PERTURBATIONS OF SELFADJOINT OPERATORS WITH DISCRETE SPECTRUM

DISSERTATION

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By
James Adduci, BS
Graduate Program in Mathematics

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Dissertation Committee:
Professor Boris Mityagin, Advisor
Professor Gregory Baker
Professor Paul Nevai
ABSTRACT

Consider a selfadjoint operator $A$ whose spectrum is a set of eigenvalues \{\lambda_0 < \lambda_1 < \ldots\} with corresponding eigenvectors \{\phi_k\}_{k=0}^{\infty}. Now introduce a perturbation $B$ and set $L = A + B$. We prove that if $\lambda_{n+1} - \lambda_n \geq cn^{\alpha-1} \forall n$ and $\|B\phi_n\|n^{1-\alpha} \to 0$ for some fixed $\alpha > 1/2$ then the spectrum of $L = A + B$ is discrete, eventually simple and the set of eigenvectors of $L = A + B$ plus at most finitely many associated vectors form an unconditional basis. As an application we consider Schrödinger operators of the form $Ly = -y'' + |x|^\beta y + b(x)y$ on $L^2(\mathbb{R})$ where $b$ is a possibly complex-valued function and $\beta > 1$. 

ii
To Helen and Anthony
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VITA

1983 ................................. Born in Geneva, Illinois

2004 ............................... B.Sc. in Applied Mathematics,

2005-Present ........................ Graduate Teaching Associate,
               The Ohio State University

PUBLICATIONS

J. Adduci, P. Djakov, B. Mityagin, Convergence radii for the eigenvalues of tri-

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# TABLE OF CONTENTS

Abstract ................................................................. ii  
Dedication ................................................................. ii  
Acknowledgments ......................................................... iv  
Vita ................................................................. v  

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
</tr>
<tr>
<td>2</td>
<td>Statement of Main Results</td>
</tr>
<tr>
<td></td>
<td>2.1 Notational conventions</td>
</tr>
<tr>
<td></td>
<td>2.2 Localization of the spectrum</td>
</tr>
<tr>
<td></td>
<td>2.3 Discrete Hilbert transform</td>
</tr>
<tr>
<td></td>
<td>2.4 Main results</td>
</tr>
<tr>
<td>3</td>
<td>Localization of the spectrum</td>
</tr>
<tr>
<td></td>
<td>3.1 A technical lemma</td>
</tr>
<tr>
<td></td>
<td>3.2 Proof of Proposition 2.2.1</td>
</tr>
<tr>
<td></td>
<td>3.3 Proof of Proposition 2.2.2</td>
</tr>
<tr>
<td>4</td>
<td>The discrete Hilbert transform</td>
</tr>
<tr>
<td></td>
<td>4.1 Technical preliminaries</td>
</tr>
<tr>
<td></td>
<td>4.2 Proof of Proposition 2.3.1</td>
</tr>
<tr>
<td>5</td>
<td>Proof of Main Results</td>
</tr>
<tr>
<td></td>
<td>5.1 Proof of Theorem 2.4.1</td>
</tr>
<tr>
<td>6</td>
<td>Application to Schrödinger operators</td>
</tr>
<tr>
<td></td>
<td>6.1 Eigenvalues</td>
</tr>
<tr>
<td></td>
<td>6.2 Eigenfunctions</td>
</tr>
</tbody>
</table>
CHAPTER 1
INTRODUCTION

Consider a selfadjoint operator \( A \) defined on a separable Hilbert space \( H \). Suppose that the resolvent
\[
R^0(z) = (z - A)^{-1}
\]
is compact for some \( z \notin \text{Sp}A \) and therefore all \( z \notin \text{Sp}A \). Then the spectrum of \( A \) consists of a denumerable set \( \{\lambda_n\}_{n=0}^\infty \) with
\[
\lim_{n \to \infty} |\lambda_n| = \infty.
\]
Denote by \( \phi_n \) the eigenvector corresponding to \( \lambda_n \): \( A\phi_n = \lambda_n \phi_n \). Expansions in the basis of eigenfunctions of \( A \) have several desirable properties. For any \( f \in H \) we have
\[
f = \sum_{n=0}^\infty f_n \phi_n \quad \text{and}
\]
the action of \( A \) on \( f \) is then determined by \( Af = \sum_{n=0}^\infty \lambda_n f_n \phi_n \). The eigenfunctions are orthogonal \( \langle \phi_m, \phi_n \rangle = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases} \) and we have the Parseval identity:
\[
\|f\|^2 = \sum_{n=0}^\infty |f_n|^2. \tag{1.0.1}
\]
The introduction of a non-selfadjoint perturbation to \( A \), \( A + B \) with \( \text{dom}A \subseteq \)
domB, can spoil these properties. Consider for example the $2 \times 2$ perturbed matrix introduced in [2]:

$$M_n(\rho) = \begin{bmatrix} n & 0 \\ 0 & n+2 \end{bmatrix} + \begin{bmatrix} 0 & \rho \\ -\rho & 0 \end{bmatrix}.$$  

For $0 < \rho < 1$ an elementary calculation shows that

$$\phi_1 = \begin{bmatrix} \sqrt{1 + \sqrt{1 - \rho^2}} \\ \rho \sqrt{2} \sqrt{1 + \sqrt{1 - \rho^2}} \end{bmatrix}, \quad \phi_2 = \begin{bmatrix} \sqrt{1 - \sqrt{1 - \rho^2}} \\ \rho \sqrt{2} \sqrt{1 - \sqrt{1 - \rho^2}} \end{bmatrix}$$

are eigenvectors corresponding to the eigenvalues $\lambda_1 = n + 1 + \sqrt{1 - \rho^2}$, $\lambda_2 = n + 1 - \sqrt{1 - \rho^2}$ respectively. Now, denoting by $\phi_1^\perp$ a vector perpendicular to $\phi_1$

$$\phi_1^\perp = \begin{bmatrix} \rho \\ \sqrt{2}\sqrt{1 + \sqrt{1 - \rho^2}} \end{bmatrix},$$

we have $\langle \phi_1, \phi_2 \rangle = \rho$ and $\langle \phi_1^\perp, \phi_2 \rangle = -(1/2) \sqrt{1 - \rho^2}$. From this we derive the decomposition:

$$\phi_1^\perp = \frac{2\rho}{\sqrt{1 - \rho^2}} \phi_1 - \frac{2}{\sqrt{1 - \rho^2}} \phi_2.$$  

So we see that Parseval’s inequality fails and as $\rho \uparrow 1$ the coefficients of $\phi_1, \phi_2$ in the expansion of $\phi_1^\perp$ approach $\infty$ and $-\infty$ respectively.
We now take this example one step further. Consider now the infinite matrices:

\[ M = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \ldots \\
0 & 3 & 0 & 0 & 0 & \ldots \\
0 & 0 & 5 & 0 & 0 & \ldots \\
0 & 0 & 0 & 7 & 0 & \ldots \\
0 & 0 & 0 & 0 & 9 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}, \quad (1.0.2) \]

\[ M' = \begin{bmatrix}
M_1(0) & 0 & 0 & 0 & 0 & \ldots \\
0 & M_5(2/3) & 0 & 0 & 0 & \ldots \\
0 & 0 & M_7(4/5) & 0 & 0 & \ldots \\
0 & 0 & 0 & M_9(6/7) & 0 & \ldots \\
0 & 0 & 0 & 0 & M_{11}(8/9) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}, \quad (1.0.3) \]

acting on \( \ell^2(\mathbb{N}) \) defined in terms of \( 2 \times 2 \) blocks. If we decompose the space \( \ell^2(\mathbb{N}) \) into the blocks \( H_0 = \text{span}(e_0, e_1), H_2 = \text{span}(e_2, e_3), \ldots \), then

\[ \ell^2(\mathbb{R}) = H_0 \times H_2 \times H_4 \times \ldots \]

and each \( H_{2n} \) is an invariant subspace for \( M' \). However, if we denote the eigenfunctions of \( M' \) in \( H_{2n} \) by \( \phi_{2n} \) and \( \phi_{2n+1} \) then by the formula above we have

\[ \phi_{2n}^+ = 2n \left( \frac{2n - 1}{4n - 1} \right) \phi_{2n} - \left( \frac{4n^2}{4n - 1} \right) \phi_{2n+1}. \]

So the eigenfunctions \( \{ \phi_n \} \) of the operator \( M' \) do not satisfy the Parseval identity (1.0.1) or even the following weakened variant of the Parseval identity:

\[ f = \sum f_k \phi_k \quad \text{implies} \quad \|f\|^2 \leq C \sum_{n=0}^{\infty} |f_n|^2. \quad (1.0.4) \]
In this work we will give conditions under which the eigenbasis of a perturbation of a selfadjoint operator is similar to the (orthogonal) eigenbasis of the unperturbed operator. The following 1967 result of Kato will play an important role in our analysis.

**Theorem 1.0.1.** [12] Let \( \{Q^0_k\}_{k=0}^{\infty} \) be a complete family of orthogonal projections in a Hilbert space \( H \) and let \( \{Q_k\}_{k=0}^{\infty} \) be a family of (not necessarily orthogonal) projections such that \( Q_j Q_k = \delta_{j,k} Q_j \). Assume that

\[
\dim (Q^0_0) = \dim (Q_0) = m < \infty \quad \text{and} \quad \sum_{j=0}^{\infty} \|Q^0_j (Q_j - Q^0_j) u\|^2 \leq c_0 \|u\|^2, \quad \forall u \in H
\]

where \( c_0 \) is a constant smaller than 1. Then there is a bounded operator \( W : H \to H \) with bounded inverse such that \( Q_k = W^{-1} Q^0_k W \) for \( k \in \mathbb{Z}_0 \).

It was observed by C. Clark that this Theorem could be used in the context of eigenprojections of non-selfadjoint perturbations of selfadjoint operators (see [4] and the comment at the end of [12]). See also [15], [20].

The following Theorem from Kato’s book reformulates the results of Clark. Let \( B(H) \) denote the space of bounded linear operators from \( H \) to \( H \).

**Theorem 1.0.2.** [13, Theorem 4.15a] Let \( T \) be a selfadjoint operator with a compact resolvent, with simple eigenvalues \( \lambda_1 < \lambda_2 < \cdots \). Let \( P_h, \quad h = 1, 2, \ldots, \) be the associated eigenprojections (so that \( P_h P_k = \delta_{k,h} P_h, \quad P^*_h = P_h, \quad \text{dim} P_h = 1 \)). Assume further that \( \lambda_h - \lambda_{h-1} \to \infty \) as \( h \to \infty \). Let \( A \in B(H) \) (not necessarily symmetric). Then \( S = T + A \) is closed with a compact resolvent, and the eigenvalues and eigenprojections of \( S \) can be indexed as \( \{\mu_{0,k}, \mu_h\} \) and \( \{Q_{0,k}, Q_h\} \), respectively, where \( k = 1, \ldots, m < \infty \) and \( h = n + 1, n + 2, \ldots \) with \( n \geq 0 \), in such a way that the
following results hold. i) $|\mu_k - \lambda_k|$ is bounded as $h \to \infty$. ii) There is a $W \in \mathcal{B}(H)$ with $W^{-1} \in \mathcal{B}(H)$ such that

$$
\sum_{k=1}^{\infty} Q_{0,k} = W^{-1} \left( \sum_{k \leq n} P_k \right) W, \quad Q_h = W^{-1} P_h W \quad \text{for} \quad h > n.
$$

In [1] it was shown that if $A$ is the harmonic oscillator operator $A = -d^2/dx^2 + x^2$ and $B$ is multiplication by any complex valued $b(x) \in L^2(\mathbb{R})$ then the eigensystem of $A + B$ is similar to the eigensystem of $A$ (the orthonormal basis of Hermite functions). This result is not covered by Theorem 1.0.2 since $\text{Sp} A = \{1, 3, 5, \ldots\}$ so that $\lambda_{n+1} - \lambda_n = 2$ $\forall n$ and furthermore $B$ is possibly unbounded. In [1] it was also shown that if $A$ is a selfadjoint operator on a Hilbert space $H$ with a compact resolvent and $\text{Sp} A = \{\lambda_1 < \lambda_2 < \ldots\}$ then the eigensystem of $A$ is similar to the eigensystem of $B$ whenever $\|B\| < \limsup_{n \to \infty} (\lambda_{n+1} - \lambda_n)$. The perturbed operator $M'$ given above shows that the condition on $\|B\|$ cannot be relaxed. These results were extended in [23] to cover perturbations of selfadjoint operators $A$ with spectra containing “clusters” and in [2] to cover the case in which $\lambda_{n+1} - \lambda_n \geq n^{\alpha - 1}$ with $\alpha > 1/2$. 

5
CHAPTER 2
STATEMENT OF MAIN RESULTS

2.1 Notational conventions

For the remainder of this work we will adopt the following notational conventions. $H$ will always denote a separable Hilbert space. Our main examples will be

\[
L^2 (\mathbb{R}) = \left\{ f : \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty \right\} \quad \text{and} \quad (2.1.1)
\]

\[
\ell^2 (\mathbb{N}) = \left\{ (a_k)_{k=0}^{\infty} : \sum_{k=0}^{\infty} |a_k|^2 < \infty \right\}. \quad (2.1.2)
\]

We will also need to consider a vector-valued analogue of the space (2.1.1):

\[
\ell^2 (H) = \left\{ (v_k)_{k=0}^{\infty} : v_k \in H \, \forall k, \quad \sum_{k=0}^{\infty} \|v_k\|^2 < \infty \right\} \quad (2.1.3)
\]

as well as the weighted $\ell^2$ spaces:

\[
\ell^2_W (H) = \left\{ (v_k)_{k=0}^{\infty} : v_k \in H \, \forall k, \quad \sum_{k=0}^{\infty} W(k)^2 \|v_k\|^2 < \infty \right\} , \quad (2.1.4)
\]

\[
W : \mathbb{N} \to (0, \infty) . \quad (2.1.5)
\]

We will always use $A$ to denote selfadjoint operators, bounded below with compact resolvent. Which specific operator $A$ represents will be made clear by the context. The eigenvalues of $A$ will be denoted by $\{\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots\}$ and the corresponding eigenfunctions $\{\phi_n\}_{n=0}^{\infty}$. Because $A$ has a compact resolvent it is necessarily true that $\text{Sp} A = \{\lambda_k\}_{k=0}^{\infty}$ and $\lim_{k \to \infty} \lambda_k = \infty$. We will use $B$ to denote a perturbation with
domA ⊆ domB and call the perturbed operator \( L = A + B \). The resolvent of \( A \), where it is defined, will be denoted \( R^0(z) = (z - A)^{-1} \); that of \( L = A + B \) will be \( R(z) = (z - A - B)^{-1} \).

We will be interested in the case in which the set of eigenvalues \( \{\lambda_n\} \) contains “clusters.” Specifically, we will suppose that the spectrum of \( A \) can be written

\[
\{\lambda_k\}_{k=0}^\infty = \bigcup_{n=1}^\infty \Lambda_n
\]

where the set of clusters \( \{\Lambda_n\} \) satisfy the following conditions:

- There is an integer \( p > 0 \) such that

\[
\#\Lambda_n \leq p \quad \forall n,
\]

- \( \Lambda_k < \Lambda_{k+1} \), i.e. \( \lambda_m < \lambda_n \) whenever \( \lambda_m \in \Lambda_k, \lambda_n \in \Lambda_{k+1} \),

- there is a constant \( c > 0 \) such that

\[
\text{dist}(\Lambda_{n-1}, \Lambda_n) > cn^{\alpha-1} \quad \forall n \quad \text{and}
\]

- \( \lambda, \lambda' \in \Lambda_n \) implies

\[
|\lambda - \lambda'| < cn^{\alpha-1}
\]

where \( \alpha \) is a positive constant.

For each \( \Lambda_n \) we will define the projector \( P_n \) onto the corresponding \( \#\Lambda_n \)-dimensional eigenspace by the Cauchy projector formula (see [5, Ch. 7, Sect. 3]):

\[
P_n = \frac{1}{2\pi i} \int_{\partial \Gamma_n} R^0(z) \, dz
\]

where \( \Gamma_n \) is defined to be the set of all \( z = a + ib \) satisfying

\[
\inf \Lambda_n - (c/2) n^{\alpha-1} \leq a \leq \sup \Lambda_n + (c/2) (n + 1)^{\alpha-1} \quad \text{and}
\]

\[
-(c/2) n^{\alpha-1} \leq b \leq (c/2) n^{\alpha-1}.
\]
By the conditions on the clusters we have $\Lambda_n \subseteq \text{int}\Gamma_n$ for all $n$ and $\Lambda_m \cap \Gamma_n = \{\}$ whenever $m \neq n$. This ensures that $R^0(z)$ is defined on $\partial\Gamma_n$ for each $n$. Also, (2.1.9) and (2.1.11) guarantee that

$$|\partial\Gamma_n| \leq \begin{cases} 
(2p + 1) c n^{\alpha - 1} & \text{if } \alpha \leq 1 \\
(2p + 1) c (n + 1)^{\alpha - 1} & \text{if } \alpha > 1.
\end{cases}$$

(2.1.12)

2.2 Localization of the spectrum

As in (2.1.10) we would like to define the projection onto the eigenspace of the perturbed operator $L = A + B$ by the Cauchy projector formula:

$$Q_n = \frac{1}{2\pi i} \int_{\partial\Gamma_n} R(z) \, dz.$$  

(2.2.1)

In order for (2.2.1) to be meaningful we must ensure that $R(z)$ is defined on $\partial\Gamma_n$. To this end we will impose growth conditions on the action of $B$ on the eigenvectors of $A$: $\|B\phi_k\|$. Interestingly, we will require similar conditions in Theorem 2.4.1. be imposed.

Define the sequence $(\beta_n)$ by:

$$\beta_n n^{\alpha - 1} = \sup_{\lambda_j \in \Lambda_n} \|B\phi_j\|, \quad \beta_\infty = \sup \beta_k.$$  

(2.2.2)

The growth conditions we shall impose on $(\beta_n)_{n=1}^\infty$ will differ depending on whether $0 < \alpha < 1$, $\alpha = 1$ or $\alpha > 1$. First we define the constant

$$v = 2^{\frac{1}{|\alpha - 1|}}$$

(2.2.3)

which will be used throughout the rest of this work. If $0 < \alpha < 1$ we suppose:

$$\limsup \beta_n < \frac{c^2 (1 - 1/v)^3}{16pv^3}, \quad \frac{3c^2}{16 (1024p\pi^2 + 2294p)}.$$  

(2.2.4)
If $\alpha = 1$ we suppose:

$$\lim \sup \beta_n < \frac{c^2}{p(64 \pi^2/6 + 96)}. \quad (2.2.5)$$

Finally, if $\alpha > 1$ we suppose:

$$\lim \sup \beta_n < \frac{c^2 (1 - 1/v)^3}{4096pv^{4\alpha}}, \frac{3c^2}{16p (1024 \pi^2 + 3c^2 + 2294)}. \quad (2.2.6)$$

Conditions (2.2.4-2.2.6) guarantee that we can find an integer $N_1 > 4$ such that the inequalities (2.2.4-2.2.6) hold without the lim sup whenever $n \geq v^{N_1}/2$ for $\alpha \neq 1$ and whenever $n \geq N_1/2$ for $\alpha = 1$. We also require $N_1$ be an integer sufficiently large to guarantee:

$$v^{N_1\alpha} > \begin{cases} 
\frac{16p\beta_\infty^2 v^3}{c^2 (1 - 1/v)^3}, & \text{if } 0 < \alpha < 1 \\
\frac{4096p\beta_\infty^2 v^3}{c^2 (1 - 1/v)^3}, & \text{if } \alpha > 1 
\end{cases} \quad (2.2.7)$$

and that

$$\sum_{j \geq N_1/2} 1/j^2 \leq \frac{c^2}{32p\beta_\infty}, \quad \text{if } \alpha = 1. \quad (2.2.8)$$

**Proposition 2.2.1.** Suppose the spectrum of $A$ satisfies (2.1.6) and $B$ satisfies (2.2.4 - 2.2.6) Then for each $n \geq \begin{cases} 
v^{N_1+1}, & \alpha \neq 1 \\
N_1, & \alpha = 1 
\end{cases}$, $R(z)$ is defined on $\partial \Gamma_n$ and

$$\dim Q_n = \dim P_n \leq p. \quad (2.2.9)$$

Furthermore, $R(z)$ satisfies:

$$\|R(z)\| \leq \begin{cases} 
(4/c) (n + 1)^{1-\alpha}, & 0 < \alpha \leq 1 \\
(4/c) n^{1-\alpha}, & \alpha > 1 
\end{cases} \quad \text{whenever } z \in \partial \Gamma_n. \quad (2.2.10)$$
The previous proposition guarantees that all but a finite number of eigenvalues of $A$ remain localized after perturbation by $B$. It is an interesting phenomenon that the first several eigenvalues cannot, in general, be localized in this way. The next proposition guarantees that as a group they can be localized and don’t interfere with the rest of the spectrum. First we define a contour which will bound them. If $n'$ is the largest integer which is smaller than $v^{N_1+1}$, define $\Gamma'$ to be the set of all $z = a + ib$ such that

$$|a|, |b| \leq \sup \Lambda_{n'} + (c/2) (n' + 1)^{\alpha-1}$$

(2.2.11)

so that $\Gamma' \cap \Gamma_n = \emptyset$ whenever $n > n'$ or equivalently whenever $n > v^{N_1-1}$ (see Prop. 2.2.1). The next proposition shows that the definition

$$Q' = \frac{1}{2\pi i} \int_{\partial \Gamma'} R(z) \, dz$$

(2.2.12)

is meaningful.

**Proposition 2.2.2.** Suppose the conditions of Proposition 2.2.1 are satisfied and $\Gamma'$ is defined in (2.2.11). Then $R(z)$ is defined on $\partial \Gamma'$ and

$$\dim Q' = \sum_{k=0}^{n'} \dim P_k.$$  

(2.2.13)

Furthermore, $\text{Sp}L \subset \Gamma' \cup \left( \bigcup_{k=n'+1}^{\infty} \Gamma_k \right)$.

### 2.3 Discrete Hilbert transform

The discrete Hilbert transform will play an important role in our proof of Theorem 2.4.1. The standard discrete Hilbert transform is defined by:

$$G : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

$$G : (a_n)_{n=1}^{\infty} \to \left( \sum_{k \neq n} a_k \frac{1}{k-n} \right)_{n=1}^{\infty}.$$
Of course care must be taken to make sure that $G$ actually maps sequences in $\ell^2(\mathbb{N})$. It is a classical result that $G$ is in fact a bounded operator from $\ell^2(\mathbb{N})$ into $\ell^2(\mathbb{N})$ with norm $\|G\|_2 = \pi$ (see [9]). An elementary proof of this fact can be found in the note [8]. Also, in the appendix of [1] an elementary proof is given to show that $G$ is in fact a bounded operator between a large class of weighted $\ell^2(\mathbb{N})$ spaces.

For our purposes in this work we will need to consider a more general class of Hilbert-like transforms. We shall state here and prove in Chapter 4 that the following generalization of the Hilbert transform is bounded.

**Proposition 2.3.1.** Suppose that $(t_k)_{k=0}^{\infty}$ is a sequence satisfying the condition:

$$t_{k+1} - t_k \geq c k^{\alpha-1}$$

(2.3.1)

where $\alpha$ is a constant satisfying $1/2 < \alpha < \infty$. Then there exists a constant $K_\alpha$ such that for every $b \in \ell^2(\mathbb{N})$

$$\sum_{n=0}^{\infty} \left| \sum_{k \neq n} k^{\alpha-1} b_k \right|^2 \leq K_\alpha \sum_{k=0}^{\infty} |b_k|^2.$$  

(2.3.2)

Let us remark that the previous proposition means that the operator

$$G_t : (a_n) \rightarrow \sum_{k \neq n} \frac{a_k}{t_n - t_k}$$

(2.3.3)

is a bounded operator between $\ell^2_W(\mathbb{N})$ and $\ell^2(\mathbb{N})$ where $W(k) = k^{1-\alpha}$. Also, in the proof of Proposition 2.3.1 we will give specific values for the constant $K_\alpha$ since this constant appears in the statement of Theorem 2.4.1. We make the disclaimer that the value we will give is very rough and could easily be improved.

The condition $\alpha > 1/2$ cannot be weakened or removed. To see this consider the sequence $(b_k)_{k=0}^{\infty} \in \ell^2$ defined by $b_k = 1$ if $k = 0$ and $b_k = 0$ if $k \neq 0$. If $t_k = k^{1/2}$ (so $\alpha = 1/2$), then the sum

$$\sum_{n=0}^{\infty} \left| \sum_{k \neq n} k^{\alpha-1} b_k \right|^2 = \sum_{n=1}^{\infty} \frac{1}{n}$$

(2.3.4)
doesn’t even converge. This is the reason we require $\alpha > 1/2$ in Theorem 2.4.1. We know of no counter-example to show that the conclusion of Theorem 2.4.1 does not hold for $0 < \alpha \leq 1/2$.

2.4 Main results

Our main theorem is the following:

**Theorem 2.4.1.** Suppose that $A$ and $B$ satisfy the conditions of Propositions 2.2.1, 2.2.2 with constant $\alpha > 1/2$ and that

$$\limsup_\beta \beta_n^2 < \frac{1}{4p^3K_\alpha^2}. \quad (2.4.1)$$

Then there is a bounded operator $W$ with bounded inverse such that

$$Q' = W \sum_{k=0}^{n'} P_k W^{-1} \quad \text{and} \quad (2.4.2)$$

$$Q_n = WP_n W^{-1} \quad \forall n > n'. \quad (2.4.3)$$

As an easy corollary we will show that the eigenprojections $Q', Q_{n'+1}, Q_{n'+2}, \ldots$ satisfy the weakened Parseval identity (1.0.4).

**Corollary 2.4.2.** For each $f \in H$, $\|Q'f\|^2 + \sum_{k=n'}^\infty \|Q_k f\|^2 \leq \|W\|^2 \|W^{-1}\|^2 \|f\|^2$

Suppose now that $f \in H$ is an arbitrary vector. Consider its expansion $f = Q'f + \sum_{k=n'}^\infty Q_k f$. We have

$$\|Q'f\|^2 = \left\| W \sum_{k=0}^{n'} P_k W^{-1} f \right\|^2 \leq \|W\|^2 \left\| \sum_{k=0}^{n'} P_k W^{-1} f \right\|^2$$

$$= \|W\|^2 \sum_{k=0}^\infty \|P_k W^{-1} f\|^2$$
and for $k > n'$

$$\|Q_k f\|^2 = \|WP_k W^{-1} f\|^2 \leq \|W\|^2 \|P_k W^{-1} f\|^2.$$

Hence,

$$\|Q' f\|^2 + \sum_{k=n'}^{\infty} \|Q_k f\|^2 \leq \|W\|^2 \sum_{k=0}^{\infty} \|P_k W^{-1} f\|^2 = \|W\|^2 \|W^{-1}\|^2 \|f\|^2.$$

This is the weak variant of the Parseval’s identity (1.0.4) with constant $\|W\| \|W^{-1}\|$. 
CHAPTER 3
LOCALIZATION OF THE SPECTRUM

3.1 A technical lemma

We first remind the reader that \( v \) was defined in (2.2.3). In what follows it will be convenient to decompose the positive real axis as

\[
[0, \infty) = \bigcup_{N=0}^{\infty} V_N
\]

where \( V_0 = [0, v) \) and \( V_N = [v^N, v^{N+1}) \) for \( N \geq 1 \). Before we prove localization of the spectrum we prove the following technical lemma.

**Lemma 3.1.1.** Suppose that \( \alpha > 0, \alpha \neq 1, n \in V_N, m \in V_M \) with \( M < N - 2 \) and \( \lambda \in \Lambda_m, \lambda' \in \Lambda_n \). Then

\[
\lambda' - \lambda \geq c \left( 1 - \frac{1}{v} \right) v^{(N-1)\alpha}.
\]

(3.1.2)

Also, if \( z \in \partial \Gamma_m \) and \( z' \in \partial \Gamma_n \) then

\[
|z - \lambda'| \geq (c/2) \left( 1 - \frac{1}{v} \right) v^{(N-1)\alpha} \quad \text{and}
\]

\[
|z' - \lambda| \geq (c/2) \left( 1 - \frac{1}{v} \right) v^{(N-1)\alpha}.
\]

(3.1.3)

(3.1.4)

**Proof.** We first prove (3.1.2). Suppose that \( 0 < \alpha < 1 \). We have

\[
\lambda' - \lambda = \text{dist} (\lambda', \Lambda_{n-1}) + \text{dist} (\Lambda_{n-1}, \Lambda_{n-2}) + \ldots + \text{dist} (\Lambda_{m+1}, \lambda)
\]

\[
\geq c \left( n^{\alpha-1} + \ldots + (m+1)^{\alpha-1} \right)
\]

(3.1.5)

(3.1.6)
Now by the mean value theorem:

$$0 < a < b \implies \alpha b^{\alpha-1} (b-a) \leq b^\alpha - a^\alpha \leq \alpha a^{\alpha-1} (b-a) \quad (3.1.7)$$

Hence,

$$\lambda' - \lambda \geq \frac{c}{\alpha} \left( ((n+1)^\alpha - n^\alpha) + \ldots + ((m+2)^\alpha - (m+1)^\alpha) \right) \quad (3.1.8)$$

$$= \frac{c}{\alpha} ((n+1)^\alpha - (m+1)^\alpha) \quad (3.1.9)$$

Now note that $0 < \alpha < 1$ implies $v > 2$ so that $m + 1 \in V_M$ or $V_{M+1}$ whence $(m + 1)^\alpha \leq v^{(M+2)\alpha} \leq v^{(N-1)\alpha}$.

Hence,

$$\frac{c}{\alpha} ((n+1)^\alpha - (m+1)^\alpha) \geq \frac{c}{\alpha} (v^{N\alpha} - v^{(N-1)\alpha}) \quad (3.1.10)$$

$$\geq \frac{c}{\alpha v^{N(\alpha-1)}} (v^N - v^{N-1}) \quad \text{(by MVT)} \quad (3.1.11)$$

$$= cv^{N\alpha} (1 - 1/v) \geq c (1 - 1/v) v^{(N-1)\alpha}. \quad (3.1.12)$$

Similarly for the case $\alpha > 1$ we have

$$\lambda' - \lambda = \text{dist} (\lambda', \Lambda_{n-1}) + \text{dist} (\Lambda_{n-1}, \Lambda_{n-2}) + \ldots + \text{dist} (\Lambda_{m+1}, \lambda) \quad (3.1.13)$$

$$\geq c \left( n^{\alpha-1} + \ldots + (m+1)^{\alpha-1} \right) \quad (3.1.14)$$

By the mean value theorem:

$$0 < a < b \implies \alpha a^{\alpha-1} (b-a) \leq b^\alpha - a^\alpha \leq \alpha b^{\alpha-1} (b-a) \quad (3.1.15)$$
Hence,

\[
\lambda' - \lambda \geq \frac{c}{\alpha} ((n^\alpha - (n - 1)^\alpha) + \ldots + ((m + 1)^\alpha - m^\alpha)) \quad (3.1.16)
\]

\[
= \frac{c}{\alpha} (n^\alpha - m^\alpha) \quad (3.1.17)
\]

\[
\geq \frac{c}{\alpha} (v^{N\alpha} - v^{(M+1)\alpha}) \quad (3.1.18)
\]

\[
\geq \frac{c}{\alpha} (v^{N\alpha} - v^{(N-1)\alpha}) \quad (3.1.19)
\]

\[
\geq \frac{c}{\alpha} \alpha v^{(N-1)(\alpha-1)} (v^N - v^{N-1}) \quad \text{(by MVT)} \quad (3.1.20)
\]

\[
= c (1 - 1/v) v^{(N-1)\alpha} (v - 1) \geq cv^{(N-1)\alpha}. \quad (3.1.21)
\]

To prove (3.1.3-3.1.4) note that the preceeding arguments hold in these cases when \(c\) is replaced by \(c/2\).

\[
\]

### 3.2 Proof of Proposition 2.2.1

Proof. It suffices to show that \(\|R^0(z)B\| < 1/2\) whenever \(z \in \bigcup_{k=n'+1}^{\infty} \partial \Gamma_k\) since this implies that \(R(z) = (z - A - B)^{-1} = (I - R^0(z)B)^{-1} R^0(z)\) is defined so that \(z \notin \text{Sp}(A + B)\). The resolvent bounds (2.2.10) will also follow since then \(\|(I - R^0(z)B)^{-1}\| \leq 2\) and for \(z \in \partial \Gamma_n\)

\[
\|R^0(z)\| \leq \sup_{\lambda \in \text{Sp}A} \frac{1}{|z - \lambda|} \leq \begin{cases} 
(2/c) (n + 1)^{1-\alpha}, & 0 < \alpha \leq 1 \\
(2/c) n^{1-\alpha}, & \alpha > 1.
\end{cases}
\]
Suppose that $\|f\| = 1$ with $f = \sum_{j=0}^{\infty} f_j \phi_j$. Then

\[
\|R^0(z)Bf\|^2 = \left| \sum_{j=0}^{\infty} f_j BR^0(z) \phi_j \right|^2 = \left| \sum_{j=0}^{\infty} \frac{f_j}{z - \lambda_j} B \phi_j \right|^2 \\
\leq \left( \sum_{j=0}^{\infty} |f_j| \frac{\|B \phi_j\|}{|z - \lambda_j|} \right)^2 \\
\leq \sum_{j=0}^{\infty} \frac{\|B \phi_j\|^2}{|z - \lambda_j|^2},
\]

where $(\beta_k)$ was defined in (2.2.2).

Suppose that $z \in \partial \Gamma_n$ with $n > n'$ (or equivalently $n \in V_N, N > N_1$). We split the sum up into four pieces:

\[
\sum_{k=0}^{\infty} \sum_{\lambda \in \Lambda_k} \frac{\beta_k^2 k^{2(\alpha-1)}}{|z - \lambda|^2} = S_1 + S_2 + S_3 + S_4 \tag{3.2.1}
\]

where

\[
S_1 = \sum_{M < N/2} \sum_{m \in V_M} \sum_{\lambda \in \Lambda_m} \frac{\beta_m^2 m^{2(\alpha-1)}}{|z - \lambda|^2},
\]

\[
S_2 = \sum_{N/2 \leq M < N-2} \sum_{m \in V_M} \sum_{\lambda \in \Lambda_m} \frac{\beta_m^2 m^{2(\alpha-1)}}{|z - \lambda|^2},
\]

\[
S_3 = \sum_{N-2 \leq M \leq N+2} \sum_{m \in V_M} \sum_{\lambda \in \Lambda_m} \frac{\beta_m^2 m^{2(\alpha-1)}}{|z - \lambda|^2},
\]

\[
S_4 = \sum_{M > N+2} \sum_{m \in V_M} \sum_{\lambda \in \Lambda_m} \frac{\beta_m^2 m^{2(\alpha-1)}}{|z - \lambda|^2}.
\]

We will first analyze these sums for $0 < \alpha < 1$. For $S_1$, the condition $M < N/2$ with (3.1.3) implies that $|z - \lambda|^2 \geq (c/2)^2 (1 - 1/v)^2 v^{2\alpha(N-1)}$ and that $m^{2(\alpha-1)} \leq$
$v^{2(\alpha-1)M} = 2^{-2M}$ whenever $\lambda \in \Lambda_m$, $m \in V_M$. Thus,

$$
\sum_{m \in V_M} \beta_m^2 m^{2(\alpha-1)} \leq \sum_{m \in V_M} \beta_m^2 2^{-2M} M \leq 2^{\alpha(N-1)}
$$

$$
= \sum_{m \in V_M} \beta_m^2 2^{-2M} M c^2 (1 - 1/v)^2 (v/2)^{2(N-1)}
\leq \frac{pv^2 (\sup_{m \in V_M} \beta_m)^2}{c^2 (1 - 1/v)^2} \left( \#V_M 2^{2(N-M)} v^{-2N} \right).
$$

Since $\#V_M \leq v^{M+1}$ we have

$$
\sum_{m \in V_M} \beta_m^2 m^{2(\alpha-1)} \leq \frac{pv^2 (\sup_{m \in V_M} \beta_m)^2}{c^2 (1 - 1/v)^2} 2^{2(N-M)} v^{M-2N+1}
$$

$$
= \frac{pv^3 (\sup_{m \in V_M} \beta_m)^2}{c^2 (1 - 1/v)^2} (v/2)^{2(M-N)} v^{-M}
$$

$$
= \frac{pv^3 (\sup_{m \in V_M} \beta_m)^2}{c^2 (1 - 1/v)^2} v^{-M} (v^{\alpha})^{2(M-N)}
\leq \frac{pv^3 (\beta_{\infty})^2}{c^2 (1 - 1/v)^2} v^{-M} (v^{\alpha})^{-N_1}
$$

where the last inequality follows because $M < N/2$ implies $2(N-M) > 2(N/2) > N_1$. So,

$$
S_1 \leq \frac{pv^3 (\beta_{\infty})^2}{c^2 (1 - 1/v)^2} (v^{\alpha})^{-N_1} \sum_{M=0}^{\infty} v^{-M}
$$

$$
= \frac{pv^3 (\beta_{\infty})^2}{c^2 (1 - 1/v)^3} (v^{\alpha})^{-N_1} < 1/16 \quad \text{by (2.2.7).}
$$

Now for $N/2 \leq M < N - 2$ the previous argument is valid up to (3.2.2) so we have

$$
\sum_{m \in V_M} \beta_m^2 m^{2(\alpha-1)} \leq \frac{pv^3 (\sup_{N/2 < M < N-2} \beta_m)^2}{c^2 (1 - 1/v)^2} v^{-M} (v^{\alpha})^{2(M-N)}
$$

$$
\leq \frac{pv^3 (\sup_{N/2 < M < N-2} \beta_m)^2}{c^2 (1 - 1/v)^2} v^{-M}.
$$

18
Thus,
\[
S_2 \leq \frac{pv^3 \left( \sup_{N/2 < M < N - 2} \beta_m \right)^2}{c^2 \left( 1 - 1/v \right)^2} \sum_{M=0}^{\infty} v^{-M} = \frac{pv^3 \left( \sup_{N/2 < M < N - 2} \beta_m \right)^2}{c^2 \left( 1 - 1/v \right)^3} \tag{3.2.6}
\]
\[
< 1/16 \text{ by (2.2.4).} \tag{3.2.7}
\]

Consider \( S_3 \). The condition \( z \in \partial \Lambda_n \) with \( n \in V_N, N - 2 \leq M \leq N + 2 \) implies by (2.1.6) that
\[
|z - \lambda|^2 \geq c^2 (v^{N+3})^{2(\alpha-1)} (|n - m| - 1)^2 = c^2 2^{-2(N+3)} (|n - m| - 1)^2
\]
for \( m \neq n-1, n, n \). Also, for \( \lambda \in \Lambda_{n-1} \cup \Lambda_n \cup \Lambda_{n+1} \), we have \( |z - \lambda|^2 \geq (c/2)^2 (n + 1)^{2(\alpha-1)} \)
so
\[
S_3 \leq \sum_{N-2 \leq M \leq N+2} \sum_{m \in V_M} \sum_{\lambda \in \Lambda_m} \frac{\beta_m^2 2^{-2(N-2)} c^2 2^{-(N+3)} (|n - m| - 1)^2}{(c/2)^2 (n + 1)^{2(\alpha-1)}} + \frac{p \beta_{n-1}^2 (n - 1)^{2(\alpha-1)} + p \beta_n^2 (n - 1)^{2(\alpha-1)} + p \beta_{n+1}^2 (n + 1)^{2(\alpha-1)}}{(c/2)^2 (n + 1)^{2(\alpha-1)}}
\]
But the first term is bounded above by \( c^{-2} 1024p \sup \beta_m^2 \pi^2 / 3 \). Also \( n - 1, n, n + 1 \in V_{N-1} \cup V_N \cup V_{N+1} \) implies that
\[
\frac{(n - 1)^{2(\alpha-1)} + (n - 1)^{2(\alpha-1)} + (n + 1)^{2(\alpha-1)}}{(n + 1)^{2(\alpha-1)}} \leq \frac{3v^{2(\alpha-1)(N-1)}}{v^{2(\alpha-1)(N+2)}}.
\]
Combining these bounds we have
\[
S_3 \leq \sup_{m \in V_M} \beta_m^2 \left( \frac{1024p \pi^2}{3c^2} + \frac{3p}{(c/2)^2 v^{6(\alpha-1)}} \right) \leq \sup_{m \in V_M} \beta_m^2 \left( \frac{1024p \pi^2}{3c^2} + \frac{3p 256}{c^2} \right) < 1/16 \text{ by (2.2.4).}
\]
Now for $S_4$, $M \geq N+3$ with (3.1.4) implies that $|z - \lambda|^2 \geq (c/2)^2 (1 - 1/v)^2 v^{2(M-1)}$ so that

$$
\sum_{m \in V_M} \sum_{\lambda \in \Lambda_m} \frac{\beta_m^2 m^{2(\alpha-1)}}{|z - \lambda|^2} \leq \frac{4p \sup_{m \in V_M} \beta_m^2}{c^2} \frac{\#V_M v^{2(\alpha-1)M}}{(1 - 1/v)^2} \frac{1}{v^{2\alpha(M-1)}}
$$

$$
= \frac{4 \sup_{m \in V_M} \beta_m^2}{c^2} \frac{\#V_M v^{-2(M-1)} v^{2(\alpha-1)}}{(1 - 1/v)^2}
$$

$$
\leq \frac{4 \sup_{m \in V_M} \beta_m^2}{c^2} \frac{\#V_M v^{M+1} v^{-2(M-1)} v^{2(\alpha-1)}}{(1 - 1/v)^2}
$$

$$
= \frac{4 \sup_{m \in V_M} \beta_m^2 v^3}{c^2} \frac{v^{2(\alpha-1)}(M+1)}{(1 - 1/v)^2} v^{-M} v^{2(\alpha-1)}
$$

Hence $S_4 < 1/16$ by (2.2.4).

We now proceed to the case $\alpha > 1$. Because this case is similar to $0 < \alpha < 1$ our presentation will be brief. Consider the decomposition (3.2.1) beginning with $S_1$. The condition $M < N/2$ with (3.1.3) implies that $|z - \lambda|^2 \geq (c/2)^2 (1 - 1/v)^2 v^{2\alpha(N-1)}$ and that $m^{2(\alpha-1)} \leq v^{2(\alpha-1)(M+1)} = 2^{2(M+1)}$ whenever $\lambda \in \Lambda_m$, $m \in V_M$. Thus,

$$
\sum_{m \in V_M} \sum_{\lambda \in \Lambda_m} \frac{\beta_m^2 m^{2(\alpha-1)}}{|z - \lambda_m|^2} \leq \frac{4p \sup_{m \in V_M} \beta_m^2}{c^2} \frac{v^{M+1} 2^{2(M+1)}}{(1 - 1/v)^2} \frac{1}{v^{2\alpha(N-1)}}
$$

$$
= \frac{16p \sup_{m \in V_M} \beta_m^2}{c^2} \frac{v^{M+2M}}{(1 - 1/v)^2} \frac{1}{v^{2\alpha(N-1)}}
$$

$$
= \frac{256p \sup_{m \in V_M} \beta_m^2 v^3}{c^2} \frac{v^{M-2N} 2^{2(M-N)}}{(1 - 1/v)^2}
$$

$$
= \frac{256p \sup_{m \in V_M} \beta_m^2 v^3}{c^2} \frac{v^{-M} (2v)^{2(M-N)}}{(1 - 1/v)^2}
$$

$$
\left(\text{which since } 2^{\frac{\alpha-1}{\alpha-1}} v = 2^{\frac{\alpha}{\alpha-1}}\right)
$$

$$
= \frac{256p \sup_{m \in V_M} \beta_m^2 v^3}{c^2} \frac{v^{-M} \left(2^{\frac{\alpha}{\alpha-1}}\right)^{2(M-N)}}{(1 - 1/v)^2}
$$

$$
\leq \frac{256p \sup_{m \in V_M} \beta_m^2 v^3}{c^2} \frac{v^{-M} \left(2^{\frac{\alpha}{\alpha-1}}\right)^{-N}}{(1 - 1/v)^2}
$$

Hence,

$$
S_1 \leq \frac{256p \sup_{m \in V_M} \beta_m^2 v^3}{c^2} \left(2^{\frac{\alpha}{\alpha-1}}\right)^{-N} < 1/16 \text{ by (2.2.7)}.
$$
For $S_2$ the preceding argument holds up to and including (3.2.8). In this case
\[
\frac{256p \sup_{m \in V_M} \beta_m^2 v^3}{c^2 (1 - 1/v)^2} v^{-M} \left( \frac{2 \alpha - 1}{\alpha^2} \right)^{2(M - N)} \leq \frac{256p \sup_{m \in V_M} \beta_m^2 v^3}{c^2 (1 - 1/v)^2} v^{-M}
\]
(3.2.8)

So that
\[
S_2 \leq \frac{256p \sup_{m \in V_M} \beta_m^2 v^3}{c^2 (1 - 1/v)^3}
\]
(3.2.9)

Hence by (2.2.6) it follows that $S_2 < 1/16$.

Consider $S_3$. The condition $z \in \partial \Lambda_n$ with $n \in V_N$, $N - 2 \leq M \leq N + 2$ implies that
\[
|z - \lambda|^2 \geq c^2 (v^{N-2})^{2(\alpha-1)} (|n - m| - 1)^2 = c^2 2^{2(N-2)} (|n - m| - 1)^2
\]
for $m \neq n - 1, n, n$. So
\[
S_3 \leq \sum_{N - 2 \leq M \leq N + 2} \sum_{m \in V_M} \sum_{\lambda \in \Lambda_n} \frac{\beta_m^2 2^{2(N+3)}}{c^2 2^{2(N-2)} (|n - j| - 1)^2}
+ \frac{p \beta_{n-1}^2 (n - 1)^{2(\alpha-1)} + p \beta_n^2 n^{2(\alpha-1)} + p \beta_{n+1}^2 (n + 1)^{2(\alpha-1)}}{(c/2)^2 (n - 1)^{2(\alpha-1)}}.
\]

By an argument similar to the one given for our analysis of $S_3$ when $\alpha < 1$ we have
\[
S_3 \leq \sup_{N - 2 \leq M \leq N + 2} \frac{\beta_m^2 p}{\beta_n^2} \left( \frac{1024\pi^2}{3c^2} + 1 + \frac{3 \cdot 256}{c^2} \right) \leq 1/16 \quad \text{by (2.2.6)}.
\]

Now for $S_4$, $M \geq N + 3$ with (3.1.4) implies that
\[
|z - \lambda|^2 \geq \left( c/2 \right)^2 (1 - 1/v)^2 v^{2(M-1)\alpha}
\]
so that
\[
\sum_{m \in V_M} \sum_{\lambda \in \Lambda_m} \frac{\beta_m^2 m^{2(\alpha-1)}}{|z - \lambda|^2} \leq \frac{4p \sup_{M \geq N+3} \beta_m^2}{c^2 (1 - 1/v)^2} \frac{\# V_M v^{2(\alpha-1)(M+1)}}{v^{2\alpha(M-1)}}
\]
(3.2.10)
\[
= \frac{4p \sup_{M \geq N+3} \beta_m^2}{c^2 (1 - 1/v)^2} \frac{\# V_M v^{-2(M+1)} v^{4\alpha}}{v^{4\alpha}}
\]
(3.2.11)
\[
\leq \frac{4p \sup_{M \geq N+3} \beta_m^2 v^{4\alpha}}{c^2 (1 - 1/v)^2} v^{-M-1}.
\]
(3.2.12)
Hence $S_4 \leq \frac{4p \sup_{m \in \mathcal{M}} \beta_m^2 v^{4\alpha}}{c^2 (1 - 1/v)^3} < 1/16$ by (2.2.6).

Finally, we consider $\alpha = 1$. In this case we have

$$
\sum_{k=0}^{\infty} \sum_{\lambda \in \Lambda_k} \beta_k^2 |z - \lambda|^2 \leq \sum_{k=0}^{\infty} p^{\beta_k^2} |z - \lambda|^2
$$

$$
= \left( \sum_{k < n/2} + \sum_{k \geq n/2} \right) \left( p^{\beta_k^2} |z - \lambda|^2 \right) + p^{\beta_{n-1}^2 + \beta_n^2 + \beta_{n+1}^2} \left( c/2 \right)^2
$$

$$
\leq p \left( \beta_\infty \sum_{j \geq n/2} \frac{1/j^2}{(c/2)^2} + \sup_{k \geq n/2} \beta_k^2 \left( \frac{2}{(c/2)^2} \sum_{j=1}^{\infty} \frac{1/j^2}{(c/2)^2} \right) \right) + 3 \frac{\beta_{n+1}^2}{(c/2)^2}
$$

$$
< 1/4 \text{ by (2.2.5), (2.2.8) since } n > N_1.
$$

\[ \square \]

### 3.3 Proof of Proposition 2.2.2

**Proof.** Suppose that $z \in \partial \Gamma'$. It follows from the definition of $\Gamma'$ (2.2.11) that all of the inequalities used in the proof of Proposition 2.2.1 for $z \in \partial \Gamma_{n'+1}$ also hold for $z \in \partial \Gamma'$. Hence, $R(z)$ is defined on $\Gamma'$. Now suppose that $z \notin \Gamma' \cup (\cup_{n'+1}^\infty \Gamma_k)$. Let $\partial \Gamma_n$ be the closest contour to the point $z$ with $n > n'$. Then once again, all inequalities from the proof of Proposition 2.2.1 hold for this $z$. Hence, $z \notin \text{Sp}(A + B)$. That is,

$$
\text{Sp}(A + B) \subset \Gamma' \cup (\cup_{k=n'+1}^\infty \Gamma_k).
$$

(3.3.1)

\[ \square \]
CHAPTER 4
THE DISCRETE HILBERT TRANSFORM

4.1 Technical preliminaries

Define $B \left( \ell^2(\mathbb{N}) \right)$ to be the space of all bounded linear operators on $\ell^2(\mathbb{N})$. Given a strictly increasing sequence of real numbers $a = (a_k)$ define the generalized discrete Hilbert transform (GDHT) by

$$(G_a \xi)(n) = \sum_{k \neq n} \frac{\xi_k}{a_k - a_n}, \quad \xi = (\xi_k)_{k=1}^{\infty}. \quad (4.1.1)$$

Of course care must be taken to ensure that the right hand side of (4.1.1) is defined. If $a_k = k \quad \forall k$ we have the standard discrete Hilbert transform $G \in B \left( \ell^2(\mathbb{N}) \right)$. Recall that $G$ is bounded from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$ with $\|G\| = \pi$. Before proving Proposition 2.3.1 we will need to prove a string of lemmas concerning slight modifications of the standard discrete Hilbert transform.

**Lemma 4.1.1.** Suppose $a_{k+1} - a_k > 0$ and $a_k \in \mathbb{N} \quad \forall k \in \mathbb{N}$. Then $G_a \in B \left( \ell^2(\mathbb{N}) \right)$ with $\|G_a\| \leq \pi$.

**Proof.** Suppose $\xi \in \ell^2 := \ell^2(\mathbb{N})$. Define the operator $I_a$ by $I_a(e_{a_k}) = e_{a_k} \quad \forall k$ and $I_a(e_j) = 0 \quad \forall j \notin a$ and define a vector $\tilde{\xi}$ by $\tilde{\xi}_{a_k} = \xi_k \quad \forall k$ and $\tilde{\xi}_j = 0 \quad \forall j \notin a$. Then $\|\xi\| = \|\tilde{\xi}\|$ and $G_a \xi = I_a G I_a \tilde{\xi}$. Thus, $\|G_a\| \leq \|I_a G I_a\| = \|G\| = \pi$. \qed

23
Lemma 4.1.2. Suppose $A$ is an operator whose matrix entries satisfy

$$|A_{k,j}| \leq \frac{C}{|k-j|^\tau}, \quad A_{k,k} = 0, \quad \text{with} \quad \tau > 1.$$  

Then $A \in B \left( \ell^2 \left( \mathbb{N} \right) \right)$ with $\|A\| \leq C \sum_{t=-\infty}^{\infty} |t|^{-\tau}$.

Proof. Decompose $A$ over its diagonals

$$A = \sum_{t=-\infty}^{\infty} A^t \quad \text{with} \quad A^t_{i,i+j} = \delta(t,j) A_{i,i+t}, \quad \forall i \in \mathbb{Z}_+, \quad j \geq -i.$$  

So $\|A^t\|_2 \leq C \max_{i \in \mathbb{Z}_+} |A^t_{i,i+t}| \leq C |t|^{-\tau}$ and it follows that

$$\|A\|_2 \leq \sum_{t=-\infty}^{\infty} C |t|^{-\tau}.$$  

Lemma 4.1.3. Suppose $a_{k+1} - a_k > \delta > 0 \quad \forall k$. Then $G_a \in B \left( \ell^2 \left( \mathbb{N} \right) \right)$ with $\|G_a\| \leq \frac{2\pi}{\delta} \left( 1 + \frac{\pi}{3} \right)$.

Proof. Write $\mathbb{R} = \bigcup_{k \in \mathbb{Z}} I_k, \quad I_k = [(k-1/2)\delta/2, (k+1/2)\delta/2]$. Then $\# (I_k \cap a) = 0$ or $1$. Enumerate $\left\{ j \left( \frac{\delta}{2} \right) : \# (I_j \cap a) = 1 \right\}$ in increasing order and call the sequence $\tilde{a}$. It follows from Lemma 4.1.1 that the GDHT $G_{\tilde{a}} \in B \left( \ell^2 \left( \mathbb{N} \right) \right)$ with $\|G_{\tilde{a}}\| \leq (2/\delta) \|G\| = \frac{2\pi}{\delta}$. Thus, with $A := G_a - G_{\tilde{a}}$ it suffices to show $\|A\| \leq \frac{2\pi^2}{3\delta}$.

Consider the matrix entries $A_{k,k} = 0 \quad \forall k$, and for $j \neq k$,

$$|A_{j,k}| = \left| \frac{1}{a_j - a_k} - \frac{1}{\tilde{a}_j - \tilde{a}_k} \right| = \frac{|\tilde{a}_j - a_j| - (\tilde{a}_k - a_k)}{(a_j - a_k) (\tilde{a}_j - \tilde{a}_k)}.$$  

Now note that $|a_j - a_k| > |j-k|\delta, \quad |\tilde{a}_j - \tilde{a}_k| \geq |j-k| (\delta/2)$, and $\tilde{a}_j - a_j, |\tilde{a}_k - a_k| < \delta/2$. Hence,

$$|A_{j,k}| \leq \frac{\delta/2 + \delta/2}{(j-k)(\delta/2)} \left| \frac{1}{|j-k|^{2\delta}} \right| = \frac{(2/\delta) \frac{1}{|j-k|^{2\delta}}}{|j-k|^{2\delta}}.$$  

So by Lemma 4.1.2, $\|A\| \leq \frac{2\pi^2}{3\delta}$.
Lemma 4.1.4. Suppose $a_k$ is a real sequence with $a_{k+1} - a_k > 2\delta > 0$ and $(z_k)$ is a complex sequence satisfying

$$|\text{Im} z_k|, |\text{Re} z_k - a_k| < \delta \quad \forall k.$$  

(4.1.2)

Then the operator $Z_a$ defined by $(Z_a \xi)(n) = \sum_{k \neq n} \frac{\xi_k}{a_k - z_n}$ is bounded in $\ell^2$ with

$$\|Z_a\| \leq \frac{\pi}{\delta} \left(1 + \frac{\pi}{2}\right).$$  

(4.1.3)

Proof. Since $a_{k+1} - a_k > 2\delta$, Lemma 4.1.3 implies $\|G_a\| \leq \frac{\pi}{\delta} \left(1 + \frac{\pi}{3}\right)$. Now, set $A = G_a - Z_a$. Consider the matrix elements $A_{k,k} = 0 \quad \forall k$ and for $j \neq k$,

$$|A_{j,k}| = \left|\frac{1}{a_k - a_n} - \frac{1}{a_k - z_n}\right| = \left|\frac{a_n - z_n}{(a_k - a_n)(a_k - z_n)}\right| \leq \frac{2\delta}{(2\delta)|k-n|(2\delta)|k-n|} = \frac{1}{2\delta|k-n|^2}.$$  

It follows from Lemma 4.1.2 that $\|A\| \leq \frac{\pi^2}{6\delta}$. \hfill \qed

Define $\ell^2(H)$ with the norm

$$\|\xi\|_{\ell^2(H)}^2 = \sum_{j=1}^{\infty} \|\xi_j\|_H^2, \quad \xi = (\xi_k), \xi_k \in H.$$  

Lemma 4.1.5. Suppose $a$, $z$, and $Z_a$ are as in Lemma 4.1.4. Consider the operator $Z_a^Y$ in $\ell^2(H)$

$$(Z_a^Y \xi)(n) = \sum_{k \neq n} \frac{\xi_k}{a_k - z_n}.$$  

Then $\|Z_a^Y\|_{\ell^2(H)} \leq \frac{\pi}{\delta} \left(1 + \frac{\pi}{2}\right).$
Proof. Suppose $\xi = (\xi_k) \in \ell^2 (H)$ with $\xi_j = \sum_{k=1}^{\infty} \xi_j^{(k)} \phi_k \in H$. Then

$$\| Z_a \xi \|^2 = \sum_{n=1}^{\infty} \| \left( Z_a \xi \right) (n) \|^2 = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\xi_j^{(k)}}{a_j - z_n} \right|^2$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \| (Z_a \xi^{(k)}) (n) \|^2 = \sum_{k=1}^{\infty} \| Z_a \xi^{(k)} \|_{\ell^2}^2$$

$$\leq \| Z_a \| \sum_{k=1}^{\infty} \| \xi^{(k)} \|_{\ell^2 (H)} = \| Z_a \| \| \xi \|_{\ell^2 (H)}.$$

\[ \square \]

4.2 Proof of Proposition 2.3.1

Proof. Let $b \in \ell^2 (N)$. First suppose $1/2 < \alpha < 1$ so that $v$ from (2.2.3) can be written as $v = 2^{2(1+\delta)}$ with $\delta > 0$. Set $\gamma = \frac{1 + 2^{2\delta}}{2}$. We have

$$\sum_{n=1}^{\infty} \left| \sum_{m \neq n} m^{\alpha-1} b_m \right|^2 = \sum_{n=1}^{\infty} \left| \sum_{m \neq n} m^{\alpha-1} b_m \right|^2$$

$$= \sum_{N=0}^{\infty} \sum_{n \in V_N} \left| \sum_{M=0}^{\infty} \sum_{m \in V_M, m \neq n} m^{\alpha-1} b_m \right|^2$$

by Cauchy’s inequality

$$\leq \left( \frac{2}{1-\gamma^{-1}} \right) \sum_{N=0}^{\infty} \sum_{n \in V_N} \sum_{M=0}^{\infty} \gamma^{|N-M|} \left| \sum_{m \in V_M, m \neq n} m^{\alpha-1} b_m \right|^2$$

$$= \left( \frac{2}{1-\gamma^{-1}} \right) (S_1 + S_2)$$

26
where

\[ S_1 = \sum_{N=0}^{\infty} \sum_{n \in V_N} \sum_{M=0}^{\infty} \gamma^{N-M} \left| \sum_{m \in V_M} m^{\alpha-1} b_m \right|^2 \]

and

\[ S_2 = \sum_{N=0}^{\infty} \sum_{n \in V_N} \sum_{M=0}^{\infty} \gamma^{N-M} \left| \sum_{m \in V_M, m \neq n} m^{\alpha-1} b_m \right|^2. \]

By an application of Cauchy’s inequality

\[ S_1 \leq \left( \frac{v}{2c (1 - 1/v)} \right)^2 \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \gamma^{N-M} \left( \frac{\#V_M}{\#V_N} \right) \left( \frac{2-2M}{v/2} \right)^{2\max(M,N)} \sum_{m \in V_M} |b_m|^2. \]  

(4.2.2)

Now, \( m \in V_M \) and \( 1/2 < \alpha < 1 \) implies \( m^{2(\alpha-1)} \leq 2^{-2M} \) and \( n \in V_N \) with \( |M - N| > 2 \) implies

\[ |t_m - t_n|^2 \geq c^2 (1 - 1/v)^2 v^{2(\max(M,N)-1)} = c^2 (1 - 1/v)^2 (v/2)^{2(\max(M,N)-1)}. \]

It follows that

\[ S_1 \leq \left( \frac{v}{2c (1 - 1/v)} \right)^2 \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \gamma^{N-M} \left( \frac{\#V_M}{\#V_N} \right) \left( \frac{2-2M}{v/2} \right)^{2\max(M,N)} \sum_{m \in V_M} |b_m|^2 \]

\[ \leq \left( \frac{v}{2c (1 - 1/v)} \right)^2 \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \gamma^{N-M} v^{M+N+2} 2^{-2M} \left( \frac{2}{v} \right)^{2\max(M,N)} \sum_{m \in V_m} |b_m|^2 \]

\[ = \left( \frac{v^2}{2c (1 - 1/v)} \right)^2 \sum_{M=0}^{\infty} \sum_{N=0}^{\infty} \gamma^{N-M} v^{-|M-N|} 2^{2(\max(M,N)-2M)} \sum_{m \in V_m} |b_m|^2. \]  

(4.2.3)

We will show that the following is uniformly bounded in \( M \):

\[ \sum_{N=0}^{\infty} \gamma^{N-M} v^{-|M-N|} 2^{2\max(M,N)-2M}. \]
We have
\[
\sum_{N=0}^{\infty} \gamma |N-M| v^{-|M-N|} 2^{2\max(M,N)-2M}
\]
\[
= \sum_{N=0}^{\infty} \gamma |N-M| 2^{-2(1+\delta)|M-N|-2M} 2^{2\max(M,N)}
\]
\[
\leq \sum_{N=0}^{\infty} \gamma |N-M| 2^{-2\max(M,N)-2\delta|M-N|} 2^{2\max(M,N)}
\]
\[
= \sum_{N=0}^{\infty} \gamma |N-M| (2^{-2\delta})^{|M-N|} \leq \frac{2}{1 - \gamma/2^{2\delta}}
\]
from which it follows that
\[
S_1 \leq \left( \frac{v^2}{2c (1 - 1/v)} \right)^2 \left( \frac{2}{1 - \gamma/2^{2\delta}} \right) \|b\|^2. \tag{4.2.4}
\]
We now analyze \(S_2\). The condition \(m, n \in V_{N-2}, V_{N-1}, V_N, V_{N+1}, V_{N+2}\) \(m \neq n\) implies by (2.3.1) that
\[
t_n - t_m = 2^{-(N+3)} (t'_n - t'_m) \quad \text{with} \quad |t'_n - t'_m| \geq c|n - m|.
\]
It follows that
\[
S_2 = \sum_{N=1}^{\infty} \sum_{n \in V_N} \sum_{M=0}^{\infty} |N-M| \sum_{m \in V_M} \left| \frac{m^{\alpha-1}b_m}{t_m - t_n} \right|^2
\]
\[
= \sum_{N=0}^{\infty} \sum_{n \in V_N} \gamma^2 \sum_{|M-N|\leq 2} \sum_{m \in V_M} \left| \frac{m^{\alpha-1}2^{N+3}b_m}{t'_m - t'_n} \right|^2
\]
\[
\leq \gamma^2 \left[ \frac{2\pi}{c} \left( 1 + \frac{\pi}{3} \right) \right]^2 \sum_{N=0}^{\infty} \sum_{m \in V_M} \left| m^{\alpha-1}2^{N+3}b_m \right|^2
\]
where the last inequality above follows from Lemma 4.1.3. Using the fact that for
\(m \in V_M, |M - N| \leq 2, m^{\alpha-1}2^{N+3} \leq 32\) It follows that
\[
S_2 \leq 5 \cdot \gamma^2 \frac{4096\pi^2}{c^2} \left( 1 + \frac{\pi}{3} \right)^2 \|b\|^2.
\]
Thus for $1/2 < \alpha < 1$ we have
\[
\sum_{n=1}^{\infty} \left| \sum_{m \neq n} m^{\alpha-1} b_m \right|^2 t_m - t_n \leq \left( \frac{2}{1 - \gamma^{-1}} \right) \left[ \left( \frac{v^2}{2c(1-1/v)} \right)^2 \left( \frac{1}{c} - \frac{2}{2\pi} \right) + 5 \cdot \gamma^2 \frac{4096\pi^2}{c^2} \left( 1 + \frac{\pi}{3} \right)^2 \right] \|b\|^2.
\]

We have already proved this result for $\alpha = 1$.

The proof for $\alpha > 1$ is similar to $1/2 < \alpha < 1$. In this case we let $\gamma = \frac{1 + v}{2}$. Our argument from the $1/2 < \alpha < 1$ case is valid up to the point (4.2.2). In this case, we have $m \in V_M$ implies $m^{2(\alpha-1)} \leq 2^{2(M+1)}$ and $n \in V_N$ implies
\[
|t_m - t_n|^2 \geq c^2 (1 - 1/v)^2 v^{2\alpha(\max(M,N)-1)} = c^2 (1 - 1/v)^2 (2v)^{2(\max(M,N)-1)}.
\]

So
\[
S_1 \leq \left( \frac{2}{c(1-1/v)} \right)^2 \sum_{M=0}^{\infty} \sum_{\substack{N=0 \\mid |N-M| > 2}} \gamma^{N-M} \left( \#V_M \right) \cdot \left( \#V_N \right) \frac{2^M}{(2v)^{2(\max(M,N)-1)}} \sum_{m \in V_M} |b_m|^2
\]
\[
\leq \left( \frac{4v}{c(1-1/v)} \right)^2 \sum_{M=0}^{\infty} \sum_{\substack{N=0 \\mid |N-M| > 2}} \gamma^{N-M} v^{M+N+2-2\max(M,N)} 2^{2(M-\max(M,N))} \sum_{m \in V_M} |b_m|^2
\]
\[
= \left( \frac{4v^2}{c(1-1/v)} \right)^2 \sum_{M=0}^{\infty} \sum_{\substack{N=0 \\mid |N-M| > 2}} \gamma^{N-M} v^{M+N-2\max(M,N)} 2^{2(M-\max(M,N))} \sum_{m \in V_M} |b_m|^2
\]
\[
= \left( \frac{4v^2}{c(1-1/v)} \right)^2 \sum_{M=0}^{\infty} \sum_{\substack{N=0 \\mid |N-M| > 2}} \gamma^{N-M} v^{-|M-N|} 2^{2(M-\max(M,N))} \sum_{m \in V_M} |b_m|^2.
\]

We will show that the following is uniformly bounded in $M$
\[
\sum_{N=0}^{\infty} \gamma^{N-M} v^{-|M-N|} 2^{2(M-\max(M,N))} \leq \sum_{N=0}^{\infty} \left( \gamma/v \right)^{|N-M|} \leq \frac{2}{1 - \gamma/v}.
\]

We conclude
\[
S_1 \leq \left( \frac{4v^2}{c(1-1/v)} \right)^2 \left( \frac{2}{1 - \gamma/v} \right) \|b\|^2. \tag{4.2.5}
\]
We now analyze $S_2$. The condition $m, n \in V_{N-2}, V_{N-1}, V_N, V_{N+1}, V_{N+2}, \quad m \neq n$ implies that

$$t_n - t_m = 2^{N-2} (t'_n - t'_m) \quad \text{with} \quad |t'_n - t'_m| \geq c|n - m|.$$  

It follows that

$$S_2 = \sum_{N=0}^{\infty} \sum_{n \in V_N} \sum_{|M-N| \leq 2} \gamma^{N-M} \left| \sum_{m \in V_M} m^{a-1} b_m \right|^2 \frac{1}{t_m - t_n}$$

$$\leq \gamma^2 \sum_{N=0}^{\infty} \sum_{n \in V_N} \sum_{|M-N| \leq 2} \left| \sum_{m \in V_M} m^{a-1} \gamma^{-(N-2)} b_m \right|^2 \frac{1}{t'_m - t'_n}$$

$$\leq \gamma^2 \left[ \frac{2\pi}{c} \left(1 + \frac{\pi}{3}\right)^2 \right] \sum_{N=0}^{\infty} \sum_{m \in V_M \atop |M-N| \leq 2} \left| m^{a-2} b_m \right|^2$$

where the last inequality above follows from Lemma 4.1.3. Using the fact that for $m \in V_M, \quad |M - N| \leq 2, m^{a-2} \leq 32$ it follows that

$$S_2 \leq 5 \cdot \gamma^2 \frac{4096\pi^2}{c^2} \left(1 + \frac{\pi}{3}\right)^2 \|b\|^2.$$

We conclude that for $\alpha > 1$,

$$\sum_{n=1}^{\infty} \left| \sum_{m \neq n} m^{a-1} b_m \right|^2 \left( \frac{2}{1 - \gamma^{-1}} \right) \left[ \left( \frac{4v^2}{c(1-1/v)} \right)^2 \left( \frac{2}{1 - \gamma/v} \right) + 5 \cdot \gamma^2 \frac{4096\pi^2}{c^2} \left(1 + \frac{\pi}{3}\right)^2 \right] \|b\|^2.$$

\[\square\]

**Lemma 4.2.1.** Suppose $\alpha \in (1/2, \infty) \setminus \{1\}$ and $\{t_k\}_{k=1}^{\infty}$ satisfies (2.3.1). Let $\{z_k\}$ be a sequence such that $|z_k - t_k| \leq (c/2) k^{a-1} \quad \forall \quad k \in \mathbb{N}$. Then there is a constant $C > 0$ depending only on $\alpha$ such that for all $(b_m) \in \ell^2 (\mathbb{N})$ we have

$$\sum_{n=1}^{\infty} \left| \sum_{m \neq n} m^{a-1} b_m \right|^2 \left( \frac{2}{1 - \gamma^{-1}} \right) \left[ \left( \frac{4v^2}{c(1-1/v)} \right)^2 \left( \frac{2}{1 - \gamma/v} \right) + 5 \cdot \gamma^2 \frac{4096\pi^2}{c^2} \left(1 + \frac{\pi}{3}\right)^2 \right] \|b\|^2.$$
Proof.

\[
\sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{m^{\alpha-1} b_m}{t_m - z_n} \right|^2 = \sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{m^{\alpha-1} b_m (z_n - t_n)}{(t_m - z_n) (t_m - t_n)} \right| + \sum_{m \neq n} \left| \frac{m^{\alpha-1} b_m}{t_m - t_n} \right|^2 \\
\leq 2 \left( \sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{m^{\alpha-1} b_m (z_n - t_n)}{(t_m - z_n) (t_m - t_n)} \right|^2 + \sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{m^{\alpha-1} b_m}{t_m - t_n} \right|^2 \right).
\]

The second sum is easily handled by the previous lemma. Consider now

\[
\sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{m^{\alpha-1} b_m (z_n - t_n)}{(t_m - z_n) (t_m - t_n)} \right|^2
\]

We will apply Lemma 4.1.4, so we consider

\[
\left| \frac{m^{\alpha-1} (z_n - t_n)}{(t_m - z_n) (t_m - t_n)} \right|.
\]

First suppose \(1/2 < \alpha < 1\). If \(m \in V_M\) and \(n \in V_N\) with \(|M - N| \leq 2\) then

\[
\left| \frac{m^{\alpha-1} (z_n - t_n)}{(t_m - z_n) (t_m - t_n)} \right| \leq \frac{(v^{\alpha-1})^{N-2} (c/2) n^{\alpha-1}}{|t_m - z_n|M} |t_m - t_n| \leq \frac{2^{2-N} (c/2) 2^{-N}}{\alpha^2 (c/2)^2 (v^{N+2})^{2(\alpha-1)} |m - n|^2} = \frac{(128 \alpha)}{|m - n|^2}.
\]

Now if \(|M - N| > 2\) then

\[
\left| \frac{m^{\alpha-1} (z_n - t_n)}{(t_m - z_n) (t_m - t_n)} \right| \leq \frac{2^{-M} (c/2) 2^{-N}}{(c/2)^2 (1 - 1/v)^2} = \left( \frac{2}{c (1 - 1/v)^2} \right) (2/v)^2 \left( \frac{2^{M-N}}{2(1+\delta)|M-N|} \right) \leq \frac{2}{c (1 - 1/v)^2} (2/v)^2 \left( \frac{1}{v^{(M-N)/2}} \right) \leq \frac{2}{c (1 - 1/v)^2} (2/v)^2 \frac{v^{3/2}}{|m - n|^{3/2}}.
\]

Hence, for \(1/2 < \alpha < 1\) a (rough) upper bound is

\[
\sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{m^{\alpha-1} b_m}{t_m - z_n} \right|^2 \leq \left( \frac{4096}{(\alpha^2 c^2 v^4)} + \frac{64 v^3}{(v-1)^4} \right) \left( \sum_{t=-\infty}^{\infty} \frac{|t|^{-3/2}}{t \neq 0} \right)^2 + C_\alpha^2 \|b\|^2
\]

\[
= \left( \tilde{C}_\alpha \right)^2 \|b\|^2.
\]
Similarly, for $1 < \alpha$, if $|M - N| \leq 2$ we have:

$$\left| \frac{m^{\alpha-1}(z_n - t_n)}{(t_m - z_n)(t_m - t_n)} \right| \leq \frac{(v^{\alpha-1})^{N+3} (c/2) n^{\alpha-1}}{|t_m - z_n||t_m - t_n|} \leq \frac{2^{N+3} (c/2) 2^{N+1}}{\alpha^2 (c/2)^2 (v^{N-2})^{2(\alpha-1)} |m - n|^2} = \left( \frac{512}{c \alpha^2} \right) |m - n|^{-2}.$$

If $|M - N| > 2$ we have

$$\left| \frac{m^{\alpha-1}(z_n - t_n)}{(t_m - z_n)(t_m - t_n)} \right| \leq \frac{2^{M+1} (c/2) 2^{N+1}}{(c/2)^2 (1 - 1/v)^2 v^{2(\text{Max}(M,N) - 1)\alpha}} = \left( \frac{8v^2}{c (1 - 1/v)^2} \right) \left( \frac{2^{-|N-M|}}{v^{2(\text{Max}(M,N) + 1)}\alpha} \right) \leq \left( \frac{8v^2}{c (1 - 1/v)^2} \right) \left( \frac{1}{|m - n|^2} \right).$$

Hence, for $\alpha > 1$ a (rough) upper bound is

$$\sum_{n=1}^{\infty} \sum_{m \neq n} \left| \frac{m^{\alpha-1}b_m}{t_m - z_n} \right|^2 \leq \left[ \left( \frac{512}{c^2 \alpha^4} + \frac{64v^4}{c^2 (1 - 1/v)^4} \right) \frac{\pi^2}{9} + C_\alpha^2 \right] \|b\|^2 = \left( \tilde{C}_\alpha \right)^2 \|b\|^2.$$

The following vector-valued version of Lemma 4.2.1 can be proven in the same manner as Lemma 4.1.5. We omit the details.

**Lemma 4.2.2.** Suppose $\alpha \in (1/2, \infty) \setminus \{1\}$ and $\{t_k\}_1^n$ satisfies (2.3.1). Let $\{z_k\}$ be a sequence such that $|z_k - t_k| \leq (c/2) k^{\alpha-1} \forall k \in \mathbb{N}$. Then there is a constant $C > 0$ depending only on $\alpha$ such that for any $(b_m) \in \ell^2(H)$ we have

$$\sum_{n=1}^{\infty} \sum_{m \neq n} \left| \frac{m^{\alpha-1}b_m}{t_m - z_n} \right|^2 \leq C \|b\|^2.$$
CHAPTER 5

PROOF OF MAIN RESULTS

5.1 Proof of Theorem 2.4.1

By Theorem 1.0.1 it suffices to show that there exists an integer $N_*$ such that

$$\sum_{n>N_*} \| P_n (Q_n - P_n) f \|^2 < 1/2 \quad \forall \| f \| = 1.$$  

To this end, suppose that $f = \sum_{k=0}^{\infty} f_k \phi_k$ with $\| f \| = 1$. By the first resolvent formula:

$$Q_n - P_n = \frac{1}{2\pi i} \int_{\partial \Gamma_n} (R(z) - R^0(z)) \, dz$$

$$= \frac{1}{2\pi i} \int_{\partial \Gamma_n} (R(z) BR^0(z)) \, dz.$$  

Hence $\| (Q_n - P_n) f \|^2 = \frac{1}{2\pi} \left( \int_{\partial \Gamma_n} R(z) BR^0(z) f \, dz \right)^2$

$$\leq \frac{1}{2\pi} \left( \int_{\partial \Gamma_n} \left\| R(z) BR^0(z) f \right\| \, dz \right)^2$$

$$\leq \frac{1}{2\pi} \left( \int_{\partial \Gamma_n} \left\| \sum_{k=1}^{\infty} \frac{f_k R(z) B \phi_k}{z - \lambda_k} \right\| \, dz \right)^2$$

Now define $z_n^* \in \partial \Gamma_n$ to be the point at which the sum

$$\left\| \sum_{k=1}^{\infty} \frac{f_k R(z) B \phi_k}{z - \lambda_k} \right\|$$
is maximized. Note that by (2.1.12) and (2.2.10) there is a constant $\kappa > 0$ such that

$$\sup_{z \in \partial \Gamma_n} \| R(z) \|^2 |\partial \Gamma_n|^2 \leq \kappa$$

which gives

$$\| (Q_n - P_n) f \|^2 \leq \frac{\kappa}{2\pi} \left\| \sum_{k=1}^{\infty} \frac{f_k B \phi_k}{z_n^* - \lambda_k} \right\|^2.$$

It follows that

$$\sum_{n > N_*} \| P_n (Q_n - P_n) f \|^2 \leq \frac{\kappa}{2\pi} \sum_{n > N_*} \left( \sum_{k \leq M_*} \left\| \frac{f_k B \phi_k}{z_n^* - \lambda_k} \right\|^2 + \sum_{m > M_*} \left\| \frac{f_k B \phi_k}{z_n^* - \lambda_k} \right\|^2 \right).$$

By Cauchy’s inequality:

$$\left\| \sum_{m \leq M_*} \frac{f_k B \phi_k}{z_n^* - \lambda_k} \right\|^2 \leq \left( \sum_{k \leq M_*} |f_k|^2 \left\| B \phi_k \right\|^2 \right) \left( \sum_{k \leq M_*} |z_n^* - t_k|^{-2} \right)$$

so that

$$\sum_{n > N_*} \left\| \sum_{k \leq M_*} \frac{f_k B \phi_k}{z_n^* - \lambda_k} \right\|^2 \leq \sum_{n \geq N_*} \beta_{\infty}^2 M_*^{\alpha-1} M_* |z_n^* - \lambda_{M_*}|^{-2}.$$

Since $\alpha > 1/2$ the sum on the right converges and can be made arbitrarily small by choosing sufficiently large $N_* >> M_*$. 

On the other hand:

$$\sum_{n > N_*} \left\| \sum_{k > M_*} \frac{f_k B \phi_k}{z_n^* - \lambda_k} \right\|^2 \leq \sum_{n > N_*} \left\| \sum_{k \geq M} \sum_{\lambda \in \Lambda_k} \frac{f_k B \phi_k}{z_n^* - \lambda} \right\|^2$$

where $M_* \in \Lambda_{\tilde{M}}$. For each $k \geq \tilde{M}$, enumerate

$$\Lambda_k = \lambda_k^{(1)} \leq \lambda_k^{(2)} \leq \ldots \lambda_k^{(|\Lambda_k|)}$$
where of course $|\Lambda_k| \leq p$. Enumerate likewise the corresponding eigenfunctions $\phi_k^{(j)}$ and coefficients $f_k^{(j)}$. Then

$$\sum_{n>N^*} \left\| \sum_{k>M^*} \frac{f_k B \phi_k}{z_n^* - \lambda_k} \right\|^2 \leq p^2 \sum_{j=1}^p \sum_{n>N^*} \left\| \sum_{k \geq M} f_k^{(j)} B \phi_k^{(j)} \right\|^2$$

$$\leq p^3 \sup_{k \geq \tilde{M}} \beta_k^2 \sum_{n>N^*} \left| \sum_{k \geq \tilde{M}} f_k^{(j)} \right|^2 \left| \frac{z_n^* - \lambda_k^{(j)}}{z_n^* - \lambda_k} \right|^2$$

$$\leq p^3 \sup_{k \geq \tilde{M}} \beta_k^2 K_\alpha^2 \| f \|^2.$$

It follows from condition (2.4.1) that for sufficiently large $\tilde{M}$ the sum is less than $1/4$ and the proof is complete.
CHAPTER 6
APPLICATION TO SCHRÖDINGER OPERATORS

6.1 Eigenvalues

As an application of Theorem 2.4.1, we consider the differential operator $A$ on $L^2(\mathbb{R})$ defined by

$$Ay = -y'' + |x|^{\beta} y, \quad \text{with} \quad \beta > 1.$$  \hspace{1cm} (6.1.1)

We will derive conditions on a potential $b(x)$ under which the eigensystem for the operator $L = A + B$, $Bf = b(x)f(x)$ is an unconditional basis for $L^2(\mathbb{R})$. In the case $\beta = 2$ (6.1.1) is the harmonic oscillator and the eigenfunctions are the Hermite functions (see [16, Sect. 10.8]). We considered this operator and its perturbations in [1]. A proof of the unconditional basis property of the eigensystem of $L$ when $\beta > 1$ was sketched in [2] and will be furnished in full here.

For our purposes here we will be interested in two related problems. To determine the growth rate of $\text{Sp}A = \{\lambda_0 \leq \lambda_1 \leq \ldots\}$ and, for a given perturbation $b(x)$, to determine the growth rate of $\|b(x)\phi_n\|_2$ where $\phi_n$ is the eigenfunction corresponding to $\lambda_n$. Once these have been determined Theorem 2.4.1 can be applied.

For $\beta = 2$ it is a classical result that the spectrum of $A$ is $\{2n + 1\}_{n=0}^\infty$ (see [16, Sect. 10.8]). In the anharmonic case ($\beta \neq 2$) the analysis of $\text{Sp}A$ is not as elementary. Techniques using the theory of analytic functions and connection formulas can be found in the work of Olver [19, Ch. 13]. Interesting comments on the general theory
and history of turning-point problems can also be found in [19]. Let us also mention the papers [21], [22] where the eigenvalues for the eigenproblem \(-y_{zz} + q(z)y = \lambda y\) are analyzed for polynomial \(q(z)\) in great detail. Here we will outline the results of the interesting 1954 paper of Titchmarsh [25].

The spectrum of \(A\) consists of an infinite set of eigenvalues

\[
\text{Sp} A = \{ \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \} \quad \text{with} \quad \lim_{n \to \infty} \lambda_n = \infty.
\]

For \(\beta > 1\) the growth of the sequence of eigenvalues is described by the formula

\[
\lim_{n \to \infty} \left[ 2 \int_0^{\lambda_n^{1/\beta}} \left( \lambda_n - |x|^{\beta} \right)^{1/2} dx - (n + 1/2) \pi \right] = 0. \tag{6.1.2}
\]

For a proof, see the last section of [25]. It follows from (6.1.2) by a change of variables that

\[
\lim_{n \to \infty} \left[ 2 \lambda_n^{2/\beta} \Omega_{\beta} - (n + 1/2) \pi \right] = 0 \quad \text{with} \quad \Omega_{\beta} = 2 \int_0^1 (1 - x^{\beta})^{1/2} dx. \tag{6.1.3}
\]

Subtracting the \(n\)th term from the \((n+1)\)st term in (6.1.3) we derive

\[
\lim_{n \to \infty} \left[ \lambda_{n+1}^{2/\beta} \Omega_{\beta} - \lambda_n^{2/\beta} \right] = \pi/\Omega_{\beta}. \tag{6.1.4}
\]

From (6.1.4) it is straightforward to show that there exist constants \(c > 0\), \(N \in \mathbb{N}\) (depending on \(\beta\)) such that

\[
\lambda_{n+1} - \lambda_n \geq cn^{\alpha-1} \quad \forall n > N, \quad \alpha = \frac{2\beta}{\beta + 2}. \tag{6.1.5}
\]

### 6.2 Eigenfunctions

We now turn to an analysis of the eigenfunctions of the operator \(A\). Because the derivation of pointwise bounds for the eigenfunctions requires its own delicate treatment we postpone our analysis to Chapter 7. Denote the eigenfunction corresponding to \(\lambda_n\) by \(\phi_n\) and define

\[
L(p; \gamma) = \left\{ b : b(x) (1 + |x|^2)^{-\gamma/2} \in L^p(\mathbb{R}) \right\}.
\]
In Chapter 7 we will establish the following pointwise bounds on $\phi_n$:

$$
|\phi_n(x)| \leq \frac{K \exp(Q(x))}{|\lambda_n - |x|^\beta|^{1/4} + \lambda_n^{(\beta-1)/6\beta}} \text{ with (6.2.1)}
$$

$$
Q(x) = \begin{cases} 
\int_{\lambda_n^{1/\beta}}^x (\lambda_n - |t|^\beta)^{1/2} dt, & x > \lambda_n^{1/\beta} \\
0, & |x| \leq \lambda_n^{1/\beta} \\
\int_{\lambda_n^{1/\beta}}^x (\lambda_n - |t|^\beta)^{1/2} dt, & x < -\lambda_n^{1/\beta}.
\end{cases}
$$

For the case $\beta = 2$, this inequality is proven in [3]. In Chapter 7 we boost this proof to cover $\beta > 1$. Such constructions for Schrödinger operators with turning points are discussed in [19, Ch 8,11].

We now turn to the growth of $\|b(x)\phi_n(x)\|_2$. Suppose that $b \in L(p;\gamma)$. Then

$$
\|b(x)\phi_n(x)\|_2 = \left\|b(x) \left(1 + |x|^2\right)^{-\gamma/2} \phi_n(x) \left(1 + |x|^2\right)^{\gamma/2}\right\|_2
\leq \left\|b(x) \left(1 + |x|^2\right)^{-\gamma/2} \|p\| \phi_n(x) \left(1 + |x|^2\right)^{\gamma/2}\right\|_q
$$

where $1/p + 1/q = 1/2$.

Our analysis of $\left\|\phi_n(x) \left(1 + |x|^2\right)^{\gamma/2}\right\|_q$ will follow the procedure outlined in [24, Sect. 1.5, p.27] and adopted in Lemma 8 of [1]. For notational convenience, $C$ will be used to denote an arbitrary constant. First break the integral up:

$$
\int_0^\infty \phi_n(x) \left(1 + |x|^2\right)^{\gamma/2} q \, dx = 
\left[\int_0^{1/2 \lambda_n^{1/\beta}} + \int_{1/2 \lambda_n^{1/\beta}}^{2 \lambda_n^{1/\beta}} + \int_{2 \lambda_n^{1/\beta}}^\infty\right] \left|ph_i n(x) \left(1 + |x|^2\right)^{\gamma/2} q \right| dx.
$$

By (6.2.1) we have:

$$
\int_0^{1/2 \lambda_n^{1/\beta}} \left|\phi_n(x) \left(1 + |x|^2\right)^{\gamma/2}\right|^q \, dx \leq C \int_0^{1/2 \lambda_n^{1/\beta}} \left|\lambda_n - |x|^\beta\right|^{-q/4} \left(1 + |x|^2\right)^{\gamma/2} q \, dx
\leq C \lambda_n^{\frac{\gamma}{2} + \frac{1}{3} - \frac{3}{4}} \leq C \left(n \frac{2\beta}{\alpha+\beta}-\frac{\gamma}{4}\right) \left(\frac{2\beta}{\alpha+\beta}-\frac{3}{4}\right).
$$

(6.2.2)
A very similar argument can be used to prove the same growth rate for the integral in the region $x \geq \frac{1}{2} \lambda_n^{1/\beta}$.

The same bounds for the integral over $\frac{1}{2} \lambda_n^{1/\beta} < x < \lambda_n^{1/\beta} - \lambda_n^{-\frac{\beta-1}{3\beta}}$ and $\lambda_n^{-\frac{\beta-1}{3\beta}} < x < \frac{3}{2} \lambda_n^{1/\beta}$ can be found by very similar arguments so we focus on the former. Note that for $\frac{1}{2} \lambda_n^{1/\beta} < x < \lambda_n^{1/\beta} - \lambda_n^{-\frac{\beta-1}{3\beta}}$ we have

$$\left(\lambda_n - |x|^\beta\right)^{-q/4} \leq C \left[\left(\lambda_n^{1/\beta} - |x|\right) \lambda_n^{-\frac{\beta-1}{3\beta}}\right]^{-q/4}$$

by an application of the mean value theorem. It follows that

$$\int_{\frac{1}{2} \lambda_n^{1/\beta}}^{\lambda_n^{1/\beta} - \lambda_n^{-\frac{\beta-1}{3\beta}}} \left|\phi_n (x) \right| \left(1 + |x|^2\right)^{\gamma/2} q \, dx$$

$$\leq C \int_{\frac{1}{2} \lambda_n^{1/\beta}}^{\lambda_n^{1/\beta} - \lambda_n^{-\frac{\beta-1}{3\beta}}} \left|\left(\lambda_n - |x|^\beta\right)^{-q/4} \left(1 + |x|^2\right)^{q\gamma/2}\right| \, dx$$

$$\leq C \lambda_n^{q/2 - \frac{q}{4} \left(\frac{\beta-1}{\beta}\right)} \int_{\frac{1}{2} \lambda_n^{1/\beta}}^{\lambda_n^{1/\beta} - \lambda_n^{-\frac{\beta-1}{3\beta}}} \left(\lambda_n^{1/\beta} - |x|\right)^{-q/4} dx$$

$$\leq C \lambda_n^{q/2 - \frac{q}{4} \left(\frac{\beta-1}{\beta}\right)-\left(\frac{\beta-1}{3\beta}\right)\left(\frac{q-4}{4}\right)}, \quad (6.2.3)$$

Note that the above inequality only holds if $q \neq 4$. For $q = 4$ the bound on the right should be replaced by:

$$C \lambda_n^{q/2 - \frac{q}{4} \left(\frac{\beta-1}{\beta}\right)} \log (n + 2). \quad (6.2.4)$$

Finally, for $|x - \lambda_n^{1/\beta}| \leq \lambda_n^{1-\frac{\beta}{3\beta}}$ it follows from (6.2.1) that $|\phi_n (x)| \leq \lambda_n^{1-\frac{\beta}{3\beta}}$. In fact, this bound holds for all $x \in \mathbb{R}$. Hence,

$$\int_{\lambda_n^{1/\beta} - \lambda_n^{-\frac{\beta-1}{3\beta}}}^{\lambda_n^{1/\beta} + \lambda_n^{-\frac{\beta-1}{3\beta}}} \left|\phi_n (x) \right| \left(1 + |x|^2\right)^{\gamma/2} q \, dx \leq C \lambda_n^{\frac{1-\beta}{3\beta}} \lambda_n^{q(1-\beta)/3\beta}. \lambda_n^{\frac{q(1-\beta)}{3\beta}}.$$

Note that this matches (6.2.3).
By (6.2.2) and (6.2.3) it is seen that if \( b(x) \in L(p; \gamma) \) then \( \|b\phi_n\|_2 \leq Cn^{\frac{2\xi}{p+2}} \) where

\[ \xi = \max \left\{ \frac{1}{3\beta} (1 - \beta + 3\gamma + (\beta - 1)/p); \frac{1}{\beta} (\gamma - \beta/4 + 1/2 - 1/p) \right\} \quad (6.2.5) \]

\[ = \begin{cases} 
\frac{1}{3\beta} (1 - \beta + 3\gamma + (\beta - 1)/p), & 2 \leq p < 4 \\
\frac{1}{\beta} (\gamma - \beta/4 + 1/2 - 1/p), & p > 4.
\end{cases} \]

In the exceptional case \( p = 4 \) we have

\[ \|b\phi_n\|_2 \leq Cn^{\frac{2\gamma}{p+2} + \frac{1-\beta}{2(\beta+2)}} \log (n + 2) \quad (6.2.6) \]

The following Proposition follows from (6.1.5), (6.2.5), (6.2.6) and Theorem 2.4.1.

**Proposition 6.2.1.** Let \( A \in (6.1.1) \), \( b \in L(p, \gamma) \), and define the operator \( B \) on \( L^2(\mathbb{R}) \) by \( Bf = b(x)f(x) \). Suppose that

\[ \begin{cases} 
\beta - 1 < p(-4 + 5\beta/2 - 3\gamma) & \text{if} \quad 2 \leq p < 4 \\
2 > p(3 - 3\beta/2 + 2\gamma) & \text{if} \quad 4 \leq p.
\end{cases} \quad (6.2.7) \]

Then the system of eigen and associated functions for the operator \( A + B \) is an unconditional basis.
CHAPTER 7
EIGENFUNCTION ASYMPOTOTICS

7.1 Pointwise bounds

In this Chapter we will prove the pointwise bounds for the eigenfunctions \( \{ \phi_n \} \) of the operator

\[
A = -\frac{d^2}{dx^2} + |x|^\beta, \quad \beta > 1
\]

(7.1.1)

which were used in the previous chapter. Equation (7.1.1) is an example of the more general eigenproblem:

\[
-\frac{d^2}{dx^2} + q(x)y = \lambda y.
\]

(7.1.2)

In the case where \( q(x) \) is monotonically increasing (7.1.2) is referred to as a turning point problem and the point where \( q(x) = \lambda \) is called a turning point.

In analyzing (7.1.1) our arguments and notations closely follow [3] in which the case \( \beta = 2 \) is considered. Our arguments will be concise since our goal in this chapter is merely to show how to modify the arguments of [3] in a few places to cover all \( \beta > 1 \).

First suppose that for a given \( \lambda \), \( y_1(x, \lambda) \) is a solution of \( Ay = \lambda y \) in \( L^2((0, \infty)) \)
and that \( y_2 \) is a linearly independent solution. Define

\[
\xi (x, \lambda) = \left( \frac{3}{2} \int_{\lambda^{1/\beta}}^{x} |t^\beta - \lambda|^{1/2} dt \right)^{2/3} \tag{7.1.3}
\]

\[
S (x, \lambda) = |\xi' (x, \lambda)|^{-1/2}, \quad K (x, y) = \frac{S'' (x, \lambda)}{S (x, \lambda)} |x^\beta - \lambda|^{-1/2} \tag{7.1.4}
\]

\[
\tilde{Q} (x, \lambda) = \begin{cases} 
Q (x, \lambda) = \int_{x}^{\lambda^{1/\beta}} (\lambda - t^\beta)^{1/2} dt, & 0 \leq x \leq \lambda^{1/\beta} \\
Q_1 (x, \lambda) = \int_{\lambda^{1/\beta}}^{x} (t^\beta - \lambda)^{1/2} dt, & x > \lambda^{1/\beta}
\end{cases} \tag{7.1.5}
\]

and set

\[
z_1 (x, \lambda) = S (x, \lambda) \text{Ai} (\xi (x, \lambda)), \quad z_2 (x, \lambda) = S (x, \lambda) \text{Bi} (\xi (x, \lambda)) \tag{7.1.6}
\]

where \( \text{Ai} \) and \( \text{Bi} \) are the Airy functions (see [16, Sect. 5.17]). Then \( y_1 \) and \( y_2 \) satisfy the integral equation

\[
y_1 (x, \lambda) = z_1 (x, \lambda) + \int_{x}^{\infty} H (x, t, \lambda) y_1 (t, \lambda) dt,
\]

\[
y_2 (x, \lambda) = z_2 (x, \lambda) + \int_{0}^{x} H (x, t, \lambda) y_2 (t, \lambda) dt
\]

where \( H (x, t, \lambda) = \frac{(z_1 (x, \lambda) z_2 (t, \lambda) - z_1 (t, \lambda) z_2 (x, \lambda)) S'' (t, \lambda)}{S (t, \lambda)} \). For a proof we refer to [19, Sect. 11.3]. We now introduce the auxiliary functions

\[
\hat{Q}_1 (x, \lambda) = \begin{cases} 
0, & 0 \leq x \leq \lambda^{1/\beta} \\
Q_1 (x, \lambda), & x > \lambda^{1/\beta}
\end{cases}
\]

\[
v_k (x, \lambda) = \left| \lambda - |x|^\beta \right|^{1/4} z_k (x, \lambda) \exp \left[ (-1)^{k+1} \hat{Q}_1 (x, \lambda) \right]
\]

\[
\psi_k (x, \lambda) = \left| \lambda - |x|^\beta \right|^{1/4} y_k (x, \lambda) \exp \left[ (-1)^{k+1} \hat{Q}_1 (x, \lambda) \right]
\]

The following is Lemma 1 in [3].

**Lemma 7.1.1.** The functions \( \psi_k (x, \lambda) \) are presented as \( \psi_k (x, \lambda) = v_k (x, \lambda) + v_k^{(1)} (x, \lambda) \), where \( \sup_{x \geq 0, \lambda > 0} |v_k (x, \lambda)| \leq C_0 \) and \( \sup_{x \geq 0, \lambda > 0} (\lambda + 1) \left| v_k^{(1)} (x, \lambda) \right| \leq C_0 \).
We state formula (28) from [3] and refer to Section 5.11 of [16] for a proof.

\[ v_1(x, \lambda) = \frac{1}{\sqrt{3}} \left( \frac{\tilde{Q}(x, \lambda)}{2} \right)^{1/6} \]

\[ \times \left[ \frac{1}{\gamma \left( \frac{4}{3} \right)} - \frac{1}{\gamma \left( \frac{2}{3} \right)} \right] \left( \frac{\tilde{Q}(x, \lambda)}{2} \right)^{2/3} \text{sgn} \left( x - \lambda^{1/\beta} \right) + O \left( \tilde{Q}(x, \lambda)^{13/6} \right) \]

as \( \tilde{Q}(x, \lambda) \to 0 \). Note that the definitions of \( v_k \) and \( \psi_k \) together with the previous lemma imply

\[ y_k(x, \lambda) = z_k(x, \lambda) \left( 1 + z^{(1)}_k(x, \lambda) \right). \]

**Lemma 7.1.2.** We have the following estimate:

\[ |y_1(x, \lambda)| \leq \frac{C \exp \left[ (-1)^{k+1} \tilde{Q}_1(x, \lambda) \right]}{|x|^\beta - \lambda} \]

where \( C \) is a constant.

**Proof.** Suppose that \( |x - \lambda^{1/\beta}| \leq \lambda^{1-\beta} \). Then

\[ \tilde{Q}(x, \lambda) = \int_{\lambda^{1/\beta}}^x (t^\beta - \lambda)^{1/2} dt = \int_{\lambda^{1/\beta}}^x \frac{(t^\beta - \lambda)^{1/2} \beta t^{\beta-1}}{\beta \left( t^{\beta-1} - \lambda \frac{\beta-1}{\beta} \right)} dt \]

\[ = \frac{1}{\beta} \int_{\lambda^{1/\beta}}^x (t^\beta - \lambda)^{1/2} \beta t^{\beta-1} \left[ \frac{\lambda^{\beta-1}}{\beta} \left( 1 + \frac{t^{\beta-1} - \lambda^{\beta-1}}{\lambda^{\beta-1}} \right) \right]^{-1} dt \]

\[ = \frac{1}{\beta \lambda^{\beta-1}} \int_{\lambda^{1/\beta}}^x (t^\beta - \lambda)^{1/2} t^{\beta-1} \left[ 1 + \frac{t^{\beta-1} - \lambda^{\beta-1}}{\lambda^{\beta-1}} + \ldots \right] dt \]

Now by the mean value theorem we have:

\[ t^{\beta-1} - \lambda^{\beta-1} = \left( t^\beta \right)^{\beta-1} - \lambda^{\beta-1} \leq \left( \frac{\beta-1}{\beta} \right) \lambda^{-1/\beta} \left[ t^\beta - \lambda \right]. \]

It follows that

\[ \tilde{Q}(x, \lambda) = \frac{2 \left( x^\beta - \lambda \right)^{3/2}}{3 \beta \lambda^{1-1/\beta}} + \frac{2 \left( x^\beta - \lambda \right)^{5/2}}{5 \beta \lambda^{2-1/\beta}} + \ldots \]

From (7.1.7) and the assumption \( |x - \lambda^{1/\beta}| \leq \lambda^{1-\beta} \) we deduce that \( |v_1(x, \lambda)| \leq C \left| x^\beta - \lambda \right|^{1/4} \lambda^{\beta-1} \) and from Lemma 7.1.1 that \( |\psi_1(x, \lambda)| \leq C \left| x^\beta - \lambda \right|^{1/4} \lambda^{\beta-1} \). Finally by the definition of \( \psi_1 \) we have

\[ |y_1(x, \lambda)| \leq C \lambda^{1-\beta} \text{ whenever } |x - \lambda^{1/\beta}| \leq \lambda^{1-\beta}. \]
Furthermore,
\[ |\lambda - x^\beta|^{1/4} = \left| (\lambda^{1/\beta})^\beta - x^\beta \right|^{1/4} \]
by the MVT
\[ \geq (\lambda^{1/\beta})^{\frac{\beta-1}{4}} |x - \lambda^{1/\beta}|^{1/4} \geq \lambda^{\frac{\beta-1}{4}}. \]
Hence \[ |\lambda - |x|^\beta|^{1/4} \geq \lambda^{\frac{\beta-1}{4\beta}} \] whenever \[ |x - \lambda^{1/\beta}| > \lambda^{\frac{1-\beta}{3\beta}} \] so that by Lemma 7.1.1 and the definition of \( \psi_1 \) we have
\[ |\lambda - |x|^\beta|^{1/4} |y_1(x, \lambda)| \exp \left[ (-1)^{k+1} \hat{Q}_1(x, \lambda) \right] \leq C. \]
Finally, for \[ |x - \lambda^{1/\beta}| > \lambda^{\frac{1-\beta}{3\beta}} \] we have
\[ |y_1(x, \lambda)| \leq \frac{C \exp \left[ (-1)^{k+1} \hat{Q}_1(x, \lambda) \right]}{|\lambda - |x|^\beta|^{1/4} + \lambda^{\frac{\beta-1}{4\beta}}}. \] (7.1.8)

**7.2 Pseudodifferential operators**

We consider now the case of a pseudodifferential operator with symbol:
\[ T_{a,b} = |\xi|^a + |x|^b. \] (7.2.1)

Note that if \( a = b = 2 \) we have the harmonic oscillator and if \( a = 2, \ b = \beta \) we have the differential operator (6.1.1). In this section we will outline a technique to use the basis of Hermite functions to approximate the smallest eigenvalue of the operator (7.2.1). As an application we refer to the paper [18] where the value of the smallest eigenvalue of \( |\xi| + 2x^4 \) is stated as approximately equal to 0.978 . . . .

The Hermite functions are the eigenfunctions of the Fourier transform, the \( n^{th} \) Hermite function \( h_n(x) \) corresponds to the eigenvalue \((-i)^n\). Thus if \( F \) denote the Fourier transform, we have \( F^{-1}h_n = ih_n \). Now the identity:
\[ x h_0(x) = \sqrt{1/2} h_1(X) \]
\[ x h_n(x) = \sqrt{n/2} h_{n-1} + \sqrt{(n+1)/2} h_{n+1} \quad n \geq 1 \]
together with $F h_n = (-i)^n h_n$ can be used to construct a matrix for $|\xi|^2$ in the basis of Hermite functions. This matrix is positive and so we can numerically compute a matrix for $(|\xi|^2)^{b/2} = |\xi|^b$ for any $b > 0$. By a similar procedure we can derive a matrix for the operation of multiplication by $|x|^b$ in the basis of Hermite functions. We can then numerically approximate the eigenvalues of the matrix for $|\xi|^a + |x|^b$. 


