DIRECT KINEMATICS OF PARALLEL MECHANISMS

A Thesis

Presented in Partial Fulfillment of the Requirements for
the degree Master of Science in the
Graduate School of the Ohio State University

by
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*****

The Ohio State University
1988

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ACKNOWLEDGEMENTS

I would like to express my gratitude to Dr. K. J. Waldron for his guidance during the course of this work. This thesis would not have been possible without his valuable suggestions and help.

I would also like to thank the Graduate School for financial support during the first year of this work. Partial financial support was also provided by the National Science Foundation grant number MSM - 87037121.
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CHAPTER I

INTRODUCTION

INTRODUCTION

Kinematics is the science of motion which treats motion without regard to the forces that cause it. The study of kinematics of manipulators consists of the study of the position variable and the time derivatives of the position variables of all the members of the manipulator. A very basic problem in the study of mechanical manipulation is that of direct kinematics. This is the static geometrical problem of computing the position and orientation of the end effector of the manipulator given a set of input joint variables. As opposed to this is the problem of inverse kinematics. This problem is as follows: Given the position and the orientation of the end effector of the manipulator, calculate all the possible set of input joint variables which could be used to attain this given position and orientation.

The kinematic behavior of parallel mechanisms shows many inverse characteristics as compared to serial mechanisms (Waldron and Hunt, 1987). Of particular interest here is that, whereas in serial mechanisms the direct kinematic problem is straightforward and the inverse kinematic problem is challenging, the converse is true of parallel mechanisms. In fact, very few solutions to the direct kinematic problem for fully parallel robotic mechanisms can be found in the literature. One exception is a three limbed structure (Figure 1.1) which has been studied by Lee and Shah (1986) and in more detail by Waldron, Raghavan and Roth (in press). For convenience, this will be referred to here as the "triple arm mechanism".

1
Another mechanism whose kinematics has been extensively studied is the Stewart Platform. The Stewart Platform (Stewart 1965) shown in Figure 1.2 is a well known, fully parallel mechanism with six controlled degrees of freedom. It is widely used in aircraft simulators and similar applications. It has received considerable attention in the literature (Hunt, 1986; Fichter, 1986; Fichter, 1984; Sugimoto 1987). Nevertheless, no solution of the direct kinematics problem of this mechanism has been presented.

Due to the complexity of the problem of the direct kinematics of parallel mechanisms, previous researchers have proposed various numerical techniques to solve the equations (Merlet, 1988; Sugimoto, 1987). But there are some distinct advantages in having the solution in a closed form. Numerical techniques are computationally expensive and so real time control becomes difficult. Also, some important aspects of design like optimization and sensitivity analysis can be easily performed if the solution is available in a closed form.

Figure 1.1 Triple arm mechanism.
THESIS OVERVIEW

In this thesis, the direct kinematics of a specific type of Stewart Platform has been presented. These mechanisms have 6 controlled degrees of freedom. The general structure of these mechanisms can be described as follows. It consists of two members connected to each other by six limbs in a specific geometry. One of the members is fixed to the ground and will be referred to as the fixed member and the other will be referred to as the free member. Each limb is a six degree of freedom limb. The limb has a spherical joint at one end and a universal joint (or a spherical joint) at the other end. These joints are passive. Between the two ends of each limb there is an actively controlled sliding joint.

The positions of the joints that connects the six limbs to the two members are arranged in a specific manner. These positions can be arbitrary on one of the members. However, on the
other member, all limbs must be connected to the member at three locations only.

![Diagram of the Stewart Platform]

Figure 1.3 Special form of the Stewart Platform.

Under the above constraints, it can be shown that only two unique mechanisms are possible. Since there are no restriction on the location of joints on one of the members, we can attach the limbs at any arbitrary location. On the other member however, for six controlled degrees of freedom we must have at least one limb attached at each one of the three locations. Therefore, at most four limbs can be attached at each location. Thus the only possible combinations are the ones given in Table 1.1.
<table>
<thead>
<tr>
<th>No.</th>
<th>No. of limbs attached to first location</th>
<th>No. of limbs attached to second location</th>
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<td>1</td>
<td>4</td>
<td>1</td>
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Combination No. 1 has four limbs attached to the first location. Clearly these four limbs cannot be actuated independently and this mechanism will have a mobility less than 6. Thus combination No.2 and combination No.3 are the only two possible mechanisms.

The first mechanism is a special form of the Stewart Platform, namely that in which the joint centers of pairs of adjacent limbs of either the upper or the lower member are coincident. This mechanism is shown in Figure 1.3. One of the hexagonal members in Figure 1.2 is reduced to a triangle. A more specific case of the abovementioned mechanism is when the joint centers of adjacent limbs of both the upper and lower member are coincident (Figure 1.4). This mechanism can be viewed as a deformable octahedron.

The direct kinematic problem for the above type of mechanisms can be stated as follows: Given the lengths of the six variable limbs, find the transformation of coordinates representing the position of the upper member relative to the lower member. Consequently, following similar arguments to those of Waldron, Raghavan, and Roth (in press), it is possible to reduce these equations to a polynomial equation in a single variable of order 24.

The other type of mechanism shown in Figure 1.5, is obtained by connecting three limb ends at one of the locations (T), two limb ends at the second location (S) and one limb
end at the third location (R) on one of the members. This mechanism will be referred to as the "3-2-1 Mechanism". Again, we can have a special case by having the above configuration on both the members (Figure 1.6). The kinematic analysis of the above cases is described in chapter 3. It is shown that for a given set of limb lengths, this mechanism can be assembled in 8 different configurations.

Figure 1.4 Octahedral Stewart Platform.
Figure 1.5. 3-2-1 Mechanism.

Figure 1.6. Special form of 3-2-1 Mechanism.
This is a special note about the elimination method used in this thesis to reduce non-linear equations. This method is known as the Bezout's method and is used to eliminate \((n-1)\) variables from \(n\) separate equations. Consider the following case:

\[
\begin{align*}
    a_1x^2 + a_2x + a_3 &= 0 \\
    b_1x^2 + b_2x + b_3 &= 0
\end{align*}
\]  
\((1.1)\)

Multiply equations \((1.1)\) by \(x\). We get the following set of equations:

\[
\begin{align*}
    a_1x^3 + a_2x^2 + a_3x &= 0 \\
    a_1x^2 + a_2x + a_3 &= 0 \\
    b_1x^2 + b_2x + b_3 &= 0 \\
    b_1x^3 + b_2x^2 + b_3x &= 0
\end{align*}
\]

If we treat these five linear equations, homogeneous in \(x^3, x^2, x\) and 1, then the condition for their consistency by the theory of linear equations is:

\[
\begin{vmatrix}
    a_1 & a_2 & a_3 & 0 \\
    0 & a_1 & a_2 & a_3 \\
    0 & b_1 & b_2 & b_3 \\
    b_1 & b_2 & b_3 & 0
\end{vmatrix} = 0
\]

The above equation is free from \(x\) and is thus the resultant equation. The above determinant is known as the Sylester's eliminant.

The above determinant equation can be further reduced to the form:
\[ \begin{vmatrix} a_{12} & a_{13} \\ a_{23} & a_{23} \end{vmatrix} = 0 \]

where

\[ |a_{ij}| = a_{ij} - a_{ji} \]

The above determinant is known as the Bezout's eliminant. The above determinant equation has been used in this thesis to reduce non-linear equations.
CHAPTER II

SPECIAL STEWART PLATFORM

INTRODUCTION

This chapter deals with the direct position kinematics of the special Stewart Platform shown in Figure 1.3. The mechanism consists of two members connected to each other by six limbs. Each limb is a six degree of freedom limb and has either one spherical joints at one end and a universal joint at the other end or two spherical joints at its ends. In the first case, the mobility of the mechanism can be calculated by the Kutzbach's equation:

\[ F = 6(n - j - 1) + \sum_{i=1}^{j} f_i \]

where

n = number of members.

j = number of joints.

f_i = connectivity of ith joint.

For the above mechanism n = 14, j = 18, and \( \sum f_i = 18 + 12 + 6 \). Therefore, the mobility of the above mechanism will be 6.

In the second case, a redundant degree of freedom in the form of rotation about an axis along its length is introduced in each limb (\( \sum f_i = 18 + 18 + 6 \)). This increases the mobility of the mechanism to 12. The direct position kinematics remains unaffected by the above
variation in the type of limb joint and can be stated as follows: Given the lengths of the six variable limbs, find the transformation of coordinates representing the position of the upper member relative to the lower member.

DIRECT POSITION KINEMATICS

Consider the mechanism shown if Figure 1.3 without the moving platform RST for the moment. In the triangle $A_1R_A2$, $R$ must be located on a sphere with center at $A_1$ and radius equal to $L_1$. Also, $R$ must be located on a sphere with center at $A_2$ and radius $L_2$. Thus the locus of $R$ will be the intersection of the two spheres. This will be a circle with center located on the line joining the centers of the two spheres i.e. line $A_1A_2$ (Figure 2.1).

![Figure 2.1 Base member with the motion of the joints in the free member.](image)

Analysis similar to the above can be done for the points $S$ and $T$. Thus $S$ will move in a circle with center located on the line joining the joints $B_1$ and $B_2$ and $T$ will move in a circle
with center located on the line joining the joints C₁ and C₂. So the Stewart Platform can be represented by the equivalent mechanism in Figure 2.2.

The linkage A₁RA₂ can be replaced by a single link with a revolute joint at one end and a spherical joint at the other end. The revolute joint axis will lie on A₁A₂ (Figure 2.2). The link can be represented by the normal from R to A₁A₂. Thus the linkages A₁RA₂, C₁TC₂, and B₁SB₂ can be replaced by links O₁R, O₁T, and O₈S respectively.

Figure 2.2. Location and description of variables and vectors used in calculations.

Now if we also consider the moving platform, additional constraints on the movement of the joints R, S, and T will be introduced. These additional constraints are that the distances between the joints R and T, R and S, and S and T should be fixed. This leads to the
mechanism shown in Figure 2.3. Comparison with Figure 1.1 shows that it has the same 
kinematic structure as the three arm mechanism. Thus, the three arm mechanism of Figure 
2.3 is kinematically equivalent to the special Stewart Platform of Figure 1.3. However, the 
lengths \( r_1, s_1, t_1 \) determine the locations of \( O_r, O_s, O_t \), respectively on \( A_1A_2, B_1B_2, C_1C_2 \), 
are variable. These are the additional degrees of freedom.

A similar analysis of the octahedral Stewart Platform will lead to the mechanism shown in 
Figure 2.4, which is clearly a special case of the above general mechanism.
Figure 2.3. Base and the limbs with the moving member.

Figure 2.4. Description of variables and vectors for the octahedral Stewart Platform.
EQUATIONS FOR DIRECT POSITION KINEMATICS

It is necessary to express $r_1$, $s_1$, $t_1$, $m_r$, $m_s$ and $m_t$ in terms of the limb lengths $L_1, \ldots, L_6$.

These expressions are common for both the mechanisms being considered in this thesis.

These expressions are:

$$r_1 = \frac{a^2 + L_1^2 - L_2^2}{2a}$$

$$r_2 = a - r_1$$

$$t_1 = \frac{c^2 + L_4^2 - L_5^2}{2c}$$

$$t_2 = c - t_1$$

$$s_1 = \frac{b^2 + L_6^2 - L_5^2}{2b}$$

$$s_2 = b - s_1$$

$$m_r = \left( L_1^2 - r_1^2 \right)^{\frac{1}{3}}$$

$$m_s = \left( L_6^2 - s_1^2 \right)^{\frac{1}{3}}$$

$$m_t = \left( L_4^2 - t_1^2 \right)^{\frac{1}{3}}$$

Figure 2.5. Positions of $O_r$, $O_s$ and $O_t$ and the lengths $m_r$, $m_s$ and $m_t$.

Consider the general case shown in Figure 2.3 first. Let us place the fixed coordinate system in the lower member with the $xy$ plane in the plane of the lower member. Then from
Figure 2.6, the position vectors of $O_t$, $O_s$, $O_u$ are given by the equations:

$$\mathbf{P}_{O_t} = \mathbf{P}_{A_t} + r_t \frac{(\mathbf{P}_{A_2} - \mathbf{P}_{A_1})}{\|\mathbf{P}_{A_2} - \mathbf{P}_{A_1}\|}$$

$$\mathbf{P}_{O_s} = \mathbf{P}_{B_t} + s_t \frac{(\mathbf{P}_{B_2} - \mathbf{P}_{B_1})}{\|\mathbf{P}_{B_2} - \mathbf{P}_{B_1}\|}$$

$$\mathbf{P}_{O_u} = \mathbf{P}_{C_t} + t_t \frac{(\mathbf{P}_{C_2} - \mathbf{P}_{C_1})}{\|\mathbf{P}_{C_2} - \mathbf{P}_{C_1}\|}$$

Also, the angles made by the lines perpendicular to $A_1A_2$, $B_1B_2$ and $C_1C_2$ with the $x$ axis are given by the equations:

$$\beta_r = \cos^{-1} \left[ \frac{\{(\mathbf{P}_{A_1} - \mathbf{P}_{A_2}) \times \mathbf{k}\} \cdot \mathbf{i}}{\|\mathbf{P}_{A_1} - \mathbf{P}_{A_2}\| \times \mathbf{k}} \right]; \quad \beta_t = \sin^{-1} \left[ \frac{\{(\mathbf{P}_{A_1} - \mathbf{P}_{A_2}) \times \mathbf{k}\} \cdot \mathbf{i}}{\|\mathbf{P}_{A_1} - \mathbf{P}_{A_2}\| \times \mathbf{k}} \right]$$
\[
\beta_s = \cos^{-1} \left[ \frac{\{(\mathbf{p}_{B_1} - \mathbf{p}_{B_2}) \times \mathbf{k}\} \cdot \mathbf{i}}{\mathbf{(p}_{B_1} - \mathbf{p}_{B_2}) \times \mathbf{k}} \right]
\]
\[
\beta_i = \cos^{-1} \left[ \frac{\{(\mathbf{p}_{c_1} - \mathbf{p}_{c_2}) \times \mathbf{k}\} \cdot \mathbf{i}}{\mathbf{(p}_{c_1} - \mathbf{p}_{c_2}) \times \mathbf{k}} \right]
\]
\[
\beta_s = \sin^{-1} \left[ \frac{\{(\mathbf{p}_{B_1} - \mathbf{p}_{B_2}) \times \mathbf{k}\} \cdot \mathbf{j}}{\mathbf{(p}_{B_1} - \mathbf{p}_{B_2}) \times \mathbf{k}} \right]
\]
\[
\beta_i = \sin^{-1} \left[ \frac{\{(\mathbf{p}_{c_1} - \mathbf{p}_{c_2}) \times \mathbf{k}\} \cdot \mathbf{j}}{\mathbf{(p}_{c_1} - \mathbf{p}_{c_2}) \times \mathbf{k}} \right]
\]

Now the above quantities can be more specifically calculated for the octahedral Stewart Platform. The dimensions of the fixed member are shown in Figure 11. Let us fix the location of the fixed coordinate system relative to the base triangle as shown in the Figure 2.8. The origin of the coordinate system is located at \( A_2 \) and the \( x \)-axis is directed along the line \( A_1A_2 \). The locations of the joints \( O_r, O_s, \) and \( O_t \) in this coordinate system are given as follows:

![Figure 2.7. Base triangle for the Octahedral Stewart Platform.](image)

![Figure 2.8. Descriptions of the angles \( \theta_r, \theta_s \) and \( \theta_t \) (image)]
\((O_sB) = s_1, \quad (O_rB) = r_2\)

\((O_1B) = \left( t_2^2 + a^2 - 2t_2a\cos{\alpha_1} \right)^{\frac{1}{2}}\)

\(\theta_s = 180 - \alpha_2\)

\(\theta_1 = 180 - \alpha_2 + \cos^{-1}\left[ \frac{b^2 + (O_1B)^2 - t_1^2}{2b(O_1B)} \right]\)

Thus the position vectors \(\mathbf{P}_{O_1}, \mathbf{P}_{O_s}, \text{ and } \mathbf{P}_{O_t}\) of joints \(O_r, O_s, \text{ and } O_t\) respectively are

\[\mathbf{P}_{O_r} = -r_2 \mathbf{i}\]

\[\mathbf{P}_{O_s} = s_1 \cos{\theta_s} \mathbf{i} + s_1 \sin{\theta_s} \mathbf{i}\]

\[\mathbf{P}_{O_t} = (O_tB) \cos{\theta_1} \mathbf{i} + (O_tB) \sin{\theta_1} \mathbf{i}\]

Angles \(\beta_s\) and \(\beta_t\) are given by the following equations

Figure 2.9. Description of the angles \(\beta_r, \beta_s\) and \(\beta_t\).
\[ \beta_s = 270 - \alpha_2 \]
\[ \beta_t = 270 + \alpha_2 \]

The rest of the calculations are common for both the mechanisms. The unit vectors \( \mathbf{w}_r \), \( \mathbf{w}_s \), and \( \mathbf{w}_t \) along the links \( O_1R \), \( O_3S \), and \( O_4T \) respectively are given by

\[ \mathbf{w}_r = \cos \beta_r \cos \phi_r \mathbf{i} + \sin \beta_r \cos \phi_r \mathbf{j} + \sin \phi_r \mathbf{k} \]
\[ \mathbf{w}_s = \cos \beta_s \cos \phi_s \mathbf{i} + \sin \beta_s \cos \phi_s \mathbf{j} + \sin \phi_s \mathbf{k} \]
\[ \mathbf{w}_t = \cos \beta_t \cos \phi_t \mathbf{i} + \sin \beta_t \cos \phi_t \mathbf{j} + \sin \phi_t \mathbf{k} \]

Figure 2.10. Description of the vectors \( \mathbf{w}_r \), \( \mathbf{w}_s \) and \( \mathbf{w}_t \).

For the special case, these vectors simplify to:

\[ \mathbf{w}_r = \cos \phi_r \mathbf{i} + \sin \phi_r \mathbf{k} \]
\[ \mathbf{w}_s = \cos \beta_s \cos \phi_s \mathbf{i} + \sin \beta_s \cos \phi_s \mathbf{j} + \sin \phi_s \mathbf{k} \]
\[ \mathbf{w}_t = \cos \beta_t \cos \phi_t \mathbf{i} + \sin \beta_t \cos \phi_t \mathbf{j} + \sin \phi_t \mathbf{k} \]
Now the position vectors of R, S, and T are

\[ \mathbf{P}_r = \mathbf{P}_{Or} + m_r \mathbf{w}_r \]
\[ \mathbf{P}_s = \mathbf{P}_{Os} + m_s \mathbf{w}_s \]
\[ \mathbf{P}_t = \mathbf{P}_{Ot} + m_t \mathbf{w}_t \]  \hspace{1cm} (2.2)

Since the distance between the joints R, S, and T is fixed, the positions vectors of R, S, and T must satisfy the following equations.

\[ | \mathbf{P}_r - \mathbf{P}_s | = b_1^2 \]  \hspace{1cm} (2.3)
\[ | \mathbf{P}_s - \mathbf{P}_t | = b_2^2 \]  \hspace{1cm} (2.4)
\[ | \mathbf{P}_t - \mathbf{P}_r | = b_3^2 \]  \hspace{1cm} (2.5)

where

\[ b_1 = \text{length of RS}. \]
\[ b_2 = \text{length of ST}. \]
\[ b_3 = \text{length of RT}. \]

**SOLUTION OF DIRECT POSITION KINEMATICS**

Substitution of the expressions for \( \mathbf{P}_r \) and \( \mathbf{P}_s \) from equation (2.2) into equation (2.3) gives
\[
\left\{ \begin{array}{l}
\left[ \left( \mathbf{P}_{o_{,y}} - \mathbf{P}_{o_{,y}} \right) \right] + m_{r} \cos \beta_{r} \cos \phi_{r} - m_{s} \cos \beta_{s} \cos \phi_{s} \\
+ \left( \mathbf{P}_{o_{,y}} - \mathbf{P}_{o_{,y}} \right) + m_{r} \sin \beta_{r} \cos \phi_{r} - m_{s} \sin \beta_{s} \cos \phi_{s} \\
+ \left( m_{r} \sin \phi_{r} - m_{s} \sin \phi_{s} \right)^2
\end{array} \right\} = b_{1}^2
\]

Upon simplification we get the following equation:

\[D_{1} \cos \phi_{r} + D_{2} \cos \phi_{s} + D_{3} \cos \phi_{r} \cos \phi_{s} + D_{4} \sin \phi_{r} \sin \phi_{s} + D_{5} = 0 \quad (2.6)\]

where

\[D_{1} = 2m_{r} \cos \beta_{r} \left\{ \left( \mathbf{P}_{o_{,y}} - \mathbf{P}_{o_{,y}} \right) \right\} + 2m_{s} \sin \beta_{r} \left\{ \left( \mathbf{P}_{o_{,y}} - \mathbf{P}_{o_{,y}} \right) \right\}
\]

\[D_{2} = -2m_{r} \cos \beta_{r} \left\{ \left( \mathbf{P}_{o_{,y}} - \mathbf{P}_{o_{,y}} \right) \right\} - 2m_{s} \sin \beta_{r} \left\{ \left( \mathbf{P}_{o_{,y}} - \mathbf{P}_{o_{,y}} \right) \right\}
\]

\[D_{3} = -2m_{r} m_{s} \cos (\beta_{r} - \beta_{s})
\]

\[D_{4} = -2m_{r} m_{s}
\]

\[D_{5} = \left\{ \left( \mathbf{P}_{o_{,y}} - \mathbf{P}_{o_{,y}} \right) \right\}^2 + \left\{ \left( \mathbf{P}_{o_{,y}} - \mathbf{P}_{o_{,y}} \right) \right\}^2 + m_{r}^2 + m_{s}^2 - b_{1}^2
\]

For the special case, these coefficients are:

\[D_{1} = -2s_{1} m_{r} \sin \theta_{s}
\]

\[D_{2} = 2r_{2} m_{s} \cos \beta_{s} + 2s_{1} m_{s} \cos (\theta_{s} - \beta_{s})
\]

\[D_{3} = -2m_{r} m_{s} \sin \beta_{s}
\]

\[D_{4} = -2m_{r} m_{s}
\]

\[D_{5} = 2s_{1} r_{2} \cos \theta_{s} + r_{2}^2 + s_{1}^2 + m_{r}^2 + m_{s}^2 - b_{1}^2
\]

Similarly, substitution from equation (2.2) into equations (2.4) and (2.5) will yield the following equations:
\[ E_1 \cos \phi_t + E_2 \cos \phi_s + E_3 \cos \phi_t \cos \phi_s + E_4 \sin \phi_t \sin \phi_s + E_5 = 0 \] (2.7)

where
\[ E_1 = 2m_s \cos \beta_s \left\{ \left( P_{o,y} \right)_x - \left( P_{o,x} \right)_y \right\} + 2m_s \sin \beta_s \left\{ \left( P_{o,y} \right)_y - \left( P_{o,x} \right)_y \right\} \]
\[ E_2 = -2m_i \cos \beta_i \left\{ \left( P_{o,y} \right)_x - \left( P_{o,x} \right)_y \right\} - 2m_i \sin \beta_i \left\{ \left( P_{o,y} \right)_y - \left( P_{o,x} \right)_y \right\} \]
\[ E_3 = -2m_s m_i \cos (\beta_s - \beta_i) \]
\[ E_4 = -2m_i m_i \]
\[ E_5 = \left\{ \left( P_{o,y} \right)_x - \left( P_{o,x} \right)_y \right\}^2 + \left\{ \left( P_{o,y} \right)_y - \left( P_{o,x} \right)_y \right\}^2 + m_i^2 + m_s^2 - b_2^2 \]

For the special case, these coefficients are:

\[ E_1 = -2s_t m_i \cos (\theta_s - \beta_s) + 2m_s (O,B) \cos (\beta_i - \theta_i) \]
\[ E_2 = 2s_i m_s \cos (\theta_s - \beta_s) - 2m_s (O,B) \cos (\beta_s - \theta_i) \]
\[ E_3 = -2m_s m_i \cos (\beta_s - \beta_i) \]
\[ E_4 = -2m_i m_i \]
\[ E_5 = s_t^2 + (O,B)^2 + m_s^2 + m_i^2 - b_2^2 - 2s_t (O,B) \cos (\theta_s - \theta_i) \]

\[ F_1 \cos \phi_t + F_2 \cos \phi_s + F_3 \cos \phi_t \cos \phi_s + F_4 \sin \phi_t \sin \phi_s + F_5 = 0 \] (2.8)

where
\[ F_1 = 2m_s \cos \beta_s \left\{ \left( P_{o,y} \right)_x - \left( P_{o,x} \right)_y \right\} + 2m_s \sin \beta_s \left\{ \left( P_{o,y} \right)_y - \left( P_{o,x} \right)_y \right\} \]
\[ F_2 = -2m_i \cos \beta_i \left\{ \left( P_{o,y} \right)_x - \left( P_{o,x} \right)_y \right\} - 2m_i \sin \beta_i \left\{ \left( P_{o,y} \right)_y - \left( P_{o,x} \right)_y \right\} \]
\[ F_3 = -2m_s m_i \cos (\beta_s - \beta_i) \]
\[ F_4 = -2m_i m_i \]
\[ F_5 = \left\{ \left( P_{o,y} \right)_x - \left( P_{o,x} \right)_y \right\}^2 + \left\{ \left( P_{o,y} \right)_y - \left( P_{o,x} \right)_y \right\}^2 + m_i^2 + m_s^2 - b_3^2 \]

For the special case, these coefficients are:
\[ F_1 = -2m_i (O_i B) \sin \theta_i \]
\[ F_2 = 2m_i (O_i B) \cos (\theta_i - \beta_i) + 2r_2 m_i \cos \beta_i \]
\[ F_3 = -2m_i m_i \sin \beta_i \]
\[ F_4 = -2m_i m_i \]
\[ F_5 = r_2^2 + (O_i B)^2 + m_i^2 + b_3^2 + 2r_2 (O_i B) \cos \theta_i \]

The method used to solve the equations from here onwards follows similar lines to the one presented in Waldron et al. (in press). Equations (2.6) and (2.7) can be solved simultaneously for \( \cos \phi_s \) and \( \sin \phi_s \) in terms of \( \cos \phi_t \), \( \sin \phi_t \), \( \cos \phi_t \) and \( \sin \phi_t \). The result is:

\[
\cos \phi_s = \frac{(D_5 + D_1 \cos \phi_t) E_4 \sin \phi_t - (E_5 + E_2 \cos \phi_t) D_4 \sin \phi_t}{(E_1 + E_3 \cos \phi_t) D_4 \sin \phi_t - (D_1 + D_3 \cos \phi_t) E_4 \sin \phi_t}
\]
\[
\sin \phi_s = \frac{(D_5 + D_1 \cos \phi_t)(E_1 + E_3 \cos \phi_t) - (E_5 + E_2 \cos \phi_t)(D_1 + D_3 \cos \phi_t)}{(D_1 + D_3 \cos \phi_t) E_4 \sin \phi_t - (E_1 + E_3 \cos \phi_t) D_4 \sin \phi_t}
\]  

(2.9)

Substitution of the expressions for \( \cos \phi_s \) and \( \sin \phi_s \) in the identity

\[ \cos^2 \phi_s + \sin^2 \phi_s = 1 \]

gives, upon simplification:

\[ G_1 \cos^2 \phi_t + G_2 \sin^2 \phi_t + G_3 \sin \phi_t \cos \phi_t + G_4 \cos \phi_t + G_5 \sin \phi_t + G_6 = 0 \]  

(2.10)

where

\[ G_1 = (E_2 D_4 \sin \phi_t)^2 + \{(D_5 + D_1 \cos \phi_t) E_3\}^2 + \{(D_1 + D_3 \cos \phi_t) E_2\}^2 - 2E_2 E_3 (D_5 + D_1 \cos \phi_t)(D_1 + D_3 \cos \phi_t) - (E_3 D_4 \sin \phi_t)^2 \]
\[ G_2 = \{ (D_5 + D_1 \cos \phi_r) E_4 \}^2 - \{ (D_1 + D_3 \cos \phi_r) E_4 \}^2 \]
\[ G_3 = -2 E_2 E_4 D_4 (D_5 + D_1 \cos \phi_r) \sin \phi_r + 2 E_3 E_4 D_4 (D_1 + D_3 \cos \phi_r) \sin \phi_r \]
\[ G_4 = 2 E_2 E_5 (D_4 \sin \phi_r)^2 + 2 E_1 E_3 (D_5 + D_1 \cos \phi_r)^2 + 2 E_2 E_5 (D_1 + D_3 \cos \phi_r)^2 - 2 (D_1 + D_3 \cos \phi_r)(D_5 + D_1 \cos \phi_r)(E_3 E_5 + E_2 E_1) \]
\[ - 2 E_1 E_3 (D_4 \sin \phi_r)^2 \]
\[ G_5 = -2 (D_5 + D_1 \cos \phi_r) E_5 E_4 D_4 \sin \phi_r + 2 (D_1 + D_3 \cos \phi_r) E_1 E_4 \]
\[ D_4 \sin \phi_r \]
\[ G_6 = (E_5 D_4 \sin \phi_r)^2 + \{ E_1 (D_5 + D_1 \cos \phi_r) \}^2 + \{ E_5 (D_1 + D_3 \cos \phi_r) \}^2 \]
\[ + \{ E_5 (D_1 + D_3 \cos \phi_r) \}^2 - 2 E_1 E_5 (D_5 + D_1 \cos \phi_r)(D_1 + D_3 \cos \phi_r) \]
\[ - \{ E_1 D_4 \sin \phi_r \}^2 \]

From equation (2.8)
\[ \cos \phi_i = \frac{-(F_5 + F_4 \sin \phi_i \sin \phi_i + F_1 \cos \phi_i)}{F_2 + F_3 \cos \phi_i} \]  
(2.11)

Substitution of the above expression in equation (2.10) and simplification gives:
\[ K_1 \sin^2 \phi_i + K_2 \sin \phi_i + K_3 = 0 \]  
(2.12)

where
\[ K_1 = -G_1 (F_4 \sin \phi_r)^2 + G_2 (F_2 + F_3 \cos \phi_r)^2 - G_3 (F_2 + F_3 \cos \phi_r) \]
\[ F_4 \sin \phi_r \]
\[ K_2 = -2 G_1 (F_5 + F_1 \cos \phi_r) F_4 \sin \phi_r - G_3 (F_5 + F_1 \cos \phi_r) \]
\[(F_2 + F_3 \cos \phi_r) - G_4 (F_2 + F_3 \cos \phi_r) F_4 \sin \phi_r\]
\[+ G_5 (F_2 + F_3 \cos \phi_r)^2\]
\[K_3 = -G_1 (F_5 + F_1 \cos \phi_r)^2 - G_4 (F_2 + F_3 \cos \phi_r) (F_5 + F_1 \cos \phi_r)\]
\[+ G_6 (F_2 + F_3 \cos \phi_r)^2\]

Also, substitution of the expression for \(\cos \phi_t\) in the identity

\[\cos^2 \phi_t + \sin^2 \phi_t = 1\]

gives:

\[L_1 \sin^2 \phi_t + L_2 \sin \phi_t + L_3 = 0\]  \hspace{1cm} (2.13)

where

\[L_1 = (F_4 \sin \phi_r)^2 + (F_2 + F_3 \cos \phi_r)^2\]
\[L_2 = 2 (F_1 \cos \phi_r + F_5) F_4 \sin \phi_r\]
\[L_3 = -(F_2 + F_3 \cos \phi_r)^2 + (F_1 \cos \phi_r + F_5)^2\]

Equations (2.12) and (2.13) can be solved simultaneously for \(\sin \phi_t\). Using Bezout's method, we get the following equation after the elimination of \(\sin \phi_t\).

\[
\begin{vmatrix}
K_1 L_2 & K_1 L_3 \\
K_2 L_3 & K_2 L_3 \\
\end{vmatrix} = 0
\]  \hspace{1cm} (2.14)

where

\[|K_i L_j| = K_i L_j - K_j L_i\]
L₁, L₂, L₃, K₁, K₂ and K₃ can be expressed in terms of \( \tan(\phi_r/2) \) as follows:

\[
K₁ = \left( \frac{K_{11} \tan^4 \left( \frac{\phi_r}{2} \right) + K_{12} \tan^3 \left( \frac{\phi_r}{2} \right) + K_{13} \tan^2 \left( \frac{\phi_r}{2} \right) + K_{14} \tan \left( \frac{\phi_r}{2} \right) + K_{15}}{S} \right)
\]

\[
K₂ = \left( \frac{K_{21} \tan^4 \left( \frac{\phi_r}{2} \right) + K_{22} \tan^3 \left( \frac{\phi_r}{2} \right) + K_{23} \tan^2 \left( \frac{\phi_r}{2} \right) + K_{24} \tan \left( \frac{\phi_r}{2} \right) + K_{25}}{S} \right)
\]

\[
K₃ = \left( \frac{K_{31} \tan^4 \left( \frac{\phi_r}{2} \right) + K_{32} \tan^2 \left( \frac{\phi_r}{2} \right) + K_{33}}{S} \right)
\]

where

\[
S = \left( 1 + \tan^2 \left( \frac{\phi_r}{2} \right) \right)^2
\]

\[
K_{11} = G₂(F₃ - F₂)^2
\]

\[
K_{12} = 2G₃F₄(F₃ - F₂)
\]

\[
K_{13} = 2G₃(F²₂ - F²₃) - 4G₁F₄²
\]

\[
K_{14} = -2G₃F₄(F₃ + F₂)
\]

\[
K_{15} = G₂(F₃ + F₂)^2
\]

\[
K_{21} = G₃(F₃ - F₂)(F₅ - F₁)
\]

\[
K_{22} = 4F₄(F₁ - F₅)
\]

\[
K_{23} = 2G₃(F₁F₃ - F₂F₅)
\]

\[
K_{24} = -4G₃F₄(F₅ + F₁)
\]

\[
K_{25} = -G₃(F₃ + F₂)(F₅ + F₁)
\]

\[
K_{31} = G₆(F₃ - F₂)^2 + G₄(F₃ - F₂)(F₅ - F₁) - G₁(F₁ - F₅)^2
\]

\[
K_{32} = 2G₆(F²₂ - F³₃) + 2G₄(F₁F₃ - F₂F₅) + 2G₁(F¹₅ - F²₅)
\]

\[
K_{33} = G₆(F₃ + F₂)^2 - G₄(F₃ + F₂)(F₅ + F₁) - G₁(F₅ + F₁)^2
\]

G₁, ..., G₆ are also functions of \( \tan(\phi_r/2) \) and are given by the equations:
\[
G_1 = \frac{G_{11} \tan \left( \frac{\phi}{2} \right) + G_{12} \tan^2 \left( \frac{\phi}{2} \right) + G_{13}}{S}
\]
\[
G_2 = \frac{G_{21} \tan \left( \frac{\phi}{2} \right) + G_{22} \tan^2 \left( \frac{\phi}{2} \right) + G_{23}}{S}
\]
\[
G_3 = \frac{G_{31} \tan^3 \left( \frac{\phi}{2} \right) + G_{32} \tan \left( \frac{\phi}{2} \right)}{S}
\]
\[
G_4 = \frac{G_{41} \tan^4 \left( \frac{\phi}{2} \right) + G_{42} \tan^2 \left( \frac{\phi}{2} \right) + G_{43}}{S}
\]
\[
G_5 = \frac{G_{51} \tan^3 \left( \frac{\phi}{2} \right) + G_{52} \tan \left( \frac{\phi}{2} \right)}{S}
\]
\[
G_6 = \frac{G_{61} \tan^4 \left( \frac{\phi}{2} \right) + G_{62} \tan^2 \left( \frac{\phi}{2} \right) + G_{63}}{S}
\]

where

\[
G_{11} = (D_5 - D_1)^2 E_5^2 + 2(D_5 - D_1)(D_3 - D_1)E_5 E_3 + (D_3 - D_1)^2 E_2^2
\]
\[
G_{12} = 2(D_5 - D_1)^2 E_3^2 + 4D_1(D_3 - D_5)E_2 E_3 + 2(2D_4^2 - D_3^2 + D_1^2)E_2^2
\]
\[
G_{13} = (D_5 + D_1)^2 E_3^2 - 2(D_3 + D_1)(D_5 + D_1)E_2 E_3 + (D_3 + D_1)^2 E_2^2
\]
\[
G_{21} = (D_5^2 - 2D_1 D_5 - D_3^2 + 2D_1 D_3)E_4^2
\]
\[
G_{22} = 2(D_5^2 + D_3^2 - 2D_1^2)E_4^2
\]
\[
G_{23} = D_5(D_5 + 2D_1) - D_3(D_3 + 2D_1)
\]
\[
G_{31} = - 4\{ (D_3 - D_1) E_3 + (D_5 - D_1) E_2 \}D_4 E_4
\]
\[
G_{32} = 4\{ - (D_3 + D_1) E_3 + (D_5 + D_1) E_2 \}D_4 E_4
\]
\[
G_{41} = 2(D_3^2 - D_1^2) E_2 E_5^2 + 2(D_5 - D_1)^2 E_1 E_4 + 2(D_5 - D_1)(D_3 - D_1)(E_2 E_5 - E_1 E_4)
\]
\[
G_{42} = 4(2D_4^2 - D_3^2 + D_1^2) E_2 E_5 + 4(D_5^2 - D_1^2 - 2D_4^2) E_1 E_3 + 4D_1(D_3 - D_5)(E_2 E_5 + E_1 E_3)
\]
\[
G_{43} = 2(D_3 + D_1)^2 E_2 E_5 + 2(D_5 + D_1)^2 E_1 E_3 - 2(D_3 + D_1)(D_5 + D_1)(E_2 E_5 + E_1 E_3)
\]
\[
G_{51} = - \{ (D_5 - D_1) E_5 + (D_3 - D_1) E_1 \}D_4 E_4
\]
\[ G_{52} = 4\{(D_5 + D_1)E_5 - (D_3 + D_1)E_1\}D_4E_4 \]

\[ G_{61} = 2(D_3 - D_1)E_1^2 + (D_5 - D_1)E_1^2 + 2(D_3 - D_1)(D_5 - D_1)E_1E_5 \]

\[ G_{62} = 4(D_4^2 - D_3^2 + D_1^2)E_5^2 + 2(D_5^2 - D_1^2 - 2D_4^2)E_1^2 + 4(D_3 - D_5)D_4E_1E_5 \]

\[ G_{63} = 2(D_3 + D_1)^2E_5^2 + (D_5 + D_1)^2E_1^2 - 2(D_3 + D_1)(D_5 + D_1)E_1E_5 \]

\[ L_1 = \frac{\left( L_{11}\tan^2\left( \frac{\Phi_r}{2} \right) + L_{12}\tan^2\left( \frac{\Phi_r}{2} \right) + L_{13} \right)}{S} \]

\[ L_2 = \frac{\left( L_{21}\tan^3\left( \frac{\Phi_r}{2} \right) + L_{22}\tan\left( \frac{\Phi_r}{2} \right) \right)}{S} \]

\[ L_3 = \frac{\left( L_{31}\tan\left( \frac{\Phi_r}{2} \right) + L_{32}\tan^2\left( \frac{\Phi_r}{2} \right) + L_{33} \right)}{S} \]

where

\[ L_{11} = (F_3 - F_4)^2 \]

\[ L_{12} = 2\left( 2F_4^2 - F_3^2 + F_1^2 \right) \]

\[ L_{13} = (F_3 + F_4)^2 \]

\[ L_{21} = 4F_4(F_5 - F_1) \]

\[ L_{22} = 4F_4(F_5 + F_1) \]

\[ L_{31} = (F_5 - F_4)^2 - (F_3 - F_2)^2 \]

\[ L_{32} = 2\left( F_5^2 + F_3^2 - F_2^2 - F_1^2 \right) \]

\[ L_{33} = (F_5 + F_1)^2 - (F_3 + F_2)^2 \]

L_1, L_2, and L_3 are fourth order polynomials in \( \tan(\Phi_r/2) \) and \( K_1, K_2, \) and \( K_3 \) are eighth order polynomials in \( \tan(\Phi_r/2) \). Thus the maximum order of \( \tan(\Phi_r/2) \) in the above equation is 24 and in the general case, there will 24 different solutions to the above equation. Since the value of \( \Phi_r \) is uniquely determined in the interval \( 0 \leq \Phi_r \leq 360 \) by \( \tan(\Phi_r/2) \) this permits unique solutions for \( \Phi_r, \Phi_3 \) and \( \Phi_1 \) can now be computed from \( \Phi_r \) using equations (2.7) and (2.8) respectively. Each of these equations has the form:
\[ P \cos \phi + Q \sin \phi + R = 0 \]

yielding two solutions of the form:

\[
\cos \phi = \frac{-PR + \sigma Q(P^2 + Q^2 - R^2)}{P^2 + Q^2}^{\frac{1}{2}}
\]

\[
\sin \phi = \frac{-QR - \sigma P(P^2 + Q^2 - R^2)}{P^2 + Q^2}^{\frac{1}{2}}
\]

where \(\sigma = \pm 1\). However, substitution of the solutions for \(\phi_s\) and \(\phi_t\) into equation (2.6) will resolve the sign ambiguity since only one of the four possible combinations of solutions will satisfy the equation. This has been proved later in the next section using a numerical example.

The positions of points R, S and T relative to the fixed frame are now given by equations (2.2) with substitution for \(w_1\), \(w_s\) and \(w_t\) from equation (2.1). This is sufficient to enable the transformation from moving to the fixed reference frame to be formulated since

\[
\frac{\vec{P}_R - \vec{P}_S}{|\vec{P}_R - \vec{P}_S|}, \quad \frac{\vec{P}_R - \vec{P}_S \times (\vec{P}_S - \vec{P}_T)}{|(\vec{P}_R - \vec{P}_S) \times (\vec{P}_S - \vec{P}_T)|}, \quad \frac{\vec{P}_R - \vec{P}_S \times (\vec{P}_S - \vec{P}_T) \times (\vec{P}_S - \vec{P}_T)}{|(\vec{P}_R - \vec{P}_S) \times (\vec{P}_S - \vec{P}_T)|} \]

form a set of orthogonal basis vectors whose directions are known relative to both the fixed and moving reference frames.

NUMERICAL EXAMPLE
Let us start with a known solution for the above problem. Let the coordinates of the members of the mechanism shown in Figure 1.4 in the coordinate system shown in Figure 1.8 be:

\[ A = (-3,0,0), \quad B = (0,0,0), \quad C = (-2,3,0) \]
\[ R = (-1.5,0.5,4), \quad S = (-0.5,0.5,3), \quad T = (-2,1.5,2) \]

With these coordinates, the dimensions of the members ABC and RST will be:

\[ a = 3.0, \quad b = 3.6056, \quad c = 3.1623 \]
\[ \alpha_1 = 71.565^\circ, \quad \alpha_2 = 56.31^\circ, \quad \alpha_3 = 52.125^\circ \]
\[ b_1 = 1.4142, \quad b_2 = 2.0616, \quad b_3 = 2.2913 \]

Using the above values in the equations, the coordinates of \( O_r, \ O_s \) and \( O_t \) will be:

\[ O_r = (-1.5,0,0), \quad O_s = (-0.3846,0.5769,0), \quad O_t = (-2.45,1.65,0) \]

Now the angles \( \phi_r, \ \phi_s \) and \( \phi_t \) will be:

\[ \phi_r = 82.875^\circ, \quad \phi_s = 87.35^\circ, \quad \phi_t = 76.657^\circ \]

Starting with the known value of \( \phi_r \), substitute it in the equations 2.6 and 2.8 to find the values of \( \phi_s \) and \( \phi_t \). These values are:

\[ \phi_s = 87.358^\circ \text{ and } \phi_t = 65.361^\circ \]
\[ \phi_s = 76.653^\circ \text{ and } \phi_t = 68.252^\circ \]

Substitute these values in equation 2.7. The results are:

With \( \phi_s = 87.358^\circ \) and \( \phi_t = 76.653^\circ \),
Error = -4.35 E-04

With \( \phi_s = 87.358^\circ \) and \( \phi_t = 68.252^\circ \),
Error = 0.7268

With \( \phi_s = 65.361^\circ \) and \( \phi_t = 76.653^\circ \),
Error = 0.7723.

With \( \phi_s = 65.361^\circ \) and \( \phi_t = 68.252^\circ \),
Error = 1.1552

The above error values clearly indicate that only one set of values of \( \phi_s \) and \( \phi_t \) will satisfy all the three equations.

**DISCUSSION**

In this chapter, the direct position kinematics of the special Stewart Platform has been presented. It has been shown that the direct kinematic equations can be reduced to a single polynomial equation in one variable of order 24. This suggests that there are 24 different solutions to this problem.

The algebra in this chapter was verified using MACSYMA. Further numerical computations were performed to check the algebra and the results have been presented in the previous section.
CHAPTER III

3-2-1 MECHANISM

INTRODUCTION

Figure 3.1. 3-2-1 Mechanism.

This chapter deals with the analysis of the mechanism shown in Figure 3.1. This mechanism consists of a fixed member and a six degree of freedom moving member connected to each other by six limbs. On the fixed member the location of the limb ends
\((A_1, A_2, B_1, B_2, C_1, C_2)\) is arbitrary. On the moving member, three limb ends are connected at one of the locations \((T)\), two limb ends at the second location \((S)\) and one limb end at the third location \((R)\). Again, we can have a special case by having the above configuration on both the members. This configuration is shown in Figure 3.2.

Figure 3.2. Special form of 3-2-1 Mechanism.

It has been shown by Waldron and Hunt (1987) that the limbs of the Stewart Platform viewed as the force generators are, in fact, duals of the revolute joints of the serial chain under the motor wrench symmetry. Using this duality the kinematic analysis of the 3-2-1 Mechanism should have some similarity with the kinematic analysis of an equivalent serial chain.

The equivalent serial chain can be constructed as follows: the three force generators at \(T\) are
duals of a three degree of freedom revolute wrist joint, the single force generator at R is
dual of a single revolute elbow joint and the two force generators at S are duals of a two
degree of freedom revolute shoulder joint. We should expect the inverse kinematics
analysis of the above serial chain to be similar to the direct kinematic analysis of the 3-2-1
Mechanism.

DIRECT POSITION KINEMATICS

Consider the mechanism shown in Figure 3.1. Following the arguments of the previous
chapter, joint T will lie on a circle with center located on the line joining joints A₁ and A₂.
Also, T will lie on a circle with center located on the line joining joints A₂ and A₃. Thus, T
will lie on the intersection of these two circles restricting T to two locations only. Note that
the position of T is determined completely by the limb lengths L₁, L₂ and L₆.

Once the position of T is fixed, then S will lie on a sphere with radius ST and center at T. S
will also lie on a circle with center located on the line joining joints C₁ and C₂. Thus S will
lie on either of the two intersection points between the circle and the sphere.

After the locations of S and T have been fixed, we can locate the position of R as follows.
R will lie on a circle with center located on the line joining joints S and T. Also R will lie on
a sphere with center located at B and radius L₁. Thus R will also lie on either of the two
intersection points of the circle and the sphere.

Since there are two locations for each of the three joints R, S and T, there are a total of
eight different configurations of the mechanism for a given set of limb lengths. The detailed
calculations follow.
Let us fix the coordinate system with the origin located in the base member and the z-axis normal to the base member. The position vectors for the joint locations $A_1$, $A_2$, $A_3$, $B$, $C_1$ and $C_2$ are given by $\mathbf{p}_{A_1}$, $\mathbf{p}_{A_2}$, $\mathbf{p}_{A_3}$, $\mathbf{p}_B$, $\mathbf{p}_{C_1}$ and $\mathbf{p}_{C_2}$ respectively. Since the position of $T$ depends only on the limb lengths $L_2$, $L_5$ and $L_6$ only, it can be calculated directly as follows.

If we pass a plane through the point $T$ and normal to line $A_1A_2$ and another plane normal to line $A_2A_3$ and also containing the point $T$, then they will intersect along the line $TT'$ which will be normal to the base member (Figure 3.3). The intersection point of the first plane with $A_1A_2$ is $O_{t1}$ and the intersection point of the second plane with the line $A_2A_3$ is $O_{t2}$.

EQUATIONS AND SOLUTION FOR DIRECT POSITION KINEMATICS

The position vectors of $O_{t1}$ and $O_{t2}$ are given by the equations:

$$
\mathbf{p}_{O_{t1}} = \mathbf{p}_{A_1} + t_{11} \frac{\mathbf{p}_{A_2} - \mathbf{p}_{A_1}}{\mathbf{p}_{A_2} - \mathbf{p}_{A_1}}
$$

$$
\mathbf{p}_{O_{t2}} = \mathbf{p}_{A_2} + t_{22} \frac{\mathbf{p}_{A_3} - \mathbf{p}_{A_2}}{\mathbf{p}_{A_3} - \mathbf{p}_{A_2}}
$$
Figure 3.3. Description of variables used for calculating position of T.

where $t_{11}$ and $t_{22}$ are determined by the equations:

$$t_{11} = \frac{|\mathbf{P}_{A_2} - \mathbf{P}_{A_1}|^2 + L_2^2 - L_6^2}{2|\mathbf{P}_{A_2} - \mathbf{P}_{A_3}|}$$

$$t_{22} = \frac{|\mathbf{P}_{A_2} - \mathbf{P}_{A_3}|^2 + L_5^2 - L_6^2}{2|\mathbf{P}_{A_2} - \mathbf{P}_{A_3}|}$$

Now the position vector of T is given by the equation:

$$\mathbf{p}_t = \mathbf{p}_{O_{t1}} + m_1 \cos \phi_1 (\cos \beta_1 \mathbf{i} + \sin \beta_1 \mathbf{j}) + m_1 \sin \phi_1 \mathbf{k}$$

(3.1)

where
\[ m_i = \left( L_2^2 - t_{ii}^2 \right)^{\frac{1}{2}} \]

\[ \phi_i = \pm \tan^{-1} \left( \frac{(m_i^2 + t_{ii})^{\frac{1}{2}}}{t_{22}} \right) \]

\[ \beta_i = \cos^{-1} \left[ \frac{\left\{ \left( \mathbf{P}_{A_2} - \mathbf{P}_{A_1} \right) \times \mathbf{k} \right\} \cdot \mathbf{i}}{\left\| \left( \mathbf{P}_{A_2} - \mathbf{P}_{A_1} \right) \times \mathbf{k} \right\|} \right] \]

\[ \beta_r = \sin^{-1} \left[ \frac{\left\{ \left( \mathbf{P}_{A_2} - \mathbf{P}_{A_1} \right) \times \mathbf{k} \right\} \cdot \mathbf{i}}{\left\| \left( \mathbf{P}_{A_2} - \mathbf{P}_{A_1} \right) \times \mathbf{k} \right\|} \right] \]

Note that since there are two possible values for \( \phi_r \), we will get two positions of \( T \) for a given set of lengths \( L_1, L_5 \) and \( L_6 \).

Once the position of \( T \) has been calculated, we can calculate the position of \( S \). From Figure 3.4, the position vector of \( O_S \) is given by:

\[ \mathbf{P}_{O_s} = \mathbf{P}_{c_1} + s_1 \frac{\left( \mathbf{P}_{c_2} - \mathbf{P}_{c_1} \right)}{\left\| \mathbf{P}_{c_2} - \mathbf{P}_{c_1} \right\|} \]

where

\[ s_1 = \frac{\left| \mathbf{P}_{c_2} - \mathbf{P}_{c_1} \right|^2 + L_2^2 - L_5^2}{2 \left| \mathbf{P}_{c_2} - \mathbf{P}_{c_1} \right|} \]

The position vector of \( S \) is given by:

\[ \mathbf{P}_s = \mathbf{P}_{O_s} + m_s \cos \phi_i \left( \cos \beta_i \mathbf{i} + \sin \beta_i \mathbf{j} \right) + m_s \sin \phi_i \mathbf{k} \]

(3.2)

where

\[ m_s = \left( L_3^2 - s_{ii}^2 \right)^{\frac{1}{2}} \]

\[ \beta_s = \cos^{-1} \left[ \frac{\left\{ \left( \mathbf{P}_{c_2} - \mathbf{P}_{c_1} \right) \times \mathbf{k} \right\} \cdot \mathbf{i}}{\left\| \left( \mathbf{P}_{c_2} - \mathbf{P}_{c_1} \right) \times \mathbf{k} \right\|} \right] \]

\[ \beta_s = \sin^{-1} \left[ \frac{\left\{ \left( \mathbf{P}_{c_2} - \mathbf{P}_{c_1} \right) \times \mathbf{k} \right\} \cdot \mathbf{i}}{\left\| \left( \mathbf{P}_{c_2} - \mathbf{P}_{c_1} \right) \times \mathbf{k} \right\|} \right] \]
In the above equation $\phi_s$ is an unknown quantity. $\phi_s$ can be calculated by the following equation:

$$|P_i - P_s|^2 = b_2^2$$  \hspace{1cm} (3.3)

where

$b_2$ = distance between the joint locations $S$ and $T$.

Substitution for $P_s$ from equation (3.5) into the above equation and simplification gives:

$$D_1 \cos \phi_s + D_2 \sin \phi_s + D_3 = 0$$  \hspace{1cm} (3.4)

where
\[ D_1 = 2(\mathbf{P}_{o_i} - \mathbf{P}_i). (m_i \cos \beta_i \mathbf{i} + m_i \sin \beta_i \mathbf{j}) \]
\[ D_2 = -2m_i \sin \phi_i (\mathbf{P}_r . \mathbf{k}) \]
\[ D_3 = |\mathbf{P}_r|^2 + |\mathbf{P}_{o_i}|^2 + m_i^2 - 2 \mathbf{P}_r \cdot \mathbf{P}_{o_i} \]

The solution for equation 3.4 is given by:

\[
\cos \phi_i = \frac{-D_1 D_3 + \sigma D_2 (D_1^2 + D_2^2 - D_3^2)}{D_1^2 + D_2^2} \]
\[
\sin \phi_i = \frac{-D_2 D_3 - \sigma D_1 (D_1^2 + D_2^2 - D_3^2)}{D_1^2 + D_2^2} \]

where \( \sigma = \pm 1 \).

From the above equation we will again get two positions for \( S \). The only variable to be determined now is the position of \( R \). The position vector of \( R \) is given by:

\[
\mathbf{P}_r = \mathbf{P}_a + L_1 \cos \phi_r \cos \beta_r \mathbf{i} + L_1 \cos \phi_r \sin \beta_r \mathbf{j} + L_1 \sin \phi_r \mathbf{k} \]

where

\( \beta_r = \) Angle made by the projection of limb \( BR \) on the \( xy \)-plane with the \( x \)-axis.

\( \phi_r = \) Angle made by the limb \( BR \) with the \( xy \)-plane.

The variables \( \beta_r \) and \( \phi_r \) can be calculated using the following equations:

\[
|\mathbf{P}_r - \mathbf{P}_i|^2 = b_3^2
\]
\[
|\mathbf{P}_r - \mathbf{P}_a|^2 = b_1^2
\]

(3.7)
where

\[ b_1 = \text{length between the locations of the joints R and S}. \]
\[ b_3 = \text{length between the locations of the joints R and T}. \]

Substitution in the above equations for the expression of \( \mathbf{P}_r \) from the equation (3.6) and simplification gives the following set of equations:

\[ E_1 \cos \phi_r \cos \beta_r + E_2 \cos \phi_r \sin \beta_r + E_3 \sin \phi_r + E_4 = 0 \]  \hspace{1cm} (3.8)

where

\[ E_1 = 2(\mathbf{P}_B - \mathbf{P}_i) \cdot (L_i \hat{\mathbf{i}}) \]
\[ E_2 = 2(\mathbf{P}_B - \mathbf{P}_i) \cdot (L_i \hat{\mathbf{j}}) \]
\[ E_3 = -2 \mathbf{P}_i \cdot (L_i \hat{\mathbf{k}}) \]
\[ E_4 = |\mathbf{P}_B - \mathbf{P}_i|^2 + L_i^2 \]

and

\[ F_1 \cos \phi_r \cos \beta_r + F_2 \cos \phi_r \sin \beta_r + F_3 \sin \phi_r + F_4 = 0 \]  \hspace{1cm} (3.9)

where

\[ F_1 = 2(\mathbf{P}_B - \mathbf{P}_i) \cdot (L_i \hat{\mathbf{i}}) \]
\[ F_2 = 2(\mathbf{P}_B - \mathbf{P}_i) \cdot (L_i \hat{\mathbf{j}}) \]
\[ F_3 = -2 \mathbf{P}_i \cdot (L_i \hat{\mathbf{k}}) \]
\[ F_4 = |\mathbf{P}_B - \mathbf{P}_i|^2 + L_i^2 \]
Equations (3.8) and (3.9) have to be solved simultaneously to the solution for $\beta_r$ and $\phi_r$.

Before proceeding with the rest of the solution, the above calculations for the special case of this mechanism shown in Figure 3.2 are presented below. Let us fix the coordinate system at B with the x-axis along AB as shown in Figure 3.5.

![Figure 3.5 Description of variables for the special form of 3-2-1 Mechanism.](image)

From equation 3.1 $\phi_i$ is given by:

$$\phi_i = \pm \tan^{-1} \left[ \frac{c_2}{t_{22}} \right]$$

The parameters in the above equation can be expressed in terms of the known limb lengths and member geometry as follows:

$$t_{11} = \frac{c^2 + \frac{L_2^2}{2} - \frac{L_6^2}{2}}{2c}; \quad t_{12} = c - t_{11}$$
\[ t_{21} = \frac{a^2 + L_5^2 - L_6^2}{2a}, \quad t_{22} = a - t_{21} \]

\[ m_i = \left( L_2^2 - t_{11}^2 \right)^{\frac{1}{2}} \]

The position vectors for O_s and O_{t1} are given by:

\[ \mathbf{PO}_s = (BO_s) \cos\theta_s \mathbf{i} + (BO_s) \sin\theta_s \mathbf{j} \]
\[ \mathbf{PO}_{t1} = -t_{12} \mathbf{i} \]

where

\[ (BO_s) = \left( c^2 + s_1^2 - 2cs_1 \cos\alpha_i \right)^{\frac{1}{2}} \]
\[ \theta_s = 180 + \cos^{-1} \left[ \frac{c^2 + (BO_s)^2 - s_2^2}{2c(BO_s)} \right] \]
\[ s_1 = \frac{b^2 + L_5^2 - L_6^2}{2b}, \quad s_2 = b - s_1 \]

The position vectors of R, S and T are given by the equations:

\[ \mathbf{P}_s = \mathbf{PO}_s + m_s \cos\phi_s \cos\beta_s \mathbf{i} + m_s \sin\beta_s \cos\phi_s \mathbf{j} + m_s \sin\phi_s \mathbf{k} \]
\[ \mathbf{P}_t = \mathbf{PO}_t - m_t \cos\phi_t \mathbf{i} + m_t \sin\phi_t \mathbf{k} \]
\[ \mathbf{P}_r = \mathbf{P}_a + L_1 \cos\phi_r \cos\beta_r \mathbf{i} + L_1 \cos\phi_r \sin\beta_r \mathbf{j} + L_1 \sin\phi_r \mathbf{k} \]

where

\[ m_s = \left( L_3^2 - s_1^2 \right)^{\frac{1}{2}} \]
\[ \mathbf{P}_a = -c \mathbf{i} \]
\[ \beta_s = 90 - \alpha_i \]
The coefficients in equation 3.4 will be:

\[ D_1 = 2 \ (BO_3) \ m_s \ \cos(\theta_s - \beta_s) + 2 \ m_s \ t_{12} \ \cos\beta_s + 2 \ m_t \ m_s \ \cos\phi_t \ \sin\beta_s \]
\[ D_2 = -2 \ m_s \ m_t \ \sin\phi_t \]
\[ D_3 = 2 \ t_{12} \ (BO_3) \ \cos\theta_s + 2 \ m_t \ (BO_3) \ \cos\phi_t \ \sin\theta_s + (BO_3)^2 + (t_{12})^2 + (m_s)^2 + (m_t)^2 - (b_2)^2 \]

![Diagram of forces and directions](image)

Figure 3.6. Description of the angle \( \beta_s \).

The solution for \( \phi_s \) is given by:

\[ \cos \phi_s = \frac{-D_1 D_3 + \sigma D_2 (D_1^2 + D_2^2 - D_3^2)^{\frac{1}{3}}}{D_1^2 + D_2^2} \]
\[ \sin \phi_s = \frac{-D_2 D_3 - \sigma D_1 (D_1^2 + D_2^2 - D_3^2)^{\frac{1}{3}}}{D_1^2 + D_2^2} \]

where \( \sigma = \pm 1 \).
The coefficients in equations (3.8) and (3.9) are given by:

\[ E_1 = 2L_1(t_{12} - c) \]
\[ E_2 = 2L_1m_t \cos \phi_t \]
\[ E_3 = -2L_1m_s \sin \phi_t \]
\[ E_4 = (t_{12} - c)^2 + (L_1)^2 + (m_t)^2 - (b_3)^2 \]

\[ F_1 = -2L_1[c + (BO_s) \cos \theta_s + m_s \cos \phi_s \cos \beta_s] \]
\[ F_2 = -2L_1[(BO_s) \sin \theta_s + m_s \sin \beta_s \cos \phi_s] \]
\[ F_3 = -2L_1m_s \sin \phi_s \]
\[ F_4 = [c + (BO_s) \cos \theta_s + m_s \cos \phi_s \cos \beta_s]^2 + (L_1)^2 + [(BO_s) \sin \theta_s + m_s \sin \beta_s \cos \phi_s]^2 + (m_s \sin \phi_s)^2 - (b_2)^2 \]

Equations (3.8) and (3.9) can be solved as follows. They can be solved simultaneously for \( \cos \phi_t \) and \( \sin \phi_t \). The resulting expressions are:

\[ \cos \phi_t = \frac{-F_4E_3 + E_4F_3}{(E_1 \cos \beta_t + E_2 \sin \beta_t)F_3 - (F_1 \cos \beta_t + F_2 \sin \beta_t)E_3} \]
\[ \sin \phi_t = \frac{-E_4(F_1 \cos \beta_t + F_2 \sin \beta_t) + F_4(E_1 \cos \beta_t + E_2 \sin \beta_t)}{-(E_1 \cos \beta_t + E_2 \sin \beta_t)F_3 + (F_1 \cos \beta_t + F_2 \sin \beta_t)E_3} \]

(3.10)

Substitution of the above equations in the equation

\[ \cos^2 \phi_t + \sin^2 \phi_t = 1 \]
and simplification gives

\[ G_1 \cos^2 \beta_r + G_2 \sin^2 \beta_r + G_3 \sin \beta_r \cos \beta_r + G_4 = 0 \]  \hspace{1cm} (3.11)

where

\[ G_1 = (F_4 E_1 - E_4 F_1)^2 - (F_3 E_1 - F_1 E_3)^2 \]
\[ G_2 = (E_2 F_4 - F_2 E_4)^2 - (F_3 E_2 - E_3 F_2)^2 \]
\[ G_3 = 2 (F_4 E_1 - E_4 F_1)(E_2 F_4 - F_2 E_4) - 2 (F_3 E_1 - F_1 E_3)(F_3 E_2 - E_3 F_2) \]
\[ G_4 = (E_4 F_3 - F_4 E_3)^2 \]

The above equation is a fourth order equation in \( \tan(\beta_r/2) \). Therefore it will give four solutions for \( \beta_r \). Since we had started with two different values of \( \phi_t \) and \( \phi_s \), we will get a total of sixteen different solutions for \( \beta_r \). This value of \( \beta_r \) can be substituted back into equation 3.10 to get the value of \( \phi_r \). As mentioned in the previous chapter, after obtaining the positions of \( R, S \) and \( T \) in the coordinate system fixed to the base member, we can determine the position of the coordinate system fixed to the moving member.

3.4 DISCUSSION

In the beginning of this chapter, it had been argued that there are only eight different possible solutions. But the equations give us sixteen different solutions. By comparing the calculations with the argument, we can clearly see that the spurious solutions are introduced in the last stage of calculations for \( \beta_r \). Equation 3.11 should give us only two solutions for
\( \beta_r \). The extra two solutions obtained are spurious solutions and will be apparent when we substitute the value of \( \beta_r \) in equation 3.10 to solve for \( \phi_r \).

Another interesting observation can be made at this point regarding the number of solutions of the above problem. We know that the number of solutions of the inverse kinematic analysis of the equivalent serial chain described in the beginning of this chapter is 8. It has been proved in this chapter that the number of solutions to the direct kinematic analysis of the 3-2-1 Mechanism is also 8.
CHAPTER IV

CONCLUSION

CONCLUSION

In this thesis, the equations for the direct position kinematics of a specific group of parallel mechanisms has been presented.

It has been shown that the kinematic equations of the special Stewart Platform can be reduced to a polynomial equation in a single variable of order 24.

For the 3-2-1 Mechanism shown in Figure 1.5, it has been shown that the direct position kinematic equations can be reduced to a polynomial equation of order 4. In this case it has been further shown that two of these solutions are spurious solutions.

Similarly it is quite possible that in the case of the special Stewart Platform, out of the 24 different solutions of $\phi_r$, some of them are spurious solutions. It should be possible to go back to the different stages of the derivation to determine the solutions that do not satisfy those intermediate equations and thus are spurious solutions.

The main difficulty in reducing the polynomial equations to a polynomial equation in a single variable is the high rate of increase of the order of the polynomials largely due to the introduction of the spurious solutions. The spurious solutions satisfy the final polynomial equation but in fact are not the correct solution to the problem. These are introduced during the various stages of the derivation mostly due to squaring of the equations in order to
simplify them. The number of spurious solutions introduced into the calculations seems to depend on the choice of the coordinate system and the variables. Ideally, the choice of the coordinate system and the variables should be such that no spurious solutions are introduced. As an example, if we try to find the intersection of a circle and a sphere in the cartesian coordinate system, we will get a polynomial equation of order 8, while from geometry we know that there are at most 2 possible solutions (unless the circle and the sphere coincide). If we cannot totally avoid the spurious solutions, then at least we must make sure that the spurious solutions are not introduced early in the derivation.

The derivations in this thesis provide us valuable insight into the direct kinematics problem of the Stewart Platform. Successful solution of the general Stewart Platform will be possible if we can come up with a suitable coordinate system and variables which inhibit the introduction of spurious solutions.
REFERENCES


