MODELING WAVES IN A HUMAN BRAIN BY SPACE-TIME CONSERVATION ELEMENT AND SOLUTION ELEMENT METHOD

M.S. Thesis

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This thesis reports the results of my M.S. research work about applying the space-time Conservation Element and Solution Element (CESE) method to calculate the wave propagation in the human brain. I am concerned with theoretical and numerical simulations of wave propagation based on two models: Fung’s model and Iatridis’ model. The thesis is divided into two parts.

First, the governing equations which include the equation of motion, the viscoelastic constitutive relations, and the equations of internal variables are cast into vector-matrix form, and eigenvalues of the Jacobian matrices of the governing equations are thoroughly derived. The system of equations is shown to be hyperbolic with real eigenvalues and diagonalizable eigenvector matrices in the equations. The treatment for source terms in the CESE method is also reported. The material response is modeled by Fung’s model and modified Fung’s model which is developed by Iatridis. I used parallel connected standard linear solid (SLS) models to discretize Fung’s model and the modified Fung’s model. The resultant relaxation functions formulated in an integral form are then transformed to be differential equations by using the method of internal variables. I then performed simulation of wave absorption in the human brain tissues. While I took conventional approach to determine the parameters in the relaxation functions by using experimental quasi-static longitudinal tests, I also
employed the measured wave absorption coefficients to determine the relaxation functions. As such, the constructed relaxation functions are inherently suitable for wave dynamics.

In the second part of the thesis, I applied the CESE method to solve newly developed model equations for the waves in the human brain.

CESE method is a generic numerical method for high-fidelity simulation of first-order, coupled, linear or nonlinear hyperbolic Partial Differential Equations (PDEs). Originally developed for solving compressible flows with complex shock waves, the CESE method had been widely used to simulate various complex compressible flows, including detonations, hypersonic flows, and shock and acoustic wave interactions. The approach is validated by the simulation of an one-dimensional impact wave.

The results in this thesis present a general theoretical framework of using the first-order, hyperbolic PDEs to model wave motion in brain. The model equations are presented in three-dimensional space. To demonstrate the capabilities of the model equations, I reported numerical solutions of one-dimensional and validated the result, and showed the two-dimensional, and three-dimensional results to create a general pictures of wave propagation in the human brain.
To my parents: Wang ZhengGuang and Fu ChaoMin, and
my grandparents: Wang CuiXiang, Lu YueLun, Fu TingShu, and Hou SuJing.
Their continuous love, supports, and encouragement keep motivating me to complete
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During this work I have collaborated with many colleagues for whom I have great regard, and I wish to extend my warmest gratitude to all of those who have helped me with my work in the Department of Mechanical Engineering at the Ohio State University, especially Mr. Lixiang Yang and Mr. Yung-Yu Chen. Last, I owe my special gratitude to my lab-mates Mr. Po-Hsien Lin who also helps me a lot.
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CHAPTER 1
INTRODUCTION

This thesis constructed a theoretical and numerical approach to model wave propagations in the human brain. In general, the thesis could be divided into two parts. For the first part, I employed the equation of motions, constitutive relations, and internal variables to derive the governing equations for modelling human brain. Then I compared the Fung’s model which is usually used to model the brain tissue and the Iatridis’ model, and showed that Fung’s model is actually not suitable to model the human brain. Furthermore, I introduced wave absorption coefficients, which are proved to be more suitable to determine the relaxation functions of the human brain. Eventually, the results are a set of coupled, first-order, nonlinear, hyperbolic equations. The Conservation Element and Solution Element (CESE) method is then used to integrate the hyperbolic equations in the space-time domain for the numerical solutions of wave propagation in the human brain.

In the second part, I performed two-dimension and three-dimension numerical simulations of the waves in the human brain. I employed Ricker’s wave as a source term to show the profile of wave propagation over the brain. This application is common around our lives, e.g. traumatic brain injury, terrorist bombing, and ultrasonic scan for the medical purpose. To know the wave profiles in the brain is going to help people obtain a general idea of what wave propagation looks like in the brain. I will
show numerical analysis of one-, two-, and three-dimensional wave propagation in the present thesis.

It is important to find the wavelength of the wave since it dominates the size of mesh cells in the computational domain. It usually needs at least 20 cells to accurately resolve one wavelength.

1.1 Background

Among all types of injuries, brain injury is the most likely to result in death or permanent disability. An estimated 1.4 million people sustain a traumatic brain injury (TBI) each year in the United States [24]. Blast traumatic brain injury (BTBI) currently represents the main cause of military TBI, but it is poorly understood on BTBI since little is known about the effects of blasts on the human head and the brain injury thresholds for blast such as loadings have not been established yet. Some researchers have recently shown that mild TBI can be caused by the early time intracranial wave motion. These researchers have shown that intracranial wave motion can generate significant intracranial pressure, negative pressure and shear stress in the brain causing TBI. The interactions of a blast wave with the head (the interaction duration were shown as about 0.05 ms with sudden peak pressure of about 4 MPa using air skull interface simulation) results in the propagations of stress waves among and within different components of the head along with dissipations of energy due to the viscosity and heterogeneity of the head tissues[50]. This wave propagation event in the head plays an important role within 2 ms to reaching maximum stresses. This part of the blast response is characterized by the peak pressure, the duration acting on the head and the wave propagation properties or the high strain rates responses of the head.

A lot of effort has been made to simulate human head response under various
blast conditions to obtain brain injury information\cite{3, 16, 33, 38, 44}, but there is a time scale mismatch between the blast loading which generates waves in the head and the current constitutive models used for human brain which are based on low frequency dynamic or low strain rate tests which can not represent the wave propagation properties of the brain. For example, the role of shock absorption (or energy dissipation) of the brain under blast waves could be very different than under quasi-static conditions \cite{1, 17, 24, 52}. Unfortunately, there are few references that take into account the energy absorption properties of the brain, and consider the high strain rate response of brain, which might be the main difference between blast TBI and normal impact TBI. To predict the mechanical response of the contents of the head during BTBI, computational modelling and analysis of the human brain are introduced in recent years \cite{5, 29, 43}. Finite element modelling is a powerful tool for the prediction of the head injury situations since current FE head models contain a detailed geometrical description of several anatomical components of the brain. However, it often lacks accurate validated descriptions of the mechanical behaviour of brain tissue. Without an accurate representation of the constitutive behaviour of the brain, the predictive capabilities of the brain models may be limited. On the other hand, since the early 1960s, researchers have been studying the material properties of brain tissue using a variety of testing techniques. The reported mechanical properties describes that linear viscoelastic behaviour are orders of magnitude different. This may be caused by the broad range of testing methods and protocols used, which make comparison of results difficult. Several authors have presented an overview of available literature on the constitutive properties of brain tissue \cite{15}. An overview of the methods and the conditions of materials tested in previous studies is also given in \cite{15}. The studies were divided into groups depending on the type of experiments.
The mechanical behaviour of brain tissue has been tested mostly in compression/tension and shear [2], [24], [25], [36], [42], [45]. Mechanical characterization of the brain has been one of the central subjects in the field of biomechanics, and still attracts a lot of attentions these days. However, it is not usual to find an experiment which conveys the compression and shear relaxation test on the same sample of brain. In this case, it is hard to determine the parameters for compression modulus and shear modulus accurately.

Also, the so called quasi linear viscoelastic (QLV) theory has been proposed by Fung [20]. This theory has become widely used in injury biomechanics and has been applied for the constitutive or structural modelling of many soft biological tissues. However, the numerical results show no relaxation effect with Fung’s model. Thus, Fung’s model is not suitable for dynamics wave problems with short time durations. Finally, we consider the brain by using Iatridis’ model which is an extended Fung’s model[26] and the numerical solutions show apparent relaxation effect.

Ultrasonic absorption coefficient is another method to obtain the properties of human brain within a really short time period[22], and[34]. However, most of the studies do not establish the connection between absorption coefficient and relaxation function of the human brain. In present thesis, starting with power law theory of absorption coefficient[7] and experiment data [19], I reconstructed relaxation function of the human brain. Then generalized standard linear solid models were used to fit the relaxation functions. By transferring absorption coefficient to relaxation function, and employing collocation method and internal variable method, we can obtain the properties of human brain within the interested frequency range. The numerical results can be compared with original experiments and validate our numerical model.

Animal brain is frequently used as a substitute for the human brain in experiments to characterize the mechanical behaviour. The main reason is that animal
brains are easily available and can be tested at short post mortem times. The mechanical response of fresh human brain tissue was reported to be similar with porcine brain tissue. The differences between human and animal brains are often considered relatively small which enables animal brains to be a good substitute for human brains.

1.2 Objectives

The objective of my thesis is to develop a new system which can directly and simultaneously captures the nonlinear wave propagation in soft tissue. This system differs from conventional methods with which people have to design the code case by case. This formulation considers velocity, stress and internal variable as unknown vectors. The Jacobian matrix is introduced for the governing equations, which makes us enable to pack all the material information into a matrix form. As a result, this new system should be adapt to any soft tissue and to calculate the wave propagation as long as we could derive the Jacobian matrix for a certain soft tissue.

In the present research work, the CESE method by Chang [10] will be employed. The CESE method is a novel numerical framework for hyperbolic conservation laws. The tenet of the CESE method is an uniform treatment of space and time for flux conservation. Based on the CESE method, a suite of one-, two-, and three-dimensional computer codes using structured and unstructured meshes have been developed. The two- and three-dimensional codes have been parallelized and can be used for large-scaled simulations. Previously, Yu and coworkers [10, 12, 13, 27, 28, 31, 40, 47, 49, 51] have reported a wide range of highly accurate solutions of hyperbolic systems, including detonations, cavitations, complex shock waves, turbulent flows with embedded dense sprays, dam breaking flows, MHD flows, and aero acoustics. Detailed algebra of the method has been extensively illustrated in the cited references. For conciseness,
only the basic ideas of the CESE method in one dimension domain will be illustrated in Chapter 2.

The code developed for solving viscoelastics is under the structure of software called SOLVCON that is developed by my lab mate Dr.Chen. The SOLVCON is a multi-physic software based on the CESE method. This software currently has the ability to deal with hypersonic flow problem, elastic solid problem and more is coming. The SOLVCON also has the function that decomposes calculation domain and distributes out to compute in parallel machine. SOLVCON has proved that it is capable to control over 500 computer nodes. The present object of SOLVCON is to run 1 billion mesh elements on the parallel machine.

1.3 Organization

The rest of this thesis is divided into six chapters. A brief description of each chapter is provided in the following. Chapter 2 illustrates the CESE method, in which a space-time integral form of the governing equations is integrated for time-marching solutions. Space-time flux conservation over Conservation Elements (CEs) is imposed. The integration is aided by the prescribed discretization of the unknowns in each Solution Elements (SE), which in general does not coincide with a CE.

In Chapter 3, Chapter 4, and Chapter 5, I reported a theoretical and numerical framework to model and validate wave propagation in the human brain.

The first part is to prove that the governing equations for human brain are hyperbolic. This conclusion is important because the CESE method was originally developed for numerical solutions of a set of coupled, hyperbolic system of PDEs. The second part is to compare Fung’s model and Iatridis’ model in order to show that Fung’s model as a traditional approach to solve high frequency viscoelastic problem actually is not suitable, while Iatridis’ model is able to model that problem. Chapter
5 deals with absorption coefficient of brain tissue, providing another good approach to solve high frequency viscoelastic problem.

In Chapter 6, I provided the numerical results for two-dimensional and three-dimensional simulations for an exhibition purpose. The profile of wave propagation based on Fung’s model and Iatridis’ model is shown.

In Chapter 7, I provided the concluding remarks and a list of tasks for further development to compute wave propagation by using CESE method, followed by a list of cited references.
Conventional finite volume methods are formulated according to a flux balance over a fixed spatial domain. The conservation laws state that the rate of change of the total amount of a substance contained in a fixed spatial domain, i.e., the control volume $V$, is equal to the flux of that substance across the boundary of $V$, denoted as $S(V)$. Consider the differential form of a conservation law as follows:

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{f} = 0$$  \hspace{1cm} (2.1)

where $u$ is density of the conserved flow variable, $\mathbf{f}$ is the spatial flux vector. By applying Reynolds’ transport theorem to the above equation, one can obtain the integral form as:

$$\frac{\partial}{\partial t} \int_V u dV + \oint_{S(V)} \mathbf{f} \cdot d\mathbf{S} = 0$$  \hspace{1cm} (2.2)

where $dV$ is a spatial volume element in $V$, $d\mathbf{S} = d\sigma \mathbf{n}$ with $d\sigma$ and $\mathbf{n}$ being the area and the unit outward normal vector of a surface element on $S(V)$ respectively. By
integrating Eq. (2.2), one has

\[
\left[ \int_V u dV \right]_{t=t_f} - \left[ \int_V u dV \right]_{t=t_s} + \int_{t_s}^{t_f} \left( \oint_{S(V)} f \cdot dS \right) dt = 0
\]  

(2.3)

The discretization of Eq. (2.3) is the focus of the conventional finite-volume methods. In particular, the calculation of the flux terms in Eq. (2.3) would introduce the upwind methods due to its nonlinearity of the convection terms in the conservation laws.

In the CESE method, we do not use the above formulation based on the Reynolds transport theorem. Instead, the conservation law is formulated by treating space and time in an equal-footing manner. This unified treatment for space and time would allow a consistent integration in calculation space-time and thus ensure local and global flux balance. This chapter only briefly illustrates the CESE method in one-dimension.

Here I will introduce CESE method for one-dimension. For multi-dimension, it can refer to Chang and Wang [14] for more detail information. Let time and space be the two orthogonal coordinates of a space-time system, i.e., \(x_1 = x\) and \(x_2 = t\). They constitute a two-dimensional Euclidean space \(E_2\). Define \(h \equiv (f, u)\), then by using the Gauss divergence theorem, Eq. (2.1) becomes

\[
\int_{\partial \Omega} h \cdot ds = 0
\]  

(2.4)

Equations (2.4) states that the total space-time flux \(h\) leaving the space-time volume through its surface vanishes. Refer to Fig. 2.1 for a schematic of Eq. (2.4), we employed the CESE method [10] to integrate Eq. (2.4).

The tenet of the CESE method is the uniform treatment of space and time in calculating flux conservation. Based on the CESE method, a suite of computer one-, two-, and three-dimensional codes using structured and unstructured meshes have
been developed. The two- and three-dimensional codes have been parallelized and can be used to perform large-scaled simulations of nonlinear stress waves in fluids and solids. In the present report, only basic ideas of the CESE method in one spatial domain will be illustrated.

In the CESE method, separated definitions of Solution Element (SE) and Conservation Element (CE) are introduced. In each SE, solutions of unknown variables are assumed continuous and a prescribed function is used to represent the profile. In the present calculation, a linear distribution is used. Over each CE, the space-time flux in the integral form, Eq. (2.4), is imposed. Figure (2.2) shows the space-time mesh and the associated SEs and CEs. Solutions of variables are stored at mesh nodes which are denoted by filled circular dots. Since a staggered mesh is used, solution variables at neighboring SEs leapfrog each other in time-marching calculation. The SE associate with each mesh node is a yellow rhombus. Inside the SE, the solution variables are assumed continuous. Across the interfaces of neighboring SEs, solution discontinuities are allowed. In this arrangement, solution information from on SE to another propagates only in one direction, i.e., toward the future through the oblique interface as denoted by the red arrows. Through this arrangement of space-time
staggered mesh, the classical Riemann problem has been avoided. Figure 2.2(b) illustrates a rectangular CE, over which the space-time flux conservation is imposed. This flux balance provides a relation between the solutions of three mesh nodes: \((j, n)\), \((j - 1/2, n - 1/2)\), and \((j + 1/2, n - 1/2)\). If the solutions at time step \(n - 1/2\) are known, the flux conservation condition would determine the solution at \((j, n)\).

In the present research, many differential equations have source terms. Thus, we consider the one-dimensional equations with source terms:

\[
\frac{\partial u_m}{\partial t} + \frac{\partial f_m}{\partial x} = s_m, \tag{2.5}
\]

where \(m = 1, 2, 3\) and the source term \(s_m\) are functions of the unknowns \(u_m\) and their spatial derivatives. For any \((x, t) \in \text{SE}(j, n)\), \(u_m(x, t)\), \(f_m(x, t)\) and \(h_m(x, t)\), are approximated by \(u^*(x, t; j, n)\), \(f^*(x, t; j, n)\), and \(h^*(x, t; j, n)\). By assuming linear distribution inside an SE, we have

\[
\begin{align*}
u^*_m(x, t; j, n) &= (u_m)^n_j + (u_{mx})^n_j(x - x_j) + (u_{mt})^n_j(t - t^n), \\
f^*_m(x, t; j, n) &= (f_m)^n_j + (f_{mx})^n_j(x - x_j) + (f_{mt})^n_j(t - t^n), \\
h^*_m(x, t; j, n) &= (f^*_m(x, t; j, n), u^*_m(x, t; j, n)).
\end{align*}
\]
where

\[ (u_{mx})_j^n = \left( \frac{\partial u_m}{\partial x} \right)_j^n, \]
\[ (f_{mx})_j^n = \left( \frac{\partial f_m}{\partial x} \right)_j^n = (f_{ml})_j^n (u_{lx})_j^n, \]
\[ (u_{mt})_j^n = \left( \frac{\partial u_m}{\partial t} \right)_j^n \]
\[ (f_{mt})_j^n = \left( \frac{\partial f_m}{\partial t} \right)_j^n = -(f_{mx})_j^n = -(f_{ml})_j^n (u_{lx})_j^n, \]

and \((f_{ml})_j^n \equiv \left( \frac{\partial f_m}{\partial u_l} \right)_j^n\) is the Jacobian matrix. Assume that, for any \((x, t) \in SE(j, n)\), \(u_m = u^*_m(x, t; j, n)\) and \(f_m = f^*_m(x, t; j, n)\) satisfy Eq. (2.5), i.e.,

\[ \frac{\partial u_m^*(x, t; j, n)}{\partial t} + \frac{\partial f_m^*(x, t; j, n)}{\partial x} = s^*_m(x, t; j, n), \quad (2.6) \]

where we assume that \(s^*_m\) is constant within SE\((j, n)\), i.e., \(s^*_m(x, t; j, n) = (s_m)_j^n\).

Eq. (2.6) becomes

\[ (u_{mt})_j^n = -(f_{mx})_j^n + (s_m)_j^n. \quad (2.7) \]

Since \((f_{mx})_j^n\) are functions of \((u_m)_j^n\) and \((u_{mx})_j^n\); and \((s_m)_j^n\) are also functions of \((u_m)_j^n\), Eq. (2.7) implies that \((u_{mt})_j^n\) are also functions of \((u_m)_j^n\) and \((u_{mx})_j^n\). Aided by the above equations, we determine that the only unknowns are \((u_m)_j^n\) and \((u_{mx})_j^n\) at each mesh point \((j, n)\).

Next, we impose space-time flux conservation over CE\((j, n)\) to determine the unknowns \((u_m)_j^n\). Refer to Fig. 2.2(b). Assume that \(u_m\) and \(u_{mx}\) at mesh points \((j - 1/2, n - 1/2)\) and \((j + 1/2, n - 1/2)\) are known and their values are used to
Fig. 2.2: A schematic of the CESE method in one spatial dimension. (a) Zigzagging SEs. (b) Integration over a CE to solve $u_i$ and $(u_x)_i$ at the new time level.
calculate \((u_m)^n_j\) and \((u_{mx})^n_j\) at the new time level \(n\). By enforcing the flux balance over CE\((j, n)\), i.e.,

\[
\oint_{S(CE(j,n))} h^*_m \cdot ds = \int_{CE(j,n)} s_m^* d\Omega,
\]

one obtains

\[
(u_m)^n_j - \frac{\Delta t}{4} (s_m)^n_j = \frac{1}{2} \left[ (u_m)^{n-1/2}_{j-1/2} + (u_m)^{n-1/2}_{j+1/2} \right. \\
+ \frac{\Delta t}{4} (s_m)^{n-1/2}_{j-1/2} + \frac{\Delta t}{4} (s_m)^{n-1/2}_{j+1/2} \left. \\
+ (p_m)^{n-1/2}_{j-1/2} - (p_m)^{n-1/2}_{j+1/2} \right],
\]

(2.8)

where

\[
(p_m)^n_j = \frac{\Delta x}{4} (u_{mx})^n_j + \frac{\Delta t}{2} (f_m)^n_j + \frac{\Delta t^2}{4\Delta x} (f_{mt})^n_j.
\]

Given the values of the marching variables at the mesh nodes \((j - 1/2, n - 1/2)\) and \((j + 1/2, n - 1/2)\), the right-hand side of Eq. (2.8) can be explicitly calculated. Since \((s_m)^n_j\) on the left hand side of Eq. (2.8) is a function of \((u_m)^n_j\), we use Newton’s method to solve for \((u_m)^n_j\). The initial guess of the Newton iterations is

\[
(\bar{u}_m)^n_j = \frac{1}{2} \left[ (u_m)^{n-1/2}_{j-1/2} + (u_m)^{n-1/2}_{j+1/2} \right. \\
+ \frac{\Delta t}{4} (s_m)^{n-1/2}_{j-1/2} + \frac{\Delta t}{4} (s_m)^{n-1/2}_{j+1/2} \left. \\
+ (p_m)^{n-1/2}_{j-1/2} - (p_m)^{n-1/2}_{j+1/2} \right],
\]

i.e., the explicit part of the solution of \((u_m)^n_j\).

The solution procedure for \((u_{mx})^n_j\) at node \((j, n)\) follows the standard \(\alpha-\epsilon\) scheme
[10] with \( \varepsilon = 0.5 \). To proceed, we let

\[
(u_{mx})^n_j = \frac{(u_{mx}^+)^n_j + (u_{mx}^-)^n_j}{2},
\]

(2.9)

where

\[
(u_{mx}^\pm)^n_j = \pm \frac{(u_m)^{n_j \pm 1/2} - (u_m)^n_j}{\Delta x/2},
\]

\[
(u_m)^{n_j \pm 1/2} = (u_m)^{n_j - \frac{1}{2}} + \frac{\Delta t}{2} (u_{mt})^{n_j - \frac{1}{2}}.
\]

For solutions with discontinuities, Eq. (2.9) is replaced by a re-weighting procedure to add artificial damping at the jump

\[
(u_{mx})^n_j = W\left((u_{mx}^-)^n_j, (u_{mx}^+)^n_j, \alpha\right),
\]

where the re-weighting function \( W \) is defined as:

\[
W(x_-, x_+, \alpha) = \frac{|x_+|^\alpha x_- + |x_-|^\alpha x_+}{|x_+|^{\alpha} + |x_-|^{\alpha}},
\]

and \( \alpha \) is an adjustable constant. The complete discussion of the one-dimensional CESE method can be found in [10, 11]. The above method with CE and SE defined as in Fig. (2.2) is useful for solving the hyperbolic PDEs with non-stiff source terms.
CHAPTER 3

VISCOELASTICITY MODELS

3.1 Viscoelasticity

Stresses in a viscoelastic medium like human brain are functions of the material deformation at the present time as well as the history of the deformation. The most general form of constitutive relations for viscoelastic media can be expressed by the Weierstrass approximation theorem [23], in which Pipkin and Rogers’ integral series [39] are used:

\[ S(t) = \sum_{n=1}^{\infty} P_n(t), \]

where \( S \) is the Piola-Kirchhoff stress and

\[ P_n(t) = \frac{1}{n!} \int_{E(-\infty)}^{E(t)} \cdots \int_{E(-\infty)}^{E(\tau_1)} dE(\tau_1) \cdots dE(\tau_n) \]

\[ \frac{\partial^n R_n [E(\tau_1), t - \tau_1; \cdots; E(\tau_n), t - \tau_n]}{\partial E(\tau_1) \cdots \partial E(\tau_n)}. \]

In the above equation, \( E \) is the Green-St. Venant strain tensor, \( R_n \) is the nth relaxation function, and \( \tau_n \) is the nth discrete time between 0 and \( t \). The Pipkin-Rogers model is valid for non-aging materials.

To proceed, we assume the material is simple in the sense that a single relaxation
function \( R_1 \) is adequate to model the material response. As such, the Pipkin-Rogers model becomes

\[
S(t) = P_1(t) = \int_{E(-\infty)}^{E(t)} dE(\tau_1) \frac{\partial R_1 [E(\tau_1), t - \tau_1]}{\partial E(\tau_1)}.
\] (3.1)

Next, we illustrate the functional form of the relaxation function \( R_1 \) in Eq. (3.1). Based on experimental observations, when \( R_1 \) is plotted against time \( t \), or an normal strain \( E \), in a logarithm plot, one finds straight lines parallel to each other. This implies that the stress relaxation modulus can be factorized into time- and strain-dependent terms, i.e.,

\[
\log R_1(E, t) = A \log t + B \log E = \log (t^A E^B),
\]

where \( A \) and \( B \) are constants to be determined by experimental data. This type of material response is commonplace for a wide range of viscoelastic media [35], including soft tissues. As such, the relaxation equation \( R_1 \) can be generalized to be

\[
R_1(E(\tau), t - \tau) = G(t - \tau) S^e[E(\tau)].
\] (3.2)

where \( G \) is a function of \( t \) only and \( S^e \) is a function of \( E \) only. By substituting Eq. (3.2) into Eq. (3.1) with \( \tau_1 \) replaced by \( \tau \), we obtain

\[
S(t) = \int_{-\infty}^{t} G(t - \tau) \frac{\partial S^e(E(\tau))}{\partial E(\tau)} \frac{\partial E(\tau)}{\partial \tau} d\tau.
\] (3.3)

To proceed, we assume \( E = 0 \) when \( \tau < 0 \), and at \( \tau = 0 \), \( E \) experiences a jump start. Thus, Eq. (3.3) can be written as

\[
S(t) = S^e(0) G(t) + \int_{0}^{t} G(t - \tau) \frac{\partial S^e(E(\tau))}{\partial E(\tau)} \frac{\partial E(\tau)}{\partial \tau} d\tau,
\] (3.4)

where \( S^e(0) G(t) \) is obtained by integrating \( S(t) \) from \( \tau = -\infty \) to \( \tau = 0 \).
To proceed, we recast Eq. (3.4) into the index form by using a Cartesian coordinate system:

\[ S_{ij}(t) = S^{e}_{kl}(0)G_{ijkl}(t) + \int_{0}^{t} G_{ijkl}(t - \tau) \frac{\partial S^{e}_{kl}}{\partial E_{mn}} \frac{\partial E_{mn}}{\partial \tau} d\tau, \]  

(3.5)

where \( S_{ij} \) is the \((i, j)\)th component of the second Piola-Kirchhoff stress tensor, \( E_{mn} \) is the \((m, n)\)th component of Green’s strain tensor, and \( G_{ijkl} \) is the component of the fourth-order relaxation function tensor. Equation (3.5) is a general form of the classical Fung’s model for modeling material response of soft tissues.

Next, we assume small deformation and adapt the constitutive relation for linear viscoelasticity. Thus, the second Piola-Kirchhoff stress tensor becomes the Cauchy stress, \( S_{ij} = \sigma_{ij} \), and the Green’s strain tensor becomes the infinitesimal strain tensor, \( E_{mn} = \epsilon_{mn} \). As such, Eq. (3.5) is changed to

\[ \sigma_{ij}(t) = \sigma^{e}_{kl}(0)G_{ijkl}(t) + \int_{0}^{t} G_{ijkl}(t - \tau) \frac{\partial \epsilon_{kl}}{\partial \tau} d\tau, \]  

(3.6)

which is equivalent to a general form of Fung’s model for linear deformations in soft tissues. Moreover, we assume the stress-free initial condition, i.e., \( \sigma^{e}_{kl}(0) = 0 \). The constitutive model Eq. (3.6) can be simplified to

\[ \sigma_{ij}(t) = \int_{0}^{t} G_{ijkl}(t - \tau) \frac{\partial \epsilon_{kl}}{\partial \tau} d\tau. \]  

(3.7)

Next, the medium is assumed isotropic. The fourth-order relaxation tensor \( G_{ijkl} \) can be written as:

\[ G_{ijkl}(t - \tau) = \lambda(t - \tau)\delta_{ij}\delta_{kl} + \mu(t - \tau)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \]  

(3.8)
where $\lambda(t)$ and $\mu(t)$ are the two viscoelastic variables. Aided by Eq. (3.8), the constitutive relation Eq. (3.6) can be simplified to

$$\sigma_{ij}(t) = \int_0^t \left( \lambda(t - \tau) \frac{\partial \varepsilon_{kk}}{\partial \tau} \delta_{ij} + 2\mu(t - \tau) \frac{\partial \varepsilon_{ij}}{\partial \tau} \right) d\tau$$

$$= \lambda \ast \frac{\partial \varepsilon_{kk}}{\partial t} \delta_{ij} + 2\mu \ast \frac{\partial \varepsilon_{ij}}{\partial t}, \quad (3.9)$$

where the symbol $\ast$ denotes the convolution integral and the overdot denotes the differentiation in time. Because of small deformation, the time rate of the strain becomes

$$\frac{\partial \varepsilon_{ij}}{\partial t} = \frac{1}{2} \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right). \quad (3.10)$$

Aided by Eq. (3.10), Eq. (3.9) becomes

$$\sigma_{ij}(t) = \lambda(t) \ast \frac{\partial v_k}{\partial x_k} \delta_{ij} + \mu(t) \ast \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right). \quad (3.11)$$

Equation (3.11) is the general constitutive relationship between stress and strain for small deformation in a linear viscoelastic medium.

### 3.2 Standard Linear Solid Models

In this section, we proceed to determine the two relaxation parameters $\lambda(t)$ and $\mu(t)$ in the above constitutive relation Eq. (3.11) for soft tissues. To proceed, we approximate the general viscoelasticity relation by a series of parallelly connected SLS models. Each SLS model includes two springs and one dash-pot. The relaxation function of the parallelly connected SLS models is a linear combination of a series of exponential functions. It is very easy to integrate or differentiate exponential functions. Thus,
the functional form of the relaxation parameters $\lambda$ and $\mu$ can be readily derived. In this process, we consider one-dimensional, uni-axial loading only. Nevertheless, the resultant formula of the two relaxation parameters can be substituted back to the general constitutive relation Eq. (3.11), which in turn can be applied to model waves in two and three spatial dimensions.

Recall Eq. (3.7), which can be rewritten as $\sigma_{ij} = G_{ijkl} \ast \partial \epsilon_{kl}/\partial t$. Aided by the Voigt notation, the Cauchy stress tensor can be written as a 6-component vector as

$$
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{bmatrix} \rightarrow 
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix} = \mathbf{\sigma} ,
$$

where $\mathbf{\sigma}$ is the 6-component Cauchy stress vector. Similarly, the rate of the Cauchy strain tensor can be written as a 6-component vector as

$$
\begin{bmatrix}
\partial \epsilon_{11}/\partial t \\
\partial \epsilon_{22}/\partial t \\
\partial \epsilon_{33}/\partial t \\
\partial \epsilon_{23}/\partial t \\
\partial \epsilon_{13}/\partial t \\
\partial \epsilon_{12}/\partial t
\end{bmatrix} \rightarrow 
\begin{bmatrix}
\partial \epsilon_1/\partial t \\
\partial \epsilon_2/\partial t \\
\partial \epsilon_3/\partial t \\
2\partial \epsilon_4/\partial t \\
2\partial \epsilon_5/\partial t \\
2\partial \epsilon_6/\partial t
\end{bmatrix} = \frac{\partial \mathbf{\epsilon}^*}{\partial t} ,
$$

where $\mathbf{\epsilon}^*$ is the 6-component vector of strain. As such, Eq. (3.7) can be rewritten as
the following vector-matrix equation:

\[
\sigma = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu 
\end{bmatrix} \ast \frac{\partial \epsilon}{\partial t},
\] (3.12)

where \(\lambda\) and \(\mu\) are functions of time.

To determine the viscoelastic parameters, \(\lambda\) and \(\mu\), we consider two stress-stain relations: (i) an uni-axial longitudinal case, and (ii) a simple shear case. To proceed, we consider the case of longitudinal along the \(x_1\) direction. We assume \(\sigma_1 \neq 0\) and \(\epsilon_1 \neq 0\). All of other stress and strain are null. Equation (3.12) becomes \(\sigma_1 = (\lambda(t) + 2\mu(t)) \ast \partial \epsilon_1 / \partial t\). For convenience, we let \(\Pi(t) = \lambda(t) + 2\mu(t)\). By using the parallelly connected SLS models, \(\Pi(t)\) can be shown as

\[
\Pi(t) = \pi \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{l}^p}{\tau_{\sigma l}} \right) e^{-t/\tau_{\sigma l}} \right] H(t)
\] (3.13)

where \(H(t)\) is the Heaviside function, and \(L\) is the total number of the SLS models involved. In Eq. (3.13), \(\pi\), \(\tau_{\sigma l}\), and \(\tau_{l}^p\) are the tension or compression modulus and relaxation parameters, respectively. The values of these parameters must be determined by the compression/tension tests. If the test datas are presented in the form of Fung’s model for soft tissues, we use the collocation method to deduce the values of these parameters. Section 3.4 illustrates this procedure. Once the values of these parameters are provided, \(\Pi\) is determined by Eq. (3.13).

Next, we consider a case of simple shear under the condition of \(\sigma_4 \neq 0\) and \(\epsilon_4 \neq 0\).
All other stress and strain components are null. Eq. (3.12) becomes the governing equations with five internal variables, Eq. (4.5), for pure shear waves in a soft tissue \( \sigma_4 = \mu(t) \ast \partial \epsilon_4 / \partial t \). For this pure shear condition, the relaxation function of the parallelly connected SLS models \([4]\) can be formulated as

\[
\mu(t) = \nu \left( 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}}{\tau_{sl}} \right) e^{-t/\tau_{sl}} \right) H(t),
\]

(3.14)

where \( \nu, \tau_{sl} \) and \( \tau_{sl}^* \) are the shear modulus and the two relaxation parameters of viscoelastic material. The values of these parameters need to be determined by experiments. If needed, the collocation method is used to deduce the data. Once \( \mu \) is determined, \( \lambda(t) \) can be calculated by \( \lambda(t) = \Pi(t) - 2\mu(t) \). To recapitulate, the parameters \( \lambda \) and \( \mu \) in the general three-dimensional isotropic viscoelasticity model, Eq. (3.12), can be obtained by (i) approximating the relaxation function by a series of parallelly connected SLS models, and (ii) two simple tests of a pure compression/tension test and a pure shear test.

To proceed, we substitute the derived functions for the viscoelasticity parameters Eq. (3.13) and Eq. (3.14) into Eq. (3.11). We then differentiate the resultant by time. To present the resultant equations, we separate the diagonal terms and the off-diagonal terms. The diagonal elements \((i \neq j)\) can be expressed as:

\[
\frac{\partial \sigma_{ij}}{\partial t} = \left( \frac{\partial \Pi}{\partial t} - 2 \frac{\partial \mu}{\partial t} \right) * \frac{\partial v_k}{\partial x_k} + 2 \frac{\partial \mu}{\partial t} * \frac{\partial v_j}{\partial x_i}.
\]

(3.15)

The off-diagonal element \((i \neq j)\) are

\[
\frac{\partial \sigma_{ij}}{\partial t} = \frac{\partial \mu}{\partial t} * \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right).
\]

(3.16)
3.3 Internal Variables

In this section, we show the transformation the constitutive relations from an integral form into a differential form. To proceed, we substitute the relaxation functions of parallel connected SLS model Eqs. (3.13, 3.14) into the constitutive relation Eq. (3.15) for the diagonal elements of the stress tensor, i.e., \((i = j)\):

\[
\frac{\partial \sigma_{ij}}{\partial t} = \int_0^t \pi \left[ 1 - \sum_{i=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) e^{-\frac{t-\tau}{\tau_{sl}}} \right] \delta(t-\tau) \frac{\partial v_k(t)}{\partial x_k} d\tau \\
+ \int_0^t \pi \left[ \left( \frac{1}{\tau_{sl}} \right) \sum_{i=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) e^{-\frac{t-\tau}{\tau_{sl}}} \right] H(t-\tau) \frac{\partial v_k(t)}{\partial x_k} d\tau \\
- 2 \int_0^t \nu \left[ 1 - \sum_{i=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) e^{-\frac{t-\tau}{\tau_{sl}}} \right] \delta(t-\tau) \frac{\partial v_i(t)}{\partial x_i} d\tau \\
- 2 \int_0^t \nu \left[ \left( \frac{1}{\tau_{sl}} \right) \sum_{i=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) e^{-\frac{t-\tau}{\tau_{sl}}} \right] H(t-\tau) \frac{\partial v_i(t)}{\partial x_i} d\tau \\
+ 2 \int_0^t \nu \left[ 1 - \sum_{i=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) e^{-\frac{t-\tau}{\tau_{sl}}} \right] \delta(t-\tau) \frac{\partial v_j(t)}{\partial x_j} d\tau \\
+ 2 \int_0^t \nu \left[ \left( \frac{1}{\tau_{sl}} \right) \sum_{i=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) e^{-\frac{t-\tau}{\tau_{sl}}} \right] H(t-\tau) \frac{\partial v_j(t)}{\partial x_j} d\tau. \tag{3.17}
\]

Aided by the properties of the delta function \(\delta(-\tau) = \delta \tau\) and \(\int_{-\infty}^{\infty} f(\tau) \delta(t-\tau) dt = f(t)\), Eq. (3.17) becomes

\[
\frac{\partial \sigma_{ij}}{\partial t} = \left\{ \pi \left[ 1 - \sum_{i=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) \right] - 2 \nu \left[ 1 - \sum_{i=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) \right] \right\} \frac{\partial v_k}{\partial x_k} \\
+ 2 \nu \left[ 1 - \sum_{i=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) \right] \frac{\partial v_j}{\partial x_j} + \sum_{i=1}^L \gamma_{ij}.
\]
where all terms involving the Heaviside function $H(t - \tau)$ are collected and defined as the internal variables:

\[
\gamma_{ij}^l = \int_0^t \pi \left( \frac{1}{\tau_{al}} \left( 1 - \frac{\tau_p^{al}}{\tau_{al}} \right) e^{\left( -\frac{t - \tau}{\tau_{al}} \right)} \right) H(t - \tau) \frac{\partial v_k(t)}{\partial x_k} d\tau \\
- 2 \int_0^t \nu \left( \frac{1}{\tau_{al}} \left( 1 - \frac{\tau_s^{al}}{\tau_{al}} \right) e^{\left( -\frac{t - \tau}{\tau_{al}} \right)} \right) H(t - \tau) \frac{\partial v_k(t)}{\partial x_k} d\tau \\
+ 2 \int_0^t \nu \left( \frac{1}{\tau_{al}} \left( 1 - \frac{\tau_s^{al}}{\tau_{al}} \right) e^{\left( -\frac{t - \tau}{\tau_{al}} \right)} \right) H(t - \tau) \frac{\partial v_j(t)}{\partial x_i} d\tau.
\] (3.18)

To proceed, we differentiate Eq. (3.18) by time to obtain

\[
\frac{\partial \gamma_{ij}^l}{\partial t} = \\
- \frac{1}{\tau_{al}} \int_0^t \pi \left( \frac{1}{\tau_{al}} \left( 1 - \frac{\tau_p^{al}}{\tau_{al}} \right) e^{\left( -\frac{t - \tau}{\tau_{al}} \right)} \right) H(t - \tau) \frac{\partial v_k(t)}{\partial x_k} d\tau \\
+ 2 \tau_{al} \int_0^t \nu \left( \frac{1}{\tau_{al}} \left( 1 - \frac{\tau_s^{al}}{\tau_{al}} \right) e^{\left( -\frac{t - \tau}{\tau_{al}} \right)} \right) H(t - \tau) \frac{\partial v_k(t)}{\partial x_k} d\tau \\
- 2 \nu \tau_{al} \int_0^t \nu \left( \frac{1}{\tau_{al}} \left( 1 - \frac{\tau_s^{al}}{\tau_{al}} \right) e^{\left( -\frac{t - \tau}{\tau_{al}} \right)} \right) H(t - \tau) \frac{\partial v_j(t)}{\partial x_i} d\tau.
\] (3.19)

Recall the definition of $\gamma_{ij}^l$ and recognize that the first three terms in the right hand side of Eq. (3.19) is $-1/\tau_{al}$ multiplied by $\gamma_{ij}^l$. Next, we apply the equality

\[
\int_{-\infty}^\infty f(\tau) \delta(t - \tau) d\tau = f(t)
\]

to the remainder three terms in Eq. (3.19), and Eq. (3.19) becomes

\[
\frac{\partial \gamma_{ij}^l}{\partial t} = - \frac{1}{\tau_{al}} \left( \gamma_{ij}^l + \pi \left( \frac{\tau_p^{al}}{\tau_{al}} - 1 \right) \frac{\partial v_k}{\partial x_k} \right) \\
- 2 \nu \left( \frac{\tau_s^{al}}{\tau_{al}} - 1 \right) \frac{\partial v_k}{\partial x_k} + 2 \nu \left( \frac{\tau_s^{al}}{\tau_{al}} - 1 \right) \frac{\partial v_j}{\partial x_i}.
\] (3.20)
To proceed, we derive the equations of the internal variables for the off-diagonal elements, i.e., \((i \neq j)\). We substitute the relaxation functions of parallel connected SLS model Eqs. (3.13,3.14) into the constitutive relation Eq. (3.16):

\[
\frac{\partial \sigma_{ij}}{\partial t} = \int_0^t \nu \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau_{sl}}{\tau_{\sigma l}} \right) e^{-\frac{t - \tau_{\sigma l}}{\tau_{\sigma l}}} \right] \delta(t - \tau) \left[ \frac{\partial v_j(t)}{\partial x_i} + \frac{\partial v_i(t)}{\partial x_j} \right] d\tau \\
+ \int_0^t \nu \left[ \left( \frac{1}{\tau_{\sigma l}} \right) \frac{L}{\sum_{i=1}^L} \left( 1 - \frac{\tau_{sl}}{\tau_{\sigma l}} \right) e^{-\frac{t - \tau_{\sigma l}}{\tau_{\sigma l}}} \right] H(t - \tau) \left[ \frac{\partial v_j(t)}{\partial x_i} + \frac{\partial v_i(t)}{\partial x_j} \right] d\tau.
\]

(3.21)

Aided by the properties of the \(\delta\) function, Eq. (3.21) is rewritten as

\[
\frac{\partial \sigma_{ij}}{\partial t} = \nu \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau_{sl}}{\tau_{\sigma l}} \right) \right] \left[ \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right] + \sum_{l=1}^L \gamma_{ij}^l,
\]

(3.22)

All terms involving the Heaviside function are collected and defined as the internal variables for the off diagonal elements:

\[
\gamma_{ij}^l = \int_0^t \nu \left[ \left( \frac{1}{\tau_{\sigma l}} \right) \sum_{i=1}^L \left( 1 - \frac{\tau_{sl}}{\tau_{\sigma l}} \right) e^{-\frac{t - \tau_{\sigma l}}{\tau_{\sigma l}}} \right] H(t - \tau) \left[ \frac{\partial v_j(t)}{\partial x_i} - \frac{\partial v_i(t)}{\partial x_j} \right] d\tau.
\]

(3.23)
Next, we differentiate Eq. (3.23) by time to yield

\[ \frac{\partial \gamma^{l}_{ij}}{\partial t} = -\frac{1}{\tau_{\sigma l}} \int_{0}^{t} \nu \left[ \left( \frac{1}{\tau_{\sigma l}} \right) \sum_{i=1}^{L} \left( 1 - \frac{\tau_{d}^{s}}{\tau_{\sigma l}} \right) e\left( -\frac{\tau_{d}^{s}}{\tau_{\sigma l}} \right) \right] \]

\[ \times H(t - \tau) \left[ \frac{\partial v_{j}(t)}{\partial x_{i}} + \frac{\partial v_{i}(t)}{\partial x_{j}} \right] d\tau \]

\[ + \int_{0}^{t} \nu \left[ \left( \frac{1}{\tau_{\sigma l}} \right) \sum_{i=1}^{L} \left( 1 - \frac{\tau_{d}^{s}}{\tau_{\sigma l}} \right) e\left( -\frac{\tau_{d}^{s}}{\tau_{\sigma l}} \right) \right] \]

\[ \times \delta(t - \tau) \left[ \frac{\partial v_{j}(t)}{\partial x_{i}} + \frac{\partial v_{i}(t)}{\partial x_{j}} \right] d\tau. \] (3.24)

Again, the first term of right hand side of Eq. (3.24) is the internal variable \( \gamma^{l}_{ij} \) itself.

Aided by the property of the \( \delta \) function, Eq. (3.24) becomes

\[ \frac{\partial \gamma^{l}_{ij}}{\partial t} = -\frac{1}{\tau_{\sigma l}} \left[ \gamma^{l}_{ij} + \nu \left( \frac{\tau_{d}^{s}}{\tau_{\sigma l}} - 1 \right) \left( \frac{\partial v_{j}(t)}{\partial x_{i}} + \frac{\partial v_{i}(t)}{\partial x_{j}} \right) \right] \] (3.25)

The Eqs. (3.20,3.25) are listed in Section 4.2.2.

### 3.4 The Collocation Method

In this section, we introduce the collocation method. To proceed, we recall the parallelly connected SLS models, Eq. (3.13),

\[ G(t) = \left( G_{e} + \sum_{l=1}^{L} G_{l} e^{-\frac{\tau_{d}}{\tau_{\sigma l}}} \right) H(t) \] (3.26)
where \( G_e = \pi \) and \( G_l = -\pi (1 - \tau_{el}/\tau_{ol}) \). As such, \( \pi \), \( \tau_{el} \), and \( \tau_{ol} \) can be determined.

The curve-fitting process is done by using the collocation method [46]. For completeness, a brief description of the method is provided in the following. To proceed, we substitute a series of collocation times into Eq. (3.26) and have

\[
G(t_m) = \left( G_e + \sum_{l=1}^{L} G_l e^{-t_m/\tau_{ol}} \right), m = 1, 2, \ldots ,
\]

where \( t \) is replaced by \( t_m \) for each discrete collocation time. The task here is to determine the parameters \( G_e \), \( G_l \) and \( \tau_{ol} \) at the right hand side of Eq. (3.27) such that \( \pi \), \( \tau_{el} \), and \( \tau_{ol} \) in the governing equations can be determined.

The idea of collocation method is to minimize the error in the non-linear curve fitting process. We first let \( t_m \) to be very large, approaching \( \infty \). By using Eq. (3.27), we can determine \( G_e \) by letting \( G_e = G(\infty) \). We then let the discrete collocation times \( t_m \) with \( m = 1, 2, \ldots \), be separated with orders of magnitudes. For example, we could let \( t_1 = 0.001 \) s, \( t_2 = 0.01 \) s, \( t_3 = 0.1 \) s, et cetera. We then choose each \( \tau_{ol} \) so that its value is in the same order of magnitude as that of \( t_m \). For example, let \( \tau_{o1} = 2 \times 0.001 \) s \( \tau_{o2} = 2 \times 0.01 \) s, and so on. We write down the linear algebraic equations for all collocation times:

\[
\begin{bmatrix}
A_{11} & A_{12} & \cdots \\
A_{21} & A_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
G_1 \\
G_2 \\
\vdots
\end{bmatrix}
=
\begin{bmatrix}
G(t_1) - G_e \\
G(t_2) - G_e \\
\vdots
\end{bmatrix},
\]

(3.28)

where \( A_{ij} = \exp(-t_m/\tau_{ol}) \). Equation (3.28) can be readily solved by inverting matrix \( A \) to obtain \( G_1, G_2, \ldots \). Finally, we substitute \( G_e, G_l, \) and \( \tau_{ol} \) with \( l = 1, \ldots , L \) back to Eq. (3.27) to obtain the values of the constants \( \pi, \tau_{el}^p, \) and \( \tau_{ol} \) for \( l = 1, \ldots , L \).
4.1 Introduction

In this chapter, a theoretical and numerical framework is developed to model wave propagation in human brain. The material response is modelled by well-established viscoelasticity relations, which are formulated in the conventional integral forms. In order to employ a modern numerical method for time-accurate solutions of waves in a time domain, the constitutive relations are transformed into Partial Differential Equations (PDEs) by using parallelly connected Standard Linear Solid (SLS) models in conjunction with the memory variables. As such, the constitutive relation is an open framework and can be readily extended to model complex soft tissues. A generic viscoelasticity relation is factorized into time- and strain-dependent terms. The strain-dependent part is formulated for elastic-like behaviour of the medium. On the other hand, hereditary integration is employed to model time-dependent/relaxation effects. Typical models for viscoelastic medium include Fung’s model [20] and many extended Fung’s models. Extensive experiments on various brain tissues have been conducted to determine the viscoelasticity relation formulated in this two-part functional form. Essentially, experimental data were used to determine the relaxation functions in the constitutive relation. However, such constitutive models are formulated in integrals,
which are cumbersome to be coupled with the equation of motion for numerical solutions. Modern numerical methods for solving wave equations are designed to solve coupled, first-order, hyperbolic PDEs. To circumvent this difficulty, we recast the constitutive relations into PDEs by the following two steps:

(i) The constitutive relations are reconstructed by using parallelly connected SLS models. By adding and tuning SLS modules, the composite SLS model can be readily adapted to various soft tissues.

(ii) The composite SLS model in an integral form is transformed into PDEs by using the method of internal variables [8, 9, 41].

The transformed constitutive relations in the form of PDEs can be directly coupled with the equation of motion. As will be shown, the complete governing equations are a set of fully coupled, first-order, hyperbolic PDEs with source terms. The eigenvalues of the Jacobian matrices of the PDEs are real and they represent wave speeds. Thus, the governing equations can be readily solved by using a modern numerical methods for time-accurate solutions.

According to M.Hrapko [24], the normalized relaxation modulus are shown for both shear and compression tests on porcine brain tissue. From this experimental data, we could obtain the relaxation function and relative parameters, and import those into the governing equations.

### 4.2 The Governing Equations

In this section, we discuss the complete governing equations for waves in viscoelastic media. The equations include the equation of motion and the constitutive relations derived in Chapter 3. To proceed, we consider the equation of motion $\nabla \cdot \sigma = \rho \partial^2 u / \partial t^2$, where $\sigma$ is the Cauchy stress tensor, $\rho$ is density, and $u$ is the displacement vector. We assume that the body force is negligible. The independent variables are
the position $\mathbf{x} = (x_1, x_2, x_3)$ and time $t$. By using the index notation, the equation of motion is rewritten as $\sigma_{ij,j} = \rho \partial^2 u_i / \partial t^2$, where a subscript following a comma denotes partial differentiation with respect to the spatial coordinate. The velocity components $v_i = \partial u_i / \partial t$ are used as the unknowns instead of the displacement. The equation of motion is then coupled with the constitutive equations, Eqs. (3.20) and (3.22) to form the complete set of the governing equations. In what follows, we list all the governing equations:

\[
\begin{align*}
\rho \frac{\partial v_1}{\partial t} &= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}, \\
\rho \frac{\partial v_2}{\partial t} &= \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3}, \\
\rho \frac{\partial v_3}{\partial t} &= \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}.
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \sigma_{11}}{\partial t} &= \frac{\pi \tau^p}{\tau^\sigma} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \\
&\quad - 2\nu \frac{\tau^s}{\tau^\sigma} \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_3}{\partial x_2} \right), \\
\frac{\partial \sigma_{22}}{\partial t} &= \frac{\pi \tau^p}{\tau^\sigma} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \\
&\quad - 2\nu \frac{\tau^s}{\tau^\sigma} \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_3}{\partial x_2} \right), \\
\frac{\partial \sigma_{33}}{\partial t} &= \frac{\pi \tau^p}{\tau^\sigma} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \\
&\quad - 2\nu \frac{\tau^s}{\tau^\sigma} \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_2}{\partial x_3} \right), \\
\frac{\partial \sigma_{12}}{\partial t} &= \nu \frac{\tau^s}{\tau^\sigma} \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right), \\
\frac{\partial \sigma_{13}}{\partial t} &= \nu \frac{\tau^s}{\tau^\sigma} \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right), \\
\frac{\partial \sigma_{23}}{\partial t} &= \nu \frac{\tau^s}{\tau^\sigma} \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right),
\end{align*}
\]
where $\pi = \lambda + 2\mu$ is the relaxation modulus for longitudinal waves, and $\nu$ is the relaxation modulus for shear waves. $\tau^p_l$ and $\tau^s_l$ are the strain relaxation times for the longitudinal and shear waves, respectively. $\tau^p_l$ and $\tau^s_l$ are the stress relaxation time for the longitudinal and shear waves, respectively [4]. $v_1$, $v_2$ and $v_3$ are the velocity components. $\sigma_{ij}$, $i, j = 1, 2, 3$ are the stress components.

4.2.1 One-Dimensional Equations with Five Internal Variables

Let $\partial/\partial x_2 = 0$ and $\partial/\partial x_3 = 0$ and Eq. (4.1) is reduced to the one-dimensional equations in the $x_1$ axis.

For viscoelastic media, the one-dimensional governing equations for the longitudinal waves Eq. (4.1) can be extended to include the internal variables. Here we model the waves by using Fung’s model [20] which is usually used to model the viscoelastic media, and we use internal variables, i.e., $L = 5$.

\[ \rho \frac{\partial v_1}{\partial t} = \frac{\partial \sigma_{11}}{\partial x_1}, \]
\[ \frac{\partial \sigma_{11}}{\partial t} = \left\{ \pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^p_l}{\tau_{sl}} \right) \right] \frac{\partial v_1}{\partial x_1} \right\} + \sum_{l=1}^{5} \gamma^l_{11}, \]
\[ \frac{\partial \gamma^l_{11}}{\partial t} = -\frac{1}{\tau_{sl}} \left[ \gamma^l_{11} + \pi \left( \frac{\tau^p_l}{\tau_{sl}} - 1 \right) \frac{\partial v_1}{\partial x_1} \right], \]
\[ l = 1, \ldots, 5. \]

The above equations are cast into a matrix-vector form:

\[ \frac{\partial \mathbf{V}}{\partial t} + \overline{A}_1 \frac{\partial \mathbf{V}}{\partial x} = \mathbf{Q}. \]  

(4.3)

where the unknown vector

\[ \mathbf{V} = [v_1, \sigma_{11}, \gamma^l_{11}, \gamma^2_{11}, \cdots, \gamma^5_{11}]^t, \]
the source term vector

\[ Q = \left[ 0, \sum_{l=1}^{5} \frac{\gamma_{11}}{\tau_{11}}, -\frac{\gamma_{11}}{\tau_{11}}, \cdots, -\frac{\gamma_{11}}{\tau_{11}} \right]^t, \]

and the Jacobian matrix

\[ \overline{A}_1 = \begin{bmatrix} 0_1 & \overline{A}_{1v} \\ \overline{A}_{1T} & 0_6 \end{bmatrix}. \]

In the Jacobian matrix, \(0_1\) and \(0_6\) denote \(1 \times 1\) and \(6 \times 6\) null matrices, \(A_{1v}\) is a \(1 \times 6\) matrix:

\[ \overline{A}_{1v} = \begin{bmatrix} -\frac{1}{\rho}, 0, \cdots, 0 \end{bmatrix}, \]

and \(A_{1T}\) is a \(6 \times 1\) matrix:

\[ \overline{A}_{1T} = \begin{bmatrix} -\pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{cl}}{\tau_{al}} \right) \right] \\ \frac{\pi}{\tau_{a1}} \left( \frac{\tau_{c1}}{\tau_{a1}} - 1 \right) \\ \frac{\pi}{\tau_{a1}} \left( \frac{\tau_{c2}}{\tau_{a2}} - 1 \right) \\ \frac{\pi}{\tau_{a2}} \left( \frac{\tau_{c3}}{\tau_{a3}} - 1 \right) \\ \frac{\pi}{\tau_{a3}} \left( \frac{\tau_{c4}}{\tau_{a4}} - 1 \right) \\ \frac{\pi}{\tau_{a4}} \left( \frac{\tau_{c5}}{\tau_{a5}} - 1 \right) \end{bmatrix}. \]

Simple derivation shows that the nonzero eigenvalues of \(\overline{A}_1\) are

\[ \bar{\beta}_{1,2} = \pm \sqrt{\frac{\pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{cl}}{\tau_{al}} \right) \right]}{\rho}}. \]
The other eigenvalues are null.

4.2.2 Two-Dimensional Equations with Five Internal Variables

We continue to derive the two-dimensional governing equations for the longitudinal and shear waves. We use 5 internal variables for both longitudinal waves and shear waves.

\[
\rho \frac{\partial v_1}{\partial t} = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2},
\]
\[
\rho \frac{\partial v_2}{\partial t} = \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2},
\]
\[
\rho \frac{\partial v_3}{\partial t} = \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2},
\]
\[
\frac{\partial \sigma_{11}}{\partial t} = \left\{ \pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^{p l}}{\tau_{\sigma l}} \right) \right] \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) - 2\nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^{s l}}{\tau_{\sigma l}} \right) \right] \frac{\partial v_2}{\partial x_2} \right\}
\]
\[
\quad + \sum_{l=1}^{5} \gamma_{11}^l,
\]
\[
\frac{\partial \sigma_{22}}{\partial t} = \left\{ \pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^{p l}}{\tau_{\sigma l}} \right) \right] \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) - 2\nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^{s l}}{\tau_{\sigma l}} \right) \right] \frac{\partial v_1}{\partial x_1} \right\}
\]
\[
\quad + \sum_{l=1}^{5} \gamma_{22}^l,
\]
\[
\frac{\partial \sigma_{33}}{\partial t} = \left\{ \pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^{p l}}{\tau_{\sigma l}} \right) \right] \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) \right\}
\]
\[
- \left\{ 2\nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^{s l}}{\tau_{\sigma l}} \right) \right] \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) \right\} + \sum_{l=1}^{5} \gamma_{33}^l
\]

(4.4)
\[
\frac{\partial \sigma_{12}}{\partial t} = \nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_s^l}{\tau_{\sigma_l}} \right) \right] \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) + \sum_{l=1}^{5} \gamma_{12}^l, \\
\frac{\partial \sigma_{13}}{\partial t} = \nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_s^l}{\tau_{\sigma_l}} \right) \right] \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) + \sum_{l=1}^{5} \gamma_{13}^l, \\
\frac{\partial \sigma_{23}}{\partial t} = \nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_s^l}{\tau_{\sigma_l}} \right) \right] \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) + \sum_{l=1}^{5} \gamma_{23}^l, \\
\frac{\partial \gamma_{11}^l}{\partial t} = -\frac{1}{\tau_{\sigma_l}} \left[ \gamma_{11}^l + \pi \left( \tau^p_{\sigma_l} - 1 \right) \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) - 2\nu \left( \frac{\tau^p_{\sigma_l} - 1}{\tau_{\sigma_l}} \right) \frac{\partial v_2}{\partial x_2} \right] \\
\frac{\partial \gamma_{12}^l}{\partial t} = -\frac{1}{\tau_{\sigma_l}} \left[ \gamma_{12}^l + \pi \left( \tau^p_{\sigma_l} - 1 \right) \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) - 2\nu \left( \frac{\tau^p_{\sigma_l} - 1}{\tau_{\sigma_l}} \right) \frac{\partial v_1}{\partial x_1} \right] \\
\frac{\partial \gamma_{13}^l}{\partial t} = -\frac{1}{\tau_{\sigma_l}} \left[ \gamma_{13}^l + \pi \left( \tau^p_{\sigma_l} - 1 \right) \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) - 2\nu \left( \frac{\tau^p_{\sigma_l} - 1}{\tau_{\sigma_l}} \right) \frac{\partial v_3}{\partial x_1} \right] \\
\frac{\partial \gamma_{22}^l}{\partial t} = -\frac{1}{\tau_{\sigma_l}} \left[ \gamma_{22}^l + \pi \left( \tau^p_{\sigma_l} - 1 \right) \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_3}{\partial x_2} \right) - 2\nu \left( \frac{\tau^p_{\sigma_l} - 1}{\tau_{\sigma_l}} \right) \frac{\partial v_3}{\partial x_2} \right] \\
\frac{\partial \gamma_{23}^l}{\partial t} = -\frac{1}{\tau_{\sigma_l}} \left[ \gamma_{23}^l + \pi \left( \tau^p_{\sigma_l} - 1 \right) \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_3} \right) - 2\nu \left( \frac{\tau^p_{\sigma_l} - 1}{\tau_{\sigma_l}} \right) \frac{\partial v_3}{\partial x_3} \right] \\
\frac{\partial \gamma_{33}^l}{\partial t} = -\frac{1}{\tau_{\sigma_l}} \left[ \gamma_{33}^l + \pi \left( \tau^p_{\sigma_l} - 1 \right) \left( \frac{\partial v_3}{\partial x_1} + \frac{\partial v_3}{\partial x_2} \right) - 2\nu \left( \frac{\tau^p_{\sigma_l} - 1}{\tau_{\sigma_l}} \right) \frac{\partial v_3}{\partial x_2} \right] \\
\frac{\partial \gamma_{12}^l}{\partial t} = -\frac{1}{\tau_{\sigma_l}} \left[ \gamma_{12}^l + \nu \left( \tau^p_{\sigma_l} - 1 \right) \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \right] \\
\frac{\partial \gamma_{13}^l}{\partial t} = -\frac{1}{\tau_{\sigma_l}} \left[ \gamma_{13}^l + \nu \left( \tau^p_{\sigma_l} - 1 \right) \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) \right] \\
\frac{\partial \gamma_{23}^l}{\partial t} = -\frac{1}{\tau_{\sigma_l}} \left[ \gamma_{23}^l + \nu \left( \tau^p_{\sigma_l} - 1 \right) \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) \right] \\
l = 1, 2, 3, 4, 5.
\]

The above equations are cast into a matrix-vector form:

\[
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}_1 \frac{\partial \mathbf{V}}{\partial x} + \mathbf{A}_2 \frac{\partial \mathbf{V}}{\partial x} = \mathbf{Q}.
\]
where the unknown vector

$$\mathbf{V} = [v_1, v_2, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}, \gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{12}, \gamma_{13}, \gamma_{23}, \gamma_{11}/\tau_{\sigma 1}, \gamma_{22}/\tau_{\sigma 2}, \gamma_{33}/\tau_{\sigma 3}, \gamma_{12}/\tau_{\sigma 4}, \gamma_{13}/\tau_{\sigma 5}, \gamma_{23}/\tau_{\sigma 6}],$$

the source term vector

$$\mathbf{Q} = [0, 0, 0, \sum_{l=1}^{5} \gamma_{11 l}, \sum_{l=1}^{5} \gamma_{22 l}, \sum_{l=1}^{5} \gamma_{33 l}, \sum_{l=1}^{5} \gamma_{12 l}, \sum_{l=1}^{5} \gamma_{13 l}, \sum_{l=1}^{5} \gamma_{23 l}, \gamma_{11 l}/\tau_{\sigma 1}, \gamma_{22 l}/\tau_{\sigma 2}, \gamma_{33 l}/\tau_{\sigma 3}, \gamma_{12 l}/\tau_{\sigma 4}, \gamma_{13 l}/\tau_{\sigma 5}, \cdots, \gamma_{23 l}/\tau_{\sigma 6}],$$

and the Jacobian matrix

$$\overline{\mathbf{A}}_1 = \begin{bmatrix} 0_3 & \mathbf{A}_{1v} \\ \mathbf{A}_{1v}^T & 0_{36} \end{bmatrix}.$$
and $A_{1T}$ is a $36 \times 3$ matrix:

$$
\overline{A}_{1T} = \begin{bmatrix}
-\pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{pl}} \right) \right] & 0 & 0 \\
m1 & 0 & 0 \\
m2 & 0 & 0 \\
0 & -\nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{sl}}{\tau_{sl}} \right) \right] & 0 \\
0 & 0 & -\nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{sl}}{\tau_{sl}} \right) \right] \\
m3 & 0_{15} & 0_{15} \\
0_{5} & m4 & 0_{5} \\
0_{5} & 0_{5} & m5 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

where

$$
m1 = \left[ -\pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{pl}} \right) \right] + 2\nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{pl}} \right) \right] \right]$$

$$
m2 = \left[ -\pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{pl}} \right) \right] + 2\nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{pl}} \right) \right] \right]
$$
\[
m_3 = \begin{bmatrix}
\frac{\pi}{\tau_{\sigma_5}} (\tau^{p}_{\varepsilon_1} - 1) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma_1}} (\tau^{p}_{\varepsilon_1} - 1) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma_2}} (\tau^{p}_{\varepsilon_2} - 1) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma_3}} (\tau^{p}_{\varepsilon_3} - 1) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma_4}} (\tau^{p}_{\varepsilon_4} - 1) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma_5}} (\tau^{p}_{\varepsilon_5} - 1) & 0 & 0
\end{bmatrix}
\]

\[
m_4 = \begin{bmatrix}
0 & \frac{\nu}{\tau_{\sigma_1}} (\tau^{s}_{\varepsilon_1} - 1) & 0 \\
0 & \frac{\nu}{\tau_{\sigma_2}} (\tau^{s}_{\varepsilon_2} - 1) & 0 \\
0 & \frac{\nu}{\tau_{\sigma_3}} (\tau^{s}_{\varepsilon_3} - 1) & 0 \\
0 & \frac{\nu}{\tau_{\sigma_4}} (\tau^{s}_{\varepsilon_4} - 1) & 0 \\
0 & \frac{\nu}{\tau_{\sigma_5}} (\tau^{s}_{\varepsilon_5} - 1) & 0
\end{bmatrix}
\]
Based on the Schur complement \cite{37}, the non-trivial eigenvalues of $A_1$ can be obtained by solving the following equation:

$$\det(A_{1v}A_{1\sigma v} - \beta^2 I_3) = 0.$$ 

where $\beta$ is the eigenvalues of $A_1$ and

$$A_{1v}A_{1\sigma v} =
\begin{pmatrix}
\pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{cl}}{\tau_{cl}} \right) \right] & 0 & 0 \\
\rho & \nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{cl}^s}{\tau_{cl}} \right) \right] & 0 \\
0 & \rho & \nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{cl}^s}{\tau_{cl}} \right) \right] \\
0 & 0 & \rho
\end{pmatrix}.$$
$A_1$ has 39 eigenvalues. 33 of them are null and the remaining 6 are

$$\beta_{1,2} = \pm \sqrt{\frac{\pi}{\rho} \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{\sigma l}} \right) \right]}, \quad \beta_{3,4} = \pm \sqrt{\frac{\nu}{\rho} \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{sl}}{\tau_{\sigma l}} \right) \right]}.$$  

(4.7)

$$\beta_{5,6} = \pm \sqrt{\frac{\nu}{\rho} \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{sl}}{\tau_{\sigma l}} \right) \right]}.$$  

(4.8)

The Jacobian matrix

$$\overline{A}_2 = \begin{bmatrix} 0_3 & \overline{A}_{2v} \\ \overline{A}_{2vT} & 0_{36} \end{bmatrix}.$$  

In the Jacobian matrix, $0_3$ and $0_{36}$ denote $3 \times 3$ and $36 \times 36$ null matrices, $A_{2v}$ is a $3 \times 36$ matrix:

$$\overline{A}_{2v} = \begin{bmatrix} 0, & 0, & 0, & -\frac{1}{\rho}, & 0, & \cdots & 0 \\ 0, & -\frac{1}{\rho}, & 0, & 0, & 0, & \cdots & 0 \\ 0, & 0, & 0, & 0, & -\frac{1}{\rho}, & \cdots & 0 \end{bmatrix}.$$
and $A_{2T}$ is a $36 \times 3$ matrix:

\[
A_{2T} = \\
\begin{bmatrix}
0 & m1 & 0 \\
0 & -\pi \left( 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^p_{kl}}{\tau_{\sigma l}} \right) \right) & 0 \\
0 & 0 & m2 \\
-\nu \left( 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^s_{kl}}{\tau_{\sigma l}} \right) \right) & 0 & 0 \\
0 & 0 & -\nu \left( 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^s_{kl}}{\tau_{\sigma l}} \right) \right) \\
m1 & 0 & 0 \\
m2 & 0 & 0 \\
m3 & 0 & 0 \\
m4 & 0 & 0 \\
m5 & 0 & 0 \\
\end{bmatrix}
\]

where

\[
m1 = \left[ -\pi \left( 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^p_{kl}}{\tau_{\sigma l}} \right) \right) + 2\nu \left( 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^p_{kl}}{\tau_{\sigma l}} \right) \right) \right] \\
m2 = \left[ -\pi \left( 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^s_{kl}}{\tau_{\sigma l}} \right) \right) + 2\nu \left( 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^s_{kl}}{\tau_{\sigma l}} \right) \right) \right]
\]
\[
\begin{bmatrix}
\pi \left( \frac{\tau_{e1}^p}{\tau_{\sigma1}} - 1 \right) & 0 & 0 \\
\pi \left( \frac{\tau_{e2}^p}{\tau_{\sigma2}} - 1 \right) & 0 & 0 \\
\pi \left( \frac{\tau_{e3}^p}{\tau_{\sigma3}} - 1 \right) & 0 & 0 \\
\pi \left( \frac{\tau_{e4}^p}{\tau_{\sigma4}} - 1 \right) & 0 & 0 \\
\pi \left( \frac{\tau_{e5}^p}{\tau_{\sigma5}} - 1 \right) & 0 & 0 \\
\pi \left( \frac{\tau_{e1}^s}{\tau_{\sigma1}} - 1 \right) - 2\mu \left( \frac{\tau_{e2}^s}{\tau_{\sigma2}} - 1 \right) - 1 & 0 & 0 \\
\pi \left( \frac{\tau_{e3}^s}{\tau_{\sigma3}} - 1 \right) - 2\mu \left( \frac{\tau_{e4}^s}{\tau_{\sigma4}} - 1 \right) - 1 & 0 & 0 \\
\pi \left( \frac{\tau_{e5}^s}{\tau_{\sigma5}} - 1 \right) - 2\mu \left( \frac{\tau_{e1}^s}{\tau_{\sigma1}} - 1 \right) - 1 & 0 & 0 \\
\pi \left( \frac{\tau_{e2}^s}{\tau_{\sigma2}} - 1 \right) - 2\mu \left( \frac{\tau_{e3}^s}{\tau_{\sigma3}} - 1 \right) - 1 & 0 & 0 \\
\pi \left( \frac{\tau_{e4}^s}{\tau_{\sigma4}} - 1 \right) - 2\mu \left( \frac{\tau_{e5}^s}{\tau_{\sigma5}} - 1 \right) - 1 & 0 & 0 \\
\pi \left( \frac{\tau_{e5}^s}{\tau_{\sigma5}} - 1 \right) - 2\mu \left( \frac{\tau_{e1}^s}{\tau_{\sigma1}} - 1 \right) - 1 & 0 & 0 \\
\end{bmatrix}
\]

\[
m3 = \begin{bmatrix}
0 & \nu \left( \frac{\tau_{e1}^s}{\tau_{\sigma1}} - 1 \right) & 0 \\
\nu \left( \frac{\tau_{e2}^s}{\tau_{\sigma2}} - 1 \right) & 0 & 0 \\
\nu \left( \frac{\tau_{e3}^s}{\tau_{\sigma3}} - 1 \right) & 0 & 0 \\
\nu \left( \frac{\tau_{e4}^s}{\tau_{\sigma4}} - 1 \right) & 0 & 0 \\
\nu \left( \frac{\tau_{e5}^s}{\tau_{\sigma5}} - 1 \right) & 0 & 0 \\
\end{bmatrix}
\]
Based on the Schur complement \[37\], the non-trivial eigenvalues of $A_2^2$ can be obtained by solving the following equation:

$$
\det(A_{2v}A_{2\sigma v} - \beta^2 I_3) = 0.
$$

where $\beta$ is the eigenvalues of $A_2$ and

$$
A_{2v}A_{2\sigma v} =
\begin{bmatrix}
\pi \left[ 1 - \sum_{l=1}^5 \left( 1 - \frac{\tau_{cl}^s}{\tau_{cl \sigma l}} \right) \right] & 0 & 0 \\
0 & \nu \left[ 1 - \sum_{l=1}^5 \left( 1 - \frac{\tau_{cl}^s}{\tau_{cl \sigma l}} \right) \right] & 0 \\
0 & 0 & \nu \left[ 1 - \sum_{l=1}^5 \left( 1 - \frac{\tau_{cl}^s}{\tau_{cl \sigma l}} \right) \right]
\end{bmatrix}.
$$
A_2 has 39 eigenvalues. 33 of them are null and the remaining 6 are

\[ \beta_{1,2} = \pm \sqrt{\frac{\pi}{\rho} \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{p l}}{\tau_{s l}} \right) \right]}, \quad \beta_{3,4} = \pm \sqrt{\frac{\nu}{\rho} \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{s l}}{\tau_{s l}} \right) \right]}, \]

\[ \beta_{5,6} = \pm \sqrt{\frac{\nu}{\rho} \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{s l}}{\tau_{s l}} \right) \right]} \] (4.9)

For three-dimensional equations, the Jacobian matrix just has one more A_3 than two-dimensional equations. For concise purpose, I do not show the derivation.

### 4.3 Results and Discussions

Generally, porcine brain tissue is chosen as a substitute for human brain because the machanical response of fresh porcine brain tissue was reported to be similar with human brain tissue. The differences between human brains and animal brains are often considered relatively small which enables animal brains to be a good substitute for human brains [24].

While many experiments provide the relaxation function of annial brain tissue, almost all of them provide the data only in longitudinal or shear direction. However, M.Hrapko [24] shows an experimental data for both longitudinal and shear direction of porcine brain tissue. With such a condition, we could obtain an accurate data of brain properties obtained from the same sample. Then we need to use the collocation method to fit this experimental data so that we could reconstruct the relaxation functions which are shown below:
\[
G(t) = 0.1480 + 0.0336e^{-\frac{t}{0.0021}}
- 0.1743e^{-\frac{t}{0.0005}} + 0.5903e^{-\frac{t}{0.0009}} + 0.2513e^{-\frac{t}{0.0011}} + 0.1493e^{-\frac{t}{2.1418}}
\]

\[
G_s(t) = 0.3726 - 0.0071e^{-\frac{t}{0.0020}}
- 0.0529e^{-\frac{t}{0.0009}} + 0.2811e^{-\frac{t}{0.0222}} + 0.2138e^{-\frac{t}{0.0205}} + 0.1955e^{-\frac{t}{2.4706}}.
\] (4.10)

Fig. 4.1 shows original relaxation function data obtained by quasi-static tension tests [24], which also could be considered to be based on Fung’s model, and Fig. 4.2 shows the reconstructed relaxation function compared to the original one by using five internal variables, which correspond to five parallel-connected SLS models. By using five internal variables, Fung’s model for porcine brain tissue has been successfully
reconstructed. The values of parameters in the reconstructed model are calculated by the collocation method and are tabulated in Table 4.1.

Now, we are going to apply the CESE method to solve Eq. (4.3). However, before doing so, it is better for us to validate the numerical method. Here we consider one-dimensional longitudinal waves in Maxwellian medium, which is a special case of the SLS model. Numerical solutions of wave in the medium is used to assess numerical accuracy by comparing the numerical results with the derived analytical solution. In what follows, we proceed to derive the analytical solution of the impact wave propagating in the Maxwellian medium.

The relation between the normal stress $\sigma$ and the normal strain $\epsilon$ of a Maxwellian medium is \( \partial \epsilon / \partial t = \partial v / \partial x = (1/E) \partial \sigma / \partial t + \mu \sigma \), where $v$ is the velocity in the uni-axial direction, $E$ and $\mu$ are two constants of the Maxwellian medium. The equation of motion is \( \partial \sigma / \partial x = \rho \partial v / \partial t \), where $\rho$ is the density of the unstrained material. By
Table 4.1: Parameters of the reconstructed experimental data by using 5 SLS models.

<table>
<thead>
<tr>
<th>Relaxation Functions</th>
<th>$G_l$</th>
<th>$G_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_0$</td>
<td>0.1480</td>
<td>0.3726</td>
</tr>
<tr>
<td>$\tau_{\sigma 1}$</td>
<td>0.0021</td>
<td>0.0020</td>
</tr>
<tr>
<td>$\tau_{\epsilon 1}$</td>
<td>0.0026</td>
<td>0.0020</td>
</tr>
<tr>
<td>$\tau_{\sigma 2}$</td>
<td>0.0053</td>
<td>0.0057</td>
</tr>
<tr>
<td>$\tau_{\epsilon 2}$</td>
<td>-0.0009</td>
<td>0.0049</td>
</tr>
<tr>
<td>$\tau_{\sigma 3}$</td>
<td>0.0209</td>
<td>0.0222</td>
</tr>
<tr>
<td>$\tau_{\epsilon 3}$</td>
<td>0.1041</td>
<td>0.039</td>
</tr>
<tr>
<td>$\tau_{\sigma 4}$</td>
<td>0.2351</td>
<td>0.2665</td>
</tr>
<tr>
<td>$\tau_{\epsilon 4}$</td>
<td>0.6345</td>
<td>0.4188</td>
</tr>
<tr>
<td>$\tau_{\sigma 5}$</td>
<td>2.1516</td>
<td>2.4904</td>
</tr>
<tr>
<td>$\tau_{\epsilon 5}$</td>
<td>4.3230</td>
<td>3.7969</td>
</tr>
</tbody>
</table>

Combining the two equations, we have the governing equation for the normal stress:

$$\frac{\partial^2 \sigma}{\partial x^2} - \frac{1}{c^2} \left( \frac{\partial^2 \sigma}{\partial t^2} + \frac{1}{\tau_0} \frac{\partial \sigma}{\partial t} \right) = 0,$$

where $c = \sqrt{E/\rho}$ is the elastic wave velocity, and $\tau_0 = 1/(E\mu)$ is the relaxation time. For the boundary conditions, at $x = 0$, $v = VH(t)$, where $H(t)$ is the Heaviside function, and at $x = \infty$, $\sigma = 0$. Equation (4.11) is a particular form of the telegraph equation. To proceed, we perform the Laplace transform to Eq. (4.11) and obtain

$$\frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{1}{c^2} \left( s^2 + \frac{s}{\tau_0} \right) \bar{\sigma} = 0,$$

where the over-bar denotes the transformed unknown and $s$ is the Laplace parameter. Equation (4.12) is a second-order ordinary differential equation. To solve the equation, two boundary conditions on the two end of the spatial domain are needed. These boundary conditions can be obtained by performing the Laplace transformation to the original boundary conditions, i.e., at $x = 0$, $\bar{v} = V/s$ and at $x = \infty$, $\bar{\sigma} = 0$. The
boundary condition at \( x = 0 \) is for \( \ddot{v} \), which can be changed to be a condition for \( \ddot{\sigma} \) by performing the Laplace transformation to the equation of motion: \( \partial\ddot{\sigma}/\partial x = \rho s \ddot{v} \). The analytical solution of \( \sigma \) can be obtained by performing the inverse Laplace transformation and was provided by Lee [32]:

\[
\sigma = -\rho c V e^{-\frac{t}{\tau_0}} I_0 \left( \frac{\sqrt{t^2 - \frac{x^2}{c^2}}}{2\tau_0} \right) H \left( t - \frac{x}{c} \right),
\]

where \( I_0 \) is the zeroth-order Bessel function of the first kind. The dimensionless form of the analytical solution is

\[
-\frac{\sigma}{\rho c V} = e^{-\frac{\tau}{\tau_0}} I_0 \left( \frac{\sqrt{\tau^2 - \frac{\xi^2}{2}}}{2} \right) H(\tau - \xi).
\]

where dimensionless time \( \tau = t/\tau_0 \) and the dimensionless distance \( \xi = x/c\tau_0 \).

For numerical solutions, we use the CESE method to solve Eq. (4.3), i.e., the one-dimensional governing equations with one internal variable for longitudinal waves. As a typical medium, we let \( \rho = 800 \text{kg/m}^3 \) and \( E = 1 \times 10^9 \text{ Pa} \). The wave speed is \( c = \sqrt{E/\rho} = 1118 \text{m/s} \). The wave speed is the eigenvalue of the Jacobian matrix \( \hat{A}_1 \). Shown in Eq. (4.9), the wave speed \( c = \sqrt{\pi \tau_p / \rho \tau_\sigma} \). The values of three constants in Eq.(4.3) are chosen to be \( \pi = 10 \times 10^7 \text{ N/m}^2 \), \( \tau_p = 0.1 \text{ s} \), and \( \tau_\sigma = 0.001 \text{ s} \). These values are chosen so that the wave speed is consistent with the wave speed of the analytical solution. In the calculations, the length of the one-dimensional domain is 20 m, \( \Delta x = 0.025 \text{ m} \) and \( \Delta t = 2 \times 10^{-5} \text{ s} \). The impact velocity is \( V = 1 \text{ m/s} \).

Figure 4.3 shows the numerical solution and the analytical solution of the moving impact wave in the Maxwellian medium. Three snapshots of dimensionless stress profiles at three different times are presented. The analytical solutions are presented by lines. The numerical solutions are presented by symbols. The solution jump at the wave fronts is resolved by about three mesh nodes. The result shows the
calculated wave speed is a constant, which matches that of the analytical solution. The amplitude of the wave decays continuously due to the relaxation effect. The calculated wave amplitudes are slightly higher than that of the analytical solution. Overall, numerical solutions compare well with the analytical solutions.

After validation, we then applied the CESE method to solve the one-dimensional governing Eq. (4.3) for human brain. Numerical results of moving waves are shown in Fig. 4.4. Three snapshots of stress profiles at three different times are presented. In this figure, we could observe that the amplitude of the wave front remains the same for all times, which means no relaxation effect is happened. However, experiments [30] shows that soft tissues exhibit wave absorption effects at high frequencies. The results imply that Fung’s model is not useful for modeling dynamics for short time durations or for waves at high frequencies since we are interested in high speed of wave propagation in the human brain.

Fig. 4.3: Comparison between the analytical solution and the numerical solution of the normal stress at $\tau = 1, 5$ and $8$. 

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To this end, Iatridis et al. [26] extended Fung’s model with the following the relaxation function:

\[ G(t) = \frac{1 + \int_0^\infty S(\tau) \exp(-t/\tau) d\tau}{1 + \int_0^\infty S(\tau) d\tau} \]  

(4.13)

where \( S(\tau) \) is the relaxation spectrum:

\[
S(\tau) = \begin{cases} 
\frac{c_1}{\tau} + \frac{c_2}{\tau^2}, & \text{for } \tau_1 \leq \tau \leq \tau_2, \\
0, & \text{for } \tau < \tau_1, \tau > \tau_2.
\end{cases} 
\]  

(4.14)

c_1 is the amplitude of the relaxation effect and \( c_2 \) is a constant representing linear increase in the relaxation effect with respect to frequencies. Similar to that in Fung’s model, \( \tau_1 \) and \( \tau_2 \) are two times corresponding to the slow and fast limits of the relaxation function.

To proceed, we substitute Eq. (4.14) into Eq. (4.13) and obtain

\[
G(t) = \left[1 + \frac{c_1 \ln \left( \frac{\tau_2}{\tau_1} \right) + c_2 \left( \frac{1}{\tau_1} - \frac{1}{\tau_2} \right)}{1 + \frac{c_1}{\tau_1} + \frac{c_2}{\tau_2} \exp(-t/\tau_2) - \exp(-t/\tau_1)} \right]^{-1} \left\{1 + \frac{c_1}{\tau} \left[ \eta \left( \frac{t}{\tau_2} \right) - \eta \left( \frac{t}{\tau_1} \right) \right] + \frac{c_2}{t} \left[ \exp(-t/\tau_2) - \exp(-t/\tau_1) \right] \right\} 
\]  

(4.15)

where \( \eta \) is defined as that in Fung’s model. By matching curve of \( G \) and curve of wave absorption coefficient of human brain, as will be discussed in Chapter 5, \( c_1 \) and \( c_2 \) could be determined. \( \tau_1 \) and \( \tau_2 \) can be determined by choosing similar frequency range in Fig. 4.1. With these values determined, Eq. (4.15) can be plotted as Fig. 4.5. We use 5 parallel-connected SLS models to reconstruct Iatridis’ model by using the collocation method. For completeness, the parameters in Fig. 4.5 are provided in Table 4.2. Fig. 4.5 shows the relaxation function plotted against the logarithm function of time. The plotted curve in the range of short time periods, e.g., \( 10^{-4} \) to
Table 4.2: Reconstructed data of relaxation function based on Iatridis’ model by using 5 SLS modules.

<table>
<thead>
<tr>
<th>Relaxation Functions</th>
<th>( G_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi )</td>
<td>0.72719</td>
</tr>
<tr>
<td>( \tau_{\sigma 1} )</td>
<td>( 2.0 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \tau_{\tau 1} )</td>
<td>( 2.1573 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \tau_{\sigma 2} )</td>
<td>( 6.7046 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \tau_{\tau 2} )</td>
<td>( 6.9582 \times 10^{-4} )</td>
</tr>
<tr>
<td>( \tau_{\sigma 3} )</td>
<td>( 2 \times 10^{-3} )</td>
</tr>
<tr>
<td>( \tau_{\tau 3} )</td>
<td>( 2.3 \times 10^{-3} )</td>
</tr>
<tr>
<td>( \tau_{\sigma 4} )</td>
<td>( 2.02 \times 10^{-2} )</td>
</tr>
<tr>
<td>( \tau_{\tau 4} )</td>
<td>( 2.26 \times 10^{-2} )</td>
</tr>
<tr>
<td>( \tau_{\sigma 5} )</td>
<td>( 6.46 \times 10^{-2} )</td>
</tr>
<tr>
<td>( \tau_{\tau 5} )</td>
<td>( 6.58 \times 10^{-2} )</td>
</tr>
</tbody>
</table>

\( 10^{-1} \) has a negative and steep slope, indicating pronounced relaxation effect in this range of time.

We then applied CESE method once again to solve the governing equations, and this time we considered impact wave propagation in brain, modeled by Ledoux et al. [30]. The model was an extension of that by Iatridis et al. [26]. Here we call Iatridis’ model modified Fung’s model. As shown in Table 4.2, 5 internal variables are used for this case. In numerical calculation, we let \( dx = 0.16 \times 10^{-2} \) m, \( dt = 1 \times 10^{-7} \) s. Numerical results are shown in Fig. 4.6. Three snapshots of stress profiles at three different times are shown. The amplitude of the wave front decays continuously as the wave propagates to the right. The results show apparent relaxation effect.
Fig. 4.4: Dispersion effect of Fung’s model

Fig. 4.5: Reconstruction of modified Fung’s model
Fig. 4.6: Dispersion effect of modified Fung’s model
4.4 Conclusions

In this chapter, a new theoretical and numerical framework has been developed to calculate waves propagating in human brain. A generalized constitutive model based on paralleling connected SLS models is employed to represent the original Fung’s model and Iatridis’ model called modified Fung’s model for material response of brain. To be coupled with the equation of motion, the constitutive relation formulated in an integral form is transformed into a differential form by using the method of internal variables. The curve-fitting process is done by using the collocation method. The complete governing equations include the equation of motion, the constitutive relation between stress and the rate of strain, and the equations of internal variables. The governing equations can be cast into a set of first-order, fully coupled, hyperbolic PDEs with source terms.

To demonstrate the capabilities of the numerical method, we considered one-dimensional longitudinal waves in Maxwellian medium, which is a special case of the SLS model. As a result, numerical solutions compare well with the analytical solutions. Then one-dimensional equations are solved by the CESE method for time-accurate solutions of propagating longitudinal waves in the human brain. Results of two cases are reported. In the first case, we employed the experimental data obtained based on quasi-static stress test which could be considered to be based on Fung’s model. The governing equations include five internal variables. The results show no relaxation effect by employing Fung’s model within the short time duration of the traveling impact wave. Since Fung’s model did not capture the relaxation effect for high frequency dynamics problem, we proceed to employ a modified Fung’s model developed by Iatridis et al. [26]. The numerical results show apparent relaxation effect in the material response, which shows that Iatridis’ model is able to capture the relaxation effect for the dynamics problem with the short time duration.
CHAPTER 5
WAVE ABSORPTION IN BRAIN TISSUES

5.1 Introduction

It is generally recognized that brain tissues are viscoelastic media, in which the material response to the applied stress is dictated by the memory or relaxation effect. This key attribute of viscoelastic media demands hereditary integration in the constitutive relation when simulating wave dynamics so that the observed wave dispersion and dissipation due to relaxation of the medium can be correctly calculated.

It is well known that the functional form of the relaxation functions in these models are particularly useful for modeling material response of soft tissues. Conventionally, parameters in the relaxation function employed have been determined by quasi-static compression/tension and/or shear tests. Such approach, however, may not be useful for dynamic problems characterized by really short time durations and cyclical loadings at high frequencies. In this chapter, instead of using quasi-static testing data, we will use the measured wave absorption coefficients, which are widely available for many soft tissues, to determine the relaxation functions in Iatridis’ model [26] which has been proven as an effective model for short time durations. Eventually, the dispersion of wave propagation could be clearly observed on the numerical results.
5.2 Absorption Coefficient and Complex Modulus

In this section, we illustrated the use of Iatridis’ model for modeling wave motion in the human brain. There are three subsections. The first subsection showed that the measured wave absorption coefficients could be connected with the Carson transform of a generic relaxation function of a viscoelastic relation. In the second subsection, Iatridis’ relaxation functions were transformed to the frequency domain to be connected to the measured wave absorption coefficients that was obtained in the experiment. The third subsection illustrated the parallel connected SLS models for numerical calculation.

5.2.1 Wave Absorption and Viscoelasticity

Based on observation, wave amplitudes decay as waves propagate in soft tissues. The wave absorption effect has been usually represented by an exponential decay function:

\[ A(x) = A_0 \exp(-\mu_A x), \quad (5.1) \]

where \( A(x) \) is the magnitude of the propagating stress wave at location \( x \) after a certain period of time, \( A_0 \) is the initial wave amplitude, and \( \mu_A \) is the wave absorption coefficient and it is positive. In experiments, \( A_0 \) is given and \( A(x) \) are recorded. The measured absorption effect were commonly presented as decrease in decibels (dB), i.e., \( 20 \log_{10}(A(x)/A_0) \). The wave absorption coefficient \( \alpha \) is then defined as

\[ \alpha = 10 \log_{10}(A(x)/A_0)^2/x = 20 \log_{10}(e)\mu_A \approx 8.7\mu_A. \quad (5.2) \]

For soft tissues, the absorption coefficients, i.e., \( \alpha \) and \( \mu_A \), strongly depends on the wave frequency. The experimental results were often presented in the form of a power law:

\[ \alpha = a f^b, \quad (5.3) \]
where \( \hat{f} \) is the frequency measured in MHz, and \( a \) and \( b \) are experimentally determined constants, depending on the condition of the medium, e.g., temperature, humidity, PH value, etc. For most of soft tissues, \( b \geq 1 \). One can find measured data of \( \alpha \) presented in the form of Eq. (5.3) for many soft tissues in the literature.

In this chapter, an extended Fung’s model by Iatridis et al. [26] is employed to model material response of brain tissues. The key contribution of the present chapter is the use of measured absorption coefficients \( \alpha \) to determine the relaxation functions in the constitutive relations. As will be shown in the following section, the essential derivation is to relate the measured \( \alpha \) to the imaginary part of the transformed relaxation function by the Fourier transformation in the frequency domain. The relation in the frequency domain can fully determine the parameters in the original relaxation function employed.

To proceed, we consider a generic viscoelastic constitutive relation with a stress-free initial condition:

\[
\sigma_{ij}(t) = \int_{0}^{t} G_{ijkl}(t - \tau) \frac{\partial \epsilon_{kl}}{\partial \tau} d\tau. \tag{5.4}
\]

where \( \sigma_{ij} \) is the stress tensor, \( \epsilon_{kl} \) is the strain tensor, and \( G_{ijkl} \) is the relaxation function, which in general is a fourth-order tensor. For one-dimensional longitudinal waves in a homogeneous medium, Eq. (5.4) is simplified to be

\[
\sigma = G(t) * \frac{\partial \epsilon}{\partial t} \tag{5.5}
\]

where \( \sigma = \sigma_{11} \) is the normal stress on the \( x_1 \) surface, \( G \) is the relaxation function, \( \epsilon = \epsilon_{11} = \partial u_1/\partial x_1 = \partial u/\partial x \) is the normal strain with \( u \) as the displacement, and \( * \) represents the convolution integral. Aided by the convolution theorem, the constitutive relation Eq. (5.5) is equivalent to

\[
\tilde{\sigma} = \tilde{G} \tilde{\epsilon} = \tilde{G} \tilde{\epsilon}, \tag{5.6}
\]

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where the symbol dot on top of a variable denotes differentiation by time, and the last equality in Eq. (5.6) is obtained with integration by part. The symbol tilde denotes the transformed variables by the Fourier transformation in time:

\[
\tilde{G}(\omega) = \int_{-\infty}^{\infty} \frac{\partial G(t)}{\partial t} e^{i\omega t} dt = G' + iG'',
\]

\[
\tilde{\epsilon} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\omega t} dt = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} u e^{i\omega t} dt = \frac{\partial \tilde{u}}{\partial x}.
\]

where \( \omega \) is the angular frequency with the unit rad/s. The transform relaxation function \( \tilde{G}(\omega) \) is a function of \( \omega \). It is complex with its real and imaginary parts denoted by \( G' \) and \( G'' \), respectively. To proceed, we consider the one-dimensional momentum equation governing the longitudinal waves in the \( x \) direction:

\[
\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}.
\]  

(5.7)

By applying the Fourier transform in time to Eq. (5.7) we obtain

\[
\frac{\partial \tilde{\sigma}}{\partial x} = \rho (i\omega)^2 \tilde{u}.
\]  

(5.8)

Next, we apply the Fourier transform to Eq. (5.8) in space. Aided by Eq. (5.6), we obtain

\[
\int_{-\infty}^{\infty} \frac{\partial}{\partial x} (\tilde{G}(\omega) \tilde{\epsilon}) e^{ikx} dx = \int_{-\infty}^{\infty} \rho (i\omega)^2 \tilde{u} e^{ikx} dx,
\]

\[
\tilde{G}(\omega) (ik)^2 \int_{-\infty}^{\infty} \tilde{u} e^{ikx} dx = \rho (i\omega)^2 \int_{-\infty}^{\infty} \tilde{u} e^{ikx} dx.
\]

After cancelling the equivalent terms on both sides, we obtain the following relation:

\[
\tilde{G}(\omega) k^2 = \rho \omega^2.
\]  

(5.9)

To proceed, we consider the solution of a plane wave with damping: \( A = A_0 \exp [i(\omega t - kx)] \), where \( A \) is the magnitude of the wave and the wave number

\[
k = \kappa - i\mu_A
\]

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is a complex with the damping effect represented by $\mu_A$. The phase speed of the plane wave $v_p = \omega/\kappa$. Aided by complex $k$, simple manipulation shows that the imaginary part of Eq. (5.9) becomes

$$\frac{2\omega^2 \kappa \mu_A}{(\kappa^2 + \mu_A^2)^2} = \frac{G''}{\rho}. \quad (5.10)$$

If $\kappa \gg \mu_A$, i.e., relatively small damping, Eq. (5.10) becomes $\mu_A = (\pi G'' f)/(\rho v_p^3)$, or $\alpha = (8.7 \pi G'' f)/(\rho v_p^3) \approx a f^b$. Finally, the dispersion relation can be related to $G''$ as

$$4.35 \frac{G'' \omega}{\rho v_p^3} = a \left(10^6 \frac{\omega}{2\pi}\right)^b. \quad (5.11)$$

We note that the frequency $f = \omega/2\pi$ in Hz is equal to $10^6 \hat{f}$, where $\hat{f}$ is the frequency in MHz. In general, the phase velocity $v_p$ is constant within one type of soft tissue. This assumption has been extensively verified by experiments [48]. Within one soft tissue, density $\rho$ is also constant. Thus Eq. (5.11) shows that $G''$ is a function of frequency $\omega$.

### 5.2.2 Transformed Relaxation Functions

In this subsection, we proceed to derive $G''$ of Fung’s model and the relaxation function $G(t)$ is defined as

$$G(t) = \frac{1 + c [\eta(t/\tau_2) - \eta(t/\tau_1)]}{1 + c \ln(\tau_2/\tau_1)}, \quad (5.12)$$

where

$$\eta(z) = \int_z^{\infty} \frac{e^{-t}}{t} dt, \quad (5.13)$$

$c$ is a dimensionless constant, and $\tau_1$ and $\tau_2$ are the slow and fast relaxation times, respectively. To proceed, we apply the Fourier transformation to the time derivative of the Fung’s relaxation function $\dot{G}$, where $G$ is defined by Eq. (5.12). This
transformation is equivalent to the Carson transformation of the relaxation function $G$ itself. We note that for a generic function $f(t)$, its Carson transform [18] is $F(s) = s \int_0^\infty f(t)e^{-st}dt$. The Carson transform is related to the Laplace transform: $F(s) = \mathcal{L}[f(t) - f(0)] = \tilde{f} - f(0)$, where $\tilde{f}(s)$ and $\tilde{f}(s)$ are the Laplace transform of $f(t)$ and $\dot{f}(t)$, respectively. Moreover, the Laplace transform is related to the Fourier transform in time by letting $s = -i\omega$. The transformed $\dot{G}$ is also available in [20] and it is

$$
\tilde{\dot{G}}(\omega) = G'(\omega) + iG''(\omega)
$$

$$
= \frac{1}{1 + c\ln \tau_2/\tau_1} \left[ \eta \left( \frac{t}{\tau_2} \right) - \eta \left( \frac{t}{\tau_1} \right) \right] + \frac{c}{1 + c\ln \tau_2/\tau_1} \left[ \tan^{-1}(\omega\tau_2) - \tan^{-1}(\omega\tau_1) \right].
$$

(5.14)

We then substitute the derived $G''$ into the dispersive relation Eq. (5.11). With measured $a$ and $b$ in Eq. (5.11), we can determine $c$, $\tau_1$, and $\tau_2$ in $G''$, which in turn would completely determine the relaxation function $G$ itself. We remark that there are only three parameters $c$, $\tau_1$, and $\tau_2$ in Fung’s model. The slow time $\tau_1$ and the fast time $\tau_2$ must be used to determine the frequency range, i.e., $1/\tau_1 \leq 2\pi f \leq 1/\tau_2$. This leave $c$ as the only parameter to determine the damping effect. As a result, the function form of Fung’s model is such that $G''$ is a linear function of $\omega$ and the damping would be insensitive to wave frequencies within the range decided by $\tau_1$ and $\tau_2$. In order words, if one uses Fung’s model, one is committed to let $b = 1$ for the wave frequencies of interest.

Next, we consider the relaxation function of Iatridis’ model

$$
G(t) = \frac{1 + c_1 \left[ \eta \left( \frac{t}{\tau_2} \right) - \eta \left( \frac{t}{\tau_1} \right) \right] + c_2 \left[ e^{-t/\tau_2} - e^{-t/\tau_1} \right]}{1 + c_1 \ln \left( \frac{\tau_2}{\tau_1} \right) + c_2 \left( \frac{1}{\tau_1} - \frac{1}{\tau_2} \right)}.
$$

(5.15)

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where \( \eta \) is identical to that in Fung's model, i.e., Eq. (5.13). \( c_1 \) and \( c_2 \) are two constants. We apply the Carson transformation to the above relaxation function to yield

\[
\tilde{G}(\omega) = G' + iG'' = \left(1 + \frac{c_1}{2} \ln \left(1 + \omega^2 \tau_2^2 \right) - \ln \left(1 + \omega^2 \tau_1^2 \right)\right) + c_2 \omega \left[\tan^{-1}(\omega \tau_2) - \tan^{-1}(\omega \tau_1)\right] \\
+ i \left(1 + c_1 \ln \left(\frac{\tau_2}{\tau_1}\right) + c_2 \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)\right) + \frac{c_2 \omega}{2} \left[\ln \left(\frac{\tau_2^2}{1 + \omega^2 \tau_2^2}\right) - \ln \left(\frac{\tau_1^2}{1 + \omega^2 \tau_1^2}\right)\right]
\]

(5.16)

The derived expression of \( G'' \) can be substitute into the wave dispersion relation Eq. (5.11) to determine the parameters in Iatridis’ relaxation function. \( \tau_1 \) and \( \tau_2 \) are chosen such that the frequency range of interest is included. \( c_1 \) and \( c_2 \) can then be determined by matching the curve of absorption coefficient of the brain with the relaxation function of the brain in desired frequency range.

### 5.3 Governing Equations

The complete governing equations for longitudinal wave propagation in a brain tissue by using \( L \) internal variables are formed by Eq. (4.2). Without losing any generality,
the number of SLS models and the internal variables are \( L = 6 \).

\[
\begin{align*}
\frac{\partial v_1}{\partial t} &= \frac{\partial \sigma_{11}}{\partial x_1}, \\
\frac{\partial \sigma_{11}}{\partial t} &= \left\{ \pi \left[ 1 - \sum_{l=1}^{6} \left( 1 - \frac{\tau_{pl}}{\tau_{\sigma l}} \right) \right] \frac{\partial v_1}{\partial x_1} \right\} + \sum_{l=1}^{6} \gamma_{11}^l, \\
\frac{\partial \gamma_{11}^l}{\partial t} &= -\frac{1}{\tau_{\sigma l}} \left[ \gamma_{11}^l + \pi \left( \frac{\tau_{pl}}{\tau_{\sigma l}} - 1 \right) \frac{\partial v_1}{\partial x_1} \right].
\end{align*}
\] (5.17)

\( l = 1, \ldots, 6 \).

Rewrite them as the hyperbolic form:

\[
\frac{\partial \mathbf{V}}{\partial t} + \overline{A}_1 \frac{\partial \mathbf{V}}{\partial t} = \mathbf{Q}. 
\] (5.18)

where \( \mathbf{V}, \overline{A}_1 \) and \( \mathbf{Q} \) are given by

\[
\mathbf{V} = [v_1, \sigma_{11}, \gamma_{11}^l]^T, \\
\mathbf{Q} = \frac{1}{\rho} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{V}.
\] (5.19)

\[
\overline{A}_1 = \begin{bmatrix} \mathbf{0}_1 & \overline{\mathbf{A}}_{1v} \\ \overline{\mathbf{A}}_{1T} & \mathbf{0}_7 \end{bmatrix},
\]

\[
\overline{\mathbf{A}}_{1v} = \begin{bmatrix} -\frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]
\[ \bar{\mathbf{A}}_{1T} = \begin{pmatrix} \pi \left[ 1 - \sum_{l=1}^{6} \left( 1 - \frac{\tau_{\sigma l}}{\tau_{\sigma l}} \right) \right] \\
\pi \left( \frac{\tau_{\sigma 1}}{\tau_{\sigma 1}} - 1 \right) \\
\frac{\tau_{\sigma 2}}{\pi} \left( \frac{\tau_{\sigma 2}}{\tau_{\sigma 2}} - 1 \right) \\
\frac{\tau_{\sigma 3}}{\pi} \left( \frac{\tau_{\sigma 3}}{\tau_{\sigma 3}} - 1 \right) \\
\frac{\tau_{\sigma 4}}{\pi} \left( \frac{\tau_{\sigma 4}}{\tau_{\sigma 4}} - 1 \right) \\
\frac{\tau_{\sigma 5}}{\pi} \left( \frac{\tau_{\sigma 5}}{\tau_{\sigma 5}} - 1 \right) \\
\frac{\tau_{\sigma 6}}{\pi} \left( \frac{\tau_{\sigma 6}}{\tau_{\sigma 6}} - 1 \right) \end{pmatrix} \]

\[ \mathbf{Q} = \begin{pmatrix} 0, \sum_{l=1}^{6} \gamma_{l 11}, -\gamma_{11}, \ldots, -\gamma_{61}^{\tau_{\sigma 6}} \end{pmatrix}^T. \quad (5.20) \]

For nonzero eigenvalues of jacobian matrix \( \bar{\mathbf{A}}_{1} \), we have

\[ \bar{\beta}_{1,2} = \pm \sqrt[\rho]{\pi \left[ 1 - \sum_{l=1}^{6} \left( 1 - \frac{\tau_{\sigma l}}{\tau_{\sigma l}} \right) \right] \}. \quad (5.21) \]

### 5.4 Results and Discussions

We proceed to employ modified Fung’s model and use the measured absorption coefficients data to determine the relaxation functions of brain tissue. According to A. Etoh et al.[19], the measured absorption coefficients in bovine brain is shown in a log-log plot of absorption coefficient versus frequency.

By using data dig software, the measured data were digitized, and via power-law fitting method, the experimental data could be transformed into analytical form as follows:

\[ \alpha = 0.082 f^{1.129}, \quad (5.22) \]
where $f$ is frequency in MHz. The measured wave speed in bovine brain tissue is 1560 m/s. The density of brain tissue is $1.10 \times 10^3 kg/m^3$. The frequency range of interest is between 1 to 7 MHz.

According to A. Etoh [19], Young’s modulus $G_0$ is $2.676 \times 10^9 Pa$ obtained by considering the brain as a elastic model. For viscoelastic model, Young’s modulus $G_0$ could be obtained when $t$ is in initial. In this case, $G_0$ is $0.98666 \times 2.676 \times 10^9 Pa$.

The four parameter values of modified Fung’s model can be adjusted to fit the values shown in Table 5.1. That is, $\tau_1$, $\tau_2$, $c_1$ and $c_2$ can be selected as $10^{-10}$, $10^{-4}$, $0.000259$ and $10^{-12}$.

Similarly, The generalized 6-parameters SLS model is used to fit the modified Fung’s model in time domain. It can be written as

$$G_p(t) = 0.98666 + 0.0011 e^{-\frac{t}{3.1567 \times 10^{-9}}} - 0.000003688 e^{-\frac{t}{1.0568 \times 10^{-8}}}$$
$$+ 0.0006136 e^{-\frac{t}{3.3761 \times 10^{-8}}} + 0.0005527 e^{-\frac{t}{3.3444 \times 10^{-7}}}$$
$$+ 0.0007203 e^{-\frac{t}{3.2638 \times 10^{-6}}} + 0.0002533 e^{-\frac{t}{3.1123 \times 10^{-5}}}.$$ (5.23)

For completeness, the parameters for $G_p$ are provided in the Table 5.1.

Correspondingly, in the frequency domain, the 6-parameters absorption modulus $G''$ are give as

$$G'' = \sum_{i=1}^{6} \left( \frac{\omega \tau_{\sigma_i} G_i}{1 + \omega^2 \tau_{\sigma_i}^2} \right),$$ (5.24)

where $G_i$ is the coefficient of Eq.(5.23), e.g., 0.0025, 0.000369, 0.000744, 0.000132, 0.000511 and 0.000753.

5.4.1 Calculation of Ricker’s Wavelet Propagation by CESE Method

The Ricker’s wavelet (Mexican hat wavelet) can be expressed as

$$R(t) = (1 - 2\pi^2 f^2 t^2) \exp(-\pi^2 f^2 t^2),$$ (5.25)
Table 5.1: Parameters of the reconstructed data by using 6 SLS modules.

<table>
<thead>
<tr>
<th>Relaxation Functions</th>
<th>$G_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>0.98666</td>
</tr>
<tr>
<td>$\tau_{\sigma_1}$</td>
<td>$6.3134 \times 10^{-9}$</td>
</tr>
<tr>
<td>$\tau_{\epsilon_1}$</td>
<td>$6.3202 \times 10^{-9}$</td>
</tr>
<tr>
<td>$\tau_{\sigma_2}$</td>
<td>$2.1016 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\tau_{\epsilon_2}$</td>
<td>$2.1016 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\tau_{\sigma_3}$</td>
<td>$6.7522 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\tau_{\epsilon_3}$</td>
<td>$6.7564 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\tau_{\sigma_4}$</td>
<td>$6.2688 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\tau_{\epsilon_4}$</td>
<td>$6.2723 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\tau_{\sigma_5}$</td>
<td>$6.5872 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\tau_{\epsilon_5}$</td>
<td>$6.5920 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\tau_{\sigma_6}$</td>
<td>$6.2246e \times 10^{-5}$</td>
</tr>
<tr>
<td>$\tau_{\epsilon_6}$</td>
<td>$6.2262 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

where $f$ is central frequency. Ricker’ wavelet has a nice property that has a very narrow frequency band. Ricker’s wavelet from 1 MHz to 7 MHz frequency will be used as testing profiles to simulate stress wave propagation in the human brain. Since there is only one central frequency in Ricker’s wavelet, the profile of the wavelet will remain the same shape when wave propagates from the source. However, the amplitude of Ricker’s wavelet will decay because of absorption effect of the brain tissue. The central frequency of Ricker’s wavelet are selected from 1 MHz to 7 MHz, spaced by 1MHz.

After employing power law to fit the curve of absorption coefficient of bovine brain in A. Etoh [19], I tried to adjust $c_1$ and $c_2$ until $G''$ and power law curve meet in the desired frequency area as will be shown in Fig. 5.1(a). After obtaining $c_1$ and $c_2$, I could obtained relaxation function for bovine brain. 6-parameters standard linear solid models are used for modified Iatridis’s model computation. The parameters of
the model are given in the section 5.4. The absorption coefficient can be obtained by

$$\alpha = \frac{20 \log_{10}(A(x_1)/A(x_2))}{x_2 - x_1}$$  \hspace{1cm} (5.26)

In all calculations, the computational domain is 0.05 m. $\Delta x = 5 \times 10^{-5}$ m, $\Delta t = 1.0 \times 10^{-7}$ s, and CFL = 0.95. Figure 5.2 to Figure 5.8 show snapshots of numerical solutions of stress profiles for 7 cases by using modified Fung’s model. Dash lines are stress profiles at $t = 5 \times 10^{-5}$s and solid lines are stress profiles at $t = 1.95 \times 10^{-4}$s. As could be observed in the figure, the wave front shows the dispersion tendency along with the x axis, also along with higher frequency, which demonstrates that wave absorption could be used for dynamic problems characterized by very short time durations. In the figure, stresses are normalized by $\rho V_p^2$. Figure 5.9 shows the calculated wave absorption effect compares with the experimental data. The calculated results compares well with experiment data in the ultrasonic range.
Fig. 5.1: Reconstruction of modified Fung’s model (a) Comparison of modified Fung’s model to power law theory. (b) Comparison of modified Fung’s model to 6-parameters standard linear solid model in time domain.
Fig. 5.2: Calculated Ricker’s wavelet with central frequency at 1Mhz

Fig. 5.3: Calculated Ricker’s wavelet with central frequency at 2Mhz
Fig. 5.4: Calculated Ricker’s wavelet with central frequency at 3Mhz

Fig. 5.5: Calculated Ricker’s wavelet with central frequency at 4Mhz
Fig. 5.6: Calculated Ricker’s wavelet with central frequency at 5Mhz

Fig. 5.7: Calculated Ricker’s wavelet with central frequency at 6Mhz
Fig. 5.8: Calculated Ricker’s wavelet with central frequency at 7Mhz

Fig. 5.9: Calculated wave absorption effect by using Iatridis’s model and its comparison with the experimental data.
5.5 Conclusion

In this section, we presented a novel framework to simulate wave motion in brain tissues. An extended Fung’s model developed by Iatridis was employed to model brain tissue response. The key contribution of the present chapter is the use of measured absorption coefficients to construct the viscoelastic relations. We showed that in the frequency domain the transformed relaxation function by the Carson transformation is directly related to the measured wave absorption coefficients. As such, the constructed relaxation function of the brain tissue would provide accurate wave attenuation effect and would be suitable for modeling wave motion.

For numerical solution, we transformed the constructed relaxation functions, which include transferring hereditary integration into PDEs that can be readily coupled with the equations of motion for numerical calculation. The transformation involved two steps: (i) approximation of the relaxation function by parallel connected SLS models, and (ii) introduction of the internal variables. The complete governing equations are a set of first-order, fully coupled PDEs with source terms. The primary unknowns are velocity, stress, and internal variables. The model equations were solved by using the space-time CESE method for simulation of Ricker’s wavelets propagating in a bovine brain. A.Etoh et al. [19] provided the measured properties of the brain tissue and the empirical relation of the wave absorption coefficient. Numerical results were obtained by employing Iatridis’ model. The frequencies range from 1 to 7 MHz. The transient solutions were processed to obtain the calculated wave absorption effect. By Iatridis’ model, the calculated wave absorption effect compared well with the measured data by A.Etoh et al. [19].
CHAPTER 6
NUMERICAL RESULTS AND DISCUSSIONS

6.1 Two-Dimensional Results

I have conducted four cases based on both Fung’s model and the modified Fung’s model. The parameters of all cases are given in Table 6.1.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Fung’s model</th>
<th>Modified Fung’s model</th>
</tr>
</thead>
<tbody>
<tr>
<td>∆t</td>
<td>0.115 × 10⁻⁷ s</td>
<td>0.19 × 10⁻⁷ s</td>
</tr>
<tr>
<td>Spatial Domain</td>
<td>0.16m×0.14m</td>
<td>0.16m×0.14m</td>
</tr>
<tr>
<td>Mesh Elements</td>
<td>0.85 × 10⁶</td>
<td>0.85 × 10⁶</td>
</tr>
<tr>
<td>Time Steps</td>
<td>4000</td>
<td>4500</td>
</tr>
<tr>
<td>CFL numbers</td>
<td>0.95</td>
<td>0.90</td>
</tr>
</tbody>
</table>

6.1.1 Fung’s model

I used Cubit, an open-sourced mesh generator developed in DoD Sandia, to generate the mesh to model a human brain. The mesh figure of the human brain used in the present calculation was provided by Brankov [6]. Given the figure of the brain, I used Windig, a software for digitalizing graphics, to digitize the brain in order to obtain
the coordinate of each points. The digitized graph for brain was used as the input to Cubit for generating the mesh. As could be seen in Fig. 6.1, the brain is composed of white matter and grey matter. In this simulation, a tumor is also included in the brain. The information about tumor was given by Gevertza [21]. The properties of the grey matter are defined in the previous sections.

I manually increased 20 percents of the values in Jacobian matrix for white matter compared to that for grey matter, and reduced 20 percents of the values in Jacobian matrix for tumor so that they could be distinguished with grey matter.

As was mentioned in chapter 1, we could use the animal brain tissue to simulate human brain tissue because the differences between human and animal brains are often considered relatively small which enables animal brains to be a good substitute for human brains.

Now I will simulate the wave propagations in two-dimensional domain of human brain.

First, I assume that an impact occurs in the top of the human brain. I still use Ricker’s wavelet in 1 MHz frequency that we interested in as testing profiles to simulate stress wave propagation in the brain. From Figs. 6.2-7, we could observe the wave profile. The wave initiates from the center of the brain, and propagates over the domain as a circle since we treat the brain tissue as isotropic material. The wave then encounters the tumor, and white matter, and the phenomenon of reflect, refract and diffract could be observed until the wave propagates out of the brain.
Fig. 6.1: Two-Dimensional mesh for brain with 0.85 million triangular elements

Fig. 6.2: Wave propagates from the center of brain at $t = 4.6 \times 10^{-6}$ s
Fig. 6.3: Wave propagates from the center of brain at $t = 9.2 \times 10^{-6}\text{s}$

Fig. 6.4: Wave propagates from the center of brain at $t = 1.38 \times 10^{-5}\text{s}$
Fig. 6.5: Wave propagates from center of brain at $t = 1.84 \times 10^{-5} \text{s}$

Fig. 6.6: Wave propagates from center of brain at $t = 2.3 \times 10^{-5} \text{s}$
Fig. 6.7: Wave propagates from center of brain at $t = 3.22 \times 10^{-5}$ s
Second, I assumed a wave starting from the right side of the brain. Figures 6.8-13 show the snapshots of wave propagation. We could observe the wave profiles starting from the side of the brain, and propagating over the domain. The wave firstly encounters the white matter, and then tumor, and the phenomenon of reflect, refract and diffract could also be observed. Moreover, from the color map, we could observe that the wave front still remain the same strength and the phenomenon of reflect does not disappear as wave propagates. It demonstrates once again that Fung’s model is not suitable for modeling the wave propagation in high frequency.

Fig. 6.8: Wave propagates from the side of brain at $t = 9.2 \times 10^{-6} s$
Fig. 6.9: Wave propagates from the side of brain at $t = 1.38 \times 10^{-5} s$

Fig. 6.10: Wave propagates from the side of brain at $t = 2.3 \times 10^{-5} s$
Fig. 6.11: Wave propagates from the side of brain at \( t = 3.45 \times 10^{-5} \text{s} \)

Fig. 6.12: Wave propagates from the side of brain at \( t = 4.6 \times 10^{-5} \text{s} \)
Fig. 6.13: Wave propagates from the side of brain at $t = 5.175 \times 10^{-5}$ s
6.1.2 Modified Fung’s model

Next I examine the propagations of waves that are generated by employing modified Fung’s model. Figures 6.14-18 show the snapshots of the Ricker wave propagating from the center of the brain. It is important to note that, compared with Figs. 6.2-7, the wave generated by using modified Fung’s model shows a different wave pattern when the wave front encounters the tumor in the brain and the white matter of the brain. This might be explained in following ways: (1) The relaxation functions for Fung’s model and modified Fung’s model are absolutely different.(2) The frequency domains for those two models are also different.(3) Since the corresponding wave absorption coefficients for those two models are different, the dispersion effects would have to vary.

Fig. 6.14: Wave propagates from the center of brain at $t = 9.5 \times 10^{-6}$s
Fig. 6.15: Wave propagates from the center of brain at $t = 1.9 \times 10^{-5} s$

Fig. 6.16: Wave propagates from the center of brain at $t = 2.66 \times 10^{-5} s$
Fig. 6.17: Wave propagates from the center of brain at $t = 3.8 \times 10^{-5} \text{s}$

Fig. 6.18: Wave propagates from the center of brain at $t = 5.7 \times 10^{-5} \text{s}$
Last, I examine the propagation of waves that are generated from the side of the brain. Figs. 6.19-23 show the snapshots of Ricker wave propagation. There is one thing need to be noticed in the figures. The magnitude of normal stress $\sigma_{11}$ generated by modified Fung’s model is smaller than that by Fung’s model, which demonstrates that the wave propagations modeled by modified Fung’s model shows dispersion effect more obviously than by Fung’s model.

Fig. 6.19: Wave propagates from the side of brain at $t = 9.5 \times 10^{-6}$ s
Fig. 6.20: Wave propagates from the side of brain at $t = 1.52 \times 10^{-5} \text{s}$

Fig. 6.21: Wave propagates from the side of brain at $t = 3.04 \times 10^{-5} \text{s}$
Fig. 6.22: Wave propagates from the side of brain at $t = 5.32 \times 10^{-5}s$

Fig. 6.23: Wave propagates from the side of brain at $t = 8.36 \times 10^{-5}s$
6.2 Three-dimensional Results

The parameters for three-dimensional case are given in the table 6.2:

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>∆t</td>
<td>$0.75 \times 10^{-7} s$</td>
</tr>
<tr>
<td>Spatial Domain</td>
<td>$0.16 \times 0.14 \times 0.14$</td>
</tr>
<tr>
<td>Mesh Elements</td>
<td>$3 \times 10^6$</td>
</tr>
<tr>
<td>Time Steps</td>
<td>4500</td>
</tr>
<tr>
<td>CFL numbers</td>
<td>0.90</td>
</tr>
</tbody>
</table>

The mesh which is generated by Cubit is shown in Fig. 6.24. For exhibition purpose, I just create 3 million unstructured tetrahedral elements to model a three-dimensional human brain. I still employed Ricker’s wave and initiated it from the side of brain.
Fig. 6.24: Three-Dimensional mesh for brain with 3 million tetrahedral elements

Fig. 6.25: Wave propagates at $t = 2.25 \times 10^{-5} s$
Fig. 6.26: Wave propagates at $t = 3.75 \times 10^{-5} s$

Fig. 6.27: Wave propagates at $t = 6 \times 10^{-5} s$
Fig. 6.28: Wave propagates at $t = 7.5 \times 10^{-5} \text{s}$
In the Figs. 6.29-31, we could observe the wave propagation in entire brain. At 
t = 3.75 \times 10^{-5}s and t = 6 \times 10^{-5}s, we obtained snapshots for x normal, y normal, 
and z normal cut to see what is really happened inside the brain. From the Figs. 
6.25-27, we could see that at t = 3.75 \times 10^{-5}s, the wave just propagates to the plane 
of x normal, while the phenomenon of reflect, refract and diffract have already been 
oberved in y normal plane and z normal plane. Furthermore, at t = 6 \times 10^{-5}s, we 
could find that the phenomenon of reflect, refract and diffract could be observed in 
all normal planes, which could be observed in Figs. 6.32-34. Plus, while the wave 
propagates over the entire brain domain, we could also obtained the wave profile in 
any directions and at any time.

Fig. 6.29: An x-normal cut at t = 3.75 \times 10^{-5}s
Fig. 6.30: A y-normal cut at $t = 3.75 \times 10^{-5}$ s

Fig. 6.31: A z-normal cut at $t = 3.75 \times 10^{-5}$ s
Fig. 6.32: An x-normal cut at $t = 6 \times 10^{-5}s$

Fig. 6.33: A y-normal cut at $6 \times 10^{-5}s$
Fig. 6.34: A z-normal cut at $6 \times 10^{-5}$s
CHAPTER 7
CONCLUSIONS AND FUTURE WORKS

In this thesis, I have reported a general framework to model stress waves in the human brain tissue by employing Fung’s model and modified Fung’s model. In the following I will summarize my contribution as the conclusions, followed by recommended future works.

7.1 Conclusions

The theoretical and numerical approach developed and reported in the present thesis consisted of a novel theoretical framework for modeling stress wave propagation in the human brain based on the use of a set of coupled, first-order, hyperbolic partial differential equations. Then, the CESE method was used to solve the couple differential equations to demonstrate the approach. The wave characteristics could be assessed by analyzing the eigenvalues and eigenvectors of the Jacobian matrices of the model equations. The contributions of present thesis are briefly discussed below:

1. Development of a generic theoretical framework based on the first-order hyperbolic partial differential equations for a wide range of waves motion in brain tissue. Numerical results reported here include simulated one-, two-, and three-dimensional stress wave propagation in the human brain.

2. Wave propagation in the human brain has been studied. Fung’s model and
modified Fung’s model were used to capture the brain’s responses. The model equations were solved by the CESE method for an impact problem. The numerical results compared well with the analytical solution, thus the model equations and the computer code for one-dimensional simulation were validated. The validated code was then used to exhibit the wave profile in two-dimensional and three-dimensional simulation.

3. I have further developed the CESE method for simulation of wave propagation. In particular, source term treatment was developed to solve the first-order wave equations with stiff source terms in model equations for waves in the human brain.

4. One advantage of the CESE method is that the method is developed to use unstructured meshes. Thus it can be applied to a complex geometries. In this work, an open-sourced CESE code, SOLVCON, has been used. The solver kernel was developed specific for viscoelastic computation. The new solver kernel was plugged into the software framework provided by SOLVCON. The whole code was written by using Python, an Objective Oriented language. A special strength of SOLVCON is massively parallel computing with options to support GPGPU. As stated in Ch. 2, the CESE method provides highly accurate transient solutions. Therefore, the method is suitable for direct calculation of wave propagation in viscoelastic material.

7.2 Future Works

Based on my current results and experience, I would like to provide the following suggestions for the future works:

1. The brain mesh could be further refined with many more mesh cells. Since
we focused on model selection, equation deduction, and CESE method, I only constructed a simply mesh of human brain to show the numerical results. For further development for the brain mesh, we could consider either construct a more detail brain mesh or purchase a commercial brain mesh to calculate full wave profiles in two-dimension and three-dimension brain domain.

2. All results shown in this thesis were obtained by running SOLVCON on workstation in CFD lab at Ohio State University. Due to the future requirements on large scale simulation, the time consuming on three-dimension calculation, the demand for high performance CPU, and the needs for storage spaces in the hard drive will largely increase. Thus, the future work for the numerical calculation could be done by using parallel computers, which are well supported in SOLVCON so that we could largely reduce the computation time on running codes while using millions of mesh elements.

3. We could extend wave propagation from brain tissue to more soft tissues which are wider subjects. Also, the future work may include new mathematical models, e.g., poro-elastic or bio-phase model. The numerical simulation work may be applied on more clinic applications and we may cooperate with some medical research groups to obtain the detail information and data inside the human brain. If that is the case, the numerical results will be more accurate.
BIBLIOGRAPHY


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