Knowledge and Understanding of Function held by Students with Visual Impairments

Dissertation

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By

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ABSTRACT

This study examined understanding of linear functions held by students with visual impairments. The purpose of this study was to determine students’ level of knowledge and type of understanding of linear function and to describe students’ abilities in using the four main representational forms of a function: a) description, equations, tables, and graphs. Other aspects studied were students’ preferred representation of function and students’ perceived influences in his or her mathematics education. Participants in this study included four high school and four college students who were receiving educational services for a visual impairment and who had completed at least one course in algebra.

Data collection and analysis followed a qualitative research design. Three instruments were used for data collection, a) the Mathematics Education Experiences and Visual Abilities (MEEVA) Interview, b) the Function Knowledge Assessment (FKA), and c) the Function Competencies Assessment (FCA). The MEEVA provided demographic information and responses provided information on students’ previous educational experiences in mathematics. The FKA and the FCA were mathematics assessments that consisted of problems related to linear functions and their applications. Student responses from the FKA and the FCA provided information on student knowledge of linear functions and student abilities when solving word problems.
involving linear functions. Instruments were given orally and responses were audio recorded. Each participant met with the researcher one-on-one on two different occasions to complete the three data collection instruments.

Data analysis followed the tenets of the *Constant Comparative Method* (Glaser & Strauss, 1967). Student responses to the MEEVA, FKA, and FCA were transcribed and coded for student understanding in the four function competencies, a) modeling, b) interpreting, c) transcribing, and d) reifying as described by O’Callaghan (1998). Students’ level of knowledge of linear function was further described by students’ ability to comprehend and apply knowledge when solving word problems, as described by Wilson (1971).

Results indicate that the understanding of modeling and interpreting problems involving linear functions of high school and college students with visual impairments was stronger than that of either translating between representational forms of a function or the ability to reify the function concept. A positive relationship was observed between students’ graphing abilities and his or her overall understanding of function. Results also show that students were most comfortable with gaining information on functions through tables and were least comfortable gaining information through graphs. The perceived influences on students’ mathematics education were that of individualized education and the use of appropriate materials that allowed for independent access to the curriculum.
DEDICATION

Dedicated to my best friend and husband:

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Publications


Field of Study

Major Field: Education

Focus: Mathematics
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The National Council of Teachers of Mathematics (NCTM) is the nation’s leading professional organization in promoting mathematics education reform. One of the major avenues of this reform has been the development of the *Principles and Standards of School Mathematics* (Standards; NCTM, 2000), a document highlighting the principles that guide mathematics learning. The first Principle of the *Standards* is the need for equity for all students. This presumes that all students, regardless of race, gender, disability, or cultural background, should be presented with opportunities to actively be engaged in learning mathematics. This charge can take on a different meaning when the students under consideration have little or no vision. Although these students should be afforded the same opportunities as their sighted peers within the mathematics classroom, students with visual impairments are not always provided with the same opportunities as their classmates. Empirical research in the area of mathematics education for students with disabilities is lacking. The research described here seeks to expand the research base in this area by describing the current understanding of linear functions held by students with visual impairments. Results of this research are intended to be used to further understanding on how to best accommodate learning for mathematics students with visual impairments.
My interest in the education of students with disabilities stemmed from my time at Wright State University (WSU). When beginning my undergraduate degree in mathematics education at WSU in 1997, I was afforded my first opportunity to work with students who have disabilities. One of the most rewarding assignments was as a reader/writer (i.e. a person who attends class with the student to take notes as well as a homework helper that reads and writes for the student) in a basic algebra course for two students who are blind. Since they had trouble with the course content and needed more than someone to take notes, I became their tutor. My research interest in the learning of mathematics by students who are blind or have visual impairments began with these two women. It was interesting how differently each student approached a problem. It became evident that part of this distinction stemmed from the way their vision loss occurred. One student, Rachel (pseudonyms are used throughout), had lost her vision early in life and had very little recollection of ever having sight. Betty, on the other hand, had lost her vision due to a disease that caused a slow loss of vision and did not completely remove her sight until she was a teenager. The fact that each of these students displayed very different approaches to mathematics and that they each struggled with the subject, even when accommodations and tutoring were provided, piqued my curiosity and has led to an interest in learning how to best address mathematics learning for this student group.

Statement of Problem

The function concept is an essential part of mathematics education (Dubinsky & Harel, 1992; O’Callaghan, 1998; NCTM, 2000). Izsak (2000) contends that, “depending on the context, one might view algebra as…the study of functions” (p. 31). The importance of student understanding of function as well as students’ continued challenges
in mastering functional reasoning (Sfard, 1991; Sfard & Linchevski, 1994; O’Callaghan, 1998) has led to a research emphasis in this area of mathematics education. A major vein of research in the area of function education is the integration of technology within the curriculum (Fey, 1989; Kaput, 1992, O’Callaghan, 1998). Research in the use of technology within mathematics education has demonstrated advantages including, providing support for student problem solving initiatives (O’Callaghan, 1998), providing support for students transitioning from arithmetic to algebraic reasoning (Tabach, Arcavi, & Hershkowitz, 2008), and “enabling students greater access to understanding the use and application of mathematical ideas and procedures” (Moreno-Armella, Hegedus, & Kaput, 2008). The advantages to use of technology in mathematics education can not, in many cases, be realized by students with visual impairments. Technologies commonly used during mathematics instruction, such as graphing calculators, Computer Algebra Systems, and computer software programs are not accessible to students with visual impairments. The prevalence of technology in the mathematics classroom and its inaccessibility for students with visual impairments serve as a reminder that students with visual impairments bring unique challenges to the classroom that must be addressed if they are to receive an equal education to their peers.

The challenges faced by students with visual impairments is a need that must be addressed in the public school districts as, according to the American Printing House for the Blind (APH), the majority of students who have vision-related needs are educated in their local schools (APH, 2010). Therefore, content teachers are now responsible for accommodating not only typically developing students, but students with special needs as well. Further, National and state regulations have increased the level of accountability
schools have regarding their students with disabilities (Individuals with disabilities education act [IDEA], 2000; No child left behind [NCLB], 2002). Schools are now held to a high standard for all of their students’ education. This means that mathematics teachers, who may have had little to no training in special education, may be responsible for the education of a student with visual impairments. Although the students’ education is supported by the teacher of the visually impaired and the special educator, the mathematics teacher is ultimately responsible for the students’ mathematics knowledge.

There has been a lack of empirical research in the area of mathematics learning for students with visual impairments. Professional works on this topic have been dominated by articles relaying experiences within a classroom with practical examples of what accommodations worked with one student. The current research aims to strengthen the empirical data in the area of mathematics education for students with visual impairments.

Rationale

The National Council of Teachers of Mathematics places great emphasis to the study of algebra and algebraic reasoning. Topics within the algebra curriculum such as recognition and extension of patterns, understanding of relationships, and describing change are highlighted in each grade band of the Standards. The NCTM indicates this emphasis is due to the observation that “Algebra is also closely linked to geometry and to data analysis. The ideas included in the Algebra Standard constitute a major component of the school mathematics curriculum and help to unify it” (NCTM, 2000, p. 37). Dubinsky and Harel (1992) view the study of functions, which is a major topic within the algebra curriculum, as a having a unifying role in mathematics education. Williams and
Molina (1998) posited that not all students will reach a level of sophistication in symbolic manipulation, but that all students should “understand how quantities depend on one another, how a change in one quantity affects the other, and how to make decisions based on these relationships” (National Council of Teachers of Mathematics & National Research Council, 1998, p. 41). A way for students to learn and organize information regarding how quantities depend on one another and how a change in one quantity affects another can be enhanced through the study of function.

Algebra is a far-reaching topic of mathematics education and contains various subcategories of content. A major component within the algebra curriculum is functions. The students’ view of function (Sfard, 1991; Dubinsky & Harel, 1992; Sfard & Linchevski, 1994; Slavit, 1997, 2006; Ronda, 2009) and their ability to work with functions in multiple representational forms are two heavily researched areas of student understanding (Fey, 1984; Chiu, Kessel, Moschkovich, & Muno-Nunez, 2001; Friedlander & Tabach, 2001; Moreno-Armella et al., 2008). This body of research has shown that learning to switch from one representation of a function to another can aid in a student’s overall understanding of function (Slavit, 1997; Ainsworth, Bibby, & Wood, 1998). Schwarz and Dreyfus (1995) contend that translation skills between various representational forms of functions are a prerequisite for having a unified understanding of function.

Research in the area of student competency in translating between representational forms of functions has received more attention by the research community in the past couple of decades. This research has found that many students are not flexible in their use of representations of algebraic forms and that this inflexibility
could be due to students’ being unaware of the connections between the various representations of a function (Brenner, Mayer, Moseley, Brar, Duran, Reed, & Webb, 1997; Keller & Hirsch, 1998; Senk & Thompson, 2006; Herman, 2007; Huntley & Davis, 2008). Research has also shown that most students’ regard the only “correct” way of solving an algebra equation is through symbolic manipulation (Friedlander & Tabach, 2001; Senk & Thompson, 2006; Huntley & Davis, 2008). Lack of understanding is partly attributed to the heavy emphasis placed on learning symbolic manipulation throughout the curriculum. The NCTM, through their 2001 Yearbook, which is devoted to the role played by representations and school mathematics, has addressed the need for well-rounded instruction of functions and their various representations. Even though the use of representations in the mathematics classroom is much broader than that of expressing algebraic expressions, this topic is highlighted throughout the book. Friedlander and Tabach (2001) discuss how students use multiple representations and why an understanding of various representations is important.

Friedlander and Tabach (2001) begin their chapter by highlighting the advantages and disadvantages of the various forms of a function (symbolic, verbal, graphical, tabular). The combination of advantages and disadvantages for each representation make it such that the greatest effect for learning the function concept is realized by understanding all representational forms of a function. Having access to all forms of a function will allow the reader to have a more complete understanding of the relationship being modeled. Their discussion of the advantages and disadvantages leads them to conclude that, “the importance of working with various representations is a result of these
and other advantages and disadvantages of each representation and of the need to cater to students’ individual styles of thinking” (Friedlander & Tabach, 2001, p. 174).

An example of advantages and disadvantages discussed by Friedlander and Tabach (2001) is that of the graphical representation. The main advantage of graphs and diagrams is their visual nature. A graph or diagram can give the reader a picture of the behavior of a graph quickly and efficiently. Disadvantages include the possible lack of accuracy and the influence of external conditions, such as scale (Friedlander & Tabach, 2001). There is additional research that indicates that even though graphs are often intended to give students a clear understanding of the behavior of a function or a data set, graphs can be difficult for students to interpret (Chiu et al., 2001; Roth & Bowen, 2003; Roth & Lee, 2004, Roth, 2004). In terms of the student population under consideration, students with visual impairments would not have the advantage of being able to quickly determine the behavior of a function and yet they will be prone to the same disadvantages of this form of representation. In all situations and with all student groups, educators need to be mindful of the difficulties students can encounter when first learning to use graphical representations. This logic extends to any of the four representations of function. The graphical representation highlights one of the more difficult ideas to adapt for students with visual impairments. A deeper understanding of student knowledge of function for this student population is needed in order to better accommodate them in the middle and high school mathematics classroom.

Research Questions

In order to strengthen the base of empirical studies in the area of mathematics education for students with visual impairments, a descriptive study was conducted. The
The purpose of the study is to better understand student comprehension of linear function. The population under consideration is students with visual impairments who have completed at least one course in algebra.

The research questions for the current study are:

1) What level of knowledge and type of understanding of linear functions is held by students with visual impairments?
2) What are the graphing skills of high school/college students with visual impairments?
3) What are the representational preferences of students with visual impairments when solving word problems involving functions?
4) What factors do high school/college students with visual impairments perceive as influencing the development of their mathematical understanding?

Students’ level of knowledge was addressed through the evaluation levels of Wilson’s Taxonomy (1971). Students’ type of understanding of linear functions was addressed through O’Callaghan’s Function Model (1998). Perceived influences on students’ mathematical education were reviewed through the lens of Lowenfeld’s Special Methodologies (1981; reprinted from 1952) for students with visual impairments. These three theories were used for the theoretical frame of this research and are described, in detail, next.

**Theoretical Framework**

Theories of how to evaluate and categorize knowledge and understanding abound (Bloom, 1956; Wilson, 1971; Skemp, 1989; Pirie & Kieren, 1994; Kilpatrick, Swafford, & Findell, 2001; Martin, 2008). The number and complexity of the various theories speak
to the fact that understanding is a complex entity that can be very difficult to accurately describe. Defining the knowledge held by an individual is complicated by the fact that knowledge is personal and is not always accessible to another person (Pape & Tchoshanov, 2001). An educational framework that has been in wide use throughout a variety of educational settings is the Taxonomy of Educational Objectives created under the leadership of Benjamin S. Bloom (Bloom, 1956). The taxonomy developed is usually referred to as Bloom’s Taxonomy, but was actually developed by a committee of educators that was chaired by Bloom (1956). In 1971 a handbook of learning objectives of formative and summative evaluation in education was written under the direction of Bloom. This body of work offered individual taxonomies for school subjects, all of which used Bloom’s Taxonomy as a basis for the proposed theory. The secondary mathematics education taxonomy developed by Wilson for the Handbook on Formative and Summative Evaluation of Student Learning (1971) is the theory of knowledge that was used for this research. Due to the fact that Wilson’s Taxonomy is directly related to Bloom’s Taxonomy, I first give a brief description of Bloom’s Taxonomy before discussing Wilson’s Taxonomy in detail.

Bloom’s Taxonomy is divided into three domains: a) Affective, b) Psychomotor, and c) Cognitive (Krathwohl, Bloom, & Masia, 1964; Bloom, 1956). The Affective domain considers the way people react emotionally and their ability to feel another living thing’s pain or joy. The Psychomotor domain describes the ability to physically manipulate a tool. The Cognitive domain involves knowledge and comprehension. The latter domain is broken into a six-level hierarchical order of: a) knowledge, b) comprehension, c) application, d) analysis, e) synthesis, and f) evaluation (Bloom, 1956).
A main tenant of the Bloom’s Taxonomy is that knowledge is described as being hierarchical. That is, if a student displays knowledge on the comprehension level, he or she has at least shown some competency within the knowledge level. This hierarchy does not imply that students on a higher level will not use knowledge from lower levels. This hierarchy allows for students to use competencies within lower levels and to move flexibly among the levels to achieve the desired outcome.

A second tenant of Bloom’s Taxonomy is the evaluation of specific educational objectives. The intended use of Bloom’s Taxonomy is to write and evaluate specific educational objects. Each objective must be measurable, address a specific topic, and evaluation should specifically seek to address one of the six levels of understanding (Bloom, 1956). Next, I describe Wilson’s Taxonomy that guided the current research by providing a theoretical background on what it means for a student to have a certain level of knowledge within the mathematics curriculum.

**Wilson’s Taxonomy**

Wilson’s Taxonomy of learning objectives was used as a theoretical model for the current research because it specifically addresses students’ current level of understanding for secondary mathematics. Also, Wilson’s Taxonomy provided an excellent resource for the classification of learning objectives for the study and emphasizes levels of knowledge that are based on observable student behaviors. Finally, Wilson’s Taxonomy was chosen to give theoretical perspective to the current research because of the use of hierarchical levels of understanding. The main objective of this research is to describe students’ current level of knowledge of linear functions. I believe the most effective way to do this is to describe student knowledge based on categories that indicate the sophistication of
the students’ understanding. Wilson’s Taxonomy of learning objectives for secondary mathematics provides a way to specifically address students’ current level of knowledge of linear functions. Next I describe, in detail, how Wilson’s Taxonomy specifically addresses student understanding in the secondary mathematics curriculum through student behavior.

Wilson’s Taxonomy uses a dual classification for assessment questions. Each question or educational task addresses a specific area of content and a specific level of behavior. The content categories used in Wilson’s Taxonomy are, a) number systems, b) algebra, and c) geometry. These categories are specific to the course or content being evaluated and are often divided into smaller subcategories throughout a course of study. For the current research the content being evaluated is linear functions, which is a subcategory of algebra. More specifically, students’ understanding in modeling, interpreting, translating, and reifying (see O’Callaghan, 1998) linear functions is evaluated through this research.

The second classification given to an evaluation item under Wilson’s Taxonomy is that of behavior level. These are consistent across all types of content and are hierarchical. The main behavior levels are a) computation, b) comprehension, c) application, and d) analysis, each of which has subcategories. Next I describe each of these levels and the various sublevels of each category.

The computation level of Wilson’s (1971) taxonomy is comprised of entry-level evaluation items that require basic understanding of mathematical language and definitions to complete. It also includes exercises of simple recall and routine manipulation. In this respect, the computation level models behaviors similar to those
described by procedural knowledge and can be learned by rote memorization. The sublevels of the computation level are expounded in Table 1.1.

<table>
<thead>
<tr>
<th>Category</th>
<th>Explanation/Types of Problems</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.1 Knowledge of specific facts</td>
<td>• The student is asked to reproduce or recognize material in almost exactly the same form as it was presented during instruction</td>
</tr>
<tr>
<td></td>
<td>• Fundamental units of knowledge that a student can be reasonably assumed to know because of exposure to them over a long period of time</td>
</tr>
<tr>
<td>A.2 Knowledge of terminology</td>
<td>• Ability to recall definition of familiar terms</td>
</tr>
<tr>
<td>A.3 Ability to carry out algorithms</td>
<td>• A most important subcategory of computation</td>
</tr>
<tr>
<td></td>
<td>• Ability to manipulate elements of a stimulus according to some learned algorithm</td>
</tr>
<tr>
<td></td>
<td>• Example: Solving a simple linear equation for a beginning algebra course</td>
</tr>
</tbody>
</table>

Table 1.1: Computation Sublevels of Wilson’s Taxonomy

The computation level places an emphasis on basic knowledge and does not require decision making or complex memory for completion. The next level of behavior is comprehension.

Evaluation items on the comprehension level are more cognitively complex than those on the computation level and require knowledge of concepts, not just isolated facts. A description of the sublevels under comprehension can be found in Table 1.2.
## B.0 Comprehension

<table>
<thead>
<tr>
<th>Category</th>
<th>Explanation/Types of Problems</th>
</tr>
</thead>
</table>
| B.1 Knowledge of concepts                     | • Knowledge of concepts  
• Specific knowledge of elements that are included or excluded from certain concepts                                                                                                                                 |
| B.2 Knowledge of principles, rules, and generalizations | • Knowledge of principles and rules within a content of study  
• Understanding of when and how to use these rules  
• If the student is asked to produce a rule or generalization the behavior is at a higher level                                                                                                                     |
| B.3 Knowledge of mathematical structure       | • Knowledge of principles governing the concepts in a particular branch of mathematics                                                                                                                                         |
| B.4 Ability to transform problem elements from one mode to another | • Ability to transform problem elements from one mode of representation to another  
• Example: Transformation from a verbal description to a pictorial representation, or translation from a verbal representation to a symbolic form, or vice versa in each case. |
| B.5 Ability to follow a line of reasoning     | • Ability to read or listen to a mathematical argument  
• Ability to receive communication about mathematics                                                                                                                                                                       |
| B.6 Ability to read and interpret a problem   | • Ability to understand what a problem is asking and to interpret problem components  
• Does not include the ability to solve a problem  
• Is a necessary first step in being successful in problem solving  
• Goes beyond simple verbal skills and reading ability                                                                                                                                                   |

Table 1.2: Comprehension Sublevels of Wilson’s Taxonomy

The third level of student behavior, application, furthers the complexity of evaluation items by including those that require a sequence of responses from the student.
Application items also include problems that are similar to ones studied, but are not identical and therefore require a minimal level of transfer of knowledge. Subcategories for the application level are in Table 1.3.

<table>
<thead>
<tr>
<th>Category</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.1 Ability to solve routine problems</td>
<td>• Involves selecting and carrying out an algorithm.</td>
</tr>
<tr>
<td></td>
<td>• Problems must be somewhat familiar to students</td>
</tr>
<tr>
<td>C.2 Ability to make comparisons</td>
<td>• Recall relevant information (concepts, rules, mathematical structure, terminology)</td>
</tr>
<tr>
<td></td>
<td>• Discover a relationship and be able to formulate a decision</td>
</tr>
<tr>
<td></td>
<td>• Ability to choose from among available alternatives</td>
</tr>
<tr>
<td>C.3 Ability to analyze data</td>
<td>• Ability to read and interpret information and to manipulate that information</td>
</tr>
<tr>
<td></td>
<td>• Ability to make decisions or draw conclusions</td>
</tr>
<tr>
<td></td>
<td>• Ability to separate a problem into its component parts</td>
</tr>
<tr>
<td></td>
<td>• Ability to distinguish relevant from irrelevant information</td>
</tr>
<tr>
<td>C.4 Ability to recognize patterns, isomorphisms, and symmetries</td>
<td>• Ability to recall relevant information</td>
</tr>
<tr>
<td></td>
<td>• Ability to transform problem elements and manipulate these elements in a sequence</td>
</tr>
<tr>
<td></td>
<td>• Ability to recognize relationships</td>
</tr>
<tr>
<td></td>
<td>• It is assumed that similar patterns, isomorphisms, or symmetries have been studied by the student and that recognition is possible</td>
</tr>
</tbody>
</table>

Table 1.3: Application Sublevels of Wilson’s Taxonomy
The final level of behavior in the cognitive domain of Wilson’s (1971) taxonomy is analysis. This level represents questions of the most complex behaviors. This level emphasizes a student’s ability to apply previously learned knowledge to nonroutine and unfamiliar problems. Another area of analysis is that of student’s ability to justify their answers and often includes formal reasoning through developing proofs. The sublevels of analysis are listed in Table 1.4.

<table>
<thead>
<tr>
<th>Category</th>
<th>Explanation</th>
</tr>
</thead>
</table>
| D.1 Ability to solve nonroutine problems | • Requires transfer of previous mathematics learning to a new context  
• Ability to solve problems unlike those that have been solved previously  
• A holistic approach is often necessary such as establishing a plan and carrying it out |
| D.2 Ability to discover relationships | • Ability to restructure problem elements in a new way to formulate a relationship  
• Differs from the last category in that the student must discover a new relationship rather than recognize a familiar relationship in the new data |
| D.3 Ability to construct proofs | • Ability to construct proofs as opposed to the ability to reproduce proofs or recall proofs |
| D.4 Ability to criticize proofs | • Ability to criticize any mathematical argument |
| D.5 Ability to formulate and validate generalizations | • Ability to discover a relationship and to construct a proof to substantiate the discovery |

Table 1.4: Analysis Sublevels in Wilson’s Taxonomy
All evaluation questions, using Wilson’s Taxonomy, can be categorized by content and level of behavior. Labeling individual items on an evaluation with these components gives an educator a tool to verify whether or not the evaluation appropriately covers the content as well as verifies the cognitive complexity of the items. Wilson (1971) found that many educators wanted to evaluate student understanding at the higher levels of the taxonomy, but when categorizing test items, found they were placing an emphasis on lower level behaviors such as those under the computation and comprehension levels. The use of context-based problems on the assessments for the current research gave opportunities for students to display knowledge in the comprehension and application levels of Wilson’s Taxonomy. Student understanding in the level of computation was demonstrated in some of the student responses. For a detailed account of the content and level of behavior used for each question of the mathematics assessments for this research refer to Chapter 3.

Student understanding of linear function was assessed for the current research using the Function Model developed by O’Callaghan (1998). This model and its implications for the current research are discussed, next.

O’Callaghan’s Function Competency Model

The Function Model developed by O’Callahan (1998) addresses student understanding in four areas of function knowledge: a) modeling, b) interpreting, c) translating, and d) reifying. O’Callaghan’s Function Model provided a theoretical perspective for the content being evaluated for the current research. That is, the current research addressed student understanding of linear function in the areas of modeling, interpreting, translating, and reifying. By addressing the overarching ideas of functions,
the O’Callaghan model provided a way to determine areas of strength and areas of weakness in student understanding of function. The O’Callaghan model was also chosen to provide theoretical perspective to the current research because it specifically addresses student understanding of function when working within a context (O’Callaghan, 1998). Thus, this particular model of function understanding worked well for the current research since both mathematical assessments used only contained problems imbedded in context. Next I describe each of the four areas of the function model.

O’Callaghan defines modeling as the ability to translate from a problem situation to a functional representation. Interpreting is considered the reverse of modeling. That is, a student who can interpret a function is able to accurately relate the information in real-life applications. A students’ ability to make smooth transitions from one form of a function to another (i.e. from an algebraic expression to a graph, from a graph to a table) addresses the translating competency. Finally, reifying is defined as by Sfard (1991) in that students who are capable of reification are able to work with a function in a holistic way and can view the function as an object and think in terms of properties of types or classes of functions.

The final theoretical perspective used in data analysis for the current research is that of Lowenfeld’s Special Methodologies for the education of students with visual impairments. Details of this theory are discussed, next.

**Lowenfeld’s Special Methodologies for the Education of Students with Visual Impairments**

Although there is not a lot of research to build upon in the area of mathematics concept development for students with visual impairments, Lowenfeld in 1952 set forth
five principles that he referred to as essential parts of a specialized methodology for students with visual impairments. The five principles of specialized instruction are a) individualization, b) concreteness, c) unified instruction, d) additional stimulation, and e) self-activity (Lowenfeld, 1981; reprinted from 1952). Lowenfeld’s principles for a special methodology were used as a foundation to analyze and organize data concerning students’ perceived influences on their mathematics education. I used Lowenfeld’s terms, but adjusted the definitions for the five principles to be more useful in the current context.

There has been much research in the general education needs of students with visual impairments since Lowenfeld put forth these ideas in 1952. Therefore, I decided to include some of the modern concepts such as “accommodations” in the definitions used for this research. Although the principles were put forth in 1952, the underlying concepts still apply today and are relevant to the current study. The principles provided a means to discuss broad categories of learning needs that are specific to students with visual impairments.

The current research used the first two categories of individualization and concreteness as initial codes for data analysis. Lowenfeld’s (1981) concept of individualization focused on the need for educators to take into consideration students’ individualized needs including the degree of the visual impairment, the cause of vision loss, the age at which the student lost vision, eye care that is required for the student, and the home environment of the student, especially in the preschool years. Lowenfeld (1981) recommends that class size for students with visual impairments should be no greater than eight for elementary students and no more than twelve for older students. The focus of the current research is neither on students’ visual needs nor on class size. However, the
idea of having individualized educational opportunities, whether that be in an inclusive setting or at a school for the blind, provided a way to discuss the more modern concepts of accommodations and modifications in the classroom. Therefore, for the purposes of this study, individualization refers to accommodations or modifications that provide for the individual needs and learning styles of the student.

Lowenfeld (1981) defines concreteness in terms of touch observation specifically used to gain knowledge about objects and their spatial characteristics such as shape, size, weight, hardness, and surface qualities. He notes that the use of concreteness in education is what helps students to avoid falling into a pattern of unreality. For the current study, the focus of concreteness is on providing educational resources, such as braille and tactile graphics, for students’ observation through touch. The definition of concreteness is also expanded to include providing access to large print materials and graphics for students who have usable vision. Therefore, for this study, concreteness is defined as any accommodation or modification that specifically provides tangible access to course content.

**Definition of Terms**

For the purpose of this study I have defined the following terms.

*Understanding* is defined in terms of Skemp’s (2006) *relational understanding* that includes student knowledge of both how to perform a mathematical procedure and the reason behind the actions taken to solve the procedure.

*Levels of understanding* include students’ relational understanding of modeling, interpreting, translating, and reifying.

*Knowledge* is the ability to perform mathematical tasks.
**Types of Knowledge** include students’ abilities in comprehension and application of mathematical tasks.

**Individualization** is defined as any accommodation or modification that provides for the individual needs and learning styles of the student.

**Concreteness** is defined as any accommodation or modification that specifically provides tangible access to course content.

**Modeling** is defined as the ability to translate from a problem situation to a functional representation.

**Interpreting** is defined as the ability to accurately gain information from descriptions, tables, equations or graphs in problem contexts.

**Translating** is defined as the ability to make smooth transitions from one representation of a function to another.

**Reifying** is defined as the ability to work with a function in a holistic way and the ability to view a function as an object.

**Students with visual impairments** are students who are blind or have limited visual abilities that affect their education.

**Summary**

Students with visual impairments should be afforded the same learning opportunities as their sighted peers within the mathematics classroom. However, the needs of students with visual impairments might differ to those of their peers in terms of effective methodologies. The goal of this research is to describe the current understanding of linear functions held by students with visual impairments. It is intended for this information to be used as a starting place for future discussions of the
methodological and pedagogical needs of students with visual impairments when learning mathematical functions.

The current research also addressed the graphing abilities of students with visual impairments, representational preferences of linear functions of students with visual impairments, and students’ perceived influences on his or her mathematical education. Theoretical underpinnings used in data procedures and data analysis included Wilson’s Taxonomy of learning objectives for secondary mathematics (1971), O’Callaghan’s Function Model (1998), and Lowenfeld’s Special Methodologies (1981) for educating students with visual impairments (1981).

The next chapter synthesizes literature that is related to the current research. Areas of research discussed in this literature review include, research on theories of understanding of functions, current instructional practices for learning mathematical functions, and general mathematics learning theories. A literature review of education for students with visual impairments was also conducted in light of the student population for this study.
A firm understanding of school mathematics is important for a student’s general knowledge and is needed, in many cases, for entrance into higher education as well as a variety of career fields. The need for a strong mathematics program cuts across all student groups and includes students who have disabilities. Education for students with visual impairments is an area that has limited empirical research. Spindler (2006) notes there are very few research articles regarding students with visual impairments in university mathematics, especially for higher-level courses. This observation is given backing by a literature review of best practices in mathematics education, which found only 10 studies related to education for students who are blind that met research criteria set forth by NCLB (Ferrell, 2006). There is a shortage of research in the area of mathematics education for students with visual impairments, but recent progress has been made in research in the area of science education for students with visual impairments. Thus, the literature review for the current research draws upon that body of knowledge and relates it to the mathematics classroom.

The current research seeks to determine the understanding of functions of this student population with an emphasis on their understanding of function through reification and their use of multiple representations. This chapter will discuss current
algebra education standards, general theories of student understanding, and specific
theories of understanding of function. The chapter will conclude with a short review of
literature regarding education for students with visual impairments.

School Algebra in the United States

The United States does not have a national curriculum in mathematics. The
curriculum choice resides with the state and individual school district. There are state and
national standards for mathematics that help guide curriculum development as well as
direct classroom instruction. The national guidelines for K-12 mathematics was
introduced by the National Council for Teachers of Mathematics (NCTM) in 1989 in a
publication entitled, *Curriculum and Evaluation Standards for School Mathematics.* This
document directed curriculum development for K-12 mathematics education throughout
the United States from 1989 until a revised edition was published at the turn of the
century. This revised version is the *Standards* document discussed earlier (NCTM, 2000).
This document clarified standards put forth in the first document as well as introduced
five principles that guide mathematics education. The influence of *Standards* has been far
reaching and has dictated much of the direction of K-12 mathematics education.

The Algebra Strand within the *Standards* highlights the importance the national
agency places on algebraic reasoning skills and knowledge. Algebra is no longer seen as
exclusive to the high school curriculum, but rather a unifying theme throughout K-12
mathematics. The enactment of these standards through elementary and middle grades
will help to prepare students for formal algebraic reasoning when they reach high school
(NCTM, 2000). The development of these skills is important because, “algebraic
competence is important in adult life, both on the job and as preparation for
postsecondary education” (NCTM, 2000, p. 37). Although algebra has been a part of formal education for generations, the aspects of algebra that are emphasized in a high school curriculum have evolved through the years.

According to Davis (1989), previous algebra curricula have overemphasized symbolic manipulation, which has contributed to a common view of mathematics as being simply a set of prescribed directions and meaningless rituals. He concludes, “it is no mystery that most students find mathematics boring and incomprehensible” (p. 269). The NCTM (2000) is trying to change this image. They believe, by the end of high school, students should “understand patterns, relations, and functions; represent and analyze mathematical situations and structures using algebraic symbols; use mathematical models to represent and understand quantitative relationships; analyze change in various contexts” (p. 37). This list of knowledge points is in part to highlight the need for students to have varied experiences with a wide range of algebraic skills. General theories of student understanding will be discussed, next.

**Definitions and Theories of Understanding**

Educators expect their students to gain knowledge and understanding through instruction and course assignments. Although this expectation is at the very heart of education, what is meant by *knowledge* and *understanding*? What does it mean to be *proficient* in mathematics? Knowledge and understanding are abstract concepts that have been difficult to define and fully describe as can be seen by the number of theories and definitions of understanding. Though there is overlap between theories of understanding, there are important differences as well. Some theories of understanding make the distinction between meaningful learning and rote memorization. There are theories that
describe knowledge as hierarchical and ordered (i.e. Wilson, 1971), while others believe different types of knowledge are interdependent and do not progress linearly (i.e. Sfard & Linchevski, 1994; Kilpatrick et al., 2001). In this section I will discuss four general theories and definitions of understanding in mathematics education, compare and contrast these four theories of understanding and provide a rationale for the use of Wilson’s taxonomy (1971) as a theoretical framework for the current study.

**Mathematical Proficiency**

The National Research Council (Kilpatrick et al., 2001) believes there is no single concept to describe the competence or level of understanding a student has in mathematics. Mathematical proficiency is a term they chose to indicate the compilation of what they believe necessary for a student to learn mathematics successfully. The components, or *strands*, are, a) conceptual understanding, b) procedural fluency, c) strategic competence, d) adaptive reasoning, and e) productive disposition (Kilpatrick et al., 2001). These strands are seen as interdependent and interwoven throughout school mathematics.

Conceptual understanding is seen as “an integrated and functional grasp of mathematical ideas” (Kilpatrick et al., 2001, p. 118) and is an understanding that is beyond knowledge of isolated facts. Procedural fluency refers to knowledge of when and how to use procedures flexibly and accurately. Historically there has been much debate on whether basic skills and procedural knowledge should be emphasized in curricula or if conceptual understanding should be stressed. However, Kilpatrick et al. (2001) emphasize the interdependence of these skills:
Understanding makes learning skills easier, less susceptible to common errors, and less prone to forgetting. By the same token, a certain level of skill is required to learn many mathematical concepts with understanding, and using procedures can help strengthen and develop that understanding. (p. 122)

A weakness in a students’ procedural understanding can prevent him or her from seeing important relationships between problem situations.

The third strand of mathematical proficiency, strategic competence, refers to students’ ability to appropriately use mathematical representations to solve problems. This strand includes understanding in problem formation as well as in problem solving. An emphasis is placed on a student’s ability to recognize solution paths, given the context of the problem, and on having a variety of strategies to solve such problems. Adaptive reasoning is a more holistic idea that refers to a students’ capacity to think logically and to justify conclusions. Kilpatrick et al. (2001) describe adaptive reasoning as,

the glue that holds everything together, the lodestar that guides learning. One uses it to navigate through the many facts, procedures, concepts, and solution methods and to see that they all fit together in some way, that they make sense. (p. 129)

The final strand, productive disposition, is holistic in that it refers to students’ perception of mathematics as being useful and worthwhile. Competency within this strand develops alongside the other strands and requires frequent opportunities to make sense of mathematics and to be successful in performing mathematical tasks and understanding mathematical connections.

**Procedural and Conceptual Knowledge**

As discussed in the NRC (Kilpatrick et al., 2001) strands, a common concept within mathematics learning is that of procedural versus conceptual knowledge. There are two distinct parts to procedural knowledge, a) having an understanding of formal
language or symbol representation and b) knowledge of how to execute predetermined linear steps to complete a problem. Knowledge of procedural operations, alone, implies an understanding of surface knowledge, not an understanding of relationships or reasons behind the symbol manipulation (Hiebert, 1986). Conceptual knowledge, on the other hand, is characterized by a deep understanding of relationships between concepts. It can be thought of as a connection of ideas that links discrete pieces of information. A concept directly linked to procedural and conceptual knowledge is that of rote learning versus learning with meaning (Hiebert, 1986).

*Rote learning* is absent of relationships and is not linked to other knowledge. This leads to knowledge that is closely tied to the context in which it is learned and is therefore not easily generalized to other situations (Hiebert, 1986). Learning with meaning implies a rich network of relationships between discrete pieces of information and leads to a more flexible use of the knowledge. By definition, conceptual understanding is learning with meaning as its emphasis is on relationships and generalizations between facts and concepts (Hiebert, 1986). This is not the case, however, for procedural knowledge. Learning the symbolic language of mathematics and the rules regulating those symbols can be understood through individual facts. This type of rote learning is not ideal for a rich understanding of mathematics. Hiebert (1986) believes:

> Mathematical knowledge, in its fullest sense, includes significant, fundamental relationships between conceptual and procedural knowledge. Students are not fully competent in mathematics if either kind of knowledge is deficient or if they both have been acquired but remain separate entities. (p. 9)

Benefits of learning with meaning are numerous and will be discussed, next.
Procedural knowledge that is not connected with conceptual knowledge has several potential downfalls for true understanding. First, this type of knowledge can deteriorate quickly and once this happens it is difficult to recover. Procedures that are learned without understanding are also subject to confusion between procedures or a mixture of partial procedures (Hiebert, 1986). A lack of understanding can also lead to problems of transfer, in that students are unable to apply the knowledge to nonroutine problems. In the absence of knowledge of relationships, students are often unable to determine if an answer is reasonable and are therefore unable to appropriately check their work (Hiebert, 1986). The current section discussed general learning theories for mathematics. Next, I will discuss specific learning theories that apply directly to an understanding of the concept of function.

**Instrumental and Relational Understanding**

Knowledge in mathematics can be viewed through the lens of two types of understanding, instrumental and relational understanding (Skemp, 2006; reprinted from 1976). **Instrumental understanding** pertains to knowing what steps to take to complete a problem, but without an understanding of why the steps are performed. **Relational understanding**, on the other hand, is both knowing what to do and why. Instrumental understanding is based on a plethora of rules while relational understanding uses fewer principles to make general applications. Skemp (2006) highlights advantages of relational understanding and discusses why instrumental understanding can seem like a viable option for learning.

Skemp (2006) favors the use of relational understanding, but in order to understand the prevalence of instrumental understanding he discusses three “advantages”
of instrumental understanding. First, instrumental mathematics, a version of mathematics that is rules without reasons, is easier because applying a formula or set of procedures is easier than understanding why the procedure works (Skemp, 2006). Thus, if the goal of the task is to accurately and quickly complete a page of mathematics problems, instrumental mathematics provides the most direct way. Second, because problems are often easy to solve, the rewards are more apparent. Instrumental mathematics is sometimes used to build confidence in students by providing an easy way for them to have success with mathematics (Skemp, 2006). Third, the correct answer can often be obtained more quickly and reliably by instrumental thinking, which is why even seasoned mathematicians will use this technique.

The four advantages of relational mathematics, a mathematics that is learned through relational understanding, highlighted by Skemp (2006) include; a) it is more adaptable to new tasks, b) it is easier to remember, c) it can be more effective as a goal in itself, and d) relational schemas are organic in quality. Learning both the how and why behind a problem solution allows for students to relate the method to the problem at hand as well as applying the method in new contexts (Skemp, 2006). On the other hand, instrumental mathematics provides only the knowledge needed to solve familiar problems. Transfer of knowledge to novel situations is problematic because the basic elements of solving problems through instrumental understanding have been memorized within the original problem context. Although relational mathematics is harder and more time consuming to learn initially, the result is longer lasting. The permanence of learning relationally allows for less need to re-learn material. When students learn with relational understanding, it provides motivation and greatly reduces the need for external rewards.
and punishments (Skemp, 2006). Finally, the organic quality of relational understanding refers to the tendency of students to gain satisfaction from learning material relationally and wanting to learn new material in the same manner (Skemp, 2006).

The benefits of relational understanding and the reasons for instrumental learning of mathematics led Skemp (2006) to conclude, “it would appear that while a case might exist for instrumental mathematics short-term and within a limited context, long-term and in the context of a child’s whole education it does not” (p. 93). Thus, relational mathematics should be emphasized throughout mathematics education.

**Knowing-About, and Knowing–To**

Mason and Spence (1999) extend a view of knowledge put forth by Ryle (1984; reprinted from 1949) and relate the view of knowledge to mathematics learning. Ryle (1984) defines three different forms of knowledge that, taken together, are referred to as *knowing-about*. First, *knowing-that* relates to knowing something is factually true. Second, *knowing-how* includes knowledge of how to carry out a procedure. Third, *knowing-why* involves having some explanation of how to account for something (Ryle, 1984). Mason and Spence’s (1999) extension of this model details *knowing-to* act in the moment. Knowing-to often comes by way of a sudden shift of focus. The sudden know-to moment is depicted by Mason and Spence as a bolt of lightning on a clear blue day.

Mason and Spence (1999) explain the knowing-to act in the moment as connecting with knowing-about a topic in the following way.

Once the moment of knowing-to takes place, knowing-how takes over to exploit the fresh idea; knowing-that forms the ground, the base energy upon which all else depends and on which actions depend; knowing-why provides an overview and sense of direction that supports connection and link making and assists reconstruction and modification if difficulties arise en route. Knowing-how
provides action, things to do, changing the situation and transforming it, and providing the various knowings with fresh situations upon which to operate. (pp. 146-147)

For educators to foster an environment to allow for knowing-to moments for students the educators must have an extra degree of awareness by both knowing-to and knowing-how to create suitable conditions to focus students’ attention on how they are working (Mason & Spence, 1999). A more theoretical approach to levels of knowledge as put forth by Mason and Spence includes responsive-knowing, active-knowing, and reflective or meta-knowing.

*Responsive-knowing* uses background experiences to respond to future events. In the classroom responsive-knowing may manifest itself through the educator needing to remind students of facts or techniques before they are able to solve the problem. *Active-knowing* is characterized by knowing-to act without having extra cues. Therefore, students are able to actively initiate the use of knowledge. Finally, *reflective or meta-knowing* pertains to being aware of knowing and the ability to reflect on that knowing. Students at the level of meta-knowing have the ability to describe how they did a problem and are able to create general questions of the same type (Mason & Spence, 1999).

One downside of having levels of knowledge is the inability for an outsider to assess what kind of knowledge is being used for any given task. Mason and Spence (1999) have observed that student knowledge is much less flexible between problem situations than he would expect. The inflexibility to apply knowledge to novel problems has led Mason and Spence (1999) to conclude that in many cases, “student concentration has actually been on behavior than rather than understanding” (p. 149). They detail several examples of how the same observable behavior could be a case of relational
understanding, as defined by Skemp (2006) or instrumental understanding through memorized behaviors that are void of meaning.

*Wilson’s Taxonomy*

The definitions and theories of understanding, discussed above, provide various ways in which to classify student learning in mathematics. A theory of learning that incorporates many of the components of these four theories is that of Wilson’s Taxonomy (1971). Based on the work by Bloom (1956), Wilson provides a general evaluation of students’ formative and summative knowledge specifically for mathematics learning. The focus of Wilson’s taxonomy is the evaluation of student’s mathematical knowledge as shown through their behaviors, which is in contrast to Mason and Spence’s (1999) theory of understanding where classification of student behavior is difficult to ascertain. Also, Wilson’s taxonomy, unlike Mason and Spence’s (1999) and the NRC’s (Kilpatrick et al., 2001) theories of understanding, provides an avenue to describe students’ level of understanding at a point in time. Finally, Wilson’s taxonomy covers many of the ideas embedded in other theories of understanding including procedural knowledge, which is expressed in the computational level of behavior in Wilson’s model, conceptual learning expressed in Wilson’s comprehension level, and the ability to apply knowledge to familiar and unfamiliar tasks, which is expressed in the application and analysis levels of Wilson’s theory.

Wilson’s taxonomy can “provide a unifying theme for considering three different types of problems: problems of mathematics curriculum, instruction, and evaluation” (Wilson, 1971, p. 648). This model stratifies outcomes of mathematics in two ways, by mathematical content and by levels of behavior. The model’s double classification allows
an educator to evaluate individual problems for both content and level of complexity. The double classification, in turn, can lead to a clear identification of the goals of the course and provides a way to verify students are being appropriately evaluated for these goals. Skemp (2006) highlights the need for educators to become aware of what their teaching methods emphasize, whether they promote instrumental or relational understanding, because lack in student learning can often be traced to a mismatch between student and teacher expectations. That is, students learning can be impeded if they are trying to understand the material on an instrumental level while the teacher aims to provide relational learning opportunities or vice versa.

Mathematical content can be specified based on the course being taught. Wilson (1971) has broad categories of number systems, algebra, and geometry with subcategories for elements under each topic. A broad look at the topics covered in today’s mathematics classroom is detailed in the Standards (NCTM, 2000). Levels of student behavior for the cognitive domain are divided into four overarching categories of: a) computation, b) comprehension, c) application, and d) analysis (Wilson, 1971). Each of these categories is further subdivided into specific behaviors. Evaluation items can be written to one of these behavior levels. A summary of these levels is provided in Table 2.1. Computation questions mainly involve memorization of facts and procedures, while comprehension items require an understanding of concepts and relationships. Successfully completing an application problem assumes the ability to choose appropriate operations to solve word problems, while analysis questions involve unfamiliar or nonroutine uses of familiar concepts.
<table>
<thead>
<tr>
<th>Level of behavior</th>
<th>Description of items</th>
<th>Example evaluation item</th>
</tr>
</thead>
</table>
| Computation      | • Emphasis on recall of basic facts and definitions  
                  • Manipulation of problem elements based on rules previously learned | • What is the slope of the line, \( y=3x+4 \)  
                  • Simplify the expression, \( 2x - 3y +4y = 5x +4 \), and write in standard form. |
| Comprehension    | • Recall or generalizations of concepts are needed  
                  • Transformations of problem elements are required  
                  • Emphasis is on understanding of concepts and their relationships | • Graph the equation, \( 3x+5=2y \) |
| Application      | • Require students to choose appropriate operations and to use those operations to compute a final response | • How much will Mary receive if she babysits for two and a half hours if we assume her pay is $10.50 per hour? |
| Analysis         | • Use knowledge to solve nonroutine problems  
                  • Find patterns and relationships to solve unfamiliar problems | • Does the system of equations, \( 2x + 7 = 5y \) and \( x=5 \) have a solution? |

Table 2.1: Levels of Behavior and Example Items for Wilson’s Taxonomy

Wilson (1971) emphasizes that the level of behavior reflects the cognitive complexity of a task and not simply the difficulty of a task. He also notes the levels of behavior are both ordered and hierarchical. That is, as the levels progress the cognitive complexity increases and to achieve a higher order task one might need to employ lower-level skills. For example, a student answering an application question might need to use
skills from the computation level such as recalling basic facts or solving routine equations. Even though there is a hierarchical nature to the levels, Wilson (1971) stresses there is no evidence to support the claim that a student needs to show mastery on any given level before being introduced to higher cognitive activities.

**Theories of Student Understanding of Function**

A function is a relation that uniquely associates members of one set with members of another set. A more formal, set-theoretic, definition of function is also used. However, educational research indicates that introducing students to functions as a relationship between two variables helps capitalize on their intuitive understanding of function (Vinner & Dreyfus, 1989). Research indicates students have difficulty in fully understanding the concept of function (Sfard, 1991; Sfard & Linchevski 1994; O’Callaghan, 1998; Izsak, 2000, 2003). This has led to a great deal of research into the causes of the lack in understanding of function concepts. Research in this area has typically addressed students’ ability to accurately perform given tasks using functions or to determine how the student views a function. I will first discuss a general model for student understanding of function and will follow this discussion with a more detailed account of student orientation-view of function.

**Function Model**

O’Callahan (1998) promotes a model of student understanding of function that addresses students’ competency in four areas: a) modeling, b) interpreting, c) translating, and d) reifying. He uses this model to compare the conceptual knowledge of function between students in a typical collage algebra course to those instructed in the computer-
intensive section of the same course. He defines modeling as the ability to translate from a problem situation to a functional representation. Interpreting is considered the reverse of modeling. That is, a student who can interpret a function is able to accurately relate the information in real-life applications. A students’ ability to make smooth transitions from one form of a function to another (i.e. from an algebraic expression to a graph, from a graph to a table) addresses the translating competency. Finally, reifying is meant in the same respect as Sfard (1991) in that students who are capable of reification are able to work with a function in a holistic way and can view the function as an object. Research is extensive in each of these areas of student understanding of function. This paper will focus on the final two competencies as they most directly relate to the current research. I begin by discussing student orientation views of function that relate to reification of function and follow with a discussion of translating and the use of multiple representations.

**Orientation Views**

Early research on function knowledge indicates two basic ways students view functions (Sfard, 1991; Dubinsky & Harel, 1992; Sfard & Linchevski, 1994; Schwarz & Dreyfus, 1995). This is typically referred to as a process-object duality and applies to many mathematical concepts, including functions. A process-oriented view of function is a student’s seeing a function as a process or action to be completed; whereas an object-oriented view of function is apparent when students are able to view a function as an object. The general consensus is that students start with an action-oriented view of function and through a long, slow, tedious process moves toward an object-oriented view of the concept (Sfard, 1991). A more detailed description of these views follows.
Process-oriented view of function. A student who has a process-oriented view of function will understand it as a non-permanent construct and relates functions to computation and arithmetic skills. This view can be likened to a student learning of function as a ‘machine’ where a value (usually a number) is inputted and an “answer” emerges (Slavit, 1997). This view of function allows the student to ‘operate’ a function without being aware of patterns or relationships between the inputs and outputs. Sfard and Linchevski (1994) refer to this stage of functional understanding as the Operational Phase and emphasizes the students’ tendency to use algebraic expressions as generalized arithmetic. An operational or action-oriented view of function gives the student a shallow view of function and should therefore be given opportunities to advance their understanding to a more object-oriented or structural view of function. Many researchers argue the dichotomy between these two orientations. That is, they believe a student either has an action-oriented view or he or she has an object-oriented view of function. Sfard and Linchevski, however, believe there is a duality between these views.

Sfard’s (1991) theory of operational and structural views of mathematics concepts stresses the complementary aspect of the operational and the structural views. She purports,

unlike ‘conceptual’ and ‘procedural’, or ‘algorithmic’ and ‘abstract’, the terms ‘operational’ and ‘structural’ refer to inseparable, though dramatically different, facets of the same thing. Thus, we are dealing here with duality rather than dichotomy (italics in original). (p. 9)

In order to fully understand the function concept, students need a deep understanding both of the process of function as well as the relationships and patterns governed by functions that require an object orientation. A student with a deep understanding of the
process and relationships represented in functions will be able to smoothly transition from one orientation to another and be able to determine which view is appropriate to use for a given situation.

**Object-oriented perspective**

An object-oriented view of function is obtained when the student is able to view and work with the function as an object. This level of understanding is called reification. Moving a student to this place of understanding can be very challenging. “Although reification itself may be difficult to achieve, once it happens, its benefits become immediately obvious” (Sfard & Linchevski, 1994, p. 198). While the action-oriented view of function is relatively straightforward, researchers have not fully embraced one analysis of the object-oriented view of function. Several theories of student understanding in this area have been addressed in the literature. Three such views will be discussed.

**Structural perspective.** According to Sfard (1991) and Sfard and Linchevski (1994) a student has an operational (process) view of the function concept when he or she speaks in terms of processes, algorithms, and actions. Progress toward a structural view of function is typified by a student advancing through hierarchical stages until he or she can fully work with the concept as both a process and as an object. A student has a structural view of a concept when she or he is able to view the concept as an object that can be manipulated as a whole without going into detail (Sfard, 1991).

Learners typically start with an operational view of a concept and develop a structural view after much time and cognitive effort (Sfard, 1991). This progression, from the operational to the structural view, is typical for understanding function. A counter example for this progression can be seen in geometry education. Certain aspects of
geometry will be viewed structurally first, usually those concerning a particular shape or object, before detailed analysis can be done by the student (Sfard, 1991). For most of mathematics learning, however, a student will initially be able to perform operations on a given structure, before he or she is able to perceive the function as a unit that can be manipulated as a whole. Sfard (1991) theorizes that students’ progress through three stages when moving from an operational to a structural view of function: a) interiorization, b) condensation, and c) reification.

The three stages of concept development detailed by Sfard (1991) are hierarchical. A student has to have experience in and be somewhat proficient with skills from the preceding stage before moving onto the next. While a learner is becoming acquainted with processes of a concept he or she is in the interiorization stage. Once the student is familiar enough with the processes of a concept that he or she can carry out the processes through mental representations he or she is ready to begin the condensation stage. Learning to transition from interiorization to condensation is an important step as this second stage is characterized by the student “squeezing” lengthy computations and sequences of processes into more manageable units. The acquisition of condensation can be seen by the students’ growing easiness in representing a concept in multiple ways. For the function concept, research indicates that students’ understanding and abilities in translating between various modes of representations can serve as a way to promote an object view of function (Schwarz & Dreyfus, 1995; Slavit, 1997). When a student has efficiently grouped key process components of a concept, he or she is ready to advance to reification.
A student who is able to reify a concept is one who can work with the unit as a whole. For example, if a student understands that solving simultaneous equations for two linear functions have three possible answer types (zero solutions, one solution, or infinite solutions) he or she has shown an ability to view the functions as objects and can think in terms of the properties of function classes. A student who has this knowledge demonstrates an understanding that the graph of a linear function is comprised of a straight line and that this property leads to the fact that two straight lines can be parallel, cross at one point, or be copies of the same line. Once a student is at this stage he or she can view the concept structurally. Reification is not easy to achieve, however, and is one that some students may never reach for certain mathematical concepts. This is because reification is, according to Sfard (1991),

an ontological shift—a sudden ability to see something familiar in a totally new light. Thus, whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into object, into a static structure. (pp. 19-20)

Although these stages of development are hierarchical, this does not mean that an operational view is replaced by a structural view. A student with a deep understanding of function will be able see them as both objects and processes. Sfard (1991) likens this ability to a physicist understanding light in terms of both particles and waves.

_Growth points._ Ronda (2009) discusses high school students’ understanding of function in equation form by addressing various “nodes” or “growth points.” These growth points follow the process-object theory of understanding. To determine growth points and to establish a typical learning trajectory of student understanding of function Ronda (2009) administered a set of written tasks that were completed by 444 high school
students across three grade levels in the Philippines. The tasks were specifically designed to draw out student knowledge, strategy use, justification and reasoning for function related questions. To further clarify student thinking a select group of students were interviewed following completion of the written tasks. Four growth points in student understanding of function in equation form were identified: a) equations are procedures for generating values, b) equations are representations of relationships, c) equations describe properties of relationships, and d) functions are objects that can be manipulated and transformed (Ronda, 2009).

The most prevalent solution method used by Ronda’s (2009) participants was to evaluate the equation for specific values. This was the most common method for students in all three grade levels and was done even when this was the most tedious method of evaluation. Through interview analysis, it became clear that the prevalence of this method was due to the students’ lack of confidence and skills in treating the functions holistically (Ronda, 2009). A student at this level of understanding is said to be working at the first growth point. The student behavior of evaluating a function for particular values is distinct from students working at growth point 2, equations are representations of relationships, because students are not specifically demonstrating their awareness of the relationship that is represented by the function. Students working at growth point 3 demonstrated knowledge of the properties of functions such as the y-intercept or the slope of the function. For a particular task that required students to compare the rate of change between two given functions, not one student used their knowledge of slope. Most reasoned through the answer by evaluating each equation for several values and comparing those results. Only a small percentage of students (approximately 12% of
students in the third year of high school mathematics) demonstrated work in growth point 4, functions are objects that can be manipulated and transformed, which is the concept of reification (Ronda, 2009).

Ronda (2009) does not see these growth points as strictly hierarchical. Although she believes students typically start with a process oriented view of function and later move to a more structural view, her research indicates that students can show understanding in growth points 1 and 3 without having a firm grasp of growth point 2. Students who were coded at growth points 1 and 3 could interpret equations as representations of functions, but could not do so in a holistic way. Growth point 3 focuses on the development of understanding of properties of given functions and classes of functions (Ronda, 2009). Understanding of properties within a given concept was also highlighted by Sfard (1991). Slavit (1997) takes this feature of function understanding to a new level with his theory of “Property-oriented” view of function.

Property-oriented view. Slavit (1997) also describes a student’s understanding of function in terms of process-object views of orientation. However, his system focuses on the students’ knowledge of local and global properties of given functions and classes of functions. Slavit’s (1997) view of function understanding differs from others in that he believes the student will typically move from a process oriented view of function to one of multiple object oriented views. He believes an object oriented view will manifest itself through: a) correspondence view b) causation dependence c) growth-oriented view and d) covariance functional view. It is the covariance view that Slavit (1997) develops and addresses in his research. Slavit (1997) defines the property-oriented view of function as:
A property-oriented view of function deals with the gradual awareness of specific functional growth properties of a local and global nature, followed by the ability to recognize and analyze functions by identifying the presence or absence of these growth properties. (p. 266)

A student’s property-oriented view of function can continue to develop as long as he or she is presented with novel functional properties (Slavit, 1997). The theories highlighted in the previous discussion have addressed student understanding of function through their orientation. Another aspect of understanding function, which is a part of O’Callaghan’s function model, is student competency in working with and translating between different forms of representations.

**Multiple Representations and Translation**

The word “representation” can have several meanings and connotations within a mathematics classroom. One characterization of representations is internal versus external. Internal representations refer to abstraction of mathematical ideas or cognitive schemata that are developed by a learner through experience (Pape & Tchoshanov, 2001). This can be thought of as the mental pictures students have of mathematical concepts. External representations consist of any symbol, graph, diagram, picture, numeral, equation, table, etc. that is external to the person representing the object and can be viewed and interpreted by another individual. The NCTM (2000) characterizes representations as both processes and products and explains the distinction, “the act of capturing a mathematical concept or relationship in some form and to the form itself” (p. 67). They also call for representations to be treated as essential elements in supporting students through mathematics learning. This research is concerned with students’ external representations of functions and their ability to transfer from one representational form to
another. Students are expected to have an understanding of several components of representations.

The Representation Strand (NCTM, 2000) calls for instructional programs in prekindergarten through grade 12 to enable students to, a) create and use representations to organize, record, and communicate mathematical ideas, b) select, apply and translate among mathematical representations to solve problems, and c) use representations to model and interpret physical, social, and mathematical phenomena. Specifically, by the end of middle school, “students should be able to judge the advantages and disadvantages of each way of representing relationships for particular purposes” (p. 38). Before graduation the NCTM (2000) calls for students to “understand relations and functions and select, convert flexibly among, and use various representations for them” (p. 296). In general, the NCTM believes, “When students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically” (p. 67).

Several researchers echo the benefits of instruction of multiple representations including Herman (2007) who found that students who used several representations to solve a problem were more likely to arrive at a correct answer. An understanding of the various representations of a function (i.e. numerical/tabular, graphical, verbal, symbolic) will allow students to have a better overall and more connected understanding of the function concept. Other researchers show the practical use of knowledge in multiple representations as in a chemistry education course where students were working with representations of chemical reactions. It was shown that students who were instructed in the use of multiple levels of representation acquired a more meaningful understanding of
changes that occur in chemical reactions (Chandrasegaran, Treagust, & Mocerino, 2007). In interview sessions regarding representation choice and preferences, students noted advantages to knowing more than one representation for a function; a) so they have a repertoire of strategies to choose from and b) for the thought processes involved, which lead to mental connections (Herman, 2007).

Friedlander and Tabach (2001) discuss the advantages and disadvantages of each representation (summarized in Table 2.2). They conclude, “The importance of working with various representations is a result of these and other advantages and disadvantages of each representation and of the need to cater to students’ individual styles of thinking” (p. 174).

<table>
<thead>
<tr>
<th>Representation</th>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Verbal</td>
<td>• Natural environment for understanding and communicating</td>
<td>• Ambiguous</td>
</tr>
<tr>
<td></td>
<td>• A tool for solving problems</td>
<td>• Elicits irrelevant or misleading associations</td>
</tr>
<tr>
<td></td>
<td>• Facilitates the presentation and application of general patterns</td>
<td>• Less universal</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Dependence on personal style</td>
</tr>
</tbody>
</table>

Table 2.2: Advantages and Disadvantages of Representational Forms of a Function as Presented by Friedlander and Tabach (2001)
Table 2.2 continued

<table>
<thead>
<tr>
<th>Numerical/Tabular</th>
<th>Graphical</th>
<th>Symbolic/Algebraic</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Familiar to students coming into algebra</td>
<td>• Effective in providing a clear picture of a real valued function</td>
<td>• Concise</td>
</tr>
<tr>
<td>• Convenient and effective bridge to algebra</td>
<td>• Intuitive and good for students with a visual approach</td>
<td>• General</td>
</tr>
<tr>
<td>• Precedes all other representations</td>
<td>• Effective in the presentation of patterns and mathematical models</td>
<td>• Effective in the presentation of patterns and mathematical models</td>
</tr>
<tr>
<td>• Can investigate particular cases</td>
<td>• Manipulation of algebraic objects can be the only way of justifying or proving a general statement</td>
<td>• Manipulation of algebraic objects can be the only way of justifying or proving a general statement</td>
</tr>
<tr>
<td>• Use of numbers is needed to acquire initial understanding of a problem</td>
<td></td>
<td>• Exclusive use of algebraic symbols can blur or obstruct mathematical meaning or the nature of the represented object</td>
</tr>
<tr>
<td></td>
<td>• Lack of generality</td>
<td>• Can cause difficulty in interpreting results</td>
</tr>
<tr>
<td></td>
<td>• Not very effective in providing a general picture</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Some important aspects or solutions can be missed</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Lack of accuracy</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Influenced by external factors</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Often presents only a section of the domain and range</td>
<td></td>
</tr>
</tbody>
</table>

Although there has been an upswing in curriculum coverage of multiple representations (Senk & Thompson, 2006) and a call for a more thorough use of representations in the K-12 education (NCTM, 2000) there is evidence that students have difficulty with mastering the various representations. Friedlander and Tabach (2001) note:
We cannot expect the ability to work with a variety of representations to develop spontaneously. Therefore, when students are learning algebra in either a technologically based or a conventional environment, their awareness of and ability to use various representations must be promoted actively and systematically. (p. 175)

A study comparing the representations used by American, Chinese, Taiwanese, and Japanese students reveals that American students, in particular, struggle with problems that require representational understanding (Brenner, Herman, Ho, & Zimmer, 1999). All students within this study were shown to have more difficulty with representational items than with solving given equations, but “Students in the U.S. sample had strikingly low performances on the representational subitems, scoring with less than 50% accuracy on more than ½ of the questions” (Brenner et al., 1999, p. 550). This research also demonstrated that even when students had computational skills within each individual representation, they did not always understand the connection between the representations. However, it has been shown that students can gain understanding and fluency in using multiple representations when their instruction is integrated within the curriculum (Brenner et al., 1997; Keller & Hirsch, 1998; Van Streun, 2000; Senk & Thompson, 2006; Chandrasegaran et al., 2007).

Students who were instructed in multiple representations through an integrated curriculum where heuristic methods were used throughout the course were found to have a higher ability to solve applied problems than those who did not have many opportunities to use multiple representations (Van Streun, 2000). The students in the latter group, those who did not receive much explicit instruction in the use of various representational forms, acquired basic knowledge, but were unable to flexibly use their understanding (Van Streun, 2000). Senk and Thompson (2006) discovered that students
tended toward the use of multiple representations whether or not they were receiving special curriculum, however, the use of graphical strategies increased dramatically for students using the University of Chicago School Mathematics Project (UCSMP) curriculum. A special curriculum used in a study conducted by Brenner et al. (1997) shows a switch in preference of representation by students within the treatment group. Students within the control group used graphs and tables 14-21% of the time while those in the treatment group chose to use these representations approximately 30% of the time (Brenner et al., 1997). Unfortunately, a curriculum that emphasizes knowledge and understanding of multiple representations will most likely sacrifice the level of understanding students have with symbolic manipulation (Brenner et al., 1997). Students in the comparison group did considerably better on solving equations than did the treatment group. Therefore, the research team concluded:

There is no evidence that instruction emphasizing problem representation skills has stronger influence on students’ equation solving behavior than does instruction emphasizing symbol-manipulation skills. If accuracy in solving equations is the goal of instruction, then conventional methods of instruction appear to be more effective than methods that emphasize multiple representations of word problems. (p. 682)

Due to the importance of using multiple representations and being able to translate between various forms of a function, much research has focused on students preferences of function representations (Brenner et al., 1997; Keller & Hirsch, 1998; Friedlander & Tabach, 2001; Akkus & Cakiroglu, 2006; Senk & Thompson, 2006; Herman, 2007; Huntley & Davis, 2008).

A preference in representations seems to shift between middle school and high school/college students. This is likely due to the fact that most students are introduced to
formal algebraic concepts in 9th grade. For students in middle school there seems to be a tendency toward using numerical/tabular representations (Brenner et al., 1997; Friedlander & Tabach, 2001) whereas for high school and college students, Herman (2007) found “tabular approaches were not used as a problem-solving technique” (p. 35). An exception to the findings from Brenner et al. (1997) is presented in research by Akkus and Cakiroglu (2006) who found that some seventh-grade students have a preference of solving problems using equations. The students of the present study, however, would not need to perform any complex symbolic manipulations to reach a conclusion. Students who have had at least an introduction to formal algebra strongly prefer to use the symbolic representation of a function (Keller & Hirsch, 1998; Senk & Thompson, 2006; Herman, 2007; Huntley & Davis, 2008).

Student preference for symbolic manipulation seems to be rooted in several factors. First, symbolic manipulation has traditionally dominated the algebra curriculum (Senk & Thompson, 2006). There is evidence that students perceive the emphasis placed on symbolic manipulation. Students who were being interviewed regarding their choice of representation answered, “We always do problems by hand first” (Herman, 2007). Research also indicates that teachers perceive symbolic and graphical forms of an equation as most important for student understanding (Herman, 2007). This emphasis has, in many cases, influenced student beliefs on what is “allowed” and what is appropriate strategies for solving algebra problems (Herman, 2007). Some students who were interviewed by Herman thought that using a calculator to solve a problem would be “cheating.” Keller and Hirsch (1998) found that student preference of representation was based on, a) nature of students’ experiences with each representation, b) students’
perceptions of the acceptability of certain representations, c) the level of the task, and d) the context of the problem. This latter reason for preference of students addresses how the problem is communicated to the student. The researchers found that when problems were presented as an equation students would typically solve (or attempt to solve) them using symbolic manipulation. However, for problems that were imbedded in context, students tended to use tables and graphs for their solution. The increased use of tables and graphs by Keller and Hirsch’s participants supports other theories that indicate the importance of presenting problems using a variety of representations (Friedlander & Tabach, 2001). The use and choice between various representations can also be influenced by a student’s spatial ability level.

There is a link between students achievement level, their spatial reasoning skills, and their usage of multiple representations (Erbilgin & Fernandez, 2004). Erbilgin and Fernandez used a purposeful selection of four students where each student fit into one of the following categories; a) high achieving-high spatial abilities, b) low achieving-high spatial abilities, c) high achieving-low spatial abilities, and d) low achieving-low spatial abilities. The results indicate that the student who was high achieving-high spatial ability had the strongest connections between the multiple representations and was flexible in using each form. The student who was high achieving-low spatial abilities had a good understanding of linear functions, but had weak connections between the various representations (Erbilgin & Fernandez, 2004). Erbilgin and Fernandez have particular relevance because of the population being considered in the current study. Students who are congenitally blind, in particular, are not necessarily going to have strong spatial reasoning skills. Erbilgin and Fernandez (2004) found that for the two students with low
spatial reasoning skills they either had difficulty in making necessary connections between representations or they avoided using multiple representations altogether. The student population of the blind and those with visual impairments has some unique educational challenges. The remainder of this chapter will discuss some of those challenges and how they relate to the mathematics classroom.

**Education for Students with Visual Impairments**

This section will cover the specialized needs of students with visual impairments in the field of education. To do this I begin with a brief history of the education of this group as well as a look at the legislation that has increased the rights of individuals with disabilities and how these laws have affected the education of these students. After reviewing the history, I review current educational practices for students with visual impairments.

**History of Education for Students with Visual Impairments**

The education of students with visual impairments has changed radically in the last century. Educational reforms have helped to improve opportunities for students with visual impairments. These educational reforms directly affect the mathematics learning of students with visual impairments as they provide greater access to equitable education in the field. Therefore, I will discuss the history of education, in general, for students with visual impairments before I discuss students’ with visual impairments specific needs for learning mathematics.

Berthold Lowenfeld (1956, 1975, 1981) has written extensively on the history of blind persons and the views of society toward these individuals. He has determined three
distinct periods in history in which blind persons have been treated in fundamentally
different ways, 1) primitive society, 2) a period of humanitarian views towards
individuals who are blind, and 3) a period of integration of blind persons into society.
Lowenfeld contends, “only in this third period was the soil prepared for the beginning of
planned educational attempts” (p. 5). For this reason, I will include only a review of the
literature for the latter period. However, I contend that this period, at least in terms of
education, was divided into a time of segregation before there were steps taken for full
integration of students who are blind into society. The following is a brief history of
education for students with visual impairments I begin with a brief overview in Europe
and move to an emphasis on the practices used in the United States.

Formal education for students with visual impairments was started in Europe in
the late 1700s by Valentin Hauy who opened a school in Paris called the Institution des
Jeunes Aveugles (Institute for Blind Youth) (Osgood, 2005). This school aimed to meet
the specific educational needs of blind children. Hauy believed that the greatest need for
this population was an independent means of reading and writing. Therefore, he invented
a printing process that embossed letters which allowed for tactile reading. The opening of
this institution and the invention of this specialized printing press marked a great
milestone in the history of educating blind children. Hauy’s methods were soon
implemented throughout Europe.

Historical events that made the opening of Hauy’s institution possible included
the precedent established for the education of the deaf by Abbe De L’Epee, the inhumane
treatment, which some blind beggar-musicians met at the hand of the Parisian populace,
and the ever increasing evidence that individuals who are blind are capable of being
educated (Lowenfeld, 1981; reprinted from 1946). Although there was increased interest in protecting and educating individuals with visual impairments in the late 18th century, “at that time the educability of blind children was by no means the generally accepted fact, and education in residential schools especially geared to the needs of blind pupils offered the best solution to the problem” (Lowenfeld, 1981, p. 11). Regardless of the merits or inherent downfalls of segregated education for blind children, the standards upon which other nations would replicate was set once the doors of Institution des Jeunes Aveugles were opened.

Germany’s early educational system for their blind population became an example of an extreme case of segregated education. Here, school aged children who were blind attended a residential school in one building, upon graduation they moved to another building on the same campus after which they would be kept “productively occupied” until they became too old or weak to continue work at which time they would be moved to yet another building that housed the aged blind population (Lowenfeld, 1981).

European countries were not the only ones to gain inspiration from the success of Hauy’s school. Samuel Gridley Howe visited several European institutions for educating blind students, including Hauy’s in Paris, and used the information from these trips to open United States’ first school for blind children in Massachusetts in 1832 called the Asylum for the Blind (now called Perkins School for the Blind). Events in the United States that precipitated the opening of such a school included the issue of overcrowding and little manpower in typical classrooms throughout the country. This overcrowding was due, in large part, to the fact that in the early decades of the country there were few cities
and much of the population lived in rural settings where formal schooling was not always available. When formal education was offered, within some of the larger school districts in the 1800s, the class size could be as large as 70 or 80 students in one classroom (Osgood, 2008). This led to an interest in institutions for students who either were seen as a threat or those who needed considerable time and materials to be included in learning. Before such institutions, the education of blind students depended on the teacher making time to individually work with them and to find materials that would allow them to learn alongside their classmates.

Massachusetts opened some of the first institutions for specific student populations, including Howe’s Asylum for the Blind and the Idiotic and Feeble-minded Youth (opened in 1848) (Osgood, 2008). Institutions like these increased in size and number throughout the United States.

The startling rise in the number and scope of separate classes for children with various exceptionalities took place during a period when an enduring, widespread, and mostly unchallenged belief held sway, which stated that the segregation and even isolation of these children was in the best interests of pedagogy, school management, and social control. (Osgood, 2005, p. 27)

In 1837, the New York Institution for the Blind, the Philadelphia Institution for the Blind and the Ohio State School for the Blind became the first schools specifically for the blind that were completely supported on public funds and by 1875 twenty-nine states had State Schools for the Blind (Linsenmeier & Moyer, 2006).

As more rural areas became increasingly populated, formal schooling was introduced at a fairly quick pace. Another change within the young educational system was the practice started in 1847 in the Boston public schools of assigning specific grades according to chronological age. “Consequently, students found themselves grouped in
classrooms with other children who were close in age but could vary dramatically in background, interests, skills, abilities, and preparation” (Osgood, 2005). By 1918 all states had compulsory education laws (Scott & Santos de Barona, 2007) and almost all educational institutions had graded classes that were based on age. The increase in population led to a more formal education that helped to magnify the visibility of and a concern for disabled persons (Osgood, 2008). All of these changes to the educational system created an environment that was unfriendly toward the pursuit of some advocates to integrate students with visual impairments (and those with other disabilities) into the general education populace.

Samuel Gridley Howe was one of the strongest advocates for public school education for students who are blind (Lowenfeld, 1981; reprinted from 1957). The residential school system had inherent problems that stemmed from the students being isolated from society and living apart from their families for much of the year. However, “Howe insisted that although [residential] institutions may be isolated from the outside world, they nevertheless should be considered an important part of the educational structure” (Osgood, 2005, p. 21). Others believed that segregated education was not the wisest approach and a few school districts attempted to implement what was then called the “Batavia Plan” or the “Winnetka Plan” where “assistant or unassigned teachers” would work with students who needed extra assistance and in some ways provided individualized instruction to pupils who were considered “laggards” or “retarded” (Osgood, 2005).

Lowenfeld (1981) discusses some of the criticisms regarding education in the residential schools including students leading a sheltered life where they were not “facing
realities” and a lack of independence due to poor training of “foot travel.” In response to these criticisms, Lowenfeld maintains that residential school leadership must cooperate with the state supervisory program in order to enable as many children with visual impairments as possible to live at home and attend regular public school. Some schools met this challenge by providing opportunities for their residential students to attend some classes at the public school or to allow certain high school pupils to attend the local public school for their last two years of instruction.

Programs designed to give students who are blind the opportunity to study alongside peers with typical sight was met with opposition. For instance, the beliefs of school administrators and educators helped to determine the role of residential schools for the education of blind persons. These professionals relied on two fundamental arguments for segregating persons with disabilities from the regular classroom; that segregation was needed to ensure efficient classroom and school operations and that separate programs for students with disabilities would be in the students’ best educational and psychological interests (Osgood, 2005). Factors that contributed to these beliefs included overcrowding in schools and a perceived need for homogeneous classrooms. At a presentation of school administrators in 1941, Lowenfeld (1981; reprinted from 1941) discussed concerns of administrators regarding educating the blind in public facilities.

The administration problems with regard to public school classes for the blind have a common root in the fact that the administration of our public school system serves a homogeneous group of children—homogeneous because the large number makes individual differences disappear—and has developed highly standardized procedures. (p. 6)
Another argument for segregated education was the supposed benefit to the more capable students, who would be able to receive more of the teachers’ time and attention (Osgood, 2005).

Despite the reluctance of some to embrace inclusive education for students with disabilities there were some who were emphatic about providing these educational opportunities for students with visual impairments.

It is not by chance that the decisive step towards a final integration of the blind child into the general process of education was taken in the United States. Here, the public school system as the fundamental instrument of education has reached its highest development (Lowenfeld, 1981, p. 6).

A public school option specifically for students with visual impairments, called “Braille classes,” became available in many public school settings. This educational placement was similar to that of “mainstreaming.” That is, students with visual impairments would attend regular classroom instruction as well as additional instruction for learning braille and other educational skills specific to their visual impairment. The establishment of these braille classes in public schools was based on the need for increasing opportunities for integration of the blind into society and the recognition of the importance of the family life for the individual child (Lowenfeld, 1981).

The Oregon Project, started in the 1940s, represented the start of the next basic development in educational facilities for children who are blind. This project promoted special classes for students with visual impairments in the local public schools and “organized a unique program of coordination of public school instruction and specialized provisions for visually handicapped children” (Lowenfeld, 1981; reprinted from 1946, p. 12). This program came about through the collaboration between the Superintendent of
Public Instruction, the director of the Division of Special Education, the Superintendent of the Oregon State School for the Blind, and the supervisor of Education of the Visually Handicapped. Three fundamental principles of this project were, a) Any child who can be educated in the public schools should not be institutionalized or even segregated, b) the residential school for the blind has as its aim the rehabilitation of visually handicapped children for public school education and is not interested in keeping children any longer than necessary, and c) each child has a right to individualized services both in regard to optimum restoration of sight as well as in regard to the use of aids which will conserve and safeguard his or her sight (Lowenfeld, 1981).

Integration into the public school system has not been fully realized and the progress toward full inclusion has been slow in coming. Seven years after Lowenfeld reported on the Oregon Project a survey was conducted on residential school policies and beliefs held by the superintendents of these schools (Buell, 1953). The fifty-five schools who participated in this survey were asked to give their policies regarding high school student’s placement and opportunities for pupils to attend local public schools. Three distinct groups emerged from the data, schools which had students remain at the residential school throughout high school, those who sent some high school pupils to the regular public school, and those who sent all pupils above a certain grade to public high school. It was found that half of the schools retained students through high school and of the remaining schools; two-thirds sent some students to public school. Only eight of the schools surveyed had policies that required students beyond a certain grade to attend local public classrooms. Three of these schools required this integration into public school for a limited number of subjects and others would send only a few of their most
talented students to the local public school. Only one state followed the Oregon Plan completely while others had a modified version of the plan. Only 14 percent of the residential schools for the blind sent all pupils to public high schools for some or all of their education (Buell, 1953).

The transformation of education being administered through segregated institutions for people with disabilities to integrated settings has been brought about, in a large part, by court cases, laws, parent and teacher advocacy, and a civil rights movement that moved the nation toward equality. These forces fostered a transition from special education and disability services being provided by private institutions to being a central component in the public schools of today. This transition took place over several hundred years and was the result of a growing awareness of the extent of disability within the United States (Osgood, 2008).

The decision to keep children who are blind in the local school system often ended in court cases as the school districts did not have or could not provide the resources to adequately educate students who had additional and costly needs. These court battles have led to legislation on equity issues for persons with disabilities. Specifically, four laws have radically changed the way in which public schools handle the education of people with disabilities. These are Public Law 94-142, the Education for All Handicapped Children Act (now named the Individuals with Disabilities Education Act (IDEA)), the Rehabilitation Act of 1973-Section 504, and the Americans with Disabilities Act of 1990. The latter two regulations mainly affect public schools by prohibiting disability-based discrimination, in general, while the IDEA directly speaks to the education of persons with disabilities (Essex, 1999).
The IDEA has given parents and educators two tools in which to guide the overall perception of disability education; a) the Individualized Educational Program (IEP) and b) the standard that students are to be educated in the Least Restrictive Environment (Essex, 1999). These two concepts have reshaped the experiences of students with disabilities in the public school system. With legal backing, students can not be turned away from receiving a free and appropriate education, for any reason. This led to an increase in the percentage of students with disabilities in the public schools, which went from 8.5% in the 1976-1977 school year to 13% in the 1995-1996 school year (Essex, 1999). Districts handled the influx of people with disabilities in different ways and with a variety of educational models, including full inclusive practices and partial inclusion with pull-out classes for any additional help for instruction on specific skills related to the visual impairment. The specific needs of students with visual disabilities have been a topic of much debate and one that has changed over several decades. A list of these specialized needs has been culminated into what is known as the Expanded Core Curriculum, which I will discuss next.

Education in the public school districts has a concept known as the Core Curriculum. Within this curriculum students must be involved in some specific school subjects, and in some instances must prove competency in before graduating high school. These subjects include English, mathematics, health, social studies, economics, fine arts, science, physical education, history, business education, and vocational education (Hatlen, 1996). For students who are blind or have visual impairments there are additional skills they need to develop to help ensure a productive, healthy life. These are skills that are particular to this group of students and are ones that should be taught early
in life and through their school-aged years. These “extra” skills are needed and, by the IDEA act of 1997 are guaranteed to be provided, no matter the educational setting of the student (Essex, 1999). This guarantee was not always given as early ideas of inclusive education held that all academic training needed could be obtained through the “regular” core curriculum and little, if any, was done on the behalf of the student to receive these additional skills (Hatlen, 1996).

The discrepancy between the needs of the student and the services available led to the development and adoption of the Expanded Core Curriculum (ECC) for students who are blind or have visual impairments. The list of skills on the ECC have been modified through the years, but to date has 9 topics that consist of: 1) accessing assistive technology, 2) career education, 3) compensatory skills, including braille, functional academics, communication and tactile graphics, 4) independent living, 5) orientation and mobility, 6) recreation and leisure, 7) social interaction, 8) visual efficiency and 9) self-determination.

**Mathematics Learning for Students with Visual Impairments**

There is a rich history of educational ventures aimed at improving the educational outcomes in mathematics and the sciences for students with visual impairments, however, little of this work utilizes empirical research. Initiatives for improving mathematics and science education for students with visual impairments can be seen throughout the last century and evidence of others who have been successful in mathematics and various science fields are documented as early as the mid-seventeenth century. A biography of persons with visual impairments who made outstanding contributions to society or to their field of study was originally published in 1821 and described more than 60
individuals, including several mathematicians. Laboratory techniques for blind students were described in a journal article in 1972 and another in 1977. A summary of mathematical learning by students with visual impairments was published in 1979 and another soon followed in 1981.

The Research and Development Institute (2006) cited the NEW BEACON journal from 1934 asserting that there were two essential needs for mathematics students who are blind. The first was a comprehensive system of notation that was capable of expressing all mathematical relationships neatly and concisely. The second was an apparatus that would enable students to have a concrete representation of drawings used for geometry. The comprehensive system of notation for expressing mathematics was not invented until the 1950’s when Abraham Nemeth was studying for his Ph.D. in mathematics at Columbia University and found that “as the mathematical concepts, and therefore the notation, became increasingly intricate, I found that the braille techniques for expressing mathematical notation were either inadequate or non-existent” (Nemeth, 1996, paragraph 6). Over time he began to improvise new braille techniques to make it possible for him to adequately write the entire mathematical notation he needed. It might not have gone beyond a personal braille code if it had not been for a blind nuclear physicist, a colleague of Nemeth, who asked if he had a table of integrals in braille. Upon learning that Nemeth did indeed have such a table and that this table was written in a code Nemeth invented, the colleague insisted that Nemeth teach him the code. The code was officially adopted as the standard mathematics code in the United States for Mathematics and the Sciences in 1953. However, even with the invention of a system for reading and writing
mathematical notations, many students still struggle to learn mathematics and scientific concepts.

The struggles in learning mathematics for students with visual impairments go beyond understanding the basic concepts of the subject. There are mechanical and language issues regarding how students with visual impairments are taught and the materials in which they have access. Some research indicates that the majority of problems faced in mathematics for students with visual impairments are with the limitations of available tools and not the cognitive understanding of the material (Cahill, Linehan, McCarthy, Bormans, & Engelen, 1996).

In the following section, I discuss the mathematics learning challenges faced by students with visual impairments, teacher preparation in the area of mathematics education for students with visual impairments, methods utilized in mathematics classrooms to help ensure student success, and resources available for mathematics educators who are instructing students with visual impairments. During this discussion I will draw upon research on learning outcomes for students with visual impairments in the science classroom, which is a research area that has grown in recent years and can shed light on the area of mathematics learning. I will also draw upon anecdotal information as well as research in the area of mathematics learning for students with visual impairments. The reason for the inclusion of anecdotal information is explained.

The No Child Left Behind Act (NCLB) initiated a renewed focus on research standards in the field of education. NCLB called for scientifically-based research that was defined as studies that a) used the scientific method, b) had results that were replicated in more than one setting and by more than one researcher, and c) resulted in clear
conclusions (Ferrell, 2006). This definition of research and its implication for educational studies involving students with visual impairments piqued the interest of Ferrell who decided to do a literature review of literacy and mathematics studies regarding students with visual impairments. Her results are disappointing. To qualify to be a part of Ferrell’s study the research had to be published between 1963 and 2003, involve children with visual impairments of ages 3 to 21, and contain an intervention and some type of comparison group. In the area of literacy 652 articles were identified, but only 32 of these met all three criteria. For mathematics, only 128 articles were identified, 10 of which met the criteria (Ferrell, 2006). This review of the literature led Ferrell to conclude, “for students with visual impairments…the best practices are more often than not based on tradition, superstition, anecdote, and common sense rather than science” (2006, p. 42). Even when research is extended to include rigorous qualitative studies, there seems to be a dearth in the area of mathematics learning research for students with visual impairments. However, there can be a lot gained by reviewing practices that educators and other professionals have found successful. Therefore, I will now review the literature, both research-based and anecdotal, as it relates to science and mathematics learning for students with visual impairments.

**Mathematics Learning Challenges**

Students in the Mathematical Access to Technology and Science (MATHS) project were asked to rate which mathematical concepts they found most challenging. Participants who are blind indicated graphs, logarithms, set notation, tables, and trigonometry were the most difficult and algebra and punctuation were the least difficult. Students with partial sight were similar in that their most difficult concepts were
logarithms and trigonometry and the least difficult concepts were algebra and punctuation. Sighted students also rated logarithms and trigonometry as the most difficult topics, but noted punctuation and set notation as the least difficult (Cahill et al., 1996). The survey also included a section that focused on difficulties students had with specific algebra topics such as semantics-understanding, control, manipulation, memory, speed, and syntax-layout. Manipulation, speed, and memory caused the most difficulty for students with visual impairments. Students who are blind also had trouble with control, which was defined as the access to controlled reading and navigation through an expression, a general overview and a detailed inspection of the problem (Cahill et al., 1996). Calculations are just a small portion of mathematics learning. Topics involving geometry and spatial reasoning are also important for all mathematics students to comprehend.

Mathematics is a field that not only includes numerical relationships, but one that works with shapes and various rotations of shapes in space. The study of spatial relationship in mathematics is often taught with a heavy reliance of visualization and mental rotation. Research has been conducted on the mental rotation abilities of students who are blind and those with visual impairments. The study found that participants who were blind performed less accurately than did the participants with residual vision. It was also shown that the success of mental rotation partly depended on the student having an accurate mental representation of the layout of the object (Koustriava & Papadopoulos, 2010). A student who is blind was asked to give accounts of how he interpreted spatial concepts. It was shown that the young man had a well-defined comprehension of his surrounding environment and that he used a variety of senses to build this knowledge.
For instance, he would use the sound of the echo of his voice to determine the approximate size of a room (Koustriava & Papadopoulos, 2010). Although there are challenges for mathematics students with visual impairments, there are also areas of mathematics learning in which students with visual impairments have been shown to outperform their sighted peers.

The development of the ability to estimate and measure the size of objects for students with visual impairments and their sighted peers was conducted by Andreou and Kotsis (2005/2006). Objects that were used for this study included, the students’ school desk (both the length and surface area), bed (as measured in meters and in number of steps), liter of milk (as measured by the number of glasses it would take to fill). Finally, students were asked which cup they would rather use if they were drinking their favorite beverage (between two distinct cups). Participants who are blind outperformed their sighted peers on every question except for the measurement in meters of the students’ bed. Fifty-eight percent of the sighted students were able to correctly estimate the length of their bed in meters, while 52% of the students who are blind were able to identify the correct length of their bed in meters. However, when it came to estimating the number of steps the length of a bed is, the students who are blind far exceeded their sighted peers (90% correct for the participants who are blind; 37% correct for sighted participants).

Another challenge for students with visual impairments when learning mathematics is understanding a two-dimensional drawing of a three-dimensional object (Fisher & Hartmann, 2005; Spindler, 2006). The efficient and accurate use of graphical materials is also a concern for students with visual impairments. Teachers of the visually impaired list “preparation of graphical material” as one of their primary concerns for
supporting students’ mathematical learning. Students with severe visual impairments rated graphical-spatial mathematical topics, such as working with tables and graphs, as more difficult than the participants who had some usable vision (Cahill et al., 1996). A particular difficulty for student with visual impairment in learning to read and produce graphs is the need for specialized instruction and skill development in tactile discrimination. Skills for reading tactile graphs and diagrams vary widely between students based on prior experiences with tactile graphs, whether or not the student received direct instruction on efficient ways to read tactile graphs, and student characteristics such as fingertip sensitivity. These factors will help determine how successful a student is when learning to read mathematical graphs. For this reason, Withagen, Vervloed, Janssen, Knoors, and Verhoeven, (2009), developed a system of charting tactual functions, called a student’s Tactual Profile. The Tactile Profile procedures were tested within a school system and it was found that administrators and special education teachers were impressed with how the profile helped them gain insight into the intervention needs of the students. Student success in the mathematics classroom depends on a multitude of professionals, all of which help to shape the student’s understanding of mathematics and his or her ability to function effectively and independently in the mathematics classroom. Therefore, teacher preparation for both mathematics educators and for teachers of the visually impaired is vital for student success.

Access to Mathematics Textbooks. Textbooks are used in almost every classroom throughout America. The textbook in many cases sets entire curriculum. This importance has given rise to several troubling points for students with visual impairments. First of all,
not all textbooks are available in Braille or large print (Dick & Kubiak, 1997). A school district may order a book brailled through the American Printing House for the Blind (APH), but this process can take one to two years to complete (Dick & Kubiak, 1997) and therefore must be decided upon early. The visual nature of mathematics adds additional complications to the accessibility of textbooks. Most mathematics textbooks are covered with diagrams, pictures, charts, and graphs, all of which have to be converted to tactile form to be included in the brailled version of the text. The use of braille or large print textbooks is also awkward. The books are larger than the print books and, often, come in multiple volumes. Students can find it difficult to gain an understanding of the graph, as a whole, since they can only view portions of the graph at a time. An alternative to Braille and large print texts is that of audio-textbooks.

The use of audio textbooks (i.e. books on tape or books on CD) has diminished through the last decade as braille transcription technology improves and digital options become more readily available for students. Mathematics textbooks in an audio format have inherent problems including how difficult and time consuming it can be to find a particular passage. Availability has also been an issue with audio versions of mathematics textbooks. Peter, a student learning calculus, found audio recordings of mathematics books difficult to find (especially at the calculus level) and hard to listen to because of the length of the tapes (Spindler, 2006). Dick and Kubiak (1997) note:

Taped mathematical texts can be difficult to understand because the readers may not pace them appropriately or read with a mathematical interpretation. As a result, a listener may not easily be able to follow how a line-by-line analysis demonstrates the steps taken to perform an operation. (p. 346)
The heavy dependence on visual content can also be difficult to translate into audio recordings. An audio version of the text provides only the reader’s description of a graph and cannot be independently analyzed by the student. However, David Moore, a mathematician who has been blind since the age of thirteen, recommends the use of audio descriptions for accessing graphical material (personal communication, 2011). He has found that accessing graphs through voiced descriptions has provided him with rich information that he would otherwise not be able to access. For Braille readers, becoming fluent with the Nemeth Code can be a further hindrance in accessing textbooks.

The Nemeth Code is a braille code that is specifically used for mathematics and science. While there are programs that can translate print English into Literary Braille and vice versa, there is no equivalent program for the Nemeth Code. A process can be taken to translate mathematics into the Nemeth Code, but it involves translating the material through Scientific Notebook (a software package for seamlessly writing mathematics text and graphics) and then converting that text into Duxbury (a software package that converts print to literary Braille). Through these translations mistakes are prevalent and the end product must be checked for accuracy. This makes producing Nemeth Code products more time consuming and tedious. Several organizations are working to solve this dilemma, including the Mathematical Access to Technology and Science (MATHS) project (Cahill et al., 1996) and the Mathematics Accessible to Visually Impaired Students (MAVIS) project (Karshmer, Gupta, Geiger, & Weaver, 1999). One constraint that makes the Nemeth Code difficult to translate is the underlying differences in structure between print mathematics and mathematics presented in braille. That is, Nemeth code is constructed linearly, while print mathematics is not (Kumagai,
For instance, in print mathematics a division problem is typically divided into a numerator that is over a denominator. This same scenario in Braille would place the numerator next to the denominator. Another aspect of the tools of mathematics education is that of models and the use of multiple representations.

**Teacher preparation**

Kapperman and Sticken (2002) believe that the reason for the poor representation of persons who are blind in technical career fields such as engineering, information technology, and computer sciences is that many students with visual impairments are “mathematically illiterate” in that they are unable to read and write the Nemeth Code. Students who cannot independently read and write mathematics are unlikely to be able to demonstrate the sophisticated mathematical knowledge that is needed to enter technical fields. In order to increase the numbers of students with visual impairments that are able to independently read and write mathematics, Rosenblum and Amato (2004) believe it is imperative for teachers of the visually impaired to receive appropriate training in the Nemeth Code. To determine the current state of teacher preparation in the Nemeth Code, Rosenblum and Amato surveyed teachers of the visually impaired on a) educators’ preparation in the Nemeth Code, b) educators’ daily use of the Nemeth Code within the workplace, and c) educator concerns regarding support for their mathematics students.

Teacher preparation in the Nemeth Code indicated that 13.4% of participants had taken a course in literary Braille, but not in the Nemeth Code and that 34.4% said, “My preservice training in the Nemeth code was so rudimentary that I learned most of what I need to know while on the job” (Rosenblum & Amato, 2004, p. 489). Of the 135 participants, 37 had help with transcribing materials into braille and 33 had teaching
assistants. The lack of understanding of the Nemeth Code by teachers of the visually impaired undermines student use of the Nemeth Code by relegating all mathematics information in the Nemeth Code to that offered in the textbook as students would not have access to classroom materials in the Nemeth Code. A lack of available materials in the Nemeth Code also undermines students’ ability to be independent in mathematics learning. Anyone who needs assistance in learning the Nemeth Code can download a free tutorial that is intended for the use of sighted individuals and is composed of 18 lessons on how to write mathematical symbols and expressions, how to read mathematical expressions in the Nemeth Code, and efficient methods of proofreading texts in produced in the Nemeth Code (Kapperman & Sticken, 2002, 2003).

The makeup of American classrooms today is such that educators have a wide variety of needs to meet in order to ensure student success, which includes the success of students with visual impairments. According to the 2009 Annual Report of the APH, of the 59,355 students registered for their services, 83.3% attend their local school. Students who have a visual disability who attend their local school usually interact with their general educator as well as a teacher of the visually impaired. This arrangement can be helpful, but it can also lead to confusion over who is ultimately responsible for students’ education. The mathematics educator has an intimate knowledge of the content, while the teacher of the visually impaired has knowledge of effective methods of teaching students who are blind. Although there are unique skills and knowledge held by individual professionals, it is important that the classroom teacher is willing to work with students with visual impairments and to accommodate them the best they can, and for teachers of the visually impaired to have an underlying basis of mathematics knowledge or to be
willing to learn such knowledge when working with a mathematics student with a visual
impairment. The exchange of knowledge between mathematics educators and teachers of
the visually impaired is important, but does not always occur.

Research indicates that special education teachers do not have sufficient
knowledge of the NCTM standards, which can undermine their support of the
mathematical learning of their students (ERIC Clearinghouse on Disabilities and Gifted
Education, 2002). Also, general education teachers do not always understand how to
make their teaching more effective for their students with visual impairments and some
are unwilling to change. Spindler’s (2006) tutee, Peter, had both sympathetic and
unsympathetic calculus professors. At one point Peter stopped going to class and solely
worked with his tutor because the educator was unwilling to change his method of
teaching to accommodate Peter into his class. For instance, Peter’s calculus professor
would use PowerPoint slides during instruction, but would not verbalize them (Spindler,
2006). This example may be an extreme case, but it shows that there is progress that still
needs to be made in how general educators work with their students with visual
impairments.

The Kentucky Teacher Academy (KTA) (Penrod, Haley, & Matheson, 2006) and
the Creating Laboratory Access for Science Students (CLASS) project (Kirch,
Bargerhuff, Turner, & Wheatly, 2005; Kirch, Bargerhuff, Cowan, & Wheatly, 2007)
found success with professional development opportunities for science educators and
special educators. KTA held teacher training opportunities specifically for teaching
students with visual impairments where CLASS worked with science educators and
special educators of students with a variety of physical and sensory disabilities.
Researchers from CLASS found that special educators became more comfortable with supporting their students in science classrooms after the professional development and that science educators were better prepared and were less anxious about having a student with a physical or sensory disability after attending the two-week professional development. An integral part of the CLASS professional development was the interaction of students with disabilities with the science educators. It was the interaction of the students and educators that came across as the most important aspect of the workshop and the element that ultimately lead to attitudinal changes (Kirch et al., 2005).

The researchers from the KTA project found that giving participants opportunities to perform tasks within a science classroom while either blindfolded or while wearing vision simulators helped participants gain empathy toward their students with visual impairments. Results from pre-tests and post-tests indicate attendees of the KTA professional development program increased their level of knowledge about science and nature studies, perception of using an outdoor setting as a classroom, and a degree of comfort when teaching science to students with visual impairments (Penrod et al., 2006).

*Educational Methods for Mathematics and Science Students with Visual Impairments*

Abraham Nemeth, a mathematician who is congenitally blind and who invented the Nemeth Code for Mathematics and Science, gave a speech to the Mathematical Association of America (MAA) in 1996 and answered a survey regarding mathematics instruction for students with visual impairment, also in 1996. In each format, Nemeth contends that he did not require special methods of instruction when learning mathematics. Cary Supalo, a blind chemist, also believes his mathematics education was
complete without the need for special methodology (personal communications, 2010). Nemeth and Supalo believe that when they were given proper access to learning materials, they were able to keep up with the content of the courses without special accommodations. Both gentlemen are braille readers and Nemeth mentioned that he “could not have reached my potential in mathematics without the Nemeth Code” (Research and Development Institute, 2006). Although Nemeth and Supalo believe there is no need for special accommodation in mathematics education, there are students who have visual impairments who struggle in the area of mathematics learning that have benefited from specific instruction. The following section addresses some of the research of mathematics and science learning methods as well as anecdotal information from individuals in the field of education for students with visual impairments.

**Promising Practices**

The work of Ferrell (2006) provides a list of “promising practices” that came from the ten mathematics studies that met the criteria of, a) being published between 1963 and 2003, b) involving children with visual impairments of ages 3 to 21, and c) having an intervention and some type of comparison group. These practices mainly address computation skills. Although Ferrell’s (2006) literature review covered a 40 year span from 1963 to 2003, all of the research resulting in promising practices for mathematics was conducted between the years 1964 and 1984, with the majority occurring in the 1970s. The promising practices include, the use of concrete mathematics aids to increase computation accuracy, the use of the talking calculator to improve concept development, the use of the English Language Grammar Method and the use of “fingermath” to improve computation skills, and the use of a braille writer may be more
accurate than mental calculations or those completed on an abacus. There is conflicting reports on the accuracy and effectiveness of calculations performed on an abacus. Mathematics is a field that goes beyond calculations.

Although there is not a lot of research to build upon in the area of mathematics concept development for students with visual impairments, Lowenfeld in 1952 set forth five principles that he referred to as essential parts of a specialized methodology for students with visual impairments. The five principles of specialized instruction are a) individualization, b) concreteness, c) unified instruction, d) additional stimulation, and e) self-activity (Lowenfeld, 1981). The following discussion of the mathematics and science education for students with visual impairments are organized around these principles.

**Individualization**

Individualization recognizes that each student is an individual with individual needs and learning styles. Classroom modifications around the idea of individualization might include having extended time for students to complete activities, modifying curriculum and classroom activities to provide access to students, providing support for student learning, having opportunities for students to learn in small groups or in one-on-one situations, and ensuring that all students have quality materials that are prepared in a timely manner. Wild and Trundle (2010a) demonstrate individualization in their research on student conception of seasonal change by modifying a curriculum to specifically meet the needs of middle school students with visual impairments and comparing the conceptual change of students using the modified curriculum with students who used traditional curriculum and materials to learn about seasonal changes. The research results
show promise for lessons that are not only modified to meet individual student needs, but for lessons that are inquiry-based as opposed to one with a heavy reliance on lecture.

The participants in Wild and Trundle’s (2010a) research group were given the modified and inquiry-based lessons while the control group, which was also composed of middle school students who attend a residential school, were taught with traditional methods including lecture and the use of a model of the Earth and a model showing the Earth, Sun, and Moon relationship. Results show a greater improvement in the number of scientific concepts held by the students who completed the inquiry-based and modified lessons than those in the comparison group. At the end of instruction, students in the inquiry-based group displayed no alternative conceptions regarding seasonal changes (Wild & Trundle, 2010a).

Wild and Trundle (2010a) modified an entire curriculum for a unit on seasonal change. Others in the field of education for students with visual impairments share specific examples of ways to make particular concepts accessible and are therefore providing ideas on how to individualize the curriculum for students with visual impairments. Project Math Access (discussed in further detail in the Resource section later in this paper) provides a plethora of examples for specific mathematics topics. For instance, a recommendation on the project’s website (http://s22318.tsbvi.edu/mathproject/) for Advanced Mathematics is;

When presenting the braille symbols and teaching the concepts of intersection (dots 4-6, 1-4-6) and union (dots 4-6, 3-4-6), note that the braille “plus” sign (dots 3-4-6) is part of the symbol for union. This is because the new set which results from combining two original sets is similar to the sum in an addition problem. (Research and Development Institute, 2006)
Specific examples of this kind can be beneficial for teachers of the visually impaired as well as for general educators who would not be aware of the particular arrangements of the braille configurations. Next, I discuss ways in which educators can make the curriculum concrete for their students with visual impairments.

**Concreteness**

Concreteness provides for opportunities for touch observation. Concrete aids not only help to improve computation skills (see Ferrell, 2006), but seem to be an essential part of all aspects of mathematics and science education for students with visual impairments. Tombaugh (1972) describes a myriad of objects used for actively engaging students with visual impairments into the science laboratory. These objects ranged from simple modeling clay to instruments for weighing chemicals to large plastic models from biology supply houses. Weems (1977) describes how several new tactile aids were utilized in his physical science course to ensure concept development for a student of his who was blind. One instrument extensively used by Weems (1977) is an Optacon, which is an instrument that can obtain optical information and display it in the form of a tactile impression. As a part of a unit on oceanography, Fraser (2008) arranged for students to handle whale bones that were typically covered by a display case and to examine small sea creatures in a local aquarium. Spindler (2006) found it helpful to use pens and rulers to model vectors and curves in space and various sheets of paper to help visualize surfaces in space. In a geology course, for a student who is blind, block models of folds and faults were used and were modified by the instructor to include strips of sandpaper that indicated the presence of unconformities (Asher, 2001). An educator of a chemistry
course noted that “I used the chalk board less and relied more on physical objects that were passed around during the lecture period” (Ratliff, 1997, p. 710).

Models. The use of a variety of models and representations should be utilized for students who are blind just as they are for students with typical sight. To discover how the use of representations for the blind can become a reality, Fisher and Hartmann (2005) contacted the Hadley School for the Blind, which is the only distance education school for students with visual impairments. After this meeting the authors were convinced that it is possible for learners who are blind to use representations successfully to learn mathematics (Fisher & Hartmann, 2005). One aspect of representations discussed was the use of two-dimensional and three-dimensional representations. It is the observation of faculty at the Hadley School for the Blind that it is extremely challenging for students who are blind (and are therefore using raised line diagrams) to comprehend a two-dimensional representation of a three-dimensional object (Fisher & Hartmann, 2005). This observation was reinforced by the observations of Spindler (2006) where he noted that the most difficult idea for his tutee was with two- and three-dimensional concepts. Even though this concept is difficult for students, it is extremely important, as they will use this knowledge to interpret the raised line drawings of various geometric figures and mathematical diagrams and charts (Fisher & Hartmann, 2005).

Hadley School for the Blind emphasized how important it is for a teacher to understand how the student interprets the information of a given representation before using it. It was also noted that the possibility of a miscommunication when using a representation is the reason that educators should be prepared to explore topics using different representational systems (Fisher & Hartmann, 2005). The idea of multiple
representations is consistent with the mathematics literature. “As students become mathematically sophisticated, they develop an increasingly large repertoire of mathematical representations and the knowledge of how to use them productively” (NCTM, 2000, p. 360). Developing the use of these representations is what a teacher needs to do in order to provide appropriate tools for student success.

Several tools are available for educators to use with students with visual impairments in the mathematics classroom. As mentioned before, there are several adapted aides of typical products such as brailled and large print rulers, large print and talking calculators, brailled and large print protractors, and the like. However, there are also specialized tools for students to read and produce mathematical representations. For instance, graphs can be made with an embossed printer where braille-type dots are used to make grid lines and graph representations and raised line diagrams that are produced with a machine called Pictures In A Flash (P.I.A.F.). This machine, using heat and specialized paper, will raise dark lines to make them tactile. Peter, Spindler’s tutee (2006), found some of these raised line diagrams very helpful, specifically in integrating over areas. There are also low-tech items for producing graphs including the use of a corkboard, string, and pushpins, and the use of Wikki Sticks, long, thin, flexible wax sticks that can provide a raised surface for the student to access (Dick & Kubiak, 1997). Peter also found a variety of props, such as pens and rulers, helpful as he was able to feel the items and to visualize the vectors and curves in space (Spindler, 2006).

The Hadley School for the Blind also mentioned various tools available for the blind community, such as the Crammer abacus, which allows students to make arithmetic calculations by manipulating beads that represent numbers in base ten (Fisher &
Hartmann, 2005). Students who are proficient with the use of the abacus have little difficulty performing most mathematical calculations that are typically required for high school curricula (Fisher & Hartmann, 2005). Computer adaptations such as screen readers and screen magnifiers can help students’ access information he or she would not ordinarily be able to obtain independently. However, there are not many mathematics programs that are accessible to students who depend on screen readers to access the information. The powerful software packages such as Mathematica, Fathom, and Geometer’s Sketchpad are not accessible to these students. The tools, materials, and models used in the classroom can be effective only when educators are properly prepared and are able to use appropriate methods to reach their students.

An important step in researching the needs of students with visual impairments in mathematics education is to find what is challenging for the students. A study that addresses this issue was conducted for the Mathematics Access to Technology and Science (MATHS) project (Cahill et al., 1996). The survey used for this research asked students with visual impairments and their sighted peers what particular difficulties they experienced when doing mathematics. The survey was given to 42 students with visual impairments and 60 sighted students in Ireland and Belgium. Many of the students with visual impairments reported having the most difficulty with mechanical aspects of mathematics. The results of this survey, therefore, do not indicate that students with visual impairments fail to experience challenges with conceptual understandings in mathematics.

Mechanical difficulties indicated by participants with visual impairments included the time needed to read each question, the length of the problem presented in either
braille or large print, and the general manipulation of mathematical symbols. To contrast, the students with typical sight noted having more difficulty with mathematical concepts (Cahill et al., 1996). These survey results led the researchers to conclude:

In comparison to their sighted colleagues, only a small proportion of blind and partially sighted students take higher-level mathematics examinations. The results of this survey suggest that this situation has more to do with the mathematical-access difficulties than with any conceptual or cognitive problem. (p. 110)

Thus, students with visual impairments not only need excellent instruction in the conceptual understanding of mathematics, but accommodations to help overcome difficulties with the mechanics of mathematics in order to be successful in learning mathematics.

**Unified Instruction**

Unified instruction speaks to the need to have various parts of the curriculum together as a unified whole. Lowenfeld (1981) notes how vision acts as a unifying sense for most children and is deprived for students who are blind. Therefore, there needs to be a more pronounced effort to help unify instruction for students with visual impairments. Individualization addresses curriculum or methodology that acknowledges that each child is an individual and provides for individual differences. Smith (2006) highlights the opportunities that Orientation and Mobility (O&M) instructors have to reinforce mathematical concepts, especially spatial and geometric ideas. For example, O&M instructors teach students how to estimate the time it will take them to travel to a predetermined place by using the formula \( d = rt \) (distance equals rate multiplied by time) (Smith, 2006). Even before using this formula to calculate the time for travel, O&M professionals teach the abstract concept of having a fixed point in space—the point the
student is traveling to (Smith, 2006). Geometric ideas that can be reinforced when teaching O&M skills include learning to read a tactile map and practicing skills involving parallel, perpendicular, point, line, curve, and distance between points (Smith, 2006). For example, when students are learning O&M skills outdoors they learn to walk parallel to traffic and to wait at an intersection until the traffic switches from being perpendicular to parallel (Smith, 2006).

Another concept that is developed through O&M instruction is that of problem solving. Students who are able to handle unique situations while traveling in unfamiliar places or where unexpected barriers occur in well-known areas are more likely to become fully independent travelers. Therefore, the general concept of problem solving is given high priority in O&M instruction (Perla & O’Donnell, 2004). Another way to promote unified instruction is to provide a myriad of activities, involving as many senses as possible, across multiple lessons within a unit of study.

An example of a comprehensive unit comes from curriculum developed by the National Wildlife Turkey Federation (NWTF), and modified for students with visual impairments by a research team (Wild & Trundle, 2010b). This unit was on the impact of the NWTF conservation efforts for the wild turkey. The lesson was both hands-on and multifaceted, allowing the active engagement of students. Activities included exploring a taxidermy model of a wild turkey and wild turkey parts such as a skull, pelts, and feathers. Students also met with champion turkey callers and state wildlife conservationists (Wild & Trundle, 2010b). The multi-sensory approach and the active engagement of the students could account for the significant increase in the scientific
knowledge gained through this curriculum. Results indicate that scientific answers given by students increased from 40.9% on the pre-test to over 90% on the post-test.

Additional Stimulation

Additional stimulation can be provided through activities that students with visual impairments would not be able to experience on their own. This might include field trips, active classroom experiences which give students actual knowledge of objects and situations, or arrangements for experiences outside the classroom that reinforce course material. Portions of the NWTF curriculum, discussed in the previous section, would qualify as providing additional stimulation. Some examples of additional stimulation included taxidermy models of a wild turkey and wild turkey parts and students meeting with champion turkey callers and state wildlife conservationists (Wild & Trundle, 2010b).

The project OceanInsight works with senior scientist, Amy Bower, of the Woods Hole Oceanographic Institution to communicate to students with visual impairments how the Earth works and to promote a level of excitement and interest in oceanographic research. Ms. Bower has been blind for 17 years. She has teamed up with students from the Perkins School for the Blind to test experiments the students had designed for the science team on the R/V Knorr (Fraser, 2008). As a part of the OceanInsight project students were able to take a tour of the ship, interview crew members, and discuss ways in which Ms. Bower conducts research while on research voyages and how she is able to adjust to light on the ship. In the classroom students are also able to ask questions of Ms. Bower and the research team and receive responses back via the project’s website. Next I
will address ways that educators can promote self-activity for students with visual impairments.

**Self-activity**

Providing opportunities for students with visual impairments to learn what sighted children do through visual imitation is the essence of self-activity. Providing activities for students to be actively engaged in the lesson is one way to promote self-activity in the classroom. A survey of the academic engagement of students with visual impairments found that students were engaged between half and most of the time (Bardin & Lewis, 2008). The survey used was a modified version of Student Participation Questionnaire and was completed by the general education teachers. This survey has six subcategories of effort, initiative, motivation, attentive behavior, nondisruptive behavior, and self-determination. The subcategory with the lowest scores for the students with visual impairments was motivation. Overall the results indicate that general educators perceive these students as being only a moderately engaged in the academic curriculum. This is of concern because other studies using this same assessment tool determined that high achieving sighted students were engaged more than 75% of the time (Frederick, 1977).

The results lead the authors to conclude that:

> Teachers of students with visual impairments may need to assist general educators with strategies for ensuring that these students can participate in the full range of educational activities that are offered throughout the school day and with the educational materials that are used, and are offered sufficient opportunities to respond. (Bardin & Lewis, 2008, p. 480)

A comprehensive approach to the lessons and the use of concrete manipulatives are both important when instructing students with visual impairments. Fraser and Maguvhe (2008) believe there are even more basic requirements when working with students with visual impairments.
impairments. “Any curriculum that is not learner-based and learner-paced will hinder the blind and visually-impaired learner from learning and actively participating in the learning mediation to her or his full potential” (p. 85).

In a discussion of developing number sense in children with visual impairments, Liedtke (1998) notes the importance of providing equal opportunities for these students to gain understanding in both conceptual and procedural knowledge. He also challenges educators to think broadly about their lessons and to determine the level at which visualization plays a role. Liedtke (1998) discovered that typical activities for developing number sense in young children have to do with visualization. Therefore, he goes through some typical concepts in number sense and suggests a list of ways an educator can change the activity to make it depend less on visualization. For example, one activity suggested is to promote the number five by having children find groups of five of different objects and to place them in a picture frame or on a plate. The teacher is then to use these objects to ask questions such as “How can you show that there are 5 of something without counting?” (Liedtke, 1998, p. 2, emphasis in original).

**Resources**

There are a multitude of resources available for educators who need practical, classroom-ready activities and ideas for inclusion of students with visual impairments. Some texts list hundreds of ways to adapt lessons and ways to improve the classroom environment for students with visual impairments. Some common themes within these texts include, making your speech more explicit by removing general words and phrases such as “this over here” or “that equation” (Dick & Kubiak, 1997; Asher, 2001; Spindler, 2006), arrange the classroom in such a way that allows for students to move easily
through, and discuss classroom procedures such as acceptable forms of homework and
class work, at the beginning of the school year (Dick & Kubiak, 1997). Another avenue
for educators to gain understanding about educating the visually impaired is through the
World Wide Web. There are several professional organizations that are dedicated to
improving the lives and education of this population. These include the Visual
Impairment Division of the Council for Exceptional Children (http://www.cecdvi.org)

This section will discuss two examples of mathematics curriculum ideas and is
meant to give the reader examples of methods promoted for students with visual
impairments. The first example explores how a typical mathematics lesson can be
adapted for students who are without sight. The second example shows how mathematics
lessons can be included alongside other ECC instruction by looking at how one educator
taught concepts such as parallel and perpendicular while instructing his students in
Orientation and Mobility.

Educators have several options when it comes to finding information on what to
do and what considerations need to be made when working with a student who has a
visual impairment. For instance, there are websites with information on how to include a
student into the mathematics classroom (two such sites are http://www.tsbvi.edu/math/
and http://s22318.tsbvi.edu/mathproject/), books (i.e. When you have a visually impaired
student in your classroom: A guide for teachers; Itinerant teaching: Tricks of the trade
for teachers of blind and visually impaired students; Making it work: Educating the
blind/visually Impaired student in the regular school), professional organizations (i.e.,
Association for Education and Rehabilitation of the Blind and Visually Impaired-
www.aerbvi.org, the National Federation of the Blind-www.nfb.org, American Federation for the Blind-www.afb.org). An article that has a nice summary of considerations for working with mathematics students with visual impairments can be found in Dick and Kubiak (1997). The following are further details on the two resource websites, mentioned above.

The website developed by the Research and Development Institute and funded by the Department of Education and the Office of Special Education and Rehabilitative Services, called Project Math Access (Research and Development Institute, 2006), aims to equip teachers, parents, and administrators with information and tools needed to successfully include students with visual impairments into the mathematics classroom. This project has summarized existing research as well as anecdotal information on strategies educators have found successful when working with mathematics students with visual impairments. The “Teaching Mathematical Concepts” section of the website has insights into teaching Number Sense, Basic Concepts, One-to-One Correspondence and Counting Skills, Place Value, and Measurement. The section dedicated to higher mathematics has some concrete examples on how to teach the following concepts: inequality symbols, intersection and union, signed numbers, equality as the same amounts, combining like terms, place value, variables as unknown amounts, isolating variables, distributive property, the concept that angles of rays are not affected by the length, correct movement of the decimal point in metric system, and hierarchal mathematics operations (Research and Development Institute, 2006).

Another prominent person in the field of mathematics education for students with visual impairments is Susan Osterhaus. Ms. Osterhaus has a website,
http://www.tsbvi.edu/math (Osterhaus, 2006), that is filled with information on how to work effectively with mathematics students with visual impairments is one by Susan Osterhaus, the secondary mathematics educator at the Texas School for the Blind. Ms. Osterhaus’ work in this area is extensive and includes 29 years of teaching secondary mathematics to students with visual impairments at the Texas School for the Blind and Visually Impaired. Ms. Osterhaus spends considerable time presenting at conferences and workshops around the nation to share her experiences with the broader community.

**Rationale for the Study Arising from the Literature**

Research indicates students have difficulty in fully understanding the function concept in school mathematics (Sfard & Linchevski, 1994; Slavit, 1997, 2006; Izsak, 2000; 2003). Studies addressing function learning have all, to my knowledge, focused on the understanding of function held by sighted students. Research efforts geared toward improving instruction in functions have centered on the integration of technology (Usiskin, 1986, 1993, 1997; Kaput, 1992; Schwarz & Dreyfus, 1995) and the use of application based learning (Usiskin, 1986, 1993, 1997; Izsak, 2000, 2003). Recommendations for the use of technology and application based learning have been recorded over the last couple of decades and have to some extent been implemented throughout mathematics education. The current study extends the previous work in that it describes the understanding of function held by students with visual impairments and reviews the learning environment and technology used by students with visual impairments in the mathematics classroom.
Conclusion

A growing awareness in our culture of the unique challenges and the incredible potential of students with visual impairments partly justify the proposed research. American education practices are under much scrutiny to better perform and to show improvement in mathematics outcomes. At the same time, schools are being held to a higher standard regarding their students with disabilities. Students with visual impairments have the right to be educated alongside their peers and to receive the same quality of education. To receive a quality education, students with visual impairments must be provided with effective learning experiences, which will require cooperation between the instructor and the teacher of the visually impaired and proper tools for learning in kinesthetic and auditory ways. Equal education also indicates students have the right to learn any subject or course content that is offered to students throughout the school, including those traditionally taught with a reliance on visual techniques.
CHAPTER 3
METHODOLOGY

This chapter describes the research design including the participants, setting, data collection procedures, and data analysis. The goal of this descriptive research is to understand and describe the knowledge of linear function held by students with visual impairments. Qualitative research methodologies were used throughout this study and the Constant Comparative Method was utilized for data analysis (Glaser & Strauss, 1967). Trustworthiness of the study will also be discussed.

Data collected for this study will help to address the following research questions.

1) What level of knowledge and type of understanding of linear functions is held by students with visual impairments?

2) What are the graphing skills of high school/college students with visual impairments?

3) What are the representational preferences of students with visual impairments when solving word problems involving functions?

4) What factors do high school/college students with visual impairments perceive as influencing the development of their mathematical understanding?
Participants and Recruitment

Participants were recruited from high schools and colleges in the central and southwest regions of Ohio. Students were qualified to participate if they were receiving educational services for a visual impairment and had taken at least one algebra course that introduced them to linear functions. There were no restrictions placed on participation due to previous or current educational setting. Therefore, participants were recruited from local schools and universities as well as a residential school for the blind. I recruited high school students through teachers of the visually impaired and college students through the university’s Office of Disability Services. There were also no exclusions due to type or cause of vision loss. Participants range from being congenitally blind to those who have had recent vision loss. Students varied in the type of vision loss they have. For instance, some students had usable vision while others had only light perception.

Although participation of all students who are receiving educational services for visual impairments and who have completed at least one algebra course were welcome, the population from which to draw participants is relatively small. The number of students with visual impairments can be estimated through the American Printing House for the Blind’s (APH) annual tallies of students registered for their services. This is because federal quota money, which can only be redeemed through the APH, is given to a school district for each student who is listed as having a visual disability as the main disability that affects their education. According to the APH 2010 Annual Report, Ohio has 1,5644 students registered with them (p. 311). This number represents students pre-Kindergarten through high school who attend either a local school or a residential school.
for the blind. In contrast, there are 102,469 students in Ohio who have specific learning disabilities (http://disabilitycompendium.org/Compendium2010.pdf). Thus visual impairments are considered a low incidence disability whereas specific learning disabilities are classified as a high incident disability. The population of high school students with visual impairments who are educated in the central and southwest regions of Ohio and who have taken at least one algebra course is very limited. Eight students participated in the study; four high school students and four college students.

Participants represented a wide range of visual abilities, educational placement and experiences. A total of four high school students, all of whom were just finishing their junior year, participated in the study. Three of the high school students are female. The three female high school participants were educated for half of a day at a local school and the other half of the day at the residential school for the blind. Each of these students was instructed in mathematics at the school for the blind. The male high school student attended his local high school full time. Four college students, two females and two males, also participated in the study. Each of these students attended a university that is known for its active Office of Disability Services offering a wide range of supports for these students. One of the college students, Yvonne, attended a state school for the blind for her high school education. The other three graduated from their local high schools.

All participants had access to technology such as screenreaders, braille notetakers, braille writers, and accessible reading materials (i.e. large print, braille, or audio). Students were also provided with educational tools and accommodations through aides, teachers of the visually impaired, concrete manipulatives, mathematics specific tools such as braille rulers and protractors, and accessible textbooks.
Participants were varied on their view of mathematics. Three students listed mathematics as one of their favorite subjects. At least one student was avidly against learning mathematics and 75% of participants said they have struggled with their mathematics courses. Specifically, three students said that graphing or anything related to geometry was most challenging for them in their mathematics education. Two stated that the most difficult for them was learning the Nemeth Code.

For Algebra, several participants said that keeping track of their steps in a long problem or writing and working with long equations was the hardest aspect of learning algebra. This difficulty is one reason that Gabe, a college student, has always used a Perkin’s Braillewriter for mathematics work. Work on the Perkin’s Braillewriter makes it easier to look back at the completed steps. The convenience of being able to easily review previous work allows Gabe to keep track of where he is with a problem.

In order to participate in this study students must have taken at least one algebra course that had introduced them to linear functions. Therefore, there was the potential to have a wide variety of mathematics backgrounds among the sample. To highlight the range in mathematics levels I list the highest mathematics level reached by each participant. Robert and Caleb have completed courses in Pre-Calculus. Caleb took Pre-Calculus in his junior year of high school and Robert took Pre-Calculus in college. Liz, Yvonne, and Gabe completed an Intermediate Algebra course in college that covers topics such as factoring, algebraic fractions, linear equations, laws of exponents, and line graphs. Madison had completed Algebra I in her high school mathematics and Amanda had taken mathematics courses through Algebra II. Finally, Wanda had taken
mathematics courses through high school geometry as well as an Ohio Graduate Test Preparation course.

All participants had lecture-based mathematics courses where homework problems were assigned on a regular basis. There was a minimal amount of group work or homework projects assigned. All students either directly or indirectly spoke to the need for one-on-one instruction, especially for mathematics. Four specifically said that this is how they learn best and discussed ways they would arrange for this instruction, if it was not already provided.

Participant’s visual abilities and experiences were also varied. Four students had vision early in life. Amanda, a student with usable vision, gradually lost sight until it stabilized at the age of six. Two students, Madisen and Yvonne, had some vision until the age of 13, but now have only light perception. Robert and Liz were born blind. Gabe and Caleb have some limited vision that has improved and diminished at various points of time. Their vision is now stable with Gabe having very limited vision in his left eye and Caleb being able to see some color, movement, and large objects. Wanda is nearsighted and thus can see things in detail when they are close to her, but cannot visually access items that are far away from her. Wanda cannot, for instance, read a road sign and has trouble reading information presented on a whiteboard if she is not sitting close to the board.

A more detailed account of study participants follow. For a summary of participant characteristics and demographics refer to Tables 3.1 and 3.2.
Participant profiles

Robert. Robert is a college student studying psychology. He was born blind, which was caused by Norrie’s Disease. Robert does not have usable vision except for light perception. Robert attended his local public schools for his K-12 education. He enjoys doing mathematics, but has struggled with some of his recent college mathematics courses, especially Statistics. Robert’s high school mathematics courses included Algebra I, Geometry, Algebra II and Pre-Calculus. Beyond these courses, he has taken Pre-Calculus, Trigonometry, and Statistical Concepts in college. Robert is an excellent braille reader and is comfortable reading and producing tactile graphics.

Liz. Liz is in her third year of a psychology degree in a public university. Although she moved several times, her K-12 education was held at her local public school. Liz attended a school for the blind for one quarter because her local school did not have the appropriate accommodations. Liz has been blind since birth and learned to read braille early in life; she is a proficient braille reader. Liz was hesitant when it came to reading tactile diagrams and remembers graphs giving her difficulties when learning mathematics in high school. Liz has had both good and bad experiences in mathematics and relates the good ones with her teacher and/or tutor. Liz took Algebra I and Algebra II in high school and a basic algebra course and a second more advanced algebra course in college.

Madisen. Madisen is a junior in high school and attends a local school for half of the day and is at a residential school for the blind the other half day. Madisen receives her mathematics instruction at the school for the blind. Madisen had a slight visual impairment all of her life, but lost the rest of her vision through a virus the summer
before her freshman year of high school. Therefore, the timing of her first algebra course corresponded with learning literary braille and the Nemeth Code. Thus, her first algebra course was abbreviated. This course went over pre-algebra topics as well as some beginning algebra concepts. The following year Madisen re-took Algebra I, which covered the typical content of this course and included a few topics from the Algebra II curriculum. Mathematics has never been Madisen’s favorite subject and the good experiences she has had with mathematics are usually tied to the teacher she had that year. Madisen is a proficient braille reader but she prefers to not use braille graphics because the information is often hard to read.

_Gabe_. Gabe will soon finish a psychology major at a public university. Although he says that he sometimes struggles with interpreting the graphics correctly, Gabe is a proficient braille reader and is comfortable with reading tactile graphics. Gabe was educated in his local school and was the first person with a visual impairment to come through that particular school system. He has no vision in his right eye and limited vision in the left eye. Vision loss was due to retinopathy prematurity and occurred at birth. There has been some fluctuation in the amount of vision in his left eye, but that stabilized after a cornea transplant that was completed when he was in high school. Gabe appreciates mathematics and understands its importance, but has not had good experiences with learning mathematics. He notes how tedious doing mathematics problems can be and at first Gabe’s teachers were not prepared to explain visual concepts effectively. Gabe struggles with thinking of mathematics both linearly (as it is presented in braille) and visually because of the time when he learned mathematics with the vision he had.
Caleb. Caleb is an 11th grader who attends his local high school. He has just completed Pre-Calculus. Caleb has been blind since birth, but regained some sight for about 7 years before high school. Toward the end of Caleb’s 8th grade year he lost what vision he had except for light and color perception and a limited ability to see movement and large objects. Caleb enjoys doing mathematics and has had a pretty good experience overall with his mathematics education, though he has also struggled with some of his classes. Caleb recalls a couple of teachers who “ruined” mathematics for him because they were unable to teach the concepts in a way he could access. Caleb has taken Algebra I, Geometry, Algebra II, and Pre-Calculus. Caleb is a proficient braille reader and is also proficient with reading tactile graphics.

Yvonne. Yvonne is a senior in university working toward a psychology degree. Yvonne’s vision loss was gradual and she was completely without vision by the age of thirteen. Yvonne reads braille well, but says she avoids it as much as she can and has always struggled with the Nemeth code. Yvonne does not enjoy mathematics and has always struggled with learning it throughout her education. She also struggles with reading tactile graphics. Yvonne still visualizes mathematics, since most of her mathematics learning occurred when she had vision. Yvonne took Algebra I, Geometry, and a proficiency preparation course in high school and two basic and one advanced algebra course in college.

Amanda. Amanda is a junior in high school and spends half of her day at a school for the blind and the other half at a local high school. All of her high school mathematics courses have been taken at the school for the blind. Amanda lost her vision gradually around the age of six, but has some usable vision and though she can read braille, she
prefers to read large print. Amanda has taken Algebra I, Geometry, and Algebra II. She enjoys mathematics, in general, and algebra specifically. Geometry was difficult for her.

*Wanda.* Wanda is an 11th grader who attends a school for the blind part of the day and a local high school the other half. Wanda is nearsighted and therefore can see things clearly when they are close to her. Wanda prefers to read regular print and only uses large print in some circumstances. She does not like mathematics and has had mixed experiences with her mathematics classes. Wanda’s mathematics classes at the school for the blind have been better than those at her local school because of the one-on-one attention she receives. Wanda recalls at her local school the teacher often just explained how to do problems by putting examples on the board, which she often could not see. The mathematics Wanda has taken includes Algebra I, Geometry, and an Ohio Graduate Test Preparation course.

Table 3.1 provides a summary of student characteristics including their preferred reading medium, level of mathematics education, high school environment, and current educational setting.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Reading Medium</th>
<th>Highest Level of Mathematics Completed</th>
<th>High School (Type)</th>
<th>Current Education Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robert</td>
<td>Braille</td>
<td>Pre-Calculus</td>
<td>Local School</td>
<td>College: 3rd year</td>
</tr>
<tr>
<td>Liz</td>
<td>Braille</td>
<td>Intermediate</td>
<td>Local School</td>
<td>College: 3rd year</td>
</tr>
<tr>
<td>Madisen</td>
<td>Braille</td>
<td>Algebra I</td>
<td>Half-day at School for the Blind; Half-day at Local School</td>
<td>HS: 11th grade</td>
</tr>
</tbody>
</table>

Table 3.1: Summary of Participant Characteristics

98
Table 3.1 continued

<table>
<thead>
<tr>
<th>Name</th>
<th>Reading Medium</th>
<th>Course</th>
<th>School</th>
<th>College: Senior</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gabe</td>
<td>Braille</td>
<td>Intermediate Algebra</td>
<td>Local School</td>
<td>College: Senior</td>
</tr>
<tr>
<td>Caleb</td>
<td>Braille</td>
<td>Pre-Calculus</td>
<td>Local School for the Blind</td>
<td>HS: 11th grade</td>
</tr>
<tr>
<td>Yvonne</td>
<td>Braille</td>
<td>Intermediate Algebra</td>
<td>Local School for the Blind</td>
<td>College: Senior</td>
</tr>
<tr>
<td>Amanda</td>
<td>Large Print (preferred); Braille</td>
<td>Algebra II</td>
<td>Half-day at School for the Blind</td>
<td>HS: 11th grade</td>
</tr>
<tr>
<td>Wanda</td>
<td>Regular Print</td>
<td>Geometry</td>
<td>Half-day at School for the Blind; Half-day at Local School</td>
<td>HS: 11th grade</td>
</tr>
</tbody>
</table>

Three comparisons of understanding are made between various student characteristics, a) time of the participants’ vision loss, b) the students’ reading medium, and c) the students’ high school placement. My intention was to compare student understanding between those who are congenitally blind and those who lost their sight later in life. However, due to the characteristics of my participants, I decided to make the comparison between three categories based on the time at which students lost their vision.

In particular, both Gabe and Caleb had a limited amount of vision for part of their life, but had little enough use of this sight that they both learned braille early in life. However, they discuss visualizing mathematics because of the early vision they had. Therefore, I have placed them into their own category of *early visual experiences* and will discuss student understanding around the three categories, a) congenitally blind, b) early visual experiences, and c) later blind.

Student reading medium is easily defined with six participants using braille as their main reading medium and two students preferring to use either large or regular.
print. High school setting was determined based on where the student attended their mathematics classes. This addresses the issue of the participants who spend half of their day in the school for the blind and half at a local school. A summary of the participants who belong to each category is given in Table 3.2

<table>
<thead>
<tr>
<th>Time of Vision Loss</th>
<th>Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Congenitally Blind</td>
<td>Robert, Liz</td>
</tr>
<tr>
<td>Early Visual Experiences</td>
<td>Gabe, Caleb</td>
</tr>
<tr>
<td>Late Blind</td>
<td>Madisen, Yvonne</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reading Medium</th>
<th>Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Braille</td>
<td>Robert, Liz, Gabe, Caleb, Yvonne</td>
</tr>
<tr>
<td>Large/Regular Print</td>
<td>Amanda, Wanda</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>High School Setting for Mathematics Instruction</th>
<th>Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>School for the Blind</td>
<td>Madisen, Yvonne, Amanda, Wanda</td>
</tr>
<tr>
<td>Local School</td>
<td>Robert, Liz, Gabe, Caleb</td>
</tr>
</tbody>
</table>

Table 3.2: Participant Characteristics by Time of Vision Loss, Reading Medium and High School Setting for Mathematics Instruction
Qualitative Design

This research is descriptive and follows a qualitative design that is influenced by the definitions and research highlighted by the work of Denzin and Lincoln (2000). As mentioned previously, the descriptive research paradigm seems best suited for certain fields where population bases are small in number and/or where participants have multiple variables that cannot be controlled (Denzin & Lincoln, 2000). For the present research, there is a limited population from which to draw participants. There is also a wide range of variables that cannot be controlled. These conditions contributed to the decision to use a Qualitative research approach. Another reason for use of qualitative methods is I wanted to provide a thick description in order to fully understand multiple perspectives within the area of education for students with visual impairments. A detailed description of these beliefs is discussed next.

Instrumentation

Qualitative research, in general, and with constant comparative analysis, in particular, the researcher is close to the data and is an indispensable part of the data analysis. The constant comparative method of data analysis was “not designed (as methods of quantitative analysis are) to guarantee that two analysts working independently with the same data will achieve the same results” (Glaser & Strauss, 1967, p. 103). The researcher is constantly comparing new data with data already analyzed. Thus, the researcher should be aware of how her perspectives may influence self as an instrument within data preparation and analysis (Strauss, 1987). To inform my audience, I describe my background and position as a researcher and educator, next.
Researcher Background

“Qualitative research is fundamentally interpretive” (Creswell, 2003, p. 182). This implies the active interpretation of research findings by the researcher. When analyzing and categorizing data the researcher filters the data through the lens of personal experience, views of researcher, in general. This personal interpretation cannot be avoided in qualitative analysis (Glaser & Strauss, 1967; Glesne, 1999; Creswell, 2003). Therefore, a description of my beliefs of research and education are appropriate.

My experiences in the classroom and those with tutoring students who have a variety of disabilities including those with learning disabilities and those with visual impairments have helped to shape my view on the construction of knowledge. My work with students with visual impairments began when I began my undergraduate program at Wright State University in 1997. One of the most rewarding assignments was as a tutor in a basic algebra course for two women who are blind. It was interesting to me how differently each student approached a problem. It became apparent that part of this distinction stemmed from the way their vision loss occurred. The differences in learning styles of these students interested me and caused me to look further into educational practices for students with visual impairments.

Through the tutoring opportunities I have had, it is apparent that misconceptions are a great barrier to progress in new understanding. Students can have difficulty with integrating new information with knowledge that is contradictory to their current understanding. Reconciling new information with the current understanding may be difficult, but I believe, this is what leads to deep understanding. My experiences with two
research projects helped me deepen my understanding of qualitative and quantitative research and when each is appropriate to use.

One research project called, Creating Laboratory Access for Science Students (Kirch et al., 2005, 2007), helped prepare and equip educators to include their students with disabilities in the science classroom and laboratory. A two-week professional development was used as our main data collection. Qualitative methods were used including educator journals, laboratory observations, and pre- and post-questionnaires using a Likert-type scale. These methods allowed for a rich body of knowledge about teacher experiences and beliefs about disability and education. It also allowed us to ascertain changes in attitudes regarding students with disabilities. The second research project took a different view on research methodology and introduced me to a quantitative design, which out of necessity moved toward a mixed methodological approach.

The second research project I am involved with is one that developed a haptic glove system for use by students with visual impairments in inclusive classrooms. The system will allow the student to access teacher gestures as he or she works with a diagram or graph. To test this system we ran four experiments that varied in complexity and purpose. Although my partners at Virginia Tech who developed the system typically work in a quantitative paradigm and the computer program was able to collect a plethora of quantitative values, a strictly quantitative approach was not possible due to the size of the population. It also became clear that the quantitative measures, when taken alone, were not enough to give a complete picture of what was happening. For instance, a result emerged from the quantitative data that suggested that two students who had not had
previous experience in using the glove outperformed the students who had used the technology previously. This contradicted an underlying assumption that participants would improve as they became more familiar with the system. However, through my research notes and observations I was able to determine that the reason for the high level in performance was not due to the students’ level of comfort in using the technology, but instead was related to an overall technique they each had when working with haptic technology. As a beginning researcher, it was interesting to me to observe some of the differences in research possibilities when working within the different paradigms.

My experiences have led me to identify myself in the postpositivistic paradigm as a social constructivist. I agree with Schwandt (2000), that “there is an inevitable historical and sociocultural dimension to [the construction of knowledge]. We do not construct our interpretations in isolation but against a backdrop of shared understandings, practices, language, and so forth” (p. 197). I believe that knowledge is constructed based on previous experiences. This belief has implications for me as an educator as well as a researcher. First, this implies individuality in understanding. That is, since experiences shape understanding, and each person has a unique set of experiences, the construction of meaning is also unique. This individuality implies the need for educators to gain understanding of students’ current knowledge including their misconceptions as these will play an integral part in understanding new concepts. For research, the individuality of knowledge implies the need for multiple perspectives. Understanding for each person, including the researcher, is influenced by a different set of experiences and therefore multiple interpretations of one event can occur.
Instruments

The instruments used for data collection in the main study were: the Mathematics Educational Experiences and Visual Abilities (MEEVA) Interview (Appendix A), the Function Knowledge Assessment (FKA; Appendix B), and the Function Competency Assessment (FCA; Appendix C). The purpose of the MEEVA is to obtain both demographic information and student experiences regarding education, particularly that in the mathematics classroom. The FKA and the FCA were used to assess student understanding in mathematics, particularly in the area of linear function knowledge for problems presented in context. The mathematics assessments were compiled through a collection of problems from the literature. In order to add credibility to the FKA and the FCA a pre-study was conducted with sighted students to test the credibility of the mathematics instruments for the main study. Details of the pre-study are discussed, next.

Development of Research Instrumentation

The Instrument Development Study (IDS) was conducted prior to data collection of the main study and served as a way to verify dependability of individual mathematics problems from the two mathematics assessments, the Function Knowledge Assessment (FKA) and the Function Competency Assessment (FCA). Results from the IDS helped to finalize the FKA and FCA. Next, I discuss the participants, recruitment, details of the instruments used, procedures, analysis and implications for the main study.

Participants and Recruitment for the IDS

Participants for the IDS were high school students with typical sight. The decision to use typically sighted participants was due to the need to keep the pool of students for
the main study, which would already be relatively small, intact. Data from the IDS was not intended for use as a comparison with data from the main study. Thus, the sample chosen for the IDS was not particularly chosen to reflect characteristics of main study participants. For this reason a convenience sample was used.

The Institutional Review Board of Ohio State University granted permission for the recruitment of students from mathematics classes at a local high school. Five students were recruited from two honors Pre-Calculus classes.

**Instruments for the IDS**

Three instruments were used in the IDS, a) the Function Knowledge Assessment: Version 1 (FKA: V1; Appendix D), b) the Function Knowledge Assessment: Version 2 (FKA: V2; Appendix E), and c) the Function Competency Assessment (FCA; Appendix F). Next I describe each of the three instruments and discuss how each instrument was developed.

*Function Knowledge Assessment: Version 1 and 2.* Two versions of the FKA were used in order to determine what questions to include and whether an emphasis should be placed on any particular portion of the FKA. The FKA: V1 and the FKA: V2 differ in only a few aspects and therefore I describe the FKA: V1 and contrast that with the FKA: V2 following this description. The FKA: V1 was divided into two parts. Part 1 of the FKA: V1 asks a series of questions about the savings accounts of four children. Descriptions of the savings accounts, with some modifications, came from an assessment used by Friedlander and Tabach (2001). I first describe the children’s savings accounts and then discuss the changes made to these descriptions.
Providing problem information in various representational forms encourages students to use different modes of representations when responding to context-based problems (Friedlander & Tabach, 2001). Thus, the savings for the four children, Adam, Heather, Diane, and Matthew, were given using the four representations of function. Adam’s savings were described in a table that showed his total amount for weeks 20 to 28. Heather’s savings were described in words. Diane had an expression to describe her savings. Matthew’s savings were shown in a scatter plot that gave his total savings for weeks zero to 20. The graph had three “holes” where values for those particular weeks are not shown (see Figure 3.1). Details of the four savings accounts were given at the beginning of the FKA: V1 with a statement indicating the children would be saving their money for one year.
The following are descriptions of four children’s savings. Use the information from these
descriptions to answer the following questions.

Adam: The table shows how much money Adam had saved at the end of each week.
The table continues in the same way for the rest of the year.

<table>
<thead>
<tr>
<th>Week</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>140</td>
</tr>
<tr>
<td>21</td>
<td>147</td>
</tr>
<tr>
<td>22</td>
<td>154</td>
</tr>
<tr>
<td>23</td>
<td>161</td>
</tr>
<tr>
<td>24</td>
<td>168</td>
</tr>
<tr>
<td>25</td>
<td>175</td>
</tr>
<tr>
<td>26</td>
<td>182</td>
</tr>
<tr>
<td>27</td>
<td>189</td>
</tr>
<tr>
<td>28</td>
<td>196</td>
</tr>
</tbody>
</table>

Heather: Heather kept her savings at $300 throughout the year.

Diane: Diane’s savings can be described by the expression 300-5x, where x stands for the
number of weeks.

Matthew: The graph describes Matthew’s savings for several weeks near the beginning of the
year. His savings continues in the same way for the rest of the year.

Figure 3.1: Descriptions of the Four Children’s Savings Accounts for the FKA: V1 and
FKA: V2

Changes made to the original savings accounts described by Friedlander and Tabach (2001) included, 1) changing the names of the children, 2) having Adam’s table
describe his savings for the middle of the year instead of providing Adam’s savings for the first 9 weeks of the year, and 3) leaving out three data points on the graph of Matthew’s savings. The original assessments were used with Israeli students. I decided to change the names to names well known in America so that attention would not be diverted from problem solving because of unfamiliar names. The decision to describe
Adam’s savings for weeks 20 to 28 instead of for the beginning of the year stemmed from a discussion with mathematics education professionals and mathematics education doctoral students. The group suggested that the change in description of Adam’s savings would provide a better way to access student reasoning. Providing information for Adam’s mid-year savings also made the questions relating to Adam’s savings slightly more challenging. I wanted to promote more challenging questions because the original assessments were given to seventh grade students. Unlike the students who originally used the descriptions of the four children’s savings accounts, my participants would be at least freshmen in high school and would have taken at least one algebra course. Finally, the decision to leave out a few data points from the graph of Matthew’s savings also stemmed from the discussion of the instruments from the mathematics educators. The reason behind the omission of data points was to determine student reasoning regarding linearity and whether linearity was preserved when data points were not shown.

Three categories of questions were used by Friedlander and Tabach (2001) to develop student understanding of the use of multiple representations through the savings accounts. The first nine questions on the FKA: V1 (see Figure 3.2) were written by me to fit within the “getting acquainted with the initial representation” category and provided students with the opportunity to “analyze each component in its original presentation and make some extrapolations or draw some conclusions” (Friedlander and Tabach, 2001, p. 177). Further, the first nine questions helped verify that participants were interpreting the savings account information correctly. The verification that students were reading the information correctly was especially important for braille readers. The braille readers’ interactions with the four savings accounts provided me with data regarding the accuracy
of the braille and the level of readability of the tactile graph for Matthew’s savings and
the table of Adam’s savings.

Function Knowledge Assessment: Version 1 - First Nine Questions

1. How much money will Diane have after the 8th week?
2. How much money will Adam have after week 22?
3. How much money does Matthew have after week 18?
4. How much money will Matthew have on week 30?
5. How much money did Adam have on the 10th week?
6. How much money did Heather have at the end of the year?
7. Matthew will have $140 on which week?
8. Which week will Adam have $287?
9. When will Diane have $200?

Figure 3.2: First Nine Questions from the Function Knowledge Assessment: Version 1 in
the Getting Acquainted with the Initial Representation Category

The next category of questions from Friedlander and Tabach (2001) are those that
ask students to work within a specific representation. Numbers 10 and 11 from the FKA:
V1 fit within this category and came from Friedlander and Tabach’s article (2001).
Question 10 (FKA: V1) asked “Describe in words how the savings of each child changes
through the year.” Question 11 (FKA: V1) asked students to, given the graphs of the four
children’s savings accounts, identify each graph and to find the meaning of the
intersection point.
I added questions 12, 13 and 14 on the FKA: V1. These questions do not fit one of the categories listed by Friedlander and Tabach (2001). Questions 12, 13 and 14 ask participants to translate from one representation to another, which represents more advanced mathematical tasks that students from Friedlander and Tabach may not be prepared to do. Through questions 12, 13, and 14 on the FKA: V1 students were asked to perform the following translations, 1) a graph to an equation (“Write an equation for Matthew’s savings.”), 2) a table to an equation (“Write an equation for Adam’s savings.”), and 3) a description to an equation (“Write an equation for Heather’s savings”).

The final four questions of Part 1 on the FKA: V1 (questions 15-18) fit within the third category of Friedlander and Tabach’s (2001), exploratory questions. Exploratory questions are those that are more complex and open-ended. Questions 15 and 16 came from Friedlander and Tabach’s work (with slight modifications in wording for question 15). Question 15 read:

Compare the savings of two out of the four children (using words like “the savings increase (or decrease),” “the savings increase or decrease at a rate of…,” “who has a larger (or smaller) amount at the beginning (or end),” and “larger (or smaller) by…, double…, equal.” Also, use tables graphs, expressions, and explanations.

Question 16 asked students to add another child to their comparison. The final two questions (Questions 17 and 18) within Part 1 of the FKA: V1 were also exploratory questions. These questions were written by me and asked students, “Whose savings is growing the fastest?” and “Whose savings is growing the slowest?”, respectively. The inclusion of questions 17 and 18 were to foster student thinking in terms of comparing the children’s savings account in a specific way. Question 18 was written with some
ambiguity as Diane’s savings was decreasing and Heather’s savings remained constant and thus there were several ways in which students could respond to whose savings was growing the slowest. The ambiguity was kept to further promote access to student reasoning behind the various savings accounts.

Figure 3.3 shows the seven questions that made up Part 2 of the FKA: V1. All of these questions came from the work of Friedlander and Tabach (2001). These seven questions were specifically related to the savings of Diane and Matthew. Students were told they could choose to use one of four representations for Diane’s and Matthew’s savings to aid in their responses. The four representations students could choose from were equations of the children’s savings, a table displaying the amount both Diane and Matthew had in the account each week of the year, a written description of the children’s savings, and a graph that displayed both Diane’s and Matthew’s savings on a single set of axes (see Appendix G). The seven questions from Part 2 of the FKA: V1 were used by Friedlander and Tabach as an assessment after course instruction on the use of multiple representations.
Function Knowledge Assessment: Version 1 Part 2

1. How much had Matthew saved after half a year? How much did Diane have at the same time?

2. After how many weeks did each of the two children have $210?

3. When was the difference between their savings $60? In whose favor was the difference?

4. Find the week with the largest difference between their savings.

5. Find the week when their savings were equal.

6. Find the week when the savings of one were double that of the other. In whose favor?

7. Diane and Matthew decided to pool their savings in order to buy a $400 walkie-talkie. Find the week in which they are able to purchase the walkie-talkie.

Figure 3.3: Questions from Part 2 of the Function Knowledge Assessment: Version 1

The seven questions that made up Part 2 of the FKA: V1 consisted of getting acquainted with the initial representation problems (questions 1, 2, and 5 for students who used the graph or table of the children’s savings) and word problems specifically related to comparing Diane’s and Matthew’s savings while working with a specific representation (the representation chosen by the student at the beginning of Part 2). Some of the questions were easier to answer when using a particular representational form. For instance, question 6 asked, “Find the week when the savings of one were double that of the other. In whose favor?” This question was relatively straightforward for students using the table. The table allowed students to read through the savings of each child and to do a subtraction between the two savings in order to find the week in which there was a difference of 60. This same question was difficult for students who chose to use the graph as using the graph required the student to rely more on estimation. The seven
questions from Part 2 of the FKA: V1 finished this assessment. I now discuss the difference between the FKA: V1 and the FKA: V2.

*Difference between the FKA: V1 and the FKA: V2.* The FKA: V2 contained the same descriptions for the four children’s savings accounts and the same 18 questions for Part 1. Part 2 still addressed questions related to Diane’s and Matthew’s savings. However, except for the FKA: V2 had only four questions instead of seven questions. The FKA: V2 also contained a third part that contained a description of a fifth child, Ellen’s, savings account. Ellen saved her money in a non-linear format. The description is followed by two questions relating to Ellen’s savings (see Figure 3.4). Question 1 of Part 3 of the FKA: V2 is a *getting acquainted with the initial representation* problem and question 2 from this part was an open-ended problem asking for a comparison of Ellen’s savings with that of the original four savings accounts. Thus, question 2 from Part 3 of the FKA: V2 was an exploratory question.

<table>
<thead>
<tr>
<th>Function Knowledge Assessment: Version 2 Part 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Another child’s savings was as described below.</td>
</tr>
<tr>
<td>Ellen received her allowance in the following way: On the first weekend, she got two cents. Every subsequent weekend, she received an amount identical to the amount she had left in her savings box the previous week. Ellen saves all the money she gets.</td>
</tr>
<tr>
<td>1. How much money had Ellen saved after 10 weekends?</td>
</tr>
<tr>
<td>2. How is Ellen’s savings different from that of the other four children?</td>
</tr>
</tbody>
</table>

Figure 3.4: Part 3 of the Function Knowledge Assessment: Version 2

*Function Competency Assessment.* The version of the FCA that was used with the IDS study contained eight questions, many of which have multiple parts, which gives a
total of 19 questions for student response (Appendix F). All questions were embedded in context. The first five problems came from research conducted by O’Callaghan (1998). Each of these five questions was specifically written to test competency in one of the four areas from O’Callaghan’s Function Model. Questions 1a and 1b were written to probe student competency in interpreting information given in tabular form. Question 1c asked students to translate between a table of values and a graph. Students’ competency with modeling was probed through questions 2a and 2b while question 2c was asked for a translation from a description to an equation. Questions 3a through 3e probed student knowledge of interpreting graphical information and question 4a asked students to interpret information from a given table. Question 4b was another translating question where participants were asked to write an equation of the table expressing car values over time. Finally, question 5a was written to probe student reasoning in interpretation and question 5b tested for students’ ability with reification.

Question 6 was taken from the work of Brenner et al. (1997) and read, “Mary Wong just got a job working as a clerk in a candy store. She already has $42 dollars. She will earn $7 per hour. How many hours will she have to work to have a total of $126?” A second part to that question originally asked students to: “Draw a diagram, chart, table, or graph to represent the problem.” This question was modified for the IDS study to read: “Draw a table or graph to represent the problem.” The current study is focused on students’ use of descriptions, tables, graphs, and equations to read and understand functions. Therefore, I decided to take out the options of drawing a diagram or a chart for this problem. Also, I wanted to give students the opportunity to choose between making a table or a graph as students were already asked specifically to make a graph for question
1c on the FCA. Giving students a choice between building a graph or making a table also informed me of the representational preferences of the participants.

Question 7 from the IDS version of the FCA was a problem used in the research by Slavit (2006). This question asked students to compare scenarios of two carnival’s entrance and ride ticket prices and to determine when two individuals (who attended different carnivals) would pay the same amount. The problem scenario was described in words and as such student responses had a greater potential of varying. Correct answers could also vary because the problem did not include the constraint that the two individuals had to ride the same number of rides. For instance, the answer of 12 dollars is an acceptable response because 12 dollars would allow one individual to ride one ride while the other individual could ride two rides. Another acceptable response to question 7 was 18 dollars, which would allow each of the individuals to ride four rides. The potential for having a variety of methods used to solve question 7 was the impetus for including this problem. It provided the opportunity to probe student thinking regarding comparing multiple scenarios. Question 7 also allowed me the opportunity to note what representation the participant chose to use when solving.

The final question, question 8, on the IDS version of the FCA was found in a mathematics textbook by Sfard and Linchevski (1994). Sfard and Linchevski described the problem, which involves a system of equations, as one of the best problems she had found for probing students’ reification of the function concept.

**Procedures for the IDS**

Five students participated in this study. Participants met with me for one session, at which time he or she completed either the FKA: V1 or the FKA: V2 and the FCA. The
FKA was given first followed immediately by the FCA. I met with one student at a classroom in her high school and the other participants I met at a local university as the students were on their spring break and we did not have the use of the high school during that time. Students were given the choice whether to take the tests orally or to take them in a standard paper/pencil format. If students hesitated when asked whether or not they wanted to take the test orally I encouraged them to take the oral version because I wanted to be able to practice giving the test in this manner. For the tests given orally, I read the questions aloud to the student and they answered orally. These participants were instructed to explain their reasoning as they worked through each problem. If there was a pause in the dialogue, I reminded them to voice their thinking. These sessions were audio recorded. For written tests I was present the entire time to answer questions or to provide clarification. Two students completed the tests orally and the other three did written answers. Data collection was completed in 9 days and through four sessions. The researcher met one-on-one with three students. Two participants came together and took the written test simultaneously. I took notes throughout all sessions.

*Materials.* All students, including those who completed the assessments orally, were given a written copy of the test, extra paper and a pencil to work out their responses. All students had the use of a calculator. Three students had their own graphing calculator with them and preferred to use these for completing the assessments. Two students did not bring a calculator with them to use. One of these students was given a scientific calculator and the other participant was given a basic four function calculator to use to complete the FKA and the FCA.
Analysis for the IDS

Participant answers and researcher notes were analyzed to determine in what ways the FKA and FCA needed to be modified before using them in the main research study. Analysis focused on, a) the time required for test completion and time related issues with how students interacted with the tests, b) questions asked by participants, and c) researcher notes of procedural matters. Notes that dealt with procedural matters included those of how and when to encourage oral responses, how to handle student questions when working a problem, the readability of the questions, the flow of the test, the set-up of materials, and the clarity of the questions.

Results from the IDS

The average completion time for both tests, together, was 70 minutes with the shortest completion time of 51 minutes and the longest at 84 minutes. For the two students who took the test orally the average time was 60 minutes and for those who took the written version the average time was 77 minutes.

Clarification questions from the students were minimal. One student questioned whether or not Heather and Diane were applicable for the question asking which child’s savings was growing the slowest. Another student asked if she could switch her choice of representation she was using after she started answering questions for the FKA: Part 2. This same student noted that her answer did not seem correct on one question because the application would not make sense.

During the two sessions where the tests were given orally, I noted the need to reassure the student about the process and of the importance of voicing their thinking and their final answers at the beginning of the test. Reminders for the students to provide
voiced answers stemmed partly from the fact that they were unaccustomed to taking assessments orally. I found that a couple of questions were difficult to read because of the use of parenthesis and/or they had a lengthy description before the problem questions. I found that there were two questions that were similar enough on the FCA that made a good combination to go back to and ask a clarification question. However, on the IDS version of the test these questions were not adjacent to each other and so it was awkward to go back and ask any follow-up questions.

The placement and use of the microphones and recording devices were tested with the oral sessions. A mistake was made on the first session where the recording device was turned on, but the external microphone was not on. Also, the secondary recording device was off to the side a little and not directly facing the student (due to room configuration). This made the recording hard to decipher at times. I also found that it worked well for the student to have a copy of the questions for the assessments and for them to have enough space to work out their answers.

I questioned the wording on the question asking whose savings is growing the slowest because there are no definitive answers on this due to the variety of savings behaviors. I also questioned the wording on a problem asking about the intersection point on the graphs of the four children’s savings. There were also a few questions that I determined were redundant.

Discussion of IDS Results. The average completion time for students taking the written test was 17 minutes more than the average for those taking it orally (77 minutes versus 60 minutes). This was due, in part, to the questions that were more open ended and
required a paragraph or two for the response. It took longer for students to write these answers than it did for them to voice their responses.

Although one of the goals of the IDS was to determine an appropriate length for the tests the results must be considered in light of the fact that participants in the main study have visual impairments. Dick and Kubiak (1997) noted that it can be expected for an educational task completed by sighted students to take one and a half to two times as long for students with visual impairments. Also, students from the main study were asked to complete a third instrument, the MEEVA, a survey regarding their mathematics education experiences and details of their vision loss. Third, most of the IDS participants were further along in their mathematics education and therefore had little “think time” as they completed the test items. These considerations went into the final modifications for the mathematics assessments.

Question 18 from the FKA: Part 1 asked students which savings account was growing the slowest. The question that asks about the savings account with the slowest growth is somewhat ambiguous because Heather’s savings is constant and Diane’s savings is decreasing. Though students seemed uncomfortable with answering this question, the ambiguity produced rich answers as they discussed the merits of each possibility. Therefore, after speaking with my advisor we decided to keep the wording the same in the final instrument.

Choosing a representation for use with questions from FKA: Part 2 was interesting in that all except one student chose to use the equations and several students were uncomfortable with making a choice before they knew what types of questions were being asked. One student even looked ahead to review the questions before making her
choice. This alerted me to the fact that the questions from FKA: Part 2 should be kept separate so that students in the main study would be unable to look ahead.

One student was thinking ahead and asked if she could change her mind after seeing the questions. She was anticipating some questions being easier to answer with one representation and others to be easier with a different representation. It was also interesting that students chose the equations even though Diane’s equation was provided in the original description and students were already asked to write an equation for Matthew’s savings. Therefore, they would have had at least some access to their equations, though there was the possibility that their equation for Matthew was incorrect. In fact, one participant, upon being handed the equations, declared that her previous answer for Matthew’s equation was incorrect.

A question from the IDS version of the FCA had a question regarding a business owner who used one equation to calculate his profit and a second equation to use the profit to determine how much he would donate to charity. One IDS participant who was taking the test orally hesitated with her answer and when questioned said that her answer did not make sense because the amount given to charity was more than the profit. After double checking the participant’s response, I saw that this was indeed the case and the question was removed.

Although students were nervous about taking the IDS orally, I was grateful for the opportunity to practice giving the test in this manner. It allowed me practice with interacting with students and answering questions. Having both written and oral answers allowed me to see the benefits behind the oral test. I assumed the participants with visual impairments in the main study would not be anxious about taking the tests orally as they
would be more accustomed to taking tests with a proctor. This was, indeed, the case, although there were a couple of participants in the main study who needed a reminder to voice their steps as they worked.

*Changes to instruments based on IDS data.* The IDS data showed that the instruments had room for improvement and that the assessments were too long. Therefore, the following changes were made before data collection began for the main study.

The first 9 questions of the FKA remained the same except the problem “Matthew will have $140 on which week?” was changed to “Matthew will have $150 on which week?” This change was made because even though the week at which Matthew has $140 is not shown, the scale goes from zero to $140 and I wanted the question to require more extrapolation.

Question 10, “Describe in words how the savings of each child changes through the year,” was redundant with Question 11 where students were asked to determine which graph went with which child. Therefore, Question 10 was removed from the final version.

Question 12, “Write an equation for Matthew’s savings” appeared to be a redundant question because all of the IDS participants had answered earlier questions regarding his savings by writing an equation for him. However, this was not a guaranteed response by participants in the main study and I wanted to be able to determine whether or not the participants were able to write an equation from a graph, so the question was retained for the final version.

Question 15 was re-worded to make it more readable and to eliminate the request to draw a graph or table. The IDS data showed that the question was awkward to read and
that because it was an open-ended question about comparing two children’s savings, none of the IDS participants chose to draw a table or graph. This portion was also eliminated because of time concerns for the main study.

Question 16 was eliminated because of time constraints and because it was similar to previous questions.

Function Knowledge Assessment: Version 1 had the full list of questions for Part 2 of the assessment and all, except one, of these questions were used in the final version. Number 3 was eliminated because it was similar to number four. Also, the problem was specific enough that it was really challenging for students who had chosen to use the graphs when answering questions from the FKA: Part 2.

The third part of the FKA: V2 was retained with the condition that the questions would only be asked if there was enough time. Table 3.3 summarizes the changes for the FKA and FCA.

<table>
<thead>
<tr>
<th>Test</th>
<th>Problem</th>
<th>Action</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>FKA: Part 1</td>
<td>7</td>
<td>Change $140 to $150</td>
<td>$140 was on the original scale; wanted it to be more of an extrapolation</td>
</tr>
<tr>
<td>FKA: Part 1</td>
<td>10</td>
<td>Eliminated</td>
<td>Time constraints; Redundant information</td>
</tr>
<tr>
<td>FKA: Part 1</td>
<td>15</td>
<td>Reworded; “Also, use tables, graphs, expressions, and explanations.” Was removed</td>
<td>Awkward to read the problem aloud; time constraints</td>
</tr>
</tbody>
</table>

Table 3.3: Summary of Changes to Instruments Based on IDS Data
<table>
<thead>
<tr>
<th>FKA: Part 1</th>
<th>16</th>
<th>Eliminated</th>
<th>Time constraints; redundant information</th>
</tr>
</thead>
<tbody>
<tr>
<td>FKA</td>
<td>General Format</td>
<td>Separated the questions to be on separate sheets of paper</td>
<td>Prevents the participants from looking ahead to see what types of questions Part 2 contains before they choose which representation to use</td>
</tr>
<tr>
<td>FKA: Part 2</td>
<td>3</td>
<td>Eliminated</td>
<td>Difficult for students who chose to use the graphs; redundant information; time constraints</td>
</tr>
<tr>
<td>FKA: Part 3</td>
<td>1 and 2</td>
<td>Retained with the condition that only participants that had enough time would complete these questions</td>
<td>Time constraints</td>
</tr>
<tr>
<td>FCA</td>
<td>1</td>
<td>Moved to question 3</td>
<td>Placed it next to question 4 as they are similar questions and provide good opportunities for clarification questions; Make it so the test did not start off with a non-linear problem</td>
</tr>
<tr>
<td>FCA</td>
<td>4</td>
<td>Made the last “T” value of the table be “8” instead of “10”; changed the question to reflect this change (i.e. “What is the value of the car after 8 years?”)</td>
<td>The “T” values go up by 2s until the last one which went up by 4; wanted to keep this more consistent as I am testing their understanding of the function, not their ability to read the problem accurately</td>
</tr>
<tr>
<td>FCA</td>
<td>5</td>
<td>Eliminated</td>
<td>The final answer did not make sense because the profit ended up to be less than the contributions to charity; time constraints</td>
</tr>
<tr>
<td>FCA</td>
<td>8</td>
<td>Replaced with similar problem that had context</td>
<td>Wanted the problem in a format that was easy to discuss; possible redundancy to number 7 (depending on how the student solved that problem)</td>
</tr>
</tbody>
</table>
Instruments for the Main Study

Three instruments were used for data collection in the main study. The Mathematics Education Experience and Visual Abilities (MEEVA) interview provided information on students’ background, visual abilities, experiences in the mathematics classroom, and demographic information. These details were especially important to obtain for participants with visual impairments because vision level and educational background can vary widely between participants and each characteristic can greatly affect their educational experiences and hence their understanding of function. The second and third instruments, the Function Knowledge Assessment (FKA) and the Function Competency Assessment (FCA) evaluated students’ specific knowledge and competencies when working with functions in the context of a problem situation. Two mathematics assessments were used for data collection because of differences in design that allowed for access to particular student reasoning. Next I discuss each of the instruments in detail.

Mathematics Education Experience and Visual Abilities (MEEVA) Interview

The MEEVA interview was semi-structured with a set of questions each participant was asked as well as some impromptu questions that arose during the interview. There were 30 questions on the MEEVA protocol (Appendix A). I wrote the questions for the MEEVA interview and based these questions on previous research regarding individuals with visual impairments. Each question on the MEEVA interview was written to inform me of the student’s demographic information, visual experiences and abilities, general school experiences, or experiences with his or her mathematics education. Thus, questions regarding the students’ educational placement, whether he or
she was educated in a school for the blind or was in inclusive educational environments type of vision loss and when the vision loss occurred were included. Also included were questions that asked about students’ access to technology, how the student was taught to produce and read mathematical graphs, mathematics topics that were challenging for the student, and accommodations that were made for the student within the mathematics classroom. A summary of the questions from the MEEVA and the purpose for each question is in Table 3.4.

<table>
<thead>
<tr>
<th>MEEVA Question</th>
<th>Type of Information Gained from Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) For High School; What grade are you currently in? For College; What year are you in your program?</td>
<td>Demographics</td>
</tr>
<tr>
<td>2) For College: What is your major? For High School: What are you interested in doing when you are finished with high school? Do you plan to attend college? What do you plan to major in?</td>
<td>Demographics</td>
</tr>
<tr>
<td>3) When did you lose your vision?</td>
<td>Visual Experiences</td>
</tr>
<tr>
<td>4) How did you lose your vision? (i.e. gradually over time or all at once)</td>
<td>Visual Experiences</td>
</tr>
<tr>
<td>5) Where do you attend school? Where have you attended in the past?</td>
<td>Demographics</td>
</tr>
<tr>
<td>6) Were you ever educated separately from your classmates? (i.e. pulled out of class to work with a Vision Intervention Specialist, took a class at the school for the blind)</td>
<td>Demographics</td>
</tr>
<tr>
<td>7) Do you read Braille? Large print?</td>
<td>Demographics</td>
</tr>
<tr>
<td>8) What were/are some of the supports the school provides that are specifically related to your visual impairment/blindness?</td>
<td>General School Experiences</td>
</tr>
<tr>
<td>9) What technology do you use? (Zoomtext, JAWS, Braille-note, etc.)</td>
<td>General School Experiences</td>
</tr>
</tbody>
</table>

Table 3.4: Categories of MEEVA Interview Questions
Table 3.4 continued

<table>
<thead>
<tr>
<th></th>
<th>Question</th>
<th>Category</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>Do you have a favorite subject or favorite class? What is it that you like about this class?</td>
<td>General School Experiences</td>
</tr>
<tr>
<td>11</td>
<td>What classes have you struggled with? Why do you think these classes were so difficult for you?</td>
<td>General School Experiences</td>
</tr>
<tr>
<td>12</td>
<td>What is your impression of mathematics? Do you enjoy doing math? Overall would you say you have had good or bad experiences in mathematics classes?</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>13</td>
<td>What tools and/or accommodations were you given specifically for helping you to learn mathematics?</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>14</td>
<td>How did your teacher(s) introduce you to graphing?</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>15</td>
<td>Do you feel you have a good understanding of linear functions?</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>16</td>
<td>Did you use many tactile graphs when learning mathematics? Are you comfortable with reading these?</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>17</td>
<td>How did you take your mathematics exams? Did you have a brailled/large print version? Did a proctor work with you to read the questions and write your responses?</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>18</td>
<td>What was most challenging for you when learning mathematics?</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>19</td>
<td>Was there anything particularly difficult for you when learning algebra?</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>20</td>
<td>How were your mathematics classes structured? (Lecture, group work, projects, papers, etc.)</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>21</td>
<td>How do you learn best?</td>
<td>Mathematics Education Experiences</td>
</tr>
<tr>
<td>22</td>
<td>In what ways did your mathematics teachers help to include you into the class? Did they make any special arrangements? How did they handle situations when they pointed to something on a graph or wrote on the board? How did you access that information?</td>
<td>Mathematics Education Experiences</td>
</tr>
</tbody>
</table>
Function Knowledge Assessment (FKA)

The final version of the FKA contained three parts. The FKA: Part 1 had 16 questions, the FKA: Part 2 had 6 questions, and the FKA: Part 3 contained 2 questions. All questions on the FKA: Part 1 focus on the savings habits of four children. Questions on the FKA: Part 2 focused on the savings accounts of two of the four children. The FKA: Part 3 introduced a fifth child’s, Ellen’s, savings plan and asked two questions regarding her savings. The information for the children’s savings accounts originated
with work from Friedlander and Tabach (2001). Some of the questions are also from their work, as mentioned in Table 3.5.

<table>
<thead>
<tr>
<th>Question</th>
<th>Source</th>
<th>Type of Problem/Reason for Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>How much money will Diane have after the 8\textsuperscript{th} week?</td>
<td>Researcher</td>
<td>Getting acquainted with the initial representation Knowledge of using equations/expressions</td>
</tr>
<tr>
<td>How much money will Adam have after week 22?</td>
<td>Researcher</td>
<td>Getting acquainted with the initial representation Knowledge of interpreting information presented in a table</td>
</tr>
<tr>
<td>How much money does Matthew have after week 18?</td>
<td>Researcher</td>
<td>Getting acquainted with the initial representation Knowledge of reading information presented on a graph</td>
</tr>
<tr>
<td>How much money will Matthew have on week 30?</td>
<td>Researcher</td>
<td>Getting acquainted with the initial representation Knowledge of extrapolating information from a graph</td>
</tr>
<tr>
<td>How much money did Adam have on the 10\textsuperscript{th} week?</td>
<td>Researcher</td>
<td>Getting acquainted with the initial representation Knowledge of extrapolating information from a table</td>
</tr>
<tr>
<td>How much money did Heather have at the end of the year?</td>
<td>Researcher</td>
<td>Getting acquainted with the initial representation Knowledge of interpreting a description of a constant function</td>
</tr>
<tr>
<td>Matthew will have $150 on which week?</td>
<td>Researcher</td>
<td>Getting acquainted with the initial representation Knowledge of extrapolating information from a table</td>
</tr>
<tr>
<td>Which week will Adam have $287?</td>
<td>Researcher</td>
<td>Getting acquainted with the initial representation Knowledge of extrapolating information from a graph</td>
</tr>
</tbody>
</table>

Table 3.5: Sources for Questions from the FKA and Problem Types
Table 3.5 continued

9. When will Diane have $200?  

10. Compare the savings of two out of the four children. Describe how the children’s savings are changing throughout the year. In your comparison you can talk about whether or not the savings is increasing or decreasing, which has a faster rate of change, which child has the largest amount at the end of the year, etc.

11. Given the graphs of the savings of all four children throughout the year, identify each graph and find the meaning and the value of each intersection point.

12. Write an equation for Matthew’s savings.

13. Write an equation for Adam’s savings.

14. Write an equation for Heather’s savings.

15. Whose savings is growing the fastest?

16. Whose savings is growing the slowest?

FKA: Part 2

| How much had Matthew saved after half a year? How much did Diane have at the same time? | Friedlander & Tabach (2001) | Work within a specific (chosen) representation Ability to gain information from the chosen representation |
Table 3.5 continued

<table>
<thead>
<tr>
<th>Problem</th>
<th>Author(s)</th>
<th>FKA: Part 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>After how many weeks did each of the two children have $210?</td>
<td>Friedlander &amp; Tabach (2001)</td>
<td>Work within a specific (chosen) representation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ability to gain information from the chosen representation</td>
</tr>
<tr>
<td>Find the week with the largest difference between their savings.</td>
<td>Friedlander &amp; Tabach (2001)</td>
<td>Work within a specific (chosen) representation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ability to use the chosen representation to solve a problem</td>
</tr>
<tr>
<td>Find the week when their savings were equal.</td>
<td>Friedlander &amp; Tabach (2001)</td>
<td>Work within a specific (chosen) representation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ability to use the chosen representation to solve a problem</td>
</tr>
<tr>
<td>Find the week when the savings of one were double that of the other.</td>
<td>Friedlander &amp; Tabach (2001)</td>
<td>Work within a specific (chosen) representation</td>
</tr>
<tr>
<td>In whose favor?</td>
<td></td>
<td>Ability to use the chosen representation to solve a problem</td>
</tr>
<tr>
<td>Diane and Matthew decided to pool their savings in order to buy a $400</td>
<td>Friedlander &amp; Tabach (2001)</td>
<td>Work within a specific (chosen) representation</td>
</tr>
<tr>
<td>walkie-talkie. Find the week in which they are able to purchase the</td>
<td></td>
<td>Ability to use the chosen representation to solve a problem</td>
</tr>
<tr>
<td>walkie-talkie.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How much money had Ellen saved after 10 weekends?</td>
<td>Friedlander &amp; Tabach (2001)</td>
<td>Getting acquainted with initial representation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ability to understand a non-linear function</td>
</tr>
<tr>
<td>How is Ellen’s savings different from that of the other four children?</td>
<td>Friedlander &amp; Tabach (2001)</td>
<td>Exploratory</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Ability to compare linear and non-linear functions</td>
</tr>
</tbody>
</table>

**Function Competency Assessment (FCA)**

The final version of the FCA consisted of seven mathematics problems, with several containing multiple parts. When taking into consideration the multiple part questions, a total of 17 individual questions were provided for student response (Appendix C). The purpose of the FCA was to evaluate student competencies within the context of O’Callaghan’s function model, therefore, all questions specifically address
student understanding in modeling, interpreting, translating, or reifying. All problems are imbedded in context. Each of the problems, with the exception of the final one, came from the mathematics education literature on assessing function knowledge. A summary of the source of each question on the FCA and information gained from each question is given in Table 3.6.

<table>
<thead>
<tr>
<th>Question from FCA</th>
<th>Source</th>
<th>Type of Problem/Reason for Inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. A truck is loaded with boxes, each of which weighs 20 pounds. If the empty truck weighs 4500 pounds, find the following:</td>
<td>O’Callaghan (1998)</td>
<td></td>
</tr>
<tr>
<td>a) The total weight of the truck if the number of boxes is 75.</td>
<td>O’Callaghan (1998)</td>
<td>Modeling-Application</td>
</tr>
<tr>
<td>b) The number of boxes if the total weight of the truck is 6,740 pounds.</td>
<td>O’Callaghan (1998)</td>
<td>Modeling-Application</td>
</tr>
<tr>
<td>c) Using W for the total weight of the truck and x for the number of boxes, write a symbolic rule (or equation) that expresses the weight as a function of the number of boxes.</td>
<td>O’Callaghan (1998)</td>
<td>Translating-Comprehension</td>
</tr>
<tr>
<td>2. The graph below gives the speed of a cyclist on his daily training ride. During his ride, he must climb a hill where he pauses for a drink of water before descending. Using this graph to answer the following questions as accurately as possible.</td>
<td>O’Callaghan (1998)</td>
<td></td>
</tr>
<tr>
<td>a) Find the speed when time equal 25 minutes.</td>
<td>O’Callaghan (1998)</td>
<td>Interpreting-Comprehension</td>
</tr>
<tr>
<td>b) Find the time when speed equals 30 mph.</td>
<td>O’Callaghan (1998)</td>
<td>Interpreting-Comprehension</td>
</tr>
<tr>
<td>c) During what time intervals was the speed increasing?</td>
<td>O’Callaghan (1998)</td>
<td>Interpreting-Comprehension</td>
</tr>
<tr>
<td>d) During which 10-minute interval did the speed decrease the most?</td>
<td>O’Callaghan (1998)</td>
<td>Interpreting-Comprehension</td>
</tr>
</tbody>
</table>

Table 3.6: Sources and Types of Problems for Questions on the FCA

132
Table 3.6 continued

3. The table below gives the average price of a new home in Smalltown, USA, for every 2 years since 1980:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>Author/Reference</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>e)</td>
<td>When was the cyclist at the top of the hill?</td>
<td>O'Callaghan (1998)</td>
<td>Interpreting-Application</td>
</tr>
<tr>
<td>b)</td>
<td>When would you predict the price to be $100,000?</td>
<td>O'Callaghan (1998)</td>
<td>Interpreting-Application</td>
</tr>
<tr>
<td>c)</td>
<td>Draw the graph of the data.</td>
<td>O'Callaghan (1998)</td>
<td>Translating-Comprehension</td>
</tr>
<tr>
<td>4.</td>
<td>Suppose that the following table gives the value (v), in dollars, of a car for different numbers of years (t) after it is purchased.</td>
<td>O'Callaghan (1998)</td>
<td>Interpreting-Application</td>
</tr>
<tr>
<td>a)</td>
<td>What is the value of the car after 8 years?</td>
<td>O'Callaghan (1998) (with slight modification)</td>
<td>Translating-Comprehension</td>
</tr>
<tr>
<td>b)</td>
<td>Write a symbolic rule expressing v as a function of t.</td>
<td>O'Callaghan (1998)</td>
<td>Translating-Comprehension</td>
</tr>
<tr>
<td>5.</td>
<td>Mary Wong just got a job working as a clerk in a candy store. She already has $42. She will earn $7 per hour.</td>
<td>Brenner et al. (1997)</td>
<td>Modeling-Application</td>
</tr>
<tr>
<td>a)</td>
<td>How many hours will she have to work to have a total of $126?</td>
<td>Brenner et al. (1997)</td>
<td>Translating-Comprehension</td>
</tr>
<tr>
<td>b)</td>
<td>Make a table or graph to represent the problem.</td>
<td>Brenner et al. (1997) (with slight modification)</td>
<td>Translating-Comprehension</td>
</tr>
<tr>
<td>6.</td>
<td>Two carnivals are coming to town. You and your friend decided to go to different carnivals. The carnival that you attend charges $10 to get in and an additional $2 for each ride. The carnival your friend attends charges $6 to get in, but each additional ride costs $3. If the two of you spent the same amount of money, how many rides could each of you have ridden?</td>
<td>Slavit (2006)</td>
<td>Modeling-Application</td>
</tr>
</tbody>
</table>

Continued
Table 3.6 continued

7. There are two children who save their money in the following way:

Helen has a set amount of money which does not change. Her savings can be described as $y=200$

Tommy has some income from a lawn service business. His savings can be described with the equation $y=5x-2$.

Without solving or graphing these equations, can you tell if there will be a time when Helen and Tommy have the same amount of money? How can you tell?

Data Collection Procedures

Participants were asked to complete the Mathematical Education Experience and Visual Abilities interview (MEEVA), the Function Knowledge Assessment (FKA) and the Function Competency Assessment (FCA) during two face-to-face sessions. The MEEVA was split between the two sessions with general demographic questions and those relating to general education experiences (questions 1-12) asked in the first session and questions specific to the students’ mathematics education, in the second (questions 13-30). This allowed students the opportunity to become comfortable with me and for the mathematics instruments to be given on two separate days, which helped to prevent fatigue. Therefore, the first session consisted of the first 12 questions from the MEEVA and the FKA while the second session consisted of the remaining MEEVA questions and the FCA assessment. Data collection was completed at the end of May and the beginning of June 2011.
Sessions were scheduled at a time and place convenient to the student and were video and audio recorded for transcription and data analysis. Each session lasted approximately one and a half to two hours. Participants were not given a written copy of the questions for the MEEVA, however, they were provided with a written copy, in their preferred reading medium, of the FKA and FCA. For the FKA and the FCA, I provided assistance to complete the tasks when it was clear the student was having difficulty progressing. This assistance was provided in the form of questions and reminders of basic facts or with prompts. Students completing tasks on the FKA and FCA were given a variety of materials.

A summary of data collection procedures are provided in Table 3.7.

<table>
<thead>
<tr>
<th>Data Collection (per participant)</th>
<th>Instruments</th>
<th>Participants</th>
<th>Duration/Setting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Session 1</td>
<td>First 12 questions of the MEEVA FKA: Sorting FKA: Savings</td>
<td>High School and college students with visual impairments</td>
<td>No more than 2 hours One-on-one with the participant and researcher in a location convenient to the participant</td>
</tr>
<tr>
<td>Session 2</td>
<td>Questions 13-30 of the MEEVA FCA</td>
<td>High School and college students with visual impairments</td>
<td>No more than 2 hours One-on-one with the participant and researcher in a location convenient to the participant</td>
</tr>
</tbody>
</table>

Table 3.7: Summary of Data Collection Procedures

**Materials**

The materials used by participants for completing the mathematics assessments (the FKA and the FCA) varied based on the reading medium and visual abilities of the
student. All students were provided with a copy of the assessments in their preferred reading medium. Thus, Robert, Liz, Madisen, Gabe, Caleb, and Yvonne were provided with the braille copy of the FKA and the FCA. Braille translation of the assessments was provided by Wright State University’s Office of Disability Services. Amanda was provided with the large print version and Wanda used the regular print copy of the FKA and the FCA. A Perkin’s braille writer was available for students’ use. Gabe used the Perkin’s braille writer to keep track of his steps as he worked through a problem and to create a table for question 5b on the FCA. Yvonne also used the Perkin’s braille writer to produce her table of information for problem 5b on the FCA. Other braille readers used their personal braille notetakers to keep track of their work. Yvonne was the only braille reader to use paper and pen to write out her steps. Yvonne commented that since she had vision until the age of 13, she found it easier to do the mathematics if she could write out her steps (even if she could no longer see what was being written).

Calculator use also varied among participants. Students who had their personal braille notetakers with them used the scientific calculator that is included with the notetaker. Amanda and Wanda had calculators they preferred to use, which were regular print scientific calculators. Amanda used a small magnifying glass to read the output from the calculator screen. When taking the FKA, Gabe and Robert did not have an accessible calculator, so any calculations that needed to be done they would tell me the steps and I would input the numbers into a regular print scientific calculator.

Graphing supplies for Amanda and Wanda included large print graph paper and pencils. Participants who produced tactile graphs were provided with a graph board that had the x-axis and y-axis marked with string and push pins of a variety of shapes.
Analysis

Data collected for this project came from four direct sources, student work on the Function Knowledge Assessment, student work on the Function Competency Assessment, student responses to the Mathematics Education Experiences and Visual Abilities interview, and researcher journal entries. These data were used to better understand the level of knowledge of linear functions held by students with visual impairments, the students’ access to and proficiency in graphing, whether or not the participants displayed a preference for a certain type of representation of function, and what characteristics students perceived as playing a role in their development of their mathematical understanding. Theoretical perspectives came from Wilson’s (1971) taxonomy of evaluation for secondary mathematics and O’Callaghan’s (1998) model of function competencies. Analysis for each of the four research questions was done using the constant comparative method. This analysis method is described next and is followed by a detailed description of the analysis process for each of the research questions.

Constant Comparative Method

The constant comparative method was utilized for data analysis. This method is particularly suited for qualitative data and was originally developed by Glaser and Strauss (1967) for use in grounded theory. The constant comparative method provides a systematic way to analyze data across all participants and to compare results among participants with similar understandings as well as participants with different understandings. To aid in the creation of conceptual categories, the constant comparative method constantly questions and compares the data (Glaser & Strauss, 1967; Strauss, 1987). Although I have found no use of the constant comparative method in mathematics
education research, there is a growing body of science education literature that has utilized this method of data analysis. The constant comparative method has been used for studies across various content including student conception of tides (Ucar & Trundle, in press), student understanding of the particulant nature of matter (Adadan, Trundle, & Irving, 2010), conceptual change of moon phases (Trundle, Atwood, & Christopher, 2002, 2006; Bell & Trundle, 2008), and student understanding of seasonal changes (Wild & Trundle, 2010a, 2010b). The constant comparative method has also been used in research with young students (Trundle, Atwood, & Christopher, 2006), high school students (Adadan, Irving, & Trundle, 2009; Ucar & Trundle, in press), and pre-service teachers (Sackes, Trundle, Bell, & O’Connell, 2011; Ucar & Trundle, in press). The constant comparative method has also been utilized in research with students with visual impairments (Wild & Trundle, 2010a, 2010b).

The constant comparative method allows for the possibility of themes to emerge from the data, especially in areas where research is rare. Thus it worked well for data analysis of the current study, even though there is a theoretical framework for identifying themes initially. In addition to the up-front theoretical framework going into the study, the constant comparative method allowed for the opportunity for themes to emerge from the data for adjustment and hence for a more solidified theoretical framework for future research. The current research is also descriptive and exploratory, which makes it well suited for the constant comparative method of analysis. Next, I describe the general technique of the constant comparative method of analysis and discuss specifics on how the constant comparative method was applied to each of the research questions.
Coding. Coding is an important part of the constant comparative method. It is through the codes that themes emerge. As Strauss (1987) summarizes, “The excellence of the research rests in large part on the excellence of the coding” (p. 27). The depth of the analysis is therefore closely linked to the process of code development. The constant comparative method gives researchers a systematic and inductive process to analyze data, which aids in the development of explanatory frameworks that specify relationships among the raw data (Denzin & Lincoln, 2000). The coding process is composed of two main phases, open and selective coding. Open coding has a sub-coding process called axial coding, and throughout these phases researchers are encouraged to practice memo-writing. These are defined and described next.

Open or initial coding is the first and most unrestricted phase in the coding process. At this stage, the data are scrutinized closely, often sentence by sentence or phrase by phrase (Glaser & Strauss, 1967; Strauss, 1987). This line-by-line coding helps to build ideas inductively, and deters the researcher from imposing her own beliefs on the data (Denzin & Lincoln, 2000). Axial coding, an essential aspect of open coding, is the intense analysis done around a single category at a time. This allows the researchers to accumulate knowledge of relationships between that category and other categories and subcategories (Strauss, 1987). The second phase of coding for the constant comparative method is called selective coding. This phase is much more restricted than open coding and is characterized by a systematic and concerted review of the results from previous codes to develop the core categories. It is through these categories that the researcher will integrate the various aspects of the data into a dense theory which is close to the data (Glaser & Strauss, 1967).
Two basic rules Glaser and Strauss (1967) promote are, a) while coding an incident for a category, compare it with the previous incidents in the same and different groups coded in the same category and b) frequently stop coding and record a memo on your ideas. Following these rules is what allows for a constant comparison of incidents, which will generate theoretical properties of the data. Glaser and Strauss (1967) summarize the relationship between the phases:

Although this method of generating theory is a continuously growing process—each stage after a time is transformed into the next—earlier stages do remain in operation simultaneously throughout the analysis and each provides continuous development to its successive stage until analysis is terminated. (p. 105)

Through this process there was reduction in the data in that the analyst will find underlying uniformities in the original categories or their properties, which will allow for integration of ideas and in turn will lead to a denser theory with a smaller number of higher level concepts (Glaser & Strauss, 1967). The constant comparative method was utilized to analyze data for answering each of the four research questions. The following discussion details the analysis procedures for each question.

**Research Question 1**

Research question 1 asked, “What level of knowledge and type of understanding of linear functions is held by students with visual impairments?” In order to assess the participants’ level of knowledge of function, transcripts were completed for each students’ responses to questions from the FKA and FCA. The students’ success-level, comfort-level and student competency in modeling, interpreting, translating, and reifying linear function problems are used to analyze this first research question.
To address student understanding of function through the constant comparative method, the following comparisons were made, a) comparison from a single interview to the coding scheme, b) codes were compared with each interview, c) comparison of codes within a single interview to levels of understanding of function, d) comparison of levels of understanding in a single interview compared with levels of understanding of other participants, e) comparison of the levels of understanding of participants who are congenitally blind, to the level of understanding of participants who had early visual experiences, to the level of understanding of students who lost their sight late in life, f) comparison of the level of understanding of participants who were taught the function concept while attending a school for the blind with the level of understanding of participants who were taught the function concept while attending their local school, g) comparison of the levels of understanding of participants whose primary literacy mode is braille to the levels of understanding of participants whose primary literacy mode is large or regular print. A summary of these comparisons and the aim of each comparison are in Table 3.8.

<table>
<thead>
<tr>
<th><strong>Comparison</strong></th>
<th><strong>Aim of Comparison</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Data from a single interview was compared to the coding scheme.</td>
<td>Code transcripts</td>
</tr>
<tr>
<td>Codes was compared within each interview.</td>
<td>Describe and add new, emergent codes to the coding framework</td>
</tr>
<tr>
<td>Codes within a single interview were compared to types of conceptual understandings.</td>
<td>Determine consistency of responses within the interview and assign participants to themes from the coding framework</td>
</tr>
</tbody>
</table>

Table 3.8: Comparisons Made of Coded Data for Analyzing Research Question 1
Table 3.8 continued

<table>
<thead>
<tr>
<th>Comparisons were made between the participants’ conceptual understanding.</th>
<th>Summarize overall types of understanding between participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comparisons were made between the understanding of students who are congenitally blind, those who had early visual experiences, and the understanding of students who became blind later in life.</td>
<td>Determine if there is a difference in the levels of understanding of function for students who are congenitally blind, those with early visual experiences, and those who lost vision after the age of two</td>
</tr>
<tr>
<td>Comparisons were made between the understanding of students who were taught the function concept in a school for the blind and the understanding of students who were taught the function concept in a regular public school.</td>
<td>Determine if there is a difference in the levels of understanding of function for students who were taught at a school for the blind and those who were taught in a public high school</td>
</tr>
<tr>
<td>Comparisons were made between the understanding of students whose primary literacy mode is Braille and the understanding of students whose primary literacy mode is large or regular print.</td>
<td>Determine if there is a difference in the levels of understanding of function for students with a primary reading medium of braille, those with the primary reading medium of large or regular print</td>
</tr>
</tbody>
</table>

Student responses on the FKA and FCA were coded for, a) success-level, b) comfort-level, and c) function competency. Coding procedures for determining student level of success, comfort, and competency in function are discussed, next.

*Coding procedures and coding criteria for Success-Level.* Student responses were coded for success for problems one through nine and problems twelve through fourteen on the FKA: Part 1, for all problems from the FKA: Part 2, and for all questions on the FCA. Success was determined by students’ initial response, students’ final answer, and the amount and type of help provided to the student. Each response was coded as having high, medium-high, medium-low, or low success. Responses coded as having high success were ones where students answered the question without assistance and with a
relatively high degree of accuracy. Some leniency was given to incorrect responses that were obtained because of calculation mistakes. Responses that were gained from reading a graph or using estimation were also coded as high if the student’s reasoning was correct, but answers were slightly off from the actual answer. Medium-high was coded for responses that were correct, but ones that students’ required prompting or learning questions from me to arrive at the final answer.

Student responses coded as medium-low included ones that the student demonstrated an understanding of the problem, but was unsuccessful in solving or completing the problem. Medium-low was also coded when students were able to eventually arrive at an accurate answer, but needed direct help with the content. Responses coded as low success were those where students’ incorrect reasoning was not resolved by the end of the problem. Low success included problems that students were unable to complete. Table 3.9 gives a summary of the codes and criteria of the codes used for success-level.

<table>
<thead>
<tr>
<th>Success-Levels</th>
<th>Criteria for Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>Successful solution with no assistance</td>
</tr>
<tr>
<td>Medium-high</td>
<td>Successful solution with some assistance on content including prompting or leading questions</td>
</tr>
<tr>
<td>Medium-low</td>
<td>Demonstrates a concept of the problem but is unable to arrive at a successful response</td>
</tr>
<tr>
<td></td>
<td>Arrives at a successful solution with direct help on content</td>
</tr>
<tr>
<td>Low</td>
<td>Displays incorrect reasoning that is not rectified by the end of the problem</td>
</tr>
<tr>
<td></td>
<td>Unable to complete the problem</td>
</tr>
</tbody>
</table>

Table 3.9: Criteria for Success-Levels
Coding procedures and coding criteria for Comfort-Level. Comfort-level displayed by students specifically speaks to students’ comfort in working with a particular representation (i.e. tables, equations, graphs, descriptions). Comfort-level does not indicate students’ success or efficiency in solving the problem. Student responses were coded for comfort for problems one through nine and problems twelve through fourteen on the FKA: Part 1, for all problems from the FKA: Part 2, and for all questions on the FCA. Comfort was coded as high, medium, or low for each response. Responses coded as demonstrating high comfort were ones where students answered the question without assistance or hesitation in using the presented representation. Medium comfort was coded for responses that displayed understanding of how to use the representation, but was accompanied by hesitancy with either the use of the representation or with the final response gained through work with the representation. Comfort was coded as low when students needed direct help with using the representation. Table 3.10 gives a summary of the codes and criteria of the codes used for comfort-level.

<table>
<thead>
<tr>
<th>Comfort-Level</th>
<th>Criteria for Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>Solved the problem with little to no assistance regarding the use of the representation</td>
</tr>
<tr>
<td>Medium</td>
<td>Has doubts regarding answer</td>
</tr>
<tr>
<td></td>
<td>Shows hesitancy in solving the problem</td>
</tr>
<tr>
<td>Low</td>
<td>Help is needed with representation, itself</td>
</tr>
</tbody>
</table>

Table 3.10: Definition of Comfort-Levels
Coding procedures and coding criteria for levels of competency. The levels of understanding described by Wilson (1971) and the four function competencies described by O’Callaghan (1998) served as the framework for coding data obtained through student responses to problems on the FKA and FCA. Wilson’s Taxonomy is a dual classification system where a students’ level of behavior for a specific content area is addressed for each evaluation item. Students’ levels of behavior addressed in this research include comprehension and application. The content being addressed is the students’ competency in the four areas of O’Callaghan’s (1998) function model; a) modeling, b) interpreting, c) translating, and d) reifying. Table 3.11 presents a graphic displaying the dual classification used for coding students’ levels of knowledge and types of understanding in linear functions and the questions from the FKA and the FCA that were used to demonstrate student knowledge in each category.

<table>
<thead>
<tr>
<th></th>
<th>Comprehension</th>
<th>Application</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Modeling</strong></td>
<td>FKA: Part 3: 1</td>
<td>FKA: 1a, 1b, 5a, 6,</td>
</tr>
<tr>
<td></td>
<td>FKA: Part 1: 1, 2, 3, 6, 9</td>
<td>FKA: Part 1: 4, 5, 7, 8, 10, 15, 16</td>
</tr>
<tr>
<td></td>
<td>FKA: Part 2: 1, 2</td>
<td>FKA: Part 2: 3, 4, 5, 6</td>
</tr>
<tr>
<td></td>
<td>FCA: 2a, 2b, 2c, 2d</td>
<td>FCA: 2e, 3a, 3b, 4a</td>
</tr>
<tr>
<td><strong>Interpreting</strong></td>
<td>FKA: Part 1: 12, 13, 14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FCA: 1c, 3c, 4b, 5b</td>
<td></td>
</tr>
<tr>
<td><strong>Translating</strong></td>
<td>FKA: Part 1: 11</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FKA: Part 3: 2</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.11: Classifications of Questions for the FKA and the FCA for Level of Student Behavior and Type of Function Competency

Codes used for students’ level of understanding in function competencies were, a) modeling-comprehension, b) modeling-application, c) interpreting-comprehension, d)
interpreting-application, e) translating-comprehension, f) translating-application, and g) reifying-comprehension. A summary of the levels of understanding and the criteria for each of the levels are given in Table 3.12.

<table>
<thead>
<tr>
<th>Levels of Understanding</th>
<th>Criteria for Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modeling-Comprehension</td>
<td>Ability to represent problems situations in a mathematical way</td>
</tr>
<tr>
<td>Modeling-Application</td>
<td>Ability to solve word problems by producing and using a mathematical model</td>
</tr>
<tr>
<td>Interpreting-Comprehension</td>
<td>Ability to read information from a given function, in any representation, and to predict values not explicitly shown in the representation</td>
</tr>
<tr>
<td>Interpreting-Application</td>
<td>Ability to solve word problems based on data interpreted from a given table, graph, description, or equation of a function</td>
</tr>
<tr>
<td>Translating-Comprehension</td>
<td>Ability to move from one mode of a function to another</td>
</tr>
<tr>
<td>Reifying-Comprehension</td>
<td>Ability to discuss and understand function as a whole</td>
</tr>
</tbody>
</table>

Table 3.12: Definition of Codes for Levels of Understanding

**Research Question 2**

Research question 2, “What are the graphing skills of high school/college students with visual impairments?” was analyzed through student responses from the MEEVA interview and student work on the FKA and the FCA. Student success-level, comfort-level when working with problems using graphs, student access to graphing materials as reported on the MEEVA, and student work when producing graphs were used to
determine the students’ a) access to graphing, b) independence with graphing, c) proficiency with reading graphs, and d) proficiency with producing graphs.

To address student access and proficiency with graphing through the constant comparative method, the following comparisons were made, a) comparison from a single interview to the coding scheme, b) codes were compared with each interview, c) comparison of codes within a single interview to levels of understanding of function, d) comparison of levels of understanding in a single interview compared with levels of understanding of other participants. A summary of the comparisons and the aim of each comparison is presented in Table 3.13.

<table>
<thead>
<tr>
<th>Comparison</th>
<th>Aim of Comparison</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data from a single interview was compared to the coding scheme.</td>
<td>Code transcripts</td>
</tr>
<tr>
<td>Codes were compared within each interview.</td>
<td>Describe and add new, emergent codes to the coding framework</td>
</tr>
<tr>
<td>Codes within a single interview were compared to types of conceptual understandings.</td>
<td>Determine consistency of responses within the interview and assign participants to themes from the coding framework</td>
</tr>
<tr>
<td>Comparisons were made between the participants’ conceptual understanding.</td>
<td>Summarize overall types of understanding between participants</td>
</tr>
</tbody>
</table>

Table 3.13: Comparisons Used for Data Analyzed for Research Question 2

Responses from the MEEVA, the FKA, and the FCA used for coding for student access and proficiency in graphing (Research Question 2) is summarized in Table 3.14.
Table 3.15: Codes and Parameter of Codes Used for Analysis of Student Access and Proficiency with Graphs

<table>
<thead>
<tr>
<th>Code</th>
<th>Parameters of Code</th>
<th>Code Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Access</td>
<td>• Education on graphing</td>
<td>• Poor</td>
</tr>
<tr>
<td></td>
<td>• Availability of accessible graphs for classroom use and on examinations</td>
<td>• Adequate</td>
</tr>
<tr>
<td></td>
<td>• Access to graphing materials</td>
<td>• Excellent</td>
</tr>
</tbody>
</table>

Table 3.14: Question Responses Coded for Student Access and Proficiency in Graphing

Access to graphing was determined by availability of graphing materials, access to tactile or large print graphs, and education opportunities promoting the efficient use of graphs. Students were coded as having poor, adequate, or excellent access to graphs. Students’ were coded as either being independent or not independent with graphing through students’ ability to independently and comfortably work with graphs. Independence with graphing also included students’ ability to use graphs during instruction and on examinations without needing assistance from others. Skills in reading and producing graphs were both coded as poor, adequate, or proficient. Codes and code parameters used to determine student access and proficiency with graphs are summarized in Table 3.15.
Table 3.15 continued

| Independence                      | • Ability to be independent when graphing in class and on examinations  |
|                                  | • Ability to read graphs without assistance from others                |
|                                  | • Comfort in working with graphs                                      |
|                                  | • Independent                                                        |
|                                  | • Not Independent                                                    |

| Proficiency with Reading         | • Knowledge of the structure of graphs                               |
|                                  | • Knowledge of the parts of a graph                                  |
|                                  | • Ability to accurately read graphical information                   |
|                                  | • Employs techniques to help increase accuracy when reading a graph  |
|                                  | • Poor                                                               |
|                                  | • Adequate                                                          |
|                                  | • Proficient                                                        |

| Proficiency with Producing       | • Knowledge of components of a graph                                 |
|                                  | • Ability to start producing a graph without assistance             |
|                                  | • Ability to successfully complete a graph                          |
|                                  | • Poor                                                               |
|                                  | • Adequate                                                          |
|                                  | • Proficient                                                        |

Research Question 3

Research question 3, “Do students with visual impairments have representational preferences when solving word problems involving functions?” was addressed through student responses to the MEEVA interview, the FKA and the FCA. The following comparisons were made: a) data from a single interview was compared to the coding scheme; b) codes was compared within each interview; c) codes within a single interview was compared to types of representational preferences; d) the representational preferences of participants who are congenitally blind was compared with the
representational preferences of participants who had early visual experiences, to the representational preferences of those who became blind after the age of two; e) representational preferences of participants who were taught the function concept at a school for the blind was compared to the representational preferences of students who were taught the function concept at their local school; f) representational preferences of participants whose primary literacy mode of Braille was compared to the representational preferences of participants whose primary literacy mode is either large or regular print.

Aims of these comparisons can be found in Table 3.16.

<table>
<thead>
<tr>
<th>Comparison</th>
<th>Aim of Comparison</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data from a single interview was compared to the coding scheme.</td>
<td>Code transcripts</td>
</tr>
<tr>
<td>Codes were compared within each interview.</td>
<td>Describe and add new, emergent codes to the coding framework</td>
</tr>
<tr>
<td>Codes within a single interview were compared to types of representational preference.</td>
<td>Determine consistency of responses within the interview and assign participants to themes from the coding framework</td>
</tr>
<tr>
<td>Comparisons were made between the representational preferences of students who are congenitally blind to that of students who had early visual experiences to those who became blind later in life.</td>
<td>Determine if there is a difference in the representational preferences for students who are congenitally blind with those who had early visual experiences to those who lost vision after age of two</td>
</tr>
<tr>
<td>Comparisons were made between the representational preferences of students who were taught the function concept in a school for the blind and that of students who were taught the function concept in their local school.</td>
<td>Determine if there is a difference in the representational preferences of students who were taught at a school for the blind and those who were taught in their local high school</td>
</tr>
<tr>
<td>Comparisons were made between the representational preferences of students whose primary literacy mode is Braille to those whose primary literacy mode is either large or regular print.</td>
<td>Determine if there is a difference in the representational preferences of students with a primary reading medium of Braille to those with the primary reading medium of large or regular print</td>
</tr>
</tbody>
</table>

Table 3.16: Comparisons Made of Codes for Research Question 3
The four representations most commonly used to represent function served as the codes for the theoretical framework when analyzing Question 3. These representations are, a) graphical, b) symbolic/equation, c) table, and d) descriptions. To determine student preference of representation student success-levels and comfort-levels (as described earlier in this chapter; see Tables 3.9 and 3.10 for a summary of codes used for success-level and comfort-level) were considered for all problems involving the use of a particular representation, student stated preference of representation, and student choices of representational use when solving problems on the FKA and the FCA. Table 3.17 gives a summary of the criteria for the four representations.

<table>
<thead>
<tr>
<th>Representational Preference</th>
<th>Criteria for Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphical</td>
<td>Students displayed high success and high comfort when using graphical information</td>
</tr>
<tr>
<td></td>
<td>When given a choice, the student used a graph to obtain problem information or to solve a problem</td>
</tr>
<tr>
<td></td>
<td>Students stated a preference for using graphs when solving problems involving linear functions</td>
</tr>
<tr>
<td>Symbolic/Equation</td>
<td>Students displayed high success and high comfort when using problems involving equations</td>
</tr>
<tr>
<td></td>
<td>When given a choice, the student used an equation to obtain problem information or to solve a problem</td>
</tr>
<tr>
<td></td>
<td>Students stated a preference for using equations when solving problems involving linear functions</td>
</tr>
</tbody>
</table>

Table 3.17: Definition of Codes Used for Analysis of Research Question 3
Table 3.17 continued

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Students displayed high success and high comfort when using information given in tabular form</td>
</tr>
<tr>
<td></td>
<td>When given a choice, the student used a table to obtain problem information or to solve a problem</td>
</tr>
<tr>
<td></td>
<td>Students stated a preference for using tables when solving problems involving linear functions</td>
</tr>
<tr>
<td>Descriptions</td>
<td>Students displayed high success and high comfort when using solving problems where information was described</td>
</tr>
<tr>
<td></td>
<td>When given a choice, the student used a description of problem information to solve a problem</td>
</tr>
<tr>
<td></td>
<td>Students stated a preference for using descriptions when solving problems involving linear functions</td>
</tr>
</tbody>
</table>

**Research Question 4**

Research Question 4: What factors do high school/college students with visual impairments perceive as influencing their development of understanding of linear functions? This question was addressed using the transcribed interview data from the MEEVA. The constant comparative method was once again utilized for data analysis and the comparisons made were: a) data from a single interview was compared to the coding scheme; b) codes was compared within each interview; c) codes within a single interview was compared to the principles of a special methodology; d) comparisons were made between the factors described as influences in understanding of the function concept; e) the factors of influence in understanding function given by participants who are congenitally blind was compared with those given by participants who had early visual experiences to those who became blind after the age of two; f) the factors of influence in
understanding function given by participants who were taught the function concept at a school for the blind was compared to those given by students who were taught the function concept at their local school. Table 3.18 provides a summary of the comparisons and the aim of each comparison for research question 4.

<table>
<thead>
<tr>
<th><strong>Comparison</strong></th>
<th><strong>Aim of Comparison</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Data from a single interview was compared to the coding scheme.</td>
<td>Code transcripts</td>
</tr>
<tr>
<td>Codes were compared within each interview.</td>
<td>Describe and add new, emergent codes to the coding framework</td>
</tr>
<tr>
<td>Codes within a single interview were compared to types of conceptual understandings.</td>
<td>Determine consistency of responses within the interview and assign participants to themes from the coding framework</td>
</tr>
<tr>
<td>Comparisons were made between the influencing factors of understanding function given by participants.</td>
<td>Summarize overall factors which influence understanding of function for the participants</td>
</tr>
<tr>
<td>Comparisons were made between the given factors of influence of understanding function by students who are congenitally blind, those who had early visual experiences, and those who became blind later in life.</td>
<td>Determine if there is a difference in the perceived factors of influence of learning function for students who are congenitally blind with those who had early visual experiences to those who lost vision after age of two</td>
</tr>
<tr>
<td>Comparisons were made between the given factors of influence of understanding of function by students who were taught the function concept in a school for the blind and those given by students who were taught the function concept at their local school.</td>
<td>Determine if there is a difference in the perceived factors of influence of the understanding of function of students who were taught at a school for the blind and those who were taught at their local school</td>
</tr>
</tbody>
</table>

Table 3.18: Comparison of Codes Used in Analysis of Research Question 4

Students with visual impairments need specific accommodations to be fully included into a classroom lesson. For instance, any material used that is normally
accessed visually, needs to be either converted to a tactile or audio representation. Also, the amount of time provided for the lesson will need to be adjusted for these students as accessing the material will typically take longer when processing tactile or audio input.

Lowenfeld (1981) outlines five principles that are essential to any special methodology used for students with visual impairments. These principles are a) individualization, b) concreteness, c) unified instruction, d) additional stimulation, and e) self-activity. These principles served as the initial theoretical framework for the coding system when analyzing data for Question 4. However, after data analysis it became apparent that the data gathered for this research addressed the first two, individualization and concreteness. These were used for the final coding process and are summarized in Table 3.19.

<table>
<thead>
<tr>
<th>Principles of a Special Methodology</th>
<th>Explanation of Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Individualization</td>
<td>Any accommodation or modification that provides for the individual needs and learning styles of the student.</td>
</tr>
<tr>
<td>Concreteness</td>
<td>Any accommodation or modification that specifically provides tangible access to course content.</td>
</tr>
</tbody>
</table>

Table 3.19: Definition of Codes Used in Analysis of Research Question 4

Trustworthiness

“The basic issue in relation to trustworthiness is simple: How can an inquirer persuade his or her audiences (including self) that the findings of an inquiry are worth paying attention to, worth taking account of?” (Lincoln & Guba, 1985, p. 290). For research in the quantitative paradigm common terms for evaluating the trustworthiness of
the research are, “internal validity,” “external validity,” “reliability,” and “objectivity.”

Some of these validity issues are not relevant for qualitative researchers, while others speak to similar needs of validity in postpositivist research. Lincoln and Guba (1985) discuss four forms of validity for the naturalist researcher, which correspond to those used in quantitative research. These are summarized in Table 3.20. Each is discussed in more detail, next.

<table>
<thead>
<tr>
<th>Qualitative Validity Terms</th>
<th>Corresponding Quantitative Validity Terms</th>
<th>Type of Validity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Credibility</td>
<td>Internal Validity</td>
<td>How to determine the “truth” of the finding</td>
</tr>
<tr>
<td>Transferability</td>
<td>External Validity</td>
<td>How to determine the extent to which the findings of a particular inquiry have applicability in other contexts</td>
</tr>
<tr>
<td>Dependability</td>
<td>Reliability</td>
<td>How to determine if the findings of an inquiry would be repeated if the inquiry were repeated with similar conditions</td>
</tr>
<tr>
<td>Confirmability</td>
<td>Objectivity</td>
<td>How to establish the degree to which the findings are determined by the subjects and not by biases, motivations, or interests of the inquirer</td>
</tr>
</tbody>
</table>

Table 3.20: Qualitative and Quantitative Validity Terms and Explanations

To add to the trustworthiness of the current study I utilized the following strategies: triangulation of multiple sources, triangulation of multiple methods, peer debriefing, member checks, a dependability audit, inter-rater reliability, and a reflexive
Credibility

Credibility differs from the notion of internal validity because of the fundamental idea of “truth” between the two paradigms. It is not possible for the qualitative researcher, who believes in multiple truths, to determine how close the findings come to the “truth” of an event being studied. However, the naturalistic inquirer needs to be confident that the conclusions reached are consistent with the data. To help ensure this for the current study I will use triangulation of multiple sources, triangulation of multiple methods, peer debriefing, and member checks.

Triangulation of multiple sources

The current study collected data from multiple participants from a variety of backgrounds. Data analysis across these participant responses helped to ensure that a credible view of the understanding of function of students with visual impairments was reached.

Triangulation of multiple methods

The triangulation of multiple methods can imply either different modes of data collection (i.e. interviews, questionnaires, observation, testing) or different designs (Lincoln & Guba, 1985). The current study utilized the former definition to address research questions 1, 2, and 3. The understanding of function held by the student (Research Question 1) is assessed through student responses to both the FKA and the FCA. The participants’ access to and proficiency with graphing (Research Question 2)
utilized student responses to questions on the MEEVA, the FKA, and the FCA. Finally, the representational preference of the participant (Research Question 3) is assessed through the MEEVA, the FKA and the FCA.

**Peer debriefing**

Lincoln and Guba (1985) describe peer debriefing as, “a process of exposing oneself to a disinterested peer in a manner paralleling an analytic session and for the purpose of exploring aspects of the inquiry that might otherwise remain only implicit within the inquirer’s mind” (p. 308). Four main purposes for peer debriefing are, a) to make the researcher aware of his or her posture and process, b) to provide an opportunity to test working hypothesis that may be emerging in the inquirer’s mind, c) provide the opportunity to develop and initially test next steps in the methodological design, and d) the opportunity to clear the mind of emotions and feelings that may be clouding good judgment (Lincoln & Guba, 1985).

I used peer debriefing both during the instrument development stage of my research and for the data analysis stage of the research. For instrument development, I presented my mathematics assessments (the FKA and the FCA) to a panel of professors and graduate students under the direction of Dr. Manouchehri at The Ohio State University and received feedback on how to improve the assessments before data collection began. I made several changes to the instruments as a result of these discussions. Changes I made to the assessments based on feedback included starting the table of Adam’s savings (from the FKA) at week 20 instead of week 0 and removing a few data points from the graph of Matthew’s savings (from the FKA).
Peer debriefing was used for data analysis through a meeting of mathematics education doctoral students and my advisor, Dr. Owens. The group of professionals reviewed my mathematics assessments and the analysis procedures I was using for coding the data. Suggestions for improvement in the coding of student success-levels and comfort-levels were used in the final coding schemes. Improvements included adding an additional level to success and clarifying the codes within the success-level. Codes for comfort-level were also discussed and definitions for comfort levels were changed to better reflect the intended purpose of these codes.

**Member checks**

Member checks involve the researcher obtaining feedback on the research findings from stakeholders from whom the data were originally collected. Lincoln and Guba (1985) believe this “is the most crucial technique for establishing credibility” (p. 314). Lincoln and Guba (1985) describe several possible methods of receiving feedback from participants and emphasize that member checks can be formal or informal. For the current study informal member checks were made during data collection by asking students to elaborate on their answers and by follow-up questions as the need arose. A more formal member check was made by sending the student a summary of his or her Participant Profile and asking for confirmation that the information was accurate and reflected what the participant intended. These profiles were sent to each participant via e-mail and I have received a 75% return. Each of the six participants who responded said that the information was accurate. Robert asked that I include in his profile that he is proficient at producing as well as reading tactile graphs and indicated that he would prefer a different pseudonym to be used. To comply with the first request I note that he is
comfortable with both reading and producing graphs. For the second request I changed his pseudonym from Roland to Robert.

**Transferability**

Transferability differs from external validity in that external validity is concerned with how the findings can be generalized across different types of participants and settings. For qualitative research, transferability requires an informed decision of the person wanting to use the information. Therefore, it is “not the naturalist’s task to provide an index of transferability; it is his or her responsibility to provide the data base that makes transferability judgments possible on the part of potential appliers” (Lincoln & Guba, 1985, p. 316). To do this the researcher needs to provide a thick description of the research including a detailed description of participant characteristics, the time and context of the research, methodology used for data collection and analysis. The current research provides a rich description of participants’ educational background, visual abilities, and mathematics experiences to help ensure transferability of the study. Methodology and data collection procedures for the current research are also provided in detail.

**Dependability**

A typical method for determining dependability of a study is to rely on techniques already utilized for testing a study’s credibility. This is because there cannot exist dependability without first having credibility. Lincoln and Guba (1985) discuss this view and ultimately conclude that dependability should be determined through additional measures. They recommend the use of an *inquiry audit* or a *dependability audit.*
Dependability audit

A dependability audit is similar to a financial audit of a business or organization. Details for such an audit come from the work of Thomas Schwandt and Edward Halpern (1988) who detail a dependability audit and items needed for an audit trail.

The audit trail consists of data from six categories, a) raw data, b) data reduction and analysis products, c) data reconstruction and synthesis products, d) process notes, e) materials relating to intentions and dispositions, and f) instrument development information. Audit trail components are described in Table 3.21.

<table>
<thead>
<tr>
<th>Audit Trail Components</th>
<th>Description/Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Raw data</td>
<td>• Electronically recorded materials such as videotapes</td>
</tr>
<tr>
<td></td>
<td>• written field notes</td>
</tr>
<tr>
<td></td>
<td>• documents</td>
</tr>
<tr>
<td></td>
<td>• records</td>
</tr>
<tr>
<td></td>
<td>• survey results</td>
</tr>
<tr>
<td>Data reduction and analysis products</td>
<td>• Write-ups of field notes</td>
</tr>
<tr>
<td></td>
<td>• Summaries and condensed notes</td>
</tr>
<tr>
<td></td>
<td>• Unitized information (i.e. 3x5 index cards)</td>
</tr>
<tr>
<td></td>
<td>• Quantitative summaries</td>
</tr>
<tr>
<td></td>
<td>• Theoretical notes (i.e. working hypothesis, concepts, hunches)</td>
</tr>
<tr>
<td>Data reconstruction and synthesis products</td>
<td>• Structure of categories (i.e. themes, definitions, relationships)</td>
</tr>
<tr>
<td></td>
<td>• Findings and conclusions (i.e. interpretations and inferences)</td>
</tr>
<tr>
<td></td>
<td>• Final report</td>
</tr>
</tbody>
</table>

Table 3.21: Summary of Audit Trail Components
Table 3.21 continued

<table>
<thead>
<tr>
<th>Process notes</th>
<th>Methodological notes (i.e. procedures, designs, strategies, rationale)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Trustworthiness notes</td>
</tr>
<tr>
<td></td>
<td>Audit trail notes</td>
</tr>
<tr>
<td>Materials relating to intentions and dispositions</td>
<td>Inquiry proposal</td>
</tr>
<tr>
<td></td>
<td>Personal notes (i.e. reflexive notes, motivations)</td>
</tr>
<tr>
<td></td>
<td>Expectations (i.e. predictions, intentions)</td>
</tr>
<tr>
<td>Instrument development information</td>
<td>Pilot forms</td>
</tr>
<tr>
<td></td>
<td>Preliminary schedules</td>
</tr>
<tr>
<td></td>
<td>Observation formats</td>
</tr>
<tr>
<td></td>
<td>Surveys</td>
</tr>
</tbody>
</table>

I used an audit trail and full audit. The auditor for this study is someone who is familiar with, a) the mathematics being researched, b) the educational needs of students with visual impairments, and c) educational and research terminology and procedures.

The auditor for the study is David Moore, who fulfills all of the requirements for this position. Mr. Moore has an undergraduate degree in Mathematics and a Masters of Arts in mathematics education. Mr. Moore is a gentleman who has been blind since the age of twelve. These qualifications allow for Mr. Moore to give a unique prospective of the current research and the ability to judge the research for its dependability.

The audit roughly followed the outlined audit procedures given by Schwandt and Halpern with an initial meeting between Mr. Moore and myself to review the research, negotiate contract obligations (see Appendix H for contract details), and to determine the needed documents for the audit trail. Communication between Mr. Moore and myself remained open throughout data collection and analysis. Mr. Moore was provided with
access to all raw data and research materials such as research notes, summaries, various drafts of categories and themes as the analysis progresses. To conclude the audit Mr. Moore wrote a report attesting to the dependability of the study (Appendix I).

**Inter-rater reliability**

To add dependability to the coding process, a second-coder was used to test for inter-rater reliability. Approximately 38% of the interviews were coded by me and one other researcher, independently. The second-coder is a doctoral student in mathematics education at the Ohio State University, Candace Joswick. Ms. Joswick has been involved with multiple research projects and has had experience with coding and reporting on mathematics education research results. She also has extensive experience in the mathematics classroom and is able to critically review student responses to mathematical tasks.

Ms. Joswich and I met twice and had open communication via e-mail throughout the coding process. At our initial meeting I provided Ms. Joswick with transcribed data of responses to the MEEVA, the FKA, and the FCA for Robert, Yvonne, and Amanda. At the initial meeting we discussed the coding procedures and logistics on when and how to present final coded data. Ms. Joswick coded for the three students’ success-levels, comfort-levels, competencies in modeling, interpreting, translating, and reifying, graphing access and proficiencies, and students’ perceived influences on mathematics learning. Ms. Joswick e-mailed questions and concerns throughout the coding process and I responded to each concern by clarifying code definitions or providing examples of responses from students for whom she was not coding.
Ms. Joswick sent her final codes via e-mail and I compared her codes with mine for any discrepancies. Initial code agreement was 75% for success-level and 70% for comfort-level. An overall agreement rate of 75% was reached for student competencies in modeling, interpreting, translating, and reifying. An 85% agreement rate was initially found for coding student access and proficiencies with graphing. Finally, a 65% initial agreement rate was reached for students perceived influences on their mathematics education. Ms. Joswick and I met to discuss the disagreements in coding and to agree on final codes.

**Confirmability**

Confirmability is also addressed through a confirmability audit. The dependability audit and the confirmability audit are combined. This audit was discussed in the last section and will serve as the confirmability check for this research.

**Additional validity measures**

**Reflexive journal**

A technique for checking the validity of a study that has “broad-ranging application to all four areas and provides a base for a number of judgment calls the auditor must make” is the reflective journal (Lincoln & Guba, 1985, p. 327). This is a type of diary that is used on a daily or as needed basis by the investigator to record information about self and method. It should, at least, include a) the daily schedule and logistics of the study, b) a personal diary that provides the opportunity for reflection on what is happening in terms of one’s own values and interests, and c) methodological log where methodological decisions and rationales for such decisions are recorded (Lincoln
& Guba, 1985). For the current study I made entries in a notebook regarding data collection following each data collection session and throughout data analysis.

**Limitations of the Study**

The participants in this study had a wide range of differences in both their mathematical backgrounds and their visual experiences. These differences account for variations in the individual students’ understanding of linear function. Participants were also recruited solely from the central and southwest regions of Ohio and therefore do not represent a wide geographic. Four of the students were instructed in mathematics by the same teacher for their high school mathematics courses while the other four participants each had a different instructor(s) for their high school mathematics classes.

The mathematics assessments used in this study were lengthy, especially for students with visual impairments. The length of the assessments led, in some cases to fatigue and caused the sessions to be divided in unintended ways such as a student finishing the FKA: Part 2 during the second session. The large print version of a graph presented in the FKA could not be clearly read and therefore four student responses (all from one participant) had to be removed from analysis.

I was not always consistent in the way I handled follow-up questions to student responses. I had worked previously with four of the participants, which allowed for a higher level of comfort between me and the student.

**Summary**

Students with visual impairments are not exempt from the standards of public education and are entitled to equal educational opportunities as their sighted peers. This
extends to the mathematics classroom. Understanding and being able to use functions is a major factor of school mathematics and is one that cuts across many disciplines. Therefore, to make mathematics education equitable to all students, those with visual impairments must be given access to instruction on functions. To determine the extent of knowledge current high school and college students with visual impairments have of function I worked with students’ individually through three assessments. These assessments helped to determine students’ understanding of function through their competency level in working with problems involving functions, students’ access to and proficiency with graphing, student representational preferences when solving problems dealing with function, and what students with visual impairments sees as influences on their current level of understanding of function. Analysis was done through the lens of two theories, Wilson’s (1971) taxonomy and O’Callaghan’s (1998) function competency model.
CHAPTER 4

RESULTS

In this chapter the results of data analyzed for this study are presented and discussed around participant responses to the Mathematics Education Experiences and Visual Abilities (MEEVA) interview and student work during completion of the two mathematics exams, the Function Knowledge Assessment (FKA) and the Function Competency Assessment (FCA). The results of an analysis on participants’ graphing access and proficiency levels will also be addressed. These results were guided by a compilation of findings from the MEEVA and the mathematics assessments and resulted in Participant Graphing Profiles.

Results from the Mathematics Education Experiences and Visual Abilities Interview

The Mathematics Education Experiences and Visual Abilities (MEEVA) interview asked students 30 questions about their mathematics education, in general, their understanding of and education in their algebra courses, specifically, and demographic questions about their vision and the history of their visual abilities. Results from the MEEVA interview are discussed by topic and include information on, a) the tools and technology used by participants for their mathematics education, b) educational methods used to help include students into the classroom, c) function representations used within the classroom and the stated preference for these representations, d) independence
opportunities provided to students in the mathematics classroom, and e) challenges faced by participants in the mathematics classroom.

**Mathematics Tools and Technology Used by Participants**

Participant use of tools and technology were largely divided between students who read regular or large print and those who were braille readers. Individuals within these groups were very consistent with the tool that they used.

**Mathematics tools**

Amanda and Wanda, the two participants who had usable vision and used large print and regular print, respectively, needed few tools to make their mathematics education accessible. Amanda used a magnifying glass and discussed being given geometric shapes when learning spatial concepts. She also had access to large print graph paper. Wanda has a monocular she uses for seeing things at a distance and was given large print graph paper, though she says this did not really help and that she prefers a regular sized grid for graphing.

Tools that were commonly used by students who read braille include a braillewriter (usually a Perkin’s Brailler), braille rulers and protractors, though these were not used on a regular basis, and graphing materials. Graphing materials included graph boards with an assortment of accessories including a variety of sizes and shapes of thumbtacks, string, and rubber bands. One participant, while in high school, used a foam pillow for graphing because a graph board was not available. Participants also used braille graph paper with accessories including Wikki Sticks and sticky dots of various textures to indicate their plotted points.
Technology

Technology used by participants was also standardized based on reading medium. Those with visual impairments used talking calculators and a screen-magnification software (such as Zoomtext). All six participants who are braille users had access to JAWS (a screen-reading program), a braille notetaker (such as a BrailleNote or a Braille Lite), and access to scanning programs, such as Kurzweil or OpenBook. All braille readers also talked about having, at one time, a talking calculator, though everyone now uses the calculator which comes standard with their braille notetaker. These calculators are scientific, which represents an upgrade from a basic four function talking calculator that many students had access to. One participant has a Victor Reader, a versatile book reader that allows the user easy access to complex textbooks and includes additional features such as a voice recorder and a text to speech translator. Caleb is the only participant to have the Audio Graphing Calculator, though he has not had appropriate training and is not currently using it. A summary of the tools and technology available to participants is presented in Table 4.1.
Table 4.1: Tools and Technology Used in Math Education

The use of JAWS was prevalent and many of the other technologies depend on some type of screen-reading program. Robert sums up the importance of such software by saying, “Without JAWS I don’t have an education.” Braille notetakers were also very familiar for these students and acted as their personal laptop computer. Braille notetakers were preferred by most students due to their convenience and added features that are not provided by a standard Perkin’s Braillewriter. Two students specifically mentioned not wanting to use a Perkin’s Braillewriter in class because of the distraction it can cause other students. Gabe is one student who mentioned this and yet he prefers to do his
mathematics work on a Perkin’s Braillewriter because it allows him an easy way to review the steps of a problem.

**Educational Methods**

Participants reported a low number of methods used specifically for them to help include them into the classroom. This was partly due to the students who were educated at the school for the blind as they perceived accommodations were standard across all students. For students who attended high school at their local school the perception was largely that no special accommodations were made for them. However, many students were given accommodations that they considered ‘standard’ such as extended test time, modified homework or class assignments, and access to various services such as having an aide and a Teacher of the Visually Impaired. Some of the educational services that were mentioned throughout the MEEVA interview are discussed next.

**One-on-one instruction**

All participants emphasized their need for one-on-one instruction. Students who attend the school for the blind discussed the extremely small class size as an advantage because of their ability to work directly with the teacher. Wanda describes how her mathematics education improved when she started taking her mathematics classes at the school for the blind because she could ask more questions and work more one-on-one with the instructor. Robert has always sought help from his teachers after class or during office hours. All four of the college students said how helpful tutoring has been in the past. For his last mathematics class, Gabe had a weekly appointment to meet with is professor to go over material from the week and to make sure he was understanding the
course material. Gabe mentioned how he would be able to have the instructor, during the weekly meeting, go over step-by-step instructions and explain why each step was important to the problem.

Support

Support for college students mainly centered on materials being provided in braille and audio formats, in class writers, and instruction outside the classroom by meeting with professors during office hours or having a tutor. The supports for high school students were more comprehensive with services for the students extended to services outside the classroom such as instruction in Orientation and Mobility and assistive technology use. High school students who attend their local school districts for their mathematics classes are provided with an aide and a Teacher of the Visually Impaired. Caleb recalls that his Teacher of the Visually Impaired would come to meet with him once a week while he was in high school, but his aide was in the building and he “knew where to find her if he needed anything.”

Five of the six braille readers mentioned how helpful it was when they had access to class notes ahead of time. Unfortunately, this did not happen often and was even less likely to occur for their mathematics classes. This limited availability to class materials was the same with the students’ access to accessible graphs. Caleb specifically mentioned how difficult his upper-level mathematics classes have been because of him not having the same access to graphs as his classmates did. In some cases the teacher would tell him to use a similar graph that was shown in the textbook and say, “don’t look at the numbers.” This was difficult for him as he would inevitably look at the numbers and get confused when solving the problem at hand. Caleb also discussed how his mathematics
teachers argued that they should not be expected to provide graphs and notes ahead of
time because they would not know what questions students would ask in class and
therefore could not be expected to have material presented ahead of time.

*Function Representations*

Students reported on which representation they used most often in the classroom,
were asked, “Do you have a preference for one representation over another when solving
problems involving functions?”, and whether or not they had a good understanding of
linear functions and their representations. Students also commented on preferences for
representations throughout the interview. Findings for these are discussed next.

The majority of students (63%) said that they were instructed on the use of
equations most often. Two other students said they worked with equations and tables or
equations and graphs the most. Only one student did not list equations as one of the most
emphasized representations.

Regarding preference of representation, two students said they prefer to use
equations, one student a preference for tables, and two students said that they prefer to
use either equations or tables. Caleb liked to use graphs with the restriction that he prefers
the use of equations when the graph under consideration is “messy.” Madisen likes to use
equations when there is a large amount of information being conveyed and descriptions
when there is a small amount of data. When asked what her preferred representation was
Yvonne said, “I don’t like math…none of the above.” After prompting she said that it
depended on what she had recently learned as to which representation she was likely to
use.
All students indicated they had a good understanding of linear functions and their representations, though Caleb’s answer was, “not as good as a lot of my classmates seem to have had.” Two students stated that they had a good understanding of it when in class, but had forgotten a lot of the material. Yvonne specifically stated that the material was “easy to be tested on” but that she could not recall some of it, now. Wanda’s answer indicated that she had not received a good understanding of linear functions and their representations when at her local school, but that she did once she started taking mathematics at the school for the blind.

Liz repeatedly brought up that she has “never been that good with graphing,” Robert stated that learning to graph was the hardest thing he had to learn throughout his mathematics education, and Amanda exclaimed “I told you I didn’t like geometry!” when given four graphs to interpret for the first mathematics assessment. Caleb has experienced great difficulty in having appropriate access to graphs when taking upper level mathematics classes. Gabe says that he “doesn’t mind graphs” and that “the interpretation gets kind of clouded” but that he knows how to read graphs. Gabe also discusses the need for a graph to be set up well in order for him to be successful in reading it. Madisen had similar thoughts regarding tactile graph production and said that she can have trouble reading them because of the types of lines used to make the graphs. She said that unfortunately, “a lot of braille production facilities are not very good at making those kinds of choices.” Yvonne does not “use too many, unless it’s really necessary” and talks about how graphs can be really confusing to read if they have too much detail included. Caleb and Gabe both indicate they use tactile graphs all the time and Caleb mentions that “it’s pretty much the only graphs I’ve ever used.”
Opportunities for Independence in Mathematics Learning

Students were varied on the amount of independence they reported having in the mathematics classroom. Several participants mentioned using proctors on examinations so that they could explain their steps and have the proctor write them out. Liz stated:

I’ve never really been that great with graphs, like usually even with here on the tests, like the math tests they give me graphs and say, you know, okay, where is, or where are the points plotted on this graph? And you’d have to find it, like I have no idea how to do that, so I would just ask the proctor to tell me, you know, can you tell me what points, or you know, how far it goes up, you know, two or whatever, because I just couldn’t tell.

Later in the interview, Liz stated that her high school teachers “just kind of showed me where things were on that particular graph, you know, that’s how I know because they told me, but you know, for each graph, you know how am I going to know?” Gabe also mentioned the use of a proctor on some mathematics exams, but he would be independent with making the graph except would need assistance for the “more visual aspects” such as shading a particular area of the graph.

Several students talked about how they were unable to independently take notes in the mathematics class. Teachers’ use of ambiguous terms and the visual nature of mathematics both contributed to the fact that students were only somewhat independent in taking mathematics notes. Gabe described two occasions where he talked with the instructors and asked them to be more descriptive when speaking what is written on the board. In these two instances, both educators became so proficient at this, he was able to independently take notes. However, he also had experiences, like two other participants, where he needed an aide to accompany him to class so that he or she could take notes at the same time he was taking notes because it was impossible for him to follow what the
teacher was saying. Caleb commented that for some of his mathematics classes “my aide was in every class.” This was because these particular teachers used overhead projectors or PowerPoint slides and did not do a good job with voicing what were on these slides. Three of the college students specifically mentioned having in-class writers to take notes. Two student has their notes translated to braille while the third has the notes scanned and e-mailed so that she can have a digital copy.

Another common comment from the braille readers was their limited access to graphs presented in class, unless the graph being used was included in the textbook. If the graph was from the textbook, then they would have a tactile version of it. If not, then access was usually limited. Robert would wait for a one-on-one meeting with his instructor to ask about the graphs used during class. Other college students recall having their aide in high school making small mock ups of the graphs with Wikki Sticks so that they would have some access to these while they were being discussed. Caleb did not have a good experience with this arrangement because by the time the aide finished producing a tactile graph, often the instructor had already moved on to the next example. Liz recalls that if graphs were used in class that she did not have access to, then she would go to the resource room after class with her Teacher of the Visually Impaired and together they would discuss what had been shown.

**Challenges in Learning Mathematics**

Students struggled in various areas of their mathematics education. Four students listed either graphing or geometry as their most challenging topic in mathematics. Robert said that learning to graph was the most difficult for him when learning mathematics and Amanda mentions having particular difficulty with finding and using angles. Two
additional students, Madisen and Yvonne, listed learning the Nemeth Code as the most challenging. Both Madisen and Yvonne lost their vision at the age of thirteen and Madisen was learning the Nemeth code while she was taking her first Algebra course. Yvonne still avoids using the Nemeth code as much as possible and has her mathematics textbooks in braille “just in case I need to see the math for myself.” Typically she has someone read her the content or mathematics problems so she can type them into her braille notetaker. This helps to guarantee that she will be able to understand the braille when she reviews it.

Caleb discussed how he started having major problems in his mathematics education when graphing calculators started to be used on a more frequent basis. In one mathematics course he recalls learning the forms of different parabolas and hyperbolas. Although he did not have trouble learning the basic forms, when the class started using the graphing calculators to view changes to the graph that occurred with modified equations, he was unable to actively participate since none of his graphing tools allowed for quick transformations of the graphs. In speaking of this Caleb said,

And I think part of my problem was I always do really well, and I found this out in Chemistry last year, I do really well if I have something I can physically move in front of me. I didn’t have that when I was trying to learn these, and as a result that got the algebra in my head confused because I didn’t have anything to fall back on graphically and everything just kind of fell apart from there.

Caleb has an Audio Graphing Calculator, but received it without any guidance on how to use it and was unable to learn to use it effectively, even with hours of additional help from his mathematics teacher. Therefore it was not helpful to him and subsequently is not currently being used at all.
Results from the Function Knowledge and Function Competency Assessments

Results from the FKA and FCA are discussed around, a) participants’ representational choices, b) participants’ success-level and comfort-level in working with various function representations, and c) students’ proficiency in the four function competencies.

Representational Choices

Participants were also given several opportunities to choose a representation in which to use. First, question 10 on the FKA: Part 1, had students pick any of the two children and asked them to compare their savings. The savings of the four children in the problem were each described using a different representation; Matthew was described as a graph, Adam’s was described as a table, Diane’s savings was described using an equation, and Heather’s savings was described in words. Therefore, when students were asked to compare two savings accounts, they were also choosing to use two different representations. Next, before starting the FKA: Part 2, students were told they could choose one representation to use in order to aid them in answering the following questions about Diane and Matthew’s savings. Again, the students were given the choice to use either the equations of the students’ savings, a graph displaying the accounts of both children, a description for each savings account, or a table showing all fifty-two weeks of their savings. Finally, question 5b on the FCA asks students to either make a table or a graph of the information provided in 5a.
**Success-Level and Comfort-Level**

The success-level and comfort-level were coded for problems using the four function representations. The questions coded for success and comfort are listed in Table 4.2. Success-level is coded as the student having high, medium-high, medium-low or low success. Levels of comfort with using a particular representation were coded as high, medium, or low. Results are discussed around each representation for problems asking students to gain information about functions through tables, graphs, equations, and descriptions. Next, results of students’ success and comfort levels are presented for problems that require a translation from one representation to another. Finally, comments are made on the success and comfort shown by participants while using their chosen representation to solve the FKA: Part 2.

<table>
<thead>
<tr>
<th>Assessment</th>
<th>Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>FKA: Part 1</td>
<td>1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14</td>
</tr>
<tr>
<td>FKA: Part 2</td>
<td>1-6</td>
</tr>
<tr>
<td>FCA</td>
<td>1-7 (the entire assessment)</td>
</tr>
</tbody>
</table>

Table 4.2: Assessment Questions Coded for Success-Level and Comfort-Level

**Solving problems involving interpreting or producing tables**

*Success-Level.* Six questions from the FKA and FCA utilize tables to convey the needed information in which to solve the problems. Information from all eight students was coded for these six problems which gives a total of 48 codes for student success levels. A summary of these results can be viewed in Table 4.3. The results are given for all participant responses for the six questions involving interpretation of information given in a table. For instance, participants were coded as having a high success-level 34
times, which indicates that there were 34 responses, across all participants, which were coded as high.

<table>
<thead>
<tr>
<th>Solving Problems Involving Interpretations of Tables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Success-Level</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>High</td>
</tr>
<tr>
<td>Medium-High</td>
</tr>
<tr>
<td>Medium-Low</td>
</tr>
<tr>
<td>Low</td>
</tr>
</tbody>
</table>

Table 4.3: Success-Level Shown by Participants When Solving Problems that Convey Information Through Tables

Participants, on the whole showed great success when solving word problems and interpreting information from a table. Only 14.6% of responses were either coded as “medium-low” or “low.” Responses showing high success when working with information presented in tabular form were just over 70 percent.

*Comfort-Level.* The six problems where tables were used to provide problem information were coded with the participants’ comfort-level. Results for all participants are given in Table 4.4.

<table>
<thead>
<tr>
<th>Solving Problems Involving Interpretation of Tables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comfort-Level</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>High</td>
</tr>
<tr>
<td>Medium</td>
</tr>
<tr>
<td>Low</td>
</tr>
</tbody>
</table>

Table 4.4: Comfort-Level of Students When Interpreting Information in Tabular Form
Comfort was shown by participants with problems involving the interpretation of tables. All participants either showed high or medium comfort when solving problems that presented information through tables.

**Solving problems involving interpretations of graphs**

*Success-Level.* Eight questions involved students obtaining information by reading a graph. However, there were a few responses that were removed from analysis. First, the large print graph of Matthew’s savings from the FKA was substandard with the numbers on the y-axis being difficult to read because of the small distance between the numbers. Therefore, I have removed the data for Amanda on the three problems involving interpretation of Matthew’s graph. Secondly, two students used previous answers to solve one of the graphing problems and did not actually interact with the graph to answer the question. Therefore, these responses were not coded for success or comfort. Taking into consideration the removal of these data, success-level and comfort-level were coded for 59 responses. Results are shown in Table 4.5.

<table>
<thead>
<tr>
<th>Solving Problems Involving Interpretation of Graphs</th>
<th>Number of times students demonstrated the various levels of Success</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Success-Level</strong></td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>22</td>
</tr>
<tr>
<td>Medium-High</td>
<td>9</td>
</tr>
<tr>
<td>Medium-Low</td>
<td>16</td>
</tr>
<tr>
<td>Low</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 4.5: Success-Levels of Participants When Solving Problems by Interpreting Information Given Through a Graph
Success-levels for graphing were more varied than they were for tables. Half of the students were more hesitant with gaining information from the graph while the other half were very familiar with the use of graphs and did a fairly accurate job with reading information from presented graphs. The varied skills for reading information from a graph led to an almost even split of success shown by students. Participants demonstrated high or medium-high success for 52.5% of the problems where information was displayed in graphical form and students showed low or medium-low success 47.5% of the time on these problems.

Comfort-Level. Fifty-nine answers were coded for comfort-level when answering questions by reading information from a graph. Results are presented in Table 4.6.

<table>
<thead>
<tr>
<th>Solving Problems Involving Interpretation of Graphs</th>
<th>Number of times students demonstrated the various levels of Comfort</th>
</tr>
</thead>
<tbody>
<tr>
<td>Comfort-Level</td>
<td></td>
</tr>
<tr>
<td>High</td>
<td>33</td>
</tr>
<tr>
<td>Medium</td>
<td>17</td>
</tr>
<tr>
<td>Low</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 4.6: Comfort-Level of Participants When Solving Problems by Interpreting Information Given Through a Graph

Participants were polarized for the comfort they showed with reading graphs. Liz was coded as having low comfort for seven of the eight problems using graphical information. She indicated her unease with graphs and stated, “I’ve never been that great with graphing.” On more than one occasion I reminded Liz how to read a graph and what the various parts of a graph represented. On the other hand, Caleb, Robert, and Gabe all
showed high comfort levels with reading graphs and interacted with them without any need for assistance.

**Solving problems involving the use of equations**

*Success-Level.* Students were given equations of functions and asked to solve problems using these on three different occasions. All participants answered these three problems to give a total number of coded responses of twenty-four. Results are presented in Table 4.7.

<table>
<thead>
<tr>
<th>Success-Level</th>
<th>Number of times students demonstrated the various levels of Success</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>14</td>
</tr>
<tr>
<td>Medium-High</td>
<td>3</td>
</tr>
<tr>
<td>Medium-Low</td>
<td>4</td>
</tr>
<tr>
<td>Low</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 4.7: Success-Levels of Participants When Solving Problems Through the Use of a Given Equation or Expression

Participants, in general, showed a high level of success in solving equations. This success was more prominent when questions asked them for a solution to a particular x-value. Problems such as number 9 on the FKA: Part 1, which asks “When will Diane have $200?” were more challenging for students and revealed various solving techniques. Madisen demonstrated low understanding of equations when she solved this problem by replacing the 300 from Diane’s expression (300-5x) with the 200 from the question and dividing by five. Gabe solved the equation, but instead of following the steps of a
“typical” solution he looked for what number he needed to subtract from 300 to get 200 and then found what number would multiply by 5 to give the final answer.

*Comfort-Level.* Twenty-four responses were coded for comfort-level for use of equations. Results are shown in Table 4.8.

<table>
<thead>
<tr>
<th>Comfort-Level</th>
<th>Number of times students demonstrated the various levels of Comfort</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>16</td>
</tr>
<tr>
<td>Medium</td>
<td>5</td>
</tr>
<tr>
<td>Low</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 4.8: Comfort-Level of Participants When Solving Problems Through the Use of a Given Equation or Expression

Students were familiar with equations and although there were misconceptions on how to use equations and several students who used equations in an ineffective, guess-and-check manner, 87.5% of students showed either high or medium comfort when solving problems that involved a given equation.

*Solving problems with information presented as descriptions*

*Success-Level.* All students participated in the five questions that used descriptions to convey the problem context. This gave a total of 40 responses that were coded for success. Results are presented in Table 4.9.
Students were successful with interpreting descriptions of problem contexts. Seventy-five percent of student responses had a high level of success. One question, “How much money did Heather have at the end of the year?” (FKA: Part 1; 6), gave students difficulty. Several students hesitated when responding to this question because Heather’s savings did not change. Students were expecting a more difficult answer or misread the phrase “throughout the year” when reading the description of Heather’s savings.

*Comfort-Level.* The comfort-level for description based problems was also coded for 40 responses. Results can be seen in Table 4.10.
Participants were very comfortable with reading a problem description and answering questions based on description. Only nine responses were coded below a high comfort-level.

**Solving problems involving translation from one representation to another**

*Success-Level.* Students were asked to translate from one representation to another on seven occasions. One response was removed because of the poor quality of the large print graph of Matthew’s savings. A total of 55 responses were coded for student success with translation. Results are given in Table 4.11.

<table>
<thead>
<tr>
<th>Solving Problems Involving Translation from One Representation to Another</th>
<th>Number of times students demonstrated the various levels of Success</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>20</td>
</tr>
<tr>
<td>Medium-High</td>
<td>4</td>
</tr>
<tr>
<td>Medium-Low</td>
<td>7</td>
</tr>
<tr>
<td>Low</td>
<td>24</td>
</tr>
</tbody>
</table>

Table 4.11: Success-Levels of Participants for Problem Solutions Involving a Translation from One Representation to Another

Participants performance on translating between representations was strikingly low when compared to the success shown when working with a single representation. Students showed low success in translating 43.6% of the time. This was due, in part, to two students who were coded as having low success for all seven translations. Although there were 24 responses that demonstrated low success, there were also 20 responses (36.3%) that showed high success.
Some translations were more difficult for students than others, which contributed to the polarization of success levels. The most challenging translation for students was translating from a graph to an equation. Students were asked to write an equation for Matthew’s savings, which was presented as a graph. Of the seven responses coded, five students demonstrated low success and two students demonstrated high success when writing an equation for Matthew’s savings. A translation that students were more successful with was writing an equation from a given table. When asked to write the equation for Adam’s savings, which was presented in tabular form, five participant responses showed a high level of success.

Comfort-Level. Fifty-five responses were coded for comfort-level. Results are shown in Table 4.12.

<table>
<thead>
<tr>
<th>Solving Problems of Translation from One Representation to Another</th>
<th>Comfort-Level</th>
<th>Number of times students demonstrated the various levels of Comfort</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>Medium</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Low</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 4.12: Comfort-Levels of Participants for Problem Solutions Involving a Translation from One Representation to Another

Participants displayed low comfort when translating between representations almost half of the time (45%). This is clearly an area that has been a struggle for most of these students. Caleb, Gabe, and Robert were three students who did relatively well in this area, with Caleb and Gabe having high or medium comfort 100% of the time and Robert having high or medium comfort for all problems except one. Robert displayed
extreme discomfort with writing an equation for Matthew’s savings, which was presented in a graph. Two students, Yvonne and Amanda, had a particularly difficult time with translations and both had low comfort on 86% and 83% of the problems, respectively.

**Success and comfort for questions from FKA: Part 2 with use of a chosen representation**

The second part of the FKA was composed of six questions, each regarding the savings accounts of Diane and Matthew. The six problems of the FKA: Part 2 are shown in Figure 4.1. Participants were told they could choose a representation of Diane’s and Matthew’s savings in which to help them answer the questions from the FKA: Part 2 (the choice of representation was made before students were given the questions).

<table>
<thead>
<tr>
<th>Function Knowledge Assessment: Part 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. How much had Matthew saved after half a year? How much did Diane have at the same time?</td>
</tr>
<tr>
<td>2. After how many weeks did each of the two children have $210?</td>
</tr>
<tr>
<td>3. Find the week with the largest difference between their savings.</td>
</tr>
<tr>
<td>4. Find the week when their savings were equal.</td>
</tr>
<tr>
<td>5. Find the week when the savings of one were double that of the other. In whose favor?</td>
</tr>
<tr>
<td>6. Diane and Matthew decided to pool their savings in order to buy a $400 walkie-talkie. Find the week in which they are able to purchase the walkie-talkie.</td>
</tr>
</tbody>
</table>

Figure 4.1: Questions from FKA: Part 2
Table 4.13 gives the success and comfort levels for each participant for responses from the six problems on the FKA: Part 2 and the representation that was being used when answering these questions.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Representation Used</th>
<th>Participant Success-Levels for Questions from FKA: Part 2</th>
<th>Participant Comfort-Levels for Questions from FKA: Part 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robert</td>
<td>Table</td>
<td>High: 6 Medium-High: 0 Medium-Low: 0 Low: 0</td>
<td>High: 6 Medium: 0 Low: 0</td>
</tr>
<tr>
<td>Liz</td>
<td>Equations</td>
<td>High: 1 Medium-High: 0 Medium-Low: 1 Low: 4</td>
<td>High: 2 Medium: 1 Low: 3</td>
</tr>
<tr>
<td>Madisen</td>
<td>Equations</td>
<td>High: 2 Medium-High: 0 Medium-Low: 2 Low: 2</td>
<td>High: 3 Medium: 1 Low: 2</td>
</tr>
<tr>
<td>Gabe</td>
<td>Equations</td>
<td>High: 4 Medium-High: 0 Medium-Low: 2 Low: 0</td>
<td>High: 6 Medium: 0 Low: 0</td>
</tr>
<tr>
<td>Caleb</td>
<td>Table</td>
<td>High: 5 Medium-High: 1 Medium-Low: 0 Low: 0</td>
<td>High: 6 Medium: 0 Low: 0</td>
</tr>
<tr>
<td>Yvonne</td>
<td>Table</td>
<td>High: 3 Medium-High: 2 Medium-Low: 1 Low: 0</td>
<td>High: 6 Medium: 0 Low: 0</td>
</tr>
</tbody>
</table>

Table 4.13: Participants’ Success-Levels and Comfort-Levels for Questions on the FKA: Part 2
Table 4.13 continued

<table>
<thead>
<tr>
<th>Amanda</th>
<th>Equations</th>
<th>High: 1</th>
<th>Medium-High: 0</th>
<th>Medium-Low: 2</th>
<th>Low: 3</th>
<th>High: 2</th>
<th>Medium: 0</th>
<th>Low: 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wanda</td>
<td>Graph</td>
<td>High: 3</td>
<td>Medium-High: 0</td>
<td>Medium-Low: 2</td>
<td>Low: 1</td>
<td>High: 4</td>
<td>Medium: 1</td>
<td>Low: 1</td>
</tr>
</tbody>
</table>

Robert, Caleb, and Yvonne chose to use the table that displayed all 52 weeks of Diane’s and Matthew’s savings. All three of these participants were comfortable with using the table and were successful in completing the six problems with either high or medium-high success, except for in one case where Yvonne had medium-low success when answering number 6 on the FKA: Part 2. Wanda was the only participant to use the graph of Diane’s and Matthew’s savings. Three of Wanda’s responses were of high success and three were either medium-low or low. Although her success rate was low, Wanda was very comfortable with reading the graph. Half of the participants chose to use the equations of Diane’s and Matthew’s savings. These students had various success levels when answering the six problems from the FKA: Part 2. Most students simply used the equations in an inefficient manner by guessing-and-checking for a variety of weeks. For instance, in number 4 (FKA: Part 2), students are asked when Diane and Matthew have the same amount of money and not one of the four students who were using the equations set the functions equal to one another to solve.
**Function Competencies**

Wilson’s (1971) Taxonomy of learning objectives is used a dual classification system to evaluate assessment questions for both a specific area of content and a specific level of student behavior. The content being reviewed for the current research are the four function competencies described by O’Callaghan (1998). The four competencies in O’Callaghan’s model are, a) modeling, b) interpreting, c) translating, and d) reifying. The levels of student behavior that were assessed with these function competencies were comprehension and application (Wilson, 1971). Thus, each mathematics assessment item from the Function Knowledge Assessment (FKA) and the Function Competency Assessment (FCA) intended to assess student knowledge in both a function competency level and a level of student behavior. Table 4.14 shows the function competency and level of student behavior used for each question.

<table>
<thead>
<tr>
<th></th>
<th>Comprehension</th>
<th>Application</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Modeling</strong></td>
<td>FKA: Part 3: 1</td>
<td>FCA: 1a, 1b, 5a, 6,</td>
</tr>
<tr>
<td></td>
<td>FKA: Part 1: 1, 2, 3, 6, 9</td>
<td>FKA: Part 1: 4, 5, 7, 8, 10, 15, 16</td>
</tr>
<tr>
<td></td>
<td>FKA: Part 2: 1, 2</td>
<td>FKA: Part 2: 3, 4, 5, 6</td>
</tr>
<tr>
<td></td>
<td>FCA: 2a, 2b, 2c, 2d</td>
<td>FCA: 2e, 3a, 3b, 4a</td>
</tr>
<tr>
<td><strong>Interpreting</strong></td>
<td>FKA: Part 1: 12, 13, 14</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FCA: 1c, 3c, 4b, 5b</td>
<td></td>
</tr>
<tr>
<td><strong>Translating</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>FKA: Part 1: 11</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FKA: Part 3: 2</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.14: Classifications of Questions for the FKA and the FCA for Level of Student Behavior and Type of Function Competency
Student responses to questions from the FKA and FCA were coded for demonstrating a high, medium, or low level of competency. The result of this analysis is given in Table 4.15. Results indicate how many times participants were coded for high, medium, and low competency for the indicated function competency. For instance, there are seven questions evaluating student level of comprehension for translating. For these questions all except for one student response was reviewed and coded, which gives a total of 55 responses coded for level of competency.

<table>
<thead>
<tr>
<th></th>
<th>Comprehension</th>
<th>Application</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Modeling</strong></td>
<td>High Competency: 5 (83%)</td>
<td>High Competency: 20 (62.5%)</td>
</tr>
<tr>
<td></td>
<td>Medium Competency: 1 (17%)</td>
<td>Medium Competency: 11 (34.4%)</td>
</tr>
<tr>
<td></td>
<td>Low Competency: 0</td>
<td>Low Competency: 1 (3.1%)</td>
</tr>
<tr>
<td><strong>Interpreting</strong></td>
<td>High Competency: 57 (65.5%)</td>
<td>High Competency: 63 (54.3%)</td>
</tr>
<tr>
<td></td>
<td>Medium Competency: 21 (24.1%)</td>
<td>Medium Competency: 43 (37.1%)</td>
</tr>
<tr>
<td></td>
<td>Low Competency: 9 (10.3%)</td>
<td>Low Competency: 10 (8.6%)</td>
</tr>
<tr>
<td><strong>Translating</strong></td>
<td>High Competency: 21 (38.2%)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Medium Competency: 10 (18.2%)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Low Competency: 24 (43.6%)</td>
<td></td>
</tr>
<tr>
<td><strong>Reifying</strong></td>
<td>High Competency: 8 (36.4%)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Medium Competency: 13 (59.1%)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Low Competency: 1 (4.5%)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.15: Number of Times and Percentage of Times Each Level of Competency is Shown for Questions from the FKA and the FCA in the Area of Function Competency and Level of Student Behavior

I discuss the results indicated in Table 4.15 around the four function competencies and levels of student behavior.
Modeling

Modeling is the ability to translate from a problem situation to a functional representation. There were a total of five questions that addressed students’ competency in modeling. Results from modeling-comprehension and modeling-application are discussed, next.

Modeling-comprehension. To show competency in modeling-comprehension a student needed to demonstrate that he or she could represent problem situations in a mathematical way. Question 1 from the FKA: Part 3 was used to assess students’ comprehension in modeling (see Figure 4.2). Questions from the FKA: Part 3 were given only if there was sufficient time. Thus, six of the eight participants completed the two questions from the FKA: Part 3.

Function Knowledge Assessment: Part 3

Ellen received her allowance in the following way: On the first weekend, she got two cents. Every subsequent weekend, she received an amount identical to the amount she had left in her savings box the previous week. Ellen saves all the money she gets.

1) How much money had Ellen saved after 10 weekends?

Figure 4.2: Problem context and question 1 from the FKA: Part 3

Students showed a high level of competency in the modeling-comprehension area with five of the six students being coded with high competency for problem. All, except for Robert, solved problem by making a table of values for the first 10 weekends of Ellen’s savings. Robert wrote the equation for Ellen’s savings. He did so without hesitation.
Modeling-application. To be coded for modeling-application students needed to show competency with solving word problems by producing and using a mathematical model. Questions that were specifically coded for the modeling-application competency were 1a, 1b, 5a and 6 from the FCA (see Figure 4.3).

<table>
<thead>
<tr>
<th>Function Competency Assessment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) A truck is loaded with boxes, each of which weighs 20 pounds. If the empty truck weighs 4500 pounds, find the following. (O’Callaghan, 1998)</td>
</tr>
<tr>
<td>a. The total weight of the truck if the number of boxes is 75.</td>
</tr>
<tr>
<td>b. The number of boxes if the total weight of the truck is 6,740 pounds.</td>
</tr>
<tr>
<td>5) Mary Wong just got a job working as a clerk in a candy store. She already has $42. She will earn $7 per hour. (Brenner et al., 1997)</td>
</tr>
<tr>
<td>a. How many hours will she have to work to have a total of $126?</td>
</tr>
<tr>
<td>6) Two carnivals are coming to town. You and your friend decided to go to different carnivals. The carnival that you attend charges $10 to get in and an additional $2 for each ride. The carnival your friend attends charges $6 to get in, but each additional ride costs $3. If the two of you spent the same amount of money, how many rides could each of you have ridden? (Slavit, 2006)</td>
</tr>
</tbody>
</table>

Figure 4.3: Questions from the FCA evaluating students’ modeling-application

Participants showed relatively high competency in solving mathematical word problems through modeling. More than 90 percent demonstrated some level of competency in modeling-application. An example of a student who showed some competency with this problem was Amanda. Amanda solved problem 5a through a guess and check system and failed to include Mary’s initial money. Though there is some indication that she used the initial amount when calculating her answer, but did not formally include it in her answer because she stopped her calculations when she reached
$84 even though the problem was asking for the number of hours Mary would work to reach $126.

For problem 6 on the FCA, students were successful mainly through a systematic guess and check or by making a more formal table of values to indicate how much each person would spend for a particular number of rides. Two students were able to solve it through reasoning about the pattern at which the carnival attendees were spending the money. Gabe noted the $12 dollar response where one person could ride one ride and the other person would have enough money to ride two rides. He then said, “The next time we would meet would be at 18 dollars.” When asked, he explained that he was following the pattern from 12 dollars and finding the next one that occurred which was divisible by both 2 and 3. On the contrary, Yvonne tried a guess and check method by the number of rides each person has ridden, but does so in a random fashion and got flustered a couple of times and asked, “Is that possible?” Madisen was the only student to choose a certain amount of money to find the number of rides each person could have ridden. However, her first guess of $20 did not work and she simply said, “Well, it wouldn’t work, it just couldn’t.”

**Student competency in Modeling.** Table 4.16 gives a breakdown of how each individual student was coded for both modeling-comprehension and modeling-application. There were five questions, total, that allowed students to demonstrate knowledge in modeling. Each of the student’s responses to these five questions was coded as either showing high, medium, or low competency. The total percentage for the number of responses that were coded as demonstrating high competency is also given in the results listed in Table 4.16.
Participants mainly showed their mathematical modeling through algebraic reasoning, but without producing a formal equation to solve the problem. However, there were a few students who used a more formal approach when solving these problems. Gabe wrote and solved an equation for 1a, 1b, and 5a on the FCA. Liz also wrote an equation to solve problem 1b and Caleb solved number 5a by writing an equation.

**Interpreting**

The interpreting competency is coded most often out of the four function competencies, with 11 questions belonging to interpreting-comprehension and 15 questions in the interpreting-application competency. This concept frequently occurs because of the nature of the assessments. For instance, questions 1-9 of the FKA: Part 1 were specifically designed to determine if the students were reading the four children’s savings accounts correctly and to analyze their ability to read and interpret functions from
various representations. Also, these mathematics tests were partly designed to bring out
the students’ understanding of the various representations of functions. Therefore, almost
all of the questions ask participants to gather information about functions in some way.
The results of the two interpreting categories are given, next.

*Interpreting-comprehension.* The interpreting-comprehension code is given to
students who show an ability to gain information from a given function and to predict
values that are not explicitly shown in the given representation. Sixty-five percent of
students showed high competency in the area of interpreting-comprehension. Ten percent
demonstrated low competency in this area.

*Interpreting-application.* The code for interpreting-application applies to
problems where participants solve word problems through the use of the information
given by a function. Table 3.12 gives the details for the codes. A range in success for
problems involving interpreting-application occurred with a high percent of students,
between 88% and 100% of students showing competency for problems 4, 5, and 6 of the
FKA: Part 2. The lowest performance occurred for question 2e, where only 38% of
participants were able to successfully find when the cyclist was at the top of the hill.
Figure 4.4 shows the questions coded as interpreting-application.
Function Knowledge Assessment: Part 1

4. How much money will Matthew have on week 30?

5. How much money did Adam have on the 10th week?

7. Matthew will have $150 on which week?

8. Which week will Adam have $287?

10. Compare the savings of two out of the four children. Describe how the children’s savings are changing throughout the year. In your comparison you can talk about whether or not the savings is increasing or decreasing, which has a faster rate of change, which child has the largest amount at the end of the year, etc.

15. Whose savings is growing the fastest?

16. Whose savings is growing the slowest?

Function Knowledge Assessment: Part 2

3) Find the week with the largest difference between their savings.

4) Find the week when their savings were equal.

5) Find the week when the savings of one were double that of the other. In whose favor?

6) Diane and Matthew decided to pool their savings in order to buy a $400 walkie-talkie. Find the week in which they are able to purchase the walkie-talkie.

Function Competency Assessment

2) The graph below gives the speed of a cyclist on his daily training ride. During his ride, he must climb a hill where he pauses for a drink of water before descending. Using this graph to answer the following questions as accurately as possible. (O’Callaghan, 1998)

\[ \text{Figure 4.4: Interpreting-Application problems from the FKA: Part 1, the FKA: Part 2, and the FCA} \]
A summary of how each participant did with interpreting is given in Table 4.17.

Gabe and Caleb were the top performing students in this area followed by Robert and Wanda. Next I will briefly discuss student success in interpreting for each representation.

<table>
<thead>
<tr>
<th>Student</th>
<th>Interpreting-Comprehension</th>
<th>Interpreting-Application</th>
<th>Total Percent for High Competency in Interpreting</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robert</td>
<td>High 8 Medium 3 Low 0</td>
<td>High 11 Medium 2 Low 2</td>
<td>73%</td>
</tr>
<tr>
<td>Liz</td>
<td>High 4 Medium 3 Low 4</td>
<td>High 6 Medium 8 Low 1</td>
<td>38.5%</td>
</tr>
<tr>
<td>Madisen</td>
<td>High 4 Medium 5 Low 2</td>
<td>High 5 Medium 7 Low 3</td>
<td>34.6%</td>
</tr>
<tr>
<td>Gabe</td>
<td>High 10 Medium 1 Low 0</td>
<td>High 10 Medium 4 Low 1</td>
<td>76.9%</td>
</tr>
<tr>
<td>Caleb</td>
<td>High 11 Medium 0 Low 0</td>
<td>High 12 Medium 2 Low 0</td>
<td>92%</td>
</tr>
<tr>
<td>Yvonne</td>
<td>High 6 Medium 2 Low 3</td>
<td>High 6 Medium 8 Low 0</td>
<td>48%</td>
</tr>
<tr>
<td>Amanda</td>
<td>High 6 Medium 3 Low 0</td>
<td>High 4 Medium 6 Low 3</td>
<td>45.5%</td>
</tr>
<tr>
<td>Wanda</td>
<td>High 8 Medium 3 Low 0</td>
<td>High 9 Medium 6 Low 0</td>
<td>65.4%</td>
</tr>
</tbody>
</table>

Table 4.17: Participant Competency Levels for Interpreting-Comprehension and Interpreting-Application and Total Percent of High Competency Displayed for Interpreting

*Interpretation for descriptions.* There were four problems from the two mathematics instruments that utilized descriptions and were classified as either interpreting-comprehension or interpreting-application. Robert, Gabe, Caleb and Wanda were coded as having high competency for all four of these problems. Liz and Madisen had competencies in three of the four problems. Yvonne showed competency for just one and Amanda did not demonstrate competency in any of the four. Gabe and Caleb were the only two participants to choose Heather (the child’s savings account that was described in words) as one of their two students to compare for question 10 of the FKA: Part 1.
Interpretation for equations. Participants were asked two questions regarding Diane’s savings, which involved the use of an equation. All participants were able to successfully complete the first question where they were asked to find the amount Diane had for a given week. All students except for Madisen were able to complete the second question which asked for the week in which Diane would have 200 dollars. Only two students, Robert and Liz, chose to discuss Diane’s savings for question 10 on the FKA:

Part 1. Four students decided to use the equations for Matthew and Diane when answering questions for the FKA: Part 2. For these six questions, Gabe was successful in answering all of the questions and Liz and Madisen were able to answer 5 of the six.

Interpretation for tables. Six problems involved reading a table and all students did very well with interpreting functions represented in this manner. All participants were successful in at least five of these questions. Also, all students chose Adam, the child whose savings was described in a table, as one of their two savings accounts to compare. Finally, three students, Robert, Caleb, and Yvonne, chose to use the table of values to answer the six questions on the FKA: Part 2 and each of them were successful in answering each problem.

Interpretation for graphs. Questions coded as interpreting were composed of 3 questions concerning Matthew’s savings account from the FKA: Part 1 and 5 problems concerning the cyclist’s daily training ride from the FCA. Of these eight problems, student competency varied widely. Liz was coded least often for this category at 13% success rate. Gabe, Caleb and Wanda were successful with 100% of the graph interpretation problems, while Yvonne and Madisen struggled more with these and only obtained 38% and 43% accuracy, respectively. Four students chose to discuss Matthew’s
savings on question 10 of the FKA: Part 1, but only one of those participants were coded for interpreting-comprehension for this problem. The other three students had incorrect reasoning when discussing this graph. Finally, only one student, Wanda, chose to work with the graph of Diane’s and Matthew’s savings to answer questions from the FKA: Part 2. She was coded for showing competency in five of the six questions.

**Translating from one representation to another**

Translating-comprehension is the ability to move from one representation of a function to another. Participants were specifically asked to perform such a translation seven times on the FKA and FCA. These questions, in general, were difficult for participants and the coding rate ranged from 25% to 75% success rate. Details of those coded for these problems can be found in Table 4.18

<table>
<thead>
<tr>
<th>Problem/Type of Translation</th>
<th>Success-Rate</th>
<th>Problem/Type of Translation</th>
<th>Success-Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1c; FCA (Description to Equation)</td>
<td>63%</td>
<td>5b; FCA (Description to Graph)</td>
<td>100%</td>
</tr>
<tr>
<td>3c; FCA (Table to Graph)</td>
<td>63%</td>
<td>12; FKA: Part 1 (Graph to Equation)</td>
<td>25%</td>
</tr>
<tr>
<td>4b; FCA (Table to Equation)</td>
<td>25%</td>
<td>13; FKA: Part 1 (Table to Equation)</td>
<td>75%</td>
</tr>
<tr>
<td>5b; FCA (Description to Table)</td>
<td>40%</td>
<td>14; FKA: Part 1 (Description to Equation)</td>
<td>25%</td>
</tr>
</tbody>
</table>

Table 4.18: Percent of Participants that Showed Competency in Translating-Comprehension
Two of the translation problems asked students to write an equation based on a problem description. Sixty-three percent of students were coded as showing competency for one of these problems (1c; FCA) and only 25% were successful in solving the other, question 14 from FKA: Part 1. All three students who chose to produce a graph for question 5b (FCA) were successful. Two of the participants who chose to produce a table for this problem were coded for translating-comprehension. Results for individual participants’ competencies in translating are shown in Table 4.19.

<table>
<thead>
<tr>
<th>Student</th>
<th>High</th>
<th>Medium</th>
<th>Low</th>
<th>Total Percent for High Competency in Translating</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robert</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>57%</td>
</tr>
<tr>
<td>Liz</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>28.6%</td>
</tr>
<tr>
<td>Madisen</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>28.6%</td>
</tr>
<tr>
<td>Gabe</td>
<td>5</td>
<td>2</td>
<td>0</td>
<td>71.4%</td>
</tr>
<tr>
<td>Caleb</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>71.4%</td>
</tr>
<tr>
<td>Yvonne</td>
<td>0</td>
<td>1</td>
<td>6</td>
<td>0%</td>
</tr>
<tr>
<td>Amanda</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>0%</td>
</tr>
<tr>
<td>Wanda</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>42.9%</td>
</tr>
</tbody>
</table>

Table 4.19: Participant Competency Levels for Translating-Comprehension and Total Percent of High Competency Displayed for Translating

**Reifying**

*Reifying-comprehension.* A student was coded under reifying-comprehension if he or she were able to demonstrate an ability to understand function as an object, as something that can be manipulated and used in its entirety. One student, Caleb, demonstrated reification of linear function through his answer to number 2b on the FCA (see Figure 4.4). This problem involved students reading the graph displaying the speed of a cyclist and asked students to find the time at which the cyclist was going 30 miles
per hour. Caleb immediately stated that there would be two answers to this problem because the cyclist’s highest speed is above 30 miles per hour, which indicates the cyclist will pass the 30 mile per hour mark again when he starts to slow.

Caleb also demonstrated a basic understanding of reification of linear functions when answering number 4 on the FKA: Part 2 (see Figure 4.4). When asked which week the savings of Diane and Matthew would be at its largest difference, Caleb immediately stated that this would occur at either the beginning or the end of the year. Caleb was able to recognize the structure of the graphs and the property that the difference between the savings accounts would get smaller and smaller until the point of intersection and that the difference would grow apart after that point. Table 4.20 summarizes individual competencies in reifying.

<table>
<thead>
<tr>
<th>Student</th>
<th>Reifying-Comprehension</th>
<th>Total Percent for High Competency in Reifying</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
<td>Medium</td>
</tr>
<tr>
<td>Robert</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Liz</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Madisen</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>Gabe</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Caleb</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Yvonne</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Amanda</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Wanda</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.20: Participant Competency Levels for Reifying-Comprehension and Total Percent of High Competency Displayed for Reifying

Next, I discuss results of function competencies in terms of student characteristics of visual experiences, reading medium, and school setting.
**Visual Experiences**

A students’ visual experience can greatly affect his or her learning. Participants from this study who had some early visual experiences showed a higher level of competency than those who were born without sight as well as those who lost their vision later in life. Table 4.21 gives a summary of all competencies combining modeling, interpreting, translating, and reifying shown by participants with various visual experiences.

<table>
<thead>
<tr>
<th>Function Knowledge</th>
<th>Congenitally Blind</th>
<th>Early Visual Experiences</th>
<th>Later Blind</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Competency:</td>
<td>45 (55%)</td>
<td>High Competency: 67</td>
<td>High Competency: 28</td>
</tr>
<tr>
<td>Medium Competency:</td>
<td>24 (29%)</td>
<td>Medium Competency: 12</td>
<td>Medium Competency: 32</td>
</tr>
<tr>
<td>Low Competency:</td>
<td>13 (16%)</td>
<td>Low Competency: 1</td>
<td>Low Competency: 19</td>
</tr>
</tbody>
</table>

Table 4.21: Levels of Competency in the Four Function Competencies by Participants Based on Visual Experiences

**Reading Medium**

Reading medium indicates whether or not a student has enough vision to read large or regular print and therefore whether they can visually access the material. Braille readers and those who read regular or large print demonstrated about the same level of competency in their knowledge of function (Table 4.22).
Table 4.22: Levels of Competency in the Four Function Competencies by Participants’ Preferred Reading Medium

<table>
<thead>
<tr>
<th>Function Knowledge</th>
<th>Regular/Large Print</th>
<th>Braille</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Competency: 140 (59%)</td>
<td>High Competency: 38 (50%)</td>
<td></td>
</tr>
<tr>
<td>Medium Competency: 63 (27%)</td>
<td>Medium Competency: 27 (36%)</td>
<td></td>
</tr>
<tr>
<td>Low Competency: 34 (14%)</td>
<td>Low Competency: 11 (15%)</td>
<td></td>
</tr>
</tbody>
</table>

School Setting

The school setting at which a student learns mathematics can have an impact on his or her overall understanding of mathematics. Participants from this study who were taught the function concept at a local school showed a higher competency than those educated at a school for the blind (high competency 67% versus 43% respectively). Table 4.23 summarizes the results for function knowledge based on school setting.

Table 4.23: Levels of Competency in the Four Function Competencies by Participants’ School Setting When Learning the Function Concept

<table>
<thead>
<tr>
<th>Function Knowledge</th>
<th>Local School</th>
<th>State School for the Blind</th>
</tr>
</thead>
<tbody>
<tr>
<td>High Competency: 112 (67%)</td>
<td>High Competency: 66 (43%)</td>
<td></td>
</tr>
<tr>
<td>Medium Competency: 40 (24%)</td>
<td>Medium Competency: 59 (38%)</td>
<td></td>
</tr>
<tr>
<td>Low Competency: 15 (9%)</td>
<td>Low Competency: 30 (19%)</td>
<td></td>
</tr>
</tbody>
</table>

Graphing Access and Proficiencies

Student access to graphing materials and graphing opportunities as well as student proficiency with reading and producing graphs was reviewed through a variety of results from the MEEVA, the FKA, and the FCA. A graphing profile was written for each participant based on students’ stated representational preference, student choices of
representational use when solving problems from the FKA and FCA, student success-levels and comfort-levels in working with the various representations, student success- and comfort-levels when solving problems from the FKA: Part 2 with their chosen representation, and student competencies in interpreting when solving problems using a particular representation. Each student was coded for having poor, adequate, or excellent access to graphs and graphing materials, whether or not they were Independent with graphing, and the Proficiency level (either poor, adequate, or proficient) for reading and producing graphs. Table 4.24 presents a summary of this information for each student.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Access</th>
<th>Independence</th>
<th>Proficiency with Reading</th>
<th>Proficiency with Producing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robert</td>
<td>Adequate</td>
<td>Independent</td>
<td>Adequate</td>
<td>Proficient</td>
</tr>
<tr>
<td>Liz</td>
<td>Poor</td>
<td>Not Independent</td>
<td>Poor</td>
<td>Adequate</td>
</tr>
<tr>
<td>Madisen</td>
<td>Excellent</td>
<td>Independent</td>
<td>Adequate</td>
<td>Adequate</td>
</tr>
<tr>
<td>Gabe</td>
<td>Excellent</td>
<td>Independent</td>
<td>Proficient</td>
<td>Proficient</td>
</tr>
<tr>
<td>Caleb</td>
<td>Excellent (except in higher mathematics courses)</td>
<td>Independent</td>
<td>Proficient</td>
<td>Proficient</td>
</tr>
<tr>
<td>Yvonne</td>
<td>Adequate</td>
<td>Independent</td>
<td>Adequate</td>
<td>Poor</td>
</tr>
<tr>
<td>Amanda</td>
<td>Excellent</td>
<td>Independent</td>
<td>Proficient</td>
<td>Poor</td>
</tr>
<tr>
<td>Wanda</td>
<td>Excellent</td>
<td>Independent</td>
<td>Proficient</td>
<td>Proficient</td>
</tr>
</tbody>
</table>

Table 4.24: Summary of Graphing Access and Proficiency for Individual Participants
Produced Graphs

Figure 4.5 shows the graphs produced by the students. All eight participants produced a graph for problem 3c on the FCA and three participants chose to produce a graph for problem 5b on the FCA.

Figure 4.5: Graphs produced by participants for FCA questions 3c and 5b
Taking into consideration the participants’ stated preferred representation, his or her choice of representation when solving problems on the mathematics assessments, and the students’ success-level and comfort-level for each representation, I have assigned the each participant a preferred representation(s).

The information is summarized in Table 4.25 and indicates that of the eight students who participated in this study, four prefer to work with equations when solving word-problems involving functions, 7 prefer the use of tables, 2 prefer the use of graphs, and one prefers the use of descriptions.

<table>
<thead>
<tr>
<th>Participant</th>
<th>Preferred Representation(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robert</td>
<td>Equations or Tables</td>
</tr>
<tr>
<td>Liz</td>
<td>Equations or Tables</td>
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<tr>
<td>Madisen</td>
<td>Equations or Descriptions</td>
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<td>Gabe</td>
<td>Equations or Tables</td>
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<tr>
<td>Caleb</td>
<td>Graphs or Tables</td>
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<td>Yvonne</td>
<td>Tables</td>
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<td>Amanda</td>
<td>Tables</td>
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<td>Wanda</td>
<td>Graphs or Tables</td>
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Table 4.25: Individual Participant Preferred Representation for Solving Linear Function Problems
Graphing Profiles

Robert. Robert enjoys mathematics and believes he has a good understanding of linear functions and their representations. When solving problems involving linear functions Robert prefers to use equations. Robert had adequate access to graphing in his K-12 education and a somewhat diminished access level in his college career. In high school Robert was provided with various graphing materials such as braille graph paper, Wikki Sticks, a foam pillow (which was used as a graph board), push pins, thumb tacks, and rubber bands. He was also provided with raised line graphs for tests and quizzes, but was not typically provided with one during class. Robert had a braille copy of the mathematics textbook. If the class discussion centered on a graph provided in the book, he had access to it. Robert’s textbooks are provided in braille for his college courses, but he rarely has a tactile version of what is being discussed in class. To gain an understanding of what was being discussed that day Robert will wait for a one-on-one meeting with the instructor. Robert also recalls that the symbolic version of an algebraic expression was used most often in instruction and that the use of tables and graphs were not emphasized.

Robert is very independent when it comes to reading and producing graphs. He is familiar with how to read a graph and uses techniques to help ensure accurate answers. For instance, when reading a graph Robert uses a two-hand method, which helps to guide his reading finger straight from a point on the graph to the x- or y-axis. Robert also gains an overall perspective of the graph before determining details of the graph.

Robert is an adequate reader of graphs and a proficient producer of graphs. When solving problems that were answered through reading a graph, Robert was successful six
out of the seven times. Robert demonstrated a certain level of comfort and accuracy in matching up the child’s savings account as described at the beginning of the FKA to the graphs provided in question number 11 of the FKA. Although Robert was very successful in answering questions based on information from a graph, he mixed up the x- and y-axis on several occasions, was very uncomfortable when asked to write an equation for Matthew’s savings (which was presented as a graph on the FKA), and persisted in reading the top of the graph in Number 2 of the FCA as the top of the hill, instead of the speed of the cyclist.

Robert produced two graphs using the graph board (for Number 3c on the FCA and Number 5b on the FCA). Both graphs are produced well without a lot of input from me. Robert was able to start the graphing process by choosing appropriate scales to use for his axes and for choosing an appropriate starting point for his origin. For the graph of 5b on the FCA he produced a very straight line and was aware of the need for the line to be straight due to the constant increase of Mary’s income.

*Liz.* Liz has always struggled with mathematics throughout her K-16 education. Liz attended her local schools except for one-quarter at a residential school for the blind. However, during her K-12 education she moved several times and across several states. Liz indicated that she prefers to work with equations when solving algebra problems and says “tables or equations beat graphs.” Throughout the two mathematics assessments Liz was very uncomfortable solving any problem where the information came from a graph. Her access to graphing has been poor and she is not independent with graphing. Liz said that all of the graphs she has worked with were already produced. However, when we got out the graph board to answer number 3c on the FCA, she recalled using one similar to it.
during her occupational therapy. In speaking of her education on graphing Liz stated, “Actually they just showed me a graph.” Exams where Liz was required to either read a graph or plot points, she would tell a proctor her steps and have him or her write the answers. Specifically, Liz said that when a test would ask, “Where are the points plotted on this graph?” Liz would, “have no idea how to do that” and would ask her proctor to tell her how far the point was from the origin in each of the directions.

Liz has poor graph reading skills. For most of the questions involving reading a graph, I had to explain to Liz how the graph worked and where to find the information. More than once she tried to find the answers to the questions based solely on the scales of the axes. I would have to point out that the line we were following was in the middle of the graph. At one point Liz equated a shorter line with one that is growing slower, but later said that the same line looks to be “more to the right than up.” Liz was able to tell where the graph of the cyclist was increasing (for question 2 on the FCA) and was able to correctly describe Helen’s savings (FCA; 7) as a horizontal line.

I have given Liz a proficiency producing graphs rating of adequate because her final product of the graph for 3c on the FCA was well constructed. This construction was not without help, however. I intervened on a significant level to get her started, but once the first pin was placed, she was able to proceed with little assistance.

Madisen. Madisen struggles with mathematics. Although it has never been her favorite subject, she doesn’t hate it, either. Madisen prefers to work with equations when there is a lot of information presented. Madisen believes this is the best representation for someone who reads braille, because she finds it is best to give as much information in the most compact form possible to individuals who are blind. Madisen has had excellent
access to graphing throughout high school. She has had access to materials such as graph boards, push pins, rubber bands, braille graphs and textbooks, and has had graphs provided to her when they are being discussed in class. Madisen is also independent in her work with graphs. She needed little instruction on how to read a graph and was comfortable reading most of the graphs provided on the mathematics assessments.

Madisen’s proficiency in reading and producing graphs is adequate. She was successful in answering 4 out of 7 questions using information from a given graph. Madisen mentioned that Graph B (FKA: Part 1: Number 11) was strange because “it does not start at week zero.” She also persisted in her belief that the highest point on the cyclist graph (FCA: Number 2) was the top of the hill. When producing a graph for 3c (FCA) she did a good job, except she put the years as her y-axis and the prices on the x-axis. Madisen organized the push pins and used the large ones to mark her axes and the rounded ones to graph the home prices. When Madisen went to place her first pin, however, she stated, “I don’t know how to do this.” So, I helped her mark her starting pin and she did a good job after that point.

Gabe. Gabe has been successful in his mathematics classes, but says that mathematics is the first to come to mind when speaking of classes with which he has struggled. Gabe appreciates math, but says he has not always had a good experience with it in the school system. Gabe prefers to work with equations and tables, but says he doesn’t mind graphs either. Although Gabe struggled somewhat because his teachers did not always know how to present the information in an appropriate manner, it seems as though Gabe had excellent access to graphing. Tactile graphs were provided to him on a regular basis and various tools and manipulatives were provided for producing graphs.
Gabe is independent and proficient with both reading and producing graphs. Gabe did not hesitate in his answer of when the cyclist was at the top of the hill (FCA: Number 2e) and accurately described each of the graphs in Number 11 (FKA: Part 1). Other than having a slight hesitation as to where to place his first thumb tack when producing the graph for 3c (FCA), he made a very good graph without comment from me.

Caleb. Caleb enjoys doing mathematics and has read tactile graphs his whole life. He prefers to work with graphs when solving algebraic problems, unless the graph is messy, than he prefers to use equations. Caleb attended local schools for all of his education and has had excellent access to graphing, until his higher mathematics courses such as pre-calculus. Caleb had access to graphing tools including a graphing board and various accessories such as push pins and rubber bands, an Audio Graphing Calculator (AGC), tactile graphs either from his aide or from his braille textbook, and in some courses an aide to produce graphs during class. Caleb’s access to graphing fell sharply when he entered courses that relied heavily on the graphing calculator. Some of the course concepts, such as manipulating matrices, were completely inaccessible to him because the instructor only taught the students to do the manipulations on the calculator, which Caleb did not have access to. He also did not find the AGC very useful and did not have adequate training on how to use it effectively. Caleb also found it difficult to gain a good understanding of how graphs shifted because all of the graphing tools he used were stationary. That is, once the graph was produced, it could not shift over on the x-axis or expand. To show the effects of a changing equation on the graph, individual graphs on the graph board would have to be produced.
Caleb is very independent when it comes to working with graphs. He was able to read the information from the given graphs and produce two graphs without any assistance. Caleb was successful in answering all but one question that required him to get the information from a graph. Caleb is proficient at both reading and producing graphs. He was able to accurately describe all of the graphs from Number 11 (FKA: Part 1) and the graphs he produced were very well done. When placing the last pin of the graph for 5b (FCA) Caleb knew something was wrong because it did not produce a straight line.

Yvonne. Yvonne has always struggled with mathematics and science. Yvonne believes that she had a good understanding of linear functions at the time she was learning about them, because testing over the information was easy. However, she has now forgotten a lot of the information. Yvonne states that her preferred representation is whichever one she has learned most recently. In high school and in college Yvonne has had adequate access to graphing. Yvonne was not introduced to graphing until she was in high school, which was not long after her vision. She was provided with the standard graphing materials of graph boards, push pins, and rubber bands and was given access to tactile graphs at least part of the time. For Yvonne’s college education she has her mathematics textbooks in braille, and therefore has some access to the graphs studied in class. If a graph is put on the board and discussed in class that she does not have access to (either she does not have her copy of the book with her or it is a graph that is not in the book) it all depends on the teacher as to whether the graph is described adequately.

Yvonne is independent with reading graphs. She knew the various pieces of the graph and was able to talk about the graphs without much assistance. Yvonne’s graph
reading skills were adequate, but her ability to produce a graph was poor. She had difficulty with question 2c (FCA) and was only able to give the intervals of increasing speed for one of the times of increase. Yvonne also incorrectly stated that Diane starts with zero when looking at the different graphs given for number 11 (FKA: Part 1). Yvonne had a difficult time determining the type of line Graph C (FKA: Part 1; Number 11) represented. Although it is a horizontal line, she was convinced that it was showing a dramatic increase. The graph Yvonne produced for number 3c (FCA) was done incorrectly by just placing separate pins for each component. This led to a final graph that looked as though she had simply marked her axes.

_Amanda._ The most difficult part of learning mathematics for Amanda has been geometry and she relates graphs with this branch of mathematics. When we reached number 11 (FKA: Part 1) she said, “I told you I don’t like geometry!” Amanda believes that she has a good understanding of linear functions and their representations, but that she has forgotten some of that information. Amanda prefers to work with equations when solving algebraic problems. She has had excellent access to graphing throughout high school with the basic supplies such as graph boards and accessories and large print graph paper being on hand. Amanda is independent when working with graphs and was comfortable with reading the graphs from the two mathematics assessments.

Amanda is proficient with reading graphs, but she has poor skills when it comes to producing graphs. She was successful in reading the graph of the cyclist’s ride from number 2e (FCA). Amanda was less successful in reading the graph of Matthew’s savings from the FKA, but I believe this was due to the way the graph was produced and not her reading skills. Amanda knew the functions of the various pieces of the graph and
inquired as to whether it mattered if she used the ‘positive’ side of the graph when making a graph for number 3c (FCA). The graph Amanda produced for 3c (FCA) was one where she put separate marks for each component and therefore her final graph basically looked as though she had marked the scale. I had asked Amanda to draw the line of the graph, and she did it by connecting each x-coordinate with the corresponding y-coordinate. This made the effect of nested triangles as the scales increased (see Figure 4.5).

*Wanda.* Wanda does not really enjoy mathematics, but likes geometry because she can see how it applies to ‘the real world’. Wanda believes her understanding of linear functions has improved since she started taking mathematics at the school for the blind. She has had excellent access to graphing as she typically just uses regular print graphs and occasionally has a need for a large print graph if there is a lot of information that needs to be plotted. Wanda is independent and comfortable with reading and producing graphs and had few questions while working through the problems involving graphs on either the FKA or the FCA. Wanda is proficient with both reading and producing graphs. She did not see that there were two answers to number 2b (FCA) and had some difficulty figuring out where the points of increase were for 2c (FCA), but other than that she was successful in all of the questions involving graphs and was able to produce good graphs for both 3c (FCA) and 5b (FCA).

**Summary**

Results of the data were organized around each of the three instruments used for data collection. Students were shown to have a wide range of knowledge in the four function competencies and demonstrated a high success in modeling and interpreting, but
a low success in translating and reification of linear function. Graphing abilities and access to graphing materials and opportunities varied among students, but was only exceptionally low for one student. Two participants demonstrated excellent skills in reading and producing graphs. Most students preferred to use tables and equations when solving word problems involving linear functions. All students noted the importance of individualized education, especially having opportunities to learn mathematics in one-on-one settings. Participants also noted a need for having access to appropriate materials in a timely manner. Students repeatedly discussed how helpful it was to have a copy of the class notes ahead of time so that they could follow along with what the teacher was presenting. Unfortunately most students also indicated that they were rarely provided with class notes before or during instruction. The conclusions drawn from the results presented in Chapter 4 are discussed in Chapter 5 and are centered on answering each research question.
CHAPTER 5

CONCLUSION

The purpose of this chapter is to summarize the study, give conclusions for the research questions based on the data analysis, discuss implications for the mathematics education for students with visual impairments, and make recommendations for future research needs in this area.

Summary

This study reviewed the current understanding of linear functions held by students with visual impairments. Graphing abilities and access were also addressed along with students’ preferred representation when solving word problems involving linear functions. Finally, an analysis of participants’ perceived influences on their mathematical education was conducted. Data collection occurred during May and June of 2011 with participants from central and southwest Ohio. Eight students participated in this study including four high school students and four college students. Three of the high school students attend a local high school for half of the day and a school for the blind the second half. The fourth high school student attends his local school. The four college students attend a university that is known for providing excellent services for students with disabilities.
Data collected for analysis included student responses on three instruments, a) the Mathematics Education Experiences and Visual Abilities (MEEVA) interview (Appendix A), b) the Function Knowledge Assessment (FKA) (Appendix B), and c) the Function Competency Assessment (FCA) (Appendix C). Participants met with me one-on-one for two sessions. All instruments were administered orally with questions 1-12 from the MEEVA and the FKA administered during the first session and the final part of the MEEVA and the FCA completed during the second session. This arrangement varies somewhat between participants, for full details of data collection procedures see Chapter 3.

There were three main theories used in the data analysis for this study. Two of these theories were specifically used to answer Research Question 1, “What level of knowledge and type of understanding of linear functions is held by students with visual impairments?” First, levels of understanding from Wilson’s (1971) taxonomy of learning for secondary mathematics, adapted from Bloom (1956), were utilized as a way to categorize the level of knowledge held by participants. Secondly, O’Callaghan’s (1998) Function Competencies Model was used as a way to address participants’ type of understanding as it relates to linear function. The third theory used for data analysis is Lowenfeld’s (1981) special educational methodologies for students with visual impairments. This model includes five methodologies. A modified version of two of these methodologies were applicable for this study and were used during the coding process. Individualization and Concreteness are the codes used and were specifically used for analyzing Research Question 4, “What factors do high school/college students
with visual impairments perceive as influencing the development of their mathematical understanding?”

The Constant Comparative Method was used during all stages of data analysis with coding processes involving open coding and selective coding, as described by Glaser and Strauss (1967). Student responses were first coded individually, and then compared across student characteristics including the participants’ high school environment, reading medium, and visual experiences. One set of codes used for analysis was a combination of Wilson’s levels of understanding and O’Callaghan’s function competencies and included modeling-comprehension, modeling-application, interpreting-comprehension, interpreting-application, translating-comprehension, and reifying-comprehension. Student responses to a set of the mathematical problems from the FKA and FCA were also coded for success-level and comfort-level. Success-level measured students’ successful completion of a problem with taking into consideration assistance they received from me. The efficiency of the solution is not represented in the given success-level. Success-level is coded for each problem as high, medium-high, medium-low, or low. Comfort-level gaaged participants’ comfort in working with a particular representation and did not take into consideration the success of the solution. Comfort-level was coded for each problem as high, medium, or low. For details of the coding process see Chapter 3.

Perceived influences on participants’ mathematics education were obtained by coding responses to the MEEVA interview. Statements regarding either positive or negative experiences in the mathematics classroom or with their education, in general, were marked as being examples of either Individualization or Concreteness. These two
special methodologies, initially described by Lowenfeld (1981) and modified for the current study, entail accommodations and methodologies that provide for the student as an individual, one who has individual needs and learning styles (Individualization) and accommodations and methodologies that arrange for content to be presented in a concrete manner for student use (Concreteness). The final coding process was done to assess student access to graphing and proficiency with graphing skills. Statements from the MEEVA interview concerning graphing were coded as either being statements about access to graphing or independence with graphing. Responses to questions involving either reading or producing graphs on the mathematical instruments were used to identify whether a student had poor, adequate, or proficient abilities in reading and producing graphs.

Results of the coding and analysis process revealed that high school and college students with visual impairments have competency in comprehension of linear functions. Students also showed competency in the areas of modeling and interpreting problems involving linear functions. However, their performance varied somewhat depending on the representation being used to communicate problem information. For example, students were most comfortable and successful when working with problems involving tables and were least accurate or comfortable when gaining information from a graph. Participants struggle the most with the function competency of translating. A few students were able to demonstrate an ability to move from one mode of representation to another on a semi-consistent basis, but on the whole students did not demonstrate competency in this area. A student having reification in a particular area can be difficult to determine as it applies to students’ use and view of a function as an object, which is
not always accessible through their voiced responses. However, there were three participants who demonstrated understanding that was classified as showing a reification of the linear function concept. For full details see Chapter 4.

All participants, except one, had good access to graphs and graphing materials. The participant with poor access to these materials and experiences was educated in her local schools, but lacked a consistency in her education due to moving several times during her elementary and high school years. This student was also the only one coded as not being independent with graphing. Students’ skills in reading and producing graphs were more varied. Three students (38%) demonstrated proficiency with both reading and producing graph, while one student had poor graph reading skills and two students demonstrated poor skills in producing graphs. Several students were able to produce a graph when assisted with placing the first pin on the graph board. Two student’s graphs were both incorrect with each of the participants “graphing” the points separately on the axes (see Figure 4.4).

Participant preference of representation when solving linear function word problems was determined by three factors, a) participants’ stated preference, b) participants’ choices of representational use when solving problems on the mathematics assessments, and c) students’ success-level and comfort-level when solving problems involving the various representations. I coded students as having one or two preferred representations. For details on individual student preference refer to table 4.25. The results indicate that 38% of students prefer to use either equations or tables, 25% prefer to use either graphs or tables, 25% prefer the use of tables and one student prefers either
equations or descriptions. Therefore, 88% of students had a preference for using tables and 50% preferred working with equations.

Accommodations that addressed Individualization were more often stated by students than those that addressed Concreteness. Under the category of individualization, all students emphasized the need for one-on-one instruction either through working with their teacher before or after class or through tutoring services. Students also indicated the importance of support services such as having an aide or a Teacher of the Visually Impaired to help ensure that materials were provided and concepts were conveyed in an accessible format. Providing materials in an accessible format is allowing the material to be concrete for students. The most common methods of making concepts accessible to students were by providing class material, such as class notes and textbooks, in braille and graphs in a tactile form. Graphing materials such as large print or braille graph paper, graph boards, and accessories, and technology were also highlighted as needed for making course content accessible. Technology commonly used by students includes braillewriters, braille notetakers, and screen reader and screen magnification software for computer use.

**Conclusions**

Results from the present study add to the literature by providing a description of the understanding of linear functions held by students with visual impairments. I found that high school and college students with visual impairments have a gap in their knowledge of working with various representational forms of a function. Students demonstrated weakness particularly in the areas of translating between representational forms and reifying the function concept. I also found a positive relationship between
students’ graphing skills and his or her overall understanding of linear function. Participants stated representational preferences were shown to correspond with those they perceived as being emphasized in the classroom and their stated representational preference did not always correspond to the representation they were most comfortable using. Next I expound on the specific conclusions drawn from the results of this study for each research question.

**Research Question 1: What level of knowledge and type of understanding of linear functions is held by students with visual impairments?**

In order to determine the level of knowledge and type of understanding of linear functions held by participants, student responses to all of the problems presented on the two mathematics instruments, the Function Knowledge Assessment (FKA: Parts 1-3) and the Function Competency Assessment (FCA), were analyzed. This analysis centered on student understanding in the four Function Competencies Model as presented by O’Callaghan (1998). That is, each student was assessed as having competency in, a) modeling, b) interpreting, c) translating, and d) reifying of functions. Further, the distinction between student comprehension and their ability to apply these competencies to word problems was reviewed.

Participants from this research demonstrated a 66%, 59%, 38%, and 36% high competency rates in modeling, interpreting, translating, and reifying, respectively. The fact that students showed high competency in the four areas of function understanding at most 66% of the time highlights the need for additional strategies and learning opportunities for students with visual impairments in the mathematics classroom. For the modeling and interpreting competencies, participants displayed high or medium
competency 98% and 91% of the time, respectively. The higher percentage of competency displayed when considering both high and medium competency rates indicate that students were able to display understanding in the area of modeling and interpreting for the majority of the problems, but that this understanding was not always complete.

Level of knowledge of linear functions held by students with visual impairments was demonstrated through Wilson’s (1971) levels of understanding in comprehension and application. Analysis from this study indicates that students showed high competency in the area of comprehension 54% of the time and in the area of application 56% of the time. When taking into consideration participants who were able to demonstrate either a high or medium competencies in problems involving comprehension skills and application skills, participants demonstrated 80% and 92% competency, respectively. Once again, the low percentage of students who were able to demonstrate high competency in comprehension and application of problems involving linear functions is something that must be addressed in the field of mathematics education.

Results from this research indicate that translation from one representation to another is an exceptionally weak area in linear function knowledge for students with visual impairments. Participants displayed high competency in this area only 38% of the time and demonstrated low competency in the area of translation almost 44% of the time. The problems involving translating a linear function from one representation to another was one of the most difficult items presented on the FKA and FCA. However, according to the Ohio Department of Education’s Academic Content Standards, by the end of the 9th grade students should be able to “Generalize patterns using functions or relationships
(linear, quadratic, and exponential), and freely translate among tabular, graphical and symbolic representations” (p. 94, emphasis added). Therefore, all participants in this study should have been able to flexibly move from one representation to another. Instruction in the translation between various representations of linear function needs to be made explicit to students with visual impairments. Instruction in translation should be given high priority throughout algebra instruction. This instruction should not be segmented, but integrated across the various representations so that students can more readily make connections between the representations.

Three students displayed high proficiency in both reading and producing graphs. These same three students demonstrated a higher degree of reification. In fact, high competency was demonstrated for the two problems specifically targeting students’ reification of linear function only by the three students who demonstrated high proficiency in both reading and producing graphs. In addition, two of the three students who showed high competency with reading and producing graphs also demonstrated high competency in translating between representations. Another student who is proficient in producing graphs and adequate at reading information presented in a graph, also demonstrated some reification of linear functions. On the other hand, the four remaining students who did not show proficiency in either reading or producing graphs also demonstrated very low and inconsistent reification of the linear function concept. Therefore, I conclude that students who have a better understanding in reading and producing graphs are more likely to reify linear functions. This result is consistent with research with sighted students conducted by Slavit (1997). Slavit determined that one
way to encourage reification of function is for students to have instruction on determining properties of functions through the functions’ graphical representation.

Sfard (1991) describes student understanding as having an operational or a structural view of function. Similar theories (Dubinsky & Harel, 1992) promote the dual nature of function by describing student view of function as either a process or an object. Results from this research are consistent with Sfard’s description of the view of function where students did not solely view a function as a process (operationally) or as an object (structurally), but instead moved between these views (with varying levels of success) as was necessary. In this study half of the participants held some level of knowledge of linear function as an object (i.e. had reified the function concept). The other four participants were firmly within the realm of viewing a function as a process and did not demonstrate the concept of function as an object. The low percentage of participants having an object view of function is consistent with previous research with sighted students indicating that the reification process is most difficult for students and that many students never reach full reification of the function concept (Sfard, 1991; Sfard & Linchevski, 1994; Slavit, 1997; Ronda, 2009).

Research Question 2: What are the graphing skills of high school/college students with visual impairments?

Results of this research indicate visual experience promotes graphing skills for students who read tactile graphs. Seventy-five percent of students who had some vision in the past (either as early visual experiences or students who became blind late in life) demonstrated either adequate or proficient skills reading graphs and 50% of students with
visual experiences were proficient at producing a tactile graph. This result does not indicate that students who are congenitally blind are unable to have proficient graphing skills, only that students who have had no visual experiences will need additional accommodations and education to ensure success with learning to read and produce graphs. Instruction should be provided through excellent access to materials and explicit instruction on techniques to read and produce graphs. All students with visual impairments should be provided with an accessible copy of every graph that is presented during instruction.

Students who attended their local school while learning the mathematical concept of function demonstrated a higher proficiency in producing graphs (75% for students who attended their local school and 25% for students who attended a state school for the blind). Caution should be exercised here due to the fact that numerous variables are at play besides the school placement of the students. For instance, two of the four students who attended their local school and who were proficient with producing graphs were college students and were therefore older and had more mathematics coursework. Also, two of the students from the school for the blind who did not show proficiency in producing graphs lost their vision at the age of 13. These students were learning about linear functions for the first time while adjusting to the loss of sight and while learning braille.

Proficiency with reading graphs was 100% for participants who attended high school at a school for the blind and 75% for participants who attended a local high school. The fact that students across the board were able to demonstrate graph reading skills and yet there was a large discrepancy between students who attended a school for
the blind and those who attended their local school in the area of producing graphs highlights the need for further research on the methods of teaching graphing at both local schools and residential schools for the blind. Graphing skills, both in reading and producing graphs, are vital for students’ understanding of linear functions. Slavit (1997) indicates that it is through the development of graphing skills that students become aware of functions as objects (i.e. students are able to reify the function concept).

This study demonstrates that students who read braille and who access graphs through tactile means were slightly less likely to have proficiency with reading graphs than students who read large or regular print (67% vs. 100%, respectively). One out of the two (50%) of participants who read large or regular print demonstrated proficiency with producing graphs while four out of the six (67%) participants who read braille demonstrated adequacy or proficiency with producing graphs. These results indicate that no matter the reading medium, students with visual impairments are capable of learning to read and produce graphs.

Research Question 3: What are the representational preferences of students with visual impairments when solving word problems involving functions?

Students were asked which representation was used most often during instruction and which representation they preferred to use when solving word problems involving linear functions. This study shows a strong connection between students’ stated representational preference and the representation they perceived as being used most often during instruction. In fact, all participants who gave a stated representational preference also listed this preference(s) as ones used most often during instruction. The
connection between the students’ stated preference and those used most often during instruction is consistent with other research that has linked student preference to what students believe to be “mathematical” based on what is emphasized by instructors (Keller & Hirsch, 1998; Friedlander & Tabach, 2001; Akkus & Cakiroglu, 2006; Herman, 2007). The stated representational preference of students did not have as clear of connection to the representations they chose to use as did the representations used most often during instruction.

The stated representational preference of students matched what they chose to use while completing mathematical problems on the FKA and FCA approximately 50% of the time. One reason for this low percentage of students utilizing their stated representational preference is that results from the current study show that participants believed it necessary to solve or attempt to solve the problem in the representation in which it was presented. The perceived need to solve a problem using the given representation is consistent with Friedlander and Tabach (2001) who believed that “presenting various parts of a problem situation in different representations encourages flexibility in students’ choice of representations in their solution path” (p. 176).

Promoting student use of multiple representations was the main tenant in developing the two mathematics assessments. Although it was expected that students would use a variety of representations when solving the mathematics problems, it was striking how persistent they were in finding a way to use the given representation. For instance, for problem number 8 on the FKA students are asked, “Which week will Adam have $287?” From previous questions regarding Adam’s savings, all participants knew that Adam’s savings increased by seven dollars each week. However, five students used the information given
on the table of values for Adam’s savings and either expanded the table or found how much additional money he needed from the end of the table to reach the goal of $287 before dividing by seven and arriving at the answer through an additional couple of steps. One student wrote an equation to solve this problem and one student started the problem by reading the table of values before saying “Why am I even looking at the table?”

The conclusion in the present study is that other influences on students’ representational preferences were related to his or her past experiences with the representations as well as the overall impression he or she had of the representation in terms of how they might access the information. First, students expressed personal experiences or ideas about the use of various representations, which is also addressed in the research of Akkus and Cakiroglu (2006). Four participants from the current study listed their most challenging topic in mathematics as learning to draw or read graphs or working with geometric (i.e. visual) components of mathematics. Robert stated that working with equations was “easier than doing graphs,” and Liz commented that “tables or equations beats the graph.” Caleb’s stated representational preference was graphs, unless “they are too messy,” and Gabe stated that his preference was to either use equations or tables but that he doesn’t mind graphs and can read graphs, but sometimes he has a hard time interpreting graphs. Second, students had overall impressions of the various representations and how accessible they were for students with visual impairments. One participant stated that for students with visual impairments it is best to give them the most information in as small of a package as possible. For this reason the student chose to use the equations for Diane’s and Matthew’s savings when solving questions from the FKA: Part 2. Another student was surprised that I would be willing or
even able to provide a table in braille of the fifty-two weeks of Diane’s and Matthew’s savings.

**Research Question 4: What factors do high school/college students with visual impairments perceive as influencing the development of their mathematical understanding?**

The data used to determine what factors influenced participants’ mathematical understanding were self-reported through the MEEVA interview. Statements made by participants that related to their mathematical understanding were coded as providing the student with either *individualization* or *concreteness*. The majority of statements made by students related to individualization and included things such as one-on-one instruction, general classroom support that was often provided by an aide or an in-class writer, the importance of having an understanding or patient teacher, and the need for clear communication by teachers and tutors. One-on-one instruction was stated as a necessity by students 23 times, which was the most any single accommodation was mentioned. Students emphasized their need to either meet with their instructor after class or during office hours and how extremely helpful it was to work with a tutor for their mathematics work.

Concreteness was also discussed by students mainly in terms of being provided with accessible materials such as textbooks in braille and tactile graphs. The need for access to appropriate tools was discussed by all students. Although participants were well equipped with standard tools for education for students with visual impairments such as
screen-reader and screen-magnification software, braillewriters, and braille notetakers, students noted limitations of these technologies.

Students with visual impairments should be given opportunities for one-on-one instruction with their teacher. The importance of individual educational opportunities cannot be overstated. Individual instruction should be provided for students who attend local schools as well as those educated at state schools for the blind. I recognize that individualized education is easier to arrange at a school for the blind, but that does not mean that students with visual impairments who attend their local school should be denied the necessity of one-on-one educational opportunities. Individual education should also not solely be relegated to teachers of the visually impaired or to students’ aides. Mathematics educators are responsible for the mathematics learning of all of their students. Therefore, general education teachers should seek ways to actively engage their students with visual impairments in the classroom and to arrange times to meet with students one-on-one.

Mathematics educators should be aware of and sensitive to the limitations of technology used by students with visual impairments. Most notably is the limitation of graphing tools for students who access graphs through touch. Arrangements should be made to fully include students with visual impairments into all aspects of the mathematics curriculum, including sections that are typically taught through the use of a graphing calculator.
Discussion

*Revealing Student Knowledge Through Translations*

Problems that have students translate from one mode of representation to another have been shown to help educators diagnose students’ learning difficulties and to identify instructional opportunities (Lesh, Post, & Behr, 1987). The current research reinforces the knowledge that student reasoning when performing translations is an excellent way to access overall understanding of the function concept. Responses to translation tasks in the current study revealed both deep understanding and incomplete or inaccurate reasoning about linear function.

*Flexibility When Translating from One Mode of Representation to Another*

Developing good problem solving skills is a key objective in mathematics education (Usiskin, 1986, 1993, 1997; NCTM, 2000) and research has shown that good problem solvers tend to be more flexible in their use of a variety of representations (Lesh et al., 1987). Thus, it is important to ensure that students are given appropriate instruction in translating linear functions from one mode of representation to another. Findings from the current study reveal that students who persisted in solving (or attempting to solve) problems within the given representation as opposed to switching representations, when appropriate, were the same students who displayed difficulty when solving word problems. On the other hand, two students were flexible in their use of representations and demonstrated a high success when solving word problems.
Conceptual Understanding

Learning with understanding, or conceptual understanding, implies knowledge that is linked to form an integrated view of a concept. Conceptual learning should be emphasized over rote memorization or learning rules and facts in isolation (NCTM, 2000; Kilpatrick et al., 2001; Heibert, 1986; Skemp, 1989). Participants in the current study demonstrated a wide range of conceptual understanding of linear function. Determining students’ conceptual understanding can be difficult sometimes due to the need to access student reasoning. Students can be successful with a response through understanding that is isolated and not conceptualized. Therefore, a correct response, in and of itself, does not imply conceptual knowledge.

The current study was well placed to access student conceptions because of the students voicing their responses and being promoted to explain their reasoning. An example of an accurate response that was not contextualized came from two students when determining Diane’s savings based on the set of graphs presented in problem 11 on the FKA. Diane’s savings decreased and therefore had a negative slope. Both students correctly identified the savings as decreasing, but did so by voicing the memorized saying that “left means less,” implying that graphs that “go up and to the left” means that the value is decreasing.

Graphing Technologies

The effectiveness of graphing technologies in teaching and learning mathematics has been debated over the last two and a half decades (Kaput, 1992; Schwarz & Dreyfus, 1995; Usiskin, 1997; O’Callaghan, 1998). However, there seems to be a consensus that, when coupled with effective pedagogy, computer and calculator use for instruction in
mathematics is not only beneficial, but necessary for today’s classrooms (Usiskin, 1986, 1993, 1997; Schwarz & Dreyfus, 1995; Moreno-Armella et al., 2008). There is also a position put forth that indicates technologies such as computer algebra systems, graphing calculators, and dynamic geometry software, is allowing access to all students, thus making mathematics education equitable. This position does not take into account students who are blind or those with visual impairments who have no access to graphing or dynamic software.

Property oriented view of function

An alternative instructional method advanced for encouraging student reification of the function concept is through awareness of growth properties, both local and global in nature, of various classes of function (Slavit, 1997). It is also put forth that the use of graphing technologies and access to graphic forms of functions encouraged knowledge of function properties across function classes. The current study revealed a lack of access for one participant, a high school student who recently finished a Pre-Calculus course. He was unable to participate in most of the course activities involving manipulating functions and determining properties and transformations occurring when equations were modified. The lack of resources in the area of graphing for students with visual impairments and the current emphasis on the use of graphing technologies that are not accessible to students with visual impairments are causing an inequitable education in mathematics for this student population.
Implications for the Mathematical Education of Students with Visual Impairments

Explicit instruction in reading graphical information is needed in the mathematics classroom for students with visual impairments. Participants in the present study demonstrated a wide range of abilities in reading graphs. Students who approached graphical representations with confidence and through the use of specific techniques demonstrated a higher level of accuracy in their responses. A technique that was effective for participants was the use of two hands when locating particular points. One hand was used to read the information and the other was used as a guide to help the student move across the page in a straight line to either the x-axis or y-axis. A second technique used by participants who were successful with reading graphs was to gain an overview of the graph before attempting to find individual points or particular details on the graph. One stumbling block for participants who read graphs tactually was obtaining the parameters of the graphs. Each of the six students in the current study who read tactile graphs went to the last point marked on the scale of the x-axis and then moved their reading finger up and found the last plotted point on the graph. The students then assumed that these two points corresponded. There was no thought to the possibility that the scale went further than the plotted data points.

Skills with producing graphs should be emphasized for students with visual impairments. Participants who were proficient at producing graphs were also the students who were able, at some level, to reify the function concept. A technique used by two participants who constructed graphs using a graph board was to mark one of their axes with pins that were of a different shape from those marking the plotted points. Having the axis marked allowed the students to have a base to return to in order to check on the
alignment of the pins. The use of a marked axis also served to provoke thought on the use of an appropriate scale for the graph being constructed.

Although the marking of the scale was beneficial to some students, the majority of participants plotted a point based on the pin that had been previously placed, as opposed to making reference to the scale. The method of placing pins based on previous points was a good technique except in the case of placing the first pin. Two participants who constructed tactile graphs were able to produce an accurate graph only after I helped them with the placement of the first point. One participant who used a point-by-point method of constructing a graph was well served by knowing the graph he was constructing would be linear and that the points would be evenly spaced. He used this information to informally measure the space between each plotted point by placing two fingers together in between two pins to determine if the pins were evenly spaced. Plotting one point based on a previous point could allow for a good understanding of slope for linear functions in terms of calculating the direction and magnitude separating each point. However, a downside to the technique of a point-to-point construction of a graph is the tendency for the student to not have a clear picture of the overall graph. Students successful with producing graphs combined a point-to-point approach with a constant “review” of the overall graph by running one hand over the entirety of the graph after each point they plotted.

Students should be encouraged to become independent with accessing and producing graphs. Independence can be promoted through the availability of appropriate materials for producing graphs, being provided with an accessible copy of any graphic being discussed during instruction, and providing a way for students to construct graphs
without the need for assistance on assignments and examinations. Materials for producing graphs may include large print or braille graph paper, a graph board with push pins of a variety of shapes and sizes, string, tracing wheel with braille paper, and Wikki Sticks. Time should also be allotted for manually creating graphs with these materials. Participants in the present study emphasized the need to have an accessible copy of the graphs being discussed in class. To allow for graphs that are not a part of a pre-planned lesson (i.e. addressing students’ questions from a homework assignment) educators should have a way to quickly produce informal graphs for their students with visual impairments. This can be done through the use of Wikki Sticks or a tracing wheel. Four participants indicated that at some point in their education the only way they were able to produce a graph for an assignment or an examination was to describe it to a sighted proctor and to have him or her draw the graph. It is unacceptable practice for students to lack appropriate tools to independently produce graphs. Time and materials should be provided in all examinations to allow for independent access to graphs.

Educators should ensure understanding in all aspects of working within a single representation of a function. Participants in the present study showed high competency in reading information given in a tabular form, but in some cases showed great difficulty in producing tables. Participants also stated that they preferred to use equations when solving word problems, yet there was a large discrepancy in students’ abilities in using the equation effectively to obtain the final result. This result is consistent with the research of Herman (2007) and Keller and Hirsch (1998). Teachers should be accepting and encouraging of using various strategies to solve problems, rather than focusing on equations.
Students with visual impairments should be provided with multiple opportunities to translate between representational forms of a function. Instruction should highlight the relationships between the representations and characteristics regarding the advantages and disadvantages of each representation. Participants who could flexibly switch from one representation to another were able to demonstrate a higher competency in modeling and solving word problems involving linear functions. To promote the use of and connections between the various representations of function, I recommend using lessons and activities that give problem information using the various forms of functions. One mathematics assessment used in the present study, the FKA, is an example of a problem based in context that promotes the use of multiple representations.

Individualized education opportunities should be promoted for supplemental instruction for students with visual impairments. I emphasize supplemental instruction because results from the current research indicate that students who attend local schools and are integrated within a “typical” mathematics classroom can succeed (and will sometimes flourish) in this environment. However, all participants in the present study noted how important it was for them to receive one-on-one instruction. Individualized instruction for participants of this study came in a variety of forms including hiring a tutor, having an aide or a teacher of the visually impaired in the classroom, arranging to meet after class with the instructor, or having close access to the instructor because of small classroom size.

Students with visual impairments should be provided with materials that are accessible, in a timely manner, for all course content. Participants of the current study discussed the advantages gained by having an accessible copy of class notes to review
during lecture. However, these same participants noted how infrequently the practice occurred. Participants noted that they would have access to the graphs drawn in class if they were included in the textbook, but otherwise they were not provided with an accessible copy.

Educators should always be complete and explicit when relaying what has been written on the board. Mathematics is a spatial language and can be read using ambiguous statements. For instance, when an educator states, “x minus one over two” this expression can be conceptualized in two different ways. The expression could mean “x minus one all over two” or “x minus one-half.” If the material being read is unclear to the student with a visual impairment, it is impossible for that student to independently take notes. One participant from the current study mentioned how his teachers’ communication skills lead him to have “a lot of messed up equations.” On the other hand, this same participant relayed a story where another teacher practiced being explicit when speaking the mathematics being written on the board and by the end of the quarter the student was able to be completely independent in his note taking.

Ensuring accessibility of materials in the mathematics classroom for students with visual impairments includes the use of concrete manipulatives and alternative instruction when inaccessible technology is being used. All instruction needs to be made accessible to students with visual impairments. Students should not be excused from learning a mathematics topic because of the difficulties involved with making the materials accessible. The heavy reliance on graphing and computer technologies within the mathematics classroom can make it difficult for educators to ensure accessibility to all course content. However, providing opportunities for students with visual impairments to
be actively engaged in learning is essential for promoting students’ mathematical understanding.

**Recommendations for Future Research**

There is a dearth of research in the area of mathematics learning for students with visual impairments. The shortage of research offers a wide range of opportunities to expand the research discussed in this paper as well as to address additional mathematics education topics for students with visual impairments. Specific suggestions follow.

The current research needs to be expanded to include students in other areas of the United States. Research for the present study was conducted in the mid-west. A wider sample of students is also needed in order to corroborate the results of the current research. Students from local schools and those from residential schools for the blind across the country could add to the depth of the research reported here.

A more detailed analysis of students’ educational needs in the specific area of linear functions is needed. The present study addressed the current understanding of linear function held by students with visual impairments. Research results indicate weaknesses in the areas of knowledge of translating between various representational forms of a function and reifying the function concept. Therefore, research is needed in order to determine effective instructional strategies and methodologies that promote skills in translating between representational forms of a function and the reification of the concept of function.

Skills in both reading and producing graphs were shown to be an important aspect in understanding functions for students with visual impairments. Participants of the current study demonstrated a wide range of approaches and techniques of reading and
producing graphs. Further research is needed to determine more effective strategies for reading and constructing graphs and effective instructional strategies to promote the use of such strategies.

A review of the current instructional methodologies used for students with visual impairments learning the concept of function would also be helpful to the field. Such a review would be enhanced by comparing the current classroom practices of students taught in local schools and those taught at residential schools for the blind. The present study addressed the issue instructional methodologies used in the mathematics classroom, but did so solely based on student perception. A full review of the current techniques used in teaching the concept of linear functions as well as whether or not an emphasis is placed on the use of multiple representations needs to include classroom observations and information from the mathematics instructors.

Techniques for promoting the reification of the function concept and the implications of reification on learning have been researched extensively for sighted students. There is a need for such research to be conducted for students with visual impairments. Some of the research with sighted students has focused on using properties of graphs to promote reification of functions (Slavit, 1997, 2006). Due to the inherent differences in accessing graphical information for students who read tactile graphs, research addressing whether the same recommendations for promoting reification apply for students with visual impairments, or if there are other techniques that would be more effective with this population would greatly add to the literature on the learning of functions for this student population.
Research on the implications for learning the function concept through accessing content via the Nemeth Code, which presents mathematical information in a linear format, is also needed. Three of the four main representations of functions, equations, tables, and graphs, all have a visual, non-linear format. Studies conducted on the linear format of braille and how this affects learning the concept of function would be enhanced through a comparison of students who are congenitally blind and have only accessed mathematics through braille with those who lost their sight after beginning school mathematics and therefore has experience with the visual/spatial aspects of mathematics as well as the linear mathematics through braille.

Through interviews regarding students’ mathematics learning experiences, the current study has revealed a need for research in the area of geometry education for students with visual impairments. Fifty percent of the students from this study indicated that either graphing or learning geometric concepts was the most difficult for them in their mathematics education. A review of effective methodologies in teaching spatial components of mathematics to this student population is clearly needed.

Another research opportunity in the area of mathematics education for students with visual impairments is a review of the accessibility of the curriculum for various high school and college mathematics courses. One student in the current study indicated a deprivation in his mathematics education due to poor accommodations and an unmodified curriculum. For this particular student the main difficulties arose in his Pre-Calculus course and closely corresponded to the amount of time instruction was centered on the use of graphing calculators. However, there are other aspects of mathematics education that are not always accessible to students with visual impairments even in
lower level mathematics courses. A review of mathematics courses and needed accommodations for students with visual impairments would be a valuable addition to the field of education for students with visual impairments.

Research opportunities in mathematics education for students with visual impairments abound. The suggestions listed here are only the beginning of what can and should be done in the field.
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APPENDIX A

Mathematics Education Experience and Visual Abilities (MEEVA) Interview Protocol

1) For High School; What grade are you currently in? For College; What year are you in your program?

2) For College: What is your major? For High School: What are you interested in doing when you are finished with high school? Do you plan to attend college? What do you plan to major in?

3) When did you lose your vision?

4) How did you lose your vision? (i.e. gradually over time or all at once)

5) Where do you attend school? Where have you attended in the past?

6) Were you ever educated separately from your classmates? (i.e. pulled out of class to work with a Vision Intervention Specialist, took a class at the school for the blind)

7) Do you read Braille? Large print?

8) What were/are some of the supports the school provides that are specifically related to your visual impairment/blindness?

9) What technology do you use? (Zoomtext, JAWS, Braille-note, etc.)

10) Do you have a favorite subject or favorite class? What is it that you like about this class?
11) What classes have you struggled with? Why do you think these classes were so difficult for you?

12) What is your impression of mathematics? Do you enjoy doing math? Overall would you say you have had good or bad experiences in mathematics classes?

13) What tools and/or accommodations were you given specifically for helping you to learn mathematics?

14) How did your teacher(s) introduce you to graphing?

15) Do you feel you have a good understanding of linear functions?

16) Did you use many tactile graphs when learning mathematics? Are you comfortable with reading these?

17) How did you take your mathematics exams? Did you have a brailled/large print version? Did a proctor work with you to read the questions and write your responses?

18) What was most challenging for you when learning mathematics?

19) Was there anything particularly difficult for you when learning algebra?

20) How were your mathematics classes structured? (Lecture, group work, projects, papers, etc.)

21) How do you learn best?

22) In what ways did your mathematics teachers help to include you into the class? Did they make any special arrangements? How did they handle situations when they pointed to something on a graph or wrote on the board? How did you access that information?

23) Is there a certain way you prefer to access the information presented on the board? (have a student read you what has been written, have a vision specialist with you in class, have the teacher describe what he or she is pointing to or writing, etc.)

24) How do you take notes in class? Do the teachers provide raised-line diagrams of what is being discussed in class that day?
25) Functions can be represented as an equation, a graph, or in tabular form (a set of (x,y) coordinates). Do you remember studying all three of these forms?

26) Which form did you work with most often?

27) Do you have a preference for one representation over another when solving problems involving functions?

28) How did you produce graphs while in your mathematics (and/or science) classes?

29) What did your teacher do to help you learn functions that he/she did not do for the rest of the class?

30) Do you feel you were able to get a good understanding of functions and their various representations in your algebra class?
APPENDIX B

Function Knowledge Assessment

Four children will save their money for one year (52 weeks). The following are descriptions of the four children’s savings. Use the information from these descriptions to answer the following questions.

Adam: The table shows how much money Adam had saved at the end of each week. The table continues in the same way for the rest of the year.

<table>
<thead>
<tr>
<th>Week</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>140</td>
</tr>
<tr>
<td>21</td>
<td>147</td>
</tr>
<tr>
<td>22</td>
<td>154</td>
</tr>
<tr>
<td>23</td>
<td>161</td>
</tr>
<tr>
<td>24</td>
<td>168</td>
</tr>
<tr>
<td>25</td>
<td>175</td>
</tr>
<tr>
<td>26</td>
<td>182</td>
</tr>
<tr>
<td>27</td>
<td>189</td>
</tr>
<tr>
<td>28</td>
<td>196</td>
</tr>
</tbody>
</table>

Heather: Heather kept her savings at $300 throughout the year.

Diane: Diane’s savings can be described by the expression 300 - 5x, where x stands for the number of weeks.

Matthew: The graph describes Matthew’s savings for several weeks near the beginning of the year. His savings continues in the same way for the rest of the year.
Part 1:

1) How much money will Diane have after the 8\textsuperscript{th} week?

2) How much money will Adam have after week 22?

3) How much money does Matthew have after week 18?

4) How much money will Matthew have on week 30?

5) How much money did Adam have on the 10\textsuperscript{th} week?

6) How much money did Heather have at the end of the year?

7) Matthew will have $150 on which week?

8) Which week will Adam have $287?

9) When will Diane have $200?

10) Compare the savings of two out of the four children. Describe how the children’s savings are changing throughout the year. In your comparison you can talk about whether or not the savings is increasing or decreasing, which has a faster rate of change, which child has the largest amount at the end of the year, etc.
11) Given the graphs of the savings of all four children throughout the year, identify each graph and find the meaning and the value of each intersection point.

Graph A:

Graph B:

Graph C:

Graph D:
12) Write an equation for Matthew’s savings.

13) Write an equation for Adam’s savings

14) Write an equation for Heather’s savings.

15) Whose savings is growing the fastest?

16) Whose savings is growing the slowest?

Part 2:

I would like for you to compare the savings of Diane and Matthew by answering some specific questions. I have with me a side-by-side table showing Diane and Matthew’s savings throughout the year, a graph that shows their savings, an equation for Diane and Matthew’s savings, and a verbal description of each of their savings. Which of these four representations would you like to use in order to answer questions about these accounts?

1) How much had Matthew saved after half a year? How much did Diane have at the same time?

2) After how many weeks did each of the two children have $210?

3) Find the week with the largest difference between their savings.

4) Find the week when their savings were equal.

5) Find the week when the savings of one were double that of the other. In whose favor?
6) Diane and Matthew decided to pool their savings in order to buy a $400 walkie-talkie. Find the week in which they are able to purchase the walkie-talkie.

**Part 3:**

Another child’s savings is as described below.
Ellen received her allowance in the following way: On the first weekend, she got two cents. Every subsequent weekend, she received an amount identical to the amount she had left in her savings box the previous week. Ellen saves all the money she gets.

1) How much money had Ellen saved after 10 weekends?

2) How is Ellen’s savings different from that of the other four children?
APPENDIX C

Function Competency Assessment (FCA)

1) A truck is loaded with boxes, each of which weighs 20 pounds. If the empty truck weighs 4500 pounds, find the following. (O’Callaghan, 1998)

a) The total weight of the truck if the number of boxes is 75.

b) The number of boxes if the total weight of the truck is 6,740 pounds.

c) Using W for the total weight of the truck and x for the number of boxes, write a symbolic rule (or equation) that expresses the weight as a function of the number of boxes.

2) The graph below gives the speed of a cyclist on his daily training ride. During his ride, he must climb a hill where he pauses for a drink of water before descending. Using this graph to answer the following questions as accurately as possible. (O’Callaghan, 1998)

a) Find the speed when time equal 25 minutes.
b) Find the time when speed equals 30 mph.

c) During what time intervals was the speed increasing?

d) During which 10-minute interval did the speed decrease the most?

e) When was the cyclist at the top of the hill?

3) The table below gives the average price of a new home in Smalltown, USA, for every 2 years since 1980: (O’Callaghan, 1998)

<table>
<thead>
<tr>
<th>Year</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>50,000</td>
</tr>
<tr>
<td>1982</td>
<td>53,000</td>
</tr>
<tr>
<td>1984</td>
<td>57,500</td>
</tr>
<tr>
<td>1986</td>
<td>62,000</td>
</tr>
<tr>
<td>1988</td>
<td>67,000</td>
</tr>
<tr>
<td>1990</td>
<td>70,000</td>
</tr>
<tr>
<td>1992</td>
<td>74,000</td>
</tr>
</tbody>
</table>

a) What would you predict for the price in 1995?

b) When would you predict the price to be $100,000?

c) Draw the graph of the data.

4) Suppose that the following table gives the value (v), in dollars, of a car for different numbers of years (t) after it is purchased. (O’Callaghan, 1998)

<table>
<thead>
<tr>
<th>t</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$16,800</td>
</tr>
<tr>
<td>2</td>
<td>$13,600</td>
</tr>
<tr>
<td>4</td>
<td>$10,400</td>
</tr>
<tr>
<td>6</td>
<td>$7,200</td>
</tr>
<tr>
<td>8</td>
<td>?</td>
</tr>
</tbody>
</table>
a) What is the value of the car after 8 years?

b) Write a symbolic rule expressing $v$ as a function of $t$.

5) Mary Wong just got a job working as a clerk in a candy store. She already has $42. She will earn $7 per hour. (Brenner et al., 1997)

   a) How many hours will she have to work to have a total of $126?

   b) Make a table or graph to represent the problem.

6) Two carnivals are coming to town. You and your friend decided to go to different carnivals. The carnival that you attend charges $10 to get in and an additional $2 for each ride. The carnival your friend attends charges $6 to get in, but each additional ride costs $3. If the two of you spent the same amount of money, how many rides could each of you have ridden? (Slavit, 2006)

7) There are two children who save their money in the following way:

   Helen has a set amount of money which does not change. Her savings can be described as $y=200$
   Tommy has some income from a lawn service business. His savings can be described with the equation $y=5x-2$.

   Without solving or graphing these equations, can you tell if there will be a time when Helen and Tommy have the same amount of money? How can you tell?
APPENDIX D

Instrument Validation Study:

Function Knowledge Assessment: Version 1 (FKA: V1)

The following are descriptions of four children’s savings. Use the information from these descriptions to answer the following questions.

**Adam:** The table shows how much money Adam had saved at the end of each week. The table continues in the same way for the rest of the year.

<table>
<thead>
<tr>
<th>Week</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>140</td>
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<tr>
<td>26</td>
<td>182</td>
</tr>
<tr>
<td>27</td>
<td>189</td>
</tr>
<tr>
<td>28</td>
<td>196</td>
</tr>
</tbody>
</table>

**Heather:** Heather kept her savings at $300 throughout the year.

**Diane:** Diane’s savings can be described by the expression 300-5x, where x stands for the number of weeks.

**Matthew:** The graph describes Matthew’s savings for several weeks near the beginning of the year. His savings continues in the same way for the rest of the year.
Part 1:

1) How much money will Diane have after the 8th week?

2) How much money will Adam have after week 22?

3) How much money does Matthew have after week 18?

4) How much money will Matthew have on week 30?

5) How much money did Adam have on the 10th week?

6) How much money did Heather have at the end of the year?

7) Matthew will have $140 on which week?

8) Which week will Adam have $287?

9) When will Diane have $200?

10) Describe in words how the savings of each child changes through the year

   Adam:

   Heather:

   Diane:

   Matthew:
11) Given the graphs of the savings of all four children throughout the year, identify each graph and find the meaning and the value of each intersection point.

Graph A:

Graph B:

Graph C:

Graph D:
12) Write an equation for Matthew’s savings.

13) Write an equation for Adam’s savings.

14) Write an equation for Heather’s savings.

15) Compare the savings of two out of the four children (using words like “the savings increase (or decrease),” “the savings increase or decrease at a rate of…,” “who has a larger (or smaller) amount at the beginning (or end),” and “larger (or smaller) by…, double…, equal.” Also, use tables, graphs, expressions, and explanations.

16) Add another child to your comparison.

17) Whose savings is growing the fastest?

18) Whose savings is growing the slowest?

Part 2:

I would like for you to compare the savings of Diane and Matthew by answering some specific questions. I have with me a side-by-side table showing Diane and Matthew’s savings throughout the year, a graph that shows their savings, an equation for Diane and Matthew’s savings, and a verbal description of each of their savings. Which of these four representations would you like to use in order to answer questions about these accounts?

1) How much had Matthew saved after half a year? How much did Diane have at the same time?

2) After how many weeks did each of the two children have $210?

3) When was the difference between their savings $60? In whose favor was the difference?

4) Find the week with the largest difference between their savings.

5) Find the week when their savings were equal.

6) Find the week when the savings of one were double that of the other. In whose favor?

7) Diane and Matthew decided to pool their savings in order to buy a $400 walkie-talkie. Find the week in which they are able to purchase the walkie-talkie.
APPENDIX E

Instrument Validation Study:

Function Knowledge Assessment: Version 2 (FKA: V2)

The following are descriptions of four children’s savings. Use the information from these descriptions to answer the following questions.

Adam: The table shows how much money Adam had saved at the end of each week. The table continues in the same way for the rest of the year.

<table>
<thead>
<tr>
<th>Week</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount</td>
<td>140</td>
<td>147</td>
<td>154</td>
<td>161</td>
<td>168</td>
<td>175</td>
<td>182</td>
<td>189</td>
<td>196</td>
</tr>
</tbody>
</table>

Heather: Heather kept her savings at $300 throughout the year.

Diane: Diane’s savings can be described by the expression 300-5x, where x stands for the number of weeks.

Matthew: The graph describes Matthew’s savings for several weeks near the beginning of the year. His savings continues in the same way for the rest of the year.

![Graph showing weekly savings increase](chart.png)
Part 1:

1) How much money will Diane have after the 8th week?

2) How much money will Adam have after week 22?

3) How much money does Matthew have after week 18?

4) How much money will Matthew have on week 30?

5) How much money did Adam have on the 10th week?

6) How much money did Heather have at the end of the year?

7) Matthew will have $140 on which week?

8) Which week will Adam have $287?

9) When will Diane have $200?

10) Describe in words how the savings of each child changes through the year

Adam:

Heather:

Diane:

Matthew:
11) Given the graphs of the savings of all four children throughout the year, identify each graph and find the meaning and the value of each intersection point.

Graph A:

Graph B:

Graph C:

Graph D:
12) Write an equation for Matthew’s savings.

13) Write an equation for Adam’s savings.

14) Write an equation for Heather’s savings.

15) Compare the savings of two out of the four children (using words like “the savings increase (or decrease),” “the savings increase or decrease at a rate of…,” “who has a larger (or smaller) amount at the beginning (or end),” and “larger (or smaller) by…, double…, equal.” Also, use tables graphs, expressions, and explanations.

16) Add another child to your comparison.

17) Whose savings is growing the fastest?

18) Whose savings is growing the slowest?

Part 2:

I would like for you to compare the savings of Diane and Matthew by answering some specific questions. I have with me a side-by-side table showing Diane and Matthew’s savings throughout the year, a graph that shows their savings, an equation for Diane and Matthew’s savings, and a verbal description of each of their savings. Which of these four representations would you like to use in order to answer questions about these accounts?

1) Find the week with the largest difference between their savings.

2) Find the week when their savings were equal.

3) Find the week when the savings of one were double that of the other. In whose favor?

4) Diane and Matthew decided to pool their savings in order to buy a $400 walkie-talkie. Find the week in which they are able to purchase the walkie-talkie.
Part 3:

Another child’s savings was as described below.

Ellen received her allowance in the following way: On the first weekend, she got two cents. Every subsequent weekend, she received an amount identical to the amount she had left in her savings box the previous week. Ellen saves all the money she gets.

1) How much money had Ellen saved after 10 weekends?

2) How is Ellen’s savings different from that of the other four children?
APPENDIX F

Instrument Validation Study:

Function Competency Assessment (FCA)

1) The table below gives the average price of a new home in Smalltown, USA, for every 2 years since 1980: (O’Callaghan, 1998)

<table>
<thead>
<tr>
<th>Year</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>50,000</td>
</tr>
<tr>
<td>1982</td>
<td>53,000</td>
</tr>
<tr>
<td>1984</td>
<td>57,500</td>
</tr>
<tr>
<td>1986</td>
<td>62,000</td>
</tr>
<tr>
<td>1988</td>
<td>67,000</td>
</tr>
<tr>
<td>1990</td>
<td>70,000</td>
</tr>
<tr>
<td>1992</td>
<td>74,000</td>
</tr>
</tbody>
</table>

a) What would you predict for the price in 1995?

b) When would you predict the price to be $100,000?

c) Draw the graph of the data.
2) A truck is loaded with boxes, each of which weighs 20 pounds. If the empty truck weighs 4500 pounds, find the following. (O’Callaghan, 1998)

a) The total weight of the truck if the number of boxes is 75.

b) The number of boxes if the total weight of the truck is 6,740 pounds.

c) Using $W$ for the total weight of the truck and $x$ for the number of boxes, write a symbolic rule (or equation) that expresses the weight as a function of the number of boxes.
3) The graph below gives the speed of a cyclist on his daily training ride. During his ride, he must climb a hill where he pauses for a drink of water before descending. Using this graph to answer the following questions as accurately as possible. (O’Callaghan, 1998)

![Graph of speed vs. time](image)

- a) Find the speed when time equals 25 minutes.

- b) Find the time when speed equals 30 mph.

- c) During what time intervals was the speed increasing?

- d) During which 10-minute interval did the speed decrease the most?

- e) When was the cyclist at the top of the hill?
4) Suppose that the following table gives the value \(v\), in dollars, of a car for different numbers of years \(t\) after it is purchased. (O’Callaghan, 1998)

<table>
<thead>
<tr>
<th>(T)</th>
<th>(V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$16,800</td>
</tr>
<tr>
<td>2</td>
<td>$13,600</td>
</tr>
<tr>
<td>4</td>
<td>$10,400</td>
</tr>
<tr>
<td>6</td>
<td>$7,200</td>
</tr>
<tr>
<td>10</td>
<td>?</td>
</tr>
</tbody>
</table>

a) What is the value of the car after 10 years?

b) Write a symbolic rule expressing \(v\) as a function of \(t\).

5) A small company determines its contribution to charity \(C\) by its profit \(p\), which is dependent on the number of items \(n\) sold according to the following formulas:

\[ C = 10(p - 1000) \quad \text{and} \quad p = 100n - n^2 \]  

(O’Callaghan, 1998)

a) What will the company contribute to charity if it sells 50 items?

b) Write a formula expressing \(C\) as a function of \(n\).
6) Mary Wong just got a job working as a clerk in a candy store. She already has $42. She will earn $7 per hour. (Brenner et al., 1997)

   a) How many hours will she have to work to have a total of $126?

   b) Draw a table or graph to represent the problem.

7) Two carnivals are coming to town. You and your friend decided to go to different carnivals. The carnival that you attend charges $10 to get in and an additional $2 for each ride. The carnival your friend attends charges $6 to get in, but each additional ride costs $3. If the two of you spent the same amount of money, how many rides could each of you have ridden? (Slavit, 2006)
8) Solve the following system of equations: (Sfard & Linchevski, 1994)

\[2(x - 3) = 1 - y\]
\[2x + y = 7\]
APPENDIX G

Representations for the Function Knowledge Assessment: Part 2

All Versions

Participants were provided with either a table, a graph, the equations, or the descriptions of Diane’s and Matthew’s savings to assist with answering questions from the FKA: Part 2. These representations are presented, below.

<table>
<thead>
<tr>
<th>Week #</th>
<th>Diane’s</th>
<th>Matthew’s</th>
<th>Week #</th>
<th>Diane’s</th>
<th>Matthew’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>300</td>
<td>30</td>
<td>26</td>
<td>170</td>
<td>160</td>
</tr>
<tr>
<td>1</td>
<td>295</td>
<td>35</td>
<td>27</td>
<td>165</td>
<td>165</td>
</tr>
<tr>
<td>2</td>
<td>290</td>
<td>40</td>
<td>28</td>
<td>160</td>
<td>170</td>
</tr>
<tr>
<td>3</td>
<td>285</td>
<td>45</td>
<td>29</td>
<td>155</td>
<td>175</td>
</tr>
<tr>
<td>4</td>
<td>280</td>
<td>50</td>
<td>30</td>
<td>150</td>
<td>180</td>
</tr>
<tr>
<td>5</td>
<td>275</td>
<td>55</td>
<td>31</td>
<td>145</td>
<td>185</td>
</tr>
<tr>
<td>6</td>
<td>270</td>
<td>60</td>
<td>32</td>
<td>140</td>
<td>190</td>
</tr>
<tr>
<td>7</td>
<td>265</td>
<td>65</td>
<td>33</td>
<td>135</td>
<td>195</td>
</tr>
<tr>
<td>8</td>
<td>260</td>
<td>70</td>
<td>34</td>
<td>130</td>
<td>200</td>
</tr>
<tr>
<td>9</td>
<td>255</td>
<td>75</td>
<td>35</td>
<td>125</td>
<td>205</td>
</tr>
<tr>
<td>10</td>
<td>250</td>
<td>80</td>
<td>36</td>
<td>120</td>
<td>210</td>
</tr>
<tr>
<td>11</td>
<td>245</td>
<td>85</td>
<td>37</td>
<td>115</td>
<td>215</td>
</tr>
<tr>
<td>12</td>
<td>240</td>
<td>90</td>
<td>38</td>
<td>110</td>
<td>220</td>
</tr>
<tr>
<td>13</td>
<td>235</td>
<td>95</td>
<td>39</td>
<td>105</td>
<td>225</td>
</tr>
<tr>
<td>14</td>
<td>230</td>
<td>100</td>
<td>40</td>
<td>100</td>
<td>230</td>
</tr>
<tr>
<td>15</td>
<td>225</td>
<td>105</td>
<td>41</td>
<td>95</td>
<td>235</td>
</tr>
<tr>
<td>16</td>
<td>220</td>
<td>110</td>
<td>42</td>
<td>90</td>
<td>240</td>
</tr>
<tr>
<td>17</td>
<td>215</td>
<td>115</td>
<td>43</td>
<td>85</td>
<td>245</td>
</tr>
<tr>
<td>18</td>
<td>210</td>
<td>120</td>
<td>44</td>
<td>80</td>
<td>250</td>
</tr>
<tr>
<td>19</td>
<td>205</td>
<td>125</td>
<td>45</td>
<td>75</td>
<td>255</td>
</tr>
<tr>
<td>20</td>
<td>200</td>
<td>130</td>
<td>46</td>
<td>70</td>
<td>260</td>
</tr>
<tr>
<td>21</td>
<td>195</td>
<td>135</td>
<td>47</td>
<td>65</td>
<td>265</td>
</tr>
<tr>
<td>22</td>
<td>190</td>
<td>140</td>
<td>48</td>
<td>60</td>
<td>270</td>
</tr>
</tbody>
</table>
Diane’s savings: 300 - 5x

Matthew’s savings: 30 + 5x

Note: x represents the number of weeks.

Diane’s savings: Diane starts the year out with $300 and each week she spends five dollars. She does not add any money to her account.

Matthew’s savings: Matthew starts the year off with just $30, but each week he saves an additional five dollars.
APPENDIX H

Independent Auditor Contract

This contract is between David Moore and Heidi Cowan for the audit of Ms. Cowan’s research for her dissertation, *Knowledge and Understanding of Linear Functions held by Students with Visual Impairments*.

I, David Moore, will review the raw data, procedures, research notes, and final report of Heidi Cowan’s dissertation work entitled, *Knowledge and Understanding of Linear Functions held by Students with Visual Impairments*, and will provide a summative audit that is an independent and empirical review of these elements. Specifically, this audit will comment on the *dependability* and the *confirmability* of the research. The audit results will be provided in the form of a written report, which will be made available to the public, and will due two weeks prior to the defense date and must be submitted no later than the eve of the defense.

I, Heidi Cowan, will provide David Moore with an audit trail including the raw data, notes on procedure and analysis decisions, and research and analysis notes in a timely manner. I will also be in contact with Mr. Moore at least three times between providing him with the raw data and the final audit report and will be available for questions and concerns between these times. The conversations will center on any questions Mr. Moore has about my process, the raw data, or my interpretations of the data. A copy of Mr. Moore’s final audit will be placed in an appendix in the dissertation.

A sum of $___.__ will be given to Mr. Moore for his work on and completion of the above mentioned audit.

Signed,

Heidi Cowan    David Moore
APPENDIX I

Independent Audit Report

David Moore
August 29, 2011

Independent Auditor’s Report

Statement of Purpose

The purpose of this audit was to evaluate the Dependability and Confirmability of research conducted by Heidi Cowan for her dissertation entitled, Knowledge and Understanding of Function held by Students with Visual Impairments. I, David Moore, was asked by Ms. Cowan to review data and related documents from her research and to determine whether the results accurately reflect the data, are void of bias, and whether the study was conducted in a manner that promoted trustworthiness. My role in this research has been as an independent auditor. I was not involved with data collection or data analysis. The audit was conducted between May and August of 2011.

Statement of Scope

I reviewed all data from Ms. Cowan’s dissertation work for this audit. Data included student responses from eight participants for three research instruments.
including, a) the Mathematics Education Experiences and Visual Abilities interview, b) the Function Knowledge Assessment, and c) the Function Competency Assessment. I accessed raw data through audio recordings of the sessions. I reviewed all data and was not involved in study details or procedures. My removal from data gathering allowed me to independently consider the raw data and compare these to Ms. Cowan’s findings.

Ms. Cowan and I met in May of 2011 to determine audit procedures and elements for the audit. We also met twice more in order for Ms. Cowan to administer the mathematics tests to me. By completing the two mathematics assessments (the Function Knowledge Assessment and the Function Competency Assessment) I was better able to understand possible stumbling blocks for participants, which allowed me additional insights as I listened to student responses. After our initial meetings, Ms. Cowan and I communicated via e-mail and through phone calls. We talked on multiple occasions to discuss our opinions of the data and for me to gain clarity for any data that was unclear. Ms. Cowan sent me analysis data and conclusions as she completed these, which I reviewed after having finished listening to the raw data.

**Statement of Findings**

First of all, I firmly agree that this study accomplished by Ms. Cowan is Dependable and Confirmable. It has, in my opinion, *Knowledge and Understanding of Function held by Students with Visual Impairments*, as prepared by Heidi Cowan, stating the findings fairly and dependably. Based on my observations of the raw data from all three instruments compared to the findings of Ms. Cowan’s results and conclusions, I find this study as being very dependable and confirmable. I will now begin to back this finding up
with several examples based on my observation of all data compared to Heidi Cowan’s results and conclusions.

I investigated students’ comfort-level when solving problems involving interpretation of graphs. I found that there was a high comfort-level. Most of the students were comfortable with playing around with Matthew’s graph. Gabe seemed to have a good time figuring out how much money Matthew would have on week 18. Braille readers had a very difficult time matching the X values to the Y values. They had to feel on the X-axis and go up with their finger as straight as they could. Many talked about this; however, they were comfortable playing around. Amanda and Wanda, the two students with some sight, read the graph immediately. Robert surprised me at how much trouble he had reading Matthew’s graph. He could not figure out why there were not as many ticks as points on the graph. He thought that points were missing when they were really there; however, there is not enough room to label all of them on a tactual graph. Robert acted like he had never had read a graph like this one.

Ms. Cowan found that the success and comfort levels of translating from one representation to another had nearly as many low codes to high ones. Robert had no idea how to write an equation for Matthew’s graph. Robert is in college having taken pre-calculus. Robert could not remember the slope intercept form for a straight line. He never mentioned slope. He does not realize for a while that Matthew’s graph is linear. Ms. Cowan tried to give him hints; however, Robert did not want to begin trying to figure out how the equation should look like.

Now, I will proceed to function competency. Interpreting comprehension was low on problem FCA; 2d. Liz did not know that increasing is shown on the graph as going up
and to the right. She did not know that increasing is written as intervals even though Ms. Cowan asked her to find the intervals of increasing. Liz kept mumbling single numbers. Again, this really surprised me, because she is in college. Sixty three percent found by Ms. Cowan I totally agreed with. With FCA; 2b, participants did not consider that there were two answers for the same speed. This surprised me, because there were two humps in the graph. At times, Ms. Cowan would ask them if that was the only answer. If she asked that question, some of the participants would think about it and attempt to give the second answer. However, most participants would give their only answer so quickly and confidently that Ms. Cowan let them go on. This also agrees with her data. Ms. Cowan found that interpreting-application was very low for problem FCA; 2e at 38 percent. I totally agreed. Most participants did not know that if the person is at the top of the hill, the speed will be 0. Now where is the speed 0? It is where the graph touches the X-axis. This graph touches three different times. Gabe was a participant that did not get mixed up with what the shape of the graph was saying. Most participants thought that the top of the hill was represented by where the tops of the humps were on the graph. They forgot that the Y-axis represented speed. Therefore, the tops of those humps represented maximum speeds instead of being at the top of the hill. This is where Gabe kept his head and realized that those maximums were speeds. Also, Gabe knew that the speed would be 0 on the X-axis. I agreed with the data concerning this as well.

When it came to translating from one representation to another, Like I mentioned earlier in this audit, this was a big problem as shown by Ms. Cowan in Table 4.21. My biggest surprise was when I listened to the interviews concerning when participants chose to represent FCA; 5b as a table. All participants had no problem reading a table;
however, when it came to constructing the table, many had no idea how to start. Liz is a prime example. She did not know how to label the columns. She started to mumble words like add 42 to 42 kept saying multiples of 42. Then she mumbled something about adding 7 to each hour. She messed and messed around with the Brailler. She never got a good start to constructing the table. This is a great disturbance when it comes to translating from one representation to another. This represents a larger problem among students with visual impairments. This sample of participants did not have a good overall sense of linear function. It is clear that they had rote learning experiences which they told Ms. Cowan in the MEEVA when they told Ms. Cowan that their mathematics classes were comprised by lecture and homework. Most of them said that they learned much better when being taught one-on-one. Many of them also said that they understood mathematical concepts much better when they could talk out their thinking with a tutor. This all agreed with Ms. Cowan’s findings. Caleb was a shining example of a participant who had great translation-application abilities. Caleb also had reification abilities. When it came to the FKA, he knew right away that Dianne’s and Matthew’s savings were decreasing and increasing at the same rate. Therefore, the largest difference between their savings would be at the beginning and at the end. Caleb knew that the equations had slope 5 with Dianne’s negative and Matthew’s positive. He was able to see Matthew’s graph as an equation. He did not say the word “slope,” but I had a good idea that he was thinking the word. He said everything that went into slope and understood that slope represents rate of increase or decrease. He was so close to saying words that hinted toward the derivative.
When I examined the data concerning the producing of graphs, I was surprised when it came to comparing Caleb’s ability to Amanda’s. Caleb has early visual experiences, but he now uses Braille to read and write mathematics. Caleb was one of just three participants to draw a graph of problem FCA; 5b. He did an excellent job. Caleb was both very able to read and produce graphs. Amanda has reading vision and prefers using large print graphing paper. However, she displayed a good ability to read graphs, but did not concerning producing graphs. When Amanda drew the graph of problem FCA; 3c, she connected each X value with each Y value. She ended up drawing many lines which feels to me like a big triangle with smaller triangles inside. I am surprised that she could not draw a diagonal line even though she has vision. I am sure that she has drawn graphs in high school. I am not sure whether the absences of certain skills relating to linear functions are related to total loss of vision or to mathematics teachers. Participants told Ms. Cowan that they learned mathematics better after they lost all sight, because they had more one-on-one teaching with their VI teachers. However, after my examination of the data, I can see a theme that the participants who had more access to graphs and read and drew them had more translation and reification ability. Caleb talked to Ms. Cowan about how graphing calculators help the sighted a lot; however, the visually impaired cannot take advantage of them. With graphing calculators, one can see how a simple graph of \( X^2 \) or \( X^3 \) can be stretched, moved up and down and left and right, and flipped over to produce graphs of much more complicated functions. For example, the graph of \( 2(X^2 + 3) – 4 \) can be understood by stretching \( X^2 \) and moving it down and to the left to produce the graph of the more involved function. One can do the same with \( X^3 \). The more that students with visual impairments are able to handle and experience graphing, the better
they will understand the other representations of functions and to translate between them. I observed that with Adam’s table on the FKA, most participants did not think of Adam’s equation as being \( Y = 7X \). They had the table in front of them and knew that Adam started with no money. They clearly saw that his savings increased by seven; however, they did not mention vocabulary like the fact that the \( Y \)-intercept would be at the origin. The slope of Adam’s equation would be 7. His graph will be steeper than Matthew’s, because Matthew’s slope is only 5. Adam’s line will increase faster. Being able to put all knowledge of linear functions together into a beautiful multiple representational picture was missing. This shows that the participants who had mathematics teachers who believed in them and spent time with them had a much better understanding of linear function independent of their visual impairment. This all agreed with Ms. Cowan’s findings.

The study procedures for this research were outstanding. First of all, Ms. Cowan met with participants one-on-one. In the MEEVA, most participants said that they are much more comfortable talking and doing mathematics when they are working one-on-one with another person. Ms. Cowan talked with her participants and made them all feel very comfortable. Robert constantly talked with Ms. Cowan about where the problems on the tests came from; what the original names were for Matthew, Dianne, Heather, and Adam; and even people who work at the university where both Ms. Cowan and Robert have attended. While administering the FKA and FCA, Ms. Cowan acted as a guide to help participants understand what the problems were asking for. She also cleared up any misconceptions that participants had. If participants were way out in left field with a problem, Ms. Cowan gently guided them on track. After that, most participants did many
of the problems at least partially accurately. In addition, Ms. Cowan asked the questions on the MEEVA before both tests. In my opinion, this made participants very comfortable before taking both tests. Ms. Cowan did not spend too much time with either of two meetings. As I listened to the interviews, participants did not seem tired or lethargic by the end of either test. I liked how Ms. Cowan let a couple of the participants finish the FKA the next time they met for the FCA. This made sure that these meetings did not interfere with their personal lives. In my opinion, Ms. Cowan did everything possible to administer these interviews in a way that made participants feel comfortable and happy that they could help. Ms. Cowan could not have treated the participants better.

Statement of Credentials

Ms. Cowan asked me to be the auditor for this study because of many reasons. I am visually impaired with only light and dark perception. I had some usable vision until the age of twelve and learned mathematics by having the teacher write it out in very large print. I did not learn how to read Braille until the age of fourteen. Because of these circumstances, I picture all mathematics I do visually in my head. I picture nothing in Braille. Because of my own visual impairment, I know what it is like to learn and understand mathematics with no vision. I was able to relate with the participants very well. I attended my regular public school system; therefore, there were no special accommodations for me to learn mathematics. My teachers would choose a student out of study hall to read the tests to me. Because I was in school in the 70s and 80s, there were no laws like at the present where a VI teacher had to be there to help me. I had no mathematics converted into Braille. I used an audio recording of the textbook from the
Recording for the Blind. I sat and listened to what the narrator of the book read and learned mathematics on my own. I learned nothing in class.

I have a Bachelor of Science in mathematics and my Master of Arts in mathematics education both from The Ohio State University. This background gave me the opportunity to understand all of the mathematical concepts that are involved with linear function. I totally understand all representations of linear functions. I have a strong calculus and advanced geometry background. In Addition, my MA in mathematics education enabled me to learn all of the best teaching and learning strategies for students in general to understand all mathematics concepts. I tutored students with visual impairments in the Office for Disability Services at The Ohio State University in the 90s. Therefore, I know what this population needs to best learn and understand all mathematical concepts including calculus and beyond. I completed a master’s thesis of my own that dealt with how students who have visual impairments best learn and understand mathematics.