APPLICATION OF PLATE AND SHELL MODELS IN THE LOADED TOOTH CONTACT ANALYSIS OF BEVEL AND HYPOID GEARS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Doctor of Philosophy in the Graduate School of the Ohio State University

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To my parents
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NOMENCLATURE

D flexural rigidity of the plate
E Young's modulus
face angle angle between the face of the gear and the axis
G shear modulus
h thickness of the shell
heel large end of the gear
p external load
r tang angle between the root and the axis
SE strain energy of the plate
t heel thickness of the gear at base of the heel
t thickness of plate
t tip tip thickness of the gear
toe small end of the gear
t toe thickness of gear at the base of the toe
t heel thickness of the gear at base of the heel
U strain energy of the plate
u r deflection in the radial direction
u θ deflection in the angular direction
W transverse deflection in z direction
WF work done by the external force

α difference between the face angle and root angle
β r shear slope in the radial direction
β θ shear slope in the angular direction
β x shear slope in the axial direction
β θ shear slope in the angular direction
ε x normal strain in the axial direction
ε r normal strain in the radial direction
ε θ normal strain in the angular direction
ε z normal strain in the z direction
γ rθ shear strain in the r-θ plane
γ rz shear strain in the r-z plane
γ θ x shear strain in the θ-z plane
γ θθ shear strain in the x-θ plane
γ xx shear strain in the x-z plane
\( \gamma_{\theta z} \) shear strain in the \( \theta-z \) plane
\( \nu \) Poisson's ratio
\( \Pi \) total potential energy
\( \sigma_z \) normal stress in the axial direction
\( \sigma_r \) normal stress in the radial direction
\( \sigma_\theta \) normal stress in the angular direction
\( \sigma_z \) normal stress in the z direction
\( \sigma_{\theta \theta} \) shear stress in the r-\( \theta \) plane
\( \sigma_{rz} \) shear stress in the r-z plane
\( \sigma_{\theta z} \) shear stress in the \( \theta-z \) plane
\( \psi_r \) bending slope in the radial direction
\( \psi_\theta \) bending slope in the angular direction
\( \tau_{x \theta} \) shear stress in the x-\( \theta \) plane
\( \tau_{\theta z} \) shear stress in the \( \theta-z \) plane
\( \tau_{xz} \) shear stress in the x-z plane
APPLICATION OF PLATE AND SHELL MODELS IN THE LOADED TOOTH CONTACT ANALYSIS OF BEVEL AND HYPOID GEARS

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A numerical procedure based on the Rayleigh-Ritz method is used to investigate the flexural behavior of an annular sector plate and a segment of circular cylindrical shell. The annular sector plate is cantilevered along the radial edge and the shell segment is cantilevered along the circular edge. The static analysis incorporates the effects of variable rigidity, including the effects of shear deformation. The Ritz method used, employs algebraic polynomial trial functions in two dimensions, to obtain the deflections and stresses. Convergence is investigated with attention being given to the number of terms taken for each coordinate direction. While it is known that an increase in number of terms improves the accuracy in the Ritz procedure and makes the assumed solution tend towards the exact solution, the solution times are seen to increase considerably. Hence the final choice of the number of terms balances the benefits of any further increase in accuracy against the corresponding increase in computation times. Two levels of shear based theories, the Mindlin and the Bhimaraddi theory are used in this study. The Mindlin theory is based on the assumption of a constant shear strain through the thickness while the Bhimaraddi theory assumes a parabolic variation of the shear strain through the thickness. It is seen that for a choice of identical number
of terms in the Ritz expansion, the Bhimaraddi theory is far more computationally intensive. The Ritz solutions for the sector plate and the cylindrical shell are shown to compare well with finite element predictions and experimental results. Additionally for a comparable accuracy, the Ritz procedure is seen to be numerically efficient compared to the finite element procedure since it requires only a limited model building effort and far fewer degrees of freedom.

The application of the sector plate and the shell model to predict the deflections and root stresses in bevel gears is demonstrated. The sector plate and the cylindrical shell are shown to be natural representations of straight and spiral bevel teeth due to their ability to include the salient features of the corresponding tooth geometries. When shear deformation and rigidity variation are also taken into account, the flexural behavior of the sector plate and the cylindrical shell closely represent the bending behavior of straight and spiral bevel teeth. The compliance calculations based on the plate and shell models can be readily incorporated into existing computer codes for bevel gear design to determine the load distribution and transmission error.

This dissertation has led to the development of mesh generators for bevel and hypoid gears to perform a full fledged finite element based loaded tooth contact analysis using CAPP. The geometry of the bevel gear is modeled as an octoid and the spiral bevel gear geometry is obtained through the simulation of the cutting kinematics. It is seen that the contact conditions are very sensitive to the geometry of the contacting surfaces and a very high degree of coordinate accuracy is required for the elements along the active profile and root. It is now possible to obtain the load distribution and static transmission error for all types of bevel and hypoid gears if the blank dimensions and machine settings are provided. Sample case studies for straight and spiral bevel gears are performed and discussed.
CHAPTER I

INTRODUCTION

1.1 Motivation

Noise and transmission error occurring in geared systems are often troublesome to gear manufacturers and users. It is widely recognized that the static transmission error describes the principal source of vibratory excitation arising from meshing gear pairs. One of the major causes of these errors is the nonuniform load distribution over the contact zone of the mating teeth as the gears are meshed under load. The existence of the nonuniform load distribution is mainly attributed to the elastic deflection of the geared system since the compliance at different points of contact is nonuniform. Load distribution factors are generally used by designers to compensate for the nonuniform load distribution. The elastic deformation of geared systems involves bending and shear deformations of gear teeth, shaft and gear body deflections, as well as the Hertzian deformations between the meshing teeth. This thesis is primarily concerned with evaluating the bending and shear deformation and root stresses of the bevel gear teeth due to a prescribed load distribution. A schematic of the different types of bevel gears is presented in Fig. 1.1.
Fig. 1.1 Types of bevel gears [ANSI/AGMA 1012-F90]
The bending and shear deformations as well as the root stresses of spur and helical gear teeth have traditionally been calculated through the use of beam and plate models. One of the early applications of plate theory in determining the deflections and root stresses in spur gears can be found in MacGregor (1935). Various researchers and designers since then have modeled the flexural behavior of plates to predict stresses and deflections of gear teeth. The early models described the gear tooth as a thin, infinite rectangular plate of constant rigidity. The root stresses in spur and helical gears were calculated from the stresses predicted by an infinite cantilever plate (Jaramillo, 1950). A correction to the infinite plate model through the use of the "moment image" method is described by Wellauer and Seireg (1960) wherein closed form solutions and analytic expressions were developed and correlated with experimental data.

With the advent of powerful computers, enhanced numerical techniques began to be widely used in the gear design process to determine tooth compliance and to predict load distribution and transmission error. Furthermore, the later models became more representative of the actual flexural behavior by taking into account the thickness taper along the tooth height and shear deformation effects. A generalized technique to evaluate the load distribution and transmission error for spur and helical gears based on a simplex type algorithm was first provided by Conry and Seireg (1972). The study used an infinite cantilever plate model (with corrections due to the finite face width of the tooth) to estimate the deflection due to the bending of the contacting teeth. Since then, the compliance calculations to determine the gear transmission error have been carried out through the use of a variety of beam/plate models and more recently the actual tooth geometries as in Vijayakar (1991). The beam and plate models, which are really one and two dimensional approximations of the actual geometry of the gear tooth, enjoy widespread use by gear designers because they provide a considerable reduction
in the compliance calculations, compared to the use of actual three dimensional geometries, without much loss in accuracy. When the gear tooth is modeled as a cantilever plate it should take into account the finite width, transverse shear deformation, tooth base rotation and variable thickness. Compliance calculations using all the above parameters were carried out by Tobe et. al. (1978) using finite element methods and by Yau (1987) using the Rayleigh-Ritz method.

A procedure similar to that developed by Conry and Seireg (1972) can be used to determine the load distribution in bevel gears by developing suitable tooth compliance models based on the bevel gear geometry. An overview of the salient features of such a bevel gear design and analysis procedure is presented in Sec. 1.5. A description of the loaded tooth contact analysis using CAPP (Contact Analysis Program Package) which uses the actual bevel and hypoid tooth geometry in the compliance calculations is presented in Chapter VI. This program developed by Vijayakar (1991), uses a combination of surface integral techniques and the finite element methods to evaluate the load distribution and transmission error. A similar, finite element based compliance calculation which uses the actual tooth geometry to predict root stress in spiral bevel gears can be found in Wilcox (1981).

A review of the literature reveals that other than the time intensive finite element based procedures, no satisfactory compliance model presently exists that takes into account the salient characteristics of the bevel and hypoid tooth geometry. A compliance model based on beam theory, for the loaded tooth contact analysis of spiral bevel gears has been used by Krenzer (1981). It is clear from Fig. 1.1 that such a model is not fully capable of representing the three dimensional characteristics of the bevel tooth. From Fig. 1.2 it is seen that the straight bevel tooth is characterized by a large reduction in tooth height from the heel (large end) to the toe (small end). The thickness
Fig. 1.2 Bevel gear nomenclature - axial plane [GWPSD3006H]
varies with the tooth height and position along the face width. The spiral bevel teeth have in addition, significant longitudinal curvature along the tooth length. In this dissertation, the application of the static analysis of the cantilevered annular sector plate in determining the straight bevel tooth compliance is demonstrated. For spiral bevel gears, a compliance model based on concepts from shell theory is developed which represents all the salient characteristics of the tooth form.

Although the contact analysis using CAPP provides excellent results, it requires a highly detailed model based on a very accurate surface description, which is time consuming to create and expensive to exercise and is not suited as a standard design tool. A full fledged three dimensional contact analysis using finite elements is too time intensive to be used on a routine basis for all except the most critical applications. This is true from both the computational point of view as well as from the model preparation point of view. This dissertation, thus presents new and improved tooth compliance models for straight and spiral bevel gears, based on two dimensional approximations, which provide FEM accuracy at a fraction of the computational effort, through the use of the Rayleigh-Ritz scheme. The models could be readily integrated into existing programs for loaded tooth contact analysis. A description of one such program, the Gleason LTCA is provided in the Appendix. Additionally, the model can be used to determine the root stresses for any predetermined load distribution.

1.2 Dissertation Objectives:

- Investigate the bending behavior of a cantilevered annular sector plate incorporating the effects of shear deformation through the use of Mindlin plate theory and Bhimaraddi’s higher order theory (which assumes a parabolic variation of shear strain through the thickness). This would be the first development of its
kind in attempting a series analytic solution modeling the flexural behavior of annular sector plates.

- To develop the cantilevered annular sector plate (CASP) model as a reliable and efficient tool for the prediction of compliance and root stresses in the design of straight bevel gears. The model would incorporate shear deformation, height taper, rigidity variation and base rotations.

- Develop a circular cylindrical shell model for the compliance and root stress prediction in spiral bevel and hypoid gears, incorporating the effects of longitudinal curvature, variable rigidity and shear deformation. The proposed strength calculations will depend primarily on the external tooth dimensions and not on the machine settings. This satisfies a long felt need to provide a universally acceptable calculation procedure that will start with basic gear parameters like tooth thickness, height, face width, taper, etc. At present, the strength calculations (finite elements) use machine settings and cutter specifications as input data. This is one step removed from the basic gear parameters.

1.3 Literature review:

Annular sector plates have been widely used in various areas of structural design like curved bridge decks, windmill blades, slabs, steam turbines diaphragms and a variety of marine and aerospace structures. Hence, the determination of deflections and stresses in transversely loaded sector plates is of both theoretical and practical interest. The flexural behavior of the sector plate has been analyzed for a variety of boundary conditions based on the application. For example, the windmill blade, requires free boundary conditions along the radial edges and the outer circular edge while the diaphragm has clamped conditions along its inner and outer circular edges. This
research will focus primarily on the static analysis of an annular sector plate cantilevered along a radial edge. The sector plate problem poses considerable mathematical difficulties to solve in closed form and approximate methods become necessary when thickness variability and shear deformation are taken into account.

An extensive review of isotropic and orthotropic sector plates has been given by Harik (1984). Sector plates, due to their versatile shape have found a wide variety of applications in structures and machines and the accurate determination of deflections and stresses has been and still continues to be a source of scientific investigation.

A review of the literature shows that the classical theory of plates has been the most popular theory in investigating the static and dynamic behavior of sector plates. The methods of analysis of sector plates can be broadly divided into analytical methods and numerical methods. The analytical methods employ the classical method of separation of variables. The basic functions in the angular direction satisfy the boundary condition of the radial edges. The corresponding transformation results in a fourth order ordinary differential equation from which the deflection expression is generated. The numerical methods include the finite element method, the finite strip method and the finite difference method.

Many sophisticated models of plate bending behavior have been developed including Reissner (1945) and Mindlin (1951) who incorporated shear deformation. Plate models still continue to be developed as in Bhimaraddi (1987). A good review of recent developments in plate theory is provided by Lo et al. (1977). However, to the best knowledge of the author, no researcher has developed a series analytic solution to the sector plate bending problem which includes the effects of shear deformation and variable rigidity. A finite element analysis of an orthogonally stiffened annular sector
plate was performed by Bhimaraddi et al. (1989) by combining an annular sector plate element and a curved beam element in which the effects of shear deformation and rotary inertia are taken into account. The following is a survey of relevant literature applicable to the static and dynamic analysis of sector plates. The section on bevel gear applications deals primarily with the literature related to geometry and stress analysis of face milled gears.

1.3.1 Annular Sector Plate

One of the earlier investigations into determining the bending moments along the edges of a uniformly loaded, fully clamped, isotropic sectorial plate can be found in Carrier (1944a) and moments and deflections for a fully clamped, cylindrical anisotropic plate loaded by a central concentrated load in Carrier (1944b). The application involved stress analysis of reinforced piston heads and circular reinforced concrete slabs. A general expression for the deflection of a sector plate clamped along the radial edges has been shown in Deverall et al. (1950) using the finite Fourier sine transform.

Several researchers have investigated the free vibration problem of a sector plate for a variety of boundary conditions. Ramakrishnan et al. (1973) has shown that a closed form solution can be obtained for the free vibration problem of an annular sector plate with simply supported radial edges for any combination of edge conditions on the circumferential edges. Mukhopadhyay (1978) used a semi-analytical method in which the basic function in the circumferential direction satisfying the boundary conditions of the radial edges is substituted into the free vibration equation of the curved plate. The resulting ordinary differential equation is solved using finite difference techniques. Srinivasan and Thiruvenkatachary (1986) investigated the free vibration analysis
including the effects of in-plane and rotatory inertia of a laminated sector plate with clamped edges.

The finite strip method has been developed and used by Cheung et al. (1981) for the static and dynamic analysis of thin and thick sectorial plates. The sectorial plates are divided into curved finite strips which satisfy the end boundary conditions. Interpolation polynomials are used to achieve compatibility of displacements along the interface of adjoining strips. The stresses and deflection are tabulated along the central radial line for various combinations of clamped and simply supported boundary conditions for fan plates (annular sectors).

Rubin (1983) showed that the sector plate could have a total of 54 possible combinations of boundary conditions out of which only 9 had solutions in the literature up to that time. His work provides solutions for all of the boundary conditions using the principle of minimum potential energy and a variable separable form for the displacements. This study was significant from the point of view of this investigation since it provides the solution to the cantilevered annular sector plate (clamped along the radial edge) subjected to a uniformly distributed load which had not been available before.

Harik (1984) obtained an analytical solution for the bending of the polar orthotropic pie and ring shaped sector plate for clamped and simply supported radial edges. The method of separation of variables was employed to convert the partial differential equation to an ordinary differential equation from which the deflection expression is generated. In structural applications, often thin sector plates are reinforced by circumferential stiffeners. Traditionally, such sector plate stiffener systems had been represented as an equivalent orthotropic plate for analysis. However,
the orthotropic plate idealization does not adequately represent the actual plate stiffener system when the stiffeners are not disposed asymmetrically with respect to the mid plane of the plate or unequally spaced. Harik and Haddad (1987) showed how the Analytical Strip Method (ASM) could be applied to model the plate stiffener system better than the traditional approximations. A semi-analytical solution for the in plane stability of a sector plate having clamped radial edges and any combination of boundary conditions on the circular edges was also provided by Harik (1985). For the free, out of plane vibration of a thin isotropic annular sector plate, Harik and Molaghaswami (1989) employed an analytical representation of functions in the radial direction along with the classical method of separation of variables to obtain an analytical solution. Sector plates of various radii and outer radius to thickness ratios along with various boundary conditions were analyzed.

A semi-analytic method called the Compound Strip Method (CSM) has been used by Puckett and Lang (1986) for the analysis of linear elastic curved plate systems that are continuous over non rigid supports. The stiffness contributions of support elements are derived and given as strip stiffness matrices which are combined with the plate strip matrices prior to assembly. The summation of the plate and support stiffness contribution forms a substructure called the compound strip. The results were compared with finite element solutions to illustrate the capability of CSM to model the static response of curved plates continuous over flexible supports.

Recently, the spline method has been used by Misuzawa (1991) to investigate the vibrations of stepped annular sector plates for arbitrary boundary conditions and by Misuzawa and Takami (1992) for tapered thickness annular sector plates. The variations of frequency with stepped thickness ratio and sector angle for stepped plates and with thickness ratio for tapered plates were studied. Thus it is seen that a variety of
techniques have been used to model the flexural behavior of sector plates based on using
the classical plate theory.

1.3.2 Wedge Plates

Wedge plates or sector plates (annular sector plates with zero inner radius) have received a lot of attention by researchers due to versatility in modeling and their wide variety of engineering applications similar to that of annular sector plates. Conway and Huang (1952) studied the bending of a uniformly loaded clamped sector plate by using superposition methods on the elementary solution for a uniformly loaded circular plate with a clamped edge. Stresses for the plate with clamped radial edges were provided by Woinowsky-Krieger (1953).

Approximate stress functions were derived by Silvermann (1955) as a solution of the Euler-Lagrange equation of variational calculus. The variational method was also used by Horvay and Hanson (1957) to determine approximate solutions of the biharmonic equation. The stress functions so derived, created the required shear and normal tractions on the circular and radial edges. Bassali and Halim (1961, 1963) investigated the transverse flexure of a sector plate clamped along the radial edges and free or elastically restrained along the circular edge. The position and magnitude of maximum deflection and bending moment are presented. Weber (1973) evaluated the deflection of a uniformly loaded clamped sector plate with 60 and 90 degree sector angles and compared the solution with that of Conway and Huang (1952).

Ben-Amoz (1959) was among the early researchers to investigate the flexural vibrations of clamped sector plates through the application of the theorem of minimum potential energy. Further investigations into the natural frequency and flexural vibrations of the isotropic wedge were carried out by Rubin (1975) for a fully clamped
case and Bhattacharya and Bhownic (1975) for the plate clamped along the radial edges. Cheung and Chan (1981) performed the static and dynamic analysis of thin and thick sector plates using the finite strip method (previously discussed). Their method could be applied to isotropic or orthotropic plates of variable thickness for different combinations of boundary conditions. Based on this literature review and earlier assessments by prior investigators like Harik (1984) and Misuzawa et al. (1992) it is clear that the transverse shear deformation has not been adequately modeled in the bending behavior of cantilevered annular sector plates.

1.3.3 Bevel gear applications

Bevel gears are the most commonly used means of transmitting motion between angularly disposed shafts. The bevel gear nomenclature in the axial plane is shown in Fig. 1.2. Spiral bevel and hypoid gears are favored over straight bevel gears in high performance transmissions in automobiles, marine and aviation industries because their curved teeth provide smoother and quieter operation along with greater bending resistance.

The geometry of straight and spiral bevel gears depends on the method of manufacture. Among the types of straight bevel gears commonly used are the Revacyle gears made by the Revacyle process and the Coniflex gears. A schematic of the Revacyle process is shown in Fig. 1.3. The Revacyle process is a broaching process which uses a Revacyle cutter (a large disk type cutter). The cutter turns continuously at a uniform rate. During each turn it is first fed in one direction along the tooth space to rough out the tooth space and then back again to finish it. The Coniflex gears are made by the Coniflex generator and the teeth are crowned in the lengthwise direction. The theory of meshing and the basic equations for straight bevel gears were
first described by Wildhaber (1945a, 1945b, 1945c). An in depth discussion on the requirements for conjugacy along with the expressions for relative curvature during mesh were provided. The bevel gears were shown to be conjugate to basic crown gears which have plane tooth sides. The crown gear has a pitch angle of 90 degrees and its pitch surface is a plane. Tsai and Chin (1987) detail the development of the tooth surface as a spherical involute and Chambers and Brown (1987) outline the procedure for obtaining an octoid tooth geometry.

The basic relationships of hypoid gears was first dealt with in a series of eight papers by Wildhaber (1946). The method of generation, tooth profile curvature calculations and the gear conjugate action are discussed in detail. The geometrical characteristics and nomenclature of spiral bevel gears have been characterized by ANSI/AGMA 1012-F90 (American Gear Manufacturers Association) and Gleason Works Publication (SD4168, SD3006H, SD3108F). A number of investigators have attempted to characterize the tooth surface and develop methods to determine the machine settings used to cut the gear. A discussion on the Helixform cutting method can be obtained in King et. al (1959, 1960). Baxter (1961) provides a detailed description on the basic geometry and tooth contact of hypoid gears. A set of vector equations and formulae to determine tooth and contact characteristics are provided in Baxter (1966). Additional work relating to hypoid geometry can be found in Shtipelman (1978) and Litvin and Gutman (1981a, 1981b, 1981c and 1981d).

When computers were introduced to the gearing industry in the 1950's, bevel and hypoid gear calculations were among its first applications. Programs were written to calculate machine settings, undercut and to analyze unloaded tooth contact. Since then, computers have played an increasing role in the design, analysis, manufacture and
inspection of bevel gears. Presently, the design, stress analysis and coordinate inspection of bevel gears can be performed using software provided by the Gleason Works. A description of the Gleason terminology can be found in the Gleason Works Publication (SD4168). The design of bevel gears includes both the strength determination and a contact analysis which involves the location and movement of the contact zone under load. A description of the tooth contact analysis can be found in GWP (1978) and guidelines for installation and assembly of bevel and hypoids in GWP (1982b).

A review of the literature (Wilcox et al. (1977), Wilcox (1981) and Vijayakar et al. (1991) etc.) shows that the finite element procedure has been the primary tool to estimate root stresses in bevel and hypoid gears. Wilcox(1981) used the HEX20 element for the stress analysis and compliance calculations while Vijayakar et al. (1991) used the FQP (finite quasi prism) element developed by Vijayakar (1987) at the Ohio State University. The finite element analysis requires a mesh generator and starts off with the basic machine settings rather than the gear parameters. A brief summary of the sequence of operations is as follows. Once the external dimensions are estimated, the machine settings used to cut the gear are calculated and an iterative procedure is used to evaluate the coordinates on the tooth surface. These are used by the mesh generator to create the input file for the finite element analysis. Any change in the blank dimensions would require going through all the intermediate procedures before the finite element run. This procedure makes it very unattractive to incorporate the finite element technique in the design sequence and has resulted in the finite element procedure primarily being used as a final analysis tool.
1.4 Contact analysis of bevel gears

The evaluation of the load distribution in gears is based on the formulation of the contact problem as a linear program and the use of a Simplex type algorithm to solve the frictional contact conditions imposed on the bodies. The contact problem is restricted to normal surface loading conditions. The deformations are assumed to be small and the two bodies are linearly elastic with smooth surfaces having continuous first derivatives. From a knowledge of the compliance at each point in the contact zone, the initial separations under no load and the applied torque the load distribution and the overall system rigid body rotation are evaluated using the Simplex type algorithm.

This section outlines the general procedure that could be used to determine the load distribution along the line of contact and the transmission error at a typical mesh position, for the straight bevel gear. Considering gear 1 as the input gear and gear 2 as the output gear, we assume that the input gear is rotated at a constant angular velocity and the output gear has a constant load torque (generalized force \( \lambda \)). The origin and the angular position of the input gear are prescribed, while the output gear is allowed to rotate about its kinematically computed position by a small angle \( \theta \) which is the transmission error. When the gears are in mesh, each instant line of contact common to the pair of teeth is discretized into a series of nodes \( N \) spaced along the line. At each node \( m \), let \( e_m \) be the initial separation of the two bodies along the common normal before the elastic deformation, \( i_m \) the increase in separation due the elastic deformation, \( d_m \) the final separation and \( p_m \) the compressive normal force. Discrete forces are taken to represent the distributed pressures over finite areas.
Compatibility conditions:

At any point \( m \) in the zone of contact, the sum of the total elastic deformation of the two bodies and the initial separation must be greater than or equal to the rigid body approach along the line of action. This condition is represented as

\[
\mathbf{e} + [\mathbf{H}] \mathbf{p} \geq [\mathbf{C}] \mathbf{\theta}
\]  

(1.1)

where

\[
\mathbf{e} = (e_1, e_2, \ldots, e_N)^T \quad N \times 1 \text{ vector of initial separations}
\]

\[
\mathbf{d} = (d_1, d_2, \ldots, d_N)^T \quad N \times 1 \text{ vector of final separations}
\]

\[
\mathbf{p} = (p_1, p_2, \ldots, p_N)^T \quad N \times 1 \text{ vector of forces}
\]

\[
\mathbf{i} = (i_1, i_2, \ldots, i_N)^T \quad N \times 1 \text{ vector of increase in separations}
\]

\[
[\mathbf{H}] \quad N \times N \text{ compliance matrix}
\]

\[
[\mathbf{C}] \quad N \times N \text{ rigid body rotation matrix}
\]

From the geometry of the system and the relative orientation of reference systems, the following linear relationship can be written.

\[
\mathbf{i} = [\mathbf{H}] \mathbf{p} - [\mathbf{C}] \mathbf{\theta}
\]  

(1.2)

The matrix \( \mathbf{H} \) is the compliance matrix which is expressed as

\[
\mathbf{H} = \mathbf{H}^1 + \mathbf{H}^2 \quad \text{where}
\]

\[
\mathbf{H}_{ij} = \\
\begin{bmatrix}
\delta_1^1 & \delta_2^1 & \ldots & \delta_N^1 \\
\delta_1^2 & \delta_2^2 & \ldots & \\
\ldots & \ldots & \ldots & \\
\delta_1^N & \ldots & \delta_N^N
\end{bmatrix}
\]  

(1.3)
and $\delta_m^o$ is the deflection is node n due to the load applied at node m. The matrix $C$ relates the increase in separation between the two gears at the contact points to the rigid body rotation (angular displacement) of the output gear.

Equilibrium conditions:

Torque balance must be achieved. The total moment about the axis of rotation of the compressive forces $p_m$ acting along the line of action must balance the applied torque. From equilibrium considerations

$$\lambda = [B]p \quad (1.4)$$

where B relates the load vector p to the applied maximum torque.

Contact criterion:

This criterion states that the two surfaces must be in contact at a point for a pressure to exist at that point.

Hence we have

$$d_m = i_m + e_m \geq 0 \text{ and } p_m \geq 0 \quad (1.5a-b)$$

and either $p_m = 0$ or $d_m = 0$.

The contact equations can thus be summarized as

$$\lambda = [B]p \quad (1.6)$$

$$e = d - [H]p - [C]\theta \quad (1.7)$$
It is required to solve for $d$, $p$ and $\theta$, given $e$ and $\lambda$, subject to the conditions $p, d \geq 0$ and at each node either $d_m = 0$ or $p_m = 0$. This procedure is repeated at different positions in the mesh cycle to obtain the transmission error plots.

### 1.5 Matrix form of the Ritz equation

The governing differential equation and the boundary conditions for the problem are usually obtained by applying the principle of virtual work. This involves setting the first variation of the appropriate functional with respect to the dependent variable to zero. The Euler-Lagrange equations are obtained by using the procedures of the calculus of variation. The Euler equations in the form of differential equations are not tractable by the exact methods of solution. A number of approximate techniques like the finite difference method, perturbation methods, etc. exist for solving differential equations. The direct methods bypass the derivation of the Euler equations and go directly from a variational statement of the problem to the solution of the Euler equation. The Rayleigh-Ritz method is one such method.

The Rayleigh-Ritz scheme is based on the minimization of the total potential energy of a linear elastic body. The total potential energy of a linear elastic body undergoing small displacements is given by

$$
\Pi(u) = \int \left( \frac{1}{2} \sigma_{ij} e_{ij} - f_i u_i \right) dV - \int \hat{t}_i u_i dS
$$

(1.8)

The functional is of the form

$$
\Pi(u) = Q(u) - L(u)
$$

(1.9)
where

$$Q(u_1, u_2, u_3) = \frac{1}{2} \sum_{i,j=1}^{3} \int \epsilon_{ij} \sigma_{ij} dV$$

(1.10)

$$L(u_1, u_2, u_3) = \sum_{i=1}^{3} \left( \int_{V} f_i u_i dV + \int_{S} t_i u_i dS \right)$$

(1.11)

Here, the stresses and strains are known in terms of the displacements through the kinematic and constitutive relations. Setting the first variation of the functional to zero gives

$$\delta \Pi = \sum_{i=1}^{n} \left[ \left( \frac{\partial Q}{\partial c_i} - \frac{\partial L}{\partial c_i} \right) \delta c_i + \left( \frac{\partial Q}{\partial c_i^2} - \frac{\partial L}{\partial c_i^2} \right) \delta c_i^2 + \left( \frac{\partial Q}{\partial c_i^3} - \frac{\partial L}{\partial c_i^3} \right) \delta c_i^3 \right]$$

(1.12)

Since $Q$ is quadratic and $L$ in linear in $c_i, c_i^2$ and $c_i^3$, their partial derivatives with respect to $c_i, c_i^2$ and $c_i^3$ are bilinear and linear functionals of the assumed functions. Thus we get

$$\frac{\partial Q}{\partial c_i} = \sum_{\beta=1}^{3} \sum_{j=1}^{3} \mathbf{B}_{\alpha \beta} \phi_i^\alpha \phi_j^\beta c_j^\beta$$

(1.13)

$$\frac{\partial L}{\partial c_i^\alpha} = \mathbf{F}^\alpha \phi_i^\alpha$$

(1.14)

so that

$$\sum_{\beta=1}^{3} \sum_{j=1}^{3} \mathbf{B}_{\alpha \beta} \phi_i^\alpha \phi_j^\beta c_j^\beta = \mathbf{F}^\alpha \phi_i^\alpha$$

(1.15)

In matrix form

$$\begin{bmatrix} [\mathbf{B}^{11}] & [\mathbf{B}^{12}] & [\mathbf{B}^{13}] \\ [\mathbf{B}^{21}] & [\mathbf{B}^{22}] & [\mathbf{B}^{23}] \\ [\mathbf{B}^{31}] & [\mathbf{B}^{32}] & [\mathbf{B}^{33}] \end{bmatrix} \begin{bmatrix} \{c^1\} \\ \{c^2\} \\ \{c^3\} \end{bmatrix} = \begin{bmatrix} \{\mathbf{F}^1\} \\ \{\mathbf{F}^2\} \\ \{\mathbf{F}^3\} \end{bmatrix}$$

(1.16)
where $b_{ij}^{qs}$ can be identified in terms of the volume integral of the material coefficients and coordinate functions.

1.6 Thesis overview

This thesis consists of six chapters. In chapter II the governing equations of the Mindlin sector plate are derived, along with the simplified forms of the energy expressions which are used in the Ritz approach. Chapter III deals with the application of the CASP model for straight bevel gears. The results from the sector plate model are compared with previously available results for spur gears and the finite element methods. The results of a convergence study are also presented. An overview of the theory of surfaces is presented in Chapter IV and the metric derived for obtaining the strain displacement relations for a circular cylindrical shell. Chapter V deals with the application of the circular shell model to represent spiral bevel gears. The deflection and stress results are compared with finite element methods. Chapter VI deals with use of the finite (FQP - finite quasi prism) element in the loaded tooth contact analysis for straight bevel and hypoid gears. The last chapter summarizes the achievements of this thesis and presents recommendations for future work.
CHAPTER II

FUNDAMENTAL EQUATIONS OF AN ANNULAR SECTOR PLATE

2.1 Introduction

The subject of plate theory is the mathematical analysis of stress and strain in thin elastic bodies which in their natural state are bounded by two parallel planes and by a cylindrical surface with generators perpendicular to the two planes and remain within the bounds of the linear theory of elasticity. The continuing interest in the theory of plates is governed by the expectation that the quality of thinness makes it possible to reach significant conclusions on the basis of two dimensional rather than three dimensional considerations. In this chapter an overview of the classical (Kirchoff) theory of plates is presented followed by the derivation of the Mindlin governing equations and strain energy expressions for the sector plate. A more complete derivation of the governing equations and stress resultants can be found in Mansfield (1964). The strain energy expression derived here will be used in Chapter III, to determine the flexural behavior of the straight bevel gear tooth. Fig. 2.1 shows the sector plate under load.

2.2 Classical (Kirchoff) plate theory

The classical small deflection theory of plates is based on the following assumptions.
Fig. 2.1 Sector plate element under load
1) The points which lie on a normal to the midplane of the plate lie on the normal to the midplane after deformation.

2) The stresses normal to the midplane of the plate due to the applied loading are negligible relative to the inplane stresses of the plate.

3) The slope of the deflected plate in any direction is so small such that its square may be neglected in comparison with unity.

Additionally, the material is assumed to be homogenous, isotropic, continuous and linearly elastic. This allows the use of stress strain relationships in terms of two elastic constants.

For the sector plate, the displacement assumptions in mathematical form are

\[ u = -z \frac{\partial w}{\partial r} \]  \hspace{1cm} (2.1)

\[ v = -\frac{z}{r} \frac{\partial w}{\partial \theta} \]  \hspace{1cm} (2.2)

\[ w = w(r, \theta) \]  \hspace{1cm} (2.3)

The final form of the governing differential equation defining the lateral deflection of the middle surface of the sector plate subjected to lateral loads is defined as

\[ \nabla^2(D \nabla^2 w) - (1 - v) \delta^4(D, w) = p \]  \hspace{1cm} (2.4)

where

\[ \nabla^2 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \]  \hspace{1cm} (2.5)

and
\[ \phi^2(D, w) = \frac{1}{2} \left\{ (\nabla^2 D)(\nabla^2 w) + \nabla^2 (D\nabla^2 w + w\nabla^2 D) \right\} + \]
\[ \frac{1}{4} \left\{ \nabla^4 (Dw) + D\nabla^4 w + w\nabla^4 D \right\} \]  
(2.6)

2.3 Derivation of the basic equations of the Mindlin sector plate

The equations defining the lateral deflections of the middle surface of a sector plate subjected to lateral loads may be formulated in different ways. In this approach, the strain displacement relations, the constitutive relations and the governing equations of equilibrium of an annular sector plate of variable rigidity are derived by utilizing the basic displacement assumptions. Due to the stubby nature of the bevel teeth, the Mindlin equations rather than the classical plate equations will be used to determine the flexural behavior.

2.3.1 Displacement assumptions

The Mindlin plate theory assumes a constant shear strain through the thickness. The displacement assumptions are

\[ u_r = -z\psi_r \]  
(2.7)

\[ u_\theta = -z\psi_\theta \]  
(2.8)

\[ w = w(r, \theta) \]  
(2.9)

with

\[ \frac{\partial w}{\partial r} = \psi_r + \beta_1 \]  
(2.10)

\[ \frac{1}{r} \frac{\partial w}{\partial \theta} = \psi_\theta + \beta_2 \]  
(2.11)
2.3.2 Strain displacement relations

The corresponding strain displacement relations from the theory of elasticity are given as

\[ \varepsilon_r = -z \frac{\partial \psi_r}{\partial r} \]  \hspace{1cm} (2.12)

\[ \varepsilon_\theta = -\frac{z}{r} \frac{\partial \psi_\theta}{\partial \theta} - \frac{z}{r} \psi_r \]  \hspace{1cm} (2.13)

\[ \varepsilon_z = 0 \]  \hspace{1cm} (2.14)

\[ \gamma_{r\theta} = -\frac{z}{r} \frac{\partial \psi_r}{\partial \theta} - \frac{z}{r} \frac{\partial \psi_\theta}{\partial r} + \frac{z}{r} \psi_\theta \]  \hspace{1cm} (2.15)

\[ \gamma_{r\zeta} = \frac{\partial w}{\partial r} - \psi_r \]  \hspace{1cm} (2.16)

\[ \gamma_{\zeta \theta} = \frac{1}{r} \frac{\partial w}{\partial \theta} - \psi_\theta \]  \hspace{1cm} (2.17)

2.3.3 Constitutive relations

In three dimensional elasticity theory, there are six components of stress which are expressed in terms of the six components of strain through Hooke's law as shown below.

\[ \sigma_r = \frac{E}{(1 - v^2)} (\varepsilon_r + v(\varepsilon_\theta + \varepsilon_z)) \]  \hspace{1cm} (2.18)

\[ \sigma_\theta = \frac{E}{(1 - v^2)} (\varepsilon_\theta + v(\varepsilon_r + \varepsilon_z)) \]  \hspace{1cm} (2.19)

\[ \sigma_z = \frac{E}{(1 - v^2)} (\varepsilon_z + v(\varepsilon_\theta + \varepsilon_r)) \]  \hspace{1cm} (2.20)

\[ \sigma_{r\theta} = G \gamma_{r\theta} \]  \hspace{1cm} (2.21)
\[ \sigma_{rz} = G \gamma_{rz} \]  \hspace{1cm} (2.22)

\[ \sigma_{\theta z} = G \gamma_{\theta z} \]  \hspace{1cm} (2.23)

The stress strain relations and the expressions for the six components of strain, in terms of three components of displacements, are then used to reduce to three, the number of unknowns in the three equations of motion. In the present plate theory there are only five plate stress components and these will be expressed in terms of the same number of plate strain components as shown below.

\[ \sigma_r = -\frac{Ez}{(1-\nu^2)} \left( \nu \frac{\partial \psi_r}{\partial r} + \frac{1}{r} \left[ \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right] \right) \]  \hspace{1cm} (2.24)

\[ \sigma_\theta = -\frac{Ez}{(1-\nu^2)} \left( \nu \frac{\partial \psi_r}{\partial r} + \frac{1}{r} \left[ \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right] \right) \]  \hspace{1cm} (2.25)

\[ \sigma_{\theta r} = -\frac{Ez}{2(1+\nu)} \left[ \frac{1}{r} \frac{\partial \psi_r}{\partial \theta} + \frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r} \right] \]  \hspace{1cm} (2.26)

\[ \sigma_{rz} = \frac{E_k}{2(1+\nu)} \left( \frac{\partial w}{\partial r} - \psi_r \right) \]  \hspace{1cm} (2.27)

\[ \sigma_{\theta z} = \frac{E_k}{2(1+\nu)} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \psi_\theta \right) \]  \hspace{1cm} (2.28)

The stress couples (moments per unit length) are obtained by multiplying the stresses by \( z \) and integrating through the thickness.

\[ M_r = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_r z dz = -D \left( \frac{\partial \psi_r}{\partial r} + \frac{\nu}{r} \left[ \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right] \right) \]  \hspace{1cm} (2.29)

\[ M_\theta = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma_\theta z dz = -D \left( \nu \frac{\partial \psi_r}{\partial r} + \frac{1}{r} \left[ \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right] \right) \]  \hspace{1cm} (2.30)
\[ M_{\theta} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_{\theta z} dz = -D \frac{(1 - \nu)}{2} \left( \frac{\partial \psi_{\theta}}{\partial r} + \frac{1}{r} \left[ \frac{\partial \psi_{r}}{\partial \theta} - \psi_{\theta} \right] \right) \]  \hspace{1cm} (2.31)

where \( D = \frac{Et^3}{12(1 - \nu^2)} \)

Similarly, the transverse shear resultants are obtained by integrating the shear stress through the thickness as

\[ Q_r = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{r z} dz = Gt k_t \left( \frac{\partial w}{\partial r} - \psi_r \right) \]  \hspace{1cm} (2.32)

\[ Q_\theta = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{\theta z} dz = Gt k_t \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \psi_\theta \right) \]  \hspace{1cm} (2.33)

Once the stress-strain relationships and the stress resultant have been developed, it is possible to write the energy expression in terms of the bending rotations and the transverse deflection, along with their spatial derivatives.

### 2.3.4 Equilibrium equations

The equilibrium equations in terms of moments and shears in polar coordinates for the Mindlin sector plate are written below using the previously derived expressions for the stress resultants.

From the theory of elasticity, the equilibrium equations in polar coordinates are

\[ \frac{\partial M_r}{\partial r} + \frac{1}{r} \frac{\partial M_{\theta r}}{\partial \theta} + \frac{M_r - M_\theta}{r} - Q_r = 0 \]  \hspace{1cm} (2.34a)

\[ \frac{\partial M_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial M_\theta}{\partial \theta} + \frac{2}{r} M_{r \theta} - Q_\theta = 0 \]  \hspace{1cm} (2.34b)
\[
\frac{\partial Q_r}{\partial \theta} + \frac{1}{r} \frac{\partial Q_\theta}{\partial \theta} + \frac{Q_r}{r} + p(r, \theta) = 0
\]  
(2.34c)

Substituting for the stress resultants in the above equations yields

\[
\frac{\partial}{\partial r} \left( \frac{1}{D} \left[ \frac{\partial \psi_r}{\partial r} + \frac{v}{r} \left( \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right) \right] + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{D(1-v)}{2} \left[ \frac{1}{r} \frac{\partial \psi_r}{\partial \theta} + \frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r} \right] \right) \right) + \frac{Q_r}{r} = 0
\]  
(2.35a)

\[
\frac{\partial}{\partial r} \left( \frac{D(1-v)}{2} \left[ \frac{1}{r} \frac{\partial \psi_\theta}{\partial \theta} + \frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r} \right] \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{D}{r} \left[ \frac{1}{r} \frac{\partial \psi_r}{\partial \theta} + \psi_r \right] + v \frac{\partial \psi_r}{\partial r} \right) \right) + \frac{Q_\theta}{r} = 0
\]  
(2.35b)

\[
G_{tk} \nabla^2 w - \phi + p(r, \theta) = 0
\]  
(2.35c)

where \( \phi = \frac{\partial \psi_r}{\partial r} + \frac{1}{r} \frac{\partial \psi_\theta}{\partial \theta} + \frac{\psi_r}{r} \)

2.4 Energy expression

The total potential energy of a body is the sum of the external potential energy and the strain energy.

\[ \Pi = U + H \]  
(2.36a)

For work done by external forces in a conservative system, then

\[-W = H \]  
(2.36b)

The strain energy of an elastic body in polar coordinates is given as

\[ U = \frac{1}{2} \int \left[ \sigma_{rr} \varepsilon_r + \sigma_{\theta\theta} \varepsilon_\theta + \sigma_{zz} \varepsilon_z + \sigma_{s\theta} \varepsilon_{s\theta} + \sigma_{erez} \varepsilon_{erez} + \sigma_{\theta\phi} \varepsilon_{\theta\phi} \right] dv \]  
(2.37)
and in expanded form by,

\[
U = \frac{1}{2} \int \frac{Ez^2}{(1-v^2)} \left( \frac{\partial \psi_r}{\partial r} + \frac{v}{r} \left[ \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right] \right) \left( \frac{\partial \psi_r}{\partial r} \right) rrd\theta dz
\]

\[
\quad + \frac{1}{2} \int \frac{Ez^2}{(1-v^2)} \left( \frac{\partial \psi_\theta}{\partial r} + \frac{1}{r} \left[ \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right] \right) \left( \frac{1}{r} \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right) rrd\theta dz
\]

\[
\quad + \frac{1}{2} \int \frac{Ez^2}{2(1+v)} \left( \frac{\partial \psi_\theta}{\partial r} + \frac{1}{r} \left[ \frac{\partial \psi_\theta}{\partial \theta} - \psi_\theta \right] \right) rrd\theta dz
\]

\[
\quad + \frac{1}{2} \int \frac{Ek}{2(1+v)} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \psi_r \right)^2 rrd\theta dz
\]

\[
\quad + \frac{1}{2} \int \frac{Ek}{2(1+v)} \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \psi_\theta \right)^2 rrd\theta dz
\]  

(2.38)

Integrating each of the above expressions with respect to \( z \) and substituting

\( D = \frac{Et^3}{12(1-v^2)} \)

and noting that \( \frac{Et}{4(1+v)} = \frac{3D(1-v)}{t^2} \), one obtains

\[
U = \frac{1}{2} \int D \left( \frac{\partial \psi_r}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right)^2 + 2v \left( \frac{1}{r} \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right) \left( \frac{\partial \psi_r}{\partial r} \right) rrd\theta
\]

\[
\quad + \frac{1}{2} \int D \frac{(1-v)}{4} \left( \frac{1}{r} \frac{\partial \psi_r}{\partial \theta} + \frac{\partial \psi_\theta}{\partial r} - \psi_\theta \right)^2 rrd\theta
\]

\[
\quad + 3D(1-v)k_\theta \left( \left( \frac{\partial w}{\partial r} - \psi_r \right)^2 + \left( \frac{1}{r} \frac{\partial w}{\partial \theta} - \psi_\theta \right)^2 \right) rrd\theta
\]  

(2.39)

The energy expression can also be written in terms of moments and shear forces. The following expressions

\[
M_r = -D \left( \frac{\partial \psi_r}{\partial r} + \frac{v}{r} \left[ \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right] \right)
\]

(2.40)

\[
M_\theta = -D \left( v \frac{\partial \psi_r}{\partial r} + \frac{1}{r} \left[ \frac{\partial \psi_\theta}{\partial \theta} + \psi_r \right] \right)
\]  

(2.41)
which have been previously derived, can be substituted in \( M_r^2 + M_\theta^2 - 2vM_rM_\theta \) and simplified to get

\[
\frac{M_r^2 + M_\theta^2 - 2vM_rM_\theta}{2D(1-v^2)} = \frac{D}{2} \left[ \left( \frac{1}{r} \frac{\partial \psi_r}{\partial r} + \frac{\psi_r}{r} \right)^2 + 2v \left( \frac{1}{r} \frac{\partial \psi_\theta}{\partial r} + \frac{\psi_\theta}{r} \right) \left( \frac{\partial \psi_r}{\partial r} \right) \right]
\] (2.42)

Similarly, it can be shown that

\[
\frac{M_\theta^2}{D(1-v)} = \frac{D}{2} \left[ \frac{1}{2} \left( \frac{1}{r} \frac{\partial \psi_r}{\partial r} + \frac{\partial \psi_\theta}{\partial r} - \frac{\psi_\theta}{r} \right)^2 \right]
\] (2.43)

Therefore, the strain energy can be compactly written as

\[
U = \int \left[ \frac{M_r^2 + M_\theta^2 - 2vM_rM_\theta}{2D(1-v^2)} + \frac{M_\theta^2}{D(1-v)} + \frac{Q_r^2 + Q_\theta^2}{2Gt_k} \right] r dr d\theta
\] (2.44)

and work done by the external force as

\[
W = \int p wr dr d\theta.
\] (2.45)

2.5 Higher order plate theory:

This theory developed by Bhimaraddi (1984) differs from the conventional shear deformation theory in that it does not use a shear correction factor and assumes a parabolic variation of shear strain through the thickness. In addition this theory satisfies the shear free conditions on the top and bottom surfaces of the plate.

2.5.1 Displacement assumptions

\[
u_r = z(1-\frac{4z^2}{3h^2})\beta_1 + z \frac{\partial w}{\partial r}
\] (2.46)

\[
u_\theta = z(1-\frac{4z^2}{3h^2})\beta_2 - z \frac{\partial w}{r \partial \theta}
\] (2.47)
\[ w = w(r, \theta) \quad (2.48) \]

From the displacement assumptions it can be seen that the strain expressions will include the derivatives of the thickness with respect to spatial coordinates, which must be accounted for in the strain energy expression. The strain displacement equations using this higher order theory are presented below.

### 2.5.2 Strain displacement relations

\[
\begin{align*}
\varepsilon_r &= z \left( 1 - 4z^2 \right) \left[ \frac{1}{r} \frac{\partial \beta_r}{\partial \theta} \right] - z \left[ \frac{\partial^2 w}{\partial r^2} \right] + \frac{8z^3}{3h^3} \beta_r \frac{\partial h}{\partial r} \\
\varepsilon_\theta &= z \left( 1 - 4z^2 \right) \left[ \frac{1}{r^2} \frac{\partial \beta_\theta}{\partial \theta} \right] - z \left[ \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right] + \frac{8z^3}{3h^3} \beta_\theta \frac{\partial h}{\partial \theta} \\
+ z \left( 1 - 4z^2 \right) \frac{\beta_r}{r} - z \left[ \frac{1}{r} \frac{\partial w}{\partial r} \right] \\
\gamma_{r\theta} &= z \left( 1 - 4z^2 \right) \left[ \frac{1}{r} \frac{\partial \beta_r}{\partial \theta} \right] - z \left[ \frac{2}{r} \frac{\partial^2 w}{\partial \theta \partial \theta} \right] + \frac{8z^3}{3h^3} \beta_r \frac{\partial h}{\partial \theta} \\
+ z \left( 1 - 4z^2 \right) \frac{\partial \beta_\theta}{\partial r} - z \left[ \frac{1}{r} \frac{\partial w}{\partial r} \right] + \frac{8z^3}{3h^3} \beta_\theta \frac{\partial h}{\partial r} - z \left( 1 - 4z^2 \right) \frac{\beta_\theta}{r} \\
+ z \left[ \frac{2}{r^2} \frac{\partial w}{\partial \theta} \right] \\
\gamma_{rz} &= \left( 1 - \frac{4z^2}{h^2} \right) \beta_r \\
\gamma_{z\theta} &= \left( 1 - \frac{4z^2}{h^2} \right) \beta_\theta \\
\end{align*}
\]
2.6 Solution method

The Rayleigh-Ritz method is used to obtain the approximate solution for the transverse deflection of the sector plate. The transverse deflections $w$ and the bending rotations $\psi_r$ and $\psi_\theta$ in the Mindlin plate and the shear rotations $\beta_r$ and $\beta_\theta$ in the Bhimaraddi plate that are to be determined, are assumed in the form of a series solution (a finite linear combination) containing unknown parameters which satisfy the geometric boundary conditions. As the number of linearly independent terms in the assumed solution is increased, the assumed solution converges to the desired solution of the Euler equation. The method is based on the minimization of the total potential energy of the system. Each of the functions used in the series solution is actually a product of two expressions, $\phi_i(r)$ a function of the radial position $r$, $\epsilon_i(\theta)$ a function of the angular position $\theta$.

For the Mindlin plate, this is expressed as

\[ w = \sum_i \sum_j A_{ij} \phi_i(r) \epsilon_j(\theta) \quad (2.54) \]

\[ \psi_r = \sum_i \sum_j B_{ij} \phi_i(r) \epsilon_j(\theta) \quad (2.55) \]

\[ \psi_\theta = \sum_i \sum_j C_{ij} \phi_i(r) \epsilon_j(\theta) \quad (2.56) \]

For the Bhimaraddi plate,

\[ w = \sum_i \sum_j A_{ij} \phi_i(r) \epsilon_j(\theta) \quad (2.57) \]

\[ \beta_r = \sum_i \sum_j B_{ij} \phi_i(r) \epsilon_j(\theta) \quad (2.58) \]

\[ \beta_\theta = \sum_i \sum_j C_{ij} \phi_i(r) \epsilon_j(\theta) \quad (2.59) \]
The functions so chosen must satisfy the following requirements.

1) \( \phi_i(r) \) and \( \epsilon_j(\theta) \) must satisfy in their actual form, the essential boundary conditions associated with \( w, \psi_r \) and \( \psi_\theta \).

2) \( \phi_i(r) \) and \( \epsilon_j(\theta) \) must be continuous (as specified by the variational principle) and be linearly independent and complete.

2.6.1 Boundary conditions

The sector plate has 4 boundaries, two radial and two circular. In this study, the following conditions are investigated: (a) the plate clamped along a radial edge and (b) elastically supported along the radial edge. Fig. 2.2 shows the two boundary conditions of the sector plate.

The clamped condition involves the boundary conditions specified as

\[
w(r, \theta = 0) = 0, \quad \psi_\theta(r, \theta = 0) = 0 \tag{2.60a-b}
\]

The appropriate choice of the functions \( \phi_i(r) \) and \( \epsilon_j(\theta) \) associated with \( w, \psi_r \) and \( \psi_\theta \) are made to obtain the boundary conditions specified above. In the Mindlin plate, for \( w \) and \( \psi_\theta \), \( \phi_i(r) \) and \( \epsilon_j(\theta) \) satisfy the free-free conditions and pinned-free conditions, respectively, and for \( \psi_r \), both \( \phi_i(r) \) and \( \epsilon_j(\theta) \) are functions satisfying free-free conditions. For the Bhimaraddi plate, for \( w \) and \( \beta_\theta \), \( \phi_i(r) \) and \( \epsilon_j(\theta) \) satisfy the free-free conditions and pinned-free conditions respectively and for \( \beta_r \), both \( \phi_i(r) \) and \( \epsilon_j(\theta) \) are functions satisfying free-free conditions.

For a plate supported on elastic foundation

\[
w(r, \theta = 0) = 0 \tag{2.61}
\]
2.2(a) Clamped boundary condition

2.2(b) Elastic support condition

Fig. 2.2 Sector plate boundary condition
This boundary condition is modeled by letting $\epsilon_i(\theta)$ satisfy the pinned-free condition and $\phi_i(r)$ the free-free condition for $w$. For $\psi_o$ and $\psi_i$, both $\epsilon_i(\theta)$ and $\phi_i(r)$ satisfy the free-free condition. Also the strain energy expression has an additional term due to the elastic support.

2.6.2 Admissible functions

For a pinned-free condition

$$\epsilon_i(\theta) = \left(\frac{\theta}{\alpha}\right)^i$$

(2.62a)

and for a free-free condition

$$\epsilon_i(\theta) = \left(\frac{\theta}{\alpha}\right)^{i-1}$$

(2.62b)

It was determined that the use of polynomial type functions possessed far greater advantages than the beam functions. The polynomials and their higher order derivatives are very easy to generate, and lend themselves to exact integration. The beam functions require a highly accurate determination of the constants used in the expressions, are not exactly integrated and their solutions tend to converge to an oscillatory shape with an increasing number of trial functions. Hence polynomials were the choice of admissible functions in the subsequent computations presented in future chapters.

2.7 Summary

In this chapter the strain energy expression for the annular sector plate including shear deformation are derived. The strain displacement relations for the higher order theory are seen to include terms containing the derivatives of the thickness with respect to the principal coordinates unlike Mindlin's theory. The procedure to represent the
cantilevered boundary condition and the elastic support condition through the use of polynomial type functions is demonstrated. The stress relations and the strain energy expression are used in Chapter III to demonstrate the application of the sector plate model to determine the flexural behavior of the straight bevel gear tooth.
CHAPTER III

APPLICATION OF THE SECTOR PLATE MODEL TO

STRAIGHT BEVEL GEARS

3.1 Introduction

In this section, the application of the sector plate model to represent the bevel tooth geometry is demonstrated. Bevel gears are characterized by a height and thickness taper along the face width. Fig. 3.1 shows the tapered tooth geometry of the bevel gear with radial lines of contact. Fig. 3.2 shows an annular sector plate of sector angle $\alpha$ cantilevered along the radial edge subject to a point load at the midpoint of the free edge. It can be seen that the distance from a point on the free edge to the clamped edge continually increases as we move along a radial line away from the origin which will be used to represent the tooth height increase along the face width. The free edge of the plate is taken to represent the topland and the clamped edge, the base of the tooth. The respective thickness variation along the angular and radial directions are taken to closely represent the actual tooth thickness variations of the bevel tooth, by assuming an appropriate functional expression form for $h(r, \theta)$. Fig. 3.3 shows the sector plate model used to describe the bevel gear geometry. Due to the stubby nature of the gear tooth, it is necessary to include shear deformations in the computations. Two levels of shear based theories are employed. The first theory is the Mindlin theory which
3.1 (a) Bevel gear dimensions (\(r_o\) - outer cone distance, \(b\) - facewidth, \(t\) - circular pitch) [Lindner]

3.1 (b) Tapered form of bevel tooth [ANSI/AGMA 2005-B88]

Fig.3.1 Bevel gear geometry
Fig. 3.2 Sector plate under a point load
Fig. 3.3 Sector plate model of the bevel tooth
assumes a constant shear strain through the thickness. The other is a higher order theory developed by Bhimaraddi which assumes a parabolic shear strain variation through the thickness. The computations take into account the height and thickness taper, the finite face width, base rotations and the shear deformation effects.

The number of terms to be used in the series approximation is obtained through a convergence study, by matching the free edge deflection curve with the finite element solution. To demonstrate the validity and capability of the sector plate model, two problems are considered: (a) free edge deflection and stresses in a spur gear whose solution is known and (b) deflection and stresses in a bevel gear calculated using finite element methods. The sector plate model can be used to model the flexural behavior of a spur gear by specifying a constant base thickness and negligible height taper along the face width. The rigidity variation of the bevel gear is modeled through the thickness variation of the plate. The thickness is assumed to vary linearly in the radial and angular directions according to the expression

\[ h = t_{oe} + \frac{(t_{heel} - t_{oe})*(r - (a_0 - f))}{f} + \frac{(t_{tip} - t_{oe})\theta}{\alpha} \]

\[ -\left(\frac{(t_{heel} - t_{oe})*(r - (a_0 - f))}{f}\right)^{\theta} \]

While a linear variation of thickness is an approximation to the actual thickness variation along the profile of the straight bevel tooth, it offers immense advantages in terms of computational efficiency, especially in the higher order theory, relative to the errors introduced due to the approximation. The model can however incorporate any user specified thickness variation in the compliance calculations.
3.2 Model definition and assumptions

Salient features

- The model used to define the bevel gear geometry is that of an annular sector plate which accounts for the tooth height taper from the small end to the large end. The tooth heights at the toe and heel are evaluated from the input blank dimensions.

- The rigidity (thickness) variation in both the angular and radial directions is taken into account by using the actual thickness at the toe and heel base and at the tip. A linear variation in thickness from the base to the tip is assumed along with a linear variation in thickness along the face width.

- Shear deformation is modeled using Mindlin plate theory and Bhimaraddi's higher order theory. The effect of base rotations is incorporated by replacing the clamped edge with an elastic support boundary condition.

- The lines of contact are modeled as radial lines from the pitch apex.

3.3 Convergence study

This study is performed to determine the appropriate number of terms to be chosen, in the radial and angular directions of the series approximation. While it is known that an increase in the number of terms improves the accuracy in the Ritz procedure and makes the assumed solutions tend toward the exact solution, the solution times increase considerably. Hence, the final choice of the number of terms balances the benefits of any further increase in accuracy against the corresponding increase in solution times. The effect of the number of terms in the approximating functions in each coordinate direction is determined by comparing the free edge deflection of the sector plate loaded at the midpoint of the free edge for different combinations of the
number of terms and comparing them with finite element predictions. Two ANSYS models are developed which use 8 noded brick elements. The finite element representations of the sector plate model of the bevel tooth is shown in Fig. 3.13. The tooth model with the support is shown in Fig. 3.14. From Fig. 3.13 it can be seen that the finite element tooth model does not include the effects of base rotation. Base rotations constitute an important part of the overall deflection of the gear tooth. The finite element tooth model with the support allows for the base rotation, as would occur in an actual bevel gear loading condition. In the subsequent plots, the results using the two models are referred to as FEM-tooth and FEM-gear respectively where the FEM-gear includes base deflections as well as tooth deflections. The dimensions of the bevel gear used in this study are given in Table 3.1.

Table 3.1. Bevel gear dimensions

<table>
<thead>
<tr>
<th>Gear Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer cone distance</td>
<td>1.722 in</td>
</tr>
<tr>
<td>Face width</td>
<td>0.625 in</td>
</tr>
<tr>
<td>Tip thickness</td>
<td>0.125 in</td>
</tr>
<tr>
<td>Toe base thickness</td>
<td>0.187 in</td>
</tr>
<tr>
<td>Heel base thickness</td>
<td>0.281 in</td>
</tr>
<tr>
<td>Face angle</td>
<td>43 degrees</td>
</tr>
<tr>
<td>Root angle</td>
<td>25.5 degrees</td>
</tr>
</tbody>
</table>
3.3.1 Mindlin plate theory

A load of 100 lbs is applied at the midpoint of the free edge. From Fig. 3.4 it is seen that 9 terms in the radial direction are adequate to model the free edge deflection. This information is then used to determine the number of terms in the angular direction. Keeping the number of terms in the radial direction equal to 9, the number of terms in the angular direction is varied to investigate convergence. From Fig. 3.5 it can be seen that 5 terms in the angular direction is adequate to obtain the deflection curve. A further increase in the number of terms in the angular direction does not influence the deflection curve as much as the number of terms in the radial direction. Hence, the number of terms in the angular direction was chosen to be 5. All subsequent computations are based on the choice of 5 terms in the angular direction and 9 terms in the radial direction.

3.3.2 Bhimaraddi plate theory

A similar loading condition is assumed for this higher order theory. From Fig. 3.6 and Fig. 3.7 it can be seen that the Bhimaraddi theory predicts larger deflections compared to the finite element method. Also the run times are considerably longer. A choice of 9 terms in the radial direction is seen to be adequate to model the free edge deflection from Fig. 3.6. Keeping the number of terms in the radial direction equal to 9, the number of terms in the angular direction is varied to investigate convergence. From Fig. 3.7 it can be seen that 5 terms in the angular direction is adequate to obtain the deflection curve. As in the Mindlin plate theory further increase in the number of terms in the angular direction did not have a significant influence on the deflection curve.
Fig. 3.4 Mindlin plate convergence study - radial
Fig. 3.5 Mindlin plate convergence study - angular
Fig. 3.6 Bhimaraddi plate convergence study - radial
Fig. 3.7 Bhimaraddi plate convergence study - angular
Hence, the number of terms in the angular direction is chosen to be 5. The flexural behavior was modeled using 5 terms in the angular direction and 9 terms in the radial direction.

The typical computation times encountered on a 486/33 personal computer for a 20 point Gauss integration, based on the number of terms selected, is presented in Table 3.2.

<table>
<thead>
<tr>
<th>No. of terms</th>
<th>Mindlin plate (seconds)</th>
<th>Bhimaraddi plate (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ang9*rad9</td>
<td>24</td>
<td>180</td>
</tr>
<tr>
<td>ang5*rad5</td>
<td>58</td>
<td>510</td>
</tr>
<tr>
<td>ang5*rad9</td>
<td>190</td>
<td>1680</td>
</tr>
<tr>
<td>ang5*rad15</td>
<td>542</td>
<td>5364</td>
</tr>
<tr>
<td>ang3*rad9</td>
<td>67</td>
<td>625</td>
</tr>
<tr>
<td>ang5*rad9</td>
<td>190</td>
<td>1680</td>
</tr>
<tr>
<td>ang9*rad9</td>
<td>634</td>
<td>6264</td>
</tr>
<tr>
<td>ang11*rad9</td>
<td>960</td>
<td>8880</td>
</tr>
</tbody>
</table>

3.4 Spur gear applications

The solutions have been tested by using the sector plate model to calculate the free edge deflections and stresses of a spur gear whose solution is available in the literature.

3.4.1 Comparison with MacGregor's solution

The sector plate is used to compute the free edge deflection and the stresses at the midpoint of the top surface of a thin rectangular plate of constant rigidity (model used to describe the helical gear geometry) and compared with the solution given by
MacGregor (1935). The plate is 8.5" long, 0.125" thick and 1.25" wide. A load of 27.4 lbs is applied 0.046" from the free edge.

The deflection at the free edge was calculated by MacGregor using the analytical expression

\[ w = K \left( \frac{Py^2}{\pi N} \right) \]  

(3.2)

where \( P \) is the applied load, \( a \) is the plate width, \( N \) is the plate rigidity, \( y \) is the distance along the face width from the midpoint of the free edge of the plate and \( K \) obtained from Table 3.3.

<table>
<thead>
<tr>
<th>Distance from load application</th>
<th>Factor K</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.527</td>
</tr>
<tr>
<td>( \frac{a}{4} )</td>
<td>0.470</td>
</tr>
<tr>
<td>( \frac{a}{2} )</td>
<td>0.380</td>
</tr>
<tr>
<td>( a )</td>
<td>0.213</td>
</tr>
<tr>
<td>( 2a )</td>
<td>0.0496</td>
</tr>
<tr>
<td>( 2.5a )</td>
<td>0.0385</td>
</tr>
</tbody>
</table>

The stresses were computed at the midpoint of the clamped end using the expressions

\[ \sigma_x = \frac{3.048P}{h^2} \]  

(3.3)

\[ \sigma_y = \frac{0.9162P}{h^2} \]  

(3.4)

\[ \tau_{xy} = 0 \]  

(3.5)
where \( h \) is the thickness of the plate. From Fig. 3.8 we can see that the deflections match well with MacGregor's results except under the point of load application, where MacGregor's deflection was larger than the plate model solution. The stresses calculated are also close to MacGregor's solution as shown in Fig. 3.9.

### 3.4.2 Comparison with Yau's solution

The sector plate model is next used to compute the free edge deflection of a spur gear tooth and compare its results with the Ritz solution calculated by Yau (1987). The model used by Yau was a shear flexible, tapered, rectangular plate model. The tooth has a base thickness of 0.38", a tip thickness of 0.12" and a face width of 0.4". A load of 200 lbs was applied at the midpoint of the free edge of the plate. Fig. 3.10 and Fig. 3.11 show the deflection at the free edge and at the middle respectively, as computed by Yau. Fig. 3.12 is the sector plate solution. A good correlation is seen with Yau's shear based deflection results.

We can thus see that the sector plate can be used to model the spur gear tooth as a special case of a bevel gear tooth.

### 3.5 Bevel gear applications

A typical differential bevel gear is analyzed using the sector plate model and the results compared with the previously described ANSYS models for deflections and stresses. The dimensions of the gear are the same as in Table I. A load of 100 lbs is applied at three different points on the free edge and for each loading condition, the sector plate results compared with the finite element predictions. The tooth thickness variation is assumed as in eqn. (3.1). The stresses are calculated at 12.5% of the tooth height from the clamped end to match with the corresponding nodal position of the finite element analysis.
Fig. 3.8 Deflection comparison with MacGregor's solution
Fig. 3.9 Stress comparison with MacGregor's solution
Fig. 3.10  Deflections across the tip face width of the narrow plate due to a tip middle plane load (Yau, 1987)
Fig. 3.11 Deflections across the middle facewidth of the narrow plate due to a tip middle plane load [Yau, 1987]
Fig. 3.12 Comparison of the sector plate solution with the tapered plate results [Yau, 1987]
The deflection and stress results, when the point load is successively applied at the toe tip, heel tip and the midpoint of the free edge are presented in Fig. 3.15-3.17. When the load is applied at the toe tip, the deflection at the toe increases with tooth height, from zero at the clamped end to a maximum under the point of load application as seen in Fig. 3.15. A similar trend for the deflection variation with tooth height can be seen with the deflection being zero at the clamped end and a maximum at the point of load application, when the load is successively applied at the midpoint of the free edge and at the heel tip respectively. The free edge deflection for the various load conditions is shown in Fig. 3.16. When the load is applied at the toe tip, the deflection is maximum at the toe tip and decreases along the free edge face width. The free edge deflections are a maximum at the point of load application. From Figs. 3.15 and 3.16 it is seen that the maximum deflection occurs when the load is applied at the heel tip. A comparison of the principal stresses for the different load conditions is shown in Fig. 3.17. The principal stress reaches a maximum at the clamped end close to the point of load application. From the principal stress plot of Fig. 3.17, it can be seen that the maximum principal stress occurs when the load is applied at the toe tip. The excellent agreement with the finite element predictions demonstrates the application of the CASP model in determining the flexural behavior of the straight bevel tooth.

3.5.1 Effect of elastic support

The effect of the elastic support condition on the deflections and stresses is shown in comparison with the finite element predictions in Figs. 3.18-3.22. The ability of the sector plate model to incorporate base rotations (as would be required in a realistic simulation of actual gear tooth flexural behavior) is demonstrated by replacing the clamped boundary condition with an elastic foundation. A load of 100 lbs is applied at
Fig. 3.13 Finite element representation of the sector plate model of the bevel tooth
Fig. 3.14 Finite element representation of the sector plate model of the bevel tooth with support
Fig. 3.15 Deflection along the straight bevel gear tooth height due to the application of a point load (100 lbs) at different positions on the free edge
Fig. 3.16 Deflection along the free edge of the straight bevel gear due to the application of a point load (100 lbs) at different positions on the free edge.
Fig. 3.17 Maximum principal stress along the straight bevel gear clamped edge due to the application of point load (100 lbs) at different positions on the free edge.
Fig. 3.18 Comparison of the free edge deflection of the elastically supported plate with the clamped plate and FEM models due to a point load at the midpoint of the free edge.
Fig. 3.19 Comparison of $\sigma$, along the clamped edge of the elastically supported plate with the clamped plate and FEM models due to a point load at the midpoint of the free edge.
Fig. 3.20 Comparison of $\sigma_0$ along the clamped edge of the elastically supported plate with the clamped plate and FEM models due to a point load at the midpoint of the free edge.
Fig. 3.21 Comparison of $\sigma_{re}$ along the clamped edge of the elastically supported plate with the clamped plate and FEM models due to a point load at the midpoint of the free edge.
Fig. 3.22 Comparison of $\sigma_{xx}$ along the clamped edge of the elastically supported plate with the clamped plate and FEM models due to a point load at the midpoint of the free edge.
the midpoint of the free edge. From Fig. 3.18 it is seen that the effect of the elastic support, is to increase the free edge deflections compared to the clamped conditions, which is expected. No significant changes in the normal stresses \( \sigma_r \) and \( \sigma_\theta \) are seen due to the elastic support condition from Fig. 3.19 and Fig. 3.20. The tensile stresses \( \sigma_\theta \) are somewhat more evened out along the face width due to the elastic support. Fig. 3.21 shows the inplane shear stress \( \sigma_{r\theta} \) of both boundary conditions which are seen to almost identical. The transverse shear stress \( \sigma_\theta \) are seen to increase markedly due to the elastic support condition in Fig. 3.22. This can be attributed to the increased shear strain, due to reduced constraint at the previously clamped radial edge. A similar, though less marked trend is seen in the finite element results.

### 3.5.2 Effect of higher order theory

In this section, the effect of a higher order theory on the flexural behavior of the sector plate is investigated. A load of 100 lbs is applied at the midpoint of the free edge. Fig. 3.23-327 show the deflection and stress results using Bhimaraddi's theory. From Fig. 3.23 it is seen that the higher order theory predicts higher deflections compared to the clamped condition and matches better with the elastic support condition. This is expected as the theory includes terms up to the third order in the thickness coordinate \( z \) in the displacement assumption. The radial and tangential normal stresses \( \sigma_r \) and \( \sigma_\theta \) are correspondingly higher than the FEM results in Fig. 3.24 and Fig. 3.25 and hence would provide a conservative estimate for design calculations. A comparison of the inplane shear stress \( (\sigma_{r\theta}) \) variation with the finite element predictions is shown in Fig. 3.26. The inplane shear stress variation along the face width has similar trends as the finite element predictions, with the stresses antisymmetric about the center and dropping to zero at the ends. The peak stresses are
Fig. 3.23 Comparison of free edge deflection (Bhimaraddi) with FEM due to a point load applied at the midpoint of the free edge.
Fig. 3.24 Comparison of $\sigma_r$ (Bhimaraddi) along the clamped edge with FEM due to a point load applied at the midpoint of the free edge.
Fig. 3.25 Comparison of $\sigma_\theta$ (Bhimaraddi) along the clamped edge with FEM due to a point load applied at the midpoint of the free edge.
Fig. 3.26 Comparison of $\sigma_{r\theta}$ (Bhimaraddi) along the clamped edge with FEM due to a point load applied at the midpoint of the free edge.
Fig. 3.27 Comparison of $\sigma_{th}$ (Bhimaraddi) along the clamped edge with FEM due to a point load applied at the midpoint of the free edge.
however higher than the finite element predictions. The through the thickness shear ($\sigma_{th}$) comparison shows the higher order theory results matching well with the FEM results in Fig. 3.27, except at the ends. It is noted that the $\sigma_{th}$ stress values presented are the values at the midplane, as the transverse shear strain goes to zero at the top and bottom surfaces.

3.6 Summary

This chapter demonstrated the ability of the cantilevered annular sector plate to model the flexural behavior of the straight bevel gear tooth. The Rayleigh-Ritz method was used to determine the deflections and stresses. The effect of the higher order plate theory and an elastic support condition were investigated. The results are verified with previously published spur gear results and finite element predictions. The higher order plate theory requires much larger computation times compared with Mindlin plate theory for the same number of terms in Ritz expansion.
CHAPTER IV

FUNDAMENTAL EQUATIONS OF A CIRCULAR CYLINDRICAL SHELL

4.1 Introduction

A thin shell is a three dimensional body bounded by two closely spaced curved surfaces, the distance between the surfaces being small in comparison with the other dimensions. The midsurface of the shell is the locus of points which lie midway between the two surfaces. The distance between the two surfaces measured normal to the midsurface of the shell is the thickness of the shell at that point. This chapter presents the derivations of the fundamental equations of shell theory for a cylindrical shell segment, along with the shear based displacement assumptions and the boundary conditions. A detailed derivation of the basic equations can be found in Leissa (1973), Kraus(1967) and Novozhilov (1964). The derivations using tensor notation can be found in Flugge (1962).

Unlike thin plate theory where a single fourth order differential equations is universally agreed upon, the motion of a given shell is described by differing sets of equations. In shell theory there are six strain displacement relations, eight force and moment resultant equations and five equations of motion leading to a total of nineteen equations in nineteen unknowns. While the usual procedure is used to reduce the number of equations and unknowns to a more manageable number, because of the
relatively large number of equations and unknowns, the derivation of final forms of the equations of motion is beyond the scope of this dissertation. The classical thin shell theory leads to a set of eighth order differential equations and to a tenth order set of differential equations when the effects of shear deformation are included. Hence, in this chapter, once the strain displacement relations are obtained starting from the theory of surfaces, the strain energy equation is written from the theory of elasticity, without writing out the equations of motion. The material is assumed to be linearly elastic, isotropic and homogenous and the thickness a function of the shell curved length and height.

4.2 Theory of surfaces

4.2.1 Coordinate system

The deformation of the shell is completely determined by the displacements of the middle surface. The equation of the undeformed middle surface can be expressed in terms of two independent parameters \( \alpha \) and \( \beta \) as

\[
\mathbf{r} = \mathbf{r}(\alpha, \beta) \tag{4.1}
\]

The set of curves on the surface obtained by keeping \( \alpha \) constant and \( \beta \) constant are called \( \beta \) curves and \( \alpha \) curves respectively. The respective tangents to the \( \alpha \) and \( \beta \) curves are denoted as

\[
\mathbf{r}_\alpha = \frac{\partial \mathbf{r}}{\partial \alpha} \tag{4.2}
\]

\[
\mathbf{r}_\beta = \frac{\partial \mathbf{r}}{\partial \beta} \tag{4.3}
\]
If the angle between the coordinate curves is given as $\chi$, and $A = |r_\alpha|$, $B = |r_\beta|$ we can write

$$\frac{r_\alpha \cdot r_\beta}{A B} = \cos \chi \quad (4.4)$$

and

$$\hat{i}_n = \frac{i_\alpha \times i_\beta}{\sin \chi} \quad (4.5)$$

where $\cdot$ and $\times$ are the vector dot product and vector cross product, respectively.

### 4.2.2 First fundamental form of the surface

Given two points $(\alpha, \beta)$ and $(\alpha + d\alpha, \beta + d\beta)$ close to each other, the increment of the vector $\mathbf{r}$ moving from the first to the second point is given as

$$d\mathbf{r} = r_\alpha d\alpha + r_\beta d\beta \quad (4.6)$$

The square of the differential of the arc length on the surface is the first quadratic form of the surface given by

$$d\mathbf{r} \cdot d\mathbf{r} = A^2 d\alpha^2 + 2AB \cos \chi d\alpha d\beta + B^2 d\beta^2 \quad (4.7)$$

This form determines the intrinsic geometry of the surface and the coefficients of the coordinate differentials are called the first fundamental quantities.

### 4.2.3 Second fundamental form of the surface

This concept is used to determine the curvature of any curve lying on the surface. Let $\mathbf{r} = \mathbf{r}(s)$ be the vector equation of a curve on the surface and $\mathbf{\hat{t}}$ the unit tangent to the curve. Then
\[ \hat{\tau} = \frac{dr}{ds} = r_\alpha \frac{d\alpha}{ds} + r_\beta \frac{d\beta}{ds} \] 

(4.8)

The Frenet's formulae in the theory of curves relates the spatial derivatives of the basic vectors of the curve to the basic vectors. From Frenet's formula,

\[ \frac{d\hat{\tau}}{ds} = \frac{N}{\rho} \] 

(4.9)

and substituting for \( \hat{\tau} \) from eqn.(4.8) leads to

\[ \frac{N}{\rho} = r_{,\alpha} \left( \frac{d\alpha}{ds} \right)^2 + 2r_{,\alpha\beta} \left( \frac{d\alpha}{ds} \right) \left( \frac{d\beta}{ds} \right) + r_{,\beta\beta} \left( \frac{d\beta}{ds} \right)^2 + r_\alpha \frac{d^2\alpha}{ds^2} + r_\beta \frac{d^2\beta}{ds^2} \] 

(4.10)

where \( N \) is the principal normal to the curve and \( \rho \) is the radius of curvature of the curve. Let \( \phi \) be the angle between the normal to the surface \( \hat{i}_n \) and the principal normal to the curve, \( N \). Taking the dot product of both sides of eqn.(4.10) with \( \hat{i}_n \) yields

\[ \frac{\cos \phi}{\rho} = \frac{L d\alpha^2 + 2M d\alpha d\beta + N d\beta^2}{ds^2} \] 

(4.11)

where

\[ L = r_{,\alpha} \cdot \hat{i}_n \] 

(4.12a)

\[ M = r_{,\alpha\beta} \cdot \hat{i}_n \] 

(4.12b)

\[ N = r_{,\beta\beta} \cdot \hat{i}_n \] 

(4.12c)

The expression, \( L d\alpha^2 + 2M d\alpha d\beta + N d\beta^2 \), is called the second quadratic form of the surface and \( L, M \) and \( N \) coefficients which can be expressed as functions of \( \alpha \) and \( \beta \). By setting \( \phi = \pi \) in Eqn.(4.11) the normal curvatures to the surface can be obtained. The normal curvature is thus given by
\[
\frac{1}{R} = -\frac{L\alpha^2 + 2Md\alpha\beta + N\beta^2}{A^2\alpha^2 + 2ABd\alpha\beta + B^2d\beta^2}
\] (4.13)

Setting \(\alpha = \text{constant and } \beta = \text{constant in Eqn.}(4.12)\) gives the curvatures of the \(\beta\) and \(\alpha\) curves respectively as

\[
\frac{1}{R_\alpha} = -\frac{L}{A^2}
\] (4.14a)

\[
\frac{1}{R_\beta} = -\frac{N}{B^2}
\] (4.14b)

### 4.2.4 Formulae of Gauss and Weingarten

These formulae express the derivatives of the unit vectors of the surface in terms of the basic vectors similar to the formulae of Frenet in the theory of curves. Assuming that the curves \(\alpha = \text{constant and } \beta = \text{constant}\) are lines of principal curvature of the undeformed surface we get

\[
\cos \chi = 0
\] (4.15)

\[
M = 0
\] (4.16)

The second derivatives of \(r\) with respect to the parameters can be expressed in terms of \(r_\alpha, r_\beta\) and \(\hat{i}_n\) as follows.

\[
r_{\alpha\alpha} = \Gamma^1_{11} r_\alpha + \Gamma^1_{12} r_\beta + L\hat{i}_n
\] (4.17)

\[
r_{\alpha\beta} = \Gamma^1_{12} r_\alpha + \Gamma^2_{12} r_\beta + M\hat{i}_n
\] (4.18)

\[
r_{\beta\beta} = \Gamma^1_{22} r_\alpha + \Gamma^2_{22} r_\beta + N\hat{i}_n
\] (4.19)
where $\Gamma^i_k$ are the Christoffel symbols which can be expressed in terms of the coefficients of the first principal quadratic form as

\[
\begin{align*}
\Gamma^1_{\alpha} &= \frac{1}{A} \frac{\partial A}{\partial \alpha} \\
\Gamma^1_{\beta} &= -\frac{A}{B^2} \frac{\partial A}{\partial \beta} \\
\Gamma^2_{\alpha} &= \frac{1}{A} \frac{\partial A}{\partial \beta} \\
\Gamma^2_{\beta} &= \frac{1}{B} \frac{\partial B}{\partial \alpha} \\
\Gamma^1_{\alpha\alpha} &= -\frac{1}{B} \frac{\partial A}{\partial \beta} \frac{\partial B}{\partial \alpha} \\
\Gamma^2_{\alpha\beta} &= \frac{1}{B} \frac{\partial B}{\partial \alpha} \\
\Gamma^2_{\beta\alpha} &= \frac{1}{B} \frac{\partial B}{\partial \alpha} \\
\Gamma^2_{\beta\beta} &= \frac{1}{B^2} \frac{\partial B}{\partial \beta}
\end{align*}
\]  

(4.20a) 

(4.20b) 

(4.20c) 

(4.20d) 

(4.20e) 

(4.20f)

Using the above equations and $\hat{i}_n \cdot \hat{i}_n = 1$ the expressions for the derivatives of the basic vectors can be written as

\[
\begin{align*}
\hat{i}_{n,\alpha} &= \frac{A}{R_\alpha} \hat{i}_\alpha \\
\hat{i}_{n,\beta} &= \frac{B}{R_\beta} \hat{i}_\beta \\
\hat{i}_{\alpha,\alpha} &= -\frac{1}{B} \frac{\partial A}{\partial \beta} \hat{i}_\beta \frac{A}{R_\alpha} \hat{i}_n \\
\hat{i}_{\alpha,\beta} &= -\frac{1}{B} \frac{\partial B}{\partial \alpha} \hat{i}_\alpha \frac{B}{R_\beta} \hat{i}_n \\
\hat{i}_{\beta,\alpha} &= \frac{1}{B} \frac{\partial B}{\partial \alpha} \hat{i}_\alpha \frac{B}{R_\beta} \hat{i}_n \\
\hat{i}_{\beta,\beta} &= \frac{1}{A} \frac{\partial B}{\partial \alpha} \hat{i}_\beta
\end{align*}
\]  

(4.21a) 

(4.21b) 

(4.21c) 

(4.21d) 

(4.21e)
\[ \hat{\mathbf{b}}_{\alpha} = \frac{1}{A} \frac{\partial A}{\partial \beta} \hat{\mathbf{a}}_{\alpha} \]  

(4.21f)

4.3 Derivation of the basic equations of the circular cylindrical shell

4.3.1 Shell coordinates and the metric

In this section, the metric used in the strain displacement relations of the circular cylindrical shell is derived. The position vector of a point on the shell is defined as

\[ \mathbf{R}(\alpha, \beta, z) = r(\alpha, \beta) + z\hat{\mathbf{n}} \]  

(4.22)

The magnitude of an infinitesimal change of the vector \( \mathbf{R} \) is given by

\[ (ds)^2 = d\mathbf{R} \cdot d\mathbf{R} = (dr + z d\hat{\mathbf{n}} + \hat{\mathbf{n}} dz) \cdot (dr + z d\hat{\mathbf{n}} + \hat{\mathbf{n}} dz) \]  

(4.23)

where

\[ d\hat{\mathbf{n}} = \frac{\partial \hat{\mathbf{n}}}{\partial \alpha} d\alpha + \frac{\partial \hat{\mathbf{n}}}{\partial \beta} d\beta \]  

(4.24)

From

\[ (ds)^2 = g_1 d\alpha^2 + g_2 d\beta^2 + g_3 dz^2 \]  

(4.25)

and substituting for \( \frac{\partial \hat{\mathbf{n}}}{\partial \alpha} \) and \( \frac{\partial \hat{\mathbf{n}}}{\partial \beta} \) in eqn.(4.24) from eqn.(4.21) we get

\[ g_1 = \left[ A(1 + z/R_\alpha) \right]^2 \]  

(4.26)

\[ g_2 = \left[ A(1 + z/R_\beta) \right]^2 \]  

(4.27)

\[ g_3 = 1 \]  

(4.28)
The components of the metric tensor $g_{ij}$ thus defined are used in the strain displacement relations.

4.3.2 Strain displacement equation

The strain displacement equations in orthogonal curvilinear coordinates are given by

$$e_i = \frac{\partial}{\partial \alpha_i} \left( \frac{U_i}{\sqrt{g_i}} \right) + \frac{1}{2g_i} \sum_{k=1}^{3} \frac{\partial g_{ik}}{\partial \alpha_k} \frac{U_k}{\sqrt{g_k}}$$

$$i = 1, 2, 3$$

$$\gamma_{ij} = \frac{1}{\sqrt{g_i g_j}} \left[ g_i \frac{\partial}{\partial \alpha_j} \left( \frac{U_i}{\sqrt{g_i}} \right) + g_j \frac{\partial}{\partial \alpha_j} \left( \frac{U_i}{\sqrt{g_i}} \right) \right]$$

$$i, j = 1, 2, 3; i \neq j$$

where $e_i, \gamma_{ij}$ and $U_i$ are the normal strains, shear strains and the displacement components, respectively, and $g_i$ are related to the metrics.

In the shell coordinates, using eqns. 4.26-4.28 we get

$$e_\alpha = \frac{1}{(1 + z/R_\alpha)} \left( \frac{1}{A} \frac{\partial U}{\partial \alpha} + \frac{V}{A B} \frac{\partial A}{\partial \beta} \frac{W}{R_\alpha} \right)$$

$$e_\beta = \frac{1}{(1 + z/R_\beta)} \left( \frac{1}{B} \frac{\partial V}{\partial \beta} + \frac{U}{A B} \frac{\partial B}{\partial \alpha} \frac{W}{R_\beta} \right)$$

$$e_z = \frac{\partial W}{\partial z}$$

$$\gamma_{\alpha \beta} = \frac{A(1 + z/R_\alpha)}{B(1 + z/R_\beta)} \frac{\partial}{\partial \beta} \left( \frac{U}{A(1 + z/R_\alpha)} \right) + \frac{B(1 + z/R_\beta)}{A(1 + z/R_\alpha)} \frac{\partial}{\partial \alpha} \left( \frac{V}{B(1 + z/R_\beta)} \right)$$

$$\gamma_{\alpha z} = \frac{1}{A(1 + z/R_\alpha)} \frac{\partial W}{\partial \alpha} + A(1 + z/R_\alpha) \frac{\partial}{\partial z} \left[ \frac{U}{A(1 + z/R_\alpha)} \right]$$
\[ \gamma_{\beta z} = \frac{1}{B(z/R_\beta)} \frac{\partial W}{\partial \beta} + B(z/R_\beta) \frac{\partial}{\partial z} \left[ \frac{V}{B(z/R_\beta)} \right] \]  

(4.36)

The displacement assumptions that satisfy either the Kirchhoff-Love hypothesis or the shear theories are substituted into the strain displacement equations. The strains are then used in the energy expression for the stiffness computations.

In the case of the circular cylindrical shell we have

\[ R_\alpha = \infty \quad R_\beta = r_c \quad \alpha = x \quad \beta = \theta \quad A = 1 \quad B = r_c \]  

(4.37a-f)

where \( r_c \) is the radius of the cutter. Substituting the above in eqns. (4.31-4.36) we obtain the strain displacement relations for the circular cylindrical shell as

\[ \varepsilon_x = \frac{\partial U}{\partial x} \]  

(4.38)

\[ \varepsilon_\theta = \frac{1}{(r_c + z)} \left( \frac{\partial V}{\partial \theta} + W \right) \]  

(4.39)

\[ \varepsilon_z = \frac{\partial W}{\partial z} \]  

(4.40)

\[ \gamma_{\theta z} = \frac{\partial V}{\partial x} + \frac{1}{(r_c + z)} \frac{\partial U}{\partial \theta} \]  

(4.41)

\[ \gamma_{z\theta} = \frac{1}{(r_c + z)} \frac{\partial W}{\partial \theta} + \frac{\partial V}{\partial z} - \frac{V}{(r_c + z)} \]  

(4.42)

\[ \gamma_{zz} = \frac{\partial W}{\partial x} + \frac{\partial U}{\partial z} \]  

(4.43)
4.3.3 Displacement assumptions

Two levels of shear based theories have been used in this study. The first set of displacement assumptions used to model the shell flexural behavior is based on the higher order shear theory developed by Bhimaraddi (1989).

\[ U = z \left( 1 - \frac{4z^2}{3h^2} \right) \beta_x - z \frac{\partial W}{\partial x} \quad (4.44) \]

\[ V = \left( \frac{r_c + z}{r_c} \right) \left( 1 - \frac{4z^2}{3h^2} \right) \beta_\theta - \frac{z}{r_c} \frac{\partial W}{\partial \theta} \quad (4.45) \]

\[ W = W(x, \theta) \quad (4.46) \]

The assumptions allow for a parabolic variation of the transverse shear strain through the thickness with zero values at the top and bottom surfaces. From the displacement assumptions it can be seen that the strain expressions will include the derivatives of the thickness with respect to the spatial coordinates which must be accounted for in the strain energy expression. The second set of displacement assumptions is based on the Mindlin type of shear theory defined as

\[ U = z \beta_x - z \frac{\partial W}{\partial x} \quad (4.47) \]

\[ V = \left( \frac{r_c + z}{r_c} \right) z \beta_\theta - \frac{z}{r_c} \frac{\partial W}{\partial \theta} \quad (4.48) \]

\[ W = W(x, \theta) \quad (4.49) \]

The Mindlin type of shear theory assumes a constant shear strain through the thickness. It is however simpler to implement as it does not have the thickness h or its derivatives appearing in the strain energy expression.
Substituting the displacement assumptions in eqn.(4.38-4.43) we get the following relations for the Mindlin shell.

\[ \varepsilon_x = z \left( \frac{\partial \beta_x}{\partial x} - \frac{\partial^2 W}{\partial x^2} \right) \]  
\[ \varepsilon_\theta = \frac{z}{r_c} \left[ \frac{\partial \beta_\theta}{\partial \theta} - \frac{1}{(r_c + z)} \frac{\partial^2 W}{\partial \theta^2} \right] + \frac{W}{(r_c + z)} \]  
\[ \gamma_{x\theta} = (r_c + z) \frac{1}{r_c} \frac{\partial \beta_\theta}{\partial x} - \frac{z(2r_c + z)}{r_c(r_c + z)} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{z}{(r_c + z)} \frac{\partial \beta_x}{\partial \theta} \]  
\[ \gamma_{x\theta} = (1 + \frac{z}{r_c}) \beta_\theta \]  
\[ \gamma_{xx} = \beta_x \]  

The corresponding strain displacement relations for the Bhimaraddi shell are

\[ \varepsilon_x = z(1 - \frac{4z^2}{3h^2}) \frac{\partial \beta_x}{\partial x} - \frac{8z^3}{3h^2} \frac{\beta_x}{dx} + \frac{8z^3}{3h^2} \frac{dh}{dx} \]  
\[ \varepsilon_\theta = \frac{1}{r_c} z(1 - \frac{4z^2}{3h^2}) \frac{\partial \beta_\theta}{\partial \theta} - \frac{z}{(r_c + z)r_c} \frac{\partial^2 W}{\partial \theta^2} + \frac{W}{(r_c + z)} + \frac{8z^3}{3h^2} \frac{\beta_\theta}{dh} \]  
\[ \gamma_{x\theta} = \frac{(r_c + z)}{r_c} z(1 - \frac{4z^2}{3h^2}) \frac{\partial \beta_\theta}{\partial x} + (r_c + z) \frac{8z^3}{3h^2} \frac{\beta_\theta}{dx} - \frac{z}{r_c} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{z}{(r_c + z)} (1 - \frac{4z^2}{3h^2}) \frac{\partial \beta_x}{\partial \theta} + \]  
\[ \frac{1}{(r_c + z)} \frac{\beta_x}{3h^2} \frac{dh}{d\theta} - \frac{z}{(r_c + z)} \frac{\partial^2 W}{\partial x \partial \theta} \]  
\[ \gamma_{x\theta} = (1 + \frac{z}{r_c})(1 - \frac{4z^2}{h^2}) \beta_\theta \]  
\[ \gamma_{xx} = (1 - \frac{4z^2}{h^2}) \beta_x \]
The stress strain relations can be written as

\[
\sigma_x = \frac{E}{(1-v^2)}(\varepsilon_x + v\varepsilon_\theta) \quad \sigma_\theta = \frac{E}{(1-v^2)}(\varepsilon_\theta + v\varepsilon_x) \quad \tau_{xz} = G\gamma_{xz} \quad \tau_{\theta z} = G\gamma_{\theta z}
\]

(4.52a-e)

### 4.4 Solution method

#### 4.4.1 Strain energy

The Rayleigh-Ritz method is used to obtain an expression for the transverse deflection and shear rotations from which the normal and shear stresses are evaluated. This method involves setting the first variation of the total potential energy to zero. The potential energy is defined as the sum of the external potential energy and the strain energy of the shell.

\[
V = SE + H
\]

(4.53a)

For work done by the external forces in a conservative system

\[
H = -WF
\]

(4.53b)

Hence

\[
V = SE - WF
\]

(4.53c)

The strain energy of the deformed shell surface is given by

\[
SE = \frac{1}{2} \left( \frac{E}{(1-v^2)} \right) \int \int \int \left( \varepsilon_x^2 + \varepsilon_\theta^2 + 2v\varepsilon_x\varepsilon_\theta \right) + \\
\frac{(1-v)}{2} \left( \gamma_{xz}^2 + \gamma_{x\theta}^2 + \gamma_{\theta z}^2 \right) \frac{(r_c + z)}{r_c} r_c dz d\theta dx
\]

(4.54)
The work done by the external force $p$ is

$$WF = \int \int_{x, \theta} p \omega_c \, dx \, d\theta$$  \hspace{1cm} (4.55)$$

Setting the first variation of the energy functional (potential energy) to zero gives

$$\left(\frac{E}{(1 - v^2)}\right) \int \int_{x, \theta} \left(\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{yy} \delta \varepsilon_{yy} + \sigma_{zz} \delta \varepsilon_{zz} + \nu (\varepsilon_{xx} \delta \varepsilon_{yy} + \varepsilon_{yy} \delta \varepsilon_{xx})\right) +$$

$$\frac{(1 - v)}{2} \left(\gamma_{xx} \delta \gamma_{xx} + \gamma_{yy} \delta \gamma_{yy} + \gamma_{zz} \delta \gamma_{zz}\right) \left(\frac{r_c + z}{r_c}\right) r_c \omega_c \, dz \, dx - \int \int_{x, \theta} p \omega_c \, dx \, d\theta = 0$$  \hspace{1cm} (4.56)$$

The transverse deflections and shear rotations are assumed in the form of a finite linear combination of algebraic polynomial trial functions which satisfy the geometric boundary conditions. The assumed trial functions are substituted in the above equation. Each term of the series solution is a product of two expressions, one a function of the axial position, $x$, and the other of the angular position, $\theta$. Specifically,

$$W = \sum_i \sum_j A_{ij} \phi_i(x) \varepsilon_j(\theta)$$  \hspace{1cm} (4.57)$$

$$\beta_x = \sum_i \sum_j B_{ij} \phi_i(x) \varepsilon_j(\theta)$$  \hspace{1cm} (4.58)$$

$$\beta_\theta = \sum_i \sum_j C_{ij} \phi_i(x) \varepsilon_j(\theta)$$  \hspace{1cm} (4.59)$$

To ensure convergence, the functions so chosen must satisfy the following requirements:

1) $\phi_i(x)$ and $\varepsilon_j(\theta)$ must satisfy in their actual form, the essential boundary conditions associated with $W, \beta_x$ and $\beta_\theta$.

2) $\phi_i(x)$ and $\varepsilon_j(\theta)$ must be continuous (as specified by the variational principle) and be linearly independent and complete.
Each of the terms in the strain energy expression are evaluated based on the kind of shear theory selected. The integrations are carried out numerically using Gauss quadrature and allocated in the respective positions in the stiffness matrix. The force vector depends on the type of loading (point load, line load etc.) and the load location. In matrix form

\[
\begin{bmatrix}
[K]^{11} & [K]^{12} & [K]^{13} \\
[K]^{21} & [K]^{22} & [K]^{23} \\
[K]^{31} & [K]^{32} & [K]^{33}
\end{bmatrix}
\begin{bmatrix}
\{A\}_{ij} \\
\{B\}_{ij} \\
\{C\}_{ij}
\end{bmatrix} =
\begin{bmatrix}
\{F\}_{1} \\
\{F\}_{2} \\
\{F\}_{3}
\end{bmatrix}
\] (4.60)

where the elements of the stiffness matrix \([K]\) are identified in terms of the volume integral of the material properties and the assumed trial functions. The problem reduces to solving for the coefficients \(A_{ij}, B_{ij}\) and \(C_{ij}\) numerically from the matrix equation, to determine the series analytic forms of the transverse deflections and shear rotations of the shell surface.

### 4.4.2 Boundary conditions

The boundary condition that represents the flexural behavior of spiral bevel gears is the shell model clamped along the circular edge. Fig. 4.1 shows the shell segment clamped along the circular edge. This involves

\[
W(x = 0, \theta) = 0, \quad \frac{\partial W}{\partial x}(x = 0, \theta) = 0, \quad \beta_x(x = 0, \theta) = 0, \quad \beta_y(x = 0, \theta) = 0
\] (4.61)

The functions \(\phi_i(x)\) and \(\varepsilon_j(\theta)\) representing \(W, \beta_x\) and \(\beta_y\) in eqns.(4.57-4.59) are appropriately chosen to satisfy the desired boundary conditions. For \(\beta_x\) and \(\beta_y\), \(\phi_i(x)\) is chosen to be pinned-free. For \(w, \phi_i(x)\) is chosen to be clamped-free. \(\varepsilon_j(\theta)\) is chosen to be free-free for \(W, \beta_x\) and \(\beta_y\). Polynomial representations are used in preference to the beam functions due to their simplicity and their ability to be exactly integrated.
Fig. 4.1 Cylindrical shell boundary conditions
The pinned-free condition is represented by the functions

\[ \phi_i(x) = \left( \frac{x}{a} \right)^i, \quad i=1,2,3, \ldots \]  

(4.62)

and the clamped free condition by

\[ \phi_i(x) = \left( \frac{x}{a} \right)^{i+1}, \quad i=1,2,3, \ldots \]  

(4.63)

where \( a \) is the tooth height.

The free-free condition for \( \varepsilon_i(\theta) \) is similarly represented by

\[ \varepsilon_i(\theta) = \left( \frac{\theta}{\alpha} \right)^{i-1}, \quad i=1,2,3, \ldots \]  

(4.64)

where \( \alpha \) is the angle subtended by the circular segment of the shell of length equal to the face width of the gear.

\subsection*{4.5 Summary}

In this chapter the strain displacement equations of the circular cylindrical shell are presented for two levels of shear based theories, the first order theory of Mindlin and the third order theory of Bhimaraddi. The thickness is assumed to be a function of the curved length of the shell and the shell height. The boundary conditions for the shell clamped along the circular edge are presented along with the functions chosen to represent them. The chosen boundary conditions represent the boundary conditions on a spiral bevel tooth during mesh. The results of the static analysis are presented in Chapter V.
CHAPTER V

APPLICATION OF THE CYLINDRICAL SHELL MODEL TO

SPIRAL BEVEL AND HYPOID GEARS

5.1 Introduction

In this chapter, the ability of the circular cylindrical shell model to represent the salient features of the spiral bevel tooth geometry to determine tooth compliance and root stresses are demonstrated. Most spiral bevel and hypoid gears are manufactured either by the face milling process or the face hobbing process. The two methods typically use different blank designs, cutting tools and contact pattern control for producing their respective tooth geometries. Each process has its advantages and disadvantages and the choice of the process is not clear cut. The older machines generate the gear by only one or the other method while the newer machines are capable of using either method. A discussion of the two types of processes and their respective tooth geometries is presented in Sec. 5.2.1. The shape and the nomenclature of a face milled spiral bevel tooth are shown in Fig. 5.1. It can be seen that the height of the tooth continually increases from the toe to the heel and the tooth is curved along the lengthwise direction. The tooth base thickness increases from the toe to the heel and the thickness decreases along the tooth height at any point on the face width. Hence, any model to determine the compliance should take into account the rigidity variation
Fig. 5.1 Spiral bevel tooth nomenclature [GWPSD6151B]
along the face width, the lengthwise curvature, as well as the tooth height taper along the face width.

Fig. 5.2 shows the spiral bevel tooth shape modeled as a segment of a thick cylindrical shell, along with the chosen coordinate system. The curved length of the shell segment corresponds to the face width of the gear and the height and thickness of the shell closely approximate the bevel tooth dimensions. The shell segment is clamped along the bottom circular edge. Due to the small radius to thickness and length to thickness ratios, the flexural behavior of the spiral bevel gear is modeled using shear deformation theories. The number of terms to be used in the Ritz series approximation is determined through a convergence study of the free edge deflection curve and the results of the static analysis verified using an ANSYS finite element model. Both the Mindlin and Bhimaraddi shear theories, previously defined in Chapter IV, are used to model the flexural behavior of the shell model and the deflections and stresses compared to the finite element model.

The height $h$ and the thickness $t$ are assumed to vary linearly according to the equations

$$h = h_{toe} + \frac{(h_{heel} - h_{toe})}{f} \cdot \varphi \cdot r_c$$  \hspace{1cm} (5.1)

$$t = t_{toe} + \frac{x}{h} (t_{tip} - t_{toe}) + \frac{(t_{heel} - t_{toe})}{f} \cdot \varphi \cdot r_c \cdot \left(1 - \frac{x}{h}\right)$$  \hspace{1cm} (5.2)

Eqn. 5.1 describes accurately, the height variation of the pinion and gear tooth with face width. In spiral bevel gears the gear is usually cut FORMATE (no generation) and all the profile curvature is put on the pinion. The assumed thickness variation is thus exact for FORMATE gears. The assumption of the linear variation of thickness with
Fig. 5.2 Shell model of bevel tooth
tooth height for the pinion provides computational advantages over a more accurate higher order assumption. The model can, however, incorporate any mathematically defined thickness variation in the compliance calculations.

5.2 Model definition and assumptions

5.2.1 Method of manufacture

The two manufacturing processes, face milling and face hobbing, produce two very different tooth geometries. The tooth shape of the spiral bevel gear is easily understood by considering the basic generating gear. The salient features of the face milling process are shown in Fig. 5.3 and Fig. 5.4 and a schematic of the face hobbing process is shown in Fig. 5.5.

The face milling process employs a circular face mill cutter. In the formate process the cutter is set into position relative to the work such that it cuts the correct spiral angle and pressure angle at the calculating point and sweeps out the tooth form of the gear as it rotates about its axis. The lengthwise tooth form is thus a circular arc of curvature equal to the curvature of the cutter. The gear tooth has straight sides in the normal plane. The pinion is generated conjugate to an imaginary gear called the crown gear. The schematic of the generating process is shown in Fig. 5.4. The workpiece is positioned relative to the cutting tool such that the teeth of the workpiece mesh with the teeth of the generating gear. The cutting tool is carried on a rotating machine called the cradle whose axis is identical to the axis of the generating gear. The cradle and workpiece roll together exactly as would the workpiece and the imaginary generating
Fig. 5.3 Machine setup of the face mill generator
Fig. 5.4 Face mill generating gear [Saginaw]
Fig. 5.5 Face hobbing generation [Keck]
gear. The blanks are designed with tapering depth such that the tooth depth is a function of the distance from the pitch apex.

The face hobbing process is a conjugate generation method which employs continuous indexing. The cutter blades of the circular face hob cutter are radially disposed about the axis of the cutter body and alternate blades cut opposite flanks of the tooth space. The cutter head is arranged in groups of blades. The work cycle motions are three continuous rotary motions: the two rotary motions of the cutter head and the gear blank to be machined and the feed motion which can be a plunge feed motion, a generating feed motion or a combination of both. A schematic that shows the emulation of a face hobbing generating gear by a face hob cutter is shown in Fig. 5.5. The lengthwise tooth curve is an extended epicycloid. The generating motion which rolls out the tooth surface is superimposed on the indexing motion. The teeth are of constant depth.

The two processes thus produce different tooth geometries, one with varying depth and circular curvature and the other with constant depth with the lengthwise tooth curve being an epicycloid. The model developed in this research describes the face milled geometry. The face hobbed gear geometry can be modeled by assuming constant depth and incorporating the radius of curvature change along the face width.

5.2.2 Salient features of the shell model

- The shell model includes the tooth height taper from the toe to the heel. The radius of curvature along the face width is assumed to be a constant equal to the radius of the cutter. The tooth heights at the toe and heel are evaluated from the input blank dimensions.
• The rigidity (thickness) variation in both the circumferential and axial directions is taken into account by using the actual thickness at the toe and heel base and at the tip. A linear variation in thickness from the base to the tip is assumed along with a linear variation in base thickness along the face width.

• Shear deformation is modeled using Mindlin plate theory and Bhimaraddi's higher order theory. The Rayleigh-Ritz solution is based on the strain energy equation of the shell. The model is clamped along the bottom circular edge.

5.3 Convergence study

The number of terms to be used in the Rayleigh-Ritz expansion is determined through this study. The shell segment is cantilevered along the base circumferential edge as in Fig. 5.2 and the free edge deflection is calculated for a point load of 100 lbs applied at the midpoint of the free edge. The results of the shell model deflection predictions are compared with finite element predictions. The dimensions of the spiral bevel gear used in the analysis are shown in Table 5.1. The finite element model of the gear tooth is shown in Fig. 5.6.

<table>
<thead>
<tr>
<th>Blank dimensions</th>
<th>Gear</th>
<th>Pinion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer cone distance</td>
<td>4.467 in</td>
<td>4.467 in</td>
</tr>
<tr>
<td>Face width</td>
<td>1.718 in</td>
<td>1.718 in</td>
</tr>
<tr>
<td>Cutter radius</td>
<td>4.5 in</td>
<td>4.5 in</td>
</tr>
<tr>
<td>Face angle</td>
<td>78 deg.</td>
<td>17.5 deg.</td>
</tr>
<tr>
<td>Root angle</td>
<td>72 deg.</td>
<td>11.5 deg.</td>
</tr>
<tr>
<td>Toe base thickness</td>
<td>0.36 in</td>
<td>0.32 in</td>
</tr>
<tr>
<td>Heel base thickness</td>
<td>0.45 in</td>
<td>0.405 in</td>
</tr>
<tr>
<td>Tip thickness</td>
<td>0.12 in</td>
<td>0.12 in</td>
</tr>
</tbody>
</table>
Different combinations of the number of terms in the axial and circumferential direction are investigated. The emphasis is to determine a sufficient number of terms that would be adequate to represent the flexural behavior while maintaining efficient computational times. Fig. 5.7 shows the effect of increasing the number of terms in the longitudinal direction for the Mindlin shell. It is seen that a choice of 3 or 5 functions in the circumferential direction would be inadequate to predict the deflection shape. When the number of trial functions is increased to 9 or 15, a better match with the finite element results is seen. It was decided to limit the number of terms to 9 to avoid the large computations times that result with the use of 15 terms, for only a marginal improvement in accuracy. When the number of terms in the axial direction are increased as in Fig. 5.8, it is seen that the deflection curve remains almost unchanged. Therefore, 5 terms in the axial direction are adequate to accurately represent the deflection curve. From the results of this convergence study, 5 terms in the axial direction and 9 terms in the circumferential direction are chosen for all subsequent computations in evaluating the flexural behavior of the Mindlin shell. Figs. 5.9 and Fig. 5.10 represent a similar study for the Bhimaraddi shell. Fig. 5.9 shows the effect of increasing the number of terms in the circumferential direction. As in the Mindlin shell, 9 terms are chosen in the circumferential direction. A choice of 3 or 5 terms is inadequate to represent the flexural behavior and the small improvement in accuracy over 9 terms by using 15 terms is greatly offset by the huge increase in computation times. Fig. 5.10 represents the effect of varying the axial terms. It is seen that the deflection shape is not altered much by an increase in the number of axial terms. As in the Mindlin shell, 5 terms in the axial direction and 9 terms in the circumferential direction are chosen to model the flexural behavior of the Bhimaraddi shell.
Fig. 5.6 Finite element model of the spiral bevel tooth
Fig. 5.7 Mindlin shell convergence study - circumferential direction
Fig. 5.8 Mindlin shell convergence study - axial direction
Fig. 5.9 Bhimaraddi shell convergence study - circumferential direction
Fig. 5.10 Bhimaraddi shell convergence study - axial direction
The effect of increasing the number of terms in the Rayleigh-Ritz expansion on the computations times is shown in Table 5.2. The computations times are based on a 20 point Gauss integration on a 486/33 personal computer.

<table>
<thead>
<tr>
<th>No. of terms</th>
<th>Mindlin shell (seconds)</th>
<th>Bhimaraddi shell (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ax5*cir3</td>
<td>50</td>
<td>147</td>
</tr>
<tr>
<td>ax5*cir5</td>
<td>140</td>
<td>415</td>
</tr>
<tr>
<td>ax5*cir9</td>
<td>457</td>
<td>1414</td>
</tr>
<tr>
<td>ax5*cir15</td>
<td>1350</td>
<td>4262</td>
</tr>
<tr>
<td>ax3*cir9</td>
<td>163</td>
<td>486</td>
</tr>
<tr>
<td>ax5*cir9</td>
<td>457</td>
<td>1413</td>
</tr>
<tr>
<td>ax9*cir9</td>
<td>1600</td>
<td>5008</td>
</tr>
<tr>
<td>ax11*cir9</td>
<td>2449</td>
<td>7544</td>
</tr>
</tbody>
</table>

5.4 Bevel gear and hypoid applications

Most spiral bevel and hypoid gears are designed for a highly localized (point) contact under no load which expands to line contact under load. The contact analysis involves the location and movement of this contact line under load, as the gears roll together. The procedure to determine the load distribution includes the compliance calculations due to tooth bending and the deformation due to Hertz contact. The load distribution is usually evaluated as a series of point loads on a set of previously chosen nodes on the line of contact. The shell model developed in this paper can be used for the compliance calculations in the contact analysis, as well as the gear strength determination once the load distribution is calculated.
5.4.1 Comparison with finite element predictions

In this section, the deflection and stress results of the static analysis of a shell model based on the Mindlin shell theory, of the gear and pinion tooth, are compared with finite element predictions. The static analysis for the gear and pinion is carried out for three different loading conditions, to demonstrate its applicability to determine gear strength and compliance. The three loading cases are: a) a point load applied at the toe tip, b) a point load applied at the midpoint of the free circular edge and c) a point load applied at the heel tip. A comparison of the deflection and stress results with finite element predictions for the gear tooth are presented in Figs. 5.11 - 5.13 and for the pinion tooth in Figs. 5.14 - 5.16. All the stresses are evaluated at 5% of the tooth height from the clamped end, as the finite element stress predictions at the nodes where the boundary conditions are applied are often unreliable. When the load is applied at any point on the free edge, the deflection is a maximum at the point of load application as shown in Figs. 5.11 and Figs. 5.12. The poor correlation of the peak deflection values with FEM is due to the inability of the finite element model to predict deflections accurately very close to the point of load application. A small oscillation in the deflection values can be seen in the Rayleigh-Ritz and FEM results along the free edge, away from the point of load application, in Figs. 5.11 and 5.12. The magnitude of these errors are small in comparison to the peak values and will have negligible effect in the compliance calculations along a contact line. From Fig. 5.13, the principal stress values are seen to be highest at the clamped end close to point of load application. A sharp reduction in the Rayleigh-Ritz stress values at the toe edge of the shell model for loading case (a) and at the heel edge for loading case (c), is seen in Fig. 5.13. The lack of restrictions on lengthwise elongation (proximity to free edge) could cause a condition
Fig. 5.11 Deflection along the spiral bevel gear tooth height due to the application of a point load (100 lbs) at different positions on the free edge.
Fig. 5.12 Deflection along the free edge of the spiral bevel gear due to the application of a point load (100 lbs) at different positions on the free edge.
Fig. 5.13 Maximum principal stress along the spiral bevel gear clamped edge due to the application of point load (100 lbs) at different positions on the free edge.
Fig. 5.14 Deflection along the spiral bevel pinion tooth height due to the application of a point load (100 lbs) at different positions on the free edge.
Fig. 5.15 Deflection along the free edge of the spiral bevel pinion due to the application of a point load (100 lbs) at different positions on the free edge.
Fig. 5.16 Maximum principal stress along the spiral bevel pinion clamped edge due to the application of point load (100 lbs) at different positions on the free edge.
approaching plane stress at the edges causing a localized stress dropoff [Wingate et al., 1989]. From Figs. 5.11 - 5.13 it can be thus be seen that the deflection and stress predictions of the shell model compare very well with FEM except for the deflection predictions under the point of load application.

The static analysis of the pinion tooth is performed using the cylindrical shell model. The dimensions of the pinion are given in Table 5.1. The results due to loading cases (a), (b) and (c) are presented in Figs. 5.14 - 5.16. As in the gear tooth, the deflections are seen to be a maximum at the point of load application in Figs. 5.14 and 5.15 for loading case (a), (b) and (c). The principal stress results are seen to compare excellently with finite element predictions in Figs. 5.16. A similar reduction in the principal stresses as in the gear tooth model, at the toe and heel edges, is seen in Fig. 5.16, for loading cases (a) and (c).

5.4.2 Comparison with higher order theory

In this section, a comparison of the Mindlin results with Bhimaraddi's theory is presented. The dimensions of the bevel gear and pinion are as in Table 5.1. A load magnitude of 100 lbs is used. Fig. 5.17 shows the deflection comparison of the shell model with the finite element results. Both of the theories compare well with FEM, with the first order (Mindlin) theory predicting slightly higher deflections. The higher order theory can be expected to predict higher stress results as the in plane displacement components include up to cubic terms of the thickness coordinate z compared to the first order terms in the Mindlin theory. This is seen in Fig. 5.18 and Fig. 5.19 for the normal stresses \( \sigma_z \) (bending) and \( \sigma_\theta \) respectively. An increase in \( \sigma_\theta \) for both theories is seen near the tooth ends which could be due to existence of the higher order terms in the circumferential direction of the Ritz expansion. This has however, negligible influence
on the principal stresses as seen previously in Fig. 5.13. The inplane shear stress $\tau_{\theta}$ for both theories is seen to match well with the finite element results in Fig. 5.20. In Fig. 5.21, the transverse shear stresses are seen to be markedly less than the finite element predictions. This is due to the fact that the maximum transverse shear stress predicted by the shell model occurs away from the 5% tooth height. It is to be noted that the Mindlin shear stress is evaluated on the shell surface and Bhimaraddi shear stress at the midplane (since this theory assumes a parabolic variation of shear strain through the thickness with zero shear stress on the free surfaces).

5.5 Summary

A good agreement with the finite element predictions is seen for the deflection and stress results using Mindlin shell theory. The higher order theory predicts higher stresses and would result in a conservative design. The deflection and stress computations based on the Mindlin theory are also much faster than the higher order theory. The application of the flexural behavior of the shell model to predict the deflection and stresses in a spiral bevel gear is thus demonstrated.
Fig. 5.17 Comparison of Mindlin's free edge deflection with Bhimaraddi's theory and FEM for a point load applied at the midpoint of the free edge
Fig. 5.18 Comparison of $\sigma_x$ (Mindlin) along clamped edge with $\sigma_x$ (Bhimaraddi) and FEM for a point load applied at the midpoint of the free edge
Fig. 5.19 Comparison of $\sigma_\theta$ (Mindlin) along clamped edge with $\sigma_\theta$ (Bhimaraddi) and FEM for a point load applied at the midpoint of the free edge.
Fig. 5.20 Comparison of $\sigma_{x_0}$ (Mindlin) along clamped edge with $\sigma_{x_0}$ (Bhimaraddi) and FEM for a point load applied at the midpoint of the free edge.
Fig. 5.21 Comparison of $\sigma_{xx}$ (Mindlin) along clamped edge with $\sigma_{xx}$ (Bhimaraddi) and FEM for a point load applied at the midpoint of the free edge.
CHAPTER VI

LOADED TOOTH CONTACT ANALYSIS USING CAPP

6.1 Introduction

An accurate solution to the loaded tooth contact problem has been one of the most challenging problems faced by gear engineers. The reasons for this are manyfold. When gears are brought into contact, the size of the contact zone is typically an order of magnitude smaller than the other dimensions of the gear. When a standard finite element model is used, it gives rise to the need for a highly refined mesh near the contact zone. As the gears roll together, the size of the contact zone changes and the contact zone moves over the surface of the gear. This would require a highly refined mesh all over the contacting surface. The sensitivity of the contact coordinates to the geometry of the contacting surfaces and the steep stress gradients in the critical regions place huge demands on the general purpose finite element models.

In the last decade, techniques have been developed that combine the strength of the finite element method with the strengths of other techniques like boundary element methods and surface integral techniques. The surface integral technique is accurate in predicting relative displacements near the contact zone while the finite element model predicts deformations well in regions away from the contact zone. This chapter deals with the use and description of one such program CAPP (Contact Analysis Program Package) developed by Vijayakar (1991) for the loaded tooth contact analysis of bevel
and hypoid gears. The method uses an FQP element (Vijayakar, 1987) in the tooth stiffness calculations. An overview of the procedure used by CAPP is presented in the next section and a description of the bevel gear mesh generators in Sec. 6.3.

6.2 Mathematical framework of CAPP

The calculations are based on the assumption that the finite element model predicts deflections away from the contact zone and surface integral technique predicts the deflections in the close neighborhood of the contact zone. In order to combine the two solutions, a reference surface imbedded in the contacting body is used. This surface is chosen far enough from the contact zone such that the finite element solution of displacement is accurate and close enough to contact zone such the finite size of the body does not affect the solutions on this reference surface with respect to the points in the region of contact. A detailed description of the procedure is given in Vijayakar (1991). A brief overview is presented below.

The contact analysis involves several steps. The first step involves locating points on the two surfaces of the two gear teeth which are closest to each other before the application of the load. The surfaces are numerically represented using the FQP element. The distance between this chosen series of grid points on the two surfaces is then minimized by solving a nonlinear set of equations using the Newton Raphson technique to locate the principal contact points. Next, the principal normal curvatures and unit tangent vectors to the surfaces at the contact points are calculated followed by the relative curvature. After the relative curvatures have been obtained near the principal contact point they are used to determine the size of the contact zone using Hertz's theory. In general, it is to be noted that the contact zone is not an ellipse and its size is different from Hertz theory predictions due to the nonconstant curvatures over the contact zones. Also, the point of maximum contact pressure will not coincide with
the principal contact point. The dimensions of the contact ellipse are estimated by assuming that all the load is borne by only one principal contact point. A grid of points is laid out on both surfaces over a size larger than the Hertzian contact ellipse around the principal contact points to allow for the fact that the actual contact zone might not be an ellipse of the predicted size. The grid must be made small to achieve maximum resolution.

The terms in the compliance matrix are calculated using a combination of surface integral form of Boussinesq solution and the finite element model of the contacting teeth. The matching interface is calculated using the following procedure. The displacement of a field point \( r \) due to a load at the surface grid point \( r_{ij} \) can be expressed as

\[
u(r_{ij}; r) = (u(r_{ij}; r) - u(r_{ij}; q)) + u(r_{ij}; q) \tag{6.1}
\]

where \( q \) is sufficiently removed from the surface. If the first two terms are evaluated using surface integral techniques and the last term by finite element methods, we obtain

\[
u(r_{ij}; r)(q) = (u^{si}(r_{ij}; r) - u^{si}(r_{ij}; q)) + u^{fe}(r_{ij}; q) \tag{6.2}
\]

The relative component (term in parenthesis) is better estimated by the Boussinesq solution and the gross deformation of the body due to the fact that it is not a half space will not significantly affect this term. The remaining term is not affected by the local stresses at the surface as \( q \) is chosen to be far below the surface. This term is thus calculated using finite element methods. The value \( u(r_{ij}; r)(q) \) thus computed will depend on the location of \( q \) due to different values of the surface integral technique and the finite element solution. A minimization technique is used to iteratively determine the set of points where the finite element solution matches with surface integral solution. The set of points so calculated would be the reference or matching interface.
Once the compliance matrix is determined, the contact force distribution and the rigid body motion is determined by setting up the contact equations (derived in Sec. 1.5) and solving them using a modified Simplex algorithm.

6.3 Mesh generation for bevel gears

A finite element mesh has to be generated before the loaded tooth contact analysis can be carried out. The independence of CAPP from any specific gear geometry has been realized by ensuring that all of the required kinematic and geometric information comes in strictly numeric form. The geometric information for each gear is contained in mesh files. The mesh files are created using special purpose mesh generators. There are two mesh files, one for the gear and one for the pinion. The mesh files contain information about the geometry of the body and the mesh information. The mesh generator program obtains the finite element mesh information from standardized template files. The template file used for the contact analysis is given in Appendix B and C. This kind of mesh generation offers the user, the flexibility to create a mesh suitable to a given problem, by developing his own template files rather than providing a few standard meshes. Different mesh generators are required for different types of gears. The mesh generators are not part of CAPP and are written by the gear designer. Two mesh generator programs were developed as part of this research, one for straight bevel gears and the other for spiral bevel and hypoid gears. The straight bevel mesh generator develops the tooth geometry as an octoid. The spiral bevel mesh generator uses a simulation of the machine kinematics of the gear generator to obtain the gear geometry. A description of the two mesh generators is presented in Sec. 6.3.1 and Sec. 6.3.2.
The contact analysis uses a special purpose finite element formulation, called the FQP element which uses Chebychev polynomials to model the gear tooth. Nodes are defined generically as the smallest bundle of finite element data. Real nodes correspond to an actual point in space while virtual nodes do not. When the data associated with a node is used to interpolate coordinates it is called a coordinate node. They have three coordinate data values associated with them. When the data associated with a node is used to interpolate for displacements or stresses it is called a displacement node. It has 12 real and 3 integer values associated with it. There are three real values for the three components of displacement, three real values for the three load components and six real values for the six components of stress. The three integer values are the constraint codes (0 representing free, 1 representing constrained) for the three degrees of freedom. An axode is defined as a bundle of nodes and the order determined by the number of nodes. An axode of order $n$ contains $n+1$ nodes.

A very high degree of coordinate accuracy is required for the coordinate elements along the active profile of the gear and in the root region. The element sides along the profile require position and normal vector information. In the interior, the coordinate elements do not require such high degrees of accuracy. Hence two types of coordinate connectivity are used. Type I connectivity is used for elements which require a low degree of accuracy. They have 4 real axodes. Type II connectivity is used for elements along the active profile and in the root region. The Type II connectivity elements have a variable number of axodes $n$, where $n$ is greater than six with a combination of real and virtual axodes where the virtual axodes contain information about the normal vectors. The displacement connectivity of the displacement element can be varied from linear to cubic through the use of optional axodes. There are a minimum of four axodes, one each at the element corners which are real.
The template file contains information about the coordinate mesh and displacement mesh. The number of coordinate and displacement axodes per tooth, the coordinate axodes along the active profile, the displacement and coordinate connectivities as well as the nodal constraints are specified in the template file. Once the position vectors of the surface axodes are located from the geometry information during mesh generation, the locations of the rest of the axodes are computed from the location of the surface axodes through a series of projection and interpolation statements in the template file.

The displacement and the coordinate axodes are numbered according to specific conventions to maintain compatibility and to identify the active profiles on the tooth. If \( N_d \) is the number of displacement axodes of tooth \( i \) and \( N_{com} \) is the number shared with an adjacent tooth, then the first \( N_{com} \) axodes are the axodes that tooth \( i \) shares with tooth \( i-1 \) and the last \( N_{com} \) axodes are those that are shared with tooth \( i+1 \). Similarly, if \( N_c \) is the number of coordinate axodes used to model one tooth and \( N_{surf} \) the number of coordinate axodes along the active profile, the first \( N_{surf} \) axodes are along the active profile of side 1 of the tooth and the next \( N_{surf} \) are along the active profile of side 2 of the tooth. The odd numbered axodes along the two active profiles contain information about the position vectors and the even numbered axodes are virtual axodes containing information about the normal vectors.

6.3.1 Straight bevel mesh generator

The straight bevel mesh generator program creates the mesh file for straight bevel gears based on the octoid tooth geometry. Bevel gears are not made to fit a simple geometric shape, but, instead are defined by the cutting tool shapes and machine motions that generate them. Hence, no analytical description of the surface presently exists, unlike the involute surface of spur and helical gears. The theoretical surface
could be defined in two distinct methods. In one, the surface is defined by the motions from an errorless cutting machine with built in modifications. In the other, the tooth surface is defined as those of a perfectly conjugate gear and pinion. The conjugate method of generation is preferred to the cutting simulation in straight bevel gears as it makes all other modifications visible and can be defined once and for all, without being influenced by continual modifications based on requirements and experience.

In this program the coordinates of the gear and pinion are calculated as true conjugates to a generating gear called the crown rack. The crown rack has a pitch angle of ninety degrees with plane tooth surfaces. The geometry thus described is the octoid geometry, the name given since the path of contact when developed on the surface of a sphere resembles the figure eight. The profile of the bevel gear is defined as the intersection of a sphere centered on the pitch cone apex or the crossing point and the side of the tooth. This is appropriate for bevel gears since the meshing action is two dimensional on the surface of this sphere, similar to the transverse plane for parallel axis gears. A description of the octoid method of generation in the coordinate measurement of bevel gear teeth is presented in Chambers and Brown (1987).

The number of points on the tooth surface is determined by the number of coordinate axodes on the active profile and the order of the axode. This information is obtained from the template file. The coordinate calculations are performed with a high degree of accuracy. The spacing between the tooth profiles across the face width corresponds to the spacing of the roots of the Chebychev polynomial between -1 and +1. The basic gear geometry parameters like the pressure angle, pitch cone angle, face angle, outer cone distance, etc. are input interactively.

The program initially calculates the coordinates and normals of the specified number of points on the active profiles. The other points are calculated based on the
interpolation and projection commands in the template file. The Chebychev coefficients which determine the coordinate and displacement axodites are now calculated and stored. No coordinates are stored in the mesh file. The nodal and element connectivity are evaluated and stored.

6.3.2 Spiral bevel mesh generator

The spiral bevel mesh generator is very similar to the straight bevel mesh generator except for the geometry computations. Here, instead of using a conjugate generation technique like the straight bevel generation, a simulation of the kinematics of the gear generator is carried out to generate the coordinates. The gear cutting process is the face milling process and the teeth are of tapered depth. Generation is the basic process in bevel and hypoid gear manufacturing as at least one member of every pair must be generated. Often the pinion is generated and the gear is form cut to achieve faster production rates. The gear generator carries a face mill cutter mounted on a rotatable machine member called the cradle, whose axis is identical to the axis of the generating gear. The generating gear and the workpiece are turned on their respective axes according to a prescribed motion which is defined by the nonslip rolling of the pitch surfaces of the generating gear and the workpiece. The motion of the cutting tool combined with the cradle rotation describes at least one tooth of the generating gear. A schematic of the generation process is shown in Fig. 5.3.

The bevel gear generator is the basic machine in the gear cutting process. There are different types of gear generators. The tilt generators allow the cutter axis to be inclined with respect to the cradle axis. The modified roll generator allows the ratio of the turning velocities between the cradle and the workpiece to be varied during the cutting cycle. The nongenerators do not have any cradle or workpiece rotation and the tooth surfaces are straight in the normal section, formed by a single sweep of the cutting
tool. In generators with helical motion, either the work or the cradle is advanced linearly during the combined rotations of both members. The mesh generator can simulate a variety of such generators to generate the tooth surface of both spiral bevel and hypoid gears.

The inputs to the mesh generator for coordinate generation are the blank data, cutter data and the machine settings. These are usually available in the summary sheets provided by the gear manufacturers. Once the coordinates and the normals are evaluated at the required number of points on the active profiles, the rest of the coordinates are obtained based on the projection and interpolation statements in the template file. As in the straight bevel generator the Chebychev coefficients defining the coordinate and displacement axodes are evaluated and stored along with the nodal and element connectivities.

6.4 Contact analysis examples

Two case studies are conducted to demonstrate the use of finite element techniques in the contact analysis. The first study involves the contact analysis of straight bevel gears. The mesh files for the gear and pinion are generated using the mesh generator, as explained in Sec. 6.3.1. A contact analysis of spiral bevel gears is performed in the second study. The gear is form cut and the pinion is generated. The spiral bevel mesh generator is used to create the mesh files for the contact analysis.

The following procedure is used to perform the contact analysis. The mesh generator is run twice, once for the gear and once for the pinion. The output of the mesh generator is a finite element model of one tooth of the gear. The tooth models are brought into contact to perform the contact analysis using the information specified in a configuration file. The configuration file contains information about the load torque,
the number of teeth in each gear and the number of teeth modeled, the orientation of gear coordinate frames with respect to the reference frame and the initial rotations. The configuration files used for the two case studies are presented in the Appendix. The gears are oriented to mesh at the desired position by specifying the corresponding initial rotations for both members.

The preprocessor is used to evaluate the gear set graphically before carrying out the contact analysis. The preprocessor is run interactively. It reads in the configuration file and the mesh files. The various graphic options like the outline displays and the sectional views can be used to view the gear set configuration and make necessary changes to the initial rotation angle before the contact analysis. The finite element computations are done in the contact analysis program and this module runs in batch mode. The contact analysis program generates several output data files. The file SUMMARY.DAT contains information about the transmission error at each roll angle step analyzed by the contact analysis program. The file LOADS.DAT contains the information about the location of the grid cell and the load and separation at the grid cell after contact. The displacement and stress information is stored in files whose specification is internal to CAPP which are used by the postprocessor. The postprocessor is used to graphically interpret the results of the contact analysis. Contour plots of stresses at any section can be drawn as well as load distribution plots on the contacting surfaces.

6.4.1 Straight bevel contact analysis

The octoid geometry description of the straight bevel gear is used for the contact analysis. The dimensions of the bevel gear are shown in Table 6.1.
Table 6.1 Blank dimensions of straight bevel gear

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Gear</th>
<th>Pinion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer cone distance</td>
<td>1.4578 in</td>
<td>1.4578 in</td>
</tr>
<tr>
<td>Face width</td>
<td>0.643 in</td>
<td>0.643 in</td>
</tr>
<tr>
<td>Pressure angle</td>
<td>24 deg.</td>
<td>24 deg.</td>
</tr>
<tr>
<td>Pitch angle</td>
<td>54.466 deg.</td>
<td>35.533 deg.</td>
</tr>
<tr>
<td>Face angle</td>
<td>60.783 deg.</td>
<td>44.933 deg.</td>
</tr>
<tr>
<td>Root angle</td>
<td>45.066 deg.</td>
<td>29.166 deg.</td>
</tr>
<tr>
<td>Number of teeth</td>
<td>14</td>
<td>10</td>
</tr>
</tbody>
</table>

The initial configuration of the gear pair is shown in Fig. 6.1. Four teeth are modeled for each member. The bore diameter can be varied to alter the rim thickness. There is only one element along the face width. The number of elements, their size and shape can be varied using the template file. The load distribution on the pinion is shown in Fig. 6.2. At the start, the entire load is taken up by the first tooth and the second tooth is not yet in contact. The movement of the load distribution as the gear roll together through one mesh cycle is shown in Fig. 6.2. It is also seen that the contact line extends radially along the full face width. The sector plate model developed in Chapter III is thus well suited to model the bevel tooth, as the coordinate lines (lines of constant sector angle) coincide with the contact lines. A variation of the transmission error with load torque over one base pitch is presented in Fig. 6.3. The numerical values of the TE with roll angle for the various load conditions are presented in Table 6.2. It can be seen that with increasing load, the mean value of the transmission error increases, which is primarily due to the increase in tooth deflections. The unloaded TE corresponds to a torque of 10 in-lbs. A sharp drop in the loaded TE from 32 to 46 degrees roll angle is seen for a load torque of 800 in-lbs due to the second pair of teeth coming into contact.
Fig. 6.1 Straight bevel gear meshing
Fig. 6.2  Load distribution in straight bevel gears
Fig. 6.3 Effect of load on the transmission error in straight bevel gears
### Table 6.2 Straight bevel gear transmission error

<table>
<thead>
<tr>
<th>Pinion roll angle (deg)</th>
<th>Gear roll angle (deg)</th>
<th>TE (rad) 10 in-lbs</th>
<th>TE (rad) 100 in-lbs</th>
<th>TE (rad) 200 in-lbs</th>
<th>TE (rad) 400 in-lbs</th>
<th>TE (rad) 800 in-lbs</th>
</tr>
</thead>
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<tr>
<td>30</td>
<td>-98.5714</td>
<td>-6.76E-03</td>
<td>-6.94E-03</td>
<td>-7.13E-03</td>
<td>-7.51E-03</td>
<td>-8.27E-03</td>
</tr>
<tr>
<td>33</td>
<td>-100.7143</td>
<td>-6.77E-03</td>
<td>-6.96E-03</td>
<td>-7.16E-03</td>
<td>-7.55E-03</td>
<td>-8.35E-03</td>
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<tr>
<td>36</td>
<td>-102.8571</td>
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<td>-6.98E-03</td>
<td>-7.15E-03</td>
<td>-7.39E-03</td>
<td>-7.86E-03</td>
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<td>39</td>
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<td>-6.95E-03</td>
<td>-7.17E-03</td>
<td>-7.61E-03</td>
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<tr>
<td>42</td>
<td>-107.1429</td>
<td>-6.70E-03</td>
<td>-6.84E-03</td>
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<td>-7.18E-03</td>
<td>-7.63E-03</td>
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<td>45</td>
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<td>-6.89E-03</td>
<td>-7.08E-03</td>
<td>-7.47E-03</td>
<td>-8.08E-03</td>
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<tr>
<td>48</td>
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<td>-6.71E-03</td>
<td>-6.89E-03</td>
<td>-7.07E-03</td>
<td>-7.45E-03</td>
<td>-8.20E-03</td>
</tr>
<tr>
<td>51</td>
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<td>-6.89E-03</td>
<td>-7.08E-03</td>
<td>-7.45E-03</td>
<td>-8.19E-03</td>
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<tr>
<td>54</td>
<td>-115.7143</td>
<td>-6.73E-03</td>
<td>-6.90E-03</td>
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<td>-7.45E-03</td>
<td>-8.18E-03</td>
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<tr>
<td>57</td>
<td>-117.8571</td>
<td>-6.74E-03</td>
<td>-6.91E-03</td>
<td>-7.09E-03</td>
<td>-7.46E-03</td>
<td>-8.20E-03</td>
</tr>
<tr>
<td>60</td>
<td>-120.000</td>
<td>-6.75E-03</td>
<td>-6.92E-03</td>
<td>-7.11E-03</td>
<td>-7.49E-03</td>
<td>-8.25E-03</td>
</tr>
<tr>
<td>63</td>
<td>-122.1429</td>
<td>-6.76E-03</td>
<td>-6.94E-03</td>
<td>-7.13E-03</td>
<td>-7.52E-03</td>
<td>-8.29E-03</td>
</tr>
<tr>
<td>66</td>
<td>-124.2857</td>
<td>-6.77E-03</td>
<td>-6.95E-03</td>
<td>-7.15E-03</td>
<td>-7.56E-03</td>
<td>-8.37E-03</td>
</tr>
<tr>
<td>69</td>
<td>-126.4286</td>
<td>-6.78E-03</td>
<td>-6.97E-03</td>
<td>-7.18E-03</td>
<td>-7.60E-03</td>
<td>-8.43E-03</td>
</tr>
<tr>
<td>72</td>
<td>-128.5714</td>
<td>-6.79E-03</td>
<td>-6.99E-03</td>
<td>-7.16E-03</td>
<td>-7.41E-03</td>
<td>-7.88E-03</td>
</tr>
</tbody>
</table>
The loaded TE error at the maximum load also shows a larger peak to peak value (0.7e-3 radians) compared to the unloaded TE error. The low peak to peak TE in the unloaded case is due to the conjugate gear geometries.

### 6.4.2 Spiral bevel contact analysis

The spiral bevel gear geometries were obtained by a simulation of the cutting kinematics of the Gleason 116 gear generator. The gears are cut using the face milling process. The blank dimensions, cutter data and the machine settings used to cut the gear and the pinion are presented in Table 6.3. The meshing gears are shown in Fig. 6.4. Four teeth are modeled for each of the members.

The load distribution plots for a load torque of 250 in-lbs, 2000 in-lbs, 20000 in-lbs and 50000 in-lbs are presented in Fig. 6.5, Fig. 6.6, Fig. 6.7 and Fig. 6.8 respectively. These plots show the location and movement of the contact lines with roll angle for the corresponding load torques. At low loads, the contact lines do not extend across the full face width of the gear, as in Fig. 6.5. This is due to the introduction of a deliberate mismatch of the curvatures in the profile and lengthwise direction of the contacting tooth surfaces, to make the contact pattern less sensitive to misalignment and assembly tolerances. It can be seen from Fig. 6.6 that as the load is increased to 2000 in-lbs the length of the contact line increases along the length of the tooth. Additionally, for the same roll angle, the next pair of teeth is seen to come into contact early, increasing the contact ratio. When the load is increased to 20000 in-lbs and 50000 in-lbs, the contact spreads over the full length of the tooth, as in Fig. 6.7 and Fig. 6.8 respectively. Additionally a minimum of two teeth are seen to share the load at any mesh position.
Table 6.3 Dimension sheet for spiral bevel gear

<table>
<thead>
<tr>
<th></th>
<th>GEAR</th>
<th>PINION</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Blank Dimensions</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of teeth</td>
<td>36</td>
<td>11</td>
</tr>
<tr>
<td>Pressure angle</td>
<td>14D 30M</td>
<td>14D 30M</td>
</tr>
<tr>
<td>Spiral angle</td>
<td>35D 60M</td>
<td>35D 00M</td>
</tr>
<tr>
<td>Face angle</td>
<td>75D 05M</td>
<td>21D 14M</td>
</tr>
<tr>
<td>Pitch angle</td>
<td>73D 01M</td>
<td>16D 59M</td>
</tr>
<tr>
<td>Dedendum angle</td>
<td>04D 15M</td>
<td>02D 04M</td>
</tr>
<tr>
<td>Outer cone distance</td>
<td>4.444 inch</td>
<td>4.444 inch</td>
</tr>
<tr>
<td>Face width</td>
<td>1.25 inch</td>
<td>1.25 inch</td>
</tr>
<tr>
<td>Addendum</td>
<td>0.116 inch</td>
<td>0.286 inch</td>
</tr>
<tr>
<td>Dedendum</td>
<td>0.330 inch</td>
<td>0.160 inch</td>
</tr>
<tr>
<td>Distance between</td>
<td></td>
<td></td>
</tr>
<tr>
<td>crossing pt. and apex</td>
<td>0.000 inch</td>
<td>0.000 inch</td>
</tr>
<tr>
<td>Mounting distance</td>
<td>1.387 inch</td>
<td>4.167 inch</td>
</tr>
<tr>
<td><strong>Cutter data</strong></td>
<td></td>
<td>Concave</td>
</tr>
<tr>
<td>Cutter diameter</td>
<td>9.000 inch</td>
<td>8.770 inch</td>
</tr>
<tr>
<td>Point width</td>
<td>0.170 inch</td>
<td>0.005 inch</td>
</tr>
<tr>
<td>Outside blade angle</td>
<td>12D 45M</td>
<td>12D 45M</td>
</tr>
<tr>
<td>Inside blade angle</td>
<td>16D 45M</td>
<td>16D 15M</td>
</tr>
<tr>
<td>Outside blade edge radius</td>
<td>0.05 inch</td>
<td>0.03 inch</td>
</tr>
<tr>
<td>Inside blade edge radius</td>
<td>0.05 inch</td>
<td>0.03 inch</td>
</tr>
<tr>
<td><strong>Machine settings</strong></td>
<td></td>
<td>Convex</td>
</tr>
<tr>
<td>Machine root angle</td>
<td>68D 46M</td>
<td>12D 54M</td>
</tr>
<tr>
<td>Machine center to back</td>
<td>0.000 inch</td>
<td>-0.061 inch</td>
</tr>
<tr>
<td>Sliding base</td>
<td>0.000 inch</td>
<td>0.043 inch</td>
</tr>
<tr>
<td>Blank offset</td>
<td>0.000 inch</td>
<td>0.087 inch</td>
</tr>
<tr>
<td>Eccentric angle</td>
<td>52D 46M</td>
<td>53D 16M</td>
</tr>
<tr>
<td>Cradle angle</td>
<td>352D 11M</td>
<td>137D 45M</td>
</tr>
<tr>
<td>Swivel angle</td>
<td>0D 0M</td>
<td>252D 15M</td>
</tr>
<tr>
<td>Cutter spindle rotation angle</td>
<td>0D 0M</td>
<td>8D 15M</td>
</tr>
<tr>
<td>NC/50 ratio gears</td>
<td>52/44x47/74</td>
<td>52/40x48/83</td>
</tr>
</tbody>
</table>
Fig. 6.4 Spiral bevel gear meshing
Fig. 6.5 Load distribution on spiral bevel pinion at 250 in-lbs
6.6 (a) Roll angle -15 deg.  
6.6 (b) Roll angle -27 deg.  
6.6 (c) Roll angle -39 deg.  
6.6 (d) Roll angle -51 deg.  

Fig. 6.6 Load distribution on spiral bevel pinion at 2000 in-lbs
Fig. 6.7 Load distribution on spiral bevel pinion at 20000 in-lbs
Fig. 6.8 Load distribution on spiral bevel pinion at 50000 in-lbs
The variation of the TE with roll angle for different load torques is shown in Fig. 6.9. An increase in the mean transmission error values is seen with increasing load torques. The unloaded TE corresponds to a load torque of 10 in-lbs. The larger peak to peak TE values of the unloaded TE curve is due to the relief on the profile and tip areas of the pinion. This results in the gear being retarded from its theoretical position during the point of motion transfer to the adjacent tooth. From Fig. 6.9 it can be seen that as the load torque is increased, the convex shape of the curve first flattens out and then becomes convex again but with a phase shift of half of the base pitch. The difference in deflection between the double tooth contact and single tooth contact smooths out the motion curve. Uniform motion is achieved at about 1000 in-lbs due to the optimum mismatch for the given load. As the load is further increased the motion error again grows but with a phase shift of half of the base pitch which is still due to stiffness difference between single and double tooth contact.

At very high loads (>8000 in-lbs) the contact ratio becomes greater than two and the motion error is due to the stiffness difference between double and triple tooth contact. The motion error curve flattens out at 30000 in-lbs similar to the earlier described effect at 1000 in-lbs. A further increase in the load torque increases the motion variation due to the contact occurring at the edges of the teeth. A similar description of the variation of the motion curve with load for a spiral bevel gear system has been presented by Krenzer (1981). The numerical values of the loaded and unloaded TE are presented in Table 6.4.
Fig. 6.9 Effect of load on the transmission error in spiral bevel gears
Fig. 6.9 Contd.

Pinion transmission error (radians)

Roll angle (degrees)

-4.50E-03
-5.50E-03
-6.50E-03
-7.50E-03
-8.50E-03
-9.50E-03
-1.05E-02
-1.15E-02

-12 -18 -24 -30 -36 -42 -48 -54

20000 in-lbs
30000 in-lbs
40000 in-lbs
50000 in-lbs
<table>
<thead>
<tr>
<th>Pinion roll angle (deg)</th>
<th>Gear roll angle (deg)</th>
<th>$TE(\text{rad})$ 10 in-lbs</th>
<th>$TE(\text{rad})$ 250 in-lbs</th>
<th>$TE(\text{rad})$ 500 in-lbs</th>
<th>$TE(\text{rad})$ 1000 in-lbs</th>
<th>$TE(\text{rad})$ 2000 in-lbs</th>
</tr>
</thead>
<tbody>
<tr>
<td>-12</td>
<td>-26.3333</td>
<td>-6.81E-04</td>
<td>-8.00E-04</td>
<td>-8.95E-04</td>
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<td>-1.33E-03</td>
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<td>-5.80E-04</td>
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<td>-23.5833</td>
<td>-5.12E-04</td>
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<td>-1.05E-03</td>
<td>-1.35E-03</td>
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</table>
Table 6.4 Continued

| Pinion roll angle (deg) | Gear roll angle (deg) | $|T_E| (\text{rad})$ 8000 in-lbs | $|T_E| (\text{rad})$ 20000 in-lbs | $|T_E| (\text{rad})$ 30000 in-lbs | $|T_E| (\text{rad})$ 40000 in-lbs | $|T_E| (\text{rad})$ 50000 in-lbs |
|------------------------|-----------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| -12                    | -26.3333              | -2.71E-03                       | -5.04E-03                       | -6.88E-03                       | -8.63E-03                       | -1.04E-02                       |
| -15                    | -25.4167              | -2.68E-03                       | -4.97E-03                       | -8.83E-03                       | -8.66E-03                       | -1.05E-02                       |
| -18                    | -24.5                 | -2.71E-03                       | -5.01E-03                       | -6.84E-03                       | -8.63E-03                       | -1.04E-02                       |
| -21                    | -23.5833              | -2.75E-03                       | -5.09E-03                       | -6.80E-03                       | -8.45E-03                       | -1.01E-02                       |
| -24                    | -22.6667              | -2.79E-03                       | -5.05E-03                       | -6.80E-03                       | -8.55E-03                       | -1.03E-02                       |
| -27                    | -21.75                | -2.89E-03                       | -5.12E-03                       | -6.83E-03                       | -8.52E-03                       | -1.02E-02                       |
| -30                    | -20.8333              | -2.98E-03                       | -5.17E-03                       | -6.85E-03                       | -8.52E-03                       | -1.02E-02                       |
| -33                    | -19.9167              | -2.99E-03                       | -5.18E-03                       | -6.84E-03                       | -8.50E-03                       | -1.02E-02                       |
| -36                    | -19                   | -2.93E-03                       | -5.14E-03                       | -6.90E-03                       | -8.64E-03                       | -1.04E-02                       |
| -39                    | -18.0833              | -2.91E-03                       | -5.32E-03                       | -7.25E-03                       | -9.17E-03                       | -1.11E-02                       |
| -42                    | -17.1667              | -2.83E-03                       | -5.21E-03                       | -7.10E-03                       | -8.95E-03                       | -1.08E-02                       |
| -45                    | -16.25                | -2.77E-03                       | -5.20E-03                       | -7.05E-03                       | -8.85E-03                       | -1.07E-02                       |
| -48                    | -15.3333              | -2.71E-03                       | -5.08E-03                       | -6.99E-03                       | -8.88E-03                       | -1.08E-02                       |
| -51                    | -14.4167              | -2.73E-03                       | -5.06E-03                       | -6.96E-03                       | -8.79E-03                       | -1.06E-02                       |
| -54                    | -13.5                 | -2.78E-03                       | -5.14E-03                       | -6.83E-03                       | -8.50E-03                       | -1.02E-02                       |
6.5. Summary

In this chapter the details of the mesh generators used in loaded tooth contact analysis of a straight bevel gear and a spiral bevel gear have been described. The loaded tooth contact analysis is performed using CAPP. Two case studies, one for straight bevel gears and the other for spiral bevel gears are presented and discussed. The shape and movement of the load distribution with roll angle for each case is investigated. The transmission error for straight and spiral bevel gears for various loads are plotted and discussed.
CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS

7.1 Summary of research contributions

The specific contributions of this research are listed below.

1) A review of the literature reveals that shear has been inadequately modeled in the flexural behavior of the annular sector plates. The analytical solution to the cantilevered sector plate based on the classical plate theory was first provided by Rubin (1983). This research provides a series analytic solution for a cantilevered annular sector plate of variable rigidity based on the Mindlin shear theory for the first time. In addition, the effect of the elastic support has been investigated by suitably modifying the trial functions in the Rayleigh-Ritz expansion. The only other known and currently available method for performing a static analysis is the finite element method, against which the results are shown to compare well.

2) This research has led to the development of a new mathematical model to evaluate the stiffness of the straight bevel gear tooth. The model takes into account the height taper and the rigidity variation along the face width. The computations include the effects of shear deformation as well as the base rotations. An accurate modeling of the gear tooth compliance is a key element in predicting the load distribution and
transmission error during meshing. The static analysis of a shear flexible annular sector plate is shown to have an important application in the design of bevel gears.

3) The application of shell theory in determining the flexural behavior of spiral bevel gears has been demonstrated in this dissertation. Spiral bevel gear teeth are characterized by a lengthwise curvature and small radius to thickness and length to thickness ratios. A new model which describes the spiral bevel gear tooth geometry as a circular cylindrical shell of dimensions matching the actual tooth dimensions has been developed. Two levels of shear based theories were used to investigate the flexural behavior. A result of this dissertation has been the development of a new and improved model to predict tooth compliance compared to the existing beam and plate models. In addition, the model depends only on the external tooth dimensions rather than the machine settings, which makes it very attractive for iterative runs during the loaded tooth contact analysis.

4) The models developed in this research can also be used to estimate root stresses in bevel and hypoid gears. The stress results have been compared to previously existing results in the literature and also with finite element predictions. A good correlation has been observed, which justifies the use of the models as an alternate method of root stress predictions with the use of suitable root stress concentration factors.

5) This dissertation has led to the development of mesh generators for bevel and hypoid gear geometries for CAPP. The straight bevel gear geometry is obtained by a simulation of the conjugate motions between the gear blank and the crown rack. The geometry of spiral bevel gears is determined by a cutting simulation of the machine kinematics of the gear generator. With this, it is possible to obtain the load distribution and static transmission error for all kinds of straight and spiral bevel gears if the blank dimensions and basic settings are provided. The contact analysis is performed using
CAPP. The loaded tooth contact analysis of straight bevel gears based on the octoid tooth geometry has been performed for the first time.

7.2 Concluding remarks

The following remarks can be made based on results of this research.

1) For a comparable accuracy, the Rayleigh-Ritz procedure is seen to be numerically efficient compared to the finite element method. It does not require a mesh generator, preprocessor or postprocessor and only a limited model building effort. It requires far fewer number of degrees of freedom compared to the finite element method. Additionally, it makes possible the implementation and investigation of higher order theories upon the flexural behavior, which is not always possible when using standard finite element packages.

2) The number of terms in the Ritz expansion was decided from a convergence study. The free edge deflection of the cantilevered plate or shell was evaluated and compared with finite element predictions. Polynomials were chosen as the admissible functions in the series expansion, due to their simplicity and ability to be exactly integrated in the stiffness calculations. While it is seen that an increase in the number of terms in the series expansion improves the accuracy of the deflection solution and makes the assumed solutions tend toward the exact solution, the solution times considerably increase. Hence, the final choice of the number of terms balances the benefits of any further increase in accuracy against the corresponding increase in solution times.

3) It is seen that the use of the higher order theories results in higher stress predictions and would lead to a conservative design. In general, plate theories can be developed by expanding the displacements in a power series of the coordinates, normal to the midplane. In principle, theories developed by this means can be made as accurate as
possible by including sufficient number of terms. In practice, however, a point of diminishing returns is reached when the complexity of resulting forms becomes too great. The displacement assumptions of Bhimaraddi have terms including third order in the thickness coordinate and sets the transverse shear strain to zero at the top and bottom surfaces. However, the computation times for stiffness evaluations increase dramatically when compared to Mindlin's theory and the strain displacement relations become lengthy and unwieldy as they include the derivatives of the thickness with respect to the principal coordinates. Based on the deflection predictions, it is concluded that the first order theory is adequate for compliance calculations.

4) The sector plate and the cylindrical shell are natural representations of straight and spiral bevel teeth due to their ability to include the salient features of the corresponding gear geometries. When shear deformation and rigidity variation are also taken into account, the flexural behavior of the plate/shell very closely represents the bending behavior of the bevel teeth.

5) The full fledged finite element based loaded tooth contact analysis is performed on the CRAY-YMP using CAPP. It is seen that the contact conditions are very sensitive to the geometry of the contacting surfaces. Hence, a very high degree of coordinate accuracy is required for the coordinate elements along the active profile of the gear and in the root region. This is accomplished by a very accurate definition of the coordinates and normals in these regions in the mesh generator. The run times increase considerably with an increase in displacement order connectivity and a choice of 4 or 5 was found to be optimum.
7.3 Recommendations for future work

This dissertation has opened up several areas for further study and development. The strength calculations in this procedure are primarily dependent upon the external blank dimensions. A parametric study could be performed to determine the effect of varying the longitudinal tooth curvature as well as the amount of thickness and height taper. Additionally, the effect of the location of the line of contact and the load distribution on the root stresses and bending deflections can be studied. A procedure to verify the stress and deflection results experimentally should be developed.

Presently, the stiffness matrix calculation takes the longest time in the compliance or strength determination calculations. This is true even in finite element computations. Efforts to increase the computational efficiency of these models through speedier integration schemes and efficient matrix manipulations could be explored to enhance their use for design calculations.

Often, base rotations constitute a significant part of the overall tooth deflections. In straight bevel gears, a procedure to model the elastic support has been developed and its effect on the deflections and root stresses investigated. A similar procedure for spiral bevel gears could be developed to model the base rotations to investigate their influence on the flexural behavior.

In the loaded tooth contact analysis, the initial separations along the line of contact are required before loading the tooth. The contact conditions are very sensitive to the geometry of the contacting surfaces. In straight bevel gears, the lines of contact can be modeled as radial lines and the initial separations calculated from the lead and profile modifications. In spiral bevel gears, a similar procedure to determine the initial
separations needs to be developed. The calculations to determine the initial separations along the line of contact depend on the gear geometry and the configuration of the gear pair. The gear geometry description must be very accurate and would require a simulation of the cutting kinematics using the actual machine settings. The location of the pinion with respect to the gear also determines the length and location of the line of contact.
References


APPENDIX A

Gleason Loaded Tooth Contact Analysis

A.1 Introduction

This section explains the loaded tooth contact analysis procedure used by the Gleason Works to determine motion transmission error and contact patterns under load for bevel and hypoid gears. A detailed description of the LTCA program is given in Krenzer (1981). The analysis is commonly used to better understand the effects of design and development parameters on performance. The output of the program shows the contact patterns under load, the motion transmission errors and the maximum contact pressures. The details of LTCA and the calculation procedure are explained in the following sections.

A.2 Overview of the program

When the gears are loaded, the load is shared between all the contacting tooth pairs at that mesh position. An iterative procedure to determine the final load distribution over all the tooth pairs in contact for any given mesh position is used. Briefly, the load is assumed and the change in the relative position of the teeth, the contact deformations and tooth bending calculated. The load distribution is continually varied until the result of surface deflection, tooth bending and change in relative position indicates that the load is properly distributed among the contacting teeth. The basic calculation blocks shown in Fig. A.1 are explained below.
Fig. A.1 Program sequence in LTCA [Krenzer, 1981]
1) **Relative deflection between the mating members** - When the gears are loaded, the shafts as well as the bearings themselves deflect, which changes the relative position between the mating members and influences the final running quality of a gear set. The relative displacements are resolved into the pinion axial displacement, the gear axial displacement, the component perpendicular to the pinion and gear axis and a change in the shaft angle. The relative displacements are calculated and stored. When deflection data are not available, estimates of the relative displacements are made, based on the gear geometry and previously established databases.

2) **Edge TCA** - In TCA, the gear and pinion tooth surfaces are mathematically described by the machine settings and cutter specifications used to make the gears. After the initial point of contact is obtained, usually at the middle of the tooth, the other points which contact as the gears roll through mesh, are calculated. The tooth engagement is incremented through mesh so that contact transverses across the full working depth, from the tip inside of the blank of one member to the tip outside on the other. At each point of contact, TCA gives the gear rotational position, the gear displacement error, the pinion rotation, the location of the point of contact on the gear surface and the size and inclination of the contact ellipse. The data stored includes position vectors to the contact points, contact normals and surface curvatures.

3) **Contact deformations** - The relative curvatures are calculated from the information provided by TCA and contact ellipses and tooth deformation evaluated, using Hertz theory. Three different contact conditions are explored. When the contact ellipse is completely within the tooth boundary, Hertz theory is applied directly. When the contact extends to the tooth boundary with the center of pressure within the tooth boundary, it is assumed that the tooth surfaces continue beyond the boundaries and the assumed force changed, until the applied force within the tooth boundaries produces the
desired load. When the equivalent center of pressure exists beyond the tooth boundary, a condition known as the edge contact occurs and the effective centre of pressure is evaluated. From this the contact pressure and tooth deformation are determined.

4) **Tooth stiffness** - The tooth stiffnesses are calculated on basis of a cantilever beam formula developed by Westinghouse. The tooth deflection per unit load is given by

\[
\delta = \frac{L^3}{P} \left[ 1 + 1.3 \frac{t}{L} + \left\{ 0.25 + 0.75(1 - \gamma) \right\} \left( \frac{t}{L} \right)^2 + 0.35 \left( \frac{t}{L} \right)^3 \right]
\]

where

- \( I = \frac{Ft^2}{12} \) rectangular moment of inertia
- \( L \) tooth height at point of loading
- \( P \) load
- \( E \) modulus of elasticity
- \( t \) normal tooth thickness in plane containing load
- \( F \) facewidth projected into the normal plane
- \( \delta \) tooth deflection

A modification factor is used to calculate the deflections at the end of the teeth.

5) **Summation of deflections and mismatches** - At any time one or more teeth are in contact depending on the load and the mismatches on the tooth surfaces. Computations are carried out to determine the angular position of each contacting tooth considering initial mismatches as well as tooth bending and contact deformations.

**A.3 Program sequence**

A description of the program sequence is given below. From the flow chart Fig. A.1, the program input consists of the basic machine settings, cutter and blank data and load information. The machine settings can be adjusted such that the contact pattern
and motion errors of the gear set resemble the final lapped form. The machine settings, cutter and blank data together contain all the information necessary to describe the tooth geometry for the contact analysis. The relative deflections between the mating members are then calculated for the given load. All subsequent calculations for the loaded tooth contact analysis are run using the displaced position of the members. It is assumed that no coupling exists between the rigid body displacement of the members and the tooth bending and tooth surface deformations. In Edge TCA the contact points are calculated for each tenth of an angular pitch through an arc of contact to ensure that all points are considered in the analysis. The local effects of the load on the contact conditions are obtained in this segment.

The next three segments are included within a large iterative loop. The initial load sharing is assumed and using Hertz theory, the contact ellipse is calculated along with the magnitude and location of the equivalent point load. Using this point load, the tooth bending deflections are calculated using beam theory in the next segment. In the final segment the angular position of each tooth is calculated from the tooth deflection and surface deformation. The tooth loading is continually adjusted and the angular tooth displacements recalculated until all the contacting teeth have the same angular displacement. At the end of this procedure the parts are rotated to the next angular position. This iterative loop is repeated until all the angular positions of roll are calculated in one mesh cycle. Thus the motion errors for a given load are evaluated. When the load torque is varied the entire procedure is repeated which requires the calculation of a new TCA.

A.4 Program output

The outputs of the program are the contact plots, the motion error plots and the maximum pressure table. Fig. A.2 shows a typical loaded TCA plot. The contact is
plotted at successive positions as the tooth rolls through mesh. The instantaneous line of contact is the major axis of the contact ellipse. The envelope of all the contact lines is the contact pattern. Two plots are shown, one representing the no load condition and the other showing the effects of the load. The motion curves are similar to the TCA plots which display the motion errors and the angular separation of the adjacent teeth. The maximum pressures at successive positions of roll are contained in the maximum pressure table.

Fig. A.2 Output of Gleason LTCA [Krenzer, 1981]
APPENDIX B

Straight bevel configuration file

10 0., INPUT TORQUE, COEFF OF FRICTION

10 -4 0.0 , NO. OF TEETH, NO. OF TEETH MODELED, AXIAL PITCH
-0.32, INITIAL GEAR ROTATION
0 0 1
1 0 0
0 1 0 , THREE UNIT VECTORS TO LOCATE THE GEAR IN SPACE
0 0 0 , ORIGIN OF THE GEAR COORDINATE SYSTEM
1 1 1 1 1 1 , BOUNDARY CONDITION CODES FOR RIGID BODY MOTION
0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0 0 0
0 0 0 0 0 0 , 6X6 STIFFNESS MATRIX FOR SUPPORT STRUCTURE
pochtoid.dat

14 -4 0.0 , NO. OF TEETH, NO. OF TEETH MODELED, AXIAL PITCH
-1.7958110, INITIAL GEAR ROTATION
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0 1 0
0 0 1 , THREE UNIT VECTORS TO LOCATE THE GEAR IN SPACE
0 0 0 , ORIGIN OF THE GEAR COORDINATE SYSTEM
1 1 1 1 1 1 , BOUNDARY CONDITION CODES FOR RIGID BODY MOTION
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0 0 0 0 0 0
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0 0 0 0 0 0
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0 0 0 0 0 0
0 0 0 0 0 0 , 6X6 STIFFNESS MATRIX FOR SUPPORT STRUCTURE
gochtoid.dat
4
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2,1,3,1
3,1,2,1
4,1,1,1
0

172
APPENDIX C

Spiral bevel configuration file

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11400, NO. OF TEETH, NO. OF TEETH MODELED, AXIAL PITCH
-1.9155, INITIAL GEAR ROTATION
100
010
001, THREE UNIT VECTORS TO LOCATE THE GEAR IN SPACE
000, ORIGIN OF THE GEAR COORDINATE SYSTEM
111111, BOUNDARY CONDITION CODES FOR RIGID BODY MOTION
000000
000000
000000
000000
000000
000000
000000
000000
000000, 6X6 STIFFNESS MATRIX FOR SUPPORT STRUCTURE

pspbevl.dat

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100
010, THREE UNIT VECTORS TO LOCATE THE GEAR IN SPACE
000, ORIGIN OF THE GEAR COORDINATE SYSTEM
111111, BOUNDARY CONDITION CODES FOR RIGID BODY MOTION
000000
000000
000000
000000
000000
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000000
000000
000000, 6X6 STIFFNESS MATRIX FOR SUPPORT STRUCTURE

gspbevl.dat

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4,1,1,1
0

173