CLASSIFICATION OF COMPLETE REAL KÄHLER EUCLIDEAN SUBMANIFOLDS IN CODIMENSION THREE

DISSERTATION

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We show that if the second fundamental form $\alpha$ of a real Kähler Euclidean submanifold $f : M^{2n} \to \mathbb{R}^{2n+p}$ of codimension $p \geq 3$ splits orthogonally as $\alpha = \alpha' \oplus \gamma$, with image of $\gamma$ spans a rank 2 subbundle of the normal bundle and satisfies some symmetry with respect to the complex structure, then the submanifold can be extended to a real Kähler Euclidean submanifold $\tilde{f} : \tilde{M}^{2n+2} \to \mathbb{R}^{2n+p}$ with 2 higher real dimensions. Using this result, we can describe a class of codimension 3 real Kähler Euclidean submanifold that can be extended to a real Kähler hypersurface. In addition, in codimension 3, we describe some of the the non-minimal situations by showing that $f$ is a cylinder over a real Kähler curve, surface or threefold.
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CHAPTER 1
INTRODUCTION

It is well-known that any complete Riemannian manifold $M$ can be isometrically embedded in an Euclidean space [N56]. The interplay between the extrinsic and intrinsic geometry as well as the underlying topology of the manifold are extensively studied in the literature. Here we are particularly interested in the special case when $M$ is a complete Kähler manifold, namely, a complete Riemannian manifold $M^{2n}$ which is also a complex manifold of complex dimension $n$, such that both the Riemannian metric and the Levi-Civita connection compatible with the complex structure. Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be the isometric embedding. Ideally, one would hope that the target space is a complex Euclidean space and $f$ is a holomorphic map. However, from Calabi’s thesis [C53], it was known that such a case is very rare. In fact, Calabi gave a complete description of what kind of Kähler manifold can be holomorphically and isometrically immersed into a complex Euclidean space (or more generally a complex space form). Most Kähler manifolds do not admit such a map. So we have to study the hybrid case of $M^{2n}$ being Kähler while $f$ is just a smooth (isometric) map into a real Euclidean space. Following terminologies in this area, we will call such a manifold a real Kähler Euclidean submanifold, or simply a real Kähler submanifold.

In the joint work with L. Florit and F. Zheng [FHZ05], we observed that the second fundamental form of a real Kähler Euclidean submanifold obeys some additional symmetry conditions, and when the codimension $p$ is relatively small, or when the
curvature tensor of $M^{2n}$ is restrictive, the manifold tends to be rather special. Following that work, Florit and Zheng continued the study and obtained classification results for complete real Kähler submanifolds in codimension 1 and 2 [FZ05], [FZ07], [FZ08]. It turns out that in codimension 1, any complete real Kähler submanifold $f : M^{2n} \to \mathbb{R}^{2n+1}$ must be a cylinder. That is, $f = g \times I$, where $g : N^2 \to \mathbb{R}^3$ is a complete surface in Euclidean 3-space and $I$ is the identity map of $\mathbb{R}^{2n-2} \cong \mathbb{C}^{n-1}$.

For codimension 2, assuming $n > 1$, they showed that, either $M$ is minimal, or it is a cylinder over a $N^4 \to \mathbb{R}^6$. Also, for $N$, under very mild additional assumptions, it is actually the product of two surfaces in Euclidean 3-spaces.

So in short, complete real Kähler submanifolds in codimension 1 or 2 are rather special: they are essentially cylinders over products of surfaces in Euclidean three spaces, unless they are minimal. Note that by a result of Dajczer and Gromoll [DG85], any minimal isometric immersion from a Kähler manifold $M$ into an Euclidean space $\mathbb{R}^{2n+p}$ must be the real part of a holomorphic map from $M$ into $\mathbb{C}^{2n+p}$. When $p = 2$ and $n \geq 3$, a celebrated theorem of Dajczer and Gromoll [DG95] states that either $M$ is holomorphic under some (isometric) identification $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$, or $M$ is the total space of a holomorphic vector bundle over a Riemann surface, and $f$ embeds each fiber onto a linear subvariety in $\mathbb{R}^{2n+2}$. In the latter case, they gave a precise description in terms of a Weierstrass type representation of the Gauss map.

The goal of this thesis is to give a systematic study of complete real Kähler submanifolds of codimension 3. We will try to give a complete classification of such manifolds. Notice that when the codimension is increased to 3, aside from the usual arising algebraic complexity, a major difficulty emerge from the new phenomenon of composition. For instance, since there are plenty of isometric embeddings from an open subset of $\mathbb{R}^{2n+2}$ into $\mathbb{R}^{2n+3}$, its composition with any codimension 2 real Kähler submanifold would automatically give a real Kähler submanifold of codimension 3.
Another type of composition that may occur in the codimension 3 case would be as follows. Let $M^{2n} \subseteq N^{2(n+1)}$ be a holomorphic isometric embedding of Kähler manifolds of complex dimensions $n$ and $n + 1$, respectively. If $N$ can be a real Kähler submanifold of codimension 1, then $M$ would be a real Kähler submanifold of codimension 3. In order to classify the codimension 3 cases, we need to detect such compositions.
CHAPTER 2
PRELIMINARIES

Let \( f : M^{2n} \to \mathbb{R}^{2n+p} \) be a real Kähler Euclidean submanifold and let \( N \) be the normal bundle. For \( x \in M \), let \( \alpha \) be the second fundamental form of \( f \) at \( x \). Extend \( \alpha \) bilinearly over \( \mathbb{C} \) and still denote it by \( \alpha \),

\[
\alpha : (T_xM) \otimes \mathbb{C} \times (T_xM) \otimes \mathbb{C} \to N \otimes \mathbb{C}.
\]

Using the complex structure, \( J \), of \( M \), we get the decomposition \((T_xM) \otimes \mathbb{C} \cong V \oplus \overline{V},\) where \( V \) (the \((1,0)\)-part) is the eigenspace of \( J \) with eigenvalue \( i \) and \( \overline{V} \) (the \((0,1)\)-part) is the eigenspace of \( J \) with eigenvalue \(-i\). Write

\[
H = \alpha|_{V \times V} \quad \text{and} \quad S = \alpha|_{V \times V},
\]

then \( S : V \times V \to N \otimes \mathbb{C} \) is a symmetric complex bilinear map, while \( H : V \times \overline{V} \to N \otimes \mathbb{C} \) is a Hermitian bilinear map, that is,

\[
H(Y, \overline{X}) = \overline{H(X, \overline{Y})}, \quad \forall X, Y \in V.
\]

for the \((1,1)\) and \((2,0)\) parts of \( \alpha \), respectively. We will use the following notations, for any \( X, Y, e_i, e_j, \tilde{e}_i, \tilde{e}_j \in V, \)

\[
S_{XY} = S(X, Y), \quad S_{ij} = S_{e_i e_j}, \quad S_{ij} = S_{\tilde{e}_i \tilde{e}_j},
\]

\[
H_{XY} = H(X, \overline{Y}), \quad H_{\overline{ij}} = H_{\overline{e}_i \overline{e}_j}, \quad H_{\overline{ij}} = H_{\overline{\tilde{e}}_i \overline{\tilde{e}}_j},
\]

and similarly for \( \hat{e}_i \), etc.
Definition 2.1. We define the kernel of $H$, $\ker H$, and kernel of $S$, $\ker S$, as follows:

\[ \ker H = \{ X \in V : H_{XY} = 0, \forall Y \in V \} \]
\[ \ker S = \{ X \in V : S_{XY} = 0, \forall Y \in V \} \]

Definition 2.2. We define the image of $H$, $\text{img} H$, and image of $S$, $\text{img} S$, as follows:

\[ \text{img} H = \text{span}\{ X,Y \in V : H_{XY} \} \]
\[ \text{img} S = \text{span}\{ X,Y \in V : S_{XY} \} \]

We also extend the inner product $\langle , \rangle$ on $N$ bilinearly over $\mathbb{C}$ to $N \otimes \mathbb{C}$, and still denote it by $\langle , \rangle$. Then the Gauss equation:

\[ R_{ABCD} = \langle \alpha(A,D), \alpha(B,C) \rangle - \langle \alpha(A,C), \alpha(B,D) \rangle \]

also holds for any $A, B, C, D \in (T_xM) \otimes \mathbb{C}$. Since $M^{2n}$ is Kähler, we have $R_{XY**} = 0$ if both $X$ and $Y$ in $V$. As in [FHZ05], we have the following:

Proposition 2.3. Let $f : M^{2n} \to \mathbb{R}^{2n+p}$ be a real Kähler Euclidean submanifold. Fix a point $x \in M^{2n}$, and consider $V, H, S$ as above. Then for any $X,Y,Z,W \in V$, it holds that

\[ \langle H_{XW}, H_{YZ} \rangle = \langle H_{YW}, H_{XZ} \rangle \]
\[ \langle H_{XW}, S_{YZ} \rangle = \langle H_{YW}, S_{XZ} \rangle \]
\[ \langle S_{XW}, S_{YZ} \rangle = \langle S_{YW}, S_{XZ} \rangle \]
\[ R_{X_{YZ}W} = \langle H_{XW}, H_{YZ} \rangle - \langle S_{XZ}, S_{YW} \rangle \]

Suppose now $M$ is complete. Let $\Delta_x = \{ X \in T_xM : \alpha(X,T_xM) = 0 \}$ be the relative nullity of $f$ at $x$, and $\nu(x) = \dim_{\mathbb{R}} \Delta_x$ be the index of relative nullity. Let $\nu_0$ be the minimum value of $\nu(x)$ on $M$ and $U = \{ x \in M : \nu(x) = \nu_0 \}$ be an open subset
of $M$. It is well-known that $\Delta$ is smooth and integrable in $U$ with totally geodesic and complete leaves in both $M^{2n}$ and $\mathbb{R}^{2n+p}$.

Define the index of pluriharmonic nullity $\nu J = \nu J (x)$ of $f$ at $x \in M$ by

$$\nu J = \dim_{\mathbb{C}} \ker H.$$  

Then we define $D_x = \Delta_x \cap J \Delta_x$, where $J$ is the almost complex structure of $M$, so $D_x$ has a linear complex structure.

Let $W' = \ker H \cap \ker S$, so that $W'$ is a complex vector subspace of $V$. Then we have $D_x \otimes \mathbb{C} = W' \oplus \overline{W'}$ and write $\nu' (x) = \dim_{\mathbb{C}} W'$. Let $\nu'_0$ be the minimum value of $\nu' (x)$ on $U$, and so $U_0 = \{ x \in U : \nu' (x) = \nu'_0 \}$ is an open subset of $U$. Since $J$ is parallel, $D$ is also smooth and integrable in $U_0$ with totally geodesic and complete leaves and $D$ is called the complex relative nullity.

Let $r = n - \nu'_0$ and fix any $x \in U_0$. Considering the complexified tangent space $T_x M \otimes \mathbb{C}$, using $J$, we get the decomposition $T_x \otimes \mathbb{C} \cong V \oplus \overline{V}$, that is $V$ (the $(1,0)$ part) is the eigenspace of $J$ with eigenvalue $i$ and $\overline{V}$ (the $(0,1)$ part) is the eigenspace of $J$ with eigenvalue $-i$. Let $W \cong \mathbb{C}^r$ be the complex linear subspace of $V$ perpendicular to $D_x$, that is $W \oplus \overline{W} = D_x^\perp \otimes \mathbb{C}$ and we have $V = W \oplus W'$.

Let $C : D \times D^\perp \to D^\perp$ be the twisting tensor defined by $C T X = - (\nabla_X T)_{D^\perp}$ where $\nabla$ is the Levi-Civita connection of $M$ and $(\cdot)_{D^\perp}$ is the orthogonal projection onto $D^\perp$. Fix $T \in D$, for a basis $\mathcal{B} = \{ e_1, \ldots, e_r \}$ of $W$, we write the complexified operator $C_T$ as

$$C_T (e_i) = \sum_{j=1}^r (A_{ij} e_j + B_{ij} \overline{e_j}).$$

With respect to the basis $\mathcal{B}$, we have $A$ and $B$ are $r \times r$ complex matrices, while $H$ and $S$ are Hermitian and complex symmetric matrices, respectively, with values in the complexification of the normal space of $f$ at $x$. Then we have the following two main results from [FZ08] (Theorem 7 and Corollary 8):
Theorem 2.4. For any complete real Kähler Euclidean submanifold $f : M^{2n} \to \mathbb{R}^{2n+p}$, the complex relative nullity $D$ in $U_0$ is a holomorphic foliation, that is $B = 0$.

Corollary 2.5. For any complete real Kähler Euclidean submanifold $f : M^{2n} \to \mathbb{R}^{2n+p}$, for any $T \in D$, $A$ is nilpotent, $AS$ is symmetric, and $AH = 0$. 
CHAPTER 3
MAIN RESULTS

3.1 Symmetric complex bilinear map, $S$, with $\text{dim}_C \text{img} S \leq 2$

In this section we will study the symmetric complex bilinear map with image dimension 2. And we will use these results in the later sections. Let $V$ be a complex vector space of complex dimension $n$ and $N_C$ be another complex vector space. Let $S: V \times V \to N_C$ be a symmetric complex bilinear map, that is,

$$S(X, Y) = S(Y, X), \quad \forall X, Y \in V,$$

and let $s$ be the complex dimension of $\text{img} S$. For $s = 1$, the diagonalization of symmetric bilinear form on complex vector space gives us,

**Lemma 3.1.** For $s = 1$, we have a basis $\{e_1, \ldots, e_n\}$ that diagonalizes $S$. i.e, there exists $k$, $1 \leq k \leq n$, such that $S_{11} = \cdots = S_{kk} \neq 0$, $S_{ii} = 0$ for $k + 1 \leq i \leq n$ and $S_{ij} = 0$ for all $i \neq j$.

Next, we will study the case when $s = 2$. For the special case $n = 2$, we have the following:

**Lemma 3.2.** For $n = 2$ and $\text{dim} \text{img} S = 2$, there exists a basis $\{e_1, e_2\}$ for $V$ such that we have one of the following two cases:

1. $\{S_{11}, S_{22}\}$ forms a basis for $\text{img} S$ and $S_{12} = 0$. 

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(2) \(\{S_{11}, S_{12}\}\) forms a basis for \(\text{img} S\) and \(S_{22} = 0\).

Proof. We pick any basis \(\{e_1, e_2\}\) for \(V\). The only case that is not on our list is all \(S_{11}, S_{12}\) and \(S_{22}\) are nonzero. First, since \(\dim \text{img} S = 2\), if \(\{S_{11}, S_{22}\}\) are linearly independent, then \(S_{12} = kS_{11} + lS_{22}\) for some \(k, l \in \mathbb{C}\), and WLOG, we can assume \(k \neq 0\). Then we consider the quadratic equation \(kx^2 + x + l = 0\), and let \(\alpha\) and \(\beta\) be the roots of it, so we have

\[
\alpha + \beta = -\frac{1}{k} \quad \text{and} \quad \alpha\beta = \frac{l}{k}.
\]

Next we consider the vectors \(\tilde{e}_1 = e_1 + \alpha e_2\) and \(\tilde{e}_2 = e_1 + \beta e_2\), then we have

\[
S_{\tilde{1}\tilde{2}} = S_{11} + (\alpha + \beta)S_{12} + \alpha\beta S_{22}
\]

\[
= [1 + k(\alpha + \beta)]S_{11} + [\alpha\beta + l(\alpha + \beta)]S_{22}
\]

\[
= 0
\]

If \(\tilde{e}_1 \neq \tilde{e}_2\), then they will form a basis for \(V\), and this happens iff \(\alpha \neq \beta\), which is equivalent to \(1 - 4kl \neq 0\). Compute \(S_{\tilde{1}\tilde{1}}\) and \(S_{\tilde{2}\tilde{2}}\):

\[
S_{\tilde{1}\tilde{1}} = (1 + 2k\alpha)S_{11} + (\alpha^2 + 2l\alpha)S_{22} \neq 0,
\]

\[
S_{\tilde{2}\tilde{2}} = (1 + 2k\beta)S_{11} + (\beta^2 + 2l\beta)S_{22} \neq 0,
\]

this reduces to our case (1). For \(1 - 4kl = 0\), we have \(\alpha = \beta = -\frac{1}{2k} = -2l\), \(\tilde{e}_1 = \tilde{e}_2\) and \(S_{\tilde{2}\tilde{2}} = 0\). Then we will consider the basis \(\{e_1, \tilde{e}_2\}\), and we get

\[
S_{11} \neq 0,
\]

\[
S_{\tilde{2}\tilde{2}} = 0,
\]

\[
S_{\tilde{1}\tilde{2}} = S_{11} + \alpha S_{12} = -\frac{1}{2}S_{11} - 2lS_{22} \neq 0,
\]

and this reduces to our case (2). Finally, if \(\{S_{11}, S_{22}\}\) are linearly dependent, then
we can rescale $e_2$ such that $S_{22} = S_{11}$ and $\{S_{11}, S_{12}\}$ must be linearly independent. Then we consider the new basis $\{\tilde{e}_1, \tilde{e}_2\}$ with $\tilde{e}_1 = e_1 + e_2$ and $\tilde{e}_2 = e_1 - e_2$, we have

\[
S_{11} = S_{11} + 2S_{12} + S_{22} = 2S_{11} + 2S_{12} \neq 0,
\]
\[
S_{12} = S_{11} - S_{22} = 0,
\]
\[
S_{22} = S_{11} - 2S_{12} + S_{22} = 2S_{11} - 2S_{12} \neq 0,
\]

and this reduces to our case (1). □

For the general case $n \geq 2$, we will have the following:

**Corollary 3.3.** For any $n \geq 2$, if $\dim \text{img}\ S = 2$, we have a pair of vectors $\{X, Y\}$ such that $\text{span}\{S_{XX}, S_{XY}, S_{YY}\} = \text{img}\ S$. Moreover, we can choose $\{X, Y\}$ such that one of the following occurs:

1. $\{S_{XX}, S_{YY}\}$ forms a basis for $\text{img}\ S$ and $S_{XY} = 0$,
2. $\{S_{XX}, S_{XY}\}$ forms a basis for $\text{img}\ S$ and $S_{YY} = 0$.

**Proof.** By Lemma 3.2, we just need to find a pair $\{X, Y\}$ such that $\text{span}\{S_{XX}, S_{XY}, S_{YY}\} = \text{img}\ S$. Takes any basis $\{e_1, \ldots, e_n\}$ for $V$, since $\dim \text{img}\ S = 2$, by renumbering the $e_i$’s, we can assume one of the following five cases happens:

1. $\text{span}\{S_{11}, S_{12}\} = \text{img}\ S$
2. $\text{span}\{S_{11}, S_{22}\} = \text{img}\ S$
3. $\text{span}\{S_{11}, S_{23}\} = \text{img}\ S$
4. $\text{span}\{S_{12}, S_{13}\} = \text{img}\ S$
5. $\text{span}\{S_{12}, S_{34}\} = \text{img}\ S$
In the first two cases, we are done. For the third case, we can assume \( S_{12}, S_{13}, S_{22} \) and \( S_{33} \) are all parallel to \( S_{11}, \) otherwise we are done. For the same reason, we can assume \( S_{22} \) and \( S_{33} \) are all parallel to \( S_{23}, \) this forces \( S_{22} = S_{33} = 0. \) Then the space \( \text{span}\{S_{22}, S_{23}, S_{33}\} \) is one dimensional, by the Lemma 3.1, we can diagonalize it and find \( Y = ae_2 + be_3 \) such that \( S_{YY} = S_{23}. \) Then, by taking \( X = e_1, \) we get our \( X \) and \( Y. \)

In the forth case, we can assume \( S_{11} \) and \( S_{22} \) are parallel to \( S_{12}, \) and \( S_{11} \) and \( S_{33} \) are parallel to \( S_{13}. \) So this reduces to \( S_{11} = 0. \) Now, if one of \( S_{22} \) or \( S_{33} \) nonzero, we can reduce to the third case. So we can now assume \( S_{11} = S_{22} = S_{33} = 0. \) Then consider \( X = e_1 + ae_2 \) and \( Y = e_3 \) with \( a \in \mathbb{C} \) to be chosen later. Then we have \( S_{XX} = 2aS_{12} \) and \( S_{XY} = S_{13} + aS_{23}. \) Now, we just choose \( a \neq 0 \) such that \( S_{13} \neq -aS_{23} \) and get our the \( X \) and \( Y. \)

In the fifth case, we can assume \( S_{11} \) and \( S_{22} \) are parallel to \( S_{12}, \) and \( S_{33} \) and \( S_{44} \) are parallel to \( S_{34}. \) Then the space \( \text{span}\{S_{11}, S_{12}, S_{22}\} \) is one dimensional, by Lemma 3.1, we can diagonalize it and find \( X = ae_1 + be_2 \) such that \( S_{XX} = S_{12}. \) Similarly, we can find \( Y = ce_3 + de_4 \) such that \( S_{YY} = S_{34}. \) And these are the \( X \) and \( Y \) we are looking for.

\[ \square \]

3.2 Symmetric bilinear map on real vector space with linear complex structure.

In this section, we shall study the symmetric bilinear map on a real vector space with linear complex structure. Let \( T \) be a real vector space of dimension \( n \) with linear complex structure \( J : T \to T, J^2 = -\text{id}_T. \) Let \( N \) be a real vector space of dimension \( p \) with an inner product \( \langle \cdot , \cdot \rangle. \) Let \( \alpha : T \times T \to N \) be a symmetric bilinear map. We
extend $\alpha$ bilinearly over $\mathbb{C}$, and still denote it by $\alpha$,

$$\alpha : (T \otimes \mathbb{C}) \times (T \otimes \mathbb{C}) \to N \otimes \mathbb{C}.$$ 

As in chapter 2, we get $V, \overline{V}, H, S$ and we will use similar notations for $H$ and $S$ here. Again, we also extend the inner product $\langle \cdot, \cdot \rangle$ on $N$ bilinearly over $\mathbb{C}$ to $N \otimes \mathbb{C}$, and still denote it by $\langle \cdot, \cdot \rangle$.

In the view of Proposition 2.3, we will mainly work with those $\alpha$ that satisfy the following symmetry:

**Definition 3.4.** The symmetric bilinear map $\alpha$ ($S$ and $H$) is called RKES-symmetric, if the corresponding $H$ and $S$ satisfies the following symmetries, for any $X, Y, Z, W \in V$

1. $\langle H_{XW}, H_{YZ} \rangle = \langle H_{YW}, H_{XZ} \rangle$  \hspace{1cm} (3.1)
2. $\langle H_{XW}, S_{YZ} \rangle = \langle H_{YW}, S_{XZ} \rangle$  \hspace{1cm} (3.2)
3. $\langle S_{XW}, S_{YZ} \rangle = \langle S_{YW}, S_{XZ} \rangle$  \hspace{1cm} (3.3)

We have the following key lemma, Lemma 7 in [FHZ05]:

**Lemma 3.5.** Suppose $\dim_{\mathbb{C}} \text{img} H = p' \leq n$ and $H$ satisfies the equation (3.1), then

1. there exists a basis $\{e_1, \ldots, e_n\}$ of $V$ such that $H_{ij} = 0$ if either $i \neq j$ or $i = j > p'$, and for $1 \leq i \leq p'$, $H_{ii} \neq 0$. Moreover, since $H_{ii} = \overline{H_{ii}}$, it is real and $\{w_i = H_{ii}/|H_{ii}| : 1 \leq i \leq p\}$ forms an orthonormal basis of a real vector subspace $\text{img} H \cap N$ of $N$.

2. if, in addition, $S$ and $H$ also satisfies the equation (3.2) and $\text{img} H = N \otimes \mathbb{C}$, then, for the same basis above, we have $S_{ij} = 0$ if either $i \neq j$ or $i = j > p'$, and for $1 \leq i \leq p'$, $S_{ii}$ collinear to $w_i$.

We have the following easy observation:
Corollary 3.6. Suppose $\dim_{\mathbb{C}} \text{img} H = p' \leq n$ and $H$ satisfies equation (3.1), then we always have the equality

$$\dim_{\mathbb{C}} \ker H = n - \dim_{\mathbb{C}} \text{img} H.$$ 

Also, from the symmetry (3.2), we have the following:

Lemma 3.7. Suppose $S$ and $H$ satisfy equation (3.2), then we have

$$S(\ker H, V) \perp \text{img} H.$$ 

In particular, we have $\dim_{\mathbb{C}} \text{img} S|_{\ker H \times \ker H} \leq p - \dim_{\mathbb{C}} \text{img} H$.

Proof. For any $X \in \ker H$ and $Y, Z, W \in V$, we have

$$\langle S_{XY}, HZW \rangle = \langle S_{ZY}, H_{XW} \rangle = \langle S_{ZY}, 0 \rangle = 0.$$ 

Next, we consider $\dim_{\mathbb{R}} N = p = 3$ and $H \neq 0$, we also assume $\alpha$ is RKES-symmetric. Our goal is to find a good basis for $V$ such that $S$ and $H$ can be described as simply as possible. Clearly, we have $\dim_{\mathbb{C}} \text{img} H \leq 3$. If $\dim_{\mathbb{C}} \text{img} H = 3$, then by Lemma 3.5, there exists a basis that diagonalizes $H$ and $S$. For $\dim_{\mathbb{C}} \text{img} H = 2$, by Lemma 3.7, we have $\dim_{\mathbb{C}} \text{img} S|_{\ker H \times \ker H} \leq 1$, and we will have the following two cases:

Lemma 3.8. Suppose $S$ and $H$ satisfy the RKES-symmetry. If $\dim_{\mathbb{C}} \text{img} H = 2$ and $\dim_{\mathbb{C}} \text{img} S|_{\ker H \times \ker H} = 0$, then there exist a basis $\{e_1, \ldots, e_n\}$ of $V$ and an orthonormal basis, $\{w_1, w_2, w_3\}$, of the real vector space $N$ such that, for some $a_1, a_3, b_2, b_3, c_3 \in \mathbb{C}$ with $a_3b_3 = c_3^2$, we have $H_{11} = w_1$, $H_{22} = w_2$, $S_{11} = a_1w_1 + a_3w_3$, $S_{22} = b_2w_2 + b_3w_3$, $S_{12} = c_3w_3$ and, for other $i, j$, $H_{ij} = 0$ and $S_{ij} = 0$. 

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Proof. The existence of a basis \( \{e_1, \ldots, e_n\} \) with \( \{e_3, \ldots, e_n\} \) a basis for \( \ker H \) and \( H_{11} = w_1, H_{22} = w_2 \) and \( H_{12} = 0 = H_{21} \) follows from Lemma 3.5. Take \( w_3 \in N \) such that \( \{w_1, w_2, w_3\} \) form an orthonormal basis of \( N \). Then

\[
\langle S_{11}, w_2 \rangle = \langle S_{11}, H_{22} \rangle = \langle S_{21}, H_{12} \rangle = 0
\]

\[
\langle S_{22}, w_1 \rangle = \langle S_{22}, H_{11} \rangle = \langle S_{12}, H_{21} \rangle = 0
\]

Thus, we have \( S_{11} = a_1 w_1 + a_3 w_3 \) and \( S_{22} = b_2 w_2 + b_3 w_3 \) for some \( a_1, a_3, b_2, b_3 \in \mathbb{C} \).

For all \( i, j \geq 3 \), we have \( S_{ij} = 0 \) and, from Lemma 3.7, \( S_{1i}, S_{2i} \in \text{span}\{w_3\} \). Fix any \( i \geq 3 \), then there exists \( k \in \mathbb{C} \) such that \( S_{1i} = kw_3 \). On the other hand, we have

\[
k^2 = \langle S_{1i}, S_{1i} \rangle = \langle S_{11}, S_{ii} \rangle = 0,
\]

so we have \( k = 0 \). Hence \( S_{1i} = 0 \) for any \( i \geq 3 \). Similarly, we have \( S_{2i} = 0 \) for any \( i \geq 3 \). Finally, since

\[
\langle S_{12}, H_{11} \rangle = \langle S_{11}, H_{22} \rangle = 0
\]

\[
\langle S_{12}, H_{22} \rangle = \langle S_{22}, H_{11} \rangle = 0,
\]

we have \( S_{12} = c_3 w_3 \) for some \( c_3 \in \mathbb{C} \) and

\[
c_3^2 = \langle S_{12}, S_{12} \rangle = \langle S_{11}, S_{22} \rangle = a_3 b_3.
\]

\[\square\]

Lemma 3.9. Suppose \( S \) and \( H \) satisfy the RKES-symmetry. If \( \dim_{\mathbb{C}} \text{img} H = 2 \) and \( \dim_{\mathbb{C}} \text{img} S|_{\ker H \times \ker H} = 1 \), then there exist a basis \( \{e_1, \ldots, e_n\} \) of \( V \) and an orthonormal basis, \( \{w_1, w_2, w_3\} \), of a real vector space \( N \) such that, for some \( a_1, b_2 \in \mathbb{C} \), we have \( H_{11} = w_1, H_{22} = w_2, S_{11} = a_1 w_1, S_{22} = b_2 w_2, S_{33} = w_3 \) and, for other \( i, j, H_{ij} = 0 \) and \( S_{ij} = 0 \).
Proof. Again we have a basis \( \{ e_1, \ldots, e_n \} \) with \( \{ e_3, \ldots, e_n \} \) a basis for \( \ker H \) such that \( H_{11} = w_1, H_{22} = w_2, H_{12} = 0 = H_{21} \). Take \( w_3 \in N \) such that \( \{ w_1, w_2, w_3 \} \) form an orthonormal basis of \( N \). By our assumption, we have \( \text{img} S|_{\ker H \times \ker H} = \text{span} \{ w_3 \} \), so by Lemma 3.1, we can find another basis for \( \ker H \) that diagonalizes \( S|_{\ker H \times \ker H} \). For simplicity, we still call this basis \( \{ e_3, \ldots, e_n \} \). By rescaling and renumbering, we can assume \( S_{33} = w_3 \). Now, by Lemma 3.7, for any \( i = 1, \ldots, n \) and \( j = 4, \ldots, n \), we have \( S_{ij} \perp \text{span} \{ w_1, w_2 \} \). But we also have
\[
\langle S_{ij}, w_3 \rangle = \langle S_{ij}, S_{33} \rangle = \langle S_{3j}, S_{i3} \rangle = 0,
\]
so it gives \( S_{ij} = 0 \). Next, fix any \( i = 1, 2 \), we have \( S_{i3} \perp \text{span} \{ w_1, w_2 \} \), so \( S_{i3} \in \text{span} \{ w_3 \} \) and we write \( S_{i3} = k_i w_3 = k_i S_{33} \) for some \( k_i \in \mathbb{C} \). Let \( \tilde{e}_1 = e_1 - k_1 e_3 \) and \( \tilde{e}_2 = e_2 - k_2 e_3 \), we have \( H_{11} = w_1, H_{22} = w_2 \) and \( H_{12} = 0 = H_{21} \). We also have
\[
S_{13} = S_{13} - k_1 S_{33} = 0
\]
\[
S_{23} = S_{23} - k_2 S_{33} = 0
\]
Similar to the argument in Lemma 3.8, we have \( S_{11} = a_1 w_1 + a_3 w_3, S_{22} = b_2 w_2 + b_3 w_3 \) and \( S_{12} = c_3 w_3 \) for some \( a_1, a_3, b_2, b_3 \in \mathbb{C} \). But, for \( i, j = 1, 2 \), we also have
\[
\langle S_{ij}, w_3 \rangle = \langle S_{ij}, S_{33} \rangle = \langle S_{3j}, S_{i3} \rangle = 0,
\]
thus we get \( a_3 = b_3 = c_3 = 0 \). Hence \( \{ \tilde{e}_1, \tilde{e}_2, e_3, \ldots, e_n \} \) is the basis we want. \( \square \)

Next, for \( \dim_{\mathbb{C}} \text{img} H = 1 \), by Lemma 3.7, we have \( \dim_{\mathbb{C}} \text{img} S|_{\ker H \times \ker H} \leq 2 \).

**Lemma 3.10.** Suppose \( S \) and \( H \) satisfy the RKES-symmetry. If \( \dim_{\mathbb{C}} \text{img} H = 1 \) and \( \dim_{\mathbb{C}} \text{img} S|_{\ker H \times \ker H} = 0 \), then there exist a basis \( \{ e_1, \ldots, e_n \} \) of \( V \) and an orthonormal basis, \( \{ w_1, w_2, w_3 \} \), of a real vector space \( N \) such that we have one of the following case:
(1) For some \(a_1, a_2, a_3 \in \mathbb{C}\) with \(b^2 \neq -1\), we have \(H_{11} = w_1\), \(S_{11} = a_1 w_1 + a_2 w_2 + a_3 w_3 + 3\) and, for other \(i, j\), \(H_{ij} = 0\) and \(S_{ij} = 0\).

(2) For some \(a_1, a_2, a_3 \in \mathbb{C}\) with \(b \neq i\), we have \(H_{11} = w_1\), \(S_{11} = a_1 w_1 + a_2 w_2 + a_3 w_3\), \(S_{12} = w_2 + iw_3\) and, for other \(i, j\), \(H_{ij} = 0\) and \(S_{ij} = 0\).

Proof. Since \(\dim_{\mathbb{C}} \text{img} H = 1\), we have a basis \(\{e_1, \ldots, e_n\}\) with \(\{e_2, \ldots, e_n\}\) a basis for \(\ker H\) such that \(H_{11} = w_1\). Take \(w_2, w_3 \in N\) such that \(\{w_1, w_2, w_3\}\) form an orthonormal basis of \(N\). By our assumption \(S|_{\ker H \times \ker H} = 0\), we have \(S_{ij} = 0\) for all \(i, j \geq 2\). By Lemma 3.7, we have \(S_{1j} \in \text{span}\{w_2, w_3\}\) for all \(j \geq 2\), we write \(S_{1j} = a_j w_2 + b_j w_3\) with \(a_j, b_j \in \mathbb{C}\). If \(S_{1j} = 0\) for all \(j \geq 2\), then we get case (1).

Suppose the collection \(\{S_{1j} : j \geq 2\}\) are not all zero, by renumbering, we can assume \(S_{12} \neq 0\). Then we have

\[
a_2^2 + b_2^2 = \langle S_{12}, S_{12} \rangle = \langle S_{11}, S_{22} \rangle = 0,
\]

so we get either \(b_2 = ia_2\) or \(b_2 = -ia_2\). Replacing \(w_3\) by \(-w_3\) if needed, we can assume \(b_2 = ia_2\) and \(S_{12} = a_2 (w_2 + iw_3)\) with \(a_2 \neq 0\). Rescaling \(e_2\) by the factor \(a_2^{-1}\) and still denote it \(e_2\) for simplicity, we have \(S_{12} = w_2 + iw_3\). Now, for any \(j \geq 3\), we have \(S_{1j} = a_j w_2 + b_j w_3\) and

\[
a_j + ib_j = \langle S_{12}, S_{1j} \rangle = \langle S_{11}, S_{2j} \rangle = 0,
\]

so we have \(b_j = ia_j\) and \(S_{1j} = a_j (w_2 + iw_3) = a_j S_{12}\). Let \(\tilde{e}_j = e_j - a_j e_2\) for \(j \geq 3\), we have \(S_{1j} = S_{1j} - a_j S_{12} = 0\), and this gives us case (2).

\[\square\]

Lemma 3.11. Suppose \(S\) and \(H\) satisfy the RKES-symmetry. If \(\dim_{\mathbb{C}} \text{img} H = 1\) and \(\dim_{\mathbb{C}} \text{img} S|_{\ker H \times \ker H} = 2\), then there exist a basis \(\{e_1, \ldots, e_n\}\) of \(V\) and an orthonormal basis, \(\{w_1, w_2, w_3\}\), of a real vector space \(N\) such that we have one of the following case:
(1) for some $a_1 \in \mathbb{C}$, we have $H_{11} = w_1$, $S_{11} = a_1 w_1$, $S_{22} = \cdots = S_{kk} = S_{1k+1} = w_2 + iw_3$ for some $k$ with $2 \leq k \leq n - 1$ and, for other $i, j$, $H_{ij} = 0$ and $S_{ij} = 0$.

(2) for some $a_1, b \in \mathbb{C}$, we have $H_{11} = w_1$, $S_{11} = a_1 w_1 + b(w_2 + iw_3)$, $S_{22} = \cdots = S_{kk} = w_2 + iw_3$ for some $k$ with $2 \leq k \leq n$ and, for other $i, j$, $H_{ij} = 0$ and $S_{ij} = 0$.

(3) for some $a_1, a_2, a_3, b, c \in \mathbb{C}$ with $a_2 + a_3 b = c^2$, we have $H_{11} = w_1$, $S_{11} = a_1 w_1 + a_2 w_2 + a_3 w_3$, $S_{12} = c w_3$, $S_{22} = w_2 + bw_3$ and, for other $i, j$, $H_{ij} = 0$ and $S_{ij} = 0$.

Proof. Again we have a basis $\{e_1, \ldots, e_n\}$ with $\{e_2, \ldots, e_n\}$ a basis for $\ker H$ such that $H_{11} = w_1$. Take $w_2, w_3 \in N$ such that $\{w_1, w_2, w_3\}$ form an orthonormal basis of $N$. By our assumption and Lemma 3.1, $S|_{\ker H \times \ker H}$ can be diagonalized, that is we can choose $\{e_2, \ldots, e_n\}$ such that $S_{22} = \cdots = S_{kk} \neq 0$ for some $k$ with $2 \leq k \leq n$ and for other $i, j \geq 2$, $S_{ij} = 0$.

Suppose $k \neq n - 1$ and the collection $\{S_{1j} : j \geq k + 1\}$ are not all zero, by renumbering, we can assume $S_{1k+1} \neq 0$ and, by Lemma 3.7, $S_{1k+1} \in \text{span}\{w_2, w_3\}$, then

$$\langle S_{1k+1}, S_{1k+1} \rangle = \langle S_{11}, S_{k+1k+1} \rangle = 0$$

implies $S_{1k+1} \in \text{span}\{w_2 + iw_3\}$ or $S_{1k+1} \in \text{span}\{w_2 - iw_3\}$. Replacing $w_3$ by $-w_3$ if needed and by rescaling $e_{k+1}$, we can assume $S_{1k+1} = w_2 + iw_3$. Then for any $j \geq 2$ and $j \neq k + 1$, we also have $S_{1j} \in \text{span}\{w_2, w_3\}$ and

$$\langle S_{1j}, S_{1k+1} \rangle = \langle S_{11}, S_{k+1j} \rangle = 0,$$

so there exists $a_j \in \mathbb{C}$ such that $S_{1j} = a_j S_{1k+1}$. Let $\tilde{e}_j = e_j - a_j e_{k+1}$. Then we have

$$S_{1j} = S_{1j} - a_j S_{1k+1} = 0$$
and, since \( S_{k+1j} = 0 \) and \( S_{k+1k+1} = 0 \), we have \( S_{ij} = S_{ij} \), for all \( i, j \geq 2 \). So, for simplicity, we still denote this new choice of basis for \( \ker H \) by \( \{e_2, \ldots, e_n\} \). Then for any \( 2 \leq j \leq k \), we have \( S_{jj} \in \text{span}\{w_2, w_3\} \) and

\[
\langle S_{jj}, S_{1k+1} \rangle = \langle S_{1j}, S_{j,k+1} \rangle = 0,
\]

so \( S_{jj} \in \text{span}\{w_2 + iw_3\} \) and, since \( S_{jj} \neq 0 \), by rescaling \( e_j \) we get \( S_{jj} = w_2 + iw_3 \).

Next, we consider \( S_{11} \), we have

\[
\langle S_{11}, S_{22} \rangle = \langle S_{12}, S_{12} \rangle = 0
\]

so \( S_{11} = a_1w_1 + b(w_2 + iw_3) \) for some \( a_1, b \in \mathbb{C} \). Let \( e_1 = e_1 - \frac{b}{2}e_{k+1} \), then, for any \( j \geq 2 \) we have

\[
S_{11} = S_{11} - bS_{1k+1} + \frac{b^2}{4}e_{k+1} = a_1w_1
\]

\[
S_{1j} = S_{1j} - \frac{b}{2}S_{k+1} = S_{1j},
\]

and we get case (1).

Next, suppose that the collection \( \{S_{jj} : j \geq k + 1\} \) are zero and \( 3 \leq k \leq n \). So, we have \( S_{33} = S_{22} \neq 0 \). But we also have \( S_{22} \in \text{span}\{w_2, w_3\} \) and

\[
\langle S_{33}, S_{22} \rangle = \langle S_{23}, S_{23} \rangle = 0,
\]

this implies, replace \( w_3 \) by \( -w_3 \) if needed, \( S_{22} = \cdots = S_{kk} = c(w_2 + iw_3) \) for some \( 0 \neq c \in \mathbb{C} \). After rescaling, we have \( S_{22} = \cdots = S_{kk} = w_2 + iw_3 \). Then for any \( 2 \leq j \leq k \), we can find \( 2 \leq l \leq k \) with \( l \neq j \) such that \( S_{ll} = w_2 + iw_3 \) and we have

\[
\langle S_{lj}, S_{ll} \rangle = \langle S_{ll}, S_{lj} \rangle = 0,
\]

thus \( S_{lj} = a_j(w_2 + iw_3) \) for some \( a_j \in \mathbb{C} \). Let \( \tilde{e}_1 = e_1 - \sum_{l=2}^{k} a_le_l \), we have

\[
S_{lj} = S_{lj} - \sum_{l=2}^{k} a_lS_{lj} = S_{lj} - a_jS_{jj} = 0.
\]
Also, we get
\[ \langle S_{11}, S_{22} \rangle = \langle S_{12}, S_{12} \rangle = 0, \]
this implies \( S_{11} = a_1 w_1 + b(w_2 + iw_3) \), and we get case (2).

Finally, the only case left is the collection \( \{S_{1j} : j \geq 3 \} \) are zero and \( k = 2 \). In this case, by switching \( w_2 \) and \( w_3 \) if needed and rescaling \( e_2 \), we get \( S_{22} = w_2 + bw_3 \) for some \( b \in \mathbb{C} \). Again we have \( S_{12} \in \text{span}\{w_2, w_3\} \), so \( S_{12} = c_2 w_2 + c_3 w_3 \). Let
\[ \tilde{e}_1 = e_1 - c_2 e_2, \]
we have
\[ S_{12} = S_{12} - c_2 S_{22} = (c_3 - c_2 b)w_3, \]
and we write \( c = c_3 - c_2 b \). Finally, we write \( S_{11} = a_1 w_1 + a_2 w_2 + a_3 w_3 \) and we have
\[ \langle S_{11}, S_{22} \rangle \langle S_{12}, S_{12} \rangle, \]
this implies \( a_2 + a_3 b = c^2 \) and we get case (3).

**Lemma 3.12.** Suppose \( S \) and \( H \) satisfy the RKES-symmetry. If \( \dim_{\mathbb{C}} \text{img} \ H = 1 \) and \( \dim_{\mathbb{C}} \text{img} \ S|_{\text{ker} \ H \times \text{ker} \ H} = 2 \), then there exist a basis \( \{e_1, \ldots, e_n\} \) of \( V \) and an orthonormal basis, \( \{w_1, w_2, w_3\} \), of a real vector space \( N \) such that we have one of the following cases:

1. for some \( a_1, b \in \mathbb{C} \) with \( b^2 \neq -1 \), we have \( H_{11} = w_1, S_{11} = a_1 w_1, S_{22} = w_2 + bw_3, S_{33} = -bw_2 + w_3 \) and, for other \( i, j \), \( H_{ij} = 0 \) and \( S_{ij} = 0 \).

2. for some \( a_1, b \in \mathbb{C} \) with \( b \neq i \), we have \( H_{11} = w_1, S_{11} = a_1 w_1, S_{22} = w_2 + bw_3, S_{23} = w_2 + iw_3 \) and, for other \( i, j \), \( H_{ij} = 0 \) and \( S_{ij} = 0 \).

**Proof.** Again we have a basis \( \{e_1, \ldots, e_n\} \) with \( \{e_2, \ldots, e_n\} \) a basis for \( \text{ker} \ H \) such that \( H_{11} = w_1 \). Take \( w_2, w_3 \in N \) such that \( \{w_2, w_3\} \) form an orthonormal basis of \( N \). By our assumption and Lemma 3.3, we can choose \( X, Y \in \text{ker} \ H \) with \( \text{span}\{S_{XX}, S_{XY}, S_{YY}\} = \text{span}\{w_2, w_3\} \). By choosing \( \{X + Y, Y\} \) instead of \( \{X, Y\} \)
if needed, we can always assume \( \text{span}\{S_{XX}, S_{XY}\} = \text{span}\{w_2, w_3\} \). Then we define \( S_X : \ker H \to N \) by \( S_X(Z) = S(X, Z) \), for any \( Z \in \ker H \). By our choice of \( X \) and Lemma 3.7, we have \( \text{img } S_X = \text{span}\{w_2, w_3\} \). Hence \( \ker S_X \) is of codimension 2 in \( \ker H \), and we get the decomposition \( \ker H = \text{span}\{X, Y\} \oplus \ker S_X \). Choose \( e_2 = X \), \( e_3 = Y \) and \( \{e_4, \ldots, e_n\} \) to be a basis for \( \ker S_X \). Then, for any any \( 1 \leq i \leq n \) and \( 4 \leq j \leq n \), we have

\[
\langle S_{ij}, H_{11} \rangle = \langle S_{i1}, H_{1j} \rangle = 0
\]
\[
\langle S_{ij}, S_{22} \rangle = \langle S_{i2}, S_{2j} \rangle = 0
\]
\[
\langle S_{ij}, S_{23} \rangle = \langle S_{i3}, S_{2j} \rangle = 0,
\]

since \( \text{span}\{H_{11}, S_{22}, S_{23}\} = N \), we must have \( S_{ij} = 0 \).

Now, we may reselect our basis for \( \text{span}\{e_2, e_3\} \), such that one of the cases in Lemma 3.3 occurs, and still call it \( \{e_2, e_3\} \) for simplicity. If we are in case (1) of Lemma 3.3, we have \( \text{span}\{S_{22}, S_{33}\} = \text{span}\{w_2, w_3\} \) and \( S_{23} = 0 \). By rescaling \( e_2 \) and \( e_3 \) and switching indexes if needed, we can assume \( S_{22} = w_2 + bw_3 \) and \( S_{33} = cw_2 + w_3 \). On the other hand, we have

\[
c + b = \langle S_{22}, S_{33} \rangle = \langle S_{23}, S_{33} \rangle = 0,
\]

so \( c = -b \), and since \( S_{22} = w_2 + bw_3 \) and \( S_{33} = -bw_2 + w_3 \) are linearly independent, we have \( b^2 \neq -1 \). We also have

\[
w_2 = \frac{S_{22} - bS_{33}}{1 + b^2}, \quad \quad w_3 = \frac{bS_{22} + S_{33}}{1 + b^2}.
\]

Then, we consider \( S_{12} \) and \( S_{13} \), by Lemma 3.7, we have \( S_{12}, S_{13} \in \text{span}\{w_2, w_3\} \) and we also have

\[
\langle S_{12}, S_{33} \rangle = \langle S_{13}, S_{23} \rangle = 0
\]
\[
\langle S_{13}, S_{22} \rangle = \langle S_{12}, S_{23} \rangle = 0,
\]
so, we get

\[ \langle S_{12}, w_2 \rangle = \frac{\langle S_{12}, S_{22} \rangle}{1 + b^2}, \quad \langle S_{12}, w_3 \rangle = \frac{b \langle S_{12}, S_{22} \rangle}{1 + b^2}, \]

\[ \langle S_{13}, w_2 \rangle = \frac{-b \langle S_{13}, S_{33} \rangle}{1 + b^2}, \quad \langle S_{13}, w_3 \rangle = \frac{\langle S_{13}, S_{33} \rangle}{1 + b^2}, \]

hence we have \( S_{12} = a_2 S_{22} \) and \( S_{13} = a_3 S_{33} \) for some \( a_2, a_3 \in \mathbb{C} \). Let \( \tilde{e}_1 = e_1 - a_2 e_2 - a_3 e_3 \), then we have

\[
S_{12} = S_{12} - a_2 S_{22} - a_3 S_{23} = 0
\]
\[
S_{13} = S_{13} - a_2 S_{23} - a_3 S_{33} = 0.
\]

Also, we have

\[
\langle S_{11}, S_{22} \rangle = \langle S_{12}, S_{12} \rangle = 0,
\]
\[
\langle S_{11}, S_{33} \rangle = \langle S_{13}, S_{13} \rangle = 0,
\]

this implies

\[
\langle S_{11}, w_2 \rangle = \langle S_{11}, w_3 \rangle = 0,
\]

and hence we get \( S_{11} = a_1 w_1 \) for some \( a_1 \in \mathbb{C} \). And this is our case (1).

Next, suppose we are in case (2) of Lemma 3.3, we have \( \text{span}\{S_{22}, S_{23}\} = \text{span}\{w_2, w_3\} \) and \( S_{33} = 0 \). By rescaling \( e_2 \) and \( e_3 \) and switching indexes if needed, we can assume \( S_{22} = w_2 + bw_3 \). Then we write \( S_{23} = cw_2 + dw_3 \). Note that

\[
c^2 + d^2 = \langle S_{23}, S_{23} \rangle = \langle S_{22}, S_{33} \rangle = 0,
\]

so \( d = \pm ic \) and, in particular, \( c \neq 0 \). By rescaling \( e_3 \) and replacing \( w_3 \) by \(-w_3\), \( b \) by \(-b\) if needed, we have \( S_{22} = w_2 + bw_3 \) and \( S_{23} = w_2 + iw_3 \). Since \( S_{22} \) and \( S_{23} \) are linearly independent, we must have \( b \neq i \), and

\[
w_2 = \frac{-i S_{22} + b S_{23}}{b - i}, \quad w_3 = \frac{S_{22} - S_{23}}{b - i}.
\]
Then, consider $S_{13}$, by Lemma 3.7, we have $S_{13} \in \text{span}\{w_2, w_3\}$ and

$$\langle S_{13}, S_{13} \rangle = \langle S_{11}, S_{33} \rangle = 0,$$

this implies $S_{13} = a_2(w_2 + iw_3) = a_2S_{23}$ for some $a_2 \in \mathbb{C}$. Let $\tilde{e}_1 = e_1 - a_2e_2$, we have

$$S_{13} = S_{13} - a_2S_{23} = 0.$$

Next, consider $S_{12}$, again we have $S_{12} \in \text{span}\{w_2, w_3\}$ and

$$\langle S_{12}, S_{23} \rangle = \langle S_{13}, S_{22} \rangle = 0,$$

this implies $S_{12} = a_3(w_2 + iw_3) = a_3S_{23}$ for some $a_3 \in \mathbb{C}$. Let $\hat{e}_1 = e_1 - a_3e_3$, then we have

$$S_{12} = S_{12} - a_3S_{32} = 0$$

$$S_{13} = S_{13} - a_3S_{33} = 0$$

Also, we have

$$\langle S_{11}, S_{22} \rangle = \langle S_{12}, S_{12} \rangle = 0,$$

$$\langle S_{11}, S_{23} \rangle = \langle S_{13}, S_{13} \rangle = 0,$$

this implies

$$\langle S_{11}, w_2 \rangle = \langle S_{11}, w_3 \rangle = 0,$$

and hence we get $S_{11} = a_1w_1$ for some $a_1 \in \mathbb{C}$. This is our case (2). \qed

### 3.3 Real Kähler Euclidean submanifolds of codimension 3

In this section, we will use the results in section 3.2 to give a classification of Real Kähler Euclidean submanifolds of codimension 3.

We have the following:
Theorem 3.13. Let \( f : M^{2n} \to \mathbb{R}^{2n+p}, \ n \geq p \geq 3, \) be a real Kähler Euclidean submanifold. Assume the second fundamental form splits orthogonally as \( \alpha = \alpha' \oplus \gamma \) with \( \gamma : TM \times TM \to L, \) where \( \text{img} \gamma = L \subset N \) is a vector subbundle of the normal bundle with rank 2, and for any \( X, Y \in TM, \) \( \gamma \) satisfies
\[
\gamma(JX,Y) = \gamma(X,JY),
\]
\[
\gamma(JX,Y) \perp \gamma(X,Y),
\]
\[
\|\gamma(JX,Y)\| = \|\gamma(X,Y)\|,
\]
Suppose there exist a smooth rank 2 vector subbundle \( \Gamma \subset TM \oplus L \) with \( \Gamma \cap TM = \{0\} \) so that
\[
\nabla_Z \mu \in TM \oplus L
\]
for all \( \mu \in \Gamma \) and \( Z \in TM. \) Then \( f \) extends uniquely to a real Kähler submanifold \( F : \tilde{M}^{2n+2} \to \mathbb{R}^{2n+p}. \)

Proof. The proof is similar to the proof of Theorem 1 in [DG97]. By assumption, we have
\[
\gamma(JX,Y) = \gamma(X,JY), \quad \forall X, Y \in TM,
\]
so, we can extend the complex structure \( J \) in \( TM \) to a complex structure, still denoted by \( J, \) in \( TM \oplus L \) by
\[
J(Z + \gamma(X,Y)) = JZ + \gamma(JX,Y),
\]
for any \( X, Y, Z \in TM \otimes \mathbb{C}. \) Now consider the total space of vector bundle \( \pi : \Gamma^{2n+2} \to M^{2n}, \) we define the map \( F : \Gamma^{2n+2} \to \mathbb{R}^{2n+3} \) by
\[
F(\xi) = f(\pi(\xi)) + \xi.
\]
Then \( F \) is an immersion in a neighborhood of the 0-section of \( \Gamma^{2n+2}. \) Now, it remains to prove that this is Kähler. First, we claim that, over \( M, \) the complex structure \( J \)
in $TM \oplus L$ is parallel with respect to the induced connection, $\nabla' = (\tilde{\nabla})_{TM \oplus L}$, where $(\nu)_{TM \oplus L}$ denotes the orthogonal projection onto $TM \oplus L$. Choose $X, Y \in TM$ such that
\[ \|\gamma(X, Y)\| = 1, \]
then $\gamma(X, Y)$ and $-\gamma(JX, Y)$ form an orthonormal basis for $L$, call them $\xi_1$ and $\xi_2$ respectively, and we have
\[ J\xi_1 = -\xi_2 \quad \text{and} \quad J\xi_2 = \xi_1. \]
For any $X, Z \in TM$,
\[ \nabla'_X JZ = \nabla_X JZ + \gamma(X, JZ) \]
\[ = J\nabla_X Z + J\gamma(X, Z) \]
\[ = J(\nabla'_X Z) \]
For the $L$ direction, fix any $X \in TM$, we have the following,
\[ \nabla'_X \xi_i = -A_{\xi_i}X + (\nabla^\perp_X \xi_i)_L \]
So, to prove $\xi_i$ are parallel, we need
\[ A_{J\xi_i}X = JA_{\xi_i}X \quad \text{and} \quad (\nabla^\perp_X J\xi_i)_L = J(\nabla^\perp_X \xi_i)_L. \]
We first compute $(\nabla^\perp_X \xi_i)_L$. Note that,
\[ \langle \nabla^\perp_X \xi_i, \xi_i \rangle = \frac{1}{2} X \langle \xi_i, \xi_i \rangle = 0 \]
and
\[ \langle \nabla^\perp_X \xi_1, \xi_2 \rangle = X \langle \xi_1, \xi_2 \rangle - \langle \nabla^\perp_X \xi_2, \xi_1 \rangle = -\langle \nabla^\perp_X \xi_2, \xi_1 \rangle \]
So, we have the following:

\[
(\nabla_X^\perp \xi_1)_L = \langle \nabla_X^\perp \xi_1, \xi_1 \rangle \xi_1 + \langle \nabla_X^\perp \xi_1, \xi_2 \rangle \xi_2
\]

\[
= \langle \nabla_X^\perp \xi_1, \xi_2 \rangle \xi_2
\]

\[
(\nabla_X^\perp \xi_2)_L = \langle \nabla_X^\perp \xi_2, \xi_1 \rangle \xi_1
\]

\[
= - \langle \nabla_X^\perp \xi_1, \xi_2 \rangle \xi_1
\]

These imply

\[
(\nabla_X^\perp J \xi_1)_L = (\nabla_X^\perp - \xi_2)_L
\]

\[
= \langle \nabla_X^\perp \xi_1, \xi_2 \rangle \xi_1
\]

\[
= \langle \nabla_X^\perp \xi_1, \xi_2 \rangle J \xi_2
\]

\[
= J(\langle \nabla_X^\perp \xi_1, \xi_2 \rangle \xi_2)
\]

\[
= J(\nabla_X^\perp \xi_1)_L
\]

Similarly, we will have

\[
(\nabla_X^\perp J \xi_2)_L = J(\nabla_X^\perp \xi_2)_L.
\]

We consider \( \xi_1 \), for any \( W \in TM \), write \( \gamma(X, W) = a_1 \xi_1 + a_2 \xi_2 \), hence \( \gamma X, JW = -a_1 \xi_2 + a_2 \xi_1 \), then

\[
\langle A_{J \xi_1} X - JA_{\xi_1} X, W \rangle = \langle A_{J \xi_1} X, W \rangle - \langle JA_{\xi_1} X, W \rangle
\]

\[
= \langle A_{-\xi_2} X, W \rangle - \langle A_{\xi_1} X, -JW \rangle
\]

\[
= - \langle \xi_2, \alpha(X, W) \rangle + \langle \xi_1, \alpha(X, JW) \rangle
\]

\[
= - \langle \xi_2, \gamma(X, W) \rangle + \langle \xi_1, \gamma(X, JW) \rangle
\]

\[
= -a_2 + a_2
\]

\[
= 0
\]
this proves $A_{J_1}X = JA_{\xi_1}X$. Similarly, we can get $A_{J_2}X = JA_{\xi_2}X$. This proves our claim. Next, note that, for any $\delta \in \Gamma$,

$$TF(\delta)F(\Gamma) = T_{\pi(\delta)}M \oplus L(\pi(\delta)),$$

which depends only on $\pi(\delta) \in M$. Thus, we have $TT = \pi^*(TM \oplus L)$ via parallel transport in $\mathbb{R}^{2n+p}$, and the Levi-Civita connection for $TT$ is the pullback of the connection of $TM \oplus L$. Also the parallel transport in $\mathbb{R}^{2n+p}$ also extends the complex structure $J$ on $TM \oplus L$ to the complex structure $\pi^*(J)$ on $\pi^*(TM \oplus L)$ which is clearly parallel. This proves that $F$ gives us a real Kähler submanifold. 

**Lemma 3.14.** Let $f : M^{2n} \to \mathbb{R}^{2n+3}$, be a real Kähler Euclidean submanifold, $n \geq 3$. Assume $\nu_J = n - 1$, we define pointwise

$$\Omega(x) = \text{span}\{\tilde{\nabla}_X \eta : X \in T_x M, \eta \in \text{img} \ H\}.$$

Let $\{e_1, \ldots, e_n\}$ be the smooth frame that gives the diagonalization in Lemma 3.5, and $H_1^{TM} = w_1$ be the unit normal vector field that spans $\text{img} \ H$. If $|\langle S_{11}, H_1^{TM} \rangle| \neq 1$, then $\text{dim} \ A_{w_1} = 2$ and $\Omega$ forms a rank 2 subbundle of $\text{img} \ A_{w_1} \oplus (\text{img} \ H)^\perp$.

**Proof.** We claim that, for any $X \in \ker A_{w_1}$, we have $\tilde{\nabla}_X w_1 = 0$. Let $X \in TM \otimes \mathbb{C}$, write

$$X = \sum_{j=1}^n a_j e_j + b_j \overline{e_j}.$$

We first compute $\nabla_{X}^\perp w_1$. Observe that

$$\langle \nabla_{X}^\perp w_1, w_1 \rangle = \frac{1}{2} X \langle w_1, w_1 \rangle = 0,$$

and, for any $Y, Z \in V$,

$$\nabla_{Z} Y \in V, \quad \nabla_{X} \overline{Y} \in \overline{V}, \quad \nabla_{Z} \overline{Y} \in V, \quad \nabla_{X} \overline{Y} \in \overline{V}.$$
Then for any $j \neq i$, using the Codazzi equation, we have

\[
\nabla_{e_j}^\perp w_1 = \nabla_{e_j}^\perp \alpha(e_1, e_1) = \alpha(e_1, e_1) + \alpha(e_1, \nabla_{e_j} e_1) - \alpha(e_1, e_1) - \alpha(e_1, \nabla_{e_j} e_1) = 0
\]

\[
\nabla_{e_j}^\perp X w_1 = 0
\]

Therefore,

\[
\nabla_X^\perp w_1 = a_1 \nabla_{e_1}^\perp w_1 + b_1 \nabla_{e_1}^\perp w_1
\]

Now, we have

\[
\langle A_{w_1} X, e_j \rangle = \langle w_1, \alpha(X, e_j) \rangle = \begin{cases} 0, & \text{for } j \neq 1 \\ a_1 s + b_1, & \text{for } j = 1 \end{cases}
\]

\[
\langle A_{w_1} X, \overline{e_j} \rangle = \langle w_1, \alpha(X, \overline{e_j}) \rangle = \begin{cases} 0, & \text{for } j \neq 1 \\ a_1 + b_1 \overline{s}, & \text{for } j = 1 \end{cases}
\]

where $s = \langle S_{ii}, H_{ii} \rangle$. Then $X \in \ker A_{w_1}$ will implies

\[
a_1 s + b_1 = 0
\]

\[
a_1 + b_1 \overline{s} = 0
\]

If $|s| \neq 1$, then $a_i = b_i = 0$ and $\nabla_X^\perp w_1 = 0$, for any $X \in \ker A_{w_1}$. This proves our claim. In fact, from the above, we also see that $\dim \ker A_{w_1} = 2n - 2$, hence $\text{img } A_{w_1}$ is of rank 2. Note that $\langle \nabla_X^\perp w_1, w_1 \rangle = 0$, $\nabla_X^\perp w_1 \subset (\text{img } H)^\perp$. So, we have $\Omega \subset \text{img } A_{w_1} \oplus (\text{img } H)^\perp$ is a rank 2 vector subbundle of $\text{img } A_{w_1} \oplus (\text{img } H)^\perp$.

Using this lemma, we have the following result:

**Theorem 3.15.** Let $f : M^{2n} \rightarrow \mathbb{R}^{2n+3}$, $n \geq 3$, be a real Kähler Euclidean submanifold that is in the case (1) or (2) of Lemma 3.11. Then, there exist an open dense
subset \( U \subset M \) such that \( f \) extends uniquely to a real Kähler hypersurface along each connected component of \( U \).

Proof. First, in the case (1) or (2), we claim that the smooth frame \( \{e_1, \ldots, e_n\} \) satisfies \( s = |\langle S_{11}, H_{11T} \rangle| \neq 1 \). Suppose not, then rescale \( e_1 \) by a factor of \( t \) with \( t^2 = s^{-1} \), we can assume \( s = 1 \). Write \( e_1 = v_1 - iJv_1 \), then we have

\[
\begin{align*}
S_{11} &= \alpha(v_1, v_1) - \alpha(Jv_1, Jv_1) - 2i\alpha(v_1, Jv_1) \\
H_{1T} &= \alpha(v_1, v_1) + \alpha(Jv_1, Jv_1)
\end{align*}
\]

since \( s = 1 \), we must have

\[
\begin{align*}
\alpha(v_1, v_1) &= H_{1T}, \\
\alpha(Jv_1, Jv_1) &= 0, \\
\alpha(v_1, Jv_1) &\in (\text{img } H)^{\perp}
\end{align*}
\]

which also means

\[
S_{11} - H_{11T} = 2i\alpha(v_1, Jv_1)
\]

Now, replacing \( e_1 \) by \( \tilde{e}_1 = e_1 + e_{k+1} \) in case (1) or \( \tilde{e}_1 = e_1 + e_2 \) in case (2) if needed, the new frame \( \{\tilde{e}_1, e_2, \ldots, e_n\} \) still satisfies

\[
|\langle S_{11}, H_{11T} \rangle| = 1
\]

and

\[
S_{11} - H_{11T} \neq 0.
\]

Write \( \tilde{e}_1 = \tilde{v}_1 - iJ\tilde{v}_1 \), and we get

\[
2i\alpha(\tilde{v}_1, J\tilde{v}_1) \neq 0
\]

On the other hand, we have

\[
S_{11} - H_{11T} \in \text{span}\{w_2 + iw_3\},
\]

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which leads to a contradiction. So, we must have $s = |\langle S_{11}, H_{1T} \rangle| \neq 1$.

Now, by Lemma 3.14, we have

$$\Omega = \text{span}\{ \nabla_X \eta : X \in TM, \eta \in \text{img} H \}$$

is a rank 2 subbundle of $\text{img} A_{w_1} \oplus (\text{img} H)\perp$. Denote $(\text{img} H)\perp$ by $L$. By assumption on $H$, this is a rank 2 subbundle of the normal bundle. And we can define $\gamma : TM \times TM \to L$ as follows, for any $X, Y \in TM$,

$$\gamma(X, Y) = (\alpha(X, Y))_L.$$

So the second fundamental form splits orthogonally as:

$$\alpha(X, Y) = (A_{w_1} X, Y) w_1 \oplus \gamma(X, Y).$$

We claim that $\gamma$ satisfies the condition in Theorem 3.13. For any $X, Y \in TM$, let $Z = X - iJX \in V$ and $W = Y - iJY \in V$. By our assumption, we have

$$(H_{Z\overline{W}})_L = 0 \quad \text{and} \quad \langle (S_{ZW})_L, (S_{ZW})_L \rangle = 0.$$

On the other hand, we have

$$(H_{Z\overline{W}})_L = \gamma(X - iJX, Y + iJY) = \gamma(X, Y) + \gamma(JX, JY) + i(\gamma(X, JY) - \gamma(JX, Y)),$$

this implies

$$\gamma(X, Y) = -\gamma(JX, JY) \quad \text{and} \quad \gamma(JX, Y) = \gamma(X, JY).$$

Also, we have

$$(S_{ZW})_L = \gamma(X - iJX, Y - iJY)$$

$$= \gamma(X, Y) - \gamma(JX, JY) - i(\gamma(X, JY) + \gamma(JX, Y))$$

$$= 2(\gamma(X, Y) - i\gamma(JX, Y)).$$
\begin{align*}
\frac{1}{4} \langle (S_{ZW})_L, (S_{ZW})_L \rangle &= \langle \gamma(X, Y) - i\gamma(JX, Y), \gamma(X, Y) - i\gamma(JX, Y) \rangle \\
&= \|\gamma(X, Y)\|^2 - \|\gamma(JX, Y)\|^2 - 2i \langle \gamma(X, Y), \gamma(JX, Y) \rangle.
\end{align*}

This implies

\[ \|\gamma(X, Y)\| = \|\gamma(JX, Y)\| \quad \text{and} \quad \gamma(X, Y) \perp \gamma(JX, Y), \]

and proves our claim. Now, let \( \Gamma \) be its orthogonal complement in \( \text{img} \, A_{w_1} \oplus L \), which can be viewed as a rank 2 subbundle of \( TM \oplus L \) and \( \Gamma \cap TM = \{0\} \). In the view of Theorem 3.13, we just need to show, for all \( \mu \in \Gamma \) and \( Z \in TM \),

\[ \tilde{\nabla}_Z \mu \in TM \oplus L, \]

By viewing \( TM \oplus L \) as a subbundle of \( TM \oplus N \), this is equivalent to the statement

\[ \tilde{\nabla}_Z \mu \perp w_1. \]

Next, we know that \( \mu \in \Gamma \subset TM \oplus L \), so

\[ \langle \mu, w_1 \rangle = 0. \]

This implies

\[ \left\langle \tilde{\nabla}_Z \mu, w_1 \right\rangle + \langle \mu, \tilde{\nabla}_Z w_1 \rangle = Z \langle \mu, w_1 \rangle = 0. \]

But from the construction of \( \Gamma \), we have

\[ \mu \perp \tilde{\nabla}_Z w_1, \]

hence

\[ \left\langle \tilde{\nabla}_Z \mu, w_1 \right\rangle = -\left\langle \mu, \tilde{\nabla}_Z w_1 \right\rangle = 0. \]

Finally, apply Theorem 3.13, we get the Kähler extension of \( f \). \( \square \)
For the following cases, we get cylinders:

**Theorem 3.16.** Let \( f: M^{2n} \to \mathbb{R}^{2n+3}, n \geq 3 \), be a real Kähler Euclidean submanifold with \( \nu_J = n - 3 \). Then, each connected component \( U_i \) of \( U_0 \) is isometric to a product \( U_i = N^6 \times \mathbb{C}^{n-3} \) and \( f|_{U_i} = f' \times \text{id} \) splits, where \( f': N^6 \to \mathbb{R}^9 \) is a real Kähler Euclidean submanifold and \( \text{id}: \mathbb{C}^{n-3} \cong \mathbb{R}^{2(n-3)} \) is the identity map.

**Theorem 3.17.** Let \( f: M^{2n} \to \mathbb{R}^{2n+3}, n \geq 3 \), be a real Kähler Euclidean submanifold with \( \nu_J = n - 2 \) and suppose \( S|_{\ker H \times \ker H} = 0 \). Then, each connected component \( U_i \) of \( U_0 \) is isometric to a product \( U_i = N^4 \times \mathbb{C}^{n-2} \) and \( f|_{U_i} = f' \times \text{id} \) splits, where \( f': N^4 \to \mathbb{R}^7 \) is a real Kähler Euclidean submanifold and \( \text{id}: \mathbb{C}^{n-2} \cong \mathbb{R}^{2(n-2)} \) is the identity map.

**Theorem 3.18.** Let \( f: M^{2n} \to \mathbb{R}^{2n+3}, n \geq 3 \), be a real Kähler Euclidean submanifold with \( \nu_J = n - 2 \) and suppose \( S|_{\ker H \times \ker H} \neq 0 \). Then, each connected component \( U_i \) of \( U_0 \) is isometric to a product \( U_i = N^6 \times \mathbb{C}^{n-3} \) and \( f|_{U_i} = f' \times \text{id} \) splits, where \( f': N^6 \to \mathbb{R}^9 \) is a real Kähler Euclidean submanifold and \( \text{id}: \mathbb{C}^{n-3} \cong \mathbb{R}^{2(n-3)} \) is the identity map.

The above cases have been observed in [FZ08]. And here are the new cases:

**Theorem 3.19.** Let \( f: M^{2n} \to \mathbb{R}^{2n+3}, n \geq 3 \), be a real Kähler Euclidean submanifold that is in the case (1) of Lemma 3.10. Then, each connected component \( U_i \) of \( U_0 \) is isometric to a product \( U_i = N^2 \times \mathbb{C}^{n-1} \) and \( f|_{U_i} = f' \times \text{id} \) splits, where \( f': N^2 \to \mathbb{R}^5 \) is a real Kähler Euclidean submanifold and \( \text{id}: \mathbb{C}^{n-1} \cong \mathbb{R}^{2(n-1)} \) is the identity map.

*Proof.* In this case, under the basis given by Lemma 3.10, \( A \) is just a 1 by 1 matrix and \( AH = 0 \) implies \( A = 0 \).

**Theorem 3.20.** Let \( f: M^{2n} \to \mathbb{R}^{2n+3}, n \geq 3 \), be a real Kähler Euclidean submanifold that is in the case (3) of Lemma 3.11. Then, each connected component \( U_i \) of \( U_0 \) is
isometric to a product \( U_i = N^4 \times \mathbb{C}^{n-2} \) and \( f|_{U_i} = f' \times \text{id} \) splits, where \( f' : N^4 \to \mathbb{R}^7 \) is a real Kähler Euclidean submanifold and \( \text{id} : \mathbb{C}^{n-2} \cong \mathbb{R}^{2(n-2)} \) is the identity map.

**Proof.** In this case, under the basis given by Lemma 3.11, \( A \) is a 2 by 2 matrix and \( AH = 0 \) implies

\[
A = \begin{pmatrix}
0 & \alpha \\
0 & \beta
\end{pmatrix}.
\]

Also, \( A \) is nilpotent implies \( \beta = 0 \). Finally, we have

\[
AS_2 = \begin{pmatrix}
0 & \alpha \\
0 & 0
\end{pmatrix} \begin{pmatrix}
a_2 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & \alpha \\
0 & 0
\end{pmatrix}.
\]

Note that it is symmetric, \( \alpha = 0 \) and hence \( A = 0 \).

**Theorem 3.21.** Let \( f : M^{2n} \to \mathbb{R}^{2n+3}, n \geq 3, \) be a real Kähler Euclidean submanifold that is in the case (1) of Lemma 3.12. Then, each connected component \( U_i \) of \( U_0 \) is isometric to a product \( U_i = N^6 \times \mathbb{C}^{n-3} \) and \( f|_{U_i} = f' \times \text{id} \) splits, where \( f' : N^6 \to \mathbb{R}^9 \) is a real Kähler Euclidean submanifold and \( \text{id} : \mathbb{C}^{n-3} \cong \mathbb{R}^{2(n-3)} \) is the identity map.

**Proof.** In this case, under the basis given by Lemma 3.12, \( A \) is a 3 by 3 matrix and \( AH = 0 \) implies

\[
A = \begin{pmatrix}
0 & \alpha & \beta \\
0 & \gamma & \delta \\
0 & \theta & \phi
\end{pmatrix}.
\]

Then

\[
AS_2 = \begin{pmatrix}
0 & \alpha & \beta \\
0 & \gamma & \delta \\
0 & \theta & \phi
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -b
\end{pmatrix} = \begin{pmatrix}
0 & \alpha & -b\beta \\
0 & \gamma & -b\delta \\
0 & \theta & -b\phi
\end{pmatrix}
\]

\[
AS_3 = \begin{pmatrix}
0 & \alpha & \beta \\
0 & \gamma & \delta \\
0 & \theta & \phi
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & b\alpha & \beta \\
0 & b\gamma & \delta \\
0 & b\theta & \phi
\end{pmatrix}.
\]

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Note that they are symmetric, \( \alpha = \beta = 0 \) and

\[-b\delta = \theta, \quad \delta = b\theta\]

this implies \(-b^2\delta = \delta\), but we have \(b^2 \neq -1\), so \(\delta = 0\) and \(\theta = -b\delta = 0\). Hence we have

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & \gamma & 0 \\
0 & 0 & \phi 
\end{pmatrix},
\]

since \(A\) is nilpotent, \(\gamma = \phi = 0\) and hence \(A = 0\). \(\square\)
BIBLIOGRAPHY


