Characteristics of Spatial Human Arm Motion and the Kinematic Trajectory Tracking of Similar Serial Chains

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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2011

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Abstract

This work studies spatial reaching motion in healthy humans. Research suggests that for individual instances of movement, the central nervous system (CNS) composes an explicit wrist path, which is transformed into joint motions in a time-invariant fashion. This is the *time invariance hypothesis* (TIH), and its validation for spatial motion is the first goal of this study. The human arm is typically modeled as a multi-link, serial chain. When one joint of a serial chain is actuated, it simultaneously causes movement at other joints because of *interaction effects*. Based on horizontal-plane reaching studies, the *leading joint hypothesis* (LJH) proposes that the interaction effects at (mostly) the proximal joint in the multi-link serial-chain model of the arm are low. Therefore, the CNS ignores this interaction effect to simplify the computation of joint torques and control of the joint trajectory. The second objective of this dissertation is to validate the LJH for spatial motion.

In a spatial reaching experiment, healthy subjects performed point-to-point reaching movements at three distinct speeds. Data analysis revealed time-invariant wrist paths only for some subjects in some reaching tasks, suggesting that the TIH is not a truly general organizing principle for spatial reaching motion. Therefore, this hypothesis needs refinement and further investigation. On the other hand, the interaction effects at the shoulder joint were small for a majority of the movements in this experiment so, the LJH was successfully extended to spatial motion.
The TIH identifies the inputs and outputs of the first stage in the process of composing the muscle activations for a given motor task. A computational algorithm that can potentially be used to execute this transformation was developed next. The algorithm, called speed-ratio control, also has beneficial applications in commercial robot control. It is demonstrated that the application of this algorithm to robotic serial chains provides greater navigational accuracy in the vicinity of certain kinds of singularities.

Speed ratio control applies to non-redundant serial chains. The simplest model of the human arm consists of three-degree-of-freedom spherical joints at the shoulder and the wrist and a revolute joint at the elbow. This yields seven degrees of freedom for the arm. For positioning and orienting the hand relative to the thorax, only six degrees of freedom are necessary. The human arm is, therefore, a redundant serial chain. The formal process of extending the algorithm to redundant serial chains is undertaken. Initial work in which three- and four-degree-of-freedom planar chains track point paths is presented. Speed ratio control allows the resolution of the redundancy in the mechanism by maximizing the output-space tracking accuracy. Examples show superior local tracking performance with this approach compared to path tracking using unweighted pseudoinverse solutions.
To my parents,

Manik and Sanjeev.
Acknowledgments

I thank my advisor, Dr. J.P. Schmiedeler foremost for the opportunity he provided to me for doing this work. His support, patience, and guidance were key to the successful completion of this dissertation. I thank my committee members for providing insightful suggestions at various stages of this work.

I thank my co-advisor, Dr. Kinzel for giving me opportunities to teach at the undergraduate and graduate level. This experience has helped me improve as a teacher enormously.

I had many remarkably useful discussions with Dr. M. Srinivasan and Dr. A. Sheets. I thank you for your kindness. Similarly, I thank Dr. M.M. Stanišić at University of Notre Dame, who was instrumental in the development of some key ideas in this work.

I thank the students from the University of Notre Dame for participating in my study. I also acknowledge the efforts of Julian Corona for helping me conduct these experiments. I get by with a little help from my friends, the Beatles sing. My journey for the past six years was rendered positively delightful by several people. Amongst them were Amod Damle, Satya Seetharaman, Sai and Sucheta Bhatwadekar, Janhavi Karandikar, Andrea Cordoba-Arenas, Arpit Mittal, Oren Costantini, Justin Persinger, Natalie Nazaryan, and Sachit Rao. You guys rock!

Waroon, Abhijit and Sangeeta Varde deserve a special mention. The Vardes are my family, and the moral, psychological, and logistic support they have provided has been invaluable.
Finally, I thank my parents. They have loved me without reservation. They have shown great patience toward me and faith in my abilities. I am fully aware that without these people and their support, the present work would not have been possible.
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**Fields of Study**

Major Field: Mechanical Engineering, Human Motor Control, Robot Kinematics and Dynamics
# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>i</td>
</tr>
<tr>
<td>Dedication</td>
<td>iii</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>iv</td>
</tr>
<tr>
<td>Vita</td>
<td>vi</td>
</tr>
<tr>
<td>List of Tables</td>
<td>xi</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xiii</td>
</tr>
<tr>
<td>List of Abbreviations</td>
<td>xx</td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Human motor control</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Human arm motion</td>
<td>4</td>
</tr>
<tr>
<td>1.2.1 The time invariance hypothesis</td>
<td>7</td>
</tr>
<tr>
<td>1.2.2 The leading joint hypothesis</td>
<td>7</td>
</tr>
<tr>
<td>1.3 Kinematic trajectory tracking using serial chains</td>
<td>8</td>
</tr>
<tr>
<td>1.3.1 Singularity navigation with serial manipulators</td>
<td>10</td>
</tr>
<tr>
<td>1.3.2 Trajectory tracking with redundant, planar manipulators</td>
<td>11</td>
</tr>
<tr>
<td>2. The kinematics of spatial human reaching</td>
<td>13</td>
</tr>
<tr>
<td>2.1 Introduction and background</td>
<td>13</td>
</tr>
<tr>
<td>2.2 Experimental methods</td>
<td>25</td>
</tr>
<tr>
<td>2.2.1 Data analysis</td>
<td>32</td>
</tr>
<tr>
<td>2.3 Results</td>
<td>42</td>
</tr>
<tr>
<td>2.4 Discussion</td>
<td>66</td>
</tr>
<tr>
<td>2.5 Conclusions</td>
<td>71</td>
</tr>
</tbody>
</table>
Appendices

A. Linearity of coordination equations ........................................... 214

B. Speed Ratios from Time-Based Joint Motions .......................... 216

C. Non-Ordinary Singularity ......................................................... 218

D. Instantaneous tracking capabilities of three- and four-DOF planar mechanisms .......................... 221
   D.1 Three-revolute (3R) mechanism .......................................... 222
   D.2 PRR Mechanism ................................................................. 224
   D.3 RPR Mechanism ................................................................. 228
   D.4 RRP mechanism ................................................................. 232
   D.5 PPR mechanism ................................................................. 236
   D.6 PRP mechanism ................................................................. 239
   D.7 RPP mechanism ................................................................. 241
   D.8 PPP mechanism ................................................................. 244
   D.9 Four-revolute mechanism, an example of a four-DOF system .......................... 245
List of Tables

Table Page

2.1 Target locations relative to the right shoulder. The target locations were not controlled precisely across subjects. Median values accurate to 5° for the elevation and azimuth angles, and radial distance from the shoulder accurate to 0.05 m are reported. .......................................................... 27

2.2 ANOVA results for the A-H linearity metric for planarized wrist paths. Tukey’s HSD test was used for pair-wise comparisons. ‘S’, ‘N’, and ‘F’ indicate slow, normal, and fast movement speed, respectively. Entries indicate the speed pairs for which there is no significant difference ($P > 0.05$) between the metric means. ‘None’ indicates that the metric means for all three speeds are significantly different ($P < 0.05$). ........................................... 45

2.3 $R^2$ values for the regression of the A-H linearity metric against average wrist speed. Entries in bold are not significant. Entries in brackets are significant at the 0.05 significance level. Other entries are significant at the 0.01 significance level. .......................................................... 46

2.4 Slope (sec / m), and intercept values for the regression of the A-H linearity metric against average wrist speed. Entries in bold are not significant. Entries in brackets are significant at the 0.05 significance level. Other entries are significant at the 0.01 significance level. .......................................................... 46
2.5 ANOVA results for the normal angles. Tukey’s HSD test was used for pair-wise comparisons. ‘S’, ‘N’, and ‘F’ indicate slow, normal, and fast movement speed, respectively. Entries in the table indicate the speed pairs that have significantly different means at the 0.05 significance level. The asterix indicates that wrist paths for the slower of the two movements in the pair loop upward compared to those of the faster movement of the pair. For other cases, wrist paths for the slower movement of the pair loop downward compared to those of the faster movement of the pair. ‘None’ indicates no significant difference \( (P > 0.05) \) between the normal angles for all three speeds.

2.6 The angular spreads for the planarized normals for all subjects. The angles are in degrees and are rounded up to the nearest integer.

2.7 The Atkeson-Hollerbach linearity metric computed for spatial wrist paths. The average value for the thirty repetitions of a task are provided.

3.1 Nomenclature. The symbols used in Figure 3.1 are defined here.

3.2 The classification of all movements as ‘S’: shoulder-led, ‘E’: elbow-led, ‘I’: independent joint control for all speeds. ‘N’ indicates that the motion could not be classified in the above three categories. For each subject / task, the set of three letters classify the slow-, normal-, and fast-speed motions in that order.
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1 Sensor placement. The sensors of the Flock-of-Birds system were used in the reaching experiment to record the positions and orientations of the arm segments.</td>
<td>26</td>
</tr>
<tr>
<td>2.2 The location of the targets relative to the subject’s right shoulder are shown. Tennis balls covered with colored paper serve as targets. The targets were mounted on light, rigid structures fabricated using PVC pipes. The dotted lines in the figure assist in the visualization of the spatial locations of the targets.</td>
<td>27</td>
</tr>
<tr>
<td>2.3 Definition of the reaching tasks. Two views of each task are provided. The first view is as seen by the subject, and the second view is as seen from above the subject’s head.</td>
<td>30</td>
</tr>
<tr>
<td>2.4 The movement-direction vectors and wrist-plane normals for thirty repetitions of Task 6 by subject FR are plotted. Note that all vectors are normalized. The variability in the movement-direction vectors is small. The movement-direction vector for the task is the average of the movement-direction vectors for the thirty repetitions and is indicated as the thick arrow. The movement-direction vector defines the plane of the normals onto which the wrist-plane normals are projected.</td>
<td>37</td>
</tr>
</tbody>
</table>
2.5 A wrist plane is defined as the best-fit plane to the wrist-path data. It is also defined by a normal to it. This normal itself lies in another plane, called plane of the normals, which is defined by the movement-direction vector. The average of the thirty movement-direction vectors from all repetitions of a given task is considered the movement-direction vector for that task. The gravity vector and the thirty path normals for all repetitions of this task are projected onto the plane of normals. The projected normals are called planarized normals. The angle of elevation \( \theta \) of the plane of normals is measured from the horizontal to the movement-direction vector in the plane defined by the gravity vector and the movement-direction vector. The angle is positive upward.

2.6 The wrist paths progress from left to right in the figure. Therefore, this figure describes Tasks 2 through 6. When the plane of the normals is viewed from the right, the planarized normal for a wrist path looping upward will appear to the right of the planarized normal for a wrist path looping downward. This relation will be reversed for Task 1 since the movement is from right to left and the viewing direction is reversed.

2.7 Percent error and goodness of fit for the best-fit plane to the wrist paths for all tasks. Subject JM. All statistical results in this chapter follow the following color code: green - slow-speed movement, red - normal-speed movement, and blue - fast-speed movement.

2.8 Box plot of the A-H linearity metric for the planarized wrist paths. Plot shows data for all tests performed by all subjects. A lower value for the metric indicates greater linearity of the wrist path.

2.9 The average of ten wrist paths of each speed for Tasks 2, 4, and 3 for subject KB are shown. From the top views for each task, it is evident that the wrist paths curve away from the subject. The front views are as seen by the subject. They suggest that the wrist paths for slower movement loop downward when the movement is performed against gravity (Task 2), and the slower movements loop upwards when the movements are performed with gravity (Task 4). There is no discernible difference in the wrist paths for Task 3 which has a wrist path almost parallel to the ground.
2.10 The \textit{planarized normals} for all repetitions of a given task are plotted. The movement-direction vector is pointing out of the plane of the paper. The thick black line represents the gravity vector, also projected onto the plane of normals. The length of the gravity vector indicates the elevation angle of the plane of normals. One-way ANOVA is performed on the angles between the dotted line and the normals. The angles are measured from the dotted line to the normals in the clockwise direction and are called \textit{normal angles}.

2.11 Plot of the angular spread of wrist-plane normals vs. mean A-H linearity metric. A strong linear correlation between the parameters is observed for mean A-H metric values up to 0.1.

2.12 Box plot for the data in Table 2.7. Wrist paths for Tasks 1 and 3 have high degree of linearity, and therefore, these tasks will likely show large angular spread in the planarized normals.

2.13 The elevation angle $\theta$ of the best-fit wrist-path plane for all subjects and tasks.

2.14 Two qualitatively different wrist-speed profiles. The unimodal and multimodal wrist-speed profiles for subject JR are plotted. The profiles are normalized for movement time, but not for peak speed. All fast-speed profiles are unimodal. Most slow-speed profiles are multimodal. Normal-speed profiles can be unimodal or multimodal. Multimodal profiles are observed for all tasks. In all subfigures, the normalized time is plotted on the $x$ axis, and the wrist speed is plotted on the $y$ axis.

2.15 Box plots of the observed number of multimodal wrist-speed profiles for tasks and subjects. Except for Task 3, the number of profiles tends to be independent of the task. The number varies with the individual subject’s interpretation of the ‘normal’ and ‘slow’ for performing those movements.

2.16 The average wrist speed is plotted against the modified movement time for all tests and subjects. Tests that exhibit unimodal wrist-speed profiles are indicated as crosses, and those with multimodal profiles are indicated as circles. The rectangle at the center of the figure defines a transition zone between tests exhibiting unimodal and multimodal wrist-speed profiles.

2.17 The variability in the normalized wrist-speed profiles within each speed. The variability in the profiles for fast-speed movements is lower than that for the normal-speed or slow-speed movements for all subjects.
2.18 The A-H linearity metric is plotted against the modified movement time and average wrist speed for all tasks and subjects. The tests that have unimodal wrist-path shape are indicated as crosses, and those with multimodal wrist-shape profiles are indicated as circles. A wrist path with a given A-H metric value can have a unimodal or a multimodal wrist-speed profile. Conversely, multimodal wrist-speed profiles are observed for wrist paths of varying degree of straightness. The figure shows that multimodal wrist-speed profiles are not seen exclusively for highly curved wrist paths.

3.1 The top figure of the human arm is borrowed from www.flashkid.org. The free-body diagrams (FBD) of the upper arm and the forearm are shown. The hand is modeled as a point mass rigidly attached to the end of the forearm. Coordinate frames \( U \) and \( F \) are fixed to the upper arm and the forearm, respectively, and the equations of motion are written in the global reference frame.

3.2 Hypothetical examples to explain the use of the three metrics to deduce the relation between MT, NT and IT.

3.3 Duration of the total movement for which MT is assistive. The figure shows mean ± SD for all tasks and speeds. The first, second and third rows of figures show data for slow-, normal- and fast-speed movements, respectively. Subject JM.

3.4 Mean ± SD of MT and IT impulses at the shoulder and the elbow for all tasks and speeds. The first, second and third rows of figures show data for slow-, normal- and fast-speed movements, respectively. Subject JM.

3.5 Mean ± SD of the vestigial impulse factor \( \Psi \) at the shoulder joint for all tasks and speeds. Subject JM.

3.6 Mean ± SD of the joint impulses for the cases that cannot be classified as either shoulder- or elbow-led or as independent joint control. The first, second and third rows of figures show data for slow-, normal- and fast-speed movements, respectively. The x-axis labels show the subjects and tasks that exhibit this behavior.
3.7 Mean ± SD of the joint excursions for the subjects that exhibit elbow-led control strategies. The elbow excursion is simply the Z Euler-angle displacement when the $Z - X' - Y''$ Euler-angle sequence is used to relate the orientation of the forearm to that of the upper arm. The shoulder joint excursion is the angle obtained from the axis-angle parametrization [145] of the initial and final orientations of the upper-arm coordinate frame. The movements that are classified as elbow-led are marked by an asterix. The number on top of each bar indicate the excursion ratio.

3.8 Mean ± SD of peak wrist speeds and joint excursions for the subjects that exhibit the independent-joint-control strategy for some motions. The movements that display the independent-joint-control strategy are marked by an asterix. For all three subjects, the slow-speed movement for Task 4 shows the independent-joint-control strategy. For subject CR, the slow- and normal-speed movements for Task 6 show the independent-joint-control strategy.

4.1 The spatial 3R mechanism is required to track the desired path. The vectors $\hat{T}$, $\hat{N}$, and $\hat{B}$ form the trihedron of the FS frame and describe the path geometry. Geometry is tracked by relating the motions of the joint variables $\mu$ and $\nu$ to that of joint variable $\lambda$.

4.2 The paths traced by point $P$ on the EE of the 3R mechanism for first-, second-, and third-order joint coordination are shown. Higher order joint coordination results in $P$ following the desired path more closely near the zero position.

4.3 A six-revolute robot. Reference frames 3, 4 and 5 are centered at the wrist. They are drawn separately for clarity. The wrist center is used as a control point, so $\Psi = 0$ and $d_6 = 0$.

4.4 The three control points on the EE form a triangular lamina. The initial, the final and an intermediate position of the lamina are indicated. The orientation of the EE must be constant during the motion, so all control points follow identical paths.

4.5 Position errors for the three control points are close to each other. The maximum error is $2.6 \times 10^{-5}$ units.
4.6 Joint velocities for the tracking task. The velocities are bounded throughout the task. The slopes of the profiles indicate the joint accelerations. The accelerations are high toward the end of the motion for \( \theta_2, \theta_3 \) and \( \theta_4 \).  

5.1 A planar 3R mechanism is shown. The coordinate frame \( XY \) indicates the ground link.  

5.2 A planar 3R mechanism is required to track the circular desired path indicated by the solid circle passing through point \( P \). The geometric and the pseudoinverse tracking solutions are shown. The curvature center of only the geometric solution matches that of the desired path. In all the figures of this chapter demonstrating results, the tracking solution is plotted until the position error reaches a predetermined value. The position error is defined as the minimum distance between the current position of the EE from the desired path.  

5.3 A planar 3R mechanism is required to track the circular desired path indicated by the solid circle passing through point \( P \). The geometric and the pseudoinverse tracking solutions are shown. The curvature center of the geometric solution is closer to that of the desired path.  

5.4 The first two coordination equations are solved to obtain the solution set for the 4-DOF tracking system. For a given value of the tracked curvature \( \kappa_d \), real first-order speed ratios that track the instantaneous tangent and curvature of the desired path exist if the intersection of the surface \( \Delta(\kappa, \phi) \) and \( \kappa = \kappa_d \), when projected onto the \( \Delta - \phi \) plane, passes through the first quadrant of the \( \Delta - \phi \) plane. The figure shows four generic solution sets based on the sign of the slope \( m \) and the \( \Delta \)-intercept \( c \). The solid lines indicate real solutions, and dashed lines indicate complex solutions.  

5.5 The planar 4R mechanism is shown. The coordinate frame \( XY \) indicates the ground link.  

5.6 Tracking results for Case 1 illustrated in Figure 5.4.  

5.7 Tracking results for Case 2 illustrated in Figure 5.4.  

5.8 Tracking results for Case 3 illustrated in Figure 5.4.  

5.9 Tracking results for Case 4 illustrated in Figure 5.4.
D.1 A planar PRR mechanism. ........................................... 225
D.2 A planar RPR mechanism. ........................................... 229
D.3 A planar RRP mechanism. ........................................... 233
D.4 A planar PPR mechanism. ........................................... 236
D.5 A planar PRP mechanism. ........................................... 239
D.6 A planar RPP mechanism. ........................................... 242
D.7 A planar PPP mechanism. ........................................... 244
# List of Abbreviations

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-H metric</td>
<td>Atkeson-Hollerbach metric</td>
</tr>
<tr>
<td>ANOVA</td>
<td>Analysis of Variance</td>
</tr>
<tr>
<td>COM</td>
<td>Center of Mass</td>
</tr>
<tr>
<td>CNS</td>
<td>Central Nervous System</td>
</tr>
<tr>
<td>DALYs</td>
<td>Disability-Adjusted Life-Years</td>
</tr>
<tr>
<td>DGi</td>
<td>The $i^{th}$ Definition for Gravity torque</td>
</tr>
<tr>
<td>DH</td>
<td>Denavit-Hartenberg</td>
</tr>
<tr>
<td>Dli</td>
<td>The $i^{th}$ Definition for Interaction torque</td>
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<tr>
<td>DMi</td>
<td>The $i^{th}$ Definition for Muscle torque</td>
</tr>
<tr>
<td>DNi</td>
<td>The $i^{th}$ Definition for Net torque</td>
</tr>
<tr>
<td>DOF</td>
<td>Degrees of Freedom</td>
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<tr>
<td>EOM</td>
<td>Equation of Motion</td>
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<td>EE</td>
<td>End Effector</td>
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<td>EMG</td>
<td>Electro-Myography</td>
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<td>FBD</td>
<td>Free Body Diagram</td>
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<td>FOB</td>
<td>Flock of Birds</td>
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<td>FS</td>
<td>Frenet-Serret</td>
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<tr>
<td>GT</td>
<td>Gravitational Torque</td>
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<tr>
<td>HSD</td>
<td>Honestly Significant Difference</td>
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<tr>
<td>II</td>
<td>Instantaneous Invariant</td>
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<tr>
<td>ISB</td>
<td>International Society of Biomechanics</td>
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<tr>
<td>IT</td>
<td>Interaction Torque</td>
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<tr>
<td>LJH</td>
<td>Leading Joint Hypothesis</td>
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<td>LHS</td>
<td>Left Hand Side</td>
</tr>
<tr>
<td>MT</td>
<td>Muscle Torque</td>
</tr>
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<td>NT</td>
<td>Net Torque</td>
</tr>
<tr>
<td>PCA</td>
<td>Principal Component Analysis</td>
</tr>
<tr>
<td>PC</td>
<td>Principal Component</td>
</tr>
<tr>
<td>PPP</td>
<td>Prismatic-Prismatic-Prismatic</td>
</tr>
<tr>
<td>PPR</td>
<td>Prismatic-Prismatic-Revolute</td>
</tr>
<tr>
<td>PRP</td>
<td>Prismatic-RevolutePrismatic</td>
</tr>
<tr>
<td>RBC</td>
<td>Rigid Body Constraints</td>
</tr>
<tr>
<td>RHS</td>
<td>Right Hand Side</td>
</tr>
<tr>
<td>RR</td>
<td>Revolute-Revolute</td>
</tr>
<tr>
<td>RRP</td>
<td>Revolute-Revolute-Prismatic</td>
</tr>
<tr>
<td>RPP</td>
<td>Revolute-Prismatic-Prismatic</td>
</tr>
<tr>
<td>RP</td>
<td>Revolute-Prismatic</td>
</tr>
<tr>
<td>RPR</td>
<td>Revolute-Prismatic-Revolute</td>
</tr>
<tr>
<td>Abbreviation</td>
<td>Description</td>
</tr>
<tr>
<td>--------------</td>
<td>-----------------------------</td>
</tr>
<tr>
<td>SD</td>
<td>Standard Deviation</td>
</tr>
<tr>
<td>SR</td>
<td>Speed Ratio</td>
</tr>
<tr>
<td>TIH</td>
<td>Time Invariance Hypothesis</td>
</tr>
<tr>
<td>3P</td>
<td>3-Prismatic</td>
</tr>
<tr>
<td>3R</td>
<td>3-Revolute</td>
</tr>
</tbody>
</table>
Chapter 1: INTRODUCTION

1.1 Human motor control

The field of human motor control engages in the discovery of computational principles, and their implementation in the central nervous system (CNS), that are involved in organizing the musculo-skeletal system into creating organized movements and skilled actions. Producing voluntary, or purposeful, movement is one of the fundamental functions of the CNS, yet it is arguably one of the most complex. Neuroscientists, biomechanists, computer scientists, mathematicians, philosophers, psychologists, cognitive scientists, roboticists, among others, have been actively contributing to enhance the understanding of motor control. A brief list of movement phenomena that have been studied so far includes locomotion [124], reaching [169], manipulating objects with the hand [9, 93], bicycling [175], and control of the eyeballs [137]. The body of literature in these areas is extensive, dating back over a century. For example, Donders’ law, which relates the torsion of the eyeball to the other two rotations (elevation-depression and left-right gaze), was formulated in the 1840’s [137]. Muybridge [109] studied the human walking cycle by photographing a man in his stride in 1887, and Woodworth [169] published a psychophysical study on hand movement in 1899. Each problem presents a different set of challenges for the CNS and, indeed, for the researchers who wish to understand them.
There are some established principles that unify motor control, however. The view of the CNS as an information processing system is now well entrenched in the field [113]. In fact, the American philosopher Daniel Dennett goes so far as to claim that consciousness is an epiphenomenon that emerges from a highly complex information processing system [29]. Stemming from evolutionary biology is the idea that the human animal performs movement in the most economical way in order to maximize its chances of survival. Based on this view, optimal control theory has been applied to understand human locomotion [146], reaching [61, 151], and jumping [116], among other phenomena. Sometimes, there also appear principles that promise to unify a specific set of disparate movement phenomena. For example, there was a concentrated effort to unify rotational motion of the eye with that of the arm using Donders’ law as a common control principle [142].

The holy grail for the field of motor control would be a theory that explains how all human movement - voluntary and involuntary - is composed, executed and controlled by the CNS. That is to say, the theory would explain how muscle activation signals corresponding to an intended motor action are composed and subsequently altered to account for external disturbances. This theory would probably consist of a hierarchical structure of a number of principles that are applicable under different task requirements or contexts. For example, it is likely that the CNS uses one control algorithm while executing a pointing movement and another for piano playing. Although both movements involve motion of the arm, the accuracy requirement, i.e., the context, is different. Furthermore, a grand theory of motor control would also identify the corresponding neural correlates in the CNS. It would specify which part of the CNS is responsible for executing an optimization calculation, for example, and how this calculation is executed by a network of neurons.
Understanding human motor control, apart from being a fascinating endeavor in and of itself, has important applications. An obvious application of the knowledge gained from the study of healthy human movement is in the diagnosis and treatment of pathological movement. Stroke has been a leading cause of movement pathologies, with disorders like apraxia (characterized by loss of the ability to execute or carry out learned purposeful movements) and ataxia (gross lack of coordination of muscle movements) common among patients. Disabilities similar to those resulting from stroke can also be caused by multiple sclerosis, Parkinson’s disease, and traumatic brain and spinal cord injury [121]. This problem cannot be overstated; Donnan et al. [31] report that in 2002, stroke-related disability was judged to be the sixth most common cause of reduced disability-adjusted life-years (DALYs - the sum of life-years lost as a result of premature death and years lived with disability adjusted for severity). However, because of the burgeoning elderly population in Western societies, the estimation is that by 2030, stroke-related disability in Western societies will be ranked as the fourth most important cause of DALYs. Worldwide, stroke consumes about 24% of total health-care costs, and in industrialised countries, stroke accounts for more than 4% of direct health-care costs. The total costs to society have been variously estimated at £7.6 billion in the UK (US$ 12 billion) at 1995 prices, AUS$1.3 billion in Australia (US$ 1 billion), and US$40.9 billion in the USA at 1997 prices [31].

Clinically-focused, robot-assisted rehabilitation is a field that has benefited directly from motor control research. Since the mid 1990s, robot-assisted rehabilitation has been investigated as a means to improve post-stroke motor recovery. It has the potential to cost-effectively deliver precisely standardized and online-adaptable therapy that is coupled with multi-sensory feedback [3]. As a tool for quantified evaluation of muscular weakness, increased muscle tone, and movement incoordination, a robotic device can serve both
diagnostic and therapeutic purposes [120]. Early work has shown that robot-assisted therapy can speed recovery compared to traditional therapy [101] and can improve recovery as a supplement to traditional therapy [37, 163]. Yadav [171] developed novel metrics for quantifying the quality of reaching motion by studying the movement patterns of healthy individuals using a robotic device. An attempt was made to compare the metric values for motions of stroke patients to quantify their progress during therapy. Shadmehr, Mussa-Ivaldi, and their colleagues have extensively investigated human ability to adapt to robot-applied force fields to gain insight into how the CNS controls movement [135, 150] and how training forces could be used for teaching and rehabilitating motor skills [117].

1.2 Human arm motion

Human arm motion has been extensively studied. Based on the pioneering work of Bernstein [17], researchers have proposed several models of trajectory generation for arm motion, which Todorov and Jordan [152] classify into three groups. The first group of models are optimal-control-based models of movement production. Harris and Wolpert [61] assume that the noise in the muscle activation signals is proportional to the magnitude of the signals. Then, they obtain the activations by minimizing the variability in the final arm pose. Other proposed models have used purely kinematic (minimum jerk [43], minimum snap [122], etc.) as well as dynamic (minimum torque rate [155], minimum work [142], minimum energy [4], etc.) cost functions, and Dul et al. [36] provide a minimum-fatigue criterion to achieve muscle forces. For a review of optimal-control models, see Todorov [151].

The second group of models simply quantify regularities in arm movement. Hand path shape in pointing or reaching movements, often roughly a straight line, has been shown to
be independent of trajectory speed [11, 107, 110, 140]. Similarly, fixed relations between instantaneous elbow and shoulder angular positions have been observed across a range of speeds and tasks [84, 88, 89, 138, 140, 141, 159]. The tangential velocity of the hand has a single, bell-shaped curve regardless of its magnitude [11, 64, 81, 107, 110]. In handwriting and drawing, tangential velocity is proportional to the radius of curvature of the trajectory such that the angular velocity takes on several distinct, constant values during the movement and the corresponding movement segmentation is independent of the total duration [161, 162].

The third group of models consists of partial models of trajectory formation. These are intermediate models that attempt to bridge the gap between providing a complete recipe for trajectory formation (a.k.a. optimal-control models) and simply quantifying intrinsic regularities of observed trajectories. Such models assume that the motor task is transformed into a compact spatio-temporal representation, involving a small number of parameters that are sufficient to generate the observed trajectory. Morasso and Mussa-Ivaldi [108] suggested that complex hand trajectories are composed of partially overlapping linear strokes (modeled as B splines with bell-shaped speed profiles) that can be scaled and positioned so as to form a desired trajectory. The parameters of the model (timing, scale, and position of each segment) initially were extracted from the experimental data and then adjusted iteratively until a good fit to the observed trajectories was obtained.

In addition to these studies, Flash and her coworkers have concentrated on studying the differential geometry of reaching motions [16, 42]. Recently, inverse optimization techniques have been introduced to identify what is being optimized in human actions with respect to various aspects of human movements and different motor tasks. From the mathematical point of view, this problem consists of finding an unknown objective function
given the values at which it reaches its minimum [149]. Sainburg, Dounskaia, and colleagues have concentrated on explaining the role of interaction effects in multi-joint, arm movement [32, 33, 47, 128]. Furthermore, Sainburg and Kalakanis [125] and Beer et al. [15] compare interaction effects in healthy subjects and hemiparetic patients. Finally, arm motion has been explained using the equilibrium-point hypothesis. This hypothesis views the muscle as a spring-mass system in which the springs possess variable length-tension relationships and final limb position corresponds to the equilibrium point of the system [38]. Trajectory generation, then, is achieved by specifying a sequence of equilibrium points.

While significant research effort has explored using optimal-control frameworks to explain movement phenomena including arm motion, the optimal control problem is computationally intense [173]. Therefore, a relevant question is whether the CNS performs optimization for planning and executing mundane, rehearsed movements. It is proposed in this work that the CNS identifies invariant patterns or characteristics in the presumably optimal behavior (motion kinematics and dynamics) and uses them to compose fresh instances of motion rather than solve the optimal control problem de novo. This strategy leads to computational savings. Thus, rather than trying to identify what cost functions describe data best, the focus here is to look for invariant patterns or characteristics in presumably optimal behavior and describe how these patterns are used to compose fresh instances of motion.

Notice that the proposed strategy constitutes a model that belongs to the third group as per Todorov and Jordan [152], specifically to address the computational complexity of the optimal-control algorithms.

The present work investigates two hypotheses that encode this approach, namely, the time invariance hypothesis (TIH) [5], and the leading joint hypothesis (LJH) [32]. These
two hypotheses are proposed based on experimental evidence gathered from a host of planar arm motion studies (mentioned below). Now, it is a sensible strategy to reduce the number of variables in phenomena as complex as human motor control so that governing principles may be identified. In the present context, restricting arm movement to a single plane allowed the identification of motion characteristics that led to the TIH and the LJH. However, having identified these principles for a restricted class of movements, it becomes vital to validate these principles for a wider class of possibly more natural movements. Therefore, the specific objectives of this work are to extend the TIH and the LJH to spatial arm motion.

1.2.1 The time invariance hypothesis

Chapter 2 aims to extend the TIH to spatial arm motion. The TIH pertains to movement kinematics, and it claims that the fundamental movement plan composed by the CNS is the wrist path that has a characteristic shape. It further claims that this movement goal is converted into a joint-space motion objective in a particular way - using a time-invariant transformation. The wrist-path shapes are characteristics of presumably optimal movement, and it is by enforcing this characteristic on new instances of motion that the CNS ensures optimality. The TIH gathers experimental evidence from several horizontal-plane [107, 110, 140] and vertical-plane [11] reaching studies, in which the wrist paths displayed characteristic shapes that were independent of movement duration.

1.2.2 The leading joint hypothesis

Chapter 3 aims to extend the LJH to spatial arm motion. The LJH pertains to movement dynamics, and it proposes a strategy to utilize the characteristics of the interaction effects in human arm motion. When one of the arm’s joints is actuated, it simultaneously causes
movement at all other joints\textsuperscript{1}. This effect is called interaction. Based on inverse-dynamics computations, several studies have reported that the interaction effects at (usually) the proximal joint for a large class of reaching movements is negligible [21, 33, 47, 75, 92, 63]. This joint is called the \textit{leading joint}, and other joints of the arm are called \textit{subordinate joints}. Based on this evidence, the LJH was proposed which claims that to achieve the joint-space motion objective, the CNS ignores the interaction effects at the leading joint. This strategy simplifies inverse-dynamics computations and joint-level control, thus yielding computational savings. Further support to the LJH is offered by muscle electro-myographical (EMG) signals that show reciprocal bursts of activity at the leading joint that are tightly coupled with its acceleration and deceleration [33, 92]. The EMG activity at the subordinate joint is more complex. All the studies mentioned above except [63] are planar motion studies.

1.3 Kinematic trajectory tracking using serial chains

The analyses presented in Chapters 2 and 3 model the human arm as a serial, kinematic chain. This is a common practice in arm motion studies [32, 33, 63, 92]. Now, the TIH claims that a reaching task is defined in terms of a specific wrist trajectory, and this information is converted into a corresponding joint-space trajectory via an inverse kinematic transformation. Furthermore, the TIH specifies the nature of the input and the output of the inverse kinematics transformation without specifying a computational scheme for executing this transformation. This process is modeled as a trajectory tracking problem using

\textsuperscript{1}Zajac [175] provides an anatomical viewpoint of this effect by explaining how a muscle can act to accelerate joints that it does not span.
the underlying serial chain. Chapter 4 develops an algorithm that executes trajectory tracking in a way required by the TIH, and, together with the TIH, provides a more complete, plausible computational account for the inverse kinematics of spatial, human reaching.

The problem of trajectory tracking has been well studied in the robotics field [24]. Traditionally, spatial trajectory tracking requires that the end-effector (EE) of a manipulator follow a trajectory that is described using time as the independent path variable. Therefore, the corresponding joint-space trajectory is also typically described as a function of time [24]. In particular, no distinction is made between the control of the output-space path geometry and the control of the independent path variable [100]. The TIH makes this exact distinction by claiming that the planned wrist trajectory has a characteristic shape. Therefore, the traditional method of solving the trajectory tracking problem prevalent in the robotics field cannot serve as the algorithm for implementing the TIH.

Kieffer and Litvin [78] developed a method for spatial trajectory tracking with serial manipulators which produces the joint-space trajectory parameterized in terms of an arbitrary independent variable. To implement any joint-space solution, the independent variable must be measured, since it is a necessary input to the trajectory tracker. The algorithm of Kieffer and Litvin [78], in its abstract form, cannot be used to describe human reaching because one must specify the independent variable based, in part, on the sensing capabilities of the CNS. Chapter 4 builds on the work of Kieffer and Litvin [78], Kieffer [76], Lloyd [95], and Stanišić and colleagues [18, 98, 99, 100, 123, 147] to develop an algorithm that uses the displacement of one joint of the kinematic chain, rather than an arbitrary variable, to parameterize the joint-space trajectory. In humans, the measurement of the leading joint position is provided by kinesthesia. Mechanoreceptors in the periarticular soft tissue
around a joint provide the sense of joint position under static as well as dynamic conditions [67, 126, 157]. This joint variable is the kinematic counterpart of the leading joint defined in the description of the LJH, and is therefore called the *leading joint variable*. The algorithm solves the problem of trajectory tracking in two steps. The solution to the *geometric tracking subproblem* is a joint-space path corresponding to a given output-space path (defined as a trajectory sans the temporal information). The solution to the *temporal tracking subproblem* provides the time-based joint motions (i.e. joint speeds and accelerations) when the output-space speed and its derivatives are specified. It thus distinguishes the control of the path geometry from the control of the independent path variable.

1.3.1 Singularity navigation with serial manipulators

In developing the trajectory-tracking algorithm, there is a shift from studying human movement to studying movement more generally of serial kinematic chains or manipulators. Because of this abstraction, the developed algorithm has applications to serial manipulators. In fact, a significant motivation for developing the algorithm in Chapter 4 comes from the problem of singularity navigation. This is an important issue in robotic manipulation that limits the usable workspace of a manipulator. Humans, however, seem to have no trouble navigating through singularities. This provides the motivation for applying to robotic manipulation the algorithm developed to explain human movement.

*Singularities* are configurations of a manipulator in which the motion capabilities of the manipulator are diminished in specific ways [145]. Furthermore, the effect of the singularity is evident in its neighborhood; typically, joint speeds become unbounded [24]. Lorenc et al. [100] pointed out that in singular configurations, the control of the path geometry can be decoupled from the control of the independent path variable, and in this
way, arbitrary geometric accuracy of tracking can be achieved with bounded joint speeds, although at the expense of temporal tracking accuracy. Several approaches tackle the problem of trajectory tracking in the vicinity of singularities, and comprehensive reviews of these methods are available [23, 95]. Most methods do not explicitly control the independent variable [24, 165, 167], and therefore, geometrical imprecision of tracking is intrinsic to these methods. In contrast, the algorithm developed in Chapter 4 does separate these aspects, and as a result, it is capable of traversing certain kinds of singular poses with arbitrarily high geometric accuracy.

1.3.2 Trajectory tracking with redundant, planar manipulators

The algorithm developed in Chapter 4 is applicable to non-redundant kinematic chains. However, most kinematic models of the human arm used to describe spatial reaching, including the one used in Chapters 2 and 3, are redundant. Therefore, the algorithm must be extended to incorporate kinematic redundancy before it can completely describe human reaching motion kinematics as per the TIH.

In the field of robotics, redundant manipulators are increasingly employed in useful practical tasks that are specified in terms of a geometric path to be followed by the EE. Redundant DOFs make it possible to achieve objectives such as avoiding collisions, joint limits and/or singular configurations. However, objective criteria need to be specified to resolve the kinematic redundancy. Kinematic performance metrics, such as locally bounded joint-space velocities, involve computation of damped least-squares solutions [24], although such pseudoinverse-based control cannot avoid singular configurations [12]. Alternatively, time-optimal control uses the manipulator dynamics to minimize the performance
time, which is a solved problem for non-redundant manipulators [136]. For kinematically redundant manipulators, numerical procedures have been proposed by Galicki [45] to achieve path-constrained time-optimal control. A computationally efficient feedback-control law is developed by Galicki [46] that provides joint forces/torques for a redundant manipulator while minimizing the output-space tracking error. These methods use information from the output-space path up to the first-order only, whereas the definition of the desired output-space path contains more geometric information in the form of higher order derivatives. Effective utilization of this path information can reduce the required feedback frequency for a desired tracking accuracy and potentially, the computational cost of path tracking.

Chapter 5 begins the task of generalizing the algorithm developed in Chapter 4 to include redundant serial chains. The focus here is on three- and four-DOF planar mechanisms. Using only joint velocity control, it is demonstrated how the best possible geometric path tracking can be achieved.
2.1 Introduction and background

Coordinating voluntary, or purposeful, movement is one of the fundamental functions of the human central nervous system (CNS), yet it is arguably one of the most complex. A reaching task is defined in the output, or extra-personal space, and this task definition must be transformed into the appropriate neural commands that activate the appropriate muscles to execute the desired task. This conversion can be modeled as a series of four inverse transformations [10], called so because they compute the causes (muscle activations) that will generate the desired effects (movement). First, the output-space task definition is converted into a joint-space plan. This is the solution to the inverse kinematics problem for the arm. Next, the joint-level plan is converted into torques that must be applied at those joints via an inverse dynamics calculation. The joint torques are then transformed into the muscle forces that produce the desired joint torques, and finally, the forces are transformed into muscle activations. Now, there are approximately 640 muscles, over 100 articulations and 206 bones in the human body. Not only does the CNS have to control this massive musculo-skeletal network, it has to do so using noisy communication channels (nerves) [61] and in the presence of uncertain information about an uncertain environment.
The motor systems in the CNS are served by neural mechanisms that are finely tuned to tackle these challenges and to plan, coordinate, and execute muscular movements. It has long been understood that these mechanisms are complex information-processing systems [113], and the commonly used strategy to understand these systems is to study them at three levels as distinguished by Marr [102]. The top level is the abstract computational theory of the studied system that defines inputs and outputs of the system and abstract properties of the mapping between them. The next level is understanding the algorithm used for the implementation of the mapping. Finally, the third level is investigation of how the algorithm revealed at the second level is realized physically. This involves revealing neuronal structures that implement the operations performed by the algorithm. The present work studies the phenomenon of human reaching, and it is restricted to Marr’s first and second levels.

A substantial body of work concentrates on developing models to describe human reaching motion that belong to Marr’s first and/or second level. Todorov and Jordan [152] classify these models into three groups. The first group of models is based on optimal-control theory. These are comprehensive models that provide a complete ‘recipe’ of trajectory generation, starting from a task description defined in the extra-personal space and leading to an observable behavior. The general strategy is to formulate the problem as a constrained dynamic optimization problem. This involves defining a cost function that must be minimized (e.g., path smoothness) and the constraints (e.g., arm dynamics, muscle forces) on the system. Then, the observed trajectory is the one that minimizes the cost function while respecting the constraints; it is an epiphenomenon, a consequence of the minimization process. Optimal-control theory is a versatile mathematical technique, and it allows formulation of the motor-control problem as a single problem encompassing the
four inverse problems starting from the output-space task description and leading to the
muscle activations. For example, Harris and Wolpert [61] assume that the noise in the mus-
cle activation signals is proportional to the magnitude of the signals. Then, they obtain the
activations by minimizing the variability in the final arm pose. The movement kinematics
and dynamics are thus a consequence of this strategy and are not explicitly planned. The
optimal-control approach also allows the formulation of each of the four inverse problems
separately. For example, proposed models have used purely kinematic (minimum jerk [43],
minimum snap [122], etc.) as well as dynamic (minimum torque rate [155], minimum work
[142], minimum energy [4], etc.) cost functions, and Dul et al. [36] provide a minimum-
fatigue criterion to achieve muscle forces. For a review of optimal-control models, see
Todorov [151].

The second group of models simply observe regularities in the observed behavior. One
example is the two-thirds power law that relates the hand speed and the trajectory curvature
for planar cyclic and drawing movements [87]. The time-invariant nature of the wrist paths
for reaching arm movements in the horizontal plane [107, 140] and the vertical plane [11]
are also models that belong to the second group.

The third group consists of partial models of trajectory formation. These are inter-
mediate models that attempt to bridge the gap between providing a complete recipe for
trajectory formation, as in the models belonging to group one, and simply quantifying in-
trinsic regularities of observed trajectories, as in the models belonging to group two. Such
models assume that the motor task is transformed into a compact spatio-temporal represen-
tation involving a small number of parameters that are sufficient to generate the observed
trajectory. Morasso and Mussa-Ivaldi [108] suggested that complex hand trajectories are
composed of partially overlapping linear ‘strokes’ (modeled as B splines with bell-shaped
speed profiles) that can be scaled and positioned so as to form a desired trajectory. The parameters of the model (timing, scale, and position of each segment) initially were extracted from experimental data and then adjusted iteratively until a good fit to the observed trajectories was obtained.

The optimal-control-based models are arguably the strongest motor-control models available. However, there is no principled explanation as to why the CNS should have evolved to minimize such complex quantities as jerk or torque change, or why human reaching movement should be smooth [61]. From an evolutionary perspective, this behavior presumably serves to enhance the fitness of the animal - a claim that is difficult to verify. Furthermore, optimal-control-based models for motor control have other serious drawbacks [134]. The primary difficulty is their lack of falsifiability. These models cannot be falsified because, in theory, the cost functions can always be modified so that simulation results match the experimental trajectories. In other words, without a priori knowledge of costs and rewards of the movement, it is not be possible to make quantitatively reliable predictions of behavior. Without such predictions, the theory cannot be experimentally falsified. The second difficulty is regarding the time scales of the optimization. Is optimization computed in the reaction time of each movement de novo? Studies have suggested that the timescale is longer than a single movement [134]. This concern may be expressed in terms of the computational effort required in solving optimal-control problems. These problems suffer from the ‘curse of dimensionality’, which causes the computation time to increase exponentially with the degree of complexity of the problem [173]. The third difficulty is regarding the timescale of system identification. Optimal-control-based models of motor control require the use of state estimators that generate internal representations of the body. However, the body changes over multiple timescales. Muscles fatigue and recover quickly;
objects are lifted and replaced rapidly; aging can produce gradual loss of motor neurons and transformation of muscle fibers. Thus, motor control algorithms must be capable of anticipating and compensating for changes in the motor apparatus that occur at significantly different time scales. Finally, it is unknown how the CNS could estimate the complex quantities that constitute the objective functions and integrate those over entire trajectories in order to arrive at an optimal solution [61].

This chapter concentrates on the second difficulty expressed above. It provides an alternative to the optimal control approach for composing and controlling individual instances of reaching motion. It is proposed that the CNS identifies invariant patterns or characteristics in the presumably optimal behavior (motion kinematics and dynamics) and uses them to compose fresh instances of motion *rather than solve the optimal control problem de novo*. This strategy leads to computational savings and forms a part of a *hybrid algorithm*. Although the present work does not verify the claim of computational savings for human reaching, hybrid strategies wherein computationally demanding components in an algorithm are replaced by memory in a way that minimally influences its performance are commonly used in robotics [119] and have been proposed in motor-control literature [10, 32, 74]. Therefore, rather than trying to identify what cost functions describe data best, the focus here is to look for invariant characteristics in motion kinematics. The work of Shadmehr and Mussa-Ivaldi [135] provides strong experimental support to this idea. They performed a study wherein healthy human subjects performed reaching motions in the horizontal plane while holding the end-effector of a robot manipulandum. The robot imposed a force field on the arm which resulted in distorted hand trajectories in the initial phase of the experiment. As the subjects adapted to the force field, hand trajectories converged to a path very similar to that observed in free space. This observation yields
two important conclusions. First, there exists a kinematic plan independent of dynamic conditions, which indicates the separation of these aspects of the motor control problem. Secondly, the wrist paths are suboptimal in terms of minimizing muscle forces. Therefore, there is convergence towards a kinematic characteristic of optimal movement under normal circumstances at the expense of optimality at the dynamic level. This is a strong indicator that the CNS observes and uses characteristics of optimal movement, rather than perform the optimization \textit{de novo}, for composing a movement plan for a fresh instance of reaching.

Notice that the objective here is to propose a model that belongs to the third group as per Todorov and Jordan [152] to address the computational complexity of the optimal-control algorithms. It is the purpose of this study to identify patterns in arm movement and describe how those can be used to execute fresh instances of reaching motion. This chapter studies the time-invariance hypothesis (TIH) [5]. In studying this hypotheses, extensive use is made of the notion of ‘\textit{internal models}’. This idea must be explained before the TIH can be elaborated.

A commonly held theory in the literature states that through experience, the CNS builds and maintains internal models of the motor apparatus and the external world [10, 74]. These are neural mechanisms that can mimic the input-output characteristics, or their inverse, of the motor apparatus. A number of internal-model concepts that are supported by neurophysiological, behavioral, and imaging data are prevalent in motor control literature [40, 74, 151]. Internal models are of two types. An inverse internal model achieves the inverse kinematic or dynamic transformation. A forward internal model uses an efference copy of motor commands and predicts the probable kinematic or dynamic outcome of those. Inverse internal models are inevitable; motion objectives must be mapped into muscle activations. Furthermore, the adaptation study of Shadmehr and Mussa-Ivaldi [135]
mentioned above implies the existence of internal inverse models. A subject’s adaptation is viewed as the inverse dynamic model changing to represent the inverse of the combined dynamics of the arm and the external force field. The most convincing data for the existence of forward (and inverse) models comes from studies on coordination between reaching and grasping. When an object is held with the tips of the index finger and thumb on either side, the grip force is precisely controlled so that, under normal conditions, it is just slightly greater than the minimum grip force needed to prevent slip [74]. Such a grip-force / load-force coupling is explained by a framework that contains both the inverse and forward models of the arm. For a point-to-point arm-reaching movement with the grasped object, the inverse model of the combined dynamics of the arm, hand and object calculates the necessary motor commands from the desired trajectory of the arm. These commands are sent to the arm muscles as well as to a forward dynamics model as the efference copy. Then, the forward model can predict an arm trajectory that is slightly in the future. Given the predicted arm trajectory, the necessary minimum level of grip force can be computed. To realize this grip force time course, motor commands are sent to hand muscles.

Now, experimental evidence supports the existence of separate kinematic and dynamic internal models in the CNS. In exposing subjects to environments with novel kinematic and dynamic transformations, Flanagan et al. [40] demonstrated that humans can compose separate internal kinematic and dynamic models, but successful decomposition from the paired transformations to isolated ones was only evident for the kinematic transformation. Scheidt et al. [131] observed that disadaptation to a removed, learned force field was influenced by both kinematic and dynamic criteria, but on two different time scales, the kinematic error minimization occurring more quickly.
Further evidence clearly suggests that the internal kinematic model may separate time-invariant and time-dependent aspects of motion. Hand path shape in pointing or reaching movements has been shown to be independent of trajectory speed in horizontal-plane [107, 140] and vertical-plane [11] studies. The paths were roughly straight in the horizontal plane and curved in the vertical plane. The tangential speed of the hand has a single, bell-shaped curve regardless of its magnitude [2, 11, 43, 107, 140]. In handwriting and drawing, tangential velocity is proportional to the radius of curvature of the trajectory such that the angular velocity takes on several distinct, constant values during the movement and the corresponding movement segmentation is independent of the total duration - the so-called two-thirds power law [87, 161]. Piano playing is another skilled task in which the set of ratios between interstroke intervals is independent of the duration [143]. Human subjects proved to be incapable of adapting to robot-applied force fields that depended explicitly on time rather than motion variables [26], but successfully generalized adaptation to motion-variable-variable-dependent fields across different movements to the same regions of the fields [25]. Based on this collective experimental evidence, this work proposes the

**TIH:** The fundamental motion plan composed by the CNS is geometric and in the output space, namely, the wrist path. Therefore, the most fundamental internal model employed by the human CNS for motor coordination is a time-invariant, inverse kinematic model that uses the output-space, geometric motion plan and the motion timing as independent inputs to produce timed joint motions as the output.

Consequently, the present study seeks to quantify characteristics of the wrist path for spatial reaching motion. The TIH, as stated, belongs to Marr’s first level - it aims at establishing the inputs and outputs of the inverse kinematics problem. The TIH does not specify an algorithm to execute the inverse kinematic transformation. Chapter 4 develops an algorithm
to achieve this transformation in a time-invariant fashion, so it belongs to the second level of Marr.

Clearly, the validity of the TIH depends critically on the presence of time-invariant features in spatial reaching motion. While the evidence for kinematic invariant patterns for planar motion is substantial and convergent, the same for spatial motion is not. For example, Schaal and Sternad [130] demonstrated systematic violations of the two-thirds power law for unconstrained, rhythmic, spatial arm movements. Other works studying motion geometry in the output space report conflicting results. Adamovich et al. [2], Nishikawa et al. [114] and van der Well et al. [156] claim that wrist-path shape for spatial reaching motion is time-invariant, whereas Breteler et al. [19] report the opposite. This conflict is due in part to the use of different metrics to quantify path linearity. However, the reaching movements in these four studies encompass a limited portion of the workspace (details are provided in the next section). Other spatial arm movement studies look at postural control during reaching [30, 50, 57, 139], trying to correlate the initial and final postures of the arm. Soechting et al. [141] try to generalize the two-thirds power law to drawing movements in various vertical planes, and Medendorp et al. [103] study the upper-arm orientations during spatial reaching. An investigation of the output-space motion geometry encompassing a large portion of the workspace is required. Therefore, the first objective of the present study is to look for regularities in movement kinematics in the output space for spatial arm motion in an effort to validate the TIH. Although the wrist path is the primary candidate for the kinematic motion plan, other regularities in the motion kinematics are also considered.

The TIH is an internal model. It must be noted that a competing view to the internal-model theory is the equilibrium-point hypothesis that views the muscle as a spring-mass system in which the springs possess variable length-tension relationships and final limb
position corresponds to the equilibrium point of the system [38]. Trajectory generation, then, is achieved by specifying a sequence of equilibrium points. A number of experimental studies have shown that an equilibrium-point controller alone cannot account for typical behavior [52, 53], and in fact, separate kinematic and dynamic internal models are more consistent with the data [40, 85, 129, 131, 161]. In particular, the equilibrium-point theory requires that viscoelastic forces increase as a movement speeds up because the dynamic forces acting on the multijoint links grow roughly in proportion to the square of the velocity. On the other hand, the internal-model hypothesis proposes the realization of a fast and accurate movement even with low viscoelastic forces. Observations of a relatively low stiffness during well-trained, horizontal-plane, point-to-point arm movements support the existence of internal models [52, 53].

Attributing to internal inverse kinematic and dynamic models the function of composing muscle activations for individual instances of motion has further implications for the role of optimization in the CNS. First, learning movement control is viewed as the activity of optimization of some criterion and the extraction of invariant characteristics of the optimal solution. Second, for performing rehearsed motion, such as reaching, optimization is relegated to a ‘background’ activity focused on updating the system parameters in addition to refining the characteristics of optimal motion that are used to compose fresh instances of the motion. These activities occur at time scales longer than those of particular reaching motions. For example, the linearity of the wrist path is a characteristic of horizontal-plane reaching motion that minimizes the jerk in the wrist path [43]. Learning in the CNS would involve computing the jerk-optimizing, linear wrist trajectories, followed by generalizing this motion characteristic for all planar reaching motions. The TIH, rather
than explicit optimization computation, would then be employed to execute individual instances of reaching motion.

The present work suggests the following structure for a portion of the motor control strategy for human reaching:

- Optimal-control-based models operate at time scales longer than those for individual instances of reaching. Their objectives are to update the system parameters and to isolate characteristics of optimal arm motion.

- Particular instances of reaching motion are achieved via separate internal kinematic and dynamic models.

- The fundamental motion plan is a geometric plan in the output space. Correspondingly, the fundamental internal model is a time-invariant, inverse kinematics internal model (TIH). This model maps the output-space plan into a corresponding joint-space plan.

- The joint-space plan is converted into joint torques using an inverse-dynamic internal model. The leading joint hypothesis, elaborated in Chapter 3, is an algorithm that explains how the inverse dynamics are computed.

Of course, the joint torques must consequently be mapped into muscle forces and then into muscle activations. However, the present work does not consider these problems. Thus, the specific objective of this chapter is to investigate the validity of the TIH for spatial motion by studying the wrist-path shape, in particular.

Finally, since investigation of human reaching at Marr’s third level is not a part of this study, this work does not make any explicit claim regarding the location or nature
of the physical realization of the developed hypotheses and algorithms. Previous work aimed at localizing the internal models within the CNS does exist, however. It is well known that the cerebellum receives kinesthetic input [71] from the sensory system and other parts of the brain and spinal chord and integrates this input to fine tune movement [27]. Kinesthetic information is obtained from various sensors in the body. For example, muscle spindles sense change in muscle length, golgi tendon organs sense muscle tension, and mechanoreceptors in the periarticular soft tissue around a joint provides the sense of joint position under static as well as dynamic conditions [67, 126, 157]. The cerebellum is also believed to be the site for internal models [27, 35, 134, 168]. Doya [35] and Shadmehr and Krakauer [134] suggest that the cerebellum contains forward internal models, whereas Wolpert et al. [168] propose that it is a cite for multiple paired forward and inverse models for the generation of motor commands and the control of movement. The evidence for these claims stem from neuropathological studies [115, 134], study of ocular control [134, 168], and from considerations of the human ability to generate accurate and appropriate motor behavior under many different and often uncertain environmental conditions [168]. Based on these studies, it may be the case that the internal models investigated here are instantiated in the cerebellar neural network.

The validity of the TIH is investigated via a behavioral experiment. The overall experimental protocol is similar to previous reaching motion studies [2, 10, 114]. Healthy subjects perform point-to-point reaching tasks at various speeds between sets of initial and final targets. Their movement kinematics are recorded and subjected to statistical analysis.
2.2 Experimental methods

Subjects
Nine subjects, five female and four male, with no history of physical or neurological disorders participated in the study. Eight subjects were right handed, and one subject was ambidextrous. Subjects were between 20 and 33 years of age (mean = 23.22 years, standard deviation (SD) = 3.9 years) and naive to the purpose of the experiment. All subjects gave their informed consent prior to their inclusion in the study. This research received approval from the appropriate Institutional Review Board.

Apparatus and testing protocol
Electromagnetic sensors were taped onto the subject’s right wrist, the right upper arm, the right scapula and the spinous process of the 7th cervical vertebra, as seen in Figures 2.1(a) and 2.1(b). The sensors are part of the Flock-of-Birds (FOB) electromagnetic tracking system developed by Ascension Technology Corporation. The sensors are capable of measuring 6 DOF: X, Y, and Z position coordinates and azimuth, roll, and elevation orientation angles. To relate the position and orientation of the sensors to the movement of the subject’s body, the subject’s body was ‘digitized’ using a protocol recommended by the International Society of Biomechanics (ISB) [170]. The procedure uses bony landmarks on the subject’s body to locate various joint centers and joint rotation axes and to assign local reference frames to various body segments. These reference frames, also called anatomical frames, were defined for the right forearm, the right upper arm, the right scapula and the thorax. The center of the glenohumeral joint was located using the rotation method developed by Veeger [158]. The subject’s weight and height were recorded to allow the estimation of the inertial properties of the arm using anthropometric relationships [28]. The inertial properties are used in the computation of the joint torques in Chapter 3.
(a) Sensors placed on the wrist and the upper arm. (b) Sensors placed on the spinous process of the 7th vertebra and the scapula.

Figure 2.1: Sensor placement. The sensors of the Flock-of-Birds system were used in the reaching experiment to record the positions and orientations of the arm segments.

After completing the digitization procedure, subjects were comfortably seated with their backs resting against the back of a chair. The subjects were strapped to the chair with a band passing over the chest and under the arms to minimize movement of the thorax. Eight targets, T1 to T8, consisting of tennis balls covered with paper of different colors were arranged around the subject as shown in Figure 2.2. The balls were mounted on stands fabricated using PVC pipes. To accommodate subjects with different heights, the stands were constructed such that the height of the targets from the ground and the placement of the targets relative to each other could be varied. However, the locations of the targets were not controlled precisely across subjects, and Table 2.1 specifies the locations of the targets relative to the subject’s right shoulder (median values). The construction of the stands also allowed swinging targets T1 and T3 away from the subject when they were not required for a particular reaching task.

A reaching task consisted of the subject pointing from an initial target to a final target with the right arm. The final targets were located out of the subject’s reach. The following reaching tasks were used in the study.
Figure 2.2: The location of the targets relative to the subject’s right shoulder are shown. Tennis balls covered with colored paper serve as targets. The targets were mounted on light, rigid structures fabricated using PVC pipes. The dotted lines in the figure assist in the visualization of the spatial locations of the targets.

<table>
<thead>
<tr>
<th>Target #</th>
<th>Distance from shoulder (m)</th>
<th>Azimuth (deg)</th>
<th>Elevation (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1</td>
<td>0.45</td>
<td>−20</td>
<td>10</td>
</tr>
<tr>
<td>T2</td>
<td>0.50</td>
<td>45</td>
<td>10</td>
</tr>
<tr>
<td>T3</td>
<td>0.55</td>
<td>0</td>
<td>−35</td>
</tr>
<tr>
<td>T4</td>
<td>0.65</td>
<td>45</td>
<td>−35</td>
</tr>
<tr>
<td>T5</td>
<td>0.90</td>
<td>−60</td>
<td>0</td>
</tr>
<tr>
<td>T6</td>
<td>1.40</td>
<td>−60</td>
<td>50</td>
</tr>
<tr>
<td>T7</td>
<td>1.0</td>
<td>−60</td>
<td>−30</td>
</tr>
<tr>
<td>T8</td>
<td>0.95</td>
<td>30</td>
<td>−25</td>
</tr>
</tbody>
</table>

Table 2.1: Target locations relative to the right shoulder. The target locations were not controlled precisely across subjects. Median values accurate to 5° for the elevation and azimuth angles, and radial distance from the shoulder accurate to 0.05 m are reported.
1. Task 1: \( T_1 \rightarrow T_8 \).

2. Task 2: \( T_3 \rightarrow T_6 \).

3. Task 3: \( T_1 \rightarrow T_5 \).

4. Task 4: \( T_2 \rightarrow T_7 \).

5. Task 5: \( T_2 \rightarrow T_6 \).

6. Task 6: \( T_2 \rightarrow T_5 \).

7. Task 7: \( T_4 \rightarrow T_7 \).

8. Task 8: \( T_3 \rightarrow T_7 \).

Tasks 7 and 8 were originally included to provide a large enough number of tasks so that subjects would not simply memorize joint motions for any individual task. Subsequent analysis, however, indicated that these tasks were not necessary, so they are not included in the analysis that follows. Tasks 1 to 6 are depicted in Figure 2.3. All six tasks require movement away from the body and full extension of the elbow to point to the final target. Each task began with the subject in the neutral position with the hands resting on the knees. The following instructions were given to the subject. The experimenter will call out the color corresponding to the initial target. Point to the initial target with your forefinger and remain in that position. Move the entire arm, i.e., the shoulder and the elbow, such that the elbow is maximally extended in the final position. The experimenter will then call out a speed cue, followed by the color corresponding to the final target. The speed cue will be either ‘slow’, ‘normal’, or ‘fast’. Interpret the speed cue in a fashion that is consistent for various tasks. On hearing the speed cue and the final target specification, move your
head to locate the position of the final target (if necessary) and then point to the final target at the appropriate speed. Remain in the final position until the experimenter says the word ‘neutral’. You should then return to the neutral position and wait for the specification of the next task. Keep the wrist rigid, and focus more on producing smooth movements rather than the accuracy of the final position of the finger.

During the course of the experiment, the subject performed 220 reaching motions. Of these, the first 20 motions were the first six tasks performed at all three speeds and were meant as practice motions to acquaint the subject with the test procedure. Recording of the data began after the first 20 motions were completed. However, the subject was unaware that the first 20 movements were not recorded. The actual experiment was comprised of 200 motions presented in a random sequence which was also unknown to the subject. Tasks 1 through 6 were executed at three speeds, each ten times, yielding 180 motions. Twenty repetitions of Tasks 7 and 8 and all three speeds were added to reduce the possibility of subjects simply memorizing joint motions for any individual task. As mentioned before, these movements are not included in the following analysis. To minimize fatigue, breaks of approximately five minutes were enforced after the 80th, 130th, 170th, and the 200th motion. None of the subjects reported feeling fatigued during the experiment.

Spatial reaching studies that explicitly study movement geometry in the output space have been conducted previously by Adamovich et al. [2], Breteler et al. [19], Haggard and Richardson [60], Nishikawa et al. [114], and van der Well et al. [156], and they all have a similar experimental setup and protocol. However, the variety in the spatial positioning of the targets was rather restricted in most of these studies. In [60], four targets were placed at the corners of a square in a horizontal plane. Subjects performed point-to-point reaching motions along the sides of the square. This study qualifies as a spatial reaching
Figure 2.3: Definition of the reaching tasks. Two views of each task are provided. The first view is as seen by the subject, and the second view is as seen from above the subject’s head.
study because the wrist paths do not lie in a single plane. In [19], screens placed in front of the subject limited the reachable volume of the subject’s workspace and constrained some movements. In [2], only one initial arm configuration was utilized. The target placement in [114] was more varied and required the subject to perform a larger variety of reaching motions. The largest linear distance between targets was roughly 0.44 m in [60], 0.78 m in [114], 0.66 m in [2], and 0.67 m in [19]. The target placement in [156] was similar to but more restricted than that in [2].

In contrast, the target placement in the present study ensures variety in the reaching movements. In terms of the portion of the subject’s extra-personal space utilized, only [114] is comparable to the present study. Linear distances traveled by the wrist from the initial target position to the final target position for Tasks 1 through 6 are (mean ± SD) 0.49(±0.06) m, 0.92(±0.07) m, 0.39(±0.05) m, 0.85(±0.08) m, 0.77(±0.07) m, and 0.83(±0.09) m, respectively. Data are pooled across subjects. There is significant variation in the lengths of the various tasks in this study, and four of the six tasks in this study are longer than any task required in the studies mentioned above.

The positions of the sensors were sampled at 100 Hz. The raw sensor data were analyzed using the The MotionMonitor software developed by Innovative Sports Training, Inc. The software filters the raw data using a fourth-order, zero-lag, Butterworth filter with a cut-off frequency of 1.5 Hz. The software is capable of performing kinematic and dynamic analyses and provides motion properties of various joints and body segments, as well as joint torques. The kinematic analysis that follows utilizes the wrist paths and velocities derived from the software.
Movement onsets and endpoints were identified as the points in time when 10% of the peak wrist speed attained during the particular trial was first reached (onset), and next reached (endpoint). All computations were confined between these two points.

2.2.1 Data analysis

Movement duration

The movement time for slow-, normal-, and fast-speed movements was pooled across tasks and subjects. This was done to ensure that significant difference between movement speeds did in fact exist.

Wrist path

A visual inspection suggested that the wrist paths were planar in nature. Therefore, principal component analysis (PCA) [56, 83] was used to fit planes to the 3D wrist-path data. The PCA approach for obtaining a linear regression model is appropriate when there is no natural distinction between predictor and response variables. This is true for the $X - Y - Z$ Cartesian coordinates of the wrist position expressed in the global reference frame. The principal components (PC’s) of the wrist path for a reaching task are the eigenvectors of the covariance matrix of the data set comprised of the spatial positions of the wrist [56]. The PC’s are linear combinations of the original Cartesian coordinates. The first PC (associated

3For studies that employ discrete point-to-point arm motions, clipping the initial and final phases of motion is commonly employed [2, 11, 19] to isolate the smoothest portion of the motion and disregard the effects of initiating and concluding the motion. Other reaching-motion studies require the subjects to trace trajectories like lines, circles, ellipses, cloverleaves, limacons, and figure eights, repeatedly so that the entire motion is continuous rather than discrete [33, 42, 141]. This reduces effects of starting and ending the movement, and the entire motion is typically studied in these works.

3PCA is a mathematical technique that transforms a set of observations of a possibly correlated set of variables into another set of values of uncorrelated variables called principal components. This transformation is defined in such a way that the first principal component has the highest possible variance (that is, accounts for as much of the variability in the data as possible) and each succeeding component in turn has the highest variance possible under the constraint that it be orthogonal to (uncorrelated with) the preceding components. PCA can be executed by performing singular value decomposition [148] of a data matrix.
with the largest eigenvalue) points in the direction of the greatest variability in the data, and the last PC (associated with the smallest eigenvalue) points in the direction of the least variability in the data. Therefore, the first two PC’s define the plane that explains the most variability in the data. This plane is also the best-fit plane in the least-squared sense. Equivalently, the third PC is the error term in the regression. Another interpretation of the PC’s is as follows. The first PC points roughly in the direction of the reaching movement, the second PC points in the direction of the greatest deviation from the direction of the reaching movement, and the third PC represents the out-of-plane’ direction of the wrist motion. In the discussion that follows, the first PC is called the *movement direction*.

The eigenvalues obtained from the PCA define the amount of explained variance for each PC. Therefore, a goodness-of-fit parameter is computed as simply the sum of the two largest eigenvalues obtained from the PCA. Another error parameter for the fit is computed as [19]

$$\text{percent error} = \frac{\text{mean[abs(normal distance of data from fit plane)]}}{\text{linear distance between first and last points on the path}}.$$ 

Since this error metric is normalized with respect to the length of the movement, comparisons can be made across tasks and subjects.

The focus now shifts to the effect of speed on the shape of the wrist paths. Wrist-path shape has been an important aspect of previous studies on human reaching motion [2, 11, 43, 86, 114, 140, 152, 155]. The earliest studies compared wrist-path shapes visually [107]. Rigorous metrics to quantify global (rather than differential) wrist-path shapes appeared soon after. For example, the linearity metric developed by Atkeson and Hollerbach [11] measures the deviation of a wrist path from a straight line, and the curvature (and torsion) metric used by Breteler et al. [19] is the average of the local curvature (torsion) values over the entire wrist path. There also exist measures that do not explicitly use the geometric
properties associated with curves. Recently, Yadav et al. [172] used a similarity metric based on PCA [83] to compare shapes of large groups of wrist paths.

The present study utilizes geometric properties of curves to study wrist-path shapes. The first-order differential-geometric property of a curve is the path tangent. The movement direction is the global version of this property. The movement direction in human reaching studies is usually determined by the placement of targets. The movement direction appears in wrist-path-shape analysis as an additional factor that potentially influences higher-order path properties [2, 19], e.g., relation of path curvature to movement direction. Other studies [11, 114] compare wrist-path shape for tasks with the same movement direction. The present study performs both these analyses. The second-order geometric property of a curve is curvature. Breteler et al. [19] and Atkeson and Hollerbach [11] developed the global curvature metrics mentioned above.

There are two third-order geometric properties of curves. The first is the rate of curvature change with respect to arc length, and the second is torsion. The rate of curvature change is a planar property, and it has not been studied for wrist paths in the past. Torsion measures the non-planarity of a curve, and therefore, it is relevant only for spatial reaching studies. Breteler et al. [19] compute a local ‘3D curvature’ for the wrist path by fitting a plane to four equidistant points on the wrist path. The fit plane was the one in which the measure of the local curvature was the greatest. Breteler et al. [19] then report that the local torsion, i.e., ‘out-of-plane curvature’ of the path, was neglected because it was 1000 times lower that the in-plane curvature. Atkeson and Hollerbach [11] study path torsion implicitly when they report on the planarity of the wrist paths. Other than these two papers, there seem to be no experimental studies that investigate wrist-path properties of order greater
than two. At the same time, most optimization-based motor-control theories specify objective functions that involve up to the third-order kinematic (jerk) [43] or dynamic (torque change) [155] quantities. Richardson and Flash [122] generalized the minimum jerk hypothesis to higher orders and showed that a minimum snap (fourth derivative of position) principle is compatible with planar arm-movement data. However, they did not investigate fourth-order path properties.

There is no a priori reason to stop wrist-path analysis at (mostly) third-order properties. This decision seems to be based on heuristics and visual inspection of wrist paths traced by healthy humans. It is plausible that the sensing capabilities of the CNS will be key in determining the order of motion properties that should be studied or used for specifying objective functions in optimization-based motor-control theories. However, this topic is not the focus of the present work, and in accordance with previous practice, the present study investigates movement direction, path curvature, and path torsion.

The first PC of the wrist-path data was defined as the movement direction for that task. Next, the spatial wrist paths were projected onto the corresponding best-fit planes, and the Atkeson-Hollerbach (A-H) linearity metric [11] is utilized to quantify the curvature of the planarized wrist paths. The A-H metric is the ratio of the maximum normal distance between a wrist path and a straight line joining the initial and the final points on the path to the length of that line. The metric value will be zero for a wrist path that is a straight line, and it will increase as the deviation of the wrist path from a straight line increases. This metric has been adopted elsewhere in the literature [2, 114]; it is simple, scale invariant, and appropriate for the wrist paths observed in the present study. Note that ‘linearity’ and ‘curvature’ refer to opposite trends describing the same characteristic of the path. A three-way (subject \( \times \) speed \( \times \) task number) ANOVA of the A-H linearity metric revealed significant
(0.01 level) main and interaction effects. Therefore, the data for each subject was analyzed separately. A two-way (speed × task number) ANOVA of the A-H linearity metric revealed significant (0.05 level) main and interaction effects for all subjects. Therefore, the effect of speed alone on the path curvature was evaluated by means of one-way ANOVA with respect to the three movement speeds, and Tukey’s HSD test was used for pair-wise comparisons. To quantify the relation between the wrist-path speed and path linearity, the A-H linearity metric was regressed against the average wrist speed.

**Orientation of the wrist-path plane and the loop direction of the wrist path**

On visual inspection, the wrist paths for some tasks displayed curvature in the vertical plane. This curvature is referred to as the loop of the wrist path. The nature of the path loop varies with task and movement speed. The following analysis is performed to quantify these observations.

The planes in which the wrist paths tend to lie are called the *wrist planes*. They are represented by (unit) vectors normal to the planes which are simply the third PC of the corresponding wrist-path data. Ideally, the thirty repetitions of a task by a subject will have the same movement direction indicated by a vector, henceforth called the *movement-direction vector*, that points along the line joining the initial and the final points on the wrist path. Furthermore, all the wrist-plane normals would lie on a single plane that is defined by this unique movement-direction vector. However, since the initial and final points on the wrist paths for all tasks do not coincide, there is a small amount of variability in the orientation of the individual movement-direction vectors, as Figure 2.4 illustrates. Therefore, the average of the thirty vectors is assumed to be the movement-direction vector representing the particular task for the subject. This average vector defines a plane called the *plane of the normals*. The thirty wrist-path normals are projected onto the plane of
Figure 2.4: The movement-direction vectors and wrist-plane normals for thirty repetitions of Task 6 by subject FR are plotted. Note that all vectors are normalized. The variability in the movement-direction vectors is small. The movement-direction vector for the task is the average of the movement-direction vectors for the thirty repetitions and is indicated as the thick arrow. The movement-direction vector defines the plane of the normals onto which the wrist-plane normals are projected.

The plane of the normals is shown in Figure 2.5 along with its projection onto the plane of the normals. It also shows the elevation angle $\theta$ of the plane of the normals as the angle between the horizontal direction and the movement-direction vector measured in the plane defined by the gravity vector and the movement-direction vector.

The relation between the direction of loop of the wrist paths and the corresponding planarized normals is illustrated in Figure 2.6. For Tasks 2 through 6, the wrist paths progress from left to right, and the direction is reversed for Task 1. For the subsequent descriptions and ensuing analysis, the plane of the normals is always viewed from a direction opposite
Figure 2.5: A wrist plane is defined as the best-fit plane to the wrist-path data. It is also defined by a normal to it. This normal itself lies in another plane, called plane of the normals, which is defined by the movement-direction vector. The average of the thirty movement-direction vectors from all repetitions of a given task is considered the movement-direction vector for that task. The gravity vector and the thirty path normals for all repetitions of this task are projected onto the plane of normals. The projected normals are called planarized normals. The angle of elevation $\theta$ of the plane of normals is measured from the horizontal to the movement-direction vector in the plane defined by the gravity vector and the movement-direction vector. The angle is positive upward.
Figure 2.6: The wrist paths progress from left to right in the figure. Therefore, this figure describes Tasks 2 through 6. When the plane of the normals is viewed from the right, the planarized normal for a wrist path looping upward will appear to the right of the planarized normal for a wrist path looping downward. This relation will be reversed for Task 1 since the movement is from right to left and the viewing direction is reversed.

the movement-direction vector, as illustrated in Figure 2.6. Therefore, the planarized normal for a wrist path looping upward will appear to the right of the planarized normal for a wrist path looping downward for Tasks 2 through 6. This relation will be reversed for Task 1 since the viewing direction is reversed. Finally, the normal angle is defined as the angle made by a planarized normal relative to a reference line, measured positive clockwise. The planarized normals for a given task are chosen⁴ such that the within-speed and between-speeds variabilities in the normal angles are minimized. ANOVA was performed on the normal angles with respect to movement speed for each task to establish the patterns.

⁴The choice is between a normal and its negative, both of which define the same wrist plane.
described in the preceding paragraphs. Tukey’s HSD test was employed for pair-wise comparisons. ANOVA was performed on the normal angles with respect to movement speed for each task to establish the patterns described in the preceding paragraphs. Tukey’s HSD test was employed for pair-wise comparisons.

The largest angle subtended by two planarized normals for a given task is defined as the angular spread. It is predicted that tasks that display highly linear wrist paths will also have greater angular spread. To understand the reason, consider that there are an infinite number of planes that contain a given straight line. Therefore, the problem of fitting a plane to a straight line is ill defined. If one tries to fit a plane to a spatial curve that is arbitrarily close to a straight line, the fitting procedure is expected to be sensitive to small ‘out-of-plane’ deviations in the curve because the problem is close to being ill defined. To identify the tasks that are significantly affected by this sensitivity of the plane fitting algorithm, the A-H linearity metric was computed for all spatial wrist paths and was averaged across speeds. Note that previously the metric was computed for the planarized wrist paths. The difference in the value of the metric for a spatial wrist path and the corresponding planarized version was small in all cases. The spatial paths are used here for thoroughness. The angular spread was plotted against the A-H linearity metric.

**Wrist speed profile**

The movement times for all tasks were computed for data trimmed at the 10% peak wrist-speed level. Therefore, the movement times underestimate the total movement time required for completing the tasks. To rectify this, the ten repetitions of a task at a given speed by a subject were considered as a unit. One test was randomly chosen from this unit, and the total time required for this representative motion was computed using The MotionMonitor software. The differential in this total time and the time computed for the
trimmed data of the same task was regarded as the error in the estimation of movement time for all ten repetitions of the unit. This error was added to the computed time for all the movements in the unit to yield the modified movement time. This procedure was executed for all subjects, tasks and speeds. The average wrist speed was plotted against the modified movement time for all tests and subjects.

A persistent, time-invariant characteristic of reaching movement observed in planar studies (e.g. [11, 107, 43, 122, 110]) as well as spatial studies (e.g. [2, 114]) has been the unimodal, bell-shaped profile of the wrist speed. This invariant profile is obtained after normalizing for peak movement speed and time of movement [11]. To investigate wrist-speed profile invariance, the variability in the wrist-speed profiles for a particular speed for every subject was quantified. First, all profiles were normalized for peak wrist speed and movement time. Next, the average of all 60 normalized wrist-speed profiles of the same speed was constructed. This average profile served as the reference profile for that speed, and the deviation of each of the 60 experimental wrist-speed profiles from the reference profile was measured. The similarity metric developed by Atkeson and Hollerbach [11] was used for this purpose. The Atkeson-Hollerbach (A-H) similarity metric for a reference profile and an experimental profile is computed using the areas under these curves. Let $A$ and $B$ be the areas under the reference profile and the experimental profile, respectively. Then $A \cup B$ is the total area contained beneath both curves, and $A \cap B$ is the area common to both curves. The similarity metric is defined as

$$w := \frac{A \cup B - A \cap B}{B}.$$  \hfill (2.1)

The value of $w$ will be zero for identical profiles, and it will increase as the profiles become more dissimilar. Since the metric $w$ measures the deviation of the experimental profiles
from a mean, the standard deviation in the values of \( w \) obtained for a particular speed-subject pair is a measure of variability in the wrist-speed profile for that pair.

**Revisiting wrist-path shape**

The A-H linearity metric is plotted against the modified movement time and average wrist speed for all tasks and subjects to investigate if multimodal wrist-speed profiles are seen exclusively for highly curved wrist paths.

### 2.3 Results

**Movement duration**

The slow-speed, normal-speed, and fast-speed movements lasted (mean ± SD) 2.27 (±0.75) seconds, 1.28 (±0.34) seconds, and 0.79 (±0.09) seconds, respectively, for data pooled across tasks and subjects. This indicates that the three movement speeds were significantly different from each other.

**Wrist path**

Figure 2.7 shows the percent error and the goodness of fit values for subject JM\(^5\) for the plane-fitting algorithm described in the previous section. The maximum percent error was less than 2.5% for all subjects, and the goodness of fit was greater than 0.996 for all subjects. This observation yields the following result.

(R1) *Wrist paths for spatial reaching movements tend to lie in planes.*

A similar observation was reported by Atkeson and Hollerbach [11] and Breteler et al. [19]. However, in [11], the target placement was responsible for arm movement being

\(^5\)In all figures describing statistical results in this section, the following color code is employed: green - slow-speed movement, red - normal-speed movements, and blue - fast-speed movements. This code holds unless a legend is presented.
Figure 2.7: Percent error and goodness of fit for the best-fit plane to the wrist paths for all tasks. Subject JM. All statistical results in this chapter follow the following color code: green - slow-speed movement, red - normal-speed movement, and blue - fast-speed movement.

predominantly in a para-saggital plane. In [19], the arm movements were less varied compared to those in this study as well as constrained by screens placed in the workspace. Therefore, the main contribution of the above result is that the planar nature of wrist paths is observed for unconstrained reaching movements that span a large portion of the subject’s extra-personal space.

The results of the one-way ANOVA on the A-H linearity metric with respect to movement speed are provided in Table 2.2. ‘S’, ‘N’, and ‘F’ indicate slow, normal, and fast movement speed, respectively. Entries in the table indicate the speed pairs for which there is no significant difference \( (P > 0.05) \) between the metric means. For example, the entry ‘SN’ for subject JM’s Task 1 indicates no significant difference in the metric means for slow- and normal-speed movements. Simultaneously, it indicates that the wrist-path
curvatures for slow- and fast-speed movements, and normal- and fast-speed movements are significantly different ($P < 0.05$). ‘None’ (e.g. Task 2, subject JM) indicates that the metric means for all three speeds are significantly different.

The ANOVA results indicated that wrist-path shape may change with movement speed depending on the task as well as the subject. Therefore, the A-H linearity metric was regressed against the average wrist speed. The $R^2$ values for the regression are provided in Table 2.3, and the slopes and intercepts of the regression lines are provided in Table 2.4. The error bar plots of the confidence intervals on the residuals showed a random distribution about a zero mean, indicating the validity of the normality assumption for all cases.

Figure 2.8 presents the box plot of the A-H metric values for all repetitions of all tasks performed by all subjects. The following observations are based on the results in Tables 2.2, 2.3, and 2.4, and Figure 2.8.

(O1) Figure 2.8 shows that wrist paths for Task 3 are the most linear.

(O2) Table 2.2 shows that wrist-path shapes for Task 3 are consistently time invariant for all subjects. The wrist-path shapes for normal and fast speeds are time invariant for all subjects. The wrist paths for slow speeds are significantly different from the other two speeds for three subjects.

(O3) According to Figure 2.8, wrist paths for Task 1 are the next most linear, with a few outliers that show more curved wrist paths.

(O4) Following Task 3, wrist-path shapes for Task 1 are the next most time invariant, with seven of nine subjects showing time-invariant characteristics (Table 2.2).

6On each box in all box plots, the central mark is the median, the edges of the box are the 25th and 75th percentiles, the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually.
Figure 2.8: Box plot of the A-H linearity metric for the planarized wrist paths. Plot shows data for all tests performed by all subjects. A lower value for the metric indicates greater linearity of the wrist path.

Table 2.2: ANOVA results for the A-H linearity metric for planarized wrist paths. Tukey’s HSD test was used for pair-wise comparisons. ‘S’, ‘N’, and ‘F’ indicate slow, normal, and fast movement speed, respectively. Entries indicate the speed pairs for which there is no significant difference ($P > 0.05$) between the metric means. ‘None’ indicates that the metric means for all three speeds are significantly different ($P < 0.05$).
### Table 2.3: $R^2$ values for the regression of the A-H linearity metric against average wrist speed. Entries in bold are not significant. Entries in brackets are significant at the 0.05 significance level. Other entries are significant at the 0.01 significance level.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
<th>Task 4</th>
<th>Task 5</th>
<th>Task 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>JM</td>
<td>0.53</td>
<td>0.88</td>
<td><strong>0.05</strong></td>
<td>0.78</td>
<td>0.75</td>
<td>0.81</td>
</tr>
<tr>
<td>AL</td>
<td><strong>0.03</strong></td>
<td>(0.17)</td>
<td>0.33</td>
<td><strong>0.00</strong></td>
<td>0.60</td>
<td>(0.17)</td>
</tr>
<tr>
<td>DE</td>
<td>0.45</td>
<td>0.62</td>
<td>(0.17)</td>
<td>0.80</td>
<td>0.64</td>
<td>0.76</td>
</tr>
<tr>
<td>FR</td>
<td><strong>0.02</strong></td>
<td>0.56</td>
<td>0.29</td>
<td>(0.18)</td>
<td>0.63</td>
<td>0.19</td>
</tr>
<tr>
<td>RR</td>
<td>0.24</td>
<td>0.40</td>
<td><strong>0.09</strong></td>
<td>0.30</td>
<td>0.41</td>
<td>0.31</td>
</tr>
<tr>
<td>CR</td>
<td>0.66</td>
<td>0.70</td>
<td>0.38</td>
<td>0.73</td>
<td>0.83</td>
<td>0.77</td>
</tr>
<tr>
<td>KB</td>
<td><strong>0.11</strong></td>
<td>0.86</td>
<td>0.26</td>
<td>0.69</td>
<td>0.70</td>
<td>0.72</td>
</tr>
<tr>
<td>JR</td>
<td>0.23</td>
<td>0.69</td>
<td>0.22</td>
<td>(0.16)</td>
<td>0.36</td>
<td><strong>0.08</strong></td>
</tr>
<tr>
<td>AS</td>
<td><strong>0.04</strong></td>
<td><strong>0.06</strong></td>
<td>0.49</td>
<td><strong>0.00</strong></td>
<td>0.84</td>
<td><strong>0.03</strong></td>
</tr>
</tbody>
</table>

### Table 2.4: Slope (sec / m), and intercept values for the regression of the A-H linearity metric against average wrist speed. Entries in bold are not significant. Entries in brackets are significant at the 0.05 significance level. Other entries are significant at the 0.01 significance level.

<table>
<thead>
<tr>
<th>Subj</th>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
<th>Task 4</th>
<th>Task 5</th>
<th>Task 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>JM</td>
<td>-0.11,0.17</td>
<td>-0.13,0.25</td>
<td><strong>-0.04,0.05</strong></td>
<td>-0.25,0.36</td>
<td>-0.42,0.47</td>
<td>-0.30,0.39</td>
</tr>
<tr>
<td>AL</td>
<td><strong>-0.03,0.07</strong></td>
<td>(-0.04,0.13)</td>
<td>-0.14,0.10</td>
<td><strong>0.00,0.14</strong></td>
<td>-0.22,0.35</td>
<td>(-0.04,0.19)</td>
</tr>
<tr>
<td>DE</td>
<td>-0.08,0.13</td>
<td>-0.11,0.28</td>
<td>(-0.14,0.11)</td>
<td>-0.24,0.36</td>
<td>-0.28,0.45</td>
<td>-0.23,0.37</td>
</tr>
<tr>
<td>FR</td>
<td><strong>-0.02,0.04</strong></td>
<td>-0.08,0.12</td>
<td>-0.07,0.08</td>
<td>(-0.05,0.07)</td>
<td>-0.29,0.33</td>
<td>(-0.06,0.08)</td>
</tr>
<tr>
<td>RR</td>
<td>-0.08,0.11</td>
<td>-0.19,0.26</td>
<td><strong>-0.10,0.09</strong></td>
<td>-0.11,0.18</td>
<td>-0.54,0.52</td>
<td>-0.18,0.29</td>
</tr>
<tr>
<td>CR</td>
<td>-0.20,0.13</td>
<td>-0.14,0.23</td>
<td>-0.16,0.12</td>
<td>-0.32,0.31</td>
<td>-0.32,0.28</td>
<td>-0.37,0.33</td>
</tr>
<tr>
<td>KB</td>
<td><strong>-0.08,0.12</strong></td>
<td>-0.18,0.27</td>
<td>-0.22,0.14</td>
<td>-0.24,0.29</td>
<td>-0.25,0.29</td>
<td>-0.31,0.34</td>
</tr>
<tr>
<td>JR</td>
<td>-0.26,0.23</td>
<td>-0.17,0.28</td>
<td>-0.21,0.14</td>
<td>(-0.10,0.22)</td>
<td>-0.28,0.31</td>
<td><strong>-0.08,0.22</strong></td>
</tr>
<tr>
<td>AS</td>
<td><strong>-0.01,0.04</strong></td>
<td><strong>-0.01,0.09</strong></td>
<td>-0.18,0.11</td>
<td><strong>-0.00,0.08</strong></td>
<td>-0.18,0.20</td>
<td><strong>-0.01,0.09</strong></td>
</tr>
</tbody>
</table>
Four of nine subjects show some degree of time invariance in the wrist-path shapes for all tasks.

Of all tasks, the largest number of subjects (four of nine) exhibited no time-invariant wrist-path shapes whatsoever in performing Task 4.

In order of decreasing time invariance, the tasks can be arranged as 3, 1, (4 and 6), 2, 5. Task 4 has a binary nature, with five subjects showing strong time-invariant characteristics, and the other four showing none at all.

For Tasks 3 and 6, the wrist (and not the entire arm) mostly moves in the horizontal plane. The mean arc length of Task 3 for all subjects at $0.3(\pm 0.04)$ m is the shortest of all tasks. The mean arc length for Task 6 is longer: $0.6(\pm 0.07)$ m. While the wrist paths for Task 3 show a strong tendency toward linear wrist paths that are time invariant, those for Task 6 are curved and time invariant for some subjects. Thus, the wrist paths for Task 3 are most like those observed for horizontal-plane arm motion.

The results presented in Tables 2.2 and 2.3 are corroborative. For ten of the eleven cases showing no significant correlation between path curvature and average wrist speed in Table 2.3, the corresponding ANOVA result indicates strong time-invariant characteristics (‘SNF’) in wrist-path shape. (Task 6 for subject JM is the counter-example.) Also, there are fourteen cases in Table 2.2 that show time-dependent wrist path shapes (‘None’) for all three speeds. The corresponding regression results show significant correlation with (mostly) high $R^2$ values.
(O10) Of the 54 cases in Table 2.4 (nine subjects × six tasks), 43 show a significant, negative correlation (P < 0.05) between the path curvature and average wrist speed.

(O11) The relation between the movement speed and path curvature in (O10) implies that path curvature increases with movement time. A regression analysis (results are not presented here) revealed that 46 of the 54 cases have a significant, positive correlation (P < 0.05) between path curvature and movement time.

These observations are consolidated into the following result:

(R2) **Wrist paths for a given reaching task tend to get more curved as movement speed decreases or as movement time increases. This relation is not always significant, and its strength depends on the nature of the reaching task as well as the subject. Therefore, wrist-path shape is not time invariant in general.**

The plane-fitting procedure outlined earlier qualifies as an analysis of torsion. The subsequent result (R1) demonstrating the planarity of wrist paths implies that path torsion can be ignored for all subjects and precludes torsional analysis of wrist paths. This observation corroborates two earlier studies on wrist-path torsion. Breteler et al. [19] found path torsion to be ignorable based on a metric that averaged local path torsion measurement over the entire wrist path. Atkeson and Hollerbach [11] comment on path torsion implicitly when they report on the planarity of the wrist paths.

**Orientation of the wrist-path plane and the loop direction of the wrist path**

Figure 2.9 shows the average of ten wrist paths of each speed for Tasks 2, 4, and 3 for subject KB. The curves are translated such that the initial points on the paths coincide. From the relative positions of the wrist paths and the thorax in the top view of each task,
it is evident that the wrist paths curve away from the subject. Next, the front views (as seen by the subject) suggest that for movements performed against gravity (Task 2), the wrist path for a slower movement loops downward compared to that for a faster movement. In contrast, for movements performed with gravity (Task 4), the wrist path for a slower movement loops upward compared to that for a faster movement. There is no discernible difference in the wrist paths for Task 3 for which the wrist traveled almost parallel to the ground. Furthermore, the variability in the wrist paths of different speeds is greater for Task 2 than for Task 4. This suggests that the variation in the wrist paths with speed depends on the elevation angle $\theta$ ($|\theta| = 74°$ for Task 2, and $|\theta| = 19°$ for Task 4).

The maximum angular rotation that any normal underwent during its projection onto the plane of normals across all subjects and tasks was $13°$, and the mean ($\pm$SD) for all subjects was $1.87(\pm1.7)$ degrees. Therefore, the loss of information associated with this projection is negligible. Figure 2.10 shows the planarized normals for all tasks for two subjects. In Figure 2.10(a), the planarized normals for the fast movements (blue) appear to the right of those for slow movements (green) for Tasks 1 and 5. This implies that compared to the fast movements, the slow movements loop upward for Task 1 and downward for Task 5.

Figure 2.10 shows the planarized normals for subjects JM and FR. In each case shown in this figure, the movement-direction vector is pointing outward from the plane of the paper. The black arrow pointing downward is the projection of the gravity vector onto the plane of the normals, and its length is related to the elevation angle of the plane of the normals ($length = \cos \theta$). The dotted vertical line in each plot is the reference from which the normal angles are measured. Table 2.5 provides the ANOVA results performed on the normal angles with respect to movement speed for all tests and subjects. ‘S’, ‘N’, and ‘F’ indicate slow, normal, and fast speed of movement, respectively. The entries in the
Figure 2.9: The average of ten wrist paths of each speed for Tasks 2, 4, and 3 for subject KB are shown. From the top views for each task, it is evident that the wrist paths curve away from the subject. The front views are as seen by the subject. They suggest that the wrist paths for slower movement loop downward when the movement is performed against gravity (Task 2), and the slower movements loop upwards when the movements are performed with gravity (Task 4). There is no discernible difference in the wrist paths for Task 3 which has a wrist path almost parallel to the ground.
Figure 2.10: The *planarized normals* for all repetitions of a given task are plotted. The movement-direction vector is pointing out of the plane of the paper. The thick black line represents the gravity vector, also projected onto the plane of normals. The length of the gravity vector indicates the elevation angle of the plane of normals. One-way ANOVA is performed on the angles between the dotted line and the normals. The angles are measured from the dotted line to the normals in the clockwise direction and are called *normal angles*.
Table 2.5: ANOVA results for the normal angles. Tukey’s HSD test was used for pair-wise comparisons. ‘S’, ‘N’, and ‘F’ indicate slow, normal, and fast movement speed, respectively. Entries in the table indicate the speed pairs that have significantly different means at the 0.05 significance level. The asterix indicates that wrist paths for the slower of the two movements in the pair loop upward compared to those of the faster movement of the pair. For other cases, wrist paths for the slower movement of the pair loop downward compared to those of the faster movement of the pair. ‘None’ indicates no significant difference ($P > 0.05$) between the normal angles for all three speeds.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
<th>Task 4</th>
<th>Task 5</th>
<th>Task 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>JM</td>
<td>SNF*</td>
<td>SF</td>
<td>SF, NF</td>
<td>SF*, NF*</td>
<td>SF, NF</td>
<td>SF*</td>
</tr>
<tr>
<td>AL</td>
<td>None</td>
<td>SF, NF</td>
<td>SF</td>
<td>SF, NF</td>
<td>SF, NF</td>
<td>SF, NF</td>
</tr>
<tr>
<td>DE</td>
<td>SN*, NF*</td>
<td>SF, NF</td>
<td>SF</td>
<td>SF, NF</td>
<td>SF, NF</td>
<td>None</td>
</tr>
<tr>
<td>FR</td>
<td>None</td>
<td>None</td>
<td>SF*</td>
<td>None</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>RR</td>
<td>SN*, SF*</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>SF, NF</td>
<td>None</td>
</tr>
<tr>
<td>CR</td>
<td>SN*, NF*</td>
<td>SF</td>
<td>None</td>
<td>SF*, NF*</td>
<td>None</td>
<td>SF*, NF*</td>
</tr>
<tr>
<td>KB</td>
<td>SF*</td>
<td>SF, NF</td>
<td>None</td>
<td>SF*, NF*</td>
<td>SF</td>
<td>None</td>
</tr>
<tr>
<td>JR</td>
<td>None</td>
<td>SNF</td>
<td>None</td>
<td>SF*, NF*</td>
<td>SF, NF</td>
<td>None</td>
</tr>
<tr>
<td>AS</td>
<td>None</td>
<td>SF, NF</td>
<td>NF</td>
<td>None</td>
<td>SF, NF</td>
<td>NF</td>
</tr>
</tbody>
</table>

Table 2.5 indicates the speed pairs that have significantly different mean values. For example, the entry ‘SF’ for subject JM’s task two indicates significantly different normal angles for slow and fast movements. Simultaneously, it indicates that the angles for normal- and slow-speed movements and the angles for normal- and fast-speed movements do not differ significantly. The entry ‘None’ (e.g. Task 1, subject AL) indicates that the angles for all three speeds were not significantly different. The asterisk indicates that wrist paths for the slower of the two movements in the pair loop upward compared to the faster movement in the pair. For other cases, wrist paths for the slower movement in the pair loop downward compared to the faster movement in the pair.

The following observations are made from the results in Table 2.5:
For tasks two and five, wrist paths loop downward as the speed of movement reduces. This characteristic is displayed by seven of nine subjects, and the pattern is consistent across subjects, i.e. wrist paths for slower movements loop downward for all cases displaying significant differences.

In contrast to (O12), wrist paths loop upward as the speed of movement reduces for tasks one and four. Furthermore, fewer subjects display the motion characteristic (five of nine for task one and six of nine for task four), and the pattern is less consistent across subjects. There are two exceptions for task four, wherein two subjects show downward looping paths for slower movement speeds.

The effect of speed on wrist-path plane for tasks three and six is not systematic across subjects. A majority of the subjects (five out of nine) show no influence of speed on wrist plane, and those that do are equally split between wrist paths for slower movements looping upward and downward.

The values for the angular spread in the planarized normals provided in Table 2.6 exhibit a wide range from $31^\circ$ to $167^\circ$ across tasks and subjects. This variability is related to the linearity of the wrist paths. The A-H linearity metric computed for all spatial wrist paths and averaged across speeds are provided in Table 2.7. The plot of the angular spread against the A-H linearity metric (Figure 2.11) reveals a linear trend between the parameters up to a mean A-H metric value of 0.1. There seems to be no significant trend beyond this value. The angular spread was regressed against the metric for values of the metric less that 0.1. The residuals showed a random distribution around a zero mean, thus validating the assumption of normality. Figure 2.11 shows the data and the fit model with two outliers deleted. A significant ($P = 0.000$) negative correlation was observed between the angular

53
Table 2.6: The angular spreads for the planarized normals for all subjects. The angles are in degrees and are rounded up to the nearest integer.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
<th>Task 4</th>
<th>Task 5</th>
<th>Task 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>JM</td>
<td>76</td>
<td>31</td>
<td>150</td>
<td>38</td>
<td>36</td>
<td>44</td>
</tr>
<tr>
<td>AL</td>
<td>167</td>
<td>75</td>
<td>145</td>
<td>45</td>
<td>77</td>
<td>58</td>
</tr>
<tr>
<td>DE</td>
<td>76</td>
<td>32</td>
<td>154</td>
<td>50</td>
<td>42</td>
<td>37</td>
</tr>
<tr>
<td>FR</td>
<td>164</td>
<td>90</td>
<td>132</td>
<td>159</td>
<td>38</td>
<td>157</td>
</tr>
<tr>
<td>RR</td>
<td>126</td>
<td>81</td>
<td>142</td>
<td>129</td>
<td>97</td>
<td>96</td>
</tr>
<tr>
<td>CR</td>
<td>112</td>
<td>44</td>
<td>105</td>
<td>75</td>
<td>42</td>
<td>53</td>
</tr>
<tr>
<td>KB</td>
<td>82</td>
<td>29</td>
<td>141</td>
<td>76</td>
<td>64</td>
<td>44</td>
</tr>
<tr>
<td>JR</td>
<td>62</td>
<td>34</td>
<td>96</td>
<td>44</td>
<td>86</td>
<td>63</td>
</tr>
<tr>
<td>AS</td>
<td>160</td>
<td>48</td>
<td>135</td>
<td>69</td>
<td>71</td>
<td>85</td>
</tr>
</tbody>
</table>

The angular spread of the planarized normals and the mean A-H linearity metric for mean A-H metric values up to 0.1. This indicates that, for this range, there is greater variability in the orientation of the fit plane for subject-task combinations that have straighter wrist paths. This high sensitivity is reflected in the large negative slope \((\text{slope} = -1510.1^\circ)\) of the correlation between the angular spread and the A-H metric. In the present context, out-of-plane deviation means deviation of the curve in any direction perpendicular to that of the movement direction.

A regression analysis for data points with mean A-H value greater than 0.1 revealed no significant linear relationship between the parameters \((P = 0.29)\), suggesting that, beyond this value, the angular spread of the planarized normals is not due to the sensitivity of the plane-fitting algorithm to the high degree of straightness of the wrist paths. Figure 2.12 suggests that Tasks 3 and 1 are likely to exhibit a large angular spread because of nearly linear wrist paths. Indeed, all subjects exhibit this motion characteristic for Task 3, and 5 subjects show the characteristic for Task 1. Furthermore, one subject each for Tasks 4 and 6 show large angular spread as well as highly linear wrist paths.
Table 2.7: The Atkeson-Hollerbach linearity metric computed for spatial wrist paths. The average value for the thirty repetitions of a task are provided.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Task 1</th>
<th>Task 2</th>
<th>Task 3</th>
<th>Task 4</th>
<th>Task 5</th>
<th>Task 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>JM</td>
<td>0.0947</td>
<td>0.1433</td>
<td>0.0367</td>
<td>0.1844</td>
<td>0.1989</td>
<td>0.1889</td>
</tr>
<tr>
<td>AL</td>
<td>0.0598</td>
<td>0.1092</td>
<td>0.0514</td>
<td>0.1522</td>
<td>0.2173</td>
<td>0.1623</td>
</tr>
<tr>
<td>DE</td>
<td>0.0890</td>
<td>0.2033</td>
<td>0.0618</td>
<td>0.2139</td>
<td>0.2932</td>
<td>0.2314</td>
</tr>
<tr>
<td>FR</td>
<td>0.0360</td>
<td>0.0693</td>
<td>0.0568</td>
<td>0.0476</td>
<td>0.1706</td>
<td>0.0456</td>
</tr>
<tr>
<td>RR</td>
<td>0.0718</td>
<td>0.1291</td>
<td>0.0620</td>
<td>0.1207</td>
<td>0.2456</td>
<td>0.1971</td>
</tr>
<tr>
<td>CR</td>
<td>0.0773</td>
<td>0.1556</td>
<td>0.0831</td>
<td>0.1723</td>
<td>0.1652</td>
<td>0.1911</td>
</tr>
<tr>
<td>KB</td>
<td>0.0990</td>
<td>0.1472</td>
<td>0.0671</td>
<td>0.1665</td>
<td>0.1580</td>
<td>0.1780</td>
</tr>
<tr>
<td>JR</td>
<td>0.1675</td>
<td>0.1985</td>
<td>0.0904</td>
<td>0.1840</td>
<td>0.1930</td>
<td>0.1914</td>
</tr>
<tr>
<td>AS</td>
<td>0.0418</td>
<td>0.0871</td>
<td>0.0632</td>
<td>0.0835</td>
<td>0.1250</td>
<td>0.0879</td>
</tr>
</tbody>
</table>

Figure 2.11: Plot of the angular spread of wrist-plane normals vs. mean A-H linearity metric. A strong linear correlation between the parameters is observed for mean A-H metric values up to 0.1.
Figure 2.12: Box plot for the data in Table 2.7. Wrist paths for Tasks 1 and 3 have high degree of linearity, and therefore, these tasks will likely show large angular spread in the planarized normals.

This analysis suggests that

(O15) It is plausible that the lack of a systematic effect in (O14) for Task 3 is a computational artifact because of the highly linear wrist paths (see Figure 2.12). In contrast, the wrist paths for Task 6 are sufficiently curved for most subjects so that sensitivity of the plane-fitting algorithm to path linearity can be safely disregarded. The lack of a systematic effect in (O14) for this task is more likely a characteristic of the movement rather than a computational artifact.

(O16) Figure 2.13 shows the elevation angle $\theta$ for all tasks. The pattern observed in (O12)-(O14) is more pronounced for larger values of elevation angle $|\theta|$.

(O17) Two subjects, DE and CR, display a spurious pattern in the normal angles for Task 1. For these cases, the mean of the normal angles for slow-speed movements is between
that for the normal-speed and fast-speed movements. In addition, four subjects show no influence of speed on the normal angles, only one of whom has a high value of the mean A-H linearity metric (JR: 0.17). Figure 2.12 shows that Task 1 also has a high level of linearity in the wrist paths. Therefore, similar to Task 3, computational issues may be responsible for the inconsistency.

(O18) The observation (O16) holds if Tasks 1 and 3 are excluded from the analysis.

Observations (O12) to (O18) yield the following result.

(R3) For reaching movements performed against gravity, wrist paths for slower movements tend to loop downward compared to those for faster movements. For reaching movements performed with gravity, wrist paths for slower movements tend to
loop upward compared to those for faster movements. This effect is more pronounced and systematic for movements that have greater change in height.

Result (R3) complements and refines result (R2). Result (R2) indicates that slower wrist paths tend to be more curved. Result (R3) describes a spatial relation between the plane in which the path will curve, the movement direction and movement speed. Result (R3) has not been reported before in literature.

Wrist speed profile

In the present study, two kinds of wrist-speed profiles were observed. Figure 2.14 shows the wrist-speed profiles for all repetitions of all tasks for subject JR. These profiles are normalized for time, but not for peak speed. All wrist-speed profiles for fast movements are unimodal, and resemble the profile shape reported in literature. Almost 50% of all wrist-speed profiles for slow movements, and some profiles for normal-speed movements, show multiple peaks (Figure 2.14(b)). These profiles are evidence of a qualitative difference in the wrist-speed trajectory for movements of different speeds. Figure 2.15 shows the box plots of the number of multimodal wrist-speed profiles observed for all subjects and tasks. A scrutiny of Figures 2.14 and 2.15 yield the following observations.

(O19) Multimodal wrist-speed profiles are observed for all tasks, suggesting that the phenomenon could potentially occur throughout the subject’s extra-personal space.

(O20) All nine subjects together performed 540 movements at each of the slow, normal, and fast speeds. Of the 540 slow movements, 251 are multimodal. There are 39 multimodal normal-speed movements, and zero multimodal fast-speed movements.

(O21) From Figure 2.15, roughly the same number of multimodal wrist-speed profiles are observed for all tasks except Task 3. The variability is also similar for all tasks.
Figure 2.14: Two qualitatively different wrist-speed profiles. The unimodal and multimodal wrist-speed profiles for subject JR are plotted. The profiles are normalized for movement time, but not for peak speed. All fast-speed profiles are unimodal. Most slow-speed profiles are multimodal. Normal-speed profiles can be unimodal or multimodal. Multimodal profiles are observed for all tasks. In all subfigures, the normalized time is plotted on the $x$ axis, and the wrist speed is plotted on the $y$ axis.

Figure 2.15: Box plots of the observed number of multimodal wrist-speed profiles for tasks and subjects. Except for Task 3, the number of profiles tends to be independent of the task. The number varies with the individual subject’s interpretation of the ‘normal’ and ‘slow’ for performing those movements.
Figure 2.15(b) indicates that the number of multimodal profiles varies with subject. Fewer multimodal profiles are observed for Task 3 because this task had the shortest arc length. Therefore, fewer movements lasted long enough so that multimodal profiles would be observed. The inter-subject difference in the interpretation of the speed cues for normal-speed and slow-speed movements can explain the variability in the number of multimodal wrist-speed profiles. Subjects JM, AL, FR, RR, and KB show similar number of multimodal profiles. The mean (±SD) of the 120 normal- and slow-speed movement times for these subjects are 1.19(±0.31), 1.6(±0.5), 1.6(±0.36), 1.54(±0.49), and 1.43(±0.51) seconds, respectively. The other four subjects, DE, CR, JR, and AS, exhibit a higher number of multimodal profiles. Correspondingly, the mean movement times for these subjects are higher at 1.86(±0.83), 2.3(±0.7), 2.0(±0.5), and 2.3(±1.2) seconds, respectively. The latter four subjects performed their tasks more slowly, and therefore have more multimodal wrist-speed profiles.

Figure 2.16 plots average movement speed for all tasks and subjects against the modified movement speed. In this figure, the data points are color coded according to movement speed, and additionally, tests that exhibit unimodal wrist-speed profiles are indicated as crosses, and those with multimodal profiles are indicated as circles. In Figure 2.16, the portion of the plot to the left of the left vertical line contains only unimodal wrist-speed profiles. Similarly, the portion to the right of the right vertical line contains multi-modal wrist-speed profiles. The portion between the vertical lines contains a mix of unimodal and multimodal wrist-speed profiles. The horizontal line indicates the largest average wrist speed at which multi-modal wrist-speed profiles were observed. The rectangle defined in the center of Figure 2.16 is a transition zone in the speed-movement duration space.
Figure 2.16: The average wrist speed is plotted against the modified movement time for all tests and subjects. Tests that exhibit unimodal wrist-speed profiles are indicated as crosses, and those with multimodal profiles are indicated as circles. The rectangle at the center of the figure defines a transition zone between tests exhibiting unimodal and multimodal wrist-speed profiles.
Figure 2.17: The variability in the normalized wrist-speed profiles within each speed. The variability in the profiles for fast-speed movements is lower than that for the normal-speed or slow-speed movements for all subjects.

wherein the unimodal bell shape of the wrist-speed profile for reaching motion begins to break down. A rough characterization of the transition zone is:

- Duration between $1.5 \text{ s} \text{ to } 3.7 \text{ s}$.
- Average wrist speed less than $0.5 \text{ m/s}$.

The A-H similarity metric $w$ quantifies the variability in the wrist-speed profiles for a particular speed for each subject. Figure 2.17 shows the standard deviations in $w$ for every speed and subject. The figure yields the following observations.

(O23) Fast-speed movements show the least variability in the wrist-speed profiles for all subjects.
Six of nine subjects show a lower variability in the wrist-speed profiles for normal-speed movements compared to the profiles for slow-speed movements.

Observations (O19) - (O24) and Figure 2.16 lead to the following result.

(R4) The variability in the shape of the normalized wrist-speed profile tends to increase as the movement speed reduces. Below an average wrist speed of about 0.55 m/s, the wrist-speed profile starts to deviate from the unimodal bell shape. In addition, if the movement duration is greater than approximately 1.5 seconds, the wrist-speed profiles may show more than one peak, and if the movement duration is greater than approximately 3.7 seconds, the wrist-speed profile shows more than one peak. This characteristic seems to be independent of the reaching task as well as subject.

This phenomenon was not observed in the previous work on spatial human reaching, possibly because previous experiments did not study reaching movements that lasted as long or were performed as slowly. The longest movements in [110] are one second long. The slow movements in [11] lasted roughly 1.2 seconds. The mean (± SD) movement duration and average peak wrist speed for the slow movements in [2] were 1.07(±0.02) seconds and 1.22(±0.05) m/s, respectively, whereas for the present study the modified movement time and average wrist speed were 2.57(±0.13) seconds and 0.46(±0.03) m/s, respectively. Nishikawa et al. [114] report peak speeds as low as 0.3 m/s, but do not mention the duration of movement in their study. It is difficult to estimate the average wrist speed from this information. Breteler et al. [19] report the slowest mean tangential finger tip speed of 1.04(±0.2) m/s but do not report movement time. It seems unlikely that movements as long and slow as performed in the present study were performed in [114] or [19].

The multiple modes in the wrist speed profiles correspond to submovements. Such submovements are commonly observed in motion studies in which the movement must be
accomplished with accuracy and in minimal time. There exists a compromise between the movement speed and accuracy which is captured by the well known Fitt’s law [105]. The normative theories explaining Fitt’s law involve choosing the duration of the submovements to optimize the total movement duration [94, 105]. The present study is of a different kind, since the movement must be accomplished in a given time, and this movement time, chosen by the subject, is not rigorously controlled. Multimodal wrist-speed profiles have been observed before in studies similar to the present one [68, 104, 156]. Isenberg and Conrad [68] and van der Well et al. [156] used targets placed atop a table. For both studies, however, movement between the targets was timed using a metronome. Messier et al. [104] conducted a spatial reaching study with target-placement in 3D space. The target placement for this study was similar to that in [2]. The subjects in this study performed point-to-point reaching motions at different speeds. Therefore, multimodal wrist-speed profiles for untimed arm movements in a large portion of the subject’s extra-personal space are reported here for the first time.

**Revisiting wrist-path shape**

It was predicted that multimodal wrist-speed profiles will be seen for movements with straight as well as curved wrist paths. This prediction is validated in Figure 2.18 which plots the A-H linearity metric against the modified movement time and average wrist speed for all tasks and subjects. The tests that have unimodal wrist-path shape are indicated as crosses, and those with multimodal wrist-shape profiles are indicated as circles. The figure yields the following observation.
Figure 2.18: The A-H linearity metric is plotted against the modified movement time and average wrist speed for all tasks and subjects. The tests that have unimodal wrist-path shape are indicated as crosses, and those with multimodal wrist-shape profiles are indicated as circles. A wrist path with a given A-H metric value can have a unimodal or a multimodal wrist-speed profile. Conversely, multimodal wrist-speed profiles are observed for wrist paths of varying degree of straightness. The figure shows that multimodal wrist-speed profiles are not seen exclusively for highly curved wrist paths.

(O25) A wrist path with a given A-H metric value can have a unimodal or a multimodal wrist-speed profile. Conversely, multimodal wrist-speed profiles are observed for wrist paths of varying degree of linearity.

This yields the following result.

(R5) *The number of modes of the wrist-speed profile does not depend on linearity of the wrist path.*

In other words, as the reaching movement becomes slower and longer in duration, the wrist-path may or may not become significantly more curved. However, it is likely that the wrist-speed profile will deviate from the unimodal bell shape.
2.4 Discussion

The first objective of this study was to investigate the validity of the TIH by studying the characteristics of the wrist trajectory. The kinematic analysis investigated if wrist paths and normalized wrist-speed profiles are time invariant. For wrist paths, the present study reports mixed results, with time invariance being a feature of wrist paths of certain subjects in certain portions of the output space (see results R2 and R3). For time-varying wrist paths, not only are the differences in the mean values of the AH linearity metric for different speeds statistically significant, but the size of the difference is large as well. For example, the metric values for subject JM, Task 5 are 0.12 ± 0.02, 0.2 ± 0.04, and 0.27 ± 0.04 for the fast, normal, and slow speed movements, respectively. This indicates that for slow motions, the departure of the wrist from a straight-line path was 25% of the straight-line-path length, whereas, for the fast movements, this departure was half that for the slow movement - 12% of the straight-line-path length. Furthermore, for the same subject, the metric value for Task 3 at slow speed is 0.04 ± 0.02; the path is close to a straight line. Thus, the wrist path shape for various tasks were significantly different.

The speed-dependence of wrist paths for some tasks and dependence of the wrist-path shapes on workspace do not provide validation of the TIH. However, time-invariant wrist paths were observed for some subjects and for some tasks. Furthermore, as discussed in Section 2.1, the evidence in favor of a planned wrist path is strong, and this remains true despite the findings of this work. Thus, the TIH is not a truly general organizing principle for spatial reaching motion. This hypothesis needs refinement and further investigation. Two alternate hypotheses are proposed below to explain the results of this study. The first hypothesis prescribes a role to the time-invariant motion plan that is consistent with the
findings of this study. The second hypothesis attributes to gravity effects the departure of the actual wrist path from a straight-line plan for slow movements.

It is plausible that a time-invariant motion plan is composed, but it serves different roles in the overall control scheme under various contexts. Consider the following contexts:

1. Performing rehearsed movement vs. learning a motor skill, and

2. Performing rehearsed movements that require varying levels of geometric accuracy.

Reaching movement performed by adults in an unaltered environment, like that of the present experiment, qualifies as a rehearsed movement with a low accuracy requirement. The motor adaptation that occurs during perturbation studies like those of Shadmehr and Mussa-Ivaldi [135] is an example of a motor learning paradigm. A baby flailing its limbs is another example of the learning paradigm, wherein the limb movement generates proprioceptive data that is utilized by the CNS to develop motor control capabilities. A professional pianist playing the piano and a professional golfer swinging a club qualify as rehearsed movements that need to be geometrically accurate.

Within the first context, the CNS does not compensate for small deviations from the time-invariant kinematic plan while performing rehearsed movement with low accuracy requirements. However, if the deviations are large, the kinematic objective becomes imperative, as was observed by Shadmehr and Mussa-Ivaldi [135]. On the other hand, for learning novel skills, like playing the piano or swinging a golf club, perhaps a time-invariant kinematic objective is initially specified. Motor commands and muscle forces necessary to achieve this objective are composed. Further training in these activities consists of refining these commands for improving kinematic performance as well as optimizing dynamic criteria such as muscle forces, in addition to updating the time-invariant kinematic plan itself.
Once learning has been accomplished, the time-invariant kinematic motion imperative may be relaxed to a degree based on the second context.

From this perspective, the variability in the path shapes with speed observed in this work could be construed as an insignificant deviation from a desired path by the CNS. In summary, the hypothesis here is that the CNS composes the wrist path for reaching tasks, but does not track it accurately since the accuracy requirement of the task is low. This conjecture can be verified by conducting a spatial version of the study of Shadmehr and Mussa-Ivaldi [135] wherein the arm is systematically perturbed while performing spatial reaching movements. The existence of a time-invariant plan would be supported if wrist-path shapes converge with practice.

While comparing the AH linearity metric for wrist paths, the slow-fast movement pair provides the strongest test of time-invariant wrist paths, since this pair has the largest velocity difference. In Table 2.8, of the 54 different subject-task combinations, 40 show significant difference in the AH linearity metric value for the fast and slow movements. Additionally, slower movements display more curved wrist paths, according to result (R2). Since the gravitational impulse for slower movements will be greater, the second hypothesis attributes the loop of the wrist path to gravity. The loop of the wrist path refers to the wrist path curvature in the vertical plane observed in Figure 2.9. This suggests that the wrist path composed by the CNS is a straight line and slower movements result in curved paths because of the inability of the CNS to accurately compensate for gravity. In the same vein, the effect of gravity appears in the wrist speed profiles in the form of multi-modal profile shapes of slow and long movements, as described in results (R4) and (R5). This hypothesis is explained in greater detail in Chapter 3 in Section 3.5 since it involves the dynamic analysis of reaching motion.
The other significant finding of this study pertains to wrist-speed profiles, as described in results (R4) and (R5). Each peak in the wrist-speed profiles in Figure 2.14 corresponds to a ‘submovement’. Liao et al. [94] define a submovement as a brief episode in which the absolute velocity of movement increases and then decreases. In this sense, the slow and long movements in this study are often concatenations of submovements. Motions with submovements were observed for spatially constrained tasks in which subjects are required to perform accurate, discrete, target-acquisition movements while simultaneously minimizing movement time. A movement speed-accuracy trade off exists for such movements, which is famously captured by Fitt’s law. Meyer et al. [105] review the existing explanations for this phenomenological law and also propose a normative theory themselves. Their stochastic, dual-submovement optimization model assumes that motor noise increases with movement speed. Then, duration of each submovement and the corresponding submovement speed are selected to simultaneously minimize the total movement duration and the variability in the final position. Subsequently, Liao et al. [94] modified this theory by showing that the first submovement duration is influenced by the target size and is chosen to achieve near optimal total movement time. Conversely, the duration of the second submovement is at an approximately constant minimum across different targets.

The tasks in the present study do not resemble spatially constrained tasks because the motions were not performed to minimize total movement time. These motions resemble temporally constrained tasks in which subjects are required to perform accurate, discrete, target-acquisition movements in a specified time. This is only a resemblance, however, since in this study, the movement times were selected by the subject and were not rigorously controlled. Occurrence of multimodal wrist speed profiles in these tasks is attributed to the workings of an internal clock, rather than the inverse dynamic algorithm.
Multimodal wrist-speed profiles in similar experiments have been observed before [68, 104, 156]. However, only van der Well et al. [156] attempt to provide any explanation of the phenomenon. Since the movements in this previous study were timed using a metronome, the explanation hinges around the motor system’s perception of time intervals and proceeds as follows. The CNS perceives longer time intervals with greater variability [69]. The CNS can establish a pattern of occurrence of a periodic, external stimulus only when the stimuli occur ‘sufficiently’ fast. Consequently, human response to the external stimulus is anticipatory for the high-frequency stimuli, and a reaction otherwise. Furthermore, the subjects in the study of van der Well et al. [156] generated several submovements at low metronome rates, which, presumably, correspond to the multiple peaks in the wrist-speed profiles. The study, however, does not propose any mechanism relating the low-frequency stimuli to the presence of submovements in the response.

The very idea of the CNS using an internal model for time estimation is consistent with the TIH, which proposes that the CNS composes movement geometry and appends movement timing to the geometric plan. The movements in the present study were not timed with a metronome; therefore, the subjects were not reacting to an external, experimental stimulus after a movement had commenced. Regardless, the subjects produced submovements that were manifested in multimodal wrist-speed profiles for several slow- and normal-speed movements. Based on the ideas in the previous paragraph, a plausible explanation for this phenomenon is as follows. The speed cue provided requires the subject to estimate the time he/she will take for the current task, and in the case of slow movements, this interval is large in several cases. Consequently, the duration of movement is estimated with difficulty, and this translates into difficulty in selecting the average motion speed. If the subject selects the average motion speed that is too low, the error may become
apparent after crossing the first peak in the wrist-speed profile when he/she realizes that the motion is slowing down, but the target is not sufficiently close. At this point, the subject may append another submovement to the initial submovement, thus adding another peak to the wrist-speed profile.

This explanation is reminiscent of the theories of Meyer et al. [105] and Liao et al. [94]. Both explanations hinge on the selection of movement time. However, the accuracy-influenced movement duration for spatially constrained tasks is near optimal whereas, in the present work, submovement timing is selected to meet a slightly vague temporal constraint.

Finally, result (R8) states that the multimodal wrist-speed profiles occur for straight as well as curved wrist movements. This simply means that the phenomenon occurs over the entire workspace. This is verified by Figure 2.14(b), which shows multimodal wrist-speed profiles for all tasks for subject JR.

2.5 Conclusions

A spatial reaching motion study with healthy human subjects was conducted aimed at extending the TIH to spatial reaching movements. Subjects performed point-to-point reaching motions at different speeds. Analysis of wrist paths revealed that wrist path shape depends on workspace and sometimes on the speed of motion. Therefore, the TIH may not be a truly general organizing principle for spatial reaching motion. Two hypotheses were proposed that describes the role of the TIH in the overall control strategy for arm motion. According to the first hypothesis, the CNS construes deviations from a planned wrist path as significant only under certain contexts. Alternatively, it was hypothesized that the deviations from a planned *straight* wrist path occur because of difficulties in handling the increased gravitational effects for slow movements.
The other significant finding of this study pertained to wrist-speed profiles. High variability in the unimodal wrist-speed profiles as well as multimodal profiles were observed for long (movement time > 1.5 seconds) and slow (average wrist speed < 0.55 m/s) untimed arm movements. Similar to wrist-path shape, this phenomenon can be attributed to the increased gravitational effects for slow movements. Alternatively, the phenomenon could be a consequence of the deterioration of the CNS’s ability to estimate longer time intervals.
Chapter 3: THE DYNAMICS OF SPATIAL HUMAN REACHING: INVESTIGATING THE LEADING JOINT HYPOTHESIS.

3.1 Introduction and background

The leading joint hypothesis (LJH) proposes an algorithm for transforming joint motion into joint torques. Therefore, it is an inverse-dynamic internal model. It belongs to Marr’s second level (see Section 2.1). It comments on the nature of the interaction effects that occur during the movement of multi-joint serial chains, and it also achieves simplification of the joint-level control required for task execution. Due to the multi-joint structure of human limbs, the control of motion at each joint is dependent on movements at the other joints because of the dynamic effects of adjacent limb segments. Inter-segmental dynamics include internal effects, such as interaction torques emerging as a result of the motion of limb segments\(^7\), effects due to hand-held objects, etc. External dynamic effects are also considered. For instance, a force applied to the hand during object manipulation affects the entire arm, generating torques at the joints. Torques that arise from such segmental and environmental interactions must be accounted for during movement control. How the descending neural commands are adjusted to these torques has intrigued researchers

\(^7\)In multi-joint serial chains, the actuation of one joint causes torques at all the joints in the chain. These are called interaction torques.
as early as Bernstein [17]. Bernstein hypothesized that the control of such ‘reactive phenomena’ would be accomplished by precise coordination of muscle timing, suggesting that these passive interactions could be specifically dampened or made use of to drive movement. Alternatively, Bernstein also suggested that these interactions could be controlled by modulating the joint stiffness via cocontraction of antagonist muscles. Hollerbach and Flash [66] first showed that the interaction effects significantly influenced motion production for human reaching. Using forward-dynamic simulations, they demonstrated that the wrist paths that result from ignoring interaction effects are highly non-characteristic of observed human behavior. Several studies followed that emphasized the role of interaction effects and showed that muscular control is adjusted to the passive effect of joint motion [1, 48, 54, 59, 72, 73, 90, 127, 128, 133, 154, 160]. There are several approaches that attempt to understand how multi-joint movements are controlled. The LJH belongs to the approach known as force control in which neural representation of inter-segmental dynamics is used to directly specify muscle force [66].

The LJH classifies the joints in a multi-joint chain as (usually) one leading joint and one (or more) subordinate joint(s). The leading joint is, in most cases, the most proximal joint in the chain. The key to the LJH is that the leading joint experiences small interaction effects. Therefore, the interaction effects at the leading joint can be ignored, or at any rate, a rough estimate of those effects is sufficient to execute the task. This results in a significant simplification of the control of the leading-joint musculature. The following description of the LJH is borrowed from Dounskaia [32]:
LJH: “Joints of a multi-articular limb play different roles in movement production according to their mechanical subordination in the joint linkage. There is one (leading) joint that creates a dynamic foundation for motion of the entire limb. Acceleration/deceleration at the leading joint is produced by reciprocal muscle activity in the same way as during single-joint movements, i.e. largely disregarding motion at the other joints. Leading joint motion generates powerful interaction torque at the other (subordinate) joints. The role of the subordinate joint musculature is to regulate interaction torque and to create net torque that results in motion of the end effector required by the task.”

The LJH gathers supporting evidence from a host of studies [21, 33, 47, 75, 92, 63]. The roles of leading and subordinate joints are assigned based on an analysis of joint torques computed using inverse-dynamics calculations. Further support is offered by muscle electro-myographical (EMG) signals that show reciprocal bursts of activity at the leading joint that is tightly coupled with its acceleration and deceleration [33, 92]. The EMG activity at the subordinate joint is more complex. Most of these studies are planar motion studies [21, 33, 47, 75, 92]. Hirashima et al. [63] conducted the only spatial study, wherein the throwing motion of skilled baseball players was analyzed. They analyzed the torque distribution at four joints: the waist, shoulder, elbow and the wrist and demonstrated the prevalence of the leading-joint strategy in the movements. However, an extension of the planar-study-based LJH to spatial reaching motion has not been done before. Therefore, the objective of this chapter is to extend the LJH to spatial reaching motion.

The time-invariance hypothesis (TIH), investigated in Chapter 2, and the LJH model different stages of the motor control problem. The TIH is an inverse kinematic model, and the LJH is an inverse dynamic model. Therefore, the output of the TIH can serve as the
input to the LJH. Both hypotheses are internal models (see Section 2.1) and, they have another strikingly common feature. Assume for a moment that the output-space motion plan is the wrist path. Then, an algorithm to perform the inverse kinematics transformation in a time-invariant fashion is developed in Chapter 4 of this dissertation. The algorithm generates a curve in the joint space that corresponds to the desired curve (the wrist path) in the extra-personal space. This joint-space solution is parameterized in terms of one of the joints of the arm. This joint is also called the leading joint, and it is the kinematic analogue of the leading joint defined in the LJH. This is the reason that these two particular hypotheses were chosen for the present study. Associated with these similarities, however, are a number of questions. How are the TIH and the LJH related? Can they be part of a coherent strategy that describes the motion generation process starting from the specification of the reaching task in the output space and the motion timing - as required by the TIH - to the generation of the appropriate joint torques according to the LJH? What kinematic characteristics of spatial motion are consequences of the LJH? Although these are important questions, they are beyond the scope of this study.

The data obtained from the experiment described in Section 2.2 were used to accomplish the analysis of this chapter.

3.2 Equations of motion and torque partitioning for the human arm

To extend the LJH to spatial motion, the appropriate equations of motion (EOM) must be developed. Additionally, a partitioning scheme is required that allows the torques at the shoulder and the elbow to be divided into the following components: ‘net torque’ (NT), ‘interaction torque’ (IT), ‘gravity torque’ (GT), and ‘muscle torque’ (MT). Once this is achieved, data analysis can be conducted. This section is divided into two parts. First,
appropriate definitions for NT, MT, GT, and IT are selected from existing literature. Next, the EOM for a three-link spatial arm are developed, and a partitioning scheme based on the appropriate definitions of the torque components is employed to obtain expressions for these components.

**Defining torque components**

The free-body diagram (FBD) of the human arm is shown in Figure 3.1, and the nomenclature used in this figure is explained in Table 3.1. Applying Newton’s second law to the forearm yields the reaction force $R_E$ at the elbow. Next, Euler’s EOM [58, 176] for the forearm yields an expression for the torque acting at the elbow joint $T_E$. A similar process for the upper arm yields the shoulder-joint torque $T_S$. The terms in Euler’s equation for a segment are partitioned into four components: NT, MT, GT, and IT. These are the torque components at the segment’s proximal joint, and they are related by the equation

$$T_{net} = T_{muscle} + T_{interaction} + T_{gravity}.$$  \hspace{1cm} (3.1)

Equation 3.1 is used to partition the torques $T_E$ and $T_S$. There is some variability in the literature regarding the choice of motion variables used to derive these equations. Furthermore, although the nomenclature used in Equation 3.1 has been extensively employed in the motor control literature for planar [32, 33, 47, 118, 125, 128] as well as spatial [39, 62] studies, there is some variability in their definitions. These issues are discussed in detail and resolved below.

The first issue is the appropriate choice of motion variables. Some authors have utilized limb or segment rotations to write the EOM for planar [118] and spatial [39] arm models where, the angle made by a limb segment with an absolute, immobile reference frame is used as the angular variable in the EOM. Others have utilized the angle made by one
The human arm

Figure 3.1: The top figure of the human arm is borrowed from www.flashkid.org. The free-body diagrams (FBD) of the upper arm and the forearm are shown. The hand is modeled as a point mass rigidly attached to the end of the forearm. Coordinate frames $U$ and $F$ are fixed to the upper arm and the forearm, respectively, and the equations of motion are written in the global reference frame.

78
Table 3.1: Nomenclature. The symbols used in Figure 3.1 are defined here.

Exercising the joint rotation rather than the segment rotation is necessary to assess the
collection of the muscle actions evoked by the CNS because muscles act directly on the
angular motion of the associated joint [128]. Therefore, joint rotations are used as the
motion variables in the analytical development in this work.

The next issue is the variability in the definitions of the torque components. Zatsiorsky
[176] provides a brief summary of the usage of the terms net, interaction, muscle and
gravity torques based on a review of planar studies. The set of definitions for these torque
components given below is provided by Schneider and Zernicke [132].

DN1. *Net Joint Torque*: The sum of all the positive and negative torque components (grav-
itational, interactive, and muscle) that act at a joint.

DG1. *Gravitational Torque*: A passive torque resulting from gravity acting at the center of
mass of each segment.
DI1. *Interactive Torques:* Passive torques arising from dynamical interactions among segments, such as inertial forces proportional to segmental accelerations or centripetal forces proportional to the square of segmental velocities.

DM1. *Generalized Muscle Torque:* A ‘generalized’ torque that includes forces arising from active muscle contractions and from passive deformations of muscles, tendons, ligaments, and other periarticular tissues. Because the effects of active muscular forces are embedded within this component, the generalized muscle torque comprises the actively-controlled elements of limb-trajectory motor programs, as well as passive effects of muscle and connective tissues.

Here, these definitions serve as a starting point for the development of the ideas related to torque partitioning. The definitions for GT and MT, DG1 and DM1, respectively, are satisfactory and are used without alteration in the analysis presented in this work. However, the definitions for NT and IT require scrutiny.

The definition of NT, DN1, is quite general, and it agrees with Equation 3.1. By describing NT as the effect of all external torques acting at a joint, this definition reflects the causal relation between torque components implied by Equation 3.1. However, it is contingent on the definition of IT. The definition DI1 is general as well. The operational phrase in the definition is ‘passive torques’. However, the definition does not explicitly define its meaning. The result of this ambiguity is that there exist multiple interpretations of NT in the literature. Dounskaia et al. [33] provide two torque-partitioning schemes that have been utilized in the literature for planar arm-motion studies. The first partitioning arises when NT includes the inertial effects of the upper arm only, and the effect of the forearm motion is expressed as the interaction torque. With this scheme, the expressions for the shoulder and elbow NT and IT for a planar, three-link arm comprising of a fixed trunk, upper arm,
and forearm jointed at the shoulder and the elbow with single-DOF revolute joints are

\[ T_{\text{net}} = I_p \ddot{\phi}, \quad (3.2) \]

\[ T_{\text{interaction}} = - \left[ I_d + m_d(l_p^2 + 2r_d l_p \cos \theta) \right] \ddot{\phi} - [I_d + r_d l_p m_d \cos \theta] \ddot{\theta} + r_d l_p m_d \sin \theta \dot{\theta}^2 + 2r_d l_p m_d \sin \theta \dot{\theta} \dot{\phi}, \quad (3.3) \]

where \( I_d \) and \( I_p \) are the moments of inertia of the distal and the proximal limb, respectively, about an axis passing through the proximal end of the corresponding segment, \( \phi \) is the shoulder-joint angle, \( \theta \) is the elbow-joint angle, \( m_d \) is the mass of the distal segment, \( l_p \) is the length of the proximal segment, and \( r_d \) is the distance of the center of mass (COM) of the distal segment from its proximal end.

The second partitioning arises from including inertial effects of the whole arm in the net shoulder torque. This yields:

\[ T_{\text{net}} = I_p \ddot{\phi} + \left[ I_d + m_d(l_p^2 + 2r_d l_p \cos \theta) \right] \ddot{\phi}, \quad (3.4) \]

\[ T_{\text{interaction}} = - [I_d + r_d l_p m_d \cos \theta] \ddot{\theta} + r_d l_p m_d \sin \theta \dot{\theta}^2 + 2r_d l_p m_d \sin \theta \dot{\theta} \dot{\phi}. \quad (3.5) \]

This partitioning scheme suggests that the purpose of the shoulder-joint muscle control is the motion of the entire arm rather than just the upper arm [13, 66, 125, 128]. This ambiguity does not occur for the elbow joint, since it is the last joint in the chain, and, therefore, there is only one segment distal to the joint. For the elbow joint, the equations are:

\[ T_{\text{net}} = I_d \ddot{\theta}, \quad (3.6) \]

\[ T_{\text{interaction}} = -(I_d + r_d l_p m_d \cos \theta) \ddot{\phi} - r_d l_p m_d \sin \theta \dot{\phi}^2, \quad (3.7) \]

where \( r_p \) is the distance of the COM of the proximal segment from its proximal end.
An alternative descriptor of net torque has been used in Dounskaja et al. [33], Dounskaja [32] and Zatsiorsky [176]:

**DN2. Net Torque**: Joint torque associated with the acceleration at the joint.

Definition DN2 is more precise than definition DN1 yet it applies to both Equations 3.2 and 3.4. Therefore, it does not eliminate the ambiguity of definition DN1 in that the final shoulder-torque expressions still depend on the definition of IT.

Definition DI1 can be made precise by defining the term ‘passive torque’. A passive torque arises at a joint due to muscle activity that is meant primarily to move other joints. This leads to an alternative definition for interaction torque which has been used by Sainburg et al. [128] and mentioned by Zatsiorsky [176].

**DI2. Interaction Torque**: Torque component at a joint associated with acceleration and velocity at other joints.

Definition DI2 does not suffer from the drawbacks of definition DI1. Adopting DI2 resolves all ambiguities in the interpretations of net and interaction torques, and it implies that the interaction torque vanishes if all other joints are stationary. Therefore, Equations 3.4 and 3.5 are the only valid equations for the shoulder-joint torques according to this definition. The definition prescribes the function of shoulder-joint muscles as the movement of the entire arm rather than just the upper arm, as suggested in [13, 66, 125, 128]. Notice that once DI2 is adopted, both DN1 and DN2 are valid definitions of NT and yield the same set of equations for the shoulder joint. The definitions DI2 and DN1 (or DN2), taken in that order, are consistent with Equations 3.6 and 3.7 for the elbow joint as well.

The discussion so far is based solely on planar motion analysis. However, the definitions mentioned above have been applied to spatial motion analysis without close scrutiny.
Definition DN2 has been employed by Hirashima et al. [62] to develop a torque partitioning scheme for 3D motion analysis. Furthermore, this scheme is used in Hirashima et al. [63] to conduct spatial motion analysis of the pitching action of skilled baseball players. Little, if any, explanation of the definition of torque components is provided by these authors.

Here, appropriate definitions for NT and IT are used after studying Euler’s equation of motion [58, 176] that describes the general motion of a rigid body in space. In vector form, the equation is

\[ \overline{M} = \mathbf{I} \cdot \overline{\alpha} + \mathbf{\omega} \times \mathbf{I} \cdot \mathbf{\omega}, \tag{3.8} \]

where \( \overline{M} \) is the sum of all external moments acting on the rigid body, \( \mathbf{I} \) is the inertia tensor, and \( \mathbf{\omega} \) and \( \overline{\alpha} \) are the angular velocity and angular acceleration, respectively, of the rigid body. Equation 3.1 for a joint is obtained by partitioning the terms in Equation 3.8 written for the limb segment distal to the joint. This equation is applicable to planar and spatial motion. The critical observation is that, in general form, the EOM for a limb segment contain terms that are proportional to the square of the angular velocity of the proximal joint. These terms arise from the second term on the RHS of Equation 3.8, called the gyroscopic torque, and from the moments created by some of the joint reaction forces.

In the case of planar motion, Equation 3.8 yields Equations 3.2-3.7. When applied to the forearm, it yields the torque components for the elbow joint. For planar motion, the angular velocity and acceleration of the forearm are parallel vectors with magnitudes \( \dot{\theta} \) and \( \ddot{\theta} \), respectively. Furthermore, the angular velocity vector is assumed to be parallel to one of the principal axes of inertia. Therefore, the gyroscopic torque vanishes. Next, the left-hand-side (LHS) of Equation 3.8 includes the moment created by the centripetal force which is proportional to \( \dot{\theta}^2 \). However, for planar motion, it is assumed that the angular velocity vector is normal to the plane of the motion, which results in the centripetal force
passing through the elbow-joint center. Therefore, the moment created by the centripetal force will also vanish. Consequently, the EOM will not contain any term proportional to the square of the angular velocity of the forearm, \( \dot{\theta}^2 \), and NT will be proportional to the angular acceleration \( \ddot{\theta} \). This can be verified from Equation 3.6. An identical argument applies to the upper arm and the EOM for the shoulder joint. In this case, the EOM will not have any term containing \( \dot{\phi}^2 \). This can be verified from Equations 3.2 and 3.4. Terms proportional to the square of the angular velocity do not feature in the discussion of torque partitioning for the planar problem, and definition DN2 applies.

In the case of general 3D motion, however, the gyroscopic torque does not vanish, and the centripetal components of the joint-reaction forces create non-zero moments. Therefore, the EOM for the limb segments will contain terms that are functions of the square of the angular velocity. These terms are isolated in the derivation given in the next section. The question is whether these terms should be classified as NT or IT. By applying definition DN2 [62, 63], the terms are included in the expressions for IT. However, this creates ambiguity about the nature of IT. Adopting DN2 for spatial motion implies that the interaction torque does not vanish if all other joints are stationary, thus violating definition DI2. Furthermore, since NT is defined prior to defining IT, it naturally creates ambiguity in its own definition of the nature pointed out by Dounskaia et al. [33]. Conversely, if these velocity-dependent terms in Euler’s equation are classified as NT, definition DI2 is respected, and consequently, definition DN1 applies as well. In this case, DN1 and DI2 can be considered as general definitions applicable to planar and spatial motion. Furthermore, the two definitions DN1 and DN2 retain their equivalence for planar motion.

In the following work, definitions DN1 and DI2 are used. This yields a simple recipe for torque partitioning at both joints. Each term in the EOM is a product of often more than
one kinematic quantity and some inertial parameters. The classification of terms is done based only on the kinematic quantities it contains. A term is classified as contributing to NT if the kinematic quantities (velocity and/or acceleration) of only the joint in question appear in the term. All other motion-dependent terms constitute IT. This scheme is used in the next section for torque partitioning.

The discussion so far resolves semantic issues associated with the use of the terms NT and IT. As a result, the definitions used in this study are reasonable and logically consistent—a characteristic that is, on the whole, absent in previous studies. The consequences of these definitions for the phenomena they potentially describe is an important discussion presented in Section 3.5.

Equations of motion

The procedure below follows the development of Gagnon and Gagnon [44], Winter [166] and Feltner and Dapena [39] and modifies it by introducing movement of the forearm relative to the upper arm. It is also similar to the development in Hirashima et al. [62], but uses a different set of definitions for NT and IT and a different torque-partitioning recipe, as described in the previous section.

In the derivation that follows, the following assumptions are made.

- The thorax is assumed to be stationary during the reaching movements. The subject’s thoracic movement was constrained by strapping him/her to the chair.

- The shoulder joint is modeled as a ball-and-socket joint allowing three degrees of rotational freedom for the humerus. The glenohumeral center was assumed to be the joint center. This center was located using the rotation method developed by Veeger [158], as mentioned before.
• The elbow joint is modeled as a single-DOF revolute joint. The axis of revolution is along the $Z$ axis of the coordinate frame fixed in the upper arm, as recommended by the ISB shoulder protocol [170]. In particular, forearm pronation-supination is ignored in this analysis. The moment of inertia of the forearm about this axis is typically small compared to the inertia about a transverse axis [28]. Therefore, the elbow torque required for this motion will be small compared to the torque about the elbow-joint axis. Furthermore, subjects were instructed to perform reaching movements with a rigid wrist. This minimizes pronation-supination movement, further reducing the torque contribution.

• The wrist joint is ignored. The subjects were instructed to move their arms with a rigid wrist. Although the wrist was not constrained, the trials wherein wrist movement was observed were immediately repeated.

• The hand was incorporated in the dynamic analysis by modeling it as a point mass at the end of the forearm. The mass and the COM of the combined ‘forearm-hand’ were computed and used in the torque calculations.

There are three reference frames: the global frame $G$, the upper-arm frame $U$, and the forearm frame $F$. Frames $U$ and $F$ are located at the center of mass of the corresponding arm segment, and the axes of the frame are aligned with the principal axes of the segment. A leading superscript $U$ or $F$ for a vector indicates the reference frame in which the vector is expressed. A vector expressed in the global frame is written without a superscript. Subscripts $f$, $u$, $h$ and $g$ indicate the forearm, upper arm, hand, and the acceleration due to gravity, respectively. Derivatives with respect to time are indicated by the dot notation. A rotation matrix expressing the orientation of frame $(..)$ in frame $(.)$ is denoted as $(^l)R_{(..)}$. 

86
The free-body diagrams for the two arm segments are shown in Figure 3.1. The symbols used in Figure 3.1 are defined in Table 3.1.

Newton’s second law is applied to the forearm (including the hand) and the upper arm. The expressions for the reaction forces are obtained in frame $G$.

$$\overline{R}_E + (m_f + m_h)g = (m_f + m_h)\ddot{r}_P.$$  
(3.9)

The absolute (i.e. measured in frame $G$) acceleration of point $P$ expressed in frame $U$ is given by [58]

$$U\ddot{r}_P = U\dddot{a}_{pf} + U\dddot{a}_{pu} + U\dddot{a}_{pint},$$  
(3.10)

where

$$U\dddot{a}_{pf} = U\dddot{r}_{P/CU},$$

$$U\dddot{a}_{pu} = U\dddot{a}_u + U\dddot{\alpha}_u \times U\dddot{r}_{P/CU} + U\dddot{\omega}_u \times (U\dddot{\omega}_u \times U\dddot{r}_{P/CU}),$$

$$U\dddot{a}_{pint} = 2U\dddot{\omega}_u \times U\dddot{r}_{P/CU}.$$  
(3.11)

In Equation 3.11, the first term is the acceleration of $P$ relative to the upper arm COM measured in coordinate frame $U$ [58]. Therefore, this component of the net acceleration of $P$ is attributed to elbow-joint movement only. In the second term, $U\dddot{\omega}_u$ and $U\dddot{\alpha}_u$ represent the absolute angular velocity and absolute angular acceleration of the upper arm and $U\dddot{r}_{P/CU}$ is the position vector of point $P$ relative to the upper-arm COM. The acceleration of the upper arm COM is $U\dddot{a}_u$. Therefore, the component $U\dddot{a}_{pu}$ of the net acceleration of $P$ arises from the movement of the shoulder joint alone. In the last term, $U\dddot{r}_{P/CU}$ is the velocity of $P$ relative to the upper arm COM measured in frame $U$. This term is an interaction term because it is influenced by both shoulder- and elbow-joint movements. Note that while the quantities $U\dddot{r}_{P/CU}$ and $U\dddot{r}_{P/CU}$ are measured and expressed in frame $U$, all other quantities

87
in Equation 3.11 are measured in frame \( G \) and expressed in frame \( U \). Substituting Equation 3.10 into Equation 3.9 yields

\[
\overline{R}_E = \overline{R}_{EF} + \overline{R}_{EU} + \overline{R}_{Eint} + \overline{R}_{Eg},
\]  

(3.12)

where

\[
\begin{align*}
\overline{R}_{EF} &= (m_f + m_h)^G R_U \overline{\alpha}_{pf}, \\
\overline{R}_{EU} &= (m_f + m_h)^G R_U \overline{\alpha}_{pu}, \\
\overline{R}_{Eint} &= (m_f + m_h)^G R_U \overline{\alpha}_{pint}, \\
\overline{R}_{Eg} &= -(m_f + m_h)g.
\end{align*}
\]

The first term in Equation 3.12 vanishes if the elbow joint has zero velocity and acceleration. The second term vanishes if the shoulder joint has zero velocity and acceleration. The third term vanishes if either of the two joints has zero angular velocity. The fourth term contains gravitational acceleration and is never zero.

Newton’s second law applied to the upper arm yields

\[
\overline{R}_S - \overline{R}_E + m_u \overline{g} = m_u \overline{a}_u,
\]

\[
\Rightarrow \overline{R}_S = \overline{R}_{SU} + \overline{R}_{SF} + \overline{R}_{Sint} + \overline{R}_{Sg},
\]  

(3.13)

where

\[
\begin{align*}
\overline{R}_{SU} &= m_u \overline{a}_u + \overline{R}_{EU}, \\
\overline{R}_{SF} &= \overline{R}_{EF}, \\
\overline{R}_{Sint} &= \overline{R}_{Eint}, \\
\overline{R}_{Sg} &= -(m_u + m_f + m_h)g.
\end{align*}
\]

As for the forearm, the first term in Equation 3.13 vanishes if the shoulder joint has zero velocity and acceleration, and the second term vanishes when the elbow joint has zero velocity and acceleration. The third term vanishes if either joint has zero angular velocity. The fourth term contains gravitational acceleration and is never zero.
To compute the joint reaction forces in the form of Equations 3.12 and 3.13, the components of the acceleration of \( P \) must be calculated. The \textit{The MotionMonitor} software provides all of the kinematic quantities in Equation 3.11 except the quantities \( \dot{U}\hat{r}_{P/CU} \) and \( \ddot{U}\hat{r}_{P/CU} \). Since the elbow joint is modeled as a single-DOF revolute joint with its joint axis passing through point \( E \), the relative velocity \( \dot{U}\hat{r}_{P/CU} \) was estimated as

\[
\dot{U}\hat{r}_{P/CU} = U\omega_{fu} \times U\hat{r}_{P/E}, \tag{3.14}
\]

where \( \omega_{fu} \) is the angular velocity of the forearm relative to the upper arm measured in frame \( U \). The quantities on the RHS of Equation 3.14 are provided by \textit{The MotionMonitor} software. The component \( U\pi_{pivot} \) can thus be computed. Finally, \( \pi_{pf} \) is obtained from Equation 3.10 by subtracting all of the other terms on the RHS from the left-hand-side (LHS). The inertial properties of the limb segments were obtained from Dempster and Gaughran [28] based on the subject’s height and weight. Knowing the inertial properties of the limb segments, the three components of the acceleration of \( P \) in Equation 3.10, and the relative positions of the limb segments, all terms on the RHS of Equations 3.12 and 3.13 can be computed.

The angular-momentum-balance equations (Euler’s equation) for the two arm segments are written next. For the forearm, the equation is written in frame \( F \). Summing moments about the COM (\( CF \)) yields

\[
\sum_{F} \vec{M}_{CF} = I_{F} \cdot F\vec{\alpha}_{f} + F\vec{\omega}_{f} \times I_{F} \cdot F\vec{\omega}_{f}, \tag{3.15}
\]

where \( F\vec{\omega}_{f} \) and \( F\vec{\alpha}_{f} \) represent the absolute angular velocity and absolute angular acceleration of the forearm, respectively, expressed in the forearm frame \( F \). \( I_{F} \) is the combined (diagonal) inertia matrix for the forearm and the hand. The angular motion of the forearm relative to the upper arm is introduced in the equation using the composition rules for
angular velocities and accelerations as follows [14]:

\[ F \omega_f = R_G \omega_f = F R_G (\omega_{fu} + \omega_u), \]  

(3.16)

\[ F \alpha_f = R_G (\alpha_u + \alpha_{fu} + \omega_u \times \omega_{fu}), \]  

(3.17)

where \( \alpha_{fu} \) is the angular acceleration of the forearm relative to the upper arm measured in frame \( U \), and \( \omega_u \) and \( \alpha_u \) are the absolute angular velocity and the absolute angular acceleration, respectively, of the upper arm expressed in frame \( G \). Note that since the thorax is assumed to be stationary, these absolute angular quantities correspond to the shoulder-joint angular motion. The LHS of Equation 3.15 is given by

\[ \sum F M_{CF} = F T_E + F \tau_{E/CF} \times [F R_G (R_{EF} + R_{EU} + R_{Eint} + R_{Eg})] + F \tau_{W/CF} \times m_h F g, \]  

(3.18)

where \( F \tau_{E/CF} \) and \( F \tau_{W/CF} \) are vectors from the forearm COM to the elbow and wrist, respectively, expressed in frame \( F \). Substituting Equations 3.16, 3.17, and 3.18 into Equation 3.15 and rearranging the terms gives

\[ F T_E = [I_F \cdot F \alpha_{fu} + F \omega_{fu} \times I_F \cdot F \omega_{fu} - F \tau_{E/CF} \times F R_{EF}] + [I_F \cdot (F \alpha_u + F \omega_u \times F \omega_{fu}) + F \omega_{fu} \times I_F \cdot F \omega_u + F \omega_u \times I_F \cdot (F \omega_{fu} + F \omega_{fu}) - F \tau_{E/CF} \times (F R_{EU} + F R_{Eint})] + [-F \tau_{E/CF} \times F R_{Eg} - F \tau_{W/CF} \times m_h F g]. \]  

(3.19)

The LHS of Equation 3.19 is the torque \( F T_{Emuscle} \) created by the muscles across the elbow joint. Note that all the terms in the first bracket on the RHS of Equation 3.19 vanish if the elbow joint has zero velocity and acceleration. Therefore, this bracket is defined as the net torque at the elbow, \( F T_{E_{net}} \). Similarly, all terms in the second bracket vanish if
the shoulder joint has zero velocity and acceleration. Therefore, this bracket describes the effect of the shoulder-joint movement at the elbow joint, and it is defined as (the negative of) the interaction torque $\mathbf{T}_{Einteraction}^T$. Finally, the terms in the third bracket are gravity terms and do not vanish. This is (the negative of) the gravity torque $\mathbf{T}_{Egravity}^T$. With these definitions, Equation 3.19 can be re-expressed as

$$\mathbf{T}_{Enet}^T = \mathbf{T}_{Emuscle}^T + \mathbf{T}_{Einteraction}^T + \mathbf{T}_{Egravity}^T.$$  

(3.20)

Thus, NT at the elbow is the cumulative effect of the torque developed by the muscles across the elbow joint, the interaction effects due to shoulder-joint movement, and the gravity forces acting on the forearm.

Having assumed that the elbow joint is a perfect hinge joint, the acceleration $\mathbf{a}_{pf}$ in Equation 3.10 can be expressed as

$$\mathbf{a}_{pf} = \mathbf{U}_f^\alpha \times \mathbf{U}_r^\alpha + \mathbf{U}_f^\omega \times \left( \mathbf{U}_f^\omega \times \mathbf{U}_r^\alpha \right).$$  

(3.21)

Substituting Equations 3.21 into Equation 3.12 and substituting the result into Equation 3.19 yields the expression for $\mathbf{T}_{Enet}^T$.

$$\mathbf{T}_{Enet}^T = I_F \cdot \mathbf{F} \cdot \mathbf{U}_f^\alpha \times \mathbf{U}_f^\omega \times \left( \mathbf{U}_f^\omega \times \mathbf{U}_r^\alpha \right) \times$$

$$\left[ \mathbf{F}_U \left( \mathbf{U}_f^\alpha \times \mathbf{U}_r^P \times \mathbf{U}_f^\omega \times \mathbf{U}_r^\omega \times \mathbf{U}_f^P \right) \right].$$  

(3.22)

The terms in $\mathbf{T}_{Enet}^T$ that are not proportional to the elbow-joint acceleration are

$$\mathbf{F} \cdot \mathbf{U}_f^\omega \times \left( \mathbf{U}_f^\omega \times \mathbf{U}_f^\omega \times \mathbf{U}_f^P \right) \times$$

$$\left[ \mathbf{F}_U \left( \mathbf{U}_f^\alpha \times \mathbf{U}_f^\omega \times \mathbf{U}_r^\alpha \times \mathbf{U}_r^\alpha \times \mathbf{U}_f^P \right) \right].$$

(3.22)

These terms are proportional to the square of the elbow-joint angular velocity $\mathbf{U}_f^\omega$. The first term belongs to the gyroscopic torque, and the second term appears from the moment caused by the elbow-joint reaction force. Recall that these terms do not feature in the
discussion of NT and IT for planar-motion studies. The inclusion of these terms in NT is according to the discussion in the previous section, and it is counter to the development in Hirashima et al. [62].

Next, Euler’s equation for the upper arm is written in frame $U$. Summing moments about the upper-arm COM ($CU$) yields

$$\sum U M_{CU} = I_U \cdot U \bar{\alpha}_u + U \bar{\omega}_u \times I_U \cdot U \bar{\omega}_u,$$

(3.23)

where $I_U$ is the (diagonal) inertia matrix for the upper arm. The LHS can be written as

$$\sum U M_{CU} = U \bar{T}_S + U \bar{T}_{S/CU} \times (U \bar{R}_{SU} + U \bar{R}_{SE} + U \bar{R}_{Sint} + U \bar{R}_{Sg})$$

$$- U R_F (F \bar{T}_{E_{net}} - F \bar{T}_{E_{int}} - F \bar{T}_{Eg})$$

$$- U \bar{\tau}_{E/CU} \times (U \bar{R}_{EU} + U \bar{R}_{EF} + U \bar{R}_{Eint} + U \bar{R}_{Eg}),$$

(3.24)

where $U \bar{T}_{S/CU}$ and $U \bar{T}_{E/CU}$ are vectors pointing from the upper-arm COM to the shoulder and the elbow, respectively, expressed in frame $U$. Substituting Equation 3.24 into Equation 3.23 and rearranging the terms gives

$$U \bar{T}_S = \left[ I_U \cdot U \bar{\alpha}_u + U \bar{\omega}_u \times I_U \cdot U \bar{\omega}_u - U \bar{T}_{S/CU} \times U \bar{R}_{SU} + U \bar{T}_{E/CU} \times U \bar{R}_{EU} \right.$$

$$+ U R_F (I_F \cdot F \bar{\alpha}_u + F \bar{\omega}_u \times I_F \cdot F \bar{\omega}_u - F \bar{T}_{E/CF} \times F \bar{R}_{EU})]$$

$$+ \left[ U R_F (F \bar{T}_{E_{net}} + I_F \cdot (F \bar{\omega}_u \times F \bar{\omega}_u) + F \bar{\omega}_u \times I_F \cdot F \bar{\omega}_u + F \bar{\omega}_u \times I_F \cdot F \bar{\omega}_u - F \bar{T}_{E/CF} \times F \bar{R}_{Eint}) - U \bar{T}_{S/CU} \times (U \bar{R}_{SF} + U \bar{R}_{Sint}) \right.$$

$$+ U \bar{T}_{E/CU} \times (U \bar{R}_{EF} + U \bar{R}_{Eint})]$$

$$+ \left[ -U R_F F \bar{T}_{E_{gravity}} - U \bar{T}_{S/CU} \times U \bar{R}_{Sg} + U \bar{T}_{E/CU} \times U \bar{R}_{Eg} \right].$$

(3.25)

The LHS is defined as the muscle torque at the shoulder joint, $U \bar{T}_{Smuscle}$. All of the terms in the first square bracket vanish if the shoulder joint has zero velocity and acceleration. All
of the terms in the second square bracket disappear if the elbow joint has zero velocity and acceleration. The third square bracket contains the gravity terms that do not vanish. Thus, the first bracketed term on the RHS is defined as the net torque at the shoulder joint, $U_T^{Snet}$.

The second bracketed term on the RHS is defined as (the negative of) the interaction torque at the shoulder joint, $U_T^{Sinteraction}$, and the third bracketed term on the RHS is defined as (the negative of) the gravity torque at the shoulder joint, $U_T^{Sgravity}$. All components are described in frame $U$. With these definitions, Equation 3.25 can be rewritten as

$$U_T^{Snet} = U_T^{Smuscle} + U_T^{Sinteraction} + U_T^{Sgravity}. \tag{3.26}$$

Substituting Equations 3.11, 3.12, and 3.13 into Equation 3.25 provides the expression for the NT, $U_T^{Snet}$.

$$U_T^{Snet} = I_U \cdot U \bar{\alpha}_u + U \bar{\omega}_u \times I_U \cdot U \bar{\omega}_u + U \cdot R_F(I_F \cdot F \bar{\alpha}_u + F \bar{\omega}_u \times I_F \cdot F \bar{\omega}_u)$$

$$- m_u U \bar{r}_{S/CU} \times U \bar{\alpha}_u + (m_f + m_h)(U \bar{r}_{E/CU} - U \bar{r}_{S/CU}) \times \left(U \bar{\omega}_u + U \bar{\alpha}_u \times U \bar{\tau}_{P/CU}\right)$$

$$+ U \bar{\alpha}_u \times U \bar{\tau}_{P/CU} + U \bar{\omega}_u \times \left(U \bar{\omega}_u \times U \bar{\tau}_{P/CU}\right)$$

$$- (m_f + m_h) U \cdot R_F \left[F \bar{r}_{E/CF} \times F \cdot R_U \left(U \bar{\alpha}_u + U \bar{\alpha}_u \times U \bar{\tau}_{P/CU}\right)\right] + U \bar{\omega}_u \times \left(U \bar{\omega}_u \times U \bar{\tau}_{P/CU}\right). \tag{3.27}$$

The terms that are not proportional to the angular acceleration of the upper arm in Equation 3.27 are

$$U \bar{\omega}_u \times I_U \cdot U \bar{\omega}_u + U \cdot R_F(F \bar{\omega}_u \times I_F \cdot F \bar{\omega}_u)$$

$$+(m_f + m_h)(U \bar{r}_{E/CU} - U \bar{r}_{S/CU}) \times \left[U \bar{\omega}_u \times \left(U \bar{\omega}_u \times U \bar{\tau}_{P/CU}\right)\right]$$

$$-(m_f + m_h) U \cdot R_F \left[F \bar{r}_{E/CF} \times F \cdot R_U \left(U \bar{\omega}_u \times \left(U \bar{\omega}_u \times U \bar{\tau}_{P/CU}\right)\right)\right].$$

As for the elbow joint, these terms are proportional to the square of the shoulder-joint angular velocity $\bar{\omega}_u$. The first two terms belong to the gyroscopic torque, and the remaining
terms appear from the moments caused by the joint reaction forces. It can be shown by explicit calculation (with the simplifying assumption that $P$ and $CF$ are coincident) that these terms vanish in the case of a planar analysis. Equations 3.19 and 3.25 reduce to the expressions provided by Sainburg and Kalakanis [125] and Dounskaia et al. [33] for a planar two-jointed, three-link arm model with the global frame fixed at the shoulder joint.

### 3.3 Data analysis

Roughly stated, the LJH claims that for most horizontal-plane arm movements, the interaction torque at the shoulder is ignorable, or, in the least, small so that a reaching task can be successfully executed using a rough estimate of it. Therefore, the role of the musculature at the shoulder is simply to create the angular acceleration of the (entire) arm. This allows significant simplification of the shoulder-joint control. The elbow joint, on the other hand, experiences significant interaction torques. However, since it is the last joint in the kinematic chain, it is easier to control, even with significant interaction effects [32, 33, 34]. The analysis required to support this claim focuses on quantifying the contributions of MT and IT to NT at each joint. The desired result is that IT contributes little to NT at one of the joints. This analysis has been developed by Sainburg et al. [128], Sainburg and Kalakanis [125], Dounskaia et al. [34], and Dounskaia et al. [33]. The analysis as performed for planar motion is briefly outlined in the next paragraph. Following that, the modifications made to the analysis for the purpose of extending the LJH to spatial motion are described.

First, the time series of the torque components for a movement are computed using the appropriate EOM and partitioning scheme. Then, an analysis of torque signs is performed. This includes computation of the portion of movement duration in which MT coincides in
sign with NT [34, 128]. When this characteristic is close to 100% of movement duration, the conclusion can be made that MT plays a dominant role in accelerating and decelerating the joint because NT is proportional to joint acceleration and, hence, NT has the same sign as joint acceleration. When MT and NT have opposite signs over a substantial portion of the movement duration, however, IT accounts for most of the acceleration/deceleration of the joint because MT necessarily opposes the movement. Next, to take into account the magnitude of the contributions of MT and IT to NT, an analysis of torque impulses is performed [128, 33]. For this purpose, the positive impulse of the torque component (MT or IT) is computed as the torque integral during intervals in which the torque component acts in the same direction as NT. Similarly, the negative impulse is computed as the integral of the torque component during the intervals in which the torque component acts in the direction opposite NT. The positive and negative impulses of each component are summed to yield the total torque impulse for each torque component.

This analysis needs to be modified for application to the current spatial study for the following reasons. First, all previous studies cited above are horizontal plane studies. Therefore, the techniques do not account for gravity torques. Second, in contrast to planar analysis, the torque components at a joint will not be aligned for spatial motion. Finally, since NT is not proportional to the joint acceleration, the conclusions from this analysis will be different in nature from those obtained for planar studies. The first two issues are discussed next. The last issue is addressed in Section 3.5.

The idea that the CNS may be computing the gravity- and motion-dependent torques separately was introduced by Hollerbach and Flash [66]. These authors observed that the gravity-independent net torque profiles required for performing reaching movements at different speeds are scaled by the square of the ratio of the movement time. They proposed
a motor-control strategy wherein the motion-dependent torques for a given movement are obtained by scaling a previous record of the torques required for the same motion, to which gravity torques are computed and added separately. This argument has since been reiterated [11, 114] and experimentally verified [41, 171]. Yadav [171] performed a study wherein different groups of healthy subjects performed reaching tasks in various externally applied force fields on two consecutive days. On the first day, one group of subjects learned a position-dependent force field. Another group learned a velocity-dependent force field, and yet another group learned a combined position- and velocity-dependent force field. Yadav [171] shows that subjects who learned the combined force field on day 1 were able to separate individual force fields on day 2 and perform smoother, more consistent arm motions than the day 1 arm movements of subjects who learned individual force fields. As gravity is a position-dependent force field, the findings of this study support the hypothesis that humans have the ability to separate gravity-dependent torques.

Based on these studies, it is assumed here that the CNS estimates the gravity torques and motion-dependent torques independently and adds them together to generate the appropriate motor signals. As a consequence, the gravity torques are removed from the discussion of the LJH. Equation 3.1 can now be modified as

\[ T_{net} = T_{muscle} + T_{interaction} + T_{gravity}, \]
\[ \Rightarrow T_{muscle}^* := T_{muscle} + T_{gravity} = T_{net} - T_{interaction}. \] (3.28)

In other words, the focus of the ensuing analysis is on quantifying the relations between \( T_{net}, T_{interaction} \) and the portion of the muscle torque that is required to generate the motion torques, \( T_{muscle}^* \). In what follows, the ‘*’ is dropped from \( T_{muscle}^* \), and the term ‘muscle torque (MT)’ as well as the notation \( T_{muscle} \) will refer to the gravity-separated muscle torque, unless specified otherwise.
The next issue is the non-planarity of the torque components at the joints. Although the torque components at both the shoulder and the elbow are non-planar, adopting the single-DOF revolute-joint model for the elbow simplifies the analysis for this joint. In fact, the torque analysis at the elbow is conducted without modification from the methods outlined in [33, 34, 125, 128]. The implication of this assumption is that the torque about the axis of elbow rotation is provided by the elbow musculature, and the other component is provided passively by the surrounding skeletal structure.

For the torque analysis at the shoulder joint, two terms are defined to facilitate the discussion that follows. When the angle between NT and MT (or IT) vectors is less than 90°, MT (or IT) is called assistive. When the angle is greater than 90°, MT (or IT) is called resistive. The analysis of signs now consists of computing the portion of the movement for which MT is assistive. This corresponds to the portion of the movement for which the dot product of MT and NT is positive. When this characteristic is close to 100%, MT is said to have played an assistive role in generating NT. Next, to quantify the magnitude of the contributions of MT and IT to NT, the analysis of torque impulses is performed, wherein the impulses are computed using the dot product between the torque components. For example, the impulse due to MT is computed as the integral of the dot product of MT and NT over the duration of the motion. The dot product operation automatically differentiates between positive and negative impulses. The two characteristics computed above consider the components of MT and IT along NT. The component of MT normal to NT is called the vestigial component. The vestigial component is spent in cancelling the component of IT normal to NT, and it is quantified by computing the vestigial impulse factor, $\Psi$. At time instant $i$, the components of MT are related by the Pythagorean theorem,

$$|\mathbf{T}_{\text{muscle}i}|^2 = |\mathbf{T}_{\text{muscle}N-i}|^2 + |\mathbf{T}_{\text{muscle}V-i}|^2,$$
where $T_{\text{muscle}N-i}$ and $T_{\text{muscle}V-i}$ are the components of $T_{\text{muscle}i}$ along and normal to NT, respectively. The vestigial impulse factor $\Psi$ is computed as

$$\Psi := \frac{\sum_i T_{\text{muscle}V-i} \cdot T_{\text{muscle}V-i}}{\sum_i T_{\text{muscle}-i} \cdot T_{\text{muscle}-i}} = \cos^2 \gamma,$$

(3.29)

where $\gamma$ is the angle between $T_{\text{muscle}V-i}$ and $T_{\text{muscle}N-i}$. Clearly, the factor $\Psi$ is bounded: $0 \leq \Psi \leq 1$. If the MT vector is mostly aligned with the NT vector, the component of MT normal to NT will be small, and therefore, $\Psi \rightarrow 0$. If the angle included by the MT and NT vectors is at about $30^\circ$, $\Psi \approx 0.25$. If the included angle is about $90^\circ$, the component of MT normal to NT will constitute most of MT, and therefore, $\Psi \rightarrow 1$. As the included angle exceeds $90^\circ$, $\Psi$ reduces in value from 1 and approaches zero as the included angle approaches $180^\circ$.

To illustrate the use of the three metrics for drawing conclusions regarding the contributions of MT and IT to NT, three hypothetical examples are provided in Figure 3.2, which shows MT and IT adding vectorially to produce NT, as per Equation 3.28 at one instant in time. For the sake of argument, suppose that the relations in these examples hold for the entire movement duration. In example (a), MT is responsible for most of NT, and MT is assistive for the entire movement. Therefore, the analysis of signs will yield a value of 100%. Next, the impulse due to the MT component along NT will be greater than the impulse due to the IT component along NT. Finally, the vestigial impulse factor will be low ($\approx 0.25$), indicating that most of MT was utilized in producing NT. In example (b), the analysis of signs yields a value of 100%, and the impulse of MT will be greater than the impulse of IT, although by a lower margin compared to example (a). However, most of MT is spent in counteracting the normal component of IT. This will be reflected in the high value of the vestigial impulse factor ($\approx 0.9$). In example (c), MT is resistive. Therefore, the analysis of signs will yield a value of 0%. The impulse of IT will be greater than the impulse of MT.
Figure 3.2: Hypothetical examples to explain the use of the three metrics to deduce the relation between MT, NT and IT.
Finally, since the component of MT normal to NT is small, the vestigial impulse factor is low ($\approx 0.3$).

Thus, a value close to 100% from the analysis of signs, accompanied by a large the MT impulse compared to IT impulse, and a low value ($< 0.3$) for the vestigial impulse factor indicates that MT was responsible for generating most of NT and that most of MT was utilized for this purpose. Together, these metric values also imply that IT was low during the movement. Conversely, if the value from the analysis of signs is low ($< 75\%$), or if the IT impulse is similar to or greater than the MT impulse, then IT contributes significantly in the production of NT. Furthermore, independent of the first two metrics, a large value ($> 0.3$) for the vestigial impulse factor by itself implies that a significant portion of MT was employed to counter the component of IT normal to NT. This reasoning is employed in the analysis of joint torques, outlined next.

The joint torques and their components were computed using the formulation in the previous section for the clipped portions of the arm movements for each repetition of each task (The tasks are defined in Section 2.2). The analysis of signs, the impulses for MT and IT, and the vestigial impulse factor were computed for each test. These quantities were averaged across the ten repetitions of a task at each speed.

### 3.4 Results

The results for one subject are provided in Figures 3.3 - 3.5. Based on these figures, the tasks are classified as shoulder- or elbow-led. Another category of movement, called independent joint control, is also introduced.

- **Task 1.** Figure 3.3 shows that MT was assistive at the shoulder for a longer duration than at the elbow. Furthermore, at the shoulder, MT was assistive for close to 100%
Figure 3.3: Duration of the total movement for which MT is assistive. The figure shows mean ± SD for all tasks and speeds. The first, second and third rows of figures show data for slow-, normal- and fast-speed movements, respectively. Subject JM.
Figure 3.4: Mean ± SD of MT and IT impulses at the shoulder and the elbow for all tasks and speeds. The first, second and third rows of figures show data for slow-, normal- and fast-speed movements, respectively. Subject JM.
of the duration. Figure 3.4 shows that at the shoulder, the MT impulse was larger than the IT impulse. Conversely, the IT impulse at the elbow is almost equal to (slow speed) or greater than (normal and fast speed) the MT impulse. Finally, Figure 3.5 shows small values for $\Psi$, indicating that most of MT was utilized in generating NT. It is therefore concluded that at the shoulder, MT was largely employed to generate the motion of the arm. Conversely, MT and IT both contributed to generate the motion of the forearm. Therefore, this movement is classified as a shoulder-led motion for all speeds.

- **Task 2.** This task is also classified as a shoulder-led motion using the same arguments as for Task 1. The contrast in the impulses at the shoulder and the elbow are greater
for this task than for Task 1. Furthermore, the values for $\Psi$ are the lowest for this task.

- **Task 3.** The shoulder and the elbow exchange roles for this task. The analysis of signs shows that MT is assistive for a longer portion of the movement at the elbow than the shoulder. MT is assistive for close to 100% of the duration at the elbow. The MT impulse is greater than the IT impulse at the elbow, but not at the shoulder. These two observations suggest that the elbow leads this motion. The large values for $\Psi$ at the shoulder indicate that a significant portion of MT at the shoulder was spent in countering interaction effects. Therefore, the shoulder acted as a subordinate joint for this motion. Thus, this motion is classified as an elbow-led motion for all speeds.

- **Task 4.** For this task, the analysis of signs shows that MT is assistive at both joints for a significant portion of the motion. The durations are similar for both joints. Figure 3.4 shows a low IT impulse and high a MT impulse at the elbow for all speeds and at the shoulder only for slow speed. For normal- and fast-speed motions, the IT impulse at the shoulder is significant. At the shoulder, the MT impulse is 17% of the NT impulse for slow speed, and 54% and 83% of the NT impulse for normal- and fast-speed motions, respectively. Furthermore, the shoulder MT impulse is directly countering the substantial negative IT impulse. This is also reflected in the low values for $\Psi$ (0.15 and 0.09 for normal- and fast-speed motions, respectively). Based on these observations, the normal- and fast-speed motions are classified as elbow-led movements. The slow-speed movement shows different characteristics. The IT impulse at both joints is low - possibly due to low movement speed (peak wrist speed $0.7\pm0.03$ m/s for slow movements compared to $1.62\pm0.01$ m/s for fast movements).
Furthermore, $\Psi$ is low as well (0.2). Therefore, it is proposed that the interaction effects at both joints are low so that both joints function independently. This motion may not be ‘led’ by either joint. Rather, the control at each joint is decoupled.

- **Task 5.** This task is classified as a shoulder-led motion using the same arguments as for Tasks 1 and 2. The only difference is that $\Psi$ for this task is greater than that for Tasks 1 and 2. However, it is still considered low (0.27, 0.26, 0.23 for slow-, normal-, and fast-speed motions, respectively).

- **Task 6.** All speeds of this task are similar to the normal- and fast-speed motions for Task 4. Therefore, this task is classified as elbow-led for all speeds.

This analysis was conducted for all nine subjects. There were a small number of motions - Tasks 1 and 3 for five subjects - that did not allow classification according to the arguments developed here. The chief reason was large IT impulses observed at both joints simultaneously. The torque impulses for these motions are presented in Figure 3.6. Note that for subject RR, only the fast-speed motion for Task 3 eluded classification.

The classification for all tasks for all subjects is provided in Table 3.2, from which the following observations are made.

(O26) The largest number of subjects (three) exhibit unclassifiable motions for Task 1, while all other subjects exhibit shoulder-led motions for this task.

(O27) Motions for Tasks 2 and 5 are classified as shoulder-led for all speeds and subjects.

(O28) Task 3 is the only task, other than Task 1, that shows unclassifiable motions. Two subjects show this characteristic. Furthermore, two subjects display elbow-led motions for this task.
Figure 3.6: Mean ± SD of the joint impulses for the cases that cannot be classified as either shoulder- or elbow-led or as independent joint control. The first, second and third rows of figures show data for slow-, normal- and fast-speed movements, respectively. The x-axis labels show the subjects and tasks that exhibit this behavior.
Table 3.2: The classification of all movements as ‘S’: shoulder-led, ‘E’: elbow-led, ‘I’: independent joint control for all speeds. ‘N’ indicates that the motion could not be classified in the above three categories. For each subject / task, the set of three letters classify the slow-, normal-, and fast-speed motions in that order.

(O29) For Task 4, two subjects show elbow-led movements, and the slow-speed movements for two subjects are classified as independent-joint-control movements. All other movements are classified as shoulder-led movements.

(O30) Task 6 shows characteristics similar to Task 4. Two subjects show elbow-led movements, and the slow- and normal-speed movements of one subject are classified as independent-joint-control movements. Six subjects show shoulder-led movements.

(O31) The majority of movements by all subjects show motion characteristics that suggest a shoulder-led control strategy. Three subjects (FR, JR, and AS) show shoulder-led motions for all tasks. Three subjects (AL, DE, and CR) show shoulder-led motions for five tasks. Subjects RR and KB show shoulder-led motions for four tasks, and subject JM shows shoulder-led motions for three tasks.

(O32) An elbow-led control strategy is observed most commonly for subject JM (three tasks), followed by subject RR (two), followed by subject KB (one).
A majority of the motions in this study can be classified as shoulder-led motions. Similar to the planar studies, a small number of motions show elbow-led characteristics. Furthermore, there are some movements characterized by significant IT at both joints. It is unlikely that the LJH would be the governing control strategy for these motions. However, the number of motions that display this characteristic is much smaller than those that may be controlled by the CNS using the leading-joint strategy. These observations suggest that

(R6) The LJH is the governing control strategy for several spatial reaching motions. Both shoulder- and elbow-led strategies are employed by the CNS for various motions. There also exist arm motions that are likely not controlled using the LJH. It is hypothesized that the class of such motions is relatively smaller than the class of motions that the CNS may control with the LJH.

Planar studies have classified arm movements into shoulder-led and elbow-led motions, and they have identified a correlation between the choice of the leading joint and the movement joint kinematics. It has been observed that elbow-led motions are typical when the shoulder-joint displacement is low. Dounskaia [32] and Levin et al. [92] observed two categories of movement studying cyclical tracing of shapes oriented in different directions on a horizontal table. The shoulder-leading control strategy was revealed during most of the analyzed movements, and the elbow-leading control was found in movements that required small shoulder excursion and substantial elbow excursion. Galloway and Koshland [47] studied discrete reaching motions in the horizontal plane. They observed that only when movements involve a shoulder excursion less than half of the elbow excursion is initial shoulder acceleration determined by a combination of MT and significant IT, whereas initial elbow acceleration is determined primarily by MT. Thus, Galloway and Koshland [47]
provide the same correlation between the choice of the leading joint and the joint kinematics for discrete arm movements.

Figure 3.7 shows the mean joint excursions for the shoulder and the elbow for all tasks performed by the subjects who displayed elbow-led strategies for some of the tasks in the present study. In this figure, the elbow excursion is simply the $Z$ Euler-angle displacement when the $Z - X' - Y''$ Euler-angle sequence is used to relate the orientation of the forearm to that of the upper arm. Recall that the $Z$-axis of the upper-arm frame is oriented along the elbow-joint rotation axis. The shoulder joint excursion is the angle obtained from the axis-angle parametrization [145] relating the initial and final orientations of the upper-arm coordinate frame for a movement. The joint excursions are averaged across all repetitions and speeds. The movements that are classified as elbow-led are marked by an asterix. The number on top of each bar is the ratio of mean shoulder excursion to mean elbow excursion. Observe that all of the motions classified as elbow-led show small excursion ratios ($\approx 0.3 - 0.4$) and simultaneously have small absolute shoulder-joint excursions ($< 40^\circ$). However, there are other motions that show similar joint kinematics but that do not show an elbow-led control strategy. For example, Task 5 for subject RR and Task 6 for subject KB are both classified as shoulder-led motions. Additionally, subject DE (not shown in Figure 3.7) shows similar joint kinematics with a mean shoulder excursion of $36^\circ$ and an excursion ratio of 0.34 for Task 3. However, this task is unclassifiable (see Table 3.2). All other tasks for all subjects show higher excursion ratios ($\geq 0.45$) and large shoulder-joint excursions ($\geq 40^\circ$). Therefore, apart from three isolated cases, the relation between the joint kinematics and the choice of the leading joint observed for planar motion is strongly reproduced for spatial motion. Dounskaia [32] has mentioned that for motion involving more than two joints, the LJH predicts that the only possibility for two
Figure 3.7: Mean ± SD of the joint excursions for the subjects that exhibit elbow-led control strategies. The elbow excursion is simply the $Z$ Euler-angle displacement when the $Z - X' - Y''$ Euler-angle sequence is used to relate the orientation of the forearm to that of the upper arm. The shoulder joint excursion is the angle obtained from the axis-angle parametrization [145] of the initial and final orientations of the upper-arm coordinate frame. The movements that are classified as elbow-led are marked by an asterix. The number on top of each bar indicate the excursion ratio.
leading joints within one movement (i.e. the independent-joint-control strategy) is when motions at the two joints are mechanically independent, for instance, when the joints move in orthogonal planes. Another instance, as mentioned above, is when the motion is slow. This will result in small IT impulses at both joints because IT is a motion-dependent torque. Figure 3.8 shows the mean ± SD of the peak wrist speed and the joint excursions for all tasks for subjects that show the independent-joint-control strategy. For all three subjects, the slow-speed movement for Task 4 shows the independent-joint-control strategy. For subject CR, the slow- and normal-speed movements for Task 6 show the independent-joint-control strategy. Figure 3.8 shows that the shoulder-joint excursion for these movements are low. Thus, the low IT impulse at the elbow is the result of small shoulder-joint velocity and acceleration. Furthermore, some component of IT at the elbow will be orthogonal to the elbow-joint axis, reducing the IT impulse along the axis further. On the other hand, the elbow-joint excursion is substantial for these motions. Therefore, the low shoulder-joint IT impulse is attributed to the smaller inertia of the forearm. However, the low IT impulse at the shoulder must also depend on the geometry of the motion. This becomes evident on comparing, for example, the slow-speed movements of Tasks 3 and 4 of subject CR. The joint excursions for both tasks are similar, and Task 3 was performed slower than Task 4 (see Figure 3.8). Still, Task 3 is classified as shoulder-led and Task 4 is classified as independent-joint-control motion. The geometry of the reaching task must be influencing the shoulder-joint IT. This is an interesting observation and potentially constitutes a subject for future study.

The discussions presented above are consolidated into the following results.

**(R7)** Spatial arm movements are typically shoulder-led. Elbow-led movements occur when the shoulder-joint excursion is low.
Figure 3.8: Mean ± SD of peak wrist speeds and joint excursions for the subjects that exhibit the independent-joint-control strategy for some motions. The movements that display the independent-joint-control strategy are marked by an asterix. For all three subjects, the slow-speed movement for Task 4 shows the independent-joint-control strategy. For subject CR, the slow- and normal-speed movements for Task 6 show the independent-joint-control strategy.
Slow arm movements may allow decoupling of joint motions because of low interaction effects. The motion geometry influences the magnitude of interaction effects at the shoulder joint and is therefore a deciding factor. The CNS may potentially employ an independent-joint-control strategy for motions that produce low interaction effects at both joints.

3.5 Discussion

The objective of this chapter was to extend the LJH to spatial motion. Result (R6) in the previous section validates the hypothesis for spatial reaching. Furthermore, according to results (R7) and (R8), most movements that were investigated in this work were shoulder-led. A small number of movements were elbow-led, or adopted independent joint control. Elbow-led movements typically occur when the shoulder-joint displacement is low. This result is in agreement with the similar observations for planar motion [32, 47, 125]. This geometric criterion for the choice of leading joint is mirrored in the development of the inverse kinematics algorithm in Chapter 4. In Section 4.2.6, the rationale for choosing a leading joint is provided. Briefly, the motion (velocity) of the subordinate joint, $\dot{\mu}$, is obtained from that of the leading joint, $\dot{\lambda}$, using the following equation.

$$\dot{\mu} = n \dot{\lambda}, \quad (3.30)$$

where $n := \frac{d\mu}{d\lambda}$ is the first-order speed ratio. Note that $n$ is simply the ratio of the joint velocities. If the leading joint velocity becomes too low, the ratio becomes large, and, although Equation 3.30 holds, small errors in the motion of the leading joint are amplified at the end effector. The problem is solved by simply switching the role of the leading and
subordinate joints. The joint motions are now related as follows.

\[ \dot{\lambda} = \frac{1}{n} \dot{\mu}. \]  

(3.31)

Thus, the leading joint must possess ‘sufficiently’ high instantaneous velocity to ensure smoothness of the wrist path. The instantaneous velocity constraint on the leading-joint motion can be re-expressed as an upper bound on the velocity ratio \( n \), which is a geometric quantity. The constraint also translates simply into a constraint on the net displacement of the leading joint for a given task. Clearly, the shoulder is the preferred leading joint. However, if the net shoulder displacement compared to the net elbow displacement for a task is low, the instantaneous shoulder velocity will be low, and the corresponding speed ratio will be large throughout the movement. Therefore, the movement will be elbow-led. The above argument provides a rationale for the choice of the leading joint. Dounskaia [32] has mentioned that the subordinate-joint interaction effects due to a slow-moving leading joint are low. Therefore, they cannot be utilized at the subordinate joint. In other words, the ‘mechanical advantage’ available to the subordinate joint is low, and therefore, the shoulder does not lead the motion when its net displacement is low [32]. Thus, there are two reasons for switching the leading joint for movements that require small net shoulder displacement.

Result (R8) introduces independent joint control as the third category of movement observed in the torque-impulse analysis. In some cases, the interaction torques at both joints are low for slow movements. Therefore, interaction effects can potentially be ignored at both joints to further simplify joint-torque composition and joint control. In addition to movement speed, the geometry of the reaching task plays an important role in producing negligible interaction effects at both joints. Characterizing the geometric features of reaching tasks that potentially allow decoupled joint control at slow speeds is a topic for future study.
The focus now shifts to the complexity of the NT expressions for the shoulder and the elbow. NT at both joints for the planar arm model is proportional to the joint acceleration. In the description of the LJH, Dounskaia [32] mentions that ‘*Acceleration/deceleration at the leading joint is produced by reciprocal muscle activity in the same way as during single-joint movements, i.e. largely disregarding motion at the other joints*’ (emphasis added). Dounskaia [32], Dounskaia et al. [33] and Levin et al. [92] support this idea by noting that antagonistic muscles at the leading joint predominantly produce pronounced reciprocal bursts of activity whose timing is tightly coupled with the timing of peak acceleration and deceleration at the joint. EMG profiles of anterior and posterior deltoids provided in [32, 33, 92] validate this idea for shoulder-led motions. This characteristic led Dounskaia [32] to propose an elegant, hybrid control scheme 8 that simplifies shoulder-joint control (at least for shoulder-led motions). According to this scheme, the shoulder-joint torque required for a particular movement may be obtained using a lookup table. This control can be organized as a simple association between a gross description of the required movement - e.g. initial and final joint position, movement time, and expected inertial resistance of the limb - and torque. This association can be learned with practice and then used to extract the necessary MT from memory before each movement. This control strategy is appropriate for the (proximal) leading joint because it replaces complex inverse kinematics and dynamic computations with simpler memory recalls. The control at the subordinate

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8Hybrid control schemes are a combination of lookup tables and structured algorithms. A lookup table replaces a computationally complex transformation with memorized particular solutions of the transformation. Structured algorithms are capable of executing transformations given a small number of input parameters. A hybrid scheme for implementing inverse dynamics typically uses a lookup table to obtain the inverse kinematics since this is the computationally most complex portion of the transformation. The output of this algorithm is input into a structured algorithm for computing the inverse dynamics. This scheme allows faster end-effector movements without sacrificing movement accuracy in industrial robots [119]. Other studies suggest that both control mechanisms - lookup tables and structured algorithms - are utilized by the CNS for motor control [74]. Atkeson [10] reviews hybrid systems in the context of the human motor-control problem.
joint employs structured algorithms that compute MT from joint kinematics using constant inertial parameters. These algorithms account for the interaction effects created by the leading joint motion and produce task-specific arm motions.

For spatial motion, NT depends linearly on the joint angular acceleration and non-linearly on the joint velocity. See Equations 3.22 and 3.27. As discussed in Section 3.2, these expressions are a result of choosing a consistent set of definitions for IT and NT. The presence of the nonlinear, velocity-dependent terms renders the control of the shoulder more complex for spatial motion as compared to planar motion. However, the hybrid control algorithm of Dounskaia [32] still applies. The shoulder-joint torque required for a particular movement may yet be obtained via a lookup table using an association between a gross description of the required movement and torque. The gross description of the required spatial task will be more complex compared to a planar reaching task. However, similar motion characteristics - initial and final positions of coordinate frames, movement time, and expected inertial resistance of the limb - will be used to construct the mapping between the task and the desired torques. Also, ignoring the interaction terms in the leading-joint torque expression is a significant simplification of the EOM for the leading joint. This can be seen from the interaction torque terms in Equations 3.19 and 3.25. Therefore, this mapping can potentially be learned with practice and then used to extract the necessary MT from memory before each movement via interpolation. There is no change in the strategy for the subordinate-joint control. With this approach, the shoulder joint is viewed as a single joint with three DOF, rather than three revolute joints with non-parallel axes. Thus, the extension of the control scheme is achieved with no modification. In essence, Dounskaia [32] used a planar-motion paradigm to outline a general, hybrid control scheme that is applicable to spatial arm motion as well.
It is difficult to predict if, for spatial motion, the muscle activity at the shoulder will resemble a simple pattern of reciprocal bursts. It is also difficult to predict how the shoulder-muscle activity would be related to the shoulder-joint kinematics. This is the principal fallout of the complexity in the NT expressions. Forward dynamic simulations can be used to address this problem, wherein specific torque profiles are input into the EOM and the resulting motion is studied. The results from the forward dynamics can be validated using EMG measurements of the shoulder musculature. These are topics for future work.

Despite these difficulties, the definitions of NT and IT adopted in this study attribute a definite, consistent role to the shoulder-joint musculature in motion production. Since the shoulder-joint MT is responsible for all torque terms involving (only) the shoulder-joint velocity and acceleration, the purpose of the shoulder musculature is the control of this single, three-DOF, spherical joint, considering the inertia of the entire arm, but disregarding the distal-joint motion.

3.5.1 Relation between the TIH and the LJH

The relation between the TIH and the LJH is briefly explored next. This discussion leads to the identification of certain aspects of the TIH and the LJH that need further elaboration. As explained in Section 3.1, the TIH and the LJH model different subproblems in the motor control scheme. The TIH converts the movement objective that must be specified in the output space into the appropriate joint-space objectives. The LJH utilizes a joint-space plan as an input and produces the required joint torques. Thus, the output of the TIH serves as the input to the LJH. The algorithm developed in Chapter 4 to perform the inverse kinematics is a local transformation and therefore, relies on reiterative computation of the solution, potentially using a forward kinematic internal model, as the movement
progresses. The control scheme proposed by Dounskaia [32] is similar, wherein the subordinate joint control uses forward internal models for kinematic predictions as well as state feedback for motion correction. Thus, both the hypotheses are local and, therefore, compatible in this sense.

The complete integration of the TIH and the LJH must include an explanation that relates the motion dynamics and the observed motion kinematics. In particular, the results (R1) - (R5) must be explained based on the TIH and LJH. Recall that the view proposed in this study is that the observed kinematic characteristics, particularly wrist paths, are not necessarily epiphenomena of some minimization procedure. Rather, they are near-optimal, i.e., they are acceptable deviations about a desired trajectory that is a solution of the optimization procedure. Since a leading joint strategy is used to implement such a solution, the kinematic deviations must be consequences of the LJH. This relationship needs to be established. For horizontal plane motions, Beer et al. [15] performed forward dynamic simulations using a two-DOF arm and compared actual hand paths with simulated hand paths when interaction torque at the shoulder or elbow was reduced or eliminated numerically. Decreasing elbow interaction torque by as little as 25% resulted in marked disturbances in hand path. In contrast, the authors noted no such changes with manipulations at the shoulder joint. Similar studies for spatial arm motion are required. This is a major subject for future work. The following paragraphs outline specific problems, and possible approaches that could be used to establish the motion kinematics as consequences of the dynamic control algorithm.

First, results (R1) - (R3) suggest that the wrist-path shape depends on the workspace. Atkeson and Hollerbach [11] first pointed out that, in contrast to horizontal-plane motion,
the wrist paths for vertical-plane motion were not straight and that curvature of the wrist-path shape depended on workspace. Haggard and Richardson [60] observed consistently curved wrist paths for point-to-point, lateral reaching movements as well. The results of the present study confirm these observations and extend them to arm motion in a large portion of the workspace. Conceivably, path curvature could result from energy considerations. On the other hand, Haggard and Richardson [60] suggest that the motor system may plan a straight wrist path, but the effects of the dynamical interactions between limb segments may lead to the actual movements being consistently curved to varying degrees depending on the workspace. The variability could also arise from difficulties in handling gravity.

Now, the variation in the wrist path shape with motion speed, when significant, is systematic. Result (R2) states that the wrist paths tend to become more curved as the movement speed decreases. Furthermore, for certain tasks, slower movements display curvature in the vertical plane. This is the wrist-path loop observed in Figure 2.9. Suppose that the desired trajectory is computed by the CNS. Then, according to the LJH, the muscle torques required to achieve this trajectory are computed by the CNS by combining the net torque, the gravity torque, and some estimate of the interaction torque as follows.

$$T_{\text{muscle}} = T_{\text{net}} - T_{\text{interaction}} - T_{\text{gravity}}.$$  \hspace{1cm} (3.32)

The movement speed will influence the RHS of Equation 3.32 as follows. As movement speed decreases, the net torque will decrease without affecting the wrist-path geometry, in accordance with the TIH. The interaction effects, which are shown to be small compared to the muscle torque, decrease further with decreasing speed. Therefore, they are unlikely to influence the wrist-path kinematics. Finally, gravity is a position-dependent force. Therefore, the impulse due to gravitational torque at a joint must increase if a given task is performed at a lower speed. This reasoning suggests that the gravitational effects, rather
than the interaction effects, lead to the variability in the observed movement kinematics. This also suggests that the fundamental motion plan composed by the CNS is a straight wrist path, and this is evident in movements performed at normal or fast speeds when the gravity effects are relatively low. This hypothesis can be explored further by computing the gravitational torques and impulses for the arm motions in this study. A possible confounding factor in this study is the geometry of reaching motion, as suggested by result (R3). The loop in the wrist path is pronounced only in some of the tasks, although the gravitational effects increase with lower speeds for all tasks. For this reason, and also in general, the relation of the wrist-path shape to the workspace is an important problem that requires systematic, experimental investigation.

Results (R4) and (R5) - the occurrence of multimodal wrist-speed profiles - were attributed to the deterioration of the CNS’s ability to estimate longer time intervals in Section 2.4. This also explains in part the increase in the variability of the profiles with a decrease in speed. In Figure 2.16, the existence of the transition zone wherein some movements show unimodal wrist-speed profiles and others show multi-modal profiles suggests a gradual degradation in the CNS’s capacity to estimate time intervals. However, an algorithm for how a ‘submovement’ is composed within the LJH and TIH frameworks is required to complete this explanation. With such an algorithm, ideally, it would be possible to make quantitative predictions regarding the variability in the wrist-speed profiles. This is simply reiterating the need for a detailed algorithm for the composition of muscle torques at the leading and subordinate joints with arbitrary initial conditions.
3.6 Conclusions

A set of consistent definitions for the net torque and the interaction torque for spatial, multi-joint mechanisms was developed, as was a torque-impulse analysis for spatial movement. This analysis, applied to reaching motions, revealed that the LJH is a valid inverse-dynamics algorithm for spatial arm reaching. Hypothetical explanations for some of the movement kinematics were provided based on the inverse dynamic formulation. Specifically, the loop of the wrist paths was attributed to difficulty in handling gravity forces.

The definitions of the joint-torque components makes it difficult to validate the LJH for spatial motion using EMG data. Furthermore, important missing aspects of the LJH were pointed out. In particular, an algorithm for composing the leading-joint torque needs to be developed. This will allow the LJH to make quantitative, verifiable predictions regarding the movement kinematics.
Chapter 4: TRAJECTORY TRACKING VIA INDEPENDENT SOLUTIONS TO THE GEOMETRIC AND TEMPORAL TRACKING SUBPROBLEMS

4.1 Introduction

Studies have shown time-invariant wrist paths for planar horizontal [107, 140] and vertical [11] reaching motion in healthy humans. The time-invariance hypothesis (TIH) [5] was prompted by this and other experimental evidence. It suggests that the central nervous system (CNS) utilizes the invariant characteristic to plan arm movement. In Chapter 2, it was shown that the TIH may be an organizing principle for spatial arm motion under certain contexts.

Now, a commonly used strategy to understand complex information-processing systems in the CNS is to study them at three levels as distinguished by Marr [102]. The top level is the abstract computational theory of the studied system that defines inputs and outputs of the system and abstract properties of the mapping between them. The next level is understanding the algorithm used for the implementation of the mapping. Finally, the third level is investigation of how the algorithm revealed at the second level is realized physically. This involves revealing neuronal structures that implement the operations performed by the
algorithm. Since the TIH establishes the inputs and outputs of an inverse kinematics algorithm, it belongs to Marr’s first level. This chapter develops the TIH further by providing an algorithm to transform the inputs into the outputs in the desired time-invariant manner. In other words, this chapter develops mathematical machinery for the TIH that belongs to the second level of Marr. The development of the algorithm, however, is situated in the robotic trajectory-tracking paradigm. Trajectory tracking is treated from a time-invariant perspective, and it is shown how the approach leads to improved navigation capability.

There is good precedent for such biologically inspired investigation into robot navigation. Perhaps inspired by dynamic interactions between arm segments, Hollerbach [65] derived a fundamental torque-scaling property for manipulators. The torque-scaling law describes the change in a manipulator’s dynamics when the timing of the joint-space kinematics is changed. This phenomenon is an instance of time scaling wherein the curve representing the evolution of a variable retains its shape, but the evolution progresses at a different speed. Therefore, the terms ‘time scaling’ and ‘time invariant’ used in the previous paragraph express the same idea. Let \( \Theta \) represent the vector of joint displacements, and \( t \) is time. Assume that the movement torque profile \( \tau(t) \), which is obtained by subtracting the torque required to compensate for gravity from the net torque required to execute the desired joint-space trajectory \( \Theta(t) \), is known. Then, the movement torque profile required to execute the same motion but at a scaled speed, expressed as a composite function \( \Theta(p(t)) \), is a linear combination of \( \tau(t) \) and the generalized momentum of the manipulator. For the special case of constant time scaling, described by the function \( p(t) = ct \), the new joint torques are given simply by \( c^2 \tau(ct) \). Thus, the dynamics are minimally affected by the change in the joint kinematics, and therefore, the torque computation is simplified for constant time scaling.
Now, a trajectory-tracking task is usually specified in the external, or output space. Therefore, the solution to the trajectory-tracking problem must map the desired output-space motion into a joint-space motion (the inverse kinematics problem), followed by a mapping of the joint-space solution into the space of joint torques (the inverse-dynamics problem). The Hollerbach model [65] points out a simplifying regularity in the second map. The time scaling can be extended to the inverse kinematics problem, using the methodology described below. This method additionally provides enhanced tracking accuracy in the vicinity of certain kinds of singularities.

The inverse kinematics problem is stated as follows: given an output-space path (defined as a trajectory sans the temporal information), construct a corresponding joint-space path. Further, when the output-space speed and its derivatives are specified, the corresponding time-based joint motions (i.e. joint speeds and accelerations) must be computed efficiently. Thus, an efficient method to compute \( \Theta(p(t)) \) from specified output-space motion is sought. What follows is the theoretical derivation of an algorithm that accomplishes trajectory tracking by obtaining separate, local solutions to two separate subproblems, namely, the geometric tracking subproblem and the temporal tracking subproblem. Time scaling is then achieved by resolving only the temporal subproblem.

This chapter develops a methodology enabling the geometric tracking of a desired planar or spatial path to any order (defined below) with a non-redundant manipulator. In contrast to previous work on geometric path tracking (e.g. [78, 98]), the equations are developed using one of the manipulator’s joint variables as the independent parameter in a fixed global frame rather than a configuration-dependent canonical frame. Both these features provide significant practical advantages, as described later in the chapter. Apart from computationally efficient time scaling, another advantage of the methodology is that
it enables the navigation of certain kinds of singularities with bounded joint motions and, consequently, bounded joint torques. A strategy for determining joint velocities and accelerations at configurations of the manipulator that belong to a certain class of singularities is provided, which allows the manipulator to approach and/or move out of these singular configurations with finite joint velocities without sacrificing the geometric fidelity of tracking. Two examples are provided to illustrate the developed method.

Spatial trajectory tracking requires that the end-effector (EE) of a manipulator follow a trajectory that is usually prescribed as a function of time. The solution to this problem involves inverting the robot’s forward kinematic map, and typically, a local second-order inverse solution (i.e. joint velocities and accelerations) is obtained via inversion of the Jacobian of this map [24]. Often, a variant, called the manipulator Jacobian [145], is used in place of the Jacobian of the kinematic map to obtain this solution. This solution, in conjunction with an appropriate feedback control law, such as resolved acceleration control [22], is utilized to track the desired output-space trajectory in real time. In another approach to trajectory tracking, the joint-space solution for the entire output-space trajectory is computed offline, and the pre-computed solution is implemented in conjunction with appropriate feedback control. One way to obtain the global joint-space solution is to discretized the output-space path and solve the inverse position kinematics at each point to obtain a discretized representation of the joint-space solution. Another, perhaps more elegant way to perform the offline computation also involves discretization of the output-space trajectory. However, a high-order, local approximation of the joint-space solution is generated, and then continuation-like predictor-corrector algorithms are used to obtain the entire solution [78]. The latter method is considered superior because it explicitly accounts for higher-order geometry of the desired output-space path.
At singular points of the kinematic map, the dimension of the tangent space of the robot is reduced\(^9\). Therefore, an arbitrary EE velocity is not achievable with finite joint velocities. Irrespective of the commanded speed (a scalar multiple of the path variable time derivative), the EE must stop at a singularity to ensure geometric tracking accuracy. Clearly, this requires explicit control of the path variable. Consequently, any parameterization of the kinematic map that does not allow for explicit control of the path variable will fail to impose the above dwell condition. This is reflected in the Jacobian in [24] becoming ill conditioned in the vicinity of singularities, and the map cannot be inverted at all at a singular configuration via this Jacobian. On the other hand, if the path variable is considered to be a dependent variable in the parameterization of the kinematic map, its inversion is well conditioned at and in the vicinity of certain singularities, and the geometric accuracy of tracking is preserved.

Several approaches tackle the problem of trajectory tracking in the vicinity of singularities, and comprehensive reviews of these methods are available [95, 23]. However, not all methods explicitly control the path variable along with the trajectory geometry. The Jacobian transpose method [167], the singular value decomposition method [24] and the damped least squares method [165] are examples. Using the Jacobian transpose instead of the inverse at singular configurations yields joint velocities that minimize the squared norm of the position error. The singular-value-decomposition technique tracks only the component of the task-space velocity that lies in the tangent-space of the singular robot. The damped least squares technique gives an approximate solution that is well-conditioned.

\(^9\)For example, the tangent space of a planar two-revolute mechanism in a non-singular pose is its plane of motion. This means that the EE can instantaneously move in an arbitrary direction in the plane. The mechanism is in a singular pose when its two links are parallel. In this pose, the EE can instantaneously move in a direction normal to the links only. That is, in the singular pose, the dimension of the tangent space has reduced to one.
and defined everywhere in the manipulator’s workspace [95, 23]. This method modifies the EE path in terms of speed and direction so that the singularity is simply avoided [112]. These techniques aim not to impose a dwell in the path variable, but to obtain bounded joint velocities in the vicinity of a singularity and at the same time, be geometrically as accurate as possible. The geometrical imprecision intrinsic to these methods is a result of not separating the control of the path variable from the control of the trajectory geometry.

The offline computation of the joint-space trajectory treats the path variable as a dependent variable [78]. In fact, the premise of the approach is that the joint-space trajectory corresponding to a given task-space trajectory is a curve in the \((N+1)\)-dimensional space of \(N\) joint displacements of the \(N\)-DOF mechanism augmented with the path variable used to parameterize the task-space path. This curve is obtained as a function of an auxiliary variable via a predictor-corrector algorithm so that dwells in none of the \(N + 1\) variables can disrupt the tracking process [78, 76]. (This is despite the fact that Kieffer and Litvin [78] provide examples that use joint variables as the independent parameter.) Other parameterizations of the joint-space curve near singularities using fractional power series also exist [96]. The inherent singularity-robustness of the approach of Kieffer and Litvin [78] and Kieffer [76] was utilized for singularity navigation by Nenchev et al. [112] and Nenchev [111], wherein the adjoint of the Jacobian and the null space of an augmented Jacobian, respectively, are used. Nenchev [111] obtained first-order joint motions for non-singular as well as singular configurations of a six-DOF mechanism. The non-zero joint velocities obtained force the EE velocity to vanish at the singularity. Therefore, the tracking solution is geometrically accurate to the zeroth order, i.e. instantaneously, the EE stays on the desired path, but when it starts to move, the direction of motion does not necessarily coincide with the desired path tangent. The applicability of this first-order approach for real-time
trajectory tracking was identified, but not implemented. Higher-order methods that address the geometric tracking accuracy of order greater than zero in singular configurations exist as well [153, 49]. In [153], only wrist-partitioned manipulators are studied, while [49] analyzes point trajectories for some serial and parallel manipulators.

Another aspect of the approach of Kieffer and Litvin [78] and Kieffer [76] is that non-temporal joint-space solutions are obtained by choosing a non-temporal, independent auxiliary variable. Timing is added by specifying the relation between the auxiliary variable and time [78]. In keeping with the offline-computation paradigm, Lloyd [95] and Lloyd and Hayward [97] assume that the entire time-invariant joint-space solution to a tracking task is available and provides an algorithm to generate temporal joint motions that allow the EE to navigate the given trajectory in near minimal time without geometric deviation at singularities. A key feature of this work, relevant here, is that it shows that a joint variable can be used to describe the solution curve in the vicinity of singular points [95] (pp. 80) and does so to obtain the time-optimal solution near singularities. However, the approach does not allow the speed of the task-space path to be user-defined.

The utility of the method of Kieffer and Litvin [78] and Kieffer [76] in describing the instantaneous kinematics of the robot is recognized. However, to the best of the author’s knowledge, a strategy for online trajectory tracking through singularities using general six-DOF serial robots has not been developed. This chapter develops the solution to the inverse kinematics problem up to any order for non-singular configurations, and up to order three explicitly for ordinary singularities\(^{10}\), and achieves trajectory tracking through singularities.

\(^{10}\)Ordinary singularities are configurations of the robot in which the Jacobian drops rank by 1 and the path tangent does not lie in the robot’s reduced tangent space. The path-variable rate is necessarily zero [95, 76]. The desired trajectory intersecting the workspace boundary non-tangentially is a typical example.
without geometric deviation for a general six-DOF serial robot, while performing the entire calculation online. Particularly, joint-velocity and -acceleration solutions at first-order singular configurations (defined later) are provided. The algorithm is used in conjunction with feedback control to correct for the inaccuracies inherent in the implementation of an approximate local solution. The geometric error in the vicinity of a singularity is an artifact of the local nature of this joint-space solution and not of the singularity in the kinematic map.

It is desirable to choose a joint variable as the independent variable to obtain the instantaneous, time-invariant, inverse kinematics solution for the real-time trajectory-tracking problem at all points, not just in the vicinity of singularities as is the approach of Kieffer and Litvin [78]. The joint variable used to parameterize the joint-space solution will be called the *leading joint variable*. Practically, it is easier to obtain measurements of the joint variables compared to an arbitrary auxiliary variable or the non-temporal path variable. However, the problem of the leading joint variable encountering a dwell (zero joint velocity) in its trajectory must be tackled. This chapter shows how to predict the occurrence of such an event, how to select a *non-dwelling* joint variable to lead the motion and that at least one joint is available to lead the motion for any non-singular or singular configuration.

Another motivation for using the joint-angle variable for solving inverse kinematics comes from human motor control. As explained earlier, the time-invariant approach described here can describe a time-invariant strategy that the human CNS may utilize to achieve control of arm motion. Other theories, such as the leading joint hypothesis (LJH) [32] independently propose that the CNS may be using the leading joint concept. It is likely that the neural instantiation of the LJH exists in the cerebellum [27, 35, 168, 134]. The LJH is examined in detail in Chapter 3. Additionally, the relation between the TIH and the LJH is explored.
in Section 3.5. Briefly, the LJH states that the dynamic interaction at (mostly) the proximal joint in a serial chain is low. Therefore, these interaction effects are roughly ignored by the CNS, which leads to simplification of the computation of joint torques from the joint kinematics and joint trajectory control. This idea of a leading joint is the dynamic counterpart of the leading joint variable defined at the beginning of this paragraph. Furthermore, kinesthesia gives humans a sense of joint position. Mechanoreceptors in the periarticular soft tissue around a joint provides the sense of joint position under static as well as dynamic conditions [67, 126, 157]. Conversely, humans are unlikely to have such a direct means of measuring quantities like arc length. This echoes the motivation for using a single joint angle as the independent variable.

The idea of treating the trajectory-tracking problem as independent geometric and temporal subproblems has another historical precedent. This begins with curvature theory [18], wherein a general characterization of planar motion is achieved using arbitrary motion parameters. Lorenc et al. [100] first generalized two-DOF curvature theory [18] to arbitrary DOFs, where the motion of a controlled point on the EE is obtained as a function of two arbitrary motion parameters in a canonical reference frame. The theory is then applied to the problem of generating the joint-space solution to a path-tracking task using a planar, two-DOF mechanism with the joint variables as the independent parameters. Taylor series are used to relate the motion of all joint variables to that of the leading joint variable, rather than to an auxiliary parameter as was the case in [78]. The coefficients of the Taylor series are called speed ratios, and the technique obtains expressions for the speed ratios as functions of the instantaneous invariants (II’s) of motion. The II’s are geometric parameters that characterize planar motion [18, 100, 123]. The theory has been applied to a planar two-revolute (RR) mechanism and a holonomic cart system in [100], as well as a planar
prismatic-revolute (RP) mechanism in [147]. However, solutions up to only the second order are obtained. Third-order solutions were obtained for the specific case of tracking planar paths with constant curvature [99]. Extension of the planar formulation [100] to second-order path tracking using three-DOF spatial systems that include a planar two-DOF subsystem was also achieved [98]. The regional structure (i.e. manipulator used only for positioning the wrist center) of the Stanford Arm (a rotating RP mechanism) was used for demonstration.

The curvature-theory approach is also concerned with finding the most parsimonious description of motion. A canonical system is first located, and general expressions for the speed ratios are obtained in this system [100, 98]. Even so, these expressions tend to get unwieldy, and the increased complexity is probably one reason why the curvature-theory approach has not been generalized to arbitrary order and to larger spatial systems. Further, the canonical system is a configuration-dependent reference frame. For a trajectory tracking application, describing the kinematics in a moving canonical frame necessitates an additional transformation between that frame and a global, fixed frame. Therefore, the canonical frame is less suitable for the application. Another observation is that, in prior work [100, 147, 99], the temporal component of the tracking problem is never explicitly solved, although, presumably, the goal is to achieve trajectory tracking and not just path tracking.

The present work utilizes ideas from the curvature-theory approach and the offline computation approach and modifies them as follows:

1. It abandons the use of an auxiliary parameter by Kieffer and Litvin [78] and uses a leading joint variable as per Lorenc et al. [100] to generate local, inverse kinematics solutions.
2. It aims to find a local representation of the joint-space curve in \( N \)-dimensional joint space, rather than the \((N + 1)\)-dimensional space used by Kieffer and Litvin [78]. As a consequence, there are fewer Taylor coefficients (the speed ratios) that are needed to describe the solution. In the process of obtaining the \( p^{th} \)–order ratios, the \( p^{th} \)–order derivatives of the path variables with respect to the leading joint variable are also obtained. These quantities are needed only for obtaining geometric solutions of order greater than \( p \). However, they can be used to reduce the computation involved in obtaining the temporal leading-joint motion of the same order.

3. It abandons the canonical frame and develops a new set of equations called coordination equations in an arbitrary fixed frame.

4. Kieffer and Litvin [78] call the speed ratios kinematic influence coefficients, and they are obtained indirectly as functions of the coefficients of the Taylor expansions of the variables in terms of the auxiliary parameter. In this work, the coordination equations yield the speed ratios directly, implying lower computational complexity.

5. With a convenient feedback control law, it achieves online execution of kinematic tracking through singularities with no a-priori knowledge of the inverse kinematics solution in any form.

### 4.2 The Spatial Three-DOF Problem

*Notation:*

For the three-DOF spatial trajectory-tracking problem, the joint variables are denoted as \( \mu \), \( \nu \), and \( \lambda \) to maintain consistency with previous curvature-theory literature, eg. [18]. The standard Denavit-Hartenberg (DH) system [145] is used for the larger, six-DOF case. A
trailing subscript(s) indicates the derivative with respect to the subscript(s). For example, \( \bar{r}_\lambda := \frac{d\bar{r}}{d\lambda} \). A zero after the subscript(s) indicates that the derivative has been evaluated in the zero position. For example, \( \bar{r}_{\lambda\mu 0} := \frac{d^2\bar{r}}{d\lambda d\mu} \bigg|_0 \). The dot notation indicates derivatives with respect to time \( t \). For example, \( \dot{\lambda} := \frac{d\lambda}{dt} \bigg|_0 \). The symbols \( n \) and \( k \) always represent the speed ratios with respect to a leading joint variable. A trailing superscript indicates the order of the ratio. For example, \( k^{(3)} := \frac{d^3\nu}{dx^3} \bigg|_0 \). The superscript is omitted for the first-order ratios, so \( n^{(1)} := n \).

The solution procedure for spatial three-DOF trajectory tracking involves writing Taylor series approximations for the desired and the generated trajectories and equating the corresponding terms of the two series. This process is also called establishing contact between the two functions [51]. Two series are written for each trajectory: one using a geometric variable, i.e. joint variable, as the independent parameter, and the second using time as the independent parameter. This way, the geometric and the temporal aspects of tracking are separated. Establishing contact between the two geometric Taylor series is equivalent to matching the Frenet-Serret frames [82] of the desired and generated paths. The solution to the geometric problem relates the motion of the mechanism’s joints to the motion of the leading joint. The solution to the temporal problem determines the motion properties of the leading joint. Singularities occur in the generated and/or desired trajectory when some derivatives in the corresponding Taylor series vanish. This chapter concentrates on singularities in the generated path that coincide with the singular configurations of the robot. The desired path is assumed to be singularity free\(^\text{11}\). In such cases, the geometric

\(^{11}\)The desired and generated paths are spatial curves. Examples of singularities of spatial curves are **cusps** and **crunodes** [20]. Visually, the curve reverses direction at a cusp, and it intersects itself at a crunode. The **cardioid** is a planar curve defined by the polar equation \( r = 1 + \cos \theta \) with a cusp at the origin. If the cardioid equation is changed to \( r = 0.5 + \cos \theta \), the resulting curve has a crunode at the origin.
tracking problem can still be solved [100]. The only error is in matching the instantaneous speeds of the desired and generated paths.

4.2.1 Geometric Path-Tracking

The current configuration of the robot is called the zero position, and the values of the joint angles in the zero position are denoted by \( \mu_0, \nu_0, \) and \( \lambda_0 \). The increments in the joint angles are denoted as \( \mu, \nu, \) and \( \lambda \), and without loss of generality, the instantaneous values of these joint variables in the zero position are taken to be zero. The forward position kinematics of the mechanism are given as \( r : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) such that \( \tau_0 = r(\mu_0, \nu_0, \lambda_0) \), where \( \tau_0 \) is the current position of a controlled point on the EE-point \( P \). For the trajectory-tracking problem, define \( \tau : \mathbb{R} \rightarrow \mathbb{R}^3 \) so that the image of the map \( \tau \) represents a spatial curve generated by point \( P \). A typical definition of this mapping yields \( \tau(t) \). However, a different parameterization is obtained here by choosing any non-dwelling joint of the mechanism (henceforth, arbitrarily \( \lambda \)) as the independent parameter to construct \( \tau = \tau(\lambda) \). Section 4.2.6 describes why a leading joint \( \lambda \) is always available. In the zero position, \( \tau(0) = \tau_0 \). Further, the function \( \tau(\lambda) \) is expressed in an arbitrary, fixed reference frame. This is in contrast to the canonical frame used in the curvature-theory-based approaches to the path-tracking problem. In constructing \( \tilde{\tau}(\lambda) \), slaving relations between the joints are imposed with the Taylor series

\[
\begin{align*}
\mu &= n\lambda + \frac{n(2)}{2}\lambda^2 + \frac{n(3)}{6}\lambda^3 + \ldots, \\
\nu &= k\lambda + \frac{k(2)}{2}\lambda^2 + \frac{k(3)}{6}\lambda^3 + \ldots
\end{align*}
\]

(4.1)
The coefficients of the series are the \textit{speed ratios}, which are defined as

\begin{align*}
    n := \frac{d\mu}{d\lambda} \bigg|_0, & n^{(2)} := \frac{d^2\mu}{d\lambda^2} \bigg|_0, & n^{(3)} := \frac{d^3\mu}{d\lambda^3} \bigg|_0, & \ldots, \\
    k := \frac{d\nu}{d\lambda} \bigg|_0, & k^{(2)} := \frac{d^2\nu}{d\lambda^2} \bigg|_0, & k^{(3)} := \frac{d^3\nu}{d\lambda^3} \bigg|_0, & \ldots.
\end{align*}

There are two ratios for every order of coordination. The generated path can now be approximated by a Taylor series.

\[
    \tau(\lambda) = \tau_0 + \tau_{\lambda_0}\lambda + \tau_{\lambda\lambda_0}\frac{\lambda^2}{2} + \tau_{\lambda\lambda\lambda_0}\frac{\lambda^3}{6} + \cdots \quad (4.2)
\]

Note that \(\tau_{\lambda_0} = \tau_{\lambda_0}(n, k)\), \(\tau_{\lambda\lambda_0} = \tau_{\lambda\lambda_0}(n, k, n^{(2)}, k^{(2)})\), and so on.

The \textit{desired path} is a spatial curve \(\overrightarrow{R}\) specified in terms of an arbitrary variable \(q\). The function \(q(t)\) is also specified. The Taylor series for \(\overrightarrow{R}\) in terms of \(\lambda\) is given by

\[
    \overrightarrow{R}(s(\lambda)) = \overrightarrow{R}_0 + (s_{\lambda_0}\overrightarrow{T})\lambda + (s_{\lambda\lambda_0}\overrightarrow{T} + s_{\lambda_0}^2\overrightarrow{N})\frac{\lambda^2}{2} + [s_{\lambda\lambda\lambda_0}\overrightarrow{T} + 3s_{\lambda_0}s_{\lambda\lambda_0}\overrightarrow{T} + s_{\lambda_0}^3\overrightarrow{N}]\frac{\lambda^3}{6} + \cdots,
\]

where \(s\) is the arc length of the desired path. Using standard relations in differential geometry, this can be rewritten as

\[
    \overrightarrow{R}(s(\lambda)) = \overrightarrow{R}_0 + \left[ s_{\lambda_0}\hat{T} \right] \lambda + \left[ s_{\lambda_0}\hat{T} + s_{\lambda_0}^2\kappa\hat{N} \right] \frac{\lambda^2}{2} + \left[ (s_{\lambda\lambda\lambda_0} - \kappa^2s_{\lambda_0}^3)\hat{T} + (3s_{\lambda_0}s_{\lambda\lambda_0}\kappa + \kappa_{s_0}s_{\lambda_0}^3)\hat{N} + \kappa_{s_0}s_{\lambda_0}^3\hat{B} \right] \frac{\lambda^3}{6} + \cdots (4.3)
\]

Equation 4.3 uses a local Frenet-Serret frame to describe the instantaneous geometric properties of the desired path [82]. The triad \(\hat{T} - \hat{N} - \hat{B}\) is the natural trihedron of the curve formed by the tangent, the normal and the binormal, respectively. The quantities \(\kappa\) and \(\tau\) are the curvature and the torsion, respectively, and \(\kappa_{s_0} = \frac{ds}{ds} \bigg|_0\). These geometric properties can be obtained from the parametric equation \(\overrightarrow{R}(q)\) (or \(\overrightarrow{R}(t)\)) using standard relations [82].
For example,

\[ \hat{T} = \frac{R_{q0}}{|R_{q0}|}, \]

(4.4)

\[ N = \overline{R}_{qq0} - (\overline{R}_{qq0} \cdot \hat{T}) \hat{T}, \quad \hat{N} = \frac{N}{|N|}, \]

(4.5)

\[ \hat{B} = \hat{T} \times \hat{N}. \]

(4.6)

Expressions for the curvature and torsion are also readily available, and higher-order properties such as \( \kappa_{so} \) are obtained by differentiating these equations. The derivatives of the arc length \( s \) with respect to \( \lambda \) also appear in Equation 4.3. These are unknown and are the terms of the Taylor series expansion of the function \( s(\lambda) \).

The \( p^{th} \)-order geometric tracking problem is solved by establishing \( p^{th} \)-order contact between the curves \( r(\lambda) \) and \( \overline{R}(\lambda) \) by equating the first \( p - 1 \) derivatives in Equations 4.2 and 4.3 [51]. Establishing contact (i.e. achieving coordination) proceeds in stages from the zeroth to the third and potentially higher orders. At each stage, it is assumed that all lower orders of coordination (or contact) have been achieved. In equating the derivatives, two things are implied: first, the geometric entities of the two paths, such as the tangent, normal, etc., are forced to be identical; and second, the arc lengths of the two paths are forced to be equal. Therefore, \( p^{th} \)-order contact is simultaneously established between the Taylor series for the arc lengths of the two paths, i.e., the function \( s(\lambda) \) is being constructed.

For each order of coordination except the zeroth order, a 3D vector equation involving two unknown speed ratios and the unknown derivative of the path variable, called the coordination equation, is obtained. The solution requires a linear inversion as shown below, where the general form of the \( p^{th} \)-order coordination equation is derived first, followed by the specific forms of this equation for the zeroth and first three orders.
4.2.2 \( p^{th} \)-Order Coordination

In Appendix A, it is shown that the \( p^{th} \)-derivative of \( \tau \) is linear in the \( p^{th} \)-order speed ratios. Therefore, for \( p = 1 \), let

\[
\bar{\tau}_\lambda = J_1 \frac{d\mu}{d\lambda} + J_2 \frac{d\nu}{d\lambda} + J_3, \tag{4.7}
\]

\[
\Rightarrow \bar{\tau}_{\lambda 0} = [J_{10} \ J_{20} \ J_{30}] \bar{\pi}_1 + J_{30} = J \bar{\pi}_1 + \bar{\Phi}_0, \tag{4.8}
\]

where the matrix \( J \) is the Jacobian of the function \( \tau(\mu, \nu, \lambda) \), \( J_i \) are the columns of the Jacobian, \( J_{10} \) are the Jacobian columns evaluated at the zero position, \( \bar{\Phi}_0 = J_{30} \), and \( \bar{\pi}_1 = [n \ k \ 0]^T \). Note that this Jacobian is the Jacobian of the kinematic mapping and not the manipulator Jacobian that also appears in the robotics literature [145]. Differentiating Equation 4.7 and evaluating the result in the zero position,

\[
\bar{\tau}_{\lambda \lambda 0} = \left\{ [J_{1\lambda 0} \ J_{2\lambda 0} \ J_{3\lambda 0}] \bar{\pi}_1 + J_{3\lambda 0} \right\} + J \begin{bmatrix} \dot{n}^{(2)} \\ k^{(2)} \\ 0 \end{bmatrix} = \bar{\Phi}_1 + J \bar{\pi}_2,
\]

where \( \bar{\Phi}_1 \) is a known quantity since \( J_{1\lambda 0}, J_{2\lambda 0}, \) and \( J_{3\lambda 0} \) are functions of the mechanism’s pose and the first-order speed ratios and \( \bar{\pi}_2 = [n^{(2)} \ k^{(2)} \ 0]^T \). With repeated differentiation,

\[
\bar{\tau}_{\lambda \lambda \lambda \ldots \lambda (p \ times) 0} = \bar{\Phi}_{p-1} + J \bar{\pi}_p, \tag{4.9}
\]

where \( \bar{\tau}_{\lambda \lambda \lambda \ldots \lambda (p \ times) 0} := \bar{\tau}_{(\lambda \lambda \ldots \lambda \ p \ times) 0} \). As before, \( \bar{\Phi}_{p-1} \) is a known quantity, and \( \bar{\pi}_p = [n^{(p)} \ k^{(p)} \ 0]^T \).

The \( p^{th} \)-order term of the series in Equation 4.3 is expressed as

\[
\frac{d^p \bar{R}}{d\lambda^p} \bigg|_0 = u_p \dot{T} + v_p \dot{N} + w_p \dot{B}, \tag{4.10}
\]

where \( u_p, v_p, \) and \( w_p \) are scalar functions of the derivatives of the arc length and the scalar-valued desired-path properties like \( \kappa, \tau, \) etc. The following structure is observed in these
scalar coefficients.

\[
\begin{align*}
    u_1 &= s_{\lambda 0}, & v_1 &= 0, & w_1 &= 0, \\
    u_2 &= s_{\lambda \lambda 0}, & v_2 &= \kappa u_1^2, & w_2 &= 0, \\
    u_3 &= s_{\lambda \lambda \lambda 0} - \kappa^2 u_1^3, & v_3 &= 3u_1 u_2 \kappa + \kappa s_0 u_1^3, & w_3 &= \kappa \tau u_1^3.
\end{align*}
\]  

(4.11)

The \( p^\text{th} \)-order derivative of the path variable \( s \) with respect to the leading joint variable is an unknown quantity. Note that only the coefficient \( u_p \) contains this derivative, and therefore, \( u_p \) is the only unknown quantity on the right-hand-side (RHS) of Equation 4.10.

To establish \( p^\text{th} \)-order contact, the RHS of Equations 4.9 and 4.10 are equated.

\[
\Phi_{p-1} + J_n p = u_p \dot{T} + v_p \dot{N} + w_p \dot{B}.
\]  

(4.12)

Rearranging Equation 4.12 gives the \( p^\text{th} \)-order coordination equation in general form.

\[
J^* \pi_p^* = v_p \dot{N} + w_p \dot{B} - \Phi_{p-1} =: \Psi_{p-1},
\]  

(4.13)

where \( J^* := [J_{10} \quad J_{20} \quad - \dot{T}] \) and \( \pi_p^* := [n^{(p)} \quad k^{(p)} \quad u_p]^T \). The matrix \( J^* \) is composed by replacing the column of \( J \) corresponding to the partial derivative of \( r \) with respect to the leading joint variable with \(-\dot{T}\). As long as \( J^* \) is full rank, Equation 4.13 can be solved for \( \pi_p^* \). Note that as long as \( J \) is full rank, it is always possible to obtain a full-rank matrix \( J^* \) by replacing one of the columns of \( J \) by \(-\dot{T}\). The trailing subscript \( p - 1 \) for \( \Psi \) indicates that the quantity depends on up to the \((p - 1)^{\text{th}}\)-order quantities.

The coefficients \( u_i, i < p \) are required to solve Equation 4.13 since \( v_p \) and \( w_p \) are functions of \( u_i \). Solving Equation 4.13 yields \( u_p \), which is critical only if a joint-space solution of order greater than \( p \) is desired. However, it can also be used for computing the \( p^\text{th} \)-order temporal motion of the leading joint, as shown in Section 4.2.4.

Finally, the solution to the coordination equation of any order requires the inversion of the modified Jacobian matrix \( J^* \). This is a consequence of the fact that the \( p^\text{th} \)-order derivative of \( \pi \) is always linear in the \( p^\text{th} \)-order speed ratios, and the coordination equation
is linear in the coefficient $u_p$. Therefore, closed-form solutions for the speed ratios of arbitrary order can be obtained for non-singular configurations.

**Zeroth-Order Coordination**

This involves achieving the appropriate pose of the mechanism by solving the inverse position kinematics problem. In the foregoing, it is assumed that the EE is on the desired path in the zero position. This is a reasonable assumption to make since the problem of path tracking mainly involves consideration of derivatives of the position, rather than of position itself. Therefore, rather than explicitly solve the inverse position kinematics problem, it is considered solved by virtue of the current pose of the mechanism.

**First-Order Coordination**

Substituting $p = 1$ in Equation 4.13 yields the first-order coordination equation.

$$J^* \pi^*_1 = \Psi_0,$$

where $\Psi_0 = -\Phi_0 = -J_{30}$. Equation 4.14 is solved for the first-order ratios and $u_1$. Note that the first-order coordination equation also implies $\tau_{\lambda_0} = s_{\lambda_0} \hat{T}$, i.e., the tangents of the desired and the generated paths are forced to be parallel.

**Second-Order Coordination**

Substituting $p = 2$ in Equation 4.13 yields the second-order coordination equation.

$$J^* \pi^*_2 = \Psi_1.$$  \hspace{1cm} (4.15)

From Equations 4.13 and 4.11, $\Psi_1 = \kappa u_1^2 \hat{N} - \Phi_1$. Note that the value of $u_1$ obtained from the first-order coordination equation is required in the second-order equation. Equation 4.15 is solved for the two second-order speed ratios and $u_2$. 

139
Third-Order Coordination

Substituting $p = 3$ into Equation 4.13 yields the third-order coordination equation.

$$J^* \pi^*_3 = \Psi_2.$$  \hspace{1cm} (4.16)

From Equations 4.13 and 4.11,

$$\Psi_2 = (3u_1u_2\kappa + \kappa s_0 u_1^2) \hat{N} + \kappa r u_1^3 \hat{B} - \Phi_2.$$

As before, $u_1$ and $u_2$ are required to solve Equation 4.16 for the two third-order speed ratios and $u_3$.

The following section uses a spatial, 3R mechanism to geometrically track a spatial path up to the third order as an example of the implementation of the theoretical development in the present section. The 3R mechanism resembles the regional structure of the 6-DOF manipulator used for the example in Section 4.3.1.

4.2.3 3-DOF Example

The spatial 3R mechanism shown in Figure 4.1 is used here to illustrate the path-tracking approach outlined in this section. The following development also appears in [6]. This system is a three-DOF system with a planar subsystem of the kind investigated in [98]. The mechanism has two revolute joints whose axes intersect orthogonally at point $A^*$ and a third revolute joint whose axis passes through point A and is parallel to the distal interior joint axis. The fixed coordinate frame is placed at $A^*$ with the $Z$ axis oriented along the joint axis denoted by joint angle $\nu$. The joint angles of the revolute joints are denoted by $\mu_0$, $\nu_0$, and $\lambda_0$, defining the current pose and the zero position of the mechanism, wherein the speed ratios are calculated. The increments in these angles are $\mu$, $\nu$, and $\lambda$, respectively, which are all zero in the zero position. The desired path is shown in Figure 4.1. The EE,
Figure 4.1: The spatial 3R mechanism is required to track the desired path. The vectors \( \hat{T} \), \( \hat{N} \), and \( \hat{B} \) form the trihedron of the FS frame and describe the path geometry. Geometry is tracked by relating the motions of the joint variables \( \mu \) and \( \nu \) to that of joint variable \( \lambda \). The desired path is denoted by point \( P \), is assumed to be on the desired path in the zero position. The natural trihedron of the desired path at point \( P \) is the unit tangent \( \hat{T} \), the unit normal \( \hat{N} \), and the binormal \( \hat{B} \). The curvature \( \kappa \), torsion \( \tau \), and quantity \( \kappa_{a0} \) are not shown in the figure. The mechanism is defined with link lengths \( l_1 = 1 \) and \( l_2 = 1.5 \), in arbitrary length units, and initial pose \( \mu_0 = 70^\circ \), \( \nu_0 = 40^\circ \), and \( \lambda_0 = 10^\circ \). The desired path is a helix given in terms of its arc length \( s \) as

\[
\bar{R}(s) = \left[ \cos \left( \frac{s}{\sqrt{2}} \right) + 1 \ \ \sin \left( \frac{s}{\sqrt{2}} \right) \right]^T + \bar{r}_0, \tag{4.17}
\]
where \( r_0 \) is the current position of point \( P \). Equation (4.17) ensures that \( P \) is on the desired path at the zero position. The curvature \( \kappa = \frac{1}{2} \), the torsion \( \tau = \frac{1}{2} \), and \( \kappa_s = 0 \). Also, \( \hat{T} = [0 \ \frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}]^T \), \( \hat{N} = [-1 \ 0 \ 0]^T \), and \( \hat{B} = [0 \ -\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}}]^T \). The leading joint is \( \lambda \).

With these values, the following results are obtained.

\[
J^* = \begin{bmatrix}
-1.1316 & -0.8004 & 0 \\
-0.9495 & 0.9539 & -0.7071 \\
0.2605 & 0 & -0.7071
\end{bmatrix},
\]

\[
\Psi_0 = \begin{bmatrix}
1.2646 \\
1.0612 \\
-1.2452
\end{bmatrix}, \quad \Psi_1 = \begin{bmatrix}
0.7064 \\
0.4508 \\
0.529
\end{bmatrix}, \quad \Psi_2 = \begin{bmatrix}
0.082 \\
-0.6959 \\
0.2924
\end{bmatrix}.
\]

Simultaneously, the coordination equations yield

\[
n = -1.4905, \quad k = 0.5272, \quad u_1 = 1.212,
\]

\[
n^{(2)} = -0.2985, \quad k^{(2)} = -0.4606, \quad u_2 = -0.858,
\]

\[
n^{(3)} = 0.3481, \quad k^{(3)} = -0.5945, \quad u_3 = -0.2853.
\]

Figure 4.2 shows the desired helical path and the tracking results for the first, second, and third orders. The generated paths are plotted for \( \lambda = 0 \) (zero position) to \( \lambda = +50^\circ \) and \( \lambda = -60^\circ \). The cross-hairs intersecting at point \( P \) indicate the natural trihedron of the helix. The position of the EE is calculated after every \( 10^{-3} \)-degree increment in \( \lambda \). Figure 4.2 clearly shows that near the zero position \( s = 0 \), a higher order of coordination results in significantly lower error.

### 4.2.4 Temporal Joint-Motion Properties

To complete the solution to the trajectory-tracking problem, the temporal joint-motion properties must be determined. Kieffer [76], mentions the possibility of completing the trajectory-tracking problem by obtaining the joint rates, and Lloyd [95] solves the problem of obtaining a time-optimal path timing assuming that the joint-space solution is available.
Figure 4.2: The paths traced by point $P$ on the EE of the 3R mechanism for first-, second-, and third-order joint coordination are shown. Higher order joint coordination results in $P$ following the desired path more closely near the zero position.

In the ongoing, the speed ratios define relations between the temporal joint motions. Therefore, the temporal motion of the leading joint is now obtained, followed by the temporal joint motion of the other ‘subordinate’ joints via the speed ratios.

Similar to the geometric problem, the approach here is to establish $p^{th}$-order contact between the time-dependent Taylor series of the desired and generated paths. Therefore, the functions $\tau(\lambda(t))$ and $R(t)$ are expressed as

\begin{align}
\tau(\lambda(t)) &= \tau_0 + \left(\dot{\lambda}\tau_{\lambda 0}\right) t + \left(\ddot{\lambda}\tau_{\lambda 0} + \dddot{\lambda}\tau_{\lambda 0}\right) \frac{t^2}{2} + \left(\lambda_2 \tau_{\lambda \lambda 0} + 3\dot{\lambda}\lambda\tau_{\lambda \lambda 0} + \dddot{\lambda}\lambda\tau_{\lambda 0}\right) \frac{t^3}{6} + \cdots \\
R(t) &= R_0 + \ddot{R} t + \frac{\dddot{R}}{2} t^2 + \frac{\ddddot{R}}{6} t^3 + \cdots
\end{align}

Note that the function $q(t)$ is specified. Therefore, the time derivatives of $R(q(t))$ can be obtained via the chain rule. The vector $\tau_{\lambda 0}$ and the coefficient $u_p$ are already available
from the solution of the \( p \)th—order coordination equation. Using the definition of the arc length of a curve, it can be shown that

\[
\tau_{\lambda_0} \cdot \tau_{\lambda_0} = u_1^2, \quad \tau_{\lambda\lambda_0} \cdot \tau_{\lambda_0} = u_1 u_2, \quad \tau_{\lambda\lambda\lambda_0} \cdot \tau_{\lambda_0} = u_1 u_3.
\]  

(4.20)

Equating the corresponding derivatives of the two series and substituting Equation 4.20 into the result yields

\[
\dot{\lambda} = \frac{\ddot{R} \cdot \tau_{\lambda_0}}{\tau_{\lambda_0} \cdot \tau_{\lambda_0}} = \frac{\ddot{R} \cdot \tau_{\lambda_0}}{u_1^2},
\]

(4.21)

\[
\ddot{\lambda} = \frac{(\dddot{R} - \dot{\lambda}^2 \tau_{\lambda_0}) \cdot \tau_{\lambda_0}}{\tau_{\lambda_0} \cdot \tau_{\lambda_0}} = \frac{\dddot{R} \cdot \tau_{\lambda_0} - \dot{\lambda}^2 u_1 u_2}{u_1^2},
\]

(4.22)

\[
\dddot{\lambda} = \frac{(\dddot{R} - 3\dot{\lambda}\ddot{\lambda} \tau_{\lambda_0} - \dddot{\lambda}^3 \tau_{\lambda_\lambda_0}) \cdot \tau_{\lambda_0}}{\tau_{\lambda_0} \cdot \tau_{\lambda_0}} = \frac{\dddot{R} \cdot \tau_{\lambda_0} - 3\dot{\lambda}\ddot{\lambda} u_1 u_2 - 3\dddot{\lambda} u_1 u_3}{u_1^2},
\]

(4.23)

and so on for higher-order joint properties. Clearly, the Taylor series for the function \( \lambda(t) \) is being constructed. The motion of the subordinate joints can be obtained by differentiating the series in Equation 4.1 as

\[
\dot{\mu} = n \dot{\lambda}, \quad \dot{\nu} = k \dot{\lambda},
\]

(4.24)

\[
\ddot{\mu} = n \ddot{\lambda} + n(2) \dot{\lambda}^2, \quad \ddot{\nu} = k \ddot{\lambda} + k(2) \dot{\lambda}^2,
\]

(4.25)

\[
\dddot{\mu} = n \dddot{\lambda} + 3n(2) \ddot{\lambda}^2 + n(3) \dot{\lambda}^3, \quad \dddot{\nu} = k \dddot{\lambda} + 3k(2) \ddot{\lambda}^2 + k(3) \dot{\lambda}^3,
\]

(4.26)

and so on for higher-order derivatives.

In the development outlined thus far, the temporal solution is added to the geometric solution to obtain the complete local solution to the tracking problem. For time-scaling applications, however, the joint-space kinematics corresponding to a desired path are available in some form. For example, Hollerbach [65] assumes the knowledge of a temporal joint-space solution. On the other hand, the method of Kieffer [76] provides a time-independent
joint-space solution. Speed-ratio control can be utilized to obtain a time-independent parameterization of the joint-space curve corresponding to the desired output-space path, and then, it would be a variant of the method of Kieffer [76]. This solution can be stored as speed ratios that are functions of the leading joint variable. Since the geometric and temporal solutions are completely separate, there is no ‘time scaling’ of the joint-space solution. One simply appends the desired timing to the geometric solution by using Equations 4.21-4.26. This translates into significant computational savings in situations in which a given trajectory needs to be traversed at different speeds.

4.2.5 Singularities

A first-order singularity is defined as a pose of the mechanism in which the Jacobian rank \((J\) of the function \(r(\mu, \nu, \lambda)\) introduced in Equation 4.8) drops by 1. The dimension of the tangent space of the manipulator is less than the dimension of the output space of the robot. Therefore, it is impossible for the robot to move along a path tangent that does not lie in its (reduced) tangent space with first-order joint control. The condition \(\hat{T} \notin \text{col}[J]\) defines an ordinary singularity\(^{12}\) [95]. The first-order motion of the EE must vanish to maintain the fidelity of geometric tracking, i.e.,

\[
u_1 = s g_{\lambda 0} = 0, \quad \Rightarrow \bar{r}_{\lambda 0} = \bar{0},\]

where \(sg\) is the arc length of the generated path. It is now necessary to draw a distinction between the arc lengths of the desired path and the generated paths. This distinction was unnecessary for the non-singular case, as the condition \(sg := s\) is implied in establishing contact between the Taylor series for the desired and generated paths. However, in the

\(^{12}\)At a non-ordinary singularity or bifurcation, \(\hat{T} \in \text{col}[J]\) [77]. The term ‘bifurcation’ represents the intersection of multiple solution branches in the joint space. These are discussed in Appendix C.
singular case, the speed of the generated path must vanish \((\dot{s}g = 0)\) irrespective of the speed of the desired path. Therefore, the condition \(sg := s\) is violated. It is well known that at singular configurations, there must be a compromise between the speed and the accuracy of tracking [95, 165]. The vanishing speed and accurate tangent tracking (demonstrated below) represent one extreme of this compromise. The vanishing speed modifies Equation 4.11 as follows.

\[
\begin{aligned}
    u_1 &= sg_{\lambda 0} = 0, & v_1 &= 0, & w_1 &= 0, \\
    u_2 &= sg_{\lambda \lambda 0}, & v_2 &= 0, & w_2 &= 0, \\
    u_3 &= sg_{\lambda \lambda \lambda 0}, & v_3 &= 0, & w_3 &= 0.
\end{aligned}
\]  
(4.27)

Note that \(u_2\) and \(u_3\) are not zero.

Since for a first-order ordinary singularity \(\text{rank} [J] = 2\) and \(\hat{T} \not\in \text{col}[J]\), it is possible to construct \(J^*\) such that \(\text{rank}[J^*] = 3\). Therefore, Equations 4.14, 4.15, and 4.16 are applicable for ordinary singularities. Further, since \(\text{rank}[J^*] = 3\), Equation 4.14 can be expressed as

\[
[J_1 \ J_2 \ -\hat{T}] \begin{bmatrix} n \\ k \\ u_1 \end{bmatrix} = aJ_1 + bJ_2,
\]  
(4.28)

where \(a\) and \(b\) are some scalars. The condition \(u_1 = 0\) is imposed by Equation 4.14, as can be seen from the solution to its variant given by Equation 4.28. This is an advantage of the present method. There is no change in the strategy for obtaining speed ratios at or in the vicinity of an ordinary singularity. Furthermore, Equation 4.28 can be expressed as

\[
J \begin{bmatrix} n \\ k \\ 1 \end{bmatrix} = 0.
\]  
(4.29)

Equation 4.29 has a unique, non-trivial solution that belongs to the null space of the singular Jacobian. Therefore, the solution obtained by Equation 4.14 is similar to the null-space solution of Nenchev [111].
In a singular configuration, since $\bar{\tau}_{\lambda_0} = 0$, Equation 4.2 becomes

$$
\bar{\tau}(\lambda) = \bar{\tau}_0 + \tau_{\lambda_0} \frac{\lambda^2}{2} + \tau_{\lambda\lambda\lambda_0} \frac{\lambda^3}{6} + \cdots .
$$

(4.30)

Furthermore, since the speed of the generated path vanishes in the singular configuration, the speed of the desired path is also assumed to be zero, $s_{\lambda_0} = 0$, and Equation 4.3 becomes

$$
\bar{R}(s(\lambda)) = \bar{R}_0 + (s_{\lambda_0} \hat{T}) \frac{\lambda^2}{2} + (s_{\lambda\lambda_0} \hat{T}) \frac{\lambda^3}{6} + \cdots
$$

(4.31)

If the desired speed in the singular configuration is non-zero, the tracking of the arc length will not be exact. However, second- and third-order coordination, as imposed by Equations 4.15 and 4.16, force the vectors $\tau_{\lambda_0}$ and $\tau_{\lambda\lambda\lambda_0}$, respectively, to be parallel to $\hat{T}$. Comparing the second and third terms in Equations 4.30 and 4.31 makes this clear. Thus, the geometry of the desired and generated paths is matched, but, in general, the evolutions of the arc lengths of the desired and generated paths are not instantaneously identical.

It is worthwhile to point out that although $s_{\lambda_0} = 0$, the definitions given in Equations 4.4-4.6 are valid as long as the desired path does not have a singularity. As expected, it is still possible to track the desired path tangent. However, notice that third-order coordination manages only tangent tracking in the singular case, compared to matching the third-order path properties for a non-singular pose. Therefore, the control of the path geometry has degraded in the singular pose.

The time-based joint motions are obtained next. The evolution of $sg$ is

$$
sg(t) = s_{g\lambda_0} \hat{T} \frac{\lambda^2}{2} + (s_{g\lambda\lambda_0} \hat{T} \lambda^3 + 3(s_{g\lambda_0} \hat{T} \lambda^2) \frac{\lambda^3}{3!} + \cdots
$$

For the non-singular case, the quantity $s_{g\lambda_0}$ is obtained by differentiating the expression for the arc length of the generated curve with respect to $\lambda$, and it is the component of $\bar{\tau}_{\lambda_0}$ along the desired tangent $\hat{T}$. Since, for the singular case, Equation 4.15 forces $\bar{\tau}_{\lambda_0}$ to be
parallel to \( \hat{T} \), define
\[
s g_{\lambda\lambda0} := \sqrt{r_{\lambda\lambda0} \cdot r_{\lambda\lambda0}}. \tag{4.32}
\]

The quantity \( s g_{\lambda\lambda\lambda0} \) is obtained by differentiating Equation 4.32 with respect to \( \lambda \) and evaluating the result in the zero position. From Equation 4.27, the quantities \( s g_{\lambda\lambda0} \) and \( s g_{\lambda\lambda\lambda0} \) are \( u_2 \) and \( u_3 \), respectively, and they are obtained from the solution to the second- and third-order coordination equations. One approach to find the leading joint motion is to ignore the mismatch in the trajectory speeds and match the tangential acceleration and tangential jerk magnitudes of the desired and generated paths by equating the second- and third-order terms of the time-based Taylor series given by Equations 4.18 and 4.19. This yields
\[
\dot{\lambda} = \pm \sqrt{\ddot{s} \cdot r_{\lambda\lambda0} \cdot r_{\lambda\lambda0}} = \pm \sqrt{\ddot{s} \cdot u_2}, \tag{4.33}
\]
\[
\ddot{\lambda} = \frac{\left( R - \dot{\lambda}^2 \tau_{\lambda\lambda\lambda0} \right) \cdot \tau_{\lambda\lambda0}}{3\dot{\lambda}^2 r_{\lambda\lambda0} \cdot r_{\lambda\lambda0}} = \frac{\ddot{s} - \dot{\lambda}^3 u_3}{3\dot{\lambda} u_2}. \tag{4.34}
\]

Equations 4.33 and 4.34 give two valid solutions. If the commanded speed at the singularity is zero, these solutions are exact. Otherwise, the error in tracking the arc length is of the order \( (\dot{s} \cdot \Delta t) \), where \( \Delta t \) is the elapsed time. Another approach to obtain the leading joint motions is to solve for the joint velocity using Equation 4.33 and then equate the arc lengths \( s \) and \( s g \) after time \( \Delta t \). The time interval can be based on the feedback frequency of the tracking system. Therefore,
\[
s(\Delta t) = s g(\Delta t) = u_2 \dot{\lambda}^2 \frac{(\Delta t)^2}{2} + (u_3 \dot{\lambda}^3 + 3u_2 \dot{\lambda}) \frac{(\Delta t)^3}{6}. \tag{4.35}
\]

Equation 4.35 is solved for \( \ddot{\lambda} \). Equation 4.35 could also be used to choose between the two solutions generated by Equations 4.33 and 4.34. Equations 4.24 are used to determine the velocities of the other joints, and they ensure that the joint velocity vector belongs to the
null space of the singular Jacobian. Next, Equations 4.25 are used to compute the joint accelerations.

It is important to mention that specifying joint velocities alone will not achieve tangent tracking. It is Equation 4.15, which requires the specification of $\vec{r}_{\lambda 0}$, that achieves tangent tracking. This imposes a constraint on the joint accelerations via the second-order speed ratios (Equations 4.25). This is consistent with previous work [100], wherein higher-order joint motions are used to obtain desired EE motion geometry of a lower order. However, note that although second-order analysis constrains the joint accelerations, it does not provide a means of computing the leading-joint acceleration itself. Here, third-order analysis is used to compute the leading-joint acceleration (via $u_3$), and it simultaneously constrains the joint jerks (via the third-order speed ratios) without providing the means to compute the leading-joint jerk. Therefore, with the suggested strategy, \textit{third-order coordination is necessary to obtain the geometric and temporal solutions to the first-order trajectory tracking problem at ordinary singularities}. From a practical standpoint, third-order coordination imposes a higher computational load on the tracking system.

In conclusion, for a first-order singular pose of the mechanism, zeroth- and first-order joint coordination together match the position properties of the desired trajectory, and second- and third-order coordination together match the first-order properties of the desired trajectory. A pattern seems to be emerging, wherein two terms of the Taylor series of the generated trajectory account for each term in the Taylor series of the desired trajectory. However, this requires further investigation.
4.2.6 Choosing a Non-Dwelling Leading Joint

An alternate expression for the $p^{th}$-order speed ratio is given by Equation B.2 in Appendix B, where the ratio is obtained as a function of the previous $p - 1$ ratios and the time derivatives of the joint variables up to the $p^{th}$-order. As the current leading joint approaches a dwell in its trajectory, $\dot{\lambda} \to 0$, and from Equation B.2, the corresponding speed ratios approach infinity. This phenomenon can be conveniently used to detect the occurrence of a dwell in $\lambda$. The speed ratios corresponding to different leading joints are related with simple algebraic expressions.

\[
\frac{d\lambda}{d\mu} \bigg|_0 = \frac{1}{n}, \quad \frac{d\nu}{d\mu} \bigg|_0 = \frac{k}{n},
\]

\[
\frac{d^2\lambda}{d\mu^2} \bigg|_0 = -\frac{n^{(2)}}{n^3}, \quad \frac{d^2\nu}{d\mu^2} \bigg|_0 = \frac{nk^{(2)} - kn^{(2)}}{n^3},
\]

and so on for higher orders, as well as for ratios with $\nu$ as the leading joint variable. Note that if $n \to \infty$, the ratios with either $\mu$ or $\nu$ as the leading joint variable will necessarily be small. With checks on the absolute magnitudes of the speed ratios, a non-dwelling leading joint can be chosen.

At least one joint of the mechanism must move for the EE to possess some velocity. Therefore, there is at least one joint capable of serving as the leading joint. A possible difficulty is in choosing the leading joint when the EE speed vanishes. This occurs at singular configurations where the EE speed is forced to vanish. However, since the Jacobian is singular, it is possible to obtain a non-zero joint velocity vector belonging to the null space of the Jacobian, thus making a leading joint available. Another scenario is when the robot begins to move from a non-singular configuration, but the EE velocity remains zero. The geometric problem is well defined, assuming that the desired path is singularity-free,
and the coordination equations can be solved. Equations 4.21 and 4.24 ensure that the generated EE velocity will vanish.

A dwell in the leading joint is governed by the current configuration of the mechanism and the direction of the desired tangent. Therefore, a dwell can occur anywhere in the workspace of the robot. The proximity of the leading joint to a dwell is reflected in the modified Jacobian $J^*$ becoming ill conditioned and the corresponding speed ratios becoming large in magnitude. Several manipulability measures, such as the smallest singular value of the Jacobian [79] and the condition number of the Jacobian [174], exist that measure distance of a robot’s configuration from a singularity. The absolute values of the speed ratios perform the function of such measures as applied to $J^*$ in order to identify proximity to a dwell in the current leading joint.

The proximity of the robot’s configuration to a singularity alone does not influence the occurrence of a dwell in the leading joint. Therefore, the speed ratios associated with the appropriate choice of a leading joint are well behaved in the neighborhood of an ordinary singularity. This provides a strategy to obtain bounded joint rates in the vicinity of an ordinary singularity. It is only at the singular configuration where a change in the computation strategy is required. The equations for obtaining the velocity and acceleration of the leading joint change as shown in Section 4.2.5. One expects that the only realistic scenario in which these alternate equations would be applicable is when the robot begins its motion from a singular configuration. Finally, it is possible that all possible variants of $J^*$ are ill conditioned. This indicates the proximity of $\hat{T}$ to the reduced tangent space of the robot, i.e., this condition is close to a nonordinary singularity which is discussed in Appendix C.
4.3 Six-DOF Manipulators

Notation:

For the spatial rigid-body guidance problem, the DH parameters [145] are used to describe the kinematics of the robot. The joint-space vector is \( \theta \) containing the joint variables \( \theta_i, i = 1 \ldots 6 \). Assuming \( \theta_1 \) to be the leading joint variable, the \( j^{th} \)-order speed ratio relating the motion of joint \( i \) to that of the leading joint when evaluated in the zero position is denoted by \( n_i^{(j)} \). Therefore, \( n_3^{(3)} := \left. \frac{d^3 \theta_3}{d \theta_1^3} \right|_0 \). For \( j = 1 \), the superscript is omitted, so \( n_2^{(1)} := n_2 = \left. \frac{d \theta_2}{d \theta_1} \right|_0 \). There will be five speed ratios for each order and five joint-coordinating Taylor series similar to those in Equation 4.1.

A rigid-body guidance problem is commonly specified by defining the time evolution of a homogeneous transformation matrix, \( H(t) \) where \( H \in SE(3) \) [145]. The task specification in this form is not suitable for employing the methods of this chapter precisely because it does not separate the geometric and the temporal tracking problems. However, there are two equivalent representations that achieve this separation, and both representations can be obtained from \( H(t) \). In the first, the translational motion of one convenient control point on the EE is obtained by constructing the function \( \overline{R}(t) \) using standard velocity, acceleration, and jerk relations. Then, the local geometric properties of the control-point path are obtained from relations of the form given in Equations 4.4-4.6. Next, the angular velocity and acceleration of the EE are related to the Darboux vector and its derivative [82]. However, computing these quantities requires higher-order derivatives \( \kappa_{s0} \) and \( \tau_{s0} \) of the control-point path, as shown by Angeles et al. [8]. Therefore, the ensuing development adopts the second representation which describes the EE translation and rotation by specifying the translation of three non-collinear control points on the EE. The use of three points on the EE
to characterize its motion was introduced by Gosselin [55] to develop dimensionally homogeneous dexterity measures. Gosselin [55] has shown that no representational singularities are introduced by this method provided the points are non-collinear. The desired translations of the three control points are obtained exactly as described in the first representation above, and the methods of the previous sections are applied with minimal modification. By writing coordination equations for three points, a sufficient number of equations are generated to obtain the five speed ratios of each order and the unknown path-variable derivatives for the paths of the three points.

Let $P_i, i = 1, 2, 3$ denote the three control points on the EE. Then, the position kinematics of the system are described as

$$r_i(\bar{\theta}) = P_i, \quad i = 1, 2, 3.$$  \hspace{1cm} (4.36)

Equation 4.36 consists of nine component equations that are subject to three rigid-body constraints (RBC) stating that the distances between the three control points are constant. The desired path is specified next by specifying the paths for the control points in terms of parameters $q_i$: $P_i = P_i(q_i)$. Equation 4.36 is modified to describe the tracking problem.

$$r_i(\bar{\theta}) - P_i(q_i) = 0, \quad i = 1, 2, 3.$$  \hspace{1cm} (4.37)

A valid task specification must also respect the RBC. Therefore, given any one of the path variables, $q_1 = q_1^*$, say, the other two can be computed using the RBC. Consequently, there is only one independent motion variable for the desired path, and Equation 4.37 represents a system of six equations in the seven unknown motion parameters $\bar{\theta}$ and $q_1$. The locus of solutions to Equation 4.37 represents a set of curves in the seven-dimensional space of the motion parameters. According to the implicit function theorem, the solution curve can be locally described in terms of one of the variable parameters in the vicinity of a given
configuration \((\theta^*, q^*_1)\) that satisfies Equation 4.37 if the Jacobian of the system of equations described by Equation 4.37 has rank six. The Jacobian is constructed as follows.

\[
J = [J' \quad \mathbf{P}'] = \begin{bmatrix}
J_1 & \frac{dP_1}{dq_1} \\
J_2 & \frac{dP_2}{dq_1} \frac{dq_2}{dq_1} \\
J_3 & \frac{dP_3}{dq_1} \frac{dq_3}{dq_1} \\
\end{bmatrix}
\]

(4.38)

where \(J_i\) is a Jacobian matrix of order \(3 \times 6\) similar to that occurring in Section 4.2.2. The order of \(J\) is \(9 \times 7\). The matrix \(J'\) can be evaluated at the given configuration \(\theta^*\). The coefficients \(\frac{dq_i}{dq_1}\) at \(q^*_1\) are obtained by differentiating the RBC with respect to \(q_1\) as follows.

For \(i = 2, 3\),

\[
\frac{d}{dq_1} \|P_i(q_i) - P_1(q_1)\| = \frac{d}{dq_1} \|P_i(q^*_i) - P_1(q^*_1)\| = 0,
\]

\[
\Rightarrow (P_i(q_i) - P_1(q_1)) \cdot \left(\frac{dP_i}{dq_i} \frac{dq_i}{dq_1} - \frac{dP_1}{dq_1}\right) = 0,
\]

\[
\Rightarrow \left. \frac{dq_i}{dq_1} \right|_{q^*_i} = \frac{(P_i(q^*_i) - P_1(q^*_1)) \cdot \frac{dP_i}{dq_i}}{(P_i(q^*_i) - P_1(q^*_1)) \cdot \frac{dP_1}{dq_1}}.
\]

(4.39)

This determines \(J\) at the current configuration.

The column vector \(\mathbf{P}'\) contains scalar multiples of the desired tangents at the three control points consistent with the RBC. The rank of \(J'\) is six, and \(\mathbf{P}' \in \text{col} [J']\) for non-singular configurations of the robot. Therefore, \(\text{rank} [J] = 6\). The robot is in an ordinary, first-order singular configuration when \(\text{rank} [J'] = 5\) and \(\mathbf{P}' \notin \text{col}[J']\).

Again, \(\text{rank} [J] = 6\). Therefore, for non-singular as well as ordinary, first-order singular configurations, the solution curve to Equation 4.37 can be described using one of the motion variables. A similar argument can be constructed for the three-DOF case developed in the previous sections.

Consistent with the development for the three-DOF case, a leading joint variable (\(\theta_1\)) is used to parameterize the solution curve. Further, the derivatives of the path variables \(u_i\) are
treated as independent unknowns, although their differentials are related to the derivatives of \( q_i \) (e.g. \( \frac{u_1}{u_2} = \frac{dq_1}{dq_2} \)). This way, the formulation developed for the three-DOF case can be utilized without alteration, and the following \( p^{th} \)-order coordination equation in the general form is obtained by simply concatenating Equation 4.13 applied to the three control points.

\[
\mathbf{J}^\ast \mathbf{\bar{n}}_p^\ast = \mathbf{\Psi}_p^{-1},
\]

where,

\[
\mathbf{J} := \begin{bmatrix} J_1^\ast & -\hat{T}_1 & 0 & 0 \\ J_2^\ast & 0 & -\hat{T}_2 & 0 \\ J_3^\ast & 0 & 0 & -\hat{T}_3 \end{bmatrix}, \quad \mathbf{\Psi}_p^{-1} := \begin{bmatrix} \mathbf{\Psi}_{1(p-1)} \\ \mathbf{\Psi}_{2(p-1)} \\ \mathbf{\Psi}_{3(p-1)} \end{bmatrix},
\]

\[
\mathbf{\bar{n}}_p^\ast := \begin{bmatrix} n_2^{(p)} & n_3^{(p)} & n_4^{(p)} & n_5^{(p)} & n_6^{(p)} & u_{1p} & u_{2p} & u_{3p} \end{bmatrix}^T,
\]

where the quantities \( u_{ip} \) and \( \mathbf{\Psi}_{i(p-1)} \) are the \( p^{th} \)-derivative of the path variable with respect to the leading joint variable and the RHS of the coordination equation as defined for the 3-DOF case applied to point \( i \), respectively, and \( \mathbf{J}_i^\ast \) is a \( 3 \times 5 \) matrix obtained by deleting the column of the Jacobian corresponding to the leading joint. Equation 4.40 is solved by computing the pseudoinverse of the \( 9 \times 8 \) matrix \( \mathbf{J} \). Equation 4.40 is used in the example provided in the next section.

Ordinary singularities in any one of the three Jacobians \( \mathbf{J}_i \) or the Jacobian \( \mathbf{J}' \) will invoke the singularity solution, and the path velocities \( u_{1i} \) will vanish. Also, the matrix \( \mathbf{J} \) drops rank if \( \hat{T}_i \) vanishes. This happens when the corresponding control point lies on the instantaneous screw axis of a pure rotational motion of the EE. If one (or two) tangent(s) vanish, the corresponding path derivatives, \( u_{ip} = 0 \) (and \( u_{jp} = 0 \)), \( \forall p \). Therefore, the column(s) containing \( \hat{T}_i \) (and \( \hat{T}_j \)) can be deleted from \( \mathbf{J} \), and the terms \( u_{ip} \) (and \( u_{jp} \)) can be deleted from \( \mathbf{\bar{n}}_p^\ast \). The resulting linear system of 9 equations with 7 (or 6) unknowns can be
solved. However, if all three control points have instantaneous dwells in their trajectories, all tangents $\dot{T}_i = 0$, $\Rightarrow \text{rank}[J] = 5$. This condition occurs when the control points lie on the instantaneous screw axis of a pure rotational motion. Choosing non-collinear control points will ensure that all tangents never vanish simultaneously. This condition on the choice of control points resonates with the result in [55] mentioned earlier in this section.

Equation 4.40 can be solved to obtain the local geometric description of the joint-space curve corresponding to the desired output-space path. This solution can be used to execute online tracking, as illustrated in the next section. However, in conjunction with continuation algorithms, this method yields the entire joint-space curve instead, akin to the method of Kieffer [76]. ‘Time scaling’ is then simply the addition of timing to the geometric solution by using Equations 4.21-4.26.

4.3.1 Rigid-body guidance example

The following example solves the complete trajectory tracking problem by generating local solutions to the geometric and temporal tracking subproblems and then using a feedback controller to compensate for tracking errors. To show how the methodology developed in this paper can be used in conjunction with existing feedback paradigms, a conventional resolved-acceleration feedback controller [22] is employed. The output of the controller is interpreted to serve as the input to the inverse kinematics problems.

A six-DOF manipulator, along with its DH parameters as per [145], is shown in Figure 4.3. The shoulder joint consists of two revolutes ($\theta_1$ and $\theta_2$) with intersecting axes, the elbow joint is a single revolute ($\theta_3$), and the wrist has three revolute joints ($\theta_4$-$\theta_6$) with axes intersecting at the wrist center. The wrist center, used as a control point, along with two more points on the EE form a triangular lamina. The robot wrist tracks a circular trajectory
parallel to the \(X_0Y_0\) plane while maintaining the orientation of the EE. This is illustrated in Figure 4.4. The initial joint angles are \(\theta_{i0} = [0\ 20\ 0\ 40\ 20\ -60]^{\circ}\). The regional structure of the robot is singular in the initial and final configurations, i.e. \(\theta_{3\text{initial}} = \theta_{3\text{final}} = 0\). The parameters of the robot in arbitrary length units are \(l_0 = 1, l_1 = 1.2, l_2 = 1.5,\) and \(d_2 = 0.25\). The desired trajectories of the control points in appropriate length units are

\[
\begin{align*}
\overline{P}_1(t) &= \begin{bmatrix} 3.3333 \\ 0.2549 \\ 1.9235 \end{bmatrix} + 0.94 \begin{bmatrix} \cos(q(t)) \\ \sin(q(t)) \\ 0 \end{bmatrix}, \\
\overline{P}_2(t) &= \overline{P}_1(t) + [0.5\ 0\ -0.25]^T, \quad \text{and}, \\
\overline{P}_3(t) &= \overline{P}_1(t) + [0.5\ 0\ 0.25]^T,
\end{align*}
\]

where \(q(t) = 0.0456t^3 - 0.2394t^2 + 3.7068\). The EE velocity at the start and end of motion is zero. The motion is executed in 3.5 seconds. The shoulder joint angle \(\theta_2\) is the leading joint variable for the entire motion.

The position and velocity gains for the controller are \(k_p = 250\sec^{-2}\) and \(k_v = 70\sec^{-1}\), respectively, and the feedback frequency is 90 Hz. The joint velocities and accelerations are limited to \(15\text{rad/sec}\) and \(15\text{rad/sec}^2\), respectively. As the robot moves, the control points generate trajectories \(\overline{r}_i\). At each time instant \(t^*\), position and velocity tracking errors are defined as \(\overline{e}_i = \overline{r}_i(t^*) - \overline{P}_i(t^*)\) and \(\dot{\overline{e}}_i = \dot{\overline{r}}_i(t^*) - \dot{\overline{P}}_i(t^*)\), respectively, for each control point. The controller corrects for these errors by maintaining the current tangent directions and redefining the desired accelerations as \(\overline{\ddot{P}}_i^* := \overline{P}_i(t^*) - k_v\dot{\overline{e}}_i - k_p\overline{e}_i\). Geometric motion properties of the control points up to the second order are obtained via Equations 4.4-4.6, and the corresponding speed ratios are obtained using Equation 4.40. The leading-joint motion is obtained via Equations 4.21-4.22 applied to the wrist, following which, the
temporal motions of the remaining joints are obtained using Equations of the form 4.24-4.25.

The configuration at $t = 0$ represents an ordinary singularity because $\text{rank}[J_1] = 2$, $\hat{T}_1 \notin \text{col}[J_1]$, $\Rightarrow \text{rank}[J] = 6$. The only change in the strategy mentioned previously is that now Equations 4.33-4.34 are used to obtain the leading-joint temporal motion. Since the desired speed at $t = 0$ is zero, matching the acceleration and jerk of the desired and generated paths provides exact solutions for the leading joint velocity and acceleration. Note that toward the end of the motion, the robot gets arbitrarily close to the singularity but does not reach the singular configuration, so that the computation strategy does not change once the robot has moved away from the singular configuration at $t = 0$. 

Figure 4.3: A six-revolute robot. Reference frames 3, 4 and 5 are centered at the wrist. They are drawn separately for clarity. The wrist center is used as a control point, so $\Psi = 0$ and $d_6 = 0$. 

<table>
<thead>
<tr>
<th>Link</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_i$</td>
<td>0</td>
<td>$l_1$</td>
<td>$l_2$</td>
<td>0</td>
<td>0</td>
<td>$(d_6)\sin(\psi)$</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>$\pi/2$</td>
<td>0</td>
<td>0</td>
<td>$\pi/2$</td>
<td>$\pi/2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$d_i$</td>
<td>$l_0$</td>
<td>$d_2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$(d_6)\cos(\psi)$</td>
</tr>
<tr>
<td>$\theta_i$</td>
<td>$\theta_1$</td>
<td>$\theta_2$</td>
<td>$\theta_3$</td>
<td>$\theta_4$</td>
<td>$\theta_5 + \pi/2$</td>
<td>$\theta_6 - \pi/2$</td>
</tr>
</tbody>
</table>
The performance of the algorithm is illustrated in Figures 4.5 and 4.6. Figure 4.5 plots the position errors $\bar{e}_i(t)$ for all control points. Note that the error around $t = 0$ is low, indicating that both the geometry and the speed of the desired motion is successfully approximated near the singular configuration. The corresponding joint velocities, given in Figure 4.6, are well below the established limit. The slope of the velocity curves indicate that none of the joint accelerations reach the acceleration limit. The accelerations are highest toward the end of motion when the robot is approaching the singularity, although the acceleration limit was reached only for joint angle $\theta_3$.

High feedback frequency and feedback gains are used in the above example to illustrate that the robot can approach the singular configuration without the joint motion becoming large. At $t = 3.5$, $\theta_3 = 0.26^\circ$, and the position error is $2 \times 10^{-5}$ units. Simulations with a feedback frequency of 30 Hz and gains $k_p = 50sec^{-2}$ and $k_v = 20sec^{-1}$ resulted in a maximum position error on the order of $10^{-3}$ length units. In conclusion, the robot successfully navigates away from the singularity ($t \approx 0$), comes arbitrarily close to the singularity ($t \approx 3.5$), and navigates the entire path with small position error (maximum position error = $2.6 \times 10^{-5}$ length units) and bounded joint motions.

4.4 Discussion

The implementation of the algorithm developed in this chapter requires description of motion with its geometric and temporal aspects separated at both the problem specification stage and the implementation stage. This requires reworking the task specifications from existing formats, and it increases the computational complexity of the algorithm. Inversion of the Jacobian of the kinematic map rather than the manipulator Jacobian is required, and therefore, it is difficult to provide a comprehensive complexity analysis here. However,
Figure 4.4: The three control points on the EE form a triangular lamina. The initial, the final and an intermediate position of the lamina are indicated. The orientation of the EE must be constant during the motion, so all control points follow identical paths.
Figure 4.5: Position errors for the three control points are close to each other. The maximum error is $2.6 \times 10^{-5}$ units.

Figure 4.6: Joint velocities for the tracking task. The velocities are bounded throughout the task. The slopes of the profiles indicate the joint accelerations. The accelerations are high toward the end of the motion for $\theta_2$, $\theta_3$ and $\theta_4$. 
the computation of the speed ratios will be roughly the most computationally expensive operation in this algorithm. The advantage of singularity navigation also comes at the cost of increased computation. Recall that third-order coordination is required to obtain the joint velocities and accelerations for singular configurations of the robot. However, the computational effort involved is not forbidding, since the simulation results presented here were developed on a standard desktop computer using MATLAB-Release 2009a. The computation of the joint motions required approximately 0.01 seconds, indicating that the algorithm can be implemented in real time with the use of dedicated, application-specific software and hardware. Finally, the controllers for the three control points employ identical gains in the presented example. The choice of these gains for various tracking tasks can be viewed as a challenge as well as design flexibility.

The matching of higher-order motion properties opens up the possibility of substantially decreasing the feedback frequency. Then, the system evolves as a combination of continuous dynamics (the evolution of the joint angles) and a sequence of discrete events (feedback-based correction). Constructing control schemes for this hybrid system (see [80], for example) and studying their stability are interesting problems for future investigation.

### 4.4.1 Relation to motor control

The algorithm developed in this chapter was inspired by time-invariant characteristics observed in human arm movement and the time-invariance hypothesis (TIH) [5]. The developed algorithm potentially serves a rather specific purpose in the motor control of arm motion, as explained below. The TIH defines the input-output characteristics of the inverse kinematics problem, and therefore, belongs to Marr’s first level, as explained in Section
4.1. Naturally, this chapter develops an algorithm to achieve this transformation, and therefore, it belongs to Marr’s second level. The algorithm provides a mathematical expression of an internal inverse model that uses a time-invariant wrist path as the starting point of the process of composing a movement plan. Internal models are neural mechanisms that can mimic the input-output characteristics, or their inverse, of the motor apparatus. For a detailed discussion of these and other ideas related to motor control of human reaching, refer to Section 2.1.

The present algorithm is differential - producing solutions that are only locally useful - and, therefore, feedback-driven. It is clearly applicable for movements that are slow enough so that sensory information from the motor apparatus and the surroundings has sufficient time to reach the CNS and serve as a feedback signal. For fast movements, feedback can be obtained from an internal feedforward model, which is a neural mechanism that uses a copy of outgoing neural signals and transforms it into predicted outcomes of the motor apparatus. This predicted outcome serves as the feedback signal to the inverse model [74]. Thus, the algorithm is viewed as a component of a larger conceptual motor machinery that allows the utilization and execution of the TIH within its framework.

The output of the algorithm serves as the input to an inverse-dynamic internal model that converts the joint motion into joint torques. This model could be the leading-joint hypothesis (LJH) [32], particularly since the classification of the joints as leading and subordinate carries over seamlessly into the dynamic regime with the LJH. The LJH treats (mostly) the shoulder as the leading joint. Given a movement objective, the shoulder produces gross torques, disregarding any elbow motion and the associated interaction effects. These torques are designed to take the arm to the general vicinity of the target and are computed using simple associations between gross description of the required movement -
e.g. initial and final joint position, movement time, and expected inertial resistance of the limb - and the torques. In contrast, the elbow motion is controlled taking into account the shoulder movement, the interaction effects that arise from it, and the specific movement objective. In particular, the computation of the elbow-joint torques requires the prediction of the shoulder and elbow-joint motions. Thus, the shoulder is said to lead the motion, and the elbow ‘follows’. This strategy allows significant simplifications in shoulder-joint control.

The variability in the time evolution of the leading-joint motion will be high as a result of the leading-joint strategy. The use of the speed ratios to encode the inverse kinematics is particularly advantageous in this situation. The speed ratios are functions of the pose of the arm and the desired path geometry, and their computation does not involve the leading-joint temporal motion. The subordinate joint movement can be obtained from the leading joint movement using the speed ratios such that the EE path is unaffected. Thus, the speed ratios shield the output-space motion geometry from the variability in the leading-joint motion. When the temporal solution is appended to the geometric solution, the wrist speed will be affected, but not the wrist path. Thus, the present algorithm serves as the mathematical machinery that computes the desired subordinate-joint movement given the output-space movement plan and the leading-joint movement. These results are used in the computation of the subordinate-joint torques.

Not all arm movements are shoulder-led, and there exists a class of motions wherein the shoulder and elbow switch roles [32, 33, 92, 47]. Galloway and Koshland [47] and Dounskaia [32] point out that movements that are not shoulder-led typically have small net shoulder displacement. Dounskaia [32] suggests that since the subordinate-joint interaction torque is proportional to the leading joint speed, a slow-moving leading joint will
not produce significant interaction torque for the subordinate joint to utilize; there is low ‘mechanical advantage’ available to the subordinate joint.

This joint-switching phenomenon can be explained based on the kinematic analysis developed in this chapter as well. It is proposed that the leading joint is switched not so much for maximizing mechanical advantage, but for preserving motion smoothness. The problem of predicting the desired movement of the subordinate joint based on the movement of the leading joint is ill conditioned when the leading joint motion is small. Section 4.2.6 describes this idea, wherein the leading joint is switched when the current leading joint approaches a dwell in its trajectory. When a leading joint approaches a dwell, the corresponding speed ratios increase in magnitude. If the leading joint is not switched in the vicinity of this singularity, a small error in its motion is greatly magnified by the large speed ratios, leading to unsteady EE movement.

The constraint on the upper bound of the speed ratios translates simply into a constraint on the net displacement of the leading joint for a given task for most arm motions. If the net shoulder displacement compared to the net elbow displacement for a task is low, the instantaneous shoulder velocity will be low, and the corresponding speed ratio will be large throughout the movement. Therefore, the movement will be elbow-led. Thus, there are two distinct explanations as to why the leading joint is switched under certain circumstances. The first explanation focuses on the magnitude of the interaction effects [32], and the second explanation focuses on the smoothness of the wrist path.

There are some aspects of this algorithm (as well as the LJH) that need to be developed further before it is fully integrated with the LJH. First, there are quantitative issues regarding feedback rates for subordinate-joint control, the specific leading-joint torque profiles, etc. The empirical data gathered during the experiment in Chapter 2, in conjunction with
the sensing capabilities of the CNS available in the motor-control literature (see [164] for example), will allow this development. This will lead to a more complete model of motor control that is capable of making quantitative, verifiable predictions of spatial, human reaching.

Another important issue is the computational complexity associated with the inverse kinematics problem, of which this chapter provides ample illustration. Atkeson [10] suggests that hybrid control schemes may be employed by the CNS to handle this computational load. Hybrid control schemes are a combination of lookup tables and structured algorithms. A lookup table replaces a computationally complex transformation with memorized particular solutions of the transformation. Structured algorithms are capable of executing transformations given a small number of input parameters. Studies suggest that both control mechanisms - lookup tables and structured algorithms - are utilized by the CNS for motor control [74]. The algorithm developed here is entirely structured, and proposing hybrid schemes for its execution is an area of future work.

Finally, the algorithm does not address redundancy in the inverse kinematics problem. Kinematic models of the human arm model the shoulder as a single-DOF revolute joint in planar studies. The simplest model of the shoulder for spatial studies would be a two-DOF joint composed of two revolute joints with non-parallel joint axes. These models are gross idealizations. Anatomically, the glenohumeral joint has three DOFs, and the center of this joint typically moves relative to the thorax. Lenarčič and Stanišić [91] provide a comprehensive analysis of the shoulder-complex kinematics. Their model of the shoulder complex gives the humerus five DOFs and is considered more realistic. Therefore, a realistic model of the human arm will possess more than six DOFs (including the wrist joint),
and is, therefore, redundant. In Chapter 2, wrist paths for spatial reaching motion are studied. Furthermore, implicit in the dynamic analysis described in Section 3.2 is a four-DOF spatial arm model, with a three-DOF shoulder and a single-DOF elbow. This, again, is a redundant, four-DOF mechanism used to position the wrist in space. The algorithm described in this chapter does not address this mechanism. Therefore, the examples provided in this chapter do not represent the model of the human arm that is utilized in Chapters 2 and 3.

Chapter 5 begins the adaptation of the ideas developed in this chapter to redundant mechanisms. The discussion there is limited to planar mechanisms with one and two degrees of redundancy. Although four-DOF spatial, regional mechanisms are not discussed, the work illustrates methods that could achieve this goal.

4.5 Conclusions

A methodology was developed to solve the geometric path-tracking problem to an arbitrary order using 3-DOF non-redundant spatial robots. The method was extended to the 6-DOF rigid-body-guidance problem by writing the kinematic equations of motion developed for the 3-DOF case for three non-collinear points in the EE. Path timing was added to the geometric solution, and in particular, a methodology was provided for obtaining the time derivatives of the joint variables for ordinary singularities. Trajectory tracking with this approach, in conjunction with resolved-rate acceleration control, was demonstrated using a spatial 6R robot. Simulation results indicated that the joint velocities and accelerations remain bounded while starting from and approaching a singular configuration. The approach can be computationally demanding; however, the simulation times indicate that
it can be effectively employed for physical systems. The method can be viewed as a structured algorithm that can execute the inverse kinematics transformation in human reaching. It integrates, to a large extent, the time-invariance hypothesis and the leading-joint hypothesis for human reaching.

Future work consists of extending the ideas in this chapter to redundant planar and spatial mechanisms. Hybrid control schemes that reduce the computational load of the algorithm also need to be designed. These schemes treat the system as a combination of continuous dynamics (the evolution of the joint angles) and a sequence of discrete events (feedback-based correction) and use various data structures (lookup tables, structured computation).
Chapter 5: APPLICATION OF SPEED-RATIO CONTROL TO REDUNDANT PLANAR SYSTEMS

5.1 Introduction

Redundant manipulators are increasingly employed in useful practical tasks that are specified in terms of a geometric path to be followed by the end-effector. Redundant degrees of freedom make it possible to achieve objectives such as avoiding collisions, joint limits and/or singular configurations. However, objective criteria need to be specified to resolve the kinematic redundancy. Kinematic performance metrics, such as locally bounded joint-space velocities, involve computation of damped least-squares solutions [24], although such pseudoinverse-based control cannot avoid singular configurations [12]. Alternatively, time-optimal control uses the manipulator dynamics to minimize the performance time, which is a solved problem for non-redundant manipulators [136]. For kinematically redundant manipulators, numerical procedures have been proposed by Galicki [45] to achieve path-constrained time-optimal controls. A computationally efficient feedback-control law is developed by Galicki [46] that provides joint forces/torques for a redundant manipulator while minimizing the output-space tracking error. These methods use information from the output-space path up to the first-order only, whereas the definition of the desired output-space path contains more geometric information in the form of higher
order derivatives. Effective utilization of this path information can reduce the required feedback frequency for a desired tracking accuracy and potentially, the computational cost of path tracking.

The present chapter illustrates how speed-ratio (SR) control can be effectively applied to redundant, planar manipulators. SR control separates the control of the path variable from that of the trajectory geometry, and concentrates solely on the latter aspect. This approach, developed in Chapter 4, generalizes the ideas of Lorenc et al. [100] and Lorenc and Stanišić [99] to path tracking with spatial, non-redundant systems using the geometric properties of the output-space path up to any order. The technique can be used to implement planar and spatial path tracking as well as rigid-body guidance.

SR control, when applied to redundant mechanisms, resolves the kinematic redundancy in the system by including higher-order geometric information from the desired path in the problem formulation. The present chapter is a preliminary investigation, with a focus on three- and four-DOF planar systems tracking point paths. The local accuracy of path tracking using planar mechanisms with one degree of redundancy can be enhanced by matching (when possible) or approximating second-order path properties with first-order joint coordination [7]. Examples show that tracking solutions with this method are locally more accurate compared to unweighted pseudoinverse solutions. Similar results were obtained for four-DOF mechanisms, wherein up to third-order path properties were matched using first-order joint coordination. With these intrinsically more accurate inverse kinematics solutions, which may be computed online (for three-, and perhaps four-DOF mechanisms) or offline, trajectory tracking can be executed with a reduced feedback rate and/or lower feedback gains.
5.2 Redundancy resolution by optimization of output-space tracking error

The general coordination equations for planar motion are obtained first. The solutions for three- and four-DOF mechanisms are then derived.

5.2.1 Coordination equations for planar mechanisms

The development and notation used in this section follows that in Section 4.3. The joint-space vector is \( \bar{\theta} := [\theta_1 \theta_2 \ldots \theta_d]^T \) containing the joint variables \( \theta_i, i = 1 \ldots d \) of a \( d \)-DOF planar mechanism. The current configuration of the robot is called the zero position, the values of the joint angles in the zero position are denoted by \( \theta_{i0} \), and the joint-variable vector is denoted by \( \bar{\theta}_0 \). Without loss of generality, the instantaneous values of the joint variables in the zero position are taken to be zero. The forward position kinematics of the mechanism are given as \( r : \mathbb{R}^d \to \mathbb{R}^2 \) such that \( r_0 = r(\bar{\theta}_0) \), where \( r_0 \) is the current position of a controlled point on the end effector (EE), point \( P \). For the trajectory-tracking problem, define \( \tau : \mathbb{R} \to \mathbb{R}^2 \) so that the image of the map \( \tau \) represents a planar curve generated by point \( P \). Choosing \( \theta_1 \) as the independent parameterizing variable leads to the construction of the function \( \tau = \tau(\theta_1) \). In this process, \( d - 1 \) joint-coordinating Taylor series similar to Equation 4.1 impose slaving relations between the joints of the mechanism.

\[
\theta_i = n_i \theta_1 + \frac{1}{2!} n_i^{(2)} \theta_1^2 + \frac{1}{3!} n_i^{(3)} \theta_1^3 + \ldots \quad i = 2 \ldots d \tag{5.1}
\]

Here, the \( j^{th} \)-order speed ratio relating the motion of joint \( \theta_i \) to that of the leading joint when evaluated in the zero position is \( n_i^{(j)} \). Therefore, \( n_i^{(3)} := \frac{d^3 \theta_i}{d\theta_1^3} \bigg|_0 \). For \( j = 1 \), the superscript is omitted. Therefore, \( n_i^{(1)} := n_2 = \frac{d\theta_i}{d\theta_1} \bigg|_0 \). There are \( d - 1 \) speed ratios of each order. Since the initial increments in the joint variables are all zero in the zero position, the Taylor series in Equation 5.1 do not have a constant term. The generated path of point \( P \)
\[ r(\theta_1) = r_0 + r_{\theta_1} \theta_1 + \frac{1}{2!} r_{\theta_1 \theta_1} \theta_1^2 + \frac{1}{3!} r_{\theta_1 \theta_1 \theta_1} \theta_1^3 + \ldots, \quad (5.2) \]

where \( \frac{dr}{d\theta_1} \bigg|_{0} := \tau_{\theta_1,0} = \tau(n_i), \frac{d^2r}{d\theta_1^2} \bigg|_{0} := \tau_{\theta_1 \theta_1,0} = \tau(n_i, n_i^{(2)}), \) and so on for higher order derivatives. A Frenet-Serret (FS) frame, consisting of a tangent, a normal and a binormal vector, is introduced to describe the generated curve. The tangent and the normal vectors define the plane of motion, and the binormal vector is perpendicular to this plane. To all vectors in Equation 5.2, zeros are appended as the third component. Then, the tangent of the frame is parallel to \( \tau_{\theta_1,0} \), and the normal is parallel to \( \tau_{\theta_1 \theta_1,0} \). Note that this description involves the unknown speed ratios.

The desired path of point \( P \) is a planar curve \( \bar{R} \). The geometry of the desired path is also described using a FS frame, but in this case, all of the quantities defining the frame are known, and Equation 4.3 describes this curve. This equation is reproduced below.

\[
\bar{R}(s(\theta_1)) = \bar{R}_0 + \left[ s_{\theta_1,0} \hat{T} \right] \theta_1 + \left[ s_{\theta_1 \theta_1,0} \hat{T} + s_{\theta_1,0}^{2} \kappa_d \hat{N} \right] \frac{\theta_1^2}{2} + \left[ (s_{\theta_1,0} \theta_1) - \kappa_d s_{\theta_1,0}^2 \right] \hat{T} \\
+ (3s_{\theta_1,0} s_{\theta_1,0} \kappa_d + \kappa_d s_{\theta_1,0}^3) \hat{N} + \kappa_d s_{\theta_1,0}^3 \hat{B} \left[ \frac{\theta_1^3}{6} \right] + \ldots \quad (5.3)
\]

where \( s \) is the arc length, \( \hat{T}, \hat{N}, \text{ and } \hat{B} \) are the tangent, normal and binormal, respectively, and \( \kappa_d, \tau, \text{ and } \kappa_{sd} \) are the curvature, torsion, and curvature rate of the desired path. Of course, the torsion for planar motion is zero, and hence, the coefficient of \( \hat{B} \) vanishes. Also, the binormal \( \hat{B} \) is assumed to be perpendicular to the plane of motion. Control of the EE path geometry is achieved by matching the geometric properties of the frames describing the generated and desired paths. For example, matching the first-order geometric property means forcing the tangent of \( \tau(\theta_1) \) to be parallel to the tangent of the desired path. This is achieved by equating the corresponding terms of the Taylor series in Equations 5.2 and 5.3. This process is also called establishing contact between the two functions \( \bar{R} \) and \( \tau \) [51].
The resulting *coordination equations* relate the speed ratios of the corresponding order, i.e., the \( n^{th} \)-order coordination equation is a linear relation between the \( n^{th} \)-order speed ratios.

The coordination equations obtained below are the same as those obtained in Section 4.2.2, however, they are expressed in a form more suitable for planar mechanisms. Equating the terms of the two Taylor series yields

\[
\tau_{\theta_1} = s_{\theta_10} \hat{T},
\]
\[
\tau_{\theta_1\theta_1} = s_{\theta_1\theta_10} \hat{T} + s_{\theta_10}^2 \kappa_d \hat{N},
\]
\[
\tau_{\theta_1\theta_1\theta_1} = (s_{\theta_1\theta_1\theta_10} - \kappa_d^3 s_{\theta_10}^3) \hat{T} + (3s_{\theta_10}^2 s_{\theta_1\theta_10} \kappa_d + \kappa_d s_{\theta_10}^3) \hat{N}.
\]

Cross multiplying these equations by \( \hat{T} \) and using Equation 4.20 yields the first-, second-, and third-order coordination equations.

\[
\tau_{\theta_1} \times \hat{T} = 0, 
\]
\[
\tau_{\theta_1\theta_1} \times \hat{T} + \kappa_d (\tau_{\theta_10} \cdot \tau_{\theta_10}) \hat{B} = 0, 
\]
\[
\tau_{\theta_1\theta_1\theta_1} \times \hat{T} + \left[ \kappa_d (\tau_{\theta_10} \cdot \tau_{\theta_10}) \right] \hat{B} = 0.
\]

The three Equations 5.4 - 5.6 yield one equation each along the direction normal to the plane of the movement. For non-redundant mechanisms \((d = 2)\), these equations can be solved for the single speed ratio of each order.

### 5.2.2 Three-DOF planar systems

For a general, planar three-DOF system, let \( \vec{\theta} = [\theta_1, \theta_2, \theta_3]^T \) be the joint-space vector, and let the leading joint be \( \theta_1 \). Assume that second-order coordination is desired. The two first- and second-order speed ratios will be \( n_2 \) and \( n_3 \), and \( n_2^{(2)} \) and \( n_3^{(2)} \), respectively. Equation 5.4 will yield an equation that is linear in the first-order speed ratios. Similarly, Equation 5.5 will yield an equation that is linear in the second-order speed ratios and
quadratic in the first-order speed ratios. These equations are

\[ A_1 n_2 + A_2 n_3 + A_3 = 0, \quad (5.7) \]

\[ B_1 n_2^2 + B_2 n_3^2 + B_3 n_2 n_3 + B_4 n_2 + B_5 n_3 + B_6 + B_7 n_2^{(2)} + B_8 n_3^{(2)} = 0, \quad (5.8) \]

where the coefficients \( A_i \) and \( B_i \) are functions of the current mechanism pose and the desired path geometry. This set of two equations in four unknown speed ratios reflects the redundancy in the mechanism. Thus, there are multiple first- and second-order speed ratios that achieve the same instantaneous geometry of the path of point \( P \). Here, the redundancy is resolved by seeking solutions with the following property:

\[ n_2^{(2)} = n_3^{(2)} = 0. \quad (5.9) \]

This is equivalent to terminating the joint-coordinating Taylor series 5.1 at the first order. On the other hand, the term in Equation 5.5 containing the second-order path property, \( \kappa_d \), does not disappear due to the vanishing second-order speed ratios. Equation 5.9 implies that joint accelerations are set to zero and the control of joint velocities potentially achieves second-order matching of the output-space path. With this constraint, Equation 5.8 reduces to

\[ B_1 n_2^2 + B_2 n_3^2 + B_3 n_2 n_3 + B_4 n_2 + B_5 n_3 + B_6 = 0. \quad (5.10) \]

Equations 5.7 and 5.10 can now be solved for the two first-order speed ratios.

Clearly, the redundancy of the mechanism is being used to match higher-order path properties with lower-order of joint control. For redundant mechanisms, assuming that the pose is non-singular, there will exist a family of joint-velocity solutions that achieve first-order tracking. A unique solution from this family is being chosen based on higher-order geometric information of the desired path. Equation 5.8 encodes the second-order path geometry, and the condition 5.9 limits joint coordination to the first order.
Since, from Equation 5.5, the second-order coordination equation is linear in the desired curvature \( \kappa_d \), a univariate polynomial can be obtained from Equations 5.7 and 5.10.

\[
\Omega(n_2) := (a_1 \kappa_d + b_1)n_2^2 + (a_2 \kappa_d + b_2)n_2 + (a_3 \kappa_d + b_3) = 0, \tag{5.11}
\]

where the coefficients \( a_i \) and \( b_i \) are derived from \( A_i \) and \( B_i \), so they are also functions of the mechanism geometry, its pose and the desired tangent. The curvature \( \kappa_d \) can be exactly matched if Equation 5.11 has real roots, the condition being

\[
\Delta := (a_2^2 - 4a_1a_3) \kappa_d^2 + (2a_2b_2 - 4a_1b_3 - 4b_1a_3) \kappa_d + (b_2^2 - 4b_1b_3) \geq 0. \tag{5.12}
\]

Explicit calculations show that the term \( a_2^2 - 4a_1a_3 \) in Equation 5.12 is zero for planar, three-DOF mechanisms of all morphologies involving only revolute and prismatic joints. Appendix D outlines these calculations. Therefore,

\[
\Delta = \kappa_d C + D, \tag{5.13}
\]

where \( C = 2a_2b_2 - 4a_1b_3 - 4b_1a_3 \) and \( D = b_2^2 - 4b_1b_3 \). Note that \( \kappa_d \) is positive by definition. Therefore, if \( C \) and \( D \) are both positive, \( \Delta \) is also positive for any \( \kappa_d \). In this case, two solutions are obtained from Equations 5.11 and 5.7. The rate of change in curvature of the desired path can be used to choose from these two solutions, as illustrated in the numerical example in the following section. If \( \Delta < 0 \), the curvature cannot be matched exactly. In this case, the tracked curvature is treated as a variable, denoted by \( \kappa \). The discriminant in Equation 5.13 is now a function of \( \kappa \), and a value \( \kappa = \kappa_t \) can be obtained from the condition in Equation 5.12 such that Equation 5.11 yields real roots for \( n_2 \). If \( C \) and \( D \) are not both negative, the solution to the equation \( \Delta(\kappa) = 0 \) gives a positive generated curvature \( \kappa_t = -\frac{D}{C} \), such that the error \( |\kappa_t - \kappa_d| \) is minimum. The minimality condition is ensured by the continuity and monotonicity of \( \Delta(\kappa) \).
When $C, D < 0$, the range of $\kappa$ for which the condition in Equation 5.12 is satisfied is given by $-\frac{D}{C} \geq \kappa$. Only negative values for $\kappa$ are possible, indicating that the mechanism can only move along the desired tangent such that the normal vector of the generated path is in the opposite direction of the normal of the desired path. In this case, the generated path with the \textit{smallest curvature magnitude} (the path with the greatest radius) will be the most accurate. Therefore, the smallest negative $\kappa_t = -\frac{D}{C}$ that satisfies the condition in Equation 5.12 is the optimal solution.

In conclusion, with first-order joint coordination, a three-DOF mechanism can track a desired path with curvature $\kappa_d$ if the quantity $\Delta$ in Equation 5.13 is greater than zero, and the speed ratios can be obtained from Equations 5.11 and 5.7. If $\Delta < 0$, the \textit{best possible} generated path has curvature $\kappa_t = -\frac{D}{C}$. This solution can be substituted into Equation 5.11 to obtain $n_2$, following which, Equation 5.7 yields the value of $n_3$.

The three-prismatic manipulator is an exception to the above scheme. For this manipulator, $C$ and $D$ are zero, and the function $\Omega$ in Equation 5.11 vanishes. This is a perhaps intuitive result indicating that the EE cannot move along curved paths with constant joint velocities for a three-DOF Cartesian robot, and therefore, path curvature cannot be tracked with first-order joint control.

5.2.3 Examples

A 3-revolute (3R) mechanism is used to demonstrate the technique developed in the previous section. The mechanism and its parameters are shown in Figure 5.1. The mechanism has link lengths $l_i$, and $\theta_{i0}$ denote the initial values of the joint variables that define the zero position. Joint angle $\theta_{i0}$ is measured counterclockwise from link $l_{i-1}$ to link $l_i$. The joint angle $\theta_{10}$ is measured with respect to a fixed reference axis. Two examples are
provided, one with a positive value for the discriminant $\Delta$, and the second with a negative value for $\Delta$.

**Example 1: Positive $\Delta$**

The link lengths are chosen as $l_1 = 1.15$, $l_2 = 1.5$, and $l_3 = 1.8$ in arbitrary length units. The initial pose, or the zero position, of the mechanism is defined by the joint-angle values $\theta_{10} = -14^\circ$, $\theta_{20} = -10^\circ$, and $\theta_{30} = -205^\circ$. The desired path is a circle passing through the controlled point on the EE, point $P$. The desired tangent is $\hat{T} = [0.8944 \ 0.4472 \ 0]^T$, and $\kappa_d$ is 1. Higher derivatives of the curvature are zero. Particularly, $\kappa_s = 0$. The joint variable $\theta_1$ is chosen as the leading joint. The system parameters yield the constants $A_i$, $a_i$
and \( b_i \) as
\[
\begin{bmatrix}
A_1 \\
A_2 \\
A_3
\end{bmatrix} =
\begin{bmatrix}
-0.5041 \\
0.4487 \\
-1.3777
\end{bmatrix},
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} =
\begin{bmatrix}
6.4670 \\
26.3920 \\
26.9267
\end{bmatrix},
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} =
\begin{bmatrix}
6.7015 \\
27.8167 \\
26.9752
\end{bmatrix}.
\]

Therefore, Equations 5.7 and 5.11 can be used to give two solutions for the speed ratios as
\[
\begin{bmatrix}
n_2 \\
n_3
\end{bmatrix} =
\begin{bmatrix}
-0.0343 \\
3.0319
\end{bmatrix},
\begin{bmatrix}
n_2 \\
n_3
\end{bmatrix} =
\begin{bmatrix}
-6.0402 \\
-3.7154
\end{bmatrix}.
\]

(5.14)

To choose from the two solutions in Equation 5.14, the rate of curvature change of the generated path with each of the above solutions is computed using the third-order coordination equation as
\[
\left. \frac{d\kappa}{ds} \right|_1 = -0.8262,
\left. \frac{d\kappa}{ds} \right|_2 = 0.4414.
\]

Since the second solution gives a value for \( \kappa_{s0} \) closer to the desired value, it is chosen for path tracking.

The speed ratios are also obtained by computing the Jacobian pseudoinverse. The first-order speed ratios are obtained as the ratios of the joint velocities obtained from the Jacobian pseudoinverse. This solution is
\[
\begin{bmatrix}
n_2 \\
n_3
\end{bmatrix} =
\begin{bmatrix}
2.9052 \\
6.3343
\end{bmatrix}.
\]

(5.15)

Note that the norm of the pseudoinverse solution will be lower than the norm of the geometric solution once the EE-velocity magnitudes for both solutions are made equal by proper choice of the leading joint velocity. In Figure 5.2, the EE paths generated by implementing the chosen geometric solution and the pseudoinverse solution are plotted until the position error, defined as the minimum distance of the EE from the desired path, reaches 0.025. Clearly, superior tracking accuracy is achieved by implementing speed-ratio control. The EE path obtained from the geometric solution follows the desired path more closely and stays close to the desired path for a longer portion of the desired path compared to the pseudoinverse solution.
Figure 5.2: A planar 3R mechanism is required to track the circular desired path indicated by the solid circle passing through point $P$. The geometric and the pseudoinverse tracking solutions are shown. The curvature center of only the geometric solution matches that of the desired path. In all the figures of this chapter demonstrating results, the tracking solution is plotted until the position error reaches a predetermined value. The position error is defined as the minimum distance between the current position of the EE from the desired path.
Example 2: Negative $\Delta$

The link lengths of the mechanism are the same as in the previous example. The initial pose, or the zero position, is defined by the joint angles $\theta_{10} = -14^\circ$, $\theta_{20} = -40^\circ$, and $\theta_{30} = -205^\circ$. The desired circular path again passes through point $P$ with the desired tangent $\hat{T} = [0.9439 \ 0.3304 \ 0]^T$ and radius of curvature 1. As for the previous example, $\kappa_s = 0$. Joint variable $\theta_1$ is again selected as the leading joint, and the constants $A_i$, $a_i$ and $b_i$ are obtained as

$$
\begin{bmatrix}
A_1 \\
A_2 \\
A_3 
\end{bmatrix} = 
\begin{bmatrix}
-0.6908 \\
-0.2595 \\
-1.6521 
\end{bmatrix},
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 
\end{bmatrix} = 
\begin{bmatrix}
19.3305 \\
102.2244 \\
135.1469 
\end{bmatrix},
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 
\end{bmatrix} = 
\begin{bmatrix}
3.4823 \\
28.8917 \\
49.2146 
\end{bmatrix}.
$$

(5.16)

This gives $C = -219.0375$, $D = 149.219$, and $\Delta = -69.8184$, indicating that the desired radius cannot be exactly matched. The tracked curvature is $\kappa_t = -D/C = 0.6812$, and the corresponding error is $|\kappa_d - \kappa_t| = 0.3188$. The geometric and the pseudoinverse-based speed ratios are

$$
\begin{bmatrix}
n_2 \\
n_3 
\end{bmatrix} = 
\begin{bmatrix}
-2.1033 \\
-0.7671 
\end{bmatrix},
\begin{bmatrix}
n_2 \\
n_3 
\end{bmatrix} = 
\begin{bmatrix}
-0.7398 \\
-4.3965 
\end{bmatrix},
$$

respectively. Figure 5.3 plots the paths generated by the geometric solution and the pseudoinverse solution until the position error reaches 0.025. The curvature centers of neither solution match the desired curvature center. However, the curvature center obtained from the geometric solution is closer to the desired curvature center. Further, with first-order coordination, the curvature center of the generated path cannot be any closer to the desired curvature center given the tangent direction and the mechanism’s pose. Therefore, the path obtained from the geometric solution is more accurate than that obtained from the pseudoinverse solution and also the most locally accurate path that can be achieved with first-order coordination.
Figure 5.3: A planar 3R mechanism is required to track the circular desired path indicated by the solid circle passing through point $P$. The geometric and the pseudoinverse tracking solutions are shown. The curvature center of the geometric solution is closer to that of the desired path.
5.2.4 Four-DOF planar systems

This section describes the tracking problem for the four-DOF planar system. For three-DOF systems, the single degree of redundancy was resolved by including in the problem formulation the second-order geometry of the output-space path. For four-DOF systems, the second- and third-order geometry of the path are included to resolve the two degrees of redundancy. A family of solutions to solve the second-order tracking problem is obtained along with bounds on the solution set. The third-order coordination problem is then solved numerically.

For a general four-DOF planar mechanism, let \( \vec{\theta} = [\theta_1 \ \theta_2 \ \theta_3 \ \theta_4]^T \) be the joint-space vector, and let \( \theta_1 \) be the leading joint. There will be three ratios of each order: \( n_i, n_i^\text{(2)}, n_i^\text{(3)} \), for \( i = 2, 3, 4 \). The first three coordination equations will be used to solve for the speed ratios. For the four-DOF system, Equation 5.4 yields a linear equation in the first-order ratios. Equation 5.5 yields one equation that is quadratic in the first-order ratios and linear in the second-order ratios. Similarly, Equation 5.6 yields one equation that is linear in the third-order ratios, quadratic in the second-order ratios, and cubic in the first-order ratios. However, to eliminate the radical in the coefficient of \( \hat{B} \), Equation 5.6 is rearranged so as to isolate the radical and then squared. This yields a polynomial equation of order six in the first-order speed ratios.

As for the three-DOF case, joint coordination is limited to the first order by imposing the conditions

\[
 n_i^\text{(2)} = n_i^\text{(3)} = 0, \quad i = 2, 3, 4. \quad (5.17)
\]
With these conditions, Equations 5.4-5.6 yield a system of three polynomial equations in the three first-order speed ratios as

\[ f_1(a_i, n_2, n_3, n_4) = 0, \quad (5.18) \]
\[ f_2(b_i, n_2, n_3, n_4) = 0, \quad (5.19) \]
\[ f_3(c_i, n_2, n_3, n_4) = 0, \quad (5.20) \]

where the functions \( f_1, f_2 \) and \( f_3 \) are polynomials of degree one, two and six, respectively. The speed ratios are the variables, and \( a_i, b_i, \) and \( c_i \) are coefficients which are functions of the mechanism’s geometry and pose and the geometry of the desired path. Note that even with the conditions in Equation 5.17, Equations 5.18-5.20 encode up to the third-order geometry (\( \kappa_{sd} \)) of the desired path. If real solutions are obtained for this system, first-order joint coordination will match the geometry of the desired path up to the third order. In other words, the redundancy in the system is resolved by incorporating two tracking accuracy constraints in the inverse kinematics problem.

It is infeasible to obtain a univariate polynomial from Equations 5.18-5.20 because of its large size. Therefore, analytical solutions to this system are not developed here. Although the entire system can be solved numerically, the approach adopted here is to solve Equations 5.18 and 5.19 to obtain a feasible solution set\(^\text{13}\) to the entire system. This can be done analytically, as shown below. A unique solution is chosen from this set that satisfies Equation 5.20 using numerical techniques. This way, the computational complexity of the problem is reduced. This approach corresponds to the geometric-tracking paradigm, wherein geometric properties of the desired and generated paths are matched in a strict ascending order. This means that higher-order geometric properties are matched only when

\(^{13}\)A system of polynomial equations that is full rank always has solutions over the complex domain. However, in kinematics, the complex solutions are of little practical utility. Therefore, the effort here is concentrated on obtaining real solutions.
lower-order matching has been achieved. Therefore, the tangent direction must be matched before curvature can be matched, and so on. Since Equation 5.18 is linear and always has real solutions (for a non-singular pose of the mechanism), the first two equations can always be analyzed together. Similarly, curvature must be matched before rate of curvature change can be matched, and conversely, if the curvature cannot be matched, then rate of curvature change cannot be matched either. This is identical to saying that if real solutions to Equations 5.18 and 5.19 do not exist, then real solutions to the system of Equations 5.18, 5.19 and 5.20 do not exist either. Furthermore, existence of real solutions to Equations 5.18 and 5.19 does not ensure the existence of real solutions to all three equations. In this case, one solution from the non-empty solution set obtained from the first two equations is chosen such that the error in the rate of curvature change is minimal. If the solution set obtained from the first two equations is empty, indicating that the curvature cannot be matched, then a solution that minimizes the error in the curvature is chosen using the techniques developed in Section 5.2.2. Here, the advantage of the second degree of redundancy is lost. This will probably occur when the mechanism is in a singular pose, although this needs to be verified.

Let

$$f_1 := a_1 n_2 + a_2 n_3 + a_3 n_4 + a_4 = 0, \quad (5.21)$$

$$f_2 := b_1 n_2^2 + b_2 n_3^2 + b_3 n_4^2 + b_4 n_2 n_3 + b_5 n_2 n_4 + b_6 n_3 n_4 + b_7 n_2 + b_8 n_3 + b_9 n_4 + b_{10} = 0. \quad (5.22)$$

Then, the speed ratio $n_2$ can be eliminated from these polynomial equations to yield

$$F_1 n_3^2 + F_2 n_3 + F_3 = 0, \quad (5.23)$$
where

\[ F_1 = a_1^2 b_2 - a_1 b_4 a_2 + b_1 a_2^2, \]  

\[ F_2 = n_4 \left( a_1^2 b_6 - a_1 b_4 a_3 - a_1 b_5 a_2 + 2 b_1 a_2 a_3 \right) \]
\[ + a_1^2 b_8 - a_1 b_4 a_4 - a_1 b_7 a_2 + 2 b_1 a_2 a_4, \]  

\[ F_3 = n_4^2 \left( a_1^2 b_3 - a_1 b_5 a_3 + b_1 a_3^2 \right) + n_4 \left( a_1^2 b_9 - a_1 b_5 a_4 - a_1 b_7 a_3 + 2 b_1 a_3 a_4 \right) \]
\[ + a_1^2 b_{10} - a_1 b_7 a_4 + b_1 a_4^2. \]  

A positive or vanishing discriminant of Equation 5.23 ensures the existence of real solutions to Equations 5.18 and 5.19. Further, from Equations 5.24, 5.25, and 5.26, it can be seen that this discriminant will be a quadratic function of the speed ratio \( n_4 \).

\[ S_1 n_4^2 + S_2 n_4 + S_3 \geq 0, \]
\[ \Rightarrow S_1 n_4^2 + S_2 n_4 + S_3 - \phi = 0, \quad \phi \geq 0, \]  

(5.27)
where $\phi$ is a non-negative scalar, and

$$S_1 = 4a_1^2b_6b_1a_2a_3 + 4a_1^3b_2b_5a_3 - 2a_1^2b_4a_3b_5a_2 - 4b_1a_2^2a_1^2b_3 - 2a_1^3b_6b_4a_3$$
$$+ 4a_1^3b_4a_2b_3 - 2a_1^3b_6b_5a_2 - 4a_1^4b_2b_3 + a_1^2b_4^2a_3^2 + a_1^2b_5^2a_2^2$$
$$- 4a_1^2b_2b_1a_3^2 + a_1^4b_6^2, \quad (5.28)$$

$$S_2 = -2a_1^3b_6b_4a_3 - 2a_1^2b_4a_3b_7a_2 - 4a_1^4b_2b_9 + 4a_1^3b_2b_5a_4 + 4a_1^3b_2b_7a_3$$
$$+ 2a_1^2b_5a_2^2b_7 + 4a_1^2b_8b_1a_2a_3 + 2a_1^4b_6b_8 - 2a_1^3b_6b_7a_2 - 4b_1a_2^2a_1^2b_9$$
$$- 8a_1^2b_2b_1a_3a_4 + 4a_1^2b_6b_1a_2a_4 + 2a_1^2b_4^2a_3a_4 - 2a_1^2b_4a_4b_5a_2 - 2a_1^3b_6b_4a_4$$
$$+ 4a_1^3b_4a_2b_9 - 2a_1^3b_8b_5a_2, \quad (5.29)$$

$$S_3 = -2a_1^2b_4a_4b_7a_2 - 4b_1a_2^2a_1^2b_10 + 4a_1^3b_2b_7a_4 + 4a_1^3b_4a_2b_10 - 2a_1^3b_8b_4a_4$$
$$- 2a_1^3b_8b_7a_2 + 4a_1^2b_8b_1a_2a_4 - 4a_1^4b_2b_10 + a_1^2b_4^2a_4^2 + a_1^2b_7^2a_2^2$$
$$+ a_1^4b_8^2 - 4a_1^2b_2b_1a_4^2. \quad (5.30)$$

Equation 5.23 will have real roots if Equation 5.27 has real roots. The condition for the existence of real solutions to Equation 5.27 is

$$\Delta := S_2^2 - 4S_1S_3 + 4S_1\phi \geq 0, \quad \phi \geq 0. \quad (5.31)$$

The second-order coordination Equation 5.5 is linear in the path curvature $\kappa$. (Path curvature $\kappa$ is treated as a variable so that $\kappa_d$ is its specific value for a given problem.) Therefore, $\kappa$ appears in Equation 5.22 in the coefficients $b_i$ as $b_i := p_i\kappa + q_i$. Substituting for the coefficients $b_i$ will yield the following form for Equation 5.31.

$$\Delta = p_1\kappa^4 + p_2\kappa^3 + (q_3\phi + p_3)\kappa^2 + (q_4\phi + p_4)\kappa + (q_5\phi + p_5) \geq 0, \quad (5.32)$$

where the coefficients $p_i$ and $q_i$ are functions of the mechanism geometry and pose and the geometry of the path of point $P$. The discriminant $\Delta = \Delta(\kappa, \phi)$ represents a surface in
the $\Delta - \kappa - \phi$ space. The solution set to the equations $\kappa = \kappa_d$ and $\Delta = \Delta(\kappa, \phi)$ is given by the intersection of this surface with the plane $\kappa = \kappa_d$. This solution set will consist of straight lines since the function $\Delta(\kappa, \phi)$ is linear in $\phi$. A point on this solution set can be mapped into the speed ratio $n_4$ using Equation 5.27. However, for real-valued speed ratios, the projection of the chosen solution point onto the $\Delta - \phi$ plane must lie in the first quadrant of the $\Delta - \phi$ plane according to conditions in Equations 5.27 and 5.31. For a given value $\kappa_d$, Equation 5.32 can be rearranged to give

$$\Delta = m\phi + c,$$

where $m$ is the slope and $c$ is the $\Delta$ intercept of the line, and both these parameters are functions of $p_i$, $q_i$ and $\kappa_d$. There are four possible generic intersection sets, as shown in Figure 5.4. In this figure, the $\kappa$ axis points out of the plane of the paper. Real-valued solutions for the speed ratios exist when one of the following conditions is satisfied:

$$m > 0,$$

$$m < 0 \quad \text{and} \quad c > 0.$$  

In Figure 5.4, Equation 5.33 describes cases one and three, whereas Equation 5.34 describes case two. Case four represents the condition wherein the desired curvature cannot be matched with first-order joint control. For case one, the solution set is bounded below by the point $B_1$ and unbounded above. Similarly, for case two, the solution set is bounded below by point $B_2$ and unbounded above. For case three, the points $B_1$ and $B_2$ provide the upper and lower bounds for the solution set. The coordinates of points $B_1$ and $B_2$ are $(0 \ c)$ and $(-\frac{c}{m} \ 0)$, respectively.

Solutions to the first two coordination equations can be obtained as follows. Given the geometry and the pose of the mechanism and the geometry of the path, $c$ and $m$ are
Figure 5.4: The first two coordination equations are solved to obtain the solution set for the 4-DOF tracking system. For a given value of the tracked curvature $\kappa_d$, real first-order speed ratios that track the instantaneous tangent and curvature of the desired path exist if the intersection of the surface $\Delta(\kappa, \phi)$ and $\kappa = \kappa_d$, when projected onto the $\Delta - \phi$ plane, passes through the first quadrant of the $\Delta - \phi$ plane. The figure shows four generic solution sets based on the sign of the slope $m$ and the $\Delta$-intercept $c$. The solid lines indicate real solutions, and dashed lines indicate complex solutions.
computed. This defines the solution set in Figure 5.4. The first check for the existence of real-valued speed ratios is to see if one of the conditions 5.33 or 5.34 is satisfied. Then, a point in the solution set in Figure 5.4 is chosen, and the corresponding $\phi$ value is substituted into Equation 5.27 to obtain two solutions for $n_4$. Each value of $n_4$ when substituted into Equation 5.23 yields two values for $n_3$, taking the total solution count to four. Finally, the linear first-order coordination equation is used to get $n_2$. Thus, there are four solutions at an arbitrary point in the solution set. Points $B_1$ and $B_2$, however, are special cases that have only two solutions. At point $B_1$, Equation 5.27 yields two distinct roots for $n_4$ since $\Delta > 0$. However, the condition $\phi = 0$ at this point implies that the discriminant of Equation 5.23 is zero, and therefore, a single repeated root for $n_3$ is obtained for each value of $n_4$, giving only two distinct solutions for all the speed ratios. Similarly, at point $B_2$, since $\Delta = 0$, there is a single repeated root for $n_4$, which, when substituted into Equation 5.23, generates two distinct roots for $n_3$. Thus, there are two solutions at point $B_2$. It is important to note that the speed ratios are continuous functions of $\phi$ or $\Delta$. For example, the solution to Equation 5.27 can be expressed as

$$n_4 = \frac{-S_2 \pm \sqrt{S_2^2 - 4S_1S_3 + 4S_1\phi}}{2S_1} = \frac{-S_2 \pm \sqrt{\Delta}}{2S_1}, \quad (5.35)$$

which, when used to solve for the other speed ratios, will also generate functions continuous in $\phi$ or $\Delta$. Consider an arbitrary point in the solution set not in the first quadrant. This point yields complex-valued solution for $n_4$. As this point is translated along the solution curve so that $\phi$ varies from a negative value to zero (for cases one and three) or to $B_2$ (case two), two real-valued solutions for $n_4$ appear. As the solution point is translated along the solution curve further, $\phi$ becomes positive, and each of the two real-valued solutions for $n_4$ split into two solution branches. A similar variation is observed with increasing $\Delta$. 

189
Once the existence of real solutions to the first two coordination equations is ascertained, a search is conducted for a solution within this set that satisfies the third-order coordination equation. The existence of multiple solutions to the first two coordination equations poses some difficulty in executing this search. It is not clear at the onset whether the third-order coordination equation can be satisfied at all or which of the four solution branches potentially satisfies the third-order equation. This is because of the difficulty associated with analyzing Equation 5.20; this is a sixth-order polynomial equation in three unknowns. Therefore, a numerical search along all four solution branches must be conducted. If none of the branches converge to a solution, a solution can be chosen that gives the least error in the rate of curvature change. Another issue is that solutions to the coordination equations 5.18-5.20 may consist of multiple isolated points, as well as larger geometric objects such as curves and surfaces. In order to find the best possible tracking solution, the entire solution set of the polynomial system should be identified, and then this solution set should be searched for a solution that matches geometric path properties of yet higher order. These numerical issues in solving the third-order coordination equation may limit the use of the algorithm described above in real-time applications. Identifying appropriate search techniques to solve the coordination equations of order greater than two is an area that requires further work.

Finally, if neither of the conditions 5.33 and 5.34 are satisfied, there exists no set of real-valued first-order speed ratios that can match the desired curvature of the path of point $P$. This is illustrated in Figure 5.4 as case four. For this case, $\phi$ and $\Delta$ cannot be simultaneously positive. Therefore, even if real solutions for $n_4$ are obtained by choosing $\Delta > 0$, Equation 5.23 will yield complex solutions for $n_3$ since $\phi < 0$. Similar to the development this set is called the algebraic set of the system of polynomial equations. Such sets can be computed using homotopy continuation methods [144].
for three-DOF mechanisms, the problem now is to find the set of first-order speed ratios that generate instantaneous path curvature \( \kappa_t \) such that \( |\kappa_d - \kappa_t| \) is minimum. Since the desired curvature cannot be matched exactly, the question of matching the third-order path geometry does not arise. This solution is obtained by shifting the plane \( \kappa = \kappa_t \) in the \( \Delta - \kappa - \phi \) space along the \( \kappa \) axis. This will continuously vary the slope and the \( \Delta \)-intercept of the solution set for case four in Figure 5.4. The continuity of the function \( \Delta(\kappa, \phi) \) ensures that one of the following five scenarios yields the best possible curvature-tracking speed ratios.

1. The slope \( m \) approaches zero and becomes positive, even as the \( \Delta \)-intercept remains negative. The optimal solution is \( (\phi, \Delta) = (\infty, 0) \).
2. The slope \( m \) and the \( \Delta \)-intercept approach negative \( \infty \) and then become positive. The optimal solution is \( (\phi, \Delta) = (0, \infty) \).
3. The slope \( m \) and the \( \Delta \)-intercept approach zero simultaneously. The optimal solution is the entire \( \phi \) axis.
4. The slope \( m \) approaches negative \( \infty \) and the \( \Delta \)-intercept approaches zero simultaneously. The optimal solution is the entire \( \Delta \) axis.
5. The slope \( m \) remains negative as the \( \Delta \)-intercept becomes zero and then becomes positive. The optimal solution is \( (\phi, \Delta) = (0, 0) \).

At the obtained solution, the generated path curvature is \( \kappa_t \). The first two scenarios yield \( n_4^* = \infty \), implying that the leading joint \( \theta_1 \) has a dwell in its trajectory. The mechanism has effectively three-DOFs, and by using the techniques in Section 5.2.2 and choosing \( \dot{\theta}_1 = 0 \), real-valued speed ratios can be obtained that achieve the best-possible curvature tracking. For cases three and four, a minimization problem can be set up over the corresponding
solution set to find a unique solution that yields the instantaneous value of \( \kappa_{at} \) closest to the desired value \( \kappa_{sd} \). Finally, the unique solution in case five gives a single real-valued solution for the speed ratios that represents the most locally accurate tracking result possible with first-order joint control.

Note that scenarios two and four require that the slope \( m \) approaches infinity. Since the slope is a polynomial function of the curvature \( \kappa \), these cases represent EE paths with vanishing radii. Such small motions may well be impractical for macro-scale robots. Therefore, scenarios one, three, and five are considered as the only possible optimal solutions for tracking path curvature.

### 5.2.5 Examples

A 4R mechanism, depicted in Figure 5.5 is used in this section to illustrate the algorithm outlined in the previous section. The link lengths are denoted by \( l_i \), and the initial pose is described by the angles \( \theta_{i0} \). Appendix D provides expressions for the position vector \( \tau_0 \), its first three derivatives with respect to \( \theta_1 \), and the coefficients \( a_i \) and \( b_i \). The coefficients \( p_1 \) and \( p_2 \) are zero for this mechanism, making the discriminant \( \Delta \) a quadratic function of the curvature \( \kappa \). The remaining coefficients \( p_i \) and \( q_i \) are not provided due to their large size.

For examples one, two, and three, \( m \) and \( c \) are obtained first, and a numerical search for one solution that best satisfies the third-order coordination equation is conducted as follows. First, the following function is constructed from the scalar Equation 5.6.

\[
\Gamma := \left( \zeta + \kappa_{sd} (\vec{\tau}_{\theta_1,0} \cdot \vec{\tau}_{\theta_1,0}) \right)^2 + 3\kappa_d (\vec{\tau}_{\theta_1,\theta_1,0} \cdot \vec{\tau}_{\theta_1,0})^2, \tag{5.36}
\]

where \( \zeta \) is the component of the term \( \vec{r}_{\theta_1,\theta_1,0} \times \vec{T} \) along the binormal vector \( \hat{B} \). The function \( \Gamma \) depends on the speed ratios and, via Equations 5.35, 5.23, and 5.21, on the variable \( \phi \). Therefore, a value \( \phi^* \) is obtained by solving the following constrained optimization

192
problem:

\[
\begin{align*}
\text{minimize} & \quad \Gamma(\phi) \\
\text{subject to} & \quad \phi_{lb} \leq \phi \leq \phi_{ub},
\end{align*}
\]

where \(\phi_{lb}\) and \(\phi_{ub}\) are the lower and upper bounds on \(\phi\) obtained from \(m\) and \(c\), as illustrated in Figure 5.4. Four versions of the function \(\Gamma\) are constructed using the positive and negative discriminants while solving for the speed ratios \(n_4\) and \(n_3\). The minimization procedure is conducted for all four functions. Each search returns a solution, and the best among those is selected. The value \(\phi^*\) is then used to compute the speed ratios.

The convexity of the functions \(\Gamma\) remains to be investigated. The \textit{fmincon} function in MATLAB is used to obtain optimal solutions for this problem. This approach will often yield local optima. The global minimum (with a vanishing objective function value) was obtained for examples one and two below. For example three, the solver yielded a non-zero residual value for the function \(\Gamma\). In this case, it is not possible to say whether the desired rate of curvature change can be accurately tracked or not. Applying appropriate search techniques to solve the coordination equations of order greater than two is an area that requires further work.

Four examples corresponding to the four cases in Figure 5.4 are provided below. For all examples, the link lengths are chosen as \(l_1 = 1.9\), \(l_2 = 1.6\), \(l_3 = 1.2\), and \(l_4 = 1\), in arbitrary length units, and the revolute joint connected to the ground link is chosen as the leading joint. For each example, the first-order speed ratios are also obtained from the Jacobian-pseudoinverse velocity solution. The tracking performance of this solution is plotted along with that of the geometric solution for comparison. Paths generated by point \(P\) in all of the examples are plotted until the position error, defined as the minimum distance of the EE from the desired path, reaches 0.025 units.
Figure 5.5: The planar 4R mechanism is shown. The coordinate frame $XY$ indicates the ground link.

Example 1

This example corresponds to case 1 in Figure 5.4. The initial pose of the mechanism is defined by $\theta_{10} = 60^\circ$, $\theta_{20} = -55^\circ$, $\theta_{30} = -64^\circ$, and $\theta_{40} = -40^\circ$. The desired path is a circle of radius 2 units. The instantaneous tangent to the path is $\hat{T} = [0.59, -0.81, 0]^T$, and $\kappa_{sd} = 0$. The system parameters are substituted into Equations D.71 and D.72 given in Appendix 1 to yield,

$$
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
\end{bmatrix} =
\begin{bmatrix}
  -2.7253 \\
  -1.9045 \\
  -0.7088 \\
  -1.9483 \\
\end{bmatrix},
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
\end{bmatrix} =
\begin{bmatrix}
  4.4402 \\
  1.3321 \\
  -0.2054 \\
  3.1188 \\
\end{bmatrix},
\begin{bmatrix}
  b_6 \\
  b_7 \\
  b_8 \\
  b_9 \\
\end{bmatrix} =
\begin{bmatrix}
  0.5084 \\
  7.7448 \\
  0.2397 \\
  -1.6524 \\
\end{bmatrix}.
$$

This gives $c = 1.8 \times 10^8$ and $m = 258.02$. Therefore, the lower bound on $\phi$ is zero, and there is no upper bound. The speed ratios that satisfy up to the third-order coordination

194
equation and the speed ratios obtained from the pseudoinverse solution, respectively, are

\[
\begin{bmatrix}
  n_2 \\
  n_3 \\
  n_4
\end{bmatrix} = \begin{bmatrix}
  -1.9956 \\
  0.9298 \\
  2.4258
\end{bmatrix}, \quad \begin{bmatrix}
  n_2 \\
  n_3 \\
  n_4
\end{bmatrix} = \begin{bmatrix}
  -0.0789 \\
  -0.7256 \\
  -0.4958
\end{bmatrix}.
\]

Figures 5.6(a) and 5.6(b) show that the path generated by the pseudoinverse solution curves in the opposite direction. Conversely, the geometric solution matches the instantaneous tangent and curvature center. Consequently, the path generated by the geometric solution is locally more accurate. However, although the solution achieves the desired rate of curvature change, this may not be the locally most accurate solution possible with first-order joint control. This is because the optimization used to obtain the speed ratios provides local minima. Therefore, there may exist other solutions that match \( \kappa_{sd} \) and higher order properties as well.

**Example 2**

This example corresponds to case 2 in Figure 5.4. The initial pose of the mechanism is defined by \( \theta_{10} = 60^\circ \), \( \theta_{20} = -55^\circ \), \( \theta_{30} = -64^\circ \), and \( \theta_{40} = -40^\circ \). The desired path is a circle of radius 2 units. The instantaneous tangent to the path is \( \hat{T} = [0.59 \quad -0.81 \quad 0]^T \), and \( \kappa_{sd} = 0 \). These parameters yield

\[
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4
\end{bmatrix} = \begin{bmatrix}
  1.1083 \\
  -0.4033 \\
  -0.5466 \\
  1.6277
\end{bmatrix}, \quad \begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4 \\
  b_5
\end{bmatrix} = \begin{bmatrix}
  5.1944 \\
  2.2497 \\
  0.1626 \\
  5.4087 \\
  1.3896
\end{bmatrix}, \quad \begin{bmatrix}
  b_6 \\
  b_7 \\
  b_8 \\
  b_9 \\
  b_{10}
\end{bmatrix} = \begin{bmatrix}
  2.1638 \\
  20.8730 \\
  12.4056 \\
  3.8827 \\
  17.4610
\end{bmatrix}.
\]

This gives \( c = -7.7 \times 10^3 \) and \( m = 62.9 \). Therefore, the lower bound on \( \phi \) is 122.4, and there is no upper bound. The speed ratios that satisfy up to the third-order coordination equation and the speed ratios obtained from the pseudoinverse solution, respectively, are

\[
\begin{bmatrix}
  n_2 \\
  n_3 \\
  n_4
\end{bmatrix} = \begin{bmatrix}
  -0.8613 \\
  -0.9250 \\
  1.9136
\end{bmatrix}, \quad \begin{bmatrix}
  n_2 \\
  n_3 \\
  n_4
\end{bmatrix} = \begin{bmatrix}
  0.2982 \\
  2.5249 \\
  1.7197
\end{bmatrix}.
\]

195
(a) The path generated using the pseudoinverse solution curves in the opposite direction. The geometric solution matches the desired tangent and curvature.

(b) The close view of the tracking solution shows that the geometric solution is locally more accurate.

Figure 5.6: Tracking results for Case 1 illustrated in Figure 5.4.
Similar to Example 1, Figures 5.7(a) and 5.7(b) show that the geometric solution is more accurate than the pseudoinverse solution. Moreover, the path generated by the geometric solution stays close to the desired path for a longer portion of the desired path than the path generated using the pseudoinverse solution.

**Example 3**

This example corresponds to case 3 in Figure 5.4. The initial pose of the mechanism is defined by \( \theta_{10} = 20^\circ, \theta_{20} = 55^\circ, \theta_{30} = 64^\circ, \) and \( \theta_{40} = 40^\circ \). The desired path is a circle of radius 1 unit. The instantaneous tangent to the path is \( \hat{T} = \begin{bmatrix} -0.5858 & 0.8104 & 0 \end{bmatrix}^T \), and \( \kappa_{sd} = 0 \). These parameters yield

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix} = \begin{bmatrix}
-2.7784 \\
-1.7685 \\
-0.5999 \\
-2.2590
\end{bmatrix},
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix} = \begin{bmatrix}
7.5795 \\
5.3513 \\
1.8001 \\
11.6118
\end{bmatrix},
\begin{bmatrix}
b_6 \\
b_7 \\
b_8 \\
b_9
\end{bmatrix} = \begin{bmatrix}
5.4387 \\
12.8880 \\
5.8535 \\
1.1169
\end{bmatrix},
\begin{bmatrix}
b_5 \\
b_6
\end{bmatrix} = \begin{bmatrix}
12.8880 \\
5.8535
\end{bmatrix}.
\]

This gives \( c = 1.4380 \times 10^5, m = -129 \), a lower bound on \( \phi \) of zero, and a higher bound of \( 1.11 \times 10^3 \). The speed ratios that satisfy the first- and second-order coordination equations and the speed ratios obtained from the pseudoinverse solution, respectively, are

\[
\begin{bmatrix}
n_2 \\
n_3 \\
n_4
\end{bmatrix} = \begin{bmatrix}
-1.1361 \\
0.5530 \\
-0.1342
\end{bmatrix},
\begin{bmatrix}
n_2 \\
n_3 \\
n_4
\end{bmatrix} = \begin{bmatrix}
-0.1678 \\
-0.8280 \\
-0.5475
\end{bmatrix}.
\]

The speed ratios however, do not satisfy the third-order coordination equation. These values yield a residual value of 0.38 in the function \( \Gamma \). Therefore, the geometric solution will match the desired tangent and curvature, but not the rate of curvature change \( \kappa_{sd} \). As expected, Figures 5.8(a) and 5.8(b) show that the path generated by the geometric solution is more accurate than that generated by the pseudoinverse solution.
(a) The curvature center of the path generated using the pseudoinverse solution curves does not match the desired center. The geometric solution matches the desired tangent and curvature.

(b) The close view of the tracking solution shows that the geometric solution is locally more accurate.

Figure 5.7: Tracking results for Case 2 illustrated in Figure 5.4.
(a) The curvature center of the path generated using the pseudoinverse solution does not match the desired center. The geometric solution matches the desired tangent and curvature.

(b) The close view of the tracking solution shows that the geometric solution is locally more accurate.

Figure 5.8: Tracking results for Case 3 illustrated in Figure 5.4.
Example 4

This example corresponds to case 4 in Figure 5.4. The initial pose of the mechanism is defined by $\theta_{10} = -20^\circ$, $\theta_{20} = -70^\circ$, $\theta_{30} = -161^\circ$, and $\theta_{40} = 70^\circ$. The desired path is a circle of radius 0.5 units. The instantaneous tangent to the path is $\hat{T} = [0.7809 \ 0.6247 \ 0]^T$, and $\kappa_{sd} = 0$. These parameters yield

$$
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
\end{bmatrix} = \begin{bmatrix} 1.3656 \\ 0.3661 \\ 0.7698 \\ 0.3774 \\
\end{bmatrix}, \quad \begin{bmatrix} b_1 \\
b_2 \\
b_3 \\
b_4 \\
\end{bmatrix} = \begin{bmatrix} 4.7873 \\ 8.2900 \\ 2.6382 \\ 9.2067 \\
\end{bmatrix}, \quad \begin{bmatrix} b_6 \\
b_7 \\
b_8 \\
b_9 \\
\end{bmatrix} = \begin{bmatrix} 6.9182 \\ 0.8082 \\ -3.7187 \\ -0.3795 \\
\end{bmatrix}.
$$

This gives $c = -419.04$ and $m = -85.32$. Therefore, there are no real-valued speed ratios that can track the path tangent and curvature. The speed ratios that achieve the best possible curvature matching and the speed ratios obtained from the pseudoinverse solution, respectively, are

$$
\begin{bmatrix} n_2 \\
n_3 \\
n_4 \\
\end{bmatrix} = \begin{bmatrix} -0.9523 \\ 0.2992 \\ 1.0567 \\
\end{bmatrix}, \quad \begin{bmatrix} n_2 \\
n_3 \\
n_4 \\
\end{bmatrix} = \begin{bmatrix} 0.1504 \\ -1.2115 \\ -0.1809 \\
\end{bmatrix}.
$$

The origin of the $\Delta - \phi$ plane furnished the optimal curvature-tracking speed ratios. With this geometric solution, the error in curvature tracking is $|\kappa_d - \kappa_t| = 0.5083$. Since path curvature is not accurately tracked, matching the rate of curvature change is irrelevant. Figure 5.9(b) shows that the path generated using the geometric solution is more accurate than that generated by the pseudoinverse solution despite the error in tracking the desired curvature.

5.3 Discussion

The examples provided in Sections 5.2.3 and 5.2.5 amply demonstrate that the geometric solutions produce EE paths that are locally more accurate. Furthermore, the inverse
(a) The curvature center of the path generated using the pseudoinverse solution does not match the desired center. The curvature center furnished by the geometric solution is the best possible using first-order joint control.

(b) The close view of the tracking solution shows that the geometric solution is locally more accurate.

Figure 5.9: Tracking results for Case 4 illustrated in Figure 5.4.
kinematics solution is time invariant, and therefore, the solutions can be scaled for movement time by simply scaling the leading joint temporal motion. The EE path geometry is preserved if the same set of speed ratios are used to implement the time-scaled motion. Time-scaling is explained in more detail in Section 4.1.

The development presented in this chapter is an initial investigation into the application of speed-ratio control to redundant mechanisms. The general framework described in Section 5.2 can be applied to planar mechanisms with higher degrees of redundancy. Furthermore, spatial regional mechanisms need to be investigated. In particular, the arm model used in Chapter 3 utilizes a four-DOF spatial regional structure to generate wrist paths. Thus, the arm has one degree of redundancy. Finally, the technique needs to be applied to planar and spatial redundant mechanisms executing rigid-body guidance. Another direction for development is truncating the joint-coordinating Taylor series at higher orders. This allows the tracking of higher-order path geometry. For example, for a three-DOF mechanism, if the series are truncated at the second order, implying second-order joint control, there will be two first-order and two second-order speed ratios. Potentially, up to the fourth order geometric properties of the output-space path can be tracked using second-order joint control. With greater local accuracy in path tracking, feedback frequency for the tracking system can be reduced. Finally, how the speed-ratio-control paradigm may improve the tracking capabilities of redundant mechanisms in singular poses needs to be studied.

The significant drawback of this approach is the increasing complexity of the polynomial systems that must be solved for obtaining real-valued speed ratios. This was clearly seen for the four-DOF mechanisms. First, the solution set of such systems potentially contain multiple, isolated solutions as well as geometric structures of various dimensions.
Identifying this set usually requires specialized numerical methods [144], and once this is achieved, the real-valued solution subset must be found. A solution from this subset can be utilized for tracking, or a unique solution that best tracks higher-order geometric path properties may be selected. These are computationally intensive tasks, and their complexity increases with the degree of redundancy of the mechanism and the order of joint control.

5.4 Conclusions

The methodology of speed-ratio control is applied to planar, redundant mechanisms. The general framework for resolving redundancy by incorporating output-space tracking accuracy into the inverse kinematics problem is outlined. The strategy is applied to three- and four-DOF planar mechanisms. Closed-form solutions for the speed ratios are obtained for the three-DOF case. For the four-DOF mechanisms, the solution set achieving second-order path tracking is obtained analytically, and the solution for third-order path tracking is obtained numerically from this solution set. The geometric path-tracking solutions are seen to be locally more accurate than Jacobian-pseudoinverse-based tracking solutions. The approach is expected to get computationally intensive for systems with higher degrees of redundancy, for spatial systems, and for higher orders of joint control.
Chapter 6: CONCLUSIONS AND FUTURE WORK

The focus of this dissertation is on validating two hypotheses regarding human motor control. Specifically, this work concentrated on reaching motion performed by healthy individuals, and Chapter 2 describes an experimental study aimed at examining the time invariance hypothesis (TIH). Chapter 3 studies the leading joint hypothesis (LJH). Chapter 4 develops an algorithm to solve the inverse kinematics problem for spatial, serial chains in a fashion required by the TIH. This algorithm provides a computational scheme for implementing the TIH. Additionally, it can be beneficially applied to the control of commercial robots. This scheme is applicable to non-redundant serial chains. Now, the simplest kinematic model for the human arm consisting of three-degree-of-freedom spherical joints for the shoulder and the wrist and a single-degree-of-freedom revolute joint for the elbow provides seven degrees of freedom for the entire structure. To position and orient the hand relative to the thorax, only six degrees of freedom are necessary. The human arm is thus a redundant kinematic structure. Therefore, Chapter 5 begins the rigorous process of extending the methodology developed in Chapter 4 to redundant mechanisms.

God keep me from ever completing anything. This whole book is but a draught - nay, but a draught of a draught. Oh, Time, Strength, Cash, and Patience!

Herman Melville
The TIH and LJH, discussed in Sections 2.1 and 3.1, respectively, were proposed based on empirical evidence from mostly planar arm motion studies. The objectives of the study were to validate these hypotheses for spatial motion. Therefore, the data analysis in Section 2.2 consists of analysis of the wrist motion, and that in Section 3.3 consists of computation of joint torques and quantification of the contribution of muscle and interaction impulses at the elbow and the shoulder joints.

The primary goal of the wrist-motion analysis is to investigate if the wrist-path shape for spatial motion is time-invariant. The presence of time-invariant wrist-path shapes for spatial motion would provide evidence supporting the TIH. However, Section 2.3 shows that wrist paths for a given reaching task tend to get more curved as movement speed decreases or as movement time increases. This relation is not always significant, and its strength depends on the nature of the reaching task as well as the subject. Therefore, wrist-path shape is not time-invariant in general. This result does not support the hypothesis that the central nervous system (CNS) composes the wrist paths as the most fundamental motion plan for reaching motions. The TIH may not be a truly general organizing principle for spatial reaching motion, and this hypothesis needs refinement and further investigation. Two alternate hypotheses are proposed in Section 2.4. According to the first hypothesis, the wrist path is composed by the CNS, but it serves different roles in the overall control scheme under various contexts. It is proposed that for rehearsed movements with low accuracy constraints, like those of the experiment in this dissertation, the CNS does not compensate for ‘small’ deviations from the time-invariant kinematic plan. However, if the deviations are ‘large’, the kinematic objective becomes imperative, and the CNS takes corrective action to adhere more closely to the kinematic plan, as was observed during perturbation experiments conducted by Shadmehr and Mussa-Ivaldi [135]. Of course, the qualifiers ‘small’
and ‘large’ here are themselves context dependent. The second hypothesis suggests that the CNS composes a straight wrist path. However, as movement speed reduces, the influence of gravitational effects increases and causes the wrist to deviate from the straight-line path. The loop of the wrist paths for certain tasks at slow speeds is considered the manifestation of this effect.

The validation of these hypotheses is a major topic of future work. The first hypothesis can be verified by conducting a spatial version of the study of Shadmehr and Mussa-Ivaldi [135] wherein the arm is systematically perturbed while performing spatial reaching movements. The existence of a time-invariant plan would be supported if wrist-path shapes converge with practice. For the second hypothesis, a methodology for quantifying the influence of gravity on the arm and ways to compare this effect to muscular effort must be developed. Movement geometry is a potential confounding factor in this analysis. This is clear from the fact that wrist paths for only some of the tasks show increased curvature in the vertical plane (loop) at slow speeds. This work can potentially be accomplished with the experimental data collected for this dissertation.

Another important area for future work is the systematic study of the relationship between the wrist-path shape and the workspace for spatial motion. The dependence of wrist path shape on the workspace was hinted at in previous work. Atkeson and Hollerbach [11] first pointed out that wrist paths in a vertical plane are curved, unlike the straight wrist paths observed for horizontal-plane arm motion [107]. In the spatial reaching experiments of Adamovich et al. [2], the wrist paths appear significantly curved as well, and Nishikawa et al. [114] report values for the Atkeson-Hollerbach linearity metric ranging from 0.2 (curved path) to 0.06 (highly linear path)\textsuperscript{15}. Similar metric values are observed in Table

\textsuperscript{15}A metric value of 0.2 indicates a departure of the wrist from the straight line connecting the final position of the wrist to its initial position of magnitude 20\% of the length of the straight line.
2.8 in this dissertation. One challenge in this experiment is to design a set of tasks that are reasonably small in number for a subject to perform without fatigue and still manage to represent a significant subset of all possible reaching motions.

The following additional findings emerge from Chapter 2. Section 2.3 reports several regularities in the wrist motion. Wrist paths are planar, and they loop progressively downward / upward as the movement speed reduces for motions performed against / with gravity. Along with the dependence of wrist-path shape on speed and task, these motion characteristics are viewed as consequences of implementing the leading-joint strategy for composing joint torques and implementing joint control. In particular, the loop of the wrist path is a departure from a planned straight-line wrist path because of gravitational effects. This departure from the planned motion may be construed as ‘insignificant’ by the CNS for the context (spatial accuracy requirement) of the experimental task. Now, Section 3.1 mentions that the TIH and LJH model two different aspects of the motor control problem, and the outputs of the inverse-kinematics calculation serve as the inputs for the inverse dynamics calculation. However, further investigations are required to establish this relation between the motion kinematics and dynamics. While Chapter 4 of this dissertation provides a rather detailed computational algorithm that could be used to compute the inverse kinematics in accordance with the TIH, a specific computational mechanism for the inverse dynamic computation needs to be developed. Dounskaiia [32] does provide such a mechanism for executing the leading-joint strategy. However, it needs to be described in more detail, and, in particular, forward dynamic computations are required to establish specific correlations between the movement kinematics and dynamics. This is the recommended next step in furthering this work.
In Section 2.3, wrist-speed profiles for motions lasting more than 3.7 seconds and with average wrist speed less than 0.5 m/s are shown to display multimodal shapes. This is attributed to the difficulty experienced in estimating long time intervals - a well-established psychological phenomenon [69]. The difficulty in estimating the duration of movement translates into selecting the average motion speed. If the subject errs on the lower side, the error may become apparent after crossing the first peak in the wrist-speed profile when he/she realizes that the motion is slowing down, but the target is not sufficiently close. At this point, the subject may append another submovement to the initial submovement, thus adding another peak to the wrist-speed profile.

Chapter 3 successfully extended the LJH to spatial arm motion. In Section 3.2, a consistent and logical set of definitions for the net torque, interaction torque, muscle torque and gravity torque is first developed. Using these definitions, the torque at the shoulder and elbow joints are partitioned, following which, the impulses created by these torque components at each joint for each reaching movement are computed. A modified version of the impulse analysis developed in [34, 128] reveals that the interaction effect at the shoulder joint is small for most of the movements in this study, suggesting that these movements are shoulder-led. Result (R7) indicates that elbow-led motion is observed typically when the shoulder-joint excursion is low. In a third class of movements, it is observed that the interaction effects at both joints are small. This occurs when the movement is slow, since the interaction torques are velocity-dependent. In result (R8), it is hypothesized that joint control for such movements may be entirely decoupled.

These results lead to the following intriguing ideas proposed here for future development. First, it is likely that movements that involve higher interaction effects at multiple joints are more difficult to learn and/or control. Various spatial body movements could
be classified using this criterion, and the methodology for quantifying interaction effects developed in this dissertation could be used for this purpose. Such a classification of movement could have potential applications in sports training as well as developing rehabilitative strategies for the treatment of pathological movement. Second, one could try and find a set of movement trajectories that minimize interaction effects, and investigate if time invariance is a characteristic of these trajectories. If true, it would suggest a new objective criterion for organizing movement and resolving redundancy in the motor apparatus.

Result (R7) in Chapter 3, states that arm movements with small shoulder excursions are elbow-led. Section 3.5 provides a convincing reason for this phenomenon based on the computational procedure developed in Chapter 4. Briefly, the motion of a subordinate joint is slaved to that of the leading joint. For example, the velocity of the subordinate joint is determined by linearly scaling the velocity of the leading joint. The scaling factor depends on the configuration of the arm and the desired path tangent and is called the (first-order) speed ratio. If the desired motion requires vanishingly small movement of the leading joint, then the speed ratio becomes large. Consequently, a small error in the leading joint motion will be amplified at the end-effector by the speed ratio. A simple solution to this problem is to reverse the slaving relation by exchanging the leading and subordinate joints. The speed ratio is now vanishingly small. This instantaneous velocity constraint on the leading-joint motion can be re-expressed as an upper bound on the speed ratio. The constraint also translates simply into a constraint on the net displacement of the leading joint for a given task. Clearly, the shoulder is the preferred leading joint. However, if the net shoulder displacement compared to the net elbow displacement for a task is low, the instantaneous shoulder velocity will be low, and the corresponding speed ratio will be large throughout
the movement. Therefore, the movement will be elbow-led. A mathematical discussion regarding the choice of a leading joint is provided in Section 4.2.6.

It has been noted that for planar motion, antagonistic muscles at the leading joint predominantly produce pronounced reciprocal bursts of activity whose timing is tightly coupled with the timing of peak acceleration and deceleration at the joint [32, 33, 92]. Muscle electro-myographical (EMG) profiles of anterior and posterior deltoids provided in these papers validate this idea for shoulder-led motions. Furthering the work in this dissertation requires validation of the LJH for spatial motion with similar use of muscle EMG activity. Now, Section 3.2 defines the interaction torque at a joint as the torque component at a joint associated with acceleration and velocity at other joints. Although this leads to the only logically consistent set of definitions for net and interaction torques, it also implies that for spatial motion, the net torque at a joint is not proportional to the angular acceleration of the joint. This fact complicates the task of validating the LJH for spatial motion via muscle EMG activities and poses a major challenge for further study.

In Chapter 4, a computational algorithm inspired by the TIH is developed. It provides a means of converting the inputs of the TIH into its outputs, i.e., it describes the inverse-kinematics transformation in the form required by the TIH. The algorithm is applicable to control of industrial robots as well. It solves the inverse kinematics of a serial, spatial robot in two steps. The solution to the geometric tracking subproblem is a joint-space path corresponding to a given a output-space path (defined as a trajectory sans the temporal information). The solution to this subproblem is obtained for an arbitrary order of geometry matching for the first time. The solution to the temporal tracking subproblem provides the time-based joint motions (i.e. joint speeds and accelerations) when the output-space speed and its derivatives are specified.
The developed technique provides four advantages over traditional trajectory tracking approaches. First, time scaling of inverse-kinematic solutions is achieved efficiently. This is rather straightforward. If the same end-effector path needs to be traversed at a different speed, only the temporal subproblem needs to be resolved for the new, time-scaled motion. Second, the algorithm allows greater navigational accuracy in the vicinity of a certain class of singular configurations. This is a consequence of separating the control of the path geometry from the control of the path variable by solving the geometric and temporal subproblems separately. Third, the extension of this algorithm to redundant mechanisms has allowed for the composition of joint-space solutions that yield highly accurate path tracking results, even with low order of joint control. Finally, tracking higher-order path geometry leads to the possibility of reducing the frequency of feedback-based correction.

The implementation of the algorithm requires description of motion with its geometric and temporal aspects separated at both the problem-specification stage and the implementation stage. This requires reworking the task specifications from existing formats, and it increases the computational complexity of the algorithm. The algorithm is also computationally intensive, with the computation of the speed ratios being roughly the most computationally expensive operation. The advantage of singularity navigation also comes at the cost of increased computation, since third-order coordination is required to obtain the joint velocities and accelerations for singular configurations of the robot (see Section 4.2.5). However, the computational effort involved is not forbidding, as simulation times provided in Section 4.3.1 suggest. Furthermore, the matching of higher-order motion properties opens up the possibility of substantially decreasing the feedback frequency. Then, the system evolves as a combination of continuous dynamics (the evolution of the joint
angles) and a sequence of discrete events (feedback-based correction). Constructing control schemes for this hybrid system (see [80], for example) and studying their stability are interesting problems for future investigation.

The shortcoming of the algorithm from the motor-control perspective is that in its present form, it is applicable to non-redundant mechanisms only, whereas, even the simplest model of the human arm is a kinematically redundant structure. Chapter 5 begins to address this drawback by starting the formal process of extending the algorithm to redundant mechanisms. Planar three- and four-degree-of-freedom mechanisms tracking point paths are studied in this chapter. A methodology to effectively utilize the geometric information available in the desired path description to choose joint velocities is developed. In other words, the redundancy in the mechanism is resolved by minimizing the end-effector tracking error. Examples provided in Sections 5.2.3 and 5.2.5 demonstrate that the local tracking accuracy of the solutions obtained with this method are superior to those obtained using unweighted pseudoinverse solutions.

From Section 5.2.4, it is evident that the computations involved in finding tracking-error-minimizing joint velocities are demanding. In fact, closed-form solutions for joint velocities are possible for three-degree-of-freedom mechanisms only. The joint velocities for a four-degree-of-freedom mechanism must be obtained using search algorithms. This complexity will only increase for larger systems. This is a limitation of this approach, which suggests that the method is well suited for offline computation of the inverse-kinematics solution. Nevertheless, the algorithm needs to be developed to a point where a spatial, redundant mechanism is capable of performing spatial rigid-body guidance tasks. This is another area of future work.
In conclusion, the experiment described in Chapter 2 showed that the TIH may not be a truly general organizing principle for reaching motion and needs to be refined and studied further. On the other hand, the LJH was successfully extended to spatial motion with all its subtleties, and a rather small class of arm motions that may be executed using independent joint control was also found. Chapter 4 developed a computational scheme that solves the inverse kinematics of serial chains in a fashion required by the TIH. The application of this algorithm to commercial robots has potential advantages. Chapter 5 extends the algorithm to redundant mechanisms and demonstrates how the mechanism’s redundancy can be exploited to minimize path-tracking error. Both the motor-control and the kinematics aspects of this dissertation can be developed further, and the above paragraphs outline the problems that need to be tackled.
Appendix A: LINEARITY OF COORDINATION EQUATIONS

It is shown that the $p^{th}$-order coordination equation is always linear in the $p^{th}$-order speed ratios. The forward kinematic map can be viewed as a composite function with a vector argument: $\mathbf{r}(\lambda) = \mathbf{r}(\mu(\lambda), \nu(\lambda), \lambda(\lambda))$, where $\mu(\lambda)$ and $\nu(\lambda)$ are the Taylor series given by Equation 4.1. When all of the appropriate derivatives are defined, the generalized Faa-di-Bruno formula is used to obtain the $p^{th}$-derivative of a component $r_m$ ($m = 1, 2, 3$) of $\mathbf{r}(\lambda)$ as [106],

$$
\frac{d^p r_m}{d\lambda^p} = \sum_0^p \sum_1^1 \sum_2 \cdots \sum_p \left( \frac{p!}{\prod_{i=1}^p (i!)^{g_i} \prod_{i=1}^p \prod_{j=1}^2 q_{ip}!} \cdot \partial^f r_m \prod_{i=1}^p \left( \mu^{(i)}(\lambda) \right)^{q_{i1}} \left( \nu^{(i)} \right)^{q_{i2}} \right) 
\times \prod_{i=1}^p \left( \lambda^{(i)} \right)^{q_{ip}},
$$

(A.1)

where the respective sums are over all non-negative integer solutions of the Diophantine equations as follows

$$
\sum_{0} g_1 + 2g_2 + 3g_3 + \ldots +/pg_p = p,
$$

$$
\sum_{1} q_{11} + q_{12} + q_{13} = g_1,
$$

$$
\sum_{2} q_{21} + q_{22} + q_{23} = g_2,
$$

$$
\vdots
$$

$$
\sum_{p} q_{p1} + q_{p2} + q_{p3} = g_p,
$$

(A.2)
and \( p_1, p_2, \) and \( p_3, \) the orders of partial derivatives with respect to \( \lambda, \mu, \) and \( \nu, \) respectively, and \( f, \) the order of the partial derivative, are

\[
p_e = q_{1e} + q_{2e} + \ldots + q_{pe}, \quad e = 1, 2, 3, \quad f = p_1 + p_2 + p_3 = \sum_{i=1}^{p} g_i,
\]

and \( \mu^{(i)} := \frac{\partial \mu}{\partial x_i}. \) Thus, only the last term in Equation A.1 consists of the speed ratios. The condition \( i = p \) gives the highest-order speed ratio. Next, note that the sum \( \Sigma_0 \) in Equation A.2 implies that \( g_p \) can have no solution other than \( g_p = 0 \) or 1. This, along with the sum \( \Sigma_p \) in Equation A.2 implies that only one of the powers \( q_{p1}, q_{p2}, \) and \( q_{p3} \) can equal 1, and the other two must be zero. Therefore, for \( i = p, \) only one of the speed ratios in the last term of Equation A.1 will remain and the others will equal 1. Therefore, the \( p^{th} \)-derivative of \( r_m \) will be linear in the \( p^{th} \)-order speed ratios.
Appendix B: SPEED RATIOS FROM TIME-BASED JOINT MOTIONS

When the $p^{th}$-order time-based solution for the joint motions exists, an alternative way to obtain the speed ratios is by differentiating the Taylor series in Equation 4.1 with respect to time $t$. The differentiation can be carried out using Faa di Bruno’s formula [70] for the $p^{th}$-order differentiation of composite functions. For example, the $p^{th}$-order differentiation of the composite function $\mu(\lambda(t))$ with respect to $t$ is given by

$$\frac{d^p \mu}{dt^p} = \sum g_1! \ldots g_p! \cdot \left(\frac{d\lambda/dt}{1!}\right)^{g_1} \ldots \left(\frac{d^p\lambda/dt^p}{p!}\right)^{g_p},$$

where the sum is over all different solutions in nonnegative integers $g_1, \ldots, g_p$ of $g_1 + 2g_2 + \ldots + pg_p = p$, and $g := g_1 + \ldots + g_p$. Note that $g \leq p$. Further, in the zero position, the term $\frac{d^p \mu}{d\lambda^p}$ is the $g^{th}$-order speed ratio. Also, for $g = p$,

$$g_1 + g_2 + \ldots + g_p = p,$$

$$g_1 + 2g_2 + \ldots + pg_p = p,$$

$$\Rightarrow g_2 + 2g_3 + \ldots + (p-1)g_p = 0,$$

$$\Rightarrow g_i = 0, \text{ for } i = 2, 3, \ldots p, \text{ and } g_1 = p.$$
Therefore, for \( g = p \), the sum in Equation B.1 reduces to \( n^{(p)}(\dot{\lambda})^p \) in the zero position.

Equation B.1 can now be written as

\[
\frac{d^p \mu}{dt^p} \bigg|_0 = \sum_{g_1! \ldots g_p!} \frac{p!}{g_1! \ldots g_p!} \cdot n^{(g)} \cdot \left( \frac{d\lambda}{dt} \right)^{g_1} \cdots \left( \frac{d^p \lambda}{dt^p} \right)^{g_p} + n^{(p)}(\dot{\lambda})^p,
\]

\[
\Rightarrow n^{(p)} = \frac{1}{(\dot{\lambda})^p} \cdot \left\{ \left. \frac{d^p \mu}{dt^p} \right|_0 - \sum_{g_1! \ldots g_p!} \frac{p!}{g_1! \ldots g_p!} \cdot n^{(g)} \cdot \left( \frac{d\lambda}{dt} \right)^{g_1} \cdots \left( \frac{d^p \lambda}{dt^p} \right)^{g_p} \right\},
\]

where \( g \leq p - 1 \). Equation B.2 obtains the \( p^{th} \)-order ratio as a function of the previous \( p - 1 \) ratios and the time derivatives of the joint variables up to the \( p^{th} \) order.
Appendix C: NON-ORDINARY SINGULARITY

A bifurcation or non-ordinary singularity occurs when $J$ is rank deficient and $\hat{T} \in \text{col}[J]$. [77] provides a means for offline planning and analysis of bifurcations. This appendix provides the rudiments of an algorithm for online trajectory tracking through non-ordinary singularities. Detailed study of this issue is a subject for future work. A non-ordinary singularity can be distinguished from an ordinary singularity by constructing the augmented matrix $J^\dagger := [J \hat{T}]$. Then, if $\text{Rank}[J^\dagger] = 3$, the singularity is ordinary. If $\text{Rank}[J^\dagger] = 2$, the singularity is non-ordinary. For non-ordinary singularities, it is possible to track the desired tangent with first-order joint coordination. Therefore, the governing coordination equation should match the direction of $\bar{r}_{x0}$ with $\hat{T}$. However, Equation 4.14 is not useful since it is not possible to construct $J^*$ that is full rank. The first-order speed ratios are obtained from Equation 4.12 with $p = 1$ as follows.

$$J\bar{\pi}_1 = u_1 \hat{T},$$

$$\therefore \hat{T} \times (J\bar{\pi}_1) = u_1 (\hat{T} \times \hat{T}) = 0, \Rightarrow [TJ]\bar{\pi}_1 = \bar{0}, \quad (C.1)$$

where $T$ is the skew-symmetrization of $\hat{T}$. It can be shown that for a non-ordinary singularity, $\text{Rank}[TJ] = 1$. Therefore, Equation C.1 yields one linear equation in the two speed ratios: $n + a_1 k + b_1 = 0$, where $a_1$ and $b_1$ are scalar coefficients determined by $T$ and $J$. 

218
The coefficient $u_1$ is obtained as

$$u_1 = J \left\{ n \begin{bmatrix} \frac{1}{a_1} \\ \frac{b_1}{a_1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -b_1 \\ 1 \end{bmatrix} \right\} \cdot \hat{T}. \tag{C.2}$$

A unique solution for the first-order speed ratios and $u_1$ can be obtained by considering the second-order coordination equation, which, after cross multiplying Equation 4.12 ($p = 2$) with the tangent vector is expressed as

$$\hat{T} \times (J\hat{n}_2) - \kappa u_1^2 \hat{B} = -\hat{T} \times \Phi_1. \tag{C.3}$$

Clearly, all three vectors in Equation C.3 belong to the $\hat{N} - \hat{B}$ plane. The RHS is a known vector. In the LHS, the first term is a vector perpendicular to the the two-dimensional column-space of $J$. Therefore, it has a fixed direction, and its coefficient is a function of the two second-order speed ratios. The second term in the LHS is also a vector with fixed direction and a variable magnitude that depends on the variable $u_1$. Equation C.3 is therefore a planar vector loop equation that can be solved for the coefficients of the vectors in the LHS. The coefficient of the first term defines a single infinity of solutions for the second-order speed ratios, and the coefficient of the second term yields a value for $u_1$. Assuming that $u_1^2 > 0$ (a negative value is not expected; however, this is a subject for future work), $u_1$ is the positive square root since, by definition, $u_1$ is the rate of change of the arc length, and a positive value of $u_1$ indicates that the EE is progressing in the desired direction. This value of $u_1$ provides a unique solution for $n$ via Equation C.2, and then, $k$ is obtained from Equation C.1. An expression for the coefficient $u_2$ can be obtained from Equation 4.12 (with $p = 2$) by taking a dot product with $\hat{T}$. As before, the third-order coordination equation will resolve the redundancy in the second-order solution and simultaneously provide a single infinity of solutions for the third-order speed ratios and the coefficient $u_3$. 

219
In the vicinity of a non-ordinary singularity, all possible matrices $J^*$ will be ill conditioned. The solution methodology described here can be implemented in a suitable neighborhood of the bifurcation.
Appendix D: INSTANTANEOUS TRACKING CAPABILITIES OF THREE- AND FOUR-DOF PLANAR MECHANISMS

Three-DOF planar mechanisms of all morphologies are analyzed here. The goal is to investigate their geometric tracking capabilities. It is shown that for a mechanism of any morphology tracking position alone, except the three-prismatic (3P) mechanism, the redundancy in the system can be resolved by implementing first-order joint coordination and optimal curvature tracking. For each mechanism, expressions for the following quantities are given

- the position vector of the controlled point on the EE, along with the first and second derivatives with respect to the leading joint. All quantities are evaluated in the zero position.

- the coefficients of the linear polynomial obtained from the first-order coordination equation, i.e. $A_i$.

- the coefficients $a_i$ and $b_i$ of the univariate polynomial obtained from the first- and second-order coordination equations.

The joint variables are denoted as $\theta_1$, $\theta_2$ and $\theta_3$ and, unless specified otherwise, $\theta_1$ is chosen as the leading joint. The initial pose of the mechanism is denoted by the joint-parameter
values \( \theta_{10} \), \( \theta_{20} \) and \( \theta_{30} \). \( l_i \) are the link lengths or offsets. ‘C’ and ‘S’ denote the \( \sin \) and the \( \cos \) functions, respectively, and subscripts denote the argument of these functions. For example, \( C_{\theta_{10}\theta_{20}} = \cos(\theta_{10} + \theta_{20}) \), and \( S_{2\theta_{20} - \theta_{30}} = \sin(2\theta_{20} - \theta_{30}) \). The desired tangent is \( \hat{T} = [t_1 \ t_2 \ t_3]^T \), and the desired curvature is \( \kappa_d \).

### D.1 Three-revolute (3R) mechanism

Figure 5.1 shows the 3R mechanism and its parameters. The position vector of the EE and its derivatives are

\[
\tau_0 = \begin{bmatrix}
l_1 C_{\theta_{10}} + l_2 C_{\theta_{10}\theta_{20}} + l_3 C_{\theta_{10}\theta_{20}\theta_{30}} \\
l_1 S_{\theta_{10}} + l_2 S_{\theta_{10}\theta_{20}} + l_3 S_{\theta_{10}\theta_{20}\theta_{30}} \\
0
\end{bmatrix},
\]

\[
\tau_{\theta_{10}} = \begin{bmatrix}
-l_1 S_{\theta_{10}} - l_2 S_{\theta_{10}\theta_{20}} (1 + n_2) - l_3 S_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3) \\
l_1 C_{\theta_{10}} + l_2 C_{\theta_{10}\theta_{20}} (1 + n_2) + l_3 C_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3) \\
0
\end{bmatrix},
\]

\[
\tau_{\theta_{10},\theta_{10}} = \begin{bmatrix}
-l_1 C_{\theta_{10}} - l_2 C_{\theta_{10}\theta_{20}} (1 + n_2)^2 - l_3 C_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3)^2 \\
-l_1 S_{\theta_{10}} - l_2 S_{\theta_{10}\theta_{20}} (1 + n_2)^2 - l_3 S_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3)^2 \\
0
\end{bmatrix}.
\]

The coefficients of Equation 5.7 are

\[
A_1 = -t_1 (l_2 C_{\theta_{10}\theta_{20}} + l_3 C_{\theta_{10}\theta_{20}\theta_{30}}) - t_2 (l_2 S_{\theta_{10}\theta_{20}} + l_3 S_{\theta_{10}\theta_{20}\theta_{30}}),
\]

\[
A_2 = -t_1 l_3 C_{\theta_{10}\theta_{20}\theta_{30}} - t_2 l_3 S_{\theta_{10}\theta_{20}\theta_{30}},
\]

\[
A_3 = -t_1 (l_1 C_{\theta_{10}} + l_2 C_{\theta_{10}\theta_{20}} + l_3 C_{\theta_{10}\theta_{20}\theta_{30}})
- t_2 (l_1 S_{\theta_{10}} + l_2 S_{\theta_{10}\theta_{20}} + l_3 S_{\theta_{10}\theta_{20}\theta_{30}}).
\]
Next, the coefficients of the univariate polynomial in Equation 5.11 are

\[
a_1 = \frac{l_2^3(t_1^2 + t_2^2)S_{\theta_{\phi0}}}{(t_1C_{\theta_{10}\theta_{20}\theta_{00}} + t_2S_{\theta_{10}\theta_{20}\theta_{00}})^2}, \quad (D.1)
\]

\[
a_2 = \frac{2(t_1^2 + t_2^2)S_{\theta_{\phi0}}(l_1S_{\theta_{20}\theta_{00}} + l_2S_{\theta_{00}})}{(t_1C_{\theta_{10}\theta_{20}\theta_{00}} + t_2S_{\theta_{10}\theta_{20}\theta_{00}})^2}, \quad (D.2)
\]

\[
a_3 = \frac{2(t_1^2 + t_2^2)(l_2S_{\theta_{00}} + l_1S_{\theta_{20}\theta_{00}})}{(t_1C_{\theta_{10}\theta_{20}\theta_{00}} + t_2S_{\theta_{10}\theta_{20}\theta_{00}})^2}, \quad (D.3)
\]

\[
b_1 = \frac{l_2}{4l_3(t_2S_{\theta_{10}\theta_{20}\theta_{00}} + t_1C_{\theta_{10}\theta_{20}\theta_{00}})^2} \left[ - 3t_2^2t_1l_3S_{\theta_{20}\theta_{30}\theta_{10}} + t_2^3C_{\theta_{20}\theta_{30}\theta_{10}} \right], \quad (D.4)
\]

\[
b_2 = \frac{l_2}{4l_3(t_2S_{\theta_{10}\theta_{20}\theta_{00}} + t_1C_{\theta_{10}\theta_{20}\theta_{00}})^2} \left[ - t_2^3l_1C_{\theta_{20}\theta_{10}\theta_{30}} + t_2^3l_1C_{\theta_{10}\theta_{30}} + t_2^3l_1S_{\theta_{10}\theta_{30}} \right], \quad (D.5)
\]
\[ b_3 = \frac{1}{4l_3(t_2S_{\theta_{10}\theta_{20}\theta_{30}} + t_1C_{\theta_{10}\theta_{20}\theta_{30}})} \left[ t_2^2l_3C_{\theta_{10}\theta_{20}\theta_{30}} + t_1^3l_3S_{\theta_{10}\theta_{20}\theta_{30}} + t_1^3l_1l_3S_{\theta_{10}\theta_{20}\theta_{30}} + t_1^3l_1l_3C_{\theta_{10}\theta_{20}\theta_{30}} \right. \\
+ 2t_1t_2^2l_3S_{\theta_{10}} - 2t_2^2l_1^2l_3C_{\theta_{10}} - 3t_1t_2^2l_1l_3S_{\theta_{10}\theta_{20}\theta_{30}} + t_1^3l_1^2S_{\theta_{10}\theta_{20}\theta_{30}} \right. \\
+ 2t_1^3l_1^2S_{\theta_{10}\theta_{20}\theta_{30}} - t_1^3l_1^2S_{-\theta_{30}\theta_{10}-\theta_{20}} - 2t_2^2l_1^2t_1C_{\theta_{10}\theta_{20}\theta_{30}} + t_2^2l_1^2C_{-\theta_{30}\theta_{10}-\theta_{20}} \right. \\
- 2t_2t_1^2l_1^2S_{\theta_{10}-\theta_{30}} - 6t_2t_1^2l_2l_1S_{2\theta_{20}\theta_{30}\theta_{30}} + 2l_2t_1^2l_2^2l_1^2S_{\theta_{20}\theta_{30}\theta_{10}} - 6l_2t_1^2l_2^2l_1C_{2\theta_{20}\theta_{30}\theta_{30}} \right. \\
- 2t_2t_1^2l_1^2l_1^2l_1C_{2\theta_{20}\theta_{30}\theta_{30}} - 2t_2t_1^2l_2^2l_1C_{-\theta_{30}\theta_{10}-\theta_{20}} - 2t_2^2l_1^2t_1C_{\theta_{10}\theta_{20}\theta_{30}} + 2l_2t_1^2l_2l_1S_{\theta_{10}\theta_{30}} \right. \\
+ t_2l_1^2l_1l_3C_{2\theta_{30}\theta_{10}\theta_{20}} - t_1^3l_1^2l_3S_{2\theta_{30}\theta_{10}\theta_{20}} - t_1^3l_1l_3S_{2\theta_{30}\theta_{10}\theta_{20}} - 2t_1^3l_1l_3C_{\theta_{10}} + t_1^3l_2^2C_{\theta_{20}\theta_{30}\theta_{30}} \right. \\
+ t_1^3l_2^2C_{\theta_{20}\theta_{30}\theta_{30}} + t_2^2l_1^2t_1S_{\theta_{10}\theta_{20}\theta_{30}} - t_2^2l_1^2t_1S_{-\theta_{30}\theta_{10}-\theta_{20}} - 3t_2^2l_1^2t_1C_{\theta_{20}\theta_{30}\theta_{30}} \right. \\
- 3t_2t_1^2l_1l_3C_{\theta_{30}\theta_{20}\theta_{30}} - 3t_2l_2^2t_1l_3S_{2\theta_{30}\theta_{10}\theta_{20}} - l_2t_2^2l_1l_3S_{2\theta_{30}\theta_{10}\theta_{20}} + 2l_2t_2^2l_1l_3S_{\theta_{10}\theta_{20}} \right. \\
- 3t_2t_2^2l_1l_3C_{\theta_{20}\theta_{30}\theta_{30}} + t_2l_2^2t_1l_3C_{\theta_{20}\theta_{30}\theta_{30}} - 2t_2^2l_1^2l_3C_{\theta_{10}\theta_{20}\theta_{30}} + 2t_1^3l_2^2S_{\theta_{10}\theta_{20}\theta_{30}} \right. \\
+ t_1^3l_2^2S_{\theta_{20}\theta_{30}\theta_{30}} - t_1^3l_2^2S_{-\theta_{30}\theta_{10}-\theta_{20}} + t_2^3l_2^2C_{\theta_{20}\theta_{30}\theta_{30}} + t_1^3l_2^2C_{-\theta_{30}\theta_{10}-\theta_{20}} \right. \\
- 2t_2^3l_1^2l_3C_{\theta_{10}\theta_{20}\theta_{30}} + l_2t_2^3l_1C_{\theta_{20}\theta_{30}\theta_{30}} + l_2t_2^3l_1C_{-\theta_{30}\theta_{10}-\theta_{20}} - 3t_2^3l_1^2S_{\theta_{20}\theta_{30}\theta_{30}} \right. \\
- 3t_2^3l_1^2C_{\theta_{10}\theta_{20}\theta_{30}} - 2t_2^3l_1^2C_{\theta_{10}\theta_{20}\theta_{30}} + t_2^3l_1^2C_{-\theta_{30}\theta_{10}-\theta_{20}} - 3t_2^3l_1^2S_{\theta_{20}\theta_{30}\theta_{30}} \right. \\
- t_2^3l_2^2t_1S_{-\theta_{30}\theta_{10}-\theta_{20}} + 2t_2^3l_2^2t_1S_{\theta_{10}\theta_{20}\theta_{30}} + l_2t_2^3l_1S_{2\theta_{30}\theta_{10}\theta_{20}} - l_2t_2^3l_1S_{2\theta_{30}\theta_{10}\theta_{20}} - l_2t_2^3l_1S_{2\theta_{30}\theta_{10}\theta_{20}} \right. \\
+ 2l_2t_2^3l_1S_{\theta_{10}\theta_{30}} + 2l_2t_2^3l_1C_{\theta_{10}-\theta_{30}} - 2l_2^2l_1^2C_{\theta_{10}\theta_{30}} + 2l_2^2l_1^2S_{\theta_{10}\theta_{30}} \right. \\
+ 2l_2^3l_1C_{\theta_{20}\theta_{30}\theta_{30}} + 2t_1^3l_1l_3S_{\theta_{10}} \right] \right] \]  

(D.6)

From Equations D.1, D.2 and D.3, it can be shown that \( a_2^2 - 4a_1a_3 = 0 \).

### D.2 PRR Mechanism

Figure D.1 shows the PRR mechanism and its parameters. Here, the joint connected to the ground link is the prismatic joint. To make the speed ratios unitless, a characteristic
length\textsuperscript{16} \( L \) is introduced such that

\[
\theta_{1p} := \frac{\theta_1}{L},
\]

where \( \theta_{1p} \) serves as the prismatic joint variable. Note that \( \theta_{1p} = 0 \) in the zero position. The first-order speed ratios are now defined as

\[
n_2 := \frac{d\theta_2}{d\theta_{1p}} \bigg|_0, \quad n_3 := \frac{d\theta_3}{d\theta_{1p}} \bigg|_0.
\]

The joint-coordination equations are given by

\[
\theta_2 = n_2 \theta_{1p} = n_2 \frac{\theta_1}{L},
\]
\[
\theta_3 = n_3 \theta_{1p} = n_3 \frac{\theta_1}{L}.
\]

\textsuperscript{16}It is desirable to define the speed ratios as ratios of dimensionless parameters. Therefore, for mechanisms that have both revolute and prismatic joints, a characteristic length \( L \) corresponding to the prismatic joint is used. For all such mechanisms, it is shown that explicit calculation of \( L \) is not necessary.
With the prismatic joint variable redefined, the position vector of the EE and its derivatives are

\[
\mathbf{r}_0 = \begin{bmatrix}
\theta_{10} + l_2 C_{\theta_{20}} + l_3 C_{\theta_{20}\theta_{30}} \\
l_2 S_{\theta_{20}} + l_3 S_{\theta_{20}\theta_{30}} \\
0
\end{bmatrix},
\]

\[
\mathbf{r}_{\theta_{1p}0} = \begin{bmatrix}
L - l_2 n_2 S_{\theta_{20}} - l_3 S_{\theta_{20}\theta_{30}} (n_2 + n_3) \\
l_2 n_2 C_{\theta_{20}} + l_3 C_{\theta_{20}\theta_{30}} (n_2 + n_3) \\
0
\end{bmatrix},
\]

\[
\mathbf{r}_{\theta_{1p}\theta_{1p}0} = \begin{bmatrix}
l_2 n_2^2 C_{\theta_{20}} - l_3 C_{\theta_{20}\theta_{30}} (n_2 + n_3)^2 \\
-l_2 n_2^2 S_{\theta_{20}} - l_3 S_{\theta_{20}\theta_{30}} (n_2 + n_3)^2 \\
0
\end{bmatrix}.
\]

The coefficients of Equation 5.7 are

\[
A_1 = t_1 (l_2 C_{\theta_{20}} + l_3 C_{\theta_{20}\theta_{30}}) + t_2 (l_2 S_{\theta_{20}} + l_3 S_{\theta_{20}\theta_{30}}), \quad (D.7)
\]

\[
A_2 = l_3 (t_1 C_{\theta_{20}\theta_{30}} + t_2 S_{\theta_{20}\theta_{30}}), \quad (D.8)
\]

\[
A_3 = -t_2 L. \quad (D.9)
\]
Finally, the coefficients in Equation 5.11 are

\[ a_1 = \frac{(t_1^2 + t_2^2)S_{\theta_20}^2}{(t_1 C_{\theta_20 + \theta_30} + t_2 S_{\theta_20 + \theta_30})^2}, \]  

(D.10)

\[ a_2 = \frac{2Ll_2(t_1^2 + t_2^2)S_{\theta_20 \theta_30} C_{\theta_20 \theta_30}}{(t_1 C_{\theta_20 + \theta_30} + t_2 S_{\theta_20 + \theta_30})^2}, \]  

(D.11)

\[ a_3 = \frac{L^2(t_1^2 + t_2^2)C_{\theta_20 \theta_30}^2}{(t_1 C_{\theta_20 + \theta_30} + t_2 S_{\theta_20 + \theta_30})^2}, \]  

(D.12)

\[ b_1 = \frac{l_2}{4l_3(t_1 C_{\theta_20 + \theta_30} + t_2 S_{\theta_20 + \theta_30})^2} \left[ -3t_1t_2^2l_3S_{\theta_20 \theta_30}^3 - 2t_1^2t_3S_{\theta_20} + 2t_1^3l_3S_{\theta_20} ight. 
\[ + t_1^3l_3S_{\theta_20 \theta_30}^3 - t_1^3l_3S_{\theta_20 \theta_30} - 2t_2l_3C_{\theta_20} + t_2^3l_3C_{\theta_20} + t_2^3l_3C_{\theta_20 \theta_30} + t_2^3l_3C_{\theta_20 \theta_30} 
\[ + t_2^3l_2C_{\theta_20 \theta_30} - t_2^3l_2C_{\theta_20 \theta_30} - 2t_2t_1^3l_3C_{\theta_20} - 3t_2^3l_1l_3C_{\theta_20 \theta_30} + t_2^3l_1l_3C_{\theta_20 \theta_30} 
\[ - 3t_2^3l_1l_2C_{\theta_20 \theta_30} + t_2^3l_1^2l_2C_{\theta_20 \theta_30} - t_2^3l_1^2l_2C_{\theta_20 \theta_30} - 3t_2^3l_1^2l_2S_{\theta_20 \theta_30} - t_2^3l_1^2l_2S_{\theta_20 \theta_30} 
\[ + 2t_1^3l_2^2S_{\theta_20 \theta_30} - t_1^3l_2^2S_{\theta_20 \theta_30} - t_1^3l_2^2S_{\theta_20 \theta_30} - t_1^3l_2^2S_{\theta_20 \theta_30} + t_2^3l_1^2l_3S_{\theta_20 \theta_30} 
\[ + t_2^3l_2C_{\theta_20 \theta_30} + 3t_2^3l_2C_{\theta_20 \theta_30} + t_2^3l_2C_{\theta_20 \theta_30} \right], \]  

(D.13)

\[ b_2 = -\frac{2Ll_2(t_1^2S_{\theta_20 \theta_30} C_{\theta_20} - t_2^2S_{\theta_20} C_{\theta_20 \theta_30} - 2t_1t_2C_{\theta_20 \theta_30})}{l_3(t_1 C_{\theta_20 + \theta_30} + t_2 S_{\theta_20 + \theta_30})^2}, \]  

(D.14)

\[ b_3 = -\frac{L^2l_2^2(t_2C_{\theta_20 \theta_30} - t_1S_{\theta_20 \theta_30})}{l_3(t_1 C_{\theta_20 + \theta_30} + t_2 S_{\theta_20 + \theta_30})^2}. \]  

(D.15)

From Equations D.10, D.11, and D.12, it can be shown that \( a_2^2 - 4a_1a_3 = 0 \). Further, note that \( L \) is a factor of coefficients \( a_2 \) and \( b_2 \), \( L^2 \) is a factor of coefficients \( a_3 \) and \( b_3 \), and \( a_1 \) and \( b_1 \) are independent of \( L \). This structure is now used to demonstrate that it is unnecessary to explicitly calculate the characteristic length.
First, observe that the discriminant $\Delta$ has $L^2$ as a factor, and therefore, its sign can be obtained without knowing $L$. When $\Delta > 0$, Equation 5.11 gives two solutions for $n_2$ that are linearly dependent on $L$. Therefore, $n_2^* = Ln_2'$, where $n_2^*$ is a solution of Equation 5.11, and $n_2'$ is a function independent of $L$ (it is not the second-order speed ratio), which can be computed. Next, the above solution, along with the coefficients $A_i$ given by Equations D.7, D.8, and D.9 are substituted into Equation 5.7 to get the ratio $n_3$ as a linear function of $L$. Therefore, $n_3^* = Ln_3'$, where $n_3^*$ is the desired solution, and $n_3'$ is a function independent of $L$, which can be computed. The coordination equations can now be rewritten as

$$\theta_2 = n_2^* \theta_1 p = Ln_2' \left( \frac{\theta_1}{L} \right) = n_2' \theta_1,$$  \hspace{1cm} (D.16)

$$\theta_3 = n_3^* \theta_1 p = Ln_3' \left( \frac{\theta_1}{L} \right) = n_3' \theta_1,$$  \hspace{1cm} (D.17)

which can be used to control the mechanism. The same argument applies when $\Delta < 0$ - the discriminant is forced to vanish. Again, Equation 5.11 provides a single (repeated) solution for $n_2$.

### D.3 RPR Mechanism

Figure D.2 shows the RPR mechanism and its parameters. Here, the joint connected to the ground link is a revolute. The prismatic joint variable $\theta_2$ is chosen as the leading variable. Let $\Psi$ be the constant angle between the link $l_1$ and the line of action of the prismatic joint. As for the PRR mechanism, a characteristic length $L$ is introduced such that

$$\theta_{2p} := \frac{\theta_2}{L},$$

and $\theta_{2p}$ serves as the joint variable. The first-order speed ratios are now defined as

$$n_1 := \left. \frac{d\theta_1}{d\theta_{2p}} \right|_0, \quad n_3 := \left. \frac{d\theta_3}{d\theta_{2p}} \right|_0.$$
The joint-coordination equations are given by

\[ \theta_1 = n_1 \theta_{2p} = n_1 \frac{\theta_2}{L}, \]
\[ \theta_3 = n_3 \theta_{2p} = n_3 \frac{\theta_2}{L}. \]

The position vector of the EE and its derivatives are

\[
\begin{align*}
\vec{r}_0 &= \begin{bmatrix} l_1 C_{\theta_1 \theta_0} + \theta_2 L C_{\theta_1 \theta_0 \psi} + l_3 C_{\theta_1 \theta_0 \psi \theta_3} \\
l_1 S_{\theta_1 \theta_0} + \theta_2 L S_{\theta_1 \theta_0 \psi} + l_3 S_{\theta_1 \theta_0 \psi \theta_3} \\
0 \end{bmatrix}, \\
\vec{r}_{\theta_{2p},0} &= \begin{bmatrix} -l_1 n_1 S_{\theta_1 \theta_0} + L C_{\theta_1 \theta_0 \psi} - n_1 \theta_{20} S_{\theta_1 \theta_0 \psi} - l_3 S_{\theta_1 \theta_0 \psi \theta_3} (n_1 + n_3) \\
l_1 n_1 C_{\theta_1 \theta_0} + L S_{\theta_1 \theta_0 \psi} + n_1 \theta_{20} C_{\theta_1 \theta_0 \psi} + l_3 C_{\theta_1 \theta_0 \psi \theta_3} (n_1 + n_3) \\
0 \end{bmatrix}, \\
\vec{r}_{\theta_{2p} \theta_{2p},0} &= \begin{bmatrix} -l_1 n_1^2 C_{\theta_1 \theta_0} - 2 n_1 L S_{\theta_1 \theta_0 \psi} - n_1^2 \theta_{20} C_{\theta_1 \theta_0 \psi} - l_3 C_{\theta_1 \theta_0 \psi \theta_3} (n_1 + n_3)^2 \\
-l_1 n_1^2 S_{\theta_1 \theta_0} + 2 n_1 L C_{\theta_1 \theta_0 \psi} - n_1^2 \theta_{20} S_{\theta_1 \theta_0 \psi} - l_3 S_{\theta_1 \theta_0 \psi \theta_3} (n_1 + n_3)^2 \\
0 \end{bmatrix}.
\]

Figure D.2: A planar RPR mechanism.
The coefficients in Equation 5.7 are

\[ A_1 = t_1 l_1 C_{\theta_{10}} + t_1 \theta_{20} C_{\theta_{10} \psi} + t_1 l_3 C_{\theta_{10} \psi \theta_{30}} + t_2 l_1 S_{\theta_{10}} + t_2 \theta_{20} S_{\theta_{10} \psi} + t_2 l_3 S_{\theta_{10} \psi \theta_{30}}, \]

\[ A_2 = l_3 (t_1 C_{\theta_{10} \psi \theta_{30}} + t_2 S_{\theta_{10} \psi \theta_{30}}), \]

\[ A_3 = L (t_1 S_{\theta_{10} \psi} - t_2 C_{\theta_{10} \psi}). \]
The coefficients in Equation 5.11 are

\[ a_1 = \frac{(t_1^2 + t_2^2)(l_1 S_{\theta_{30}} + \theta_{20} S_{\theta_{30}})^2}{(t_1 C_{\theta_{10}} \Psi_{\theta_{30}} + t_2 S_{\theta_{10}} \theta_{30})^2}, \]  

(D.18)

\[ a_2 = \frac{L(t_1^2 + t_2^2)(l_1 S_{\Psi} + l_1 S_{2 \theta_{30}} + \theta_{20} S_{2 \theta_{30}})}{(t_1 C_{\theta_{10}} \Psi_{\theta_{30}} + t_2 S_{\theta_{10}} \theta_{30})^2}, \]  

(D.19)

\[ a_3 = \frac{L^2(t_1^2 + t_2^2)C_{\theta_{30}}}{(t_1 C_{\theta_{10}} \Psi_{\theta_{30}} + t_2 S_{\theta_{10}} \theta_{30})^2}, \]  

(D.20)

\[ b_1 = \frac{1}{4l_3(t_1 C_{\theta_{10}} \Psi_{\theta_{30}} + t_2 S_{\theta_{10}} \theta_{30})^2} \left[ -6 t_2 t_1 \theta_{20} C_{3 \theta_{10} \theta_{30} 2 \Psi} - 6 t_1 \theta_{20} t_2 l_1 S_{3 \theta_{10} \theta_{30} 2 \Psi} - t_1^3 \theta_{20}^2 S_{\theta_{30} \theta_{10}} + 2 t_1^3 \theta_{20}^2 S_{\theta_{10} \theta_{30}} + t_1^2 \theta_{20}^2 C_{3 \theta_{10} \theta_{30} 3 \Psi} + 2 \theta_{10} t_2 l_1^3 S_{2 \theta_{30} \theta_{10}} - t_1^2 t_2 l_1 S_{2 \theta_{30} \theta_{10}} - 2 t_1^2 l_1 \theta_{20} S_{3 \theta_{10} \theta_{30}} - t_1^2 l_1 \theta_{20}^2 S_{\theta_{30} \theta_{10}} \Psi_{\theta_{30}} - t_1^2 l_1 \theta_{20}^2 C_{\theta_{10} \theta_{30}} \Psi_{\theta_{30}} \right. \]

(D.21)
From Equations D.18, D.19, and D.20, it can be shown that

\[
b_2 = -\frac{L}{2l_3(t_1 C_{\theta_{10}\Psi_{\theta_{30}}} + t_2 S_{\theta_{10}\Psi_{\theta_{30}}})^2} \left[ -t_2^3 \theta_{20} S_{3\Psi_{30}\theta_{30}} - t_2^3 \theta_{20} C_{-\theta_{30}\theta_{10}} - t_2^3 l_1 S_{\theta_{10}\theta_{30}} \\
+ t_2^3 l_1 C_{\theta_{10}\theta_{30}} + t_2^3 l_1 S_{\theta_{10}\theta_{30}} + t_1^3 \theta_{20} C_{3\Psi_{30}\theta_{10}} + t_2^3 l_1 C_{\theta_{10}\theta_{30}} + t_1^3 \theta_{20} C_{3\Psi_{30}\theta_{10}} \\
+ 2t_2^3 l_3 S_{\theta_{10}\theta_{10}} - t_2^3 l_1 C_{\theta_{10}\theta_{30}} - t_2^3 \theta_{20} S_{-\theta_{30}\theta_{10}} + t_1^3 \theta_{20} C_{\theta_{10}\theta_{30}} - t_2^3 l_1 S_{\theta_{10}\theta_{30}} \\
- t_1^2 t_2 l_1 C_{\theta_{10}\theta_{30}} - t_1 t_2^2 l_1 C_{\theta_{10}\theta_{30}} - t_1 \theta_{20} t_2^2 C_{-\theta_{30}\theta_{10}} - t_2^2 l_1 S_{\theta_{10}\theta_{30}} - t_2 t_2^2 l_1 S_{\theta_{10}\theta_{30}} \\
- t_2 t_1^2 \theta_{20} S_{-\theta_{30}\theta_{10}} - 2t_1^2 t_2 l_3 S_{\theta_{10}\theta_{10}} - 3t_1 \theta_{20} t_2^2 C_{3\Psi_{30}\theta_{10}} + 3t_2 t_2^2 l_1 S_{3\Psi_{30}\theta_{10}} \\
+ t_2^2 t_1 l_1 S_{\theta_{10}\theta_{30}} + t_1^2 t_2 l_3 C_{\theta_{10}\theta_{10}} + t_1 t_2^2 l_3 C_{2\theta_{30}\theta_{10}} - 3t_1^2 t_2 l_3 C_{3\Psi_{30}\theta_{10}} \\
+ 3t_1^2 t_2 l_3 S_{3\Psi_{30}\theta_{10}} + t_2^2 t_2 l_3 S_{3\Psi_{30}\theta_{10}} + t_1^2 t_2 l_1 C_{2\theta_{10}\theta_{30}} \\
- 3t_1^2 t_2 l_1 C_{2\Psi_{30}\theta_{10}} + 3t_2^2 l_1 S_{3\Psi_{30}\theta_{10}} \right].
\]  

(D.22)

\[
b_3 = -\frac{L^2}{4l_3(t_1 C_{\theta_{10}\Psi_{\theta_{30}}} + t_2 S_{\theta_{10}\Psi_{\theta_{30}}})^2} \left[ -3t_1^2 t_2 C_{3\Psi_{30}\theta_{30}} + 2t_1^2 t_2 C_{\theta_{10}\Psi_{30}} + t_1^2 t_2 C_{-\theta_{30}\theta_{10}} \\
- t_1 t_2^2 S_{-\theta_{30}\theta_{10}} - 2t_1 t_2^2 S_{\theta_{10}\Psi_{\theta_{30}}} - 3t_1 t_2^2 S_{3\Psi_{30}\theta_{10}} - 2t_1^3 S_{\theta_{10}\Psi_{\theta_{30}}} - t_1^3 S_{-\theta_{30}\theta_{10}} \\
+ t_1^3 S_{3\Psi_{30}\theta_{10}} + t_1^3 S_{3\Psi_{30}\theta_{10}} + t_2^3 C_{-\theta_{30}\theta_{10}} + 2t_2^3 C_{\theta_{10}\Psi_{\theta_{30}}} \right].
\]  

(D.23)

From Equations D.18, D.19, and D.20, it can be shown that \(a_2^2 - 4a_1 a_3 = 0\). Further, note that the coefficients \(A_i, a_i, \) and \(b_i\) have the same dependence on the characteristic length as that for the PRR mechanism. Therefore, the argument presented in the previous section applies for this case, and it is unnecessary to explicitly calculate \(L\).

### D.4 RRP mechanism

Figure D.3 shows the RRP mechanism and its parameters. Here, the joint connected to the ground link is a revolute, and it is chosen as the leading variable. Let \(\Psi\) be the constant angle between the link \(l_2\) and the line of action of the prismatic joint. Again, a characteristic
length \( L \) is introduced such that
\[
\theta_{3p} := \frac{\theta_3}{L},
\]
and \( \theta_{3p} \) serves as the joint variable. The first-order speed ratios are now defined as
\[
n_2 := \left. \frac{d\theta_2}{d\theta_1} \right|_0, \quad n_3 := \left. \frac{d\theta_{3p}}{d\theta_1} \right|_0.
\]

The joint-coordination equations are given by
\[
\theta_2 = n_2 \theta_1,
\]
\[
\theta_3 = n_3 L \theta_1. \tag{D.24}
\]
The position vector of the EE and its derivatives are

\[
\begin{align*}
\tau_0 &= \begin{bmatrix} l_1 C_{\theta_{10}} + l_2 C_{\theta_{10}\theta_{20}} + \theta_{30} C_{\theta_{10}\theta_{20} \Psi} \\ l_1 S_{\theta_{10}} + l_2 S_{\theta_{10}\theta_{20}} + \theta_{30} S_{\theta_{10}\theta_{20} \Psi} \\ 0 \end{bmatrix}, \\
\tau_{\theta_{10}} &= \begin{bmatrix} -l_1 S_{\theta_{10}} - l_2 S_{\theta_{10}\theta_{20}} (1 + n_2) + L n_3 C_{\theta_{10}\theta_{20} \Psi} - \theta_{30} S_{\theta_{10}\theta_{20} \Psi} (1 + n_2) \\ l_1 C_{\theta_{10}} + l_2 C_{\theta_{10}\theta_{20}} (1 + n_2) + L n_3 S_{\theta_{10}\theta_{20} \Psi} + \theta_{30} C_{\theta_{10}\theta_{20} \Psi} (1 + n_2) \\ 0 \end{bmatrix}, \\
\tau_{\theta_{1\theta_{10}}} &= \begin{bmatrix} -l_1 C_{\theta_{10}} - l_2 C_{\theta_{10}\theta_{20}} (1 + n_2)^2 - 2 L n_3 S_{\theta_{10}\theta_{20} \Psi} (1 + n_2) \ldots \\ -l_1 S_{\theta_{10}} - l_2 S_{\theta_{10}\theta_{20}} (1 + n_2)^2 + 2 L n_3 C_{\theta_{10}\theta_{20} \Psi} (1 + n_2) \ldots \\ \ldots - \theta_{30} C_{\theta_{10}\theta_{20} \Psi} (1 + n_2)^2 \\ \ldots - \theta_{30} S_{\theta_{10}\theta_{20} \Psi} (1 + n_2)^2 \end{bmatrix}
\end{align*}
\]

(D.25)  
(D.26)  
(D.27)

The coefficients of Equation 5.7 are

\[
\begin{align*}
A_1 &= t_1 (l_2 C_{\theta_{10}\theta_{20}} + \theta_{30} C_{\theta_{10}\theta_{20} \Psi}) + t_2 (l_2 S_{\theta_{10}\theta_{20}} + \theta_{30} S_{\theta_{10}\theta_{20} \Psi}), \\
A_2 &= L (t_1 C_{\theta_{10}\theta_{20} \Psi} - t_2 C_{\theta_{10}\theta_{20} \Psi}), \\
A_3 &= t_1 (l_1 C_{\theta_{10}} + l_2 C_{\theta_{10}\theta_{20}} + \theta_{30} C_{\theta_{10}\theta_{20} \Psi}) + t_2 (l_1 S_{\theta_{10}} + l_2 S_{\theta_{10}\theta_{20}} + \theta_{30} S_{\theta_{10}\theta_{20} \Psi}).
\end{align*}
\]

(D.28)  
(D.29)  
(D.30)
The coefficients of Equation 5.11 are

\[
a_1 = \frac{(t_1^2 + t_2^2)(\theta_{30} + l_2 C_\Psi)^2}{(t_1 S_{\theta_{10}\theta_{20}} - t_2 C_{\theta_{10}\theta_{20}})^2},
\]

\[
a_2 = \frac{2(t_1^2 + t_2^2)(l_1 \theta_{30} C_{\theta_{20}} + l_1 l_2 C_{\theta_{20}20} C_\Psi + 2t_2^2 C_\Psi^2 + 2\theta_{30}l_2 C_\Psi + \theta_{30}^2)}{(t_1 S_{\theta_{10}\theta_{20}} - t_2 C_{\theta_{10}\theta_{20}})^2},
\]

\[
a_3 = \frac{4l_1 l_2 C_{\theta_{20}} C_\Psi + t_1^2 C_{\theta_{20}20} + t_2^2 C_{\theta_{20}} + t_2^2 l_2^2 + 2\theta_{30}^2 + 2\theta_{30}l_2 C_\Psi + 4l_1 \theta_{30} C_{\theta_{20}20}}{2(t_1 S_{\theta_{10}\theta_{20}} - t_2 C_{\theta_{10}\theta_{20}})^2},
\]

\[
b_1 = \frac{1}{2(t_1 S_{\theta_{10}\theta_{20}} - t_2 C_{\theta_{10}\theta_{20}})} \left[ t_1^2 \theta_{30} C_{2\theta_{10}2\theta_{20}2\Psi} + 3t_2^2 \theta_{30} - t_2^2 \theta_{30} C_{2\theta_{10}2\theta_{20}2\Psi} 
+ 2t_2 t_1 l_2 S_{\theta_{20}2\theta_{20}2\Psi} + 2t_2 t_1 \theta_{30} S_{\theta_{10}2\theta_{20}2\Psi} + 3t_2^2 \theta_{30} t_1^2 l_2 C_{\theta_{20}2\theta_{20}} 
- t_2^2 l_2 C_{\theta_{20}2\theta_{20}} + 3C_\Psi t_2^2 l_2 + 3C_\Psi t_2^2 l_2 \right],
\]

\[
b_2 = \frac{1}{t_1 S_{\theta_{10}\theta_{20}} - t_2 C_{\theta_{10}\theta_{20}}} \left[ t_1^2 \theta_{30} C_{2\theta_{10}2\theta_{20}2\Psi} + 3t_2^2 \theta_{30} - t_2^2 \theta_{30} C_{2\theta_{10}2\theta_{20}2\Psi} 
+ 2t_2 t_1 l_2 S_{\theta_{20}2\theta_{20}2\Psi} + 2t_2 t_1 \theta_{30} S_{\theta_{10}2\theta_{20}2\Psi} + 3t_2^2 \theta_{30} - l_2^2 C_{\theta_{10}2\theta_{20}} 
+ t_1^2 l_2 C_{\theta_{20}2\theta_{20}} - t_2^2 l_2 C_{\theta_{20}2\theta_{20}} + C_{\theta_{20}} l_1 t_1^2 + C_\Psi l_1 t_2^2 + 3C_\Psi t_1^2 l_2 
+ 3C_\Psi t_2^2 l_2 + 2t_2 t_1 l_1 S_{\theta_{20}2\theta_{20}} + l_2^2 C_{\theta_{10}2\theta_{20}} \right],
\]

\[
b_3 = \frac{1}{2(t_1 S_{\theta_{10}\theta_{20}} - t_2 C_{\theta_{10}\theta_{20}})} \left[ t_1^2 \theta_{30} C_{2\theta_{10}2\theta_{20}2\Psi} + 3t_2^2 \theta_{30} - t_2^2 \theta_{30} C_{2\theta_{10}2\theta_{20}2\Psi} 
+ 2t_2 t_1 l_2 S_{\theta_{20}2\theta_{20}2\Psi} + 2t_2 t_1 \theta_{30} S_{\theta_{10}2\theta_{20}2\Psi} + 3t_2^2 \theta_{30} - l_2^2 C_{\theta_{10}2\theta_{20}} + t_2^2 l_2 C_{\theta_{20}2\theta_{20}} 
- t_2^2 l_2 C_{\theta_{20}2\theta_{20}} + 3C_{\theta_{20}} l_1 t_1^2 + 3C_\Psi l_1 t_2^2 + 3C_\Psi t_1^2 l_2 + 3C_\Psi t_2^2 l_2 
+ 2t_2 t_1 l_1 S_{\theta_{20}2\theta_{20}} + l_1^2 C_{\theta_{10}2\theta_{20}} \right].
\]

From Equations D.31, D.32, and D.33, it can be shown that the function \( a_1^2 - 4a_1a_3 = 0 \). Further, note that the coefficients \( a_i \) and \( b_i \) are independent of the characteristic length \( L \). Therefore, the speed ratio \( n_2 \) can be calculated. Next, the coefficients \( A_i \) from Equations D.28, D.29, and D.30 are substituted into Equation 5.7 to get the ratio \( n_3^4 = \frac{n_3'}{L} \), where \( n_3^4 \) is the desired solution, and \( n_3' \) is a function independent of \( L \), which can be computed.
Substituting this result into Equation D.24 gives

\[ \theta_3 = n_3' \theta_1, \]

which can be used to control the mechanism.

### D.5 PPR mechanism

Figure D.4 shows the PPR mechanism and its parameters. Here, the joint connected to the ground link is a prismatic, and it is chosen as the leading variable. Let \( \Psi \) be the constant angle between the lines of action of the two prismatic joints. Two characteristic lengths, \( L_1 \) and \( L_2 \), corresponding to the two prismatic joints are introduced such that

\[ \theta_{1p} := \frac{\theta_1}{L_1}, \quad \text{and} \quad \theta_{2p} := \frac{\theta_2}{L_2}, \]
and $\theta_{1p}$ and $\theta_{2p}$ serve as the variables for the prismatic joints. The first-order speed ratios are now defined as

$$n_2 := \frac{d\theta_{2p}}{d\theta_{1p}} \bigg|_0, \quad n_3 := \frac{d\theta_{3p}}{d\theta_{1p}} \bigg|_0.$$

The joint-coordination equations are given by

$$\theta_{2p} = n_2 \theta_{1p} \Rightarrow \theta_2 = n_2 \left( \frac{L_2}{L_1} \right) \theta_1, \quad \text{(D.37)}$$

$$\theta_{3p} = n_3 \theta_{1p} \Rightarrow \theta_3 = n_3 \left( \frac{1}{L_1} \right) \theta_1. \quad \text{(D.38)}$$

The position vector of the EE and its derivatives are

$$\bar{r}_0 = \begin{bmatrix} \theta_{10} + \theta_{20} C_{\Psi} + l_3 C_{\Psi \theta_{30}} \\ \theta_{20} S_{\Psi} + l_3 S_{\Psi \theta_{30}} \\ 0 \end{bmatrix},$$

$$\bar{r}_{\theta_{1p}0} = \begin{bmatrix} L_1 + L_2 n_2 C_{\Psi} - l_3 n_3 S_{\Psi \theta_{30}} \\ L_2 n_2 S_{\Psi} + l_3 n_3 C_{\Psi \theta_{30}} \\ 0 \end{bmatrix},$$

$$\bar{r}_{\theta_{1p},\theta_{1p}0} = \begin{bmatrix} -l_3 n_3^2 C_{\Psi \theta_{30}} \\ -l_3 n_3^2 S_{\Psi \theta_{30}} \\ 0 \end{bmatrix}.$$

The coefficients of Equation 5.7 are

$$A_1 = L_2 (t_1 S_{\Psi} - t_2 C_{\Psi}), \quad \text{(D.39)}$$

$$A_2 = l_3 (t_1 C_{\Psi \theta_{30}} t_2 S_{\Psi \theta_{30}}), \quad \text{(D.40)}$$

$$A_3 = -t_2 L_1. \quad \text{(D.41)}$$
The coefficients of Equation 5.11 are

\[ a_1 = \frac{L_2^2(t_1^2 + t_2^2)C_{\theta_{30}}}{(t_1C_{\psi_{30}} + t_2S_{\psi_{30}})^2}, \]  
\[ a_2 = \frac{2L_1L_2(t_1^2 + t_2^2)C_{\psi_{30}}C_{\psi}}{(t_1C_{\psi_{30}} + t_2S_{\psi_{30}})^2}, \]  
\[ a_3 = \frac{L_2^4(t_1^2 + t_2^2)C_{\phi_{30}}^2}{(t_1C_{\psi_{30}} + t_2S_{\psi_{30}})^2}, \]  
\[ b_1 = -\frac{L_2^2}{4l_3(t_1C_{\psi_{30}} + t_2S_{\psi_{30}})^2}, \]  
\[ \frac{t_2^2C_{\theta_{30}}^3\psi + t_2^2C_{\psi - \theta_{30}} + 2t_1^2C_{\psi_{30}} + t_1^3S_{\theta_{30}}^3\psi}{t_1^3S_{\theta_{30}}^3 - 2t_1^3S_{\psi - \theta_{30}} + 2t_1^2C_{\psi - \theta_{30}} + 2t_1^2C_{\psi_{30}} - 3t_2^2C_{\theta_{30}}^3\psi} \]  
\[ b_2 = -\frac{2L_1L_2}{l_3(t_1C_{\psi_{30}} + t_2S_{\psi_{30}})^2}, \]  
\[ -t_2^2S_{\psi_{30}}^3\psi + \frac{t_2^2C_{\psi_{30}}C_{\psi} + t_1^3S_{\theta_{30}}^3\psi}{t_1^2S_{\theta_{30}}^3 - 2t_1^2S_{\psi - \theta_{30}} - t_1^2t_2S_{\psi - \theta_{30}}} \]  
\[ b_3 = -\frac{t_3^2L_2^2(t_2C_{\psi_{30}} - t_1S_{\psi_{30}})}{l_3(t_1C_{\psi_{30}} + t_2S_{\psi_{30}})^2}. \]  

From Equations D.42 - D.47, it can be shown that the function \( a_2^2 - 4a_1a_3 = 0 \) and \( b_2^2 - 4b_1b_3 = 0 \). From the dependence of the coefficients \( A_i, a_i, \) and \( b_i \) in Equations D.39 - D.47 and from Equation 5.11, it can be seen that \( L_1^2L_2^2 \) is a factor of \( \Delta \), and the speed ratios can be expressed as \( n_2^* = \frac{L_1}{L_2}n_2' \) and \( n_3^* = L_1n_3' \), where \( n_2^* \) and \( n_3^* \) are the desired ratios and \( n_2' \) and \( n_3' \) are functions independent of the characteristic lengths \( L_1 \) and \( L_2 \).

Therefore, the sign of \( \Delta \) can be obtained, and \( n_2' \) and \( n_3' \) can be computed. The coordination equations D.37 and D.38 can be rewritten as

\[ \theta_2 = n_2'\theta_1, \quad \theta_3 = n_3'\theta_1, \]

which can be used for controlling the mechanism. As before, explicit calculation of \( L_1 \) and \( L_2 \) is unnecessary.

238
D.6 PRP mechanism

Figure D.5 shows the PRP mechanism and its parameters. Here, the joint connected to the ground link is a prismatic, and it is chosen as the leading joint variable. Let $\Psi$ be the constant angle between the lines of action of the prismatic joint and link $l_2$. Two characteristic lengths, $L_1$ and $L_3$, corresponding to the two prismatic joints are introduced such that

$$\theta_{1p} := \frac{\theta_1}{L_1}, \quad \text{and} \quad \theta_{3p} := \frac{\theta_3}{L_3},$$

and $\theta_{1p}$ and $\theta_{3p}$ serve as the variables for the prismatic joints. The first-order speed ratios are now defined as

$$n_2 := \frac{d\theta_2}{d\theta_{1p}} \big|_0, \quad n_3 := \frac{d\theta_{3p}}{d\theta_{1p}} \big|_0.$$
The joint-coordination equations are given by

\[ \theta_2 = n_2 \theta_1 p \Rightarrow \theta_2 = n_2 \left( \frac{1}{L_1} \right) \theta_1, \quad (D.48) \]

\[ \theta_3 = n_3 \theta_1 p \Rightarrow \theta_3 = n_3 \left( \frac{L_3}{L_1} \right) \theta_1. \quad (D.49) \]

The position vector of the EE and its derivatives are

\[ \tau_0 = \begin{bmatrix} \theta_{10} + l_2 C_{\theta_{20}} + \theta_{30} C_{\theta_{20}} \Phi \\ l_2 S_{\theta_{20}} + \theta_{30} S_{\theta_{20}} \Phi \\ 0 \end{bmatrix}, \]

\[ \tau_{\theta_{1p0}} = \begin{bmatrix} L_1 - l_2 n_2 S_{\theta_{20}} + L_3 n_3 C_{\theta_{20}} \Phi - \theta_{30} n_2 S_{\theta_{20}} \Phi \\ l_2 C_{\theta_{20}} n_2 + L_3 n_3 S_{\theta_{20}} \Phi + \theta_{30} n_2 C_{\theta_{20}} \Phi \\ 0 \end{bmatrix}, \]

\[ \tau_{\theta_{1p1p0}} = \begin{bmatrix} -l_2 n_2^2 C_{\theta_{20}} - 2 L_3 n_2 n_3 S_{\theta_{20}} \Phi - \theta_{30} n_2^2 C_{\theta_{20}} \Phi \\ -l_2 n_2^2 S_{\theta_{20}} + 2 L_3 n_2 n_3 C_{\theta_{20}} \Phi - \theta_{30} n_2^2 S_{\theta_{20}} \Phi \\ 0 \end{bmatrix}. \]

The coefficients in Equation 5.7 are

\[ A_1 = t_1 (l_1 C_{\theta_{20}} + \theta_{30} C_{\theta_{20}} \Phi) + t_2 (l_1 S_{\theta_{20}} + \theta_{30} S_{\theta_{20}} \Phi), \quad (D.50) \]

\[ A_2 = L_3 (t_1 S_{\theta_{20}} \Phi - t_2 C_{\theta_{20}} \Phi), \quad (D.51) \]

\[ A_3 = -t_2 L_1. \quad (D.52) \]

The coefficients in Equation 5.11 are

\[ a_1 = \frac{(t_1^2 + t_2^2)(\theta_{30} + l_2 C_{\Psi})^2}{(t_1 S_{\theta_{20}} \Psi - t_2 C_{\theta_{20}} \Psi)^2}, \quad (D.53) \]

\[ a_2 = \frac{2 L_1 (\theta_{30} S_{\theta_{20}} \Psi + l_2 S_{\theta_{20}} \Phi C_{\Psi})}{(t_1 S_{\theta_{20}} \Psi - t_2 C_{\theta_{20}} \Psi)^2}, \quad (D.54) \]

\[ a_3 = \frac{S_{\theta_{20}} \Psi}{(t_1 S_{\theta_{20}} \Psi - t_2 C_{\theta_{20}} \Psi)^2}, \quad (D.55) \]

\[ b_1 = \frac{1}{2(t_1 S_{\theta_{20}} \Psi - t_2 C_{\theta_{20}} \Psi)} \left[ 3(t_1^2 + t_2^2)(\theta_{30} + l_2 C_{\Psi}) - l_2 t_2^2 C_{\Psi} \theta_{20} + l_2 t_1^2 C_{\Psi} \theta_{20} 
- t_2^2 \theta_{30} C_{\theta_{20}} \Phi + t_1^2 \theta_{30} C_{\theta_{20}} \Phi + 2 t_1 t_2 l_2 S_{\Psi} \theta_{20} + 2 t_2 t_1 \theta_{30} S_{\theta_{20}} \Phi \right], \quad (D.56) \]

\[ b_2 = \frac{-t_2 L_1 (t_1 C_{\theta_{20}} \Psi + t_2 S_{\theta_{20}} \Psi)}{(t_1 S_{\theta_{20}} \Psi - t_2 C_{\theta_{20}} \Psi)}, \quad (D.57) \]

\[ b_3 = 0. \quad (D.58) \]
From Equations D.53, D.54, and D.55, it can be shown that $a_2^2 - 4a_1a_3 = 0$. Substituting $a_3 = \frac{a_2}{4a_1}$ and $b_3 = 0$ into Equation 5.12 for $\Delta$ gives

$$\Delta = 2a_2b_2 - \frac{b_1a_2^2}{a_1},$$

which has $L_1^2$ as a factor, and hence the sign of the determinant can be obtained without knowing $L_1$. Further, from Equations D.50 - D.58, 5.7, and 5.11, the speed ratios can be expressed as $n_2^* = L_1n_2'$ and $n_3^* = \left(\frac{L_1}{L_2}\right)n_3'$, where $n_2^*$ and $n_3^*$ are the desired ratios and $n_2'$ and $n_3'$ are functions independent of the characteristic lengths $L_1$ and $L_3$. Therefore, $n_2'$ and $n_3'$ can be computed. The coordination equations D.48 and D.49 can be rewritten as

$$\theta_2 = n_2'\theta_1, \quad \theta_3 = n_3'\theta_1,$$

which can be used for controlling the mechanism. As before, explicit calculation of $L_1$ and $L_3$ is unnecessary.

### D.7 RPP mechanism

Figure D.6 shows the RPP mechanism and its parameters. Here, the joint connected to the ground link is a revolute. Let $\Psi_2$ be the constant angle between link $l_1$ and the line of action of the first prismatic joint, and let $\Psi_3$ be the angle between the lines of action of the two prismatic joints. Two characteristic lengths, $L_2$ and $L_3$, corresponding to the two prismatic joints are introduced such that

$$\theta_{2p} := \frac{\theta_2}{L_2}, \text{ and } \theta_{3p} := \frac{\theta_3}{L_3},$$

and $\theta_{2p}$ and $\theta_{3p}$ serve as the variables for the prismatic joints. Let $\theta_1$ be the leading joint angle. The speed ratios are defined as

$$n_2 := \frac{d\theta_{2p}}{d\theta_1} \bigg|_0, \quad n_3 := \frac{d\theta_{3p}}{d\theta_1} \bigg|_0.$$
The joint-coordination equations are given by

\[
\theta_{2p} = n_2 \theta_1 \Rightarrow \theta_2 = L_2 n_2 \theta_1, \tag{D.59}
\]

\[
\theta_{3p} = n_3 \theta_1 \Rightarrow \theta_3 = L_3 n_3 \theta_1. \tag{D.60}
\]

The position vector of the EE and its derivatives are

\[
\bar{r}_0 = \begin{bmatrix}
l_1 C_{\theta_10} + \theta_{20} C_{\psi_2 \psi_{10}} + \theta_{30} C_{\psi_3 \psi_{10}} \\
l_1 S_{\theta_10} + \theta_{20} S_{\psi_2 \psi_{10}} + \theta_{30} S_{\psi_3 \psi_{10}} \\
0
\end{bmatrix},
\]

\[
\bar{r}_{\theta_10} = \begin{bmatrix}
-l_1 S_{\theta_10} + L_2 n_2 C_{\theta_{10} \psi_2} - \theta_{20} S_{\theta_{10} \psi_2} + L_3 n_3 C_{\theta_{10} \psi_3 \psi_2} - \theta_{30} S_{\theta_{10} \psi_3 \psi_2} \\
l_1 C_{\theta_10} + L_2 n_2 S_{\theta_{10} \psi_2} + \theta_{20} C_{\theta_{10} \psi_2} + L_3 n_3 S_{\theta_{10} \psi_3 \psi_2} + \theta_{30} C_{\theta_{10} \psi_3 \psi_2} \\
0
\end{bmatrix},
\]

\[
\bar{r}_{\theta_1 \psi_{10}} = \begin{bmatrix}
-l_1 C_{\theta_10} - 2L_2 n_2 S_{\psi_2 \psi_{10}} - \theta_{20} C_{\psi_2 \psi_{10}} - 2L_3 n_3 S_{\psi_3 \psi_2 \psi_{10}} - \theta_{30} C_{\psi_3 \psi_2 \psi_{10}} \\
l_1 S_{\theta_10} + 2L_2 n_2 C_{\psi_2 \psi_{10}} - \theta_{20} S_{\psi_2 \psi_{10}} + 2L_3 n_3 C_{\psi_3 \psi_2 \psi_{10}} - \theta_{30} S_{\psi_3 \psi_2 \psi_{10}} \\
0
\end{bmatrix}.
\]
The coefficients of Equation 5.7 are

\[ A_1 = L_2(t_1 S_{\psi_2 \theta_{10}} - t_2 C_{\psi_2 \theta_{10}}), \]  
(D.61)

\[ A_2 = L_3(t_1 S_{\psi_2 \psi_3 \theta_{10}} - t_2 C_{\psi_2 \psi_3 \theta_{10}}), \]  
(D.62)

\[ A_3 = t_1(l_1 C_{\theta_{10}} + \theta_{20} C_{\psi_2 \theta_{10}} + \theta_{30} C_{\psi_3 \psi_2 \theta_{10}}) + t_2(l_1 S_{\theta_{10}} + \theta_{20} S_{\psi_2 \theta_{10}} + \theta_{30} S_{\psi_3 \psi_2 \theta_{10}}). \]  
(D.63)

The coefficients of Equation 5.11 are

\[ a_1 = \frac{L_2(t_1^2 + t_2^2) S_{\psi_3}^2}{(t_1 S_{\theta_{10} \psi_2 \psi_3} - t_2 C_{\theta_{10} \psi_2 \psi_3})^2}, \]  
(D.64)

\[ a_2 = -\frac{L_2(t_1^2 + t_2^2)(2l_1 C_{\psi_2 \psi_3} S_{\psi_3} + \theta_{20} S_{\psi_3} + 2\theta_{30} S_{\psi_3})}{(t_1 S_{\theta_{10} \psi_2 \psi_3} - t_2 C_{\theta_{10} \psi_2 \psi_3})^2}, \]  
(D.65)

\[ a_3 = -\frac{(t_1^2 + t_2^2)}{2(t_1 S_{\theta_{10} \psi_2 \psi_3} - t_2 C_{\theta_{10} \psi_2 \psi_3})^2} \left[ -\theta_{20}^2 C_{\psi_3} - l_1^2 - l_2^2 C_{\psi_2 \psi_3} - \theta_{20}^2 \
- 4\theta_{30} l_1 C_{\psi_3 \psi_2} - 2\theta_{30}^2 - 4\theta_{30} C_{\psi_3} \theta_{20} - 2\theta_{20} l_1 C_{\psi_2 \psi_3} - 2\theta_{20} l_1 C_{\psi_2} \right], \]  
(D.66)

\[ b_1 = 0, \]  
(D.67)

\[ b_2 = -\frac{2L_2(t_1^2 + t_2^2) S_{\psi_3}}{t_1 S_{\theta_{10} \psi_2 \psi_3} - t_2 C_{\theta_{10} \psi_2 \psi_3}}, \]  
(D.68)

\[ b_3 = \frac{1}{2(t_1 S_{\theta_{10} \psi_2 \psi_3} - t_2 C_{\theta_{10} \psi_2 \psi_3})} \left[ 2t_2 t_1 l_1 S_{2 \theta_{10} \psi_3 \psi_2} + 2t_2 t_1 \theta_{20} S_{2 \theta_{10} \psi_3 \psi_2} \
+ 2t_2 t_1 \theta_{30} S_{2 \psi_2 \psi_2 \theta_{10}} + l_1^2 l_1 C_{2 \theta_{10} \psi_3 \psi_2} + t_1^2 \theta_{20} C_{2 \theta_{10} \psi_2 \theta_{10}} - t_2^2 \theta_{30} C_{2 \psi_2 \psi_3 \theta_{10}} \
+ 3l_1^2 l_1 C_{\psi_3 \psi_2} + 3l_1^2 \theta_{20} C_{\psi_3} + l_1^2 \theta_{30} C_{2 \psi_2 \psi_2 \theta_{10}} + 3l_1^2 C_{\psi_3 \psi_2} l_1 + 3C_{\psi_3} l_1^2 \theta_{20} \
+ 3l_2^2 \theta_{30} + 3l_1^2 \theta_{30} - t_2^2 l_1 C_{2 \theta_{10} \psi_3 \psi_2} - t_2^2 \theta_{20} C_{2 \theta_{10} \psi_3 \psi_2} \right]. \]  
(D.69)

As for previous cases, it can be shown that \( a_3^2 - 4a_1 a_3 = 0 \), and the explicit calculation of the characteristic lengths is not necessary. The coordination equations D.59 and D.60 can be rewritten in terms of functions \( n'_2 \) and \( n'_3 \), independent of the characteristic lengths as

\[ \theta_2 = n'_2 \theta_1, \quad \theta_3 = n'_3 \theta_1, \]

which can be used for controlling the mechanism.
Figure D.7: A planar PPP mechanism.

D.8 PPP mechanism

Figure D.7 shows the PPP mechanism and its parameters. Here, the joint connected to the ground link is a prismatic. Let $\Psi_1$ and $\Psi_2$ be the constant angles between the lines of action of the prismatic joints. The position vector of the EE is given by,

$$\mathbf{r} = \begin{bmatrix} \theta_{10} + \theta_1 + (\theta_{20} + \theta_2)C_{\Psi_1} + (\theta_{30} + \theta_3)C_{\Psi_1,\Psi_2} \\ (\theta_{20} + \theta_2)S_{\Psi_1} + (\theta_{30} + \theta_3)S_{\Psi_1,\Psi_2} \\ 0 \end{bmatrix}. \quad (D.70)$$

Note that no matter which prismatic joint is chosen as the leading joint, the second-order differential of the position vector is $0$. This implies that the curvature of the generated path cannot be controlled with first-order joint coordination. This result is consistent with the intuition that since there is no revolute joint in the mechanism, the rotation of the EE cannot be controlled without higher-order joint coordination.
D.9 Four-revolute mechanism, an example of a four-DOF system

The four-revolute mechanism is analyzed as an example of a four-DOF system. The position vector of the controlled point on the EE and its derivatives with respect to the leading variable, $\theta_1$, are

$$\mathbf{r}_0 = \begin{bmatrix} l_1 C_{\theta_{10}} + l_2 C_{\theta_{10}\theta_{20}} + l_3 C_{\theta_{10}\theta_{20}\theta_{30}} + l_4 C_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} \\ l_1 S_{\theta_{10}} + l_2 S_{\theta_{10}\theta_{20}} + l_3 S_{\theta_{10}\theta_{20}\theta_{30}} + l_4 S_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} \\ 0 \end{bmatrix},$$

$$\mathbf{r}_{\theta_10} = \begin{bmatrix} -l_1 S_{\theta_{10}} - l_2 S_{\theta_{10}\theta_{20}} (1 + n_2) - l_3 S_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3) \ldots \\ l_1 C_{\theta_{10}} + l_2 C_{\theta_{10}\theta_{20}} (1 + n_2) + l_3 C_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3) \ldots \\ 0 \end{bmatrix},$$

$$\mathbf{r}_{\theta_1\theta_10} = \begin{bmatrix} -l_1 C_{\theta_{10}} - l_2 C_{\theta_{10}\theta_{20}} (1 + n_2)^2 - l_3 C_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3)^2 \ldots \\ -l_1 S_{\theta_{10}} - l_2 S_{\theta_{10}\theta_{20}} (1 + n_2)^2 - l_3 S_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3)^2 \ldots \\ 0 \end{bmatrix},$$

$$\mathbf{r}_{\theta_1\theta_1\theta_10} = \begin{bmatrix} l_1 S_{\theta_{10}} + l_2 S_{\theta_{10}\theta_{20}} (1 + n_2)^3 + l_3 S_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3)^3 \ldots \\ -l_1 C_{\theta_{10}} - l_2 C_{\theta_{10}\theta_{20}} (1 + n_2)^3 - l_3 C_{\theta_{10}\theta_{20}\theta_{30}} (1 + n_2 + n_3)^3 \ldots \\ 0 \end{bmatrix}. $$
The coefficients $a_i$ of Equation 5.18 are

$$a_1 = -t_1 \left( l_2 C_{\theta_{10}\theta_{20}} + l_3 C_{\theta_{10}\theta_{20}\theta_{30}} + l_4 C_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} \right)$$
$$\quad - t_2 \left( l_2 S_{\theta_{10}\theta_{20}} + l_3 S_{\theta_{10}\theta_{20}\theta_{30}} + l_4 S_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} \right),$$

$$a_2 = -t_1 \left( l_3 C_{\theta_{10}\theta_{20}\theta_{30}} + l_4 C_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} \right) - t_2 \left( l_3 S_{\theta_{10}\theta_{20}\theta_{30}} + l_4 S_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} \right),$$

$$a_3 = -l_4 \left( t_2 S_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} + t_1 l_4 C_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} \right),$$

$$a_4 = -t_1 \left( l_1 C_{\theta_{10}} + l_2 C_{\theta_{10}\theta_{20}} + l_3 C_{\theta_{10}\theta_{20}\theta_{30}} + l_4 C_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} \right)$$
$$\quad - t_2 \left( l_1 S_{\theta_{10}} + l_2 S_{\theta_{10}\theta_{20}} + l_3 S_{\theta_{10}\theta_{20}\theta_{30}} + l_4 S_{\theta_{10}\theta_{20}\theta_{30}\theta_{40}} \right).$$  \hspace{1cm} (D.71)
The coefficients $b_i$ of Equation 5.19 are,

\[ b_1 = 2\kappa l_3 C_{\theta \rho_{30}} + 2\kappa l_2 l_4 C_{\theta \rho_{30} \theta_{40}} + 2\kappa l_3 l_4 C_{\theta_{40}} + t_1 l_3 S_{\theta_{10} \theta_{20} \theta_{30}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} - t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} - t_2 l_3 C_{\theta_{10} \theta_{20} \theta_{30}} + t_1 l_2 S_{\theta_{10} \theta_{20}} - t_2 l_2 C_{\theta_{10} \theta_{20}} + \kappa \left( l_3^2 + l_3^2 + l_4^2 \right), \]

\[ b_2 = -t_2 l_3 C_{\theta_{10} \theta_{20} \theta_{30}} - t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + t_1 l_3 S_{\theta_{10} \theta_{20} \theta_{30}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + 2\kappa l_3 l_4 C_{\theta_{40}} + \kappa \left( \frac{l_3^2}{l_4^2} + l_4^2 \right), \]

\[ b_3 = -t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + \kappa l_4^2, \]

\[ b_4 = -2 \left[ t_2 l_3 C_{\theta_{10} \theta_{20} \theta_{30}} - t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + t_1 l_3 S_{\theta_{10} \theta_{20} \theta_{30}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + \kappa \left( l_3^2 + l_4^2 \right) + \kappa l_2 l_3 C_{\theta_{30}} + \kappa l_2 l_4 C_{\theta_{30} \theta_{40}} + 2\kappa l_3 l_4 C_{\theta_{40}} \right], \]

\[ b_5 = -2 \left[ t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + \kappa l_4^2 + \kappa l_2 l_4 C_{\theta_{30} \theta_{40}} + \kappa l_3 l_4 C_{\theta_{40}} \right], \]

\[ b_6 = -2 \left[ t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + \kappa l_4^2 + \kappa l_3 l_4 C_{\theta_{40}} \right], \]

\[ b_7 = 2 \left[ 2kl_2 l_3 C_{\theta_{30}} + 2kl_2 l_4 C_{\theta_{30} \theta_{40}} + 2kl_3 l_4 C_{\theta_{40}} + t_1 l_3 S_{\theta_{10} \theta_{20} \theta_{30}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} - t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} - t_2 l_3 C_{\theta_{10} \theta_{20} \theta_{30}} + t_1 l_2 S_{\theta_{10} \theta_{20}} - t_2 l_2 C_{\theta_{10} \theta_{20}} + \kappa \left( l_2^2 + l_3^2 + l_4^2 \right) \right. \]

\[ + \kappa l_1 l_2 C_{\theta_{20}} + \kappa l_1 l_3 C_{\theta_{20} \theta_{30}} + \kappa l_1 l_4 C_{\theta_{20} \theta_{30} \theta_{40}} \right], \]

\[ b_8 = 2 \left[ \kappa l_2 l_3 C_{\theta_{30}} + \kappa l_2 l_4 C_{\theta_{30} \theta_{40}} + 2\kappa l_3 l_4 C_{\theta_{40}} + t_1 l_3 S_{\theta_{10} \theta_{20} \theta_{30}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} - t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} - t_2 l_3 C_{\theta_{10} \theta_{20} \theta_{30}} + \kappa \left( l_3^2 + l_4^2 \right) + \kappa l_1 l_3 C_{\theta_{20} \theta_{30}} + \kappa l_1 l_4 C_{\theta_{20} \theta_{30} \theta_{40}} \right], \]

\[ b_9 = 2 \left[ -t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} + \kappa \left( l_2^2 + l_3^2 + l_4^2 + l_1 l_4 C_{\theta_{20} \theta_{30} \theta_{40}} \right) \right], \]

\[ b_{10} = 2\kappa l_2 l_3 C_{\theta_{30}} + 2\kappa l_2 l_4 C_{\theta_{30} \theta_{40}} + 2\kappa l_3 l_4 C_{\theta_{40}} + t_1 l_3 S_{\theta_{10} \theta_{20} \theta_{30}} + t_1 l_4 S_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} - t_2 l_4 C_{\theta_{10} \theta_{20} \theta_{30} \theta_{40}} - t_2 l_3 C_{\theta_{10} \theta_{20} \theta_{30}} + t_1 l_2 S_{\theta_{10} \theta_{20}} - t_2 l_2 C_{\theta_{10} \theta_{20}} - t_2 l_1 C_{\theta_{10}} + t_1 l_1 S_{\theta_{10}} + \kappa \left( l_1^2 + l_2^2 + l_3^2 + l_4^2 \right) + 2\kappa l_1 l_2 C_{\theta_{20}} + 2\kappa l_1 l_3 C_{\theta_{20} \theta_{30}} + 2\kappa l_1 l_4 C_{\theta_{20} \theta_{30} \theta_{40}}. \]
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261


