Elementary Teachers’ Understanding and Use of Cognition Based Assessment Learning Progression Materials for Multiplication and Division

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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2012

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Abstract

Teachers’ knowledge of mathematical content and children’s mathematical thinking have been identified as critical elements related to teachers’ ability to effectively teach mathematics (Fennema & Franke, 1992; Kazemi & Franke, 2001; Ma, 1999; Peterson, Carpenter, & Fennema, 1989). Literature on teachers’ knowledge suggests that teachers need not only to hold a deep understanding of the mathematics they teach, but also have detailed knowledge of the common correct and incorrect mathematical conceptions their children and research-based knowledge of the progression of the development of children’s mathematical ideas. Research-based learning trajectories and learning progressions of children’s mathematical development represent a potentially valuable resource for teachers to help make sense of children’s thinking, and research on teachers’ use and understand of such materials is much needed.

This study investigated teachers’ understanding and use of the Cognition Based Assessment (CBA) learning progression materials for multiplication and division concepts. The data sources included structured clinical interviews based around teachers’ use of the CBA materials in analyzing written student work episodes, determining learning goals based on student work, and proposing instructional plans to help students progress in their mathematical understanding. Additionally, three case studies of teachers using the CBA materials in live one-on-one teaching and assessment situations are
investigated. Analysis of teachers’ use and understanding of the CBA materials consisted of grounded theorizing and retrospective analysis, involving iterations of hypothesis/conjecture formation based on data, hypothesis/conjecture testing, and hypothesis/conjecture revision.

The study had several key findings. First, as complexity of either the CBA level, or a student’s mathematical reasoning within the CBA framework increased, so did the inconsistency and variation in teachers’ interpretations of the student thinking. This finding led to the conceptualization of teachers’ consistent, partially consistent, and inconsistent use of CBA materials. A second important finding was that CBA language and research-based descriptions of children’s thinking were occasionally quite challenging for teachers to interpret or understand. The terminology and conceptual ideas embedded in the CBA multiplication and division framework occasionally involved mathematical or technical descriptions that led to inconsistent teacher conceptualizations and misinterpretations of the CBA framework. A third important finding was that CBA consistent conceptualizations of student thinking within the CBA framework were frequently related to learning goals and instructional plans that were more informed by children’s thinking. This relates to similar findings by the research related to Cognitively Guided Instruction (CGI) (Fennema et al., 1996; Franke, Carpenter, Levi, & Fennema, 200; Kazemi & Franke, 2001; Peterson, Carpenter, & Fennema, 1989). Overall, the data demonstrated that the CBA multiplication and division learning progression materials could become a powerful framework for teachers to analyze their own instruction and assessment of children’s mathematical thinking.

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Dedication

Dedicated to Ayden Parker Harrison. Born still on April 18th 2011.
Acknowledgments

My most sincere thanks to:

My advisor, Dr. Michael Battista, for including me in his research plans with the Cognition Based Assessment project, his feedback and support throughout the writing of the dissertation, and his ability to see things that others cannot. He challenged me to always try to think more deeply and carefully about the ideas I was investigating, and helped me learn about in-depth qualitative research.

My committee, Dr. Patti Brosnan, and Dr. Karen Irving for their support, insights, and belief in me.

My fellow graduate students. I had the joy of seeing many of my colleagues graduate before me and also mentoring some of the newer graduate students as they progress through their exams and proposals.

My parents, Barbara and Timothy Gegg-Harrison, for supporting me unconditionally, and for providing me with opportunities to grow and learn. I would not be a college graduate,
let alone have completed this dissertation if they had not provided me with a household that valued learning and education.

My sister, Whitney, for setting a very high bar for me to always strive for.

My best friend and wife, Jennifer, for always being there for me. The best decision I ever made in my life was shyly and nervously introducing myself to the only female in my first undergraduate Computer Science course.

My cats, Hobbes, Snoopy, Pippin, and Gypsy, for being great companions.

My son, Ayden, who was taken from us too soon for helping me learn to be more compassionate, and grateful for how precious life is. I miss you every day.
Vita

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Chapter 1: Introduction

“To empower students with mathematical thinking, teachers should be empowered first.”
-Liping Ma, 1999

This simple sentiment conveyed by Liping Ma, that teachers of mathematics must possess certain mathematical knowledge needed for teaching, continues to be a major issue in mathematics education research and reform. The goal of this chapter is not to attempt to address all of the challenges in elementary school mathematics education, but rather to consider some of the available research in an effort to explore areas where elementary teacher development could stand to improve.

The majority of elementary school teachers are “generalists”. They are tasked with the responsibility of teaching not just mathematics, but also reading and language arts, social studies and history, as well as science, often times in a self-contained classroom. According to the National Council of Teachers of Mathematics (NCTM), they must also do this based on the established principles and standards for teaching mathematics for conceptual understanding (NCTM, 1989, 2000). The expectations of the generalist elementary teacher are expansive, and it is hard to expect that these teachers would have a very solid understanding of mathematics or students’ conceptions of mathematical ideas. In her research on American and Chinese elementary school teachers, Liping Ma found substantial evidence of missing and incomplete conceptual
mathematics understandings of American elementary school teachers, even among elementary teachers characterized as “more dedicated and more confident mathematically” (Ma, 1999). In order to effectively meet the professional standards for teaching mathematics, it is essential that teachers have a deep and connected understanding of the mathematical content they are responsible for teaching. However, according to the National Commission on Teaching and America’s Future (1997), “most schools and teachers cannot achieve the goals set forth in new educational standards, not because they are unwilling, but because they do not know how, and the systems they work in do not support them in doing so” (p.1). Ma (1999) also found that in spite of elementary teachers’ claims that they wanted to teach conceptually, their attitudes and explanations concerning mathematics suggested that they viewed elementary mathematics as a simple collection of arbitrary facts. This is completely antithetical to the type of teaching and learning prescribed by NCTM’s principles and standards. In this chapter, research on teacher knowledge is explored to contextualize the many different types of knowledge teachers need to effectively teach mathematics. Next, research is explored that investigates the challenges teachers have in learning about students’ mathematical thinking, and some of the reasons teachers struggle to enact mathematics teaching that better helps students learn mathematics. Next, research that has shown promise in helping teachers learn and grow towards enacting more effective teaching strategies is explored, and hypothetical learning trajectories are proposed as a potential mechanism to promote this growth. Finally the research questions for this proposed inquiry are provided.
Teacher Knowledge and the Complexity of Teaching

The topic of what teachers need to know and be able to do in order to effectively teach mathematics is not simple. Initial conceptions of what constituted a knowledge base for teaching mathematics were limited to explorations of teachers’ level of formal mathematics training (Begle, 1972). Attempts to find a connection between student achievement and teacher quality, as defined by exposure to higher level of mathematics content, were unsuccessful. Monk (1994) found a small correlation between the number of advanced mathematics courses taken by a teacher and the corresponding student achievement, but this only appeared to be true for up to five courses in advanced mathematics. These early conceptualizations of mathematics knowledge necessary for teaching focused primarily on teachers’ content knowledge, without alignment to other aspects of teaching (e.g. – assessment), leaving open the question of how teachers were using their content knowledge to inform instructional decisions.

The lack of evidence that content knowledge preparation is a sufficient form of the mathematical knowledge necessary to effectively teach, paired with student’s continued struggles with learning mathematics motivated further exploration into understanding what a knowledge-base for effectively teaching mathematics would look like. Mathematics education researchers have developed detailed models for how students think about and understand various mathematical concepts. For example, researchers working on the cognitively guided instruction (CGI) project built models of children’s learning of addition and subtraction, and identified that teachers’ understanding of children’s thinking in these areas was incomplete (Peterson, Carpenter,
& Fennema, 1989). Clearly, knowledge about how children reason about and make sense of mathematics could play a role in the mathematical knowledge that impacts instructional and assessment decision-making. Research projects such as CGI, and many others, have helped to complicate the notion of what it means to know mathematics for teaching beyond simple content knowledge.

Shulman’s conceptualization of pedagogical content knowledge (PCK) emphasized the wide array of knowledge teachers must have, including knowledge of curriculum, general pedagogical principles such as classroom management, learners, and specialized knowledge of how specific content is understood and learned by students (Shulman, 1987). In line with Monk’s findings that subject matter preparation was not very predictive of teaching that impacted student learning, the theory of PCK suggests that subject matter knowledge alone is not enough to effectively teach students mathematical content. Much like PCK, Hill, Ball and Schilling (2008) conceptualized a more mathematically focused elaboration of PCK named mathematics knowledge for teaching (MKT). Hill, Ball and Schilling (2008) add that knowledge of content and students (KCS) is an essential addition to the concept of PCK. They state that KCS represents “content knowledge intertwined with knowledge of how students think about, know, or learn particular content” (p. 375). Knowledge of content and teaching (KCT) represents the knowledge that informs appropriate teaching moves to help students make progress in their mathematical understanding, and represents the other main component of the MKT elaboration of PCK.
Included in the conceptualization of KCS is the knowledge of common mathematical conceptions and misconceptions that children hold, but KCS of this type does not necessarily imply a connection to making appropriate teaching moves to remedy student errors (Hill, Ball, Schilling, 2008). MKT identifies both KCS and knowledge of content and teaching (KCT) as crucial components of effective teaching, and research indicates that both forms of knowledge represent areas where teachers struggle. In order for teachers to build solid models of students’ mathematical conceptions, it would be necessary for them to have exposure to the variety of mathematics strategies that students commonly use. Without detailed KCS, it is hard to imagine that teachers would be able to design appropriate teaching moves for students. Since KCS and KCT are important areas where teachers seem to struggle, it is important to investigate exactly what makes KCS and KCT so challenging for teachers to develop.

Although there seems to be a logical connection between the acts of assessing what students understand and the instructional decision-making process, these two responsibilities are discussed separately in this chapter because, for the most part, instruction and assessment do not often coincide for teachers (Even, 2005, p. 46). The act of assessment and making instructional decisions is extremely complex, and generally teachers are better at inferring student understanding than determining future instruction (Heritage, Kim, Vendlinski, & Herman, 2008, p. 3).

The process by which teachers elicit feedback from students in order to determine what students understand has important consequences to both assessment and instructional decision-making. Even (2005) found that teachers tend not to know when
and where to look for details about student thinking from student communication and written work. Teachers needed to believe there was something worth hearing in order to attend to student feedback. Even (2005) described this as hearing through, or the process by which teachers filter out student thinking through personal biases and resources. If students do not share the desired answer sought by the teacher, the answer provided by the textbook or curriculum resources, or something the teacher perceives as relevant feedback, the student feedback is largely filtered out and overlooked. Van Es and Sherin (2008) cite inattentional blindness research suggesting that teachers often do not identify or attend to certain student feedback because it does not fit with what they expect to see.

Therefore, the type of feedback that teachers use to learn about student thinking and understanding is greatly impacted by what they are open to seeing. Teachers’ prior experiences as a teacher and student impact the vision they hold for the type of discourse that is expected in a classroom. Regardless of the variety of assessments teachers use in their classroom instruction, a teacher’s classroom paradigm limits the quality of interpretations that can be drawn based on student feedback. Therefore, a teacher with a narrow set of expected student responses is more likely to be limited in the type of evidence that can be observed. In a sense, teachers tend to learn what students know and are able to do by comparing what they see in class to what they expect to see. Although this is a fairly intuitive and logical concept, the limitations that these types of teachers’ beliefs and classroom conceptions place on the type of evidence that is available for interpretation is evident.
While the beliefs and paradigms teachers hold filter the type of evidence that is used to analyze student understanding, the specific types of assessments that are used also serve to define the available resources teachers have to determine student learning.

*Hearing through*, as described by Even (2005), is related to teachers’ tendency to utilize evaluative listening techniques. This form of listening encourages teachers to evaluate student feedback simply for mistakes and computational accuracy. Assessments that are intended as *formative* (i.e. – to promote further learning) are often implemented in a *summative*, evaluative fashion. Cross (2009), found that teachers most commonly use IRE (Initiate, Respond, Evaluate) questioning techniques that essentially serve as summative assessments. This differs from the ESRU (elicit, student responds, teacher recognizes, and teacher uses feedback) model of questioning Ruiz-Primo and Furtak (2006) identified to be a successful and effective cycle of formative assessment.

Research findings indicate that although teachers value the role of questioning strategies in assessment of student learning, the overwhelming majority of teachers’ questions emphasize the memorization and recall of facts (Brown, Lake, & Matters, 2008, p. 11). This finding is similar to Adams and Hsu (1998), suggesting that teachers may value a diverse array of assessment strategies, but implement them in a way that only provides evidence of low-level, surface understanding. In addition to the emphasis placed on memorization and recall, teachers frequently move instruction at a fast pace, asking questions, evaluating responses and moving quickly on to another topic (van Es & Sherin, 2008). These ‘snap-judgments’ make it less likely for student feedback to
substantively inform instruction or provide teachers with meaningful evidence of student learning.

Beliefs and assessment techniques restrict the type of evidence that teachers can collect and interpret on students’ mathematical understanding. Teachers interpret this evidence differently in order to determine what students understand. Brown et al. (2008) found that teachers, especially secondary teachers, tended to hold conceptions of learning that were aligned with reproducing behaviors (p. 3). Therefore, teachers are more likely to hold reproducing based, quantitative assessments in higher esteem when making judgments about what students know.

Studies on what teachers notice in classroom observations indicate that teachers tend to identify general pedagogical principles (e.g. – good classroom management or student engagement) as significant evidence of learning, and rarely identify specific examples of evidence of student mathematical thinking in their analysis of classroom situations (Sherin & van Es, 2005, p. 483). Although research shows that teachers may have access to a significant amount of feedback about what students know and understand about mathematics, beliefs and evaluative listening can filter out much of this feedback. Teachers tend to rely on general observations of classroom events, low-level questioning strategies, textbook-based and summative evaluations, as well as intuitions about students’ mathematical understanding. While teachers do tend to value the worth of formative assessment strategies, they often implement them in a fashion that aligns closer to the goals of summative assessment.
Just as it is complicated to understand how teachers learn about what their students know and are able to do, teachers’ instructional decision-making processes involve many factors. In general, teachers choose instructional activities and strategies based on what they determine to be helpful for students. Most teachers care very much about the well-being of their students, and hold nurturing views towards them, identifying that they want their students to learn in an emotionally and academically safe environment (Brown, Lake, & Matters, 2009). In general, teachers typically address in their instruction what they perceive to be as their students’ needs. All teachers cite roughly the same restrictions (e.g. – amount of time) that limit their ability to teach, but still choose instructional strategies that mirror their beliefs about mathematics (Cross, 2009; Frost, 2009). Teachers often attribute their own personal feelings and beliefs about mathematics learning (e.g. – dislike of group work) to their students in an attempt to shelter them from having negative experiences in the mathematics class. It is also often the case that teachers choose to become a teacher because of a specific role model teacher from their experiences as a student, and will base instructional decisions off of this model of teaching because it represents what they deem as a successful exemplar of an effective classroom.

One of the most dominant factors in determining a teacher’s instructional choices are the beliefs held with respect to mathematics teaching and learning (Cross, 2009; Frost, 2009; Adams & Hsu, 1998). Teachers’ perspectives on the discipline of mathematics as well as the teaching of mathematics play important roles in determining the instructional decisions teachers make. Cross (2009) described teachers who viewed
mathematics as a set of procedures and rules and whose instruction mirrored this belief by placing no emphasis on communication, collaborative activity, or active engagement (p. 332). Beliefs about what students are cognitively capable of have also been shown to impact teachers’ instructional decisions, with teachers commonly viewing cognitively demanding tasks as exclusively for gifted students while struggling students should be protected and given low-cognitive responsibilities (Cross, 2009). Although providing low-achieving students with low-cognitive tasks is likely to reinforce a student’s struggles in mathematics, teachers tend to hold protective beliefs about the emotional well being of struggling students. Teachers may simultaneously hold the belief that mathematics should be based on problem solving and reasoning through communication, but also believe that direct instruction and carefully crafted teacher-centered lessons are necessary for students (Friel & Carboni, 2000), especially those who struggle with mathematics the most.

A major contributing factor to teachers’ instructional decisions is the impact of classroom activities on classroom management (Superfine, 2009). Many teachers have low efficacy for serving as a facilitator in a classroom that is student-centered, and therefore avoid group work, inquiry-based explorations, and the use of manipulatives. Teachers tend to rely heavily on fixed routines that are primarily teacher-centered in order to prevent behavior issues (Superfine, 2009). Many teachers cite past experiences in classrooms that were poorly organized as justification for such highly structured classroom environments, and choose to emphasize classroom order in an attempt to be
sensitive to the emotional impact classrooms can have on children (van Es & Conroy, 2009).

With such a heavy emphasis placed on classroom management and organization, teachers tend to focus their attention on general observations of the classroom (e.g. – student engagement and behavior) and less on students’ conceptions of mathematics. In-class, on-the-spot, decisions are often made without a lot of time for teachers to reflect, and often align with teachers’ observations of management related issues. This view of the classroom does not place heavy emphasis on identifying how students understand or think about the mathematical content, and it is commonly the case that teachers struggle to notice important features of students’ mathematical thinking without explicit guidance through professional development (Jacobs, Lamb, & Phillipp, 2010; Sherin & van Es, 2005).

In stark contrast to the evidence that teachers struggle to identify and make sense of children’s thinking in a way that helps inform instruction, the Cognitively Guided Instruction (CGI) project provided strong evidence that knowledge of children’s thinking is a powerful tool that enables teachers to transform this knowledge and use it to change instruction. (Fennema et al., 1996, p. 432). CGI developed research-based organizing frameworks for teachers to make sense of children’s thinking. Simon (1995) conceptualized the idea of a hypothetical learning trajectory (HLT) of children’s learning as a means by which teachers could theoretically benefit from research. According to Simon, the teacher must create psychological models as a basis for pedagogical decisions.
Ideally, the teacher does not have to create these models from scratch, but rather can create models from available research-based models (Simon, 1995).

The conceptualization of the HLT (Simon, 1995), as well as other research projects’ development of a variety of research-based models of children’s mathematical thinking, has provided a way for research-based findings to help encapsulate what is known about not just how children think about and represent mathematics, but how that learning develops. Such organized knowledge of children’s thinking could serve as a beneficial framework for teachers who lack the knowledge of common student strategies, and knowledge of how children develop increasingly sophisticated mathematical ideas. Therefore, in addition to the creation of research-based models of children’s mathematical thinking, it seems sensible that such frameworks could be useful in the hands of teachers of mathematics.

Need for Research

In the time since Simon’s conceptualization of the HLT, many research projects have taken up the task of designing research-based models of children’s mathematical thinking in the form of learning trajectories and/or progressions. Although there is some debate about terminology and meaning of progressions versus trajectories (Battista, 2010), there has been significant progress made in designing research-based models of children’s mathematical thinking, especially in the primary and elementary grades (Battista, 2007; Confrey, Maloney, Nguyen, Mojica & Myers, 2009; Daro, Mosher & Corcoran 2011; Maloney & Confrey, 2010). The models of children’s mathematical thinking vary on whether or not they are directly connected to a particular set of tasks and
curriculum materials. The models of children’s mathematical thinking developed by the different research teams also vary widely in grain size, as children’s thinking can be viewed on a macro or micro scale (e.g. – Van Hiele levels represent a macro view of children’s learning of geometric shapes, and Battista’s elaboration represents a micro view).

As is evidenced by the current Common Core State Standards (CCSS), learning progressions terminology and research are becoming more influential in mathematics education Standards and curriculum design. According to the CCSS, “the development of these Standards began with research-based learning progressions detailing what is known today about how students’ mathematical knowledge, skill, and understanding develop over time” (CCSS). The emphasis placed on the role of research-based learning progressions in the CCSS shows a trend towards standards and curriculum development that is based off of research-based models of children’s thinking. Although many researchers disagree that the CCSS represent an accurate depiction of the research on children’s thinking, it is clear that the CCSS claim that the use of learning progression research informed elements of the standards design.

Although there is a significant amount of research in the HLT realm, there is limited research that has investigated teachers’ learning of HLT’s and learning progressions. Research indicates that teachers need to have a solid understanding of the mathematics that children are capable of, understand the common conceptions children hold about mathematics, and understand how to help children make connections to increasingly sophisticated mathematical ideas (Fennema & Franke, 1992, Ma, 1999,
NCTM, 2000). Research also shows that teachers struggle with exactly these things, and that professional development focused on children’s thinking can help them progress in these areas.

Mathematics education research has effectively connected teacher knowledge of children’s mathematical thinking to teaching in ways that promote higher student achievement, so there may be an implicit assumption that research-based frameworks can help provide this type of knowledge for teachers, and therefore influence student achievement. However, few studies have investigated exactly how teachers learn about and use research-based learning progressions. There is a clear need for research that critically investigates how teachers understand and use research-based frameworks of children’s mathematical thinking, especially as learning progressions and trajectories are becoming far more influential in standards and curriculum design, as well as professional development.

Purpose and Research Questions

The goal of this study is to carefully investigate teachers’ use and understanding of a particular set of Cognition Based Assessment learning progression materials to better understand how teachers use learning progressions and what learning progressions can provide teachers. The focus of investigation for this research is on how teachers utilize the Cognition Based Assessment (CBA) learning progression (LP) framework to conceptualize and judge students mathematical reasoning as well as pedagogical implications in the context of student work analysis, teaching experiments with one or more students, and other instances in which student mathematical thinking occurred. For
this study, a general research question is stated along with two particular questions that will be addressed:

**General Research Question**

How do teachers understand and use information on research-based learning progressions in mathematics to conceptualize students' mathematical thinking and subsequently needed mathematics instruction?

**Particular Research Questions Investigated in This Study:**

1. In CBA2 teachers' analysis of student work, how do they conceptualize student reasoning on multiplication and division problems, and what relationships exist between CBA2 teachers’ knowledge and understanding of CBA Multiplication and Division materials, their conceptualizations of student thinking, and their determination of instructional learning goals and tasks?

2. While working with students using CBA Multiplication and Division in live one-on-one teaching experiments, how are CBA2 teachers’ instructional decisions informed by knowledge of CBA Multiplication and Division materials?

**Clarifying Notes**

Note 1. CBA2 teachers have learned about a particular set of learning progressions in mathematics, namely those developed in the CBA project.

Note 2. What clearly distinguishes Questions 1 from Question 2 is that in the former context, teachers' are only making judgments about student work, while in the latter; they are making decisions in-the-moment of teaching or assessing.
Chapter 2: Literature Review

In this chapter, three primary areas of research that serve as a theoretical and empirical basis for the proposed study are reviewed. First, research on teacher knowledge and frameworks for evaluating the types of knowledge necessary for teaching mathematics are reviewed. This portion concludes with a synthesized framework that serves as a working theoretical framework for analyzing teacher knowledge for the proposed inquiry. The second portion of the literature review addresses research on what has been done to help teachers learn about children’s mathematical thinking and gain other helpful knowledge for teaching mathematics. Finally, research on learning trajectories/progressions is briefly addressed, and the limited research on teachers’ understanding and use of learning trajectories/progressions is explored.

Frameworks for Evaluating Teacher Knowledge

Research investigating the knowledge necessary for effectively teaching mathematics has taken a variety of forms. Some research projects focus primarily on teachers’ personal mathematics content knowledge, some focus on components of the responsibilities of teachers, and still others delve more deeply into knowledge related to student mathematical thinking. These do not represent the only perspectives and
Frameworks, but rather represent a small sample of the variety of approaches available for discussing the complex notion of teacher’s knowledge for teaching mathematics.

**Organization of Mathematical Knowledge**

The form and quality of mathematical knowledge held by teachers is of great interest in the discussion about the knowledge needed to effectively teach mathematics. Ma (1999) approached this issue by interviewing elementary teachers and analyzing their responses to questions about elementary mathematics concepts and the mathematics of student errors and non-standard algorithms. She characterized teachers based on their profound understanding of fundamental mathematics (PUFM), a construct described as “an understanding of the terrain of fundamental mathematics that is deep, broad, and thorough” (Ma, 1999, p. 120). Accordingly, teachers who have PUFM do not simply have knowledge of mathematics as a set of rules and procedures, but instead have a well-connected knowledge of elementary mathematics that is both conceptual and procedural. Such knowledge affords the teacher with the potential to not only understand and execute the mathematics they are tasked with teaching, but also understand its relationship to future mathematical concepts as well as alternative representations.

Another perspective to teacher’s mathematical knowledge is through the Structure of Observed Learning Outcomes (SOLO) taxonomy. The SOLO taxonomy theorizes that levels of thinking lie within different modes of representation. Biggs (1999) described five hierarchical levels in the taxonomy: 1) prestructural, 2) unistructural, 3) multistructural, 4) relational, and 5) extended abstract. Progress through the five levels is shown through increasing understanding of task demands, as well as sophistication of
responses and relational connections between task components. Applying SOLO to teachers of mathematics can give a perspective on the depth of their own understanding of the mathematics they are tasked to teach. The goal of both PUFM and SOLO when applied to teachers is similar, as both seek to characterize the sophistication of a teacher’s mathematical knowledge and organization as well as depth of representational understanding.

Another approach that focuses on the quality and form of mathematical knowledge of teachers is that of example spaces introduced by Zazkis and Leiken (2007). Teachers frequently rely upon the use of examples of mathematical ideas and concepts in their instruction. Therefore their understanding of appropriate use and selection of examples could serve as a fruitful place for exploration. Accordingly, Zazkis and Leiken (2008) argued for “definitions of mathematical concepts, the underlying structures of the definitions and the process of defining to be fundamental components of the subject matter knowledge of teachers of mathematics” (p. 133). As defined by Watson and Mason (2005), example spaces are the collections of examples that fulfill a specific function (p. 132). They described several different forms of example spaces: situated/local example spaces, personal/experiential example spaces, conventional/traditional example spaces, and collective/situated example spaces. The variety in example spaces is intended to represent the notion that a teacher’s personal mathematical experiences and knowledge does not represent the only place from which examples can be drawn or understood. Zazkis and Leiken (2007) argued for a framework investigating teachers’ example spaces based on their accessibility and accuracy,
richness, and generality. Proponents of such a model argue that the depth of example spaces held by teachers may shed light on better understanding curricular decision making, and pedagogical conceptions of the ways in which students may or may not learn concepts. Research in this area appears to be in a relatively early state, so it is unclear how useful the perspective will be in analyzing mathematical knowledge for teaching.

Frameworks emphasizing the structure of mathematical knowledge of teachers recognize the importance for teachers to have a firm understanding of the content they are responsible for teaching. Without a well-connected and rich understanding of the mathematics content to be taught, it is difficult to expect teachers to be able to effectively create instructional materials, interpret student responses, and assess understanding. Frameworks focusing on content knowledge of teachers afford researchers with the opportunity to deeply explore the scope of mathematics that is understood by teachers, but lack in the ability to identify other forms of knowledge that are required to make content knowledge usable in the various acts of teaching.

Mathematical Knowledge for Teaching

One of the initial conceptions of specialized knowledge needed for teaching came from Shulman’s conception of pedagogical content knowledge (PCK) (1986). The conceptualization of PCK called for a different form of knowing content “for oneself” that was different from the type of knowledge necessary in order to teach someone else the subject. Shulman also specified that a teacher’s subject matter knowledge (SMK) must not simply allow teachers to define accepted truths in a domain, but also elaborate the meaning of propositions and the relationship between propositions. Shulman’s third
category of teacher knowledge included knowledge of curriculum. This included the knowledge of the variety of curriculum materials available for teaching specific concepts, as well as an understanding of the vertical curriculum, or the past and future content that students will be exposed to. Shulman’s framework for thinking about specialized teacher knowledge was highly influential as an organizational framework for thinking about the different forms of knowledge teachers needed and used.

PCK has long been assumed to be an important component of what is necessary for effective teaching. The work of Ball, Thames and Phelps (2008) and Hill, Ball and Schilling (2008) has more acutely conceptualized the domain of mathematical knowledge for teaching (MKT), especially focusing on mathematical concepts commonly taught in elementary and early middle grades. Ball et al. (2008) utilized Shulman’s PCK and focused on the “work of teaching” to inform areas of teachers’ profession that demand mathematical reasoning, insight and understanding. This integration of PCK, mathematics, and the work of teaching led to the development of the MKT constructs of common content knowledge (CCK), specialized content knowledge (SCK), knowledge of content and students (KCS) and knowledge of content and teaching (KCT). CCK is defined as the type of knowledge that overlaps with how mathematics is used in other occupations that use mathematics, while SCK is the knowledge that allows teachers to engage in specific teaching tasks such as representing a mathematical idea, or providing an explanation for a rule or procedure (Ball et al., 2008). KCS represents the knowledge teachers have related to how students will interpret tasks, and common conceptions students hold related to the mathematical ideas. Finally, KCT represents the knowledge
necessary to evaluate the order in which instruction might best suit learning, as well as an understanding of the advantages of different representations of a concept. Ball et al. (2008) also identified curriculum knowledge and knowledge at the mathematical horizon as component of MKT, and this was conceptualized in much the same way as Shulman’s curriculum knowledge. These categories are helpful to describe the types of knowledge that might play a role in the teacher decision-making process during planning and instruction.

Further exploration of the MKT construct, through factor analysis conducted by Schilling (2007) led to the explicating of seven separable components of MKT:

1) knowledge of the mathematical content taught in elementary schools; 2) providing explanations for mathematical ideas and procedures; 3) representing ideas and procedures using number lines, area models, or word problems; 4) determining the correctness of alternative or non-standard mathematical errors; 5) understanding typical student errors; 6) assessing the degree to which student responses to questions indicate understanding of mathematical concepts; and 7) ordering problems in terms of student difficulty. (p. 98)

Schilling argued that components one through four represented the content knowledge of teachers, with component one representing CCK, and components two through four representing SCK. Components five through seven were proposed to be representative of a teacher’s KCS. Schilling’s (2007) analysis of MKT measurement techniques indicated that while the CCK measures tended to be one-dimensional, the same did not hold true for measures of KCS. Ball et al. identify that the complex
relationships between categories of MKT made separability of different sub-constructs complicated.

With a similar slant to the conceptualization of MKT, Tsamir and Tirosh (2008) proposed a framework explicitly integrating the ideas of Shulman’s SMK and PCK with Fischbein’s theory of mathematical knowledge. Fischbein’s theory viewed mathematical knowledge as containing three main components: algorithmic, formal, and intuitive. Algorithmic knowledge was described as the rules and processes for developing mathematical solutions, while formal knowledge represented the definitions and theorems about how mathematics works. Fischbein’s notion of intuitive knowledge was as the mathematical knowledge that individuals take as being obvious or assumed. Such knowledge could either be developed independent of instruction, coined primary intuitions, or could evolve from explicit experience and instruction converting formal knowledge into intuition, coined secondary intuitions.

Tsamir and Tirosh’s (2008) integrated framework crossed Shulman’s SMK and PCK with Fischbein’s algorithmic, formal, and intuitive mathematical knowledge to create six components for analyzing teachers’ mathematical knowledge and understanding related to teaching: 1) algorithmic-SMK, 2) formal-SMK, 3) intuitive-SMK, 4) algorithmic-PCK, 5) formal-PCK, and 6) intuitive-PCK. Each of the six components encompasses a variety of different forms of mathematical and pedagogical knowledge that teachers use in their teaching of mathematics. In many ways, the SMK categories of the Shulman-Fischbein framework align well with aspects of CCK and SCK as they speak to the depth of mathematical knowledge held by a teacher. Tsamir and
Tirosh (2008) argued that *algorithmic, formal, and intuitive* forms of PCK relate to teachers’ understanding of different forms of mathematical errors, and students’ mathematical misconceptions, which overlaps significantly with KCS.

Frameworks such as MKT (Ball et al., 2008; Hill et al., 2008) and the Shulman-Fischbein framework (Tsamir & Tirosh, 2008) offer detailed accounts of the varied forms of knowledge that teachers of mathematics may draw upon in their instructional planning, execution, and assessment. A potential drawback to the framework proposed by Hill et al. (2008) is the issue of separating sub-constructs of MKT. As identified by Schilling’s (2007) factor analysis, MKT constructs are difficult to separate out, especially KCS. It is also unclear whether or not the MKT framework is intended to be a model of individual teacher’s knowledge, or if there are components of MKT that can function in a distributed fashion where expertise is not required from each individual teacher, but access to expertise in the various areas of MKT could be beneficial. Therefore, it seems that the strength of the MKT framework lies in its organization and conceptualization of key factors for the type of knowledge helpful for teaching mathematics.

*Professionally Situated Knowledge*

Sherin and van Es have been instrumental in developing frameworks for analyzing teacher knowledge with respect to a construct defined as *noticing* (Sherin & Han, 2004; Sherin & van Es, 2005; van Es & Sherin, 2002; van Es & Sherin, 2008). Their framework for analyzing teachers’ ability to notice classroom events has three parts that include 1) identifying noteworthy aspects of a classroom situation, 2) using knowledge about the context to reason about the classroom interactions, and 3) making
connections between the specific classroom events and broader principles of teaching and learning (van Es & Sherin, 2008). Jacobs, Lamb and Philipp (2010) elaborated a similar framework, investigating teachers’ ability to attend to children’s strategies, interpret children’s understandings, and decide how to respond on the basis of children’s understandings. Each framework emphasizes the important knowledge necessary for teachers to observe, reflect on, and determine instruction based on classroom events. Within the noticing framework, it is implicit that teachers’ prior knowledge, beliefs, and experiences impact what they are able to attend to during classroom teaching.

Another framework for analyzing teacher knowledge was developed by Kahan, Cooper, and Bethea (2003), and was intentionally designed to integrate the acts of teaching that research suggests will rely on MKT the most. Similar to the Shulman-Fischbein framework, Kahan et al. (2003) integrated what they coined to be elements of teaching with processes of teaching. Their six elements of teaching included, 1) selecting goals and objectives, 2) selecting tasks and representations, 3) motivation of content, 4) development: connectivity and sequencing, 5) allocation of time and emphasis, and 6) discourse. Their four processes of teaching were, 1) preparation, 2) instruction, 3) assessment, and 4) reflection. The two-dimensional framework essentially maps out 24 different categories within which teachers may require different forms of mathematical knowledge to inform their decision-making. This particular framework is most useful in its characterizing of the variety of instances and facets that mathematical knowledge plays a role in teaching, and encourages investigations of teachers’ mathematical knowledge to be done within the practice of teaching itself.
Powell and Hanna (2006) see the notions of PCK and MKT as being inextricably linked, and argue for the measurement of a teacher’s knowledge for teaching mathematical ideas through observations of in-the-moment interactions with students in particular situations. It is their philosophy that a teacher’s MKT is truly revealed during the practice of teaching, and can be observed in interactions with students, questions they ask, issues they make salient to students, student artifacts they use, as well as post-lesson analyses they perform on their actions, plans, and students’ work (p. 370). Powell and Hanna identify one of the most significant components of a teacher’s MKT as his/her understanding of the status of a student’s knowledge (or level of sophistication), as this enables a teacher to pose new problems to advance their construction of mathematical ideas. Through observations, Powell and Hanna (2006) formulated a three-part framework for analyzing teachers’ MKT. The first component was a teacher’s existing conceptions and knowledge about the mathematics he/she was teaching. The second was an inferential awareness of the students’ existing and evolving knowledge, or the teacher’s ability to make sense of students’ representations and arguments. The third was essentially the knowledge informing decisions intended to assist students in developing their mathematical understandings.

An assumption of frameworks related to observation of teachers practice and noticing is that teachers utilize their knowledge and abilities at observing, analyzing, and determining classroom instruction in their ‘in-the-moment’ decisions in the classroom. Therefore, these frameworks make specific components of teaching explicit, and the construct of noticing represents a form of teachers’ mathematical and pedagogical
knowledge. It is not clear to what degree such frameworks interpret teachers’ knowledge of children’s mathematical thinking (KCS) in their formulation of MKT as related to teacher actions. Pellegrino, Chudowsky & Glaser (2001) argue for a similar three-part triangle framework with cognition, observation, and interpretation as the key elements needed in designing and implementing classroom assessment. In this model, cognition corresponds to teachers’ knowledge of how students think about and develop mathematical ideas. While Pelligrino et al.’s (2001) framework might lack a component that involves teachers’ use of their classroom interpretations; the element of cognition, or teachers’ knowledge of how students think about and develop mathematical ideas, might be a valuable addition to the noticing framework.

Mathematical Development Knowledge

According to Steffe et al. (1983), teachers, as well as researchers may construct models of students’ knowledge in order to explain their observation of student work or activities. These models could serve to inform a teacher’s understanding of the level of thinking a student displays. In order to determine progress in learning, it seems reasonable to assume that teachers could utilize understandings of mathematical development to help determine progression of instruction, or assessing learning.

Franke, Carpenter, Levi, and Fennema (2001) developed a detailed framework for analyzing teachers’ change in their cognitively guided instruction (CGI) program. These levels represented a combination of a teacher’s knowledge and beliefs about children’s problem solving. Franke et al. (2001) placed an emphasis on the importance of beliefs in sustaining a teacher’s emphasis on student thinking in instruction. The framework has
five levels, progressing from minimal eliciting and use of student mathematical thinking in classroom instruction, toward the integration of what individual children know and understand mathematically into research-based understandings of children’s mathematical development. As is evidenced by Levels 3, 4a, and 4b in the framework, teachers must have a deep understanding of the development of mathematical ideas in order to effectively integrate student mathematical thinking into classroom instruction that supports learning. Franke and Kazemi (2001) described that teachers in the CGI program viewed the research-based models of student thinking as a way to make sense of children’s thinking.

Silverman and Thompson (2008) utilized Simon’s notion of a key developmental understanding (KDU), or a conceptual “change in the learner’s ability to think about and/or perceive particular mathematical relationships” (p. 993), to argue for a framework for analyzing teachers’ mathematical knowledge related to the development and use of KDU’s in specific areas of mathematics. As stated by Silverman and Thompson (2008), “teachers who develop KDU’s of particular mathematical ideas can do impressive mathematics with regards to those ideas, but it is not necessarily true that their understandings are powerful pedagogically” (p. 8). The framework therefore accounts for not only the development of KDU’s with respect to specific mathematics content, but also how such knowledge can help teachers to understand students’ learning of related ideas, as well as actions that might support students’ development of mathematical ideas and why such actions might be effective.
The proposed framework contends that a teacher has developed knowledge that supports the effective conceptual teaching of particular mathematical topics when he or she:

(1) has developed a KDU within which that topic exists, (2) has constructed models of the variety of ways students may understand the content (decentering); (3) has an image of how someone else might come to think of the mathematical idea in a similar way; (4) has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea; and (5) has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas. p. 17-18.

Silverman and Thompson’s framework focuses on the constructs of KCS and KCT as described by Ball et al. (2008), and emphasize a deep understanding of children’s mathematical development. This framework is based on the idea that teachers can develop KDU’s, but that knowledge alone does not represent sufficient knowledge to be pedagogically powerful. Citing Steffe (1994) as a basis for the notion that developing MKT involves separating one’s own personal understanding from the hypothetical learning of another, Silverman and Thompson (2008) designed their framework to account for the type of knowledge that would allow teachers to turn their KDU of a specific mathematical topic into something pedagogically powerful for a learner. The framework is influenced by both the teaching of mathematics as well as developmental
trajectories for children’s mathematics learning. It is intended to account for not just
teacher development of MKT, but also the sustainability of developing MKT through
KDU’s, reflection on KDU’s, and utilizing that information in a model of students’
learning (Silverman & Thompson, 2008, p. 20).

The CGI framework as well as the Silverman and Thompson framework place
greater focus on teacher knowledge emphasizing understanding of children’s thinking.
As is demonstrated by the first component of Silverman and Thompson’s framework, it is
crucial for teachers to have personally developed deep understanding of the mathematics.
Both frameworks also identify it as crucial knowledge that teachers not only understand
children’s mathematical thinking, but also have an understanding of how it develops and
the mechanisms that promote growth.

Analysis of Framework Discussion Towards an Integrated Framework

Before analyzing the diversity of perspectives of thinking about teachers’
mathematical knowledge, it is important to consider an additional factor in teacher
knowledge that has been largely overlooked thus far: teacher beliefs and dispositions.
What teachers know is inevitably linked to their sense of purpose as a teacher, philosophy
of teaching and learning, and concept of the discipline they are teaching. Teachers can
hold nurturing and protective views towards their students and unintentionally restrict the
scope of mathematical explorations in an effort to be mindful of students’ emotions.
Some argue that teacher beliefs are observable only in the act of teaching, while others
argue that a teacher’s MKT is also observable only in the act of teaching (Powell &
Hanna, 2006). However, beliefs and dispositions as well as MKT impact teacher
decision making profoundly, and it is impossible to completely disentangle the two. Results from Wilkin’s (2008) study showed that Ernest’s theoretical model of knowledge, beliefs, and attitudes were all found to be influential in teachers’ instructional practice. Specifically, teachers’ attitude toward mathematics and student centered mathematics teaching had a positive effect on teachers’ use of inquiry-based instructional practices, while teachers’ mathematical content knowledge had a negative relationship with teachers’ use of inquiry-based instructional practices. Cooney (1999) argued that “whatever lens we use to describe teachers' knowledge, that lens must account for the way in which knowledge is held and the ability of the teacher to use that knowledge in a reflective, adaptive way” (p. 171).

In the sense that knowledge is only valuable if it is in some way usable, it seems reasonable to conclude that teacher beliefs can play a significant impact on the type of knowledge that is actually used in teaching. A teacher may well have the ability to interpret children’s problem solving approaches, but might view mathematics as a fragmented, rule-based system and therefore discount children’s thinking. In a sense, many of the previous frameworks account for this implicitly due to the fact that the use of teacher knowledge is dependent on the classroom events that teachers are able to observe and interpret. The noticing frameworks make this more explicit, by identifying that what teachers can notice is impacted by past experiences and beliefs. In essence, beliefs and dispositions impact the paradigm through which teachers see classroom interactions, and affect the quality of reflection that can occur on such interactions. Therefore, it is
important to recognize that beliefs and dispositions color every component of a teacher’s knowledge for teaching mathematics.

Although the discussed frameworks by no means represent an exhaustive look at research frameworks and perspectives investigating teachers’ mathematical knowledge used in teaching, they do capture the complexity of the construct of MKT. A unifying theme across the frameworks is the emphasis on the need for teachers to possess potentially pedagogically powerful forms of mathematical knowledge. Ma (1999) speaks about this as PUFM, Zazkis and Leikin (2007) identify expert example spaces, Ball et al. (2008) refer to CCK and SCK, and Silverman and Thompson (2008) discuss KDU’s. All also identify deep content knowledge as being insufficient knowledge for teachers of mathematics. Therefore, teachers of mathematics require a deep mathematical knowledge of the content they are to teach, knowledge that is well connected and meaningful beyond the rote use of rules and procedures.

Several of the frameworks identify the understanding of common student errors and ways of thinking as being significant knowledge for teachers. Ball et al. (2008) identify this as KCS, Tsamir and Tirosh (2008) refer to algorithmic, formal, and intuitive PCK, and Silverman and Thompson refer to the ways students may understand content. The knowledge of common conceptions of students could impact task selection, implementation of lessons, and a teacher’s ability to identify an event in class as significant. This form of knowledge therefore represents a significant component of teachers’ MKT. Therefore, teachers need to have knowledge of the common correct and incorrect conceptions of the mathematics they teach. In order to meaningfully assess
student understanding, teachers must have a sense for what complete and incomplete student understanding looks like. This is not to say that teachers must have an understanding of *every* possible conception students might hold about a mathematical idea, but rather that teachers would need to have the ability to analyze and make sense of a wide variety of student conceptions.

Although not all frameworks identify instruction as a demonstration of MKT, the work of teaching is recognized as a key factor in the frameworks. The *noticing* framework makes it explicit that teachers must *attend* to mathematically significant events in order to interpret and use student thinking in their planning and instruction. It is therefore highly important that teachers have the knowledge to identify or prepare tasks that elicit student thinking that allows them to make determinations about student learning and understanding. Silverman and Thompson (2008) suggest that teachers must have an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of a mathematical idea. It is not clear, however, that this knowledge must be knowledge within the teacher alone. Fennema and Franke (1992) theorized that positive results in their CGI project were due to the fact that the research-based frameworks provided to teachers were specific and well organized. They cite one of their teachers as saying “I have always known that it is important to listen to kids, but before I never knew what questions to ask or what to listen for” (p. 156). It seems plausible to believe that teachers need the ability to discriminate which tasks would be helpful for eliciting student thinking, but do not need to create and construct all of the tasks themselves. Therefore, teachers need to have
discriminating knowledge of eliciting and assessment tasks that allows them to determine which activities and tasks will elicit meaningful feedback about children’s thinking. Teachers need not have the obligation to create all eliciting and assessment tasks from their own personal understanding of mathematics and children’s thinking about mathematics, but need to have the knowledge to determine which activities and assessment tasks allow for deep exploration into what their students know and understand.

Another commonality that is either implicit or explicit in many of the frameworks is a teacher’s ability to make instructional decisions based on student’s level of mathematical development. Ball et al. (2008) identifies KCT as the knowledge that informs ordering of instructional activities, CGI identifies it as significant for teachers to have the ability to integrate student thinking into instruction (Franke et al. 2001), and Silverman and Thompson (2008) suggest that teachers need to have an image of how students have come to think about a mathematical idea in the specified way. This type of knowledge requires teachers to not only know and understand common conceptions students’ hold relative to mathematical ideas, but also a meaningful order in which mathematical ideas develop. In order to effectively use assessment feedback to inform future tasks, teachers need to have some concept of children’s psychological development. Therefore, teachers need to have knowledge of the progression of development of children’s mathematical ideas.

As a whole, the integrated components of the various perspectives and frameworks emphasize that teachers need to have: 1) a disposition towards mathematics
teaching that encourages reflection, 2) to hold beliefs about the value of conceptual and procedural components of mathematics, 3) deep understanding of the mathematics they teach and related mathematical knowledge for teaching mathematics, 4) knowledge of the common correct and incorrect conceptions of the mathematics they teach, 5) discriminating knowledge of tasks that reveal and elicit for assessment of student thinking, and 6) detailed, structured, research-based knowledge of the progression of development of children’s mathematical ideas.

Efforts to Improve Teachers’ Knowledge Base for Teaching Mathematics

Increased knowledge specific to teaching mathematics should correlate to gains in student achievement; as such knowledge should assist teachers in their instructional planning, questioning, and assessment strategies. Utilizing the MKT framework of teacher knowledge significant positive relationships between a teacher’s MKT measure and student achievement have been found (Hill, Rowan, & Ball, 2005; Hill, Ball, Blunk, Goffney, & Rowan, 2007). As stated in Hill et al. (2007):

Comparing a teacher who achieved an average MKT score and a teacher who was in the top quartile of scores, we saw that the above-average teacher “added” an effect equivalent to that of 2 to 3 extra weeks of instruction to her students’ gain scores. (p. 109)

Analyses of classroom teaching allowed Hill et al. (2008) to determine that teachers who scored high on measures of MKT tended to avoid incorrect mathematical representations, and provide instruction that contained deeper explanations, reasoning and meaning (p. 117). In their CGI project, Peterson et al. (1989) found that teachers’
knowledge of student thinking was significantly positively related to students’ problem-solving achievement. Peterson et al. (1989) described teachers with less knowledge of students’ thinking as being more prone to observe students’ “non-verbal responses or ‘solutions’ and explain the problem-solving process to the students, thereby also doing the thinking for the students” (p. 568). Teachers with less knowledge of student thinking gave fewer opportunities to explain their own problem-solving processes, likely contributing to fewer meaningful opportunities to make sense of the mathematical ideas.

Although there is not an abundance of studies linking teachers’ knowledge of mathematics for teaching to student achievement, studies such as those based on MKT and CGI show the potential impact that increasing MKT has on student learning. What follows is a brief summary of different research projects that have sought to impact components of teachers’ knowledge for teaching mathematics. Select findings from these research projects are reported and analyzed through components of the frameworks for teachers’ understanding of children’s mathematical thinking.

*Conceptual Mathematics*

There is no question that teachers need to understand the material they are tasked with teaching. However, a deep content knowledge is not enough to predict a teacher’s use of inquiry-based instructional practices (Wilkins, 2008). Wilkins argued that a teacher’s disposition towards mathematical processes and problem-solving as well as a positive attitude about mathematics might be necessary to encourage him/her to employ more student-focused teaching. Many teachers originally learned mathematics in a context that focused on the learning of rules and procedures, and did not emphasize
problem solving and connections. This lack of exposure to meaningful, conceptual mathematics instruction likely contributes to a tendency for teachers to view mathematics from a highly procedural perspective. Ball (1988), argued for the reformulation of experiences that teachers have with mathematics as a means to complicate their views on what it means to do mathematics. Silver, Clark, Ghousseini, and Charalambous’s (2007) BIFOCAL project found that when teachers engaged in facilitated professional learning tasks, artifacts of practice such as viewing teaching episodes or analyzing samples of student work, they had opportunities to work on and learn mathematics.

Giving teachers experiences that provide opportunities to engage in mathematical thinking and problem-solving allows them to experience mathematics in a more meaningful and connected fashion than they were likely exposed to in their own education. These experiences can serve to increase teachers’ deep understanding of the mathematics they teach by exposing them to a variety of problem-solving approaches and multiple representations. Since teachers tend to teach in the ways in which they learned mathematics, reformulating their experiences as a mathematical learner through professional development could serve to deepen the way in which they view mathematics teaching and learning.

*Focus on Student Thinking*

Many studies have focused on the construct of *noticing*, or teachers’ ability to interpret, analyze, and use information from classroom events to inform classroom instruction. The ability to *notice* is not something that develops from teaching experience alone (Jacobs et al., 2010), but rather requires professional development related to
children’s thinking (Jacobs et al. 2010; Sherin & Han, 2004; Sherin & van Es, 2005; van Es & Sherin, 2002; van Es & Sherin, 2008). In these studies, teachers engaged in collaborative group meetings structured around watching videos of classroom events, or analyzing student work, with researchers emphasizing a focus on student mathematical thinking. Kazemi and Franke (2004) also identified that teacher learning with respect to analyzing student thinking had the potential to be generative, as teachers involved in group discussions became increasingly engaged and reflective without the need for focused researcher-led discussions.

Many of these studies also identified that teachers benefited from professional development focused on student thinking that was integrated with their experiences in classes. Student work and video samples were used from teachers’ classrooms to help formalize the understanding of children’s thinking that emerged through the professional development (van Es and Sherin, 2002). Heckaman (2008) found that integrating teachers’ learning about learning theories with specific instructional tasks helped teachers to attend to more specific components of children’s thinking. Even (2005) found that when teachers received professional development related to assessment emphasizing process rather than product, teachers began modifying the tasks they had previously used in their instruction so that they were more open-ended and allowed students to explain their solutions (p. 54).

Morris, Hiebert, and Spitzer (2009) identify understanding learning goals, or the learning required to achieve the goals of instruction, as essential for judging whether instruction was effective. In studying pre-service teachers, Morris et al. (2009) focused
on building teachers’ SCK by engaging them in the unpacking learning goals of mathematical lessons. They hypothesized that this type of knowledge would be worthwhile to focus on because “these competencies are essential for all further analyses of mathematics teaching and are potentially learnable during teacher preparation” (p. 495). Results from their study showed that although many of the pre-service teachers displayed the necessary SCK, it was often not used in evaluating teaching and learning. Morris et al. (2009) see their work with pre-service teachers as representing a possible path for teachers to learn through their teaching. Research indicates that attending to children’s mathematical thinking is not something that is likely to emerge from teaching experience alone (Jacobs et al., 2010). Professional development, and pre-service education targeted at exposing teachers to opportunities to engage in intentionally investigating the learning goals of their lessons, and making attempts to collect and interpret evidence of student learning seem to offer teachers with a possible lens through which to view teaching and learning. As was displayed by several research projects, with intentional and focused emphases on identifying and reflecting on student mathematical thinking, teacher learning of this kind could be generative. It can also promote a desire to develop new eliciting knowledge, targeted at accessing deeper forms of student understanding.

Modified Teacher Learning Environments

One of the main influences on the instructional choices teachers make is past experiences in mathematics classrooms and prior models of teaching strategies. As traditional teacher-centered instructional strategies are extremely common, it is highly
likely that teachers are at least partially influenced by a model of mathematics instruction that
does not emphasize problem-solving, representations, communication, connections, and reasoning and proof. It is also likely that past mathematical experiences have placed heavy emphasis on symbolic manipulations, rote computation, and formula-based procedures. To counter some of these potentially limiting models of mathematics instruction, some research projects and teacher preparation programs have aimed to introduce in-service and pre-service teachers to coursework and professional development that serves as a model of student-centered learning (Cady, Meier, & Lubinski, 2006; Darling-Hammond, 2006; Darling-Hammond & Bransford, 2005; Tom, 1997). Such models give teachers a new set of experiences as students in a setting that more closely resembles the type of instruction being espoused by mathematics education researchers.

Studies investigating the impact of these new learning environments on teachers have shown that these more student-centered experiences encouraged more frequent use of physical and virtual manipulatives in instruction, and greater readiness and preparedness for diverse student thinking (Cady et al, 2006). Explicit modeling of reflection in these new learning environments also encouraged teachers to more carefully analyze their instruction, and allowed teachers to slow down decision-making in order to more carefully analyze and incorporate student feedback (van Es & Sherin, 2008). While it is unclear the extent to which the new learning environment itself impacted changes in teacher practice, teachers regularly cite professional development and teacher preparation as a source for models of effective teaching (Frost, 2009).
Changes in Assessment Strategies

Fuchs et al. (1999) made the case that teachers tend to teach towards the content of tests regardless of the style or format of the test, and researchers have investigated how specific and intentional changes to assessment strategies impact teachers’ beliefs and instruction. Even (2005) found that as teachers learned about and changed their assessment to include portfolios and other novel strategies, the teachers’ quality and depth of questioning and problem posing improved (p. 53). Similarly, Fuchs et al. (1999), found that when teachers learned about and implemented performance assessments, authentic problem-solving dilemmas that require students to develop solutions involving applications of multiple skills and strategies, teachers were led to include markedly more problem solving activities in class, with significant changes in the emphasis of the process standards (p. 631). These studies provide evidence that teachers are capable of implementing teaching strategies with greater emphasis on student-centered, reform-based principles, and that encouraging a broader range and understanding of assessment strategies can lead teachers to focus instruction on deeper learning.

Heckaman, Thompson, Hull, and Ernest (2008) provided evidence that similar changes can be effective in initial teacher preparation as well. In their study, pre-service teachers were required to implement an evidence-based strategy in field experience that aligned with course readings and university-based learning. They found that given multiple opportunities to practice and study evidence-based strategies, pre-service teacher candidates were able to begin building in evidence-based strategies into their daily
teaching routines. In my own pilot study (Harrison & Harrison, 2010), pre-service teachers were required to prepare a lesson in which they intentionally encouraged greater student communication while limiting the amount of teacher-talk in order to focus intensely on observations of student thinking. Although not all pre-service teachers responded similarly to the task, the opportunity to observe students more carefully inspired several to consider increasing classroom opportunities for students to share their thinking in order to better inform their understanding of their students’ mathematical thinking.

Video and Student Work Analysis

The most substantial set of research related to teacher learning and instructional change is based around classroom video and student work analysis. Elizabeth van Es and Miriam Sherin have done extensive work related to the construct of noticing in mathematics education, focusing on teachers’ ability to describe, analyze, and interpret student thinking. The bulk of their work focuses on teachers’ engagement in video clubs that are designed to give in-service teachers an opportunity to watch and analyze episodes of their own classroom instruction. Participants in the video clubs engaged in regular meetings facilitated by one of the researchers. As teachers’ participation in the video clubs progressed, there was a noticeable shift away from descriptions towards analysis of classroom events (Sherin & van Es, 2005). This was accompanied by a shift away from a focus on general pedagogy and classroom management towards specific analysis of student mathematical thinking. The use of videos allowed teachers to begin to focus more carefully on specific events, and teachers’ began to cite more concrete forms of
evidence as opposed to vague, misremembered recollections of classroom events (van Es & Sherin 2002; van Es & Sherin, 2008). One of the most significant findings of their research on noticing was the noticeable shift in teachers’ use of evidence to inform instructional decision-making (Sherin & van Es, 2005, p. 487). This provides strong evidence that the teacher learning from the video clubs with respect to noticing not only increased teacher knowledge, but also resulted in changed practice.

Jacobs et al. (2010) recently explored this construct and definitively showed that noticing is not something most teachers engage in regularly. Jacobs et al. showed that teachers’ ability to analyze and interpret classroom events was not learned from teaching experience alone, but rather through the support of professional development focused on student mathematical thinking. As the number of years of sustained professional development on student mathematical thinking increased, so too did teachers’ ability to attend to and analyze classroom events, as well as make evidence-based instructional decisions. Although Jacob et al.’s study was disconnected from classroom practice; it provides evidence that advancements in teachers’ understanding of noticing do not take place independent of professional development and through teaching experience alone.

Kazemi and Franke (2004) observed similar progressions of teacher learning as teachers participated in a workgroup of ten other teachers analyzing written and oral student work. Initially, teachers exhibited a lack of awareness and attention to students’ mathematical thinking and strategies. This progressed over time, as teachers began to more carefully structure tasks to elicit student thinking in their own classrooms, and eventually culminated in discussions about how to utilize the knowledge of student
thinking to design instructional trajectories. Kazemi and Franke (2004) used interpretations based on the notion of *transformation of participation* in the workgroups, identifying that teacher communication and discourse about the eliciting and interpretation of student thinking was crucial to the development of new perspectives (p. 229).

Further research by van Es and Sherin (2008) found even more detailed accounts of the shifts in teachers’ ability to notice as a result of *video club* participation. Initially, teachers’ attention focused almost exclusively on teacher behaviors, but as analyses became more focused and specific, this changed to a student-centered focus. Possibly of most interest is that teachers’ analyses shifted away from interpretations based upon little to no evidence, to an emphasis on identifying specific and targeted evidence of student thinking. Van Es and Sherin (2008) connected this to classroom practice with the finding that teachers adapted their classroom questioning to be far more broad and diverse in order to better facilitate their collection of useful and substantive evidence of understanding (p. 266).

Further elaborating some of the classroom practice-based findings, Sherin and van Es (2009) argued that video study encouraged the development of selective attention, or a teacher’s ability to more carefully focus on specific form of evidence of student thinking. This was evidenced in teachers’ shift away from simple restatements of student ideas towards follow-up questioning such as “what do you mean by that?” that allowed for more student-based evidence to be collected (p. 31). This shift was also evidenced by teachers’ utilization of student ideas in the classroom as opportunities for open inquiry.
Teachers became more analytical and more curious about student thinking, almost as though the depth and substance of the video club discussions gave the teachers a new lens to view classroom interactions through. This notion is supported by von Glasersfeld (1993) who suggested, “If one succeeds in getting teachers to make a serious effort to apply some of the constructivist methodology, even if they don’t believe in it, they become enthralled after five or six weeks” (p. 37).

In a similar fashion to van Es and Sherin (2008), Dieker et al. (2009) investigated the impact of video versus transcript analysis on teacher learning about student thinking and its impact on classroom practice. Their study suggested that those teachers who participated in video-based groups as opposed to transcript-based groups showed more substantive advances in noticing abilities as well as implementation. Dieker et al. argued that the video of actual classroom practice from participating teacher’s classrooms led to increased confidence and efficacy in the ability to implement teaching strategies that elicited useful student thinking. As it is commonly the case that teachers espouse beliefs about teaching and learning that differ from their actual enacted beliefs, it seems reasonable that connecting teacher learning to actual classroom practice might serve to bring these disconnected perspectives closer together.

As is evidenced by this portion of the review, there is a significant amount of evidence that teachers can learn a significant amount about children’s mathematical thinking in a variety of forms. Professional development can be beneficial in helping teachers build the knowledge of children’s mathematical thinking that can become pedagogically powerful. The next section of this literature review investigates some of
the research on how teachers can come to learn about children’s thinking through the use of research-based frameworks of children’s thinking.

Research on Teacher Use of Research-Based Frameworks and Learning Trajectories/Progressions

Beyond placing an emphasis on attending to student thinking in planning, instruction and reflection, some research projects have investigated the impact that research-based models of student thinking have on teachers. Recent work by researchers in mathematics education has focused on carefully understanding students’ thinking about mathematical ideas and the development of mathematical concepts (Bardsley, 2006; Battista, 2003; Battista, 2004; Carpenter, Fennema, Franke, Levi & Empson, 2000; Franke, Carpenter, Fennema, Ansell & Behrend, 1998; Wilson, 2009). Cognitively guided instruction (CGI) research (Carpenter et al., 2000; Franke et al, 1998) assisted teachers in identifying what students know and understand about elementary mathematical concepts, and building instruction directly from children’s abilities. The CGI project developed a framework for the general levels that students pass through when learning the concepts and procedures in addition and subtraction (Peterson et al., 1989, p. 559). Peterson et al. (1989) found that CGI teachers often lacked the kind of research-based knowledge of student thinking contained in the researcher developed frameworks. Peterson et al. theorized that “perhaps it is not surprising that most teachers do not have this explicit and precise in depth knowledge, given that it took researchers many years to specify this knowledge clearly” (p. 559). CGI research has been connected to gains in student achievement (Carpenter et al, 2000), and has also been shown to be an
activity that teachers can engage in as an ongoing inquiry into students’ mathematical thinking that allows teacher growth and development to be sustained and generative (Franke et al, 1998).

In describing aspects of the CGI project, Kazemi and Franke (2001) suggested that the research-based frameworks provided teachers with a way of thinking about how children learned mathematics, and that knowing a sequence of strategies for student’s thinking allowed teachers to interpret why a particular problem may be difficult for children (p. 103). Kazemi and Franke (2001) identified that as teachers learned more about the research-based frameworks, they began listening to their students more carefully, and transformed teachers into students of teaching, learning about their practice. As teachers came to better understand student mathematical thinking, the researchers observed teachers generative growth, suggesting, “when individuals learn with understanding, they can apply their knowledge to learn new topics and solve new and unfamiliar problems” (p. 105).

However, it should be noted that access to research-based frameworks or curriculum materials based on children’s development alone does not imply that teachers have a very deep understanding of children’s mathematical development. Steinberg (2004) found that a research-based understanding of children’s development was not sufficient to promote teacher growth, and it was not until teachers conducted practical inquiries into children’s thinking built around “concrete particulars” of their own practical situations. Gearhart and Saxe (2004) argued that access to good curriculum materials alone is insufficient, and that teachers are more likely to teach in ways that
promote learning when they have a deeper understanding of the mathematics they are teaching.

Hypothetical learning trajectories (HLT) also represent research-based models for how mathematical ideas develop that can assist in defining goals for children’s mathematical progress (Simon, 1995). Such research-based frameworks could serve as support for teachers to better understand how children think about mathematical concepts, and to assist in framing learning goals for students. Bardsley (2006) suggested that teachers’ use of Pre-K learning trajectories differed by teachers’ approach to learning. Teachers who sought mathematics activities and quality curriculum materials tended to simply try to implement the activities and move children to higher levels. However, teachers who sought to better understand Pre-K mathematics development began making predictions about their students’ behavior and developing ways to shape instruction to fit their needs. From a practical perspective, Bardsley also identified that the teachers seeking quality curriculum materials often felt overwhelmed by the learning trajectory documents, while those seeking better understanding developed mechanisms to help make the learning trajectory documents meaningful and useful. Bardsley also found that teachers generally utilized pre-kindergarten HLT’s based on the goals they hoped the HLT would help them attain. Those teachers seeking to use HLT’s as a means to accessing more mathematics activities generally tried to move students linearly through the curriculum, while teachers who sought to use the HLT’s to better understand the development of early childhood mathematics tended to use the materials to support instruction. Wilson (2009) also investigated teachers’ use of elementary learning
trajectories, with findings similar to Bardsley (2006). Wilson argued that teachers need a diverse array of instructional and assessment tasks to support the use of trajectories of children’s development. Examples of student work exemplifying different levels of children’s thinking were helpful for teachers, but assessment tasks were necessary to help keep teachers informed about how their students’ mathematical thinking was progressing. Wilson also found that the use of an HLT for equipartitioning assisted teachers in developing more accurate models of children’s thinking, deepened their own understanding of the mathematics content, and helped teachers facilitate instructional decisions by informing them of what students should learn next.

Mathematics education research has contributed significantly to the resources available for professional development of teachers with respect to research-based models of student mathematical thinking. Learning progressions of children’s mathematical development could serve as a mechanism to provide teachers with a lens through which to interpret student thinking. In order for teachers to make informed decisions on how best to instruct children, it is important to have an understanding for how children’s mathematical sense-making develops. Therefore, learning progressions and other research-based models of children’s thinking can serve as a source for teachers’ understanding of the types of tasks that can be useful in eliciting and assessing student understanding.

The approach that researchers take to helping teachers learn about research-based frameworks can differ between projects. Projects like Kazemi and Franke (2004) approach exposure to student thinking in the form of a professional learning community
where teachers experiment with different eliciting tasks in their classrooms. They found that over the course of time, teachers were able to identify increasingly more sophisticated and specific components of children’s mathematical thinking, and even were able to generate potential instructional trajectories for student learning (Kazemi & Franke, 2004). Jacobs, Lamb, and Phillipp (2010) focused more on evaluating the impact that professional development focused on children’s thinking had on teachers’ ability to observe and analyze student work, finding that more experience with professional development of this kind helped teachers better perceive of how children were thinking about mathematical problems.

Other research projects, like North Carolina State University (Confrey, Maloney, Nguyen, Mojica & Myers, 2009; Maloney & Confrey, 2010), and the CBA project have taken a more direct approach, seeing that the teachers are given a version of the research-based frameworks initially, and provided professional development on understanding the learning progressions and trajectories prior to working with students. The thought behind this approach is that the researcher designed frameworks are complex and detailed enough that it would be unlikely for teachers to be able to construct them through simply observing children’s thinking in action. Instead, these projects focus on elaborating and explaining their research-based findings, to help construct a framework through which teachers can analyze children’s work.

Significance of Study

As was discussed in this chapter, the knowledge necessary to teach mathematics is not a simple construct. There is a significant body of research literature documenting
efforts to help teachers learn and develop aspects of this diverse array of knowledge for teaching mathematics. HLT’s often provide a very detailed, research-based, organizational framework that can support teachers’ efforts to make sense of children’s mathematical thinking, design eliciting and assessment tasks to expose children’s mathematical conceptions, and understand children’s mathematical development.

The qualitative nature of this study allows for a detailed exploration into how teachers understand and use a learning progression of children’s mathematical thinking for multiplication and division. The proposed inquiry addresses teachers’ conceptualizations of children’s thinking using the CBA learning progression, investigates how learning progressions can inform learning goal determination as well as instructional design, and explores the potential learning opportunities learning progressions can provide teachers. All of these areas are important to understand as learning progressions begin to become a more prominent fixture in teachers’ lives, and research on how teachers can learn from and use learning progressions will be beneficial.
Chapter 3: Methodology

As research on teachers’ understanding and use of learning progressions is in an early, exploratory stage, qualitative, descriptive research can serve to help explicate some of the core issues and modes of inquiry in this realm of teacher knowledge and learning research. Although quantitative methodology is not uncommon in the research design of many projects focused on designing learning progressions, the available research on teachers’ learning of learning progressions is largely qualitative in nature (Bardsley, 2006; Battista, 2007; Mojica, 2009; Wilson, 2009). In consideration of the complex construct of teacher knowledge, and the largely unexplored field of teacher learning of learning progressions, a qualitative and exploratory research project that allows for grounded theorizing and retrospective analysis is called for.

This chapter begins with a brief description of the CBA2 research project that informed the research inquiry, and is followed by a brief description of the qualitative research design and description of participants in the study. Next, the collected data from the CBA2 project that informed the research inquiry is described in detail, with samples of interview protocols included within the chapter. Then, an elaboration of each research question is provided, detailing the data that informed answers to each question. Finally,
data analysis procedures are explored, along with issues of validity, reliability, and trustworthiness.

Description of the CBA2 Project

The larger project that this research is a subset of is the CBA2 for Cognition Based Assessment, Phase 2 (An Investigation of Elementary Teachers' Learning, Understanding, and Use of Research-Based Knowledge about Students' Mathematical Thinking, ESI 0554470). CBA2 began in June of 2006, with the goal to investigate how elementary teachers make sense of and use research-based knowledge about the development of students’ reasoning about particular mathematical topics (like multiplication and division, place value, or length), as represented by teacher-friendly materials such as the Cognition Based Assessment (CBA) materials written by principal investigator (PI) Michael Battista.

CBA Materials Development

In Phase 1 of the Cognition Based Assessment (CBA) project, the principal investigator developed CBA materials, which include four critical components.

1. Descriptions of core mathematical ideas and reasoning processes that form the foundation for students' sense making and understanding of elementary school mathematics.
2. For each core idea, research-based descriptions of the cognitive constructions students make in their attempts to understand the idea.
3. For each core idea, coherent sets of assessment tasks that enable teachers to investigate their students' mathematical thinking and precisely locate students' positions in the cognitive terrain for learning that idea.
4. For each core idea, advice on instructional activities that can help students progress to higher levels of sophistication.

The CBA project has developed assessment tasks and interpretative frameworks for eight major mathematical topic strands.

<table>
<thead>
<tr>
<th>NUMBER</th>
<th>GEOMETRY</th>
</tr>
</thead>
<tbody>
<tr>
<td>addition and subtraction of whole numbers</td>
<td>length</td>
</tr>
<tr>
<td>whole-number place value</td>
<td>area</td>
</tr>
<tr>
<td>multiplication and division of whole-numbers</td>
<td>volume</td>
</tr>
<tr>
<td>fractions</td>
<td>geometric properties of shapes</td>
</tr>
</tbody>
</table>

Figure 1 – CBA topic strands

_CBA2 Project Activities June 2006 – February 2008_

_Summer 2006_

During the summer of 2006, the principal investigator conducted a workshop on CBA for 20 teacher leaders in Ohio. This was an excellent opportunity to collect data for the CBA2 project and to pilot test CBA2 instructional and interview tasks that would be used in subsequent academic ‘Year 1’ teacher case-studies. Data was collected on teachers' understanding of and thinking about the CBA ideas discussed in the workshop.

_Academic Year 1: August 2006 – June 2007_

In the fall of 2006, 14 case studies of teachers’ learning of CBA ideas were initiated —5 in Ohio, 9 in Michigan. The teachers had a range of teaching experience, from two Michigan State University interns, to teachers with 30 years of teaching.
experience. The school curricula that teachers used ranged widely, with some using reform and others traditional curricula. The school districts of the case-study teachers ranged widely too, from suburban to urban schools. As an introduction to CBA ideas, all teachers first studied only the CBA Length document. Following an initial concentration on the length document, teachers were free to study and use any two additional CBA units they thought would be most helpful to them (some teachers chose to use three or more CBA units).

Because teachers were in different schools and states, using different curricula, at different grades, and chose different CBA units, each case was quite different. However, the case studies had several common components. One commonality was that each teacher read a CBA document, then participated in a one-on-one interview on the content of that document with a CBA researcher. Detailed structured interviews were created for each CBA levels document. Of importance was what teachers understood and did not understand, and how they constructed understanding of the CBA ideas.

A second commonality was that all teachers used CBA materials (assessment tasks and CBA interpretive framework documents) to examine their own students’ learning. Almost all teachers did individual interviews of their students and some whole-class assessment. Again, detailed structured interviews were constructed for interviewing teachers about how they interpreted their own students’ work on CBA assessment tasks. Questions in the interview protocol included: “What CBA level of reasoning is exhibited by this student on this task? What level of reasoning do you think this student should move to next? What instructional tasks do you think might help this student?”
Additionally, some teachers conducted small-scale teaching experiments that attempted to move a student, a small group of students, or an entire classroom of students, forward on their learning based on CBA analyses. As part of this component of the research, all teachers sat with CBA2 researchers and discussed videotapes and written work of their students on CBA tasks. During these CBA2 interviews, which also were videotaped, researchers had teachers discuss their interpretations of their students’ work, using CBA conceptual frameworks.

*Academic Year 2: August 2007 – June 2008*

During the final interviews of CBA2 teachers in June 2007, all but one teacher expressed an interest in continuing with the CBA2 project the following year. A common feeling expressed by teachers was that during the first year of the project, they learned a tremendous amount about CBA, about their students' mathematics learning, and about their teaching, but they did not have an opportunity to fully incorporate what they had learned into their teaching. They wanted to continue with CBA so that they had a chance to use CBA ideas to change their teaching during this second year. Of the 14 participating first-year teachers, 13 wanted to continue. Unfortunately, even though these 13 teachers had committed to continuing through August, two of them did not have jobs in September, and one had to drop out for personal reasons. So the project continued with 10 of the original teachers. (Note that all 10 of these teachers knew how much of a time commitment participation in the CBA2 project was but still wanted to continue their participation.)
The fact that Year 1 participating teachers wanted to participate for a second year offered the CBA2 project a wonderful opportunity to learn a tremendous amount about teachers' learning, an opportunity that could not be passed up. Instead of enlisting a whole new set of teachers to participate in the CBA2 project, it was decided that the research goal of the CBA2 project would be best served by continuing these case studies into the second year. Seven additional new teachers were enlisted to participate in the CBA2 case studies (all in Ohio: two additional teachers in each of the original two participating districts, and a cluster of three teachers in an urban Ohio district). In total then, there were 17 teacher case studies this year\(^1\) (which started with the classroom teaching experiments in the summer workshops).

Year 2 began with workshops in August, attended by new and veteran CBA2 teachers, one workshop in Michigan, one in Ohio. These workshops introduced and reviewed CBA ideas and materials, and allowed veteran teachers to share what they did and learned, what they struggled with, and how they were planning to use CBA this year.

Participating teachers in Year 2 were required to select 3 CBA documents. For each document, after reading the document and being interviewed on that reading, teachers implement one type of instructional use of the ideas in the document:

(a) use CBA to assess and teach their whole class;

(b) use CBA to assess and teach a small group of students;

\(^1\) Note that going into September, there were an additional 5 case studies, for a total of 22. However, at the last minute, 5 teachers dropped out, 2 because they didn't get teaching jobs, 3 because of personal reasons. At that point, it was too late to add additional case studies because the project had already conducted the August CBA2 workshops.
(c) use CBA to assess and teach several individual students.

One difference between the case studies in Years 1 and 2 is that in Year 2 teachers were required to do some work with their own students on each CBA document they choose to use (not just one). Allowing teachers to choose which CBA document to use and when to use it enables teachers to connect their CBA work directly with their teaching. This plan (a) gave teachers the opportunity to try CBA ideas with their own students (which teachers in Year 1 told investigators was critical for their learning of CBA ideas), and (b) connected CBA work directly to what teachers are teaching (which teachers told investigators was critical for their use of CBA ideas).

During Year 1 (and this is above and beyond the scope of the CBA2 project), the principal investigator rewrote all of the CBA documents in ways that the research from Year 1 indicated would help teachers learn CBA ideas. First, re-writes attempted to make the language and ideas in CBA more teacher-friendly. Second, hints for instructional activities were added, keyed to CBA levels. That is, the principal investigator described what kinds of instructional activities could help students at each CBA level.

*Activities, Academic Year 2: August 2007 – February 2009*

The project continued with case studies of 17 practicing teachers. Some teachers are in the third year of CBA, some in their second year, some in the first year. The project also has some teachers working together—a new teacher working with a CBA teacher with at least one year of experience. The project examined not only the individual teachers in these collaborations, but the interaction of the teachers in the collaborations.
Design of the Study

Research literature on teachers’ understanding and use of learning progressions has yet to become more prevalent, therefore the goal of this research project is designed to add to the existing literature by providing detailed accounts of teachers’ mental processes, actions, and reasoning while using learning progression materials. By utilizing clinical interviews, written interviews, and video-taped teaching experiment interview data; detailed descriptions, grounded theories, and vignettes of teachers’ understanding and use of learning progressions were developed, as well as models of teachers’ ways of reasoning about children’s mathematical thinking while utilizing CBA research-based learning progression framework for multiplication and division. For the sake of manageability, and to help focus the research more acutely, only the CBA multiplication and division materials (CBA MD) (primarily those related to the distributive property) were investigated in the proposed research.

Participants

Participants for this study included as many as 17 elementary teachers (depending on the type of data investigated), in the state of Ohio or Michigan, who were involved in the CBA2 research project. All teachers had experience with at least 3 CBA content domains, but not all had the same experiences, and some had not spent much if any time with the CBA multiplication and division materials. For clinical interview tasks and online interview tasks involving analysis of student work, it was not necessary for teachers to have had experience using the multiplication and division CBA document (i.e.
teachers could analyze student work in multiplication and division without having had practical experience with the CBA materials for multiplication and division). However, for teaching experiments and classroom episodes conducted on multiplication and division, the teachers included in the research had greater experience with the CBA multiplication and division materials. This was necessary, because in order to conduct a one-on-one interview with a student on multiplication and division, a higher level of engagement with the CBA materials was needed.

Description of Collected Data

The CBA2 research team collected a significant amount of data from participating teachers from 2006 – 2010. The data described for the research inquiry represented only a portion of the collected data from the CBA2 project. Much of the collected data was in the form of electronic QuickTime video clips of interviews between CBA2 researchers and CBA2 teachers. A variety of collected data sources were utilized to allow for thorough data analysis that allowed for triangulation of data to help strengthen the trustworthiness of the research (Lincoln & Guba, 1985).

Clinical Interviews

There are a total of five different forms of clinical interviews that were used in analysis that occurred between 2006 and 2009. The interview protocols were for: 1) the Multiplication and Division Interview 1 from 2006 – 2007 (Appendix A), 2) the Multiplication and Division 4 interview from 2007 – 2008 (Appendix B), 3) the Multiplication and Division Interview 1 from 2007 – 2008 (Appendix C), 4) the Multiplication and Division 2 interview from 2008 – 2009 (Appendix D), and 5) the
Multiplication and Division Interview 1 from 2008 – 2009 (Appendix E). Each interview protocol is described separately, and samples of interview questions from each attached appendix are provided.

Multiplication and Division Interview 1 (Appendix A), was administered to seven teachers during the 2006 – 2007 school year. Teachers were asked to evaluate student work on multiplication and division problems, as well as analyze elements of the CBA multiplication and division materials. A sample of student work and subsequent interview protocol follows: Sample of Appendix A.

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>•</td>
<td>What CBA level for multiplication and division is demonstrated by this student?</td>
</tr>
<tr>
<td>•</td>
<td>What makes you think that [say teacher’s answer] is the CBA level?</td>
</tr>
<tr>
<td>•</td>
<td>Tell me what part of the CBA document convinces you that you are correct.</td>
</tr>
<tr>
<td>o</td>
<td>The CBA author’s level is MD3.1. Using Known Facts with Numbers NOT Decomposed by Place Value.</td>
</tr>
<tr>
<td>o</td>
<td>[If the teacher’s level is different from author’s]</td>
</tr>
<tr>
<td>o</td>
<td>Look at the CBA descriptions for your level and the author’s level. What do you think? Which level do you think is correct and why?</td>
</tr>
<tr>
<td>•</td>
<td>What CBA level do you think this student should move to next? Why?</td>
</tr>
<tr>
<td>o</td>
<td>(Author: MD3.2. Using the Distributive Property with Numbers Decomposed by Place Value; MD3.2.1 first, then MD3.2.2)</td>
</tr>
<tr>
<td>•</td>
<td>What instructional activity or task do you think might move this student to this next level?</td>
</tr>
</tbody>
</table>

Figure 2 – Appendix A Sample

The Multiplication and Division 4 interview (Appendix B) was administered to 14 teachers during the 2007 – 2008 school year. Teachers were asked to evaluate student work on a specific multiplication problem, determine the CBA level of sophistication, determine a learning goal, and design instruction. A sample of the student work and subsequent interview protocol follows: Sample of Appendix B.
Task. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?

RX: 12, 24, 36, 48, 60, 72 [raising one finger after reciting each number]. So, there’s 72 players.

• What CBA level for multiplication and division is demonstrated by this student?
• What makes you think that this is the CBA level?
• What part of the CBA document convinces you that you are correct?
• What CBA level do you think this student should move to next? Why?
• What instructional activity or task do you think might move this student to this next level?

Figure 3 – Appendix B Sample

Multiplication and Division Interview 1 from 2007 – 2008 (Appendix C), was administered to eight teachers during the 2007 – 2008 school year. Teachers were asked to evaluate student work on multiplication and division problems, as well as analyze elements of the CBA multiplication and division materials. This interview protocol very closely matches the interview protocol from 2006 – 2007 (Appendix A).

Multiplication and Division 2 interview (Appendix D), was administered to fourteen teachers during the 2008 – 2009 school year. Teachers were asked to evaluate student work on a specific multiplication problem, determine the CBA level of sophistication, determine a learning goal, and design instruction. A sample of the student work and subsequent interview protocol follows: Sample of Appendix D.
Problem: 45 × 23 = ______

<table>
<thead>
<tr>
<th>Sally:</th>
<th>45 times 10 is 450. 45 times another 10 is 450; that’s 900.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

(a) is Sally’s reasoning correct or incorrect? Explain why.
(b) what CBA level of sophistication do you think Sally’s strategy is?
(c) what type of reasoning do you think Sally should move to next? Explain why?
(d) What would you do instructionally to move Sally to this next type of reasoning?

Figure 4 – Appendix D Sample

Multiplication and Division Interview 1 from 2008 – 2009 (Appendix E), was administered to five teachers during the 2006 – 2007 school year. Teachers were asked to evaluate student work on multiplication and division problems, as well as analyze elements of the CBA multiplication and division materials. This interview protocol very closely matches the interview protocol from 2006 – 2007 (Appendix A).

*Online Interviews*

The online interviews (Appendix F) were administered yearly from 2007 until 2010, and were completed by a total of 11 or more CBA teachers. The online interviews were completed by CBA teachers on their own time, and returned to the CBA researchers. The online interviews included general questions about how CBA teachers used the CBA multiplication and division materials, analysis of student work, and questions investigating CBA teachers understanding of CBA levels of sophistication. A sample of the online interview (Appendix F) is provided.
Please read this CBA Multiplication and Division document passage.

**CBA Multiplication and Division Level 2. Iterates Numbers**

Students progress to iteration when they no longer need to count the ones within groups. To **iterate** a number means to construct an ordered list of its multiples. For example, when 2 is iterated, we get 2, 4, 6, 8, and so on. In this case, the multiples of 2 accumulate subtotals in successively more groups of 2. That is, one group of 2 is 2, two groups of 2 is 4, three groups of 2 is 6, four groups of 2 is 8, and so on. To use iteration with understanding, students must coordinate two counting schemes, one for the subtotals, and one for the number of groups.

<table>
<thead>
<tr>
<th>Count of Groups</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count of Subtotals in Groups</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>

Students iterate numbers in various ways. Most commonly they use repeated addition or skip-counting.

- Describe in your own words what this CBA passage means.
- Does the passage make sense to you? If not, describe what is confusing about it.
- How would you reword the passage to make it more understandable?

Figure 5 – Appendix F Sample

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**Teaching Experiment Interviews**

As part of participation in the CBA research project, CBA teachers were encouraged to create an interview protocol that included a diversity of CBA multiplication and division tasks to be used while working with a single student or a small group of students. CBA teachers videotaped their teaching experiments in which they worked with students on multiplication and division problems, with the goal of helping the students make progress in their thinking. CBA teachers provided the CBA research team with the videotaped teaching experiments, as well as copies of the student work from the teaching experiments. Descriptive vignettes were constructed from three detailed teaching experiments and assessments. Specific teaching experiments were selected based on a preliminary viewing of all teaching experiments, and the three chosen
were based on the richness of the case, and they will be selected for maximal variation (Patton, 1990). The three selected cases represent one teacher’s conducting a one-on-one CBA assessment with one student, the second teacher conducting a one-on-one teaching experiment with two students individually, and the third teacher selected conducted a full class CBA assessment and designed instruction and posttest assessments. Each case represented a different use of CBA, and allowed for a more varied, and rich analysis.

Connection Between Data and Research Questions

The following table summarizes the various data sources and how they were used to inform answers to the research questions. For analysis of student work, some data sources were used for both research questions. The purpose of this table is to help clarify what parts of interviews of teachers’ analysis of student work were used to help inform which research question.

Research Question 1: In CBA2 teachers' analysis of student work, how do they conceptualize student reasoning on multiplication and division problems, and what relationships exist between CBA teachers’ knowledge and understanding of CBA Multiplication and Division materials, their conceptualizations of student thinking, and their determination of instructional learning goals and tasks?
<table>
<thead>
<tr>
<th>Data Source</th>
<th>Sample Size</th>
<th>Description of Data Source and Sample of Interview Protocol</th>
<th>Affordances for Research Inquiry</th>
</tr>
</thead>
</table>
| Face to Face Interviews Appendices A, C, and E | 6 | **Task.** $46 \times 5$  
RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230. | Allows for the interpretation of teachers’ choice of CBA level for student thinking, as well as interpretation of CBA learning goal, and instructional tasks and plans. |

What CBA level for multiplication and division is demonstrated by this student?

What makes you think that [say teacher’s answer] is the CBA level?

Tell me what part of the CBA document convinces you that you are correct.

The CBA author’s level is **MD3.1. Using Known Facts with Numbers NOT Decomposed by Place Value.**

Look at the CBA descriptions for your level and the author’s level. What do you think? Which level do you think is correct and why?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?

---

Table 1 – Summary of Data for Research Question 1

Continued
### Table 1 Continued

<table>
<thead>
<tr>
<th>Data Source</th>
<th>Sample Size</th>
<th>Description of Data Source and Sample of Interview Protocol</th>
<th>Affordances for Research Inquiry</th>
</tr>
</thead>
</table>
| MD4 07.08 – Appendix B | 14 | What CBA level for multiplication and division is demonstrated by this student?  

**Task.** There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?  

SX: 12, 24, 36, 48, 60, 72 [raising one finger after reciting each number]. So, there’s 72 players. | Allows for the interpretation of teachers’ choice of CBA level for student thinking, as well as interpretation of CBA learning goal, and instructional tasks and plans. |

What makes you think that this is the CBA level?  

What part of the CBA document convinces you that you are correct?  

What purpose did raising fingers serve for SX?  

What CBA level do you think this student should move to next? Why?  

What instructional activity or task do you think might move this student to this next level?  

Continued
### Table 1 Continued

<table>
<thead>
<tr>
<th>Data Source</th>
<th>Sample Size</th>
<th>Description of Data Source and Sample of Interview Protocol</th>
<th>Affordances for Research Inquiry</th>
</tr>
</thead>
<tbody>
<tr>
<td>MD2 08.09 – Appendix D</td>
<td>14</td>
<td><em>Problem</em> 45 × 23 = _____</td>
<td>Allows for the interpretation of teachers’ choice of CBA level for student thinking, as well as interpretation of CBA learning goal, and instructional tasks and plans.</td>
</tr>
<tr>
<td></td>
<td></td>
<td><em>Sally:</em> 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(a) is Sally’s reasoning correct or incorrect? Explain why. (b) what CBA level of sophistication do you think Sally’s strategy is? (c) what type of reasoning do you think Sally should move to next? Explain why. (d) What would you do instructionally to move Sally to this next type of reasoning?</td>
<td></td>
</tr>
<tr>
<td>Online Interviews – Appendix F</td>
<td>8</td>
<td>See Face-to-Face Interviews and Appendices A, C, E, as well as MD4 07.08 and Appendix B</td>
<td>Allows for the interpretation of teachers’ choice of CBA level for student thinking, as well as interpretation of CBA learning goal, and instructional tasks and plans.</td>
</tr>
</tbody>
</table>
Research Question 2: While working with students using CBA Multiplication and Division in live one-on-one teaching experiments, how are CBA2 teachers’ instructional decisions informed by knowledge of CBA Multiplication and Division materials?

<table>
<thead>
<tr>
<th>Data Source</th>
<th>Sample Size</th>
<th>Description of Data Source</th>
<th>Affordances for Research Inquiry</th>
</tr>
</thead>
<tbody>
<tr>
<td>T6 Reflection on CBA Assessment Interview Transcript</td>
<td>1</td>
<td>Transcribed interview between T6 and a CBA researcher after watching T6’s CBA assessment with a single student. CBA researcher asks T6 to determine a CBA level of thinking for each problem that the student completes, as well as what CBA level the student should move to next and what instructional tasks would be helpful.</td>
<td>Provides a detailed account of a teacher’s thinking about the CBA levels while working with a student. This interview emphasizes the process of determining a student’s level, and primarily focuses on how the CBA materials can be used for formative assessment purposes.</td>
</tr>
<tr>
<td>T6 CBA Assessment Transcript</td>
<td>1</td>
<td>Transcribed Interview between T6 and a student working on a CBA mathematics assessment for multiplication and division. T6 does not ask follow up questions other than to probe the student’s thinking.</td>
<td>Allows the present research to investigate the student thinking and T6’s conceptualizations of the student’s thinking.</td>
</tr>
</tbody>
</table>

Table 2 - Summary of Data for Research Question 2
<table>
<thead>
<tr>
<th>Data Source</th>
<th>Sample Size</th>
<th>Description of Data Source</th>
<th>Affordances for Research Inquiry</th>
</tr>
</thead>
<tbody>
<tr>
<td>T8 Reflection on Teaching Experiment Interview Transcript</td>
<td>1</td>
<td>Transcribed interview between T8 and the CBA author while watching T8’s CBA teaching experiments with two students. CBA researcher asks T8 to think about the student’s thinking, the mathematics of different strategies, and what constitutes good questioning within the CBA framework</td>
<td>Provides a detailed account of a teacher’s thinking about the CBA levels while working with a pair of students. This interview emphasizes the process of determining a student’s thinking, and primarily focuses on how the CBA materials can be used for questioning and instruction for individual students.</td>
</tr>
<tr>
<td>T8 Teaching Experiment Transcript</td>
<td>1</td>
<td>Transcribed Interview between T8 and two students (individually) working on a CBA mathematics assessment for multiplication and division. T8 does ask follow up questions other than to probe the student’s thinking, and attempts to help the students develop more sophisticated ideas</td>
<td>Allows the present research to investigate the student thinking and T8’s conceptualizations of the student’s thinking. Also allow the present research to interpret T8’s follow-up questioning through the CBA MD framework.</td>
</tr>
<tr>
<td>Data Source</td>
<td>Sample Size</td>
<td>Description of Data Source</td>
<td>Affordances for Research Inquiry</td>
</tr>
<tr>
<td>-------------</td>
<td>-------------</td>
<td>----------------------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>T18 Reflection on CBA Whole Class Teaching Experiment Transcript</td>
<td>1</td>
<td>Transcribed interview between T18’s and a CBA researcher after watching portions of T18’s episode of teaching using the CBA framework a class of roughly 20 students. CBA researcher asks T18 to determine the CBA level of thinking for several problems that students in the video complete, as well as what CBA level the student should move to next and what instructional tasks would be helpful. Interview also includes a discussion of how CBA materials were used to develop instruction.</td>
<td>Provides a detailed account of a teacher’s thinking about the CBA levels while working with a classroom of students. This interview emphasizes the process of determining a student’s level, and primarily focuses on how the CBA materials can be used as pre and post assessment tools and also inform instruction for a whole class setting.</td>
</tr>
<tr>
<td>T18 CBA Whole Class Teaching Experiment Transcript</td>
<td>1</td>
<td>Transcribed Interview between T18 and a classroom of students working on multiplication and division problems. Transcript is only partially transcribed as the video quality make aspects of it difficult to distinguish voices and/or verbalizations. T18 does ask follow up questions other than to probe the student’s thinking, and attempts to help the students develop more sophisticated ideas.</td>
<td>Allows the present research to investigate the student thinking and T18’s conceptualizations of the student’s thinking. Also allow the present research to interpret T8’s follow-up questioning and task selection through the CBA MD framework.</td>
</tr>
</tbody>
</table>
Data Analysis

Qualitative analysis of the various data sources followed retrospective analysis procedures, as well as grounded theory analysis and constant comparison analysis techniques. All data for the research inquiry, except for online interviews, were videotaped which allowed the researcher to revisit the data to analyze based on evolving theories. As the goal of data analysis was to provide explanations of the learning and thinking of CBA teachers using research-based learning progression materials, videotaped historical accounts of their thinking and decision-making include moment-to-moment episodes of their reasoning, allowing for such analysis techniques to occur.

According to Goldin (2000), by analyzing verbal and nonverbal behavior or interactions in clinical task-based interviews,

"the researcher hopes to make inferences about the mathematical thinking, learning, and/or problem solving of the subjects. From these inferences, we hope to deepen our understanding of various aspects of mathematics education. We may aim to test one or more explicit hypotheses, using qualitative analyses of the data; we may seek merely to obtain descriptive reports about the subjects' learning and/or problem solving; or we may hope to achieve an intermediate goal, such as refining or elaborating a conjecture.” (p. 533)

Additional probing questions and tasks were occasionally intertwined within clinical interview protocols (see appendices) between CBA researchers and teachers. The ways that CBA teachers react to these probes were revealing of their conceptualizations, beliefs, and reasoning about children’s mathematical thinking and development, and shed
light on possible instances in which teacher learning occurred. Although these instances were not uniform across all clinical interviews, they represented a form of teaching experiment that can be used to investigate how teachers construct new knowledge, or elaborate and refine current knowledge. In the analysis of CBA teachers’ interviews and written work, the researcher identified regularities, patterns, and significant episodes that revealed individuals' conceptualizations, ways of reasoning, learning mechanisms, and beliefs. The researcher looked for significant episodes and patterns both within and across individuals, in-the-moment and across time, and illustrated particularly important conceptualizations, ways of reasoning, and learning mechanisms with episodes (vignettes) that effectively conveyed important ideas. The approaches of retrospective analysis, grounded theorizing, and constant comparative analysis are explored as well as thoughts on issues of reliability and validity.

Retrospective Analysis

Data analysis utilizing a retrospective approach as described by Steffe and Thompson (2000) calls for the use of videotaped interviews, and individuals’ written work in order to build an historical account of the individuals’ actions and interactions. From these accounts, the researcher formulated observational and analytic categories to help distinguish individuals’ ways of reasoning and conceptualizing. Hypotheses based on these analytic categories were then formulated for how teachers might conceptualize, reason, and use the CBA materials. The use of the terms hypotheses and ‘conjectures’ may at times be used interchangeably throughout this dissertation. Cobb, Gresalfi and Hodege (2009) refer to qualitative hypotheses that are formulated through data analysis.
and exploration as conjectures, with the same overall a posteriori implications that theory is developed through data analysis and subsequently checked or verified through further data analysis of additional data.

Evolving theories were made explicit throughout the data analysis with analytic commentary, and data and researcher inferences were provided to help describe these theories. Data was analyzed and re-analyzed in ways that can explicitly support or disconfirm the evolving theories and hypotheses. This retrospective analysis informed the continuation of model building, hypothesis/conjecture generation, and analytic category construction. Finally, data were reported in the form of illustrative vignettes, with sufficient data and explanation provided so that other researchers can evaluate inferences and theoretical interpretations. Within the results section, the reader will find a significant amount of interview transcript data, with researcher commentary and analysis intertwined to help connect the research theory with the illustrative vignettes.

In talking about the building of models and evolving theories of students, Thompson (1982) argued that a responsible researcher should ask themselves: "What can this [student] be thinking so that his actions make sense from his perspective?" (p. 161). Adapting the idea for research on teachers, the researcher should evaluate what it would take in order for teachers to behave as they did in the described event. In other words, a researcher utilizing a retrospective approach should make an effort to understand and conceptualize ideas from the perspective of those they are investigating. For the proposed research, the detail of the data sources afforded the researcher the ability to
apply a retrospective approach as teachers involved in the study were given opportunities to elaborate and clarify their thinking.

*Grounded Theory and Constant Comparative Analysis*

According to Pandit (1996), the three basic elements of grounded theory are concepts, categories, and propositions. Pandit describes concepts as potential markers of occurrences that should be given a conceptual label or observational code. Categories represent a higher level of abstraction than concepts, and can represent concepts that are grouped into classes of similar lower level codes. Finally, propositions indicate generalized relationships between categories and concepts. Glaser and Strauss (1967) refer to the constant comparative analysis approach by stating the “basic, defining rule for the constant comparative method” is that, while coding an incident, the researcher should compare it with all previous incidents so coded, a process that “soon starts to generate theoretical properties of the category” (p. 106). Glaser and Strauss refer to a similar process of coding as Pandit, instead using the terms open coding, axial coding, and selective coding.

Grounded theory is not developed a priori and then subsequently tested, but rather it is “discovered, developed, and provisionally verified through systematic data collection and analysis of data pertaining to that phenomenon” (Strauss & Corbin, 1990, p. 23). Strauss and Corbin (1990) elaborate that data collection, analysis, and theory have a reciprocal relationship to each other, as the researcher does not begin with a theory in the hopes to prove it, but instead begins with a topic and allows relevant theory to emerge (p. 23). Glaser and Straus (1967) identify the criterion for determining when to conclude
sampling for grounded theory and constant comparative analysis and the category’s ‘theoretical saturation’ point, or the point at which no new categories emerge from the data.

Coding procedures and analysis techniques for the proposed research followed the approach of beginning analysis from a lower, observational, level coding approach. As patterns and themes emerged in the data, more abstract analytic categories were constructed. Vignettes and portions of data were included to explicate the process by which the researcher constructed theories.

Issues of Validity

Grounded theory should conclude with a re-evaluation of existing literature, according to Eisenhardt (1989); “Overall, tying the emergent theory to existing literature enhances the internal validity, generalisability, and theoretical level of the theory building from case study research ... because the findings often rest on a very limited number of cases” (p.545). Goldin (2000) summarized the concept of reliability in qualitative research as including the regularity that inferences are made from the observations and data using specific criteria. This commands the need for triangulated data sources that demonstrate a wide variety of sources to make comparisons across different occurrences of a concept. This also commands data sources that address the concept over a sustained period of time. Connecting and relating the proposed research to existing literature in qualitative, emergent, research is an ongoing process. The literature review for the proposed dissertation represents an initial overview of the relevant research, and much like the iterative process of retrospective analysis and constant comparative analysis
procedures, the review of literature was revisited throughout the inquiry, and supplemented following the analysis of data. Data sources came from a wide variety of interview protocols over the course of several years, and encounter teachers’ understanding and use of learning progression frameworks in several facets of the work of teaching (e.g. analyzing written student work, working with a student one-on-one). Therefore, the data provided accounts of teachers’ reasoning over a sustained period of time, and utilizing triangulated data sources.
Chapter 4: Results and Findings

In this chapter, three main topics of data are explored that serve to help explore the research questions of the inquiry. First, Topic A presents data on teachers’ understanding and CBA classification of episodes of student thinking. In this section, data are presented that looks at different conceptualizations teachers have of students’ mathematical reasoning, the different ways that teachers can use and make sense of the CBA materials, and some of the misconceptions and challenges teachers have with using the CBA framework. Next, Topic B explores teachers’ choices about next steps in learning goals and instruction. Within this section, data are presented to help relate conceptualizations of student thinking on a particular mathematical problem to subsequent learning goals and instructional plans. Then, data are presented to investigate CBA teachers’ approach to goal setting and instruction, and general trends in teachers’ use of the CBA materials for determining learning goals are explored. Finally, the chapter concludes with Topic C, in which three individual cases of teachers working with students in different contexts are presented. The first is a teacher working with a single student on a CBA assessment for MD. The second is a teacher working with two students (individually) in a teaching experiment, and the third is a teacher working with a full classroom of students using the CBA materials for pre and post assessments.
Topic A: Teachers' understanding and CBA classification of episodes of student reasoning

In this section, different topics related to teachers’ classification of student reasoning are explored. CBA LP can serve as an organizing framework for analyzing students’ thinking about mathematics. However, teachers’ understanding of the CBA materials can vary quite widely based on experiences, and background knowledge. These differences play out in how teachers interpret written and verbal student work in order to determine what CBA MD level a students’ reasoning demonstrated.

This section begins with a discussion of teachers’ responses to a specific sample of student work by a student named Sally who was solving the problem of 45x23. This section looks at the various ways that teachers’ conceptualized Sally’s reasoning. Within this discussion it is made clear that a single sample of student work can evoke a wide variety in conceptualizations of how children think about a multiplication problem, even with teachers utilizing the same research-based learning progression framework.

Next, a grounded theory is proposed based on CBA teachers’ responses to student work on a variety of multiplication problems. The grounded theory that is explored investigates instances where CBA teachers struggle to correctly classify episodes of student thinking and hypothesizes that struggles in properly classifying episodes of student thinking arise based on the complexity of the CBA level descriptions (complexity of concept and complexity of the description of the concept), and the complexity of student work. This discussion is focused on the depth of understanding required by teachers in order to understand student thinking.
The final portion of Topic A explores instances where CBA LP language is interpreted by teachers in a variety of ways, occasionally in ways that differ from the intended meaning of the CBA MD materials. This piece focuses on the various ways teachers’ conceptualize CBA’s terminology and concepts (e.g. – ‘Algorithms’) and how these terms can impact the ways in which teachers classify student thinking.

*Teachers’ Understanding of a Student’s Reasoning About Multiplication*

Teachers participating in the CBA2 project were given the following task in one-on-one, videotaped interviews. Some teachers had used CBA MD materials the previous year, whereas others had not. (Generally, teachers of grades 4-5, and sometimes 3, chose to use CBA Multiplication and Division materials, while teachers of grades 1 and 2 chose to use CBA Addition and Subtraction materials.) While doing this problem, teachers were given the CBA MD Quick Reference Sheet (see Appendix G). Note, however, that the actual CBA MD document was 82 pages long and contained numerous CBA assessment tasks, numerous examples of student work illustrating each CBA level, and instructional suggestions for students at each CBA level.

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = ______</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035</td>
<td></td>
</tr>
</tbody>
</table>

(a) Is Sally's reasoning correct or incorrect? Explain why.
(b) What CBA level of sophistication is Sally's strategy? Explain why.
(c) What type of reasoning do you think Sally should move to next? Explain why.
(d) What would you do instructionally to move Sally to this next type of reasoning? Explain why.

*Figure 6 – Sally Interview Protocol*
For Sally’s thinking, the CBA author would characterize CBA Levels 3.2 and 3.3 as being the most reasonable descriptions for the thinking demonstrated. Sally clearly can decompose numbers by place value, as she demonstrates an example of 2 partial products multiplication (Level 3.3) exactly as described in the CBA MD framework when she computes 45 times 3 as 40x3 plus 5x3. Sally also demonstrates a further decomposition of this strategy, and a uses number properties to help compute 45x20 as 45x10 and 45x10. There is not clear demonstration of a 4 partial product (Level 3.4) approach, as Sally does not explicitly decompose both numbers into tens and ones (i.e. – 45x23 is not done as 40x20, 5x20, 40x3, 5x3).

*Teachers with Straightforward Correct Evaluation of Sally's Reasoning*

Some teachers quickly and correctly made sense of Sally's explanation. However, teachers differed in the reasoning they used for judging mathematical validity (formal versus informal).

Teacher 13 – Sally Problem

*[no use of CBA MD materials]:*

T13: [Pause] Her thinking's correct.
I: How did you decide that her thinking is correct?
T13: Cause it totally made sense. So she knows her tens and she's just breaking apart that number to get it the way she knows how to handle it.
I: Okay. All right.
T13: I mean just that everything the way she did it totally made sense. She’s breaking apart the 23 to answer her question. So she’s doing level 3.
I: Okay.
T13: Combining and Separating.
I: Okay.
T13: Cause she knows, she ultimately knows her ten’s. Which is what you teach them in addition. So she knows her tens and she’s just breaking apart that number to get, to get it the way she knows how to handle it.

I: Okay. So what type of reasoning do you think Sally should move to next? Explain why.

T13: I would guess using expanded algorithms. Because she's half way there. … I think …she'll very quickly move to level 5 because she truly understands the expanded notation. … I think she should move to level 4 next.

I: I see.

T13: But I think she would skip quickly to level 5. … She's just going to collapse it pretty quickly to a traditional algorithm. And it's just going to make sense for her.

T13 is a primary grade teacher who used the CBA materials for Addition and Subtraction, and for Place Value, but not for Multiplication and Division. Even though T13 recognized the validity of Sally's reasoning, she did not provide a formal reason, such as the distributive property and place value decomposition, to justify her validity judgment. Also, T13’s description of instructional goals for Sally omits several difficult, but critical, intermediate levels—again probably because she had not actually used the CBA MD materials and because she does not teach these topics. Interestingly, the transition from Level 3 to an expanded algorithm is easier for addition than it is for multiplication—so perhaps she was making a judgment based on her classroom experience working with CBA addition materials.

Teacher 19 – Sally Problem

[used CBA MD materials]:

T19: Well, she's using [the] distributive property first. You know 45 times 10 and 45 times 10 again is the same as 45 times 20, as long as you're adding them back together, which she's doing. So you've got that. [Points
towards the 3rd sentence] Then she's doing 40 times 3, and 5 times 3, to get the 120 and the 15. So again distributive. And [slight pause] yeah.

T19: Well, it's not necessarily 2 partial products but she's not coming to the 4 partial products that would be the more traditional 4 that you would start with and end with. She's definitely breaking it down more so. But she's still coming to partial products. She puts them together in a way which may put her at a [Level] 3.2. But, she's definitely using her number properties well. She's using the distributive property. She's breaking down the numbers. You know she's basically getting 2 partial products but she's doing more steps to get there.

T19: I think she could probably jump that step a little bit to a more expanded algorithm, based on the way she's explained this. Because it seems to me she's writing out a lot of things she's doing in her head… And the goal is to transition them [students] into the traditional algorithm which in a lot of ways has more steps. But as far as an effective mental strategy I think she has one.

T19: I think this is a kid who you could show the 4 partial products to her once and then go into an expanded algorithm. I don't think she's going to need a lot of instruction on seeing 4 partial products or need me to give her 5 problems and tell her to do these 5 problems using 4 partial products. I don't think for this particular child, based on the number sense showing here, but I think she does need the expanded algorithm before you go to the traditional algorithm because otherwise it's going to look like voodoo.

T19, who had used the CBA MD materials, explicitly cited the distributive property to justify her assessment that Sally's reasoning is correct. Also, T19 accurately identified the CBA level for Sally's reasoning—she saw that Sally did extra expansions within a 2 partial products approach, but that Sally did not use the 4 partial products described in the CBA document. That is, T19's CBA level determination is correct, and it is consistent with the complete CBA MD levels document. Interestingly, T19 thinks that Sally can easily transition to using 4 partial products. CBA data show, however, that this transition is actually very difficult. But, because Sally actually performs a mental computation that is more difficult than 2 partial products, T19's assessment might in fact be correct.
Teachers with Initial Uncertainty but Later Success in Understanding Sally's Reasoning

Some teachers initially struggled to make sense of Sally's reasoning, perceiving Sally's statement "45 times 3 is 120, plus 15" as an error as opposed to an accurate conception, although informally stated, of a decomposition of $45 \times 3$ as $40 \times 3$ (120) plus $5 \times 3$ (15). But, on further reflection, these teachers were able to make sense of Sally's reasoning and judge it as correct.

Teacher 23 – Sally Problem

[used CBA MD materials]:

T23: I think she's correct. It makes sense to me that she, you could do it.
I: So it's correct?
T23: I think so.
I: Okay. And explain why you think it's correct.
T23: [Pointing at various portions of Sally's dialogue] Well, what she did … instead of 45 times 20, she just broke it apart into 2 groups of 10, which would be the 900. And then 45 times 3 is 120, plus the 3 times. Oh but maybe not. Hum. I don't know. I'm not sure where she got the 15. I mean it makes sense but I'm not sure. I mean you'd have to add that 15 back in. But I'm not sure where she got that. …
I: Okay. So do you still think it's correct?
T23: Well, she got the right answer, but I'm not sure that she's correct in her thinking.
I: All right. So what CBA level of sophistication would you say is Sally's strategy? …
T23: I think she's trying to use, she's going towards level 4, you know, expanding the computational algorithm, maintains the values of the place value parts [reading level 4 information on the Quick Reference sheet], which she did throughout the sequence of the steps. But it didn't quite work out; just the same as I was having a problem with that, cause I was trying to break it down to do it that way too.
I: Okay. So what type of reasoning do you think Sally should move to next?
T23: Probably continue [points to level 4—expanded algorithm]; I would work on level 4 a little more with her in breaking down the numbers [points to level 5-traditional algorithm] and then go to level 5 with her.

I: Okay. And so what would you do instructionally with Sally?

T23: Well, I'd probably give her more opportunities to multiply some numbers and play. Basically a lot of times we don't give kids enough time to just experiment with multiplying numbers and breaking it apart [motions on Sally's strategy]. I think I would give her that opportunity and I would ask her [circles plus 15 in the 3rd sentence] where did you get the 15? You need that [taps on 1035 in the 4th sentence] to get the right answer [points to plus 15 in the 3rd sentence]. But she didn't really know that was the right answer though did she [taps on the beginning of the 4th sentence]. Hum, I'd probably ask her where she got the 15 [points towards the middle of 120, and plus 15 in the 3rd sentence]. Okay, I see what she did. She did 3 [points to 3 in the 3rd sentence] times 40 [points to 4 in 45 in the 3rd sentence] and got the 120 [points to 120 in the 3rd sentence] and then 3 [points to 3 in the 3rd sentence] times 5 [points to 5 in 45 in the 3rd sentence] is 15 [points to the 15 in the 3rd sentence]. So she is right.

I: Okay. All right.

T23: So she is you know breaking it apart [points towards level 4]. And I would just give her more opportunities to do that. And then maybe see if I could get her just to do the standard algorithm for 2-digit multiplication.

T23, who used the CBA MD materials, first judged that Sally's reasoning was correct, then said she was unsure, then, after further analysis, decided that Sally's reasoning was correct. But T23 incorrectly classified Sally's reasoning as Level 4 (using an expanded algorithm), when Sally was actually using Level 3 reasoning. T23 also did not correctly specify where Sally should go next instructionally. Essentially, T23 suggested that after Sally played around with more problems like the one given in this episode, Sally could move to the traditional algorithm. T23 correctly understood that Sally needed more work with the expanded algorithm, but she did not understand the difference between Sally's strategy and an expanded algorithm.
There are several issues of note here. First, T23 kept analyzing Sally's reasoning until she figured out where the 15 came from, finally seeing that it was one of the parts of multiplying 45 times 3. Persisting in attempts to make sense of students' reasoning is an essential part of the instructional philosophy of CBA, that students' reasoning makes sense to them, and we, as teachers, need to figure out the nature of that sense making. However, T23 did not correctly identify Sally's CBA level of reasoning. One reason for this is that T23, like several CBA teachers, may not have clearly distinguished between the mental decomposition strategies of Level 3 and the expanded algorithms of Level 4. This seems to be a subtle distinction that many teachers do not make. Perhaps as a consequence of this "mis-leveling," but maybe also because most curricula go directly from mental strategies to traditional algorithms, T23 skipped the essential steps of having Sally move next to mental strategies involving 4 partial products, then to solidifying her mental reasoning into expanded algorithms, and only then, moving to a traditional algorithm.

Also of note is that T23, and other teachers, seem to identify that 45x20 is 45x10 and 45x10 more easily than 45 times 3 is 120, plus 15. This might occur due to the difference in format of the distributive property. The actual decomposition, including the sub problems that produce the partial products, occurs in the first instance, while in the second instance, only the partial product results are given. This could have also played into the difficulty that T23 has with understanding Sally’s reasoning and characterizing it with the CBA levels.
Teachers Who Were Unable to Make Sense of Sally's Reasoning

Another subset of teachers were confused by Sally's reasoning; they seemed unable to make sense of Sally's "plus 15," perceiving it as an error.

Teacher 21– Sally Problem

[no use of CBA MD materials]:

T21:   It is incorrect.
I:   Okay, explain why.
T21:   Because she has already done the 45 up here times 20. So all that is left…do you mind if I …
I:  Oh no, go ahead.
T21:   [Multiplies 45 times 23 using the traditional algorithm and gets 1035] She does have it right… Okay so 45 times 10, so she is taking out one of those and then she is taking out the other one so it is going to be 3 times 45. 3 time 45 [again finds answer using traditional algorithm finds answer to be 135] so here she has an error there [pointing to 45 times 3 is 120]…45 times 3 is one hundred….oh she did…3 times 40 plus 3 times 5. She did it right it is just…okay…that came to 135. So she added 135 to 900…1020…but where did the 15….okay 900 and 120 so 1020...[pause]. She is on the right track, but I think she's got a problem here. Okay so she did her 45 times 10 is 450 and her 45 times 10 is another 450 and that is true it does give her 900. Then here she says 45 times 3 is 120 plus 15. So here 45 times 3 is actually 135 but I see where she combined that, but it should be 135 [circling 45 times 3 is 120 plus 15] but that is taking care of that 15, so 135 plus 900 is 1035. So somehow, she added this in [pointing to the 15] but she didn't really add it in because she ended up with the right number. So she is definitely on the right track and she has the right answer but she is throwing in the extra 15 in her explanation. But she didn't really need it.

T21 judges Sally's reasoning as incorrect because she seems unable to understand Sally's reasoning mathematically. T21 is unable to use the distributive property to make sense of how all the parts of Sally's computation fit together to make a correct solution.
Teacher 20 – Sally Problem

[no use of CBA MD materials]:

T20: Wow!
I: Why do you say that?
T20: Well it just seemed really confusing, but actually I mean this part makes sense to me [pointing to the sentence with 45 times 10 is 450. 45 times another 10 is 450; that's 900], but this part I am not sure of. This makes sense, 45 times 10 and then 45 times another 10, because this is a 20. [Pause] What's this, 45 times 3 is 120? … I am thinking of this because 45 times 3, I see here is 120, but where's the plus 15 come from? Just because 5 times 3 is fifteen?
I: Is there anyway you can figure it out.
T20: [Pause] I mean that's the right answer. But I don't know if I would see that 45 times…. Maybe this is not a good one for me because I don't do it that way. But maybe... this is what you are supposed to do. … I never learned it this way, I just know you can. So maybe this is the way to do it. You times the 45 times the 3 and then the 5 times the 3 [long pause, looks at the quick reference sheet].
I: Is Sally's reasoning correct or incorrect? And explain why.
T20: I don't know. I don't really understand that. Why she would do that, 45 times 3 and then plus 15. It doesn't really seem right. But yet she got the right answer. So I don't know if she just got lucky or she just added plus 15. But this does make 15 here. I think the first part is correct; 45 times 10 and 45 times another 10 because it is 45 times 20 something. And I do I actually understand the 45 times the 3 too because it was not just 20 it was 23. [Long pause]. But I guess in my thinking she already she already accounted for this. Like she accounted for the 45 twice plus the 20 and then and the 45 times 3 but I feel like it must be right because she got the right answer. Cause if you didn't have the 15 in there then it would be wrong. So I am going to say's that she that her reasoning is correct.
I: Okay. What CBA level of sophistication is Sally's strategy? Explain Why.
T20: [Pause and reviews the CBA quick reference] Can you tell me what this partial products means? Just parts?
T20: I think it's here [tapping on Level 4]. … I think it is definitely expanded what she did. Multiply and divide numbers, maintain the value of place value parts throughout the sequence of steps, which she does because she knows that this is two tens. So I think that she is at a Level 4.
I: Okay. What type of reasoning do you think Sally should move to next and explain why.

T20: You know, I really have to say that I am not sure. I mean I guess, that she would move on to Level 5 but honestly I don't have enough understanding…. I am one of those people that learned it this way [pointing to traditional algorithm] and I don't even know because I don't teach it. But my guess is that this [pointing to Sally's method] is more sophisticated than this [pointing to traditional algorithm] because this is just memorizing how to do it. I don't know that to be certain though. It seems like this has more thinking involved than what I did. I just kind of did it. … This [Sally's method] definitely involves more thinking about it. Thinking about the place value parts and thinking about which numbers you need to put together where as this is just… just me remembering how to multiply and move numbers and puts zeros and all those things.

I: What would you do instructionally to move Sally to this next type of reasoning and explain why?

T20: [Pause] Well Level 5, the difference is that the place value ideas and the algorithms are hidden. So I would want her to be able to do that. But I really don't exactly know what I would do to have them hidden. But that would be the goal as far as I can see from the levels.

I: Okay… So instructionally, you are really unsure of how you would go about it?

T20: I'm not sure.

Ultimately T20 judges Sally's reasoning as correct because Sally got the correct answer. But T20 did not understand why Sally's strategy was correct. Because T20 did not study the CBA MD document, she did not know what "partial product" means, nor did she understand that Level 5 requires understanding why the traditional algorithm works [so T20 is NOT at Level 5]. Interestingly, T20 thinks that Sally's reasoning is more sophisticated than using the traditional algorithm rotely, as she identifies herself of doing. Ultimately, because T20 does not understand Sally's reasoning, T20 admits that she does not know what do to with Sally instructionally. But in some way (perhaps
because of her use of CBA materials for addition, subtraction, and place value) T20 may understand that moving Sally directly to the traditional algorithm, without understanding how the "place value ideas are hidden," might be unwise at this time.

CBA Teachers’ Consistency with the CBA MD Framework

Recognizing and understanding of children’s mathematical thinking in terms of CBA levels is an important component of making sense of the CBA materials. It is not difficult to conceive of a teacher who is able to recognize the accuracy of a student’s strategy by checking a final answer for correctness, while simultaneously misinterpreting or failing to understand the details of the student’s strategy in terms of the CBA levels. In fact, several such examples have been detailed earlier in this text. In consideration of this, an investigation into how teachers handle the analysis of student reasoning when the student reasoning exactly matches a CBA level description that describes a single well-defined kind of reasoning versus a level that describes a large category of ways of reasoning is explored.

Certain CBA levels afford a teacher with the ability to use features of a student’s work (e.g. – visual appearance of a strategy) or concrete actions to identify and characterize a students’ thinking within the CBA levels. Other CBA levels do not afford a teacher with the ability to use solely the features of a student’s work (e.g. – visual appearance of a strategy) or concrete actions to identify and characterize a students’ thinking within the CBA levels. For these more conceptually complex levels, teachers must explicitly recognize the critical characteristics of a child’s thinking within the CBA MD framework, even when identifying the key mathematical and/or cognitive processes
involved might not perfectly match the CBA descriptions. As an example, in evaluating RR’s reasoning to the following problem, T2 relies heavily on the numeric keywords describing the number of decomposed numbers, as opposed to the emphasis on place value decomposition. RR’s reasoning does not neatly fit into a single CBA level, and could be described as level 3.2 reasoning, or possibly as a variation of level 3.3 reasoning (2 partial products) with additional steps. This complexity makes this student thinking more challenging to characterize within the CBA levels.

<table>
<thead>
<tr>
<th>Task.</th>
<th>$46 \times 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>

Figure 7 – RR and QR Student Work

[used CBA MD materials]:

T2: (Reads problem with RR’s strategy) See that’s interesting. Because, there is no 3 parts…but they broke it up into 3 parts almost, so that kinda interesting. The difference between them is that they broke down the 200 into smaller parts, and maybe that is because it is a single digit on the other number, so they couldn’t break it down any further. I would definitely put them at least at an MD 3.3, but I would say that they…well I would pursue whether they were at the 4 part level.

In T2’s case, interpretation of the CBA meaning of *partial products* focuses on the 3 concrete computations from decompositions, but does not explicitly include the integration of the distributive property and the specific place value partial products described in levels 3.3 and 3.4. T2 seems to indicate that the students’ strategy might not
exactly match the description of level 3.3 by stating that ‘The difference between them is that they broke down the 200 into smaller parts’ but does not explicitly recognize or articulate the fact that it is not representative of the technical definition of partial products in CBA (in this case, 2 partial products would mean 46x5 would be computed as 40x5 and 6x5 only). In general, CBA LP levels that describe easily recognizable characteristics and concrete actions could constitute levels that are more easily recognized by teachers because they perfectly match the CBA descriptions of such levels. T2 demonstrates how this can be challenging when there are many elements to either a CBA level or a students’ mathematical thinking.

In another example, while evaluating Sally’s work T19 recognizes that Sally did not do only a Level 3.3 or Level 3.4 partial products decomposition, but did utilize the distributive property and an approach that was very similar in nature to level 3.3 thinking.

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 \times 23 = ______</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035</td>
</tr>
</tbody>
</table>

Figure 8 – Sally Student Work

[used CBA MD materials]:

I: Okay. All right. So (b) what CBA level of sophistication is Sally’s strategy?
T19: [Pause] She’s in the 3.3. She’s using partial products.
I: Okay. Okay. And how did you decide what level?
T19: Well, it’s not necessarily 2 partial products but she’s not coming to the 4 partial products that would be the more traditional, 4 that you would start
with and end with. She’s definitely breaking it down more so. But she’s still coming to partial products. She puts them together [points towards plus 15 in the 3rd sentence] together [points towards 1035 in the 4th sentence] in a way which may put her at a 3.2. But [waves hands if unsure then puts them back towards the beginning of the 3rd sentence and the end of the 4th]. She’s definitely using her number properties well. She’s using the distributive property. She’s breaking down the numbers. You know she’s basically getting 2 partial products but she’s doing more steps to get there.

I: Okay.

T19: So, ah 3.2 instead of 3.3 probably [but/would?]. You know as far as her method goes it, it’s a heavy use of the distributive property. And you know I can’t necessarily say she’s got 2 partial products. So that would probably put her at the 3.2 instead of 3.3

Understanding more complex CBA levels (or complex student work) is more involved, as it would require teachers to recognize conceptual details about the strategy that help describe how the child is conceiving of the mathematical thinking strategy. This is intended to reflect that less complex CBA levels might only require a teacher to recognize a strategy based on how it typically appears visually, or keywords that might be associated with a strategy. On the other hand, conceptually more involved levels would require a teacher to not just recognize the concrete characteristics, but also the mathematical and cognitive underpinnings of the thinking strategy (e.g. – use of the distributive property, or decomposition of numbers by place value parts versus non-place value parts). CBA MD levels that describe students’ thinking that are not as easily recognized by concrete actions or without deeper understanding of the thinking of the conceptual components of the mathematical strategy would be potentially more challenging for teachers to identify in student thinking.
As an example of how these two types of use could play out, looking at four samples of student work can help to frame the argument for this type of teacher use of the CBA MD materials. The following three tasks represent different situations that CBA teachers were asked to evaluate students’ mathematical thinking about multiplication.

**Task 1.** There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?

SX: 12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.

Figure 9 – SX Student Work

**Task 2.** $4 \times 5$

RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.

QR: 40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.

Figure 10 – RR and QR Student Work

**Task 3.** $45 \times 23 = ______$

Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.

Figure 11 – Sally Student Work

In the first task, the student uses a fairly uncomplicated strategy that involves the iteration/skip-counting of a two-digit number. This strategy is a relatively simple strategy to recognize based on the concrete actions, as the CBA MD level for ‘skip-counting all’ will always involve a student writing or saying all multiples of the skip count sequence without decomposing numbers in any way.
The second task has two samples of student thinking. QR’s strategy represents using the distributive property to decompose by place value parts into 2 partial products. Much like the ‘skip-counts all’ CBA MD level, 2-partial products in the CBA framework will always involve a student decomposing one factor strictly by place value and the use of the distributive property. This strategy could be recognized simply by identifying the concrete action of decomposition of one number into 2 partial products. RR’s strategy, however, represents using properties in addition to strict place value decomposition of numbers. Because students at this level do not use strict place value decomposition of numbers, their decompositions can take unique forms suited specifically based on the particular students’ cognitive comfort with certain numbers. For this reason, this level could be more challenging for teachers to learn at a conceptual level because there are many variations. Essentially, learning a concept with 1 attribute (2 partial products) would likely be easier than learning a concept with 2 attributes (2 partial products, and place value decomposition). For RR’s work, RR may decompose strictly by place value into 40x6, but this is not clear because RR further decomposes the problem into 20x6 and 20x6, and this is an important difference in the CBA levels.

The third task represents a fairly complicated description of student thinking about multiplying a 2-digit by 2-digit multiplication problem. In consideration of the CBA MD levels, Sally’s thinking is quite complex, as it involves instances where Sally does decompose by place value (40 times 3 is 120, plus 15…where Sally decomposes 45x3 into 40x3 and 5x3 implicitly), and instances where Sally does not decompose strictly by place value (45 times 10 is 450 and another 10 is 450; that’s 900…where Sally
decomposes 45x23 into 45x10 and 45x10 instead of simply 45x20). It could be that this further decomposition of 45x20 into 45x10 and 45x10 is hard to recognize as being related to the partial products approach in the CBA framework. In addition to this nuance, Sally’s thinking creates what appears to be a hybrid of the CBA LP levels for ‘2 partial products’ and ‘4 partial products’. Identifying that Sally did not consistently decompose numbers strictly using place value is not obvious when reading/hearing Sally’s thinking, and therefore represents a situation in which it would be expected that teachers would struggle to make sense of her thinking by matching with CBA descriptions and examples of student work alone. Sally’s thinking is best described as applying 2-partial products twice, with further decomposition of one of the partial products (which is not an explicit CBA level, but rather a combination of components of multiple CBA MD levels; 3.2 and 3.3). Misinterpreting Sally’s approach for using a traditionally applied 2 or 4 partial products would be an expected error if looking at the outcome of her strategy without consideration of place value combined with the distributive property. Interpreting Sally’s work by looking primarily at the number of computations would represent an oversimplification of Sally’s thinking within the CBA MD levels.

A quantitative investigation into teacher responses to each of these sets of student work indicated that, as expected, the problems that represented situations that were easiest to match up with CBA MD descriptions of levels showed higher correct identification of the students’ CBA level. Those situations that required deeper analysis and exploration of the student thinking showed higher variation in teachers’
determination of students’ CBA level. Below are the distributions of teacher responses to each question:

Teacher Task 1: What CBA level of sophistication is SX’s strategy?

<table>
<thead>
<tr>
<th>Task 1. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX: 12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.</td>
</tr>
</tbody>
</table>

Figure 12 – SX Student Work

To operationalize teachers’ responses to SX’s CBA level of sophistication, three categories were considered: (1) completely consistent with CBA definition, (2) partially consistent with CBA definition, and (3) inconsistent with CBA definition. Teachers categorizing SX’s thinking as Level 2.3 were considered completely consistent. Responses of Level 2.2 or 2.4 represented different approaches to skip counting compared to SX, but were relatively close in the CBA framework and therefore were considered partially consistent. However, if a teacher chose ‘MD level 2’ because it represented skip counting, this was also considered partially correct even though MD level 2 is not a sublevel. Finally, any other responses were considered inconsistent with the CBA definition. Results in the tables are also delineated by whether or not teachers had used the CBA MD materials in their work as a teacher.
### Context 1: MD4 07.08 (SX)

### Context 2: Online Interview 07.08 and 08.09 (SX)

<table>
<thead>
<tr>
<th>CBA Level MD 1.3</th>
<th>CBA Level MD 2</th>
<th>CBA Level MD 2.1</th>
<th>CBA Level MD 2.3</th>
<th>CBA Level MD 2.4</th>
<th>Percent Completely Consistent</th>
<th>Percent Partially Consistent</th>
<th>Percent Inconsistent</th>
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<tbody>
<tr>
<td>Used CBA MD</td>
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<td>Used CBA MD</td>
<td>No CBA MD Use</td>
<td>Used CBA MD</td>
<td>No CBA MD Use</td>
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<td>Number of Teachers Task 1: Context 1</td>
<td>0 0 0 0 1 0 5 7 0 1</td>
<td>86% (12/14)</td>
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<td>7% (1/14)</td>
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<td></td>
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<td>Number of Teachers Task 1: Context 2</td>
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<td>11% (1/9)</td>
<td>11% (1/9)</td>
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<td></td>
<td></td>
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<tr>
<td>Total of Teachers with No CBA MD use</td>
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<td>78% (7/9)</td>
<td>11% (1/9)</td>
<td>11% (1/9)</td>
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<td></td>
<td></td>
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<tr>
<td>Total of Teachers with CBA MD use</td>
<td>0 1 1 12 0 0</td>
<td>86% (12/14)</td>
<td>7% (1/14)</td>
<td>7% (1/14)</td>
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<td></td>
<td></td>
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<td>Total of ALL Teachers</td>
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<td>83% (19/23)</td>
<td>8.5% (2/23)</td>
<td>8.5% (2/23)</td>
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</table>

Table 3 – Teachers' CBA Categorization of SX's Reasoning
Teacher Task 2: What CBA level of sophistication are RR and QR’s strategies?

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
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<tbody>
<tr>
<td>Student #1 RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
<td></td>
</tr>
<tr>
<td>Student #2 QR: 40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
<td></td>
</tr>
</tbody>
</table>

Figure 13 – RR and QR Student Work

To operationalize teachers’ responses to QR’s CBA level of sophistication, the same three categories were considered: (1) completely consistent with CBA definition, (2) partially consistent with CBA definition, and (3) inconsistent with CBA definition. Teachers categorizing QR’s thinking as level 3.3 were considered completely consistent. Responses of Level 3.2 or 3.4 represented different approaches to either partial products multiplication or decomposition and use of the distributive property and therefore were considered partially consistent. Finally, any other responses were considered inconsistent with the CBA definition.
Context 1: Online Interview 07.08 and 08.09

Context 2: Face-to-Face Interview 06.07

<table>
<thead>
<tr>
<th></th>
<th>CBA Level MD 3.1</th>
<th>CBA Level MD 3.2</th>
<th>CBA Level MD 3.3</th>
<th>Percent Completely Consistent</th>
<th>Percent Partially Consistent</th>
<th>Percent Inconsistent</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Used CBA MD Use</td>
<td>No CBA MD Use</td>
<td>Used CBA MD Use</td>
<td>Used CBA MD Use</td>
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<td>0</td>
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<td>Task 2: Context 1 - QR</td>
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<td>88% (7/8)</td>
<td>12% (1/8)</td>
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<tr>
<td>Total of Teachers</td>
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<td>N/A</td>
</tr>
<tr>
<td>with No CBA MD use</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total of Teachers</td>
<td>0</td>
<td>1</td>
<td>13</td>
<td></td>
<td>93% (13/14)</td>
<td>7% (1/14)</td>
</tr>
<tr>
<td>with CBA MD use</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0% (0/14)</td>
<td></td>
</tr>
<tr>
<td>Total of ALL Teachers</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>13</td>
<td>93% (13/14)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>7% (1/14)</td>
<td>0% (0/14)</td>
</tr>
</tbody>
</table>

Table 4 – Teachers' CBA Categorization of QR’s Reasoning

To operationalize teachers’ responses to RR’s CBA level of sophistication, the same three categories were considered: (1) completely consistent with CBA definition, (2) partially consistent with CBA definition, and (3) inconsistent with CBA definition. Teachers categorizing RR’s thinking as level 3.2 or 3.3 (with a recognition that RR’s reasoning did not demonstrate partial products exactly as described in CBA MD) were considered completely consistent. A response of Level 3.3 represented a 2 partial
products multiplication or decomposition and use of the distributive property and therefore was considered partially consistent. Finally, any other responses were considered inconsistent with the CBA definition.

Context 1: Online Interview 07.08 and 08.09
Context 2: Face-to-Face Interview 06.07

<table>
<thead>
<tr>
<th></th>
<th>CBA Level MD 3.1</th>
<th>CBA Level MD 3.2</th>
<th>CBA Level MD 3.3</th>
<th>Percent Completely Consistent</th>
<th>Percent Partially Consistent</th>
<th>Percent Inconsistent</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Used CBA MD Use</td>
<td>No CBA MD Use</td>
<td>Used CBA MD Use</td>
<td>No CBA MD Use</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of Teachers</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>38% (3/8)</td>
<td>38% (3/8)</td>
</tr>
<tr>
<td>Task 2: Context 1 - RR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>50% (3/6)</td>
<td>50% (3/6)</td>
</tr>
<tr>
<td>Number of Teachers</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Task 2: Context 2 - RR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total of Teachers</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>with No CBA MD use</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total of Teachers</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>with CBA MD use</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total of ALL Teachers</td>
<td>2</td>
<td>6</td>
<td>6</td>
<td>0</td>
<td>43% (6/14)</td>
<td>43% (6/14)</td>
</tr>
</tbody>
</table>

Table 5 – Teachers’ CBA Categorization of RR’s Reasoning
Teacher Task 3: What CBA level of sophistication do you think Sally’s strategy is?

<table>
<thead>
<tr>
<th>Task. 45 × 23 = _____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900</td>
</tr>
<tr>
<td>45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035</td>
</tr>
</tbody>
</table>

Figure 14 – Sally Student Work

Finally, to operationalize teachers’ responses to Sally’s CBA level of sophistication, the same three categories were considered: (1) completely consistent with CBA definition, (2) partially consistent with CBA definition, and (3) inconsistent with CBA definition. Teachers categorizing Sally’s thinking as level 3.2, 3.3, or both 3.3 and 3.4 were considered completely consistent. A response of Level 3.1 or 3.4 represented either using number properties or 4 partial products multiplication by decomposition and use of the distributive property and therefore was considered partially consistent. One teacher responded that Sally was at level ‘3’ because she was decomposing and multiplying numbers. This was categorized as partially consistent. Finally, any other responses were considered inconsistent with the CBA definition.
Although the sample sizes are small, this simple quantitative exploration of teachers’ interpretations of various complexity of children’s thinking supports the idea that mathematical and cognitive complexity of the CBA levels as well as complexity of student thinking could play a role in teachers’ ability to appropriately level students according to the CBA MD framework.

For QR’s sample of work, the CBA author deemed the student to be working at Level 3.3, and 13 out of 14 teachers (93%) leveled the student completely consistent with CBA, while the remaining teacher (7%) was partially consistent with CBA. Also, for SX’s sample of work, the CBA author deemed the work to be at level 2.3, which 19 out of 23 teachers (83%) leveled completely consistently with CBA, and two out of 23
teachers (8.5%) were partially consistent. Only two teachers (8.5%) were inconsistent with CBA for this problem. Level 3.3 specifies precisely the type of decomposition that students use for their reasoning, as they must separate a number like 46 into 40 and 6. This level describes exactly one category of student reasoning. Level 2.3 corresponds to skip counting all, which is another level that describes exactly one category of student reasoning. This makes the complexity of the student thinking potentially far less difficult to make sense of, and could be judged correctly using the CBA framework by simply attending to concrete actions such as the certain types of computations.

However, in the instances where the student thinking was either far more complex, or where the CBA level referred to a class of reasoning with far more variation (such as Level 3.2), CBA teachers struggled more to accurately level the students’ reasoning in the CBA MD framework. One factor that plays in this difficulty is that understanding Level 3.2 requires understanding the range of number properties that are applicable, rather than just two; distributive property and place value decomposition. Even the term place value decomposition could be interpreted in several ways (only one of which is used in CBA's technical definition). For example both decompositions below could be considered place value decompositions (but only the latter is satisfies the CBA definition): 26x5=10x5+10x5+6x5 versus 26x5=20x5+6x5.

For the sample of student work where the CBA author deemed the student to be working at Level 3.2 (RR), only six out of 14 teachers (43%) were completely consistent in leveling the student. Also, for the sample of student work completed by Sally, the CBA author deemed the student to be working at multiple levels (3.2 and 3.3, with either
3.2 or 3.3 counted as completely consistent), only 7 of 14 (50%) were completely consistent with the CBA framework, and there was significant variation in the levels chosen by the CBA teachers. Level 3.2 refers to a varied set of possibilities that a student could enact, all linked to the broad description of “uses number properties (but not the distributive property with place value decomposition).” Whereas Level 3.3 requires teachers to understand the distributive property applied to place value decomposition, Level 3.2 requires teacher to recognize a number of properties, including the distributive property. For teachers’ responses to Sally’s work, interestingly 6 of the 14 teachers (43%) recognized that Sally might have applied some form of partial products method, but that it did not match their expectations for partial products multiplication in some form. T15 typified this type of thinking in responding to Sally’s thinking by saying: ‘it’s interesting though how she did it. It’s not quite the way you would think of 4 [partial products]’. This gave indication that although teachers might not have explicitly stated an understanding of the technical CBA definition of partial products, or the explicit delineation between level 3.2 and levels 3.3/3.4, they may have understood Sally’s thinking as being ‘close’ but not ‘exactly the same as’ certain CBA levels. This recognition that Sally’s thinking did not perfectly match the CBA levels for partial products is a positive indication that some CBA teachers might be able to identify key components of students’ reasoning even when they do not fit perfectly within the CBA framework. It should be noted that only one teacher, T8, argued that Sally might actually be doing a 2 partial products approach with an extra step. ‘Technically it is two partial products because she found the 20 that is 10 and 10 so…I would put her at a 3.3.’ T8
identified that the key element in a 2 partial products approach with CBA would require strict place value decomposition (20), but that 10 and 10 constitute this, meaning it could be considered a 2 partial products approach.

There appears to be at least two key ideas at play in this discussion: 1) there is variation in the depth of teacher understanding of the CBA MD levels, and 2) there is variability in the types of student thinking that fall into a CBA level category. Both ideas are likely related to teachers’ struggles to effectively conceptualize the nature of student thinking. Neither one of these ideas is especially surprising, as one would expect that the more complex and varied that student work is (either in the CBA levels or in the students’ actual utterances), the more challenging it would be to properly conceptualize the student thinking. Of importance is that these ideas seem to play out in the data.

Additionally, although only nine out of 14 teachers (64%) provided at least partially consistent CBA levels for Sally’s work, seven of those nine (78%) were teachers who had used the CBA MD materials. The four teachers who provided inconsistent CBA levels for Sally’s work, however, had all not used the CBA MD materials previously. The CBA MD use appeared to be especially helpful for evaluation of Sally’s work that represented a complex example of student work that actually demonstrated thinking at multiple CBA levels, as five out of seven teachers were completely consistent and every one of the seven teachers with CBA use was at least partially consistent. This is in contrast to only one of seven teachers without CBA MD use providing a completely consistent level for Sally, and only two of seven even providing a partially consistent level.
The student work for QR and RR were far less complex, and although every teacher episode dealing with either QR or RR’s work involved a teacher who had used the CBA materials, it turned out that teachers were far more successful at properly leveling QR’s work compared to RR. While 13 of 14 teachers (93%) were completely consistent in choosing QR’s level, only six out of 14 teachers (43%) were completely consistent in choosing RR’s level. RR’s thinking was demonstrative of a more complex CBA level than QR’s, as level 3.2 includes multiple similar ways of reasoning that are grouped conceptually a higher level of abstraction, while thinking at level 3.3 includes just two attributes-- strict place value decomposition and use of the distributive property. Also, there is some ambiguity in the level classification of RR’s reasoning—technically, it is 3.2, but one could argue that it is level 3.3, with additional decompositions. The ambiguity makes the categorization more complicated.

Detailed Accounts of CBA Teachers’ Consistency with the CBA MD Framework

To further investigate situations where teachers analyzed more conceptually complicated student work involving complex, or several, CBA levels requires looking at more than just what a teacher determines a students’ level of reasoning to be. The following paragraphs detail several instances where teachers were asked to determine the level of students' reasoning. Details are provided in order to more clearly understand how teachers can correctly or incorrectly characterize students thinking in the CBA levels. Specifically, several cases look at CBA MD levels 3.2, 3.3, and 3.4 because these involve the distributive property and various decomposition strategies. Level 3.2 requires recognition of various number properties along with the distributive property, and cannot
be easily described by a single example as numbers can be decomposed in more than just one way. Level 3.3 and 3.4 constitute place-value based partial product decomposition, or distributive property based place value decomposition, and do not have the variation of a level like 3.2. It would be expected that a student demonstrating level 3.2 could be misinterpreted as operating at level 3.3 or 3.4 because of a failure to recognize the conceptual difference of the decompositions. The following tables give an overview of the patterns observed in teachers’ understanding of Sally’s reasoning (one of the most challenging samples of student work) as well as teachers’ CBA classification consistency.

The data is delineated by teachers’ use of CBA MD, and demonstrates that teachers without CBA MD use struggled more to make sense of Sally’s reasoning (2 of 7 teachers never made sense of her work), and offered CBA classifications that were not consistent with the CBA MD framework (only 3 of 7 teachers were at least partially consistent). In contrast, those teachers with CBA MD use eventually made sense of Sally’s reasoning (except in one instance in which T18 never technically confirmed that Sally’s thinking was correct) and offered more consistent CBA classifications of Sally’s reasoning.

<table>
<thead>
<tr>
<th>Teachers who had NOT used the CBA MD materials</th>
<th>Understanding of Sally’s Reasoning</th>
<th>CBA Classification Consistency</th>
</tr>
</thead>
<tbody>
<tr>
<td>T3</td>
<td>Straightforward Correct Evaluation</td>
<td>Partially Consistent</td>
</tr>
<tr>
<td>T13</td>
<td>Straightforward Correct Evaluation</td>
<td>Consistent</td>
</tr>
<tr>
<td>T16</td>
<td>Straightforward Correct Evaluation</td>
<td>Inconsistent</td>
</tr>
<tr>
<td>T20</td>
<td>Unable to Make Sense of Sally’s Reasoning</td>
<td>Inconsistent</td>
</tr>
<tr>
<td>T21</td>
<td>Unable to Make Sense of Sally’s Reasoning</td>
<td>Consistent</td>
</tr>
<tr>
<td>T25</td>
<td>Initial Uncertainty but Later Success</td>
<td>Inconsistent</td>
</tr>
<tr>
<td>T27</td>
<td>Initial Uncertainty but Later Success</td>
<td>Inconsistent</td>
</tr>
</tbody>
</table>

Table 7 – Teachers’ Understand of Sally’s Reasoning without CBA
The following cases demonstrate examples of teachers’ use of the CBA materials. Each case is analyzed with commentary throughout to bring attention to the instances that help to illustrate how this type of use can be important for correctly characterizing student thinking. Each of the cases discussed are of situations that involve the distributive property, as the cases involving skip counting did not contain rich enough data to conclude how teachers were using CBA materials.

Teacher 3 – Sally Problem

[no use of CBA MD materials] – Straightforward Correct Evaluation of Sally’s thinking

Task.

\[ 45 \times 23 = \_\_\_\_\_ \]

Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.
I: Okay. So (b) what CBA level of sophistication do you think Sally’s strategy is?
T3: [Goes down the left side of level 2 and level 3 on the QR MD.]
I: I know it’s hard because you haven’t gone through this.
T3: Uh huh.
I: But just see what you can come up with.
T3: Let’s see 3.3 or 3.4 [goes from 3.3 to 3.4]. The only thing I’m wondering about is the partial products what that means.
I: Uh huh.
T3: Did [that?] 4 times is that 1 [points to the 1st sentence for 2a]? Okay she breaks it up once [points to the 1st sentence again] is that what you mean?
I: Hum.
T3: Twice [points to the 2nd sentence] okay 3 [points to the 3rd sentence] 4 points to the 4th sentence. So it would actually be 4 partial products.
I: Okay.
T3: So 3.4 [points to 3.4 on the QR MD].

As is evidenced by the interviewer’s comments, T3 has limited experience with the CBA MD document. T3 explicitly states that she is not sure what the term partial products means, indicating that the clear delineation between levels is not completely understood. T3 rhetorically asks the interviewer if the decomposition and subsequent computation is what is meant by the term ‘partial-products’ used in the level, and continues to use this interpretation of partial products to conclude that there must be 4 partial products. This is an example of a teacher who might have limited experience in reading and using the CBA MD document, and therefore utilizes an incomplete understanding of the technical definition of partial products in CBA. T3 interprets partial products as a term meant to signify certain decompositions and computations qualifying as partial products, but not the specific form of place value decompositions in CBA’s
technical use of the term. This leads to the conclusion that a 4 partial product approach was taken, when in fact Sally’s approach differs from the distributive property, place-value based approach of a 4 partial products strategy.

Teacher 15 – Sally Problem

[used CBA MD materials]: Initial Uncertainty but Later Success in interpreting Sally’s thinking

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = _____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
<td></td>
</tr>
</tbody>
</table>

Figure 16 – Sally Student Work

T15: I think she’s definitely breaking numbers up which is [looks at QR MD] distributive property [points to 3.3 and runs a finger over 2 Partial in 3.3] 2 partial products. Well, she had a little more than 2 [points to 3.4 and moves a finger down to 4 Partial in 3.4]. I’d say 4 partial products. Well, she has a little more than 4. Cause she has 1 [possibly points the 1st sentence] 2 [points towards the 2nd sentence] 3 [points towards the 45 times 3 is 120 in the 3rd sentence] ah 4 [points towards the plus 15 in the 3rd sentence]. I guess there is 4. So it’s interesting though how she did it. It’s not quite the way you would think of 4. Because she broke the 10 up [points to the 10 in the 1st sentence and then another finger points towards the 4th sentence]. The 20 [points to 23] in 23 up into 2 tens [points to the 10 in the 1st sentence and then another finger points towards the 4th sentence and then moves a finger down to the 3 in the 3rd sentence]. Instead of breaking the 45 up [goes over the 1st and 2nd sentences and then points to 45 in the 3rd sentence] for the first distributive part [goes back and forth on the 1st sentence then points to 45 in the 2nd sentence]. But there are 4 partial products.

I: So what are the 4?

T15: 1 [points to the 1st sentence] 2 [points to the 2nd sentence] 3 [points to the 3rd sentence] 4 [points to the plus 15 in the 3rd sentence]. Well the par, the partial, the 1st partial product is the 450 [circles 450 in the 1st sentence] and then 450 again [circles 450 in the 2nd second]. So that’s 2.
And then the next partial product of the 120 [circles 120 in the 3rd sentence]. And the final one is 15 [circles 15 in the 3rd sentence].

I: Okay.

T15: So I would say [points to QR MD 3.4] uses distributive property to decompose numbers by place-value into 4 partial products. Which is 3.4.

T15 initially identifies Sally’s level to be level 3.4 by simply counting the number of computations that result from distributive property decompositions. However, T15 appears to recognize that Sally’s work does not constitute what is expected for level 3.4 reasoning by stating that ‘It’s not quite the way you would think of 4’. This indicates that T15 conceptualizes 4 partial products differently than is demonstrated in Sally’s work, but still persists in believing that Sally’s approach would qualify as a 4 partial product approach. Although T15 does appear to recognize that Sally’s approach does not match the characterization of 4 partial products as described by the CBA MD framework, the special decompositions computed by Sally seems to be the over-riding factor for T15 in the delineation between levels 3.3 and 3.4 from other decomposition strategies. T15 does not identify that Sally actually does apply a ‘2 partial products’ approach when thinking about 45 times 3 as 40x3 plus 5x3. Identifying this may have helped T15 recognize that a partial products approach as defined in CBA depends on specific distributive property, place value based decompositions. It is not clear if T15 realized that the particular decompositions described in CBA levels 3.3 and 3.4 are singled out as critically important, or if any place value decompositions (and not those specifically described in CBA levels 3.3 and 3.4) are considered equivalently to T15.
Teacher 18 – Sally Problem

[used CBA MD materials]: Unable to Make Sense of Sally’s thinking

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = ______</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 17 – Sally Student Work

T18: I think she is [brackets 3.1 to 3.4 on QR MD]. She’s recalling, deri [derive] she’s recalling [goes over 3.3] 2 p, it’s this one, 2 partial products.

I: Which is what level?

T18: 3.3. She’s de, decomposing the number into 2 partial products. In this case it’s [points to 10 in the 1st sentence, 4 in 45] she’s not doing 4 partial products. She’s doing 2 [points to 45 and 23 then 5 in 45].

I: What are the 2 that’s she’s doing?

T18: Maybe she’s using number properties [3.2]. Because I was thinking if she was doing 2 partial products it would be 45 times 20 [writes horizontally 45 x 20] and then plus [writes +] 45 times 3 [continues writing 45 x 3 horizontally]. And that would be 2 partial products. [Points to her written + sign.] So she’s even breaking [points towards 23] this apart [points to 10 in the 1st sentence] so she’s using, she’s using [points towards the right side of Level 3 on QR MD and reads part] facts to derive answers using various properties of numbers. So she’s using number properties [points to 3.2 on QR MD]. I take it back. So then I think she’s a 3.2 [underlines 3.2 on QR MD]. She’s breaking the number apart but [points to 2 in 23] that she’s using what she knows about tens to arrive at her number.

T18 initially characterizes Sally’s thinking as employing a 2 partial products approach, but upon probing from the interviewer to explain the ‘2 that she’s doing’ determines that Sally’s approach does not match a Level 3.3 partial products approach. T18 then correctly demonstrates an example of what a Level 3.3, 2 partial products approach could look like for 45x23 by showing 45x20 and 45x3 (decomposing the
number 23 by place value parts). After demonstrating what qualified as a 2 partial products approach, T18 clarified that Sally broke apart the factor of 20 into two tens which differed from the technical definition of place value partial products as defined by the CBA levels partial products approach. While T18 does recognize the overall strategy as not being 4 partial products, T18 does not explicitly address Sally’s approach to resolving 45x3 as 40x3 and 5x3, which represents a correct application of a 2 partial products approach. Had T18 been asked to look more carefully at just this component of Sally’s reasoning T18 might have been able to elaborate an accurate conceptualization. T18 might not have noticed or commented on this, however, because many of the CBA teachers struggled to completely comprehend Sally’s comment that ‘45 times 3 is 120, plus 15’ with many teachers, at least temporarily, interpreting Sally to have made a mistake. Although Sally’s work can be characterized as level 3.2, there is also a correct demonstration of level 3.3 reasoning within Sally’s work that T18 does not describe. Overall, T18 has a more detailed understanding of the delineation between the partial products levels and number property levels than T3 and T15.

Teacher 19 – Sally Problem

[used CBA MD materials]: Straightforward Correct Evaluation of Sally’s thinking

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = ____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 18 – Sally Student Work
T19: Well, she’s using distributive property first. You know 45 times 10 and 45 times 10 again is the same as 45 times 20. As long as you’re adding them back together which she’s doing. So you’ve got that. [Points towards times in the 3rd sentence.] Then she’s doing 40 times 3. And 5 times 3 [points towards 120 in the 3rd sentence] to get the 120 and the 15 [points towards 15 in the 3rd sentence]. So again distributive. And [slight pause] yeah.

I: Okay. All right. So (b) what CBA level of sophistication is Sally’s strategy?

T19: [Pause] She’s in the 3.3. She’s using partial products.

I: Okay. Okay. And how did you decide what level?

T19: Well, it’s not necessarily 2 partial products but she’s not coming to the 4 partial products that would be the more traditional, 4 that you would start with and end with. She’s definitely breaking it down more so. But she’s still coming to partial products. She puts them together [points towards plus 15 in the 3rd sentence] together [points towards 1035 in the 4th sentence] in a way which may put her at a 3.2. But [waves hands if unsure then puts them back towards the beginning of the 3rd sentence and the end of the 4th]. She’s definitely using her number properties well. She’s using the distributive property. She’s breaking down the numbers. You know she’s basically getting 2 partial products but she’s doing more steps to get there.

I: Okay.

T19: So, ah 3.2 instead of 3.3 probably but. You know as far as her method goes it, it’s a heavy use of the distributive property. And you know I can’t necessarily say she’s got 2 partial products. So that would probably put her at the 3.2 instead of 3.3 but.

Much like T18, T19 initially wonders if Sally is using a CBA place value distributive property partial products approach. In reasoning through Sally’s thinking, T19 identified that Sally is not really executing a ‘more traditional’ form of 4 partial products or 2 partial products. T19 even goes as far as to say that Sally is essentially doing 2 partial products in more steps. This represents a highly conceptual interpretation of Sally’s representation of 20x45 as 10x45 and 10x45, indicating that T19 has a fairly
complex understanding of both the student thinking and the CBA levels. T19 identifies Sally’s ‘heavy use’ of the distributive property, but also recognizes the lack of strict place value decomposition. Because of this lack of place value decomposition, T19 concludes ‘I can’t necessarily say that she’s got 2 partial products.’ This is a conservative interpretation of Sally’s work as Sally does show evidence that she can complete the thinking required for a 2 partial products approach, however, this represents a very detailed and highly conceptual use of CBA MD to analyze Sally’s thinking. It should also be noted that although T19 observed elements of several CBA levels, T19 wanted to have clear evidence that Sally was at a specific level and not make assumptions that she might be higher.

Teacher T8 – Sally Problem

[used CBA MD materials]: Initial Uncertainty but Later Success in interpreting Sally’s thinking

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = _____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 19 – Sally Student Work

I: What CBA level of sophistication is Sally’s strategy?
T8: Humm…she is decomposing the number, she [doesn’t have]… you have this as two partial products. I don’t know what the number properties is. If that would be taking the 20 and breaking that into parts? But she is definitely breaking down the numbers. She is decomposing it,…
I: So she is level 3?
T8. Right. Technically it is two partial products because she found the 20 that is 10 and 10 so…I would put her at a 3.3.
T8 seems to interpret Sally’s work similarly to T19, wondering if taking the 20 and breaking it into parts would be a demonstration of the number properties sublevel in Major level 3. However, instead of characterizing the thinking as level 3.2 like T19, T8 articulated that Sally is technically doing a two partial products approach because she is really using the 20 but as 10 and 10. Although T8 is making an assumption that Sally is thinking about the decomposition as 23 as 20 and 3, and then further decomposing into 10 and 10, T8 demonstrates understanding of the partial products approach that is helpful for delineating between place value and non place value based decompositions. In some sense, the issue here is when teachers are forced to attempt to classify a strategy that has not been explicitly described in the CBA levels. T18 and T19 seem to understand the levels, but they have difficulty because Sally is actually using: a) Level 3.2 and 3.3, and b) she uses Level 3.3 twice.

Teacher T2 – RR and QR problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>

Figure 20 – RR and QR Student Work

T2: Ok see this student here. This example in MD 3.2 page #37, does the same exact thing. The question here though is 96/4 = 24 and this person says four 20’s is 80. They broke it into something meaningful to them. And four 4’s is 16, and 80 and 16 is 96. Very similar – broke it apart into manageable parts and then added them together…but actually they didn’t
add them together. This student said 80 and 16 is 96 – but this student said 20x5 is 100 another 20x5 is 200, they didn’t say 100 plus 100 is 200. Even though the idea is there – they didn’t come out and say it. I would probably say MD 3.2.

I: Ok. Do you want to check the next one and make sure you wouldn’t put him up a level. Just to make sure?

T2: Well it looks like 3.2.2 – it says 4 partial products. There are NOT 4 partial products. Let me see what 3.2.1 says – 2 partial products by place value. Is 3.2.1. 3.2 says - uses the distributive property with numbers decomposed by place value.

I: I’m not telling you you’re right or you’re wrong – I’m just saying – is that your final answer?

T2: I would probably say that MD 3.2 would be my final answer.

I: 3.2.1? or 3.2.2? Because 3.2 means it has two sublevels.

T2: Ok well I was looking up here. So I would say 3.2.1. T2: Ok see this student here. This example in MD 3.2 page #37, does the same exact thing. The question here though is 96/4 = 24 and this person says four 20’s is 80. They broke it into something meaningful to them. And four 4’s is 16, and 80 and 16 is 96. (45:30) Very similar – broke it apart into manageable parts and then added them together…but actually they didn’t add them together. This student said 80 and 16 is 96 – but this student said 20x5 is 100 another 20x5 is 200, they didn’t say 100 plus 100 is 200. Even though the idea is there – they didn’t come out and say it. I would probably say MD 3.2.

I: Ok. Do you want to check the next one and make sure you wouldn’t put him up a level. Just to make sure?

T2: Ok. ‘cause if I was to put it in there – I would say they did 3 partial products. But I guess this is 2 because he took forty and broke it down to twenty. So he did 2 products to that one, but he did 2 products to get the sum, essentially – so I think that might be where his reasoning is there. Because you have 2 partial products and 4 partial products. There is not 3 partial products which is what he did here.

I: It feels closer to this (points to 2 partial products) doesn’t it?

T2: So he did not do 4. So maybe that’s where he says it must be 2. Because he broke the one up into 2 parts which gave him 3.

I: OK. Well that makes sense to me – 40 is broken into 20 and 20.
T2: Ok. ‘cause if I was to put it in there – I would say they did 3 partial products. But I guess this is 2 because he took forty and broke it down to twenty. So he did 2 products to that one, but he did 2 products to get the sum, essentially – so I think that might be where his reasoning is there. Because you have 2 partial products and 4 partial products. There is not 3 partial products which is what he did here.

I: It feels closer to this (points to 2 partial products) doesn’t it?

T2: So he did not do 4. So maybe that’s where he says it must be 2. Because he broke the one up into 2 parts which gave him 3.

I: OK. Well that makes sense to me – 40 is broken into 20 and 20. What CBA level would you move him to next?

T2: Well. With the understanding that was shown here in the breaking apart – I would say it’s time for the traditional algorithm.

I: And why?

T2: Just because they showed me the understanding of breaking this numbers apart the place value is there – because he says ok I’m going to break it into something easy 20. Or just for example if they said 40x5 is manageable for me, I understand this is 40. Then place value is sound – I’m ready to go with the algorithm. The concept of multiplying is there because the place value, they’re able to separate the place value.

In this episode, T2 struggles to connect the meaning of the CBA levels to RR’s reasoning. T2 does identify that RR’s thinking is clearly not a 4 partial products approach, but becomes very attached to the idea of RR’s approach as ‘3 partial products’. This type of conception of partial products as solely defined by the number of distributive property based decompositions, although not inaccurate, does not align with CBA’s technical use of the term in the CBA levels. The number of decompositions is not the sole requirement for the partial products method, as place value decomposition is an important component. T2 alludes to this by saying ‘So he did not do 4. So maybe that’s where he says it must be 2. Because he broke the one up into 2 parts which gave him 3.’ T2 implicitly seems to recognize that one of the place value components in the 2 partial
product approach is decomposed further, but does not explicitly connect the thinking to
the place value basis. When considering instruction, T2 explicitly states that he would
move RR to an algorithms level, in the process skipping several CBA levels, because of
RR’s understanding of place value. T2 states that ‘if they said 40x5 is manageable for
me, I understand this is 40…Then place value is sound – I’m ready to go with the
algorithm’. In stating this, it becomes less clear that T2 recognizes that RR’s
decomposition is not strictly by place value, and that distributive property, place value
based decomposition is not clearly demonstrated in RR’s thinking in a way that is aligned
with CBA level 3.3 and the technical definition of partial products for CBA. In this case,
by categorizing RR’s work as being a form of partial products based predominately on
number of decompositions, T2 is unable to see the conceptual components of the
reasoning and that place value decomposition might be an area to focus instruction for
RR. A 2 partial products approach might be well within the reach of a student like RR,
but T2’s suggestion to move to an expanded algorithm indicates that T2 is likely grossly
over-estimating RR’s overall use of place value based multiplication.

[NOTE: The next portion of the interview with T2 occurred roughly 2 weeks
after the initial portion of the interview that was just detailed. The interview had to be
broken into 2 parts due to time restrictions for T2. Because of this, T2 responded to
questions regarding RR’s reasoning 2 weeks prior to seeing QR’s reasoning]
Task.  

RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.

QR: 40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.

Figure 21 – RR and QR Student Work

T2: Next Task. 46 multiplied by 5. The response was 40 times 5 is 200, 6 times 5 is 30. So it is 200 plus 30 which is 230. That’s pretty good understanding of place value.

I: What CBA level for MD would you say that that’s at? Maybe I should ask you why you think that’s good at showing you understanding of place value.

T2: Because they were able to separate the numbers. To separate it into 40 and 6, the 4 tens and 6 one or 6 units. And they basically broke it down into a manageable problem for themselves so that is a higher level of thinking also. I don’t see any algorithm ‘cause it was done mentally.

I: Good – so you know not to look there.

T2: Um Off the top of my head – they broke it down by the distributive property, is that correct.

I: Yeah. That’s what I’d say.

T2: So they also decomposed the number. Numbers decomposed by place value – 40 and 6. So I would say that probably I’d pick that off the cuff. But I’m going to read it (CBA level)

I: I want you to read it because some of them have sublevels that aren’t listed.

T2: ok. (2:25) It’s kind of hard to get this organized. (Reads CBA levels). I would probably MD 3.3 because it is 2 – they break it up into 2.

I: Ok. Final Answer? Because they break it up into 2 what?

T2: 2 multiplication problems

I: What part of the CBA document convinces you that you are correct?

T2: Page 36. Under MD 3.2 using the distributive property and number decomposed by place value, which is what caught my attention because they did break it apart by place value. So I went there first. But it is broken down into 2 levels, one where they break it into 2 parts and one where they break it into 4 parts, and this problem they break it into 2 parts.
I: Good. CBA Author agrees with you – 2 partial products.

In moving to QR’s reasoning, T2 again identifies place value as strength in the students’ reasoning. After being prompted, T2 elaborates that QR separates into 40 and 6, 4 tens and 6 ones, demonstrating explicitly the place value nature of QR’s decomposition. T2 determines that it must be level 3.3 because the sublevels have two choices, ‘one where they break it into 2 parts and one where they break it into 4 parts’. Even though T2 identifies place value as crucial, it is not explicitly stated in the choice for the level, as 3.3 is chosen because in ‘this problem they break it into 2 parts’. T2 did recognize the place value nature of the decomposition, and the number of distributive property based decompositions appears to also be an important factor in T2’s decision-making.

I: Describe the differences between RR and QR’s strategies
T2: QR is the same as what we just did, right?
I: It is, yes
T2: (Reads problem with RR’s strategy) See that’s interesting. Because, there is no 3 parts…but they broke it up into 3 parts almost, so that kinda interesting. The difference between them is that they broke down the 200 into smaller parts, and maybe that is because it is a single digit on the other number, so they couldn’t break it down any further. I would definitely put them at least at an MD 3.2.1, but I would say that they…well I would pursue whether they were at the 4 part level.
I: For which one?
T2: For QR. Sorry, for RR. I would say they would at least be at that level because they showed they broke it into 3 parts. If that makes sense.

T2 is re-introduced to RR’s reasoning, and comes to the same conclusion (2 weeks later) that RR is doing a ‘3 parts’ approach. T2 determines that this must mean
that RR is ‘at least at MD 3.3’ (2 partial products), and wonders whether or not to ‘pursue whether they were at the 4 partial products level’. This is a demonstration of how the 2 partial products versus 4 partial products delineation can be confused when introducing what appears to be a 3 partial products approach. Although there are 3 decompositions in RR’s work, there is no evidence that they were decomposed strictly by place value, which is an important component of any partial products approach in the CBA framework. T2 appears to conceptualize the levels as being based on number of decompositions alone. This perspective might lead to a teacher perceiving a student who breaks 46 into 10, 10, 10, 10 and 6 as being 5 partial products, and therefore being even more sophisticated than RR and QR, when in fact it would really be a different version of RR’s strategy. It almost seemed like T2 considered a ‘3 partial products strategy’ to be between 2 and 4 in sophistication, which is not in line with the CBA framework.

I: Can I ask you a question to push your thinking on this?

T2: Yes.

I: I know you’re just getting back into it, and this is actually one of the harder examples. Did they do it by place value?

T2: No they did it by landmark numbers.

I: Then let me have you look at the document. Because, I think that you’re just warming up and I don’t want you to say ‘Oh this is what the authors says’ and be wrong.

T2: Well it was by a landmark number. At least 100 is a landmark number and they repeated that twice. That’s the number they were using. Their knowledge of 100.

I: Now you need to look and see is there anything that looks like that. I don’t know if (the CBA author) uses that exact language.

T2: (Reads) Uses known facts with numbers not decomposed by place value. But this was decomposed by place value, still. No. Kind of.
Based on prompts by the CBA researcher interviewing T2, the issue of place value decomposition is made a point of investigation. Upon being asked about place value decomposition, T2 recognizes that RR might not be decomposing strictly by place value. T2 reads the CBA document and obviously is wrestling with the thinking RR used, saying ‘(Reads from CBA) Uses known facts with numbers not decomposed by place value. But this was decomposed by place value, still. No. Kind of.’ At this point, T2 is uncertain about the role of place value in RR’s thinking, possibly because RR’s approach appears at least partially place value based (instead of decomposing 46 into 40 and 6, RR decomposes into 20, 20 and 6, which still separates tens and ones).

I: Maybe if you looked at. Um. Maybe 37 would be helpful. Just to get a feel for this.

T2: (Reads example in CBA). If the student knows 20x20 is 400 that’s more than a landmark number.

I: Ok

T2: ‘Cause 100 is a landmark number, to me. In my understanding.

I: So maybe level 2 somewhere?

T2: I’m gonna look in level 2. I will look at 2.3 first. (Reads). (10:00)

Students combine iterations of original composites into bigger composites. This levels includes the doubling and build up strategy. They didn’t really double. So I’m going to go back even further MD 2.2 which is operating on composites. (Reads) Students determine a product using skip counting. No. There was no skip counting. Or is there? They didn’t do it.

I’m narrowing it down – but I’m going to end up going back to MD 2.3. (Reads).

Ok. I’m in a quandary right now.

I: Between?

T2: I didn’t see anything in MD 2.
I: Oh. That looked like it? So you’re coming back.

T2: I’m coming back to MD 3 to try and make some sense out of it. (Reads)
OK. Task 8 on page 35. MD 3.1 says a container contains 12 eggs, Emily has 5 cartons. How many eggs does Emily have altogether. 60. How can you prove that? 5x5 is 25 5x5 again is 25 and 5x2 is 10 add those together its 60. I think that’s what RR did. I would say MD 3.1.

I: Deriving answers from known facts?

T2: Yeah. It is Using known facts using numbers not decomposed by place value. ‘cause they did not do this by place value.

I: and then 2 partial products you already said for the other one.

T2: Right.

I: (The CBA Author) agrees with you.

T2: Good!

T2 searches through the CBA document, finding nothing representing RR’s thinking in level 2. Upon re-reading the level 3 portions of the document, T2 settles in on 3.1 deriving answers from known facts, based upon a sample of student work. T2 even explains that this sample of student work is representative of RR’s thinking because ‘they did not do this by place value’. Note the heavy use of matching student thinking to examples in the CBA MD materials, this was a common strategy used by teachers to identify a student’s CBA level. This also proved to be more challenging for teachers when investigating thinking at level 3.2. Because level 3.2 has many variations, it is difficult to describe the level adequately and provide sufficient examples to represent all of the different ways that students can reason at that level.

I: Why do you think that QR’s reasoning is rated as more sophisticated that RR’s in the CBA document?
T2: The concept of place value. Um. And. Yeah. They have the concept down. They have the place value concept, and the concept that they are using groups of. They just broke it down just into smaller parts that are manageable to them, which they both broke it down into smaller parts, but RR used a number that was familiar instead of a place value concept. So that’s why I would say.

With help and leading questioning from the interviewer, T2 articulates a more detailed understanding of the difference between RR and QR’s reasoning that makes it clear that place value is an important distinction. T2 now says that ‘RR used a number that was familiar instead of a place value concept’ as the delineation between RR and QR’s reasoning, which represents a more cognitive perspective than simply recognizing the number of decompositions. T2’s case represents an interesting instance where a CBA researcher was able to encourage a CBA teacher to more carefully explore the CBA materials to make more accurate decisions about student’s thinking. What is especially positive is that after encouraging T2 to more carefully explore RR’s thinking he was able to find additional support in the CBA materials to better make sense of the student thinking. This is an important component of the CBA approach, as it is essential that teachers utilize the details of the CBA documents to reflect on student work and persist towards more detailed understanding of that work.

Teacher T6 – RR and QR Problem

[used CBA MD materials]:

125
Task. 46 × 5

RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.

QR: 40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.

Figure 22 – RR and QR Student Work

I: Please read episode aloud and tell me what you are thinking

T6: (Reads Verbatim).

I: What CBA level of MD is demonstrated?

T6: Well. They would have to be at least in the known facts because they knew 20x5 is 100. Which I wish my students would remember stuff like that. So I would skip to 3 for sure. And I can’t remember what 4 is about. So I may have to look at that. This is sort of what we’ve been doing with cluster problems only this student is a little more advanced than a few of mine. Because a few of mine would not be able to do 20x5, they would have to start with maybe 10x5 and double it. But as far as…looking at the examples…it looks like 3.2. I might go with…I’m also going to look at the 4’s. He didn’t really turn it into a partial product. I guess I’m going with the 3.2. Yeah. Like the clusters. But it truly isn’t a ‘cluster-cluster’. I think I’ll stick with the 3.2

I: What makes you think that 3.2 is the CBA Level? (19:42)

T6: Um. I guess because I even looked at the first example, and it is 20x30. She says I know 20x20 is 400, which this student did similar. She gives another fact that this student knows in the document and she adds them together which is pretty similar to what this student did. But he does have a subtotal in there. And it says they decompose and recompose. He took 46 and made it into 20 and 20, two 20’s and a 6. I think 3.2

I: What part of the CBA document convinces you that you are correct?

T6: I guess I have to go again with. I look at the examples to see if they are close, and then I go back to the description to compare that way.

I: Ok. The CBA level that (the CBA author) chose was 3.2 Deriving answers from known facts.

T6 represents an example where a deeper conception of what the student is thinking is provided by the CBA document. T6 knew, but did not elaborate on why, that RR ‘didn’t really turn it into a partial product’. T6 appears to use her experience from
working with her own students to relate the approach to ‘cluster’ type strategies. By utilizing the examples, as well as the text describing the examples, T6 is able to effectively select the appropriate CBA level.

I: So what level after the 3.2 would you want to go to then?

T6: I think I would have to see more examples of their work to see are they able to settle into the 2 partial products or 4 partial products. So level (3.3) and (3.4) rather easily [2 or 4 partial products], maybe. ‘cause if they are already at 3.2 I think they could slide into the 3.3’s rather easily. And then see how they handle the 3.4 and on from there. But you’d have to see their thinking more than once. Maybe give them some examples, or challenge them to use the strategies shown in 3.3 and 3.4, and if they were successful, then move them on.

I: (The CBA author) did say 3.3 and 3.4. What instructional activity or task might move this student to the next level

T6: I’m kinda assuming that the student has a pretty good understanding of number sense if they are already at 3.2. So if they do have truly they’re able to look at numbers and say that 20x5 is 100, that’s slightly sophisticated and they could maybe even be shown examples of 2 or 4 partial products in the 4.1’s and 4.2’s. And say here is a problem – solve it with this strategy. ‘Cause I think if they are that sophisticated they should be able to change their thinking.

T6 argued that thinking at level 3.2 is fairly sophisticated and that they would be able to move into thinking about 2 partial products fairly easily. T6 does not explicitly state the transition from 3.2 to 3.3 as progressing towards more place-value based distributive property approaches, but does explicitly state that she would like to see RR’s thinking on more than one problem to see which type of thinking they settle into. T6 wonders if RR was shown a demonstration of the 2 partial products strategy by a teacher, if they would not be able to just internalize the process and use it.

I: Please read this next episode and tell me what you are thinking.
T6: This student wrote 40x5 is 200 and 6x5 is 30 so it is 200 + 30 is 230. Um. Well they broke it apart. Did 40x5 and they knew the fact. So here we are – they are operating on known facts. Um. Just kinda wonder – let me read the 2 partial products in 4.1. (Reads) I’m gonna think that they did the 2 partial products. Because it looks like exactly what he did. He took the 46 and broke it apart. He solved 40x5, so he has to be higher in level 3 because they know the fact, and they went up to the 2 partial products. So I’ll have to go with the 4.1.1

I: What makes you think that 4.1.1 is the CBA level? (32:25)

T6: Well. They didn’t really cluster much. Well – I guess it could be considered clustering…but I think it is more sophisticated than clustering because he didn’t have to do the 20x5 and 20x5. This one went straight in to breaking the 46 into 2 parts. Which is what that says ‘2 partial products’. I read the description of course, but I just glanced through these examples. And I found task 16 where they describe how the student solves it and the student who answered talks about breaking it apart. And that’s what I said in my mind, well that’s what this student did here (QR). They broke it apart. And that’s what they did on this task 8 too. She said ‘I just took off the 2 and added 5 tens, so that’s 50 and then 5x 2 is 10.’ She broke up 12 into 10 and 2, and that’s what QR did with the 46.

Again, T6 used the CBA text as well as the samples of student work to determine the level. In addition to explaining that QR broke 46 into 2 parts, T6 elaborates ‘that’s what this student did here. They broke it apart. And that’s what they did on this task 8 too. She said ‘I just took off the 2 and added 5 tens, so that’s 50 and then 5x 2 is 10.’ She broke up 12 into 10 and 2, and that’s what QR did with the 46’.

I: What part of the CBA document convinces you that you are correct? (34:03)

T6: I think this one is one of the easier ones to look at the title. And then just operating on known facts. He or she started there, so it was kind of an easy starting point to go. At least at 3 or above. And having just talked about the other student, and knowing clusters already helps too. And then just looking at the title, 2 partial products. That narrowed the search down. And then I looked at the examples and the descriptions.
T6 identifies that the 2 partial products level is ‘one of the easier ones to look at the title’ which is an indication that it is a level that can be interpreted with relative ease, and potentially even rote. However, T6 also looked at the examples and read the descriptions to better understand and conceptualize QR’s work. It is important to note that using a CBA level, such as 2 partial products, in a rote fashion by only looking at computations and not for the specific CBA place value and distributive property based decompositions can occasionally lead to correct leveling of students within the CBA MD framework. However, if the conceptual underpinnings of the strategy are not completely understood, it would be expected that teachers who rote, but correctly, level a student within the CBA framework might struggle to conceive of a proper learning goal or instruction due to this potentially incomplete understanding of the student’s thinking.

I: Describe the difference between RR and QR’s strategies (37:51)

T6: Well. I guess in our classroom. And they’ve even said it in here a little…I don’t know if it was the students, or The author’s writing about where they break it apart. We talk about that in the classroom. Looking at QR he is more sophisticated because either he knows his facts better or he’s able to look at 46 and turn it into the tens and the ones and then combine it. Whereas RR, which is also sophisticated, to be able to look at 46 and know that 40 breaks into 20 and 20, and do 20x5, 20x5, and then combine them with the 6. They are both what we are calling in my class, clustering. It’s just that RR had to break 46 down more than QR had to. Maybe because of his familiarity with facts, or…

T6: Well they are solving it by breaking them apart. It’s just that it looks like to me in 3.3 they take the actual number and break it apart. QR did 40 and 6. But RR had to break it down further. It’s the cluster problem. He or she had to do that. RR wasn’t able to do 40x5 and 6x5 RR had to do 20 and 20. So they had to break the number down – further decompose.
Here, T6 explicitly identifies that the delineation between RR and QR’s thinking according to the CBA levels is that QR broke the ‘actual number’ apart into 40 and 6, which RR ‘had to break the number down – further decompose’. This constitutes a very detailed account of using CBA materials to more fully understand a child’s thinking.

*Teachers’ Misconceptions and Challenges with CBA Terminology*

The CBA MD materials are intended to help guide teachers in their interpretations of children’s thinking about mathematics, how to develop key concepts and skills, and provide a framework for understanding the mathematical and cognitive progression that typical children follow. As is the case with any research-based learning progression, a significant amount of research findings must be elaborated, consolidated, and presented in a form that is meaningful for teachers. Inevitably, learning progression materials, and realistically any research-based curricula, will have elements that seem to resonate and make sense to teachers, while other portions will lead to misinterpretations, confusion, and misuse. In the case of learning progressions, it is especially important that different levels of thinking or levels of sophistication are clearly delineated for teachers. If the thinking at two different levels is difficult for teachers to distinguish between, it is important to find ways to help teachers make sense of the variety of thinking strategies children may use in a way that is clear and specific.

The following section is in no way intended as a criticism of teachers, but rather is intended to summarize some of the most challenging elements of the CBA MD materials and framework for teachers. Many of the challenges teachers encountered related primarily to confusion with research-based vernacular. Whether it was the use of an
unfamiliar term, or that certain terms and ideas related to challenging concepts for
teachers to make sense of, there were several elements of the CBA framework that
elicited differing interpretations from CBA teachers. Instances of the variation of
interpretations of certain key terms and concepts from the CBA MD framework are
explored through short cases with analytic commentary. The three key terms that
teachers demonstrated difficulty in understanding within the CBA MD framework were
in the use of the term ‘hidden’ in reference to place value ideas used in an algorithm, the
term ‘partial products’ as a multiplication strategy, and the term ‘algorithm’ as a written
organizational strategy. The variety of interpretations of each term is explored with
several instances of each described in short cases.

_CBA Teachers’ Conception of the Term ‘Hidden’ in Language for Traditional
Algorithms in Multiplication and Division_

The CBA MD document details the transition children make from skip-counting
strategies, to decomposition by place value, and use of the distributive property, and
moving into the use of systematic algorithms of expanded or traditional form. An
important component of the CBA framework is that the levels of sophistication represent
a strong hierarchy, meaning that understanding at level 4 implies the ability to use, but
not necessarily the explicit demonstration of, thinking strategies at lower levels.
Therefore, moving forward from an expanded multiplication algorithm to a traditional
algorithm might imply that students no longer explicitly models understanding of place
value concepts, but such understanding would still exist.
Consequently, the language of the CBA materials emphasized this by writing about the traditional multiplication algorithm: “Students use their understanding of place value and other properties of numbers to conceptually understand traditional algorithms for whole-number multiplication and division, even though place value ideas in these algorithms are hidden.” As is the case with any set of materials for teachers, the language that is used can be misinterpreted, misunderstood, or simply confusing. The word ‘hidden’ was intended to mean that in traditional algorithms, place value ideas are not explicit; they are implicit in the placement of digits. Furthermore, in reciting the guiding language for performing a traditional algorithm, the language refers only to digits, not to the place values of the digits. The concept of ‘hidden’ is used to convey that algorithms might hide place value concepts, and students might not show written or verbal evidence of place value concepts without being probed about their thinking. Some teachers struggled to interpret this concept in a way that aligns with CBA.

Teacher 20 – Sally Problem

[no use of CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
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<tbody>
<tr>
<td>$45 \times 23 = \underline{ }$</td>
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Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.

Figure 23 – Sally Student Work

I: Okay. What type of reasoning do you think Sally should move to next and explain why.

T20: You know I. I really have to say that I am not sure. I mean I guess, (pause) I guess that she would move on to level 5 but honestly I don’t have
I am one of those people that learned it this way (pointing to traditional algorithm) and I don’t even know because I don’t teach it and I don’t know. But My guess is, my guess is that this (pointing to Sally’s method) is more sophisticated than this (pointing to traditional algorithm) because this is just memorizing how to do it. I don’t know that to be certain though. It seems like this has more thinking involved than what I did. I just kind of did it.

I: Okay, so you think that what you said for this is more sophisticated than what you did. Is that

T20: I don’t know I mean… This (Sally’s method) definitely involves more thinking about it. Thinking about the place value parts and thinking about which numbers you need to put together where as this is just… just me remembering how to multiply and move numbers and puts zeros and all those things.

I: What would you do instructionally to move Sally to this next type of reasoning and explain why.

T20: (Pause) Well level 5, the difference is that the place value ideas and the algorithms are hidden. So I would want her to be able to do that. But I really don’t exactly know what I would do to have them hidden. But that would be the goal as far as I can see from the levels.

I: Okay..So instructionally, you are really unsure of how you would go about it

T20: I’m not sure

T20 verbally states that they do not have a concept of how to move the student forward instructionally to ensure that ‘place value ideas and algorithms are hidden’. This is interesting that a teacher wanted the place value ideas to be hidden, when in fact the goal of Level 5 is for those ideas to be explicit and not hidden in the child’s reasoning. T20 clearly demonstrates a desire to apply a LP consistent viewpoint on instruction by recognizing that the CBA levels can be used as goals for students to make progress towards. However, T20 seems to overly attribute the word ‘hidden’ to be applied not just to place value, but also to algorithms. In the context of the traditional algorithm, place
value ideas may be hidden, but the algorithm itself is not hidden – the algorithm is what hides the place value concepts. This may cause confusion for T20 in determining steps forward, as teaching such that place value ideas and the algorithms are hidden might seem to be a strange thing to do. T20 represents an instance where a term can be confusing to a teacher, and not serve to help emphasize the key components of this type of thinking. In a sense, T20 seems to interpret the CBA level as a goal that should be honored without understanding exactly the consequences or meaning. T20 also seems to equate more reasoning with higher sophistication, which is not necessarily consistent with CBA.

Teacher T25 – Sally Problem

[no use of CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 \times 23 = ____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 24 – Sally Student Work

T25:  Yep. I don’t know it look like to me just off hand that she is using a level three. Like she is combining and she is doing things hum…because she knows like 23 there is two groups of 10 so she is taking 45 and times one group huh getting huh 450, 45 times the other group then taking it times….so she is separating and doing things with it hum…and she hum…and she knows her facts hum…level 4 says she is using expanded algorithms and maintain the value of place value parts throughout the sequence of steps. So she is doing that too even though she said like she understands 45 times 3 [even though] she said 120 plus the 15 so she understands the different place value and where they need to go so. I guess I would I think level 4. Cause hum…I don’t understand what they are saying in level 5 even though place value ideas and these algorithms are hidden that phrase I don’t really understand. Understanding of place value
and other properties of numbers to conceptually understand. Even though place value ideas and these algorithms are hidden no I don’t know if that would be where that 45 times 3 is 120 plus 15. Now if that’s the case she might be at a level 5 because she knows even though she said 45 times 3 is 120, hum…plus 15…so she didn’t really throw in that 40 times 3 plus 5 times 3 it is kind of all in her train of thought so if that’s what that means then she possible could be at a level 5. I started at 3…

I: okay so, you would like to have that word hidden defined is that what heard is

T25: Yea I mean you know it might be hum…you know and as far as for me it is all hidden when it comes to these algorithms. But hum yea I don’t really understand that that term there.

I: So what type of reasoning do you think Sally should move to next and why do you think that?

T25: Well I think if…I know she uses and understands expanded algorithms. Hum…now whether if she is doing you know with the hidden algorithms you know level 5 is definitely where she should be working. Cause I think that is what she possibly is doing here.

Much like T20, the CBA statement about place value concepts being hidden in the traditional algorithm confuses T25. T25 elaborates further, exposing a potentially problematic consequence of misinterpretations of learning progression language. T25 explicitly articulates confusion about the meaning of the CBA language, but also expressed confusion or uncertainty about Sally’s work on a mathematics problem. T25 had identified Sally’s statement that “45 times 3 is 120, plus 15” as being confusing, and identified this as an element of Sally’s thinking that T25 did not understand. T25 seems to conflate these misconceptions by stating “Even though place value ideas and these algorithms are hidden no I don’t know if that would be where that 45 times 3 is 120 plus 15.” This appears to be a situation where there were two unknowns (Sally’s thinking and CBA language) in T25’s mind that were equated. T25’s short case exposes a potentially challenging aspect to any LP materials, in that teachers who may not have experience
with certain strategies or types of mathematical thinking could inadvertently equate novel student work to an improper term in the CBA document.

*CBA Teachers’ Conception of ‘Partial Products’ Language for Multiplication and Division*

The term ‘partial products’ has a specialized mathematical meaning in CBA to correspond to the decomposition by place value and multiplication of place value parts. For example, 45x23 could be written using 4 partial products as (40x20) + (40x3) + (5x20) + (5x3) or using 2 partial products as (45x20) + (45x3). ‘Partial products’ is defined by children’s decomposition by place value and their use of the distributive property. The term partial products in the CBA document was challenging for many teachers to make sense of, as several teachers explicitly stated that they did not know what the term meant. Several other teachers demonstrated implicit misconceptions or incomplete conceptions about the CBA definition of the term in their analysis of a students’ work, by assuming that a student was utilizing partial products when the decomposition was not strictly by place value and therefore not the special ‘partial products’ as describe in the CBA document. This represents potentially an everyday understanding of a term, rather than a specialized CBA meaning of the term.

Occasionally with student work it is not clear whether or not a student applied a place-value based partial product conception, or simply decomposed numbers into recognizable smaller numbers. In these instances, it is important to identify the place-value decomposition as a key characteristic of the thinking, and the term partial products is intended to help teachers identify and recognize this. CBA does this in a very specific
way, emphasizing the strict place value decomposition and use of the distributive property in its definition of the term partial products. The following cases detail situations that illustrate these challenges for teachers. This section concludes with contrasted cases in which teachers demonstrate conceptions of partial products that are more closely aligned with CBA.

Teacher 3 – Sally Problem

[no use of CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 ( \times ) 23 = ____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 25 – Sally Student Work

I: Okay. So (b) what CBA level of sophistication do you think Sally’s strategy is?

T3: [Goes down the left side of level 2 and level 3 on the QR MD.]

I: I know it’s hard because you haven’t gone through this.

T3: Uh huh.

I: But just see what you can come up with.

T3: Let’s see 3.3 or 3.4 [goes from 3.3 to 3.4]. The only thing I’m wondering about is the partial products what that means.

I: Uh huh.

T3: Did [that?] 4 times is that 1 [points to the 1st sentence for 2a]? Okay she breaks it up once [points to the 1st sentence again] is that what you mean?

I: Hum.

T3: Twice [points to the 2nd sentence] okay 3 [points to the 3rd sentence] 4 points to the 4th sentence. So it would actually be 4 partial products.

I: Okay.

T3: So 3.4 [points to 3.4 on the QR MD].
I: Okay. All right. And (c) what type of reasoning do you think Sally should move to next?

T3: [Puts a hand on Level 4 QR MD.]

I: Explain why.

T3: Level 4 [points on level 4 QR MD with the other hand].

I: Okay.

T3: Because she understands [taps on level 4 QR MD]. She can break it all apart [motions over b with one hand and the other still on level 4 QR MD]. But now that we’re sure she understands [puts hand under b] what she’s doing [puts hand under b again].

I: Uh huh.

T3: And thinking about the numbers she could go to a [goes back to level 4 QR MD with the other hand still there] simpler way [moves a hand back to b with a hand still on level 4 QR MD] of doing it.

I: Uh huh.

T3: Recognizing the place value.

T3 clearly states that ‘partial products’ is not a term that is well understood. T3 then rhetorically asks if when ‘she breaks it up once’ is a partial product, indicating that decomposition into other multiplication problems is what T3 is connecting the term to. T3 then continues to count the total numbers created after decomposition (of which there are four), but does not attend to the lack of strict place value use in the students’ thinking. Although T3 appears to not fully understand the use of the term partial products in a CBA aligned fashion, in discussing the next level to take the student, T3 articulates that the student should be moved forward to doing the problem in a more simple way that involved the recognition of place value. So, in spite of the fact that the term partial products did not appear linked to place value recognition for T3, the overall trajectory that T3 set for the student indicates that place value is an important component for the...
student to master. This case represents how misinterpretation of a term in the CBA MD framework does not always have negative consequences on the trajectory that a teacher sees for a student. In this case, while T3 did not explicitly connect both place value decomposition and the distributive property to the partial products terminology, analyzing another student’s work helped T3 articulate that place value was an important missing component for the student.

Teacher 20 – Sally Problem

[no use of CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
</tr>
</thead>
<tbody>
<tr>
<td>45 × 23 = ____</td>
</tr>
</tbody>
</table>

Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.

Figure 26 – Sally Student Work

T20: (Pause and reviews the CBA quick reference) Can you tell me what this partial products means? Just parts?
I: Okay, hum.. You know what. I will let you go ahead and let you look into your document if you would like.
T20: I think it’s here (tapping on level 4). I mean I’ll look it up, but I think she’s umm…(pause, looks through the CBA document) is it back here? Oh, I don’t think I have those two.
I: You don’t have multiplication and division?
T20: Uh hum.... No, I only have the two I did.
I: Oh..Okay
T20: Oh wait no. Maybe I do. There is too much [here] (Continues to scan through CBA). Pause. cause I knew I would never do that one because we don’t teach multiplication. But I think…students expanded (reading from CBA Quick reference).I think it is definitely expanded what she did. Multiply and divide numbers maintain the value of place value parts
throughout the sequence of steps which she does because she she uh she knows that this is two tens. So I think that she is at a level 4.

T20 also clearly indicates that partial products is not a term that is well understood, and also identifies that multiplication and division is not a CBA document they have read or used. This lack of experience is evident with the interpretation that a student who decomposed 23 into $10 + 10 + 3$ ‘maintain[ed] the place value parts throughout the sequence of steps because she, she uh she knows that this is two tens’. This indicates an incomplete understanding of ‘maintaining place-value parts’ as a key element of the partial products approach, or the expanded algorithm (the level T20 actually chooses for the student). It is true that the student decomposed into tens and ones, but the student did not decompose 23 into 20 and 3 as would be necessary using a CBA partial products approach. The delineation between the level defined by decompositions that are not by place value, and the partial products levels is especially difficult for teachers. Sally’s work demonstrates exactly when this delineation can be so tricky, as Sally does maintain separation between tens and ones (10, 10 and 3), but does not group all tens together to maintain the original number of tens. T20’s limited experience with student work in multiplication and division could definitely contribute to her lack of technical understanding of the term ‘partial products’ as used in CBA.
**Teacher 24 – Sally Problem**

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 $\times$ 23 = ______</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

**Figure 27 – Sally Student Work**

T24: I guess I am going more towards the upper end right away because I know that she is doing it correctly it is just a question of how she is using… is there more? I thought that there was one that said using the distributive property. Oh here it is ok that is why it is a 3.1. Cause I thought distributive property because she is using the distributive property. Totally using the distributive property, overusing it. Using the distributive property to decompose numbers by place value in two partial products. So is this four partial products are we talking like that like 2 and 2? I guess I am not sure what partial products means.

I: Yeah those four. Yeah so this is basically four partial products.

T24: So this is four partial products that I did. So does she do 4 partial products at least 1,2. Does need to be exact? Let’s see what she did 1, 2, 3 and then she broke this in 120 and 15 which would be four. So I guess she used 3.4 she used the distributive property decomposing numbers into 4 partial products just not systematically.

I: And when you say not systematically what do you mean?

T24: I guess what I am saying when I say systematically… like what I mean is that if I am looking at this what I would verbally say to her is you are totally correct and this working for right now, but is this always going to work for you? Are you always going to know for sure that you have taken care of all of the numbers? That she is taking care of the three in the ones place. Multiplying the 20 both by the 5 and by the 40. Because my systematic that I am teaching see I am just a control person. [She is doing it the expanded way on her paper]. That I know by going 3 times 5 that I have already taken care of that one. And by 3 times 40 done so lets move on. So now we are tens place that is 20 and now we have taken care of the 5 and we are taking care of the 40. So now check, check, check, check, check [writes checks next to her partial products] and then I am taking care of all four of those {partial products}. And I know that I haven’t missed anything. This way [points to Sally’s work] makes total sense and
she has got it and I can see what her brain is thinking, but it is so random to me that I am more worried that she might miss something.

T24 also indicates an incomplete understanding of ‘partial products’, but talks through the students’ way of thinking while comparing to the method for doing 4 partial products. In doing so, T24 explains that the student is utilizing a ‘non-systematic’ 4 partial product approach. While Sally’s work is not truly a demonstration of a ‘4 partial product approach’ and T24 may fail to align perfectly to the CBA vernacular, it is still the case that T24 effectively articulates one of the key differences between the non-partial product, and non-algorithmic levels of thinking. T24’s suggestion to address the lack of systematicity in Sally’s thinking is a crucial transition in the CBA levels when moving children from simply using decomposition and the distributive property towards routine, place-value based strategies.

Teacher 15 – Sally Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = ____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
<td></td>
</tr>
</tbody>
</table>

Figure 28 – Sally Student Work

I: Okay, (b) And what CBA level of sophistication is Sally’s strategy? And explain why

T15: I think she’s definitely breaking numbers up which is [looks at QR MD] distributive property [points to 3.3 and runs a finger over 2 Partial in 3.3] 2 partial products. Well, she had a little more than 2 [points to 3.4 and moves a finger down to 4 Partial in 3.4]. I’d say 4 partial products. Well,
she has a little more than 4. Cause she has 1 [possibly points the 1st sentence] 2 [points towards the 2nd sentence] 3 [points towards the 45 times 3 is 120 in the 3rd sentence] ah 4 [points towards the plus 15 in the 3rd sentence]. I guess there is 4. So [?] it’s interesting though how she did it. It’s not quite the way you would think of 4. Because she broke the 10 up [points to the 10 in the 1st sentence and then another finger points towards the 4th sentence]. The 20 [points to 23] in 23 up into 2 tens [points to the 10 in the 1st sentence and then another finger points towards the 4th sentence and then moves a finger down to the 3 in the 3rd sentence]. Instead of breaking the 45 up [goes over the 1st and 2nd sentences and then points to 45 in the 3rd sentence] for the first distributive part [goes back and forth on the 1st sentence then points to 45 in the 2nd sentence]. But there are 4 partial products.

I: So what are the 4?

T15: 1 [points to the 1st sentence] 2 [points to the 2nd sentence] 3 [points to the 3rd sentence] 4 [points to the plus 15 in the 3rd sentence]. Well the par, the partial, the 1st partial product is the 450 [circles 450 in the 1st sentence] and then 450 again [circles 450 in the 2nd second]. So that’s 2. And then the next partial product of the 120 [circles 120 in the 3rd sentence]. And the final one is 15 [circles 15 in the 3rd sentence.

I: Okay.

T15: So I would say [points to QR MD 3.4] uses distributive property to decompose numbers by place-value into 4 partial products. Which is 3.4.

I: Okay. And (c) what type of reasoning do you think Sally should move to next? Explain why.

T15: Ah, the expanded algorithm, yeah I think [points towards level 4 expanded algorithms]. Yeah, which is basically what she’s doing almost. It’s very close. Yeah, and I think really just multiplying [points to 10 in the 1st sentence] getting her to look at [points to 23] instead of 10 [points to 10 in the 1st sentence] and 10 [points to 10 in the 2nd sentence] looking at this as 20 [points to the 2 in 23]. So combining this step here [goes over the 1st and 2nd sentences]. Okay, then that would make 3 partial products. Where would we put that one?

T15 follows the same line of reasoning as T3 and T20 in arguing that Sally has completed partial products. In spite of failing to explicitly recognize the lack of strict place value decomposition when identifying partial products, in discussing the type of
reasoning to move Sally to next T15 identifies moving towards combining 10 and 10 to make 20 (which would constitute strict place-value decomposition without further decomposition). T15 then wonders if that would create ‘3 partial products’ and rhetorically asks where that level would fall in the CBA framework. T15’s question is well received considering that a decomposition of 45x23 into 45x20 + 40x3 + 5x3 involves the thinking required to complete both ‘2 partial products’ and ‘4 partial products’. Whether or not this is a case of a teacher identifying a potential sub-level or not, it does represent a teacher who sees the possibility of helping move a student to a next level that does not appear to be explicitly part of the CBA framework. For many of the CBA levels, teachers do occasionally recognize sub-levels that some students skip over, and other students might move through very quickly. It should be stated too, that although it might not be a level included in the CBA MD framework, it is completely believable that some students might apply this sort of hybrid approach to certain 2-digit times 2-digit problems as they are transitioning away from a 2 partial product approach onto a 4 partial product approach. However, referring to it as a ‘3 partial product’ approach might be confusing given the CBA use of partial products for 2 and 4 partial products.
Teacher T2 – RR and QR problem

[used CBA MD materials]:

Task.  46 × 5

RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.

QR: 40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.

Figure 29 – RR and QR Student Work

T2: Well it looks like 3.2.2 – it says 4 partial products. There are NOT 4 partial products. Let me see what 3.2.1 says – 2 partial products by place value. Is 3.2.1. 3.2 use the distributive property with numbers decomposed by place value.

I: I’m not telling you you’re right or you’re wrong – I’m just saying – is that your final answer?

T2: I would probably say that MD 3.2 would be my final answer.

I: 3.2.1? or 3.2.2? Because 3.2 means it has two sublevels.

T2: Ok. Well I was looking up here. So I would say 3.2.1.

I: Ok. And (The CBA Author) says – MD 3.2.

T2: Ok. ‘cause if I was to put it in there – I would say they did 3 partial products. But I guess this is 2 because he took forty and broke it down to twenty. So he did 2 products to that one, but he did 2 products to get the sum, essentially – so I think that might be where his reasoning is there. Because you have 2 partial products and 4 partial products. There is not 3 partial products which is what he did here.

I: It feels closer to this (points to 2 partial products) doesn’t it?

T2: So he did not do 4. So maybe that’s where he says it must be 2. Because he broke the one up into 2 parts which gave him 3.

T2 quickly identifies that the students’ strategy looks like a partial products approach, but that it is not the 4 partial products approach. In the context of the task (46 × 5), 4 partial products is not even possible as it is a 2-digit times and 1-digit problem, and cannot be decomposed into tens and ones for each number. T2 characterizes the
additional decomposition of 40 into 20 and 20, as resulting in ‘3 partial products’.
Although T2 maintains the perspective that the student would truly be doing ‘3 partial products’, T2 states that ‘I guess this is 2 because he took forty and broke it down to twenty. So he did 2 products to that one, but he did 2 products to get that sum.’ This suggests that T2 implicitly recognizes that the students’ strategy is a variation on the 2 partial product approach that involves further decomposition. Stating that the student did ‘2 products to that one’ appears to be an informal way of stating that an additional decomposition occurred from the standard partial products approach.

Teacher T5 – RR and QR (pt 1)

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>

Figure 30 – RR and QR Student Work

The following case represents an interesting discussion between a CBA teacher (T5) and a CBA research team member, where the term partial products and the delineation between levels are discussed in a teaching experiment-like setting. The CBA researcher engages in a conversation to get at the core of the delineation between the partial products levels. The entire discussion is provided below, with commentary throughout to frame the discussion.
T5:  (Scans document) MD 3.3. 2 partial products by place value.
I:  What makes you think that 3.3 is the CBA Level?
T5:  ‘Cause it sounded like a good thing to say.  Ha
I:  Now you sound like your kids!
T5:  Yep. Well it’s in between ok. Because you’ve got MD 3.3 which is 2 partial products by place value or 3.4 which is 4 partial products… and THIS is 3 partial products. So, it’s gotta be somewhere in between.
I:  OK. And what part of the CBA document convinces you that you are correct?
T5:  Task 25 on 3.3 that says 5x32 = 160.
T5:  Or task 8 where he writes out the 3 different problems. To come up with the 3 different sums.
I:  The 5x12. Ok.

The author’s CBA level is MD 3.2 deriving answers from known facts. But that’s not what 3.2 is called is it? (T5 looks at document).
T5:  Sure, why is that the level?
I:  You know, I’m not supposed to…I’m supposed to let you make sense of it. So let me step back out of the teacher mode and let you have a chance.
T5:  (Considers CBA Author’s level) Well this is almost exactly like task 22. On page 34. Under MD 3.2. (Reads) ‘I remember that 20x5 is 100. So 20x6 is 20 more – that’s 120, and 2 more sixes I get 132.’
I:  So that’s what’s helping you believe that this answer is the correct one?
T5:  So. This is being considered as – extending basic single digit multiplication facts to corresponding multiple of 10 or 100. So we’re considering these multiples of ten. And then here is my single digit multiplication. Ok. If I pressed hard enough – I could still justify 3.2.1 – but I’m not going to.
I:  It’s a subtle difference. Do you see the difference? (47:38) It’s in what you do with this 4. Look at your 2 partial products. Remember your example here. This task. Ok. If you take this one. It is 12, they break up the ten and the 2. Separately right? Or here for 32 they break up the 30 and the 2. So they do it strictly by place value. But this student, how does that differ from what this student did here?
T5:  Oh I see. I see what you’re saying.
I:  It’s pretty subtle isn’t it?
T5: Yeah it is.
I: They take the tens and then say – well I don’t know 40 x 5 but 20 x5 that’s 100. So they’re not lining it up with place value.
T5: So it is not attached to the place value of the number.
I: Exactly.
T5: The reasoning in breaking it apart is not attached to the place value of the number.
I: Exactly. So it is a small difference.
T5: Ok. Well I get it. At least I wasn’t on the opposite end of the spectrum.
I: No. You were just right next door. Ok. What CBA level do you think this student should move to next? (49:11)
T5: Well jeez. Maybe 3.3. Because I thought he was almost there. But I can get him there – because he’s basically breaking it down already!

In this encounter, T5 interprets the student work to be at a partial products CBA level. However, the CBA author disagreed with that answer, as the author would put the student at level 3.2. After revisiting the proposed level from the CBA author, T5 does not appear convinced in spite of the fact that T5 found a sample of student work in the CBA descriptions of level section describing essentially the same thinking as the work in question. At this point, the interviewer’s comments become instructive, and T5 is given details on how the delineation between the levels depends on the ‘strict’ use of place value decomposition. T5 eventually claims to understand the delineation, and is relieved that her interpretation of the student work was not drastically different from the CBA author’s. T5 also quickly identifies the ‘next level’ as the goal for instruction based on the discussion, and even articulates that she believes she could easily move this student to that level because their thinking is very close to that of the next level.
I: Ok. Please read this episode aloud and tell me what you are thinking (QR)

T5: 46x5. 40x5 is 200 and 6x5 is 230. Ok then. So we’ve moved into the place value aspect of it. So it is 2 partial products. 3.3.

I: ok. What makes you say that? What makes you think that 3.3 is the CBA level.

T5: Because he’s broken the number into 2 products, but he’s not broken down the 40 like he did before. Here 40 is the place value of the tens times 5, and 6 is the number in the ones times the five. There’s 2 separate products and then he adds them together.

I: Good. So you’re looking at the example?

Or are you looking at it abstractly?

T5: I really am.

I: That’s what the CBA author would say. 3.3.

T5 demonstrates that she does in fact see the delineation between level 3.2 and level 3.3, by identifying that the new sample of student work demonstrates a strict use of place value decomposition that differs from the first students’ work.

Teacher T5 – RR and QR problem (pt2)

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>

Figure 31 – RR and QR Student Work

T5: (Reads Task). Where is that wonderful distributive property (searches CBA document) 3.2 derives answers without distributive property place value decomposition.

I: What makes you think it is 3.2?
T5: Because there is no real mention of place value…they are just breaking the numbers apart. That’s my answer and I’m sticking to it.

I: Can you tell me what part of CBA document convinces you?

T5: Well ok. Maybe I’m being too literal with this. ‘Cause I think it is more like 3.3. Because I think they’re using the distributive property. I just don’t think that it’s being said. So they are decomposing it by place value – they’re just not saying it. So now you have to decide is it 3.3 or 3.4?

I: How did you decide it was not 3.2?

T5: Because of the delineation between 3.2 and 3.3 is not just place value – but whether or not they’ve used the distributive property. And they did in fact use the distributive property. So, And… It’s not 4 partial products, it is 3…so is that right in between? Um. Probably 3.3 because the first 2 steps are probably considered 1. ‘Cause they are still using the same composite.

T5 clearly recognizes the distributive property in the initial determination of a CBA level, but seems uncertain about whether or not the student used the distributive property AND decomposition by place value parts. T5 accurately views both components as crucial in moving from level 3.2 (Uses Properties Other Than Distributive-Property Place-Value Decomposition) to the partial products levels. Once T5 settles in on the partial products levels, she identifies that maybe the student is doing the distributive property and place-value decomposition, but not saying it. T5 then makes a very important observation, that even though it appears that it is ‘3 partial products’ which T5 wonders if it ‘is right in between’ 2 and 4 partial products, T5 states that ‘the first 2 steps are probably considered 1. ‘Cause they are still using the same composite.’ This shows that T5 has a fairly complex understanding of the student thinking with respect to the partial products terminology. It should be noted that this interview took place following a teaching experiment interaction between a CBA researcher and T5 discussing some of the key characteristics of different MD levels (the previous case
represents a partial transcript of this teaching experiment). This is positive evidence that discussions between CBA researchers and teachers about important delineations between levels and characteristics of student thinking can be meaningful to CBA teachers, and can be internalized in some way over time. T5 initially struggled with the concept and the delineation, but demonstrated during the teaching experiment that she recognized the delineation between the levels. In this interview, T5 no longer needs the assistance of a CBA researcher to make sense of this delineation.

Teacher 18 – Sally Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 \times 23 = ____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 32 – Sally Student Work

I: Okay. So (b) what CBA level of sophistication is Sally’s strategy? And explain why.

T18: I think she is [brackets 3.1 to 3.4 on QR MD]. She’s recalling, deriving, she’s recalling [goes over 3.3] 2 p, it’s this one, 2 partial products.

I: Which is what level?

T18: 3.3. She’s de, decomposing the number into 2 partial products. In this case it’s [points to 10 in the 1st sentence, 4 in 45] she’s not doing 4 partial products. She’s doing 2 [points to 45 and 23 then 5 in 45].

I: What are the 2 that’s she’s doing?

T18: Maybe she’s using number properties [3.2]. Because I was thinking if she was doing 2 partial products it would be 45 times 20 [writes horizontally 45 x 20] and then plus [writes +] 45 times 3 [continues writing 45 x 3 horizontally]. And that would be 2 partial products. [Points to her written + sign.] So she’s even breaking [points towards 23] this apart [points to
10 in the 1st sentence] so she’s using, she’s using [points towards the right side of Level 3 on QR MD and reads part] facts to derive answers using various properties of numbers. So she’s using number properties [points to 3.2 on QR MD]. I take it back. So then I think she’s a 3.2 [underlines 3.2 on QR MD]. She’s breaking the number apart but [points to 2 in 23] that she’s using what she knows about tens to arrive at her number.

Teacher T6 – RR and QR Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>

Figure 33 – RR and QR Student Work

I: Please read episode aloud and tell me what you are thinking
T6: (Reads Problem Verbatim).
I: What CBA level of MD is demonstrated?
T6: Well. They would have to be at least in the known facts because they knew 20×5 is 100. Which I wish my students would remember stuff like that. So I would skip to 3 for sure. And I can’t remember what 4 is about. So I may have to look at that. This is sort of what we’ve been doing with cluster problems only this student is a little more advanced than a few of mine. Because a few of mine would not be able to do 20×5, they would have to start with maybe 10×5 and double it. But as far as...looking at the examples...it looks like 3.2. I might go with...I’m also going to look at the 4’s. He didn’t really turn it into a partial product. I guess I’m going with the 3.2. Yeah. Like the clusters. But it truly isn’t a ‘cluster-cluster’. I think I’ll stick with the 3.2

I: What makes you think that 3.2 is the CBA Level? (19:42)
T6: Um. I guess because I even looked at the first example, and it is 20×30. She says I know 20×20 is 400, which this student did similar. She gives another fact that this student knows in the document and she adds them together which is pretty similar to what this student did. But he does have
a subtotal in there. And it says they decompose and recompose. He took 46 and made it into 20 and 20, two 20’s and a 6. I think 3.2.

For both T18 and T6, the proper level is chosen, and partial products are not interpreted to represent simply decompositions made by the student. In the case of T18, it is explicitly stated that it doesn’t really match the proper 2 partial products approach, and T18 even writes out what that would look like. T18 elaborates to say that this must mean that the student is even breaking apart one of the composites further than expected in a partial products perspective, so therefore it must be a lower level. In the case of T6, less is explicitly stated about what the student did or did not do, but T6 effectively utilizes other samples of student work in the CBA document to determine that the student did not ‘really turn it in to a partial product’. By finding examples that helped to illustrate this difference, T6 was able to effectively identify that a place-value partial products approach was not exactly used by the student.

Teacher 19 – Sally Problem

[used CBA MD materials]:

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<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 34 – Sally Student Work

I: Okay. All right. So (b) what CBA level of sophistication is Sally’s strategy?
T19: [Pause] She’s in the 3.3. She’s using partial products.
I: Okay. Okay. And how did you decide what level?
T19: Well, it's not necessarily 2 partial products but she's not coming to the 4 partial products that would be the more traditional, 4 that you would start with and end with. She's definitely breaking it down more so. But she's still coming to partial products. She puts them together [points towards plus 15 in the 3rd sentence] together [points towards 1035 in the 4th sentence] in a way which may put her at a 3.2. But [waves hands if unsure then puts them back towards the beginning of the 3rd sentence and the end of the 4th]. She’s definitely using her number properties well. She’s using the distributive property. She’s breaking down the numbers. You know she’s basically getting 2 partial products but she’s doing more steps to get there.

I: Okay.

T19: So, ah 3.2 instead of 3.3 probably [but/would?]. You know as far as her method goes it, it’s a heavy use of the distributive property. And you know I can’t necessarily say she’s got 2 partial products. So that would probably put her at the 3.2 instead of 3.3 but.

I: Okay. All right.

T19: Ah!

Teacher T8 – Sally Problem

[used CBA MD materials]:

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<tr>
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</tbody>
</table>

Figure 35 – Sally Student Work

T8: Humm…she is decomposing the number, she [doesn’t have]… you have this as two partial products. I don’t know what the number properties is. If that would be taking the 20 and breaking that into parts? But she is definitely breaking down the numbers. She is decomposing it,…

I: So she is level 3?

T8: Right. Technically it is two partial products because she found the 20 that is 10 and 10 so…I would put her at a 3.3.
Both T19 and T8 recognize that the student is essentially using 2 partial products for this problem. T19 says ‘You know she’s basically getting 2 partial products but she’s doing more steps to get there’ while T8 says ‘technically it is two partial products because she found the 20 that is 10 and 10’. However, these cases demonstrate that even with an accurate conceptualization of a students’ thinking, and alignment with the terminology used in the CBA document, it is very reasonable to come to different determinations of what a students’ level is. In the case of T19, a more cautious approach is used to leveling the student as T19 says ‘And you know I can’t necessarily say she’s got 2 partial products. So that would probably put her at the 3.2 instead of 3.3’. However, T8 believes that the student has demonstrated the appropriate conceptual understanding of the 2 partial products approach and credits her for 3.3. Both are reasonable conclusions.

CBA Teachers’ Conception of ‘Algorithms’ Language for Multiplication and Division

The CBA LP for Multiplication and Division identifies the hallmark transition from decomposing numbers using the distributive property (level 3) to the use of expanded algorithms (level 4) as being dependent upon students moving towards systematic, formalized approaches to their thinking. Level 3 for CBA MD is characterized by children’s thinking that involves the distributive property and decomposition of numbers into partial products, but this can occur for students as simply a mental thinking strategy, and not as a written, organized framework for multiplying numbers. Level 4 for CBA MD may involve the same thinking strategies as Level 3, but
represents an important shift towards formal algorithmic thinking that provides children with a systematic framework to keep track of increasingly large and challenging multiplication problems.

CBA LP’s conceptualization of ‘algorithms’ is an important component of understanding the CBA LP and is important for understanding the CBA LP perspective on how to help children develop more sophisticated ideas in Multiplication and Division. Teachers’ responses to student work involving the distributive property indicate that this distinction is not clearly made for many teachers.

Teacher 15 – Sally Problem

[used CBA MD materials]:

\[
\begin{array}{c}
\text{Task.} \\
45 \times 23 = \\
\end{array}
\]

Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.

Figure 36 – Sally Student Work

T15: I think she’s definitely breaking numbers up which is [looks at QR MD] distributive property [points to 3.3 and runs a finger over 2 Partial in 3.3] 2 partial products. Well, she had a little more than 2 [points to 3.4 and moves a finger down to 4 Partial in 3.4]. I’d say 4 partial products. Well, she has a little more than 4. Cause she has 1 [possibly points the 1st sentence] 2 [points towards the 2nd sentence] 3 [points towards the 45 times 3 is 120 in the 3rd sentence] ah 4 [points towards the plus 15 in the 3rd sentence]. I guess there is 4. So [?] it’s interesting though how she did it. It’s not quite the way you would think of 4. Because she broke the 10 up [points to the 10 in the 1st sentence and then another finger points towards the 4th sentence]. The 20 [points to 23] in 23 up into 2 tens [points to the 10 in the 1st sentence and then another finger points towards the 4th sentence and then moves a finger down to the 3 in the 3rd sentence]. Instead of breaking the 45 up [goes over the 1st and 2nd sentences and then points to 45 in the 3rd sentence] for the first
distributive part [goes back and forth on the 1st sentence then points to 45 in the 2nd sentence]. But there are 4 partial products.

I: So what are the 4?

T15: 1 [points to the 1st sentence] 2 [points to the 2nd sentence] 3 [points to the 3rd sentence] 4 [points to the plus 15 in the 3rd sentence]. Well the par, the partial, the 1st partial product is the 450 [circles 450 in the 1st sentence] and then 450 again [circles 450 in the 2nd second]. So that’s 2. And then the next partial product of the 120 [circles 120 in the 3rd sentence]. And the final one is 15 [circles 15 in the 3rd sentence].

I: Okay.

T15: So I would say [points to QR MD 3.4] uses distributive property to decompose numbers by place-value into 4 partial products. Which is 3.4

Teacher 15 very clearly understands what the student did, and also recognizes that it isn’t ‘4 partial products’ in the traditional sense when saying “It’s not quite the way you would think of 4”. However, T15 fails to recognize that the term ‘partial products’ is very much place value based, and that Sally’s reasoning might not be best described by 4 partial products.

I: Okay. And (c) what type of reasoning do you think Sally should move to next? Explain why.

T15: Ah, the expanded algorithm, yeah I think [points towards level 4 expanded algorithms]. Yeah, which is basically what she’s doing almost. It’s very close. Yeah, and I think really just multiplying [points to 10 in the 1st sentence] getting her to look at [points to 23] instead of 10 [points to 10 in the 1st sentence] and 10 [points to 10 in the 2nd sentence] looking at this as 20 [points to the 2 in 23]. So combining this step here [goes over the 1st and 2nd sentences]. Okay, then that would make 3 partial products. Where would we put that one?

Teacher 15 suggests that Sally is “essentially” doing an expanded algorithm, and suggests that Sally really just needs to not decompose 20 into two tens. T15 appears to identify the separation between Sally’s approach and the expanded algorithm as being the
emphasis on maintaining place value. The CBA materials emphasize ‘algorithms’ as
serving as an organizational purpose for children, to systematically write out their
thinking. Teacher 15 seems to get confused when re-thinking where Sally is when asking
where they would place a ‘3 partial products’ approach, and appears to be somewhat
confused about much of the language in levels 3 and 4 used in the CBA document.

I: And (d) what would you do instructionally to move Sally to this next type
of reasoning?
T15: Hum.
I: Explain why.
T15: Well, I wonder if we could just ask Sally if she could [circles the 1st and
2nd sentences] instead of doing this in 2, in 2 steps. Is there a way you
could do that in 1 step?
I: Okay.
T15: And maybe she would realize well instead of doing the 10 [points to 10 in
the 1st sentence] and 10 [points to 10 in the 2nd sentence] I could do the
20 [appears to point towards the 1st sentence].
I: Okay. So that’s the 1 step?
T15: One-step that I would use? Is that what you [kinda mean?]?
I: Well, you said that she would do it in one step.
T15: Oh yes, I see, yes [points to the 10’s in the 1st and 2nd sentences at
the same time]. She has 2 distinct steps here [with 2 fingers goes back and
forth and circles the 1st and 2nd sentences at the same time].
I: Okay. Okay.
T15: So instead of the 2 I’d say can you do that in just 1 step? What would
you, how would you be able to do that?
I: Okay. Is there anything else you would do to help her?
T15: If, well if she doesn’t come across it on her own [points back and forth to
the 1st and 2nd sentences] maybe ask her how she could break up [points
and circles 23] this into 2 parts. Cause she’s breaking it up into [points to
23] Sally has really good understanding about multiplication and division,
she does. [Points and underlines 23] well, how can she break this up into
2 parts instead of, you know using the 20 [points to 2 in 23] and the 3
[points to 3 in 23 and then points to 2 in 23 and 4 in 45] so. I think yeah that might help too. If she doesn’t get it on her own which I think she probably would cause she seems to have a pretty good idea of division or multiplication.

I: Okay. So how would you do the 20 and the 3 that you were talking about?
T15: Well and see what, well if you multiply [points to 2 in 23] instead of [goes over times 10 in the 1st sentence] the [points to 2 in 23] but see this really it makes me wonder cause then you’re only getting [points towards 23] 2 partial products if you do the 45 [points to 45] times the 20 [points to the 2 in 23] and then do the 45 [points to 45] times the 3 [points to the 3 in 23]. That’s actually going down to the 3.3 [points to QR MD 3.3].

I: Hum.

T15: Hum. Well, maybe asking her to do 2 separate problems her [taps back and forth on the blank part of the sheet]. How can we break this up [points to the 2 in 23] into 2 separate problems [points on the blank part of the sheet]? But see I don’t, I still I don’t know I’m a little. I remember the expanded algorithm I think. Well the [?] I know the traditional algorithm [points towards level 5] I’m trying to think, the expanded algorithm was [points on the blank part of the sheet]. I’m trying to think if the difference between these 2 [points towards 3.4 - 4 partial products then level 4 expanded algorithm].

I: Okay.

T15: So that’s what I, I’m struggling with here [points to 3.4]. So I’m thinking probably, hum, it’s been awhile since I’ve looked at this. [Reads from level 4] expanded algorithms maintain the value of place-value, so I would imagine [points to 4 in 45] you could do [points to the 3 in 23] and then back to the 4 in 45]. But see I can’t, I’m having trouble understanding the difference between these 2 [points back and forth near the ends of level 4 and 3.4]. Yeah cause I’m seeing you could do 40 [points to the 4 in 45] times 20 [points to the 2 in 23] and then [points to the 4 in 45]. But see that’s the 4 partial [points on the blank part of the sheet] under 4 partial in 3.4] partial products. She’s got the 45 [points to 45 in the 1st sentence] times the ten [points to 10 in the 1st sentence]. Which was actually 45 [points to 45] times the 20 [points to the 2 in 23]. But she I mean, the 4 partial products [points possibly to 23 and then 45] would be what? The 3 [points to the 3 in 23] well I guess the expanded algorithm would be multiplying the 20 times the 40 [points to the 4 in 45] the 5 [points to the 5 in 45] times the 23 [points to 23] and the 3 [points to the 3 in 23] times 45 [points towards 45] and then I’m missing one. And then the 5 [points to 5 in 45] times the 3 [points to 3 in 23]. So that, that’s where we would want her to be is coming up with those, with those I think. Isn’t that right.
I: Okay. All right.
T15: Yeah okay. I think so.

Teacher 15 reasons through instruction with relatively close alignment to the CBA document, proposing to move to a place value based approach to the partial products method, and then moving to decomposing to create 4 partial products. Teacher 15 makes it clear that they do not see the difference between levels 3.4 and 4. T15 is likely looking only at the appearance of the outcomes of the two strategies, which apply very similar cognitive processes. However, what teacher 15 seems to be overlooking is CBA’s emphasis on formalizing a systematic approach as the defining feature of algorithmic thinking. Teacher 15 is a prime example of how CBA can serve to facilitate a teachers’ thinking about children’s mathematical thinking even when all aspects of the CBA materials are not clear to the teacher. In this particular case, Teacher 15 views level 3.4 and 4 as essentially equivalent, but even that perspective allows Teacher 15 to effectively think about moving a child forward from a lower level. Teacher 15 has a misconception about the meaning of CBA level 4, but this misconception does not limit their ability to think about children’s thinking below level 4.
Teacher 24 – Sally Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = _____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 37 – Sally Student Work

I: So you are saying that it is correct then?

T24: It is correct according to my calculations. I guess I would have to think on it more. What she did was correct and each of those steps were correct but I haven’t taken the time to go back and see if according to the total product it is correct. You know I broke it down into yes she did this section. Each of these sections are correct, but I don’t know officially if these sections all add up to \{1035\}. So maybe I should \{do it again\}. So 3 times 5 and then 3 times 40 and then 20 times 5 and 20 times 40. So that would be 15, 120, 100, 800. So that is 5, 8, 9, 1035. So we do match so she did come up with the correct answer just not the way I would have thought to have done it. But it does make sense I mean she did not do anything weird or crazy that just accidently gave her the right answer. She has a understanding of place value. So with that in mind you want to know which level she is at.

I: Yes.

T24: I guess I am going more towards the upper end right away because I know that she is doing it correctly it is just a question of how she is using… is there more? I thought that there was one that said using the distributive property. Oh here it is ok that is why it is a 3.1. Cause I thought distributive property because she is using the distributive property. Totally using the distributive property, overusing it. Using the distributive property to decompose numbers by place value in two partial products. So is this four partial products are we talking like that like 2 and 2? I guess I am not sure what partial products means.

I: Yeah those four. Yeah so this is basically four partial products.

T24: So this is four partial products that I did. So does she do 4 partial products at least 1,2. Does need to be exact? Let’s see what she did 1, 2, 3 and then she broke this in 120 and 15 which would be four. So I guess she used 3.4 she used the distributive property decomposing numbers into 4 partial products just not systematically.
I: And when you say not systematically what do you mean?

T24: I guess what I am saying when I say systematically… like what I mean is that if I am looking at this what I would verbally say to her is you are totally correct and this working for right now, but is this always going to work for you? Are you always going to know for sure that you have taken care of all of the numbers? That she is taking care of the three in the ones place. Multiplying the 20 both by the 5 and by the 40. Because my systematic that I am teaching see I am just a control person. [She is doing it the expanded way on her paper]. That I know by going 3 times 5 that I have already taken care of that one. And by 3 times 40 done so lets move on. So now we are tens place that is 20 and now we have taken care of the 5 and we are taking care of the 40. So now check, check, check, check, check [writes checks next to her partial products] and then I am taking care of all four of those {partial products}. And I know that I haven’t missed anything. This way [points to Sally’s work] makes total sense and she has got it and I can see what her brain is thinking, but it is so random to me that I am more worried that she might miss something.

T24 articulates an informal version of understanding the importance of algorithms that aligns with the CBA perspective. Although T24 incorrectly labels Sally’s thinking as level 3.4, T24 is able to identify the lack of systematicity as a potential future limitation of the thinking.

T24: This is where that whole traditional algorithm comes in. That is still ok to use the traditional algorithm, the traditional algorithm isn’t wrong it is just wrong when you aren’t understanding what you are carrying. You know when they say carry the one. You are not carrying a one you are carrying a ten. Well she understand place value so she really could go to the traditional algorithm. I don’t think I need to have her go to the expanded algorithm of you have to do this first you have to do this second. It is hard for one problem was she lucky did she remember all of them in one problem if she was consistent and then all the problems she did were always being right. Then I know that she is always going to remember for the most part. Being human she will always or 99% of the time will get if she did them all like this [points to Sally’s work]. Then I would move to the traditional because she understands place value. Obviously right here [points to something on the paper].

I: So you wouldn’t move her to the expanded.

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Teacher 24 clearly understands that it is important for kids to have a systematic approach, but the ‘stepping stone’ of having kids do an expanded algorithm does not seem to be valuable for T24. In this respect, T24 has an incomplete perspective on the concept of transition students into using algorithms within CBA. Although Level 4 represents essentially the same thinking strategy as levels 3.3 and 3.4, the different between levels 3.3/3.4 and level 4 is that systematicity is added to the child’s thinking process. Level 4 serves to act as a bridging level, giving students an opportunity to think about the distributive property and apply a systematic approach before moving towards the traditional algorithm in which place value understanding is less explicit. T24 interprets the evidence from Sally’s work to mean that Sally truly understands place value, and therefore is ready to move onto the standard algorithm. This seems to indicate that T24 believes that the expanded algorithm level is only meant to imply that children understand the place value components of multiplication, and not as a means to practice using an organizing framework.

I: What would you do instructionally to move Sally this next type of reasoning?

T24: I would actually set up the traditional algorithms and talk her through it.

I: So how would you do that?

T24: Let’s say we would write down 45 times 23. So I would say Sally, in the past you were thinking about 3 going into 40 and then 3 going into 5. So instead of having to do all the additional writing over here let’s condense our writing down and not have to do as much side work. I might say a this {points Sally’s solution} is safe way to do it and you know that you are working it out, but now that you know this let’s not write this all out. Let
try to do a lot more of it just with our thinking. So if we start with the ones place and we do the 3 times 5 groups we would get 15. I always tell the kids that there is not enough room for two kids in one square. So we would leave the five here and then we would carry the ten groups of one over here so now we have ten groups, but we would just write as 1 cause we are over in the tens place. And kind of talk her through it like that. Then we have 3 groups of 4, but it is really 3 times 4 is 12 and we have one more group of tens so it would be 13. So just kind of talking her through it like that. Now we are done with the ones place so we have to leave a placeholder of the ones place. But I think that if she was do this she be able to understand the placeholders and place value.

T24 describes both procedural and conceptual components of the standard algorithm for multiplication when discussing instruction for Sally. T24 appears to make the assumption that Sally would not struggle with understanding the procedural components of the standard algorithm, and states that she would be able to understand the placeholders and place value. Part of the need for students to use the expanded algorithm is not just to emphasize place value understanding, but also to emphasize procedural competence alongside meaningful steps. Moving to the standard algorithm too quickly could easily impose a procedure that would be completed without meaning.
Teacher 27 – Sally Problem

[no use of CBA MD materials]:

<table>
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<tr>
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<td></td>
</tr>
</tbody>
</table>

I: Still do you think she is correct or incorrect?
T27: Ok I would say that I think that she got the basics and she has got the skill correct but she is not explaining it well. That would be my {opinion} because even me having to sit here and why did she get hundred 120. If this was my student I would rather her say that this is a 120 plus a 15 because she knew that was 120 and that 3 times 5 is 15. Just to be a little thorough. Yeah I would say her thought make sense, what I can see that she is breaking apart the numbers into expanded form and she is using combinations that are easy like a 10 and 10 she realizes that would make a 20. And then she is breaking this [points to the 45 times 3] down into a manageable unit for her. So I would go with that. Yeah I would go with that as long as she could explain this more because right away I had think about it.

I: When you say that you would go with it do you mean that she was correct?
T27: Yeah, I would say that would work.

I: Ok. What CBA level of sophistication is Sally’s strategy?
T27: Let’s see um, I would kind of say she is unless I am off is she is here where she is able to use it with expanded algorithms, she is taking apart the numbers into expanded form as is able to multiply them.

I: Ok so level 4.

T27 clearly states that she believes Sally is using an expanded algorithm because she is ‘taking apart the numbers into expanded form and is able to multiply them’. There does not seem to be any recognition by T27 that Sally is not actually strictly
decomposing by place value, or that the partial products Sally produces do not match those that would be expected. To T27, decomposing numbers and properly multiplying appears to be all that is necessary in order to qualify as an expanded algorithm.

T27: Yeah, then I would move her on to this being able to that a little more efficiently and understand the algorithm would be where I think…

I: So what do you mean by that?

T27: As far as you know not having to write this out I guess to do it a little… she is able to do this obviously. My thought would be if she can do this on paper she can problem do it in her head pretty quickly.

I: This what she said.

T27: This what she said, oh ok. I would then just take her into showing her how to do with the {traditional} algorithm and why that makes sense and works.

I: So you use that [points to T27’s traditional algorithm on the paper]. Ok

T27: That would be probably what I do. Yeah because I think she obviously she understands what she is doing as far as pulling the numbers out what the algorithm is for essential doing it in expanded form. So I think that she can understand that and be able to use it then. Whether she would choose to use this {the traditional algorithm} or not, I don’t know. That is one thing I notice about doing this especially using investigations is that a lot of times when you get to that algorithm and you to that point where you show them that they don’t want to use it because they have a more efficient way. Do you notice that anywhere?

When the interviewer emphasizes that Sally did not write anything down, T27 does not connect this idea to the CBA conception of algorithm as a written, formal, systematic approach to multiplication. This provides more evidence that T27 has a misconception about the CBA meaning of algorithm.

I: Yeah.

T27: I notice that a lot. I was wondering if that was just me but I haven’t really talked to anyone about that. But that is one thing I tell the parents a lot of
times is that is surprising when they get to this point when we are saying get read for it {the traditional algorithm} and they say great but I am not going to use it that way.

I: I think a lot of students if they are trying to make sense of things then they use the thing that they feel most comfortable with. And for some of them they are not quite ready. It is like a security blanket.

T27: It is and I know and this is kind off topic, but I know we have here a 20 question double digit test that they have to do as part of our fluency program. There is 10 addition and 10 subtraction and they have a time limit to do it. They get to use a hundreds chart in January and in May they take the hundreds chart away and they have to do with just the strategies that they know. And there is regrouping involved and I always had just because I never want them to feel like I am the mean ogre that is making them do this because I always take it too. I will say I will do and you guys can time me. I did it last year and I started to do it while they all were watching using the algorithm. And I thought that if I am going to do this I am going to do the way we taught them which is expand form add your tens and add your ones put them together. And did that and I was much faster doing it that way then even using the algorithm. And carrying the number and that was kind of interesting it was the first time I thought wow this is actually more efficient for me even to pull apart using the expanded form. So I don’t know. I found that some of my really high kids this year when I taught the algorithms they ended up that they could use them and couple of them did if I would make them show me several different ways they use it. But they didn’t use 90% of the time as there first thing. But I would say with her on this [points to the CBA questions sheet] unless I am really off base that is where I would probably go next cause I think she understands enough of pulling apart the numbers and multiplying them for it to make sense.

I: So like if she did this {traditional algorithm} what would you sort have said to her?

T27: Um I know there was a really great thing when I taught investigations in fourth grade from the year. But I can’t remember how it went, you would probably know where they taught them the little box or square that show them when they are multiplying the place value how that works. Do you know what I am talking about?

I: Oh yeah.

T27: I might just for a visual reference if this child were ready for that, I might do something like before I just went to this {the traditional algorithm}. Just to say here is what this is and here is what the numbers are and show her how they add together. And then kind of gradually go into this what it
looks like to write it in a number that way. This is what you are doing. Or pull apart these numbers kind like how she has done but in number form and show her how to transfer that over to this {traditional algorithm} and that this is what that algorithm is.

I: So tell me more about that.

T27: Like perhaps taking like ok you know you could do it where you split up your 40 and your 5. But she kind of understands more so I guess saying to her ok 45 is 40 plus 5. We can do this and take both of those numbers times three and then both of them times 20. Then kind of show that and that is kind of what that box thing was that I am not remembering how to use. You know where you say ok 3 times 5 is 15 and 3 times 40 is 120. So you put a 120 and 15 and you do your 20 times 5 is a hundred and then 20 times 40 is 800 and you can see all those added up and there is your answer. Then show her how to then that is essentially what you are doing here. You know when you carry this {the one that is over the 4} it is not a one and is always a bad thing to say that is one. It is going there because it’s a ten. Does that make sense?

I: Yes.

T27: That is what I would do with her if I had her sitting in front of me. Just kind of start working with her on that. My rule always with the algorithm in addition and subtraction or even with this is they can’t use it unless they can verbally explain why it works. So as I was doing this I would also be talking that out with her what is that why did you do that? Make sure that they understand that reasoning. So that is kind of where I would go.

I: Ok that is good.

T27: Wow I am glad that I don’t teach fourth grade anymore {laughs}. It has been awhile it seems in that stuff you do everyday it seems like it has been forever now that I haven’t done it in years.

T27 is definitely struggling with the idea of what the algorithm means for CBA. However, her instruction does not seem to match her desire to move to the traditional algorithm. Her instruction actually seems to be targeted at getting Sally to do 4 partial products. T27 is an interesting example of a teacher saying that her plans were one thing (that was not CBA aligned) but what she actually planned to do was quite well aligned with CBA. This also seems to be an example of a teacher with experience with a
curriculum that is similar to many of the CBA MD ideas, like *Investigations*, having a fairly solid sense of direction for instruction even when there appear to be misconceptions with CBA.

Teacher T2 – RR and QR Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>

Figure 39 – RR and QR Student Work

T2: 46 multiplied by 5. 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230. Ok. So what this student did – without writing anything down is they broke apart 40 into 20x5 cause they knew that was manageable. This goes back to one of the things I talked about earlier is that it would be cool if a student did that

I: Yeah yeah. I remember that.

T2: I’m going to say this student probably falls in the higher functioning thought process. And they’ve found numbers that they can recognize – but they did not do a traditional algorithm because it seems to me that they did it mentally. So I’m going to stay away from the algorithms. But I’m going to move into at least operating on known facts. So I’m going to start at at least MD 3, but I’m having an…at looking at the way it is written here – I may even move into MD 4 because it says meaningful – and there is definitely a lot of meaning coming out of this.

T2 explicitly states that the students’ work does not demonstrate algorithmic thinking because of the fact that it was done mentally. T2 even identifies a key component of the students’ thinking, that the numbers were decomposed into manageable parts.
I:   ok
T2:   They deliberately used things here that tells me that it is a higher level of sophistication.
I:   Good
T2:   So I’m actually going to start at MD 4.
I:   So you’re going to look around in here (points to CBA MD document)
T2:   Yes. I’m going to look in MD 4 but I’m not opposed to looking into MD 3.
I:   Great
T2:   But they definitely have an understanding of facts – so I’m not going to go lower than MD 3.
I:   Right – we looked at all of those, and it didn’t look like any of those
T2:   Yeah. So in MD 4 it says students use computational algorithms or fixed sequences of steps to multiply or divide numbers generally they learn algorithms for multiplication before they learn those for division. Ok. Meaningful use of conceptual (reads). See it says here that it says they used algorithms and this was mostly mental. Um. I’m assuming that when they show this here – that students are using an algorithm in MD4 – but they didn’t do that here, so I’m going to go back ‘cause I’m assuming the higher, MD4.2 see alg4 below. So moving into algorithms, and I remember all of those they wrote out the traditional way. So I’m moving into MD3.

Even though T2 initially eliminated looking at algorithms, MD 4 (expanded algorithms) was explored anyway. T2 scans that CBA document and again opts to ignore the algorithm levels because of the recognition that the student is doing everything mentally and does not necessarily have a systematic strategy from organizing the multiplication. T2 is an example of a teacher with a CBA-aligned conception of algorithm.
Teacher T4 – RR and QR Problem

[used CBA MD materials]:

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<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>
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Figure 40 – RR and QR Student Work

I: What CBA level do you think student should move to next? And why?

T4: Gosh, I think that’s really good thinking

I: ok

T4: You probably want them to move to...you know, I don’t know, because they could do the regular algorithm – but this is really a higher level thinking to me. This kind of thinking is higher to me than doing the regular algorithm. So, I guess MD 4. So, it’s just another way of doing it, it’s a harder problem, you know? But I think that’s good thinking, that makes me know that they really understand what they’re doing. I mean, they could do the regular algorithm and explain it – so that could be the next. But this to me is more thinking than that. I know that sounds weird.

T4 states the students’ thinking ‘is higher to me than doing the regular algorithm’, which seems to imply that T4 views moving towards algorithms as a move away from ‘good thinking’. Although T4 does later state that the student ‘could do the regular algorithm and explain it’, T4 represents a perspective held by some teachers who believe algorithms lack meaning or do not involve good thinking. This is likely because many children, and adults, rote use strategies such as the traditional multiplication algorithm, and therefore the overly rote use of the traditional algorithm gets conflated with the CBA conception of the traditional algorithm that requires meaning and understanding. CBA
does not state that using the traditional algorithm rotely is a higher level than other levels, but it can be misinterpreted to mean just that. In order to be at level 5, or the traditional algorithm, a student must be able to demonstrate that they do in fact understand the underlying place value concepts of the multiplication.

Teacher 19 – Sally Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 \times 23 = ____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 41 – Sally Student Work

I: Okay. All right. So (b) what CBA level of sophistication is Sally’s strategy?

T19: [Pause] She’s in the 3.3. She’s using partial products.

I: Okay. Okay. And how did you decide what level?

T19: Well, it’s not necessarily 2 partial products but she’s not coming to the 4 partial products that would be the more traditional, 4 that you would start with and end with. She’s definitely breaking it down more so. But she’s still coming to partial products. She puts them together [points towards plus 15 in the 3rd sentence] together [points towards 1035 in the 4th sentence] in a way which may put her at a 3.2. But [waves hands if unsure then puts them back towards the beginning of the 3rd sentence and the end of the 4th]. She’s definitely using her number properties well. She’s using the distributive property. She’s breaking down the numbers. You know she’s basically getting 2 partial products but she’s doing more steps to get there.

I: Okay.

T19: So, ah 3.2 instead of 3.3 probably [but/would?]. You know as far as her method goes it, it’s a heavy use of the distributive property. And you know I can’t necessarily say she’s got 2 partial products. So that would probably put her at the 3.2 instead of 3.3 but.
I: Okay. All right.

T19: Ah!

I: So (c) what type of reasoning do you think Sally should move to next?

T19: I think she’s definitely ready to look at this number based on its place values alone. You know, even to take it as 45 times 20 and then 45 times 3. She obviously has no problems with the 45 times 3 [points towards times in the 3rd sentence]. And she just explains it more fully [points towards 15 and possibly 3 and back to 15 in the 3rd sentence]. It’s something that she could definitely do in her head based on the way she’s explained it here. So you know you would see that. And then the 45 times the 20, you know I don’t know that she would need to stop at 4 partial products. I think she could probably jump that step a little bit to a more expanded algorithm. Based on the way she’s explained this. Because it seems to me she’s [points towards times 3 in the 3rd sentence] writing out [points towards 120 in the 3rd sentence] a lot of things she’s doing in her head. So I, I think her effective level of understanding is higher than the steps here indicate [points towards is 120 then plus in the 3rd sentence and near the end of the 4th sentence]. Because I think [runs a finger over the 3rd and 4th sentences] you’re seeing a lot of her mental strategy on paper. So you know I, I think she’s probably ready to see the more expanded algorithms. Looking at those and maybe not necessarily. (d) I think if I showed her you know 40 times 20, 5 times 3 all that. If I showed that right now she’d go oh yeah that makes sense but that’s too complicated. Cause really what she has [puts fingers on Sally’s strategy] here in her way, the way she’s thinking is probably more understandable to her. So and I think that’s, I think that’s where [possibly points on QR MD level 3-out of sight] these last few steps in this 3 [goes up and down over 3.1 through 3.4 on QR MD] level 3 range are hard for kids because it seems like you’re making more steps instead of less. And the goal is to transition them into the traditional algorithm which in a lot of ways has more steps. But as far as an effective mental strategy I think she has one.

I: Okay.

T19: That I wouldn’t want to mess with too much. So I would hope to preserve [taps on Sally’s strategy] this mental strategy as I started to teach those expanded algorithms. Because based on the way she’s explained this [puts a hand on the sheet below Sally’s strategy] she really only needs paper to keep track of the numbers and keep from forgetting them.
T19 has both an accurate conceptualization of Sally’s work, and explicitly recognizes that she has the potential to move towards an expanded algorithm based on her thinking strategy. T19’s learning goals for Sally align quite well with CBA’s MD framework, and especially identify the key characteristic of algorithmic thinking and moving towards making a thinking strategy more formal.

I: Okay. Okay. So what would you do? [So I mean?] suppose she just did that problem in front of you and you were able to work with her one on one.

T19: I would want to talk to her a lot more about how she did this problem [possibly points towards the problem]. You know I would say did you do this in your head [appears to go back and forth over the 1st sentence] or did you write it because you needed to write it? If she’s doing this in her head [arcs around Sally’s strategy and ends at 1035]. That’s pretty amazing. You know that’s, that’s an impressive problem to be able to do in your head that easily. If she’s not doing it in her head and she needs to write it out like that then that’s a little bit different. Then I would show her definitely [taps under QR MD 3.4] the distributive property decompose and show her the 4 partial products and that kind of thing. And move her through those levels instead. If she’s doing that directly in her head I would go to the more expanded algorithms. I would look at the place value parts and show her how to put that in an algorithm that is maybe closer to the traditional algorithm. Not because I think she needs to solve it that way to get it right but because I think that that’s kind of the accepted way of doing it. You know if this strategy is accurate for her it’s not going to work for much larger numbers because she’s not going to be able to keep it in her head. Which is why she needs the more traditional algorithms and things like that. But you know the fact that she can do this strategy shows me she has good number sense, a good understanding of where she’s going and what to do with the numbers when she has [them?]. So I, I think she’s definitely ready to see some of those algorithms and probably make the choice of which one makes the most sense for her at this point.

I: Okay.
Again, T19 demonstrates alignment with CBA in discussion about algorithmic thinking. In addition to recognizing the importance of algorithms, T19 also uses plural ‘algorithms’ when discussing what to expose Sally to. This is important to recognize, as there is not always just a single expanded algorithm. T19 possibly not only plans to expose Sally to the traditional expanded algorithm, but possibly other expanded algorithms to help her develop algorithm thinking.

T19: Ah, that’s what I would like to do next I think.
I: So do you see, do you think its important, suppose your, suppose your goal was to get her to use the traditional algorithm.
T19: Uh huh.
I: I mean that might not be your goal. Maybe your goal is just the expanded algorithm. But suppose your goal is to get her to use the traditional algorithm or at least to understand it.
T19: Uh huh.
I: Do you think that going from where she is to an expanded algorithm, the 4 partial products, is an important intermediary, intermediate step?
T19: I do. I, I think, I think this is a kid who you could show the 4 partial products to her once and then go into an expanded algorithm. I don’t think she’s going to need a lot of instruction on seeing 4 partial products [motions down the sheet possibly 4 times] or need me to give her 5 problems [motions over the rectangle with the problem and Sally’s strategy] and tell her to do [with?] these 5 problems using 4 partial products. I don’t think [?] for this particular child based on the number sense showing here [goes back and forth over the inside of the rectangle and then circles the rectangle with the problem and Sally’s strategy] but I think she does need the expanded algorithm before you go to the traditional algorithm because otherwise it’s going to look like voodoo. You know, and that, that also is in the way you explain it. You know if, if you’re telling her that you’re multiplying you know in the traditional algorithm you stack them up [points to 45 with one finger and 23 with another finger in the problem]. And you would say now this is 3 [points towards the 3 in 23] times 5 [moves that finger next to the other finger and points to the 5 in 45 then back to the 3 in 23] and 4 [moves that finger towards the 4 in 45 next to the other finger]. That’s not going to make
sense to her but [moves that finger back to 23] if you actually say that’s 3
times 40 [moves that finger towards 45 next to the other finger]. It will
make more sense to her and she’ll be able to handle that information. So I
think it depends on how that traditional algorithm is explained. I think if
whoever would explain it to her would call them ones instead of tens and
things like that. It’s going to screw her up. Ah [tape stops]…

T19: But I, I think that whoever explains the traditional algorithm to her
definitely has to use place value language that’s appropriate to transition
her correctly I think. Well, that’s one of the things that I run into many
times with parents. They try to describe them as all ones, or you carry the
one or whatever and the kids don’t get it because it’s not a one. And it
kind of messes them up. And I think that would happen with her. She’s
got a very strong understanding of what place value is and how it works.
How tens affect other numbers things like that. So if you’re not
explaining that correctly in a way that’s going to be accessible to her it’s
going, it’s gonna frustrate and frankly it’s going to make it very difficult
for her to truly understand what she’s doing. Which is why I think the
expanded algorithm needs to be there in-between. Because it supports her
understanding of the numbers at this point.

I: Okay. Okay. [Collects sheets.]

T19: But no I don’t think traditional algorithm is the goal all the time.

I: [Gives T19 AS problem 3.] Yeah, neither do I.

T19: I think if some of these kids can do this stuff in their head without it.
There’s no reason I should teach them much else.

I: Yeah.

T19: Other than higher numbers and things like that. But to be honest a lot of
them are going to be using calculators for anything over a hundred
anyway.

I: Yeah, I, I personally I use more expanded algorithms now. I mean since I,
I’ve spent so much time watching kids use these algorithms it, it actually
seems easier at times to me.

T19: They make more sense to me.

I: Yeah.

T19: I’ve always done things a little strangely, more into subtraction. But you
know when I was younger I would subtract left to right, none of my
teachers understood what I was doing. So I would do it in my head that
way and then I would do it [points towards problem 3] their way on paper
just so I didn’t get hassled. So you know it’s.

I: Yeah. Yeah.
T19: I don’t know.

Teacher 24 – Sally Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = _____</th>
</tr>
</thead>
</table>
| Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.

Figure 42 – Sally Student Work

I: So what type of reasoning do you think Sally should move to next?

T24: This is where that whole traditional algorithm comes in. That is still ok to use the traditional algorithm, the traditional algorithm isn’t wrong it is just wrong when you aren’t understanding what you are carrying. You know when they say carry the one. You are not carrying a one you are carrying a ten. Well she understand place value so she really could go to the traditional algorithm. I don’t think I need to have her go to the expanded algorithm of you have to do this first you have to do this second. It is hard for one problem was she lucky did she remember all of them in one problem if she was consistent and then all the problems she did were always being right. Then I know that she is always going to remember for the most part. Being human she will always or 99% of the time will get if she did them all like this [points to Sally’s work]. Then I would move to the traditional because she understands place value. Obviously right here [points to something on the paper].

I: So you wouldn’t move her to the expanded.

T24: No, she wouldn’t need to I would say. Especially seeing this the 3 times 40 and the 3 times 5. I mean that right there shows you she gets it.

T24 demonstrates a partially aligned conception of algorithm with CBA by identifying that algorithmic thinking is important because it provides organization, as well as recognizing that Sally might lose track of some of the numbers without a more
formalized process. However, T24 does not align with CBA in the perspective that based on Sally’s understanding of place value, Sally is ready for the traditional algorithm. The key characteristic in moving from level 3.4 to level 4 (4 partial products to the expanded algorithm) is not a change in mental strategies, but rather a change in focus on being systematic and organized, and keeping track of one’s steps in writing. Moving Sally to the expanded algorithm would not necessarily be needed to help Sally build an understanding of place value, but rather to build the understanding of an algorithm with meaning. That is, following steps that have meaning to Sally. Moving directly to the traditional algorithm runs the risk of providing Sally with a set of steps that are not necessarily meaningful, and therefore would not align with CBA MD.

I: What would you do instructionally to move Sally this next type of reasoning?

T24: I would actually set up the traditional algorithms and talk her through it.

I: So how would you do that?

T24: Let’s say we would write down 45 times 23. So I would say Sally, in the past you were thinking about 3 going into 40 and then 3 going into 5. So instead of having to do all the additional writing over here let’s condense our writing down and not have to do as much side work. I might say a this [points Sally’s solution] is safe way to do it and you know that you are working it out, but now that you know this let’s not write this all out. Let try to do a lot more of it just with our thinking. So if we start with the ones place and we do the 3 times 5 groups we would get 15. I always tell the kids that there is not enough room for two kids in one square. So we would leave the five here and then we would carry the ten groups of one over here so now we have ten groups, but we would just write as 1 cause we are over in the tens place. And kind of talk her through it like that. Then we have 3 groups of 4, but it is really 3 times 4 is 12 and we have one more group of tens so it would be 13. So just kind of talking her through it like that. Now we are done with the ones place so we have to leave a placeholder of the ones place. But I think that if she was do this she be able to understand the placeholders and place value.
T24 demonstrates through discussion of instruction that she would encourage Sally to still rely on her thinking strategy to help her create meaning ‘this is a safe way to do it and you know that you are working it out’ but wants to encourage her to move quickly to efficiency by stating ‘but now that you know this let’s not write this all out’. T24 assumes that Sally’s understanding of place value will effectively transfer to understanding placeholders and place value in the traditional algorithm, which may or may not be true for Sally.

Teacher T5 – RR, QR and EB Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
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<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
<tr>
<td>EB:</td>
<td>Write: 5x6 = 30. 5x40 = 200. 30 + 200 = 230.</td>
</tr>
</tbody>
</table>

Figure 43 – RR, QR, and EB Student Work

T5: The third student is EB. (reads student work). 5x6 = 30. 5x40 = 240. 30 + 200 = 230.

(Scans document). Well. The confusion for me here sometimes comes into play as whether or not I still want to stay with the distributive property with what’s going on – or if I want to switch to algorithms. Because of the way – If she had just said ’30 + 200 = 230’ I would be still looking at 3.3.1, but writing down the algorithm that way…(scans document)…Alg 3? Correct performance of a symbolic algorithm with no indication of understanding of underlying place value concepts.

I: ok. Um.

T5: Maybe not.

T5: Last time around we didn’t get into this section – so he may not want me to be here.
I: Yeah. Well, I don’t know if that’s true.
T5: ‘Cause writing down the algorithm like that changes it up.

T5 clearly identifies that if EB had simply verbalized the strategy, they would still only be doing 3.3.1 – 2 partial products, but identifies that writing it down might constitute it as an algorithm. T5 chooses Alg 3 – and reads ‘Correct performance of a symbolic algorithm with no indication of understanding of underlying place value concepts’ which does not necessarily reflect the fact that the student did in fact show evidence of understand the place value concepts. The CBA Interviewer took this as an opportunity to help expand T5’s thinking about the CBA Algorithm levels.

I: It does. And can I remind you of and example of Alg 3. Remember how KK was really good at going ‘well 5x6 is 30, put down a 6 and carry a zero.’ And I would ask her what is the 3…and she would say ‘well it’s just a 3’. And I asked her where it came from and she would say ‘it’s just a 3’. There was no real understanding that it was 3 tens. So that’s kind of the Alg 3 thing because they don’t quite get the whole place value thing…they’re just following the algorithm very tightly.

T5: What this is – is this is conceptually explicit. That’s what this is.
I: It is.
T5: So I guess that’s 4a. Meaningful use of conceptually explicit algorithms.
I: Why do you say 4a (19:59)
T5: Because that’s what I flipped to!

The CBA interviewer gives a verbal example of what a student at Alg 3 might do, and T5 is able to then flip through the CBA document to identify that EB must be using algorithms differently than the example. T5 jokingly says that 4a is the level because ‘that’s what I flipped to’ in the CBA document, but the interviewer continues to probe to determine T5’s understanding of this selection.
I:    I can give you a hint.  4b is the same as Alg 4.  So if that helps
T5:   So it’s 4a.
I:    why?
T5:   Because there is not place value understanding in the algorithm.
I:    There’s not?
T5:   Well there is.  Well it is.  But it is not like 5 times 6.  What she did was do
them independently.  Whereas I would do it as 5times 6 put the 3 carry the
3.  Do it one line.  What they did was they did it independently.  So I
guess their place value is understood but not meshed.  I don’t know if that
makes sense.
I:    It’s not integrated into the standard algorithm….OK.  Can I play (the CBA
author) for a second?
T5:   Sure.
I:    Usually for the CBA author, meaningful use of computational algorithms.
Usually the Algs are for the standard algorithm.  So.  Because students
usually learn the standard algorithm, and they may learn it with meaning
or without.  For 4a and 4b this is standard abbreviated, and this is
conceptually explicit.  Here, he would say that this shows place value
understanding because the student actually know that this isn’t a 5 times a
40.  Because he/she knows that the 4 is 40, and they multiply it out and get
200 they understand place value.  Now a student that does 5x4 and then
line it up correctly does not necessarily
T5:   Ok.  But that’s not standard abbreviated.
I:    Good.
T5:   It’s conceptually explicit.
I:    So it’s not 4b.  It is…
T5:   It is 4a
I:    Exactly.
T5:   Ok. ‘Cause this is the way I teach kids that struggle with the concept…I
break it down (pointing to EB’s work).  I say do it this way and then add it
up.  Look at each number individually because they have a difficult time
with the ‘put the 3 up there and that means it is 30’ and so it’s – I guess it
is conceptually explicit.
The interviewer continues to give non-examples, helping T5 to eliminate certain choices for levels. T5 shows a certain level of understanding of the mathematical approach used by EB by recognizing that they are not doing a standard abbreviated algorithm, but rather a conceptually explicit algorithm.

I: What makes EB more sophisticated than QR’s? And QR’s more than RR’s?

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
<tr>
<td>EB:</td>
<td>Write: 5x6 = 30. 5x40 = 240. 30 + 200 = 230.</td>
</tr>
</tbody>
</table>

Figure 44 – RR, QR, and EB Student Work

T5: As I said earlier. Hopefully I’m staying consistent. Because I went back and figured out that this was 3.2. Where QR actually uses the distributive property place value by place value and then adds them up. RR is breaking the number into manageable units – but it is not distributing the number. It’s not distributing the number in terms of 5 x 40 and 5 x 6 it’s breaking the 46 into a known fact that’s easier.

I: That’s really clear.

T5: And then EB. Without the written algorithm. The reasoning is very similar to QR.

I: I agree.

T5: And in a sense, really all the algorithm adds is just a written format. But it’s not using the standard algorithm it’s using the conceptually explicit algorithm.

I: That’s exactly what (The CBA Author) would say. He’d say the child has a better chance for consistent success because he/she has a consistent organizational tool.

This discussion is a great example of how with the help of someone experienced in the CBA materials, non-examples of different levels can provide key teaching advice.
in helping teachers like T5 learn the delineations between different CBA levels. T5 already had the a CBA aligned conception of algorithm, in that T5 believed it was for situations where students had written out, organized strategies for solving mathematics problems. However, T5 was either not used to using, or was uncertain how to use the Algorithm levels as designed by the CBA author. The CBA Interviewer was able to provide contexts for when the algorithm levels made sense, and T5 was able to deduce the proper level and conceptualization of EB’s work. T5 even notes that ‘without the written algorithm, the reasoning is very similar to QR’. In fact, this is essentially the only separation between QR and EB.

Topic B: Teachers’ determination of learning goals and instruction

Within this topic, teachers’ conceptualizations of next steps in determining a learning goal and subsequent instruction are explored. Beyond classifying where students’ reasoning falls within the CBA framework, it is important to look at how teachers’ determination of learning goals and instruction functions within the CBA MD framework. It cannot be assumed that if a teacher properly determines the level of sophistication for a student that proper learning goals and subsequent instruction will be proposed. It also cannot be assumed that if a teacher improperly determines the level of sophistication for a student that improper learning goals and instruction will be proposed.

This first part of this topic explores teachers’ responses to Sally’s complex work on the problem 45x23. As was explored in Topic A, teachers struggled to make sense of Sally’s thinking within the CBA MD framework. In this section, teachers’ subsequent learning goals and proposed instruction for Sally are explored.
The second portion of this topic explores how teachers handled what was considered by many to be an intermediate level in the CBA MD framework. The CBA levels can be interpreted as a strong hierarchy, where each level occurs in sequential order. However, in reality students rarely follow the levels perfectly linearly, but do tend to roughly follow the proposed hierarchy. How long students spend at each level is also highly variable. For teachers to value a CBA level, it is important that they recognize the importance of a level as a learning goal. This requires understanding the level, including how the level differs from previous and subsequent levels. An example of such a level that is important to some teachers but unimportant to others is explored in this section.

The third section within this topic explores the patterns and themes the CBA teachers followed in determining learning goals and instruction based on student work. A quantitative and qualitative account of the various approaches taken by teachers is explored. In particular, teachers’ alignment with a learning progression perspective on setting learning goals and determining instruction is explored (i.e. – teachers’ alignment with 1) determining proper level, 2) selecting the next reasonable CBA level as a goal, and 3) aligning instruction with the next CBA level).

The final portion of this topic explores comments made by several CBA teachers who indicate that experience and use of CBA frameworks in a teaching context can impact teachers’ understanding and use of the CBA materials. Several teachers’ comments are explored to investigate how the use of CBA materials in a teaching context can help teachers learn about or understand the CBA MD materials.
Teachers’ Understanding of a Students’ Reasoning About Multiplication (45x23) and Subsequent Determination of Learning Goals and Instruction

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = _____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 45 – Sally Student Work

CBA teachers were presented with the above student work, and asked to analyze whether Sally’s reasoning was correct or incorrect, and assess the thinking in terms of the CBA Multiplication and Division levels of sophistication. As was discussed earlier, many teachers struggled to interpret exactly what Sally was doing, especially finding Sally’s statement that ‘45 times 3 is 120, plus 15’ to be confusing. Teachers also showed significant diversity in their conceptualizations of Sally’s thinking within the CBA MD framework. The table summarizing teachers’ responses to Sally’s conceptualization appears below.
What should be noted is that of the nine teachers who were at least partially consistent with CBA when leveling Sally’s thinking in the CBA MD framework, seven had used the CBA MD materials. Additionally, of the five teachers who leveled Sally’s thinking inconsistently with the CBA MD framework (in this case, leveled Sally too high) in the CBA MD framework, all five had not used the CBA MD materials. This section intends to explore some of the differences within these groups when considering teachers’ choice of subsequent learning goals and instruction for Sally.

With respect to learning goal determination, one overall trend was for teachers to identify Sally’s thinking as ‘advanced’ and therefore conclude that the traditional multiplication algorithm was either a short-term or long-term goal. Overall, out of the 14 teachers, 10 articulated the traditional multiplication algorithm (level 5 – the highest CBA MD level) as the eventual goal. However, there was a noticeable difference between many of the teachers who initially leveled Sally partially consistently with CBA,
and those teachers who initially were inconsistent. For the five teachers who were initially inconsistent with CBA in leveling Sally, four of them proposed moving Sally directly to the traditional algorithm with no intermediate or ‘stepping stone’ instructional moves. However, for the nine teachers who were initially partially consistent with CBA in leveling Sally, four of them proposed moving Sally to the traditional algorithm eventually, but in every case the teacher provided CBA learning progression consistent sub goals to justify this long-term goal.

In addition to differences between those teachers who leveled Sally at least partially consistently with CBA and those teachers who were inconsistent with CBA taking different approaches to determining learning goals for Sally, there were also noticeable differences in how instruction was approached in accordance with the CBA MD framework. Since Sally was primarily demonstrating elements of CBA MD levels 3.2 and 3.3 (decomposing numbers by place value parts, using distributive property and 2 partial products), the CBA MD framework would suggest attempting to move Sally to level 3.4 (4 partial product multiplication) and possibly further to Level 4 (Expanded Algorithms) to formalize and organize her thinking in a written and systematic fashion. Such instruction would be most likely to help Sally progress in her thinking, and it would be less likely that moving directly to a traditional algorithm in which place value ideas are hidden would be a jump that could be meaningfully made by Sally. Therefore, if a teacher’s proposed instruction is targeted at developing Sally’s place-value based partial products approaches, and potentially moving Sally towards written expanded algorithms, instruction can be considered at least partially CBA MD LP aligned instruction. If
instruction targets moving Sally towards implementing the traditional multiplication
algorithm, it would be expected that Sally would struggle to make this step meaningfully,
and such instruction can be considered Non-CBA MD LP aligned instruction.

As might be expected, the instructional plans differed greatly for those teachers
who initially leveled Sally at least partially consistent with CBA from those teachers who
initially leveled Sally inconsistent with CBA. Of the nine teachers who initially leveled
Sally at least partially consistently with CBA, seven proposed instruction that emphasized
building Sally’s thinking strategy of decomposing by place value, using the distributive
property, and possibly working towards formalizing these steps into an expanded
algorithm. Only one teacher who initially leveled Sally at least partially correct proposed
instruction that targeted moving Sally towards using the traditional algorithm with no
intermediate instruction. The ninth teacher in this group did not offer instruction, and
could not be categorized.

For the five teachers who initially leveled Sally inconsistently with CBA, three
proposed instruction that emphasized the traditional algorithm, and did not offer
instruction that indicated that Sally would be encouraged to consider expanded
algorithms in advance of work with the traditional algorithm. Only one teacher who
initially leveled Sally incorrectly proposed instruction that emphasized work on
decomposition by place value and the use of the distributive property. The final teacher
who initially leveled Sally incorrectly was not able to propose instruction, but indicated
that their goal was to teach to encourage CBA level 5.
Teachers Providing Intermediate Goals and CBA Aligned Instruction

Teacher 19 – Sally Problem

[used CBA MD materials]:

T19: So you know I, I think she’s probably ready to see the more expanded algorithms. Looking at those and maybe not necessarily. I think if I showed her you know 40 times 20, 5 times 3 all that. If I showed that right now she’d go oh yeah that makes sense but that’s too complicated. Cause really what she has [puts fingers on Sally’s strategy] here in her way, the way she’s thinking is probably more understandable to her. So and I think that’s, I think that’s where [possibly points on QR MD level 3-out of sight] these last few steps in this 3 [goes up and down over 3.1 through 3.4 on QR MD] level 3 range are hard for kids because it seems like you’re making more steps instead of less. And the goal is to transition them into the traditional algorithm which in a lot of ways has more steps. But as far as an effective mental strategy I think she has one.

I: Okay.

T19: That I wouldn’t want to mess with too much. So I would hope to preserve [taps on Sally’s strategy] this mental strategy as I started to teach those expanded algorithms. Because based on the way she’s explained this [puts a hand on the sheet below Sally’s strategy] she really only needs paper to keep track of the numbers and keep from forgetting them.

T19 clearly justifies the fact that he recognizes that the reasoning at the ‘last few steps in the level 3 range’ are very difficult for students. T19 states that the ultimate goal is to get the student to the traditional algorithm, but clearly recognizes that there are steps in between Sally’s reasoning and the traditional algorithm. He articulates the importance of helping Sally transfer her mental strategy into an algorithm so that she can keep from forgetting steps or terms.

I: Okay. Okay. So what would you do? [So I mean?] suppose she just did that problem in front of you and you were able to work with her one on one.
T19: I would want to talk to her a lot more about how she did this problem [possibly points towards the problem]. You know I would say did you do this in your head [appears to go back and forth over the 1st sentence] or did you write it because you needed to write it? If she’s doing this in her head [arcs around Sally’s strategy and ends at 1035]. That’s pretty amazing. You know that’s, that’s an impressive problem to be able to do in your head that easily. If she’s not doing it in her head and she needs to write it out like that then that’s a little bit different. Then I would show her definitely [taps under QR MD 3.4] the distributive property decompose and show her the 4 partial products and that kind of thing. And move her through those levels instead. If she’s doing that directly in her head I would go to the more expanded algorithms. I would look at the place value parts and show her how to put that in an algorithm that is maybe closer to the traditional algorithm. Not because I think she needs to solve it that way to get it right but because I think that that’s kind of the accepted way of doing it. You know if this strategy is accurate for her it’s not going to work for much larger numbers because she’s not going to be able to keep it in her head. Which is why she needs the more traditional algorithms and things like that. But you know the fact that she can do this strategy shows me she has good number sense, a good understanding of where she’s going and what to do with the numbers when she has [them?]. So I, I think she’s definitely ready to see some of those algorithms and probably make the choice of which one makes the most sense for her at this point.

T19 again identifies that it is important to eventually move Sally to the traditional algorithm, and also indicates the need for Sally to possibly move away from doing the entire problem in her head. T19 demonstrates a CBA MD aligned conception of learning goal for Sally, and suggests both short and long term goals. However, it is not yet clear how T19 sees the goals fitting together for Sally.

I: Okay.

T19: Ah, that’s what I would like to do next I think.

I: So do you see, do you think its important, suppose your, suppose your goal was to get her to use the traditional algorithm.

T19: Uh huh.
I: I mean that might not be your goal. Maybe your goal is just the expanded
algorithm. But suppose your goal is to get her to use the traditional
algorithm or at least to understand it.

T19: Uh huh.

I: Do you think that going from where she is to an expanded algorithm, the 4
partial products, is an important intermediary, intermediate step?

T19: I do. I, I think, I think this is a kid who you could show the 4 partial
products to her once and then go into an expanded algorithm. I don’t
think she’s going to need a lot of instruction on seeing 4 partial products
[motions down the sheet possibly 4 times] or need me to give her 5
problems [motions over the rectangle with the problem and Sally’s
strategy] and tell her to do [with?] these 5 problems using 4 partial
products. I don’t think [?] for this particular child based on the number
sense showing here [goes back and forth over the inside of the rectangle
and then circles the rectangle with the problem and Sally’s strategy] but I
think she does need the expanded algorithm before you go to the
traditional algorithm because otherwise it’s going to look like voodoo.
You know, and that, that also is in the way you explain it. You know if, if
you’re telling her that you’re multiplying you know in the traditional
algorithm you stack them up [points to 45 with one finger and 23 with
another finger in the problem]. And you would say now this is 3 [points
towards the 3 in 23] times 5 [moves that finger next to the other finger and
points to the 5 in 45 then back to the 3 in 23] and 4 [moves that finger
towards the 4 in 45 next to the other finger]. That’s not going to make
sense to her but [moves that finger back to 23] if you actually say that’s 3
times 40 [moves that finger towards 45 next to the other finger]. It will
make more sense to her and she’ll be able to handle that information. So I
think it depends on how that traditional algorithm is explained. I think if
whoever would explain it to her would call them ones instead of tens and
things like that. It’s going to screw her up.

T19 clarifies that it is important for Sally to work on the expanded algorithms
prior to seeing the expanded algorithm in order to promote sense-making for Sally. This
represents an aligned perspective to the CBA MD framework.

I: Okay.

T19: But I, I think that whoever explains the traditional algorithm to her
definitely has to use place value language that’s appropriate to transition
her correctly I think. Well, that’s one of the things that I run into many
times with parents. They try to describe them as all ones, or you carry the one or whatever and the kids don’t get it because it’s not a one. And it kind of messes them up. And I think that would happen with her. She’s got a very strong understanding of what place value is and how it works. How tens affect other numbers things like that. So if you’re not explaining that correctly in a way that’s going to be accessible to her it’s going, it’s gonna frustrate and frankly it’s going to make it very difficult for her to truly understand what she’s doing. Which is why I think the expanded algorithm needs to be there in-between. Because it supports her understanding of the numbers at this point.

I: Okay. Okay. [Collects sheets.]
T19: But no I don’t think traditional algorithm is the goal all the time
I think if some of these kids can do this stuff in their head without it. There’s no reason I should teach them much else.

I: Yeah.

T19 clearly emphasizes Sally’s understanding of place value in her work, and the fact that moving Sally to a traditional algorithm might serve to confuse or frustrate Sally in her attempts to make sense of such an approach. T19 throughout the interview demonstrates a desire to move Sally through the intermediate level of expanded algorithms and 4-partial product multiplication, emphasizing the need for a systematic approach, and eventually making it to the traditional algorithm. T19 also has a CBA aligned conception of the traditional algorithm, as he emphasizes the fact that it would need to be introduced to Sally in a way that emphasized the place value components of the algorithm, which the tradition algorithm hides.
Teacher 13 – Sally Problem

[no use of CBA MD materials]:

MB: Okay. All right. So what type of reasoning do you think Sally should move to next? Explain why.

T13: I would, I would guess using expanded algorithms. Because she’s half way there. You know. And then, and I, I think just if, if that’s truly exactly the way Sally said it. She’ll very quickly move to level 5 because she truly understands the expanded notation. Which I think is.

I: To level 5 or level 4?

T13: To level 5.

I: Oh okay.

T13: I think she should move to level 4 next.

I: I see.

T13: But I think she would skip quickly to level 5.

I: Oh I see, I see.

T13: Kind of like what we were talking about here [points to the 4th written problem]. She’s just going to collapse it pretty quickly to a traditional algorithm. And it’s just going to make sense for her.

I: Okay. What would you do instructionally to move Sally to this next type of reasoning?

T13: I might ask her, I don’t know in all honesty. I might ask her something like, well you, you broke down that, that 23 into 2 tens and a 3. Could you have broken it down into 20 and 3 instead of the 2 tens and a 3? And, and would that have made sense to you if you did it that way? Like let’s try it with another number, break it apart. And can you break it apart the tens and the ones instead of breaking into each ten? But break it into the tens and ones. And, and how does that work? Does that you know, does that feel as comfortable? And then if they can do that with that comfort because that’s all the, always the biggest thing is do it comfortably. Then after she did that you could show her how to write it in the expanded form.

T13 suggests that Sally would be able to move to level 5 eventually, but first would ‘quickly’ be shown expanded algorithms at level 4. Whether or not T13 is correct in stating that Sally would actually move quickly through the expanded algorithms is
unknown, but T13 does recognize that Sally has a decent understanding of decomposition-based multiplication and believes that Sally’s understand would allow her to make sense of a traditional algorithm quickly. T13’s instruction actually appears to emphasize a transition of Sally to utilizing level 3.4 thinking, by emphasizing the specific 4 partial product multiplication proposed in the CBA framework. Although T13 does not explicitly state that moving Sally to an expanded algorithm would require her to formalize her thinking into written steps, T13 does propose instruction that emphasizes utilizing a 4 partial product approach and then states ‘after she did that you could show her how to write it in the expanded form’. This indicates that T13 might see some separation between the partial products approach and an expanded algorithm.

*Teachers Proposing Moving Sally to Traditional Algorithm without CBA Aligned Instruction*

**Teacher 20 – Sally Problem**

[no use of CBA MD materials]:

I: Okay. What type of reasoning do you think Sally should move to next and explain why.

T20: You know I… I really have to say that I am not sure. I mean I guess, (pause) I guess that she would move on to level 5 but honestly I don’t have enough understanding… I am one of those people that learned it this way (pointing to traditional algorithm) and I don’t even know because I don’t teach it and I don’t know. But my guess is, my guess is that this (pointing to Sally’s method) is more sophisticated than this (pointing to traditional algorithm) because this is just memorizing how to do it. I don’t know that to be certain though. It seems like this has more thinking involved than what I did. I just kind of did it.

I: Okay, so you think that what you said for this is more sophisticated than what you did. Is that what you mean?
T20: I don’t know I mean… This (Sally’s method) definitely involves more thinking about it. Thinking about the place value parts and thinking about which numbers you need to put together where as this is just…just me remembering how to multiply and move numbers and puts zeros and all those things.

I: What would you do instructionally to move Sally to this next type of reasoning and explain why.

T20: (Pause – Reads from CBA) Well level 5, the difference is that the place value ideas and the algorithms are hidden. So I would want her to be able to do that. But I really don’t exactly know what I would do to have them hidden. But that would be the goal as far as I can see from the levels.

I: Okay…So instructionally, you are really unsure of how you would go about it

T20: I’m not sure

I: Not sure. That’s fine

T20: Sorry

I: No that is perfectly fine. Okay next one.

T20 struggles with her lack of experience with the CBA MD materials and lack of experience teaching MD. T20 proposes moving Sally directly to the traditional algorithm, and suggests that she is uncertain about where else to go because the traditional algorithm is how she was taught. T20 demonstrates confusion about the CBA MD framework by suggesting that Sally’s way of thinking was actually more sophisticated than the traditional algorithm. This conception of the CBA levels indicates that T20 might not interpret the traditional algorithm level as implying understanding of the place value components of the traditional algorithm, but rather a more rote use of a traditional algorithm. T20 also attempts to propose instruction that would align with moving to level 5, but again gets confused by the CBA MD materials in trying to interpret the meaning of ‘place value ideas and the algorithms are hidden’.
Teacher 24 – Sally Problem

[used CBA MD materials]:

I: So what type of reasoning do you think Sally should move to next?

T24: This is where that whole traditional algorithm comes in. That is still ok to use the traditional algorithm, the traditional algorithm isn’t wrong it is just wrong when you aren’t understanding what you are carrying. You know when they say carry the one. You are not carrying a one you are carrying a ten. Well she understand place value so she really could go to the traditional algorithm. I don’t think I need to have her go to the expanded algorithm of you have to do this first you have to do this second. It is hard for one problem was she lucky did she remember all of them in one problem if she was consistent and then all the problems she did were always being right. Then I know that she is always going to remember for the most part. Being human she will always or 99% of the time will get if she did them all like this [points to Sally’s work]. Then I would move to the traditional because she understands place value. Obviously right here [points to something on the paper].

I: So you wouldn’t move her to the expanded.

T24: No, she wouldn’t need to I would say. Especially seeing this the 3 times 40 and the 3 times 5. I mean that right there shows you she gets it.

T24 clearly demonstrates that expanded algorithms are not a necessary component of the learning goal in moving Sally to the traditional algorithm. In fact, T24 even appears to believe that Sally is demonstrating the understanding of expanded algorithms in her work on the problem 45x23. This ignores the importance placed on helping students transition to using algorithms by emphasizing systematic thinking to help organize a partial products approach, as T24 even states that Sally does not need ‘to [go to] the expanded algorithm of you have to do this first you have to do this second’ indicating that the systematic component of the algorithm is not highly valued.

I: What would you do instructionally to move Sally this next type of reasoning?
T24: I would actually set up the traditional algorithms and talk her through it.

I: So how would you do that?

T24: Let’s say we would right down 45 times 23. So I would say Sally, in the past you were thinking about 3 going into 40 and then 3 going into 5. So instead of having to do all the additional writing over here let’s condense our writing down and not have to do as much side work. I might say a this {points Sally’s solution} is a safe way to do it and you know that you are working it out, but now that you know this let’s not write this all out. Let try to do a lot more of it just with our thinking. So if we start with the ones place and we do the 3 times 5 groups we would get 15. I always tell the kids that there is not enough room for two kids in one square. So we would leave the five here and then we would carry the ten groups of one over here so now we have ten groups, but we would just write as 1 cause we are over in the tens place. And kind of talk her through it like that. Then we have 3 groups of 4, but it is really 3 times 4 is 12 and we have one more group of tens so it would be 13. So just kind of talking her through it like that. Now we are done with the ones place so we have to leave a placeholder of the ones place. But I think that if she was do this she be able to understand the placeholders and place value.

T24 then continues to describe instruction, which emphasizes the traditional algorithm. Although T24 does appear to use place-value based language in her explanations of the traditional algorithm, T24 also refers to the concept of ‘placeholders of the ones place’ which might not represent a simple concept for Sally to make sense of in the traditional approach. T24 is a nice example of a teacher who intends to move Sally directly to the traditional algorithm without going through an intermediate step of using expanded algorithms or partial products.
CBA Teachers' Treatment of an ‘Intermediate’ CBA Level

Task. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?

SX: 12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.

Figure 46 – SX Student Work

CBA teachers were presented with the above student work, and asked to analyze whether SX’s reasoning was correct or incorrect, and assess the thinking in terms of the CBA Multiplication and Division levels of sophistication. The most accurate characterization using CBA levels to describe the students’ thinking, as determined by the author of the CBA levels, was level 2.3 “Skip Counts All”. In the 2007-2008 interviews, the vast majority of teachers (12 out of 14, or 85.7%) were completely consistent with CBA in determining that SX’s reasoning was at level 2.3. Since virtually every teacher who analyzed SX’s work considered it to be a demonstration of the proper level, it is important to look into how teachers then conceptualized subsequent learning goals for SX.

If teachers implemented a CBA learning progression approach to determining a learning goal, it would be expected that they implement a ‘next level’ approach and choose the next appropriate CBA level (in this case: 2.4 - Skip counts groups of groups) as the targeted goal for moving the student forward in their thinking. The ‘next level’ approach would not apply in situations where the next CBA level represents instances of invalid student thinking that should never be an explicit goal of instruction, but arises in students’ thinking no matter what instruction occurs. Thus, another option for teachers not directly applying a ‘next level’ approach that would be consistent with the CBA MD
framework could be for teachers to describe an effective path to move towards more sophisticated forms of skip counting, such as skip counting by place value parts (e.g. – instead of 12, 24, 36, … a student might do 10, 20, 30, 40, 50, 60 and then 2, 4, 6, 8, 10, 12), as a mechanism to strengthen students’ understanding of skip counting in order to move them forward to considering basic facts and the distributive property in level 3 reasoning.

This ‘next level’ approach, however, did not turn out to be the case for most teachers for this student work as only five of the 12 (42%) teachers who accurately placed SX at the proper level chose level 2.4, and seven of 12 (58%) teachers chose levels beyond level 2.4. Level 2.4 compared to levels 3 and above, represents starkly different approaches to moving SX forward. Level 2.4 ‘skip counting groups of groups’ represents a type of reasoning that many students struggle with, however, seeking to move a student to level 2.4 would seek to deepen SX’s skills and understanding through skip counting, while moving to level 3 would change the focus to decomposition of numbers and the use of the distributive property. It should be noted that in a more recent version of CBA MD materials, single digit and multi digit multiplication levels were separated, which led to the removal of level 2.4 – Skip counts groups of groups, and the introduction of the level ‘Skip counts by place value parts’. Overall, level 2 reasoning relates to skip-counting and iterating strategies, while level 3 reasoning corresponds to reasoning that does not utilize skip counting and depends on knowledge of combining and separating parts of numbers (e.g. – the distributive property).
Although choosing a level beyond the ‘next level’ in CBA might imply an inconsistent perspective with the LP, it is possible for teachers to say their intention is to move children to a level far beyond the ‘next level’, but still plan to provide effective instruction that builds off of the current student conceptualization in an LP consistent manner. While a total of 12 teachers accurately placed SX at the proper level (2.3), there was wide variation in where teachers planned to move SX according to the CBA levels. The following table summarizes the number of teachers who planned to move SX from 2.3 to levels 2.4, ‘3’, 3.1, 3.2, 3.3, and even level 4. The table also investigates the number of teachers who verbalized instructional plans that were aligned with CBA. It was not a requirement that a teacher select level 2.4 in order to have CBA learning progression consistent instruction, as teachers’ comments on instruction were analyzed separately from their chosen level for SX. Instruction was considered to be one of the following; effective instructional alignment with CBA, partial instructional alignment with CBA, poor instructional alignment with CBA, was unclear or vague. For a teacher’s instructional suggestions to be considered effectively aligned with CBA, instruction needed to be targeted at advancing SX’s skip counting abilities (e.g. – skip counting by place value parts 10, 20, 30, 40, 50, 60 and 2, 4, 6, 8, 10, 12) or moving towards an emphasis on mental facts. If any instruction was proposed that did not align with these suggestions, the teacher must have stated instruction that fit with a long-term CBA goal (e.g. – 2 partial products multiplication) in order to be characterized as ‘effective alignment’. If teachers made multiple instructional suggestions such that some instructional tasks aligned with the aforementioned description but other tasks were not,
this was considered partial instructional alignment. If the teacher offered no instruction that targeted the aforementioned descriptions, and was clearly demonstrative of CBA learning goals beyond level 3.1 (without sub-goals specified), this instruction was considered poorly aligned to the CBA framework. Any instructional suggestions that could not be characterized by these descriptions were considered of unclear alignment.

Teacher Task: What type of reasoning do you think SX should move to next?

| Task. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether? |
| SX: 12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players. |

Figure 47 – SX Student Work

Context: MD4 07.08

<table>
<thead>
<tr>
<th>Move SX to 2.4</th>
<th>Move SX to 3*</th>
<th>Move SX to 3.1</th>
<th>Move SX to 3.2</th>
<th>Move SX to 3.3</th>
<th>Move SX to 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Used CBA MD</td>
<td>No CBA MD</td>
<td>Used CBA MD</td>
<td>No CBA MD</td>
<td>Used CBA MD</td>
<td>No CBA MD</td>
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<tr>
<td>Number of teachers Initially choosing 2.3 as SX’s level</td>
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<td>1</td>
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<td>2</td>
<td>0</td>
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<tr>
<td>Number of Teachers with Effective Instructional Alignment with CBA</td>
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<tr>
<td>Number of Teachers with Partial Instructional Alignment with CBA</td>
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<td>0</td>
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<tr>
<td>Number of Teachers with Poor Instructional Alignment with CBA</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<tr>
<td>Number of Teachers with Unclear Instructional Alignment with CBA</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 10 – Teacher Instructional Alignment SX Next Level Chart

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*Level 3 does not actually exist as a level on its own, level 3 actually has a series of sublevels. In spite of this, some teachers chose to describe level 3 reasoning more generally.

Levels 2.4 and 3.1 represent reasonable selections for moving SX to next in the CBA framework. Although Levels 3.2, 3.3, and even 3.4 could represent reasonable learning goals as long as there is mention of appropriate intermediate goals, there were no teachers who proposed instruction above level 3.1 that appeared to have meaningful sub-goals in mind. In spite of the fact that teachers could set a learning progression inconsistent goal for SX but still offer learning progression consistent instruction, only two out of seven teachers were able to offer instructional suggestions that were partially aligned with CBA when they did not offer a consistent learning goal for SX. The remaining five of seven teachers had both learning progression inconsistent goals and learning progression inconsistent instruction for SX. For those teachers with learning progression consistent goals, four out of five offered learning progression consistent instruction that integrated SX’s current skip counting way of thinking into formulating next instruction. The only teacher in this group that did not offer LP consistent instruction did not offer any instructional suggestions, and therefore was categorized as offering ‘unclear’ instruction. This is not to say that teachers who set appropriate LP consistent goals will always suggest and offer better instruction than teachers who do not, but this small data set provides evidence that it might be difficult or unexpected for teachers to conceive of instruction that effectively builds off a students’ way of thinking if they set learning progression inconsistent goals for that student. Teachers in each profile are analyzed below, with transcripts to demonstrate the details of their reasoning.
Teachers with CBA LP Consistent Perspectives

Teacher 7 – SX Problem

[no use of CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX:</td>
<td>12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.</td>
</tr>
</tbody>
</table>

Figure 48 – SX Student Work

I: Here is a quick reference chart and our favorite student SX. SX says 12, 24, 36, 48, 60, 72. So, there’s 72 players. Is SX’s reasoning correct or incorrect? If it is incorrect, what is wrong with it?

T7: It is correct. Yeah no. What is 6 times 12? Is it? Oh yeah it is. I don’t do good at twelve’s. Ok. It probably skip counts all which should be 2.3 right off the bat. Yeah because if he was recalling facts he would say 12 times 6 so.

... 

I: Ok. So what type of reasoning do you think SX should move to next?

T7: Well I think you definitely want them at 3.1 and then he is well on his way so 2.4 is just being able or it is just a stepping stone in between our understanding that here are you groups and they can be added together. Then that would be one group, two groups, three groups, its more investigations within the ultimate goal of 3.1. But I don’t know enough to tell you if that is good or bad or not.

I: So what do you think you would do…

T7: Without having 2.4 sitting there in front of me I would say 3.1. But I don’t know.

I: But how would you get them to 3.1?

T7: Well the kid can already skip count with twelve’s they can probably skip count everything else up to 12. And nobody goes past 12 and do know that. So it would just simply be a manner of understanding 12, 24, 36 and equating that. I would think that they would be easy. You know how many numbers do you have there? 1, 2, 3, 4, 5, 6 and each one is 12 so it is funny to me that if they can skip count by twelve that they already know the algorithm or I mean the facts. But again it is way out my league. Multiplication.
Teacher 7 identified SX to be at level 2.3 by recognizing a ‘skip-counts all’ approach. T7’s conceptualization of SX’s thinking correctly identifies that SX does not appear to be recalling any multiplication facts, which would be evidence for level 3.1. T7 seems to convey perceived importance for maintaining a CBA LP perspective by articulating that moving SX towards iterating groups of groups could serve as a ‘stepping stone’ that works towards the ultimate goal of ‘3.1’. Even though T7 clearly states that level 3.1 is in fact the long-term goal, the intermediate CBA level of 2.4 is considered a valuable ‘stepping stone’ towards that goal. T7’s articulation of intended instruction is less clearly aligned with a CBA LP perspective, most likely because T7 has not worked at great length teaching with the multiplication and division materials, saying “it is way out of my league. Multiplication”.

Teacher 15 – SX Problem

[used CBA MD materials]:

Task. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?

SX: 12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.

Figure 49 – SX Student Work

I: So what CBA level of sophistication is SX’s strategy?

T15: Ok this one is using iterations. Lets see, before skip counting students decompose numbers… ok I think skip counts all so I think 2.3 because student SX is iterating the numbers by reciting all the even multiples in the skip counting sequence. So they are skip counting by 12 six times.

I: What type of reasoning do you think SX should move to next?
T15: The next one is… I think, well obviously you want to get him up to level 3. But I think level 4 is important. I mean not 4 but 2.4 because I think if they don’t learn that then they are not going to be able to manipulate numbers at higher levels. So I think 2.4.

I: What would you do instructionally to move SX to this next type of reasoning?

T15: Kind of like, if I think about isn’t kind of like this [points to one of her strategies on the previous worksheet]. Where they are combining. Although this one {SX’s strategy} is lets see… I don’t know. Students shorten the iteration process by cumulating iterations or subtotals or combining iterations of the original number into bigger numbers. Well Oh I see like counting by 24’s three times. Is that kind of what that means. See I kind of see it as that or even doing this too where they are writing three 12’s but you could probably get them to see that way by having them write six 12’s. And then showing that two of the 12’s together is 24 and then realizing it that then you are only doing that three times instead of six times. So I can kind of see it both ways. I would probably have them combine using that method. Showing that there are other ways to get …kind of thinking that this is 24 times 3 instead of 12 times 6. And show them that there is other ways to find the answer. That is a tough one. Yeah by two groups together.

T15 very fluidly reasons through what SX’s reasoning was for the problem, and correctly places SX at level 2.3 on the CBA levels of sophistication. T15 articulates a very similar long-term goal as T7 (eventually moving SX up to level 3), but also recognizes an importance in helping SX move into level 2.4 reasoning. T15 even articulates that SX might struggle if level 2.4 is skipped because they will not have built strength in number manipulation and number sense by combining groups of groups. The CBA levels are intended to indicate that movement to level 2.4 is a more abstract, and efficient conception of skip-counting, one that may help students to progress to level 3 reasoning. T15 continues to articulate instruction that builds off of SX’s current level of reasoning, and appears to be effectively aligned with the proposed CBA goal. T15
represents an instance in which the CBA levels appear to be well understood, and the importance and purpose of CBA sublevels seems to be meaningful for T15. Alignment of proposed instruction with the CBA framework shows how CBA materials can effectively inform teachers not only of where student are, but where they are going.

Teacher 17 – SX Problem

[no use of CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX:</td>
<td>12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.</td>
</tr>
</tbody>
</table>

Figure 50 – SX Student Work

I: What CBA level of sophistication is SX’s strategy?

T17: 2. [points to reference sheet]

I: Level 2.3. is that what you are saying?

T17: Yeah. Students iterate numbers by reciting or writing all needed multiples in the skip-count sequence.

I: What type of reasoning do you think SX should move to next?

T17: Well, I just 2.4 where you would say well what are three groups of 12 and they say 36. Then can you double 36? Which just shortens it a little bit. So that would be 2.4. You know you would be working with recalled facts. So depending on where you were in that you maybe want them to move to that so they don’t always have to skip count. But skip count is ok and they could be clear up here [points to the highest level on the quick reference sheet.] in the algorithms and with a certain problem still have to skip count. Like if they had 6 times 9 and say it was like 36 times 9. They might not remember 6 times 9 is 54 and you will still see them go back and do 9, 18, 27, 36, on up to 54.

I: Which, well let me ask you the last question here. What would you do instructionally to move SX to this next type of reasoning?
T17: I would probably just simply say do you know how much 12 times 3 is? Could we double it? And see how they do. But I would probably be asking that on problems that didn’t get so high.

I: Let me ask you this, one strategy is 12, 24, 36, and say another strategy is 6 times 10 is 60 and 6 times 2 is 12 and 60 plus 12 is 72. So which of those is more sophisticated?

T17: Probably the latter because they are decomposing the factors. And they do decomposed all different kinds of ways. Ways that I wouldn’t never decompose. But it makes perfect sense to them because they are familiar with that type of decomposition.

Similar to Teachers 7 and 15, T17 effectively levels SX at level 2.3. Also like T7 and T15, T17 identifies moving away from skip counting as a goal. However, T17 also articulates that skip counting is an approach that makes a lot of sense to many children, and proposes moving SX to level 2.4 to help shorten the skip counting process. T17’s instruction aligns with the proposed move, and articulated instruction is consistent with level 2.4, but also considers the challenges of using large numbers and proposed instructing at level 2.4 with more manageable numbers. This seems to be quite a reasonable instructional suggestion considering that skip counting groups of groups can be a difficult and sometimes confusing strategy for many kids. T17 also demonstrates understanding of CBA levels by arguing that a decomposition strategy would be considered more sophisticated than skip counting.
Task. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?

SX: 12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.

Figure 51 – SX Problem

I: Ok. What CBA level of sophistication is SX’s strategy?
T1: 2.3.
I: What type of reasoning do you think SX should move to next?
T1: I would say 3.2 or 3.3.
I: OK. What would you do instructionally to move SX to this next type of reasoning?
T1: Well I think that I would show him that is a great strategy that he did, but what if it would have been 24 times 12. And then he would have had count or like 9 times 12 I guess I would keep it a single digit like 9 times 12 he would have had to count a lot higher. So to show him that you can break down that 12 by place value or it doesn’t necessarily have to be by place value but you can break it down so you can do the quick facts in your head. Because he can do already do some quick facts you know counting by 12s shows that he probably has the ability to do… to know some of his multiplication facts. So I guess to show him that it might be more efficient to do that because if you get into larger numbers you are not going to be able to do that. Well you might be able to but it will be more complicated just to count or skip count all the way to the answer. So just show him that 6 times 12 is the same thing as 6 times 10 and 6 times2 and adding them together.
I: Ok.

T1 correctly levels SX at level 2.3, but does not apply a ‘next-level’ approach by choosing level 3.2 or 3.3 as the targeted goal. T1’s goal moves from a skip counting approach to a place value based decomposition of numbers, which represents a
significant change in reasoning strategy for SX. Although T1 recognizes that
decomposition strategies do not have to be place value based, the articulated instruction
demonstrates that T1 intends to instruct based on place value. Instruction appears
consistent with the proposed CBA goal, but T1 does not completely align with a CBA LP
perspective on teaching and learning in this instance.

T1’s comment that decomposition can also be based on known facts demonstrates
a less ambitious jump based on the CBA levels, and T1 states: “So to show him that you
can break down that 12 by place value or it doesn’t necessarily have to be by place value
but you can break it down so you can do the quick facts in your head. Because he can
already do some quick facts, you know, counting by 12s shows that he probably has the
ability to do…to know some of his multiplication facts”. This appears to be T1
indicating that doing facts mentally, level 3.1 is an important step for the student to take.
T1 appears to be making the assumption that since SX demonstrates a knowledge of
addition and subtraction facts, that they also would either know, or easily learn, some of
the multiplication facts. T1 may have the correct long-term goal in mind, but does not
seem to have an effective way to transition SX to level 3.2 reasoning. T1 does not
mention giving the student an opportunity to work with skip counting by place value
parts, which would represent a CBA aligned short-term goal that might more effectively
move SX to level 3.2. T1 is, however, consistent with CBA learning progression
suggestions by not completely bypassing level 3 reasoning in forcing an algorithm on SX,
as it is quite likely that such a large jump would lead to rote use of an algorithm, and
therefore not the true achievement of CBA levels 4 or 5.
Teacher 8 – SX Problem

[used CBA MD materials]:

**Task.** There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?

**SX:** 12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.

**Figure 52 – SX Student Work**

I: Alright. So here is the quick reference sheet and here is student X. So X says 12, 24, 36, 48, 60, 72, raising a finger after reciting each number. So there are 72 players. Is X’s reasoning correct or incorrect? If it is incorrect what is wrong with it?

T8: It is correct.

I: And what level?

T8: So he did, oh so he is just skip counting. And he writes it down?

I: No let us assume he doesn’t and he just says it aloud. But raises a finger 12, 24, 36.

T8: OK. So he is still needing… he is 2.3. Because he is skip counting by 12’s. Right so a student can read in a skip count sequence [reading the quick reference sheet]. So 2.3.

I: What type of reasoning do you think X should move to next?

T8: Well, I want to move him to level 3. Because I would not want and algorithm on here. And I like to get him to 3.3. He wouldn’t need the two partial products. First and for most I like him to know that 6 times 12 is 72. But if he, well where would that go? Recalls facts that is not what you mean by that is it? Know that 6 times 12 is 72?

I: Yeah I am assuming… well for me it sort of that some schools like go up to tens. Like 10 times 10 is the largest basic fact for multiplication. But some school go up to 12 times 12 so it sort of depends on whether do you consider that a basic fact. So you memorize it or is not a basic fact and you have to derive it or something like that.

T8: So if these were my students they would need to know up through 12’s. So that would be a basic fact. But I could see… if I had a student that need to do this [points to X’s strategy] 12, 24, 36 and so on, I would ask them is there any part of or what could you start with? What is something
that you know time 6 or something that you know times 12? And have
them work with that. Often when they are trying to learn their
multiplication facts to 12 we will break it down to 6 times 10 and then the
6 times 2. But many like to do 6 times 11 which is 66 and add 6 on.

I: Ok. So this is like a derived fact.

T8 accurately levels SX at level 2.3. In addition to passing over level 2.4 – skip
counts groups of groups, T8 even states that SX would not need ‘2 partial products’, but
later comments do discount this. T8 seems heavily influenced by the school curriculum,
and the fact that students need to know multiplication facts ‘through 12’s’. Recalling
facts is the primary goal for T8, and alignment with a CBA LP perspective does not seem
to be influencing T8’s goals and instruction for SX. This could be in large part to the fact
that interpreting ‘basic multiplication facts’ as being through 12 is different than CBA’s
perspective and emphasis on facts through 10. T8 even contrasts students who would
learn in T8’s classroom against those who may not know their basic facts through 12. In
this comparison, T8 demonstrates that instead of encouraging counting groups of groups,
T8 would focus on working on multiplication facts for 6 and 12.

T8 also makes the statement that it would not be necessary to move the student to
‘2 partial products’ but rather go straight to 4 partial products. However, T8 later
discussed how it would be useful for SX to view 6x12 as 6x10 and 6x2, representing an
instance of ‘2 partial product reasoning’. It is not clear if this is simply a
misunderstanding of the partial products language of CBA, an oversight of when 2 partial
products can be used, or something else. T8 clearly has mixed messages about the goal
for SX and the type of reasoning to move to next. T8 really seems to be talking about
moving a student at Level 2.3 to somewhere in Levels 3.1, 3.2, 3.3; to have some
understanding that decomposing by place value is important (Level 3.3). She mentions using known facts, and decomposition as next steps to accomplish this, but there is limited evidence to show how well this fits in with the CBA MD framework.

Teacher 9 – SX Problem

[no use of CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX:</td>
<td>12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.</td>
</tr>
</tbody>
</table>

**Figure 53 – SX Student Work**

I: Ok. Here is a reference sheet {and the new task}

T9: Oh! Sure counting by 12’s that would be good. {looking and the new task}. I didn’t even to think about that one.

I: Is SX’s reasoning correct or incorrect? If it is incorrect, what is wrong with it?

T9: I think it is correct and I like the… that is a pretty sophisticated strategy. So what level is it? Umm… well they’re skip counting. So I would say it is a level 2.3 where he is skip counting.

I: What type of reasoning do you think SX should move to next?

T9: Well I would say level 3… skip count groups of groups (reads 2.4), I don’t think that is necessary. I am assuming that is like 2 twelve’s.

I: Yeah.

T9: I don’t know if that is really… If he can do this, if he can skip count by 12’s I would say that you are pretty ready to go into just knowing the facts. That is what I would say. See that is where I… and you can’t answer this I know, to me recalling fact is I guess I don’t think of that as much of a strategy, but I guess it is strategy.

I: What would you do instructionally to move SX to this next type of reasoning?

T9: I think just drill and practice. You know recalling facts is not a lot of strategy. It kind of depends on how much understanding SX actually had
with other skipped counting kind of things. If this is just a fluke which counting by 12’s is not really fluky. I mean if you can count by 12’s like that is pretty high level I think. I think you could very easily just move to a number.

T9 also correctly places SX at level 2.3. When determining the next level of reasoning to move SX, T9 explicitly states that level 2.4 is unnecessary. T9 states that a student with reasoning like SX would be ready to go into recalling facts. The explanation given by T9 is that recalling facts is not much of a strategy, and that it is learned through drill and practice. T9 also seems to believe that counting by 12’s is a high level of thinking that could lead easily into recalling facts. Aside from drill and practice, there is no indication for how T9 would transition SX from skip counting to recalling facts. It is important to recognize that if T9 were referring to knowing multiplication facts of 12 as a ‘basic fact’, then this suggestion could in fact be considered consistent with CBA LP – this classification could fall into the ‘gray area’ of CBA classification. It is important to note that initially the CBA MD level distinguished between single-digit and multi-digit levels. These were later combined in this CBA version (because of teacher requests), which caused some teachers to get confused; T9 might be an example of this.

Patterns and Themes in Consideration of CBA Teachers’ Conceptualization of Student Thinking and Subsequent Instruction

In this section, data is investigated from a big picture perspective, as trends in the relationship between teacher’s conceptualization of student thinking, determination of learning goals, and proposed instruction are considered. The overall data on CBA teachers’ conceptualization of student thinking and subsequent instruction indicates that
for the MD levels involved in all of the student work problems evaluated for this research, teachers have a tendency to set goals that are in line with the CBA LP framework. Regardless of the level chosen by teachers, the vast majority of teachers offer the ‘next’ reasonable CBA level as their goal for instruction. In addition to teachers applying predominately ‘next level’ goal-setting for students, CBA teachers’ instruction largely matched their selected CBA goal. This pattern occurred regardless of whether or not the correct CBA level was chosen for the student thinking that was being evaluated. Teachers’ ‘next level’ approach to goal setting and instruction is evaluated in the following section, and short cases are offered to help characterize the patterns described.

*CBA Teachers’ Approach to Goal-Setting and Instruction*

The CBA learning progression materials provide teachers with an organized framework for thinking about children’s mathematical thinking that can be used to assist teachers in trying to locate and identify current conceptions children have, as well as guide future goal-setting and determination of instruction for students. A CBA LP consistent method for teaching could consist of a teacher identifying the proper level for a student based on their reasoning on a mathematical task, determination of the next reasonable level, and creation of instruction to help build on the students’ current way of thinking to develop more sophisticated mathematical ideas similar to those exhibited in the next CBA level. In order to enact this, teachers would need to be able to: 1) properly select an appropriate CBA level for a student based on a sample or samples of work, 2) determine a reasonable goal for that student, and 3) construct or locate instructional tasks that matched the goal and the level of the student.
Proper Selection of CBA Level for a Student

The data suggests that proper selection of a CBA level for students was the most challenging for teachers on the samples of student work for multiplication and division involving non-standard place value based decompositions and the distributive property (Sally, and RR), where only 10 out of 28 teacher episodes (36%) leveled the student completely consistently with CBA, while 11 out of 28 were partially consistent (39%), and seven out of 28 (25%) were inconsistent. However, on problems that involved skip counting or a place-value decomposition of numbers with the distributive property (SX and QR), 25 out of 28 teacher episodes (89%) leveled the student completely consistent with CBA, two out of 28 teacher episodes (7%) leveled the student partially consistent with CBA, and only 1 teacher episode out of 28 (4%) was inconsistent with CBA. This demonstrates that for more complex student work that requires more careful examination and conceptual recognition of student strategies, properly leveling a student can be quite challenging. Overall, utilizing data from four separate samples of student work, a total of 35 out of 56 teacher episodes (65%) selected a completely consistent CBA level based on the student work, 13 out of 56 teacher episodes (23%) selected a partially consistent CBA level, and 8 out of 56 teacher episodes (14%) were inconsistent with CBA. The complex student work was especially difficult for those teachers without CBA multiplication and division work, as all five teachers who were inconsistent in analyzing Sally’s work were those that had not used the CBA materials. There was also a slight overall difference between those teachers who had used CBA multiplication and division materials and those who had not, as teachers with experience with the materials were more likely to be
completely consistent with CBA than those without experience. It was also far more likely that teachers without experience struggled to choose an even partially consistent level, as 33% those teachers without CBA multiplication and division use were inconsistent with CBA while only 7% of teachers with use of the materials were inconsistent with CBA in their chosen level.

<table>
<thead>
<tr>
<th>Context</th>
<th>Teachers Choosing a Completely Consistent CBA Level</th>
<th>Teachers Choosing a Partially Consistent CBA Level</th>
<th>Teachers Choosing an Inconsistent CBA Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Used CBA MD</td>
<td>No CBA MD Use</td>
<td>Used CBA MD</td>
</tr>
<tr>
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<td>7</td>
<td>0</td>
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<tr>
<td>45x23 (Sally) 08.09</td>
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<td>0</td>
<td>3</td>
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<tr>
<td>46x5 (RR) Online</td>
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<td>3</td>
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<tr>
<td>46x5 (RR) Face to Face</td>
<td>3</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>46x5 (QR) Online</td>
<td>7</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>46x5 (QR) Face to Face</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>TOTAL of teachers with CBA Use</td>
<td>28 (68.3%)</td>
<td>-</td>
<td>10 (24.4%)</td>
</tr>
<tr>
<td>TOTAL of teachers without CBA Use</td>
<td>-</td>
<td>7 (46.7%)</td>
<td>-</td>
</tr>
<tr>
<td>TOTAL of teachers Overall</td>
<td>28 (50%)</td>
<td>7 (12.5%)</td>
<td>10 (18%)</td>
</tr>
</tbody>
</table>

Table 11 – Teacher CBA Consistency Chart

**Determination of a Reasonable Learning Goal for a Student**

The data on teachers’ determination of learning goals can be interpreted in a number of ways. If looking at all teachers, regardless of if they initially leveled the student consistently with CBA, the data indicates that 39 out of 56 (70%) determined a reasonable goal for the student based on their determination of the initial level. In order to be categorized as choosing the next appropriate CBA level, teachers must have chosen
the next appropriate CBA level according to the CBA MD framework as their target for students to move to next, or provided a reasonable ‘long-term’ goal with appropriate CBA sub-goals. This implies that in order for teachers learning goal to be deemed as the next appropriate level, they must not propose skipping levels of sophistication that describe reasonable targets of instruction (i.e. – any level that does not describe incorrect student thinking). In cases where teachers chose learning goals that differed from this description of the appropriate next level in the CBA MD framework, teachers were only deemed to be applying a ‘next appropriate CBA level’ approach if they explicitly mentioned a sub-goal or ‘stepping stone’ goal that aligned with an appropriate CBA level in the sequence.

<table>
<thead>
<tr>
<th>Context</th>
<th>Teachers Choosing Next Appropriate CBA Level</th>
<th>Teachers NOT Choosing Next Appropriate CBA Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Used CBA MD</td>
<td>No CBA MD Use</td>
</tr>
<tr>
<td>6x12 (SX) 07.08</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>45x23 (Sally) 08.09</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>46x5 (RR) Online</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>46x5 (RR) Face to Face</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>46x5 (QR) Online</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>46x5 (QR) Face to Face</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>TOTAL of teachers with CBA Use</td>
<td>29 (71%)</td>
<td>-</td>
</tr>
<tr>
<td>TOTAL of teachers without CBA Use</td>
<td>-</td>
<td>10 (67%)</td>
</tr>
<tr>
<td>TOTAL of teachers Overall</td>
<td>29 (52%)</td>
<td>10 (18%)</td>
</tr>
</tbody>
</table>

Table 12 – Teacher CBA Next Level Chart
Interestingly, applying a ‘next level’ approach was most common on samples of student work for those involving non-standard place value based decompositions and the distributive property (Sally and RR). Although teachers only leveled these students completely consistent with CBA 36% of the time, they proposed a ‘next’ level-learning goal for the student 75% of the time (21 out of 28 teacher episodes). The other 25% that did not propose ‘next’ level-learning goals was dominated by teachers who had not used CBA MD (5 out of 7), suggesting that experience with the CBA MD materials facilitated teachers’ thinking about learning goals for the more complex, non-standard place value based decompositions and use of the distributive property.

![Context Used and CBA Level Chart](image)

Table 13 – Teacher CBA Level Complex Work and Level Chart

*Comparison of problems involving more complex student work and more conceptually complex CBA levels

Conversely, applying a ‘next level’ approach was least common on samples of student work that involved skip counting or a place-value decomposition of numbers with
the distributive property (SX and QR). Although teachers leveled these students completely consistent with CBA 89% of the time, they proposed a ‘next level’ for these students only 57% of the time (16 of 28). This can potentially be partially explained by the fact that on Sally’s sample of student work, many teachers inaccurately placed Sally too high at level 4, which is the second highest level in the CBA framework for Multiplication and Division, and therefore may have seen moving to the next level (Level 5) as the only viable option since they were at the ‘ceiling’, or the highest possible level. However, there did not exist an instance in the analyzed data where a teacher explicitly stated that they were moving a student to the next level because there was no other place in the CBA framework to move them.

<table>
<thead>
<tr>
<th>Context</th>
<th>Teachers Choosing a Completely Consistent CBA Level</th>
<th>Teachers Choosing a Partially Consistent CBA Level</th>
<th>Teachers Choosing an Inconsistent CBA Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Used CBA MD</td>
<td>No CBA MD Use</td>
<td>Used CBA MD</td>
</tr>
<tr>
<td>6x12 (SX) 07.08</td>
<td>5</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>46x5 (QR) Online</td>
<td>7</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>46x5 (QR) Face to Face</td>
<td>6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>TOTAL of teachers</td>
<td>18</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>with CBA Use</td>
<td>(90%)</td>
<td></td>
<td>(5%)</td>
</tr>
<tr>
<td>TOTAL of teachers</td>
<td>-</td>
<td>7</td>
<td>-</td>
</tr>
<tr>
<td>without CBA Use</td>
<td></td>
<td>(87.5%)</td>
<td></td>
</tr>
<tr>
<td>TOTAL of teachers</td>
<td>18</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>Overall</td>
<td>(64.5%)</td>
<td>(25%)</td>
<td>(3.5%)</td>
</tr>
</tbody>
</table>

Table 14 – Teacher CBA Level Less Complex Work and Level Chart

*Comparison of problems involving less complex student work and less conceptually complex CBA levels
There also appeared to be a trend of teachers interpreting SX’s work who sought to move SX completely away from skip-counting and on to using the distributive property, skipping over recalling facts and using number properties other than the distributive property and constituting a non ‘next level’ approach. Since ‘skip-counts all’ is one of the more sophisticated sublevels of the ‘repeated addition and skip counting’ major level, it is quite possible that many teachers saw SX as being ‘close enough’ to moving towards a completely different major level (Level 3) and to move towards thinking involving decomposition of numbers and the distributive property. Since the distributive property is one of the key markers of level 3 reasoning this could encourage skipping directly to an emphasis on the distributive property. For T15, a higher level is chosen as a goal because it is deemed that SX is ready to move forward to a different approach.

[used CBA MD materials]:

T15: I would move the student to level 3.1. SX is counting by a rather large basic number (12). SX is ready to be able to recall basic facts.

Another, more likely, explanation for why teachers overwhelmingly chose the ‘next’ level for the more complex student work and CBA levels could have to do with uncertainty about the exact nature of the child’s thinking. For SX and QR, teachers very readily identified and categorized the student work, and may have felt a greater sense for what the student knew and how to build on that knowledge. However, with less clarity in interpreting the student thinking might come uncertainty about exactly what the student knows and how to build on that knowledge. This could lead teachers to lean more heavily on the CBA framework, and more closely follow the proposed LP in order to help
move Sally or RR forward. In analyzing Sally’s work, T20 demonstrated confusion about what exactly the work demonstrated (in part because multiplication is not taught by T20), and chose to move to the next CBA level even though the next level caused T20 to feel conflicted.

<table>
<thead>
<tr>
<th>Task.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(45 \times 23 = ) _____</td>
</tr>
</tbody>
</table>

Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.

Figure 54 – Sally Student Work

[no use of CBA MD materials]:

T20: I think…students expanded (reading from CBA Quick reference)...I think it is definitely expanded what she did. Multiply and divide numbers maintain the value of place value parts throughout the sequence of steps which she does because she, she uh she knows that this is two tens. So I think that she is at a level 4.

I: Okay. What type of reasoning do you think Sally should move to next and explain why?

T20: You know I, I really have to say that I am not sure. I mean I guess, (pause) I guess that she would move on to level 5 but honestly I don’t have enough understanding. I am one of those people that learned it this way (points to traditional algorithm) and I don’t even know because I don’t teach it and I don’t know. But my guess is, my guess is that this (pointing to Sally’s method) is more sophisticated than this (pointing to traditional algorithm) because this is just memorizing how to do it. I don’t know that to be certain though. It seems like this has more thinking involved than what I did. I just kind of did it.

If looking only at those teachers who initially picked a level for the student that was at least partially consistent with CBA, you find that 34 out of 48 (71%) applied a ‘next level’ approach to goal-setting, while 14 out of 48 (29%) proposed skipping at least
one reasonable CBA level. This means that out of 56 teacher episodes, 34 of them (43%) leveled the student at least partially consistently within the CBA framework and chose a goal that was consistent with a CBA LP perspective. If looking only at those teachers who initially leveled the student inconsistently with CBA, you find that three out of 8 (37.5%) applied a ‘next level’ approach to goal setting, while 5 out of 8 (62.5%) proposed skipping at least one reasonable CBA level. This means that out of 56 teacher episodes, only five out of the 56 (9%) enacted both steps 1 and 2 inconsistently with a CBA LP perspective.

It should be noted that of the 21 teacher episodes that leveled the students either inconsistently or partially consistently, only four (19%) leveled the student too low, while the vast majority, 17 (81%), leveled the student too high. The obvious dilemma with leveling a student too high, is that the goal setting and instruction is likely to be based off of an initial conception of the students’ thinking that over-represents what that student might understand mathematically, running the risk for instruction to promote less meaningful mathematical understanding. This is especially problematic for those teachers who incorrectly picked too high of a level for the student, and subsequently proposed a goal that skipped one or more levels. In this situation, the goal would likely constitute a large jump from where the students’ thinking is, that could be very difficult for a student to make meaningful sense of.

_Determination of Instructional Tasks Based on Children’s Mathematical Thinking_

The data for the determination of subsequent instruction can be analyzed in a number of ways as well. One way to look at the data is to investigate the instructional
tendencies of all teachers, regardless of the chosen initial level. To do this, each teacher’s selected learning goal was interpreted as the intended goal of instruction. Essentially, if a teacher’s goal is to help a child learn to decompose numbers by place value parts and then skip count by place value parts, one would expect instruction to either directly encourage the student to follow this strategy, or to directly relate to the concepts required for children to learn this concept. To analyze the data, four categorizations for teachers’ instructional choices were considered: a) Consistent to Learning Goal Instruction, b) At Least Partially Inconsistent with Learning Goal Instruction, c) Inconsistent with Learning Goal Instruction and d) Vague or Missing Instruction. For teachers to be considered ‘Consistent to Learning Goal Instruction’, the professed instructional plans needed to be detailed enough to show evidence of a connection between the learning goal and instruction. That is, if a teacher sought to move students to skip counting groups of groups, they needed to specify instruction that encouraged students to count in this fashion and/or make specific references to the types of student thinking strategies encouraged in their instruction that were aligned to their learning goal. This required teachers to not just state instruction, but do so in a fashion that made it clear that their CBA learning goal was informing or connected to their instructional plans. For teachers to be considered ‘At Least Partially Inconsistent with Learning Goal Instruction’, teachers must have made references to multiple instructional tasks; some of instruction that appeared aligned to instruction goal (much like the aforementioned ‘Consistent to Learning Goal Instruction’ group) and other instruction that emphasized student thinking strategies that did not coincide with their chosen learning goal. It was also necessary that
the instructional tasks be presented in a non-LP consistent sequence (e.g. – I’d do work at level 3.1 first, and then later on do work at level 3.2 with the student), because this would be considered ‘Consistent to Learning Goal Instruction’. In order to be considered ‘Inconsistent with Learning Goal Instruction’, a teacher must make one or more instructional suggestions that do not explicitly link to their chosen learning goal, and show no evidence of presenting any tasks that were aligned to their chosen CBA learning goal. An example would be a teacher who proposed moving a student to skip counting groups of groups, but describes instruction that encourages the student to do partial products multiplication. Finally, for a teacher to be considered having ‘Vague or Missing Instruction’ there must not be detail enough for inclusion in any of the other three groups.

Although all interviews with CBA teachers had a formal script that explicitly asked teachers to clarify and explain their choice of instructional tasks, occasionally teachers were either uncertain or provided insufficient detail in their responses to determine instructional tasks or plans. Looking at all 56 teacher episodes, the data indicated that 25 (45%) proposed instruction consistent to learning goals, nine (16%) proposed partially inconsistent instruction to learning goals, and eight (14%) proposed instruction that was inconsistent with the proposed learning goals, while 14 (25%) either did not provide enough detail to determine instructional plans, or were uncertain about what to do. If only considering the 42 teachers who provided sufficiently detailed instructional suggestions to determine alignment with goals, 34 (81%) proposed instruction that at least partially consistent with their proposed learning goal, and only
eight (19%) proposed instruction that was completely inconsistent with their learning goal.

When delineating by CBA MD use, this data also shows that teachers with experience using the CBA MD materials are far more likely to offer instructional plans that are at least partially consistent with a professed learning goal (71.4% of teachers with CBA MD use, compared to 47.4% of those without CBA use). Teachers with CBA MD use were also far less likely to offer instructional plans that were inconsistent with a professed learning goal (11.4% of teachers with CBA MD use, compared to 21.1% of those without CBA use).

<table>
<thead>
<tr>
<th>Numbers of Teachers with CBA MD use</th>
<th>Offered instructional plans that were consistent with professed learning goal</th>
<th>Offered instructional plans that were at least partially inconsistent with professed learning goal</th>
<th>Offered instructional plans that were inconsistent with professed learning goal</th>
<th>Did not offer instructional plans or did not offer clear instruction plans</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numbers of Teachers without CBA MD use</td>
<td>20 (57.1%)</td>
<td>5 (14.3%)</td>
<td>4 (11.4%)</td>
<td>6 (17.2%)</td>
</tr>
<tr>
<td>Overall Number of Teachers</td>
<td>25 (4546.3%)</td>
<td>9 (16.7%)</td>
<td>8 (14.8%)</td>
<td>12 (22.2%)</td>
</tr>
</tbody>
</table>

Table 15 – Instructional Consistency Chart

This pattern indicates that the determination of instructional tasks in the context of analysis of written work utilizing the CBA framework was overwhelmingly consistent
with a learning progression approach to instruction. This is especially encouraging when looking at the 24 teachers who accurately chose the appropriate CBA level, and determined a CBA LP consistent learning goal. Out of the 24 teachers who accurately chose the appropriate CBA level, 14 (58%) chose instruction that was consistent with their learning goals and five (21%) more chose instruction that was at least partially consistent with their learning goals, while 0 (0%) chose instruction that was inconsistent and five (21%) provided insufficient detail to determine. Thus, of the original total of 56 teacher episodes, 19 out of the 56 teachers (34%) effectively followed an LP consistent process and correctly leveled the student, chose an appropriate learning goal, and proposed instructional tasks that were at least partially aligned with that learning goal. It should also be noted that there were no teachers who correctly leveled Sally and proposed a reasonable CBA learning goal that went on to propose instruction that did not align at least in part with their intended goal. Some teachers, however, were unclear or uncertain about where to proceed with instruction, but cited the learning goal as informing the intended instruction. This provides evidence that CBA teachers at least verbalize tendencies towards instruction that builds off of students’ current understandings towards higher CBA levels of sophistication in most cases. The fact that instruction is so often aligned with learning goals is also encouraging considering that designing activities based off of children’s thinking can be a very difficult task. It can also be hypothesized that there are a significant number of teachers who are unclear in their determination of tasks because conceiving of instruction based on student thinking is not a simple task. It is believable that some of the teachers who are unclear or uncertain might have
conceptually appropriate goals and a desire to match instruction with these goals, but not know how to do this without help.

Obviously, it could be problematic from a CBA LP perspective for teachers to initially level a student too high, and move to the next level (or skip levels), and then provide instruction that is actually targeted at the goal. This would most likely represent a situation that would be challenging for the student to make meaningful sense of the instruction. There were 16 CBA teachers who initially leveled students too high, and therefore set goals that were beyond the desired CBA goal for the student based on their level. Of these 16 teacher episodes, eight (50%) proposed instruction that was aligned with their goals, and the instructional suggestions offered by these eight teachers appeared less guided by the students’ current ways of thinking, and more by the desired outcome or algorithm.
Offered instructional plans that were consistent with professed learning goal

Offered instructional plans that were at least partially inconsistent with professed learning goal

Offered instructional plans that were inconsistent with professed learning goal

Did not offer instructional plans or did not offer clear instruction plans

<table>
<thead>
<tr>
<th></th>
<th>Used CBA MD</th>
<th>No CBA MD Use</th>
<th>Used CBA MD</th>
<th>No CBA MD Use</th>
<th>Used CBA MD</th>
<th>No CBA MD Use</th>
<th>Used CBA MD</th>
<th>No CBA MD Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teachers with a correct initial CBA level for student</td>
<td>11 (45.8%)</td>
<td>3 (12.5%)</td>
<td>3 (12.5%)</td>
<td>2 (8.3%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>3 (12.5%)</td>
<td>2 (8.3%)</td>
</tr>
<tr>
<td>Teachers with an incorrect initial CBA level for student</td>
<td>6 (37.5%)</td>
<td>2 (12.5%)</td>
<td>2 (12.5%)</td>
<td>2 (12.5%)</td>
<td>1 (6.3%)</td>
<td>3 (18.8%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
</tr>
</tbody>
</table>

Table 16 – Instructional Consistency and Level Chart

In other words, instruction by these eight teachers was predominately of the style of demonstrating a more efficient way to conduct the strategy. As a short example, T24 determined that Sally should move all the way up to the standard algorithm.

<table>
<thead>
<tr>
<th>Task.</th>
<th>45  × 23 = ______</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 55 – Sally Student Work

[used CBA MD materials]:

I: What would you do instructionally to move Sally this next type of reasoning?
T24: I would actually set up the traditional algorithms and talk her through it.
I: So how would you do that?
T24: Let’s say we would right down 45 times 23. So I would say Sally, in the past you were thinking about 3 going into 40 and then 3 going into 5. So instead of having to do all the additional writing over here let’s condense our writing down and not have to do as much side work.

Of the remaining eight teachers who set goals well beyond the desired CBA goal, six (37.5%) provided instructional tasks that were actually targeted at levels below the professed goal. Two teachers seemed to realize while considering instruction that the goals may be too advanced, and the other four teachers offered instruction that was targeted at deepening the students’ understanding at the initially stated level of understanding. Either of these two situations could constitute LP consistent instruction, in spite of the fact that the teachers initially leveled the students improperly, and set potentially ambitious goals.

*Comparing Different Profiles of Goal Setting and Instruction*

Although it might be considered ideal for teachers to completely align with a CBA LP approach to determining the level of a child, setting a learning goal, and designing instructional activities, realistically teachers are going to vary in their interpretations of student work as well as their determination of goals and instruction. To better get a sense for the general tendencies, and to better describe the common ways teachers approached these steps, each teachers’ analysis of student work was characterized based on how teachers interpreted each step. The three steps described earlier suggested that alignment with a CBA LP perspective would imply teachers would need to be able to: 1) properly select an appropriate CBA level for a student based on a
sample or samples of work, 2) determine a reasonable goal for that student, and 3) construct or locate instructional tasks that matched the goal and the level of the student.

Each step of this process can be considered using a binary system (Y or N), step 1 can be categorized as ‘leveled student at least partially correctly’ (Y) or ‘leveled student incorrectly’ (N), step 2 can be categorized as ‘chose next appropriate CBA level’ (Y) or ‘skipped at least one appropriate CBA level’ (N), and finally step 3 can be categorized as ‘proposed instruction that aligned with learning goal’ (Y) or ‘proposed instruction that did not align with learning goal’ (N). As an example, a teacher categorized as ‘YNY’ would have correctly leveled the student, skipped at least one appropriate CBA level, and provided instruction that was aligned with that learning goal.

Theoretically, with 3 binary categories, there are 8 possible profiles that such a classification system produces: {(YYY), (NYY), (YNY), (NNY), (YYN), (NYN), (YNN), (NNN)}. However, there were absolutely no teachers who fell the YYN or YNN categories. This is encouraging; as these categories would represent instances in which a teacher would likely struggle to make progress with a student because they would properly diagnose a student and then proceed with poorly aligned goals and/or instruction. Also, profile NNN had only a single teacher, and profile NNY had only two teachers. These, too represent instances in which a teacher would likely struggle to make progress with a student based on their responses.

<table>
<thead>
<tr>
<th>Student Work Context</th>
<th>YYY</th>
<th>NYY</th>
<th>YNY</th>
<th>NNY</th>
<th>YYN</th>
<th>NYN</th>
<th>YNN</th>
<th>NNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>46x5 RR</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
The profile for NYN had 4 teachers. Interestingly, all four teachers who fell into this category were those that responded to the most complex student work (Sally’s 45x23). This particular episode of student work had by far the most variation in interpretations. These teachers struggled to make sense of the student’s thinking, but still attempted to maintain alignment with the CBA MD framework for determining a learning goal. What was especially difficult for those teachers in this group was to conceive of instruction that aligned with the proposed learning goal. For these four teachers, instructional plans were largely unclear and seemed to be difficult for teachers to think about what would be effective to help the student learn.
Table 18 – Specific Profiles of Teacher LP Alignment

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>0</th>
<th>2</th>
<th>1</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>46x5 RR (Online)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>46x5 QR (Online)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45x23 Sally 08.09</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>6x12 SX 07.08</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>TOTAL of teachers with CBA Use</td>
<td>16 (76.2%)</td>
<td>-</td>
<td>3 (14.3%)</td>
<td>-</td>
<td>2 (9.5%)</td>
<td>-</td>
</tr>
<tr>
<td>TOTAL of teachers without CBA Use</td>
<td>-</td>
<td>3 (25%)</td>
<td>-</td>
<td>3 (25%)</td>
<td>-</td>
<td>6 (50%)</td>
</tr>
<tr>
<td>TOTAL of teachers Overall</td>
<td>16 (48.5%)</td>
<td>3 (9.1%)</td>
<td>3 (9.1%)</td>
<td>3 (9.1%)</td>
<td>2 (6.1%)</td>
<td>6 (18.1%)</td>
</tr>
</tbody>
</table>

The final three groups were by far the most common for teachers, as there were six teacher episodes in the NYY profile, eight teacher episodes in the YNY profile, and 19 teacher episodes in the YYY profile. As it was stated earlier, the vast majority of teachers proposed instruction that was aligned with their CBA learning goal. Therefore, it was not surprising to find such large representation in these groups, as all three groups share in common that they chose aligned instruction to their CBA goals.

The YYY category was by far the largest (19 teacher episodes), and can best be characterized by the complete alignment with a CBA LP perspective. For the sake of clarity in this discussion, this group will be referred to as *Consistent CBA Level and Goal*. The NYY and YNY groups together constituted nearly the same number of teachers (14) as the ‘*Consistent CBA Level and Goal*’ group, and are best characterized
by their close alignment with a CBA LP perspective, but imprecise leveling or skipping of a reasonable level for the student thinking. Even though these groups are fundamentally different (NYY initially leveled at least partially inconsistently with CBA but then followed a CBA LP perspective, while YNY initially leveled consistently and then skipped a level when determining a learning goal) they ended up at the same point after determining a learning goal. Those teachers who improperly leveled the student in this category all leveled the student too high. Pairing this initially overstated level with a ‘next level’ approach would lead teachers to determine an overstated learning goal, ending at the exact same place as those teachers who skip a reasonable level. Therefore, these groups will be considered together in considering instructional decisions, and for clarity this group can be referred to as ‘Overestimated CBA Level or Goal’.

Comparing these two main groups, 1) Consistent CBA Level and Goal, and 2) Overestimated CBA Level or Goal, based on their instructional plans can help to distinguish the main differences that arise from different diagnoses of student needs based on the CBA framework. In consideration of the verbalized instructional intentions of all of these teachers, the pattern and tendency for those in the Overestimated CBA level or goal group was to propose instructional activities that focused on the actions of the teacher, and was often characterized by teacher demonstrations or explanations of a concept. In total, 10 out of 14 teachers (71%) in the Overestimated CBA level or goal group proposed instruction characterized by demonstrating a strategy for the student, explanation of a standard or expanded algorithm, or encouragement of a strategy that is not targeted at meaning-making. The remaining four teachers (29%) described more
student-centered approaches that emphasized encouraging students to engage with specific mathematical tasks. In some instances of teacher-centered demonstrations or explanations, teachers made claims that they would try to explain how the students’ strategy related to the described strategy. In contrast, the Consistent CBA level and goal group had 16 out of 19 teachers (84%) propose instructional activities that emphasized engaging the student in some mathematical activity, use of manipulatives to model an idea, or attempt at a different type of problem. Only three of the 19 (16%) articulated instruction that emphasized teacher explanations, demonstrations, or modeling of a new strategy.

It should be noted that of the 19 teacher episodes included in the Consistent CBA level and goal group, 16 (84%) represented teachers who had used the CBA MD materials, while only three (16%) had not. For those teachers in the Overestimated CBA level or goal the data were nearly opposite. Out of the 14 teachers episodes, 9 (64%) involved teachers who had not used CBA MD materials, while the other five (36%) had used CBA MD materials. These data suggest that teachers with CBA MD experience were far more likely than those without experience to demonstrate CBA aligned actions throughout the determination of CBA level, CBA learning goal, and instruction. It also suggests that teachers without experience with the CBA MD materials were far more likely to skip a level, or focus instruction far above a reasonable CBA aligned target for the student.
Offered instructional plans that were student-focused, driven by student actions

<table>
<thead>
<tr>
<th>Used CBA MD</th>
<th>No CBA MD Use</th>
<th>Used CBA MD</th>
<th>No CBA MD Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teachers in Consistent CBA Level and Goal</td>
<td>15 (78.9%)</td>
<td>1 (5.3%)</td>
<td>1 (5.3%)</td>
</tr>
<tr>
<td>Teachers in Overestimated CBA Level or Goal</td>
<td>2 (14.3%)</td>
<td>2 (14.3%)</td>
<td>3 (21.4%)</td>
</tr>
</tbody>
</table>

Table 19 – Instruction Type versus Learning Goal Chart

One possible explanation for this stark contrast in proposed teacher strategies is that by targeting instruction at levels well beyond the current thinking of a student in the CBA framework, like those in the Overestimated CBA level or goal group, it can be difficult to conceive of how to engage the student in the new strategy because it is drastically different than their current thought process. Since skipping levels and targeting ambitious learning goals can often constitute a stark difference in thinking, teachers in this group may have felt the need to propose instruction that emphasized demonstration and direct instruction to first model a different type of thinking for the students in the hopes that they could internalize the strategy. This could also help explain why those in the Consistent CBA level and goal group might have more commonly chosen student-centered tasks that emphasized the student engaging in some task to help them move forward. Because the proposed learning goals were far less ambitious, and more represented only slight modifications and advancements in the students’ thinking strategies, teachers in this group might have found it easier to build off of what the student was doing.
Another possible explanation for the different results between the two groups could be that those in the *Overestimated CBA level and goal* group might have chosen a more advanced level that would be unlikely for students to learn without explicit demonstrative instruction. For example, a teacher who plans to skip levels based on distributive property, place-value based decomposition, and move towards the traditional multiplication algorithm might wonder how to help a student learn to use the traditional algorithm without explicit demonstration. Use of the traditional algorithm is not something that a teacher could expect a student to spontaneously learn on their own by engaging with different manipulatives or various structures of multiplication problems. For teachers who were in the *Consistent CBA level and goal* group, this would not be as likely to influence their instruction, as none of the samples of student work used to collect this data involved students at the highest CBA levels for MD.

It should again be noted that prior use of the CBA materials appeared to be a factor. Out of the 20 teacher episodes that demonstrated student-centered instructional choices, 17 (85%) involved teachers who had prior experience with the CBA MD materials. Additionally, out of the 13 teacher episodes that demonstrated teacher-centered instructional choices, nine (69%) involved teachers who did not have experience with the CBA MD materials.

*Instructional Suggestions from Teachers with Consistent CBA Level and Goal*

Teachers in the *Consistent CBA level and goal* profile were much more likely than other teachers to choose instructional tasks that promoted student engagement, and/or activities that explicitly built off of the way a student was currently thinking about
multiplication. This section details several examples of what this looked like in the data, with commentary to bring attention to specific components of each teachers’ discussion about instruction

Teacher T6 – QR Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>

Figure 56 – QR Student Work

T6: This student wrote 40x5 is 200 and 6x5 is 30 so it is 200 +30 is 230. Um. Well they broke it apart. Did 40x5 and they knew the fact. So here we are – they are operating on known facts. Um. Just kinda wonder – let me read the 2 partial products in 4.1. (Reads) I’m gonna think that they did the 2 partial products. Because it looks like exactly what he did. He took the 46 and broke it apart. He solved 40x5, so he has to be higher in level 3 because they know the fact, and they went up to the 2 partial products. So I’ll have to go with the 4.1.1

I: What CBA level do you think this student should move to next and why?

T6: I think I would give them a problem in which it could be broken into 4 partial products and see how they did with the two double digit numbers. Just to see how they did, and if they were successful in doing that…and if not, we could work on that. And then I don’t know if we would need to move this student to 4.2 or 4.3 – They’re pretty sophisticated where they are – but just as they move along…sometimes kids come up with these on their own. We could see if he/she could do this on their own, or if I had the time, I should say I would try to move them into 4.2 or 4.3. Just by giving them examples. Probably with these kids because they are so sophisticated, I think some of them could look at an example, and see if they could turn it into their own. Like I use the strategy and ask if they can use it.

I: We’ve gone from 4.1.1. – so what would be your next step?
T6: To go to 4 partial products. 4.1.2. And see if they could be successful there. Because it is a little more confusing, and there is more and more room for error.

I: So What instructional activity or task might move this student to this next level?

T6: I think this is just one where they just kind of trial and error, and just practice it. I mean, they already know to break it into 2. So for them to look at a 4 partial, to turn it into 4, look at task 13 example (in CBA document). Can they know to takes the tens and multiply by the ones for each group? Would they do that? And if not – then we would kind of dissect the problem apart and talk about it. This seems like the kind of student you could do this with. They are at a higher level of sophistication already. Just kind of let them try it on another problem, and let them talk about it.

T6 articulates clearly that she believes QR is operating at the 2 partial products level and that as a short-term goal, would want to move the QR to 4 partial products. T6 mentions more than once that ‘sometimes kids come up with these on their own’ in reference to students moving towards using more sophisticated strategies. T6 plans to provide QR with a 2 digit by 2 digit multiplication to see if QR could apply a 4 partial products approach, T6 wants to engage QR in more mathematical thinking to ‘Just kind of let them try it on another problem, and let them talk about it’. This is an approach that focuses very much on the student engaging and thinking about their own mathematical strategies, and a teacher who is working to better assess and interpret a students’ thinking. T6 does mention that for instruction beyond the current goal for QR, she may choose to show QR these new strategies ‘Just by giving examples’. This is in line with the teachers in the Overestimated CBA level or goal group that determine ambitious learning goals and then provide instruction that is more in the form of demonstration.
Teacher 2 – SX Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX:</td>
<td>12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.</td>
</tr>
</tbody>
</table>

Figure 57 – SX Student Work

I: What level of reasoning of sophistication?
T2: It is not a recalled fact. It is not a partial product. I would say level 2.4.
I: What type of reasoning do you think the student should move to next?
T2: Let me think here. I am sorry I change this to 2.3. Which is to skip count all. By skip count sequence. I would probably move them to level 2.4 next to group them into groups.
I: What would you do instructionally to move this student to this next type of reasoning?
T2: I would begin by having them build multiple towers and looking for patterns within that multiple tower. Maybe just use the same one here it is kind of easier to find patterns when you have more numbers involved so I might have them extend this even further. Or even use the six instead of the 12.

Once T2 settles on the proper level and learning goal for SX, instruction that emphasizes student actions are provided. T2 even suggests that using the same problem would be appropriate, but to use ‘multiple towers’ to look for patterns to help encourage a slightly different approach to skip counting. T2 even suggests making the problem involve the number 6 instead of 12, possibly to make the numbers less intimidating for a student to work with skip counting groups of groups.
Teacher T10 – SX Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX: 12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.</td>
</tr>
</tbody>
</table>

Figure 58 – SX Student Work

T10: CBA Level: MD2.3.1
I: What makes you think that this is the CBA level?
T10: I think this level because the student is skip counting and keeping track of the 12’s on his fingers.
I: What CBA level do you think this student should move to next? Why?
T10: I think the level SX could easily move to MD2.3.2 because he/she could begin clustering the 12’s into groups and then add them.
I: What instructional activity or task do you think might move this student to this next level?
T10: I have noticed with my own students they begin using pyramids to cluster and then add the larger amounts. SX could do 12+12, 12+12, 12+12

| 24 | 24 | 24 |
| 48 | 24 | 60+12 |

Although T10’s instructional suggestions are not especially explicit about the exact nature of the instruction, it is clear that T10 would encourage SX to engage in using pyramids and clustering strategies to find the values of larger amounts. T10 explained that she has noticed her own students begin to use these strategies when they add larger amounts, and this is used as a target for what SX could engage in. Much like T6 and T2,
T10 does not emphasize teacher actions, but rather the actions that SX would need to engage in to make progress and move forward.

*Instructional Suggestions from Teachers with Overestimated CBA Level or Goal*

Teachers in the *Overestimated CBA level or goal* profile were much more likely than other teachers to choose instructional tasks that focused on teacher actions and/or activities that were not targeted at promoting sense-making for the student. This section details several examples of what this looked like in the data, with commentary to bring attention to specific components of each teachers’ discussion about instruction.

Teach T2 – RR Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>46 × 5</th>
</tr>
</thead>
</table>

RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.

Figure 59 – RR Student Work

T2: Ok. Well I was looking up here. So I would say (RR is at) 3.2.1.

I: What CBA level would you move him to next?

T2: Well. With the understanding that was shown here in the breaking apart – I would say it’s time for the traditional algorithm.

I: And why?

T2: Just because they showed me the understanding of breaking this number apart the place value is there – because he says ok I’m going to break it into something easy 20. Or just for example if they said 40×5 is manageable for me, I understand this is 40. Then place value is sound – I’m ready to go with the algorithm. The concept of multiplying is there because the place value, they’re able to separate the place value.

OK. So what types of instructional activities or tasks do you think might move this student to the next level?
T2: Um. If I was to pull this student who is at an MD 3.2.1 to try and get them to MD 4, the person already says what he’s doing. So I guess I might, since they already did this (points to student work) – I might just say ok. We’re going to carry the 1 up here. I would probably show them. I would probably do a problem and have them watch me. Or I might even do a problem without them watching me and ask them – what do you think I did here? Can you explain to me what I did here?

I: Ok. Have them explain what you did after you did it?

T2 proposed skipping a level in moving QR to the traditional algorithm from a 2 partial product approach. To help QR learn the traditional algorithm, T2 suggests that he would have QR watch him perform the algorithm or actually perform the algorithm and then ask if QR can make sense of it. This approach is different than many of the approaches discussed earlier because it emphasizes what the teacher needs to do to either demonstrate or explain the strategy. T2’s instructional suggestions are not completely without student engagement, as T2 asks questions of the QR based on the teacher-demonstrated computations. This type of approach emphasizes students internalizing another’s approach as opposed to adapting what makes sense to them to a new context or strategy. This type of instructional approach might make building off of QR’s understanding of place value and multiplication difficult for QR, but could also serve as an opportunity for QR to adopt another’s strategy.
Teacher 24 – Sally Problem

[used CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = ______</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 60 – Sally Student Work

I: What would you do instructionally to move Sally this next type of reasoning?

T24: I would actually set up the traditional algorithms and talk her through it.

I: So how would you do that?

T24: Let’s say we would write down 45 times 23. So I would say Sally, in the past you were thinking about 3 going into 40 and then 3 going into 5. So instead of having to do all the additional writing over here let’s condense our writing down and not have to do as much side work. I might say this {points Sally’s solution} is safe way to do it and you know that you are working it out, but now that you know this let’s not write this all out. Let’s try to do a lot more of it just with our thinking. So if we start with the ones place and we do the 3 times 5 groups we would get 15. I always tell the kids that there is not enough room for two kids in one square. So we would leave the five here and then we would carry the ten groups of one over here so now we have ten groups, but we would just write as 1 cause we are over in the tens place. And kind of talk her through it like that.

In this situation, T24 proposes skipping to the traditional algorithm for Sally, which constitutes a large jump from Sally’s actual level. T24 verbalizes that she would set up the traditional algorithm and talk Sally through it. Although T24 does appear to make some effort to relate the traditional algorithm to Sally’s approach, the emphasis is on teacher demonstration. The language that T24 uses emphasizes concepts such as
carrying that might be difficult for Sally to connect to her strategy. T24 identifies Sally’s method as a ‘safe way to do it’ but believes that Sally should not write everything out. T24’s frequent use of the term ‘we’ while articulating instruction indicates that T24 would be heavily involved in the thinking that occurred during instruction. Again, this varies quite starkly from the typical instruction offered by teachers with aligned CBA level and goal.

Teacher 9 – SX Problem

[no use of CBA MD materials]:

<table>
<thead>
<tr>
<th>Task.</th>
<th>There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?</th>
</tr>
</thead>
<tbody>
<tr>
<td>SX:</td>
<td>12, 24, 36, 48, 60, 72 [raising one finger after each number]. So, there’s 72 players.</td>
</tr>
</tbody>
</table>

Figure 61 – SX Student Work

I: Is SX’s reasoning correct or incorrect? If it is incorrect, what is wrong with it?
T9: I think it is correct and I like the… that is a pretty sophisticated strategy. So what level is it? Umm… well they’re skip counting. So I would say it is a level 2.3 where he is skip counting.
I: What type of reasoning do you think SX should move to next?
T9: Well I would say level 2.4 skip count groups of groups, I don’t think that is necessary. I am assuming that is like 2 twelve’s.
I: Yeah.
T9: I don’t know if that is really… If he can do this, if he can skip count by 12’s I would say that you are pretty ready to go into just knowing the facts. That is what I would say. See that is where I… and you can’t answer this I know, to me recalling fact is I guess I don’t think of that as much of a strategy, but I guess it is strategy.
I: What would you do instructionally to move SX to this next type of reasoning?
T9: I think just drill and practice. You know recalling facts is not a lot of strategy. It kind of depends on how much understanding SX actually had with other skipped counting kind of things. If this is just a fluke which counting by 12’s is not really fluky. I mean if you can count by 12’s like that.

T9 correctly chooses the CBA level for SX’s reasoning, and even identifies the next level in the CBA framework. However, T9 does not see the value in attempting to move SX to this level, and identifies SX’s strategy as being sufficient to move beyond that level. T9 articulates that the level of thinking they hope to move SX to is not conceived of as a true strategy, and T9’s suggestion of using ‘drill and practice’ indicates that memorization of facts might be how the level is conceived of.

Topic C: CBA Teachers’ Use of CBA MD LP in Live Teaching Situations

In this section, CBA teachers’ use of CBA in three different teaching contexts are explored to help characterize how the CBA MD materials can be used by teachers to assess students, determine next steps for instruction, design instruction, and evaluate instruction. T6’s case is explored first. In this case, T6 reflects upon a videotaped CBA assessment that was conducted with a single student. T6’s analysis of the students’ thinking, and thoughts about plans for future goals and instruction are investigated. Following this is T8’s case, in which T8 reflects on a two brief teaching moments with students working on division problems. T8 discusses her teaching, instruction, and her struggles to put CBA MD to use in working with certain types of student thinking. Finally, T18’s case details T18’s reflection on a pre-assessment, instructional intervention, and post-assessment phase delivered to a class of students utilizing the CBA
materials. In T18’s case, the outcomes of instructional intervention are discussed based on the CBA materials.

One Teachers’ Use of CBA MD LP in Assessing a Student – T6

In this case, T6 reflects upon a CBA assessment conducted with a male, 5th grade student that was videotaped. The interview progresses as T6 and a member of the CBA research team watch a videotape of the pre-recorded CBA assessment between T6 and the student, with the author of the CBA materials present during the interview. The assessment consisted of seven CBA assessment questions, and after each question T6 and the interviewer discuss what they saw in the recording of the student’s work. The interview concludes with summary questions about the student’s thinking and future plans for work with the student. Analytic commentary is included throughout the transcript, and a case synthesis and summary is provided after the transcript concludes.

T6 Interview about CBA Assessment
[used CBA MD materials]:

I: Tell what CBA level of MD reasoning he used on each of the problems and explain how you decided what CBA level of reasoning he used on each of the problems.

T6: My favorite stuff…. (Discusses how long it took to complete an online CBA survey)…

I: Ok. I guess we could go ahead since I asked the question.

(Watches Video of Question 1)

Transcript of Video Assessment between T6 and Student

T6: I would like to know what you think while you solve these problems. So tell me everything you think as you do the problems. Try to think out loud. Tell me what you’re doing and why you’re doing it. I will also ask you questions to help me understanding what you are thinking. For
instance: if you say something that I don’t understand I will ask you
questions about it. I am videotaping this interview so I’m sure to record
exactly everything you say and do. Do you have any questions?

S: Uh uh.

T6: Ok. You may begin. Question #1 – I have 4 containers like this, which I
don’t have 4 containers – but pretend I have cups – there are 3 cubes in
each of those containers. How many cubes are there all together.

S: Um. (Thinking) 12.

T6: Ok. Can you tell me how you got 12?

S: Um. I thought about the boxes, and I knew there were 3 in each, and I
knew how to count by 3’s – so I counted 3, 6, 9, 12. (Student draws 4
boxes on paper and is putting 3 dots in each)

T6: Ok. So you counted by 3’s. How did you know to stop at 12?

S: Um. Because after I do 4 multiples of 3, I knew to stop here because
there’s only 4 containers.

1. I have 4 containers like this. There are 3 cubes in each container. How
many cubes are there altogether?

Figure 62 – T6 Teaching Experiment Student Work Question 1

(Video of Question 1 Ends)

T6: What do they call that – iterating composites? I know I can’t ask you that
– you’re not going to say. I just wanted to make sure though. (Searches
through the document). Do you have these all memorized?

I: No

I2: I don’t even have them all memorized!

T6: Oh, you don’t even have them all memorized – good! 2.1.1 verbal
iteration.
I: What document are you using?

T6: You mean what dates? I don’t have the front cover. (searches)

I: ‘cause I think you may have an older one. It should be 7.16.07

T6: I bet it’s older ‘cause I don’t even have the date. (Muddled discussion about which set of levels is most up to date. T6 is given CBA Author’s newest set of levels)

T6: So I started to say 2.1. Is there a verbal thing still?

I: The table of contents will help you. Tell me what he did.

T6: This student didn’t count by ones – so I know it’s not this level. So I just need to find out what skip counts all or parts.

(Reads document)

I: What level?

T6: 2.1 – uses repeated addition or subtraction. I’ll give him 2.1

I: What in the document helped you decide?

T6: Well he didn’t add though – so now I’m confused. Let me double check (Reads) No he’s 2.3 for sure – skip counts all.

I: Ok. And why?

T6: ‘cause he just listed the multiples…3, 6, 9, 12. And he knew to stop at 12 because he said in his head I think that’s the one he said he counted. So he basically just skip counted.

I: Was there an example in there that made you think it was 2.3?

T6: I keep getting confused with what he said and what’s in the document. He counted in his head.

I: Was there anything on this page that convinced you?

T6: I looked basically at the description – students iterate numbers…reciting the multiples. Basically at the description.

Once T6 settles into using the newest CBA levels, she works to determine exactly how to characterize the students’ thinking. Initially, T6 levels the student at 2.1 (uses repeated addition or subtraction), but then changes her mind after realizing that the student had not added, but rather skip counted. T6 clearly delineates between levels 2.1 and 2.3 by identifying that the student did not add, and therefore would not be a
demonstration of level 2.1. It should be noted that repeated addition and skip counting are often treated as the same in much literature, but the CBA MD framework recognizes them as different strategies and levels of sophistication.

Interestingly, T6 argues that she did not use the examples of student work to help make the determination of what level her student was working at (which was common by other CBA teachers), but rather used the CBA descriptions. This coincides with the earlier discussion about conceptual use of CBA materials. In order to deeply understand the delineation between levels that might appear similar, understanding the CBA text and descriptions about the levels can be quite helpful. T6 seems to be using CBA to help her distinguish between some of the more fine-grained details of the levels.

(Watches Video of Question 2)

Transcript of Video Assessment between T6 and Student

T6: Excellent. Ok. #2 – I have 20 cubes, I want to put them into containers so that there are 5 cubes in each container. How many containers do I need?
S: 20 cubes. There would be 4 containers.
T6: Ok. And how do you know that there would be 4 containers?
S: Because if you have 20 cubes and you go by 5, 10, 15, 20 – there would be 4 containers and there would be 5 cubes in each.
T6: Ok. Can you just write the number 12 up here too?
S: Yeah. (Writes number 12 next to pictures of boxes on problem 1)
T6: So you’re just going to draw pictures even though you understood how to do it without pictures, right?
S: Yeah. (Draws 4 boxes and makes 5 dots in each). And I should write 4.
2. I have 20 cubes. I want to put them into containers so there are 5 cubes in each container. How many containers do I need?

Figure 63 – T6 Teaching Experiment Student Work Question 2

(Video of Question 2 Ends)

T6: I should have asked him what the 4 meant. He just wrote a 4 there and I just didn’t think of it.

Ok. Well I think again he just did the same thing I believe. He just skip counted all. Well he just started with just saying 4 containers. And I said how did you know and he said 5, 10, 15, 20.

I: Well his first answer was 4, so how do you think he got that so quickly?

T6: He counted out loud and knew to divide, and counted by fives because he knew there was four containers. I wonder if he would – is there a ‘he just knows the facts’ level? (Reads) Recalls a derived – facts. I guess that would probably be the first answer.

Well he did use reasoning, I mean it was meaningful rather than just rote. He was able to do 5, 10, 15, 20. So it was different than the first one. Well that’s kind of strange, that he can divide higher than he can multiply.

Ok I’m going to give it the level 3.1

In discussing the student’s solution to the problem, T6 once again noticed her student’s tendency to skip count. T6 identifies that her student initially said ‘4 containers’ as the solution, and the justified selecting ‘skip counts all’ as the CBA by recognizing that the student verbally said 5, 10, 15, 20 to justify the answer. The interviewer probes T6 by following up on this statement and asking how T6 thinks the
student might have arrived at the answer so quickly. T6 reconsiders her skip counts level, and wonders if the student used reasoning that involved recalling facts. In this case, both of T6’s answers could be correct about the students’ thinking, as a recalled fact might have been known by the student, while skip counting was clearly demonstrated in justifying the solution. T6 finds it especially interesting that the student might have demonstrated a higher level for division than multiplication, in finding this strange it seems that T6 is potentially a little skeptical about the level choice of level 3.1 (uses derived facts) but still maintains that this is the level she is choosing for this problem. T6 also identifies that the students doesn’t just simply recall a fact rotely, but can justify it with meaning. This is an important distinction in the CBA levels, and provides addition evidence that T6 understands many of the conceptual components of the CBA MD materials.

(Watches Video of Question 3)

Transcript of Video Assessment between T6 and Student

T6: Ok. #3 is 6 x 8.
S: Um. If you count by sixes – eight times. 6, 12, 18, 24,…(quiet)…and then 24, 32, 38, 44?, forty, no 50.
T6: Ok.
S: I think that’s it.
T6: So 6 times 8 is?
S: 50?
T6: Ok write down 50. And you counted by sixes…
S: eight times
T6: How did you know to stop at 8? How did you know you wanted 8 times that one?
S: I knew I had to eight times out because if it was farther than 8 then I would count farther than 8. I would stop at 8.

T6: How did you know when you got to 8 though? When you hit 8 times 6.

S: I was counting in my head while I was counting this.

T6: Ok. So you kind of did 6 and then thought 1.

3. \[6 \times 8 = 50\]

Figure 64 – T6 Teaching Experiment Student Work Question 3

(Video of Question 3 Ends)

I: What was your last question to him

T6: I said did you count by 6 (1) and then 12 (2). He really struggled and it was interesting ‘cause he started out with 12, 24, and 38…so I don’t know if he was trying to count by 2 sets of 6’s and got confused. Then he switched to 6’s. So do I go with how he got his answer or how he started? By trying to count by 12’s 4 times. He started out by saying 12, 24, 38. ‘Cause he was completely baffled. And then counted by 6’s and messed up in the 30’s. ‘cause he might have been nervous.

Hmm. So where does that put him. I guess he did repeated addition…No. What if he got it wrong? Gosh. We need like osmosis for these levels so they sink right in! I just looked at the algorithms to remind myself of what they looked like. I guess it would be 2.3? Skip counts all again. No he just has incorrect multiples but that’s what he was doing.

I: What helped you decide that?

T6: I was thinking back to his previous examples. I don’t know with the incorrect answers…is this the document with correct and incorrect algorithms…oh there I see it. But did it used to be divided into correct and incorrect? But I think it is 2.3.

In this account, T6 appears to have initially either misheard or misinterpreted the students’ approach when she suggests that the student might have started counting by
12’s. Based on the video, it is clear that the student counts quietly by 6’s from the beginning, but it is possible that the student’s counts were so quiet that it seemed like the student was trying to skip count by groups of groups. T6 focuses on the incorrect final product of the student work, but eventually settles in on level 2.3 ‘Skip counts all’ as the level.

T6 seems to once again recognize that the student is in fact using a skip counts all approach, but wants to be able to identify that the student has some incorrect multiples in the skip count sequence. This is a valid point, considering that the student does not seem to make any errors until moving from 24 to 32 (when it should be 30) in the skip count sequence. T6 attributes additional confusion to the student by saying that he got completely baffled in trying to count by 12’s, and had to restart and count by 6’s. However, T6 appears to misinterpret the students’ actions based on her recollection of the entire interview. Later in the interview, in problem 4, the student does in fact struggle with counting 12’s, and it appears that T6 might be conflating the student’s responses to questions 2 and 3 in her analysis of this problem. However, T6 does ultimately settle in on the most reasonable CBA level for the student thinking, and even identifies that although level 2.3 is the best description of the student thinking, there is also an error in the skip count sequence that leads to an incorrect answer.

(Watches Video of Question 4)

Transcript of Video Assessment between T6 and Student

T6: Ok. So you kind of did 6 and then thought 1. Ok. Go ahead and flip the page to #4. #4 – a carton contains 12 eggs. Emily has 5 cartons. How many eggs does Emily have altogether?
S: (Student mumbles while re-reading portions of the problem). She would have 12, 24, 12, 24, 12, 24, thirty? (student sounds confused or uncertain)

T6: You’re going to count by 12’s? Is that what you were starting? I’m sorry…

S: Yeah. Um. I could count by 5’s

T6: Mmhmm.

S: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65…65?

T6: Think!

S: (restarts skip counting) 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60. She’d have 60 eggs.

T6: Ok. So how did you know to stop at 60? Because at first you stopped at 65 and then you tried it again and stopped at 60.

S: I was counting by 5’s and um, I was counting how many times I did it…and I realized that I went one too far.

T6: Ok. So, you kinda did it in your head, each 5.

S: Mmhmm.

T6: So 60 eggs. Ok. Good.

4. A carton contains 12 eggs. Emily has 5 cartons. How many eggs does Emily have altogether?

60

Figure 65 – T6 Teaching Experiment Student Work Question 4

(Video of Question 4 Ends)
T6: He tried to count by 12’s again. Then he stopped. He can’t get past 12x3. I should work on 12’s with him. Well he did start – what was that one called – skip count by parts. He started with 2.4 – skip counts part, groups of groups, and regressed to 2.3 skip counts all. Yeah – he started at 2.4 counting groups of groups, and then regressed to 2.3. Do you want me to write just one?

I: You can write both.

T6: I just think that it’s interesting to see…I mean he knew to count by 5’s instead of just by 12’s. That was interesting.

I: Was he keeping track on his fingers?

T6: That’s the one I couldn’t tell.

Problem 4 again shows that T6 misinterpreted the student to have tried to count by 12’s on the previous problem. This misinterpretation does not interfere with T6’s recognition that the student struggles with getting past 12x3, and that he could benefit from specific work ‘on 12’s’. The misinterpretation does; however, seem to confuse T6 into believing that the student is once again trying to execute a level 2.4 ‘counting groups of groups’ approach. Such an approach would be demonstrated by a student combining skip counts (e.g. – in this case instead of 12, 24, 36, 48, 60, a student might say ‘I know two 12’s is 24, and twice that is 48, and one more 12 is 60’) which is not demonstrated in this case. However, it could be that T6 misinterpreted the students’ approach on problem 3, believing that the student had combined skip counts to try and count by 12. T6 might be conflating ‘counting by 12’s’ with level 2.4 or skip counts groups of groups because on problem 3, ‘counting by 12’s’ would have been demonstration of this level.

Regardless, T6 once again properly recognizes that the student is utilizing a strict ‘skip counts all’ approach. T6 even recognizes that it was interesting that the student demonstrated difficulties with skip counting by 12’s, but understood that they could
count by 5’s instead. This demonstrates pretty flexible use of level 2.3 by the student, and implicit recognition of the commutative property (i.e. – 5x12 = 12x5). Note that T6 does ask a question in addition to the CBA assessment task to inquire about the student strategy.

(Watches Video of Question 5)

Transcript of Video Assessment between T6 and Student

T6: So 60 eggs. Ok. Good. #5 – Mary has 84 cookies. She wants to divide them equally among 4 people. How many cookies does each person get?
S: Well. 84 divided by 4 is…well I would divide 84 divided by 4 people and it would equal…um…24?
T6: You’re allowed to write stuff out.
S: Yeah
T6: If that would help.
S: Ok. So 84 (writes 84 and division symbol and 4) divided by 4, equals. (Student appears to try to use a written algorithm). I’m thinking it is 24.
T6: You’re thinking 24? How did you do that math problem? Just now
S: Well – I was just thinking about how 24 divided by 4…no, I mean 84 divided by 4…I was counting down and I got into the 60’s in there, and I knew it was somewhere around 66 or 62 or something. And then I went down farther I got into the 40’s, around 46 and then I went down and I got into the twenties and I got 24.
T6: So you counted backwards by 24’s?
S: Yeah.
5. Mary has 84 cookies. She wants to divide them equally among 4 people. How many cookies does each person get?

(Figure 66 – T6 Teaching Experiment Student Work Question 5)

(Video of Question 5 Ends)

T6: I have no idea!!! I mean he started out with the incorrect algorithm with the way he did the problem. And then he started talking about counting backwards. So I think he did the incorrect algorithm first Alg1, and then he took that answer and did the skip counting thing or something and counted backwards. Yeah. It’s like he found the answer inaccurately and then skip counted to justify it.

I: So you’re saying Alg1. And then did you look back in the document?

T6: That’s what I was thinking – the majority of the thing was that it is incorrect.

I: Is there anything in the document that would help you decide for sure?

T6: (Reads) Just says they incorrectly perform. I think I’m going on memory…from what they did in the past. When they incorrectly perform algorithms.

I: But did he write everything down?

T6: Well he wrote down the problem 84/4 which he didn’t even use, I’m not even sure how he solved that. ‘Cause he said he got 24 before he even counted backwards. So he kinda did a combo of the incorrect and the 2.3 with the counting. He justified his answer by using 2.3

On problem 5, T6 struggles with exactly what the student did. Initially, T6 thinks that the student must be using an incorrect algorithm (referred to as Alg1) based on the
student’s written work. T6 identified that Alg1 was her initial selection based on her previous understanding in using the CBA materials, and that Alg1 related to incorrectly performing algorithms. Upon being prompted by the interviewer about what exactly the student wrote down, T6 subtly identifies that the written work does not necessarily seem related to what the student used to solve the problem (of which T6 is understandably uncertain about). From the video, it appears that the student writes the problem down in some form of written algorithm. Since the student does not appear to use what is written in any way, it could be that the student has been exposed to some numeric strategies that he can remember visually, but does not fully understand. Therefore, the actual writing of the problem could be simply be a recognition of what has been shown to the student.

T6 is very confused by this particular problem, and this could easily be because the student’s work and justifications do not seem complete enough to clearly, and neatly fit into a single CBA level. The students’ main justification for how he solved the problem is “Well – I was just thinking about how 24 divided by 4…no, I mean 84 divided by 4…I was counting down and I got into the 60’s in there, and I knew it was somewhere around 66 or 62 or something. And then I went down farther I got into the 40’s, around 46 and then I went down and I got into the twenties and I got 24.” The student seems to be using some sort of estimation combined with counting backwards. How the student ultimately arrived at the number 24 is not completely clear in this justification, but T6 appears to identify that the core of the student’s thinking seems to be built off of skip counting, whether it was precise or imprecise/estimated. T6 definitely appears to want to characterize both the correct and incorrect characteristics of the student’s thinking, and
this even takes the form of multiple CBA levels. As it was unclear in the student
explanation how he arrived at his answer, T6 should have asked about how the student
counted. The fact that T6 didn't is inconsistent with the CBA approach, as it is important
to have a detailed picture of the student’s cognition.

(Watches Video of Question 6)

Transcript of Video Assessment between T6 and Student

T6: And the last page. #6 – is 10 x 6.
S: Well. 10 times 6, if you just multiply ten by six it would equal 60.
T6: Ok. And how do you know that for sure?
S: Um. Well. 10 times 6 would be if you just keep going 10, 20, 30, 40, 50, 60. Stop at the 6 and there is your answer.

6. 10 \times 6 = \underline{60}

Figure 67 – T6 Teaching Experiment Student Work Question 6

(Video of Question 6 Ends)

T6: So it was basically he memorized it. ‘Cause he said 60 and then he
justified his answer. So is that 3.1 Yeah. Recalls derived facts.
I: Ok

Question 6 is straightforward for T6, as she recognizes that the student’s response
was a recalled fact. But T6 also articulates that not only did the student say that the
answer was 60 from memory, but also that he could justify it. This is an important
characteristic of the CBA framework, that level 3.1 ‘recalls derived facts’ is not simply
about memorization, but is also must be meaningful. This is additional evidence that T6 uses the MD framework in a very conceptually aligned fashion.

(Watches Video of Question 7)

Transcript of Video Assessment between T6 and Student

T6: Ok. #7 – 20 x 8.
S: I have four, hundred and…40, 60,…(mumbles), 100…(mumbles)… 160.
T6: And how did you get to that?
S: I counted by 20’s up to 100 and then I just added three more twenties and I got my answer.
T6: Ok.

7. 20 × 8 = 160

Figure 68 – T6 Teaching Experiment Student Work Question 7

(Video of Question 7 Ends)

T6: So that’s all the farther we got. He was interesting ‘cause it’s like he knew 20x5 and knew that he needed 3 more groups of 20. But did he say that 20x8 is 160?
I: I don’t think so.
T6: Broke it down into a couple of different ways. (Reads) I think that might be 2.4 Because he didn’t just. I’m thinking it might be a combo of 2.4 – skip counts groups of groups…well no. Or he recalls…hmm.. skip counts parts, let me check that out. (reads) I think it’s the skip counts parts. ‘Cause he went to 100 and needed 3 more groups and I think did he even say 20, 40, 60? So I would say skip counts part.
I: So what on the page is helping you decide that?
T6: I just read the explanation and just skimmed through the examples.
I: Is there an example that helped you when you skimmed through it?
T6: Not really. ‘Cause he was single…I mean double digit. He didn’t really say plus, and I don’t think that’s much of a. They don’t all say plus in here. Um. He just said I knew I needed 3 more groups of 20. So the skip count parts. 2.2

Question 7 is interesting, because up to this point T6 has been pretty accurate in her assessment of her student’s thinking. However, T6 seems to veer away from what the student did and said in question 7 to conclude that they knew 20x5 was 100 and possibly that 20x8 is 160. This is potentially the case because the student mumbles quite a bit in the video, making it somewhat unclear that he is once again skip counting all. In the video, the student does seem to miss skip counts, potentially saying them inaudibly or thinking them, giving T6 the indication that the student knew 20x5 is 100 and then skip counted on. It could very well be the case that the student did know this fact, but the justification that the student gives is that he ‘counted by 20’s up to 100 and then I just added three more twenties and I got my answer.’ This is very much indicative of the general pattern of reasoning, utilizing skip counting all. In this case, T6 does not seem to have a good feel for the student’s thinking, and therefore ends up categorizing it at a level that does not make as much sense. T6’s justification for why level 2.2 is chosen does not seem to be related to the CBA document’s explanation of the level chosen (2.2), but rather based on the fact that T6 does not believe the student is demonstrating the key features of other CBA levels. This episode is consistent with the previous discussions related to teachers having difficulty with more complex examples of student work.

I: What CBA level do you think instruction should try to move this student to next?
T6: He’s mostly 2.3. So I would start with 2.3 and talk to him about skip counting and see if he really needs to use it. Because he wrote on a paper from class that 6x8 is 48, but here verbally with me he had to…So I would probably give him similar tasks. To see if 2.3 is his real level or nervous level because he was on tape. Even moving him to 2.4 skip count groups of groups – because he tried to do that. Maybe he needs to learn his facts and that’ll bump him up into level 3. Yeah ‘cause with 3.3 and 3.4 those would be the clustering types of ones. (searches) yeah – those are the clustering so that’s where we have to teach in 5th grade so I’d like to eventually get him into the 3.3’s 3.4’s which can eventually lend itself to the expanded algorithms. It would be nice to move him to level 4 but we’d have to see

T6 clearly has an overall understanding of what the student is doing, and identifies that over the course of the CBA assessment the student demonstrates mostly 2.3 level thinking. T6 offers CBA aligned goals for instruction, identifying that she would first want to give him additional tasks that were similar to make sure that the added stress of being on camera didn’t impact his thinking. This is a reasonable thing to wonder, as it would be expected that students that are nervous or anxious might demonstrate different thinking strategies than they would under more normal conditions.

Although there is only minimal evidence from the videotaped assessment that the student was ever attempting to skip count groups of groups or skip count by different numbers than the original two (Student says I counted by 20’s up to 100 and then I just added three more twenties and I got my answer), T6 perceives the student to have attempted this type of thinking. This clearly weighs on her plans to move the student forward, as she identifies skip counting groups of groups as a goal of instruction because ‘he tried to do it’. Even though this might be an example of a situation where a teacher overstates a student’s use of a strategy, it is clear that T6 is trying to build off of where
she perceives the student to be operating. In this case, she perceives evidence to indicate that the student is possibly ready for skip counting groups of groups because it appeared to her that the student already attempted to do this, but is doing so incorrectly.

T6 even articulates that the student could work on learning facts, and this was something that T6 mentioned in other parts of the interview as well. T6 had identified that the student did know some facts with understanding, but needed help in learning more. In considering long-term learning goals, T6 relies on the curriculum for the 5th grade instead of where the student is. This makes sense, as T6 employs a LP consistent view on short-term goal setting for the student based on what was demonstrated in the assessment, but a long term goal was determined more by curricular expectations.

I: What kinds of instructional activities do you think the student needs?
T6: I need to look at the facts – does he know them well or not? If he knows them well – you would think he wouldn’t have to skip count. I would check that. Instructionally, with similar tasks with different numbers, I would see if he is familiar with his multiplication facts – try to get him to solve them in different ways. Like 20x8. He knew that 20x5 is 100, but he didn’t know his 12’s. So it is kind of confusing with him. I’d like to see where we could start and find his comfort level. Give him similar tasks, and see if I can get him to move into the groups of groups or levels 3’s – but I’m not sure how I could do that without working with him first. It’s almost kind of strange how he started with different levels.

T6’s instructional goals align with her learning goals and the CBA framework in attempting to move the student to either level 2.4 or 3.1. T6 first articulates that she believes he could benefit from instruction addressing knowledge of facts. T6 rightly assumes that if the student knows certain facts well; that he might not need to lean on skip counting so heavily. T6 uses an instance where she perceived the student to have
known 5x20 is 100 as evidence that he might know certain facts but not others, such as facts with 12’s. These are, however, different types of facts. T6 states that she wants to see if she can get him to level 2.4 skip counting groups of groups, but does not know how to help the student make this jump.

I: What additional did you learn about students thinking about MD by working with students as opposed to just reading the documents? (6:30)

T6: You get more familiar with it. I was just thinking the iteration – he’s counting by. I’m familiar with the document I’ve gotten better. I remember last year thinking about how you move them up a level – and thinking I have no idea. And now I can use tasks to help figure out where they are. But by practicing and talking to them about the tasks you can tell if they are moving up. Even though I’m not sure what instruction to do.

T6 identifies that making sense of the CBA MD framework has taken time, and that identifying levels for students has become something that she feels more comfortable and confident at doing. T6 believes that she can work with students, with CBA tasks and assessments to identify their level and determine if they have made progress based on the CBA framework. This is an important addition to T6’s assessment knowledge, as it gives her a different way to assess and interpret the sophistication of her students’ thinking. T6 does, however, identify that conceiving of instruction is much more complicated for her. She may be able to tell what level a student is operating at, or if they have progressed a level, but is still trying to make sense of how to use instruction to explicitly help students make this progress. Although the CBA materials do include instructional hints for helping move students from level to level, T6 is either unaware of these, has not used them, or does not reference them in her discussion about instruction using the CBA MD framework.
Brief Case Synthesis and Summary

Overall, T6 appears to effectively use the CBA descriptions of levels to make sense of the student’s work. She uses written information in the CBA materials that helps to delineate between potentially similar looking reasoning strategies, providing several instances of evidence indicating that she was able to use the CBA materials to more deeply understand the conceptual components of the student’s thinking. T6 levels the student quite accurately, and although she slightly misinterprets certain actions of the student, they do not seem to outweigh her conceptions of what exactly the student knows how to do. The only time that T6 appeared to mis-level her student was a case in which she based the determination of the level on what the student was not doing instead of they were doing. In this instance, had T6 read the description of the CBA level, she might have changed her mind and determined that it was not an appropriate level.

T6 clearly attempts to base her learning goals and instructional plans for this student based on the general tendencies that were demonstrated during the CBA assessment interview. Although the student did not appear to demonstrate instances of skip counting by groups of groups, T6 perceives this and intends to try to move this student to this CBA level because ‘they are already trying to do it’. According to T6, her effectiveness at leveling students took roughly one year to hone, and she admits that she still struggles to think about how to move students to more advanced levels. However, even though instruction does not seem to be a component of the CBA MD framework that T6 has fully made sense of, her ability to properly level students afford her with the
ability to assess her own instruction by utilizing the CBA MD framework to assess her students’ learning.

One Teachers’ Use of CBA MD LP in One-on-one Teaching with Two Students – T8

In this short case, T8 works with two students, individually and at separate times, to give the students a CBA assessment question and work to help them make steps forward in their understanding. The transcripts of T8’s conversations with student 1 and student 2 are provided, followed by transcripts of T8’s conversation with the CBA author about the short teaching experiments. These cases are intended to demonstrate a teacher’s reflection on using the CBA materials in the context of a true teaching experience. In addition to initially asking a CBA assessment question, T8 asks follow up questions to try and probe each students’ thinking or to try and deepen or elaborate their current conception of a mathematical idea. In this case, the transcripts of T8’s teaching experiments are separate from the transcripts of the reflective conversation with the CBA author because T8 and the author talked with the tape of the student work happening in the background, making it very difficult to clearly display the conversations simultaneously in writing. Analytic commentary is included throughout the interviews between T8 and the CBA researcher.

Teaching Experiment – T8 with Student 1
[used CBA MD materials):

T8: The first problem I want you to try – I want you to write on here real big. One hundred and sixty eight (168) divided by twelve (12). I want you to
solve that whatever way you think to solve it, and then we’ll talk about what you did.

S1: So can I start?

T8: Yeah – go ahead and start.

S1: (Student writes 168 / 12 and splits up the 10 and 2 for the 12.) (Student then reaches for manipulatives)

\[
\begin{align*}
168 \div 12 &= 8 \\
168 \div 2 &= 84 \\
168 \div 10 &= 16 R8
\end{align*}
\]

Figure 69 – T8 Teaching Experiment Student1 Work part 1

T8: You can use anything there that you want.

S1: (Grabs 3 tens sticks, and a hundreds block, then grabs 3 more tens sticks) (Writes 168 / 2 = 57 AND 168 / 10 = 16 R8) (Student then writes 57 + 16 R8 = 73 R8)
Figure 70 – T8 Teaching Experiment Student1 Work part 2

T8: Ok. So your answer is? The 73 remainder 8…or?

S1: This. So added those together to get.

T8: Ok. So you took 168 and you divided it by 2, and tell me why you did that.

S1: Because the 2 in the twelve. And then I divided by 10. So then it’s 12. And then I got the answers and I added them together. And since you can put 10 in 8, with the 168. I put the remainder 8. And then I added them together and added the remainder 8 and I got 73 remainder 8.

T8: Ok. So you took this twelve and you broke it up into this 10 and 2. And that’s what you did on there. Ok what I’d like you to try right here…let me just draw a line right here (Draws a line underneath S’s work across paper).
T8: A very squiggly line. What I’d like you to try for me is to tell me something times 12 that you know right off the bat that’s a basic fact for you. And we’re trying to get to landmark numbers.

S1: Um. (Writes 8x12).

T8: And what is 8x12?

S1: 32.

T8: Try that again. Remember we talked about in class breaking this number apart into 10 and 2, and doing 8 times 10 and 8 times 2? What is 8 x10?

S1: 80.

T8: See you can tell it’s gonna be higher than 32. Did you say 32?

S1: (Nods)

T8: What is 8 x 2?

S1: 16.

T8: Ok so you had 80 and 16. How much would that be?

S1: 92.

T8: Um. Ok. You write it down so you can...uh...
S1: It’s 96 (writes 96). I think I added it up wrong.
T8: Ok so we’re at 96 and we need to get up to 168, so we still have some room to go. Because this shows us where to go (points to the 168) and this shows us how much we’ve used (points to the 96). Right?
S1: (Nods)
T8: So what else times 8 do you know that we can try next? And Keep in mind we’re at 96 and we have to get to the number 168.
S1: 11 - I know it’s 8 higher than that. So can I just do that?
T8: Sure.
S1: (writes 8x11 = 88)
T8: Ok. So if multiplying 8x12 gives us 96 and 8x11 gives us 88. What can we do with these numbers to see how much we’ve used?
S1: Add ‘em.
T8: Ok.
S1: (Adds 88 and 96 to get 184)
T8: Ok. So we have 184. But we need 168. So what happened there?
S1: We used too much.
T8: We used too much! Alright, so what else could we try then? You can put it down at the bottom here.
S1: 8x9.
T8: Ok. And what is 8x9?
S1: (uses fingers) 62.
T8: What are you doing with the fingers?
S1: (mumbles seemingly about configuration of fingers that tells her 8x9)
T8: So if I’ve got these fingers down. How many fingers do you have up?
S1: 7
T8: And how much are these worth?
S1: 70.
T8: Ok 70. And then?
S1: 2.
T8: Ok. So let’s put the 72 down there…and then let’s put the 96. Let’s try that one.
S1: (Writes 96 + 72 = 240).
T8: And what did you get?
S1: 240.
T8: 240? Alright, let me take a look at it.
S1: (Crosses out work) (redoes work). 168.

![Image of Student's work]

Figure 72 – T8 Teaching Experiment Student1 Work part 4

T8: Ok. Now that this number matches this number now. So what do we do here now to get our answer?
S1: Um. We add how many numbers you need. We add the...
T8: Ok. Tell me what you did here. So we had 96 and 72 and then..
S1: That was 168. And then I said that 8 times 12 we multiplied 8 times 12 and 8 times 9, so I added the 12 and the 9 and then I got 21.
T8: So which answer do you think is more accurate?
S1: The 21.

**Teaching Experiment – T8 with Student 2**

T8: I have a problem written on the paper that I want you to try to solve. A hundred sixty eight divided by twelve. And what I would like you to do is to use any way that you can think of to solve it, any type of pattern that you want to use. And I even have cubes here if you would need to use those for any reason. So here’s a marker. And let me have you go ahead and work it out and then I’ll ask you to explain it to me.

S2: Can I start.
T8: Yeah. Start.
S2: ( Writes 100 / 2 = 50, 50 / 2 = 25, 18 / ) (Mumbles) Ah 2. Uggh. (Crosses out work).

(Then writes 100 / 12 = 120 / 12 = 10 48 / 12 = 4.) (Uses fingers = Writes 14).

Figure 73 – T8 Teaching Experiment Student2 Work part 1
T8: Ok. Will you point to it an explain it as you’re thinking this through.

S2: I was trying to get down to like 10 (points to 100/12 line) but I knew I couldn’t do 100 divided by 12. So I did 120 divided by 12 is 10. And then 48 divided by 12 is 4. And then So I added the 10 and the 4 together and got 14.

T8: Ok. Now why did you add the 10 and the 4 together?

S2: Because If I did just 120 divided by 12, then each person or whatever would have gotten 10. And then I did 48 divided by 12 each person would have gotten 4. But 120 plus 48 is 168.

T8: And that was our original problem.

S2: Yes.

T8: Now let me have to try…let me draw a line right here (Draws vertical line next to S2’s work). On this side I want you to take a look at 168 divided by 6. ( Writes 168 / 6). And I want you to take a look at what you’ve just done and see if there’s any that what you’ve just done can help you solve this next one.

S2: Ok. (Quickly writes 120 / 6 = 20, and 48 / 6 = 8 28)
Figure 75 – T8 Teaching Experiment Student2 Work part 3

T8: Alright.
S2: What I did was. Since you asked me if I could figure it out with using that. So I did 120 divided by 6 but since I knew it was twelve…I could just double the 10 to 20.
T8: Ok
S2: And I did the same thing over here. I did 6 and doubled the 4.
T8: And then you added those two together.
S2: And I got 28.

**Interview Regarding Teaching Experiment – T8 and Student 1**

T8: Here’s (S1). She’s one of the students in the higher section…but she’s probably one of the one’s towards the bottom. She struggles a lot with understanding what we’re doing. And this is a problem that she missed.
T8 and I Watch Video of S1 – Tape is running in the background as T8 and I talk)

T8: It was very interesting what she did. This was very interesting because I had those out for her (base ten blocks) but she wasn’t quite sure what to do with them. We’ve used them. So this was. With her work there is just no understanding whatsoever, even as far as. I’m looking at this thinking that she doesn’t even understand the place value of it, or how this should all work back to 168.

I: Well she broke it. The 12 into 2 and 10. (Laughs).

T8: Right.

I: But that’s not very useful in this situation.

T8 clearly demonstrates that S1’s thinking is without understanding, and that even though she attempts to use base ten blocks to help represent the problem S1 does not seem to know how to use them help her think through the problem. T8 points out that S1 does not even seem to understand place value concepts, but the author points out that although S1 might make decisions that are not helpful for division – that there is evidence of place value understanding of some kind.

T8: Her explanation was interesting. The 57???. So after her explanation I’m at a point where I was thinking ‘What do I do with her’? (Laughs). So I guess I said ‘Let’s just get rid of what you did’ (draws squiggly line).

I’ve been really trying to hit with all the kids, landmarks. And using that terminology to have a base from which to work with. So I thought she’d go with 8 times 10. Because most of the kids are doing that.

T8 demonstrates the challenges of working with a student that struggles to demonstrate understanding. T8 did not see much evidence of work with understanding, or a strategy that made sense. In the teaching experiment, T8 asks S1 several questions targeted at getting her to explain her reasoning, but unfortunately this does not seem to
help T8 make sense of S1’s work in a helpful way. It is very possible that T8 did not see obvious elements of a CBA level of reasoning with S1’s work, as S1 did produce solutions that seemed to be based off confused attempts to implement a numeric strategy without meaning. T8 identifies that in order to help S1 make progress, she decided to just draw a line below her work and ‘just get rid of what (S1) did’ and essentially start by working with landmark numbers. This is an attempt to move a student that is clearly confused to an alternative strategy that T8 is hoping might be accessible to the student. While this move is consistent with a CBA LP perspective, unfortunately T8 inadvertently changed the problem from 168 divided by 12 to 168 divided by 8. This mistake possibly occurred out of T8’s attempts to reformulate the problem to be simpler for S1. A teaching move that might have been more appropriate would have been for T8 to demonstrate for S1 that the ‘distributive property’-like strategy she was using was incorrect. However, T8 might have deemed this to be an ill-advised approach given that S1 was very confused, and a counter-example of her strategy might not have made sense. T8 seems to be taking the approach of trying to get S1 to use a different strategy, which does seem to be something that S1 needs. The difficulty arises in whether or not the new strategy that T8 chose will help S1 make progress.

I: Yeah.
T8: So we stress with them that facts should be something that you know right away. So if 8 times 12 is something you don’t know – then that’s not the one you should start with!
I: Right, right right.
T8: (In reference to T8’s comment in the video ‘Remember we talked about in class breaking this number apart into 10 and 2, and doing 8 times 10 and 8 times 2? What is 8 x10?’)

That was one where I thought. I don’t know. Is it ok to show her that or should I have gone to just making her start down to a lower number like 8 times 10? I don’t know. I think I’m thinking too much.

I: Yeah.

T8 reflects on her decision to suggest a strategy (in this case, 2 partial products multiplication) upon S1 as she is working to solve the problem 8x12. T8 identifies that S1 is not truly using a recalled fact if she doesn’t already know the product of 8x12, but then suggests that S1 complete the multiplication by doing a 2 partial products approach. This approach is consistent with CBA. Because S1 did know the facts for 8x10 and 8x2, a 2 partial product approach was the best strategy to suggest in this case for finding 8x12. Even though partial products multiplication represents a higher level in the CBA MD levels of sophistication, T8’s handling of basic facts and the number 12 is quite well aligned to CBA. The CB MD framework does not consider 12’s as basic facts is, as treating 12’s as such appears to be inconsistent with research. In fact, if memorizing 12’s is considered to be a basic fact, one way to make it a meaningful fact is by using a 2 partial products approach. So in this case, a bad curriculum choice to consider factors of 12 as basic facts would invert the LP sequence, changing it to 2 partial products for simple problems before 12’s basic facts. T8 wonders if her choice to show her this strategy was effective or not, hypothesizing that it might have been better to just focus on 8x10. This suggestion does seem to be aligned to a CBA LP perspective for helping a student use recalled facts to compute.
T8: (After asking S1 in video if there is anything else she knows that multiplies by 8)

So she does 8 times 11. So my question to her…led her to go to 8 times 11 cause that’s another one she knows. But she wasn’t thinking of we’re already at 96 and we need to get to 168. So she just went right to here (points to 8x11).

T8: She’s another student that tested out at 130 – but her gift isn’t in math. But this was just like painful trying to get this out of her.

(Pause – Watches ending of the student explanation)

Did she actually come up with the answer? She wrote 21. Hmmm. It wasn’t 21? She should have put 8 plus 9…it should have been 17? Huh?

I: 168 divided by 8 that’s what this is.

T8: Right. Oh!!! that’s because I told her to use 8. That’s my fault!!!

Because for most of the other kids the problem was to find 168 divided by 8. OH! I did that to her. Well then this isn’t her messing up on the bottom. Up at the top I don’t account for – but the bottom. So this changed to 168 divided by 8. Oh my gosh! Ok.

T8 appeared to have expected S1 to take into consideration that the goal of the strategy that T8 was suggesting was to build up by 8’s to 168. It is not surprising that S1 does not necessarily take this into account, because S1’s original strategy was not based on a build-up strategy for division. Unfortunately, S1 does not seem to understand the goal of the new strategy that T8 is suggesting.

As T8 reflects upon the ultimate answer that she guides S1 towards, she wonders if S1 even got the correct answer. T8 is confused by the answer that is given (21), but the author of the CBA levels helps T8 realize that she had actually changed the problem from its original formulation of 164 divided by 12 to 164 divided by 8. In trying to help S1 move towards a different strategy, T8 inadvertently changed the actual problem that S1
was working on. T8 understands the consequences of this mistake, and proclaims that

‘That’s my fault…I did that to her’. In this case, either accidentally, or because of initial
confusion about how to apply a different strategy to the original problem, T8 attempts to
shift S1’s thinking away from a particular strategy and in the process completely changes
the problem.

(T8 Stops Tape After S1 completes problem)

I: So what do you think?

T8: Well. Wow. She’s up here for the division. I think the main reason she
did this divided by 2 and divided by 10, she’s trying to break down the
numbers like we did for multiplication. And we’ve talked about how
multiplication can be used to solved division, so I think she’s trying to use
that strategy here. Um. I also know she sits beside another little girl that
in order to do this would do 168 divided by 2 and get the answer and then
divide by 2 and get the answer, and take that and divide it by 2…and she
comes up with the right answer every time. I don’t know how she knows
when to stop. I was going to ask you what has she figured out that I don’t
know. So I think that’s one of the reasons she started with this. ‘Cause I
know that she has sat by her for the last two weeks and Casey just says
‘just divide it by 2 right away’ …which at first is what I thought she was
doing. So I thought that was a good sign though because she was trying to
find something that she could work with. Unfortunately she broke down
the wrong number. Which it would have been better had she tried to do 60
divided by 12 or something divided by 12 that she knew. And here again
(points to the bottom half of the paper) I’m saying what do you know what
times 8 do you know. So I’m trying to talk her through this. So is my
talking her through this showing her this strategy without her really
getting an understanding of why she’s really doing it?

I: Well. In this case. I would say probably yeah.

T8: Mhmm.

T8 reflects on what led S1 to approach the problem the way that she did, and
identifies two potential sources for S1’s attempts to apply a strategy without meaning.

T8 states that she has demonstrated a two partial products approach to multiplication,
which is very similar to the approach that S1 applies to multiplication and could be a
source of confusion for S1. T8 even identifies that decomposing numbers in the order
that S1 did it was the source of the mathematical problem. In spite of the fact that S1
appeared to not be considering the meaning of her actions mathematically, T8 identifies
that the splitting up of numbers was evidence of ‘a good sign though because she was
trying to find something that she could work with’. T8 might have benefitted from
asking S1 why she chose to split up the 12 into 10 and 2 for the division to help
determine if the decomposition was in fact meaningful for S1 or a strategy for
multiplication that was inappropriately being applied to division. T8 even wonders if in
the teaching experiment, her demonstration of a new strategy for S1 was likely resulting
in S1 completing the problem without any new meaning of why she’s doing it. The CBA
author agrees that this is likely the case based on the evidence from the student work.

I: See I guess I don’t understand how she’s…like the 168 divided by 12.
When she did this (points to decomposition of 12 to 10 and 2), I guess that
suggests to me that she’s not – she doesn’t clearly have in mind what
dividing by 12 really means.

T8: Mhmm.

I: Whether it’s like 168 divided into 12 equal groups – how much in each
group? Or how many times can I subtract 12 from 168? Or what number
times 12 gives me 168? In some senses if you sort of. At least for me –
when I think of those meanings, when I get to this (points to
decomposition of 12 into 10 and 2) I don’t get how this is going to help.
And so that part seemed to be missing with her

T8: Right.

I: She seemed to go off using a strategy but without keeping in mind ‘where
am I really ultimately trying to get?’

T8: Right.

I: And that comes from the meaning of division it seems to me.
T8: Now in other problems I’ve given her. Where it is not as large. Like 55 divided by 12 – and this is so interesting with kids. How many can do it with this (points to smaller numbers) and now just because I made the first number larger...you know from 55 to 168 suddenly they cannot do it. You know there’s the ‘I can only do it with smaller numbers’. So then do they really understand what division is? So with this little girl – one of the things I would like to do is have more concrete materials. And having her demonstrate to me 12. Here is my 12 groups, and put them out. Or draw pictures. Because on the division problems that she’s done when the numbers are smaller and where I required that you must draw a picture. She really gets into the drawing. And her answers are correct.

I: yeah.

T8: So it’s. She’s not ready to move to this yet – she needs more of show me what that looks like.

I: Yeah. Yeah.

T8: So that’s what I need to do more with her.

I: And I think – I’m just guessing…but I would suspect that she’s seen a lot of sort of numeric strategies sort of floating around.

T8: Mhmm.

I: In the discussions. And she’s paying attention to those, but she hasn’t sort of put them all together.

T8: Right.

I: And she hasn’t related those to the meaning of division, which she might know.

T8: Mhmm.

I: I mean, she might actually know – When she thinks about the meaning of division she might know that. But when she gets the big numbers then she doesn’t think about the meaning of division. She thinks about ‘oh I’m supposed to use a strategy ‘cause the number is big’

T8: Right.

I: So she goes off and the strategy isn’t sort of supported by the meaning of the operations.

T8: Right.

I: But you know.

T8: And then she has a teacher that totally changes the problem on her. (Both laugh).
I: well I didn’t notice that at first either!!!

Here, the CBA author is trying to help advise T8 on how to try to get S1 to focus on something that might help her make progress forward: considering the meaning of division. T8 identifies that for smaller numbers, S1 can in fact comprehend the meaning of division, and does quite well at solving division problems with pictures. The CBA author helps T8 see that S1 may in fact understand the meaning of division, but might be distracted by a variety of numeric strategies for doing division. It is suggested that S1 might not even be considering the meaning of division for situations with larger numbers, but rather could be thinking that she needs to do the division with a strategy that is independent of meaning. In consideration of this conversation, a CBA aligned instruction for S1 might have been to encourage her to consider the meaning of division and to see if she adjusted her strategy.

T8: ‘cause I was like 168 divided by 12 is not 21…but she solved everything right. What did I do!?
I: But she did. If I remember correctly – she did add those together (points to the 12 and 9)
T8: Mhmm.
I: So there was some meaning at that point. I mean she did actually know. I mean she knew that to find let’s see. 8 times 12 and 8 times 9 come out to when you add those its 168. But really you were doing the problem 8 times 12 plus 9. I mean that shows a certain understanding of breaking apart multiplication.
T8: Right, Right.
I: So, I’m guessing that if she can make sense. I’m guessing that she can make sense of some of these numeric strategies for division – but she just hasn’t done it yet. I mean she has the mental capacity certainly.
T8: Right.
I: But, it is one of those things that I think that – like for multiplication, you can break apart either number.
T8: Right.
I: But for division – you can’t.
T8: Exactly.
I: So you know that’s one of those things.
T8: That’s an understanding the rule. Which one and which works.
I: Yeah. One of the properties. And it, I mean understanding the properties. And I think in this algebraic reasoning kinds of emphasis then – talking specifically about those properties. Like ‘when can you do this and when can you not do this’ would be something that – like that would come up anyhow.
T8: Right. Yes, I can see that coming up especially in their group – and talking about this. ‘Cause that’s one of those things that – I’d like to just bring this up to them.
I: Yeah. ‘Cause there are times – especially with division when like when I first looked at that (points to 12 decomposed into 10 and 2) I thought – well that’s wrong. But then I thought well why is it wrong? And then I thought – well give me a moment.
T8: yeah. And what is she doing?
I: Right. So, you can run into number properties that…and one of the things that people are doing more and more these days is in pre-service education, the mathematics courses for elementary teachers – they are sort of saying ‘well here’s an algorithm – a strategy that a student used: will this always work?’ You know – like some strange kind of way of
T8: Right Right
I: Multiplying numbers and will it always work. And I mean – if you see something…the first time you see some of these it’s like ‘woah – I don’t know, I have to think about this’. And it’s not trivial at all sometimes to figure that out.
T8: Right.
I: So the fact that kids can not know which one works and which one doesn’t makes total sense to me. ‘cause it doesn’t take…I mean if I see something strange then I have to think about too.
Here, T8 and the CBA author discuss how difficult it can be to truly understand some of the strategies that students use, and more importantly that it can be really difficult to know when certain strategies do and do not work mathematically. The CBA author identifies an instance in S1’s reasoning that provides evidence that S1 has at least a preliminary understanding of ‘break-apart’ multiplication, and that unfortunately for S1 this understanding doesn’t translate perfectly to division. T8 identifies that this topic is something that could come up with her students, and she thinks it is important to discuss with them when they can and cannot use break-apart strategies. T8 earlier demonstrated that she understands that breaking apart the first number (dividend) will work for division, but breaking apart the second number (divisor) as S1 did will lead to incorrect conceptions of division.

Overall, T8 demonstrates just how difficult it can be to provide help to students when their thinking is not well understood. In talking about this teaching experiment with the CBA author, T8 reflects about the types of questions that she asked and if they would or would not be meaningful to this student. Although particular CBA levels of sophistication do not come up in this conversation, T8 attempts to apply a CBA MD LP consistent line of questioning for the student. Unfortunately the application of this approach was complicated by the fact that the initial student thinking was quite confusing and complicated making it difficult to understand exactly what was wrong with S1’s thinking. This approach was also challenged by the fact that T8 inadvertently changed the problem because she was focused on making the decomposition accessible to S1. T8 was attempting to interpret a strategy that she had never seen before that appeared to be a
combination of numerical strategies and partial understandings of division. This was obviously a difficult teaching experiment for T8, and represents the challenge in building off of children’s thinking that can occur when children’s thinking is difficult to interpret.

**Interview Regarding Teaching Experiment – T8 and Student 2**

T8: Let me see who the next one is. ‘Cause I had them do the same problem. But this young man. He tried to use the landmark of 100. So he divided that by 2 and got 50, and 50 divided by 2..so he’s trying to use Casey’s strategy (repeatedly dividing by 2). But the he realized that ‘wait a minute – I’m not up here…this doesn’t make any sense’. And then he was like ‘well wait – this is 2 and now I need 12’. So then he realized this wasn’t going to work. He was trying to get an answer of 10, he knew he wanted to multiply by 10 because it gives you what you started with. So that’s how he came up with. I think he probably figured that 10 times 12 is 120 so divide by 12 and that’s 10. And then he used what was left of it to come up with the answer.

T8 recognizes the strategy that S2 initially attempts, and notices that S2 eventually decides to apply a different strategy. She recognizes that he is using known facts to derive an answer, and that he is also decomposing the numbers (168 into 120 and 48) to make the problems easier for him. In the video, after seeing this work, T8 asks the follow up question “On this side I want you to take a look at 168 divided by 6. (Writes new problem: 168 / 6). And I want you to take a look at what you’ve just done and see if there’s any that what you’ve just done can help you solve this next one.” By asking this follow up question, T8 asks a CBA MD LP consistent question because it builds off of the students current way of thinking, and attempts to encourage S2 to use a more efficient strategy that is based off his original thinking. For S2, T8 seems to understand the mathematics and the thinking far better than for S1. This is likely because S2 is using a more common, correct strategy.

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T8: And then he knew the next one right away…and this is in his explanation…that well if this 12 and it was 10, then this is 6 it has to be 20. ‘Cause this is twice. That’s really hard!

I: So wait a second. What did he do?

T8: Well I said – well then here’s 168 divided by 6. So can you use this information to help you solve this? And then he knew. In the video he describes how he though through that. So it was 28.

I: I mean – did he actually double these?

T8: Yep. I think that’s what he said. Let’s see.

I: That’s pretty interesting.

(WATCHES Student 2 in the background)

I: You’ve got to keep on your toes with the top kids. They come up with all kinds of neat things.

T8: They do and I love it. And I say – ok. Everybody stop. We’ve gotta figure out what this person did. And they are like ‘NOT AGAIN!!!’.

I: (laughs)

T8: Yes, Again!

T8: (When asking follow-up question in S2 Video) I think my questioning has gotten a lot better, too.

I: That’s a perfect question – that’s what I thought.

T8: That’s gotten much better through doing this.

T8 even seems to recognize that her follow up question was effective for S2 by identifying that her questioning has improved. The CBA author concurs, and T8 elaborates to suggest that her questioning has improved based on her use with CBA. Just in these two short teaching experiments, T8 demonstrates a willingness and ability to probe into student thinking in order to try to better understand what the student is thinking. Clearly, T8 struggles far more with making sense of S1’s work, and ultimately
struggles with her questioning and helping S1 make progress. This is not the case when T8 deals with S2 and a thinking strategy she is more familiar and confident with.

**Brief Case Synthesis and Summary**

Overall, T8 demonstrated the difficulty that arises when working with students when their thinking is not well understood. T8’s reflections with the CBA author reflect the fact that T8 knows her work with S1 was likely not to result in meaningful progress for the student due to mathematical mistakes, and a focus on a new strategy without emphasizing the meaning of the actions. T8’s short case shows how instances when student thinking is complex, unique, or containing misconceptions and rote procedural use, it can be quite difficult to build instruction off of student thinking. In attempting to get S1 to think about the division problem in a more accurate fashion, T8 imposed a strategy incorrectly that likely was difficult for S1 to see the value in. However, with S2, T8 seemed to readily understand how the student was thinking about the multiplication, and asked an appropriate, LP consistent, question to help further his thinking.

**One Teachers’ Use of CBA MD LP in a Classroom Setting – T18**

In this case, T18 sits down to discuss an episode of classroom instruction and examples of student work with a CBA researcher. T18 and the CBA researcher watch short clips of students working in T18’s classroom, and T18 discusses the CBA levels of the students. This case differs from the previous two because it represents an example of CBA MD materials being used to pre-assess students (on May 29th), implement instruction, and post-assess students (on June 3rd) to determine growth. T18 discusses several students’ work, but details more carefully two students’ pre and post test work on
two multiplication problems. T18 details her approach to a short week-long instructional plan, and how she aligned her instructional activities closely with the CBA MD framework, ultimately leading to positive results for these students. Analytic commentary is provided throughout the transcribed conversation between T18 and the CBA researcher, as T18’s perspectives on leveling students in the CBA framework, using CBA to determine learning goals, and utilizing CBA MD instructional hints to help move kids forward, are all investigated.

**Transcript of conversation between T18 and CBA researcher**

[used CBA MD materials]:

I: So, you are going to tell me what CBA MD level of reasoning each student used on each problem. And then each time that you say that a teacher used a particular CBA level, explain how you decided what CBA level the student used.

T18: Can I just run through here (referring to the video of students working in class) and just pick a kid and say this is what I saw this is what I think?

I: Whatever’s going to be very easy for you. Yes.

(T18 Starts Video of Classroom Teaching)

T18: This is some students’ work in the video. I don’t know which kid this is. This is the girl’s work that I have.

(Problem 37x10 - Done as 37x2 + 37x2 + 37x2 + 37x2 + 37x2)

Figure 76 – T18 Teaching Experiment Student Work 1

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T18: I don’t know if you could hear in the video what she did.
I: Why don’t you show me on her paper.
T18: On her paper. (T18 retrieves student work) So this is repeated addition which is level 2.1 according to the CBA document. She just took the number added it up and added that up and that’s how she got the answer. So, that would be – well what level was she at? That’s blatant! That’s right off the examples that are in here – 2.1 Called ‘level 2.1 – used repeated addition or some skip counting’…That’s totally it! And can I say…after our instruction, she was able to not use repeated addition but use the distributive property breaking apart the number 10 times 30 plus 10 times 7, and I don’t have this on video, but she was able to ditch her old strategy and use this more efficient strategy after giving instructional hints. To me that’s really cool.

Figure 77 – T18 Teaching Experiment Student Work 2

T18 selects a student’s work to analyze and assess based on CBA level. Based on the written student work, it is clear that the student is in fact doing some repeated addition to arrive at their answer. However, this student appears to be doing so in a fashion that is grouping larger groups together, and is not really using repeated addition to add the number 37 repeatedly, but rather 37x2 repeatedly. T18 references the CBA examples of student work to say that it is definitely level 2.1. T18 definitely has a correct conception of the student work.
T18 states that following instruction; the student was able to successfully use the distributive property to employ a 2 partial products approach. T18 does not provide video or written work as evidence, but if this student was able to move towards a 2 partial product multiplication strategy, it represents a significant jump according to the CBA framework for MD. Since the pretest and posttest were only a few days apart, there were only roughly 3 class periods of instruction, which is represents quite a significant gain over a short period of time.

I:  Yeah.
T18:  So that was a good one. (Restarts video to demonstrate another boy). This is too easy – he skip counted all!

![Image](image.png)

Figure 78 – T18 Teaching Experiment Student Work 3

T18:  I just wanted to make a point to get students to explain what those numbers mean (referring to 1, 2, 3, 4, 5…along the top), which is what the CBA author has right in his book – to help them keep track. So you know when you look at the level that’s skip counting by all which is 2.3
I:  Ok
T18:  And then after our instruction – this person was able to use this problem, and use the distributive property to break that 37 into 30 plus 7. So, what is that? That’s a whole level 2.3 to 3.3.
T18 displays another students’ work, and argues that it represents skip counting all (level 2.3) and even identifies that she has helped students deepen their understanding of this particular strategy by making sure to keep track of their skip counts (with numbers 1, 2, 3, … to help keep track of the number of skip counts). Again, T18 quickly identifies some of the key aspects of the students’ strategy within the CBA framework, and much like the previous student T18 states that this student moved to 2 partial products multiplication, a jump from 2.3 to 3.3. What is not clear at this point, is whether or not the instruction that these students received was focused primarily on the use of the 2 partial products strategy. Since two students in the class that began around levels 2.1 to 2.3 seemed to make progress up to level 3.3, it is possible that instruction focused on this strategy. Later in the interview, T18 discusses her approach to instruction more carefully. Although much of the actual instruction itself was not taped or observed by the CBA research team, T18 does not provide evidence that her instruction focused exclusively on 2 partial products multiplication, but rather indicates that instruction was targeted below level 3.3 (2 partial products multiplication) as well as at level 3.3.

8. 10 x 37 = 370

Figure 79 – T18 Teaching Experiment Student Work 4
I: What kind of instruction did you give these students you know that moved them up?

T18: Just looking at making them do problems together. Like I would make them do. Like we did the Candy Jar problems, we did several of those. And then we did like harder problems, like if I four groups of 25 – use what you know to figure that out. Let’s break this apart like this other person did…so I made them watch each others’ strategies and then I made them go through several of the CBA instructional hints and to try and say I know you can draw this with a picture but let’s look at the number. Let’s do some breaking apart with the number and actually walked through it with them to make them do it. So here’s a Candy Jar problem (points to work in front of her)
Figure 80 – T18 Teaching Experiment Student Work 5
T18: And I did another Candy Jar problem with a higher number... and I did the instructional hint about the 25’s. In tried to relate it to money (points to another problem)

Figure 81 – T18 Teaching Experiment Student Work 6

T18: So they figured that problem out kind of fast actually. So once you just kind of gave them a focus to look at the problem differently, then they were able to practice that and so I used the CBA instructional hints basically. I said ‘this is where you are at – let’s practice using and doing these kinds of problems’. The instructional hints I chose were based on trying to get them to move from...I did this problem sequence 1 (points to CBA instructional hints in CBA document) to get people to move from – moving to level 2.2 Instructional hints for students at level 2, to try and get them to move up and I just used the problems on this page and these problems on this page and I was going to even do some of this (points to successive pages in the CBA document). Does that make sense?

I: Mhmm.

T18: I really just truly used the instructional hints and then using actual student work and making them explain ‘well – how did you get that problem like this, and how did you get that problem like this’ (points to two different students’ work). You know what I mean?

I: Sort of like what we’re seeing right now in the video?
T18: Yes. ‘Cause this is the work they produced (referring to written work that is being shown in the video).

(Returns to watching some of the video)

(Stops Video)

T18: So that’s what I did for instruction – I made, I had kids present problems that were efficient and then I’d say ‘ok let’s do this one’. Then I called kids over to work with them.

T18 details the instruction that she provided the students, which appears to be heavily influenced by the CBA MD materials. T18 makes several references to using some of the instructional suggestions that were included in the revised CBA materials for MD. T18 articulates her approach to instruction being based on the fact that many of the kids were at or around level 2.2. Earlier, it was mentioned that T18 seemed to be employing a more global perspective on interpreting student work, by using bigger picture approach to identifying children’s tendencies in mathematical thinking. This seems to be a believable approach that a teacher would want to employ in a larger class setting as opposed to an individual CBA assessment.

T18 makes several references to the CBA materials, and also to asking students to build off of what they know. This represents a LP consistent viewpoint on learning and teaching, suggesting that T18 not only uses the CBA MD framework to help analyze children’s thinking, but also views learning to be built off of previous cognition. T18 also utilizes a variety of children’s thinking strategies in her teaching by allowing her students to present solutions to mathematics problems. The videotape that T18 and the CBA researcher are watching in the background represents a lesson that is primarily
focused around T18 calling up different students to present their unique strategies to MD problems.

T18: (Returns to watching video of student working on 49x10 - Student writes 40x10 is 400, 9x10 is 90. 400 + 900 = 490. In Video T18 asks if he meant to write it that way – asks him to rethink it. T18 is trying to get the student to recognize that they should not write 400+900, but rather 400+90)

T18: Some of these kids in the video I don’t have papers for.
I: Ok.

(Returns to watching video of students presenting solutions to different problems. T18 occasionally asks for explanations. When students made mistakes – T18 asked them to think about it again and fix it)

T18: So they’re all trying problems.

(Returns to watching video – T18 asks students to come up to the present their changes)

(In video – T18 asks: Suppose I have 4 containers with 3 candies in each container? How many candies do I have in each container? Show me how you would write it or draw it)

T18: This is where I was using the instructional hints.

(In video –T18 asks: I want you do this next problem based on how you figured out the last one. Now you know that 4 containers with 3 candies is 12 candies altogether, alright? So think about this – 4 containers with 20 candies in each. How many candies would there be? Students present solutions says he counted 20, 40, 60, 80)

T18: So that would have been 20, 40, 60, 80 – skip counting. For that particular student, that would have been um – Let’s see…I almost get these! 2.3 – skip counting all. So I think that was (BM). But she would have known that 4 20’s is 80 – she’s just giving me a picture. So in her work that I have – I don’t know if I have this on the tape.
T18: We did in class – 3 times 4…then 4 times 20, and then it was 4 times 23. Which is her problem right down here.

T18: So what I was trying to do was get them to use the fact that these were already broken apart – so they just need to add these together to come up with the 92 is what I was trying to get them to do. And so she said 23 times 4 is 92, and 92 candies. I’m not sure if she looked at the other parts or not – but this is the work that was being shown in the video. I’m sure
she can just do this problem – if she’s just doing it then that’s great…but you can’t tell from what she did. You know what I’m saying – you can tell what level it is without talking to her. Which I didn’t do this time. I didn’t sit down and probe and ask ‘what do you think about this – and what do you think about this?’. So this is a different way for me to think about doing an intake assessment. But for this one – it was pretty concrete because they were pretty. The discussion comes out that if you just add these two you can get that whole thing – so that kind of plays out in the tape of the class.

This discussion provides even more evidence that T18 is basing her decisions for instruction and helping kids move forward based on the CBA MD framework. T18 consistently makes comments to relying on the CBA MD framework, and appears to share a view on learning that aligns with the CBA MD framework. T18 represents a teacher who appears to have a good global understanding of the CBA levels for analyzing children’s thinking, a philosophy on teaching and learning that aligns well with a LP framework, and support through the CBA MD materials, levels and instructional hints. Based on the results that T18 has shared thus far in the interview, it appears that such a formula can lead to high impact teaching that leads to significant student learning.

(Re-Starts video. Video plays in background as T18 comments)

T18: This student is just explaining their reasoning. See she skip counted by 20’s which would be like a 2.3. Right? At this level? Skip counting is what most of them were doing…because that was comfortable. And then I think I gave them the 4 times 23 and this is where their discussion comes in.

(Student in video says that she did 23 plus 23 to get 46, and kept doing it several times)

T18: See so she skip counted still – she didn’t relate it to the other problems.
(Student in video says that since it was 23 times 4, he refers to the problems where it is 4 times 20 and 4 times 3, and says he just added them)
(Student in video skip counts to get to 4 times 23)

(Student in video seems to skip count by 20’s and 3’s on 4 times 20 and 4 times 3, 20, 40, 60, 80, and 3, 6, 9, 12. But the students says that he can just add those together for 4 times 23)

(T18 in the video says that this is exactly the relationship she wanted someone to discover! T18 demonstrates 4 x 23 numerically. Demonstrates 4 x 23 as 4x20 and 4x3.)

(Stops Tape)

T18: So basically that’s what I trying to get them to do. And then we went on to do other problems.

T18’s instruction and commentary is interesting at this point. The first student in this says that she counted by 23’s to get the answer for 4 times 23, which T18 correctly identifies as skip counts all. However, the second student skip counts, but in a different way. The second student skip counts by place value parts, first counting the tens and then the ones digits. T18 in the video states that this is exactly the relationship she had hoped student would identify, and writes down that 4x23 can be written as 4x20 and 4x3. Although the second student did demonstrate an understanding of the relationship that can exist between skip counting and place value, T18’s instruction is actually targeted at moving the student potentially completely away from skip counting and onto partial product multiplication. In this instance, the second student did relate place value and skip counting. T18 seemed to help students make the transition from skip counting by place value parts to decomposition by place value. In fact, T18 seems to mention parts of each of the following CBA levels: skip counting all, skip counting by place value parts, using derived facts (3x40 derived form 3x4) and 2 partial products multiplication.
(Restarts Tape – Student says first I knew 5x5 is 25 so I know that 5 times 6 is 5 more is 30.)

T18: That is CS’s work. And I have that right here.

![Image of student work showing 5x5 = 25 and 6x5 = 30]

Figure 84 – T18 Teaching Experiment Student Work 9

T18: After doing that one – we did another one with different numbers so for instance his work would have been. This was his he did 5 times 5 and he knew that was 25 (points to his work), and then 6 times 5 was 30 so he used a known fact to come up with that which is for that particular one ‘using known facts’ – 3.1 Ok. So then we did because it was 15, I was trying to get at 6 times 15 was what I was getting at – which was my ultimate problem. So then the second one is 6 containers with 10 in each, so he skip counted.

![Image of student work showing skip counting]

Figure 85 – T18 Teaching Experiment Student Work 10

T18: So this is – what did I say. 3.1 (referring to the previous problem). So his level…he said 60 first, but he justified it by skip counting. So skip counting all would be a level 2.3, and would be less sophisticated for this particular problem. If you skip count by all – that is 2.3. And then this was the question I was trying to get at: Can you use what you know about these two to come up. But he did 6 times 10, and then 30 more. So he did try it this way…and that way is let’s see – that would be level 3.3 ‘using the distributive property by place value to do 2 partial products’, right? Let me think. Uses number properties – using what he knows…he’s not
using what he knows, No. He broke the number apart. His spelling is horrible…But I don’t know that he could do that on his own – if he would choose to do that on his own. But for here – just for this problem, I would say he’d be 3.3.

T18 seems to recognize something important here; that students might sometimes be able to comprehend and use a strategy, but might not really think about problems that way on their own. T18 seems to identify that this student might be able to decompose numbers by place value and exhibit level 3.3 thinking, but that it might be more of a reflection on the way the problems were structured and the focus of instruction than it is a reflection of the way the student would solve the problem. T18’s approach to instruction seems to be focused on increasing students’ comfort level with more advanced strategies, and then giving them practice at using the more advanced strategies to help them internalize the strategies and get to where the new strategy is the preferred way to complete a problem.

T18: Anyway that was his thinking. With what I was trying to get them to do. And then if you look back on what he was doing (refers to posttest) – yes, he was doing that on his own…so I would say 3.3. A 2 digit times a 1 digit was his reasoning.

What else do you want to see? It’s pretty much all the same things. I mean, I went through this…you know what let’s just watch a little bit more.

(Restarts Video – fast forwards through parts)

T18: This is my student that draws everything out.

T18: You know they’re still, a lot of them, stuck on the skip counting. And I tell you that they’re doing the lower of the levels – but now they’re trying to do more advanced skip counting.

(In video – T18 demonstrates the connection between 6x5 =30, 6x10 = 60 and 6x15. Video has T18 asking ‘what is 12 groups of 25’?)

T18: So this is just me trying to hit home about the show the connection about what we were trying to get them to do. And then I moved on to this
question (12x25) and we talked about their answers. And they were like Uhhhh! So then I went to here (in Video T18 writes 4 groups of 25 is 100).

T18: Here they are skip counting in larger groups – which is more sophisticated than skip counting all – which is where they kind of were.

(T18 turns off video of girl explaining solution to a problem)

T18: I think that this girl does understand it but I haven’t spent enough time to sit down with her, and say, uh, yeah she gets it. But that girl (SF) would be an example of a level 5. Let me look. Here it says - Uses and understands the traditional algorithm. Well that’s her having it correct up there. And If I listen to it again – I think she explains it. That would be a level 5 I think. But let me look. Yeah. There is an example that kind of does that exact same thing. On page 42.

I: And who was that up there?

T18: That was SF. I have her’s. Here is SF’s work pre…and let’s see where that work was. Was that not SF? Maybe I called up somebody else up and that’s not SF. I thought I heard SF’s name in this example. Is it this one? (Reads). This would have been hers. This isn’t hers. I thought I had it – but it’s not.

(Fast forwards through portions of the video)

T18: I think that the video after this is just looking at this – all the different ways of doing this problem (60x40). This is the kid who skip counts – here is his work right here. So I was trying to get him to explain…but here’s a kid that painstakingly did skip counting. That’s amazing.

I: Oh yeah.

T18: If you’re cool – I think I’m done – the rest of the video is just more of the same.

Once again, T18 demonstrates that her analysis of teaching is very much based on the CBA MD framework. Earlier in the interview, T18 made a reference that she ‘I almost get these!’ suggesting that she is trying to not just understand the CBA levels with the documents in front of her, but also internalize the levels of sophistication. T18’s fluid conversations about the rough levels that students are operating at, and how these levels influence her instruction indicates that T18 has a fairly internalized conception of CBA.
She does not need to refer to the materials to still make decisions that are influenced by CBA.

I: Ok. Maybe if you can pick one student from that pile.
(T18 chooses the student who painstakingly skip counts)

I: Find some of the problems that you’ve done and (hands T18 a sheet to document CBA levels)

T18: This one I really…well let’s just focus on problem 8 and 9. For this student, let’s see CBA levels. For problem 8, he’s definitely a 2.3 because he skip counts all (PRE TEST on May 29th). Clear level. And then even on this one even on 9 (40x60) – he’s definitely, same kid, 2.3 because he is skip counting.

8. 10 × 37 = 370

9. 40 × 60 = 2400

I: And both of those are pre-test.
T18: Yes, Pre-test.

T18: And really I focused on these two questions for this assessment – to keep it the same. So then…there was instruction, so I’ve called this like the post-test. And the student how he took the numbers and broke them apart into the 2…um I don’t remember what it’s called, the partitive? Hmm…3.3 – uses the distributive property by place value to decompose the number into two partial products. Was able to do 2 partial products which is 3.3. I’m just going to write 2 partial products – for problem 8. For problem 9, though we went to 40 times 60 and a 2 digit times a 2 digit and I don’t know if it made a difference that one of them wasn’t 10, but he went back to skip counting but he did it with um, a larger number

I: Right

T18: So he didn’t skip count all. I wrote down 2.4 which is skip counts groups of groups. Which is a more efficient strategy than what he started out with. So 2.4 I’ll just put groups of groups. So, he was at least able to say that I know that 10 times 40 is 400, and I’m just going to iterate that 6 times to get my answer instead of doing it all by ones. The chunk and go. Which is improvement. So, those are the two questions that I went on.

Figure 87 – T18 Teaching Experiment Student Work 12

T18 provides an example of pretest and posttest data for a single student that demonstrates progress from a skip counting all strategy, to either a partial products...
strategy or skip counts groups of groups strategy that represents a shortened version of skip counting. This is also a good example of student that has been introduced to decomposing numbers but still relies on skip counting of some form to complete a multiplication problem. Over a few days of instruction, the student implements a decomposition strategy that shows significant gains, but also reverts back to skip counting using a slightly more sophisticated strategy, indicating that instruction might have helped the student rethink their approach to multiplication in a more efficient manner.

I: What CBA MD level would you want him to move to next?

T18: I think because this is such a short period of time…and TE made a big jump as far at least as from 2.3 to even 2.4 but also to see him consistently recall facts and use known facts to help him, instead of relying on the skip counting part…even if he doesn’t break numbers apart…I would like him to be solidly like on a 3.2 would be good. Instead of having to drop back and rely on skip counting. So I would like him to, because it was more efficient for him. I mean, he was all over the board and can do it with instruction, but I would like to see him at a 3.2 would be good. Which, again, is higher than where we expect them to go.

I: Oh yeah.

T18: So. Maybe shoot for a 3.2 consistently?

I: What kind of Instructional Activities does he need?

T18: More of this, more of looking at the numbers and breaking apart the numbers, or perhaps for this student practicing skip counting by larger groups. Saying perhaps 10 times 40 is this, is 400 and I can skip count this, and maybe being more comfortable with skip counting larger chunks, and then working on a 2 digit times a 1 digit and trying to just break those 2 numbers apart. I think between those two things, like the Level 2.4 and well maybe we’ll need to do a little ‘if you know this, then can you use what you know about this with this?’ So maybe, I didn’t use instructional hints for moving from 2.4 to 3.1…but if I were to check this out I would probably use those instructional hints from 2.4 to 3.1. (Finds hints) Well that may be something is moving 2.4 to level 3. (Reads) ‘they should be
encouraged to start memorizing the multiplication facts for numbers of 10 or less (which we already do) you can help by using games or flashcards. So I would be doing that also, and Then making him work with the numbers. If you know 4 groups of 25, how can you use that to find 12 groups. If you know 3 groups of 12 is, like those kinds of things.

T18 identifies that the evidence of this student’s post-test does not indicate that the student has already ‘achieved’ that level of thinking. Instead, T18 interprets the students’ work to indicate that although the student has shown some evidence of moving towards a higher level of thinking, the student likely reverted to skip counting because they are not solidly at a higher level. Children often times demonstrate several CBA levels on multiplication problems, but often are disposed to a certain type of thinking. It seems that T18 identifies that this student has made progress to a more sophisticated form of skip counting, but that her goal is to move them to being disposed to different approaches.

T18 also addresses instruction in the same way that she has throughout the interview, by relying heavily on the CBA framework. T18 even reads the CBA instructional hints and identifies certain instruction that she already provides, and additional ways to help the student move towards deriving known facts. T18 suggests instruction based on the CBA instructional hints ‘If you know 4 groups of 25, how can you use that to find 12 groups?’ This demonstrates that T18 can use the CBA hints not just as her sole means of instruction, but also create her own questions based off of the instructional suggestions.

I: Do you have another student that we could look at? Is that alright?
T18: Uh huh. Totally let’s do a girl. Let’s look at SF’s. Pretest, she was at – she was doing repeated addition ‘cause she took 37 and she added it ten times and she just kept doing that until she got her answer. So for the pretest (PRETEST on may 29th) it was 2.1 – repeated addition.

![Image of student work](image)

Figure 88 – T18 Teaching Experiment Student Work 13

T18: And then I’m just gonna stay with problem 8. If you look at the posttest…on this particular problem she takes the 10 times 37 and does 10 times 30 is 300 and 10 times 7 is 70 and then recombines and adds to get 370. So that is using the distributive property and the 2 partial products. So that was a 3.3 after instruction (POST TEST on June 3rd).
T18: Even though she’s vacillating in between the two…you see this was a skip counting, this was a skip counting, (points to other problems other than 8 and 9 on post-test). But that’s one where she tried and it worked for her and that’s really cool. So the pre-test for problem 9 she did the same thing – repeated addition, she very painstakingly – I mean she had to go onto a whole other piece of paper. And that was definitely the same – 2.1 for the pretest. And then for that same question (problem 9) – she ended up using a known fact. She said ‘I know that 4 times 6 is 24 and I know I need to add 2 zeroes so it is going to be 2400’ but she’s using a known fact to get her answer which is a 3.2 in the levels. So that’s moving from a 2.1 to a 3.2 using known facts. So I mean – there’s two kids that are operating at a 2.3 and a 2.1 that jump more than 1 level to a 3. Let’s see a 3.3, 3.2, and a 3.3, 2.4. So that one is still a jump – but they all made progress.

I: Ok. So what MD level would you like her to move to next?

T18: Probably for her, um, man…she’s like, she’s like where I would want her to be probably. She’s above where she needed to be. I would probably try to get her to use known facts. So to do the instructional hints from 3.2 – using what you know and practicing doing those so that this strategy is solid for her, so she chooses this strategy, and instead of reverting back to skip counting. That’s where I would want her to be.
T18’s analysis of the second student provides even more evidence that T18 recognizes that students do not completely discard their previous ways of understand a mathematical idea, and suggests that she can only truly believe that a student has achieved level 3.2 reasoning if they do not feel the need to revert back to skip counting. T18’s approach to determining a goal is consistent with the CBA MD framework, as it seems that T18 wants to build depth at the student’s current CBA level before trying to move forward.

Overall, T18’s approach seems to be that of using assessments to diagnose students, then using appropriate CBA instructional hints and student presentations of other strategies to help student move forward to a new way of thinking, and finally giving students experiences to help deepen their understanding at the new level. This is consistent with a CBA MD LP perspective, and the positive results that were found were attributed by T18 to the support that CBA provided T18.

I: What Kinds of instructional activities do you think she needs in MD?
T18: I would use this fabulous guide back here (referring to the CBA hints). Instructional hints for MD, specifically, play number games, memorizing facts. Looking for patterns as they solve problems for properly chosen sequence of problem. Like 5x6 is 30 and 6x10 is 60 so 5 times whatever. Doing those kind of problems to get at patterns to get them ready for the break apart.

I: Anything else before I ask the last part?
T18: No, I think that’s good. I was pleased you know. You know, it’s right there in front of you – I mean they did it, they made the jumps, they were showing you how they could skip count and do repeated addition which is not as sophisticated as where they ended up. No, I’m pleased – it works.

I: This is the last part – and what additional understanding of CBA MD ideas did you develop by working with students as opposed to just reading the CBA document?
T18: Working with students you can actually see that they rely on what they know. And at this level what they knew, for the kids that I was working with, they knew addition and they knew repeated addition. And they knew that they could use that strategy however inefficient it was to get to the answer they need. And, you know if you just show them the steps – then they can change that strategy. Um, And also, like I said, when I read the levels – I think ‘Oh yeah – you can get them from 2.2 to 3. No sweat’. But when you work with them like they need to just transition, some of them really little steps at a time and they cannot do that big jump. And even though you’re telling them – you know, do it this way, they are like no no no, I’ve gotta see it this way. And like that’s apparent in their work, and the one kid who went from the 2.3 to the 2.4 saw the difference, but still needed that skip counting.

I: So you saw that with a student, but would not have seen that with just the document?

T18: Right.

**Brief Case Synthesis and Summary**

T18 represents an interesting case of a teacher very heavily adopting the CBA MD framework into her work with students. T18 appears to have internalized a big-picture, global conception of the MD materials, and although her leveling of students is not poor, it is imprecise and is not fine-grained. This appears to be sufficient for her work with an entire classroom, as it informs T18 of the general thinking that is happening in her class, to help her select instruction to help move them forward. T18 effectively uses CBA multiplication tasks to pre-assess and post-assess students to determine if her instruction had impacted the students’ learning. Earlier, in T6’s case it was shown that assessing students in the CBA MD framework does not imply that appropriate instructional tasks can be created, but T18 is able to use the CBA instructional hints that are included in the CBA materials to design activities and provide instruction to help kids progress.
Chapter 5: Discussion

The overall goal of this study was to carefully investigate the general research question of how teachers understand and use information on research-based LP in mathematics to conceptualize students’ mathematical thinking and subsequently needed mathematical instruction. Thus, the results of the study are intended to provide a rich set of data that describe the particulars of teachers’ understanding and use of a particular LP for multiplication and division of whole numbers, the CBA materials. Included throughout this rich data set was analytic commentary intended to help elaborate some of the important ideas that emerged. Within this discussion chapter, these ideas will be explored more fully and expanded upon to help frame them within the greater context of teacher knowledge, and teacher development. This chapter begins with a short reflection on the research questions and their relationship to the data. Next, some of the key connections between the results and the literature are explored; delineated by research question. Finally, the chapter ends with a conclusions section that discusses what can be drawn from the research, as well as research limitations and future research implications.

Revisiting the Research Questions

The data on CBA teachers’ conceptualizations of student thinking and determination of instructional learning goals and tasks make it quite clear that LP’s can
be useful for teachers as an organizational framework for analyzing children’s thinking. However, it is also clear that learning and using research-based LP to analyze student reasoning, especially complicated cases, is difficult, but achievable by teachers to different degrees. The research questions for this inquiry were as follows:

1. In CBA2 teachers' analysis of student work, how do they conceptualize student reasoning on multiplication and division problems, and what relationships exist between CBA teachers’ knowledge and understanding of CBA Multiplication and Division materials, their conceptualizations of student thinking, and their determination of instructional learning goals and tasks?

2. While working with students using CBA Multiplication and Division in live one-on-one teaching experiments, how are CBA2 teachers’ instructional decisions informed by knowledge of CBA Multiplication and Division materials?

Clarifying Notes

Note 1. CBA2 teachers have learned about a particular set of learning progressions in mathematics, namely those developed in the CBA project.

Note 2. What clearly distinguishes Question 1 from Question 2 is that in the former context, teachers' are only making judgments about student work, while in the latter; they are making decisions in-the-moment of teaching or assessing.

The results section provided evidence that for the teachers involved with the CBA2 project investigated in this study, LP’s can become a meaningful component of their thinking about teaching. Data from the results section provided overall trends and
percentages of CBA teacher episodes in which teachers interpreted the CBA materials in varying ways, but also provided detailed cases of the particulars of teachers’ conceptualization of children’s mathematical thinking as well as use of CBA materials in teaching. Each research question will be analyzed further based on the results and relevant literature to help clarify the findings of the research.

Analysis of Student Work

The results of CBA2 teachers’ analysis of student work led to several valuable findings. In topic A of the results section, teachers’ conceptualizations of complex student work (Sally) were investigated. Generally, those teachers with experience using the CBA MD materials were more effective at developing accurate conceptualizations within the CBA framework. Also in topic A, several of the key challenges teachers have with terminology (such as ‘hidden place value concepts’, ‘algorithms’, and ‘partial products’) in the CBA framework were discussed. Especially in the case of ‘partial products’ it was evident that some teachers had incomplete conceptual understanding of the CBA materials and how they related to the student thinking. In topic B, a key finding indicated that those teachers with more accurate conceptualizations of Sally’s thinking were able to use CBA to inform short term as well as long term learning goals. Additionally in topic B, the data indicated that teachers with prior CBA MD use were able to conceive of far more LP aligned sequences (determining an appropriate level, setting an appropriate learning goal, and providing aligned instructional suggestions) than those teachers without past experience with the CBA MD materials.
In the first part of this discussion chapter, teachers’ understanding of children’s cognition is explored further. Next, the discussion of teachers’ understanding and use of CBA materials is explored in more detail. Following this is an overview of CBA teachers’ understanding of the CBA LP materials, including elaborations of areas of difficulty for teachers in the CBA framework. Finally, connections and relationships between analyzing student’s thinking and instruction are analyzed, connecting to prior research to the findings presented in this inquiry.

Cognitive Understanding of Children’s Thinking.

In CBA teachers’ analyses of Sally’s work on the problem 45x23, it was evident that there were many different ways that CBA teachers conceptualized her work.

<table>
<thead>
<tr>
<th>Task.</th>
<th>45 × 23 = _____</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sally:</td>
<td>45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.</td>
</tr>
</tbody>
</table>

Figure 90 – Sally Student Work

What was especially interesting was that many teachers struggled with interpreting Sally’s thinking due to a perceived mathematical/computational error in her work. When Sally says ‘45 times 3 is 120, plus 15’, several teachers focused in on the first part of Sally’s statement, and interpreted it to be a mistake because 45 times 3 is not 120. While it is true that 45 times 3 is not 120, this is not part of Sally’s conceptualization of the problem and does not represent an incorrect computation or mathematical step that she is making. Sally had simply broken 45 times 3 into two pieces 40x3 and 5x3 to compute and find 120 plus 15. What might have been particularly
challenging about this episode of student work for teachers was the nature of the task itself. Teachers were asked to respond to Sally’s thinking based on written transcripts of her work. Having written transcripts of student work might emphasize pauses in verbalizations that otherwise would not be challenging for teachers to interpret. For example, hearing a student say ‘45 times 3 is 120, plus 15’ might be very different than reading it and might help teachers identify that Sally sees the 120 plus 15 as connected and does not believe that 45 times 3 is only 120.

However, the idea still remains that there can be a separation between the mathematics of a problem (i.e. - understanding that 45 times 3 is not 120, or that 45 times 3 is in fact 120 plus 15), and the cognition behind the problem. A major component of the CBA framework is targeted at not only helping teachers recognize the strategies that students use, but also the meaning behind those strategies. In this case, teachers struggled to connect the mathematics with the proper thinking strategy that Sally demonstrated, failing to recognize a partial products decomposition strategy within her reasoning. Although all of the teachers responding to Sally’s work could verify mathematically that 45 times 3 was not 120, or that 45 times 3 was in fact 120 plus 15, many could not connect this to a proper CBA level representing the conceptual strategy that Sally was using.

In the literature review, key components of teacher knowledge were synthesized from various frameworks of teacher knowledge, culminating in the statement that teachers need to have: 1) a disposition towards mathematics teaching that encourages reflection, 2) to hold beliefs about the value of conceptual and procedural components of
mathematics, 3) deep understanding of the mathematics they teach and related mathematical knowledge for teaching mathematics, 4) knowledge of the common correct and incorrect conceptions of the mathematics they teach, 5) discriminating knowledge of tasks that reveal and elicit for assessment of student thinking, and 6) detailed, structured, research-based knowledge of the progression of development of children’s mathematical ideas. Discussion around teachers’ interpretations of Sally’s work opens up the conversation about the category ‘knowledge of the common correct and incorrect conceptions of the mathematics they teach’. In instances like Sally’s work, where the student’s strategy is quite complicated, both mathematically and cognitively, a teacher must have a deep knowledge of the conceptions students might possess about solving certain mathematics problems. In the two instances in the data where teachers (T20 and T21) did not recognize Sally’s thinking strategy as valid, the teachers had not used the CBA materials, and were not responsible for teaching multiplication and division. The CBA framework provides exactly the kind of background knowledge about reasoning like Sally’s and other common correct and incorrect conceptions children might hold, so it is believable that these teachers did not have the experience with teaching the content or using the CBA materials to have gained knowledge of student conceptions about decomposition of numbers and the use of the distributive property necessary to truly make sense of Sally’s work. In fact, this is a very believable conjecture in light of Steinberg’s (2004) comments that a research-based understanding of children’s thinking was not sufficient to encourage teacher growth. It was not until after teachers had
engaged in inquiries into children’s thinking built around concrete experiences in teaching situations when elements of the research-based frameworks became meaningful. This is important, as understanding the cognition of a student can be extremely important in making sense of how to determine feasible instructional learning goals and instructional tasks. Misinterpreting the cognitive approach of a student like Sally could have profound consequences for a student, as instruction will almost surely be based on an inaccurate picture of the student’s thinking. Therefore the knowledge about children’s correct and incorrect conceptions about mathematics must include not only the background knowledge of such conceptions, but the ability to effectively identify examples of the thinking. A teacher might know and understand the CBA levels, or another research-based framework for children’s thinking, but struggle to identify instances of certain concepts. A teacher might personally understand the distributive property and decomposition strategies, but might not be able to understand how to identify when others know and understand the distributive property and decomposition strategies. Such knowledge can be referred to as cognitive knowledge of common conceptions of students’ mathematical thinking.

Beyond the data investigated for this study, there is additional evidence from the CBA research project in other mathematical domains (measurement, and addition/subtraction) that demonstrate instances in which teachers struggle to make sense of a child’s cognition on a mathematics problem. On the home to school problem below, StudentX counts squares along each path to determine which path is longer. This strategy will not consistently be effective for paths that are not straight, because in
counting squares the student is attending to an attribute that better describes area than it
does unit length. However, several teachers characterize the student’s thinking to be such
that the student is counting sides of squares incorrectly or should be counting sides of
squares. Instead of interpreting the student’s strategy to be problematic, many teachers
suggested that the student could simply count sides of squares. This problem
demonstrates another instance in which a child’s strategy is not fully understood from a
conceptual level. In this instance, it would likely be most beneficial for StudentX to be
instructed in ways that helps to emphasize unit lengths, possibly through the use of
manipulatives that are closer to 1-dimensional than squares.

<table>
<thead>
<tr>
<th>Problem 1. Consider StudentX who used the strategy below to determine which path from home to school was longer.</th>
</tr>
</thead>
<tbody>
<tr>
<td>StudentX: [Counts squares along the gray path 1-14, then along the dotted path 1-15.] The gray path is shorter because it has less squares.</td>
</tr>
<tr>
<td>Is StudentX’s reasoning correct or incorrect? If it is incorrect, what is wrong with it?</td>
</tr>
<tr>
<td>What would you do instructionally to help StudentX?</td>
</tr>
</tbody>
</table>

Figure 91 – CBA Measurement Home to School Student Work

On the subtraction problem below, the student utilizes a strategy that makes it seem like they are utilizing negative numbers in computing 6 – 8 equals negative 2.

However, the student seems to actually conceptualize of the negative 2 instead as minus, or a cue to subtract. Teachers occasionally struggled with this, and conceptualized of her work as involving deep understanding of negative numbers, which is difficult to know based on this episode of student work.
Problem: 76 - 48

Susie: First do 70 minus 40 equals 30. Then do 6 minus 8, which equals negative 2 because 8 is bigger than 6. Then minus 2 from 30 and you get 28.

Figure 92 – CBA MD Subtraction Student Work

CBA Aligned LP Use

There is no doubt that with any LP materials, or any materials intended to help teachers with the process of teaching, certain components of the materials will be related to more complex components of mathematics or children’s thinking about mathematics. In the results section, the different ways that teachers utilized the CBA materials to better understand children’s thinking were explored. One main separation in CBA materials use was between those teachers who focused on only certain elements of the CBA framework for MD while analyzing the work of QR, RR, SX, and Sally, such as the number of computations, and those teachers who focused on all conceptual components within the CBA framework, such as use of the distributive property by place value. As was anticipated, CBA teachers struggled far more to effectively characterize student work that was more complex and that represented more detailed CBA levels, or did not fit into the CBA level descriptions in a straightforward way. One interesting finding in the data was that teachers with experience using the CBA MD materials seemed to more effectively choose CBA aligned levels than teachers without use of the CBA MD materials on the student work sample for Sally’s thinking that was most complex. Specifically, of the 14 teachers that analyzed the most complex work by Sally, only nine
were at least partially consistent with CBA. Of the nine that were partially consistent with CBA, seven had used the CBA MD materials while only 2 had not. Of the five teachers that were completely inconsistent with CBA, all had not used the CBA MD materials. Within the synthesized teacher knowledge framework, the knowledge that teachers would need to complete this type of task would fall into the ‘detailed, structured, research-based knowledge of the progression of development of children’s mathematical ideas’ category. The CBA MD materials appear to provide those teachers who had previously use the materials with some much needed assistance in this area.

In order to deeply understand student thinking with a research-based framework such as CBA, teachers would need to have discerning knowledge of the delineation between different levels. That is, it would not be expected that teachers have the entire LP memorized, but rather have the tools to differentiate effectively between two types of student thinking, even if they appear to be similar. In many cases, this might require a teacher to refer to the CBA materials as a reference, to read about what constitutes thinking at a certain level or to look at examples of student work that demonstrate a particular level. There were instances in the data described in the results section that indicated that while analyzing student work for RR, QR, SX, and Sally, some teachers needed to reference the full-length text of the CBA materials in order to identify the conceptual components of a student’s thinking or to locate examples of student thinking at a particular level (e.g. – T2 analyzing RR and QR problems frequently referenced the full length CBA MD text to read descriptions of levels). However, it was also often the case that teachers only used a shortened version of the CBA levels to evaluate student
work, and based their determination of a student’s CBA level off of the written title of the level alone.

In cases where CBA levels described instances that were not computationally varied or complex (e.g. – ‘2 partial products’) teachers’ level of understanding of the CBA levels at the time of the interview appeared to be sufficient to effectively level a student (32 out of 37 or 86.5% of teacher episodes involving simpler work and CBA levels were consistent with the CBA framework). However, in instances where the student work represented elements of multiple levels, or computationally varied or complex levels, determination of level was not as simple (13 out of 28 or only 46.4% of teacher episodes involving more complex work and CBA levels were consistent with the CBA framework). This was evidenced by the drastically lower CBA alignment on conceptually more involved problems. This could potentially help explain the differences found between teachers with CBA MD use and those without, as teachers with CBA MD use might have more background knowledge of the levels through more carefully reading the documents and/or experience with student’s thinking in MD prior to the interview. This additional background knowledge might have helped to supplement a teacher who was discerning a student’s level using the titles of the CBA levels alone.

Research on concept learning suggests that if teachers were applying a ‘prototype’ approach to learning the meaning of the CBA levels, they might base the determination of a student’s level on examples of student work from that level. In instances where the student work is only different in ways that might be difficult to notice, this type of approach might make learning the delineations between certain levels especially difficult.
Other research suggests that people form two kinds of categories: fuzzy or formal (Pinker, 1997). Fuzzy or natural categories, such as games, are formed in everyday activity. Such categories generally have no clear definitions, have fuzzy boundaries, and are conceptualized mainly in terms of stereotypes. Identification of instances seems to be the major goal, which is very much like ‘prototyping’, and aligns quite closely with how many teachers approached the categorization of children’s thinking within the CBA framework. The second type of category consists of formal, explicitly defined categories, such as odd numbers, in which all instances are logically equivalent as representative of the class. Formulating and studying precise definitions is the goal. There seem to be several competing theories for how people develop fuzzy categories. The most prominent seems to be that people develop prototypes or mental images or examples that include, usually implicitly, features that are relatively common among members of a category. People decide whether an object belongs to a category by determining if it is sufficiently similar to their conception of such prototypes (Smith, 1995). Vygotsky’s (1986) work sheds further light on the relationship between fuzzy and formal concepts. He argued that children become conscious of their spontaneous (i.e., fuzzy) concepts relatively later, with the ability to define them in words appearing long after they have “acquired” the concepts. In contrast, the development of scientific (formal) concepts, such as many of those in CBA, usually begins with their verbal definition.

The data on teachers’ use of CBA indicate that there are probably three types of concepts represented in teachers' understanding of CBA levels. The first is informal/fuzzy/spontaneous as described earlier. This type occurs as teachers try to make
sense of students' mathematical thinking and behavior, without benefit of psychological/educational research theories. A second type is scientific/formal/technical concepts explicitly defined in psychological/educational research theories. This type is exemplified in CBA level descriptions. The third, and new, type is partially learned formal concepts, in which some, but not all, of the attributes of formal concepts are learned. For instance, when T3 (had not used CBA MD) identifies Sally's reasoning as L3.4 because she counts 4 partial products; she uses two attributes of the Level 3.4 concept, 1) partial products (which implicitly require the distributive property), and 2) searching for 4 decomposed numbers. However, she omits a third critical attribute, namely the place-value-based nature of the required partial products. T3's concept might be viewed as a best-guess interpretation of the short description of L3.4 on the Quick Reference sheet (Appendix G or H) and is indicative of how difficult LP materials can be to make sense of without time to learn and make sense of them.

In considering how fuzzy categories of shapes are formed, Gentner and Medina (1998) argued that there is a developmental/experiential shift from perceptual similarity (e.g., recognizing a mobile that is a very close perceptual match to one seen before) to relational similarity (e.g., recognizing the similarity of a mitten covering a hand and a shoe covering a foot). These researchers argued that the perception of similarity is accomplished by a process of “alignment or structure-mapping.” Structure mapping is a central learning mechanism that enables individuals to notice and store abstract relational properties, derive abstract knowledge from instances, and extend that knowledge to new cases. Informal concepts tend to focus on more perceptually salient (perhaps easily
recognizable in the episode of student thinking) properties (attributes) and are generally imprecise, and often wrong. This was demonstrated in the data when teachers used CBA in partially consistent manners while analyzing especially RR and Sally’s work, failing to identify all of the critical aspects of the child’s thinking within the CBA framework and consequently choosing an incorrect CBA level. Partially learned concepts generally consist of some informal properties and some formal properties; but the formal properties are insufficient to formally define the concept.

The CBA materials afford teachers with the ability to employ varying levels of conceptual understanding, and there are instances in which different approaches are justified. Obviously, it would be ideal for teachers to work to obtain as detailed an account of every students’ thinking as possible, and therefore a formal conceptual understanding and use of CBA would be desirable. Understanding the conceptual and computational delineations between each level could allow teachers to have fine-grained knowledge about a child’s mathematical thinking that could serve to inform future instructional decisions. This could be especially important for a teacher working with a student one-on-one who is struggling to make progress, for whom conventional instruction does not seem to benefit. If working with an individual student, especially a struggling student, it can be very important to have a detailed picture of how they are thinking about mathematics. As was evidenced in T8’s (used CBA MD) teaching experiment with Student 1, it was quite difficult for T8 to get a fine-grained account of what this student was thinking, and T8 struggled to come up tasks that might help this
student make progress. The more detail a teacher has about a student’s thinking affords them with more flexibility in crafting tasks that can help them improve.

However, most teachers do not have the time or schedule to work with all of their students individually to utilize CBA to its fullest. Therefore, using CBA to identify in a more efficient fashion that leads to identify big-picture conceptualizations of children’s thinking (potentially using less formal and more fuzzy conceptions of CBA) could serve a valuable purpose as well, especially for whole-class LP use. T18 (used CBA MD) appeared to utilize this approach quite well in her work in assessing a whole class. T18 did not conduct one-on-one CBA assessment interviews with all of her students. T18 did; however, consistently and accurately recognize changes in children’s thinking in a more macro fashion within the MD framework based on general trends within her classroom. T18 was able to successfully move many of her students forward within the levels using this approach, using overall patterns in children’s thinking within the class along with CBA instructional hints to give meaningful tasks to help students learn.

Fuzzy conceptions of children’s thinking can lead to misinterpretations of the precise nature of children’s thinking within the CBA MD framework. However, in the analyzed cases, it was frequently an outcome that although teachers demonstrating fuzzy or partially fuzzy conception of the CBA levels of student thinking were occasionally incorrect in their leveling of students, they were only slightly off within the MD framework. There were twenty instances in which teachers were not completely consistent with the CBA framework, but only six (6 out of 20 or 30%) were more than one CBA level away from the actual CBA level for the student. That is, in very few
instances teachers vastly overstated a student’s thinking within the CBA MD framework without evidence of understanding all components of the student thinking within the CBA framework.

In the case of T18, conducting lengthy one-on-one CBA assessments with every student was unnecessary to help work towards moving students forward, as a big picture lens of general thinking strategies in the class appeared sufficient for informing whole class instructional tasks. But in the case of T8, it was challenging for her to make progress with a student when she did not have a good sense for the precise nature of the student’s thinking. This is not to imply that T8 was poorly using the CBA materials, as there was much in T8’s analysis that was CBA aligned, and accurate. However, T8’s case indicates just how difficult it can be to help provide meaningful tasks to a child when the student thinking is difficult to interpret. Overall, there is definitely value in the various types of LP understanding and use demonstrated by the CBA2 teachers, whether it is a formal understanding or a informal or fuzzy conception of children’s thinking. However, teachers may need additional help in learning to discriminate between levels when the student thinking or level of thinking is complex in nature. The goal of the CBA materials is to help teachers gain the clearest picture possible of their students’ mathematical thinking, and if the complex nature of the levels makes it difficult to clearly identify what the student seems to understand and what the student needs to work on, then this represents an area for further exploration. The core of this discussion is built around what it means to understand learning progressions. Because LP’s are formal descriptions of the development of complex learning, understanding them requires
understanding of technical psychological concepts and mathematical concepts to which psychological concepts are connected. Failure to understand the LP concepts can come from failure to understand either mathematical or psychological concepts, both of which are technical scientific domains.

Teacher Understanding of Components of CBA LP Materials.

CBA materials can be used to different levels of conceptual understanding of student thinking, and CBA materials can also be misinterpreted or confusing based on the use of research-based language to describe difficult concepts. The results section detailed instances in the CBA MD materials where teachers struggled with terms such as ‘hidden’, ‘partial products’, and ‘algorithm’. In the presented cases, it was evident that the use of such terminology did not consistently lead to CBA-aligned conceptions of their meanings. Especially with the term ‘algorithm’, many teachers did not connect the use of the term to the CBA meaning of a formal, organized, written set of steps that students apply to multiplication and division problems. The term algorithm is intended similarly to how Bass (2003) talked about algorithms

Seeking a single, clearly described, generic solution method that works in all cases makes sense for mathematical problems that occur repeatedly, in more or less standard forms. Such a general solution method, for a general class of problems, is called and algorithm. An algorithm consists of a precisely specified sequence of steps that will lead to a complete solution for a certain class of computational problems. p. 323
Instead, the term was often interpreted simply as a thinking strategy that did not have a systematic or written component, and therefore was frequently confused with different levels of reasoning that were not considered ‘algorithmic thinking’ in the CBA framework.

It is inevitable that any set of materials will have areas that are difficult for teachers to use and understand, and the CBA materials have undergone regular and frequent revision to account for the struggles that teachers have encountered. However, certain concepts in the CBA framework will represent new ideas to many teachers, and it is difficult to connect these new ideas to their everyday vernacular. Therefore, it is important to identify areas in any LP materials that might be challenging to understand, and might lead to misuse of the materials. The findings from this study indicated that the concept of partial product multiplication elicited many different perspectives from teachers analyzing the work of QR, RR, and Sally. The term ‘partial products’ by itself was not sufficient for teachers to have a CBA aligned understanding of the meaning of the term. In the cases detailed in the results section, the teachers without experience with the CBA MD documents had much more demonstrative struggles with understanding the partial products terminology. However, even those teachers who had used the CBA MD materials that were discussed in the results section struggled to have a CBA aligned conception of the partial products when analyzing RR, QR, and Sally’s work, in spite of the fact that they showed greater fluency in interpreting instances of the term than teachers without experience with CBA MD. The terms described in this research, along with other formally defined research-based terminology represent instances where
research-based frameworks must encapsulate important concepts from research literature in ways that can be learned in alliance with their meaning, but also are not too formal and abstract that everyday language cannot describe them. To help with this it might be beneficial to provide teachers with concrete examples as well as non-examples of what constitutes partial product multiplication and division. Many of the CBA teachers benefited greatly from analyzing samples of student work within the CBA materials that represent examples of student thinking at various levels of sophistication.

Although there were areas where the CBA teachers detailed in the results section struggled to understand the language or meaning of certain concepts in the CBA framework, it was also clear that the LP framework could provide a meaningful organizational tool for many teachers. The vast majority of CBA teachers responding to episodes of student work adopted ‘next level’ approaches to instructional learning goal determination, a positive indication that these teachers hold views that are consistent with an LP view of learning. The fact that a ‘next level’ approach occurred regardless of the accuracy of teacher’s initial level for a student indicates that the LP facilitated teachers’ determination of learning goals.

In many cases, teachers utilized the CBA MD framework to help to conceive of short term goals based on student work, while long term goals were often shaped by the curriculum they were teaching or the expectations based on the grade levels they taught. T8’s assessment interview that is detailed in Topic C in the results section is a prime example of this, as T8 determined short term goals with the help of the CBA framework but kept in mind the overall goals based on the curriculum. This is a positive outcome
considering that without the CBA MD framework, it would be anticipated that short terms goals would be more difficult for teachers to determine. A major component of the CBA MD framework is based upon the idea that children’s learning of mathematics occurs in relatively small increments that can be measured using proper assessments. Mathematics curricula are not necessarily designed based on such a developmental approach, but rather might be designed based on a logical approach of the development of a mathematical concept. Although such curricula might provide teachers with valuable objectives, they do not necessarily provide teachers with an appropriate path towards meeting those objectives. The data in this research suggests that it is possible that the CBA MD framework helps to provide some teachers with a more effective path towards accomplishing already established curricular goals. In fact, in evaluating Sally’s work on the problem 45x23, those teachers who established Sally’s level correctly in the CBA MD framework determined both short and long term goals within the CBA framework in an LP aligned fashion.

<table>
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<tr>
<th>Task.</th>
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<td>45 × 23 = _____</td>
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**Sally:** 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.

Figure 93 – Sally Student Work

*CBA and Instruction.*

Much like the previous discussion related to teachers’ determination of learning goals, it was overwhelmingly the case that teachers based their instructional plans on
their learning goals. In the 40 teacher episodes with sufficient detail to determine both a teacher’s learning goal and intended instruction, 35 (87.5%) of the episodes had aligned learning goals and instruction. However, it was also the case that when teachers did not initially level students consistently within the CBA MD framework, they overwhelmingly (86% of the time) overestimated the student’s CBA level. This is an obvious concern, as instruction that is targeted too high can be far more difficult for students to make sense of than instruction that might be targeted at their current level or slightly below. Clearly, underestimating a student’s reasoning can be detrimental to helping a student make progress, but it is less likely to result in instruction that might encourage rote use of strategies without meaning.

Interestingly, teachers who overestimated a student’s understanding in the CBA framework were rather inconsistent in choosing instruction that aligned with their goals. In half of such cases, teachers proposed instruction that was actually targeted at a lower CBA level than their proposed learning goal. This is encouraging, as it demonstrates that although some teachers may struggle with initially determining a level and subsequent learning goal and inadvertently overestimate a student’s CBA level, many of them either implicitly or explicitly identify that their chosen learning goal might be too high or ambitious when they consider designing instructional tasks for the student. This finding is not completely unexpected, considering that in practice, many teachers realize during instruction that they are struggling to meet the needs of students. It would be anticipated that some teachers would follow the same process with the CBA levels, and realize during instruction that they may need to adapt instruction to meet the needs of the
students and lower the desired learning goal. For those teachers who chose overly ambitious learning goals and also proposed tasks that were aligned with those goals, it might be the case that elements of the learning progression were not understood well enough to emphasize the importance of certain missing conceptual markers in the child’s thinking. It is also possible that teachers who determine overly ambitious learning goals might be basing learning goals less upon the student’s thinking and more upon the objectives and goals of their own school curriculum.

Another interesting finding in consideration of the relationships between teachers’ choice of CBA level, learning goal, and instructional task, was that teachers seem to base instruction off of the student’s current conceptions only when they diagnose correctly in the beginning. When teachers diagnosed a student properly and set an appropriate learning goal, 16 out of 19 (84.2%) episodes demonstrated student centered instruction based off of the student’s current conceptions. On the other side, when teachers diagnosed a student improperly and/or set an inappropriate learning goal, only 4 out of 15 (26.7%) demonstrated student centered instruction based off of the student’s current conceptions. When teachers leveled students too high and set overly ambitious learning goals, proposed instruction was primarily of a teacher-centered transmission model or demonstration approach. This could be partially related to the particular location in the CBA MD document, as teachers who leveled students too high might choose learning goals that are based on algorithms that are unlikely to arise spontaneously without teacher demonstration. However, it is also the case that in targeting instruction well beyond a student’s current way of thinking it is unlikely that a student spontaneously changes their
way of thinking without explicit teacher demonstration of a new strategy. These teachers appeared to focus more carefully on a desired outcome and modeling a certain way of thinking, with the intention that students were to internalize the new strategy on their own. The assumption in this approach was that students understood the mathematical concepts deeply enough to mimic the teacher’s approach with meaning and understanding. In contrast, teachers who properly leveled a student proposed instruction that aligned with the CBA aligned learning goal, and the instructional tasks were more frequently student centered and based on elements of the student’s current way of thinking. These tasks tended to be based less on teacher demonstration and more on students engaging in solving varied problems, allowing students to use manipulatives to help create meaning for the mathematics, or encouraging students to slightly adapt their current strategy to solve more complex problems.

This evidence runs in line with the idea that teachers with deeper understanding of CBA and ability to properly level students in the CBA framework have more flexibility in thinking about short term and long term plans for a student. Teacher-centered approaches to instruction often emphasize a particular strategy or approach to thinking about a problem, and do not always take in to consideration the current conceptualizations of a student. Those teachers who proposed overly ambitious learning goals were proposing to move a student to a level of reasoning that would be unlikely to occur quickly. Therefore, it is not surprising that instruction would be largely teacher-centered, as it would be unexpected that the student could quickly make sense of any ‘stepping stone’ reasoning that would be necessary to complete the task on their own.
This is important because when teachers appropriately level student within the CBA framework, they often can think about how to move children from where they perceive them to be to a ‘next reasonable level’ in ways that are based off of the child’s cognition. This is very much in line with the findings from the research on Cognitively Guided Instruction, demonstrating that as teachers learned more about children’s thinking they became more effective at basing instruction off of how the children thought about mathematics. Additionally, Kazemi and Franke (2004) found that over time as teachers began to more carefully focus on children’s thinking, they began to focus less on teacher actions and more on student actions and evidence of understanding. A potential explanation for why student-centered instruction was more prevalent for teachers with more accurate conceptualizations of children’s thinking is that as teachers learn more and attend more to student thinking, their conception of the role of the teacher might also shift. Kazemi and Franke (2004) discussed the idea of transformation of participation in teachers’ participation in a work group, suggesting that teacher discourse and analysis of student thinking was crucial to the development of new perspectives (p. 229). It is also possible that such a transformation can have implications beyond simply attending to and analyzing student thinking, but also into instructional considerations.

Live One-on-one Teaching and Assessment

In T6’s CBA assessment of a student, it was demonstrated that the CBA MD framework effectively informed T6 of how her student was thinking about the multiplication and division problems she was assessing. T6 showed how CBA could inform a pre-assessment and diagnosis, and determination of learning goal. In spite of
the fact that T6 might have misinterpreted certain components of the students’ verbalizations, she was able to determine an accurate ‘composite’ level for the student’s mathematical tendencies, and a reasonable CBA instructional learning goal. Although T6 demonstrated that she could reason through the CBA levels, showing a conceptual understanding of the levels that her student was demonstrating, she struggled to determine instructional tasks based off of her learning goal. T6 stated that the CBA framework was not easy for her to use initially, even for assessment, but that she was able to become familiar with the CBA document and learn to use it to help determine what level a student was at and if they were making progress.

T6: You get more familiar with it. I was just thinking the iteration – he’s counting by. I’m familiar with the document I’ve gotten better. I remember last year thinking about how you move them up a level – and thinking I have no idea. And now I can use tasks to help figure out where they are. But by practicing and talking to them about the tasks you can tell if they are moving up. Even though I’m not sure what instruction to do.

T6 clearly stated that the CBA framework did not necessarily help inform her instruction, but can be used to evaluate the impact of instruction. There are several believable conjectures for interpreting T6’s comments. It is completely possible that learning to use the CBA materials to assess students takes a significant amount of time for teachers to learn. T6’s prior work with the CBA materials and her students could have been enough for her to make sense of assessment. In considering instruction, it is completely possible that for some teachers conceiving of instruction based off of knowledge of the CBA levels takes additional time to learn. Although current iterations of the CBA materials have been designed to include more instructional suggestions for
teachers, teachers may still need significant time to experiment with different instructional tasks and activities to determine their effectiveness at moving students forward within the CBA framework. It is also possible that conceiving of instruction requires a deep knowledge of not only the CBA levels and materials, but also knowledge of eliciting tasks, varied mathematics curricula, and available manipulatives and resources. Ideally, teachers would be able to construct instructional tasks based on their increasing knowledge of children’s thinking obtained from use of the CBA materials. However, as T6 demonstrates, this might be a difficult task for teachers that takes significant time, and might require additional support or resources. Understanding learning goals is essential for judging whether instruction was effective. Morris, Hiebert, and Spitzer’s (2009) research found that evaluating learning goals, although something that was learnable, was actually quite difficult for the pre-service teachers involved in the study. T6’s case shows that the CBA framework can help provide a framework for evaluating learning goals, and even in instances where instruction is not easy to conceive of, the CBA framework can help inform a teacher’s determination of the effectiveness of any instruction.

In her work with two students, T8 showed that when student thinking was understood and could be fit meaningfully into the CBA LP, a high quality LP informed question was asked. In this case, T8 articulated that she felt her questioning had improved as her work and experience with the CBA materials increased. Although it is impossible to know if T8 would have produced LP aligned questioning without the use of CBA materials, T8 does attribute her ability to help kids learn through better questioning to her
knowledge and experience with the CBA materials. T8 also showed that when student thinking was not well understood, conceiving of appropriate questions was much harder and her questions were not consistent with the CBA LP. Despite T8’s mathematical mistakes, T8 attempted to adhere to a CBA consistent approach to teaching a student even though the student’s thinking was not well understood. T8’s instructional moves were similar to the findings from the data on teachers’ analysis of student work, showing that when student thinking was not well understood and ambitious learning goals were set for students, it was difficult to base future instruction off of the student’s current ways of thinking. In T8’s work with her first student, when T8 did not understand how the student was representing the mathematics, T8 asked her to write the problem in a different way, encouraging the student to use a different strategy. Instead of asking the student to try to think about the problem in a different way or providing an example of why the student’s strategy was problematic, T8 provided the student with a set of steps to lead to an answer. This is markedly different than the case when T8 understood the second student’s thinking much better and asked a question that allowed the student to try to create meaning through his or her own experimentation and exploration.

T18’s case showed that CBA can be used to help teachers think about whole class instruction as well as individualized instruction. Similar to T6, T18 seemed to have the perspective that although she had made sense of using the CBA levels for assessment of student thinking she struggled to construct instruction on her own, relying heavily on the CBA instructional suggestions to move kids forward. T18 did, however, discuss her efforts to create new tasks based on the instructional suggestions provided in the CBA
materials. This demonstrates an example of how teachers can use knowledge of the CBA levels for assessment purposes along with resources that provide suggestions about how to align instruction to the CBA levels to construct new tasks that can help children progress mathematically. As was proposed for T6’s case, it is completely believable to think that teachers with understanding of how to use the CBA LP framework to assess student thinking may be able to conceive of appropriate instruction when given addition support with respect to creating instructional tasks that align with the CBA LP framework.

T18’s case also demonstrated that when instructional hints are provided to help teachers move students from level to level, they could be put into practice to help move students forward. T18 seemed to approach assessment of students’ thinking within the CBA levels from a more macro perspective, looking for general patterns of thinking and focusing less on the conceptual specifics of different strategies. This was likely based on the necessity of utilizing less focused and individualized assessment since the instruction was designed for an entire classroom of students. Using this overall information to conceive of what instructional hints would be more beneficial, T18 provided students with learning opportunities that closely aligned with their current ways of thinking. T18 also stated that for those students who were not making progress as quickly as others, or did not fit the overall thinking patterns of the class as a whole, she was able to work individually with them to provide different instructional hints that targeted their needs. T18 demonstrated student-centered teaching in her teaching by encouraging students to lead class discussions about their thoughts about how to solve multiplication problems in
a variety of ways. This, again, is additional evidence that when student thinking is well understood within the CBA framework, it is more likely that teachers propose, and implement LP aligned instruction that is student-centered and focused on the current mathematical ideas of students.

Limitations

As this study was qualitative in nature, one of the main limitations of this research stems from the small sample nature of the data. While the data do allow for more rich and descriptive analysis, they do not allow for generalizability. The advantage of small sample work is the depth with which the data can be investigated and explored, opening up future areas for research and concepts to be explored.

In addition to the overall small sample nature of this study, much of the data are based upon teachers’ responses to written samples of student work and do not represent an account of their use of LP’s in their everyday work as teachers. Although the data included three cases of teachers working with actual students, the remaining data do not involve teachers in the act of teaching or assessing, but rather reflecting on the work of a student who they have never met. Many of the CBA teachers identified that analyzing such student work was challenging because they were not able to ask follow-up questions, or actually watch or hear the student think through the mathematics, making it difficult for them to develop a profile of what the student understood and what the students did not. This limits the scope with which the data can be interpreted, as teachers who struggle to use the CBA materials under these conditions might demonstrate far better understanding and use of the materials in their actual work with students.
Therefore, this research should be viewed as exploratory in nature, intended to develop grounded theory based on data, and not to be viewed as conclusive and generalizable.

Another limitation to the research is that although teachers provided instructional plans for how to respond to a student based on episodes of student work, these instructional plans were never executed. It is quite possible that some teachers espouse intentions to instruct children in a certain way, while the enacted plans end up looking different than anticipated. Therefore, the results in this study are largely based on situations where espoused instruction is analyzed as opposed to actual instruction. The obvious limitation is that some teachers may have a good conceptual idea of what instruction should or could look like, but might actually struggle to put the ideas into practice in working with children. Therefore, the analysis in this research should be interpreted through the lens of teachers’ instructional planning, as opposed to actual instruction.

The focus of this dissertation has been centered on teachers’ use and understanding of the CBA materials, specifically this research has only zeroed in on issues involved in the CBA LP for multiplication and division. Although certain ideas that have emerged in this research may be found in other domains of mathematics LP (e.g. - challenges with research-based language, teachers implementing an LP consistent approach to instructional learning goals), these ideas should be investigated in other mathematical domains as well. It should also be noted that the particular LP materials that are the focus of this research, namely the CBA materials, are a very specialized LP that represent one research-based perspective on how children develop concepts of
multiplication and division. Other research-based frameworks of children’s thinking likely share similar ideas or components with the CBA framework, but it should not be assumed that these LP would be used or interpreted similarly by teachers.

Additionally, the CBA materials represent a very specialized set of LP materials, and the teachers who participated in this study represent only those teachers who were willing and able to participate in the CBA2 project. Although the CBA2 teachers teach a variety of different age groups, it is unfair to assume that these teachers are representative of all elementary teachers. As is the case with any research project, those that choose to participate may share characteristics in common, and may have backgrounds that dispose them to learning differently than other teachers.

Implications and Future Research

As was discussed in the results and discussion, teachers may need additional support when using research-based frameworks to better understand the delineating components between levels. This is especially true for conceptually complex levels that require a deep understanding of the different features of the student thinking that characterize the level. The CBA MD LP materials are a relatively fine-grained set of materials that are intended to help teachers interpret student thinking within sublevels. The level of detail in the CBA framework can clearly lead to difficulties for teachers when interpreting the differences between sometimes very similar looking levels of thinking. Moving forward, it is important to consider that LP’s are not necessarily easy for teachers to understand and use, and especially when levels of thinking or student work are complex it is might be necessary to provide teachers with support in learning to
delineate between different types of thinking. This could occur through the creation of a professional learning community (PLC) led by an experienced CBA teacher or CBA researcher who has a detailed understanding of the CBA levels. In these PLC’s, experienced teachers or researchers could help teachers analyze student work that represents some of the more complex levels or thinking. The data presented in the results demonstrated a small-scale instance of this, when a CBA researcher used leading questioning to help T5 recognize the delineation between two levels of reasoning related to decomposition strategies and partial products. Kazemi and Franke (2004) did work with a professional learning community of teachers focusing on student thinking with very positive outcomes for teachers’ ability to more effectively identify elements of student thinking. In light of the findings of Jacobs et al. (2010), suggesting that teachers do not learn to notice key elements of children’s thinking with teaching experience alone, it is clear that some form of professional development could serve to help teachers improve in their abilities to learn to discriminate between varying forms of student mathematical reasoning.

Another important implication of this research is the overwhelming pattern of teachers who chose the correct level and then an appropriate next level for instructional learning goal, chose student centered instruction that was based on the child’s thinking. This appears to support the conjecture that understanding of CBA levels and a LP view on choosing a reasonable instructional learning goal, tends to match up to intentions to teach to help student move to that next level. However, what is clear in the data is that many teachers are not certain how to craft instruction to make this move possible.
Therefore, it is important for the CBA researchers to continue to work to provide instructional suggestions, as well as identify and include mathematical tasks that align with the CBA framework as part of the CBA materials to help give teachers a starting point for converting good learning goals into good learning goals with aligned instruction. In addition to supplemental instructional suggestions for teachers, it is important for CBA researchers, and other researchers interested in LPs to work to better understand how teachers come to learn how to use knowledge of LP to develop tasks for students. In research on teachers’ learning of learning trajectories, Wilson (2009) found that teachers needed instructional tasks that supported the learning trajectories. This provides additional evidence that formulating instructional tasks based on a learning progression can be quite challenging for teachers. It also relates to the idea of a potential delineation between learning progressions and learning trajectories. Battista (2010) posited that one main difference between the two was that “trajectories include descriptions of instruction but that progressions do not” (p. 61). Theoretically, learning progressions offer far more powerful formative assessment potential, as they can be responsive to children’s thinking ‘on the fly’ and do not depend on a fixed set of curricula. This is not to say that trajectories cannot be useful for similar purposes, but rather that they are not necessarily targeted at building teachers’ understanding of children’s mathematical development in a fashion that allows teachers to construct their own mathematical tasks from this knowledge. The CBA materials do not represent a fixed set of curriculum materials that teachers move students through, and are intended as a LP. However, it is clear that becoming an expert user of LP materials is not a simple
process, and the data indicates that conceiving of reasonable instruction is one of the most challenging things for CBA2 teachers.

A further implication of this research is that improper initial selection of a student level of reasoning by teachers can be extremely problematic. Aside from the obvious fact that improper leveling suggests that a teacher might be struggling with some elements of the student’s reasoning or the CBA levels, the data in this research indicated two patterns that when they occur together can be problematic: 1) When teachers improperly level students, they almost always overestimate and choose too high of a level, and 2) No matter what level is chosen, CBA teachers tended to apply a ‘next level’ approach to learning. For a teacher to level a student too high is not necessarily a problem if that teacher chooses instruction targeted at deepening understanding at that level. In this case, a teacher might inadvertently overestimate a student’s reasoning, but choose to instruct in a way that might help a student make progress because instruction is not targeted beyond the overstated level. However, when teachers choose too high of a level and then also apply a ‘next appropriate level’ approach to determining goals and instruction, instruction is less likely to be meaningful to a student. It is therefore highly important to emphasize to teachers that proper leveling is important, and that the consequences of overestimating can lead to instruction that is quite difficult for children to make sense of.

A final implication of this research is that there is evidence from T18’s case that the CBA materials were used effectively to pre-test all students in a class, design instruction for whole-class discussion, and post-test all students with positive results. In the case of T18’s classroom teaching experiment, T18 provided evidence of multiple
students that made significant progress within the CBA levels within a relatively short (5 day) series of lessons targeted at getting students to model and explain different decomposition strategies of multiplication. In this case, T18 did not seem to be using the CBA materials to deeply understand every student’s strategy and precise CBA level, but rather to get a sense for the overall tendencies of the class. This is encouraging data as it demonstrates that LP’s can provide a powerful structure for teachers to approach teaching in whole class situations differently. It also runs in line with Fuchs et al.’s (1999) finding that regardless of assessment type, teachers tend to teach towards the assessment. In this case, T18 used CBA tasks and materials along with the CBA levels of sophistication as the method of assessment. Instead of placing emphasis only on the answer students produced, such assessments emphasize the type of thinking that students use. T18 effectively used the CBA assessments to teach to promote growth within the CBA framework, with positive results.

Future research that more deeply explores teacher’s understanding of cognition, what could be termed cognitive or psychological knowledge, is merited based on the results of this study. In this research, there are instances where teachers improperly attribute thinking to a student, which makes CBA diagnosis and understanding of not just mathematical correctness, but cognitive representation, difficult. This was especially true for Sally’s work, which was by far the most mathematically and conceptually complex of all of the student work analyzed in this research. As was mentioned earlier, there are other instances of student work that were not considered for this research that elicited teacher misconceptions about how a child was making sense of mathematical problems.
This represents a specialized type of knowledge of content and students (KCS) within the mathematical knowledge for teaching (MKT) framework that deserves more careful exploration.

Another fruitful area for future research relates to the CBA materials and their relationship to teacher knowledge growth. The current research did not explore growth, but rather reflected on teacher knowledge at specific points in time during clinical interviews and teaching experiments. Future research could explore the ways in which the CBA materials act to support the areas of teacher knowledge established earlier in this research that stated that teachers need to: 1) have a disposition towards mathematics teaching that encourages reflection, 2) to hold beliefs about the value of conceptual and procedural components of mathematics, 3) deep understanding of the mathematics they teach and related mathematical knowledge for teaching mathematics, 4) knowledge of the common correct and incorrect conceptions of the mathematics they teach, 5) discriminating knowledge of tasks that reveal and elicit for assessment of student thinking, and 6) detailed, structured, research-based knowledge of the progression of development of children’s mathematical ideas.

Is it the case that strength in one area can build another, and therefore someone who might have a deeper understand of the mathematics they teach might be able to more effectively build on this knowledge using the CBA framework to gain knowledge of the common correct and incorrect conceptions of the mathematics they teach? In this respect, the question could be investigated whether CBA or other LP materials can serve in a distributed cognition fashion, acting not as something that the teachers need to
memorize but rather can use as a resource or reference guide. Another reasonable conjecture is that CBA can provide teachers with a more detailed account of the strategies that children use, and in the process of learning about new strategies, teachers could actually work towards a deeper understanding of the mathematics they teach and related mathematical knowledge for teaching mathematics. Ball (1988), made the case for the variety of experiences that teachers need to have with mathematics as a means to challenge their perspective on what it means to do mathematics. Silver, Clark, Ghousseini, and Charalambous’s (2007) work found that when teachers engaged in analyzing artifacts of practice such as viewing teaching episodes or analyzing samples of student work, they had opportunities to work on and learn mathematics. CBA could facilitate exactly this type of learning experience by engaging teachers in thinking about episodes of student mathematical thinking.

As the CBA framework provides support for many of the components described in this teacher knowledge framework, it would be important to investigate the impact that working with the CBA materials has on teachers’ knowledge and beliefs in each of these categories.
References


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Part 3. Identifying CBA Levels

I'm now going to ask you to describe what CBA Multiplication and Division Levels are demonstrated in several episodes of student work.

I'll ask you to explain your answers.

To help you better understand the CBA levels, after you give me your answers, I will tell you the CBA levels author’s answers and ask you if his levels make sense to you.

After we discuss the levels, I will ask you questions about how you would deal with the students instructionally.
Please read this episode aloud and tell me what you are thinking as you read it.

*Give teacher the HO with transcription of student episode—see next page.*

<table>
<thead>
<tr>
<th>HO</th>
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</thead>
<tbody>
<tr>
<td><strong>Task.</strong> There are 6 soccer teams in a tournament. Each team has 11 players. How many players are there altogether?</td>
</tr>
<tr>
<td>RR: 11, 22, 33, 44, 55, 66 [raising one finger after each number].</td>
</tr>
<tr>
<td>So, there’s 66 players.</td>
</tr>
</tbody>
</table>

What purpose did raising fingers serve for RR?

What CBA level for multiplication and division is demonstrated by this student?

What makes you think that [say teacher’s answer] is the CBA level?

Tell me what part of the CBA document convinces you that you are correct.

The CBA author’s level is **MD2.1.2 Verbal Iteration.**

*If the teacher’s level is different from author’s*

Look at the CBA descriptions for your level and the author’s level. What do you think? Which level do you think is correct and why?

What CBA level do you think this student should move to next? Why?

**MD2.3.1 Accumulating iterations into manageable subtotals, MD2.3.2 Iterating composites of composites**

What instructional activity or task do you think might move this student to this next level?

**Hand Out Given to Teachers:**

**Task**

There are 6 soccer teams in a tournament. Each team has 11 players. How many players are there altogether?

RR: 11, 22, 33, 44, 55, 66 [raising one finger after each number]. So, there’s 66 players.
Please read this episode aloud and tell me what you are thinking as you read it.

[Give teacher the HO with transcription of student episode—see next page.]

<table>
<thead>
<tr>
<th>HO</th>
<th>Task. 46 × 5</th>
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<tbody>
<tr>
<td>RR:</td>
<td>20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.</td>
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</tbody>
</table>

What CBA level for multiplication and division is demonstrated by this student?

What makes you think that [say teacher’s answer] is the CBA level?

Tell me what part of the CBA document convinces you that you are correct.

The CBA author’s level is MD3.1. Using Known Facts with Numbers NOT Decomposed by Place Value.

[If the teacher’s level is different from author’s]
Look at the CBA descriptions for your level and the author’s level. What do you think? Which level do you think is correct and why?

What CBA level do you think this student should move to next? Why?

(Author: MD3.2. Using the Distributive Property with Numbers Decomposed by Place Value; MD3.2.1 first, then MD3.2.2)

What instructional activity or task do you think might move this student to this next level?

Hand Out Given to Teachers:

Task
46 × 5

RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.
Please read this episode aloud and tell me what you are thinking as you read it. [Give teacher the HO with transcription of student episode—see next page.]

<table>
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<tbody>
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<td><strong>Task.</strong></td>
</tr>
<tr>
<td><strong>QR:</strong></td>
</tr>
</tbody>
</table>

What CBA level for multiplication and division is demonstrated by this student?

What makes you think that [say teacher’s answer] is the CBA level?

Tell me what part of the CBA document convinces you that you are correct.

The CBA author’s level is **MD3.2.1 Two partial products.**

*If the teacher’s level is different from author’s*

Look at the CBA descriptions for your level and the author’s level. What do you think? Which level do you think is correct and why?

What CBA level do you think this student should move to next? Why? *(Author: MD3.2.2 Four partial products)*

What instructional activity or task do you think might move this student to this next level?

Hand Out Given to Teachers:

<table>
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<tr>
<th>Task</th>
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<tbody>
<tr>
<td>46 \times 5</td>
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</table>

QR: 40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.
Describe the difference between RR’s and QR’s strategies.

*Give teacher the HO with transcription of student episode—see next page.*

<table>
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</table>

What are the CBA levels for these students’ strategies?

RR: **MD3.1. Using Known Facts with Numbers NOT Decomposed by Place Value.**

QR: **MD3.2.1 Two partial products by place value.**

Why do you think QR’s reasoning is rated as more sophisticated than RR’s in the CBA system?

*[To interviewer: Both MD3.1 and MD3.2.1 use known facts. But MD3.2.1 does it by decomposing numbers by place value, which is moving in the direction of the decomposition that is used in multiplication algorithms (four partial products). Understanding such place-value decompositions, especially with four partial products) is thus essential for understanding multiplication algorithms.]*

Hand Out Given to Teachers:

*Task*

46 × 5

RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.

QR: 40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.
Part 3. Identifying CBA Levels

What CBA level for multiplication and division is demonstrated by this student?

Task. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?
RX: 12, 24, 36, 48, 60, 72 [raising one finger after reciting each number]. So, there’s 72 players.

What makes you think that this is the CBA level?

What part of the CBA document convinces you that you are correct?

What purpose did raising fingers serve for RX?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?
Appendix C - 07.08 MD Interview 1

**Part 3. Identifying CBA Levels**

What CBA level for multiplication and division is demonstrated by this student?

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<td>RX:</td>
<td>12, 24, 36, 48, 60, 72 [raising one finger after reciting each number]. So, there’s 72 players.</td>
</tr>
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</table>

**CBA Level:**

What makes you think that this is the CBA level?

What part of the CBA document convinces you that you are correct?

What purpose did raising fingers serve for RX?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?
What CBA level for multiplication and division is demonstrated by this student?

**Task.** 46 × 5

**RR:** 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.

**CBA Level:**

What makes you think that this is the CBA level?

What part of the CBA document convinces you that you are correct?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?
What CBA level for multiplication and division is demonstrated by this student?

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<tbody>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
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</table>

CBA Level:

What makes you think that this is the CBA level?

What part of the CBA document convinces you that you are correct?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?
Whose reasoning do you think is more sophisticated, QR’s or RR’s?

<table>
<thead>
<tr>
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</thead>
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<td>RR:</td>
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</tr>
<tr>
<td>QR:</td>
<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>

Explain why.

*Explain how to use each students’ reasoning on the problem 84 \times 6.*

(a) How do you think RR would use the reasoning he used on 46 \times 5 to do the problem 84 \times 6?

(b) How do you think QR would use the reasoning he used on 46 \times 5 to do the problem 84 \times 6?
Appendix D - 08.09 Final MD2

Problem

\[ 45 \times 23 = \boxed{} \]

Sally: 45 times 10 is 450. 45 times another 10 is 450; that’s 900. 45 times 3 is 120, plus 15. So it’s 1020 plus 15, or 1035.

(a) is Sally’s reasoning correct or incorrect? Explain why.

(b) what CBA level of sophistication do you think Sally’s strategy is?

(c) what type of reasoning do you think Sally should move to next? Explain why.

(d) What would you do instructionally to move Sally to this next type of reasoning?
Part 3. Identifying CBA Levels

What CBA level for multiplication and division is demonstrated by this student?

Task. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?

RX: 12, 24, 36, 48, 60, 72 [raising one finger after reciting each number]. So, there’s 72 players.

CBA Level:

What makes you think that this is the CBA level?

What part of the CBA document convinces you that you are correct?

What purpose did raising fingers serve for RX?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?
What CBA level for multiplication and division is demonstrated by this student?

<table>
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<tbody>
<tr>
<td>RR:</td>
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</tbody>
</table>

**CBA Level:**

What makes you think that this is the CBA level?

What part of the CBA document convinces you that you are correct?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?
What CBA level for multiplication and division is demonstrated by this student?

**Task:** $46 \times 5$

**QR:** 40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.

**CBA Level:**

What makes you think that this is the CBA level?

What part of the CBA document convinces you that you are correct?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?
Whose reasoning do you think is more sophisticated, QR’s or RR’s?

<table>
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<tbody>
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<td>QR:</td>
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</tr>
</tbody>
</table>

Explain why.

Explain how to use each students’ reasoning on the problem 84 × 6.

(a) How do you think RR would use the reasoning he used on 46 × 5 to do the problem 84 × 6?

(b) How do you think QR would use the reasoning he used on 46 × 5 to do the problem 84 × 6?
Appendix F – Online Interview

Part 3. Identifying CBA Levels

What CBA level for multiplication and division is demonstrated by this student?

Task. There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?

RX: 12, 24, 36, 48, 60, 72 [raising one finger after reciting each number]. So, there’s 72 players.

What makes you think that this is the CBA level?

What part of the CBA document convinces you that you are correct?

What purpose did raising fingers serve for RX?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?
Task. $46 \times 5$

RR: 20 times 5 is 100. Another 20 times 5 is 200. Plus 6 times 5 is 230.

What makes you think that this is the CBA level?

What part of the CBA document convinces you that you are correct?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?

Task. $46 \times 5$

QR: 40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.

CBA Level: 3.3

All the answers to these questions were the same as the previous question.

What makes you think that this is the CBA level?
What part of the CBA document convinces you that you are correct?

What CBA level do you think this student should move to next? Why?

What instructional activity or task do you think might move this student to this next level?
Whose reasoning do you think is more sophisticated, QR’s or RR’s?

<table>
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<td>40 times 5 is 200; 6 times 5 is 30. So it’s 200 plus 30 equals 230.</td>
</tr>
</tbody>
</table>

Explain why.

Explain how to use each students’ reasoning on the problem 84 × 6.

(a) How do you think RR would use the reasoning he used on 46 × 5 to do the problem 84 × 6?

(b) How do you think QR would use the reasoning he used on 46 × 5 to do the problem 84 × 6?
4c. Consider a student SX who uses the strategy below on this problem.

**Problem**
*There are 6 soccer teams in a tournament. Each team has 12 players. How many players are there altogether?*

**SX:** 12, 24, 36, 48, 60, 72 [raising one finger after reciting each number]. So, there’s 72 players.
CBA Reference Sheet: Levels of Sophistication in Students' Reasoning about *Multiplication and Division*

**Level 0. Cannot Count Groups**
Students can count single objects, but not groups of objects. A group of objects is not considered countable.

**Level 1. Counts Total by Ones**
Students can count sets of groups. To determine the total in a set of equal groups, students count all the ones within the groups.

1.1 *Counts Physical Objects*
Students repeatedly form equal groups out of things they can see or touch. They count units within the groups.

1.2 *Counts Visualized Objects*
Students visualize or imagine equal groups rather than examine something they can see or touch. They count individual units within the imagined composites.

1.3 *Counts Count-Words*
Students form equal groups out of counting-by-ones words.

**Level 2. Iterates Numbers Greater than One**
Students no longer need to count the ones within groups.

2.1 *Uses Repeated Addition/Subtraction*
Students iterate numbers by adding (for multiplication) or subtracting (for division).

2.2 *Skip-Counts Parts*
Before skip counting, students decompose the number to be iterated into parts.

2.3 *Skip-Counts All*
Students iterate numbers by reciting or writing all needed multiples in the skip-count sequence.

2.4 *Skip Counts Groups of Groups*
Students curtail or shorten the iteration process by (a) accumulating iterations into subtotals, or (b) combining iterations of the original number into bigger numbers.

**Level 3. Operates on Numbers without Counting or Iteration**
Students determine answers without counting or iteration.

3.1 *Recalls Facts*

3.2 *Uses Number Properties*
Students use known multiplication/division facts and number properties to derive answers. Students do not use distributive property together with place-value decompositions.

3.3 *Uses Distributive Property and Place-Value (2 partial products)*
To find a product, students decompose one of the factors into two parts by place value, then use the distributive property.

3.4 *Uses Distributive Property and Place-Value (4 partial products)*
To find a product, students decompose both factors into two parts by place value, then use the distributive property.

**Level 4. Understands Algorithms**
Students use and understand computational algorithms, or fixed sequences of steps, to multiply and divide numbers.

4.1 *Expanded Algorithms*
Students meaningfully use expanded, and place-value explicit, algorithms that specify manipulations of numbers with their place-values intact.

4.2 *Traditional Algorithms*
Students use traditional computational algorithms. But they can explain why these algorithms work by using place value concepts and appropriate properties of numbers.
Problem Statement Sheet  2a. [multiplication/division]
Describe ALL the different strategies you think students in grades 1 through 5 might use to solve this problem. Describe them in order, from least sophisticated to most sophisticated.

<table>
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<tr>
<th>Problem</th>
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<tr>
<td>45 × 23 = _____</td>
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</table>
Level 0. Does Not Understand Multiplication/Division Situations
   Students cannot comprehend multiplication and division situations. Often, this lack of
   comprehension is due to the fact that students cannot count groups of objects. The
   student does not consider groups of objects as countable.

Level 1. Operates on Numbers as Collections of Ones (No Skip Counting)
   Students understand multiplication and division situations, and they can count groups of
   objects. For instance, they can determine that there are three sets of 2. To determine the
   total in a set of equal groups, students count all objects in all groups by ones.
   1.1 Counts Physical Objects
   1.2 Counts Visualized Objects
   1.3 Counts Count-Words

Level 2. Operates on Numbers by ITERATING/SKIP-COUNTING
   Students progress to iteration (skip-counting) when they no longer need to count the ones
   within groups.
   2.1 Uses Repeated Addition/Subtraction or Some Counting-by-Ones
   2.2 Skip-Counts Parts
   2.3 Skip-Counts All
   2.4 Skip Counts Groups of Groups

Level 3. Operates on Numbers by COMBINING/SEPARATING (without Counting or
   Skip-Counting)
   Students determine answers without counting or iteration. Instead, they recall known
   multiplication/division facts or use such facts to derive answers using various properties
   of numbers.
   3.1 Recalls Facts
   3.2 Uses Number Properties
   3.3 Uses Distributive Property to Decompose Numbers by Place-Value into 2 Partial
       Products
   3.4 Uses Distributive Property to Decompose Numbers by Place-Value into 4 Partial
       Products

Level 4. Uses and Understands Expanded Algorithms
   Students use and understand expanded computational algorithms to multiply and divide
   numbers. Expanded algorithms maintain the values of place-value parts throughout the
   sequence of steps in the algorithms.

Level 5. Uses and Understands Traditional Algorithms
   Students use their understanding of place value and other properties of numbers to
   conceptually understand traditional algorithms for whole number multiplication and
   division, even though place value ideas in these algorithms are hidden.