ECO-INSPIRED ROBUST CONTROL DESIGN FOR LINEAR DYNAMICAL SYSTEMS WITH APPLICATIONS

DISSERTATION

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ABSTRACT

Recently, the idea of using Ecological Sign Stability approach for designing robust controllers for engineering systems has attracted attention with promising results. In this work, continued research on this topic is presented. It is well known that, in the field of control systems, key to a good controller design is the choice of the appropriate nominal system. Since it is assumed that the perturbations are about this nominal, the extent of allowed perturbation to maintain the stability and/or performance very much depends on this ‘nominal’ system. Therefore, it is evident that this nominal system must have superior robustness properties. Incorporating certain robustness measures proposed in the literature, control design techniques have been realized in state space framework. However, the variety of controllers in state space framework is not as large as that of robust control design methods in frequency domain. Even these very few methods tend to be complex and demand some specific structure to the real parameter uncertainty (such as matching conditions). Overall, the success of all these methods for application to complex aerospace systems is still a subject of debate. Hence, there is still significant interest in designing robust controllers which can perform better than the existing controllers. Addressing these issues, current research proposes that the stability robustness measures for parameter perturbation are considerably improved if the ‘nominal’ system is taken (or driven) to be a ‘sign stable’ system. Motivated by this observation, a new method for designing a robust controller for linear uncertain state space systems is proposed. The novelty of this
research lies in the incorporation of ecological principles in order to design robust controllers for engineering systems. It is observed that an ecological perspective gives better understanding of the dynamics of the open and closed loop system (nominal) matrices. One of the attractive features of this controller is that the robustness measure, enters the control design in an explicit manner. The result of implementing controllers inspired by ecological principles simplifies the control algorithm and for certain dynamic systems, greatly reduces computational effort required in the synthesis of the controller. Accurate synthesis of the control algorithms results in ‘most robust’ nominal system (closed loop system). Variations of this control design method that address different categories of uncertainty are presented. The resulting control design methods are illustrated with application to aircraft and spacecraft flight control and aircraft turbine engine control.
Na Guror Adhikam Na Guror Adhikam
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VITA

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TABLE OF CONTENTS

Abstract ................................................................. ii
Dedication .................................................................... iii
Acknowledgments .......................................................... v
Vita ........................................................................ vii
List of Figures ............................................................ x

CHAPTER ........................................ PAGE
1 Introduction .............................................................. 1
2 Motivation and Problem Formulation .............................. 5
   2.1 Robust Control of Uncertain Systems ...................... 5
      2.1.1 Robust Control: State Space Methods ............... 8
      2.1.2 Robust Control: Frequency domain methods ....... 13
   2.2 Motivation ....................................................... 16
   2.3 Problem Formulation .......................................... 18
3 Ecology and Engineering ............................................ 20
   3.1 Introduction to Ecology ....................................... 20
   3.2 Predator-prey Models .......................................... 24
   3.3 Sign Stability ................................................... 26
   3.4 Engineering perspective of Ecology ....................... 30
      3.4.1 Characteristic equation ................................. 30
      3.4.2 Eigenvalue distribution ................................. 33
      3.4.3 Normality/Condition Number ......................... 45
4 Sign Stability and Robustness .................................... 50
   4.1 Qualitative Robustness ....................................... 50
   4.2 Directional Robustness ....................................... 54
   4.3 Quantitative Robustness ..................................... 59

viii
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Classification of uncertainties</td>
</tr>
<tr>
<td>2.2</td>
<td>Frequency domain robust control (Unstructured perturbation)</td>
</tr>
<tr>
<td>2.3</td>
<td>Frequency domain robust control (Structured perturbations)</td>
</tr>
<tr>
<td>2.4</td>
<td>Block diagram for $H_2$ and $H_\infty$ control design methods</td>
</tr>
<tr>
<td>2.5</td>
<td>Frequency domain robust control</td>
</tr>
<tr>
<td>3.1</td>
<td>Food Pyramid</td>
</tr>
<tr>
<td>3.2</td>
<td>Various interactions in an ecosystems</td>
</tr>
<tr>
<td>3.3</td>
<td>Ecological System</td>
</tr>
<tr>
<td>3.4</td>
<td>Ecological System of 5 species</td>
</tr>
<tr>
<td>3.5</td>
<td>Ecological System of 6 species</td>
</tr>
<tr>
<td>3.6</td>
<td>Interactions in a ecosystem</td>
</tr>
<tr>
<td>3.7</td>
<td>Interconnection in an ecosystem</td>
</tr>
<tr>
<td>3.8</td>
<td>Nonecological System</td>
</tr>
<tr>
<td>3.9</td>
<td>Ecological System</td>
</tr>
<tr>
<td>3.10</td>
<td>Classification of all sign patterns</td>
</tr>
<tr>
<td>3.11</td>
<td>Eigenvalue distribution of a SS matrix along real axis</td>
</tr>
<tr>
<td>3.12</td>
<td>Eigenvalue distribution of a SS matrix along imaginary axis</td>
</tr>
<tr>
<td>3.13</td>
<td>Eigenvalue distribution of a SS matrix</td>
</tr>
<tr>
<td>3.14</td>
<td>Eigenvalue distribution of a HS matrix</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>----------------------------------------------------------------------</td>
</tr>
<tr>
<td>3.15</td>
<td>Subcomponents of a digraph</td>
</tr>
<tr>
<td>4.1</td>
<td>Sufficient condition for stability/instability</td>
</tr>
<tr>
<td>4.2</td>
<td>Effect of $\beta_1$ on stability when $\beta_2 = \beta_3 = 0$</td>
</tr>
<tr>
<td>4.3</td>
<td>Effect of $\beta_2$ on stability when $\beta_1 = \beta_3 = 0$</td>
</tr>
<tr>
<td>4.4</td>
<td>Effect of $\beta_3$ on stability when $\beta_1 = \beta_2 = 0$</td>
</tr>
<tr>
<td>4.5</td>
<td>Classification of Target matrices</td>
</tr>
<tr>
<td>5.1</td>
<td>Flowchart for testing SAC (sign assignability conditions)</td>
</tr>
<tr>
<td>5.2</td>
<td>Flowchart of the control algorithm</td>
</tr>
<tr>
<td>5.3</td>
<td>Trajectories of satellite dynamics</td>
</tr>
<tr>
<td>5.4</td>
<td>Trajectories of aircraft lateral dynamics</td>
</tr>
<tr>
<td>5.5</td>
<td>Trajectories of satellite angular velocities</td>
</tr>
<tr>
<td>5.6</td>
<td>target PS matrix interval family</td>
</tr>
<tr>
<td>6.1</td>
<td>Representation of large system as a set of subsystems</td>
</tr>
<tr>
<td>6.2</td>
<td>Closed loop system matrix of a system of subsystems</td>
</tr>
<tr>
<td>6.3</td>
<td>Aircraft turbine engine expressed as a system of subsystems</td>
</tr>
<tr>
<td>6.4</td>
<td>Predator-prey interaction between compressor and turbine</td>
</tr>
</tbody>
</table>
Current nominal controllers designed for linear systems, especially with aerospace applications in mind, include traditional frequency domain based methods (such as PID controllers), time domain based methods (such as Eigenstructure Assignment and Linear Quadratic Regulator (LQR methods) and for nonlinear systems, include controllers based on feedback linearization (such as dynamic inversion) methods. Though all these controllers perform very well under the nominal operating conditions, in the real world situation where perturbations and uncertainties are unavoidable, most of these controllers either fail or perform poorly due to high sensitivity to modeling errors. Invariably, robustness, which is a measure of tolerance to perturbations, uncertainties and failures, becomes a crucial consideration in the design of controllers intended for application to complex aerospace systems we currently encounter. It is well known that, in the field of control systems, the key to a good controller design is the choice of the appropriate nominal system. Since it is assumed that the perturbations are about this nominal, the extent of allowed perturbation to maintain the stability and/or performance very much depends on this ‘nominal’ system. The more perturbation this nominal system tolerates, the more robust the system is deemed to be. In the present day control design for complex systems, robustness is not only a desirable feature but also a necessity. Thus it is clear that a nominal system should be such the bound on the uncertainty is maximized.
Realizing the importance of robustness, there has been considerable research in this area and robust control design methods such as $H_{\infty}$ control [1] and $\mu$-synthesis [2, 3] have emerged. However, these frequency domain based robust controllers tend to be very complex and of high order and more importantly do not explicitly address robustness to real parameter uncertainty. They become very conservative when applied to the problem of accommodating real parameter uncertainty [4]. On the other hand, there are very limited robust control design methods in time domain that explicitly address real parameter uncertainty [5, 6, 7, 8, 9, 10, 11]. Even these very few methods tend to be complex and demand some specific structure to the real parameter uncertainty (such as matching conditions). Overall, the success of all these methods for application to complex aerospace systems is still a subject of debate. Hence, there is still significant interest in designing robust controllers which can perform better than the existing controllers.

The fields of population biology and ecology deal with the analysis of growth and decline of populations in nature and the struggle of species to predominate over one another. Many mathematical population models were proposed over the last few decades with the most significant contributions coming from the work of Lotka and Volterra. The predator-prey model models of Lotka and Volterra, studied extensively by ecologists and population biologists consists of a set of nonlinear ordinary differential equations and stability of the equilibrium solutions of these models has been a subject of intense study for students of life sciences. For example many standard text books on mathematical models in biology such as [12] cover these issues.

The existence or extinction of a species, apart from its own effect, depends on its interactions with various other species in the ecosystem it belongs to. Hence the type of interaction is very critical to the sustenance of species. This research attempts to study these interactions and their nature thoroughly and investigate the
effect of these qualitative interactions on the quantitative properties of matrices. The study of these quantitative properties is further extended to robustness aspect as it is one of the important features of an engineering system. Examining various issues pertaining to the field of control systems such as stability, eigenvalue distribution and measures of robustness etc with an ecological perspective brought forth many fascinating properties of dynamic systems. Based on these properties, a systematic method to determine what may be termed as the ‘best nominal system’ is proposed. Towards this objective, we consider in particular, the interesting aspect of qualitative stability in ecological systems.

It is well recognized that natural systems such as ecological and biological systems are highly robust under various perturbations. On the other hand, engineered systems can be made highly optimal for good performance but they tend to be non-robust under perturbations. Thus, it is natural and essential for engineers to delve into the question of as to what are the underlying features of natural systems that make them so robust and then try to apply these principles to make the engineered systems more robust. The research reported here is an attempt to make a contribution in this aspect.

The novelty of this research is two-fold. Firstly, in addition to the conventional definition of robustness of uncertain system in engineering systems, we bring out a new perspective of this robustness aspect. Referring to robustness based on quantitative information as ‘Quantitative Robustness,’ we present a new framework labeled ‘Qualitative Robustness’ to underscore the importance of the qualitative nature of the robustness assessment. It is shown here that the robustness measure based on this new perspective provides deeper insights into the nature of the dynamics (such as interactions and interconnections) of the system thus facilitating effective control design. Secondly, motivation for this new perspective comes from the principles of
natural systems (such as eco/biosystems) which are known to be inherently robust to various disturbances and perturbations occurring in nature. Therefore, in this research, we borrow certain ideas from the natural ecosystem dynamics and implement them in the study of robustness of engineering systems. The key contribution of this research is a new robust control design method that address real parameter uncertainty, thus adding to the existing, relatively small bank of robust controllers in state space framework. This control design differs from currently available methods in that a nominal systems (closed loop system) is determined \textit{apriori} and then the controller that achieves this nominal system is computed. The novelty of this research lies in incorporating ecological principles in order to design robust controllers in state space framework. Finally, it is observed that incorporation of principles of ecosystems, which are inherently robust, results in a straightforward control design technique which reduces the computational effort involved in typical robust control design techniques. This robust control methodology thus promises to be a desirable alternative to the other robustness based controllers encompassing many fields of application.
2.1 Robust Control of Uncertain Systems

The mathematical model of a dynamical system is given by the equation

\[ \dot{x} = f(x) \]  

(2.1.1)

where \( x \) is a vector of the states of the system. In real world every such dynamical system is constantly subject to perturbations. These perturbations or uncertainties can be broadly classified as [13]

1. Unmodeled dynamics: Discrepancies that occur when certain states are not considered in the dynamic model such as in model reduction techniques are considered as uncertainties and classified as unmodeled dynamics.

2. Neglected nonlinearities: When dynamic systems are linearized using Taylor series expansion, nonlinear effects occurring in the form of higher order terms are neglected in order to obtain the linear model. This leads to discrepancy between the actual physical system and the mathematical model.

3. Real parameter variations: While modeling a system, certain physical parameters such as mass, etc are assumed to be of a certain value. However, these
parameters may differ from the actual values either due to inaccuracy in determination or change over a period of time. Such a variation in the real parameters can also be considered as an uncertainty in a dynamic system.

4. Neglected external disturbances: Every dynamic system, in the real world is constantly subject to disturbances that cannot always be modeled. In the absence of modeled disturbances, their effect is considered to be an uncertainty classified as external disturbance.

Therefore, a proper representation of a real dynamical system in the presence of uncertainties is

\[ \dot{x'} = f'(x') \]  \hspace{1cm} (2.1.2)

which clearly models all four classes of uncertainty. In this research, uncertainty in the form of real parameter variation, highlighted in 2.1 is addressed, in particular.

During 1960s and 1970s, there was little emphasis on robustness [13] of controllers as stability and performance (optimal control problems) played a more significant role. However, since the onset of application in industry, beginning from 1980s, there
has been significant interest in robustness of dynamical systems involving robustness analysis and control of systems.

The problem of maintaining the stability of a nominally stable linear time invariant system subject to linear perturbation has been an active topic of research for quite some time. The recent published literature on this ‘robust stability problem can be viewed mainly from two perspectives, namely i) transfer function (input/output) viewpoint and ii) state space viewpoint. In the transfer function approach, the analysis and synthesis is carried out in frequency domain, whereas in the state space approach it is carried out in time domain.

Even though in typical control problems, these two theories are intimately related and qualitatively similar, it is also important to keep in mind that there are noteworthy differences between these two approaches (polynomial vs matrix). In fact, some of these can be better captured in one framework than in another. For example, it can be argued convincingly that real parameter variations are better captured in time domain state space framework than in frequency domain transfer function framework. Similarly, it is intuitively clear that unmodeled dynamics such as exclusion of states can be better captured in the transfer function framework. By similar lines of thought, it can be safely agreed that while neglected nonlinearities can be better captured in state space framework, neglected disturbances can be captured with equal ease in both frameworks. Stability and performance are two fundamental characteristics of any feedback control system. Accordingly, stability robustness and performance robustness are two desirable (sometimes necessary) features of a robust control system. Since stability robustness is a prerequisite for performance robustness, it is natural to address the issue of stability robustness first and then the issue of performance robustness. Extensive research in the past decades, has led to numerous control design methods that can keep the system robust in the presence
of the above mentioned perturbations/uncertainties, both in the frequency as well as time domain. In what follows these approaches are discussed in a concise manner to make a case for present research.

2.1.1 Robust Control: State Space Methods

In the design of controller for linear systems, state space methods are used when we desire to work in time domain. In the state space framework, the mathematical model is a set of first order differential equations in the states of the dynamic system. This model represents the relation between the states, inputs and outputs in the following manner. The state equation is as follows:

\[
\dot{x} = Ax + Bu
\]  

(2.1.3)

The output equation is as follows:

\[
y = Cx + Du
\]

(2.1.4)

where \( x, u \) and \( y \) are vector representations of the state, input and output variables while \( A, B, C \) and \( D \) are matrices representing the linear relation between these vectors. There are several advantages of designing controllers in statespace framework. Firstly, visualization of the dynamical system is much better as it relates more to the physics of the system. Thus it is not surprising that most of the robustness studies of uncertain dynamical systems with real parameter variations are being carried out in time domain state space framework and hence in this chapter, we emphasize the aspect of robust stabilization and control of linear dynamical systems with real parameter uncertainty. Secondly, it is well suited for control of multiple input multiple output (MIMO) systems.
Uncertainty characterization

Real parameter variations are captured in the time domain or state space framework in the form of matrix variations. For robust stability analysis of linear systems in state space representation in the presence of uncertainty, the system is modeled as

\[ A_0 + E \]  \hspace{1cm} (2.1.5)

where \( A_0 \) is the nominal, Hurwitz stable matrix and \( E \) is the perturbation matrix. The two aspects of characterization of the perturbation matrix \( E \) which significantly influence the scope and methodology of any proposed analysis and design scheme are

i) Temporal nature:
   - Time invariant error vs. Time varying error
     \( (E=\text{constant}) \quad E=E(t) \)

ii) Boundedness nature:
   - Unstructured vs. Structured
     \( (\text{Norm bounded}) \quad (\text{Elemental bounds}) \)

When \( E \) is a function of time, the perturbation is categorized as time varying and when the perturbation matrix is a constant, it is termed as time invariant perturbation. In the category of so called unstructured or ‘norm bounded’ perturbation, it is assumed that one cannot clearly identify the location of the perturbation within the nominal matrix and thus one has simply a bound on the norm of the perturbation matrix. When information regarding location of perturbation is known, the uncertainty is categorized as structured perturbation. This approach is labeled as Elemental Perturbation Bound Analysis. Structured perturbation in state space framework is expressed as follows.

\[ A = A_0 + E(q) \]
\[ q_i < q_i < \overline{q_i} \]

\[ A = A_0 + \sum_{i=1}^{r} q_i E_i \]

(2.1.6)

where \( A_0 \) and \( E \) are as defined above and \( q_i \) is a real parameter varying with a range defined by lower \((q_i)\) and upper \((\overline{q_i})\)

**Robust Stability Analysis**

Typically, stability of linear time varying systems is assessed using Lyapunov stability theory using the concept of quadratic stability whereas that of a linear time invariant system is determined by the Hurwitz stability. This distinction in the nature of perturbation profoundly affects the methodologies used for stability robustness analysis. In this research, we address the case of time varying perturbation since time invariant perturbation becomes a special case of time varying perturbation. Here, robustness is proposed as a bound on the norm of perturbation matrix \( ||E|| < r \). Robust Control involves robust stability analysis followed by robust control design. During 1980s, there was significant interest in proposing measures of robustness and bounds on permissible perturbations. Research reported in [14] introduced new bounds for unstructured perturbation. Further research involved newer and improved robustness bounds. Recent advances in this field are surveyed in [15] and [16].

The stability robustness problem for linear time invariant systems in the presence of linear time invariant perturbations (i.e. robust Hurwitz invariance problem) is basically addressed by testing for the negativity of the real parts of the eigenvalues (either in frequency domain or in time domain treatments), whereas the time varying perturbation case is known to be best handled by the time domain Lyapunov stability analysis. The robust Hurwitz invariance problem has been widely discussed in the literature essentially using the polynomial approach [17],[18]. In this research,
we address the time varying perturbation case, mainly motivated by the fact that any methodology which treats the time varying case can always be specialized to the time invariant case but not vice versa. However, we pay a price for the same, namely conservatism associated with the results when applied to the time invariant perturbation case. A methodology specifically tailored to time invariant perturbations is discussed in a separate publication [19]. From the definitions of unstructured and structured uncertainty, it can be infered that unstructured uncertainty analysis assumes no prior information regarding the location of perturbation while structured uncertainty takes location of perturbation intp consideration. Whether unstructured norm bounded perturbation or structured elemental perturbation is appropriate to consider depends very much on the application at hand.

**Robust Control in State space framework**

There are not many control design techniques reported in literature. Robust control can be broadly classified into control design techniques that address stability robustness and control design techniques that address performance robustness. These methods are based on Lyapunov, Kronecker based matrix methods which address structured, real parameter variation. Mixed $H_2/H_\infty$ theory addresses combined uncertainty in state space framework. There is a considerable amount of literature on the aspect of designing linear controllers for linear time invariant systems with small parameter uncertainty. However, for uncertain systems whose dynamics are described by interval matrices (i.e., matrices whose elements are known to vary within a given bounded interval), linear control design schemes that guarantee stability have been relatively scarce. In comparison to frequency domain methods, there are not many robust control design techniques in state space framework.

In these methods, the parameter uncertainty is typically assumed to enter linearly
and restrictive conditions are imposed on the bounding sets. In [20], norm inequalities on the bounding sets are given for stability but they are conservative since they do not take advantage of the system structure. Other issues associated with robust control design in state space framework is that there is no guarantee that a linear state feedback controller exists. Reference [21] utilizes the concept of Matching conditions (MC) which in essence constrain the manner in which the uncertainty is permitted to enter into the dynamics and show that a linear state feedback control that guarantees stability exists provided the uncertainty satisfies matching conditions. By this method large bounding sets produce large feedback gains but the existence of a linear controller is guaranteed. But no such guarantee can be given for general mismatched uncertain systems. References [22] and [24] present methods which need the testing of definiteness of a Lyapunov matrix obtained as a function of the uncertain parameters. In the multimodel theory approach [25] considers a discrete set of points in the parameter uncertainty range to establish the stability. Research presented here addresses the stabilization problem for a continuous range of parameters in the uncertain parameter set (i.e. in the context of interval matrices). The proposed approach attacks the stability of interval matrix problem directly in the matrix domain rather than converting the interval matrix to interval polynomials and then testing the Kharitonov polynomials [26]. Towards this direction bounds on the individual elements of the perturbation matrix are developed in [27],[28] which are shown to be less conservative than the existing measures since the method utilizes the structural information of the uncertainty. In Yedavalli [27], a control design method is presented in which the linear state feedback gain is obtained by the standard Riccati equation and then the robustness of the gain is investigated by the resulting elemental perturbation bound. In the following, we design the gain such that its determination directly uses the uncertainty structure and this is done by parameter optimization.
In other words, the controllers gains are determined such that they maximize (in a certain sense) the elemental perturbation bound for a given uncertainty structure.

2.1.2 Robust Control: Frequency domain methods

Uncertainty categorization

In frequency domain, where system dynamics are represented in the form of a transfer function, unstructured dynamics are categorized as multiplicative and additive uncertainty as shown in Figure 2.2.

\[ G_0(s) + \Delta G(s) \]

Structured uncertainty, classified into complex and real uncertainty, is represented in the following way. Figure 2.3 depicts complex structured uncertainty. Real structured uncertainty is expressed as follows:

\[ P(s, q) = \frac{N(s, q)}{D(s, q)} \quad (2.1.7) \]

where \( q \) is the vector of uncertain parameters. In the case of real parameter variations, the varying parameters appear as polynomial variations i.e., variation in the coefficients of the polynomial. One of the methods of robust stability analysis is
the Kharitonov theorem and its extensions. According to this theorem, given a stable polynomial with independently variation coefficients, stability of the entire set of polynomials formed by these variable coefficients can be determined based on the stability of 4 particular polynomials. This theorem gives the necessary and sufficient conditions for stability of a family of polynomials with independently varying coefficients. This theory was later extended to the study of uncertain polynomials with dependent perturbations. However, in the case of dependent perturbations, Kharitonov theorem is reduced to a sufficient condition for stability.

Robust Control Design

There are several control design techniques in frequency domain. $\mu$-synthesis that addresses structured and unstructured uncertainty, Quantitative feedback control that addresses structured uncertainty, Kharitonov based polynomial methods that address structured and real parameter uncertainties and finally, the more popular $H_2$ and $H_\infty$ theory that address unstructured uncertainty are the basic approaches. As an example, a concise description of $H_2$ and $H_\infty$ is given below. In the basic case of a single-input single-output (SISO), the system under consideration is represented in the following manner. Firstly, a sensitivity function is defined thus.
\[ \epsilon = \frac{e}{d - r} = \frac{1}{1 + pc} \quad (2.1.8) \]

where \( p \) is the plant transfer function and \( k \) is the controller transfer function.

1. \( H_2 \) Control Design:

The \( H_2 \) controller minimizes the internal system energy for a particular input or equivalently it minimizes the 2-norm (“average”) of \( \epsilon \) weighted by the input as given by the following expression

\[ \min_c \int_0^\infty e^2 dt = \min_c ||w||_2^2 \quad (2.1.9) \]

2. \( H_\infty \) Control Design:

This controller minimizes the \( \infty \)-norm or peak value of the sensitivity function \( \epsilon \) weighted by \( w \). This is expressed as follows.

\[ \min_c ||w||_\infty = \min_c \sup_\omega |\epsilon w| \quad (2.1.10) \]

Therefore, the essence of these control design techniques is to propose bounds on the norm of a transfer function \( G(s) \) such that the effective dynamical system in frequency domain has poles with negative real parts i.e., \( Re(s) < 0 \) \( \forall \) \( s \). The above
control design techniques, initially proposed for single-input single-output (SISO) systems can be extended to multiple-input multiple-output (MIMO) systems using state space representation. In MIMO systems, the block diagram of the dynamic system is modeled as shown in Figure 2.5. Reference [29] proposes state space solutions to standard $H_2$, $H_\infty$ control problems. In order to obtain the gain matrix, one needs to solve a set of riccati equations. An $H_\infty$ gain matrix exists only if these riccati equations satisfy certain conditions. Finally, the gain matrix itself is computed using the solution of a riccati equation which is a function of the robustness bound $\gamma$. The process of determination of this robustness bound is an iterative one which is another reason why the $H_\infty$ controller is complex. While attaining robustness in SISO systems is relatively simple, one can easily infer from this discussion that it is not so in the case of MIMO systems.

2.2 Motivation

Disadvantages of $H_\infty$ methods include the level of mathematical understanding needed to apply them successfully and the need for a reasonably good model of the system to be controlled. Problem formulation is important, since any controller synthesized will
only be ‘optimal’ in the formulated sense: optimizing the wrong thing often makes things worse rather than better. In frequency domain, it is difficult to comprehend the mathematical model as a physical system especially in the case of higher order systems. Additionally, these methods are computationally intensive as several Riccati equations need to be solved in order to arrive at the solution. These controllers also tend to be of high order. Another disadvantage of these methods is that they are restricted to zero initial conditions. Most importantly, considering the specific case of real parameter variation, in frequency domain, real parameter variation is not addressed explicitly and these methods become conservative in such cases. There is a considerable amount of literature on the aspect of designing linear controllers for linear time invariant systems with small parameter uncertainty. However, for uncertain systems whose dynamics are described by interval matrices (i.e., matrices whose elements are known to vary within a given bounded interval), linear control design schemes that guarantee stability have been relatively scarce. Vinkler and Woods [30] compare several techniques for designing linear controllers for robust stability for a class of uncertain linear systems. Among the methods considered are the standard linear quadratic regulator (LQR) design, Guaranteed Cost Control (GCC) method of Chang and Peng [31], Multistep Guaranteed Cost Control (MGCC) of [?]. In these methods, the weighting on state in a quadratic cost function and the Riccati equation are modified in the search for an appropriate controller. Also, the parameter uncertainty is assumed to enter linearly and restrictive conditions are imposed on the bounding sets. Similar to robust controllers in frequency domain, these methods are quite complex as well.

Considering the drawbacks of these robust control techniques, in this research, a simple and straightforward control methodology that attempts to address these shortcomings is proposed. Most importantly, this control methodology is compatible with
real parameter uncertainty since bounds on the real parameter variation appear explicitly in the control design problem formulation. Thus, this controller adds to the bank of existing robust control design techniques in state space framework.

### 2.3 Problem Formulation

The problem formulation for this robust control design methodology that addresses real parameter variation is as follows: Consider the following linear time-invariant dynamical system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  

(2.3.1)

where \( A \) and \( B \) are constant matrices, \( x \in \mathbb{R}^n \) is the \( n \)-dimensional state vector and \( u \in \mathbb{R}^m \) is the \( m \)-dimensional control vector. In the presence of uncertainty, equation (2.3.1) is expressed as

\[ \dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t) \]  

(2.3.2)

The objective is to design a robust full-state feedback control \( u = Gx \) for the nominal system given in equation (2.3.1) such that it is robust in the presence of uncertainty \( \Delta A \) and \( \Delta B \). Therefore, the nominal closed loop system matrix is given by

\[ A + BG = A_{cl} \]  

(2.3.3)

Now, the entire uncertain system with the feedback controller can be written as

\[ A + BG + \Delta A + (\Delta B)G = A_{cl} + \Delta A + (\Delta B)G \]  

(2.3.4)

Therefore, structure of the uncertainty matrix is given as

\[ E = \Delta A + (\Delta B)G \]  

(2.3.5)

The objective of robust control is to design a \( G \) in order to maximize \( E \). Maximization of \( E \) depends on its characterization. In later chapters, we discuss characterization...
of perturbation and the corresponding robustness measures.

The philosophy of controller design proposed in this research differs from the currently available methods in that a closed loop system matrix $A_{cl}$ is determined *apriori* and then the corresponding controller gain $G$, which achieves that desired closed loop system matrix is computed. On the other hand, in current methods, the controller gain $G$ is determined first and then the resulting closed loop system matrix is accepted for further analysis. This is because, as will be emphasized in rest of the chapters, it is the structure and various properties of the closed loop system matrix resulting from the structure that make it a desirable candidate for control design. The novelty of present work lies in addressing the problem of robust stability analysis and control design from a new and refreshing perspective of ecological sciences. In the next chapter, various features and properties of such matrices are arrived at from an ecological perspective. Towards this objective, in the next chapter, a brief introduction to ecology is presented followed by establishing relation between engineering and ecology.
CHAPTER 3
ECOLOGY AND ENGINEERING

3.1 Introduction to Ecology

Ecology is the scientific study of the relations that living organisms have with respect to each other and their natural environment. An ecosystem is a biological environment consisting of all the organisms living in a particular area, as well as all the nonliving, physical components of the environment with which the organisms interact, such as air, soil, water and sunlight. Ecological studies deal with the analysis of growth and decline of populations in nature and the struggle of species to predominate over one another. The existence or extinction of a species, apart from its own effect, depends on its interactions with various other species in the ecosystem it belongs to.
Its location in the food pyramid determines various characteristics of a species such as population density and breeding rate. Figure 3.1 represents the hierarchy of producers and consumers. Hence interactions are very critical to the sustenance of species. In a complex community composed of many species, numerous interactions take place. These interactions in ecosystems can be broadly classified as i) Mutualism, ii) Competition, iii) Commensalism/Ammensalism and iv) Predation (Parasitism). Mutualism occurs when both species benefit from the interaction. When one species benefits/suffers and the other one remains unaffected, the interaction is classified as Commensalism/Ammensalism. When species compete with each other, that interaction is known as Competition. Finally, if one species is benefited and the other suffers, the interaction is known as Predation (Parasitism). These interactions are represented in matrix and digraph notation as shown in Figure 3.2. Many mathematical population
models were proposed over the last few decades to study the dynamics of eco/bio systems, which are discussed in [12],[33]-[43]. The most significant contributions in this area come from the works of Lotka and Volterra.

In ecological models, the magnitudes of the mutual effects of species on each other are seldom precisely known, but one can establish with certainty, the types of interactions that are present. This means that technically, when the ecological model is linearized, in the Jacobian matrix, one does not know the actual magnitudes of the partial derivatives but their signs are known with certainty. The linearized model has no ‘quantitative’ information. Instead, the model is a ‘qualitative’ model. Thus the qualitative information about the species is represented by the signs +, - or 0. Thus, the \((i,j)^{th}\) entry of the state space (Jacobian) matrix simply consists of signs +, -, or 0, with the + sign indicating species \(j\) having a positive influence on species \(i\), - sign indicating negative influence and 0 indicating no influence. The diagonal elements give information regarding the effect of a species on itself. Negative sign means the species is self-regulatory, positive means it aids the growth of its own population and zero means that it has no effect on itself. Alternately, the qualitative Jacobian matrix can also be represented as a ‘digraph’ in terms of ‘nodes’ and ‘paths’. For higher order systems, consider the generalized representation of an n-species Lotka-Volterra system [100]

\[
\frac{dR_i}{dt} = R_i \left[ b_i + \sum_{j=1}^{n} a_{ij}R_j \right], \ i = 1, 2, \cdots n
\]  

(3.1.1)

where \(R_i\) represent the various species. Now, for example, consider the \(3 \times 3\) matrix \(A\) in Figure 3.3 corresponding to a 3-species ecosystem. The numbered circles are the nodes (species) while the lines connecting the nodes are the paths (the off-diagonal elements representing the effect of one species on another). The sign (+, − and 0) of a path indicates the nature of the effect and arrow represents the direction.
Similarly, Figures 3.4 and 3.5 depict linearized models and corresponding digraphs of ecosystems of 5 and 6 species.

Furthermore, as shown in Figure 3.1, products of paths (in matrix notation) with ordered indices are classified as follows [44]:

i. Product of off-diagonal elements of the form $a_{ij}a_{ji}$ connecting only two distinct nodes (indices) are known as ‘l-cycles’. Since the product forms a cycle of indices, products $a_{12}a_{21}$ and $a_{21}a_{12}$ are considered to be the same.

ii. Product of off-diagonal elements of the form $a_{ij}a_{jk}...a_{mi}$ connecting three or
more nodes (indices) are known as ‘\(k\)-cycles’. As is the case with \(l\)-cycles, the
\(k\)-cycles \(a_{12}a_{23}a_{31}\) and \(a_{23}a_{31}a_{13}\) are considered to be the same.

In this research, we refer to the \(l\)-cycles as *interactions* and to the \(k\)-cycles as *interconnections* between species as depicted in Figures 3.6 and 3.7. The distinction between interactions and interconnections is that the former pertains to the relation between two species while the latter pertains to that between a group (3 or more) of species.

### 3.2 Predator-prey Models

There are numerous models based on these interactions that have been studied with both ecological and engineering perspectives such as in [45], [46], [42], [39] and [43]. An extensively studied class of models is the class of predator-prey models [33], [35]. The most significant contributions in this area come from the works of Lotka and
Volterra. Following is the fundamental representation of a predator-prey interaction where $x$ is the prey density and $y$ is the predator density.

\begin{equation}
\dot{x} = xf(x, y) \\
\dot{y} = yg(x, y)
\end{equation}

where it is assumed that $\partial f(x, y)/\partial y < 0$ and $\partial f(x, y)/\partial x > 0$. This means that the effect of $y$ on the rate of change of $x(\dot{x})$ is negative while the effect of $x$ on the rate of change of $y(\dot{y})$ is positive. For example,

\begin{align*}
\dot{R}_1 & = R_1(b - pR_2) \\
\dot{R}_2 & = R_2(rR_1 - d)
\end{align*}

is a two species predator-prey model where $R_1$ is prey and $R_2$ is predator. Predator-prey interactions give rise to oscillations. Extensive research on predator prey modeling [47], [48], [49], [50] and [54] of ecological systems shows that these kind of
interactions have significant effect on the dynamics and sustenance of an ecosystem. For example, it is easily proved for a $2 \times 2$ system with non-positive diagonal elements, that oscillations occur iff there is a predator-prey interaction. It is of interest to note that there is considerable research being carried out on non predator-prey type models such as compartmental models as in [42], [43], [51] and [52], which attest to the growing interest in the interrelationship between life sciences research and engineering sciences research. It is to be noted that in these models, more emphasis is on the signs of elements of the Jacobian and therefore the qualitative information is known for certain. However, there is ambiguity in the quantitative information of such models.

### 3.3 Sign Stability

Research in [53]-[65] provides detail analysis of such qualitative linear systems to determine existence of solutions (sign solvability). Since traditional mathematical tests [67], [68], [69] and [70] for stability fail to analyze the stability of such ecological models, an extremely important question arises as to whether it be concluded, just from this sign pattern, whether the system is stable or not. The motivation for this study comes from the field of economics. The paper by Economists Quirk and Ruppert[71] was later followed by further research and application to ecology by [72] [73] and [74]. For communities of five or more species, the order of these matrices is high enough to cause difficulties in assessing the stability. For this reason, to circumvent these difficulties, alternative concepts of reduced computation have been proposed and one of these concepts is that of Qualitative Stability’.

Qualitative stability, which is independent of magnitudes, implies Hurwitz stability in the ordinary sense of engineering sciences. In other words, once a particular sign matrix is shown to be qualitative (sign) stable, the non-zero elements of this matrix
can assume *any* value and for all those values the matrix is automatically Hurwitz stable. Therefore, 

*a matrix \( A \) is called ‘Qualitative stable’ if each matrix \( B \) of the same sign pattern as \( A \) (\( sgn(b_{ij}) = sgn(a_{ij}) \forall i, j \)) is Hurwitz stable regardless of the magnitudes of \( b_{ij} \).*

The qualitative stability concepts of ecology published in [71], [72], and [73] present ‘necessary and sufficient conditions in ecological terms involving the ‘color test. Criteria for ecological sign stability including the color test in matrix theory notation were first published in [75], [76] as a computer amenable algorithm. These conditions in matrix theory notation are given below

1. \( a_{ii} \leq 0 \forall i \)
2. \( a_{ii} < 0 \) for at least one \( i \)
3. \( a_{ij}a_{ji} \leq 0 \forall i, j, i \neq j \)
4. \( a_{ij}a_{jk}a_{kl}...a_{mi} = 0 \) for any sequence of three or more distinct indices \( i, j, k...m \)
5. \( \text{Det} \ (A)=0 \)
6. Matrix must *fail* the color test (elaborated on in [75], [76])

In some literature, this concept is also labeled as ‘sign stability’. In what follows, these two terms are used interchangeably. It is important to keep in mind that the systems (matrices) that are qualitatively (sign stable) stable are also stable in the ordinary sense. That is, qualitative stability implies Hurwitz stability (eigenvalues with negative real part) in the ordinary sense of engineering sciences. In other words, once a particular sign matrix is shown to be qualitatively (sign) stable, any magnitude can be inserted in those entries and for all those magnitudes the matrix is automatically Hurwitz stable. This is the most attractive feature of a sign stable matrix: Stability independent of magnitude. However, the converse is not true. Systems that are not
qualitatively stable can still be stable in the ordinary sense for certain appropriate magnitudes in the entries. From now on, to distinguish from the concept of qualitative stability of life sciences literature, the label of quantitative stability’ for the standard Hurwitz stability in engineering sciences is used. The book [77] briefly discusses sign stability in the context of matrix diagonal stability in systems and computation and provides a few other references within the book. However, such references touch upon the sufficient conditions for sign stability and do not allude to the ‘color test’ conditions, which are part of the ‘necessary and sufficient’ conditions provided in the ecology literature. In [71], [72], and [73], necessary and sufficient conditions for qualitative stability of an ecosystem were given. These ecological sign stability conditions, stated in terms of ecology, were interpreted in matrix theory notation and these conditions were transformed into an algorithm to test the sign stability of a given sign matrix in [75] [76]. With this algorithm, all matrices that are sign stable can be stored apriori as discussed in [78]. If a sign pattern in a given matrix satisfies the conditions given in the above papers (thus in the algorithm), it is an ecological stable sign pattern and hence that matrix is Hurwitz stable for any magnitudes in its entries. A subtle distinction between ‘sign stable’ matrices and ‘ecologically sign stable’ matrices is now made, emphasizing the role of nature of interactions. Though the property of Hurwitz stability is held in both cases, ecosystems sustain solely because of interactions between various species. In matrix notation this means that the nature of off-diagonal elements is essential for an ecosystem. Consider a strictly upper triangular $3 \times 3$ matrix $B$ as shown in Figure 3.8. From quantitative viewpoint, it is seen that the matrix is Hurwitz stable for any magnitudes in the entries of the matrix. This means that it is indeed (qualitatively) sign stable. But when we analyze the system from a digraph or ecological perspective, we see that species 3 is not
connected to species 1 and 2. This means that the 3 species ‘together’ do not form an ecological system or a predation community built on the basis of interactions.

On the other hand consider matrix $A$ in Figure 3.9. From its digraph representation we see that all the three species are connected and hence this system is indeed an ecological system. Therefore, sign stable matrices representing ecosystems are labeled ‘ecologically sign stable matrices’. In matrix theory notation, this implies that all ‘ecologically sign stable matrices’ are essentially *irreducible* matrices. Figure 3.10 shows the various sets of matrices. Note that ecologically sign stable matrices form a subset of the entire set of sign stable matrices. Having discussed the fundamentals of ecosystems, we proceed to analyze ecosystems from an engineering perspective.
3.4 Engineering perspective of Ecology

In this section, certain properties of ecological systems that are of relevance in control engineering are brought to light.

3.4.1 Characteristic equation

The first feature of ecological systems is the expression for determinant which also gives the expression for characteristic equation. It is known that every term in the expression for determinant is a combination of the l-cycles, k-cycles and/or diagonal elements [44]. This means that it is a combination of diagonal elements, interactions and interconnections. From this definition an interesting feature of the expression of a determinant comes to light. According to this definition, elements with ‘stray’ indices do not appear in the expression for determinant i.e., every element in every term is part of a cycle (k-cycle or l-cycle). Expression for determinant of a matrix in terms of matrix elements for a $4 \times 4$ matrix is as follows:

$$
\Delta = \prod_{p=1}^{4} a_{pp} - \sum_{i=1,j=1,p=1}^{4} \left\{ \prod_{p=1} a_{pp} \right\} + \sum_{i=1,j=1,k=1,p=1}^{4} \left\{ (a_{ij}a_{jk}a_{ki}) \prod_{p=1} a_{pp} \right\}
$$

Figure 3.10: Classification of all sign patterns
\[
+ \sum_{i=1, j=1, k=1, l=1}^{4} \left\{ (a_{ij}a_{ji})(a_{kl}a_{lk}) \right\} + \sum_{i=1, j=1, k=1, l=1}^{4} (a_{ij}a_{jk}a_{kl}a_{li}) \quad (3.4.1)
\]

Then, by its definition, the characteristic equation becomes

\[
\Delta(\lambda) = + \prod_{p=1}^{4} (\lambda - a_{pp}) - \sum_{i=1, j=1, p=1}^{4} \left\{ (a_{ij}a_{ji}) \prod (\lambda - a_{pp}) \right\} + \sum_{i=1, j=1, k=1, l=1}^{4} \left\{ (a_{ij}a_{jk}a_{kl}a_{li}) \prod (\lambda - a_{pp}) \right\} (3.4.2)
\]

Similarly, for a 5 \times 5 matrix, expression for a determinant is

\[
\Delta = + \prod_{p=1}^{5} a_{pp} - \sum_{i=1, j=1, p=1}^{5} \left\{ (a_{ij}a_{ji}) \prod a_{pp} \right\} + \sum_{i=1, j=1, k=1, l=1, p=1}^{5} \left\{ (a_{ij}a_{jk}a_{kl}a_{li}) \prod a_{pp} \right\} - \sum_{i=1, j=1, k=1, l=1, p=1}^{5} \left\{ (a_{ij}a_{ji}a_{kl}a_{li}) \prod a_{pp} \right\} + \sum_{i=1, j=1, k=1, l=1, m=1}^{5} \left\{ (a_{ij}a_{jk}a_{ki})(a_{lm}a_{ml}) \right\} (3.4.3)
\]

The characteristic equation of a 5 \times 5 matrix becomes

\[
\Delta(\lambda) = + \prod_{p=1}^{5} (\lambda - a_{pp}) - \sum_{i=1, j=1, p=1}^{5} \left\{ (a_{ij}a_{ji}) \prod (\lambda - a_{pp}) \right\} + \sum_{i=1, j=1, k=1, l=1, p=1}^{5} \left\{ (a_{ij}a_{jk}a_{ki})(a_{ml}a_{ml}) \right\} - \sum_{i=1, j=1, k=1, l=1, p=1}^{5} \left\{ (a_{ij}a_{ji}a_{kl}a_{lk}) \prod (\lambda - a_{pp}) \right\} - \sum_{i=1, j=1, k=1, l=1, p=1}^{5} \left\{ (a_{ij}a_{jk}a_{ki}a_{li}) \prod (\lambda - a_{pp}) \right\}
\]
Characteristic equation of sign stable matrices

For the property of sign stability to hold,

1. Finally, species must be either self-regulatory or must not affect their growth rate. This is means that all diagonal elements have to be non-positive with at least one negative diagonal element ( \( a_{ii} \leq 0 \forall i \) and \( a_{ii} < 0 \) for at least one \( i \))

2. Another necessary condition for sign stability is that all interactions must be either predator-prey or ammensal/commensal in nature. This means that all the interactions, i.e., all the \( l \)-cycles have to be either negative or zero (\( a_{ij}a_{ji} \leq 0 \) \( i \neq j \)).

3. There cannot be any omnivorous species in the ecosystems. This implies that all interconnections i.e., all \( k \)-cycles must be zero (\( a_{ij}a_{jk}...a_{mi} \), \( i \neq j \neq k... \neq m \) for number of indices \( \geq 3 \)).

Therefore, characteristic equations of sign stable matrices, which now have only diagonal elements and \( l \)-cycles, gets simplified to the following form.

For a 4\( ^{th} \) order system,
\[
\Delta(\lambda) = + \prod_{p=1}^{4} (\lambda - a_{pp}) - \sum_{i=1,j=1,p=1}^{4} \{ (a_{ij}a_{ji}) \prod_{p=1}^{4} (\lambda - a_{pp}) \} \\
+ \sum_{i=1,j=1,k=1,l=1}^{4} \{ (a_{ij}a_{ji})(a_{kl}a_{lk}) \}
\]  \hspace{1cm} (3.4.5)
Similarly, the characteristic equation of a $5 \times 5$ matrix simplifies to
\[
\Delta(\lambda) = + \prod_{p=1}^{5} (\lambda - a_{pp}) - \sum_{i=1,j=1,p=1}^{5} \left\{ (a_{ij}a_{ji}) \prod_{p=1}^{5} (\lambda - a_{pp}) \right\} \\
+ \sum_{i=1,j=1,k,l=1,p=1}^{5} \left\{ (a_{ij}a_{ji})(a_{kl}a_{lk}) \prod_{p=1}^{5} (\lambda - a_{pp}) \right\} 
\] (3.4.6)

From these expressions, since $a_{ii}$ and $a_{ij}a_{ji}$ are always less than or equal to zero, it can be clearly seen as to how the self-regulatory nature of species and the predator-prey interactions result in sign invariance (positive sign) of the coefficients of the characteristic equation, which is a necessary condition for Hurwitz stability.

### 3.4.2 Eigenvalue distribution

Eigenvalues of a general matrix are complex functions of its elements. Therefore, determination of eigenvalues is a computation intensive process. Instead of the exact eigenvalues, in certain cases, distribution of eigenvalues along the real and/or imaginary axes, known as the field of values, can be helpful. However, the determination of these bounds on the real and imaginary parts still involves computation of eigenvalues of the symmetric part of the parent matrix. But, unlike general Hurwitz stable matrices, in the case of sign stable matrices, the eigenvalue distribution along the real and imaginary axes can be determined in a relatively easy manner. The following theorem and its corollary clearly bring forth this feature for the real part of the eigenvalues.

**Bounds on real part of eigenvalues**

**Theorem 3.4.1.** For all $n \times n$ ecologically sign stable matrices $[a_{ij}]$, non-positive diagonal elements,

\[
a_{ii\min} \leq Re(\lambda)_{\min} \leq Re(\lambda)_{\max} \leq a_{ii\max} \quad (\forall a_{ii} < 0)
\] (3.4.7)
\begin{align*}
a_{ii_{\text{min}}} \leq \text{Re}(\lambda)_{\text{min}} \leq \text{Re}(\lambda)_{\text{max}} < a_{ii_{\text{max}}} (= 0) \quad (\text{for some } a_{ii} = 0) \tag{3.4.8}
\end{align*}

**Proof.** The characteristic equation for an \( n \times n \), real, Hurwitz stable matrix \( A \) is given by

\[ \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n = 0 \]

where \( a_1 = \text{Trace}(A) \) and the other coefficients \((a_2, a_3, \ldots)\) satisfy the positive Hurwitz determinant conditions (Routh Hurwitz Conditions).

Note that in a sign stable matrix, \( a_1 = -\text{trace}(A) = \sum_{i=1}^{n} a_{ii} = -\sum_{i=1}^{n} |\text{Re}(\lambda_i)| \) where \( a_{ii} \leq 0 \) and equal to 0 for at least one \( a_{ii} \)

In a sign stable matrix, where Hurwitz stability is satisfied independent of the magnitudes of the elements, it is clear that the real parts of the eigenvalues are always negative for any magnitudes in the entries of the matrix. The absolute values of the real parts of the eigenvalues are solely dependent on the magnitudes of the diagonal elements and the imaginary parts of these eigenvalues are decided by the off-diagonal elements of the sign stable matrix.

Hence equations 3.4.7 and 3.4.8.

Clearly, theorem (3.4.1) significantly reduces the computation involved in determining the distribution of eigenvalues. In fact, the distribution can be determined by mere observation.

It is also seen that theorem 3.4.1 also gives the lower bound on stability degree \(-\alpha_s(A)\), where \( \alpha_s \) is defined as the real part of dominant eigenvalue or rightside bound on real part of eigenvalues. Additionally, this bound is superior to the bound derived using field of values. Based on this observation, the following theorem is stated

**Theorem 3.4.2.** Consider an \( n \times n \) ecologically sign stable matrix \( A \) whose

\begin{itemize}
  \item[a.] diagonal elements are all negative i.e., \( a_{ii} < 0 \ \forall \ i \)
\end{itemize}
b. symmetric part is either negative semi-definite or indefinite i.e., \( \text{Real}(\lambda_i(A_{sym})) \leq 0 \) or \( \text{Real}(\lambda_i(A_{sym})) <, r > 0 \)

Then, lower bound on stability degree \(-\alpha_s\) obtained from theorem 3.4.1 is always higher than bound obtained from field of values.

Proof. The theorem is proved separately for the negative semi-definite and indefinite cases.

i. \( A_{sym} \) is negative semi-definite (NSD): For matrix A, lower bound on \(-\alpha_s\) from theorem 3.4.1 can assume only negative values. Therefore, lower bound on stability degree is always positive. But if \( A_{sym} \) is NSD, at least one of its eigenvalues is 0 and hence rightside bound derived from field of values is 0. Therefore, bound obtained using theorem 3.4.1 is always higher.

ii. \( A_{sym} \) is indefinite: For matrix A, lower bound on \(-\alpha_s\) from theorem 3.4.1 can assume only negative values. Therefore, lower bound on stability degree is always positive. But if \( A_{sym} \) is indefinite, it implies that its eigenvalues spread into both left and right half of the complex plane. Due to this distribution, lower bound on \(-\alpha_s\) cannot be determined (because \(-\alpha_s\) has to be positive). Therefore, it can be said that lower bound on stability degree determined using theorem 3.4.1 is higher.

Though the phrase ‘sign stability’ refers to the nature of eigenvalues along the real axis, it is observed that sign stability affects the imaginary parts as well. Therefore, similar to the bound on real parts, we propose simplified bounds on the imaginary parts of eigenvalues of sign stable matrices.
Bounds on imaginary part of eigenvalues

Before stating the theorem on bounds on imaginary parts, we present the following lemma.

**Lemma 3.4.3.** Consider the following characteristic equation of an $n \times n$ Hurwitz stable matrix $[a_{ij}]$, $\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n = 0$ A bound on the imaginary part of the roots of the above characteristic equation is given by

$$|\text{Imag}(\lambda_i)|_{\text{max}} \leq \sqrt{a_2} \quad (3.4.9)$$

where $a_2 = \sum_{i,j=1, i<j}^n -a_{ij}a_{ji} + \sum_{i,j=1, i<j}^n a_{ii}a_{jj} \quad \forall \ i, j$

**Proof.** Since getting an upper bound on the imaginary parts is our objective, we start by considering a characteristic polynomial with all with complex conjugate pair roots. Now, consider a second degree characteristic polynomial,

$$\lambda^2 + 2\zeta \omega_n \lambda + \omega_n^2$$

It is of interest to note that the coefficient $\omega_n^2$ is seen to be $|\lambda|^2$. In other words, the coefficient of $\lambda^{n-2}$ when $n=2$ is equal to $|\lambda|^2$. Similarly, for a 4th degree characteristic equation is given by

$$\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 = 0$$

with two complex conjugate pair roots, the coefficient $a_2$ turns out to be

$$a_2 = \omega_{n_1}^2 + \omega_{n_2}^2 + (2\zeta_1 \omega_{n_1})(2\zeta_2 \omega_{n_2})$$

Thus $(|\lambda_1|^2 + |\lambda_2|^2) < a_2$.

Since $\text{Re}(\lambda) = -\zeta \omega_n$ and $\text{Imag}(\lambda) = \omega_n \sqrt{1 - \zeta^2}$, it can be seen that

$$|(\text{Imag part})_1| + |(\text{Imag part})_2| < \sqrt{a_2}$$

and hence $|\text{Imag}(\lambda_i)|_{\text{max}} \leq \sqrt{a_2}$.
The above argument can easily be extended to the higher order case.

Based on this Lemma, we now state the theorem for bounds on imaginary parts.

**Theorem 3.4.4.** For all $n \times n$ ecologically sign stable matrices,

$$|\text{Imag}(\lambda_i)|_{\text{max}} \leq \sqrt{n \sum_{i, j=1, i<j} a_{ij}a_{ji}}$$  \hspace{1cm} (3.4.10)

*Proof.* The eigenvalues of any general matrix with $a_{ij} = 0 \forall i, j \neq j$ are pure real. However, the converse is not true. But in the case of sign stable matrices, if $a_{ij} = 0 \forall i$ all eigenvalues have zero real parts. This is so for the following reasons. Consider the set of matrices with the following properties:

1. $a_{ii} \leq 0 \ \forall i$

2. $a_{ij}a_{ji} \leq 0 \ \forall i, j, i \neq j$

3. $k_i$ cycles $\equiv 0$ for $i = 3, 4, \ldots n$

Such matrices are known as sign semi-stable matrices [79]. This implies that such sign semi-stable matrices *always* have non-positive real part eigenvalues. From the conditions for sign stability given in [71], [72] and [73], it is seen that sign stable matrices are a special case of sign semi-stable matrices. The conditions for sign stability convert the property of non-positivity of real part into negativity. Thus, in the discussion on stability based on signs, it is clear that sign semi-stability is a limiting case of sign-stability while transitioning to instability (not *sign* instability). Hence, for a sign stable matrix, all eigenvalues are pure real in the absence of off-diagonal elements and all eigenvalues have zero real part in the absence of diagonal elements. From this, we infer that eigenvalue distribution of sign stable matrices about the real and imaginary axes can be decoupled as follows:
1. $|\text{Real}(\lambda_i)|_{\text{max}}$ is as large as possible when $a_{ij} = 0 \forall i, j, i \neq j$

2. $|\text{Imag}(\lambda_i)|_{\text{max}}$ is as large as possible when $a_{ii} = 0 \forall i$ except one $i$

We consider case (2) since bound on the imaginary parts of eigenvalues is of interest here. (2) implies that $|\text{Imag}(\lambda_i)|_{\text{max}}$ of eigenvalues (of $A_{ss}$) is

$$\sqrt{a_{2ss}}$$

where $a_{ssii} = 0 \forall i$ except one where $a_{2ss}$ is the $2^{nd}$ coefficient of the characteristic equation of a sign stable matrix.

Invoking Lemma (4.4.2), since the second term in the summation $\sum_{i,j=1,i<j}^{n} a_{ii}a_{jj}$ $\forall i, j, i \neq j$ and $i \leq j$ is always 0, in case (2), $a_{2ss} = -\sum_{i,j=1,i<j}^{n} a_{ij}a_{ji}$ where it is noted that for sign stable matrices, the argument in the summation is always positive (since $a_{ij}a_{ji} \leq 0 \forall i, j, i \neq j, i \leq j$).

Hence the bound

$$|\text{Imag}(\lambda_i)|_{\text{max}} \leq \sqrt{a_{2ss}} = \sqrt{-\sum_{i,j=1,i<j}^{n} a_{ij}a_{ji}}$$

Theorems 3.4.1 and 3.4.4 are now illustrated with the following examples.

**Example 1:** In this example, bound on the real parts of the eigenvalues of a sign stable matrix is illustrated. Consider the sign stable pattern

$$S = \begin{bmatrix}
- & - & - & - \\
0 & - & - & 0 \\
0 & + & - & 0 \\
+ & 0 & + & - \\
\end{bmatrix}$$

It is desired to show how distribution of the real part of eigenvalues is independent of the magnitudes of the off-diagonal elements. In order to do so, a set of 10 quantitative matrices with the same sign stable sign pattern is considered. Magnitudes of the diagonal elements are chosen such that the minimum value is 0.4 and the maximum
value is 5.0 while the off-diagonal elements, can assume any magnitude (with signs unaltered) as in matrix $M$.

$$
M = \begin{bmatrix}
-a_{11} & -( ) & -( ) & -( ) \\
0 & -a_{22} & -( ) & 0 \\
0 & +( ) & -a_{33} & 0 \\
+( ) & 0 & +( ) & -a_{44}
\end{bmatrix}
$$

where $0.4 \leq a_{11}, a_{22}, a_{33}, a_{44} \leq 5$.

Eigenvalues of all these 10 matrices plotted in Figure 3.11 clearly show that real part of every eigenvalue is independent of the magnitudes of the off-diagonal elements. Similarly, example 2 illustrates the bound on the imaginary parts of the eigenvalues.

![Figure 3.11: Eigenvalue distribution of a SS matrix along real axis](image)

**Example 2:**

Consider the sign stable pattern $S$ as in example 1. It is desired to show how distribution of the imaginary part of eigenvalues is independent of the magnitudes of the diagonal elements. This is done by keeping the bound on the imaginary part constant with no restriction on the magnitudes of the diagonal elements. Again, for a set of
10 quantative matrices with sign pattern $S$, bound on the imaginary part is taken to be 3.6606 with no restriction on the the diagonal elements as in matrix $N$.

$$N = \begin{bmatrix}
-\left( \right) & -a_{12} & -a_{13} & -a_{14} \\
0 & -\left( \right) & -a_{23} & 0 \\
0 & +a_{32} & -\left( \right) & 0 \\
+a_{41} & 0 & +a_{43} & -\left( \right)
\end{bmatrix}$$

where $$-\sum_{i,j=1}^{n} a_{ij}a_{ji} \leq 3.6606 \ \forall \ i, j, \ i \neq j$$

Figure 3.12: Eigenvalue distribution of a SS matrix along imaginary axis

Plotting the eigenvalues of these 10 matrices, it is clear from Figure 3.12 that bound on imaginary parts given by theorem 2 is independent of the diagonal elements. While examples 1 and 2 show the bounds for a set of sign stable matrices, the following example compares these bounds for a sign stable and a non-sign stable, Hurwitz stable matrix. This is done by calculating the bounds for both matrices. In addition to these bounds, a rectangle (formed by the maximum and minimum eigenvalues of the symmetric and skew-symmetric parts of the parent matrix) overbounding the field of
values is also plotted to illustrate the tightness of the bound on real part.

**Example 3:**

Consider the following sign matrices

\[ S = \begin{bmatrix} - & - & - & - \\ 0 & - & - & 0 \\ 0 & + & - & 0 \\ + & 0 & + & - \end{bmatrix}, \quad H = \begin{bmatrix} - & - & + \\ 0 & - & - & 0 \\ 0 & + & + & 0 \\ + & 0 & + & - \end{bmatrix} \]

where \( S \) is ecologically sign stable and \( H \), obtained by changing the sign of \( S(1,4) \) from \(-\) to \(+\) and the sign of \( S(3,3) \) from \(-\) to \(+\), is not ecologically sign stable. Then, consider the quantitative matrices \( A_{ss} \) and \( A_{hs} \) such that

1. \( S \) is the sign pattern of \( A_{ss} \) and \( H \) is the sign pattern of \( A_{hs} \).

2. \(|A_{ss}(i,j)| = |A_{hs}(i,j)|\).

3. \( \text{Re}(\lambda(A_{ss})) < 0 \) and \( \text{Re}(\lambda(A_{hs})) < 0 \) \( \forall \ i \) i.e., both \( A_{ss} \) and \( A_{hs} \) are Hurwitz stable.

\[ A_{ss} = \begin{bmatrix} -3 & -1 & -0.8 & -4.8 \\ 0 & -1.2 & -1.5 & 0 \\ 0 & 2 & -0.4 & 0 \\ 2.6 & 0 & 4 & -5.0 \end{bmatrix}, \quad A_{hs} = \begin{bmatrix} -3 & -1 & -0.8 & 4.8 \\ 0 & -1.2 & -1.5 & 0 \\ 0 & 2 & 0.4 & 0 \\ 2.6 & 0 & 4 & -5.0 \end{bmatrix} \]

In Figure 3.13, the eigenvalues, bounds given by theorems 3.4.1 and 3.4.4 and the over-bounding rectangle of the field of values of a sign stable matrix are plotted. It is clearly seen that the bound on real parts of eigenvalues is tighter when compared to the bound obtained from the rectangle. In the numerical example illustrated by Figure 3.14, the eigenvalues of a non-sign stable, Hurwitz stable matrix lie outside the new bound for real parts while the bound for imaginary part is not even quantifiable in this case (it is an imaginary number). Thus, sign stable matrices allow for easy determination of eigenvalue distribution which is highly desirable while determination of the same for general Hurwitz stable matrices is computationally more intensive.

A corollary to theorem 3.4.4 is stated as follows:
Corollary 3.4.5. If there are no predator-prey interactions in a sign stable matrix, then

1. The eigenvalues are pure real

2. They are simply the diagonal elements

Proof. Proof is divided into the following two parts.

1. The eigenvalues are pure real: Commensal/Ammensal interaction implies that in the pair \((a_{ij}, a_{ji})\), either \(a_{ij}\) or \(a_{ji}\) must be 0. Then \(a_{ij}a_{ji} = 0\ \forall\ i, j, i \neq j\). Since all the terms in the summation given in equation (5) are zero, the bound on imaginary part = 0 i.e., all eigenvalues are pure real.

2. They are simply the diagonal elements: This statement can proved with both engineering (matrix theory) as well as ecological perspective. Both the proofs are discussed here.

Ecological perspective:
Consider the \(l\)-cycles and \(k\)-cycles discussed in previous section. As shown in
Figure 3.14: Eigenvalue distribution of a HS matrix

Figure 3.15, $l$-cycles (a) and (b) and $k$-cycles (c) are the strongly connected subcomponents of a digraph while the acyclic (d) and (e) are the weakly connected subcomponents. Since all $k$-cyclic components in sign stable matrices are always zero [73], absence of $l$-cycles i.e., pure ammensal and commensal
interactions implies that all subcomponents are acyclic. The matrix interpretation of cyclic/acyclic subcomponents can be expressed as follows:

Cyclic terms $\Rightarrow$ strongly connected components $\Rightarrow$ irreducibility.
Acyclic terms $\Rightarrow$ weakly connected components $\Rightarrow$ reducibility.

Therefore, matrices with pure ammensal/commensal interactions are reducible. For these matrices, since every block diagonalized submatrix (leading principal minors) is further block diagonalizable (as there are absolutely no cyclic terms), it is evident that the eigenvalues of these matrices are simply their diagonal elements.

*Engineering/Mathematical perspective:*

If there are no predator-prey interactions, then $a_{ij}a_{ji} = 0 \ \forall \ i, j \ i \neq j$. This means that the bound in theorem 3.4.4

$$|\text{Imag}(\lambda_i)|_{\text{max}} \leq \sqrt{n \sum_{i,j=1, i<j}^n -a_{ij}a_{ji}} \equiv 0 \ \forall \ i, j \ i \neq j$$

which means that the maximum possible imaginary part is 0 and hence all eigenvalues are real.

In the absence of predator-prey interactions, from the expressions for characteristic equations previously discussed, the characteristic equation simplifies to

$$\prod_{i=1}^n (\lambda - a_{ii})$$

(3.4.11)

Therefore, the eigenvalues are simply the diagonal elements.

Further examination reveals a subtle yet insightful feature of the effect of these commensal/ammensal interactions. If $a_{ii} = 0$ for one $i$, according to Corollary (3.4.5),
one of the eigenvalues is zero. Moreover, if $a_{ii} = 0$ for two or more $i$s, the matrix becomes unstable. This implies that result of corollary (3.4.5) is not affected by the signs of the diagonal elements. That is, there is no restriction on the signs of the diagonal elements for the above corollary to hold. Hence, this statement can be generalized to any square matrix (irrespective of the nature of its qualitative information or its stability) and stated as below.

**Theorem 3.4.6.** For all $n \times n$ matrices (stable and unstable), with all $k$-cycles being zero and with only commensal or ammensal interactions, the eigenvalues are simply the diagonal elements.

**Proof.** Identical to proof for Corollary (3.4.5). □

Therefore, the property of sign stability greatly reduces computational effort in determination of eigenvalue distribution of the matrix. Following this, we present the result on the property of normality.

### 3.4.3 Normality/Condition Number

In order to study normality of sign stable matrices, we consider sign stable systems possessing the following properties.

1. All self-regulatory species: $a_{ii} < 0$
2. Pure predator-prey interactions: $a_{ij}a_{ji} < 0 \forall i, j, i \neq j$
3. Identical self-regulation intensities: $|a_{ii}| = d, d > 0 \forall i$
4. Equal interaction strengths: $|a_{ij}| = |a_{ji}|$ for each $i, j$ pair, $i \neq j$

Matrix $A_{ss}$ is an example of such a qualitative system, which is sign stable.
The matrix $A_{ss}$ and $A_{mag}$ are as follows:

$$A_{ss} = \begin{bmatrix} - - 0 0 \\ + - - 0 \\ 0 + - + \\ 0 0 - - \end{bmatrix} \quad A_{mag} = \begin{bmatrix} -1 -3 0 0 \\ +3 -1 -4 0 \\ 0 +4 -1 +5 \\ 0 0 -5 -1 \end{bmatrix}$$

The matrix $A_{mag}$ is a corresponding quantitative matrix. For such matrices, the following theorem can be stated.

**Theorem 3.4.7.** An $n \times n$ matrix $A$ with identical diagonal elements and equal predator-prey interaction strengths for each predation-prey link is a normal matrix ($A^T A = AA^T$).

**Proof.** Necessary and sufficient condition for normality of a $2 \times 2$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is $a_{11} = a_{22}$, $a_{21}^2$ ($\Rightarrow a_{21} = a_{12}$ or $a_{21} = -a_{12}$). For example, the $2 \times 2$ predator-prey model

$$\begin{bmatrix} -x & y \\ -y & x \end{bmatrix}$$

satisfies the above condition. Therefore, with the given condition on the diagonal elements, a single predator-prey link with identical self-regulation rates and equal interaction strengths is always normal.

Extension to higher order systems:

Consider the $3 \times 3$ matrix with pure-predator link structure

$$A = \begin{bmatrix} a & a_{12} & a_{13} \\ -a_{12} & a & a_{23} \\ -a_{13} & -a_{23} & a \end{bmatrix}$$

A can be partitioned as
$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ A_2 = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}, \ A_3 = \begin{bmatrix} -a_{13} & -a_{23} \end{bmatrix}, \ A_4 = [a]$$

Here $A_1$ and $A_4$ are normal matrices and $A_3 = -A_2^T$ (due to predator-prey interaction).

For $A$ to be normal, $A^TA = AA^T$

$$\Rightarrow \begin{bmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} A_1^T & A_3^T \\ A_2^T & A_4^T \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A_1^TA_1 + A_3^TA_3 & A_1^TA_2 + A_3^TA_4 \\ A_2^TA_1 + A_4^TA_3 & A_2^TA_2 + A_4^TA_4 \end{bmatrix} = \begin{bmatrix} A_1A_1^T + A_2A_2^T & A_1A_3^T + A_2A_4^T \\ A_3A_1^T + A_4A_3^T & A_3A_2^T + A_4A_4^T \end{bmatrix}$$

Since $A_1$ and $A_4$ (here $A_4$ is a scalar quantity) are normal matrices and $A_3 = -A_2^T$, the above equation becomes

$$\Rightarrow \begin{bmatrix} A_1^TA_1 + A_2A_2^T & A_1^TA_2 - A_2A_4 \\ A_2^TA_1 - A_4A_2^T & A_2^TA_2 + A_4A_4^T \end{bmatrix} = \begin{bmatrix} A_1A_1^T + A_2A_2^T & -A_1A_2 + A_2A_4^T \\ -A_2A_1^T + A_4A_2^T & A_2A_2^T + A_4A_4^T \end{bmatrix}$$

In each of the above matrices, the submatrix $(.)_{21} = (.)^T_{12}$. Hence the property of symmetry is satisfied. For both the matrices to be equal, $A_1^TA_2 - A_2A_4 = -A_1A_2 + A_2A_4^T$

By virtue of the equal strength predator-prey interactions, $(A_1^T + A_1)$, $(A_4^T + A_4)$ are diagonal matrices with identical diagonal elements. Hence the above equality becomes

$$\text{diag}[2a, 2a] A_2 = A_2[2a]$$

Therefore, a $3 \times 3$ matrix with pure predator-prey interactions of equal strength is always normal. Similarly, a $4 \times 4$ pure predator-prey link matrix $M$ of equal interaction strengths can be written as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$
such that A, D are normal and of order 2. Then,

\[
\begin{bmatrix}
A^T & C^T \\
B^T & D^T
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
= \begin{bmatrix}
A^T A + C^T C & A^T B + C^T D \\
B^T A + D^T C & B^T B + D^T D
\end{bmatrix}
\]

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
A^T & C^T \\
B^T & D^T
\end{bmatrix}
= \begin{bmatrix}
A A^T + B B^T & A C^T + B D^T \\
A C^T + D B^T & C C^T + D D^T
\end{bmatrix}
\]

In order to establish the property of normality, \(P = Q\) must hold. Since \(A^T A = A A^T\),

\[
\begin{bmatrix}
A^T A + B B^T & A T B - B D \\
B^T A - D^T B^T & B^T B + D^T D
\end{bmatrix}
= \begin{bmatrix}
A A^T + B B^T & -A B + B D^T \\
-B^T A^T + D B^T & B^T B + D D^T
\end{bmatrix}
\]

Therefore, \(P_{(1,1)} = Q_{(1,1)}\) and \(P_{(2,2)} = Q_{(2,2)}\)

The other condition on the blocks of \(P\) and \(Q\) matrices for normality of \(M\) is

\[
P_{(2,1)} = P_{(1,2)} = Q_{(2,1)} = Q_{(1,2)}
\]

\[
B^T A - D^T B^T = (A^T B - BD)^T \Rightarrow P_{(2,1)} = P_{(1,2)}^T
\]

\[
-B^T A^T + D B^T = (-A B + BD^T)^T \Rightarrow Q_{(2,1)} = Q_{(1,2)}^T
\]

The condition is satisfied if

\[
P_{(1,2)} = Q_{(1,2)}
\]

\[
\Rightarrow A^T B - BD = -A B + B D^T
\]

\[
\Rightarrow A^T B + AB = B D^T + BD
\]

\[
\Rightarrow (A^T + A) B = B (D^T + D)
\]

Due to the equal strength predator-prey interactions,

\[
(A^T + A) = (D^T + D) = a_{ii} I_{2 \times 2}
\]

Since identity matrices commute with every matrix, the equality \((A^T + A) B = B (D^T + D) B\) holds. Hence \(P = Q\) and therefore the matrix \(M\) is always normal.

By induction, this proof can be extended to any \(n \times n\) pure predator-prey interaction matrix that satisfies the conditions of identical self-regulation rates and equal interaction strengths.

\[\square\]
The property of normal matrices that is of interest to us is that condition number of its modal matrix is always 1 i.e., \( \kappa \equiv 1 \). The following numerical example demonstrates the effect of sign pattern on the condition number \( \kappa \).

**Example 4**

Consider a normal, sign stable matrix \( M_1 \) as defined in theorem 4.5.2.

\[
M_1 = \begin{bmatrix}
-1 & -2 & 0 & 0 & 0 \\
+2 & -1 & +3.5 & 0 & 0 \\
0 & -3.5 & -1 & -5.4 & 0 \\
0 & 0.0 & +5.4 & -1 & +3.8 \\
0 & 0 & 0 & -3.8 & -1
\end{bmatrix}
\]

Condition number of the modal matrix, \( \kappa_1 = 1 \). Now consider the matrix \( M'_1 \) formed by changing the sign of element \( M_1(2,3) \) from positive to negative.

\[
M'_1 = \begin{bmatrix}
-1 & -2 & 0 & 0 & 0 \\
+2 & -1 & -3.5 & 0 & 0 \\
0 & -3.5 & -1 & -5.4 & 0 \\
0 & 0.0 & +5.4 & -1 & +3.8 \\
0 & 0 & 0 & -3.8 & -1
\end{bmatrix}
\]

Condition number of the modal matrix, \( \kappa'_1 = 301.52 \).

It is seen that by simply changing the sign of only one element of \( M_1 \) from postive to negative, the condition number \( \kappa \), goes from 1 to 301.52 and therefore loses its property of normality! This example clearly illustrates the significance of predator-prey interactions. The property of normality has significant relevance to the results reported in this work. In the following chapter we see its importance in the study of robust stability of linear uncertain systems.
CHAPTER 4
SIGN STABILITY AND ROBUSTNESS

This chapter links the concepts of sign stability from ecological sciences and robustness from engineering sciences. In this chapter, new measures of robustness are proposed. Based on these robustness measures, the class of matrices that is most robust is identified. These matrices are such that all the proposed measures of robustness are maximized.

Taking advantage of the insights gained in previous chapters, the following measures of robustness will be considered in assessing and identifying matrices with superior robustness properties.

1. Qualitative Robustness: Robustness based only on qualitative information

2. Directional Robustness: Robustness based on qualitative and quantitative information

3. Quantitative Robustness: Robustness based only on quantitative information

4.1 Qualitative Robustness

As mentioned before it is known that natural systems such as ecological and biological systems are highly robust under various perturbations. On the other hand, engineered systems can be made highly optimal for good performance but they tend
to be non-robust under perturbations. Therefore, it is natural and essential for engineers to delve into the underlying features of natural systems which make them so robust and then apply these principles to make the engineered systems more robust. Toward this direction, the problem of assessment of qualitative robustness for linear uncertain systems is addressed from a completely new and refreshing framework of ecosystem dynamics. In these models, the interrelationship between various interactions and interconnections between states (species) play an extremely important role in making the overall system highly robust. It is proved and realized that certain interactions/interconnections enhance robustness while certain other interactions/interconnections destabilize the system [71], [72] and [77]. Exploiting the property of qualitative stability derived from the study of interactions and interconnections in ecosystems, in what follows, we propose new qualitative robustness assessment metrics namely [80],

\[ \beta_1 = \frac{\text{no. of } a_{ii} > 0}{\text{total no. of } a_{ii}(= n)} \]

(we omit the case of \(a_{ii} = 0\))

\[ \beta_2 = \frac{\text{no. of } l \text{-cycles} > 0}{\text{total no. of } l \text{-cycles}} \]

(we omit the case of \(a_{ij}a_{ji} = 0\))

\[ \beta_3 = \frac{\text{no. of } k \text{-cycles} \neq 0}{\text{total no. of } k \text{-cycles}} \]

Using the above, we can also consider a cumulative measure \(\beta\) given by

\[ \beta = \beta_1 + \beta_2 + \beta_3 \quad (4.1.1) \]

Note that under this definition, the ‘higher’ this index, the ‘less robust’ the system is qualitatively. It is also to be noted that it is relatively very easy to determine these robustness indices as the information needed for computing these indices is directly
obtained from the dynamics matrix $A$. From the definition of $\beta$s, the minimum value of the cumulative index $\beta$ is 0 while the maximum value $\beta_{\text{max}} = 3$. Thus the farther this cumulative index is from zero, the less robust the system is. This implies that a matrix is qualitatively most robust when $\beta = 0(\beta_1 = \beta_2 = \beta_3 = 0)$. Similarly, a matrix is least robust and is in fact unstable, when $\beta = \beta_{\text{max}} = 3$. Based on this definition a sufficient condition for stability is proposed. A quantitative matrix $A$ is always Hurwitz stable(unstable) if $\beta = 0(\beta = \beta_{\text{max}}=3)$ respectively.

Figure 4.1 is a pictorial representation of the above condition. It is well known that $\beta_1 = 1$ is a sufficient condition for instability. Therefore, individual contribution of each $\beta_i$ also plays a significant role in qualitative robustness. Plots in Figures 4.2, 4.3 and 4.4 show the extent of criticality of each $\beta_i$ to the robust stability of matrices. Figures 4.2, 4.3 and 4.4 are plotted varying only one $\beta_i (0 \leq \beta_i \leq 1)$ while keeping the other $\beta_i$s=0.
Clearly, the most critical of $\beta_i$s is $\beta_1$ and least critical is $\beta_3$. Though each $\beta_i$ has a distinct, individual effect on robust stability of a matrix, $\beta$ (cumulative) is a suitable measure for qualitative stability.

Qualitative robustness of sign stable matrices

As mentioned in previous chapter, sign stable matrices have

\[ a_{ii} \leq 0 \forall i \]

\[ a_{ij}a_{ji} \leq 0 \forall i \neq j \text{ and} \]

\[ a_{ij}a_{jk}a_{kl}...a_{mi} \equiv 0 \forall i \neq j \neq k, ... \neq m \]

From the definition for $\beta_i$s, we see that matrices with $\beta \equiv 0$ are sign stable! Therefore, from qualitative robustness point of view, sign stable matrices are most robust. Since robustness implies tolerance to perturbations which are quantified by numerical values, it is desirable to define robustness measures that involve numerical or quantitative information. Therefore, the following measure of robustness is defined based on qualitative as well as quantitative information.
4.2 Directional Robustness

In mathematical sciences, the aspect of ‘robust stability of families of matrices has been an active topic of research for many decades. This aspect essentially arises in many applications of system and control theory. When the system is described by linear state space representation, the plant matrix elements typically depend on some uncertain parameters which vary within a given bounded interval. In early eighties and nineties, widespread research on robust stability of linear state space systems with structured real parameter uncertainty was reported in the literature [?],[81] and [82]. The problem formulation in that research was that given a Hurwitz stable matrix, how much perturbation E can be tolerated to maintain the stability of the perturbed matrix $A + E$. When bounds on the norm of $E$ are given to maintain stability, it is labeled as robust stability for unstructured, norm bounded uncertainty. When bounds on the individual elements of the matrix are important to maintain stability, it is labeled as robust stability for structured real parameter uncertainty. The interval matrix problem or more generally the linear interval parameter matrix
family problem in which the given uncertain parameters vary within a given interval range with a lower and upper bound on the parameters, then became a special case of this structured uncertainty problem formulation. The family of interval matrices is represented as follows:

Robust Stability Analysis of a Class of Interval Matrices:

Consider the interval matrix family in which each individual element varies independently within a given interval. Thus the interval matrix family is denoted by $A \in [A^L, A^U]$ as the set of all matrices $A$ that satisfy

\[(A^L)_{ij} \leq A_{ij} \leq (A^U)_{ij} \quad \forall \ i, j \quad (4.2.1)\]

Many sufficient conditions were given throughout the literature, which were summarized in [16]. In this area, it was extremely difficult to give a necessary and sufficient condition in a finitely computable manner (like using only vertex matrix information, where vertex matrices are those matrices formed at the vertices of the interval parameters) but after intense research of many years, it was only recently that a method was presented that gives a necessary and sufficient vertex solution for checking the
robust stability of a linear interval parameter matrix family [83]. All these techniques involve considerable computation to arrive at the robust stability bounds but these techniques never delved into the sign pattern of the elements of the matrix and thus never exploited this sign structure. But now with the ecological sign stability concept, it is clear that by paying attention to the sign pattern of the given matrix element variations, much more can be said about the robust stability of the perturbed matrices. Consider a special class of interval matrix family’ given by equation (4.2.1) in which for each element that is varying, the lower bound i.e. \((A^L)_{ij}\) and the upper bound i.e. \((A^U)_{ij}\) are of the same sign.

For example, consider the interval matrix given by

\[
A = \begin{bmatrix}
0 & a_{12} & a_{13} \\
a_{21} & 0 & 0 \\
a_{31} & 0 & a_{33}
\end{bmatrix}
\]

with the elements \(a_{12}, a_{13}, a_{21}, a_{31}\) and \(a_{33}\) being uncertain varying in some given intervals as follows:

\[2 \leq a_{12} \leq 5, \quad 1 \leq a_{13} \leq 4, \quad -3 \leq a_{21} \leq -1, \quad -4 \leq a_{31} \leq -2\text{ and } -5 \leq a_{33} \leq -0.5\]

In order to assess robustness of this family of matrices, we define the measure ‘Directional Robustness’. Directional Robustness is a measure of permissible ‘sign invariant’ perturbation in elements of the matrix in spite of which stability is guaranteed.

**Example 5:**

It is seen that perturbation has the same sign as the nominal element. Once it is recognized that the signs of the interval entries in the matrix are not changing (within the given intervals), the sign matrix can be formed. The ‘sign’ matrix for this interval matrix is given by

\[
A_s = \begin{bmatrix}
0 & + & + \\
- & 0 & 0 \\
- & 0 & -
\end{bmatrix}
\]
It is of interest, now, to pursue the effect of this sign pattern on the test for robust stability.

**Directional robustness of sign stable matrices**

It is clear that ecologically sign stable matrices have the interesting feature that once the sign pattern is a sign stable pattern, the stability of the matrix is independent of the magnitudes of the elements of the matrix. That this property has direct link to stability robustness of matrices with structured uncertainty was recognized in earlier papers on this topic [75],[76].

In [18], a viewpoint was put forth that advocates using the qualitative stability’ concept as a means of achieving robust stability’ in the standard uncertain matrix theory and offer it as a sufficient condition’ for checking the robust stability of a class of interval matrices.

The above sign matrix $A_s$ is known to be qualitative (sign) stable. Since sign stability is independent of magnitudes of the entries of the matrix, it can be concluded that the above interval matrix is robustly stable in the given interval ranges. If the robust stability of this interval matrix is to be ascertained by the methods of robustness theory of mathematical sciences, one needs to resort to the extreme point solution offered in [83] which would have been computationally expensive, because it involves checking the Hurwitz stability of the $2^5 = 32$ vertex matrices first and then following the algorithm to check the virtual stability of the 32 KN (Kronecker Nonsingularity) matrices in the higher dimensional Kronecker Lyapunov matrix space. But in the above matrix, once it is realized that the sign of the matrix entries is not changing within the given intervals, the qualitative stability concept can readily be applied and it can be concluded that the above interval matrix is robustly stable, because with only signs replacing the entries, it is observed that the above matrix is Hurwitz stable irrespective of the magnitudes of those entries. Thus the robust stability of
the entire interval matrix family is established without resorting to any algorithms related to robust stability literature. Incidentally, if the vertex algorithm of [83] is applied for this problem, it can be also concluded that this interval matrix family is indeed Hurwitz stable in the given interval ranges.

In fact, more can be said about the robust stability of this matrix family using the sign stability application. This matrix family is indeed robustly stable, not only for those given interval ranges above, but it is also robustly stable for any large interval ranges in those elements as long as those interval ranges are such that the elements do not change signs in those interval ranges. Thus elements $a_{12}$ and $a_{13}$ can vary along the entire positive real line and elements $a_{21}$, $a_{31}$ and $a_{33}$ can vary along the entire negative real line simultaneously and still the resulting matrices are all stable. In other words, if this matrix were the plant matrix for a linear state space system, that particular linear system has infinite bound for robust stability in the specific sign preserving variations in the elements of that matrix. It could not have been possible to conclude this but for the usefulness of the sign stability concept. Therefore, by its definition, directional robustness is maximized for sign stable matrices. Therefore, from directional robustness point of view, sign stable matrices are most robust.

Till now, emphasis was on exploiting the sign pattern of a matrix in robust stability analysis of matrices. Thus, the tolerable perturbations are direction sensitive. Also, no perturbation is allowed in the structural zeroes of the ecologically sign stable matrices. Taking into consideration the fact that robustness conventionally refers to tolerance to perturbation of a certain magnitude in both directions, in what follows, it is shown that ecologically sign stable matrices can still possess superior robustness properties even under norm bounded perturbations, in which perturbations in structural zeroes are also allowed.
4.3 Quantitative Robustness

Consider the following representation of a linear uncertain system.

\[ A = A_0 + E \] \hspace{1cm} (4.3.1)

where \( A_0 \) is the nominal Hurwitz stable system, \( E \) is the perturbation matrix and \( A \) is the resultant perturbed system. In the following sections, robustness bounds are proposed based on the nature of \( E \).

4.3.1 Norm bounded or Unstructured Perturbation

When bounds on the norm of the perturbation matrix \( E \) are given to maintain stability, it is labeled as robust stability for norm bounded or unstructured uncertainty. This means that only the norm of the perturbation is known while precise location of perturbation in the matrix is unknown. Following are two such bounds [14].

i. Bound based on stability degree:

\[ \| E \| < \mu_d = -\frac{\alpha_s}{\kappa} \] \hspace{1cm} (4.3.2)

where, \( \alpha_s \) is the real part of dominant eigenvalue of \( A \) and \( \kappa \) is the condition number of the modal matrix of \( A \)

ii. Bound based on solution of Lyapunov equation can be given by

\[ \| E \| < \mu_p = \frac{1}{\sigma_{max}(P)} \] \hspace{1cm} (4.3.3)

where, \( P \) is the solution of the Lyapunov equation \( A^T P + PA + 2I = 0 \) and \( \sigma(\cdot) \) is the maximum singular value of a matrix
4.3.2 Structured Perturbation

When bounds on the elements of $E$ are given to maintain stability, it is labeled as robust stability for structured uncertainty. Two measures of robustness available in the literature for robust stability of time varying real parameter perturbations are given below.

i. Bound based on stability degree:

$$\epsilon < \frac{\mu_d}{n}$$

(4.3.4)

where, $\mu_d$ is as defined equation (4.3.2) and $n$ is the dimension of matrix $A$

ii. Bound based on solution of Lyapunov equation can be given by

$$\epsilon < \mu_Y = \frac{\mu_p}{n}$$

(4.3.5)

where, $\mu_p$ is as defined equation (4.3.3) and $n$ is the dimension of matrix $A$

Clearly, these bounds are extremely conservative as they do not take into consideration the exact location of perturbation in the matrix. That is, if it is known beforehand, that there is no perturbation in a particular element, the bounds proposed in equations (4.3.4) and (4.3.5) do not take such information into consideration. In order to remove such conservatism, bounds based on the structure of perturbation were proposed. Structured perturbation is further classified into independent perturbation and dependent perturbation. Following are the bounds for structured, independent and dependent parameter perturbation.
Independent Perturbation

When perturbation in matrix elements are independent of each other, such perturbations are known as independent perturbations. Therefore, the perturbation matrix $E$ is defined as follows

$$E = \sum_{i=1}^{m} k_i E_i$$

such that only one element varies in each $E_i$. For example, the perturbation matrix $E$ of a $2 \times 2$ system with perturbation only in the diagonal elements is expressed as

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = k_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Before, discussing bounds for independent perturbation, we first need to define the matrix $U_e$. $U_e$ is defined such that $U_{eij} = 0$ if $E_{ij} = 0$ and $U_{eij} = 1$ if $E_{ij} \neq 0$. Therefore, $U_e$ comprises of 0s or 1s. Using this matrix, the following, less conservative bounds are defined.

i. Bound based on stability degree or real part of the most dominant eigenvalue[81]:

$$\epsilon < \mu_{ds} = -\frac{\alpha_s}{\kappa \sigma_{max}(U_e)}$$

where $\alpha_s$ and $\kappa$ are as defined in equation (4.3.2) and $U_e$ is a $n \times n$ matrix whose entries are such that $U_{eij} = 0$ if the perturbation in $A_{ij}$ is known to be zero and $U_{eij} = 1$ if the perturbation in $A_{ij}$ is known to be nonzero and thus $\epsilon$ is the maximum modulus perturbation expected in the entries of $A$.

ii. Bound based on solution of Lyapunov equation [84]:

$$\epsilon < \mu_Y = \frac{1}{\sigma_{max}[P_m U_e]_s}$$

where $P$ and $U_e$ are as defined above. $P_m$ is the matrix formed by considering only the magnitudes of the elements of the $P$ matrix.
Dependent Perturbation

When there is functional dependence between the perturbation in two or more elements, such perturbation is known as dependent perturbation. The perturbation matrix $E$ is defined below

$$E = \sum_{i=1}^{m} k_i E_i$$  \hspace{1cm} (4.3.9)

such that $E_i$ can have more than one non-zero elements. Following is an example of such dependent perturbation.

$$\begin{bmatrix} k_1 + \alpha k_2 & \beta k_1 \\ \gamma k_2 & 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix} + k_2 \begin{bmatrix} \alpha & 0 \\ \gamma & 0 \end{bmatrix}$$

**Theorem 4.3.1.** Consider the linear system (??) with $A$ stable and $E$ as in (4.3.9). Let $P$ be the solution of the Lyapunov equation $A^T P + PA + 2I = 0$, and define $P_i$ and $P_e$ as

$$P_i := (E_i^T P + PE_i)/2 := [PE_i]_s, 1, 2, \ldots, m;$$

$$P_e := [P_1 P_2 \ldots P_m]$$

Robustness bounds for dependent perturbation [85]:

The system (??) is stable if

1. $$\sum_{i=1}^{m} k_i^2 < \frac{1}{\sigma_{\text{max}}(P_e)}$$  \hspace{1cm} (4.3.10)

   or

2. $$\sum_{i=1}^{m} |k_i| \sigma_{\text{max}}(P_i) < 1$$  \hspace{1cm} (4.3.11)

   or

3. $$|k_j| < \frac{1}{\sigma_{\text{max}}(\sum_{i=1}^{m} |P_i|)}, \hspace{1cm} j = 1, 2, \ldots, m$$  \hspace{1cm} (4.3.12)

The objective of ecologically inspired robust stability analysis is to determine those matrices that are *most robust* or those matrices for which quantitative robustness bounds are maximized.
4.4 Target Sign Stable (SS) Matrices

We now look at a particular class of sign stable matrices which have improved robustness with respect to the bound $\mu_d$

1. Sign(S) is qualitatively stable

2. $s_{ij} = -s_{ji}$

3. $s_{ii} = -d$ where $d$ is a positive real constant.

Two important properties of this class of matrices are:

1. The upper and lower bounds on the real parts of the eigenvalues a sign stable matrix are simply the maximum and minimum diagonal elements of that sign stable matrix.

2. All target sign stable matrices are normal. This means that their modal matrices are normal as well and therefore $\kappa \equiv 1$ for all target SS matrices.

Based on the above two properties, the following theorem is stated.

**Theorem 4.4.1.** The norm bound $\mu_d$ on a target SS matrix $S$ is $d$ where $d$ is the magnitude of diagonal element of $S$ i.e.,

$$\mu_d = -\frac{\alpha_s}{\kappa} = d \quad (4.4.1)$$

**Proof.** Since all diagonal elements have magnitude $d$, $\alpha_s = d$.

Since the modal matrix is normal, $\kappa \equiv 1$

Therefore, $\mu_d = -\frac{\alpha_s}{\kappa} = d \quad \square$

Another property of the Target SS matrices related to the Lyapunov based bound is presented. First the following lemma is stated and proved.
Lemma 4.4.2. The solution matrix $P$ of the Lyapunov equation $S^T P + PS + 2I = 0$ of every target SS matrix $(S)$ is equidiagonal.

Proof. The Lyapunov equation for the matrix $S$ is $S^T P + PS + 2I = 0$.

Now, let $S = S_o + S_d$ where $S_o$ is the off-diagonal part and $S_d$ is the diagonal part.

Invoking conditions ii and iii we can write $S^T = -S_o - dI$.

The Lyapunov equation now becomes $(-S - dI)P + P(S_o - dI) + 2I = 0$.

For the above equality to hold, $P$ must be equidiagonal i.e. $P = pI$.

We know that since $S$ is stable, $P$ is a positive definite matrix. Therefore, $p$ has to be a positive real constant.

Based on 4.4.2, the following result is presented.

Theorem 4.4.3. The norm bound $\mu_p$ on a target SS matrix $S$ is $d$, where $d$ is the magnitude of diagonal element of $S$ i.e.,

$$\mu_p = \frac{1}{\sigma_{\text{max}}(P)} = d \quad (4.4.2)$$

Proof. : From 4.4.2, $P = pI$.

Substituting the result into the Lyapunov equation, we get $(-S_o - dI)pI + P(S_o - dI) + 2I = 0$.

$\Rightarrow -pS_o + pS_o - 2dpI = -2I$

$\Rightarrow -2dpI = -2I$

$\Rightarrow p = 1/d$

But $\sigma_{\text{max}}(P) = p$.
Therefore \( \mu_p = \frac{1}{\sigma_{max}(P)} = d \)

We label these matrices as ‘Target SS matrices’ (where SS stands for Sign Stable). An interesting property of these Target SS matrices is that \( \mu_d \equiv \mu_p \). It is well known that the largest possible bound on unstructured perturbation can at the most be \(-\alpha_s\). Theorem 4.4.1 states that \( \mu_d = -\alpha_s \). This implies that, in the case of Target SS matrices, not only do both the bounds coincide but they are also the maximum possible bounds for unstructured perturbation. Therefore, Target SS matrices are most robust from quantitative robustness (for unstructured perturbation, for structured independent perturbation when \( U_e \) is identity matrix and dependent perturbation in the off-diagonal structure) point of view and are most desirable.

### 4.5 Target Pseudosymmetric (PS) Matrices

In theorem 4.4.3, the simplification in determination of bound is possible only because a) all negative diagonal elements are of equal magnitude and b) \( S \) is such that \( S_o^T = -S_o \). Hence all stable matrices that possess properties a) and b) have \( P \) matrices of the form \( pI \).

It turns out that these matrices are pseudosymmetric in nature. That is, \( S_o^T = -S_o \) or \( s_{ij} = -s_{ji} \) is a characteristic of pseudosymmetric matrices. For example,

\[
S_1 = \begin{bmatrix}
-1 & 3 & 5 & -4.7 \\
-3 & -1 & 2.5 & -3.9 \\
-5 & -2.5 & -1 & -1.5 \\
4.7 & 3.9 & 1.5 & -1
\end{bmatrix}
\]

is a pseudo-symmetric matrix of dimension 4.

Having made this observation, we state the following theorem.

**Theorem 4.5.1.** For a pseudo-symmetric equidiagonal (negative diagonal elements of equal magnitude) matrix (such as \( S_1 \)), the norm bound \( \mu_p \) is \( d \).
Proof. Same as for 4.4.1

Similarly, we state another theorem.

**Theorem 4.5.2.** For a pseudo-symmetric equidiagonal (negative diagonal elements of equal magnitude) matrix (such as $S_1$), the norm bound $\mu_d$ is $d$.

Proof. Same as for 4.4.3.

Therefore, in the case of these specific pseudosymmetric matrices, the norm bounds become equal and, as in the case of Target SS matrices, are maximized. Based on theorem 4.5.2, the following corollary is stated:

**Corollary 4.5.3.** For a Target PS matrix $A$, norm bounds $\mu_P$ and $\mu_Y$ are such that

$$a_{iimin} \leq \mu_P \leq a_{iimax}$$

$$a_{iimin} \leq \mu_Y \leq a_{iimax}$$

This implies that the robustness bound (unstructured uncertainty) for any pseudosymmetric matrix with negative diagonal elements lies between the magnitudes of minimum and maximum diagonal elements of the matrix. While existence of zero elements is imperative for target SS matrices described in [86], this condition is relaxed in the case of pseudosymmetric matrices such as $S_1$. Due to increased degrees of freedom (more non-zero elements), going forward ‘Target PS matrices’ will refer to matrices satisfying conditions a) and b).

Even though these target PS matrices do not satisfy the conditions for sign stability [73], since their sign structure partially imparts stability, sign stability can still be attributed to them. While sign (qualitative) stability is independent of magnitudes, the very definition of pseudo-symmetric matrices implies magnitude specificity. (It is this restriction on magnitudes that relaxes the requirement of zeros in the matrix). In
addition to this restriction on magnitudes, since constraints a) and b) include certain qualitative constraints, we coin the phrase ‘constrained sign stability’ to identify the qualitative nature of stability of target PS matrices. But due to the nature of the target SS matrices discussed in [73], adding the above constraints helps in expanding the class of matrices resulting in the new and larger set of target (or desired) matrices that have been redefined in this. The following Venn diagram represents the classification of stable matrices where ‘SS’ implies sign stable and ‘PS’ implies stable pseudo-symmetric. ‘Target SS matrices’ refers to the class of target matrices discussed in [86] while ‘Target PS matrices’ refers to the class of target matrices defined above (pseudosymmetric matrices with negative diagonal elements of equal magnitude). These target matrices can hence be visualized as a class of matrices that captures the best features of both sign stability and pseudosymmetry. That is, they are sign stable (in a constrained manner) without the requirement of zero elements but at the same time possess the property of a diagonal P matrix.

It is to be noted that the property of normality holds for the pseudosymmetric matrices as well. This property is relevant to current research because the condition
number of the modal matrix of a normal matrix is always 1 and hence the bound \( \mu_d \) is maximum. It also turns out that the bound for dependent perturbation given by equations 4.3.10, 4.3.11 and 4.3.12 goes to infinity for the pseudosymmetric structure (variation only in off-diagonal elements)! This infinity bound identifies with the ‘constrained sign stability’ nature of Target PS matrices where stability is assured for any magnitude as long as elements satisfy the condition \( a_{ij} = -a_{ji} \).

From theorems 4.4.3 and 4.5.2 we infer that for certain matrices, the norm bound is determined easily directly by the sign structure of the matrices. And in the case of these target PS matrices, the bound can be easily determined to be magnitude of diagonal element. Therefore, for any given (desired) bound, we can easily determine a corresponding stable matrix. This result is identical to the problem formulation discussed in the previous section.
CHAPTER 5
ROBUST CONTROL DESIGN METHOD

In this section we present control design techniques that address perturbations of the following nature:

1. Directional perturbation: Perturbations should be such that sign pattern of the perturbed system is the same as sign pattern of nominal system

2. Norm bounded or unstructured perturbation: Perturbations can occur anywhere in the system matrix. However, the norm of the perturbation matrix is bounded.

3. Structured perturbation: Perturbations are defined as occurring in specific locations in the matrix and bounds are proposed for such perturbation in each element separately.

These control design methods are then illustrated with examples that are reflective of the superior features of the proposed control methodologies.

5.1 Control Design for general sign stable matrices

5.1.1 Control Design Philosophy

Consider the linear time invariant system with the following uncertainty structure

\[ \dot{x}(t) = Ax(t) + Bu(t) \] (5.1.1)
where $x \in \mathbb{R}^n$ is the $n$-dimensional state vector and $u \in \mathbb{R}^m$ is the $m$-dimensional control vector.

In the presence of uncertainty, equation (5.1.1) is expressed as

$$\dot{x}(t) = (A + \Delta A)x(t) + (B + \Delta B)u(t)$$

(5.1.2)

The objective is to design a robust full-state feedback control $u = Gx$ for the nominal system given in equation (5.1.1) such that it is robust in the presence of uncertainty $\Delta A$ and $\Delta B$. Therefore, the nominal closed loop system matrix is given by

$$A + BG = A_{cl}$$

(5.1.3)

The full-state feedback controller is designed such that the closed loop system matrix $A_{cl}$ is ecologically sign stable. Now, the entire uncertain system with the feedback controller can be written as

$$A + BG + \Delta A + (\Delta B)G = A_{cl} + \Delta A + (\Delta B)G$$

(5.1.4)

In this control design, matrix $E$ defined in the previous section is of the form.

$$E = \Delta A + (\Delta B)G$$

(5.1.5)

Our philosophy of controller design is different from the currently available methods in the sense that we first establish a sign stable closed loop system matrix $A_{cl}$ apriori and then determine the corresponding controller gain $G$ which achieves that desired sign stable pattern. In contrast, in the current methods, the controller gain $G$ is determined first and then the resulting closed loop system matrix is accepted for further analysis.

In our design philosophy, prior to control design, it is important to establish whether this ‘sign assignability’ can be achieved or not. Since assigning signs is the primary objective, the conditions derived in a later section for this sign assignability problem
are termed as ‘Sign Assignability Conditions’.

Significance of Sign Assignability Conditions

Before discussing the Sign Assignability Conditions, it is appropriate to discuss its relevance and connection to the currently available alternative robust control design methods for linear uncertain systems. In particular, its similarity to the popular robust control design method based on matching conditions is noteworthy. In that connection, we now briefly recall the robust control design method under matching conditions [87].

Consider the uncertain dynamical system

\[
\dot{x}(t) = \begin{bmatrix} A + \Delta A(q(t)) \end{bmatrix} x(t) + \begin{bmatrix} B + \Delta B(q(t)) \end{bmatrix} u(t), \quad t \in [0, \infty) \tag{5.1.6}
\]

where \( x \in \mathbb{R}^n \) is the state vector, \( u \in \mathbb{R}^m \) is the control and \( q(t) \in Q \subset \mathbb{R}^k \) is the uncertainty For existence of a controller, assumptions (A1-A5) \([87]\) need to be made. Of these, assumption A5, known as ‘matching conditions’ are given below. A5. Matching conditions. There exist continuous matrix functions

\[
D(\cdot) : Q \rightarrow \mathbb{R}^{m \times n}
\]

and

\[
F(\cdot) : Q \rightarrow \mathbb{R}^{m \times m}
\]

such that

1. \( \Delta A(q) = BD(q) \), for any \( q \in Q \)

2. \( \Delta B(q) = BF(q) \), for any \( q \in Q \)

3. \( ||E|| < 1 \), for any \( q \in Q \)

That is, a control gain \( G \) exists only if \( \Delta A \) and \( \Delta B \) matrices have the above specified structures. In the following section it will be shown that Sign Assignability Conditions also are based on the specific structure of perturbation. From the above information, it can easily be inferred that these matching conditions are functionally similar to Sign Assignability Conditions since
1. They address similar uncertainty structure i.e., real parameter variation in both A and B matrices.

2. Most importantly, these conditions have to be satisfied in addition to controllability for the existence of a gain matrix G.

3. The extremely restrictive nature of these conditions renders the perturbations highly structured.

Clearly, Sign Assignability Conditions (SAC) play as important a role as matching conditions in robust control design for linear uncertain systems. Having established their importance, we now proceed to derive the SAC.

**Sign assignability conditions for control design**

We present the ‘Sign Assignability Conditions’ required for the design method that implement the concept of ecological sign stability. As mentioned in the previous section, these conditions in conjunction with controllability determine the existence of a sign stable controller.

When B is square and non-singular, there is no need to resort to any Sign Assignability Conditions because the control gain G can always be obtained as

$$G = B^{-1}(A_{cf} - A)$$  \hspace{1cm} (5.1.7)

Hence, there always exists a G that makes the closed loop system matrix A_{cf} sign stable. But when B is non-square, existence of G is determined based on the structure of A and B matrices. In what follows, we delve deep into their structure and summarize these in the form of ‘Sign Assignability Conditions’. These tests are based on the conditions for qualitative stability of ecological systems discussed in [?], [72] and [73].
Let matrices \( A \) and \( B \) be represented by the compact sets \([a_{ij}]\) and \([b_{ij}]\) respectively. The existence of \( G \) is determined by the zero rows of \( B \) where ‘zero row’ refers to a row with all zero elements. When the \( B \) matrix has zero rows, the 7 tests (a) to (g) given below need to be performed in order to determine the existence of a control gain \( G \). In the absence of zero rows, it is sufficient to perform test (h) on \((A,B)\) in order to determine the existence of \( G \).

a. For sign stable matrices, \( a_{ii} \leq 0 \forall i \). Therefore, if \( B(i,:) = 0 \), and \( a_{ii} > 0 \) for \( i \geq 1 \), no controller exists. For example, consider the following \( A \) and \( B \) matrices:

\[
A = \begin{bmatrix}
-1 & -0.3 & -1.8 & 2.1 \\
3 & -0.2 & -2.148 & -2 \\
0 & 1 & 1 & 0 \\
2.5 & -0.9 & 3 & -1.1
\end{bmatrix}
\quad B = \begin{bmatrix}
-1 & 2 \\
3 & 1 \\
0 & 0 \\
0 & 2
\end{bmatrix}
\]

The zero row of \( B \) matrix is \( B(3,:) \). Now, since the diagonal element corresponding to this row i.e., \( a_{33} \) is positive, no controller exists.

b. For sign stable matrices \( a_{ij}a_{ji} \leq 0, \forall i,j, i \neq j, i, j = 1, 2, \ldots, n \). Therefore, if \( B(i,:) = B(j,:) = 0 \) and both \( a_{ij}, a_{ji} > 0 \) or both \( a_{ij}, a_{ji} < 0 \) where \( i \neq j \) for at least one \( i,j \) pair, no controller exists. For example, consider the following \( A \) and \( B \) matrices:

\[
A = \begin{bmatrix}
-1 & -0.3 & -1.8 & 2.1 \\
3 & -0.2 & -2.148 & -2 \\
0 & 1 & 1 & 3 \\
2.5 & -0.9 & 0.5 & -1.1
\end{bmatrix}
\quad B = \begin{bmatrix}
-1 & 2 \\
3 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

Here, \( i = 3 \) and \( j = 4 \). Since the product \( a_{34}a_{43} = 1.5 > 0 \), a sign stable controller does not exist.
c. Consider the set \( \{ A_{ss} | A_{ss} \in A_S \} \) where \( A_{ss} \), any sign stable sign pattern is a member of \( A_S \), the set of all sign stable sign patterns that can be generated by the algorithm given in \([12]\). If \( \text{sign}(A(i,:)) \neq A_{ss}(i,:) \), for \( \text{sign}(A_d(i,:)) \) to be equal to \( A_{ss}(i,:) \), \( B(i,:) \) should not be equal to 0 for every such \( i \). Therefore, Consider \( A(j,:) \), where \( j \) is the index of \( B(j,:) = 0 \) and \( A(k,:) \) where \( k \) is the index of \( B(k,:) \neq 0 \). With \( A(k,:) \) fixed \( \forall k \), check with \( \{ A_S \} \) whether \( \text{sign}(A(j,:)) \equiv A_{ss}(j,:) \ \forall j \) for at least one \( A_{ss} \). If no such \( A_{ss} \) exists, no controller exists. For example,

\[
A = \begin{bmatrix}
-1 & 0.3 & -1.8 & -2.1 & 5 \\
-3 & 0 & -2.148 & 0 & 9.4 \\
0 & 1 & -4 & 0 & 1 \\
2.5 & -0.9 & 3 & -1.1 & 0 \\
0 & 0 & -2.1 & 9 & -4 \\
\end{bmatrix} \quad \text{B} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
6 & 10 \\
0 & 0 \\
7 & 5 \\
\end{bmatrix}
\]

Consider

\[
A_{fixed} = \begin{bmatrix}
-1 & 0.3 & -1.8 & -2.1 & 5 \\
-3 & 0 & -2.148 & 0 & 9.4 \\
* & * & * & * & * \\
2.5 & -0.9 & 3 & -1.1 & 0 \\
* & * & * & * & * \\
\end{bmatrix}
\]

‘*’ implies that the element is variable. After comparing with the entire set of \( 5 \times 5 \) sign stable matrices, it is concluded that there does not exist a sign stable matrix with the above pattern \( A_{fixed} \). This is because \( A_{fixed} \) does not satisfy the condition ‘M4’ \([71]\) given as one of the five necessary conditions for qualitative stability of a sign pattern. Therefore, no controller exists.

d. Let \( Q_{ss} \) (which need not be the final \( A_d \)) denote a quantitative sign stable matrix satisfying the above three conditions. In addition to this, \( Q_{ss}(j,:) \) should
be equal to $A(j,:)$ $\forall j$. (Note that this condition holds for any general control design method. That is, every row of the open loop matrix $A$ corresponding to a zero row of $B$ cannot be altered and will remain as such in the closed loop system matrix $A_{cl}$). Then, if the number of zero rows is $(n - m)$ and all rows are consecutive, $A + BG = Q_{ss}$ can be written as

$$ \begin{bmatrix} A_{uz} + B_{uz}G \\ A_t + B_tG \\ A_{lz} + B_{lz}G \end{bmatrix} = \begin{bmatrix} \text{zero matrix} \\ Q_{ss_t(m \times m)} \\ \text{zero matrix} \end{bmatrix}$$

where the rows of $B_t$ are the only non-zero rows of $B$. Then $B_t$ is a square matrix of dimension $m$. $A_{uz} + B_{uz}G$ and $A_{lz} + B_{lz}G$ represent matrices corresponding to the zero rows of $B$. These matrices are of variable dimensions and depending on the structure of $B$, one of the two may not exist. Such a representation implies that $B_t$ may be located anywhere in the $B$ matrix. For example, $B$ can be of the form

$$ \begin{bmatrix} 6 & 10 \\ 7 & 5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 6 & 10 \\ 0 & 0 \end{bmatrix} $$

When $B_t$ is invertible, the gain matrix is

$$ G = B_t^{-1}(Q_{ss_t} - A_t) \quad (5.1.8) $$

When $B_t$ is non-invertible, $Q_{ss}$ should be equal to $A_{cl}$ (desired closed loop system) and $G$ exists only if $\text{rank } \left\{ [\tilde{B}_t | (a_t - q_{ss})] \right\} \leq \text{rank } \left\{ \tilde{B}_{t1} \right\}$ where $\tilde{B}_t$ is formed such that
\[
\begin{bmatrix}
A_t(:,1) \\
A_t(:,2) \\
\vdots \\
A_t(:,m)
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
Q_{ss}(;1) \\
Q_{ss}(;2) \\
\vdots \\
Q_{ss}(;m)
\end{bmatrix}
\]

For example, consider the following \(A, B\) and \(Q_{ss}\) matrices.

\[
A = \begin{bmatrix}
-1 & 0.3 & 0 & -2.1 & 5 \\
-3 & 0 & -2.148 & 0 & 9 \\
0 & 1 & -4 & -1.8 & 0 \\
0 & 0 & 3 & -1 & -3 \\
0 & 0 & 0 & 9 & -4
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
6 \\
7 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[
Q_{ss} = \begin{bmatrix}
-1 & 0.3 & 0 & 0 & 0 \\
-3 & -3 & -2.148 & 0 & 0 \\
0 & 1 & -4 & -1.8 & 0 \\
0 & 0 & 3 & -1 & -3 \\
0 & 0 & 0 & 9 & -4
\end{bmatrix}
\]

Here, \(A_t = \begin{bmatrix}
-1 & 0.3 & -1.8 & -2.1 & 5 \\
-3 & 0 & -2.148 & 0 & 9.4
\end{bmatrix}\) \(B_t = \begin{bmatrix}
6 \\
7
\end{bmatrix}\)

According to equation (5.1.8), the gain matrix

\[
G = B_t^{-1}(Q_{ss} - A_t) = \begin{bmatrix}
0 & -0.75 & 0 & -0.2625 & -1.625 \\
0 & 0.45 & 0 & 0.3675 & 0.475
\end{bmatrix}
\]

Therefore, a control gain \(G\) exists. Note that \(G\) is non-unique as \(Q_{ss}\) is variable.
e. If the number of zero rows is \((n - m)\) but are not consecutive (distributed) 
\[ \mathbf{A} + \mathbf{B} \mathbf{G} = \mathbf{Q}_{ss} \] 
can also be written as 
\[
\begin{bmatrix}
\mathbf{A}(1,:) + \mathbf{B}(1,:)\mathbf{G} \\
\vdots \\
\mathbf{A}(n,:) + \mathbf{B}(n,:)\mathbf{G}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{Q}_{ss}(1,:) \\
\vdots \\
\mathbf{Q}_{ss}(n:)
\end{bmatrix}
\]

\[ \Rightarrow \mathbf{B}(i,:)\mathbf{G} = \mathbf{Q}_{ss}(i,:) - \mathbf{A}(i,:) \]

Let the index of a zero \(\mathbf{B}\) row be \(j\) i.e., \(\mathbf{B}(j,:) = 0\) and the number of zero rows be \(p\). Similarly, let the index of non-zero \(\mathbf{B}\) row be \(k\) i.e., \(\mathbf{B}(k,:) \neq 0\) and the number of non-zero rows be \(q\). The matrices can be rearranged as

\[
\begin{bmatrix}
\mathbf{A}_{d1} + \mathbf{B}_{d1}\mathbf{G} \\
\mathbf{A}_{d2} + \mathbf{B}_{d2}\mathbf{G}
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{Q}_{ssd1} \\
\mathbf{Q}_{ssd2}
\end{bmatrix}
\]

where \(\mathbf{A}_{d1} = \mathbf{A}(k,:)\), \(\mathbf{B}_{d1} = \mathbf{B}(k,:)\), \(\mathbf{Q}_{ssd1} = \mathbf{Q}_{ss}(k,:) \forall k, 1 \leq k \leq q\) \(\mathbf{A}_{d2} = \mathbf{A}(j,:)\), \(\mathbf{B}_{d2} = \mathbf{B}(j,:)\), \(\mathbf{Q}_{ssd2} = \mathbf{Q}_{ss}(j,:) \forall j, 1 \leq j \leq p\) When \(\mathbf{B}_{d1}\) is invertible,

\[
\mathbf{G} = \mathbf{B}_{d1}^{-1}(\mathbf{Q}_{ssd1} - \mathbf{A}_{d1}) 
\]

(5.1.9)

When \(\mathbf{B}_{d1}\) is non-invertible, \(\mathbf{Q}_{ss}\) should be the same as \(\mathbf{A}_{cl}\) and \(\mathbf{G}\) exists only if rank \(\begin{bmatrix} \mathbf{B}_{d1} | (\mathbf{a}_{d1} - \mathbf{q}_{ssd1}) \end{bmatrix}\) \(\leq\) rank \(\begin{bmatrix} \tilde{\mathbf{B}}_{d1} \end{bmatrix}\) where

\[
\mathbf{a}_{d1} = 
\begin{bmatrix}
\mathbf{A}_{d1}(;1) \\
\mathbf{A}_{d1}(;2) \\
\vdots \\
\mathbf{A}_{d1}(;m)
\end{bmatrix} \quad \text{and} \quad \mathbf{q}_{ss} = 
\begin{bmatrix}
\mathbf{Q}_{ssd1}(;1) \\
\mathbf{Q}_{ssd1}(;2) \\
\vdots \\
\mathbf{Q}_{ssd1}(;m)
\end{bmatrix}
\]

For example, consider the following \(\mathbf{A} \mathbf{B}\) and \(\mathbf{Q}_{ss}\) matrices.
Now, \( A_{d1} = \begin{bmatrix} -1 & 0.3 & 0 & 0 & 9 \\ -3 & -3 & -2.148 & 0 & 0 \\ 0 & -3 & -4 & -1.8 & 1.1 \\ 0 & 0 & 3 & -1 & -3 \\ 0 & 0 & 0 & 9 & -4 \end{bmatrix} \) and \( B_{d1} = \begin{bmatrix} 6 & 10 \\ 0 & 0 \\ 7 & 5 \end{bmatrix} \)

According to equation (5.1.9), the gain matrix

\[
G = B_{d1}^{-1}(Q_{ss} - A_{d1}) = \begin{bmatrix} 0 & 1 & 0 & 0 & -0.275 \\ 0 & -0.6 & 0 & 0 & 0.165 \end{bmatrix}
\]

Therefore, a control gain \( G \) always exists and, as discussed in condition (d), is non-unique.

f. If number of zero rows > \((n-m)\)(distributed or consecutive), Let \( A_{cl} \), a quantitative sign stable matrix) be a member of the set of all candidate (also quantitative sign stable) closed loop system matrices \( A_C \) i.e., \( \{A_{cl} | A_{cl} \in A_C \} \). Let \( A_n \) and \( A_{cn} \) denote the open loop and closed loop system matrices corresponding to \( B_n \) matrix where \( B_n \) all non-zero rows. Then \( B_n \) has dimensions \( p \times m \) where \( p < m \). Since the set of simultaneous equations given by \( B_n G = A_{cn} - A_n \) is under determined, the solution \( (G) \) always exists and is non-unique. Consider the following \( A, B \) and \( Q_{ss} \) matrices.
\[ \mathbf{A} = \begin{bmatrix} 1 & -0.3 & 0 & 0 \\ 3 & -0.2 & -2.148 & 0 \\ 0 & 1 & -1 & -3.9 \\ 0 & 0 & 3 & -1.1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{Q}_{ss} = \]

\[
\begin{bmatrix}
-1 & -0.3 & 0 & 0 \\
3 & -0.2 & -2.148 & 0 \\
0 & 1 & -1 & -3.9 \\
0 & 0 & 3 & -1.1
\end{bmatrix}
\]

Then, it can be seen that

\[ \mathbf{G} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \]

This verifies the existence of a gain matrix.

g. If number of zero rows < \((n - m)\), \(\mathbf{B}_n\) has dimensions \(r \times m\) where \(r > m\).

Then, \(\mathbf{B}_n \mathbf{G} = \mathbf{A}_{cl_n} - \mathbf{A}_n = \mathbf{A}_{aa_n} \mathbf{G}\) exists only if \(\text{rank}\{\begin{bmatrix} \mathbf{B}_n | \mathbf{a}_{aa_n} \end{bmatrix}\} \leq \text{rank}\{\mathbf{B}_n\}\)

where \(\tilde{\mathbf{B}}_n\) and \(\mathbf{a}_{aa_n}\) are such that

\[
\begin{bmatrix}
\mathbf{G}(\cdot, 1) \\
\mathbf{G}(\cdot, 2) \\
\vdots \\
\mathbf{G}(\cdot, n)
\end{bmatrix} = \mathbf{a}_{aa_n} = \begin{bmatrix}
\mathbf{A}_{aa_n}(\cdot, 1) \\
\mathbf{A}_{aa_n}(\cdot, 2) \\
\vdots \\
\mathbf{A}_{aa_n}(\cdot, n)
\end{bmatrix}
\]

For example,
\[
A = \begin{bmatrix}
-1.5 & 0 & 0 & 3 \\
-1 & -3.5 & 1 & 0 \\
-0.75 & -5.25 & -2.5 & 1 \\
-4 & 0 & -0.5 & -1.5 \\
\end{bmatrix}
B = \begin{bmatrix}
1 \\
2 \\
3 \\
0 \\
\end{bmatrix}
A_{cl} = \begin{bmatrix}
-1.5 & 0 & 0 & 3 \\
0 & -2 & 1 & 0 \\
0 & -3 & -2.5 & 1 \\
-4 & 0 & -0.5 & -1.5 \\
\end{bmatrix}
\]

It can be verified that this system satisfies equation (5.1.10). Therefore, for such a \((A, B)\) pair, a \(G\) always exists.

h. When there are no zero \(B\) rows, since the elements of \(G\) can be expressed as the solution of a set of simultaneous linear equations, existence is determined by the rank condition. Therefore, \(G\) exists only if

\[
\text{rank } \left\{ \tilde{B} | a_{a} \right\} \leq \text{rank } \tilde{B} \tag{5.1.10}
\]

where \(\tilde{B}\) and \(a_{a}\) are such that

\[
[\tilde{B}] \begin{bmatrix}
G(:, 1) \\
G(:, 2) \\
\vdots \\
G(:, n) \\
\end{bmatrix} = a_{a} = \begin{bmatrix}
A_{a}(; 1) \\
A_{a}(; 2) \\
\vdots \\
A_{a}(; n) \\
\end{bmatrix}
\]

For example,

\[
A = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
1 & -7 & -4 & 0 \\
-0.5 & -1 & -1 & -2.5 \\
-1.2 & -2.4 & 5.2 & -6 \\
\end{bmatrix}
B = \begin{bmatrix}
-1 \\
3 \\
0.5 \\
1.2 \\
\end{bmatrix}
A_{cl} = \begin{bmatrix}
-2 & -2 & 0 & 0 \\
4 & -1 & -4 & 0 \\
0 & 1 & -1 & -2.5 \\
0 & 0 & 5.2 & -6 \\
\end{bmatrix}
\]
It can be verified that this system satisfies equation (5.1.10). Therefore, a \( G \) exists.

Figure (5.1) is a pictorial depiction of the test for existence of the controller \( G \). In cases where \( A_{cl} \) is non-unique, \( A_{cl} \) is chosen such that it best suits the design requirements.

Therefore, for a completely controllable pair \((A,B)\) to have a sign stable controller, it is necessary that the pair satisfy the above sign assignability conditions.

From the above set of tests it can be inferred that if \( B \) has \( p \) non-zero rows, only \( pn \) elements of the open loop matrix \( A \), can be varied and that too with a specific pattern of only \( p \) elements in each column. This feature holds for any controller design. That is, given a \((A_{n\times n}, B_{n\times m})\), only \( pn \) elements can be changed to get an \( A_{cl} \). Such insightful properties that can simplify control design are not utilized in traditional control design techniques.

Sign Assignability Conditions are very useful as they greatly reduce the computational effort required for the algorithm. There are 8 \( 2 \times 2 \) sign stable sign patterns, 427 \( 3 \times 3 \) sign stable sign patterns, over 86000 \( 4 \times 4 \) sign stable sign patterns and so on. Consider, for instance, the case of a 3 dimensional system. In the absence of SAC, technically, one would have to check with all the 427 sign stable sign patterns in order to determine the existence of a sign stable controller. As discussed above, SAC helps in easy elimination of non-achievable sign stable sign patterns or in easy determination of the existence of a sign stable controller, thus reducing the computational effort.

Therefore, if a completely controllable system \((A,B)\) satisfies the sign assignability conditions, there always exists at least one \( G \) such that the closed loop system matrix \( A_{cl} \) is sign stable.
5.1.2 Control Design Algorithm

A distinct feature of the robust control design method proposed here is that the desired closed loop system matrix $A_{cl}$ is determined prior to the determination of the controller. This is so because the test for sign assignability requires qualitative and/or quantitative information of candidate closed loop system matrices to verify the existence of a controller.

By virtue of sign stability, there is considerable flexibility in the choice of an appropriate closed loop system matrix. In fact, $A_{cl}$ is formed element-wise as opposed to control design methods reported in literature where the closed loop system matrix generated by the algorithm has no specific structure or properties apart from the control objective it is intended to achieve.

To summarize, the control design method for a full state feedback controller for a LTI system is carried out in three steps:

1. Test for existence of gain matrix $G$
   
   (a) Based on $A$, $B$ and $A_S$ matrices
   
   (b) Based on $A$, $B$ and $A_C$ (which is generated from $A_S$) matrices

2. Choice of appropriate $A_{cl}$ (which is a member of $A_C$)

3. Determination of gain matrix $G$

We now discuss each of these steps in detail.

1. Test for existence of $G$ (SAC): In order to determine the existence of a gain matrix, $(A, B)$ must be tested for conditions (a)-(g) in a sequential order. As can be seen in the following flow chart, once the existence of $G$ is confirmed, test process for sign assignability can be terminated to proceed to the step (2) of the design algorithm.
(a) This part consists of tests (a)-(c). Test (a) and (b) are conducted based only on \( \mathbf{A} \) and \( \mathbf{B} \) matrices while test (c) requires the set of qualitative sign stable matrices. Therefore, \( \mathbf{A}_S \) is the additional input to the system \((\mathbf{A}, \mathbf{B})\). Part (c) consists of a loop since every \( \mathbf{A}_s \) has to be tested for before the final decision \((G \text{ exists}/G \text{ does not exist})\) is given.

(b) The second part consists of tests (d)-(g) or (h). Each of these tests involves a loop since every member of \( \mathbf{A}_C \) has to be tested.

2. Choice of appropriate closed loop system matrix: Choice of appropriate closed loop system matrix and test for existence of \( \mathbf{G} \) are interconnected in tests (d)-(f) or (g) as these tests involve iterative processes. Additionally, when \( \mathbf{G} \) is non-unique, another iterative process is required to determine the final closed loop system matrix that achieves the design objective. For example, if \( \mathbf{G} \) is non-unique, \( \mathbf{A}_{cl} \), the final closed loop system matrix will be that specific \( \mathbf{A}_{cl} \) that best satisfies the design requirement.

3. Determination of \( \mathbf{G} \) matrix:

After establishing the existence of a controller, we proceed to the determination of the controller.

Let the closed loop matrix be \( \mathbf{A}_{cl} \)

Then,

\[
\mathbf{A}_{cl} = \mathbf{A} + \mathbf{BG}
\]

is written as

\[
\mathbf{BG} = \mathbf{A}_{cl} - \mathbf{A} = \mathbf{A}_a \tag{5.1.11}
\]

For such a system, the design algorithm is as follows:

The above matrix equation can be rewritten as
\[
\bar{B}_{n^2 \times mn} g_{mn \times 1} = a_{cl} - a = a_{a_{n^2 \times 1}} \quad \text{where} \quad g_{mn \times 1} = \begin{bmatrix} G(:,1) \\ G(:,2) \\ \vdots \\ G(:,n) \end{bmatrix} \quad \text{and} \quad a_{cl} \quad \text{and} \quad a_{a_{n^2 \times 1}}
\]

are the vectors formed accordingly.

Solution for the above set of simultaneous equations is the set of the gain matrix elements [90]. Thus, once a feasible \( A_{cl} \) is arrived at, the gain matrix can easily be determined. The flowchart in Figure (5.2) concisely depicts the control design algorithm for completely controllable \((A,B)\) plant matrices. As defined previously, \( A_S \) denotes the entire set of qualitative stable sign patterns \( A_{ss} \) and \( A_C \) refers to the entire set of quantitative sign stable matrices, \( A_{cl} \) is the matrix that can be generated from the set \( A_S \). The dotted lines in the flow chart represent iterative processes. We now illustrate the above algorithm with an application in aircraft flight control.

5.1.3 Application

Satellite Formation Flying Control

In [91], a control design for the satellite formation flying problem was discussed. For this system, in [86], a controller was designed using the concept of sign stability but with no formal procedure and justification for the resulting closed loop system matrix.

The above control algorithm is now illustrated for the same example. Following is the satellite attitude dynamics and control problem discussed in [86] and [92]

\[
\begin{align*}
\dot{x} &= 0 \quad 0 \quad 1 \quad 0 \\
\ddot{x} &= 0 \quad 0 \quad 0 \quad 1 \\
\dot{y} &= 0 \quad 0 \quad 0 \quad 2\omega \\
\ddot{y} &= 0 \quad 3\omega^2 \quad -2\omega \quad 0
\end{align*}
\]

A controller such that the closed loop system is ecological sign stable with magnitudes decided by the analysis
and the algorithm described in the previous sections is to be designed. Accordingly, an ecological sign stable closed loop system is chosen such that

1. The closed loop matrix has as many pure predator-prey links as possible.

2. It also has as many negative diagonal elements as possible.

Taking the above points into consideration, the following sign pattern is chosen which is appropriate for the given $A$ and $B$ matrices:

\[
\begin{bmatrix}
0 & 0 & + & 0 \\
0 & 0 & 0 & + \\
- & 0 & - & + \\
0 & - & - & -
\end{bmatrix}
\]

The magnitudes of the entries of the above sign matrix are decided by the stability robustness analysis theorem discussed in previous chapters i.e.,

1. All non-zero $a_{ii}$ are desired to be identical

2. $a_{ij} = -a_{ji}$ for all non-zero $a_{ij}$. If $a_{ij} = 0$, then $a_{ji}$ should also be equal to 0.

The magnitudes of the entries, therefore, are

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & -1 & +2 \\
0 & -1 & -2 & -1
\end{bmatrix}
\]

From the algorithm discussed in previous section, the gain matrix is

\[
G_{es} = \begin{bmatrix}
-1 & 0 & -1 & 0 \\
0 & -4 & 0 & -4
\end{bmatrix}
\]
The closed loop matrix $A_c(= A + BG_{es})$ is sign-stable and hence can tolerate any amount of variation in the magnitudes of the elements with the sign pattern kept constant.

In this application, it is clear that all non-zero elements in the open loop matrix (excluding elements $A_{13}$ and $A_{24}$ since they are dummy states used to transform the system into a set of first order differential equations) are functions of the angular velocity $\omega$. Hence, real life perturbations in this system occur only due to variation in angular velocity $\omega$. Therefore, a perturbed satellite system is simply an $A$ matrix generated by a different $\omega$. This means that not every randomly chosen matrix represents a physically perturbed system and that for practical purposes, stability of the matrices generated as mentioned above (by varying $\omega$) is sufficient to establish the robustness of the closed loop system. It is only because of the ecological perspective that these structural features of the system are brought to light. Also, it is the application of these ecological principles that makes the control design for satellite formation flying this simple and insightful.

In order to demonstrate the magnitude independence of stability of the closed loop system, keeping the given $B$ matrix and the above designed $G_{es}$ (designed for $\omega = 1$) constant, time histories of the four states in each of the perturbed cases are plotted in Figure 5.3.

**Aircraft Flight Control Problem**

Consider the problem of Aircraft Lateral Dynamics from [92, ]. An approximate linear model of the lateral dynamics of an aircraft for a particular set of flight conditions is given by
\[
\begin{bmatrix}
\dot{p} \\
\dot{r} \\
\dot{\beta} \\
\dot{\phi}
\end{bmatrix} = 
\begin{bmatrix}
-10 & 0 & -10 & 0 \\
0 & -0.7 & 9 & 0 \\
0 & -1 & -0.7 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} 
\begin{bmatrix}
p \\
r \\
\beta \\
\phi
\end{bmatrix} + 
\begin{bmatrix}
20 & 2.8 \\
0 & -3.13 \\
0 & 0 \\
0 & 0
\end{bmatrix} 
\begin{bmatrix}
\delta_a \\
\delta_r
\end{bmatrix}
\]

where \(p\) (roll rate), \(r\) (yaw rate), \(\beta\) (sideslip angle) and \(\phi\) (roll angle) are the state variables and \(\delta_a\) (aileron deflection) and \(\delta_r\) (rudder deflection) are the control variables.

As established in chapter 4, it is desired to have a pure predator-prey interaction closed loop matrix. The following is a desired closed loop sign pattern that can be achieved, given the structure of \(B\).

\[
\begin{bmatrix}
- & - & 0 & - \\
+ & - & + & 0 \\
0 & - & - & 0 \\
+ & 0 & 0 & 0
\end{bmatrix}
\]

Considering the logic provided in Chapter 4, the magnitudes are chosen such that the pure predator-prey interactions have equal magnitudes and the self-regulatory intensities are identical. Therefore the corresponding quantitative closed loop system matrix is given by

\[
\begin{bmatrix}
-0.7 & -1 & 0 & -1 \\
+1 & -0.7 & +1 & 0 \\
0 & -1 & -0.7 & 0 \\
+1 & 0 & 0 & 0
\end{bmatrix}
\]

Applying the algorithm with the above \(A_{cl}\), the control gain is obtained as

\[
G_{es} = \begin{bmatrix}
0.5097 & -0.05 & 0.1421 & -0.05 \\
-0.3194 & 0 & 2.5559 & 0
\end{bmatrix}
\]

This closed loop matrix \(A_{cl}(= A + BG_{es})\) therefore, tolerates perturbation of any magnitude as long as the sign pattern remains unchanged. The eigenvalues of the
closed loop system are
\[-0.6152 + j1.5827 \quad -0.4348 + j0.5725\]
\[-0.6152 - j1.5827 \quad -0.4348 - j0.5725\]
which corresponds to a damping ratio of \(\zeta = 0.6048\) and to a natural frequency of \(\omega_n = 0.7189\).

In what follows, the usefulness of the above controller as a robust controller is demonstrated by assuming various sign preserving perturbations in the closed loop system matrix and the resulting time histories of states guaranteeing stability under these perturbations. For brevity, simulations are shown for two specific realizations of perturbations in Figure 5.4. These perturbations may be interpreted as variations in various stability derivatives that appear as entries in the state space matrices.

\[
A_{cl0} = \begin{bmatrix}
-0.7 & -1 & 0 & -1 \\
+1 & -0.7 & +1 & 0 \\
0 & -1 & -0.7 & 0 \\
+1 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\Delta_1 A_{cl} = \begin{bmatrix}
0.2 & 0.5 & 0 & 0.25 \\
-0.1 & 0.22 & -0.7 & 0 \\
0 & 0.3 & 0.3 & 0 \\
-0.4 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\Delta_2 A_{cl} = \begin{bmatrix}
-0.2 & -0.48 & 0 & -0.8 \\
0.9 & -0.3 & 0.82 & 0 \\
0 & -3.9 & -1 & 0 \\
0.73 & 0 & 0 & 0
\end{bmatrix}
\]

In both of the above examples the emphasis was on describing the control design procedure based on ecological sign stability. It can be seen that there is considerable flexibility in deciding the magnitudes of the resulting sign stable closed loop system matrix. In fact, in addition to magnitudes, even the sign stable sign patterns may also be taken as design variables to achieve various control design objectives.
5.2 Control Design for Norm Bounded Uncertainty

5.2.1 Methodology

In this section, a systematic procedure to determine a robust controller addressing unstructured uncertainty is presented. The algorithm is then illustrated by application in satellite attitude control and aircraft dynamics.

Consider the LTI system and its corresponding closed loop system given by equation (5.1.11). It is proposed that the desired closed loop system matrix be a pseudo-symmetric matrix with negative diagonal elements of equal magnitude defined. We denote this matrix as $A_t$. Ideally, we would like $A_t$ to be the eventual closed loop system matrix. However, it may be difficult to achieve this objective for any given controllable pair $(A, B)$. Therefore, it is proposed to achieve a closed loop system matrix that is close to $A_t$. Thus the closed loop system is expressed as

$$A_{cl} = A + BG = A_t + \Delta A$$

Noting that ideally the aim is for $\Delta A = 0$, this condition is imposed. Then, $A_{cl} = A_t = A + BG$

1. When $B$ is square and invertible:

   $$A_{cl} = A_t$$ and $G = B^{-1}(A - A_t)$

2. When $B$ is not square, but has full rank:

   Consider $B^\dagger$, the pseudo inverse of $B$

   where, for $B_{n \times m}$, if $n > m$, $B^\dagger = (B^TB)^{-1}B^T$

   Then $G = B^\dagger(A - A_t)$

Because of errors associated with pseudo inverse operation, the expression for the closed loop system is as follows:

$$A_t + \Delta E = A + BG$$

89
\[ A_t + \Delta E = A + B(B^T B)^{-1}B^T (A_t - A) \]

Let \( B(B^T B)^{-1}B^T = B_{aug} \)

Then \( \Delta E = (A - A_t) + B_{aug}(A_t - A) \)
\[ = -(A_t - A) + B_{aug}(A_t - A) \]
\[ = (B_{aug} - I)(A_t - A) \]
\[ \Rightarrow \Delta E = (B_{aug} - I)(A_t - A) \]

Therefore, the final closed loop system matrix is given by

\[ A_{cl} = A + B B^\dagger (A_t - A) + (B_{aug} - I)(A_t - A) \] (5.2.1)

The aim is to minimize the norm of \( \Delta E \). Thus, for a given controllable pair \( (A, B) \), we use the elements of the desired closed loop matrix \( A_t \) as design variables to minimize the norm of \( \Delta E \). In this research we adopt the Lyapunov based bound \( \mu_p \) as the robustness measure. Hence in what follows we drop the subscript ‘\( p \)’ and denote ‘\( \mu \) ’ as our robustness measure. It is clear that in the proposed design procedure at this stage of our research, we use the desired robustness measure as a design variable and for a given controllable pair \( (A, B) \) , design a robust control gain \( G \) such that the resulting closed loop system possesses a robustness measure , as close to the desired robustness measure as possible. While designing a controller to achieve any given desired robustness measure , is the ultimate objective of the research, the above control design procedure offers to achieve a robustness measure for the closed loop system that is greater than the robustness measure , of the open loop system matrix where , the unstructured bound discussed in previous sections is a positive scalar for a stable open loop system matrix and is taken as zero for an unstable open loop system matrix. We now apply this control design method to satellite attitude control problem and aircraft longitudinal dynamics problem. In these two examples, realization of the desired robustness bounds and the simplicity of this procedure is clearly illustrated.
5.2.2 Application

Aircraft Dynamics

Consider the following short period mode of the longitudinal dynamics of an aircraft [94].

\[
A = \begin{bmatrix}
-0.334 & 1 \\
-2.52 & -0.387 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-0.027 \\
-2.6 \\
\end{bmatrix}
\]

The open loop matrix properties are as follows:

Open loop matrix \( A \):

\[
\begin{bmatrix}
-0.334 & 1 \\
-2.52 & -0.387 \\
\end{bmatrix}
\]

Eigenvalues of open loop matrix:

\[
\begin{bmatrix}
-0.3605 + j1.5872 \\
-0.3605 - j1.5872 \\
\end{bmatrix}
\]

Norm bound: 0.2079

Note that the open loop system matrix is stable and has a Lyapunov based robustness bound \( \mu_{op} = 0.2079 \).

Now for the above controllable pair \((A, B)\), we proceed with the proposed control design procedure discussed before, with the target PS matrix elements as design variables, which very quickly yields the following results:

\( A_t \) is calculated by minimizing \( \sigma_{max}(\Delta E) \).

Here, \( \sigma_{max}(\Delta E) = 1.2381 \times 10^{-4} \)

For this value, following are the properties of the target matrix.

Target Matrix \( A_t \):

\[
\begin{bmatrix}
-0.3181 & 1.00073 \\
-1.00073 & -0.3181 \\
\end{bmatrix}
\]

Eigenvalues of \( A_t \):

\[
\begin{bmatrix}
-0.3181 + j1.00073 \\
-0.3181 - j1.00073 \\
\end{bmatrix}
\]
Norm bound $\mu_t = 0.3181$

From the expression for $G$, we get

$$G = \begin{bmatrix} -0.5843 & -0.0265 \end{bmatrix}$$

With this controller, the closed loop matrix $A_{cl}$ given below is determined.

Close loop matrix $A_{cl}$

$$
\begin{bmatrix}
-0.3182 & 1.00073 \\
-1.00073 & -0.319
\end{bmatrix}
$$

Eigenvalues of $A_{cl}$:

$$
\begin{bmatrix}
-0.31816 + j1.000722 \\
-0.31816 - j1.000722
\end{bmatrix}
$$

Norm bound $\mu_{cl}$: 0.3181426

It is clear that the eventual closed loop system matrix is extremely close to the target $PS$ matrix and hence the resulting robustness bounds can be simply read off from the diagonal elements of the target PS matrix, which in this example is also equal to the eventual closed loop system matrix. As expected, this robustness measure of the closed loop system is appreciably greater than the robustness measure of the open loop system. Now let us compare the eigenvalues of the open and closed loop system which is typically a feature analyzed by flight control engineers to assess the damping ratio and natural frequencies of the short period mode. It can be seen that the stability degree of the open loop system is 0.3605 with damping ratio and natural frequency. It is interesting to note that with our control design the closed loop system stability degree which is equal to 0.3181 can be read off directly as the diagonal matrix
of the Acl matrix. Another point to note is that the closed loop stability degree at a first glance is seen to be lower than the open loop stability degree. However it is important to realize that the stability robustness bound (0.3181) of the closed loop system is considerably higher than (= 0.2079) indicating that the controller worked towards achieving a better condition number of the modal matrix of the closed loop system.

Admittedly, the proposed analysis suggests that the eventual closed loop system may possess a robustness bound which may be lower or higher than the target bound. It is interesting to note that in this particular example, the eventual stable closed loop system possesses slightly higher robustness bound than the target PS matrix bound!

**Satellite attitude control**

Consider the following linear range dynamics of an axisymmetric satellite spinning about one principal axis (z axis in this example)

\[
\begin{bmatrix}
\dot{\omega}_x \\
\dot{\omega}_y \\
\dot{\omega}_z
\end{bmatrix} =
\begin{bmatrix}
0 & (J_y - J_z)\Omega & 0 \\
(J_z - J_x)\Omega & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\omega_x \\
\omega_y \\
(\omega_z - \Omega)
\end{bmatrix} +
\begin{bmatrix}
1/J_x & 0 & 0 \\
0 & 1/J_y & 0 \\
0 & 0 & 1/J_z
\end{bmatrix}
\begin{bmatrix}
T_x \\
T_y \\
T_z
\end{bmatrix}
\]

where angular velocities \(\omega_x, \omega_y\) and \(\omega_z\) are the state variables and torques \(T_x, T_y\) and \(T_z\) are the control variables.

With a constant angular velocity \(\Omega (= 2\pi)\) rad/sec, the following plant matrices are obtained.

\[
A = \begin{bmatrix}
0 & -0.7199 & 0 \\
1.1479 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
\frac{1}{96} & 0 & 0 \\
0 & \frac{1}{104} & 0 \\
0 & 0 & \frac{1}{115}
\end{bmatrix}
\]

In this problem, the open loop matrix is clearly unstable and as \(B\) is square and invertible, pair \((A, B)\) is controllable. Since \(B\) is invertible, controller is designed by
method (i) given in the algorithm. Let the desired $\mu_t = 0.5$. We proceed with the proposed control design procedure with the elements of $A_t$ as design variables. Since $B$ is invertible, the gain matrix is

$$G = B^{-1}(A_t - A)$$

$A_{cl}(= A_t)$ and $G$ are as follows:

$$A_{cl} = A_t = \begin{bmatrix}
-1.0 & -0.7199 & 0 \\
0.7199 & -1.0 & -0.5 \\
0 & 0.5 & -1.0
\end{bmatrix}$$

Eigenvalues of closed loop matrix:

$$\begin{bmatrix}
-1.0 \\
-1.0 + j0.8765 \\
-1.0 - j0.8765
\end{bmatrix}$$

Norm bound $\mu_{cl}(= \mu_t)=1.0$

$$G = \begin{bmatrix}
-96 & 0 & 0 \\
-44.512 & -104 & -52 \\
0 & 57.5 & -115
\end{bmatrix}$$

It is important to note that in this special case of $B$ being square and invertible, $A_t$ matrix is a complete design variable and as such there is considerable flexibility in picking the elements of $A_t$ matrix which also turns out to be the actual closed loop system matrix ($A_{cl}$). In other words it is possible to assign arbitrary robustness measure to the closed loop system which was the original intent of this research. In addition, since there is absolutely no restriction on the magnitudes of the off-diagonal elements (apart from pseudosymmetry), they can be chosen to satisfy other performance specifications such as damping ratio, natural frequency etc. Since the condition number ($\kappa$) of the modal matrix of any target PS matrix $A_t$ is always 1, the bound based on stability degree, is the largest possible norm bound on $E$ and is simply given by the diagonal element of that $A_t$. And as proved previously, this is the same as $\mu_p$. Therefore, in the case of target matrices, the Lyapunov based bound
is itself the maximum possible bound.

We see that though bound proposed on unstructured perturbation allows perturbation in any element of the matrix, owing to the structure of the plant matrices, perturbation occurs only in elements $A_{12}$, $A_{21}$, $B_{11}$, $B_{22}$ and $B_{33}$. Therefore, the perturbation matrix $E$ is such that if there is no variation in $B$ matrix, only elements $E_{12}$ and $E_{21}$ affect stability of the system. On the other hand, since the second term in $E$ is $\Delta B G$, where $G$ has no specific structure, the structural specificity of $B$ cannot be taken advantage of. From the mathematical model of the satellite dynamics it can be said that if perturbation is only in angular velocity $\Omega$, it is sufficient to consider perturbations in elements $A_{12}$ and $A_{21}$. In Figure 5.5, $\Delta E_1$ allows perturbation only in the non-zero elements of $A$ while $\Delta E_2$ and $\Delta E_3$ are random matrices. Perturbation matrices are chosen such that $||\Delta E_i|| = 0.99 \forall i$. It is seen that though norm of all the three perturbation matrices is the same, location of perturbation results in significance difference in time histories of the states. Time histories of states with perturbation $\Delta E_2$ and $\Delta E_3$ are quite distinct from the nominal time histories. Since $\Delta E_1$ takes the location of perturbation into consideration, by virtue of the physics of the system, the states do not deviate as much from the nominal, which is in fact, the realistic behavior of the system. Since the proposed method does not take into consideration the location of perturbation, in the case of such applications, it is reasonable to look for control design methods that incorporate the structure of perturbations. This leads us to control design addressing structured uncertainty.
5.3 Control Design for Structured Uncertainty

5.3.1 Methodology

From previous discussion, it can be safely argued that ‘structured real parameter perturbation situation has extensive applications in many engineering disciplines as the elements of the matrices of a linear state space description contain parameters of interest in the evolution of the state variables and it is natural to look for bounds on these real parameters that can maintain the stability of the state space system.

Focusing on these bounds, in this section, matrices with a special structure are identified for which these bounds are maximized and can easily be read off from their elements.

In previous sections, motivation for the current approach towards control design was clearly delineated. In this section, a systematic procedure to determine the controller is presented.

Consider the LTI system and its corresponding closed loop system given by (2.3.1) and (5.1.3). It is proposed that the desired closed loop system matrix be a target PS matrix. Let us denote this matrix as $A_t$. Ideally, we would like $A_t$ to be the eventual closed loop system matrix. However, it may be difficult to achieve this objective for any given controllable pair. Therefore, we propose to achieve a closed loop system matrix that is close to the matrix defined in [95]. Thus the closed loop system is expressed as

$$A_{cl} = A + BG = A_t + \Delta E$$  (5.3.1)

This control algorithm holds well (i.e. produces a control gain that exactly matches the given desired robustness measure) when B is invertible or $||E||$ is very small.
Since the equidiagonal pseudosymmetric structure of the desirable closed loop system is very restrictive, magnitude of $||E||$ may sometimes deviate from being sufficiently small. Therefore, one possible way to overcome this impediment is to relax the equidiagonal condition for the eventual closed loop system matrix. This means that $\text{diag}(A_t) \neq -dI$ which in turn implies that $P \neq pI$. Accordingly, the control design is formulated in such a way that it promises a $\mu_{cl}$ such that $\mu_L \leq \mu_{cl} \leq \mu_U$ where $\mu_L$ and $\mu_U$ are specified.

In order to achieve this modified goal, we resort to the approach that relies on the theory of robust stability analysis of matrix families. We briefly recall this analysis. We begin by considering a linear state space system with uncertain interval parameters $q_i (1, 2, \ldots, r)$ given by

$$\dot{x}(t) = H(q)x(t)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $q \in \mathbb{R}^r$ is the vector of $r$ parameters varying in the prescribed compact set $Q$. Treating $H(q)$ as an interval matrix family, let the parameters $q_i$ be given apriori bounds as

$$0 \leq q_i \leq q_U \quad (5.3.3)$$

In other words, all the varying parameters have the same upper bounds. The matrix family $H(q)$ with a specific structure that is of relevance to current research can be written as

$$H(q) = H^L + \sum_{i=1}^{r} q_i E_i$$

where $H^L$ is the matrix with all lower bound elements and $E_i$ are matrices with 1 as the only non-zero entry implying linear independent variations in the entries of $H(q)$.

Following is an example of such a matrix family.

$$H = H^L + q_1 E_1 + q_2 E_2 + q_3 E_3$$
where

\[
H^L = \begin{bmatrix}
-h_{11}^L & h_{12} & h_{13} \\
h_{21} & -h_{22}^L & h_{23} \\
h_{31} & h_{32} & -h_{33}^L
\end{bmatrix},
\]

\[
0 \leq q_1 \leq q_U \\
0 \leq q_2 \leq q_U \\
0 \leq q_3 \leq q_U
\]

and

\[
E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}
\]

For illustration purposes, we consider an interval matrix family with two parameters \(q_1\) and \(q_2\) varying independently from -3 to -1 i.e.,

\[-3 \leq q_1 \leq -1 \]

\[-3 \leq q_2 \leq -1 \]

These parameters lie on the diagonal of the matrix while the off-diagonal matrix has an invariant, pseudosymmetric structure. Figure 5.6 is an illustration of such a Target PS family. As mentioned, the off-diagonal elements of all the matrices of this family are identical while the diagonal elements vary independently in an identical interval. This matrix family can also be represented by a non parametric convex combination of a set of 'vertex' matrices, i.e., as a matrix family represented by

\[
H = \left\{ H = \sum_{i=1}^{h} \alpha_i A^{vi}, \alpha_i \geq 0, \sum \alpha_i = 1 \right\}
\] (5.3.5)

The problem of analyzing the stability of matrix families represented by a convex combination of the 'vertex matrices' has attracted considerable amount of research as it arises in many applications of systems and control theory. One of the reasons is that stability of the vertex matrices is not a sufficient condition for the stability of the entire matrix family. For general matrix families, stability of vertex matrices does not guarantee stability of the entire family. But in the case of this pseudosymmetric structure, by virtue of constrained sign stability, if the vertex matrices are Hurwitz
stable, the family is automatically Hurwitz stable. Therefore, elaborate tests, as
given in [83] need not be performed to determine the stability of this family. As this
is highly desirable, we further develop the control design in an attempt to incorporate
this feature into the closed loop system matrix. Since the vertex matrices are pseudo-
symmetric in nature with an invariant off-diagonal pattern, the elemental bound
of any member of the previously defined matrix family will satisfy the following con-
dition. $\mu(A^v_L) \leq \mu(H) \leq \mu(A^v_U) \Rightarrow \mu_L \leq \mu_H \leq \mu_U (= \mu_t)$ where
$A^v_L$ is the vertex matrix with the minimum magnitude diagonal element
$H$ represents any member of the matrix family and
$A^v_U$ is the vertex matrix with the maximum magnitude diagonal element
$\mu_L$ is the minimum bound
$\mu_H$ represents the bound of any member of the family and
$\mu_U$ is the maximum bound (= $\mu_t$)
Tolerate perturbation in off diagonal elements as well as long as they are same. Con-
sider an interval matrix family with perturbations of the following structure
$q^L_{ii} \leq q_{ii} \leq q^U_{ii}$
$q^L_{ij} \leq q_{ij} \leq q^U_{ij}$
$-q^L_{ij} \geq q_{ji} (= -q_{ij}) \geq -q^U_{ij}$
For such a family, $\mu$ always lies between maximum and minimum diagonal elements.
Motivated by this apriori knowledge, we build the closed loop matrix such that it is a
member of a family that has an invariant off-diagonal structure possessing constrained
sign stability (pseudosymmetric in nature with negative diagonal element structure).
Thus the final control design algorithm is to enforce the following equality condition
$$\sum_{i=1}^{r} \alpha_i A^{vi} = A_{clv} = A + BB^\dagger(A_{td} - A) + (B_{aug} - I)(A_{td} - A)$$
(5.3.6)
where $A^{vi}$ are the vertex matrices and $A_{td}$, a member of the family of matrices dis-
cussed above, is a design variable matrix. This way, since $A_{clv}$ belongs to the stable
pseudosymmetric family as defined earlier, stability is assured and by virtue of the family structure, its elemental bound always lies in a predetermined interval. Note that all the off-diagonal elements of \( A_v^i \) and \( A_{td} \), the equidiagonal elements of (which are constrained to take on magnitudes within the prescribed range of the robustness bounds) and the convex combination coefficients are all design variables, making the control gain determination very feasible. The key idea of this control design lies in the fact that given a bound, we can define a matrix family such that this given bound is and hence the solution of equation (5.3.6) ensures that \( \mu_{cl} > \mu_L \).

5.3.2 Application

Consider the following reduced order model of a jet engine [96].

\[
A = \begin{bmatrix}
-3.245 & -2.158 & -915.5 & 0.5731 & 134.2 \\
1.642 & -5.941 & -281.6 & 0.1897 & 57.05 \\
0.01685 & -0.02554 & -10.03 & 0.007994 & 0.5807 \\
0.0 & 0.0 & 0.0 & -100 & 0.0 \\
-2.163 & 6.862 & -740.5 & 1.195 & -171.58
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-0.002469 & -103.0 & 0.6333 & -0.3213 & -74.18 \\
0.01432 & -355.3 & -99.06 & -15.49 & 22200 \\
0.2871 & 728.6 & 25.14 & -64.87 & 8122 \\
-100 & 0 & 0 & 0 & 0 \\
100 & 0 & 0 & 0 & 0 \\
-0.1311 & 329.5 & -25.0 & 62.57 & -64450
\end{bmatrix}
\]

State variables:

\( x_1 = \) Fan speed (rpm)
\( x_2 = \) Compressor speed (rpm)
\( x_3 = \) Augmentor Pressure (psia)
\( x_4 = \) Fuel flow (lb./hr.)
$x_5 =$ Burner flow (psia)

Control variables

$u_1 =$ Main burner fuel ratio, WFMB-lb/hr

$u_2 =$ Nozzle jet area ft$^2$,

$u_3 =$ Inlet guide vane position, CIVV-deg

$u_4 =$ High variable stator position, RCVV-deg

$u_5 =$ Customer compressor bleed flow, BLC-

We design a controller such that closed loop system matrix $A_{cl}$ (with perturbation bound 1) is as follows

$$A_{cl} = \begin{bmatrix}
-1 & -355.3 & 0.002469 & -15.49 & 2220 \\
355.3 & -1 & 25.14 & 0 & 8122 \\
-0.002469 & -25.14 & -1 & -0.3213 & -74.18 \\
15.49 & 0 & 0.3213 & -1 & -62.57 \\
-2220 & -8122 & 74.18 & 62.57 & -1
\end{bmatrix}$$

The corresponding gain matrix obtained using the equation 5.3.6 is

$$G_{es} = \begin{bmatrix}
0.1549 & 0 & 0.0032 & 0.9900 & -0.6257 \\
0.0149 & 0.2240 & -0.1030 & 0.0038 & 1.0616 \\
7.1339 & 30.6250 & -11.6129 & -0.0617 & -30.1387 \\
1.6533 & 32.7352 & -13.0573 & -0.0997 & -139.1906 \\
0.0333 & 0.1472 & -0.0213 & -0.0010 & -0.1207
\end{bmatrix}$$

It is important to note that in this special case of $B$ being square and invertible, $A_t$ matrix is a complete design variable and as such there is considerable flexibility in picking the elements of $A_t$ matrix which also turns out to be the actual closed loop system matrix ($A_{cl}$). In other words it is possible to assign arbitrary robustness measure to the closed loop system which was the original intent of this research. In addition, since there is absolutely no restriction on the magnitudes of the off-diagonal elements under dependent perturbations (apart from pseudosymmetry), they can be
chosen to satisfy other performance specifications such as damping ratio, natural frequency etc.

Since the condition number ($\kappa$) of the modal matrix of any target PS matrix $A_t$ is always 1, the bound based on stability degree, is the largest possible norm bound on $E$ and is simply given by the diagonal element of that $A_t$. Therefore, in the case of target matrices, the Lyapunov based bound is itself the maximum possible bound.

Such desirable properties are achievable solely due to the invertible nature of $B$. However, when $B$ is not square we can make the closed loop system matrix $A_{cl}$ to be as close to $A_t$ as possible. Focus of future research will be on derivation of conditions for existence of robust controller as described above.
Figure 5.1: Flowchart for testing SAC (sign assignability conditions)
Figure 5.2: Flowchart of the control algorithm
Figure 5.3: Trajectories of satellite dynamics
Figure 5.4: Trajectories of aircraft lateral dynamics
Figure 5.5: Trajectories of satellite angular velocities

Figure 5.6: Target PS matrix interval family
6.1 Extension of theory

In the control methodologies proposed in previous chapter, an ecosystem was mimicked by making an individual state comparable to an individual species. Therefore, the interactions and interconnections were between various states of the given dynamic system. As an extension to the research reported here, another novel control design methodology is proposed, where, the basic philosophy is to mimic an ecosystem by making each subsystem of a system of subsystems comparable to an individual species. This implies that the interactions and interconnections are between subsystems. A pictorial depiction of a system built by such interacting subsystems is given in Figure 6.1. The scalar description of nature of interactions/interconnections in the previous control design method cannot be applied here since the interactions/interconnections are now in the form of matrices. Therefore, in order to extend the stability concepts of ecosystems to such systems, we first need to define the nature of interactions/interconnections in matrix notation.

The notion of positivity/negativity of a scalar quantity, when extended to a matrix, transforms to definiteness of the matrix. Therefore, a predator prey interaction between two sub systems means that one interaction matrix is negative definite while the other is positive definite. Similarly, a self-regulating species, that is, a negative
diagonal element in scalar dimension implies that the block diagonal submatrix is symmetric negative definite and a zero element corresponds to a zero matrix. Having defined the ecosystem principles in matrix notation, we proceed to design of the controller.

Control Methodology

Consider linear dynamic systems in time domain with state space description. A control methodology for cooperative stabilization of a group of subsystems is presented below

\[
\dot{x}_i = A_{ii}x_i + B_{ii}u_i
\]  

(6.1.1)

<table>
<thead>
<tr>
<th>Subsystem 1 (SS₁)</th>
<th>Subsystem 2 (SS₂)</th>
<th>Subsystem 3 (SS₃)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS₁ ←→ SS₁</td>
<td>SS₁ ←→ SS₂</td>
<td>SS₁ ←→ SS₃</td>
</tr>
<tr>
<td>SS₂ ←→ SS₁</td>
<td>SS₂ ←→ SS₂</td>
<td>SS₂ ←→ SS₃</td>
</tr>
<tr>
<td>SS₃ ←→ SS₁</td>
<td>SS₃ ←→ SS₂</td>
<td>SS₃ ←→ SS₃</td>
</tr>
</tbody>
</table>

Figure 6.1: Representation of large system as a set of subsystems
For example, if there are three subsystems, then, the A and B matrices in state space representation of the entire system will be

\[
A = \begin{bmatrix}
A_{11} & 0 & 0 \\
0 & A_{22} & 0 \\
0 & 0 & A_{33}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & 0 & 0 \\
0 & B_{22} & 0 \\
0 & 0 & B_{33}
\end{bmatrix}
\]  \hspace{1cm} (6.1.2)

We assume full state feedback where \( u = Gx \) so that the closed loop system matrix is given by

\[ A_{cl} = A + BG \]

The philosophy of the control design method is to build interactions between subsystems such that the desired objective, which, in this case is stabilization of the entire system, is achieved. In the subsystem representation, the closed loop system matrix \( A_{cl} \) can be expressed as shown in Figure 6.2 Note that the off-block-diagonal submatrices, which are not present in the open loop matrix, form the cooperative control and are also the design variables. In order to apply the laws of ecological systems, we form the definiteness matrix where each element represents the definiteness of the corresponding submatrix. For example, if \( D \) is the definiteness matrix of \( A \), \( d_{11} \) is the definiteness of submatrix \( A_{cl11} \)

![Figure 6.2: Closed loop system matrix of a system of subsystems](image)
The interactions and interconnections are designed based on the ecological principles discussed in previous section. In the ideal case of sign stability, all diagonal elements must be negative, all non-zero interactions must be predator-prey interactions and interconnections must be zero. Additionally, in [86], [98], [99] and [100], a specific class of matrices was identified to be most desirable closed loop system matrices. In order to reproduce these desirable qualities for the large system, we require that the definiteness matrix be sign stable and also satisfy the conditions presented in [86]. If the definiteness matrix of the closed loop system satisfies the conditions, the closed loop system matrix will have the following structure.

\[
D = \begin{bmatrix}
  d_{11} & d_{12} & d_{13} \\
  d_{21} & d_{22} & d_{23} \\
  d_{31} & d_{32} & d_{33}
\end{bmatrix}
\]

The interactions and interconnections are designed based on the ecological principles discussed in previous section. In the ideal case of sign stability, all diagonal elements must be negative, all non-zero interactions must be predator-prey interactions and interconnections must be zero. Additionally, in [86], [?], [98], [99] and [100], a specific class of matrices was identified to be most desirable closed loop system matrices. In order to reproduce these desirable qualities for the large system, we require that the definiteness matrix be sign stable and also satisfy the conditions presented in [86]. If the definiteness matrix of the closed loop system satisfies the conditions, the closed loop system matrix will have the following structure.

\[
A_{cl} = \begin{bmatrix}
  -A_{cl11} & A_{cl12} & 0 \\
  -A_{cl21} & -A_{cl11} & A_{cl23} \\
  0 & -A_{cl23} & -A_{cl11}
\end{bmatrix}
\]

where \( A_{cl11}, A_{cl12} \) and \( A_{cl23} \) are symmetric positive definite matrices.

Clearly, there is ample flexibility in the choice of these matrices as the matrix elements are assumed to be the design variables. The off-block diagonal sub-matrices form the cooperative control structure, and such flexibility implies that choice of a controller could be based on a specific design requirement.

### 6.2 Potential applications

#### 6.2.1 Cooperative control of multi-robot systems

One of the primary concerns in developing a multi-robot system is the design of a controller that achieves perfect coordination between individual robots. Though
multi-robot systems consist of relatively simple and inexpensive robots, given the complexity and large range of tasks assigned, coordination is critical to successful execution. There are numerous approaches to solving this problem of which, approaches inspired by biology and sociology such as behavior-based robotics, collective robotics and evolutionary robotics are of particular interest to us. The reason these approaches have gained popularity over the traditional AI methods is because they apply some simple rules from biological societies especially ants, bees and birds in the development of algorithms for cooperative control of multi-robot systems. Implementing these algorithms, the cooperative multi-robot systems demonstrate behaviors such as flocking dispersing, aggregating, foraging and following trail. For instance, swarm robotics is a field with successful implementation of biological principles [102], [103], [104], [105] and [106]. The advantage of these bioinspired methods is that they are much simpler and more effective to implement. In the same vein, we propose a new control methodology for multi-robot systems inspired by ecological societies. The key difference between biological and ecological societies is that a biological society is inhabited by a single species such as ants, bees, birds etc, while an ecological society or ecosystem is inhabited by several distinct species. This means that all members of the interacting community are need not be of the same kind. In application to multi-robot systems, this could mean that the generated algorithm is easily applicable to systems with non-similar robots as well. The essence of the proposed control design is to build a closed loop system that mimics a stable ecosystem. This is achieved by modeling individual robots as individual species and building the controller along the lines of a stably interacting ecosystem.

Reference [107] clearly mentions that the view that animal behavior is best described as a number of interacting innate motor patterns has inspired the presently popular approaches to multi-robot control [108]. Therefore, it is evident that interactions
play an important role in bio-inspired control design methods. References such as [109] cite biological systems as inspiration or justification for evolutionary robotics of the simplicity of the methods. Reference[110] that involved population biology and ecological modeling discussed the dynamics of resultant ecosystem and how its long-term behavior depends on the interactions among the constituent entities which can be associated with concepts of stability discussed in the previous section.

In [111], cooperative behavior of large-scale systems was demonstrated based on the dynamics of ecosystems. This correlation of ecological principles and ecological systems to cooperative robotics provides ample motivation to pursue ecologically inspired cooperative control of multi-robot systems. Preliminary results in this application have been published in [101]

### 6.2.2 Aircraft turbine engine control [96]

Turbine Engine modeling, simulation, control, diagnostics, prognostics and health management are very active areas of research and development in many academic institutions and industries. The task of developing mathematical models for turbine engine dynamics has received considerable attention from researchers. Traditionally, the turbine engine system is represented by a system of differential equations which are derived from first principles. These equations represent the functional relations that exist between various engine parameters, pressures, temperatures, rotor speeds and mass flow rates. The engine simulation model developed solves this set of differential equations to obtain output variables of interest such as rotor speeds and temperatures as a response to a set of control variables such as mass flow rates. Thus, the model contains a set of control variables which we can manipulate (such as mass flow rates, variable bleed valves and variable stator vanes) so that the set of state variables (such as rotor speeds, temperatures, pressures) can be affected which
in turn are related to output variables such as thrust. There are two main approaches for thermodynamic modeling of gas turbine engines: the transfer function (frequency domain based) approach and the nonlinear component models based approach (in time domain state space framework). Clearly there is preference given to the time domain state space based models over the transfer function approach because transfer function based models do not capture the component nonlinearities and other secondary effects. Thus in this research we emphasize the time domain state space modeling approach.

Given the availability of numerous methods for engine control, it is natural to question the need to break a system down into subsystems and analyze the interactions/interconnections to design a controller. In order to get deeper insight into interactions between subsystems and their significance, consider Figure 6.4. As depicted, it is known that the turbine runs the compressor. Therefore, for an engine

<table>
<thead>
<tr>
<th>Fan Dynamics</th>
<th>Coupling (SS1→SS2)</th>
<th>Coupling (SS1→SSi)</th>
<th>Coupling (SS1→SS2)</th>
<th>Coupling (SS1→SSi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compressor Dynamics</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
</tr>
<tr>
<td>Burner Dynamics</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
</tr>
<tr>
<td>Turbine Dynamics</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
</tr>
<tr>
<td>Augmentor Dynamics</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
<td>Coupling (SS1→SS2)</td>
</tr>
</tbody>
</table>

Figure 6.3: Aircraft turbine engine expressed as a system of subsystems
in steady state operation, we can say that the turbine has a positive effect on the compressor as it supplies energy to the compressor while the compressor, on the other hand, has a negative effect on the turbine as it utilizes the energy generated by the turbine. This mutual positive-negative effect is what is referred to as predator-prey interaction in ecology. Drawing inspiration from the manifestation of ecology based predator-prey interactions in the stabilization of a physical system, here, an aircraft turbine engine, we propose an eco-inspired control design methodology. In this methodology, we thoroughly exploit such features by retaining the stabilizing interactions (such as discussed above) and discarding/altering only the destabilizing interactions/interconnections. Since the control methodology aims at only nullifying the undesirable dynamics, we can say that ecologically inspired controllers work with the dynamics of the system rather than fight with it. Preliminary results in this application have been published in [97].

Figure 6.4: Predator-prey interaction between compressor and turbine
CHAPTER 7
CONCLUSIONS

The main objective of this work is to show that the robustness measures for real parameter perturbation are considerably improved if the ‘nominal system is taken (or driven) to be a ‘sign stable system. Efforts are made to highlight the role of ecological system principles such as sign stability and its implications in quantitative engineering system. In this research, some fundamental qualitative features of ecological sign stability are studied and these principles of ecology are transformed into to a set of mathematical results in matrix theory with quantitative information, which is usually encountered in engineering sciences are thoroughly reviewed. This type of cross fertilization of ideas from life sciences and engineering sciences is deemed to be highly beneficial to both fields. In particular, what effect the signs of elements of a matrix have on the matrix properties such as eigenvalues and condition number are shown. For example, new bounds on the real and imaginary parts of the eigenvalues of a sign stable matrix are derived. The novelty of these bounds is that they can simply be read off from the elements of the matrix with minimum calculation. These desirable features, along with the qualitative nature of stability motivated ecoinspired robust control. Towards this direction, sign assignability conditions are derived that guarantee the existence of a full state feedback controller for sign stability. Aided by these conditions, a control algorithm that yields a sign stable closed loop system is proposed. The proposed control design methodology is illustrated with applications
in satellite attitude control and aircraft flight control. The special nature of sign stable matrices is exploited to show that a certain class of ecological sign stable matrices is more robust that the general non ecological sign stable Hurwitz stable matrices. Similarly, it is also shown that under some assumptions on the magnitudes of the elements how predator-prey phenomenon in ecology renders some special properties like 'normality to matrices. The results presented in this research can assist in the use of ecological system principles to build highly robust engineering systems. Motivated by this observation, a new method for design of a robust controller for linear uncertain time invariant state space systems, subject to unstructured and structured uncertainty, using Ecological Sign stability approach is presented along with conditions under which such a controller exists. The robustness index is a norm bound for real parameter variation in linear systems. The distinctive feature of this method is that it specifically offers robustness guarantees to real parameter uncertainty. It was shown that sign stable matrices possess superior robustness properties and a class of matrices was identified to have the best robustness bound. With this motivation, a control design method to drive the closed loop system to such matrices was proposed. In the research reported here, exploiting the structure of these matrices, restriction on the structure of the closed loop matrices is relaxed thereby expanding the class of desirable closed loop system matrices. It is shown that equidiagonal pseudosymmetric matrices (target PS matrices) possess the property that the norm bound on real parameter perturbation can be determined without solving the Lyapunov equation. Since the bound can simply be read off the matrix, given a norm bound, we can very easily determine a corresponding stable matrix. Another notable feature of this method is that it is simple and straightforward and does not involve any intensive computation. Since the approach is remarkably distinct when compared to any of the conventional methods, we offer this as an addition to the bank of available control
design methods for linear state space systems. The resulting controller design method is illustrated with examples in the flight control area including aircraft lateral flight dynamics control and satellite attitude control. Since this closed loop system has superior robustness properties in addition to qualitative stability, future research in this direction involves further development of this control design method. Further research is being carried out on many other issues such as critical parameter selection in linear state space systems and trade off between nominal performance and robust stability in engineering systems. Research also involves extension of the algorithm in order to accommodate the discrepancies that may occur between the desired target PS bound and the bound generated by the actual closed loop system. In addition, existence of such controllers needs to be further investigated and if required, conditions for existence need to be developed. The presented examples clearly demonstrate how simple and how desirable a controller designed using the principle of qualitative stability is. Admittedly, there are certain limitations to this control design method, one of the most important being its applicability to a possibly small set of systems as the existence of controller comes with a few restrictions. But as mentioned by Doyle and coauthors in [112], if such a closed loop system is possible, manufacturing of the components for sign specific systems for control will be much easier than with precise absolute values. Clearly, control design methods based on principles of population ecology have superior robustness properties. Additionally, it is seen that there is a subtle correlation between dynamics of engineering systems to that of ecological systems which results in the ecological controller working along with the dynamics of the system rather than fight with it. Finally, versality of the control philosophy is brought out by detailed discussions on potential application in fields not restricted to Aerospace Engineering.
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