objects and the three morphisms \( \{\sqcup, \sqcap, \sqcup\} \) of \( TGL \) to their corresponding counterparts in \( BEAD_H \). The positive crossing in \( TGL \) gets sent to the crossing in \( BEAD_H \) “beaded” with \( R^{-1} \) below the crossing. (As is standard in our definition of \( BEAD_H \) the \( \sum \) signs get dropped.) The negative crossing gets sent to the immersion crossing “beaded” with \( \tau(R) \) below the crossing.

**Lemma 26.** [8] \( G \) is well defined.

We note that our functor is slightly different that the one that appears in [8]. For the positive crossing we apply the \( \tau \) action first followed by the R-matrix action, whereas Kauffman and Radford apply the R-matrix first. Since our relations in \( BEAD_H \) are also different, it isn’t difficult to show that the formulations are equivalent.

**Lemma 27.** For any link \( L \), \( G(L) \) can be represented as a number of disjoint unknots, each with a single bead.

**Proof.** After applying the functor \( G \) we are left with an immersed beaded link with a number of framing twists. The next step is to use the combination move ([B5]) along with the cup and cap moves ([B6] and [B7]) to combine all of the beads on each component of the link. We can now separate all of our components into disjoint loops using [F2] and [F3], possibly using [B1] and [B2] if the bead appears in the relevant local picture. We use the \( G \)-bead birth rule ([B8]) to replace all of the right and
left-hand loops with beads labeled with $G$ or $G^{-1}$. We once again use [B5], [B6], and [B7] to combine all the beads on each component of the link. Finally we can use [B4] to ensure that each component has a single bead (which might be the identity).

It should be noted that the Hopf algebra tensor assigned to each of the beads in the above lemma is not unique. This is because one can still apply the multiplication move ([B5]) and the cap and cup moves ([B6] and [B7]) without altering the unknot. This forces us to define an equivalence relation between Hopf algebra elements.

**Definition 28.** Let $x, y \in H$. Define a relation $\sim$ on $H$ by:

$$x \sim y \iff \text{there exists } p, q \in H \text{ such that } x = pq \text{ and } y = S^i(S^2(q)p)$$

for some $i \in \mathbb{Z}$.

**Lemma 29.** [8] The relation $\sim$ is an equivalence relation and the bead described on each unknot in Lemma 27 is unique up to $\sim$.

Using the above lemmas we can assume $G(L)$ gives a morphism in $\mathcal{BEAD}_H$ that topologically is $n$ uniquely beaded unknots with an associated class of Hopf algebra element $a = a_1^* \otimes \cdots \otimes a_n^* \in H^n$ where each $a_i$ is a representative of the equivalence class under $\sim$.

The final step is to fix an integral $\mu$ in our Hopf algebra. It turns out that $\mu$ is invariant under our equivalence relation.

**Lemma 30.** Let $H$ be a unimodular Hopf algebra, with $p, q \in H$. Then

$$\mu(S^i(S^2(q)p)) = \mu(pq)$$

**Proof.** Since $H$ is unimodular, $\alpha = \varepsilon$. Thus we use Lemma 12 twice to show:

$$\mu(S^i(S^2(q)p)) = \mu(S^2(q)p) = \mu(pq)$$

36
Finally we define the element $I^n = \mu \otimes \mu \otimes \cdots \otimes \mu$ in $(H^*)^\otimes n$. Then, by Lemma 30, $I^n(a)$ is a scalar in the field $k$. This scalar is our link invariant, $Henn(L, H)$.

In general, the above operations are not preserved by the Kirby moves. In order to make a proper 3-manifold invariant, we have to scale the link invariant to account for some extra factors that appear after a Kirby move is preformed. Namely, for a 3-manifold $M_L$ given by surgery in $S^3$ along a link $L$, we have:

$$Henn(M_L, H) = [\mu(\nu)\mu(\nu^{-1})]^{-c(L)/2}[\mu(\nu)/\mu(\nu^{-1})]^{-\sigma(L)/2}Henn(L, H) \quad (3.1.1)$$

where $\nu$ is the ribbon element, $c(L)$ is the number of components of $L$, and $\sigma(L)$ is the signature of the linking matrix of $L$. For a proof of the invariance of $Henn(M, H)$, see [8].

3.1.1 A Hennings Invariant example: $L(p, 1)$

As an example, we will compute the Hennings Invariant for 3-dimensional Lens spaces $L(p, 1)$, where $p$ is a prime number. For the following we will assume that $H$ is a unimodular, ribbon Hopf algebra with R-matrix $R = \sum \nu a^\nu \otimes b^\nu$ and ribbon element $v$.

$L(p, 1)$ is given by surgery in $S^3$ along the unknot with framing integer $p$. In the blackboard framing, this can be represented by an unknot with $p$ right-hand loops all with negative crossings. (see picture below)

Now we focus in on one of the loops. Under our functor $G$, $\{X\}$ gets sent to $\{\otimes k \otimes k+1, a_\nu \otimes S(b_\nu)\}$. Using the relations in $\mathcal{BEAD}_H$, we see that the image under $G$ of the loop is equivalent to $\{k, G^{-1}S(b_\nu)a_\nu\} = \{k, v\}$. (Figure 3.2 describes the geometric version of this calculation)
Using the fact that $S(v) = v$, we can choose the representative for the class under the relation $\sim$ to be the Hopf algebra element $v^p$. Now we can see that our link is simply an unknot labeled with this $v^p$. Thus we have that $Henn(L, H) = \mu(v^p)$.

Finally since $L$ has only a single component and the framing is positive we have $\sigma(L) = c(L) = 1$. Thus we can conclude that:

$$Henn(L(p, 1), H) = \frac{1}{\mu(v)} \cdot \mu(v^p) \quad (3.1.2)$$

In particular since, $L(1, 1) = S^3$ and $L(0, 1) = S^1 \times S^2$ we can conclude:

$$Henn(S^3, H) = 1 \quad (3.1.3)$$

and

$$Henn(S^1 \times S^2, H) = \frac{1}{\mu(v)} \cdot \mu(1) \quad (3.1.4)$$
3.2 The Involutory Kuperberg Invariant

We will now describe how to construct the Kuperberg invariant with an involutory Hopf algebra. For simplicity of exposition, we will also assume that $H$ is sem-simple and co-semi-simple, although it is not strictly necessary. We will begin by discussing Heegaard diagrams which we will then use to form Hopf algebra tensors. The following description will mainly be based on the description of the general Kuperberg invariant given in [15], even though we will be using involutory Hopf algebras. It should be noted that all of the framing information for the 3-manifold described in that paper does not affect the value of the invariant when $H$ is involutory, so we can freely omit it and simply work with unframed manifolds as in [14].

A Heegaard decomposition of a 3-manifold $M$ consists of two handlebodies $H_1$ and $H_2$ of the same genus and a homeomorphism $f : \partial H_1 \to \partial H_2$ such that $H_1 \cup_f H_2 = M$. It is well known that all 3-manifolds have such a decomposition. We will refer to $H_1$ as the lower handlebody and $H_2$ as the upper handlebody. The surface $\partial H_1 \sim \partial H_2$ is called the Heegaard surface of the splitting.

A handlebody diagram is a convenient pictorial device that shows us how to glue a handlebody of genus $n$ onto a surface $\Sigma_n$. It is formed by drawing a family of disjoint circles $\{C_i\}$ onto $\Sigma_n$ such that $\Sigma_n - \{\cup_i C_i\}$ is homeomorphic to a number of disjoint discs. We can reconstruct the handlebody with the following method. First we form a family of ‘thickened’ discs, namely a copy of $D^2 \times [-\varepsilon, \varepsilon]$ (where $\varepsilon$ is chosen to be sufficiently small). We then glue each disc onto $\Sigma_n$ by mapping the boundary piece $S^1 \times [-\varepsilon, \varepsilon]$ onto a neighborhood of each circle $C_i$ in $\Sigma_n$. Now we are left with an object whose boundary has a number of $S^2$ components. We eliminate the boundary by gluing in 3-balls along all of these boundary components. We are then left with a handlebody.

Finally a Heegaard diagram is a pair of handlebody diagrams on a single surface
Figure 3.3: A genus 2 Heegaard diagram representing $S^3$

$\Sigma_n$ that give a Heegaard splitting for a 3-manifold $M$. The circles corresponding to the handlebody $H_1$ will be called lower circles and those corresponding to $H_2$ will be upper circles. A lower circle can intersect an upper circle or circles but not other lower circles (and vice-versa). We will refer to any point where a lower circle meets an upper circle as an intersection point. We additionally require that the circles are arranged in general position, where all intersection points are transverse and there are finitely many of them.

**Lemma 31.** [14] Two different Heegaard diagrams will produce the same 3-manifold if and only if they are related by the following set of moves:

- Isotopies of the diagram
- Adding or deleting a circle that bounds a disc
- The two-point move, which creates or removes two consecutive intersections between circles. (See Figure 3.4)
- The circle slide of an upper circle past another upper circle, or a lower circle past another lower circle.
- Changing the genus by a stabilization move. This consists of removing a disc from $\Sigma_n$ that does not meet any circles and gluing in a ‘trivial handle’. (A trivial handle is a copy of $\{\Sigma_1 - D^2\}$ with a lower circle equivalent to the longitudinal
homology generator and an upper circle equivalent to the latitudal homology generator.) We can also reduce the genus by replacing a trivial handle with a disc.

• Switching the labelings of \( H_1 \) and \( H_2 \).

We are now ready to begin describing the invariant. First we take our 3-manifold \( M \) and we form a Heegaard splitting, with a corresponding Heegaard diagram. We will need to work with oriented Heegaard diagrams, where each of the lower and upper circles have both an orientation and a fixed base point. We can choose the orientations randomly and we can make any point that is not an intersection point a base point, and neither choice will affect the outcome of the invariant (See [14]).

Let \( \{L_i\} \) and \( \{U_i\} \) be the set of lower circles and upper circles respectively. We will say that an intersection point is positive if the tangent vectors to the two oriented circles from a positive basis for \( \mathbb{R}^2 \) with respect to the orientation of the surface. If an intersection point is not positive, then we say it is negative.

Since our circles are oriented, it is important to note that we have an ordering for the the intersection points of a fixed circle. To form this ordering we start at the base point of a circle \( \{L_i\} \) and trace around in the direction of the orientation, we can label the first intersection point as \( L_{(i,1)} \), the second as \( L_{(i,2)} \) and so on. We can
of course do the same for all of the upper circles, so each intersection point will have two different labels: \( L_{(i,j)} \) and \( U_{(k,l)} \).

Since \( H \) is both involutory and semi-simple, we know that \( H \) has integrals and co-integrals and it is unimodular and co-unimodular. We then fix a choice of \( \Lambda \) and \( \mu \) so that \( \mu(\Lambda) = 1 \).

We will now use the Heegaard diagram information to form a closed tensor in the algebra \( H \). To each upper circle \( \{ U_i \} \), we assign a copy of the integral \( \mu_i \), and to each lower circle \( \{ L_j \} \) we assign a co-integral \( \Lambda_j \). (All of these copies of \( \mu \) and \( \Lambda \) are identical, the labels \( i \) and \( j \) only exist to differentiate them from each other.)

Now if a particular lower circle \( \{ L_i \} \) has \( m \) intersection points. We then take the corresponding copy of \( \Lambda \), and apply the coproduct \( \Delta \) in \( H \) \( m \) times to produce the element: \( \Delta^m(\Lambda_i) = \Lambda_i^{(1)} \otimes \Lambda_i^{(2)} \otimes \cdots \otimes \Lambda_i^{(m)} \in H^{\otimes m} \). We then label the intersection point \( L_{(i,j)} \) with the element \( \Lambda_i^{(j)} \) if the intersection point is positive and with \( S(\Lambda_i^{(j)}) \) if it is negative. We apply the same process to all of the lower circles.

For an upper circle \( \{ U_j \} \) with \( l \) crossing points we apply the co-product \( \Delta^* \) in \( H^* \) \( l \) times to form the element \( (\Delta^*)^l(\mu_j) = \mu_j^{(1)} \otimes \mu_j^{(2)} \otimes \cdots \otimes \mu_j^{(l)} \) in \( (H^*)^{\otimes l} \). We then label the intersection point \( U_{(j,k)} \) with the element \( \mu_j^{(k)} \) in \( H^* \).

Now each crossing point of our diagram is labeled with exactly one element of \( H \) and exactly one element of \( H^* \). To form our invariant, we simply use our usual evaluation morphism to contract all of these and produce a scalar in \( k \), call it \( Z(M,H) \).

Finally, \( \text{Kup}(M,H) \) is defined to be:

\[
\text{Kup}(M,H) = Z(M,H)(\dim H)g(\Sigma)(\mu(1))^{-n(U)}(\varepsilon(\Lambda))^{-n(L)}
\] (3.2.1)

where \( g(\Sigma) \) is the genus of the surface \( \Sigma \), \( n(U) \) is the number of upper circles and \( n(L) \) is the number of lower circles.

**Lemma 32.** ([14],[15]) \( \text{Kup}(M,H) \) depends only on the manifold \( M \) and the Hopf algebra \( H \), and is independent of the choice of the oriented Heegaard diagram.
3.2.1 A Kuperberg invariant example: \( L(p,1) \)

It is easy to construct a Heegaard diagram for the Lens space \( L(p,1) \). We take the surface to be a torus with the standard longitudinal homology generator as the lone upper circle. For the lower circle, we take the torus knot \( T(p,1) \). We then choose orientations of the circles such that all of the crossings are positive. (See the example below, with the upper circle in black and the lower circle in grey)

![Figure 3.5: A two-dimensional L(3,1) Heegaard diagram](image)

Now there are \( p \) crossing points on the diagram, all of which are positive. Also, we can choose the basepoints such that the crossings occur in the same order on both circles (as in Figure 3.5). Hence we have that:

\[
Kup(L(p,1), H) = \mu(\Lambda^{(1)} \cdot \Lambda^{(2)} \ldots \Lambda^{(p)})
\]  

Choose \( \Lambda \) and \( \mu \) such that \( \Lambda(\mu) = 1 \), \( \mu(1) = 1 \), and \( \varepsilon(\Lambda) = (\dim H) \). Now, if \( p = 1 \) we get

\[
Kup(S^3, H) = \mu(\Lambda) = 1
\]
Finally, if we assume that $\Lambda^{(0)} = 1_H$ and $\mu^{(0)} = \varepsilon$ we get:

\[
Kup(S^1 \times S^2, H) = \mu(1_H) \cdot \varepsilon(\Lambda) = (\dim H)
\]  

(3.2.4)

### 3.2.2 The Kuperberg Invariant of $L(3, 2)$

We now will look at another example where we calculate the Kuperberg invariant, except this time we will be able to see how a twist in the diagram changes the order of the multiplication in the Hopf algebra.

Consider the genus 1 Heegaard diagram for $L(3, 2)$ given in Figure 3.6. Once again, the upper circle is in black and the lower circle is in grey.

![Figure 3.6: A two-dimensional L(3,2) Heegaard diagram](image)

If we trace along the upper circle starting at the basepoint, we see that the first crossing on the upper circle is the second crossing for the lower circle. Similarly the second crossing for the upper circle is the first for the lower. Thus we have the following:

\[
Kup(L(3, 2), H) = \mu(\Lambda^{(2)} \Lambda^{(1)} \Lambda^{(3)})
\]  

(3.2.5)
However, we know from the classification of Lens spaces that $L(3, 2)$ is homeomorphic to $L(3, 1)$. Thus by invariance we have shown the following non-obvious involutory Hopf algebra relation:

$$\mu(\Lambda^{(1)}\Lambda^{(2)}\Lambda^{(3)}) = \mu(\Lambda^{(2)}\Lambda^{(1)}\Lambda^{(3)})$$

(3.2.6)

Thus we have used topology to answer a question about algebra. It is an interesting question whether these invariants can quickly produce other Hopf algebra identities that are not readily apparent.
CHAPTER 4
THE HENNINGS IN Variant AS A TANGLE FUNCTOR

Our first step in showing the equivalence of the two invariants will be to define the Hennings Invariant in terms of a functor \( \mathcal{F}_H : TGL \to VECT \), where \( VECT \) is the category of vector spaces, and \( H \) is an involutory ribbon Hopf algebra. The image of the functor is contained in the tensor algebra of this \( H \). In fact, \( \mathcal{F} \) has the property that a tangle of the form \((k,l)\) under our functor will get mapped to an element in \( \text{Hom}(H^\otimes k, H^\otimes l) \) in \( VECT \). In particular, a \((0,0)\) tangle, (i.e. a link) will get sent to a morphism from \( k \) to itself. This element of \( k \) will be shown to be closely related to the value of the original Hennings invariant for links.

For the remainder of this chapter we will assume that \( A \) is an involutory Hopf algebra over a field \( k \) such that \( \text{char}(k) \) does not divide \( \text{dim}(A) \). Thus by Lemma 10, \( A \) is both unimodular and co-unimodular.

**Definition 33.** We will define a map \( \mathcal{H} : BEAD_H \to VECT \) on the generators of \( BEAD_H \) by:

- \( \mathcal{H}(n) = H^\otimes n \)
- \( \mathcal{H}(|) = id : H \to H \)
- \( \mathcal{H}(\cdot, a) = L_a : H \to H \)
- \( \mathcal{H}(\cap)[1] = \Delta(\Lambda) : k \to H \otimes H \)
\( H(\cup) : H \otimes H \to k : (x \otimes y) \mapsto \mu(xS(y)) \)

\( H(\times) : H \otimes H \to H \otimes H : (x \otimes y) \mapsto (y \otimes x) \)

We then extend \( H \) to all of \( \mathcal{B}_{\mathcal{EAD}} \) by tensoring and composing the generators.

**Theorem 34.** Let \( H = D(A) \) where \( A \) is a unimodular and co-unimodular Hopf algebra. Then \( H \) is a well-defined functor.

**Proof.** In order to prove the theorem we must show that \( H \) factors through all of the relations in \( \mathcal{B}_{\mathcal{EAD}} \). For the following, let \( x, y, a, b \in H \)

[B2] It is clear that \( \tau \circ (L_x \otimes L_y) \) and \( (L_y \otimes L_x) \circ \tau \) are equivalent functions on \( H \otimes H \).

[B5] This just follows from \( L_y \circ L_x = L_{yx} \).

[B6] Assume that the bead represents the Hopf algebra element \( S(x) \). Then \( H \) applied to the left hand side gives the element \( S(x) \Lambda' \otimes \Lambda'' \). Using Lemma 11, this is the same as \( \Lambda' \otimes x \Lambda'' \), which is what you get when you apply \( H \) to the right hand side.

[B7] The left hand side gives \( \mu \circ M \circ (L_x \otimes S) \) in \( (H \otimes H)^* \). Under this map, \( (a \otimes b) \mapsto \mu(xaS(b)) \). The right hand side gives \( \mu \circ M \circ (id \otimes S) \circ (id \otimes L_{S(x)}) = \mu \circ M \circ (id \otimes R_x) \circ (id \otimes S) \). So, \( (a \otimes b) \mapsto \mu(aS(b)x) \). Since \( A \) is unimodular and co-unimodular, we know that \( \alpha_D = \varepsilon \otimes 1_A \). Combing this with Lemma 12 and the fact that \( S^2 = id \) gives us \( \mu(xaS(b)) = \mu(aS(b)x) \). Thus the maps given by applying \( H \) on the left and right hand sides of the relation are the same.

[B8] \( H \) applied to a right-hand loop gives the tensor \( \Lambda' \otimes S(\Lambda'') \triangleright \mu \). By Lemma 11, this reduces to the identity. Now by Lemma 13, \( G = 1_H \), so the relation is satisfied.
[FT1] $\mathcal{H}$ applied to the left hand side gives $S(\Lambda') \triangleright \mu \otimes \Lambda''$. The right hand side gives $\Lambda' \otimes S(\mu) \triangleleft (\Lambda'')$. It follows from Lemma 11 that these are the same.

- The relations [B1], [B3], [B4], [F2], [F3], [FT2] and [FT3] are all immediate.

\[\square\]

**Definition 35.** $\mathcal{F} = \mathcal{H} \circ \mathcal{G}$ where $\mathcal{G}$ is the Kauffman and Radford functor described in Chapter 3.

![Diagram](image)

Figure 4.1: A pictorial definition of $\mathcal{F}$

Now $\mathcal{F} : T\mathcal{GL} \rightarrow \mathcal{VECT}$. Let $x, y \in H$ and let $\mathcal{R} = \sum_{\nu} a^\nu \otimes b^\nu$ If we compose $\mathcal{G}$ and $\mathcal{H}$ along the generating morphisms for $T\mathcal{GL}$, we get the following explicit formula for $\mathcal{F}$:

\[
\begin{align*}
\mathcal{F}(\text{ }) &= \text{id}_H \\
\mathcal{F}(\cap) &= 1 \mapsto \Delta(\Lambda) \\
\mathcal{F}(\cup) &= \mu \circ M \circ (id \otimes S) := (x \otimes y) \mapsto \mu(xS(y)) \\
\mathcal{F}(\times) &= L_{\mathcal{R}} \circ \tau := (x \otimes y) \mapsto \sum_{\nu} (a^\nu y \otimes b^\nu x) \\
\mathcal{F}(\times) &= \tau \circ L_{\mathcal{R}^{-1}} := (x \otimes y) \mapsto \sum_{\nu} (S(b^\nu)y \otimes a^\nu x)
\end{align*}
\]
Now we will show that $\mathcal{F}$ is related to the Hennings invariant. This is the only known description so far of the Hennings invariant as a functor of tangles, although it is limited to the case where $H = D(A)$ for an involutory and unimodular $A$.

**Theorem 36.** Let $H = D(A)$ for some involutory, unimodular, and co-unimodular Hopf algebra $A$. Let $L$ be a link with $c(L)$ components. Then $\mathcal{F}(L) = (\dim(A))^{c(L)} \cdot \text{Henn}_H(L)$.

**Proof.** First we apply the functor $G$. We can use Lemma 27 to write $G(L)$ as $m$ disjoint unknots $\{U_1, U_2, \ldots, U_m\}$, all with a single bead. The Hopf algebra element associated to each of these beads is unique up to the equivalence relation $\sim$. Let $x_i$ be a representative for this class for the bead on each unknot $U_i$. Applying $H$ to a particular one of these components $U_i$ gives us:

$$\varepsilon_H(\Lambda_H) \cdot \mu_H(x_i) = \varepsilon_A(\Lambda_A) \cdot \mu_A(1) \cdot \mu_H(x_i) = (\dim A) \cdot \mu(x_i)$$

Applying $H$ to all of these components simultaneously then gives:

$$(\dim A)^{c(L)}(\mu_H(x_1) \otimes \cdots \otimes \mu_H(x_{c(L)})) = (\dim(A)^{c(L)}) \cdot I^{c(L)}(x_1 \otimes \cdots \otimes x_{c(L)}) = (\dim(A)^{c(L)})\text{Henn}(L, D(A))$$

as desired. \hfill $\square$

Combining equation (3.1.1) and Theorem 36 gives us the following:

**Theorem 37.** Let $H = D(A)$ for some involutory Hopf algebra $A$. Let $L$ be a link and $M_L$ be the manifold given by surgery on $L$. Then:

$$\mathcal{F}(L) = [\dim(A)]^{c(L)}[\mu(\nu)\mu(\nu^{-1})]^{c(L)/2}[\mu(\nu)/\mu(\nu^{-1})]^{\sigma(L)/2} \cdot \text{Henn}(M_L, H)$$

In summary, when $H = D(A)$, for an involutory, unimodular and co-unimodular Hopf algebra $A$, our functor $\mathcal{F}$ applied to a tangle of type $(0,0)$, produces the Hennings invariant defined by Kauffman and Radford times the scalar multiple $(\dim(H))^{c(L)}$. 

49
Due to the surgery correspondence between links and 3-manifolds, all 3-manifold invariants are indirectly link invariants as well. In this chapter, we will construct a version of the Kuperberg invariant for links. This will allow us to easily compare the Kuperberg and Hennings invariants without having to first construct the 3-manifold. We will do this by considering the link $L$ to be a morphism in the category $\mathcal{TGL}$, and then we will assign a specific Hopf algebra morphism in $\mathcal{VECT}$ to each of the five generating tangle pieces. When all of these morphisms are composed we will be left with a map from the field $k$ to itself. This scalar will be the Kuperberg invariant of the 3-manifold given by surgery along $L$.

5.1 Blackboard framed links as braid closures

Computing the Kuperberg invariant requires orientations on all of the circles in the Heegaard diagram. However, since the Hennings invariant does not depend on an orientation of the link $L$, we would like to keep the correspondence between the generating pieces of $\mathcal{TGL}$ and the morphisms in $\mathcal{VECT}$ independent of link orientation as much as possible. In order to do this, we must first express our blackboard framed link $L$ as a braid closure.

First we give a geometric definition of a braid.
Definition 38. A braid of order $n$ is a morphism in $\mathcal{TG\mathcal{L}}$ of the form $(n,n)$ generated by only $\{|,\times,\times\}$

Braids and links are closely related. If we take a braid of order $n$ and connect the opposite open ends using disjoint arcs we clearly have a link. Formally we define:

Definition 39. Let $B$ be a braid of order $n$. We define the closure of $B$ to be the following closed morphism in $\mathcal{TG\mathcal{L}}$

- We start with a copy of $B \otimes \{\underbrace{| \otimes | \otimes \cdots \otimes |}_n\}$
- Next we will draw $n$ disjoint copies of $\{\cap\}$. We draw the first $\cap$ so it connects the top of the $n$-th strand of $B$ with the top of the first copy of $\{|\}$ on the right. The second $\cap$ will connect the $(n-1)$-st strand of $B$ with the second $\{|\}$ and so on, until all of the $n$ strands of $B$ are connected to the copies of $\{|\}$ on the right at the top.
- Finally we draw $n$ disjoint copies of $\{\cup\}$ the connect the bottom of $B$ to the copies of $\{|\}$ on the right in the same manner.

See the Figure 5.1 for a diagram of a closed braid. In general we will refer to the $n$ morphisms of the form $\{\cup \circ \ldots \circ | \circ \cap\}$ on the right side of the diagram as disjoint arcs. The $\{|\}$ piece connects with the open ends of $\cap$ and $\cup$ on the right hand side only, so each disjoint arc is a tangle of the form $(1,1)$.

Later it will also be useful to define a height function on these closed braids. For a point $x$ on a closed braid $L$, we define the function $\text{Height}(x)$ to be the distance from the a horizontal line tangent to the bottom of the braid. The exact values of $\text{Height}$ are irrelevant, we only want a way to precisely distinguish the relative height of two points on the diagram. This function is also depicted in Figure 5.1.
Definition 40. An oriented link $L$ is said to be in braid-link form if $L$ is the closure of a braid and the orientations on the braid piece are decreasing with respect to the height function. (i.e., the braid strands are all oriented downward)

Definition 41. An oriented link in braid-link form equipped with a blackboard framing is said to be in framed braid-link form.

One advantage of describing an oriented link in this fashion is that at any crossing in $L$, there is only one possibility for the orientation of the arcs. This will allow us to greatly simplify our description of the Kuperberg invariant for links.

The Hennings invariant is dependent on a blackboard framing of the link projection. Since the ultimate goal is to prove equivalence between the two invariants, we need to show that any link $L$ with a blackboard framing can be written in an equivalent framed braid-link form.

Theorem 42. Any oriented link is equivalent to a link written in braid-link form.
This was originally a result of Alexander. A more modern proof can be found in [2]. Unfortunately, these algorithms do not preserve a blackboard framing. However, it is not so difficult to adjust the algorithm in order to accommodate the framing.

**Theorem 43.** Any oriented blackboard framed link is framed equivalent to a link in framed braid-link form.

**Proof.**

- First apply Alexander’s theorem to the original blackboard framed link. We obtain an equivalent link in braid-link form but with possibly a different framing on each component. We adopt the convention that the braid appears on the left hand side of the link and the orientation is counterclockwise.

- For each component of the link that has an incorrect framing, we choose an arbitrary strand of this component and add in the desired number of right hand framing loops above the braid. (See Figure 5.2) A right hand loop with a positive crossing raises the framing by 1 and one with a negative crossing decreases the framing by 1. Now we have an equivalent blackboard framed link but is now no longer completely in braid link form.

- Consider the rightmost strand of the braid that contains framing loops. Choose the loop directly above the braid. Then pull it over all the strands that appear to the right of it. This gives us a new braid with one more strand, but leaves the framing and the link unchanged. A picture of this step is given in Figure 5.2.

- We then repeat the previous step. Since there are only a finite number of framing loops, eventually obtain an equivalent blackboard framed link in braid-link form.
Figure 5.2: Step 3 of the algorithm
In light of Theorem 43 we can now assume that all of our blackboard framed links are in framed braid-link form with a counterclockwise orientation.

5.2 Constructing a Heegaard Diagram from a Link

For the duration of this section, let $L$ be a framed link in $S^3$ and let $M_L$ be a 3-manifold given by surgery along $L$. We will also assume that $L$ is oriented and in framed braid-link form. We will describe a method for constructing a Heegaard diagram for $M_L$ directly from $L$. This algorithm is loosely based on one described in [14].

First, we can consider $L$ as a morphism in $\mathcal{TGL}$ by the construction described in Chapter 2. This gives us a planar projection of $L$ generated entirely by the five pieces $\{\text{\normalfont|}, \cap, \cup, \times, \xum\}$ in general position.

Next we assume that this projection is lying on top of a large 3-ball. At every place where two different generating pieces meet, we glue a 1-handle that reaches over the link projection at the meeting point. The boundary of the 3-ball and its handles will be the lower handlebody for our splitting. See Figure 5.3.

Next we will glue in disks that will become the upper circles in the Heegaard diagram. Each of the three pieces $\{\text{\normalfont|}, \cap, \cup\}$ has two meeting points and thus each one has two 1-handles associated with it. Now we glue in a thick disk (by thick disk, we mean a copy of $D^2 \times [0, 1]$) that connects both of the 1-handles. We do this in such a way that the link projection runs underneath a ‘tunnel’ formed by the disk. See Figure 5.4.

For each of the two crossing pieces $\{\times, \xum\}$, there are four meeting points and thus four associated 1-handles. Two of these 1-handles are associated to the under-arc and two are associated with the over-arc. First, we glue in a thick disk that connects the two under-arc 1-handles. This makes a ‘tunnel’ for the under-arc. We then glue in
Figure 5.3: In the diagram, the link projection lying on the 3-ball is the thick black line. The black dot represents a point where two different tangle pieces come together. The 1-handle rises over this meeting point and attaches to the 3-ball on either side.

another disk that connects all four of the 1-handles. See Figure 5.5. This is the same topologically as constructing a tunnel for the over-arc that runs over the tunnel for the under-arc.

The manifold constructed so far is the complement of a neighborhood of the link lying in a collar of the 3-ball. Now we will glue another 3-ball to the $S^2$ component of the boundary. If we consider $L$ as a link embedded in $S^3$, it is not hard to see that this new manifold is homeomorphic to $S^3 - \{L \times D^2\}$.

For the final step in our construction, we glue in thick disks along the components of the link in $\partial(S^3 - (L \times D^2))$ that run along the original projection on the 3-ball except for the over-crossing arcs where it passes over the lower disc that covers the under-crossing arc. We take care that these discs are consistent with a normal section pointing into $D^3$ (i.e. with the assumed blackboard framing). It is clear that when we glue in this disk and fill in the boundary components we get our desired 3-manifold $M_L$, as this operation is exactly the definition of surgery along a link. The only
problem is that we do not want upper circles to intersect on our diagram. At all of the non-crossing pieces the image of the link does not cross any of the upper circles we have drawn so far, so we glue the disk directly along the trace of the link. By trace of the link, we mean that we draw a copy of $L$ along the boundary of $S^3 - \{L \times D^2\}$. For the crossing pieces, we slide the trace of the link past the upper circle corresponding to the under-arc. This move is an isotopy of the 2-handle and thus does not change the topological type of the manifold. See Figure 5.5.

Thus we have produced a Heegaard diagram for the manifold $M_L$. The pieces of the diagram corresponding to each of the five tangle pieces are shown below in Figures 5.4 and 5.5.

In the diagrams, lower circles are dotted lines, the ‘local’ upper circles (i.e. the circles contained entirely within one Heegaard diagram piece) are black dashed lines, and the ‘global’ upper circles (the upper circles that span multiple Heegaard diagram pieces corresponding to the link components) are dashed gray lines.

5.2.1 Basepoints, Orientations, and Orderings on Framed Braid Links

As discussed in Chapter 3, the Kuperberg Invariant requires that all circles on the Heegaard diagrams are oriented with fixed basepoints. In this section, we will explain how the basepoints and orientations are defined for each of the Heegaard diagram pieces defined above. We emphasize that the choices of basepoints and orientations on these circles do not affect the value of the invariant, but they are essential for its computation. We will also use these basepoints and orientations to define orderings on the components of the link and on the link crossings. These orderings will allow us to more easily compute the invariant.

First, we will assume that all of the lower circles are oriented in a positive direction
Figure 5.4: The Heegaard diagram pieces for \(|\), \(\cap\), and \(\cup\) respectively
Figure 5.5: The Heegaard diagram pieces for the positive and negative crossings
with respect to the right hand rule. The base point for all lower circles will be on the underside of each handle. (See Figure 5.6)

Choices of basepoints and orientations on local upper circles for the are given in Figures 5.4, 5.7 and 5.8.

Since the global upper circles span more than one diagram piece, it is important that we make consistent choices for the orientations and basepoints. Since the global upper circles on the diagram match the outlines of the components of the link, we will assign orientations and basepoints to the link itself and assign matching choices to the circles on the diagram. Since the original link $L$ is in braid link form, we can give the link a consistent orientation that points from the top of the braid to the bottom. See Figure 5.9. This orientation is reflected in the pictures for the crossings above.

Now we need to assign a basepoint to each component of the link. Recall the height function $\text{Height}(x)$ defined earlier for each point in our link. This height function increases as we go from the bottom of the drawing to the top. Each component of the link has a unique point that is maximal with respect to this height function. We assign this point to be the basepoint of this component.

As mentioned earlier, it will be useful to define an ordering of the components of
Figure 5.7: Oriented diagram for the positive crossing

Figure 5.8: Oriented diagram for the negative crossing

61
the link. This ordering is defined by the height function. Namely, the component of
the link that has the highest basepoint with respect to this function will be labeled
with 1, and in general we will refer to this component as $K_1$. Similarly, the component
with the second highest basepoint will be labeled with 2 and referred to as $K_2$. This
continues until every component has a label.

Finally we will also wish to have a consistent labeling of the crossings in our di-
agram. Each crossing will be labeled with an ordered pair in $\{N \times N\}$. The first
coordinate is the component label of the over-arc at the crossing. The second co-
dordinate will be a counter that denotes the order of the over-arcs of this particular
component with respect to the base point and orientation. For example, take the
component $K_1$. We start at the basepoint and move in the direction of the orienta-
tion. The first time $K_1$ occurs as an over-arc at a crossing, we label that crossing
with $(1,1)$. The second time $K_1$ is an over-arc is labeled $(1,2)$. If $K_1$ occurs as an
under-arc, we do not label anything. We continue this process following along the
orientation until we reach the base point for $K_1$. Then we repeat the process for $K_2$,
labeling the first over arc as $(2,1)$. Eventually we exhaust all of the components and
all of the crossings are labeled. See Figure 5.9 as an example.

**Definition 44.** Any link $L$ written in framed braid-link form with basepoints, orien-
tations, labeled components, and labeled crossings as described above will be called an
ordered link.

We will end this section by describing some useful terminology. Any link in
braid-link form consists of both a braid and a number of disjoint ‘arcs’ of the form
$\{\cup\} \circ \{\cdots \otimes \} \circ \{\cap\}$ that connect the top of the braid to the bottom. From the
description above, we know that some of these arcs contain basepoints for components
of the link. We will refer to an arc that contains a basepoint as a *distinguished arc*,
and we denote it by $\widetilde{A}$. An arc that does not have a basepoint will be called an
undistinguished or regular arc, denoted by $\mathcal{A}$. We will later define a function that assigns morphisms in $\mathcal{VECT}$ to different pieces of the link. However, this function will assign different tensors to distinguished arcs and regular arcs, so it important to differentiate between the two.

5.3 A Kuperberg algorithm for each tangle piece

Normally, the Kuperberg algorithm is applied to a complete Heegaard diagram which then produces a large Hopf algebra tensor that will contract to a scalar. This scalar is the Kuperberg Invariant. In the previous section, we constructed a specific Heegaard diagram for our manifold $M_L$. We can think of this large diagram as being constructed out of five different Heegaard diagram pieces corresponding to the five generators of
\( \mathcal{TGL}, \{\vert, \cap, \cup, \times, \times\} \). In this section, we will apply the Kuperberg algorithm to each of these five pieces. Each of the five will produce a separate Hopf algebra tensor. The Hopf algebra tensors for \( \{\vert, \cap, \cup\} \) will be thought of as linear morphisms in \( H^* \). We will also make the tensors for the crossings into morphisms in \( H^* \), but only after applying an evaluation map. We will then contract these tensors together in a specific way that will reproduce the large Hopf algebra tensor corresponding to the Heegaard diagram for \( M_L \). This computational scheme will allow us to compute \( Kup(M_L, H) \) as the contraction of simpler local morphisms.

Now the goal of this section is to compute the Hopf algebra tensor given by applying the Kuperberg algorithm to each of the five Heegaard diagrams constructed above. As noted in Chapter 3, the Kuperberg algorithm gives a correspondence between each lower circle on the Heegaard diagram with an instance of the co-integral \( \Lambda \) in the the Hopf algebra \( H \). When we split our Heegaard diagram for \( M_L \) into the five generating Heegaard diagram pieces described above, each lower circle in the global Heegaard diagram for \( M_L \) spans two different local diagrams. This means that if we assign a tensor to each of the five local diagram pieces, we must make an arbitrary choice about which piece receives this co-integral corresponding to the lower circle. Since our link \( L \) is an ordered link, we have an established height function that runs opposite of the orientation of the braid. We will then choose to associate the element \( \Lambda \) to the lower Heegaard diagram piece with respect to this height function.

Pictures of this operation are depicted in Figures 5.10 and 5.11. In Figure 5.10, we show a 1-handle in our diagram with a dotted line representing the lower circle. We have a piece of an upper circle \( U1 \) pictured as a solid grey line. \( U1 \) is some upper circle in the diagram piece that is higher with respect to the height function. Another upper circle \( U2 \) is depicted as a solid black line and it belongs to the diagram piece that is lower with respect to the height function. Recall that under the Kuperberg algorithm,
Figure 5.10: The local picture at a lower circle and the corresponding tensor
Figure 5.11: The two tensors corresponding to the higher and lower diagram piece at a joint
that a lower circle with two intersection points gets assigned to the tensor $\Lambda' \otimes \Lambda''$. A vertical diagram version of the tensor given by this local diagram picture is written at the the bottom of the figure, where $T(U1)$ and $T(U2)$ represent the Hopf algebra tensors given by applying the algorithm to the rest of $U1$ and $U2$ respectively. Figure 5.11 shows how this tensor breaks up. Namely the diagram piece containing $U1$ does not receive a co-integral, but the piece on the bottom does. Moreover, if we contract the tensors associated to the diagram pieces that belong to this meeting point, we obtain the original tensor. The contraction can be interpreted as the composition of associated linear maps.

As a further illustration, since the $\{\cap\}$ tangle piece is always on top with respect to this height function, so its corresponding Hopf algebra tensor will have no instances of $\Lambda$. On the other hand, the tensor corresponding to $\{\cup\}$ will always have two instances of $\Lambda$ (at least prior to any simplification via Hopf algebra identities). Likewise, a $\{\mid\}$ piece will have one instance of $\Lambda$ and each of the crossings will have two.

Similarly under the Kuperberg algorithm each upper circle is associated with an integral $\mu$ in $H^*$. One can see from the pictures of the diagrams that most of the upper circles are self contained within each local diagram piece, except for the global upper circle that runs along the entire component of the link. Since the Kuperberg algorithm only records information about intersections of upper and lower circles, these ‘global’ circles will only appear in the tensors corresponding to the two crossing pieces, $\{\times, \times\}$, as this component circle runs under the handles at the other three diagram pieces. This means that the tensors assigned by the algorithm to $\{\mid\}$, $\{\cap\}$, and $\{\cup\}$ will simply turn out to be linear maps on $H^*$.

We would like to make the output of our map a linear morphism at the crossing pieces as well, but the situation is more complicated and will require some extra structure as well. First we will fix an ordered link $L$. Let $L$ have $n$ components
\{K_1, K_2, \ldots, K_n\} with the ordering as above. Also, let each component \(K_i\) has \(n(i)\) over-crossings. (Thus \(L\) will have \(m = \sum_{i=1}^{n} n(i)\) crossings). Fix an element \(\Phi = \bigotimes_{i=1}^{n} n(i) \bigotimes_{j=1}^{n} \phi_{ij} \in (H^*)^\otimes m\). Now the number of intersections of these global upper circles is equal to \(m\), as they are in 1-1 correspondence with the crossings. To a crossing labeled with \((k, l)\), we will assign the element \(\phi_{kl} \in H^*\). Now instead of evaluating the open-ended tensor against the global integral \(\mu\), we will evaluate it against \(\phi_{kl}\). This gives us linear maps \((H^* \otimes H^*) \to (H^* \otimes H^*)\) at crossings as well. (See Definition 45 below.)

Now we are ready to define the correspondence \(K\) for ordered links. We will first define a more general function \(\tilde{K}\). We will then define \(K\) as a special case and then we will show it produces the Kuperberg Invariant.

**Definition 45.** Let \(\gamma, \delta \in H^*\). Let \(\Phi = \bigotimes_{i=1}^{n} n(i) \bigotimes_{j=1}^{n} \phi_{ij} \in (H^*)^\otimes m\). Then we define five morphisms in \(\mathcal{VECT}\) by:

\[
\tilde{K}(\gamma, \delta) : H^* \otimes H^* \to H^* \otimes H^* : \quad \mu' \otimes \mu''
\]

\[
\tilde{K}(\cap, \Phi) : k \to H^* \otimes H^* : \quad \tilde{K}(\cap, \Phi)(1) = \mu' \otimes \mu''
\]

\[
\tilde{K}(\cup, \Phi) : H^* \otimes H^* \to k : \quad \tilde{K}(\cup, \Phi)(\gamma \otimes \delta) = \gamma(\Lambda') \cdot \delta(S(\Lambda''))
\]

\[
\tilde{K}(^{(k,l)} \times, \Phi) : H^* \otimes H^* \to H^* \otimes H^* : \quad \delta(S(\Lambda'))(\gamma(\phi_{kl})) \otimes \gamma''
\]

\[
\tilde{K}(^{(k,l)} \times, \Phi)(\gamma \otimes \delta) = (\delta(S(\gamma'))(S(\phi_{kl}))(\gamma'') \otimes \gamma''
\]

then we define \(\tilde{K}(L, \Phi)\) by applying \(\tilde{K}\) to each of the generators and then tensoring or composing the morphisms accordingly.
Under the original Kuperber construction, each global integral is associated with a particular component $K_i$. It will be convenient to denote these special integrals in formulas and pictures as $\mu_{K_i}$, as this will allow us to differentiate them from the ‘local’ integrals, and it will also help us keep track of the ordering on the link. Now let $\hat{\mu} = \bigotimes_{i=1}^{n} \Delta^{n(i)}(\mu_{K_i}) \in (H^*)^\otimes m$. Then we can define:

**Definition 46.** We define $K(L)$ by $K(L) := \tilde{K}(L, \hat{\mu})$

We will now prove that $K(L)$ gives the Kuperberg invariant of $M_L$.

**Theorem 47.** Let $L$ be an ordered link. Fix an involutory and semi-simple Hopf algebra $H$. Fix a co-integral $\Lambda$ and an integral $\mu$ such that $\mu(\Lambda) = 1$, $\mu(1) = 1$, and $\varepsilon(\Lambda) = (\dim H)$. Then $K(L) : k \rightarrow k$ and $K(L)(1) = Kup(M_L, H)$

**Proof.** It is clear by the construction of the Heegaard diagram for $M_L$ and the choices described at the beginning of this section that applying to Kuperberg algorithm to each of the five generating pieces of the Heegaard diagram and composing those tensors in order along the diagram will produce:

$$(Kup(M, H))(\dim H)^{-g(S)}(\mu(1))^{n(U)}(\varepsilon(\Lambda))^{n(L)}$$

Since $\mu(1) = 1$, $\varepsilon(\Lambda) = (\dim H)$ and $g(S) = n(L)$ this operation will give us $Kup(M, H)$ exactly. We thus merely need to show that the tensor produced by the Kuperberg algorithm for each of the five pieces simplifies to the morphism described by $K$.

- We will begin by applying the Kuperberg algorithm to the Heegaard diagram piece corresponding to $[\text{]}$. An oriented and pointed version of the diagram is below.
This piece of the diagram has one upper circle and two lower circles. By the convention described above, the Λ corresponding to the lower circle on the bottom handle will *not* be included in this tensor, but instead in the tensor in the Heegaard diagram piece lying below this one. This means our tensor will have one Λ and one µ. The first crossing with respect to the upper circle is negative and the second crossing is positive. On the algebra side, this means the first factor multiplying into µ receives an antipode and the second does not. In vertical diagram form, our tensor looks like:

![Diagram](image)

The two open ended pieces of the diagram will compose with the tensors corresponding to the tangle pieces lying above and below this instance of {[]}.
using Lemma 11 and unimodularity, we have that $\Lambda' \otimes (\mu \leftarrow S(\Lambda'')) = id_H$.

This means our Hopf algebra tensor reduces to

$$(id_{H^*} \otimes \tilde{coev}) \circ (id_{H^*} \otimes id_H \otimes id_{H^*}) \circ (\tilde{ev} \otimes id_{H^*}) = id_{H^*}.$$ 

Thus $\mathcal{K}()$ is equal to the Kuperberg tensor for this diagram piece, as desired.

• Next we will consider the Heegaard diagram for $\{\cap\}$. The oriented and pointed diagram is shown below.

Now because nothing lies above this piece, the tensor will not have any instances of $\Lambda$. Both of the crossings here are positive, so there are no antipodes either.

Thus the diagram for our tensor is simply:

which reduces to the morphism $1 \mapsto \mu' \otimes \mu'' = \mathcal{K}(\cap)$ by the definition of comultiplication in $H^*$

• In the case of the $\{\cup\}$ piece, there are no tangle pieces lying below it, so its tensor will receive two copies of $\Lambda$. 

71
As one can see from the diagram, both of the crossings are positive, so there are no antipodes. The original diagram for the tensor is given below on the left. We then apply Lemma 11 to obtain the simplified diagram on the right. It is clear that this simple diagram matches the value of $K(\cup) = \Lambda' \otimes S(\Lambda'')$.

Next is the Heegaard diagram corresponding to the positive crossing, labeled with $(i,j)$. The first step is to perform a circle slide move on the upper circle representing the over crossing. This simplifies the calculation of the tensor and does not change the manifold. The orientations of the global upper circles in the diagram are both pointing downward, since that is the orientation of the link pieces in the braid. The diagram after the circle slide and with the orientations and base points is depicted in Figure 5.12.

Now we will compute the tensor given by this diagram. There are four lower circles in the diagram and there are three upper circles that intersect these lower circles: the local circle given by the over-arc, the local circle given by the
under-arc, and the global circle corresponding to the over-arc. The global upper circle corresponding to the under arc appears but does not intersect anything, and thus does not contribute to the tensor.

By our convention we will only assign a copy of $\Lambda$ for the two lower circles appearing at the top of the diagram. The top left lower circle only has one intersection: there is a negative crossing with the local over-arc circle. The top right lower circle also has only crossing: a negative crossing with the local under-arc circle. These tensors appear on the left side of the tensor diagram in Figure 5.13.

The bottom left lower circle has 4 intersections: First positively with the local under-arc, then negatively with the local over-arc, then negatively with the global over-arc, and finally positively with the local over-arc. Finally the bottom left lower circle has one positive intersection with the local over-arc. The
complete diagram is pictured in Figure 5.13. The integrals $\mu$ are labeled corresponding to their upper circles. In particular, $\mu_{K_i}^{(j)}$ is the $j$-th factor in the co-multiplication of $\mu u_{K_i}$, the integral corresponding to the global circle that traces out the link component $K_i$.

Once again we can simplify this tensor by applying Lemma 11 multiple times. Then we can use Lemma 5 to rewrite it purely as a morphism in $H^*$. The tensor then appears in the simplified form depicted in Figure 5.14.

Now by translating the diagram into a formula we see that this is equal to the tensor described by the map $\mathcal{K}(X)$, as desired.
Finally we compute the tensor for a negative crossing. We start once again by performing a circle slide move on the diagram. This is pictured (along with the chosen orientations and base points) in Figure 5.15.

Once again there are four lower circles and three relevant upper circles: the local over-arc, the local under arc, and the global over-arc. The top left lower circle crosses the local under arc negatively and the top right circle crosses the local over-arc negatively. These two receive the co-integrals by convention and are depicted on the left hand side of the tensor diagram in Figure 5.16.

The bottom left lower circle intersects the local over arc positively. Finally, the bottom left lower circle intersects with upper circles in four places. In order, they are: positive with the local under arc, negative with the local over arc, positive with the global over arc, and positive with the local over arc. Following the Kuperberg algorithm, this generates the tensor that you see in Figure 5.16.

Finally we can simplify this tensor, once again by applying Lemma 11 and
Figure 5.15: The oriented negative crossing Heegaard diagram.

Figure 5.16: The unsimplified tensor for a negative crossing
Lemma 5. The simplified tensor is shown in Figure 5.17 and is equivalent to $\mathcal{K}(\mathcal{X})$. 

□
CHAPTER 6
PROOF OF THE MAIN THEOREM

6.1 Defining the tensor $\mathcal{A}$

The map $\mathcal{K}$ is reminiscent of tangle functor. Unfortunately, the dependence on $\Phi$ prevents it from being functorial in an obvious way. We can however work around this problem by encoding the image of $\mathcal{K}$ as morphisms in $\mathcal{T}(H^* \otimes H)$ instead of morphisms in $\mathcal{T}(H^*)$. We will do this by defining a parallel function $\mathcal{A}$ on $L$ where the link pieces are sent to morphisms in $\mathcal{T}(H)$. We will show that evaluating $\mathcal{A}$ against $\tilde{\mathcal{K}}$ will reduce to the Kuperberg invariant.

For the remainder of the chapter, we will assume that $L$ is an ordered link. Let the braid of $L$ be a tangle of the form $(p,p)$. Let this $L$ have $m$ components $\{K_1, K_2, \ldots, K_m\}$ and $n$ total crossings. Also, let each component $K_i$ have $n(i)$ over-crossings. We will also fix an involutory and semi-simple Hopf algebra $H$ with co-integral, $\Lambda$ and integral, $\mu$ scaled such that $\mu(\Lambda) = 1$, $\mu(1) = 1$, and $\varepsilon(\Lambda) = (\dim H)$ (These scalings are possible by Lemma 11, see [14]). Let $\{e_i\}$ be a basis for $H$ and $\{f_i\}$ a corresponding dual basis for $H^*$.

The first step in the proof is to define another tensor $\mathcal{A} \in H^\otimes n$. This new tensor will be defined as a map that sends the product of an oriented tangle generator and an element of $H^\otimes n$ to a morphism in $\mathcal{VECT}$. Unlike $\tilde{\mathcal{K}}$, the map $\mathcal{A}$ is dependent on the orientation and location of basepoints in $L$. In particular, the value of $\mathcal{A}$ is
different on distinguished arcs and regular arcs. In the definition below, we denote
the orientations of the 5 tangle pieces, and we indicate that a piece is part of a
distinguished arc by marking it with a “^” sign. The definition of \( A \) also depends
on an element \( X \in H_{\otimes n} \). We can think of this \( X \) as an \( H \)-labeling of \( L \) at the
crossings.

**Definition 48.** Let \( L \) be an ordered link. Let \( X = \bigotimes_{i=1}^{m} \bigotimes_{j=1}^{n(i)} x_{ij} \in H_{\otimes n} \). Let \( a, b \in H \)
and \( \gamma \in H^* \). Define a tensor \( A_H(L, X) \) in the following way:

\[
\begin{align*}
A(\uparrow, X) & : H^* \rightarrow H^* := \text{id}_{H^*} \\
A(\downarrow, X) & : H \rightarrow H := \text{id}_H \\
A(\cap, X) & : k \rightarrow H \otimes H^* := \text{coev} \\
A(\cup, X) & : H \otimes H^* \rightarrow k := \text{ev} \\
A(\hat{\cap}, X) & : k \rightarrow H \otimes H^* := 1 \mapsto 1_H \otimes \varepsilon \\
A(\hat{\cup}, X) & : H \otimes H^* \rightarrow k := (a \otimes \gamma) \mapsto \mu(a) \otimes \gamma(1_H) \\
A((k,l), X) & : H \otimes H \rightarrow H \otimes H := a \otimes b \mapsto (b \otimes S(x_{kl})a) \\
A((k,l), X) & : H \otimes H \rightarrow H \otimes H := a \otimes b \mapsto (S(x_{kl})b \otimes a)
\end{align*}
\]

The tensors in \( H \) are composed at the meeting points of the pieces on the link. We
define the composition of these maps for a link diagram to be \( A(L, X) \).

Notice that in the above definition, the tensor \( A \) is dependent on the orientation,
but because \( L \) is already in braid-link form, the orientations for all of the pieces other
than the vertical strands are forced: orientations around \( \{\cap\} \) and \( \{\cup\} \) run counter-
clockwise and the orientations of the crossings point downwards. Also note that the
orientations for the link match the earlier convention for diagrams, namely upward
arrows get mapped under \( A \) to morphisms of \( H^* \) and downward arrows get mapped
to morphisms of $H$. Because of this convention, we see that $A(L,X)$ must contract to a scalar.

First we will consider what happens when we apply $A$ to the three different large pieces of our ordered link diagram: namely the braid, the distinguished arcs and the regular arcs. This will allow us to compute $A$ more easily.

**Lemma 49.** The map $A$ applied to the braid gives a map $B : H^\otimes p \to H^\otimes p$. Furthermore, $B = Y \cdot \circ \mathcal{P}$, where $\mathcal{P}$ is a permutation map and $Y \cdot \circ$ represents a left multiplication action of an element $Y \in H^\otimes p$

**Proof.** By convention we have a downward orientation on everything that is part of the braid with respect to the height function. From the definition of $A$, all downward strands map to a tensor factor of $H$. Thus a braid with $p$ strands will get mapped to an element of $\text{Hom}(H^\otimes p, H^\otimes p)$.

In the definition of $A$, the only non-identity local Hopf algebra actions on the braid are given by equations 6.1.7 and 6.1.8. Each of these is a permutation map followed by a left multiplication by an element of $H$. It is clear that we could permute all of the factors first followed by a single left multiplication action of $H^\otimes p$.

**Lemma 50.** The map $A$ applied to any undistinguished arc gives the tensor $\sum_i e_i \otimes f_i$

**Proof.** Applying $A$ to an undistinguished arc gives a map $k \to H \otimes H^*$. Namely:

$$A(\mathcal{A})[1] = \sum_i \sum_j (e_j \otimes f_j(e_i) \otimes f_i) = \sum_i e_i \otimes f_i$$

**Lemma 51.** The map $A$ applied to a distinguished arc gives the tensor $1_H \otimes \mu$

**Proof.** Applying $A$ to a distinguished arc gives a map $k \to H \otimes H^*$. Namely:

$$A(\widetilde{\mathcal{A}})[1] = 1_H \otimes \mu \otimes \varepsilon(1) = 1_H \otimes \mu$$
Theorem 52. Let $X = \bigotimes_{i=1}^{m} \bigotimes_{j=1}^{n(i)} x_{ij} \in H^{\otimes n}$. Then:

$$A_H(L, X) = \hat{\mu}(X) = \prod_{i=1}^{m} \mu(\prod_{j=1}^{n(i)} x_{ij})$$

Proof. Let us begin the calculation by considering the strand in first position at the top of the braid. By our conventions, this strand belongs to the component $K_1$ and also is the strand that is connected to the distinguished $\{\hat{\times}\}$ piece of $K_1$. This means our input tensor into the function $B$ is of the form $1_H \otimes y_2 \otimes \cdots \otimes y_p$ for $y_i \in H$. We will explicitly calculate how $B$ acts on this identity element in the first tensor factor by tracing along this first strand and applying the proper local Hopf algebra actions as we go.

There are three possibilities for the first time this strand is involved in a crossing:

1. This strand is the over-arc for the positive crossing $(1, 1)$ in our link.

2. This strand is the under-arc for the negative crossing $(k, l)$ for some $k$ and $l$ in our link

3. This strand is never involved in a crossing.

Case 3 is a somewhat trivial case that will be addressed in full below. In Case 1 the first action on our tensor is:

$$(1_H \otimes z_2 \otimes z_3 \cdots \otimes z_p) \mapsto (z_2 \otimes S(x_{11})1_H \otimes z_3 \otimes \cdots \otimes z_p)$$

In Case 2 the first action is:

$$(1_H \otimes z_2 \otimes z_3 \cdots \otimes z_p) \mapsto (S(x_{kl})z_2 \otimes 1_H \otimes z_3 \otimes \cdots \otimes z_p)$$
(Here the $z_i$ are the output of the actions on the $y_i$ that were first with respect to the height function.)

Notice that in both Cases 1 and 2 that the position of our special strand in the braid still matches the position of its corresponding tensor in $H^\otimes p$. Namely, after the crossing, our strand is in the second position within the braid and the relevant tensor is in the second factor of the element in $H^\otimes p$. This will always be the case.

In general, there are four possibilities for our strand to be involved in a crossing, with four corresponding local Hopf algebra actions. For the following, let $T \in H$ denote the product so far for our relevant tensor factor and let $Y, Z \in H$ be abstract tensor factors that happen to be adjacent.

- If our strand is the over-arc in a positive crossing labeled with $(1, k)$, the action on our tensor is: $(\cdots \otimes Y \otimes T \otimes Z \otimes \cdots) \mapsto (\cdots \otimes Y \otimes Z \otimes S(x_{1k})T \otimes \cdots)$

- If our strand is the over-arc in a negative crossing labeled with $(1, k)$, the action on our tensor is: $(\cdots \otimes Y \otimes T \otimes Z \otimes \cdots) \mapsto (\cdots \otimes S(x_{1k})T \otimes Y \otimes Z \otimes \cdots)$

- If our strand is the under-arc in a positive crossing labeled with $(k, l)$, the action on our tensor is: $(\cdots \otimes Y \otimes T \otimes Z \otimes \cdots) \mapsto (\cdots \otimes T \otimes S(x_{kl})Y \otimes Z \otimes \cdots)$

- If our strand is the under-arc in a negative crossing labeled with $(k, l)$, the action on our tensor is: $(\cdots \otimes Y \otimes T \otimes Z \otimes \cdots) \mapsto (\cdots \otimes X \otimes S(x_{kl})Z \otimes T \otimes \cdots)$

Eventually our strand reaches the bottom of the braid. The strand has moved (potentially) from position 1 at the top of the braid, to position $q$, for some $q \in \{1, \ldots, p\}$. Our corresponding tensor factor is also in position $q$ in the tensor product, and it is of the form:

$$(\cdots \otimes \underbrace{S(x_{1m}) \cdots S(x_{11})}_{q}1_H) \otimes \cdots)$$
for some $m \in \{0, 1, \ldots, n(1)\}$.

Now we will evaluate this tensor against the $H^*$ element determined by the arcs. There are now two cases:

1. $q = 1$

If $q = 1$ then we have that the $K_1$ component of $L$ contributes only one strand to the braid. This means that $n(1) = m$. Since we know that there is a distinguished $\{ \bigcirc \}$ connected to position 1 at the bottom of the braid, we will evaluate our product against a copy of $\mu$. We obtain:

$$\mu((S(x_{1m}) \cdots S(x_{11}))_1) = \mu(S(x_{11}x_{12} \cdots x_{1m})) = \mu(x_{11}x_{12} \cdots x_{1m})$$

$$= \Delta^{n(1)}(\mu)[x_{11} \otimes \cdots \otimes x_{1n(1)}]$$

where the second equality follows from Lemma 12 and the third by Lemma 5.

2. $q \neq 1$

If $q \neq 1$ we know that position $q$ is connected to an undistinguished arc, as each component can have only one distinguished arc, and for $K_1$ it is in position 1. This means that we will evaluate our product over the sum of the dual basis, namely:

$$\sum_i f_i((S(x_{1m}) \cdots S(x_{11}))_1)$$

We now will repeat the process outlined above. This time, we will work with the strand that begins in position $q$, and the initial tensor will be $\sum_i e_i$ instead of $1_H$. When this strand reaches the bottom of the braid, it will be in a new position $q_2 \neq q$. The tensor factor in position $q_2$ will look like:

$$\sum_i (S(x_{1m_2}) \cdots S(x_{1(m+1)}e_i))$$
for some \( m_2 \in \{m, m + 1, \ldots, n(1)\} \). If \( q_2 \neq 1 \) we start the process again with the strand in position \( q_2 \) at the top of the braid. Eventually \( q_d = 1 \) for some \( d \). We then evaluate the product against the integral \( \mu \). The evaluated tensor factor in position 1 looks like:

\[
\sum (\mu(S(x_{1m}) \cdots S(x_{1(m_d-1)+1})e_{i_d})) \otimes \cdots
\]

In general, the evaluated tensor factor in position \( q_j \) looks like:

\[
\cdots \otimes \left( f_{i_j}(S(x_{1m_j}) \cdots S(x_{1(m_j-1)+1})e_{i(j-1)}) \right) \otimes \cdots
\]

But we can always contract \( f_{i_j} \) with \( e_{i_j} \). After these \( d \) contractions we get:

\[
\mu((S(x_{1m_d}) \cdots S(x_{11})1_H)) = \Delta^{n(1)}(\mu)[x_{11} \otimes \cdots \otimes x_{1n(1)}]
\]

Thus the component \( K_1 \) contributes the value \( \Delta^{n(1)}(\mu)[x_{11} \otimes \cdots \otimes x_{1n(1)}] \) to \( A(L, X) \).

But the algorithm can be repeated for \( K_2, K_3 \) etc. Thus we see that:

\[
A(L, X) = \bigotimes_{i=1}^{m} (\Delta^{n(i)}(\mu)[\bigotimes_{j=1}^{n(i)} x_{ij}]) = \hat{\mu}(X)
\]

as desired \( \square \)

In the above proof, each distinguished arc contributes also an extra factor of \( \varepsilon(1_H) = 1 \). This obviously does not affect the value of \( A \).

**Definition 53.** Let \( L \) be an ordered link with \( n \) components and \( n(j) \) overcrossings on each component \( K_j \). Let \( e_{i(j,k)} \) be a basis for \( H \) with dual basis \( f_{i(j,k)} \) for \( j \in \{1, \ldots, n\} \) and \( k \in \{1, \ldots, n(j)\} \). Define a tensor \( L_H \) in the following way:

\[
L_H^{(j,k)}X = \sum_i \tilde{K}(X, f_{i(j,k)}) \otimes A(X, e_{i(j,k)}) \quad (6.1.9)
\]

\[
L_H(X^{(j,k)}) = \sum_i \tilde{K}(X, f_{i(j,k)}) \otimes A(X, e_{i(j,k)}) \quad (6.1.10)
\]

84
and if \( P \) is a piece of \( L \) with no crossings:

\[
\mathcal{L}_H(P) = \tilde{\mathcal{K}}(P) \otimes \mathcal{A}(P)
\]  

(6.1.11)

We extend \( \mathcal{L}_H \) to the entire link by composing and tensoring the generating pieces. The tensors in \( H \) and \( H^* \) are composed at the meeting points on the link respectively.

Since applying either \( \tilde{\mathcal{K}} \) or \( \mathcal{A} \) to a link produces a scalar, we have the following:

**Lemma 54.** Let \( L \) be an ordered link with \( n \) crossings. Then \( \mathcal{L}_H \) is a scalar and it is equal to:

\[
\mathcal{L}_H(L) = \sum_i \tilde{\mathcal{K}}(L, f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_n}) \cdot \mathcal{A}(L, e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n})
\]

where each \( e_{ij} \) is a basis for \( H \) with corresponding dual basis \( f_{ij} \).

We have immediately that:

**Theorem 55.** \( \mathcal{L}_H(L) = Kup(M_L, H) \)

*Proof.* By Theorem 52, we have

\[
\mathcal{L}_H(L) = \sum_i \tilde{\mathcal{K}}(L, f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_n}) \cdot \hat{\mu}(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n})
\]  

(6.1.12)

By Lemma 5, 6.1.12 is equal to:

\[
\tilde{\mathcal{K}}(L, \hat{\mu}) = \mathcal{K}(L) = Kup(M_L, H)
\]

by Theorem 47.

We end this section by writing out the precise tensors produced by \( \mathcal{L}_H \) on the standard pieces of our braid-link. These all follow immediately from the definitions of \( \mathcal{A}, \tilde{\mathcal{K}}, \) and \( \mathcal{L}_H \).
Lemma 56. Let $x, y \in H$ and $\gamma, \delta \in H^*$.

$$L_H(\downarrow) : (H^* \otimes H) \rightarrow (H^* \otimes H) = (id_{H^*}) \otimes (id_H)$$

$$L_H(ab) : (H^* \otimes H) \rightarrow (H^* \otimes H) = (\mu' \otimes \Lambda' \otimes \mu''(S(\Lambda'')) \otimes (id_H)$$

$$L_H(\tilde{ab}) : (H^* \otimes H) \rightarrow (H^* \otimes H) = (\mu' \otimes \Lambda' \otimes \mu''(S(\Lambda'')) \otimes (1_H \otimes \mu \otimes \varepsilon(1))$$

$$L_H((i,k)\times) : (H^* \otimes H) \otimes (H^* \otimes H) \rightarrow (H^* \otimes H) \otimes (H^* \otimes H) := L_H((i,k)\times)[(\gamma \otimes x) \otimes (\delta \otimes y)] = \sum_i ((\delta S(\gamma')S(f_{i,(j,k)})\gamma''' \otimes y) \otimes (\gamma'' \otimes S(e_{i,(j,k)})x))$$

$$L_H(\times(i,k)) : (H^* \otimes H) \otimes (H^* \otimes H) \rightarrow (H^* \otimes H) \otimes (H^* \otimes H) := L_H(\times(i,k))[\gamma \otimes x \otimes (\delta \otimes y)] = \sum_i ((\delta'' \otimes S(e_{i,(j,k)})y) \otimes (\gamma S(\delta''')f_{i,(j,k)}\delta' \otimes x))$$

6.2 Proof of Theorem 1

In the previous section, we just showed that $L_H$ applied globally to an ordered link $L$ gives the Kuperberg invariant. Now we will show that the local formulas for $L_H$ applied to the generating pieces of the ordered link detailed in Lemma 56 are closely related to the functor $F_{D(H)}$. In fact, $F_{D(H)}$ and $L_H$ are equal for all the ordered link pieces except distinguished arcs, where they differ only by a factor of $(\dim H)$. After a quick counting exercise, this turns out to be exactly the factor that $F$ differs from $Henn(M,H)$, thus proving our main theorem.

For this entire section, we will fix a finite dimensional, involutory, and semi-simple Hopf algebra $H$. We will fix a co-integral $\Lambda$ and an integral $\mu$ such that: $\mu(\Lambda) = 1$, $\mu(1) = 1$, and $\varepsilon(\Lambda) = (\dim H)$. We will let $\{e_i\}$ be a basis for $H$ with dual basis $\{f_i\}$.

By Lemmas 8 and 14 we have that $D(H)$ is involutory, unimodular, and quasi-triangular. Formulas for the integral, co-integral and R-matrix in $D(H)$ are given by:
\( \Lambda_D = \mu \otimes \Lambda \quad (6.2.1) \)

\( \mu_D = \Lambda \otimes \mu \quad (6.2.2) \)

\[ R = \sum_{i} ((\varepsilon \otimes e_i) \otimes (f_i \otimes 1)) \quad (6.2.3) \]

Also we note that we can choose the special group-like element \( G \) to be 1, by unimodularity and Lemma 13.

**Lemma 57.** Under the conditions outlined at the beginning of this section, \( \mu_D(v) = \mu_D(v^{-1}) = 1 \)

**Proof.** First we fix a basis \( \{e_i\} \) for \( H \) and let \( \{f_i\} \) be its corresponding dual basis. Then \( v = \sum_i S(f_i) \otimes e_i \). We compute:

\[ \mu_D(v) = \mu_D\left(\sum_i S(f_i) \otimes e_i\right) = \sum_i f_i(S(\Lambda)) \otimes \mu(e_i) \]

But this simplifies to:

\[ \sum_i f_i(S(\Lambda)) \otimes \mu(e_i) = \sum_i \mu(e_i \cdot f_i(S(\Lambda))) = \mu(S(\Lambda)) = 1 \]

as desired. Since \( v^{-1} = \sum_i f_i \otimes e_i \) we have \( \mu_D(v^{-1}) = 1 \) as well. \( \square \)

**Corollary 58.** Let \( M_L \) denote the 3-manifold obtained by surgery along a link \( L \). Then under the above scaling conditions we have:

\[ \text{Henn}(M_L, D(H)) = \text{Henn}(L, D(H)) \]

**Proof.** We defined \( \text{Henn}(M_L, D(H)) \) to be:

\[ \left[ \mu_D(\nu)\mu_D(\nu^{-1}) \right]^{-c(L)/2}[\mu_D(\nu)/\mu_D(\nu^{-1})]^{-\sigma(L)/2}\text{Henn}(L, H) \]

But by Lemma 57, the scalar factor is simply 1. \( \square \)
Recall our functor $\mathcal{F} : \mathcal{TGL} \to \mathcal{VECT}$ from Chapter 4. When we apply $\mathcal{F}$ to $D(H)$ we get the following:

\[
\begin{align*}
\mathcal{F}(|) &= \text{id}_{D(H)} \\
\mathcal{F}(\cap) &= \Delta(\Lambda_D) \\
\mathcal{F}(|)[x \otimes y] &= \mu_D(M(id \otimes S_D)) \\
\mathcal{F}(X)[x \otimes y] &= \mathcal{R} \cdot (\tau(x \otimes y)) \\
\mathcal{F}(Y)[x \otimes y] &= \tau(\mathcal{R}^{-1} \cdot (x \otimes y))
\end{align*}
\]

Now let $L$ be a link written in framed braid-link form as a morphism in $\mathcal{TGL}$. We now prove the following theorem:

**Theorem 59.** $\mathcal{F}_{D(H)}(L) = (\dim H)^{c(L)} \mathcal{L}_{\text{Henn}}(L)$

Before proving Theorem 59, we will show that it implies Theorem 1.

**Proof of Theorem 1.** Let $M$ be a closed 3-manifold. Choose an ordered link $L$ that gives $M$ by surgery in $S^3$. By Theorem 36, we have that

\[
\mathcal{F}_{D(H)}(L) = (\dim H)^{c(L)} \text{Henn}(L, D(H)) = (\dim H)^{c(L)} \text{Henn}(M, D(H))
\]

where the last equality comes from Corollary 58. By Lemma 55, we know that $\mathcal{L}_{\text{Henn}}(L) = \text{Kup}(M, H^{cop})$. Thus by Theorem 59, we have $\text{Henn}(M, D(H)) = \text{Kup}(M, H^{cop})$. But we have that $H^{cop}$ is isomorphic to $H$, and thus both algebras give the same Kuperberg invariant. Hence, $\text{Henn}(M, D(H)) = \text{Kup}(M, H)$, as desired. \hfill \Box

For the proof of Theorem 59 we will need the following lemma about integrals and co-integrals in $H^{cop}$:

**Lemma 60.**

1. $\mu_{H^{cop}} = \mu$
2. $\Lambda_{H^{\text{cop}}} = \Lambda$

3. $\Delta^{\text{op}}(\Lambda) = \Delta(\Lambda)$

Proof. Statement 1 is a direct consequence of unimodularity. Multiplication in $H^{\text{cop}}$ is the same as in $H$, so Statement 2 is immediate. Statement 3 follows from unimodularity and Lemma 12.

Before the proof, we should make a remark about some minor difficulties with working in $H^{\text{cop}}$ instead of $H$. When $L_{H^{\text{cop}}}$ is applied to a strand in the output is a morphism in $H^{\text{op}} \otimes H^{\text{cop}}$. Hence, when working with the output of $L_{H^{\text{cop}}}$, we have to invert the order of multiplication of the elements in $H^{\text{op}}$ from the definition given above. Then, combined with the results above, we will consider this inverted version of the map simply as a morphism of $H^{\text{op}} \otimes H$. This will allow us to make an exact comparison with $F_{D(H)}$, which we also will consider simply as morphisms of $H^{\text{op}} \otimes H$.

Proof of Theorem 59. Let $L$ be an ordered link. We emphasize that only the map $L_{H^{\text{cop}}}$ uses the orientation and the distinguished edges, the functor $F$ does not treat pieces with different orientations or distinguished parts differently. We will prove the theorem by comparing the outputs of $F_{D(H)}$ and $L_{H^{\text{cop}}}$ on the different pieces of the ordered link. The actions from the first four pieces will match exactly and the scaling factor will be given by the last piece.

- First we consider the piece $\{\downarrow\}$. We have:

$$L_{H^{\text{cop}}} (\downarrow) = \text{id}_{H^{\text{op}}} \otimes \text{id}_{H^{\text{cop}}} = \text{id}_{H^{\text{op}}} \otimes \text{id}_{H} = \text{id}_{D(H)} = F_{D(H)} (\downarrow)$$

as desired.

- Next we will consider undistinguished arcs $\{A\}$. By Lemmas 50 and 60 we
have that \( \mathcal{L}_{H^{\text{cop}}} \mathcal{A} = (\mu' \otimes \Lambda' \otimes \mu''(S(\Lambda'')) \otimes (id_H) \). On the other hand, we have:

\[
\mathcal{F}_{D(H)}(\cap \circ \ldots | \circ \cup) = \Lambda'_D \otimes S(\Lambda''_D) \triangleright \mu_D = id_{D(H)} \]

\[
= id_{H^*} \otimes id_H = (\mu' \otimes \Lambda' \otimes \mu''(S(\Lambda''))) \otimes (id_H)
\]

where the second equality follows from Lemma 11 applied to \( D(H) \) and the fourth equality from Lemma 11 applied to \( H^* \).

- Now we will look at the positive crossings \( \{ \times \} \). Let \( \gamma, \delta \in H^* \) and \( x, y \in H \). We will compare the maps given on \( (H^* \otimes H) \otimes (H^* \otimes H) \to (H^* \otimes H) \otimes (H^* \otimes H) \) given by \( \mathcal{L}_{H^{\text{cop}}} \) and \( \mathcal{F}_{D(H)} \). We have that:

\[
\mathcal{L}_{H^{\text{cop}}} \mathcal{X}^k (\gamma \otimes x) \otimes (\delta \otimes y) = \sum \left( (\gamma'^{''''} S(f_{ik}) S(\gamma') \delta \otimes y) \otimes (\gamma'' \otimes S(e_{ik}) x) \right) = \sum \left( (\gamma'^{''''} f_{ik} S(\gamma') \delta \otimes y) \otimes (\gamma'' \otimes (e_{ik}) x) \right)
\]

where is second equality is because \( S \otimes S \circ \text{coev} = \text{coev} \) in an involutory Hopf algebra. On the other hand:

\[
\mathcal{F}_{D(H)}(\mathcal{X}^k) (\gamma \otimes x) \otimes (\delta \otimes y) = R \cdot (\tau (\gamma \otimes x) \otimes (\delta \otimes y))
\]

\[
= \sum (f_i \otimes 1)(\delta \otimes y) \otimes (\varepsilon \otimes e_i)(\gamma \otimes x)
\]

\[
= \sum ((f_i \delta \otimes y) \otimes ((e'_i \triangleright \gamma S(e''_i)) \otimes e''_i x))
\]

Now instead of writing the above formula with the \( \triangleleft \) ‘action’ notation, we will write it instead using a co-multiplication in \( H^* \). This gives us:

\[
= \sum (f_i \delta \otimes y) \otimes (\gamma''(e'_i) \gamma''(S(e''_i)) \otimes e''_i x)
\]
Now by Lemma 5 we can switch a \(c\text{coev}\) followed by a co-multiplication in \(H\) to three separate instances of \(c\text{coev}\) followed by a multiplication in \(H^*\). Performing this operation gives us:

\[
= \sum_{j,i,k} (f_j f_i f_k \delta \otimes y) \otimes (\gamma'''(e_j) \gamma'(S(e_k)) \otimes e_i x)
\]

Finally, now we have two instances of \(c\text{coev}\) followed immediately by \(ev\). (Specifically, the ones indexed with \(j\) and \(k\).) We can contract these, and doing so gives:

\[
= \sum_i ((\gamma''' f_i S(\gamma') \delta \otimes y) \otimes (\gamma'' \otimes e_i x))
\]

which is exactly what we wanted.

- The next step is to compare what goes on at the negative crossings \(\{\mathcal{X}\}\). Once again, let \(\gamma, \delta \in H^*\) and \(x, y \in H\). We have:

\[
\mathcal{L}_{H\text{cop}}(\mathcal{X}^m)[(\gamma \otimes x) \otimes (\delta \otimes y)] = \sum_i ((\delta''' \otimes S(e_{im}) y) \otimes (\delta' f_{im} S(\delta'') \gamma \otimes x))
\]

Now for the functor \(\mathcal{F}_{D(H)}\) we get:

\[
\mathcal{F}_{D(H)}(\mathcal{X}^m)[(\gamma \otimes x) \otimes (\delta \otimes y)]
= \tau(\mathcal{R}^{-1} \cdot ((\gamma \otimes x) \otimes (\delta \otimes y)))
= \sum_i ((\varepsilon \otimes e_i)(\delta \otimes y) \otimes (S(f_i) \otimes 1_H)(\gamma \otimes x))
= \sum_i ((e'_i \triangleright \delta \triangleleft S(e''_i) \otimes e'''_i y) \otimes (S(f_i) \gamma \otimes x))
\]

Again we switch from ‘\(\triangleright\)’ notation to co-product notation to obtain:

\[
\sum_i ((\delta'''(e'_i) \delta'' \delta'(S(e''_i)) \otimes e'''_i y) \otimes (S(f_i) \gamma \otimes x))
\]
Then we use Lemma 5 and the fact that $S$ is an anti-automorphism and we get:

$$\sum_{i,j,k}((\delta''(e_j)\delta''\delta'(S(e_k)) \otimes e_i y) \otimes (S(f_k)S(f_i)S(f_j)\gamma \otimes x))$$

Now we again contract the consecutive $coev$ and $\tilde{ev}$ and we get:

$$\sum_i((\delta'' \otimes e_i y) \otimes (S^2(\delta')S(f_i)S(\delta''\gamma) \otimes x))$$

Finally we can use the fact that $H$ is involutory and the fact that $(id \otimes S)coev = (S \otimes id) \circ coev$ to get:

$$= \sum_i((\delta'' \otimes S(e_i m) y) \otimes (\delta' f_i m S(\delta''\gamma) \gamma \otimes x))$$

$$= \mathcal{L}_{H^{op}}(\mathcal{Y}^m)[(\gamma \otimes x) \otimes (\delta \otimes y)]$$

as desired.

- Finally we will consider what happens on distinguished arcs. From the calculation above, we have that:

$$\mathcal{F}_{D(H)}(\cap \circ \ldots \mid \circ \cup) = (\mu' \otimes \Lambda' \otimes \mu''(S(\Lambda''))) \otimes (id_H)$$

and by Lemma 60 we have:

$$\mathcal{L}_{H^{op}}(\mathcal{F}) = (\mu' \otimes \Lambda' \otimes \mu''(S(\Lambda'')) \otimes (1_H \otimes \mu)$$

Clearly they are equal in the first three tensors. So now we wish to compare $(id_H)$ and $(1_H \otimes \mu)$ By Lemma 11 we have $id_H = \Lambda' \otimes S(\Lambda'') \triangleright \mu$. This means after we trace out the entire tensor, $\Lambda'$ will be the first factor plugged into the integral. hence we have that:

$$\mathcal{F}_{D(H)}(\cap \circ \ldots \mid \circ \cup) = S(\Lambda'') \triangleright \mu \triangleleft \Lambda' = S(\Lambda'')\Lambda' \triangleright \mu$$
which by Lemma 60 and the axiom of the antipode is equivalent to

$$\varepsilon(\Lambda) \otimes 1_H \otimes \mu = (\dim H)1_H \otimes \mu$$

Hence at each distinguished arc, $$(\dim H)\mathcal{L}_{H^{\text{cop}}} (\widetilde{\mathcal{F}}) = \mathcal{F}(\cap \circ \cup)$$. Since there are exactly $c(L)$ distinguished arcs, our theorem is proven.


