MULTI-BODY AND NONLINEAR DYNAMICS OF PLANETARY GEAR DRIVETRAINS CONSIDERING BEARING CLEARANCE AND ACOUSTICS

DISSERTATION

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ABSTRACT

Planetary gearboxes generate considerable noise and vibration due to high bearing loads and nonlinear dynamic behaviors. Excessive vibration leads to gear tooth and bearing failures. Despite of archival gearbox research, there are a few research aspects that are not fully-developed. They are: 1) tooth wedging (tight tooth mesh) and its connection with planet bearing failures; 2) nonlinear dynamics of planetary gears with bearing clearance; 3) stiffness determination of rolling element bearings; and 4) multi-body gearbox modeling linking gearbox vibration and noise radiation.

Analyzing dynamic bearing and tooth loads of planetary gears is essential to improve gearbox reliability. Chapter 2 focuses on tooth wedging in wind turbine drivetrains, which is a potential source for gearbox failures. Lumped-parameter and finite element/contact mechanics models of planetary gears with tooth wedging, gravity, aerodynamic excitations, and mesh stiffness variation are developed to compute dynamic bearing or tooth forces. Tooth wedging elevates planet bearing forces and disrupts load sharing. A method to quantify tooth wedging is developed and verified.

Bearing clearance in planetary gears introduces nonlinear behaviors and bifurcations, which have not been investigated in the past. In Chapter 3, numerical integration and harmonic balance are applied to the analytical model, while Newmark method is used for the finite element model. Floquet theory determines solution stability. Complicated nonlinear effects, coexisting solutions, bifurcations, and chaos are
exhibited in the dynamic response. Input torque can potentially eliminate some of the nonlinear effects caused by bearing clearance.

Bearings are critical components in drivetrains. Accurate modeling of rolling element bearings is essential to assess vibration and noise of drivetrain systems. In Chapter 4, a full fidelity finite element model of rolling element bearings is developed. A combined surface integral and finite element algorithm is used to analyze the rolling element contact. A numerical method based on finite difference formulation is developed to determine the full bearing stiffness matrix. This method is validated against experiments. Techniques that reduce computational effort and improve result accuracy are discussed.

Vibration and noise caused by gear dynamics at the meshing teeth propagate through power transmission components to the surrounding environment. Chapter 5 is devoted to developing computational tools to investigate the vibro-acoustic propagation of gear dynamics through a gearbox using different bearings. Detailed finite element/contact mechanics and boundary element models of the gear/bearing/housing system are established to compute the system vibration and noise propagation. Both vibration and acoustic models are validated by experiments including the vibration modal testing and sound field measurements. The effectiveness of each bearing type to disrupt vibration propagation is speed-dependent. Housing plays an important role in noise radiation. It, however, has limited effects on gear dynamics.
Dedicated to my parents: Mengdong Guo and Tonge Zhou;
my husband: Chuanbo Yang;
and beloved children: Lillian Young and Ethan Young
Thanks for your unconditional love.
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Nomenclature

\( k_p \)  Planet bearing stiffness

\( k_{sj}, k_{rj} \)  Drive-side Mesh stiffness

\( k_{sj}, k_{rj} \)  Back-side mesh stiffness

\( f_{sj}^d, f_{sj}^b \)  Drive- and back-side tooth force at the \( j^{th} \) sun-planet mesh

\( f_{il}^x, f_{il}^y, f_{il}^u \)  Linear bearing forces in \( x, y, u \) directions

\( f_{cij}, f_{cij}^y, f_{cij}^u \)  Nonlinear bearing forces on the carrier

\( f_{pj}^x, f_{pj}^y \)  Nonlinear bearing forces on planet bearings

\( f_{cr}, f_{cr}^y \)  Nonlinear carrier-ring bearing forces

\( f_{vg}^x, f_{vg}^y, f_{vg}^u \)  Gravity forces

\( f_a, f_a^y, f_a^u \)  Aerodynamic forces acting on the carrier

Subscript

c  Carrier

r  Ring

s  Sun

p  Planet

m  Mesh

g  Gravity

b  Bearing

a  Aerodynamic
Superscript

$b$ Back-side contact

d Drive-side contact

$R$ Radial

$w$ Tooth wedging
CHAPTER 1
INTRODUCTION

1.1 Motivation and Problem Description

Planetary gears are widely used in wind turbines, helicopters, automotive transmissions, aircraft engines, and other machinery. Despite their accurate power transmission, high power density, high efficiency, and compactness, planetary gears generate considerable noise and vibration. Vibration causing high dynamic loads may result in gear tooth and bearing failures. Fatigue failures are a concern in long life-cycle applications. Analyzing the dynamics of planetary gear drivetrains is crucial in reducing noise, vibration, and subsequent failures.

This dissertation aims to fill a few research gaps: 1) Tooth wedging occurs when gear teeth contact on drive-side and back-side simultaneously. Tooth wedging has significant effect on planet bearings and could cause bearing failures; 2) Bearing clearance is unavoidable and changes gear dynamics. Dynamic effects caused by bearing clearance have not been studied previously; 3) Reliable estimates of rolling element bearing stiffness are essential for overall accuracy of the gearbox modeling. Existing theoretical bearing models differ from each other in the stiffness estimate. 4) Understanding gearbox vibration is important for predicting the noise radiation. Limited work has linked the gearbox vibration originated from the meshing teeth to the noise propagation.
Wind turbine drivetrains have experienced high gearbox failure rates [11]. Investigation on tooth wedging is motivated by field observations on wind turbine gearboxes. Planetary gears from in-service wind turbines showed likely tooth wedging. These gearboxes also experienced premature planet bearing failures at an unacceptable rate. Tooth wedging in planetary gears leads to unequal load sharing and excessive planet bearing loads by disturbing the planetary gear symmetry. This may cause bearing failure and tooth damage. Despite the breadth of literature on planetary gears, no work has examined tooth wedging.

Bearings in central members of planetary gears might have deliberate clearance to achieve a “floating” support. Bearing clearance elevates system vibration and could cause instability. Dynamic behaviors induced by bearing clearance in planetary gears have, however, not investigated in the past. Chapter 3 focuses on the effects of bearing clearance on the dynamics of planetary gears. Bearing clearances in central gears and carriers induce distinctive dynamic behaviors compared to clearances in planets.

Accurate stiffness estimation of rolling element bearings is crucial in the vibration of machinery. Theoretical models of rolling element bearings in the literature make different assumptions about the rolling element contact. These assumptions lead to discrepancy in their stiffness estimates. Available experimental data is limited to a few bearings. In addition, the determination of bearing stiffness by experiments is time-consuming and its accuracy depends on the flexibility of the connecting structures. The proposed method in Chapter 4 can provide a fast and accurate way to estimate the stiffness matrix of rolling element bearings. This computational method is applicable for all rolling element bearings.

Gearbox vibration is strongly connected with the structural-borne noise propagation from the gear teeth to surrounding environment. There is, however, limited work linking the gear vibration to noise radiation. Chapter 5 is devoted to establish
the vibration and acoustic models of the gear/bearing/housing system to address the vibro-acoustic propagation. The bearings connecting the gear shafts to the housing are a primary factor in this noise path as an energy dissipation mechanism. This vibro-acoustic model is used to investigate the effectiveness of novel fluid film wave bearings in reducing gearbox noise.

1.2 Literature Review

Dynamic modeling of planetary gears is based on the extensive research on gear pairs. Various spur gear pair models have been established including variable tooth stiffness [12, 13, 14], flexibility of shafts and bearings [15, 16, 17, 18], and transmission error and tooth separation [19, 20]. Considerable progress has been made on the dynamic modeling of planetary gears in the last a few decades. Botman [21] and Cunliffe et al. [22] studied the eigenvalue problems of planetary gears based on lumped parameter models with constant mesh stiffness. Kahraman [23, 24] built torsional and transverse-torsional models of planetary gears and studied the modes and mode shapes. Lin and Parker [25, 26] included gyroscopic effects and time-varying mesh stiffness. Kahraman [27] introduced tooth separation and manufacturing error to the model in [23]. Vaishya and Singh [28, 29] and Velex et al. [30, 31] modeled tooth friction and analyzed the resulting nonlinearity. Kahraman and Vijayakar [32] showed that a compliant thin ring can improve the load sharing and reduce vibration in planetary gears. Wu and Parker [33] and Kahraman et al. [34] investigated the influence of an elastic ring on the eigensolutions and dynamics of planetary gears. Precise and comprehensive models of gear tooth contact addressing complex nonlinear behaviors, however, have not been established. The transverse-torsional model of [25]
is extended in Chapter 2 to capture all possible tooth contact situations: drive-side contact, back-side contact, tooth separation, and tooth wedging.

Three techniques used in Chapters 2 and 3 are numerical integration, harmonic balance, and finite element analysis.

Numerical integration is widely-used to simulate gear dynamics. Ambarisha and Parker [35, 36] employed numerical integration to study nonlinear dynamics and the effects of mesh phasing on vibration reduction in planetary gears. Velex and Flamand [37] used numerical integration as a benchmark to verify a Ritz method in a planetary gear with time-varying mesh stiffness. Kahraman [38, 39, 40] investigated the dynamic response of spur gears, mechanical oscillators, and geared rotor-bearing systems using numerical integration.

The harmonic balance method [41] provides fast solution of steady state dynamic response in many applications. Zhu and Parker [42] used harmonic balance to study clutch engagement loss of a belt-pulley system. Blankenship and Kahraman [38] observed complex nonlinear behaviors of a mechanical oscillator with clearance nonlinearity and time-varying stiffness using harmonic balance. The results were correlated with experimental data in [43]. Al-shyyab and Kahraman [44, 45] investigated primary resonances, subharmonic resonances, and chaos exhibited in a multi-mesh gear train with fluctuating gear mesh stiffness. Bahk and Parker [46] employed harmonic balance to analyze dynamic behaviors of planetary gears based on a purely rotational lumped-parameter model. This approach uses arc-length continuation [47] to trace complicated nonlinear resonances in dynamic response. Floquet theory [48] is widely used for stability analysis [38, 49, 50]. Solution stability is analyzed by Floquet multipliers. The harmonic balance method with arc-length continuation and stability analysis are compared against numerical integration and finite element analysis.
Due to the scarcity of experimental data on planetary gears, finite element analysis is taken as a benchmark to validate analytical models. Vijayakar [51] developed a combined surface integral and finite element method to capture precise tooth deformation and contact in geared systems. This method allows a coarse finite element mesh at tooth contact areas. This makes dynamic analysis feasible for multi-mesh gears. The reliability of this method has been demonstrated. The finite element results in [52] agree with gear pair vibration experiments by Blankenship and Kahraman [53]. Kahraman [32] used this method to study the effects of ring gear flexibility on planet load sharing. Parker et al. [52, 54] used the method to investigate tooth separation and suppression of certain mesh frequency harmonics in the dynamic response. The analytical models used in Chapters 2 and 3 are validated against this benchmark finite element analysis.

Research on tooth separation is well-established. Tooth separation was observed in spur gear pair experiments [53]. Botman [21] experimentally observed tooth separation in planetary gears. Using finite element and lumped-parameter models, Ambarisha and Parker [35] predicted tooth separation and other nonlinear phenomena in a planetary gear. Velex and Flamand [37] investigated tooth separation at critical speeds. Bahk and Parker [46] derived closed-form solutions for the dynamic response of planetary gears with tooth separation based on a purely torsional model. No study, however, examines tooth wedging.

Mesh phasing of planetary gears can be used to suppress vibration. Lin and Parker [55] studied the effects of mesh phasing rules on the dynamics of spur planetary gears. Kahraman and Blankenship [56] investigated the effects of mesh phasing on the dynamics of helical planetary gears. Parker and Lin [57] mathematically formulated the mesh phasing relationships in planetary gears. Parker [58] and Ambarisha and Parker [36] examined the effectiveness of suppressing certain mesh frequency harmonics of
vibration modes using mesh phasing. Vangipuram-Canchi and Parker [59] showed that mesh phasing can suppress certain parametric instabilities induced by fluctuating mesh stiffnesses in planetary gears. These important mesh phasing rules do not apply when tooth wedging or tooth separation occurs.

Clearance in mechanical systems causes instability and nonlinear behaviors. Blankenship and Kahraman [38] and Kahraman [60] investigated the dynamic response of a mechanical oscillator with clearance. Kahraman and Singh [39] observed chaos in the dynamic response of a geared rotor-bearing system with bearing clearance and backlash. Gurkan and Ozguven [61] studied the effects of backlash and bearing clearance in a geared flexible rotor and the interactions between these two nonlinearities. Tiwari and Gupta [62] and Bai et al. [63] investigated chaos and instability caused by bearing clearance in a balanced horizontal rotor. Kim and Noah [64] has studied various bifurcations caused by bearing clearance in a Jeffcott rotor. Limited work has, however, studied the effects of bearing clearance in gear dynamics. No study has analyzed the dynamics of bearing clearance in planetary gears.

Complicated bifurcations and nonlinear behaviors have been discovered in forced oscillators with impacts at rigid stops. The discontinuous nature of instantaneous impacts has led to unconventional bifurcations which are not found in smooth dynamical systems. These bifurcations include grazing bifurcation when the oscillator comes into impact, chatter bifurcation due to infinite impacts, Feigenbaum cascades, Smale Horseshoes, etc. [65, 66, 67, 68, 69]. In vibro-impacting and floating systems, multiple coexisting solutions are present. These solutions have different numbers of impacts during one period [70, 71, 72]. These interesting behaviors have, however, not been discussed in planetary gears, despite the thorough study in mechanical oscillators. Chapter 3 demonstrates the numerical evidence of those nonlinear behaviors,
specifically, grazing bifurcation, secondary Hopf bifurcation, coexisting solutions, and chaos due to repeated impacts between bearing races.

Extensive study has been performed on the analytical estimation of the stiffness of rolling element bearings in the past. Early studies on rolling element bearings were performed by Harris [4], Jones [7], Palmgren [73], and Brandlein et al. [74]. They investigated the nonlinear relation between bearing deflection and applied load. Gargiulo [5] gives empirical formulae for the load-stiffness and deflection-stiffness relations by assuming rigid bearing races. Those formulae apply for radial and axial stiffnesses for a few bearing types. The above theoretical studies are limited to a few types of bearings. Furthermore, they can not determine tilting and cross-coupling stiffnesses between the radial, axial, and tilting deflections of bearings. Several of these studies lack validation. Lim and Singh [75] proposed a theoretical model to estimate diagonal and cross-coupling terms in the stiffness matrix. Bourdon et al. [76] developed an alternative method to estimate the stiffness matrix by dividing the rolling element surface and raceways into slices and computing individual contact stresses. Liew and Lim [77] extended the prior study to include the time-varying stiffness caused by orbital motion of the rolling elements. Those analytical models, however, provide stiffness estimates of rolling element bearings with significant discrepancy.

Kraus et al. [2] experimentally measured radial and axial stiffnesses and damping of a radial ball bearing with varying axial preload by modal tests on a transmission test stand. They concluded that static bearing stiffnesses sufficiently represented rolling element bearings by showing little difference of measured stiffnesses when the bearing was stationary and spinning. Royston and Basdogan [3] measured radial and axial stiffnesses of a self-aligning ball bearing with combined radial and axial preloads. Walford and Stone [78] developed a test rig to measure the radial stiffness of a pair
of angular contact ball bearings under oscillating conditions. Recently, techniques to experimentally determine bearing dynamic coefficients were developed through vibratory response measurements of rotor-bearing systems. Tiwari et al. [79, 80] used identification techniques to estimate certain dynamic bearing stiffnesses from unbalance and impulse response measurements of rotor-bearing test rigs. Goodwin [81] reviewed the experimental approaches to identify bearing stiffness and damping. Experiments are, however, expensive and need to be reassembled for different bearing. The accuracy of the stiffness estimate is highly sensitive to the boundary conditions applied to the bearing (flexibility of the connecting structures).

Fluid film wave bearings are a special type of journal bearings, that have waved inner diameters of the stationary bearing sides. The wave bearing technology used for gas lubrication [82, 83] is recently applied to the planet bearings used in aviation planetary gears [84, 85, 82]. This technology provides higher stiffness and better lubrication for the bearings. Experiments on an aviation gearbox [84] with wave bearings demonstrate 25% higher load capacity of wave bearings compared to plain journal bearings. Dimofte [82] compared the load capacity between wave and journal bearings through an analytical formulation. He concluded that wave bearings are more stable than plain journal bearings under light-load or unloaded conditions in any operating regime. Furthermore, wave amplitude and starting positions of the waves are important parameters affecting wave bearing performance.

1.3 Scope of Investigation

Chapter 2 details tooth wedging and its effects on planetary gear reliability. Lumped-parameter and finite element models are established to include tooth separation, back-side contact, tooth wedging, and bearing clearances. Results show significant
influences of tooth wedging on planet bearing forces for a wide range of operating speeds. To develop a physical understanding of the tooth wedging mechanism, connections between planet bearing forces and tooth forces are studied by investigating physical forces and displacements acting throughout the planetary gear. A method to predict tooth wedging based on geometric interactions is developed and verified. The major causes of tooth wedging relate directly to translational vibrations caused by gravity forces and bearing clearance.

Chapter 3 investigates the dynamics of planetary gears where nonlinearity is induced by bearing clearance, which has not been discussed before. The harmonic balance method with arc-length continuation is formulated to analyze the dynamic response of planetary gears. Floquet theory determines the solution stability. Bearings with clearance act as impact oscillators with clearance when vibration is significant. Bearing clearance induces nonlinear dynamic behaviors, coexisting solutions, various bifurcations, and chaos. Distinctive effects of bearing clearance in central components and planets on system dynamics are investigated and physically explained. Effects of input torque on suppressing nonlinear behaviors caused by bearing clearance are discussed and proven numerically.

Chapter 4 is devoted to the determination of symmetric, fully-populated stiffness matrices of rolling element bearings. Current theoretical bearing models differ in their stiffness estimates because of different model assumptions. A finite element/contact mechanics model is developed for rolling element bearings with the focus of obtaining accurate bearing stiffness for a wide range of bearing types and parameters. This model captures the time-dependent characteristics of the bearing contact due to the orbital motion of the rolling elements. A numerical method is developed to determine the full bearing stiffness matrix corresponding to two radial, one axial, and two angular coordinates; the rotation about the shaft axis is free by design. This proposed
stiffness determination method is validated against experiments in the literature and compared to existing analytical models and commercial software.

Chapter 5 seeks to fully describe the vibro-acoustic propagation of gear dynamics through a power-transmission system using rolling element and fluid film wave bearings. Fluid film wave bearings, which have higher damping than rolling element bearings, could offer an energy dissipation mechanism that reduces the gearbox noise. The effectiveness of each bearing type to disrupt vibration propagation is explored using multi-body computational models. These models include gears, shafts, rolling element and fluid film wave bearings, and the housing. Radiated noise is mapped from the gearbox surface to surrounding environment. The effectiveness of rolling element and fluid film wave bearings in breaking the vibro-acoustic propagation path from the gears to the housing is investigated.
CHAPTER 2
DYNAMIC MODELING AND ANALYSIS OF A SPUR PLANETARY GEAR INVOLVING TOOTH WEDGING

2.1 Summary

Planetary gears from in-service wind turbines showed likely tooth wedging and experienced premature planet bearing failures at an unacceptable rate. These bearing failures and the connection to tooth wedging in a wind turbine gearbox are described in [86, 87, 88]. Tooth wedging is a major source of gearbox failures [11]. Figure 2.1 shows a typical wind turbine gear train configuration. A heavy, compliantly supported ring gear and relatively low operating speeds are common features of wind turbine planetary gears [89, 90, 91]. Furthermore, gearboxes in wind turbines are mounted horizontally instead of vertically. Due to these features, gravity excitation, which is time-dependent in the rotating carrier frame, is a fundamental vibration source for wind turbine planetary gears. Simulation reveals tooth wedging in a representative wind turbine planetary gear, which occurs when a tooth comes into contact on both the drive-side and back-side simultaneously. The study shows gravity is the dominant excitation source for tooth wedging. Tooth wedging in planetary gears leads to unequal load sharing and excessive planet bearing loads by disturbing the symmetry of planet gears. This may cause bearing failure and tooth damage. Bearing clearance contributes to tooth wedging because it allows in-plane translation of the connected
components. Despite the breadth of literature on the dynamics of planetary gears, no work has examined tooth wedging.

The objectives of the present work are to: introduce tooth wedging, tooth contact loss, and bearing clearance into a lumped-parameter model; investigate the interplay between tooth wedging and bearing clearance; physically explain the mechanism of tooth wedging and its impact on dynamic response.

2.2 Modeling and Equation of Motion

2.2.1 Model Overview

The present model extends the two-dimensional lumped-parameter one in Lin and Parker [25]. The carrier, ring, sun, and planets are rigid bodies each having two translational and one rotational degrees of freedom. The model has $3(N + 3)$ degree of freedom, where $N$ is the number of planets. This model, depicted in Figure 2.2, is extended to include back-side contact, bearing clearance, mesh stiffness variation, gravity, and other excitation sources. Bearings are modeled as springs with
clearance nonlinearity. Gear meshes are modeled as nonlinear lumped springs that act only when the teeth are in contact on the drive-side, back-side, or both sides simultaneously for the case of tooth wedging. This model captures the tooth separation nonlinearity. Excitations consist of gravity, externally applied loads, and the parametric excitations from fluctuating gear mesh stiffnesses. The gear and carrier eccentricities as well as tooth spacing, indexing, and pitch-line run-out are not believed to be the primary concern for tooth wedging. These errors are not included in the model. Because friction is usually small for well-lubricated gears, it is modeled through modal damping. Particularly for the low-speed cases examined in the current study, the specific dissipation model will not significantly affect the results.

Figure 2.2: Planetary gear lumped-parameter model.
2.2.2 Coordinates and Geometric Description

The coordinates are shown in Figure 2.2. Translational displacements \( x_l, y_l, (l = c, r, s) \) are assigned to the carrier, ring, and sun, respectively, with respect to the basis \( \{ E_i \} (i = 1, 2, 3) \) that is fixed to the carrier as shown in Figure 2.3. The origin \( O \) is at the center of the planetary gear. The radial and tangential displacements of the planets are denoted by \( \xi_j, \eta_j, j = 1, ..., N \) with respect to the basis \( \{ e_i \} \) rotating with the carrier and oriented for each planet as shown in Figure 2.3. The rotational displacements are \( u_v = r_v \theta_v, v = c, r, s, 1, ..., N \), where \( \theta_v \) is the rotation in radians and \( r_v \) is the base circle radius for the sun, ring, and planets and the radius to the planet center for the carrier. Throughout this chapter, the subscript \( j = 1, \cdots, N \) denotes planets 1 to \( N \); \( l = c, r, s \) denotes the carrier, ring, and sun; and the superscripts \( d, b, w \) denote drive-side tooth contact, back-side tooth contact, and tooth wedging.

2.2.3 Tooth Contact Model

The tooth contact model captures four possible situations: drive-side tooth contact, back-side tooth contact, no contact, and tooth wedging (simultaneous drive-side and back-side contact). Drive-side tooth contact is modeled by a mesh stiffness along the line of action shown in Figure 2.4. When the drive-side mesh deflection becomes negative, a pair of teeth lose contact. The drive-side mesh deflections \( \delta_{sj}^d \) and \( \delta_{rj}^d \) of the \( j^{th} \) sun-planet (S-P) and ring-planet (R-P) mesh are [25]

\[
\begin{align*}
\delta_{sj}^d &= -x_s \sin \psi_{sj}^d + y_s \cos \psi_{sj}^d - \xi_j \sin \alpha_s - \eta_j \cos \alpha_s + u_j + u_s \\
\delta_{rj}^d &= -x_r \sin \psi_{rj}^d + y_r \cos \psi_{rj}^d + \xi_j \sin \alpha_r - \eta_j \cos \alpha_r - u_j + u_r \\
\psi_{sj}^d &= \psi_j - \alpha_s, \quad \psi_{rj}^d = \psi_j + \alpha_r
\end{align*}
\] (2.1)
Figure 2.3: Coordinates for the $j^{th}$ sun-planet mesh.

The drive-side tooth force $f^d_{qj}$ ($q = s, r$) is calculated by

$$f^d_{qj} = h^d_{qj} k^d_{qj}(t) \delta^d_{qj}$$  \hspace{1cm} (2.2)

where $h^d_{qj}$ tracks whether a tooth is in contact at the drive-side. Fluctuating mesh stiffness as the number of teeth in contact changes periodically between one and two pairs at each mesh is an important vibration source that can cause parametric instability in planetary gears [55]. This parametric excitation is introduced through time varying mesh stiffnesses $k^d_{qj}(t)$ at the S-P and R-P meshes. They are calculated by finite element analysis [92] fully discussed in section 4.5, which includes the elastic deformation of the gear teeth and gear bodies.

The back-side mesh deflection $\delta^b_{sj}$ at the $j^{th}$ S-P mesh is derived from the relative displacement between two points A and B on the sun and planet $j$ base circles, respectively, along the back-side line of action as shown in Figure 2.3. The back-side
line of action and the line of action have equal pressure angles. The displacements of A and B and the back-side mesh deflection are

\[ d_A = x_s E_1 + y_s E_2 - u_s e^b_1 \]

\[ d_B = \xi_j e_1 + \eta_j e_2 + u_j e^b_1 \]  

(2.4)

\[ \delta_{sj}^b = (d_A - d_B)e^b_1 = x_s \sin \psi_{sj}^b - y_s \cos \psi_{sj}^b - \xi_j \sin \alpha_s + \eta_j \cos \alpha_s - u_j - u_s, \]

\[ \psi_{sj}^b = \psi_j + \alpha_s \]  

(2.5)

where the basis \{e\}^b is oriented as shown in Figure 2.3, \( e^b_1 \) is along the back-side line of action, \( \alpha_s \) is the pressure angle of the S-P meshes, and \( \psi_j \) is the fixed angular position of planet \( j \) in the carrier reference frame. Similarly, the back-side mesh deflection at the \( j^{th} \) ring-planet mesh is

\[ \delta_{rj}^b = x_r \sin \psi_{rj}^b - y_r \cos \psi_{rj}^b + \xi_j \sin \alpha_r + \eta_j \cos \alpha_r + u_j - u_r, \]

\[ \psi_{rj}^b = \psi_j - \alpha_r \]  

(2.6)
Back-side tooth contact takes place when the back-side mesh deflection exceeds the backlashes $b_q (q = r, s)$ for the sun and ring. The back-side tooth force is

$$f^b_{aqj} = h^b_{aqj} k^b_{aqj} (\delta^b_{aqj} - b_q), \ q = r, s \quad (2.7)$$

$$h^b_{aqj} = \begin{cases} 1 & \text{if} \ \delta^b_{aqj} > b_q \\ 0 & \text{if} \ \delta^b_{aqj} < b_q \end{cases}, \ j = 1, \ldots, N \quad (2.8)$$

Whether a tooth is in back-side contact is tracked by $h^b_{aqj}, \ q = r, s$. The quantity $k^b_{aqj}$ denotes the back-side mesh stiffnesses. They are approximated as the average value of the periodic mesh stiffnesses on the drive-side.

Tooth wedging occurs when the relative radial motion of two mating gears exceeds a specified tooth radial gap $\Delta_q (q = r, s)$ as shown in Figure 2.4. The tooth radial gap is the radial clearance (before tooth wedging occurs) between a pair of mating gears when they move radially relative to each other. It is a specified quantity estimated from the backlash and tooth geometry. Here $\Delta_q$ are approximated as $b_q / (2 \tan \alpha_q)$ such that the backlash $b_q$ is the projection of $\Delta_q$ into the pitch circle shown in Figure 2.4. For the S-P and R-P meshes, the relative radial deflections are

$$\delta^R_{sj} = x_s \cos \psi_j + y_s \sin \psi_j - \xi_j, \ q = r, s, \ j = 1, \ldots, N \quad (2.9)$$

$$\delta^R_{rj} = \eta_j - x_r \cos \psi_j - y_r \sin \psi_j$$

To obtain the back-side mesh deflection when tooth wedging occurs, it is important to remove the portion of relative motion along the back-side line of action caused by the relative radial motion prior to tooth wedging; this portion of back-side relative motion is not resisted by back-side contact. Thus, the back-side mesh deflection is modified by subtracting the line of action component of the relative radial displacement between the sun and planet and between the ring and planet up to the tooth wedging point, giving

$$\delta^b_{aqj, \text{mod}} = \delta^b_{aqj} - 2 \Delta_q \sin \alpha_q, \ q = r, s, \ j = 1, \ldots, N \quad (2.10)$$
When tooth wedging occurs, the drive- and back-side tooth forces at the $j^{th}$ S-P and R-P meshes are

$$ f^d_{qj} = h^w_{qj} k^d_{qj} (t) \delta^d_{qj} \quad f^b_{qj} = h^w_{qj} k^b_{qj} \delta^{b,\text{mod}}_{qj} \quad q = r, s, \ j = 1, \ldots, N \quad (2.11) $$

where $h^w_{qj}$ tracks whether tooth wedging occurs according to

$$ h^w_{qj} = \begin{cases} 
1 & \text{if} \ \delta^R_{qj} > \Delta_q, \ q = r, s, \ j = 1, \ldots, N \\
0 & \text{if} \ \delta^R_{qj} < \Delta_q
\end{cases} \quad (2.12) $$

### 2.2.4 Bearing Model with Clearance

The linear bearings or fixed supports without clearance are modeled as one torsional and two translational springs. The nonlinear bearings are modeled as circumferentially distributed radial springs with uniform clearances as shown in Figure 2.5. Forces develop only when the relative displacement between the connected bodies exceeds a specified clearance. For the $j^{th}$ planet bearing with bearing clearance $\Delta_{cp}$,

![Figure 2.5:](image)

Figure 2.5: (a) Planet bearing model with clearance; (b) Diagram of bearing force with clearance.
as an example, the relative displacement between the carrier and planet $j$ is

$$
\delta_{cj} = \left[ (x_c \cos \psi_j + y_c \sin \psi_j - \xi_j)^2 + (-x_c \sin \psi_j + y_c \cos \psi_j + u_c - \eta_j)^2 \right]^{1/2}
$$

(2.13)

The direction of the developed force is determined by the contact angle $\vartheta_{cj}$ between $e_1$ (Figure 2.3) and the direction of relative motion between the carrier and planet $j$

$$
\vartheta_{cj} = \tan^{-1} \left( \frac{-x_c \sin \psi_j + y_c \cos \psi_j + u_c - \eta_j}{x_c \cos \psi_j + y_c \sin \psi_j - \xi_j} \right)
$$

(2.14)

The bearing forces that are the projections of this bearing force in the $x_c, y_c, u_c, \xi_j, \eta_j$ directions are

$$
\begin{align*}
    f^x_{cj} &= \mu_{cj} k_p (\delta_{cj} - \Delta_{cp}) \cos(\vartheta_{cj} + \psi_j) \\
    f^y_{cj} &= \mu_{cj} k_p (\delta_{cj} - \Delta_{cp}) \sin(\vartheta_{cj} + \psi_j) \\
    f^u_{cj} &= \mu_{cj} k_p (\delta_{cj} - \Delta_{cp}) \sin \vartheta_{cj} \\
    f^\xi_{pj} &= -\mu_{cj} k_p (\delta_{cj} - \Delta_{cp}) \cos \vartheta_{cj} \\
    f^\eta_{pj} &= -\mu_{cj} k_p (\delta_{cj} - \Delta_{cp}) \sin \vartheta_{cj}
\end{align*}
$$

(2.15)

The planet bearing stiffness is denoted by $k_p$. The variable $\mu_{cj}$ tracks if the bearing is in contact according to

$$
\mu_{cj} = \begin{cases} 
1 & \text{if } \delta_{cj} > \Delta_{cp} \\
0 & \text{if } \delta_{cj} < \Delta_{cp} 
\end{cases}, \quad j = 1, \ldots, N
$$

(2.16)

The carrier-ring bearing with clearance $\Delta_{cr}$ and the sun-ring bearing with clearance $\Delta_{sr}$ are modeled similarly. The relative displacements between the carrier/ring and sun/ring are

$$
\delta_{hr} = \left[ (x_h - x_r)^2 + (y_h - y_r)^2 \right]^{1/2}, \quad h = c, s
$$

(2.17)

The contact angles are

$$
\vartheta_{hr} = \tan^{-1} \left( \frac{y_h - y_r}{x_h - x_r} \right), \quad h = c, s
$$

(2.18)
The bearing forces \( f_{hx} \), \( f_{hy} \) that are the projections of these bearing forces in the \( x_h, y_h \) directions are

\[
\begin{align*}
    f_{hx} &= \mu_{hr} k_{hr} (\delta_{hr} - \Delta_{hr}) \cos(\vartheta_{hr}) \\
    f_{hy} &= \mu_{hr} k_{hr} (\delta_{hr} - \Delta_{hr}) \sin(\vartheta_{hr})
\end{align*}
\]

where \( k_{hr} (h = c, s) \) are the bearing stiffnesses. The variable \( \mu_{hr} \) tracks if the bearings are in contact according to

\[
\mu_{qr} = \begin{cases} 
1 & \text{if } \delta_{hr} > \Delta_{hr} \\
0 & \text{if } \delta_{hr} < \Delta_{hr} 
\end{cases} \quad h = c, s
\]

\[2.20\]

### 2.2.5 Equations of Motion

Based on the derivations above, the equations of motion for the carrier are

\[
\begin{align*}
    m_c \ddot{x}_c + k_c x_c - \sum_{j=1}^{N} k_p \mu_{cj} (\delta_{cj} - \Delta_{cp}) \cos(\vartheta_{cj} + \psi_j) + \\
    k_{cr} \mu_{cr} (\delta_{cr} - \Delta_{cr}) \cos \vartheta_{cr} &= f_{cx}^c + f_{ax}^c (t) + f_{cxg}^c (t) \\
    m_c \ddot{y}_c + k_c y_c + \sum_{j=1}^{N} k_p \mu_{cj} (\delta_{cj} - \Delta_{cp}) \sin(\vartheta_{cj} + \psi_j) + \\
    k_{cr} \mu_{cr} (\delta_{cr} - \Delta_{cr}) \sin \vartheta_{cr} &= f_{cy}^c + f_{ay}^c (t) + f_{cyg}^c (t) \\
    (I_c/r_c^2) \ddot{\vartheta}_c + k_{cu} u_c + \sum_{j=1}^{N} k_p \mu_{cj} (\delta_{cj} - \Delta_{cp}) \sin \vartheta_{cj} &= f_{cu}^c + f_{au}^c (t)
\end{align*}
\]

\[2.21\]
The equations of motion for the ring are

\[ m_r \ddot{x}_r + k_r x_r - \sum_{j=1}^{N} t^{d}_{rj}(t) h^{d}_{rj} \delta^{d}_{rj} \sin \psi^{d}_{rj} + \sum_{j=1}^{N} k^{b}_{rj} h^{w}_{rj} \delta^{b;\text{mod}}_{rj} \]

\[ \sin \psi^{b}_{rj} - k_{cr} \mu_{cr} (\delta_{cr} - \Delta_{cr}) \cos \vartheta_{cr} + \]

\[ k_{sr} \mu_{sr} (\delta_{sr} - \Delta_{sr}) \cos \vartheta_{sr} = f_{xg}(t) \]

\[ m_r \ddot{y}_r + k_r y_r + \sum_{j=1}^{N} t^{d}_{rj}(t) h^{d}_{rj} \delta^{d}_{rj} \cos \psi^{d}_{rj} - \sum_{j=1}^{N} k^{b}_{rj} h^{w}_{rj} \delta^{b;\text{mod}}_{rj} \]

\[ \cos \psi^{b}_{rj} - k_{cr} \mu_{cr} (\delta_{cr} - \Delta_{cr}) \cos \vartheta_{cr} + \]

\[ k_{sr} \mu_{sr} (\delta_{sr} - \Delta_{sr}) \sin \vartheta_{sr} = f_{yg}(t) \]

\[ (I_r/r^2_r) \ddot{u}_r + k_{ru} u_r + \sum_{j=1}^{N} t^{d}_{rj}(t) h^{d}_{rj} \delta^{d}_{rj} - \sum_{j=1}^{N} k^{b}_{rj} h^{w}_{rj} \delta^{b;\text{mod}}_{rj} = 0 \]

The equations of motion for the sun are

\[ m_s \ddot{x}_s + k_s x_s - \sum_{j=1}^{N} k^{d}_{sj}(t) h^{d}_{sj} \delta^{d}_{sj} \sin \psi^{d}_{sj} + \sum_{j=1}^{N} k^{b}_{sj} h^{w}_{sj} \delta^{b;\text{mod}}_{sj} \]

\[ \sin \psi^{b}_{sj} - k_{sr} \mu_{sr} (\delta_{sr} - \Delta_{sr}) \cos \vartheta_{sr} = f^{x}_{s} + f^{x}_{sg}(t) \]

\[ m_s \ddot{y}_s + k_s y_s + \sum_{j=1}^{N} k^{d}_{sj}(t) h^{d}_{sj} \delta^{d}_{sj} \cos \psi^{d}_{sj} - \sum_{j=1}^{N} k^{b}_{sj} h^{w}_{sj} \delta^{b;\text{mod}}_{sj} \]

\[ \cos \psi^{b}_{sj} - k_{sr} \mu_{sr} (\delta_{sr} - \Delta_{sr}) \sin \vartheta_{sr} = f^{y}_{s} + f^{y}_{sg}(t) \]

\[ (I_s/r^2_s) \ddot{u}_s + k_{su} u_s + \sum_{j=1}^{N} t^{d}_{sj}(t) h^{d}_{sj} \delta^{d}_{sj} - \sum_{j=1}^{N} k^{b}_{sj} h^{w}_{sj} \delta^{b;\text{mod}}_{sj} = f^{u}_{s} \]
The equations of motion for planet \( j \) are

\[
\begin{align*}
  m_p \ddot{\xi}_j - k_{p \mu} c_j (\delta_{c_j} - \Delta_{cp}) \cos \vartheta_{c_j} - k_{d_{s_j}} (t) h_{d_{s_j}} \delta_{s_j} \sin \alpha_s + \\
  k_{r_j} (t) h_{r_j} \delta_{r_j} \cos \alpha_{s_r} - k_{b_{w_{s_j}}} h_{w_{s_j}} \delta_{s_j}^{mod} \sin \alpha_{s_r} + \\
  k_{b_{w_{s_j}}} h_{w_{s_j}} \delta_{s_j}^{mod} \sin \alpha_{r} = f_{jg} (t)
\end{align*}
\]

\[(2.24)\]

\[
\begin{align*}
  m_p \ddot{\eta}_j - k_{p \mu} c_j (\delta_{c_j} - \Delta_{cp}) \sin \vartheta_{c_j} - k_{d_{s_j}} (t) h_{d_{s_j}} \delta_{s_j} \cos \alpha_s + \\
  k_{r_j} (t) h_{r_j} \delta_{r_j} \cos \alpha_{s_r} - k_{b_{w_{s_j}}} h_{w_{s_j}} \delta_{s_j}^{mod} \cos \alpha_{s_r} + \\
  k_{b_{w_{s_j}}} h_{w_{s_j}} \delta_{s_j}^{mod} \cos \alpha_{r} = f_{jg} (t)
\end{align*}
\]

\[(2.25)\]

\[
\begin{align*}
  (I_p / r_p^2) \ddot{\omega}_j - k_{d_{s_j}} (t) h_{d_{s_j}} \delta_{s_j} + k_{d_{r_j}} (t) h_{d_{s_j}} \delta_{r_j} + k_{b_{w_{s_j}}} h_{w_{s_j}} \delta_{s_j}^{mod} - \\
  k_{b_{w_{s_j}}} h_{w_{s_j}} \delta_{s_j}^{mod} = 0
\end{align*}
\]

The masses and moments of inertias of the carrier, ring, sun, and planets are denoted by \( m_k, I_k \) (\( k = c, r, s, p \)). Quantities \( k_{lu} r_i^2 \) (\( l = c, r, s \)) denote the torsional stiffnesses of the carrier, ring, and sun supports, where \( k_{lu} \) has units of force/length. Quantities \( f_{lg}^x, f_{lg}^y \) (\( l = c, r, s \)) and \( f_{jg}^\xi, f_{jg}^\eta \) (\( j = 1, \ldots, N \)) denote the gravity force acting on the carrier, ring, sun, and planets 1 to \( N \) in the \( E_1, E_2, e_1, e_2 \) directions. These are

\[
\begin{align*}
  f_{lg}^x &= -m_l g \sin(\Omega_l t) \\
  f_{lg}^y &= -m_l g \cos(\Omega_l t), \quad l = c, r, s
\end{align*}
\]

\[(2.25)\]

\[
\begin{align*}
  f_{jg}^\xi &= -m_j g \sin(\Omega_{c} t + \psi_j) \\
  f_{jg}^\eta &= -m_j g \cos(\Omega_{c} t + \psi_j), \quad j = 1, \ldots, N
\end{align*}
\]

\[(2.26)\]

where the variable \( \Omega_{c} = \omega_m / N_r \) (if the ring is fixed) denotes the carrier rotation frequency. \( N_r \) denotes the number of teeth on the ring.

Other time-varying applied forces are denoted by \( f^i_a \), such as aerodynamic forces transmitted from the turbine blades to the planetary gear. The steady loads acting on the carrier (input) and sun (output gear) in the \( E_1, E_2 \) directions are denoted by \( f^i_c, f^i_s \) (\( i = x, y \)). Quantities \( f^u_c, f^u_s \) denote the steady torques acting on the carrier and sun.
The nonlinear forces fall into three categories: the drive-side tooth force, the back-side tooth force, and the bearing force. Assembling these forces in vector form yields the drive- and back-side tooth mesh force vectors

\[
f^d_m = \begin{bmatrix}
[0, 0, 0]^T \\
\sum_{j=1}^{N} f^d_{rj} \sin \psi_{rj}, - \sum_{j=1}^{N} f^d_{rj} \cos \psi_{rj}, - \sum_{j=1}^{N} f^d_{rj} \\
\sum_{j=1}^{N} f^d_{sj} \sin \psi_{sj}, - \sum_{j=1}^{N} f^d_{sj} \cos \psi_{sj}, - \sum_{j=1}^{N} f^d_{sj}
\end{bmatrix}^T \\
\vdots \\
\begin{bmatrix}
[f^d_{s1} \sin \alpha_s - f^d_{r1} \sin \alpha_r, f^d_{s1} \cos \alpha_s + f^d_{r1} \cos \alpha_r, \\
-f^d_{s1} + f^d_{r1} \\
\vdots \\
[f^d_{sN} \sin \alpha_s - f^d_{rN} \sin \alpha_r, f^d_{sN} \cos \alpha_s + f^d_{rN} \cos \alpha_r, \\
-f^d_{sN} + f^d_{rN}]
\end{bmatrix}
\end{bmatrix}
\]

(2.27)

\[
f^b_m = \begin{bmatrix}
[0, 0, 0]^T \\
-\sum_{j=1}^{N} f^b_{rj} \sin \psi_{rj}, \sum_{j=1}^{N} f^b_{rj} \cos \psi_{rj}, \sum_{j=1}^{N} f^b_{rj} \\
-\sum_{j=1}^{N} f^b_{sj} \sin \psi_{sj}, \sum_{j=1}^{N} f^b_{sj} \cos \psi_{sj}, \sum_{j=1}^{N} f^b_{sj}
\end{bmatrix}^T \\
\vdots \\
\begin{bmatrix}
[f^b_{s1} \sin \alpha_s - f^b_{r1} \sin \alpha_r, -f^b_{s1} \cos \alpha_s - f^b_{r1} \cos \alpha_r, \\
f^b_{s1} - f^b_{r1} \\
\vdots \\
[f^b_{sN} \sin \alpha_s - f^b_{rN} \sin \alpha_r, -f^b_{sN} \cos \alpha_s - f^b_{rN} \cos \alpha_r, \\
f^b_{sN} - f^b_{rN}]
\end{bmatrix}
\end{bmatrix}
\]

(2.28)
and the bearing force vector \( \mathbf{f}_B \) as

\[
\mathbf{f}_B = \begin{bmatrix}
\left[ f_{c}^{x} + \sum_{j=1}^{N} f_{c}^{x j} + f_{c}^{y} + \sum_{j=1}^{N} f_{c}^{y j} \right] + \sum_{j=1}^{N} f_{c}^{u j} \\
\left[ f_{r}^{x} + f_{rs}^{x} + f_{r}^{y} + f_{rs}^{y} + f_{r}^{u} \right]^{T} \\
\left[ f_{s}^{x} + f_{sr}^{x} + f_{s}^{y} + f_{sr}^{y} \right]^{T} \\
\left[ f_{p1}^{x}, f_{p1}^{y}, 0 \right]^{T} \\
\vdots \\
\left[ f_{PN}^{x}, f_{PN}^{y}, 0 \right]^{T}
\end{bmatrix}
\]

\[
(2.29)
\]

Including damping, the matrix form of these \( 3N + 9 \) equations of motion is

\[
\mathbf{M} \ddot{\mathbf{x}} + \mathbf{C} \dot{\mathbf{x}} + \mathbf{f}_d (t, \mathbf{x}) + \mathbf{f}_m (t, \mathbf{x}) + \mathbf{f}_B (t, \mathbf{x}) = \mathbf{F}(t)
\]

\[
(2.30)
\]

\[
\mathbf{x} = \left\{ \begin{array}{c}
x_c, y_c, u_c, x_r, y_r, u_r, x_s, y_s, u_s, \xi_1, \eta_1, u_1, \ldots, \xi_N, \eta_N, u_N \\
\text{Carrier} & \text{Ring} & \text{Sun} & \text{Planet 1} & \text{Planet N}
\end{array} \right\}
\]

\[
(2.31)
\]

The damping matrix \( \mathbf{C} \) is calculated from

\[
\mathbf{C} = (\mathbf{U}^{-1})^T \text{diag}(2\zeta_n \Omega_n) \mathbf{U}^{-1}
\]

\[
(2.32)
\]

where \( \zeta_n \) are the damping ratios and \( \Omega_n \) are the natural frequencies of the linear system where all bearings are in contact and the mesh stiffnesses are averaged over a mesh cycle; \( \mathbf{U} \) is the orthonormalized modal matrix \( (\mathbf{U}^T \mathbf{M} \mathbf{U} = \mathbf{I}) \).

The applied force vector \( \mathbf{F}(t) = \mathbf{F}_g(t) + \mathbf{F}_a(t) + \mathbf{F}_s \) contains the periodic gravity force \( \mathbf{F}_g(t) \) acting on the sun, ring, and planets, other forces \( \mathbf{F}_a(t) \) applied to the carrier (such as aerodynamic forces in a wind turbine), and the steady loads \( \mathbf{F}_s \) acting on the input and output members.

To improve numerical accuracy and reduce computation, the equations of motion are non-dimensionalized by introducing \( \tau = \omega_c t \) and \( z = x/L \), where \( \omega_c \) is the sixth
natural frequency as a characteristic frequency and $L = 1 \mu m$ is the characteristic length, giving

$$Mz'' + \tilde{C}z' + \tilde{f}_m^d(\tau, z) + \tilde{f}_m^b(\tau, z) + \tilde{f}_B(\tau, z) = \tilde{F}(\tau) \quad (2.33)$$

$$\tilde{C} = \frac{C}{\omega_c^2}, \quad \tilde{f}_m^d = \frac{f_m^d}{L\omega_c^2}, \quad \tilde{f}_m^b = \frac{f_m^b}{L\omega_c^2}, \quad \tilde{f}_B = \frac{f_B}{L\omega_c^2}, \quad \tilde{F} = \frac{F}{L\omega_c^2} \quad (2.34)$$

### 2.3 Comparisons to Finite Element Simulations

Due to the scarcity of experimental data on planetary gears, a finite element solution is taken as the benchmark to evaluate the analytical model. Vijayakar [51] developed a combined surface integral and finite element method to capture precise tooth deformation and contact loads in geared systems. This method allows a more coarse finite element mesh at tooth contact areas than traditional methods. The software developed by Vijayakar [92] intrinsically evaluates time-varying tooth contact forces that are specified externally with conventional simulation tools. The reliability of this code has been demonstrated [52, 54, 32], including comparisons with gear vibration experiments.

The equally spaced 3-planet system is defined in Tables 2.1 and 2.2. The carrier is modeled as a rigid body with a lumped inertia. The large ring gear mass results from a typical arrangement of wind turbines whereby much of the gearbox mass is rigidly connected to the ring, which is supported on a relatively compliant foundation. All gears are steel spur gears with involute gear teeth having modulus of elasticity $207 \times 10^9 N/m^2$ and Poisson’s ratio 0.3. A constant torque of $180 kN\cdot m$ is applied to the carrier as the input, and the sun gear is the output. Gravity acts at the center of mass of all component gears. For the ring and sun, it is a periodically varying external excitation in the carrier reference frame in which the model is formulated. Besides fluctuating mesh stiffness over a mesh cycle (shown in Figure 2.6), tooth wedging and
<table>
<thead>
<tr>
<th></th>
<th>Sun</th>
<th>Planet</th>
<th>Ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Teeth</td>
<td>16</td>
<td>26</td>
<td>68</td>
</tr>
<tr>
<td>Outer Diameter (mm)</td>
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<td>383</td>
<td>930.6</td>
</tr>
<tr>
<td>Root Diameter (mm)</td>
<td>202</td>
<td>329</td>
<td>980</td>
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<tr>
<td>Pitch Diameter (mm)</td>
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<td>364</td>
<td>952</td>
</tr>
<tr>
<td>Transverse Tooth Thickness (mm)</td>
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<td>18.5</td>
<td>25.0</td>
</tr>
<tr>
<td>Module (mm)</td>
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<td>14.0</td>
<td>14.0</td>
</tr>
<tr>
<td>Facewidth (mm)</td>
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</tr>
<tr>
<td>Backlash (mm)</td>
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<td>0.482</td>
<td></td>
</tr>
<tr>
<td>Pressure Angle</td>
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<td>24.6°</td>
<td></td>
</tr>
<tr>
<td>Tooth Radial Gap (mm)</td>
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<td>0.526</td>
<td></td>
</tr>
<tr>
<td>Sun/Pinion Center Distance (mm)</td>
<td></td>
<td>294</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1: Gear parameters for the example planetary gear.

tooth separation are included naturally in the finite element model. Using precise information on the tooth geometry, backlash, root shape, and root clearance, the contact algorithm used in the software identifies tooth contact on both the drive- and back-sides. This is important for validating the lumped parameter contact model for tooth wedging. The finite element model includes clearance nonlinearity at the carrier-ring bearing. Other bearings are modeled as linear stiffnesses without clearance in the finite element and analytical models. A global Newton iteration scheme with line search technique [93] is used to obtain accurate numerical solutions of the analytical model. The line search technique makes the predicted solution at each iteration always closer to the exact solution by adjusting the iteration step size for
Table 2.2: System parameters for the example planetary gear system. The ring mass includes the gearbox.

\[
\begin{array}{|c|c|c|c|c|}
\hline
 & Sun & Ring & Carrier & Planet \\
\hline
m (kg) & 51 & 28125 & 1330 & 114 \\
I (kg \cdot m^2) & 61.1 & 2484 & 314.7 & 51.9 \\
k_{qp} (N/m) & 3.55 \times 10^9 & 4.56 \times 10^9 & & \\
k_B (N/m) & 0 & 126 \times 10^6 & 3.95 \times 10^9 & 5.29 \times 10^9 \\
k_{cr} (N/m) & & & 3.59 \times 10^9 & \\
\Delta_{cr} (mm) & & & 1 & \\
\Delta_{cp} (mm) & & & 0 & \\
k_u (N \cdot m) & 3.02 \times 10^6 & 24.4 \times 10^6 & 0 & 0 \\
\hline
\end{array}
\]

The analytical model uses smoothing functions to approximate the nonlinearities at the gear teeth and bearing contact. The smoothing function \( f_s \) used to describe the nonlinear bearing contact is

\[
f_s(\delta) = \delta + \frac{1}{2}[1 + \tanh(\sigma - \Delta_{cr})](\delta - \Delta_{cr}) - \frac{1}{2}[1 + \tanh(\sigma + \Delta_{cr})](\delta + \Delta_{cr})
\]

(2.35)

where \( \delta \) denotes the deflection between the inner and outer bearing races. The smoothing function \( f_s \) used to describe the nonlinear gear tooth contact for both the drive- and back-side is

\[
f_s(\delta) = \frac{1}{2}(\delta - b)[1 + \tanh(\sigma - b)],
\]

\[
b = \begin{cases} 
\Delta_q & \text{if back-side contact} \\
0 & \text{if drive-side contact}
\end{cases}
\]

(2.36)
where $\delta$ denotes the drive- or back-side mesh deflection, $q = r, s$, and $\sigma$ controls how close the smoothing functions are to the original piecewise nonlinear functions. Larger $\sigma$ makes the approximation closer to the piecewise functions. Extremely high values of $\sigma$ ($10^5$), however, may cause numerical instability. This study uses $\sigma = 500$, a reliable value with deviation $<5\%$ from the nonlinear piecewise functions.

Static analysis is performed on the example planetary gear in Tables 2.1 and 2.2. One carrier period is analyzed, giving 68 mesh cycles. Each mesh cycle is divided evenly into ten intervals. For analytical solutions, the convergence tolerance is $100 \times 10^{-12} \mu m$, while the finite element solutions have single precision accuracy.

Figures 2.7 and 2.8 compare the analytical and finite element drive- and back-side tooth forces at the 2\textsuperscript{nd} S-P mesh and the 2\textsuperscript{nd} R-P mesh at mesh frequency 34 $Hz$ (which is the wind turbine operating mesh frequency). Results of both models show tooth wedging at the carrier cycle interval [0.6, 0.9] for the S-P mesh, and intervals
[0, 0.08] and [0.78, 1] for the R-P mesh. The extreme loads at 0.10 and 0.42 carrier cycles in Figure 2.7 are caused by the occurrence of tooth wedging at other meshes than those of planet 2. Figures 2.9 and 2.10 compare the planet 3 bearing force and its developed contact angle under the same conditions as Figures 2.7 and 2.8. Figure 2.11 compares the amplitude of carrier-ring bearing force at mesh frequency $34 \, Hz$ when $\Delta_{cr} = 0.7 \, mm$. The excellent comparison between these two solutions indicates that the analytical model captures the nonlinear contact behavior as well as the finite element model.

![Figure 2.7: Drive- and back-side tooth forces at the 2\textsuperscript{nd} S-P mesh for mesh frequency $\omega_m = 34 \, Hz$ with gravity and gear data given in Tables 2.1 and 2.2 from finite element (dashed line) and analytical models (solid line).](image)

Part of the excellent agreement between the mathematical and finite element models is because the mathematical model takes the mesh stiffness information from the finite element model, including the gear teeth and blanks (shown in Figure 2.6). That is of secondary importance, however. The main point of the excellent correlation is
Figure 2.8: Drive- and back-side tooth forces at the 2nd R-P mesh for mesh frequency $\omega_m = 34 \text{ Hz}$ with gravity and gear data given in Tables 2.1 and 2.2 from finite element (dashed line) and analytical models (solid line).

that it builds confidence in the analytical bearing clearance and tooth wedging models, which are handled quite differently in the lumped-parameter and finite element approaches. Thus, the finite element results are an independent benchmark for these crucial effects.

2.4 Results and Discussion

The dynamic response of the example three-planet system (defined in Tables 2.1 and 2.2) is analyzed using Runge-Kutta numerical integration with order $h^4 \sim h^5$ global accumulated error, where $h$ is the step size. All components are allowed to vibrate. The backlashes $b_r, b_s$ are chosen according to Table 3.9 in Dudley [94] and confirmed by the Fairfield gear design software [95]. The natural frequencies are given in Table 2.3. For systems with three planets, the rotational-translational planetary model yields two types of modes: rotational modes with distinct natural frequencies
Figure 2.9: Amplitude of bearing force at the 3rd planet for mesh frequency $\omega_m = 34$ Hz with gravity and gear data given in Tables 2.1 and 2.2 from finite element (dashed line) and analytical models (solid line).

and translational modes with degenerate natural frequencies of multiplicity two. The properties of these mode types are described in [25] and [26].

To examine which of the different excitations are dominant, the effects of gravity, fluctuating mesh stiffness, and aerodynamic forces on the tooth force at the 1st S-P mesh are compared in Figure 2.12. Using estimates from measurements on the example wind turbine, the aerodynamic forces $f_a^i = c_i + d_i \sin(3\Omega_c t)$, $i = x, y, u$, are transmitted from the three-bladed wind turbine to the carrier of the planetary gear, where $c_i, d_i$ are the force amplitudes in the $x, y, u$ directions. Gravity elevates the force fluctuation amplitude sharply and causes tooth wedging to occur at the 1st S-P mesh over approximately 25% of a carrier cycle. Aerodynamic forces or mesh stiffness variations do not cause tooth wedging. Tooth loads excited by them are close to the nominal forces. Gravity is the dominant excitation leading to tooth wedging. Besides gravity, bearing clearance plays an important part in tooth wedging. The effect of $\Delta_{cr}$ on the percentage of a carrier cycle in which tooth wedging (considering all meshes)
Figure 2.10: Bearing contact angle between e_1 and the motion direction between the carrier and planet 3 for mesh frequency ω_m = 34 Hz with gravity and gear data given in Tables 2.1 and 2.2 from finite element (dashed line) and analytical models (solid line).

occurs is shown in Figure 2.13. For the nominal tooth radial gap, no tooth wedging occurs when Δ_{cr} is smaller than the point L; the behavior is the same as the linear system without Δ_{cr}. Above this point, tooth wedging occurs and its occurrence in a carrier cycle increases gradually until the point H. After Δ_{cr} reaches point H, the tooth wedging occurrence percentage does not change; the behavior is the same as the linear system without the carrier-ring bearing (infinite Δ_{cr}). Bearing clearance allows greater system translational vibration by softening the support bearing. Thus, the percentage of tooth wedging in a carrier cycle increases when bearing clearance is enlarged. Besides bearing clearance, the tooth radial gaps Δ_r, Δ_s affect tooth wedging. Comparing the three curves in Figure 2.13, reducing the tooth radial gap (i.e., the backlash) increases the occurrence of tooth wedging even for small values of bearing clearance Δ_{cr}. The foregoing analysis indicates that the occurrence of tooth wedging is the combined effect of gravity and bearing clearance. Bearing clearance
admits greater translational motions that cause tooth wedging, and gravity acts as the external excitation.

Tooth wedging changes the tooth load amplitudes at both the S-P and R-P meshes. Figure 2.14 illustrates all the tooth loads at an instant when tooth wedging occurs at the 1\textsuperscript{st} S-P mesh with $\omega_m = 34\,Hz$. When tooth wedging occurs at this mesh, tooth loads increase on both the drive-side and back-side. The drive-side tooth load of 235\,kN increases to 210\% of its nominal tooth load (without tooth wedging the nominal tooth load is 112\,kN), and there is large back-side contact force. Because the movement of the sun toward planet 1 is stopped by tooth wedging, the distance between the sun and planet 2 (numbered counter-clockwise) is reduced. The distance between the sun and planet 3 increases compared to when tooth wedging is not allowed to occur. As a result, tooth loads at the 2\textsuperscript{nd} S-P and 2\textsuperscript{nd} R-P meshes increase, and tooth loads at the 3\textsuperscript{rd} S-P and 3\textsuperscript{rd} R-P meshes decrease. The drive-side tooth load
<table>
<thead>
<tr>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_1 = \Omega_2 = 42.202 (T)$</td>
</tr>
<tr>
<td>$\Omega_3 = 69.491 (R)$</td>
</tr>
<tr>
<td>$\Omega_4 = 301.46 (R)$</td>
</tr>
<tr>
<td>$\Omega_5 = \Omega_6 = 305.59 (T)$</td>
</tr>
<tr>
<td>$\Omega_7 = \Omega_8 = 700.06 (T)$</td>
</tr>
<tr>
<td>$\Omega_9 = 1022.4 (R)$</td>
</tr>
<tr>
<td>$\Omega_{10} = \Omega_{11} = 1134.4 (T)$</td>
</tr>
<tr>
<td>$\Omega_{12} = 1257.0 (R)$</td>
</tr>
<tr>
<td>$\Omega_{13} = \Omega_{14} = 1830.2 (T)$</td>
</tr>
<tr>
<td>$\Omega_{15} = 1983.0 (R)$</td>
</tr>
<tr>
<td>$\Omega_{16} = \Omega_{17} = 2431.2 (T)$</td>
</tr>
<tr>
<td>$\Omega_{18} = 2769.9 (R)$</td>
</tr>
</tbody>
</table>

Table 2.3: Natural Frequencies of the planetary gear described in Tables 2.1 and 2.2. R (Rotational Mode), T (Translational Mode).

of 61 kN at the 3rd S-P mesh is 54% of the nominal value. Tooth wedging elevates some tooth loads significantly and destroys their symmetry. Planet bearing failures are found in this wind turbine gearbox. The connection between bearing failure and tooth wedging is studied by looking at the dynamic bearing reaction forces when tooth wedging does and does not occur and the interaction between tooth loads and bearing forces. The dynamic tooth loads and bearing forces in a carrier cycle when tooth wedging occurs are shown in Figures 2.15 and 2.16. To remove response at mesh frequency that is minor in this case and impairs clarity of the figures, mesh
Figure 2.12: Tooth forces at the 1\textsuperscript{st} S-P mesh: (1) with gravity forces only; (2) with aerodynamic forces only; and (3) with mesh stiffness variations only. $c_x = 2385.9$, $c_y = 340.00$, $d_x = 51.244$, $d_y = 254.88 N$. $c_u = 0$, $d_u = 232.21 N\cdot m$. $\omega_m = 34 Hz$. The gear data is in Tables 2.1 and 2.2.

stiffness variation excitation is removed in Figures 2.15 and 2.16. Shown in Figure 2.15, tooth wedging at the 1\textsuperscript{st} S-P mesh occurs at 0.90 carrier cycle. The radial planet bearing force at this mesh increases sharply at this moment because of the additional back-side tooth load. Without tooth wedging, the radial bearing forces at all planets are almost zero. At the same instant, the tangential bearing reaction force on planet 2 (Figure 2.15) reaches its maximum of 75\% larger than in the absence of tooth wedging. This is because tooth forces increase at the 2\textsuperscript{nd} S-P and 2\textsuperscript{nd} R-P meshes (Figure 2.16). The tangential bearing reaction force on planet 3 (Figure 2.15) decreases to its minimum because tooth forces decrease at the 3\textsuperscript{rd} S-P and 3\textsuperscript{rd} R-P meshes (Figure 2.16). The destroyed load sharing pattern continues similarly as the carrier rotates as shown in Figure 2.15. The planet bearing force peaks when tooth wedging occur at the 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd} S-P meshes are summarized in Table 2.4. The highly elevated planet bearing forces, caused by the elevation of tooth loads from tooth wedging, can result in
Figure 2.13: Percentage of tooth wedging considering all meshes in a carrier cycle vs. carrier-ring bearing clearance when $\Delta_s = \Delta_r = 0.05 \text{mm}$ (dotted line), $\Delta_s = \Delta_r = 0.50 \text{mm}$ (dashed line), and $\Delta_s = \Delta_r = 0.75 \text{mm}$ (solid line). $\omega_m = 34 \text{ Hz}$, and gear data is in Tables 2.1 and 2.2.

Table 2.4: Planet bearing forces when tooth wedging occurs at the 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd} S-P meshes for three-planet planetary gears. max=maximum, min=minimum, t=tangential, r=radial, BF=bearing force.

<table>
<thead>
<tr>
<th>Tooth Wedging</th>
<th>Planet 1 BF</th>
<th>Planet 2 BF</th>
<th>Planet 3 BF</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} S-P mesh</td>
<td>max (r)</td>
<td>max (t)</td>
<td>min (t)</td>
</tr>
<tr>
<td>2\textsuperscript{nd} S-P mesh</td>
<td>min (t)</td>
<td>max (r)</td>
<td>max (t)</td>
</tr>
<tr>
<td>3\textsuperscript{rd} S-P mesh</td>
<td>max (t)</td>
<td>min (t)</td>
<td>max (r)</td>
</tr>
</tbody>
</table>

bearing failure. The mechanism of tooth wedging is analyzed by studying planetary gear translational displacements. The rotational displacements of gear components are much less influential to tooth wedging. The translational displacements of the
carrier, ring, sun, and planet 1 with tooth wedging at the S-P and R-P meshes are shown in Figure 2.17. Translations of the ring and sun are significant, while other component dynamic translations are small. This suggests the mechanics of tooth wedging can be analyzed without knowing planet displacements. To develop and explain a method to predict tooth wedging, an idealization of zero translational vibration of the carrier and planets is imposed based on the fact that their vibration is small compared to that of the sun and ring. Tooth wedging occurrence at the S-P and R-P meshes is decided by the direction and amplitude of the relative motion between the ring and planets and between the sun and planets. Thus, according to the idealization, tooth wedging at the R-P meshes is predicted by the ring motion. The ratio $y_r/x_r$ defines the ring direction of motion, and $\psi_j$ denotes the planet $j$ location in the carrier frame. When $\psi_j = \tan^{-1} \frac{y_r}{x_r} + \pi$, where $\tan^{-1} \frac{y_r}{x_r} \in [0, 2\pi]$ denotes the two-argument arctan function, the ring moves toward planet $j$ (given the above
assuming no planet vibration). With the large ring motion, tooth wedging occurs at the $j^{th}$ R-P mesh. Figure 2.18 shows the translational displacements of the sun and ring in a carrier cycle when tooth wedging is not modeled. At 0.42 carrier cycle (point E in Figure 2.18b) $\tan^{-1} \frac{0.90}{0.52} + \pi \approx \frac{4\pi}{3}$, and the ring moves toward planet 3. If the amplitude of the ring motion is large enough compared to the tooth radial gap, this ring motion causes tooth wedging at the $3^{rd}$ R-P mesh. Tooth wedging at the $1^{st}$ and the $2^{nd}$ R-P meshes is predicted similarly at locations F and D in Figure 2.18b. These specified tooth wedging locations at the R-P meshes agree with the actual tooth wedging peaks (marked by ◦) in Figure 2.16b. Tooth wedging at the S-P meshes also occurs in this system. The translation of the sun is not primarily caused by the gravity force acting on the sun (the mass of the sun is small), but indirectly by the vibration of the ring. Shown in Figure 2.14, the ring motion toward the left-bottom
Figure 2.16: Dynamic tooth forces at the (a) S-P and (b) R-P meshes on planet 1 (solid line), planet 2 (dashed line), and planet 3 (dotted line) with \( \omega_m = 34 \text{ Hz} \). \( \Delta_s = \Delta_r = 0.90 \text{ mm} \). Other gear data is given in Tables 2.1 and 2.2. ○ on the horizontal axis indicates tooth wedging peaks. Drive-side tooth forces at the S-P and the R-P meshes are fluctuating around 112 kN, and back-side tooth forces are up to 66 (S-P meshes) and 120 (R-P meshes) kN.

(same as the gravity direction) drives planet 2 to vibrate counter-clockwise (opposite to its vibration direction in the absence of gravity). At this instant, the floating sun moves 692 \( \mu m \) toward planet 1 (see arrow in Figure 2.14) driven by the elevated S-P tooth load at planet 2. This large sun motion causes tooth wedging at the 1st S-P mesh. As the gears rotate, the motions of the sun toward other planets are similarly caused by the ring through reversing the rotational vibration of the next adjacent planet in the sequence.

Tooth wedging at the S-P meshes is predicted by the motion of the sun (the motions of planets are assumed negligible). The ratio \( y_s/x_s \) indicates the direction of sun motion. When \( \tan^{-1} \frac{y_s}{x_s} = \psi_j \), the sun vibrates toward planet \( j \) so that tooth
Figure 2.17: Translational displacements of the planetary gear with tooth wedging. ξ_j, η_j are the radial and tangential translations of planets. ω_m = 34 Hz and gear data given in Tables 2.1 and 2.2. Where tooth wedging (TW) occurs is labeled at the figure bottom (S-P meshes) and at the figure top (R-P meshes).

Wedging may occur. At 0.22 carrier cycle (point A in Figure 2.18a), \( \tan^{-1} \frac{y_s}{x_s} = \tan^{-1} \frac{0.87}{-0.50} \approx \frac{2\pi}{3} \) and the sun moves toward planet 2. Tooth wedging occurs at the 2nd S-P mesh. Tooth wedging at the 1st and the 3rd S-P meshes is predicted at locations C and B in Figure 2.18a. These specified tooth wedging locations match with the actual tooth wedging peaks at the S-P meshes shown in Figure 2.16a (marked by o).

The effects of the mass of the carrier, sun, and ring on tooth wedging are shown in Figure 2.19. Whether tooth wedging occurs is indicated by the percentage of tooth wedging at any of the meshes over a carrier cycle. The percentage of tooth wedging over a carrier cycle is calculated at various values of the mass of the carrier, sun, and ring. Tooth wedging occurs for \( m_r \geq 5500 \text{ kg} \) for the system with changing \( m_r \). This minimum value of \( m_r \) for tooth wedging to occur is much smaller than the actual ring mass (28145 kg). With increasing \( m_r \), the percentage of tooth wedging over a
carrier cycle increases gradually. For the system with a heavy carrier or sun, meshing
teeth engage in simultaneous drive-side and back-side contact (tooth wedging) at
\( m_c \geq 4156\, \text{kg} \) or \( m_s \geq 1764\, \text{kg} \). Tooth wedging can occur in planetary gears with
other heavy components besides the ring. The mesh phasing rules [58, 96, 36] that
reduce certain vibrations are derived based on the symmetry of planets and the
periodicity of the mesh forces. They can suppress certain mesh frequency harmonics
in the dynamic response. For the system described in Tables 2.1 and 2.2 with no
gravity (inset of Figure 2.20), only the \( 3^{\text{rd}} \) harmonics of mesh frequency are excited
in the rotational response of the sun/ring/carrier, where \( l = 1, 2, \ldots \). By disrupting
the planet symmetry, gravity forces excite the \( 1^{\text{st}}, 2^{\text{nd}}, 3^{\text{rd}}, 4^{\text{th}}, \) and \( 5^{\text{th}} \) harmonics of
mesh frequency in the ring’s rotation as shown in Figure 2.20. Even the amplitudes of
corresponding harmonics change significantly when gravity is introduced. The mesh
phasing rules do not apply in planetary gears when gravity is significant.
Figure 2.19: Percentage of tooth wedging (at any mesh) in a carrier period with varying mass of the sun (solid line), carrier (dashed line), and ring (dotted line) when $\omega_m = \Omega_7$, given $k_c = 126 \times 10^6 \, \text{kg-m}^2$, $m_l = 100 \, \text{kg}$ ($l = c, r, s$), $\Delta_{cr} = 0.5 \, \text{mm}$. Other gear data is in Tables 2.1 and 2.2.

### 2.5 Conclusions

A 2D dynamic model is developed to examine the dynamics of spur planetary gears including tooth wedging, tooth separation, bearing clearance, mesh stiffness variation, gravity excitation, and other external excitations. Considering tooth wedging is induced by significant in-plane translational gear component motions, the proposed two-dimensional mathematical model may be sufficient to study tooth wedging for helical planetary gears with similar behavior.

Tooth wedging elevates planet bearing forces dramatically and destroys load sharing among the planets. Tooth wedging is a possible source of bearing failure. For the wind turbine system modeled in this chapter, the results show significant tooth wedging. In general, tooth wedging is prone to occur in planetary gears with a heavy component.
Tooth wedging is the combined effect of gravity and bearing clearance nonlinearity. Bearing clearance admits greater translational vibration, while gravity is the dominant excitation source causing the large motions that lead to tooth wedging. For planetary gears with one or more heavy components, the fundamental impact of gravity highlights the need to model it in such systems.

The mechanics of gravity driven gear motions lead to guidelines that predict when tooth wedging occurs. The guidelines apply when any of the ring, sun, or carrier is the heavy component.

Gravity disrupts the planet symmetry and thus breaks the mesh phasing rules that can reduce planetary gear vibration in selected harmonics of mesh frequency.

Figure 2.20: Frequency spectrum of the ring’s rotational displacement with gravity (solid line) and without gravity (dashed line). $\omega_m = 34 \text{ Hz}$. Gear data is in Tables 2.1 and 2.2.
CHAPTER 3
DYNAMIC ANALYSIS OF PLANETARY GEARS WITH
BEARING CLEARANCES

3.1 Summary

Bearing clearance is unavoidable, especially in systems where bearings are large or
cost concerns prevent use of high precision bearings. Supports for the central mem-
bers (sun, ring, and carrier) in planetary gears might have deliberate clearance to
achieve a “floating” support intended to have minimal stiffness restricting transla-
tional motion. Floating supports are frequently used to improve planet load sharing.
Bearing clearance in planetary gears has, however, not been investigated in the past.

Various bifurcations, coexisting solutions, chaos, and other complicated nonlinear
behavior have been studied in vibro-impacting systems [67, 69, 97, 98, 65, 66, 68,
70, 71, 72]. These nonlinear effects have not been discovered in planetary gears.
This study demonstrates evidence of these nonlinear behaviors, specifically, grazing
bifurcation, secondary Hopf bifurcation, coexisting solutions, and chaos.

The softening effect caused by gear tooth separation has been well-studied [43, 52,
21, 99, 54, 100, 101, 35, 46]. This study extends the foregoing studies by investigating
the nonlinear response of planetary gears when both bearing clearance and tooth
separation are present and, furthermore, examining the interaction between these
two nonlinearities.
The harmonic balance method with arc-length continuation is an efficient tool to obtain forced dynamic response in mechanical systems [46, 42, 38, 44, 45]. In this study, the semi-analytical harmonic balance method is formulated and applied to the lumped-parameter model with bearing clearance and tooth separation. This approach employs arc-length continuation [47] to trace amplitude-frequency branches in the nonlinear resonance regions. Solution stability is determined by Floquet multipliers [49, 50, 48]. The formulated harmonic balance is verified by finite element analysis and numerical integration.

In summary, the major objectives of the present work are to: establish lumped-parameter and finite element models of planetary gears including bearing clearance, tooth separation, and mesh stiffness variation; apply the harmonic balance method with arc-length continuation and stability analysis to obtain the dynamic response; investigate nonlinear behavior, bifurcations, and chaos caused by bearing clearance; study and explain the interplay between tooth separation and bearing clearance.

3.2 Dynamic Modeling of Planetary Gears with Bearing Clearance

3.2.1 Two-dimensional Analytical Model

The lumped-parameter model of a planetary gear in Fig. 3.1 considers the carrier, ring, sun, and planets as rigid bodies with each having two translational and one rotational degree of freedom. Translational displacements $x_l, y_l \ (l = c, r, s)$ are assigned to the carrier, ring, and sun. The radial and tangential displacements of the planets are denoted by $\xi_i, \eta_i \ (i = 1, 2, \ldots, N)$ with respect to the carrier-fixed reference frame. $N$ is the number of planets. The rotational displacements are $u_v = r_v \theta_v$.
Figure 3.1: Translational-rotational, lumped-parameter model of a planetary gear.

\( v = c, r, s, 1, \ldots, N \) where \( \theta_v \) is the rotation in radians, and \( r_v \) is the base circle radius for the sun, ring, and planets and the radius to the planet center for the carrier. The model has \( 3(N + 3) \) degrees of freedom.

**Nonlinear Bearing Model**

The bearings are modeled as circumferentially distributed radial springs with radial clearance as shown in Fig. 3.2. Forces develop only when the relative displacement between the connected bodies exceeds a specified clearance.

For the \( i^{th} \) planet bearing with clearance \( \Delta_p \) as an example, the relative displacement between the carrier and planet \( i \) is

\[
\delta_i = \left[ (x_c \cos \psi_i + y_c \sin \psi_i - \zeta_i)^2 + (-x_c \sin \psi_i + y_c \cos \psi_i + u_c - \eta_i)^2 \right]^{1/2} \tag{3.1}
\]

where \( \psi_i \) denotes the angular position of planet \( i \). The direction of the developed force is determined by the contact angle \( \vartheta_i \)

\[
\vartheta_i = \tan^{-1} \left( \frac{-x_c \sin \psi_i + y_c \cos \psi_i + u_c - \eta_i}{x_c \cos \psi_i + y_c \sin \psi_i - \zeta_i} \right) \tag{3.2}
\]
For a planet bearing with stiffness $k_p$, the amplitude of the bearing force equals

$$f_i = \mu_i k_p (\delta_i - \Delta_p)$$

(3.3)

where $\mu_i$ determines whether the bearing is in contact according to

$$\mu_i = \begin{cases} 
1 & \text{if } \delta_i > \Delta_p \\
0 & \text{if } \delta_i < \Delta_p
\end{cases}, \ i = 1, \ldots, N$$

(3.4)

Figure 3.2: Stiffness model of the bearings with clearance in [1].

**Tooth Separation Model**

The gear meshes are modeled as lumped springs along the lines of action that act only when the teeth are in contact. Tooth separation is identified through the sign of the mesh deflection.
The mesh deflections $\delta_{si}$ and $\delta_{ri}$ of the $i^{th}$ sun-planet and ring-planet meshes are [25]

$$\delta_{si} = -x_s \sin \psi_{si} + y_s \cos \psi_{si} - \zeta_i \sin \alpha_s - \eta_i \cos \alpha_s + u_i + u_s$$

$$\delta_{ri} = -x_r \sin \psi_{ri} + y_r \cos \psi_{ri} + \zeta_i \sin \alpha_r - \eta_i \cos \alpha_r - u_i + u_r$$

$$\psi_{si} = \psi_i - \alpha_s, \quad \psi_{ri} = \psi_i + \alpha_r$$

(3.5)

where $\alpha_s$ and $\alpha_r$ are the pressure angles. The gear teeth separate when $\delta_{si}$ or $\delta_{ri}$ are negative.

Mesh stiffness fluctuates as the number of teeth in contact changes periodically between, typically, one and two at each mesh. The mesh stiffness variation is an important excitation source that can cause parametric instability in planetary gears [55]. This parametric excitation is introduced through time-varying mesh stiffnesses $k_{si}(t)$ and $k_{ri}(t)$ at the sun-planet and ring-planet meshes. It is the only excitation source considered in this model. The fluctuating mesh stiffness is determined by quasi-static finite element analysis of the individual sun-planet and ring-planet gear pairs at 200 points over a mesh cycle as shown in Fig. 3.3 for the example system defined in Tables 2.1 and 2.2.

**Equations of Motion**

To improve numerical accuracy and reduce computation effort, the equations of motion in [1] are non-dimensionalized by introducing $\tau = t/\omega$ and $z = x/d$, where $\omega$ is the sixth natural frequency as a characteristic frequency and $d = 1\mu m$ is the characteristic length, giving

$$z'' + \ddot{C}z' + \ddot{f}_m(\tau, z) + \ddot{f}_b(\tau, z) = \ddot{F}(\tau)$$

(3.6)

$$\ddot{C} = \frac{M^{-1}C}{\omega}, \quad \ddot{f}_m = \frac{M^{-1}f_m}{d\omega^2}, \quad \ddot{f}_b = \frac{M^{-1}f_b}{d\omega^2}, \quad \ddot{F} = \frac{M^{-1}F}{d\omega^2}$$

(3.7)
Figure 3.3: Time-varying mesh stiffness of the sun-planet (−) and ring-planet (−−) meshes of the planetary gear calculated by finite element analysis.

\[
\mathbf{x} = \left\{ \begin{array}{c}
x_C, y_C, u_C, x_r, y_r, u_r, x_s, y_s, u_s, x_1, y_1, u_1, \ldots, x_N, y_N, u_N \\
\text{Carrier} & \text{Ring} & \text{Sun} & \text{Planet 1} & \ldots & \text{Planet } N
\end{array} \right\} \tag{3.8}
\]

where \( \mathbf{f}_m \) denotes the tooth force vector and \( \mathbf{f}_b \) denotes the bearing force vector. Formulations of \( \mathbf{M}, \mathbf{C}, \mathbf{f}_m, \mathbf{f}_b, \) and \( \mathbf{F} \) are detailed in [1].

### 3.2.2 Finite Element Model

The finite element model of the examined planetary gear with a fixed ring gear is established in the software Calyx developed by Vijayakar [92]. This software provides efficient dynamic analysis compared to conventional finite element methods by employing a semi-analytical contact formulation that allows a rather coarse mesh near the contact surfaces on the gear teeth. This model incorporates precise tooth geometry including tooth profile and lead modifications, backlash, and root shape.

A unique contact analysis is used to determine dynamic forces at gear teeth. It combines the surface integral method in the near-field of contact and classical finite element method in the far-field at a matching plane underneath the contact surface.
detailed in [52, 51, 54]. Tooth contact is analyzed at each time step while the gears rotate according to nominal gear kinematics. Thus, mesh stiffness variation and tooth separation are inherently included with no extra modeling effort. Results computed by this software have been successfully correlated with experiments in geared systems [52, 54, 32].

Bearings in the finite element model are modeled by a set of discrete, circumferentially distributed springs. Clearance in the bearing is modeled by radial gaps connected to each spring. To fully represent the nonlinear bearings, infinite spring/gap pairs are required. Many static analyses are performed on the examined planetary gear (described in Tables 2.1 and 2.2) to determine a sufficient number of spring/gap pairs that balances the computational effort and result accuracy. Fig. 3.2 shows the sun bearing deformation $d_s$ with nominal torque (1130 Nm) and 5 $\mu$m clearance (0.01% of the bearing outer diameter) in this bearing for various numbers of equally-spaced spring/gap pairs. The bearing deformation with 72 sets of spring/gap pairs $d_{s,0}$ is considered as the comparison baseline because the exact solution is unknown. The relative error of the bearing deformation $\frac{d_s - d_{s,0}}{d_{s,0}}$ with two spring/gap pairs is 6%. The error drops to 1% when five spring/gap pairs are used. In this study, six spring/gap pairs are used for all bearings with clearance.

This finite element analysis provides a benchmark to validate the lumped-parameter model.
Figure 3.4: The relative error of the sun bearing deformation \( \frac{d_s - d_{s,0}}{d_{s,0}} \) in the \( x \)-direction for varying number of bearing spring/gap pairs between races. The nominal torque 1130 Nm is applied, and the sun bearing clearance is 5 \( \mu \)m.

### 3.3 Harmonic Balance Method

#### 3.3.1 Harmonic Balance Formulation

The harmonic balance method [41] is used to obtain the dynamic response of the model in (3.6). The proposed method calculates the steady state response in the frequency domain.

Each component \( z_h(t) \) in \( z \) is expanded in a Fourier series containing \( R \) harmonics of the excitation frequency \( \Omega \),

\[
z_h = q_{h,1} + \sum_{j=1}^{R} \left[ q_{h,2j} \cos j\Omega t + q_{h,2j+1} \sin j\Omega t \right]
\] (3.9)

The Fourier coefficient vector \( \mathbf{q} \) is then

\[
\mathbf{q} = \begin{bmatrix}
q_{x_0,1}, \cdots, q_{x_0,2R+1}, \cdots, q_{u_N,1}, \cdots, q_{u_N,2R+1} \\
q_{x_0(\Omega)}, \cdots, q_{x_N(\Omega)}
\end{bmatrix}^T
\] (3.10)
The time domain is discretized into \( n-1 \) evenly-distributed time intervals \( t_1, \ldots, t_n \).

Each \( z_h \) in the response \( z \) is extended into a vector in the time domain as \( [z_h(t_1), \ldots, z_h(t_n)]^T \).

By defining the operators \( L, D, \) and \( E \), which are transformation matrices detailed in the Appendix, the response vector and its derivatives transform into \( z = Lq, z' = \Omega LDq, \) and \( z'' = -\Omega^2 LEq \).

The nonlinear force and excitation vectors are transformed as

\[
\tilde{f}_m = Lg_m; \quad \tilde{f}_b = Lg_b; \quad \tilde{F} = Lg_0
\]  

Substitution of these results into (3.6) yields

\[
L[(-\Omega^2 E + \Omega \tilde{C}D)q + g_b + g_m - g_0] = 0
\]  

By defining

\[
K = -\Omega^2 E + \Omega \tilde{C}D
\]  

Eqn. (3.12) becomes

\[
Kq = g_0 - g_b - g_m
\]  

Equation (3.14) requires much less computational effort to solve than Eqn. (3.6).

The Newton method is used in this study. The analytical effort of this method is concentrated in the formulation of the Jacobian matrix

\[
J = \frac{\partial(Kq - g_b - g_m - g_0)}{\partial q}
\]  

To obtain \( J \), the inverse transformation operator \( H \) is defined, mapping from the time domain to the frequency domain such that \( q = Hz \). The explicit form of \( J \) is

\[
J = K + K_{TI} + \frac{H}{\partial q}g_b L + \frac{H}{\partial q}g_m L
\]  

where \( K_{TI} \) contains all the linear, time-invariant stiffnesses, which are only the rotational support stiffnesses of each body. Components in \( J \) are described in the Appendix; they depend on time.
Using Newton iteration, the solution $\mathbf{q}$ after $\nu$ iterations is given by

$$
\mathbf{q}^{\nu+1} = \mathbf{q}^\nu + \Delta \mathbf{q}^\nu = \mathbf{q}^\nu + (\mathbf{J}^\nu)^{-1} \cdot \mathbf{f}^\nu(\mathbf{q})
$$

(3.17)

The criterion to determine convergence is satisfaction of both of

$$
\frac{\Delta \mathbf{q}^\nu}{1 + \mathbf{q}^\nu} < \epsilon, \quad \mathbf{E}^\nu(\mathbf{q}^\nu, \Omega) = \mathbf{J} \cdot \Delta \mathbf{q}^\nu - \mathbf{f}^\nu(\mathbf{q}^\nu, \Omega) < \epsilon
$$

(3.18)

where $\epsilon = 10^{-6}$ is the specified tolerance. The iteration approach using Newton method is summarized in a flow chart in Fig. 3.5. A line search technique [93] for Newton’s method is used to improve the convergence rate.

![Flow chart of the iteration solver in the harmonic balance method.](image)

Figure 3.5: Flow chart of the iteration solver in the harmonic balance method.
Smoothing functions are used to approximate the piecewise nonlinear bearing and tooth forces as shown in Fig. 3.6. The smoothing function of the planet $i$ bearing force is

$$w_i = \delta_i + \frac{1}{2}[1 + \tanh \sigma(\delta_i - \Delta_p)](\delta_i - \Delta_p) - \frac{1}{2}[1 + \tanh \sigma(\delta_i + \Delta_p)](\delta_i + \Delta_p)$$  \hfill (3.19)

The smoothing function describing the nonlinear tooth force is

$$w_{q_i} = \frac{1}{2} \delta_{q_i}[1 + \tanh \sigma(\delta_{q_i})] \, , \, q = s, r$$  \hfill (3.20)

where the quantity $\delta_{q_i}$ denotes the mesh deflection at the $i^{th}$ sun-planet or ring-planet mesh. The parameter $\sigma$ controls the deviation of the smoothing functions from the piece-wise bearing force or tooth load as shown in Fig. 3.6. With large $\sigma$, the smoothing functions converge to the piece-wise forces. Extremely large $\sigma$, however, disrupts the iteration convergence. $\sigma$ is chosen as 100 here based on a convergence study.

### 3.3.2 Arc-length Continuation

Arc-length continuation [47] is employed to trace the solution branches near nonlinear resonances.

A proper step size $ds$ in the arc-length direction is essential for the iteration convergence, especially in high-dimensional systems with multiple nonlinearities. Away from resonances, $ds$ is large as the response curve is flat. Near and at resonances, $ds$ is reduced significantly as the response curve makes sharp turns. In this study, $ds$ is automatically adjusted at each speed based on the convergence rate at the last speed and curvature of the response branch at the present speed. Thus,

$$\|ds\| = ds_{\min} 2^{1 - \max(\frac{S_p}{S_{\text{max}}}, \frac{12s}{\sigma})}$$  \hfill (3.21)
Figure 3.6: (a) Smoothing functions for the bearing reaction force with clearance; (b) smoothing functions for the tooth load with backlash.

where \( ds_{\text{min}} \) is the minimum arclength step (0.1 \( \mu m \)) on \( \|ds\| \). The variables \( \beta \) and \( \beta_{\text{max}} \) denote the number of iterations for the last speed and the maximum number of iterations (90, predefined). \( \alpha \) is the angle of the lines passing through the last three iterations. Other ways to determine \( ds \) can be found in [48] depending on the complexity of the system.

### 3.3.3 Stability

Solution stability is determined using Floquet-Liapunov theory. First, a small perturbation \( \Delta z \) to the calculated periodic response \( z \) is introduced and the full equations of motion (3.6) are linearized for small \( \Delta z \) to give

\[
\begin{pmatrix}
\Delta z' \\
\Delta z''
\end{pmatrix} =
\begin{bmatrix}
0 & I \\
-[J(t) - K] & -\tilde{C}
\end{bmatrix}
\begin{pmatrix}
\Delta z \\
\Delta z'
\end{pmatrix}
\tag{3.22}
\]

where the time-dependent \( J \) is already available from (3.16) as a by-product of the Newton solution technique. \( K \) is defined in (3.13).
The monodromy matrix $\Phi(T, 0)$ of the linear system (3.22) is computed numerically by using the fourth-order Runge-Kutta scheme and integrating the transformation matrices in multiple time intervals [102]. $T$ is the period of the excitation frequency $\Omega$. The time intervals are sufficiently small to avoid truncation error during the integration. Extremely small intervals introduce round-off error and require long computation time. The time interval in this study is 0.1% to 0.01% of the examined period of frequency $\Omega$.

Eigenvalues of the monodromy matrix are Floquet multipliers that characterize the solution stability and bifurcation. When the magnitude of any of the Floquet multipliers is larger than one, the solution becomes unstable. Three bifurcations can be identified by the Floquet multipliers:

1. Period-doubling bifurcation: A real Floquet multiplier escapes the unit circle, which is the boundary between stability and instability, at negative one. At this point, the fundamental frequency of the stable solution is divided by two.

2. Saddle-node bifurcation: A real Floquet multiplier escapes the unit circle at positive one. The solution loses stability, or gains it if the multiplier enters the unit circle.

3. Secondary Hopf bifurcation: A pair of complex Floquet multipliers leave the unit circle from complex numbers. A quasi-chaotic regime is generated afterwards.

### 3.4 Results and Discussion

The examined planetary gear detailed in Tables 3.1 and 3.2 is a variation of that in a helicopter drive train. This planetary gear includes two pairs of diametrically opposed
planets. The planets are mounted at $0, \frac{32\pi}{63}, \pi, \text{ and } \frac{95\pi}{63}$ measured counter-clockwise with respect to the $x_c$ axis in Fig. 3.1. The lowest seven natural frequencies without considering any bearing clearance are translational modes at 1761, 1771, 3282, and 3302 Hz; rotational modes at 2181 and 5281 Hz; and a planet mode at 3582 Hz. The mode types are defined in [25, 26].

<table>
<thead>
<tr>
<th></th>
<th>Sun</th>
<th>Planet</th>
<th>Ring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Teeth</td>
<td>27</td>
<td>35</td>
<td>99</td>
</tr>
<tr>
<td>Outer Diameter (mm)</td>
<td>84.07</td>
<td>105.0</td>
<td>304.8</td>
</tr>
<tr>
<td>Root Diameter (mm)</td>
<td>70.49</td>
<td>91.44</td>
<td>284.2</td>
</tr>
<tr>
<td>Base Diameter (mm)</td>
<td>70.40</td>
<td>91.26</td>
<td>258.1</td>
</tr>
<tr>
<td>Transverse Tooth Thickness (mm)</td>
<td>4.470</td>
<td>4.470</td>
<td>3.124</td>
</tr>
<tr>
<td>Module (mm)</td>
<td>2.870</td>
<td>2.870</td>
<td>2.780</td>
</tr>
<tr>
<td>Fillet Radius</td>
<td>−</td>
<td>−</td>
<td>1.473</td>
</tr>
<tr>
<td>Hob Tip Radius</td>
<td>1.041</td>
<td>0.457</td>
<td>−</td>
</tr>
<tr>
<td>Pressure Angle</td>
<td>24.60°</td>
<td>20.19°</td>
<td></td>
</tr>
<tr>
<td>Sun/Planet Center Distance (mm)</td>
<td>88.89</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Geometric parameters of the planetary gear.

Damping in the finite element model includes material damping and bearing damping. Material damping is represented as $\mathbf{C}_m = \alpha \mathbf{M} + \beta \mathbf{K}$, where $\alpha = 10 \text{sec}$ and $\beta = 10^{-8} \text{sec}^{-1}$. The damping matrices of the sun and carrier bearings are
<table>
<thead>
<tr>
<th></th>
<th>Sun</th>
<th>Carrier</th>
<th>Planet</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass (kg)</td>
<td>1.640</td>
<td>21.82</td>
<td>1.330</td>
</tr>
<tr>
<td>$I/r^2$ (kg)</td>
<td>1.560</td>
<td>24.80</td>
<td>2.460</td>
</tr>
<tr>
<td>Bearing Stiffness (N/m)</td>
<td>$2.190 \times 10^9$</td>
<td>$2.190 \times 10^9$</td>
<td>$2.190 \times 10^9$</td>
</tr>
</tbody>
</table>

Table 3.2: System parameters of the planetary gear.

<table>
<thead>
<tr>
<th></th>
<th>$c_{r,s}$</th>
<th>$c_{\theta,s}$</th>
<th>$c_{r,p}$</th>
<th>$c_{\theta,p}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3152 Ns/m</td>
<td>0.452 Nms</td>
<td>5253 Ns/m</td>
<td>0.282 Nms</td>
</tr>
</tbody>
</table>

Table 3.3: Bearing damping used in the finite element model.

$C_{b,s} = C_{b,c} = \text{diag}(c_{r,s}, c_{r,s}, c_{\theta,s})$ The damping matrices of the planet bearings are $C_{b,p} = \text{diag}(c_{r,p}, c_{r,p}, c_{\theta,p})$. These damping coefficients are given in Table 3.3.

Constant modal damping ratios are used in the lumped-parameter model. These ratios are obtained from numerical torque impulse tests on the finite element model to find the frequency response function, with damping ratios obtained from the half-power points [103]. The damping ratios corresponding to the aforementioned natural frequencies are given in Table 3.4.

The harmonic balance and numerical integration methods are applied to the lumped-parameter model to obtain the dynamic response. Numerical integration verifies the solution accuracy of the harmonic balance method. Numerical integration and finite element analysis include results for increasing and decreasing speeds. The steady state response at the prior speed is used as the initial guess of the solution at next speed. The harmonic balance method uses arc-length continuation to trace
Table 3.4: Damping used in the analytical model.

<table>
<thead>
<tr>
<th>Natural frequency</th>
<th>Damping ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1761 Hz (Translational)</td>
<td>1.25%</td>
</tr>
<tr>
<td>1771 Hz (Translational)</td>
<td>1.25%</td>
</tr>
<tr>
<td>2181 Hz (Rotational)</td>
<td>1.01%</td>
</tr>
<tr>
<td>3282 Hz (Translational)</td>
<td>0.84%</td>
</tr>
<tr>
<td>3302 Hz (Translational)</td>
<td>0.84%</td>
</tr>
<tr>
<td>3582 Hz (Planet)</td>
<td>0.74%</td>
</tr>
<tr>
<td>5281 Hz (Rotational)</td>
<td>2.49%</td>
</tr>
</tbody>
</table>

Numerical integration and finite element analysis compute only stable response.

The nondimensional quantities $\rho_n = \frac{2\Delta_n}{a_n}$, $n = c, s, p$ represent the relative amplitudes of the bearing clearances $\Delta_n$ in the carrier, sun, and planet supports compared to characteristic static translational deflections $a_n$ in a mesh cycle. $a_c$ and $a_s$ are the mean amplitudes over a mesh cycle of the radial deflections of the carrier and sun with infinite bearing clearance. $a_p$ is the mean amplitude of the radial deflection of the planet bearings restrained by a bearing stiffness ($k_p$) without clearance. For the system considered, $a_c = 2.0 \mu m$, $a_s = 3.8 \mu m$, and $a_p = 6.7 \mu m$.

3.4.1 Nonlinear Dynamics with Bearing Clearance in Central Members

Bearing forces of the central members are small because they are nearly (or sometimes exactly) balanced by almost self-equilibrating tooth loads at the sun-planet and ring-planet meshes. Clearances in such bearings create distinctive dynamic behavior.
Natural Frequency Reduction

Bearing clearance reduces the natural frequencies of any mode shapes having meaningful translation of the member with bearing clearance. This affects only the translational modes in which the central members have translational motions but no rotation. Fig. 3.7(a) shows the dynamic response of $x_c$ for mesh frequencies from $300\ Hz$ to $2000\ Hz$ with various carrier bearing clearances $\rho_c$. The resonance at the first translational mode shifts from $1767\ Hz$ (limiting system without clearance) to $1080\ Hz$ (another limiting system with infinite clearance) when $\rho_c$ increases from 0 to 30. The mode shapes of the two limiting systems are different. With extremely small or large $\rho_c$, the response converges to one of these two limiting cases. Central member bearing clearances ($\rho_c, \rho_s$, and $\rho_r$) do not affect the rotational and planet modes because the central members have no translational motion in those modes.

Figure 3.7(b) shows the percent of bearing contact in a mesh cycle under the conditions of Fig. 3.7(a). When $\rho_c$ is small, the bearing vibration is large compared to the clearance gap. Therefore, the bearing is in contact for a wide speed range. The frequency of this translational mode is slightly reduced. With large $\rho_c$, the bearing comes into contact only when the vibration is significant. The frequency of the mode significantly decreases. $\rho_c$ is the key parameter characterizing the natural frequency reduction and the strength of the bearing clearance nonlinearity.

Nonlinear Effects

Nonlinear effects are present in Fig. 3.7(a) when $\rho_c$ is between 1 and 15. The response backbone curves bend to the right, showing hardening at the resonances. More detail of the strong hardening effect at the $1^{st}$ translational mode near $1500\ Hz$ is shown in Fig. 3.8. Distinctly different nonlinear resonance occurs at the $2^{nd}$ translational mode near $2600\ Hz$ with $\rho_s$ (natural frequency of $3282\ Hz$ without clearance) in Fig.
3.9. These hardening effects are induced by the transition from no bearing contact to contact. For instance, as shown in Fig. 3.8, away from the resonance the vibration amplitude is too low to cause bearing contact. The bearing with clearance is entirely floating. When vibration is elevated near the primary resonance, the bearing comes into partial contact at 870 Hz. The bearing evolves gradually from partial contact to full contact near 1065 Hz. The hardening effect reflects the increase in the bearing stiffness that results from the increased bearing contact. Additional stable response curves between the upper and lower response curves appear at the resonances in Fig. 3.8 and Fig. 3.9. They are associated with partial bearing contact and tooth separation, as discussed later.

Gear tooth separation is a major nonlinear behavior in geared systems [52, 43, 38, 35, 46]. It introduces a softening nonlinearity that competes with the aforementioned hardening in systems with bearing clearance. This leads to unique resonances. Figure 3.10(a) shows the dynamic response of the rotational vibration of planet 1 with $\rho_c = 5$. Resonances at 870 Hz and 1065 Hz are the 4th harmonic of mesh frequency exciting the 1st planet mode and the 2nd harmonic exciting the 1st rotational mode, respectively. The carrier bearing comes into full contact from partial contact at the resonance of 1500 Hz (hardening effect). This corresponds to the stable branch bending to the right at the resonance at 1500 Hz (branch A). No tooth separation occurs along this branch. In the range 1570 Hz to 1610 Hz, significant rotational vibration causes tooth separation, resulting in zero mesh stiffness for parts of each mesh cycle (softening effect). The sometimes vanishing tooth loads at the sun-planet and ring-planet meshes with tooth separation are shown in Fig. 3.10(b). This softening corresponds to another stable branch bending to the left at the resonance (branch B). The branch connecting those two stable branches is unstable. This nonlinear effect
with a closed loop at the resonance reflects the competition between the softening and hardening.

Results in Fig. 3.8, 3.9, and 3.10(a) are calculated using lumped-parameter and finite element models. The agreement between these two models is excellent.

**Coexisting Solutions**

Coexisting orbits are frequently exhibited in the dynamic response of this planetary gear. Shown in Fig. 3.8, four stable branches are present at the primary resonance of the 1st translational mode within the speed range from 1380 Hz to 1610 Hz. From the top curve to the bottom, these curves include the stable solutions with the bearing and teeth in full contact, bearing in full contact and tooth separation, bearing in partial contact and teeth in full contact, and bearing in no contact and teeth in full contact. Arc-length continuation successfully traces the short but stable middle branches, which are difficult to obtain using numerical integration or finite element analysis. Figures 3.9 and 3.10(a) show similar stable response curves associated with partial bearing contact in the middle of the nonlinear resonances.

Two distinct response regimes (labeled A and B) are present with \( \rho_s = 2.63 \) within the speed range from 1200 Hz to 1850 Hz in Fig. 3.11(a) and Fig. 3.11(b). The low amplitude curves (regime A) are the responses with zero and partial bearing contact, which have low vibration amplitude. The 1st translational mode resonance of regime A equals 1170 Hz, a 34% reduction from that with \( \rho_s = 0 \) (1767 Hz). The high amplitude branch (regime B) is the response where the bearing is in partial and full contact. In this case the resonance of the 1st translational mode is near 1700 Hz, which is close to that with \( \rho_s = 0 \). Obtaining all responses in regimes A and B requires use of multiple initial conditions. Regime B only appears when the damping is low. Figure 3.11(b) shows the complicated response details in Fig. 3.11(a) within
the speed range from 1200 Hz to 1280 Hz. Three stable branches are associated with:
no bearing contact (lower branch, regime A), partial contact (middle branch, < 50% bearing contact in a mesh cycle, regime A), and partial contact (upper branch, > 50% bearing contact in a mesh cycle, regime B). The lumped-parameter and finite element models agree well in the responses of regimes A and B in Fig. 3.11(a) and Fig. 3.11(b) given the complexity of the nonlinear behavior.

With grazing bifurcation, the response in Fig. 3.11(a) jumps from regime A to regime B at 1575 Hz for increasing speed. Grazing bifurcation occurs when the system evolves from a non-impacting to an impacting situation or vice versa as a system parameter varies smoothly [67, 68]. With bearing clearance, grazing bifurcations occur at the transitions from no bearing contact to partial contact, partial contact to full contact, and vice versa (Fig. 3.11(a) and Fig. 3.11(b)). When these transitions take place, the solution gains or loses stability.

Coexistence of many periodic orbits is one feature of vibro-impacting dynamic systems [70]. These coexisting solutions for a given mesh frequency indicate the presence of multiple periodic orbits associated with different tooth and bearing contact situations as evident in Fig. 3.8, Fig. 3.9, Fig. 3.10(a), Fig. 3.11(a), and Fig. 3.11(b). Different initial conditions can lead to multiple different responses.

3.4.2 Nonlinear Dynamics with Planet Bearing Clearance

Planet bearing forces are orders of magnitude larger than those in the sun and carrier bearings because the tooth loads at the sun-planet and ring-planet meshes create significant tangential bearing forces. These large bearing forces cause nearly full bearing contact. As shown below, planet bearing clearance, therefore, has different impact on the system dynamics than clearances in the sun, carrier, and ring bearings.
Effects on Dynamic Response

Figures 3.12(a) and 3.12(b) compare the dynamic response of planet 1 rotational vibration without and with $\rho_p = 1.5$ in the planet 1 bearing using lumped-parameter (numerical integration and harmonic balance) and finite element models. Good agreement is apparent. The response curves without and with bearing clearance (Fig. 3.12(a) and Fig. 3.12(b)) are similar to each other at the resonances.

Planet bearing clearance actively interplays with tooth separation. Figure 3.13(a) shows the nonlinear resonance at the 1st translational mode with $\rho_p = 0, 1.5, \text{and } 3.0$ in the planet 1 bearing. The softening is caused mainly by tooth separation. With increasing $\rho_p$, the resonance backbone curve bends more to the left than without it. This extra softening is caused by bearing contact loss. Figure 3.13(b) shows the time history of planet bearing forces with planet bearing contact loss occurring at portions of a mesh cycle. The system vibration overcomes the normally large steady tangential force on the planets, which allows the bearings to lose contact.

A number of turning points appear on the upper response in Fig. 3.13(a). Each of these turns is associated with a change of tooth or bearing contact states, which illustrates the progression of tooth separation at the resonances. For instance, for decreasing mesh frequency without clearance tooth separation occurs at a pair of planets (1795 Hz) and progresses to both pairs (1750 Hz) when the response approaches the resonance peak. With $\rho_p$, tooth separation starts with one planet, then a pair of them, then three planets (the planet bearing loses contact), and finally both pairs when the vibration amplitude reaches the resonance peak. $\rho_p$ disturbs the symmetry among the dynamic tooth loads, which breaks the symmetry normally present in planetary gears.
Effect on Routes to Chaos

Two different routes to chaos in the chaotic regions indicated in Fig. 3.12 are present when $\rho_p$ is included and excluded. Figures 3.14(a) and 3.15(a) show Poincare maps of these routes computed through numerical integration.

Without bearing clearance, the period-1 solution becomes chaotic nearly instantly at $1775 \text{ Hz}$ with decreasing speed as shown in Fig. 3.14(a). Stability analysis confirms this bifurcation by tracking the evolution of Floquet multipliers when speed is varying. At $1775 \text{ Hz}$, a pair of complex Floquet multipliers leave the unit circle as shown in Fig. 3.14(b). This is a secondary Hopf bifurcation. Chaos appears nearly instantly after a short quasi-chaotic range.

When $\rho_p$ is introduced, the period-1 motion becomes the period-2 motion after the bifurcation at $1762 \text{ Hz}$ with decreasing speed as shown in Fig. 3.15(a). The period-2 motion then changes into a short period-4 motion. The fundamental frequency of the solution continues to be halved until the motion is long-term periodic through a cascade after $1747 \text{ Hz}$. The motion is entirely chaotic at $1734 \text{ Hz}$. Stability analysis reveals a Floquet multiplier crosses the unit circle at negative one at $1762 \text{ Hz}$ as shown in Fig. 3.15(b), indicating a period-doubling bifurcation. Chaos in Fig. 3.14(a) and Fig. 3.15(a) is stabilized by saddle-node bifurcations at $1692 \text{ Hz}$ (Fig. 3.14(a)) and $1684 \text{ Hz}$ (Fig. 3.15(a)).

Swift and Weisenfeld [104] investigated the suppression of period-doubling bifurcation by symmetry in damped, time-dependent mechanical systems. Bearing clearance in the planets destroys the symmetry of tooth loads so that period-doubling bifurcation can occur. Symmetry breaking caused by planet bearing clearance changes the bifurcations and routes to chaos.

The above agreement between Poincare maps calculated by numerical integration
and stability analyses embedded in the harmonic balance method further verifies the harmonic balance formulation.

### 3.4.3 Effect of Input Torque

The static deflections of the sun, carrier, and planets ($a_n, n = s, c, p$) increase monotonically with input torque as shown in Fig. 3.16. Adding input torque increases the bearing reaction forces in nearly all planetary gears (in some cases, the net central member bearing forces are zero) by elevating tooth loads at the sun-planet and ring-planet meshes. The increased bearing force elevates the translational vibration of the bearing and $a_n$. As a result, increasing input torque reduces $\rho_n$ for constant bearing clearance $\Delta_n$.

The effect of input torque on the dynamics is shown in Fig. 3.17(a), given constant $\Delta_c = 5 \mu m$. Increasing torque significantly suppresses the hardening effect caused by $\Delta_c$, although the vibration amplitude also increases similar to observations in [46]. Fig. 3.17(b) shows the percent of bearing contact of the dynamic response under the conditions of Fig. 3.17(a). With $5650 \, N.m$ torque, the bearing with clearance is in full contact at the resonance and partial contact far away from the resonance, which leads to a nearly linear resonance near $1760 \, Hz$. Input torque also increases the percent of bearing contact with $\rho_s$ and $\rho_p$.

In general, higher input torque can suppress some of the nonlinear behavior discussed earlier by reducing the numerical value and influence of the dimensionless bearing clearance ($\rho_n$).
3.5 Conclusions

A two-dimensional, lumped-parameter model of planetary gears with bearing clearance is validated against finite element analysis. The harmonic balance method with arc-length continuation is applied to this analytical model to compute the dynamic response. Solution stability is determined using Floquet theory.

Bearing clearance in the central members markedly changes the dynamic response of planetary gears. This is especially true at the translational mode resonances, where the mode shapes have translations in the bearing with clearance. The resonant frequencies of these modes decrease with increasing clearance as the time of bearing contact loss increases.

A hardening effect caused by the transition from no bearing contact to contact is often present in the dynamic response. The softening caused by tooth separation competes with this hardening, resulting in a unique crossed loop at the resonance. Multiple coexisting solutions are present in wide speed ranges associated with different bearing contact and tooth separation situations. Different initial conditions lead to multiple different responses. Grazing bifurcation changes the solution stability and induces the response to jump between these solution curves.

Clearance in planet bearings has less impact on the dynamics than that in the central members because significant planet bearing forces cause nearly full bearing contact. It, however, interacts with tooth separation, which further softens the system with eventual contact loss of the planet bearing at portions of a mesh cycle. Chaos occurs with planet bearing clearance. Period-doubling and secondary Hopf bifurcations are the routes to chaos. With planet bearing clearance, the quasi-chaos to chaos route caused by the secondary Hopf bifurcation turns into a period-doubling cascade.
Planet bearing clearance affects the bifurcation and routes to chaos by disturbing the symmetry of the dynamic tooth loads.

Input torque can partially suppress the nonlinear behavior by reducing the numerical value, and so the effective strength, of the dimensionless parameter $\rho_n$. 
Figure 3.7: (a) The rms (root mean square, mean removed) carrier $x_c$ for varying speeds with carrier clearance $\rho_c = 0, 1, 3, 5, 7, 15, 30$, and infinity. Unstable solutions are denoted by $\color{red}--$. (b) Percent of bearing contact in a mesh cycle for varying speeds with carrier clearance $\rho_c = 0, 1, 3, 5, 7, 15, 30$. Results are calculated by harmonic balance method. The modal damping ratio of this mode equals to 2.5%.
Figure 3.8: The rms (root mean square, mean removed) carrier $x_c$ for varying speeds with $\rho_c = 5$ clearance in the carrier bearing. Results are compared among finite element analysis (□), numerical integration (○), and harmonic balance (−). Unstable branches are denoted by (−−).

Figure 3.9: The rms (root mean square, mean removed) sun $y_s$ for varying speeds with $\rho_s = 2.63$ clearance in the sun bearing. Results are compared among finite element analysis (□), numerical integration (○), and harmonic balance (−). Unstable branches are denoted by (−−).
Figure 3.10: (a) The rms (mean removed) planet 1 rotational displacement $u_1$ for varying speeds with $\rho_c = 5$. The dynamic response is compared among the finite element analysis (□), numerical integration (○), and harmonic balance (−). Unstable branches are denoted by (−−). (b) Dynamic tooth loads at the (upper) sun-planet and (lower) ring-planet meshes when mesh frequency equals $1620\, Hz$. The response is at the upper branch of the resonance in (a). Tooth loads at 1$^{st}$, 2$^{nd}$, 3$^{rd}$, and 4$^{th}$ meshes are denoted by −, −−, · · ·, − · −, respectively.
Figure 3.11: The rms (mean removed) value $x_s$ with $\rho_s = 2.63$ in the sun bearing (a) within speed range 1000 to 2000 Hz; (b) within speed range 1100 to 1450 Hz. Results are compared among finite element analysis (□), numerical integration (○), and harmonic balance (−). Unstable branches are denoted by (−−).
Figure 3.12: The rms (mean removed) planet 1 rotational displacement $u_1$ for varying speeds (a) without clearance and (b) with $\rho_p = 1.5$ clearance in the planet 1 bearing. Results are compared among finite element analysis (---), numerical integration (· · ·), and harmonic balance (−).
Figure 3.13: (a) The rms (mean removed) planet 1 rotational displacement $u_1$ for varying speeds when $\rho_p = 0$, 1.5, 3.0 in the planet 1 bearing. Unstable branches are denoted by (−−). (b) Planet bearing forces of planets 1 (−), 2 (−−), 3 (· · ·), and 4 (− · −) when mesh frequency equals to 1760 Hz and $\rho_p = 3.0$ in the planet 1 bearing. System damping ratios are 1.25, 1.25, 1.003, 1.68, 1.68, 3.37, 2.46%, which are increased to eliminate chaos. The occurrences of tooth separation (TS) and bearing contact loss are labeled.
Figure 3.14: (a) Poincare map of the velocity of planet 1 rotation without clearance computed by numerical integration; (b) Real and imaginary parts of Floquet multipliers without bearing clearance from mesh frequency $1692\;Hz$ to $1775\;Hz$. The multipliers are symmetric about the real axis. The response is at the upper branch of the resonance in Fig. 3.12(a).

Figure 3.15: (a) Poincare map of the velocity of planet 1 in rotation with $\rho_p = 1.5$ in the planet 1 bearing computed by numerical integration; (b) Real and imaginary parts of Floquet multipliers of the planetary gear with $\rho_p = 1.5$ in the planet 1 bearing from mesh frequency $1684\;Hz$ to $1762\;Hz$. The multipliers are symmetric about the real axis. The response is at the upper branch of the resonance in Fig. 3.12(b).
Figure 3.16: Sensitivity of the static deflection measures $a_n, n = s, c, p$ of the sun (−−), carrier (⋯), and planets (−) to the input torque.
Figure 3.17: (a) The rms (mean removed) carrier $y_c$ for varying speeds with $\Delta_c = 5\, \mu m$ when input torque varies from 1130 (nominal torque) to 5650Nm. Unstable branches are denoted by (−−); (b) percent of bearing contact in a mesh cycle for changing speeds under the conditions of (a). Results are calculated by the harmonic balance method.
CHAPTER 4
STIFFNESS MATRIX CALCULATION OF ROLLING ELEMENT BEARINGS USING A FINITE ELEMENT/CONTACT MECHANICS MODEL

4.1 Summary

Accurate stiffness estimation of rolling element bearings is essential to analyze the static and dynamic deformation of rotating mechanical systems.

Existing theoretical models of rolling element bearings make different assumptions about the contact between the rolling elements and races, leading to the dramatic discrepancy in their stiffness estimations discussed later in this chapter. Furthermore, theoretical models have not included some bearing details known to significantly affect bearing stiffness, such as, internal radial and axial clearances, roller and race crownings, race width and thickness, diameter of the inner raceway, and bore of the outer raceway. Theoretical models of rolling element bearings, therefore, provide stiffness estimates with limited accuracy and many limitations on when they can be realistically applied.

Widely-used commercial software analyzing rolling element bearings have also demonstrated discrepancy in the bearing stiffness estimate.

The objectives of this study are to: establish a finite element/contact mechanics
model of rolling element bearings in [92] and investigate the contact characteristics
between the rolling elements and races; propose a numerical method to compute the
fully-populated stiffness matrix of rolling element bearings based on quasi-static finite
element/contact mechanics analysis; validate the proposed stiffness determination
method by comparing results against published experiments [2, 3]; investigate the
importance of cross-coupling terms in the stiffness matrix by analyzing their effects
on the vibration propagation of an example gearbox from [8].

4.2 Modeling Assumptions

Rotation speed has limited effect on the stiffness of rolling element bearings for mod-
erate speeds. In one experiment, measurements show only slight difference in bearing
stiffness throughout the entire tested speed range with a maximum deviation of 6.89%
from the static stiffness [2]. The rolling element bearings under investigation are as-
sumed to operate at moderate speeds, and thus the centrifugal and gyroscopic effects
on the rolling elements due to high rotation speeds are ignored.

The major friction forces in rolling element bearings include sliding and rolling
friction. The sliding friction force is negligible for rolling element bearings [73]. The
rolling friction force is dominant only for heavily loaded systems where plastic de-
formation occurs in the contact area [105]. The loads applied to the bearings are
assumed to be moderate, so friction and related tribology effects are not considered.

While lubrication, including elastohydrodynamic lubrication (EHL), can signifi-
cantly affect bearing damping, it has mild effect on bearing stiffness. Brandlein found
that EHL prevails at high speed rolling conditions [74] or with heavy bearing loads
[105]. Consequently, the influence of lubrication on the bearing stiffness is expected
to not be significant considering the moderate rolling speed and thin fluid film (about 0.1 micrometer, 0.002% of bearing outer diameter).

Bearing cages are assumed to maintain a constant angular position of each rolling element relative to each other.

### 4.3 Finite Element/Contact Mechanics Bearing Model

The finite element/contact mechanics models of rolling element bearings (Figure 4.1) include important design details such as accurate roller and race crownings, internal radial and axial clearances, contact angle, roller length, bearing width, length and diameter of raceway for ball bearings, and so on. These parameters affect bearing stiffnesses significantly. They are, however, not included in many theoretical bearing models [4, 7, 73, 74].

![Figure 4.1: Cut-away finite element mesh of a radial ball bearing (dimensions detailed in Table 4.1) and a double-row cylindrical bearing (from a helicopter application).](image)

In addition, the finite element/contact mechanics model applies to most bearing
types, including cylindrical, tapered, and spherical roller bearings, radial and angular contact ball bearings, thrust bearings, and needle bearings.

The contact solution is a critical feature. This approach uses a combined surface integral and finite element method detailed in [51, 106, 107], which is implemented in the software Calyx [92]. Summaries of its key features for gear tooth contact mechanics, which also apply to bearing mechanics, are given in [52, 54].

The surface integral method analyzes the rolling element near-field contact mechanics by integrating, in the style of a Green’s function, the solution for a point load on a half space over the contact area. This assumption normally requires the size of the contact area to be orders of magnitude smaller than the contacting bodies. The rolling element contact considered in this study does not satisfy this condition because the sizes of the contact areas are not sufficiently small compared to the bearing dimensions. This assumption is, however, not needed to accurately predict relative displacements of points in the near-field contact area.

Finite element analysis reliably predicts far-field elastic deformations starting a small distance away from the contact area. Matching of the surface integral and finite element solutions at the matching surface (Figure 4.2) yields a combined contact solution for near-field surface deformations and far-field rolling element and race deformations due to the rolling element contact.

The displacement $d_{r,c}$ at point $r$ inside the contact area due to a load applied at a three-dimensional grid point $c$ on the contact surface

$$d_{r,c} \approx \{d^S_{r,c} - d^S_{r,m}\} + d^{FE}_{r,m}$$

(4.1)

where point $m$ (Figure 4.2) is located on the matching surface $\Gamma$ that defines the boundary between the near- and far-field regions. The relative displacement $d^S_{r,c} - d^S_{r,m}$ in the near-field is calculated by the surface integral method based on the Bousinesq
half-space solution; it is not significantly affected by the body deformation outside of the near-field. The last term $d_{r,m}^{FE}$ is obtained by conventional finite element analysis; it depends on full system elastic deformation and is not significantly affected by the near-field contact deformations. $d_{r,c}$ is obtained by minimizing the spatial discretization error according to [51]

$$\sum_{r} [d_{r,c} - (d_{r,c}^{S} - d_{r,m}^{S}) - d_{r,m}^{FE}]^2 \approx 0 \quad (4.2)$$

Figure 4.2: Pair of contacting elastic bodies with matching interfaces (---).

Essential contact parameters control the shape and size, and hence the accuracy, of the contact pressure at all contacting surfaces. These parameters include the separation tolerance ($\epsilon$), number ($N$) and width ($\delta$) of contact patches in the rolling element profile direction (that is, the direction in the plane of the bearing), and number of contact patches ($M$) in the axial direction. $\epsilon$ controls the size of the area where potential contact is searched in the deformed state. The size of the searched area increases with $\epsilon$. Correct selection of these contact parameters is crucial to balance computational efficiency against solution efficiency.
Calculated contact pressures in the contact zones between each rolling element and bearing race are used to check the correctness of the contact parameters. The shape of the contact pressure is expected to be parabolic in the profile direction. The shape of the contact pressure in the axial direction depends on the loading conditions. For instance, if a cylindrical bearing has tilting motion, the contact pressure in the axial direction is similar to a triangle. With radially applied load, a nearly rectangular contact pressure is expected in the axial direction. Examples of correct contact pressure of a radially loaded cylinder of a cylindrical bearing with full crowning and ball of a radial ball bearing are demonstrated in Figure 4.3. A thorough parametric study is necessary to determine proper choices of contact parameters.

![Figure 4.3: Contact pressure distributed over the contact zones of a (a) cylinder of a cylindrical roller bearing, and (b) ball of a radial ball bearing when radial loads are applied.](image)

The contact solver searches for the contact at every time step as the bearing rotates, which makes this approach capable of solving the time-dependent rolling element contact.
This finite element/contact analysis approach [51, 106, 107] has been validated against gear experiments and analysis [52, 54, 32]. The bearing contact force variation due to internal clearances calculated by this method has been correlated with analytical model numerical integration in [1].

4.4 Numerical Method of Bearing Stiffness Estimation

Traditional models of rolling element bearings include only radial and axial stiffnesses. These models do not provide the tilting stiffness, which is important when helical gears, shaft bending, or other effects related to relative tilting are present. For some example transmission systems, out-of-plane housing vibrations closely connected to tilting deformation at the bearings are dominant in experiments [108, 109]. To explain such motions, off-diagonal cross-coupling terms in the stiffness matrix must be included in the bearing model. Fully-populated stiffness matrices are necessary to properly address the vibration transmissibility through rolling element bearings under the above and other similar circumstances.

The stiffness matrix of rolling element bearings, where $x$ and $y$ denote axes in the plane of the bearing and $z$ is the axial direction, is

$$K = \begin{bmatrix}
  k_{xx} & k_{xy} & k_{xz} & k_{x\theta_x} & k_{x\theta_y} & 0 \\
  k_{yx} & k_{yy} & k_{yz} & k_{y\theta_x} & k_{y\theta_y} & 0 \\
  k_{zx} & k_{zy} & k_{zz} & k_{z\theta_x} & k_{z\theta_y} & 0 \\
  k_{\theta_x x} & k_{\theta_x y} & k_{\theta_x z} & k_{\theta_x \theta_x} & k_{\theta_x \theta_y} & 0 \\
  k_{\theta_y x} & k_{\theta_y y} & k_{\theta_y z} & k_{\theta_y \theta_x} & k_{\theta_y \theta_y} & 0 \\
  0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

(4.3)

$\theta_x, \theta_y$ are the out-of-plane angular deflections about the $x$ and $y$ axes, respectively. Diagonal terms in the stiffness matrix include radial stiffnesses $k_{ii}$, axial stiffness $k_{zz}$,
and tilting stiffnesses $k_{\theta_i}$, where $i = x, y$. The off-diagonal cross-coupling terms fall into four categories: coupling between radial and axial deflections $k_{iz}$, $k_{zi}$, $i = x, y$; coupling between radial and out-of-plane angular deflections $k_{i\theta_j}$, $k_{\theta_i j}$, $i, j = x, y$; coupling between axial and out-of-plane angular deflections $k_{z\theta_i}$, $k_{\theta_i z}$, $i = x, y$; and other coupling terms $k_{ij}$, $k_{\theta_i \theta_j}$, $i \neq j = x, y$. The sixth row and column are zeros because of the free rotation along the shaft axis.

Without any prior assumption about the load-deflection relation, the relative bearing deflection $q = \{x, y, z, \theta_x, \theta_y\}$ between the inner and outer races with the applied forces and moments $F = \{F_x, F_y, F_z, M_{\theta_x}, M_{\theta_y}\}$ is obtained computationally through finite element and contact analyses in this study. Contact between the rolling elements and races is nonlinear. The bearing stiffness is thus defined as $K = \frac{\partial F}{\partial q}$ (instead of $F q$) and calculated for given steady loading vector $F_0$, giving

$$\mathbf{K} = \begin{bmatrix}
\frac{\partial F_x}{\partial x} & \frac{\partial F_x}{\partial y} & \frac{\partial F_x}{\partial z} & \frac{\partial F_x}{\partial \theta_x} & \frac{\partial F_x}{\partial \theta_y} & 0 \\
\frac{\partial F_y}{\partial x} & \frac{\partial F_y}{\partial y} & \frac{\partial F_y}{\partial z} & \frac{\partial F_y}{\partial \theta_x} & \frac{\partial F_y}{\partial \theta_y} & 0 \\
\frac{\partial F_z}{\partial x} & \frac{\partial F_z}{\partial y} & \frac{\partial F_z}{\partial z} & \frac{\partial F_z}{\partial \theta_x} & \frac{\partial F_z}{\partial \theta_y} & 0 \\
\frac{\partial M_{\theta_x}}{\partial x} & \frac{\partial M_{\theta_x}}{\partial y} & \frac{\partial M_{\theta_x}}{\partial z} & \frac{\partial M_{\theta_x}}{\partial \theta_x} & \frac{\partial M_{\theta_x}}{\partial \theta_y} & 0 \\
\frac{\partial M_{\theta_y}}{\partial x} & \frac{\partial M_{\theta_y}}{\partial y} & \frac{\partial M_{\theta_y}}{\partial z} & \frac{\partial M_{\theta_y}}{\partial \theta_x} & \frac{\partial M_{\theta_y}}{\partial \theta_y} & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad (4.4)$$

The stiffness matrix is symmetric because the bearing system is conservative. The symmetry of the stiffness matrix is used later as one check to validate the proposed stiffness determination method.

The stiffness matrix $\frac{\partial F}{\partial q}$ is calculated numerically through finite differences to first, second, fourth, and sixth order, respectively, by

$$\mathbf{F}(\mathbf{q}_0 + \delta \mathbf{q}) - \mathbf{F}(\mathbf{q}_0)$$

$$\delta \mathbf{q} \quad (4.5)$$

85
\[
\frac{F(q_0 + \delta q) - F(q_0 - \delta q)}{2\delta q} = 8F(q_0 + \frac{\delta q}{2}) - 8F(q_0 - \frac{\delta q}{2}) - F(q_0 + \delta q) + F(q_0 - \delta q) \tag{4.6}
\]
\[
\frac{8F(q_0 + \frac{\delta q}{2}) - 8F(q_0 - \frac{\delta q}{2}) - F(q_0 + \delta q) + F(q_0 - \delta q)}{6\delta q} \tag{4.7}
\]
\[
\frac{256F(q_0 + \frac{\delta q}{4}) - 256F(q_0 - \frac{\delta q}{4}) - 40F(q_0 + \frac{\delta q}{2}) + 40q_0 - \frac{\delta q}{2})}{90\delta q} + F(q_0 + \delta q) - F(q_0 - \delta q) \tag{4.8}
\]

where \(q_0\) is the calculated bearing deflection vector at given load \(F_0\) about which the stiffness matrix is desired and \(\delta q = \{\delta q_x, \delta q_y, \delta q_z, \delta q_{\theta x}, \delta q_{\theta y}\}\) is the specified small disturbance vector at \(q_0\). The calculated reaction bearing force vector is denoted by \(F\). The computational effort is concentrated in these calculations of \(F\). Each finite difference formula requires a number of \(F\) evaluations equal to its accuracy order except the first order difference formula, which requires two evaluations of \(F\). Consequently, the total number of \(F\) evaluations is 25 times the formula accuracy order, or 25 times two for first order finite difference. Using the above formulae, each diagonal and cross-coupling stiffness element in the fully-populated stiffness matrix is determined. For example, the cross-coupling stiffnesses determined below are based on the second order difference formula

\[
k_{xy} \approx \frac{[F_x(q_0 + \delta q_y e_y) - F_x(q_0 - \delta q_y e_y)]}{2\delta q_y} \tag{4.9}
\]
\[
k_{\theta_xz} \approx \frac{[M_z(q_0 + \delta q_z e_z) - M_z(q_0 - \delta q_z e_z)]}{2\delta q_z}
\]

where \(e_y, e_z\) are the unit vectors along the \(y, z\) directions.

The accuracy of the determined stiffnesses relies on the accuracy of the selected finite difference formula and the finite element/contact mechanics analysis. If the finite element/contact analysis has low order accuracy, use of high order finite difference to calculate the bearing stiffness does not improve the result accuracy although it requires more \(F\) evaluations. The chosen finite difference order should have the same accuracy order as the finite element/contact mechanics analysis.
To obtain the accuracy order of the finite element analysis, we examine the discretization error $e_h = V_h - V$, where $h$ denotes the average element size and $V, V_h$ denote measures of the exact and finite element solutions, respectively. $V_h$ is approximated as

$$V_h = V + c_1 h^{p_1} + c_2 h^{p_2} + ...$$

(4.10)

where $c_1 h^{p_1}$ represents the principle truncation error of $V_h$ and the exponents $p_1, p_2$ denote the orders of accuracy. $c_1, c_2$ denote the coefficients of the corresponding accuracy orders, which are independent of $h$. Because $V$ is unknown, the accuracy order is studied by evaluating $e_h$ with different meshes when $h_1 > h_2 > h_3 \cdots$. By ignoring the higher order truncation errors of $V_{h_1}, V_{h_2}, V_{h_3}$, the following equation is derived

$$\frac{e_{h_3} - e_{h_2}}{e_{h_2} - e_{h_1}} = \frac{V_{h_3} - V_{h_2}}{V_{h_2} - V_{h_1}} \approx \frac{h_2^{p_1} - h_3^{p_1}}{h_1^{p_1} - h_2^{p_1}}$$

(4.11)

Consequently, the leading order of accuracy is [110, 111, 112]

$$p_1 = \frac{\log\left(\frac{V_{h_3} - V_{h_1}}{V_{h_3} - V_{h_2}}\right)}{\log\left(\frac{h_1}{h_2}\right)}$$

(4.12)

By reducing $h$ successively according to $h_2/h_1 = h_3/h_2 = r$, the above equation simplifies to

$$p_1 = \frac{\log\left(\frac{V_{h_3} - V_{h_1}}{V_{h_3} - V_{h_2}}\right)}{\log(r)}$$

(4.13)

This method yields the algorithm accuracy without knowing $V$.

Following the above steps, two separate studies have been conducted to determine the order of solution accuracy: a) finite element analyses of a disk deforming, and b) finite element/contact analyses of a four-roller rolling element bearing. Average element sizes are used here because the finite element meshes are not evenly-distributed in the actual disk and bearing models. By reducing the element size in steps by factors of two, $p_1$ converges to 1.11 when $h_3 \approx 0.983 \text{mm}$ and the disk outer diameter and
thickness are $27.0\ mm$ and $10.0\ mm$, respectively. Similarly, finite element/contact analysis is performed on the rolling element bearing while the average element size is halved at each iteration. In this case, $p_1 = 1.94$ when $h_3 \approx 0.470\ mm$ and the outer diameter of the bearing is $52\ mm$. Based on the above analyses, the results are estimated to have second order of accuracy. Therefore, the second-order finite difference formula is appropriate to determine $K$ in this study.

![Asymmetry Factor vs. Magnitude of Vector](image)

Figure 4.4: Asymmetry factor, the maximum of $\frac{2\|k_{ij} - k_{ji}\|}{\|k_{ij} + k_{ji}\|}$, $i \neq j = x, y, z, \theta_x, \theta_y$, of cylindrical bearing C for various $\frac{\delta q}{\|q_0\|}$ at $q_0 = (10, 26, 0, -0.7, 0.3)\mu m$. Bearing C is described in Table 4.3.

It is crucial to use proper $\delta q$ in the finite difference formula Equation 4.8. Ideally, the magnitude of $\delta q = (\delta q_x, \delta q_y, \delta q_z, \delta q_{\theta_x}, \delta q_{\theta_y})$ is as small as possible. If it is too small, however, round off error corrupts the approximation. The step size is selected based on $\frac{\delta q}{\|q_0\|} = \epsilon^{\frac{1}{m+1}} = 5 \times 10^{-6}$ [113] when $F$ is evaluated to full machine precision, where $m$ is the formula order and $\epsilon$ is the unit roundoff ($10^{-16}$). Numerical
study has been performed to confirm this selection of $\delta q$ for equal amplitudes in the $x$, $y$, $z$, $\theta_x$, $\theta_y$ directions. Figure 4.4 demonstrates the effect of $\delta q$ on the asymmetry factor, defined as the maximum of $\frac{2\|k_{ij} - k_{ji}\|}{\|k_{ij} + k_{ji}\|}$, $i \neq j = x, y, z, \theta_x, \theta_y$ of the bearing stiffness matrix. The stiffness matrix should be symmetric because rolling element bearing systems are conservative. Using improper $\delta q$, the calculated stiffness matrix becomes asymmetric. In Figure 4.4, the asymmetry factor drops significantly from 40% to 2% when $\frac{\delta q}{\|q_0\|}$ is smaller than $3 \times 10^{-5}$. Round off error comes into play when $\frac{\delta q}{\|q_0\|}$ is smaller than $0.1 \times 10^{-6}$. The optimal range of $\frac{\delta q}{\|q_0\|}$ to minimize numerical error is from $0.1 \times 10^{-6}$ to $30 \times 10^{-6}$, which includes the above theoretical estimate of $5 \times 10^{-6}$. If higher solution accuracy is required, individual $\delta q_m$, $m = x, y, z, \theta_x, \theta_y$ can be adjusted around the optimal $\delta q$ determined above by performing similar step size analyses.

The above stiffness determination method gives stiffness estimates for many possible boundary conditions. For instance, the bearing races can be connected to the housing/shaft or fixed to ground. Thus, this method provides stiffness estimates when the bearings are parts of an overall system or are isolated.

### 4.5 Comparison Against Experimental Data

To validate the proposed stiffness determination method, radial and axial stiffnesses of bearings A and B have been compared against published experiments [2, 3] that give only diagonal bearing stiffnesses. Dimensions of these bearings are listed in Tables 4.1 and 4.2.

Radial and axial stiffnesses of radial ball bearing A calculated by the proposed method under axial preloads are compared against experiments conducted by Kraus.
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<th>Values (mm, degree)</th>
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</tr>
<tr>
<td>Outer race crown curvature</td>
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</tr>
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</table>

Table 4.1: Radial ball bearing A parameters (based on SKF EEB3-2Z bearing).

et. al. [2] as shown in Figure 4.5. The experimental data used in Figure 4.5 are average values of measured stiffnesses. The measurement was taken with shaft speed of 1000 rpm. Because of the lack of information on internal radial clearance, two clearance values of 0 and 0.01 mm are used to compute the computational results. These values are the extreme clearance design values under regular operating conditions as given in the SKF product information. Radial and axial stiffnesses determined by the proposed method match the experiments well.
Figure 4.5: Comparison between the proposed method with zero (⋯○⋯) and 0.01 mm (−□−) radial clearances and Kraus et al.’s [2] experiment (−○−) for radial and axial stiffnesses of radial ball bearing A under axial preloads. Dimensions of bearing A are described in Table 4.1.

Radial and axial stiffnesses of self-aligning ball bearing B calculated by the proposed method are compared against the Royston and Basdogan experiment [3] under radial and axial preloads, respectively (Figure 4.6). Good agreement is evident between the proposed method and the experiments for both radial and axial stiffness. The slight differences between the computed and measured stiffnesses might be caused by different bearing boundary conditions. In the finite element bearing models, the outer surfaces of the bearing outer races are considered as ground, which is the ideal operating condition. Bearings in the experiments have slightly different boundary conditions depending on the material of the housings, shafts, bearing spacers, and other connecting bodies. The elasticity of these connecting bodies affects the experimental bearing stiffness measurements.
<table>
<thead>
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<th>Parameters</th>
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</tr>
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<tr>
<td>Axial offset between rows</td>
<td>10.70</td>
</tr>
<tr>
<td>Contact angle</td>
<td>12.70</td>
</tr>
<tr>
<td>Angular position of the 1\textsuperscript{st} rollers (1\textsuperscript{st} row)</td>
<td>0</td>
</tr>
<tr>
<td>Angular position of the 1\textsuperscript{st} rollers (2\textsuperscript{nd} row)</td>
<td>12.00</td>
</tr>
<tr>
<td>Pitch diameter</td>
<td>35.24</td>
</tr>
<tr>
<td>Bore diameter</td>
<td>17.00</td>
</tr>
<tr>
<td>Ball diameter</td>
<td>7.131</td>
</tr>
<tr>
<td>Bearing width</td>
<td>14.00</td>
</tr>
<tr>
<td>Outer diameter</td>
<td>47.00</td>
</tr>
<tr>
<td>Outer diameter of inner raceway</td>
<td>28.90</td>
</tr>
<tr>
<td>Inner diameter of outer raceway</td>
<td>41.00</td>
</tr>
<tr>
<td>Radial clearance</td>
<td>$15.00 \times 10^{-3}$</td>
</tr>
<tr>
<td>Inner race crown curvature factor</td>
<td>0.526</td>
</tr>
</tbody>
</table>

Table 4.2: Self-aligning ball bearing B parameters (based on SKF 1303EM bearing).

### 4.6 Comparison Against Commercial Software

Commercial programs such as Program X, REBM [75, 77], and COBRA [114, 115] are widely used to estimate the stiffness of rolling element bearings. Program X is commercial software that is used by industries worldwide and provides bearing stiffness estimates. We are not free to state its name because of license restrictions.
Figure 4.6: Comparison between the proposed method (• • • • • •) and Royston and Basdogan [3] experiment (− ◦ −) for radial and axial stiffnesses of self-aligning ball bearing B under radial and axial preloads, respectively. Dimensions of bearing B are described in Table 4.2.

for academic use. REBM-NASA denotes the stiffness estimate computed by REBM included in [116].

Diagonal stiffnesses of the cylindrical and radial ball bearings C and D (listed in Tables 4.3 and 4.4) calculated by these programs are compared against the current approach in Tables 4.5 and 4.6 when combined loads are applied. Stiffnesses calculated by Program X, REBM, REBM-NASA, and COBRA are presented as the deviations from the stiffness determined by the current approach. The discrepancy among these programs is apparent, showing the disagreement in state-of-the-art commercial codes.

The bearing models in REBM-NASA are slightly different from the bearings in the other programs. The race crown curvature factor of bearing D (SKF Explorer 6205, Table 4.4 below) is 0.575 in REBM-NASA whereas it is 0.52 in the current approach and the other programs. The roller diameter of bearing C (FAG N205E, Table 4.3 below) in REBM-NASA is 0.5% smaller than that in the current approach and the other programs. Internal clearances used in [116] are not clearly specified.
The large differences between REBM and REBM-NASA are significant considering they use the same stiffness determination method with only slight differences in the bearing dimensions. This indicates that bearing/race micro-geometry strongly affects overall stiffness. Thus, a model that captures these effects is essential for accurate bearing stiffness.

Comparisons to Gargiulo’s [5] and Harris’s [4] methods are not included in Tables 4.5 and 4.6 because they do not permit moment and combined force/moment loading. While Harris calculates contact pressure for misaligned rollers as occurs with moment loading, he underestimates the contact pressure [117]. In that case, the bearing stiffnesses, although not actually calculated by Harris for this case, will not be accurate.

4.7 Comparisons Against Published Theoretical Models

Theoretical bearing models [4, 7, 73, 74] make different assumptions to derive the bearing load-deflection relation. These formulations are built starting from the predicted behavior of an individual rolling element contact. The load-deflection relation of a single rolling element is assumed as

\[ P_e = k_e \delta_e^l \]  

(4.14)

where \( P_e \) is the applied normal load on the rolling element and \( k_e \) is the effective stiffness constant for the contact between the rolling element and the bearing race. The exponent \( l \) describes the nonlinear load-deflection relation. \( k_e \) and \( l \) are essential quantities to estimate the overall stiffness for theoretical bearing models. Theoretical models, however, use different \( k_e \) and \( l \) based on their assumptions about the rolling element contact. Different assumptions on the bearing race elasticity are
<table>
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<tbody>
<tr>
<td>Number of rows</td>
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</tr>
<tr>
<td>Number of rolling elements</td>
<td>13</td>
</tr>
<tr>
<td>Contact angle</td>
<td>0</td>
</tr>
<tr>
<td>Pitch diameter</td>
<td>39.00</td>
</tr>
<tr>
<td>Bore diameter</td>
<td>25.00</td>
</tr>
<tr>
<td>Roller length</td>
<td>8.600</td>
</tr>
<tr>
<td>Roller diameter</td>
<td>7.500</td>
</tr>
<tr>
<td>Bearing width</td>
<td>15.00</td>
</tr>
<tr>
<td>Outer diameter</td>
<td>52.00</td>
</tr>
<tr>
<td>Outer diameter of inner raceway</td>
<td>31.50</td>
</tr>
<tr>
<td>Inner diameter of outer raceway</td>
<td>46.40</td>
</tr>
<tr>
<td>Radial clearance</td>
<td>$40.00 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 4.3: Single row cylindrical bearing C parameters (based on FAG N205E bearing).

made to simplify the bearing stiffness formulations. For instance, Gargiulo [5] ignored the elasticity of the races so that the rolling element contact can be considered as springs. By making this assumption, direct formulae relating bearing deflection and stiffness are obtained. Nevertheless, the elasticity of the races significantly affects bearing stiffnesses. Figure 4.7 shows the nonlinear radial stiffness-load relation of the cylindrical and the radial ball bearings (labeled C and D, described in Tables 4.3 and 4.4) obtained by the Harris [4], Gargiulo [5], and While [6] theoretical models. Remarkable discrepancy is present among these models.
<table>
<thead>
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<tr>
<td>Contact angle</td>
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<tr>
<td>Pitch diameter</td>
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<tr>
<td>Bore diameter</td>
<td>25.00</td>
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<tr>
<td>Ball diameter</td>
<td>7.900</td>
</tr>
<tr>
<td>Bearing width</td>
<td>15.00</td>
</tr>
<tr>
<td>Outer diameter</td>
<td>52.00</td>
</tr>
<tr>
<td>Outer diameter of inner raceway</td>
<td>34.40</td>
</tr>
<tr>
<td>Inner diameter of outer raceway</td>
<td>46.30</td>
</tr>
<tr>
<td>Radial clearance</td>
<td>$20.00 \times 10^{-3}$</td>
</tr>
<tr>
<td>Inner race crown curvature</td>
<td>0.520</td>
</tr>
<tr>
<td>Outer race crown curvature</td>
<td>0.520</td>
</tr>
</tbody>
</table>

Table 4.4: Radial (deep groove) ball bearing D parameters (based on SKF Explorer 6205 bearing).

Many theoretical models, such as the Harris [4] and Jones [7] models, employ Hertzian contact theory to approximate the rolling element contact. Hertzian contact theory is valid when the extent of the contacting bodies is much larger than the dimension of the contact area. This contact area must be sufficiently far from the other boundaries of the bodies so that the contacting bodies can be considered as elastic half-spaces. These conditions are not satisfied by the rolling element contact. Thus, theoretical models based on the Hertzian contact theory apply for bearings with
<table>
<thead>
<tr>
<th></th>
<th>$k_{xx}, N/mm$</th>
<th>$k_{yy}, N/mm$</th>
<th>$k_{zz}, N/mm$</th>
<th>$k_{\theta,\phi_z}, Nmm/\text{rad}$</th>
<th>$k_{\theta,\phi_y}, Nmm/\text{rad}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current</td>
<td>113,149</td>
<td>200,320</td>
<td>0</td>
<td>1,843,453</td>
<td>1,550,232</td>
</tr>
<tr>
<td>Program X</td>
<td>-16.3%</td>
<td>+66.0%</td>
<td>0</td>
<td>+20.0%</td>
<td>-59.2%</td>
</tr>
<tr>
<td>REBM</td>
<td>-66.9%</td>
<td>-12.9%</td>
<td>0</td>
<td>+0.20%</td>
<td>-73.0%</td>
</tr>
<tr>
<td>REBM-NASA</td>
<td>-55.8%</td>
<td>+24.3%</td>
<td>0</td>
<td>-17.5%</td>
<td>-80.2%</td>
</tr>
<tr>
<td>COBRA</td>
<td>+10.2%</td>
<td>+74.6%</td>
<td>0</td>
<td>+2.86%</td>
<td>-57.1%</td>
</tr>
</tbody>
</table>

Table 4.5: Diagonal stiffnesses for the bearing C calculated by the current approach, Program X, REBM, REBM-NASA, and COBRA. The applied load vector is $\mathbf{F} = (323N; 908N; 0; -825Nmm; 293Nmm)$.

<table>
<thead>
<tr>
<th></th>
<th>$k_{xx}, N/mm$</th>
<th>$k_{yy}, N/mm$</th>
<th>$k_{zz}, N/mm$</th>
<th>$k_{\theta,\phi_z}, Nmm/\text{rad}$</th>
<th>$k_{\theta,\phi_y}, Nmm/\text{rad}$</th>
</tr>
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<td>95,013</td>
<td>3,955</td>
<td>940,694</td>
<td>506,869</td>
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<td>Program X</td>
<td>-14.5%</td>
<td>+0.40%</td>
<td>-17.1%</td>
<td>-4.40%</td>
<td>-33.7%</td>
</tr>
<tr>
<td>REBM</td>
<td>+1.40%</td>
<td>+18.5%</td>
<td>-19.6</td>
<td>-1.10%</td>
<td>-51.0%</td>
</tr>
<tr>
<td>REBM-NASA</td>
<td>+1.04%</td>
<td>-29.8%</td>
<td>-75.8%</td>
<td>-74.7%</td>
<td>-77.1%</td>
</tr>
<tr>
<td>COBRA</td>
<td>-22.7%</td>
<td>+4.30%</td>
<td>-17.8%</td>
<td>-3.37%</td>
<td>-36.2%</td>
</tr>
</tbody>
</table>

Table 4.6: Diagonal stiffnesses for the bearing D calculated by the current approach, Program X, REBM, REBM-NASA, and COBRA. The applied load vector is $\mathbf{F} = (317N; 894N; 0; -9.7Nmm; 3.6Nmm)$.

unrealistically thick and wide races. Experiments on bearing stiffness [118] exhibit significant deviation between measured and calculated stiffness using Hertzian contact theory, showing that Hertzian models are suitable for qualitative but not quantitative
Figure 4.7: Radial stiffness of cylindrical bearing C and radial ball bearing D vs. applied radial loads calculated by the Harris (−) [4], Gargiulo (··−·) [5], and While (−−) [6] models. The While [6] model is modified to use \( \frac{\Delta F}{\Delta q} \) to calculate the stiffness instead of \( \frac{F}{q} \). Bearings C and D are described in Tables 4.3 and 4.4.

Predictions of bearing stiffness. In addition, stiffnesses determined by the various Hertzian models have large differences as shown in Figure 4.7.

Different from Hertzian contact theory, the combined surface integral and finite element approach used in this study does not require assumptions about bearing dimensions or unrealistic race thickness and width. This is because the near-field contact solution mates with a far field finite element solution that accurately captures realistic geometry. The near-field contact solution with precise geometry definition allows for accurate micro-geometry of the rolling elements and races. This approach provides a more accurate stiffness estimate than Hertzian contact theory.

The Harris [4], Jones [7], and Gargiulo [5] models of rolling element bearings are now compared against the proposed method for bearings C, D, and E (listed in Tables 4.3, 4.4, and 4.7). Figure 4.8 shows radial stiffnesses of cylindrical bearing C
and radial ball bearing D estimated by the proposed method and Harris model at a
variety of applied radial loads. Radial stiffness from the finite element/contact
mechanics models is calculated two ways. One way uses the design bearing dimen-
sions (circles), which are nearly identical to commercial products; and the other uses
unrealistic dimensions with much thicker/wider races chosen to match assumptions
of Hertzian contact theory (semi-infinite space). With the unrealistic dimensions,
excellent agreement is evident between the proposed method and Harris model, es-
pecially for bearing D. This serves as one validation of the proposed method. It also
shows the limitation of Harris’ method: bearing stiffnesses with the designed bearing
dimensions are up to 38% lower than those with the unrealistic dimensions chosen to
mimic Harris’ assumptions.

![Graph showing radial stiffness for cylindrical bearing C and radial ball bearing D.](image)

Figure 4.8: Radial stiffness calculated by the proposed method with design (○○○○) and
unrealistic (○○□○) bearing dimensions and Harris’s model [4] (−) of
cylindrical bearing C and radial ball bearing D. Dimensions of bearings
C and D are described in Tables 4.3 and 4.4. Radial clearance is not
considered to match the Harris model assumption.

Also based on Hertzian contact theory, Jones [7] proposed a theoretical model
to study bearing deflection and axial load for angular contact ball bearings. Harris [4] extended this model to determine radial and axial stiffnesses of angular contact ball bearings, noted as the Jones-Harris model. Figure 4.9 compares axial stiffnesses of angular contact ball bearing E (described in Table 4.7) with contact angles of 20 and 35 deg calculated by the proposed method, Jones-Harris model, and Gargiulo [5] model. To compare against these theoretical models, the finite element/contact mechanics solution is given for design dimensions and with unrealistically enlarged bearing width (240% of the design value) chosen to match the theoretical model assumptions under axial preloads. With the unrealistic dimensions, results of the proposed method match the Jones-Harris model well, while axial stiffness calculated by Gargiulo’s model deviates dramatically from the other models. Again, this helps validate the computational model by comparing with published results. The results also highlight the limitations of the Jones-Harris model. Bearing stiffnesses with design dimensions are up to 33% lower than those calculated by the Jones-Harris model. Gargiulo’s model is worse.

Furthermore, these theoretical bearing models generally do not include important bearing details such as the crowning on the rollers for roller bearings, race width, length and diameter of the chord that subtends the raceway for ball bearings, clearances, etc. These parameters highly affect bearing stiffness. The finite element/contact mechanics approach captures these details using a precise description of all geometric features. The discrepancies in stiffness estimates, application limitations of the contact theory, and lack of important bearing details listed above suggest that the bearing stiffness estimate of the theoretical models [4, 7, 73, 74] is lack of accuracy.
Figure 4.9: Comparison between the proposed method with design (○) and unrealistic (□) bearing dimensions, the Harris-Jones model [4, 7] (−), and the Gargiulo’s [5] model (−−) for axial stiffnesses of angular contact ball bearing E with contact angles of (a) 20 degree and (b) 35 degree. Dimensions of bearing E are in Table 4.7.
### Table 4.7: Single row angular ball bearing E parameters (based on SKF ALS28ABP bearing)

<table>
<thead>
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<th>Values (mm, degree)</th>
</tr>
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<tbody>
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<tr>
<td>Number of rolling elements</td>
<td>16</td>
</tr>
<tr>
<td>Contact angle</td>
<td>20, 35</td>
</tr>
<tr>
<td>Pitch diameter</td>
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<tr>
<td>Bore diameter</td>
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</tr>
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<td>Ball diameter</td>
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</tr>
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<td>Bearing width</td>
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<tr>
<td>Outer diameter</td>
<td>165.0</td>
</tr>
<tr>
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</tr>
<tr>
<td>Inner diameter of outer raceway</td>
<td>138.5</td>
</tr>
<tr>
<td>Inner race crown curvature</td>
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</tr>
<tr>
<td>Outer race crown curvature</td>
<td>0.520</td>
</tr>
</tbody>
</table>

**4.8 Cross-coupling Terms in the Stiffness Matrix**

Cross-coupling terms in the stiffness matrix indicate interactions between radial, axial, and tilting motions of rolling element bearings. They demonstrate the coupling between shaft tilting motion, flexural motion of the structure connected to the outer race, and shaft radial and axial motions. While common in many systems, this coupling was specifically seen in two experiments related to the transmission systems motivating this study [108, 109].
The importance of cross-coupling terms is investigated by comparing amplitudes of these terms to diagonal ones in the stiffness matrix. Figures 4.10 and 4.11 show amplitudes of the cross-coupling terms in the stiffness matrix of cylindrical bearing C and radial ball bearing D in their ball passing periods. The ball pass period 

\[ T = \frac{2}{Z\omega(1 - \frac{D_e}{D_p})} \]

is defined as the amount of time between one rolling element leaving a reference point and the next rolling element arriving. Here, \( Z \) is the number of rolling elements, \( \omega \) denotes the shaft speed, and quantities \( D_e \) and \( D_p \) denote the diameter of the rolling elements and pitch diameter of the bearing, respectively. Off-diagonal bearing stiffnesses in Figures 4.10 and 4.11 are nondimensionalized as

\[ s_{ij} = \frac{k_{ij}}{\sum_{n=1}^{5} k_{nn}/5} \times 100 \]

where \( i, j, n = x, y, z, \theta_x, \theta_y \), which are divided by the mean amplitude of diagonal stiffnesses in the ball passing period of each bearing. Amplitudes of the cross-coupling terms are significant compared to diagonal ones. The maximum of \( s_{x,y} \) is 34% of the average diagonal stiffness. \( s_{\theta_x,y} \) is the dominant coupling effect of cylindrical bearing C (Figure 4.10). \( s_{\theta_x,y} \) is, however, ignorable for radial ball bearing D (Figure 4.11). The maximum of \( s_{z,\theta_y} \) is 19% of the average diagonal stiffness for bearing D, showing that coupling between the shaft tilting motion and housing flexural motion is important for radial ball bearing D. In contrast, \( s_{z,\theta_y} = 0 \) for cylindrical roller bearing C.

The effects of cross-coupling terms in the stiffness matrix on vibration transmissibility through rolling element bearings are investigated on an example gearbox from [8]. This gearbox includes two spur gears with shafts mounted on two cylindrical bearings C and two radial ball bearings D. Gear dynamic transmission error is defined as \( r_1\theta_1 - r_2\theta_2 \), where \( r_{1,2} \) are radii of the contacting gears and \( \theta_{1,2} \) denote rotational deflections. As the primary excitation source of gearboxes, gear transmission error is an important measure of gearbox vibration. Figure 4.12 shows the spectra of dynamic transmission error in the frequency range from 1500 to 4000 Hz from numerical torque.
Figure 4.10: Nondimensionalized off-diagonal stiffness of cylindrical bearing C in a ball pass period. The applied load vector equals $\mathbf{F} = (323 \text{N}; 908 \text{N}; 825 \text{Nmm}; 293 \text{Nmm})$. $s_{ij} = \frac{k_{ij}}{(\sum_{n=1}^{5} k_{nn})/5} \times 100$, $i, j, n = x, y, z, \theta_x, \theta_y$. $s_{zz}$ and its coupling terms vanish within the period with zero crowning on the cylinders.

Impulse cases. Bearing models with fully-populated and diagonal stiffness matrices are compared in Figure 4.12. Distinctive differences in resonant frequencies and amplitudes are evident between these two bearing models. This stresses the importance of including cross-coupling terms in bearing models.

All stiffnesses including cross-coupling terms in the stiffness matrix fluctuate with time as the number of rolling elements in contact and the carrying load changes when the rolling elements rotate about the shaft axis in this ball pass period. Radial stiffness of the radially-loaded cylindrical bearing C and ball bearing D in their ball pass periods is shown in Figures 4.10 and 4.11. This bearing stiffness variation can potentially excite vibration of the connected structures.
Figure 4.11: Nondimensionalized off-diagonal stiffnesses of ball bearing D in a ball pass period. The applied load vector equals $\mathbf{F} = (317 \text{N}; 894 \text{N}; 0; -9.7 \text{Nmm}; 3.6 \text{Nmm})$. $s_{ij} = \frac{k_{ij}}{(\sum_{n=1}^{5} k_{nn})/5} \times 100$, $i, j, n = x, y, z, \theta_x, \theta_y$. $s_{zz}$ is close to 1.25% within the period and so not shown in the figure.

4.9 Time Varying Property of Bearing Stiffness

In addition, the locations of the rolling elements and the surface portions in contact change as the bearing rotates. When loaded rolling elements rotate about the shaft axis, the number of rolling elements in contact and the load carried by individual rolling elements change periodically. This leads to periodically varying bearing stiffness matrices as shown in Figures 4.10 and 4.11.

The locations of the rolling elements and the surface portions in contact change as the bearing rotates as shown in Figure 4.13. When loaded rolling elements rotate about the shaft axis, the number of rolling elements in contact and the load carried by individual rolling elements change periodically with a ball pass period. The ball pass period $T = \frac{2}{Z\omega(1 - \frac{D_p}{D_e})}$ is the amount of time between one rolling element leaving a reference point and the next rolling element arriving. Here, $z$ is the number of rolling
Figure 4.12: Numerical torque impulse response of gear dynamic transmission error with fully-populated (−) and diagonal (−−) stiffness matrices of the rolling element bearings mounted in the gearbox from [8]. The input torque equals \( 84.74 \, N \cdot m \).

elements, \( \omega \) denotes the shaft speed, and quantities \( D_e \) and \( D_p \) denote the diameter of rolling elements and pitch diameter of the bearing, respectively.

Figures 4.14(a) and 4.15(a) show radial stiffness of the cylindrical and radial ball bearings in a ball pass period with \( 1000 \, N \) applied load. Radial stiffness fluctuates with 16\% deviation from the mean stiffness, while the number of rolling elements in contact alternates between four and five. Figures 4.14(b) and 4.15(b) show tilting stiffness of the cylindrical and ball bearings with \( 1000 \, N \cdot mm \) applied moment. The time-varying tilting stiffness deviates from the mean value nearly 6\%. These stiffness fluctuations introduce an additional vibration source, which can excite gear vibration.
Figure 4.13: Time-varying property of the orbital motion of radially loaded rolling element bearing stiffness: (a) rolling elements in contact (shaded circles) from time 0 to \(t\); (b) expected time-varying radial stiffness.

### 4.10 Conclusions

A finite element/contact mechanics bearing model is established based on a contact algorithm suited to high-precision elastic bodies and well-established for gear tooth contact mechanics. The load-dependent rolling element contact for all roller-race contacts throughout the entire bearing is solved through a combined solution based on surface integral (for the near-field) and finite element approaches (for the far-field). The computational model includes all the important bearing details besides basic geometry, such as, internal clearance, roller and race crowning, race width and thickness, and dimensions of the raceway shoulders. These parameters significantly affect bearing stiffness.

Using the finite element/contact mechanics model, a stiffness determination method based on finite difference analysis is developed to provide accurate stiffness estimates.
for rolling element bearings. The accuracy order of this method depends on the finite element/contact mechanics analysis and the step size selected for the finite difference formulae. Without knowing the exact solution, an iteration scheme is introduced to estimate the accuracy order of the finite element analysis. The selected step size of the finite difference formulae strongly affects accuracy of the results, which stresses the importance of investigating the optimal size range.

This stiffness determination method is validated against experiments in the literature. Comparisons against published models provide further validation and expose shortcomings of analytical models.

This method determines the fully-populated stiffness matrix of rolling element bearings. The computed stiffness matrix captures the coupling between radial, axial,
and tilting deflections of rolling element bearings. These cross-coupling terms are significant compared to diagonal ones for the bearings considered in this study. The importance of these cross-coupling terms is demonstrated by their significant effects on the dynamics of an example NASA gearbox.

The contact characteristics change under bearing rotation as rolling elements enter and leave the region of loaded rollers. In consequence, all elements in the stiffness matrix fluctuate periodically about the ball pass frequency. These fluctuations influence quasi-static system response and can potentially excite vibration of the connected structures.
Figure 4.15: (a) Radial and (b) tilting stiffness of the ball bearing D over a ball pass period. The bottom figures show the number of rolling elements in contact over a ball pass period. The applied load and moment are 1000 N and 1000 N·mm, respectively.
CHAPTER 5

VIBRATION PROPAGATION OF GEAR DYNAMICS IN A GEAR-BEARING-HOUSING SYSTEM

5.1 Summary

Gear vibration is a major source of noise in helicopters. Limited work has, however, investigated the relationship between the gearbox noise and vibration. The vibro-acoustic models of geared systems are sparse [119] in the literature. Structural-borne noise calculations of geared systems are, therefore, semi-empirical due to the complexity of these problems [120, 121]. These acoustic models are not able to capture realistic dimensions of gearboxes, which affect the noise estimate. This work develops a multi-dynamics computational model to fully describe the vibro-acoustic propagation of gear dynamics through a power-transmission system shown in Figures 5.1(a) and 5.1(b).

Dynamic forces at the meshing teeth drive a system vibration through power transmission components to the fuselage. The bearings linking the gear shafts to the housing are a primary factor in this noise path. Most machinery applications use rolling element bearings that do not create meaningful damping to reduce the transmitted structural-borne noise. Novel wave bearings have higher damping and could offer an energy dissipation mechanism that reduces the noise transmitted from
gearboxes. Fluid film wave bearings are a special type of journal bearings, which have waved inner diameters of the stationary bearing sides shown in Figure 5.2.

The major objectives of this study include:

1. develop a finite element gearbox model which includes the detailed contact analysis of the gear tooth mesh and individual bearing rolling elements;
2. build up analytical (lumped parameter) model of the gear/bearing/housing system, which provides fast dynamic analysis;
3. establish a boundary element model of the gearbox housing. Map the radiated noise from the gearbox surface through acoustic analysis;
4. validate the vibration and acoustic models of the examined gearbox against measurements performed by NASA personnel and theoretical solutions in the literature;
5. understand the effectiveness of rolling element and fluid film wave bearings in breaking the vibro-acoustic propagation path from the gears to the housing.

5.2 Steps to Analyze the Gearbox Vibro-Acoustic Propagation

The vibro-acoustic gearbox modeling consists of four parts. They are: 1). full-fidelity finite element/contact mechanics model of the entire gearbox including gears, shafts, bearings, and the housing in Calyx [92]; 2). mathematical lumped-parameter gearbox model in Matlab; 3). finite element housing model in ANSYS; and 4). boundary element model of the housing in Coustyx [122].

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Figure 5.1: (a) Outside and (b) inside of the NASA Glenn Research Center (GRC) gearbox.

The finite element gear/bearing/housing model includes realistic dimensions of rolling element bearings, shafts, and the housing. The unique contact algorithm seeks the gear and rolling element contact in three-dimensional space. This algorithm has been validated against experiments [54, 52, 35, 46, 123, 124, 125]. Output of this detailed finite element model includes dynamic bearing force, transmission error, and bearing stiffness, which are crucial information for the acoustic analysis and mathematical modeling.

Three-dimensional mathematical model provides fast dynamic analysis of the gearbox. Bearings are modeled through the stiffness matrices computed by the detailed finite element/contact analysis. Housing is included through its compliance matrix,
computed by finite element analysis. The gear contact is modeled through a network of springs along lines of action. The mesh model captures the non-nominal load distributions due to shaft tilting, misalignment, and profile/lead modification.

Isolated finite element housing model is established in ANSYS. The housing excitation source is dynamic bearing forces calculated using the detailed finite element/contact or simplified analytical model. The housing response caused by dynamic bearing forces is analyzed to obtain its surface velocity. This step can not be committed because it connects the vibration analysis of the entire gearbox and the housing noise radiation computation in the following step.

The full-fidelity boundary element model of the gearbox housing is established in the software Coustyx [126, 122]. The acoustic model employs a multipole method (detailed in [126]) to provide fast noise calculation with the realistic housing dimensions. This housing surface velocity calculated by ANSYS is inputted into the acoustic model as its boundary condition. This acoustic model computes the radiated noise from the housing.

Major steps of the gearbox vibro-acoustic analysis is depicted in Figure 5.3 and summarized as below:

1. Dynamic bearing force estimate through vibration analysis using the detailed
finite element/contact mechanics or mathematical lumped-parameter model of the entire gearbox in Calyx and in-house program, respectively;

2. Determination of the housing surface velocity through forced response analysis on the gearbox housing in ANSYS;

3. Gearbox housing noise radiation analysis with the surface velocity as the boundary condition in Coustyx.

Figure 5.3: Key steps to perform the gearbox acoustic analysis.
5.3 Multibody Dynamics Gearbox Modeling and Analysis

5.3.1 Finite Element/Contact Analysis Gearbox Model

The examined NASA GRC gearbox (shown in Figures 5.1(a) and 5.1(b)) includes a pair of spur gears with webbed rims, two staged shafts, four rolling element bearings, and the housing. The gear parameters are listed in Table 5.1. Two types of rolling element bearings are used: cylindrical bearings and deep groove ball bearings (labeled C and D, described in Tables 4.3 and 4.4). Product designations are SKF N205ECP and SKF 6205, respectively. The basic dimensions of the gearbox housing are listed in Table 5.2. Details of the gearbox are described in [127, 116].

The finite element/contact mechanics software (Calyx [92]) is well-suited for this work. The unique contact solver seeks contact on tooth mesh and bearing rolling elements. Mesh stiffness variation, transmission error, and tooth separation are inherently included. This specialized finite element/contact mechanics software allows dynamic simulations with greater modeling fidelity than conventional finite element tools. It is validated against benchmark studies of complex gear dynamics problems [54, 52, 35, 46, 123, 124]. In experimental comparisons, it has proven accurate in capturing the complex tooth mesh forces leading to strong nonlinearity in the dynamics of single gear pairs [52], idler [123, 128], and planetary gears [54, 35, 46, 129]. The rolling element contact in multiple bearings has been validated against experiments in [125].

The realistic bearing model captures detailed bearing mechanics. It includes rolling elements, inner/outer races, and the cage as shown in Figure 4.1. This detailed bearing model is used to determine the full 6×6 stiffness matrix between the shafts and housing. That matrix is used in the mathematical gearbox model to perform fast dynamic simulations, as discussed later.
The fluid film wave bearings are included in the gearbox model through the stiffness and damping matrices calculated by the program developed by Hanford and Campbell [130, 131]. This wave bearing model uses a perturbation method based on the Reynold equation to calculate the dynamic stiffness, damping, pressure distribution, and load capacity of the fluid film. The program, however, is limited to be two-dimensional.

The gearbox housing is modeled by importing the full fidelity mesh established in commercial finite element software, PATRAN. The housing is then assembled into the gear/bearing/shaft system as shown in Figure 5.4. The housing plays an important role in the noise radiation.

Figure 5.4: Assembly of the gear-bearing-shaft-housing model.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values (mm, degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of teeth</td>
<td>28</td>
</tr>
<tr>
<td>Outer diameter</td>
<td>95.25</td>
</tr>
<tr>
<td>Root diameter</td>
<td>79.73</td>
</tr>
<tr>
<td>Facewidth</td>
<td>6.350</td>
</tr>
<tr>
<td>Diametral pitch</td>
<td>0.3150</td>
</tr>
<tr>
<td>Pressure angle</td>
<td>20</td>
</tr>
<tr>
<td>Center distance</td>
<td>88.90</td>
</tr>
<tr>
<td>Tooth thickness</td>
<td>4.851</td>
</tr>
<tr>
<td>Backlash</td>
<td>0.1778</td>
</tr>
<tr>
<td>Cutter edge radius</td>
<td>1.270</td>
</tr>
<tr>
<td>Linear tip relief</td>
<td>0.1778 starting at 24 degrees</td>
</tr>
</tbody>
</table>

Table 5.1: Dimensions of the Spur Gear Pair

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>279.4</td>
</tr>
<tr>
<td>Width</td>
<td>254.0</td>
</tr>
<tr>
<td>Length</td>
<td>330.2</td>
</tr>
<tr>
<td>Wall thickness</td>
<td>6.350</td>
</tr>
<tr>
<td>Lid thickness</td>
<td>6.350</td>
</tr>
</tbody>
</table>

Table 5.2: Dimensions of the Gearbox Housing.
**Unique Contact Solver**

The geometric surface descriptions of the contacting bodies must be precise to fully address the contact characteristics. Additionally, the contact area is narrow and travels over the entire body surface. Conventional finite element analysis requires a prohibitively refined mesh to address these problems; a complete dynamic response analysis becomes impossible in that case.

The finite element/contact mechanics model used here addresses these issues by using a combination of the Bousinesq solution near the contacting surfaces and traditional finite element analysis far away from the contact zones to exploit the advantages of each. The details about this contact solver can be found in [51].

To accurately address the contact properties on these tiny regions, the contact zone is discretized into many small patches (grid cells), which are the basic contact elements. Sufficient number of grid cells within the contact region is essential to obtain reliable contact pressure and load distribution. The shape of the contact pressure is expected to be parabolic in the profile direction and is sensitive to the loading conditions in the axial direction. The finite element model of the gear pair and contact pressure on individual tooth over a mesh cycle are shown in Figure 5.5(a) and Figure 5.5(b). Figure 5.6(a) shows the finite element model of a double-row cylindrical bearing. Contact patches on one of the radially loaded cylinders are shown in Figure 5.6(b).

When the gears and rolling element rotate, the number of teeth and rolling elements in contact change, as do the location, size, and shape of the contact areas. Modeling these changes is important to obtain accurate bearing forces, gear tooth loads, and transmission error calculations. The contact solver addresses these issues by determining and analyzing the instantaneous gear and bearing contact conditions at every time instant.
Figure 5.5: (a) Mating gears in the NASA GRC gearbox; (b) contact pressure on gear teeth over one mesh cycle;

5.3.2 Gear Transmission Error

Transmission error is the major excitation source in geared systems. Accurate transmission error estimate is crucial. With the precise contact solver, the finite element/contact analysis of the gearbox provides reliable transmission error estimate.

Transmission error is computed according to the tooth mesh deflection $TE = (x_1 - x_2)sin(\alpha) + (y_1 - y_2)cos(\alpha) + r_1\theta_1 + r_2\theta_2$, where $x_i, y_i, \theta_i, i = 1, 2$ are the coordinates of the mating gears as shown in Figure 5.7. $\alpha$ is the pressure angle and $r_i, i = 1, 2$ are the radii of the meshing gears. This formulation includes the shaft and bearing compliance.

Transmission error of this gear pair without shaft or bearing compliance has been compared among Program X, Load Distribution Program, NASA DANST [132], and the current approach in Figure 5.8. Program X is multi-body dynamics software that is used by industries worldwide. We are not free to state its name because of
license restrictions for academic use. The agreement on the peak to peak amplitude is reasonable. Differences among the mean amplitudes are present. These differences are mainly caused by different rim models these programs have used. Gear rims introduce compliance into the system, leading to high amplitude of the mean transmission error. The current model includes the realistic rim (as shown in Figure 5.5(a)). Others model the rims differently by excluding the rim shoulders. Thus, their estimates of transmission error are lower. The Harris map of transmission error at various torques is shown in Figure 5.9. The minimum transmission error without including the shaft compliance is at 67.79 Nm torque, which matches the torque the gear teeth are modified at. This further validates the transmission error estimate.

In addition, the minimum of the peak to peak value of transmission error is at lower torque (57.62 Nm) when flexible shafts are included as shown in Figure 5.9.
This suggests shaft compliance needs to be considered to estimate transmission error and modify gear teeth.

5.3.3 Stiffness Determination of Rolling Element Bearings

Bearings are critical components in geared systems. Theoretical bearing models [4, 7, 73, 74] make different assumptions to formulate the load-deflection relation. These assumptions include different race elasticity, different property of the rolling element contact, and the ignorance of microgeometry dimensions of bearing raceways. Figure 5.10 shows the nonlinear stiffness-load relations of the cylindrical and radial ball bearings calculated using the Harris [4], Gargiulo [5], and While [6] models. Significant discrepancy is present among them.

The method developed to determine bearing stiffness does not make any assumptions about the load-deflection relation. Instead, it calculates the $6 \times 6$ bearing stiffness matrix by partial derivatives of applied forces and moments related to six degrees of freedom. This stiffness determination approach is detailed in Chapter 4.

Radial and axial stiffness of radial ball bearing in [3] calculated by the proposed method under axial preloads is compared against the experiment conducted by Kraus et. al. in [3] as shown in Figure 5.11. Because of the lack of information on internal...
Radial and axial stiffness of self-aligning ball bearing in [2] calculated by the proposed method is compared against the Royston and Basdogan [2] experiment under radial and axial preloads, respectively (Figure 5.12). Good agreement is evident between the proposed method and the measurements for both bearings.

The number of rolling elements in contact oscillates when they rotate, which indicates bearing stiffness is periodic about the ball pass frequency. The ball pass period $T = \frac{2}{Z \omega (1 - \frac{D_e}{D_p})}$ is the amount of time between one rolling element leaving a reference point and the next rolling element arriving. Here, $z$ is the number of rolling elements, $\omega$ denotes the shaft speed, and quantities $D_e$ and $D_p$ denote the diameter of rolling elements and pitch diameter of the bearing, respectively.

Figures 5.13(a) and 5.14(a) show radial stiffness of the cylindrical and radial ball bearings in a ball pass period with 1000 N applied load. Radial stiffness of the cylindrical bearing fluctuates with 16% deviation from the mean stiffness, while the
Figure 5.9: Peak to peak amplitude of static transmission error of the gear pair with 
\((-\times-\)) and without \((-\circ-\)) shafts at different torques.

number of rolling elements in contact alternates between four and five over the ball 
pass period. Figures 5.13(b) and 5.14(b) show tilting stiffness of the cylindrical and 
bearings when \(1\, Nm\) moment is applied. These stiffness fluctuations can excite 
gearbox vibration.

Traditionally, diagonal stiffness matrices are used to represent rolling element 
bearings. These stiffness matrices, however, include both diagonal and off-diagonal 
terms. Equation 5.1 demonstrates the stiffness matrix structure. The quantities 
k_{xx}, k_{yy} denote radial stiffness. \(k_{zz}\) denotes axial stiffness. The quantities \(k_{\theta_x\theta_x}, k_{\theta_y\theta_y}\) denote tilting stiffness, which prevents the tilting motion of the shafts. The off-
diagonal stiffness falls into four categories: the coupling between radial and rotational 
displacements \((k_{x\theta_x}, k_{x\theta_y}, k_{y\theta_x}, k_{y\theta_y})\), the coupling between radial and axial displace-
ments \((k_{xx}, k_{yx})\), the coupling between axial and rotational displacements \((k_{z\theta_x}, k_{z\theta_y})\), 
and other coupling terms \((k_{xy}, k_{\theta_x\theta_y})\). Rolling elements are free to rotate in \(\theta_z\) direc-
tion so that stiffness in \(\theta_z\) direction and other related matrix components are zeros. 
The stiffness matrix is symmetric because rolling element bearings are conservative
Figure 5.10: Radial stiffness of examined cylindrical bearing and radial ball bearing vs. applied radial loads calculated by the Harris (−) [4], Gargiulo (−⋯−) [5], and While (−−) [6] models. The While [6] model is modified to use $\frac{\Delta F}{\Delta q}$ to calculate the stiffness instead of $\frac{F}{q}$.

Cross-coupling terms in the stiffness matrix indicate interactions between radial, axial, and tilting motions of rolling element bearings. They demonstrate the coupling between the shaft tilting motion, the flexural motion of the structure connected to the outer race, and the shaft radial and axial motions. The effects of cross-coupling terms on the gearbox vibration transmissibility through rolling element bearings are investigated. As the primary excitation source, gear transmission error is an important
Figure 5.11: Comparison between the proposed method with zero (··· □ ···) and 0.01 mm (− · −) radial clearances and Kraus et al.’s [3] experiment (− ○ −) for radial and axial stiffness of the ball bearing in [3] under axial preloads.

measure of gearbox vibration. Figure 5.15 shows the spectra of dynamic transmission error in the frequency range from 1500 to 4000 Hz from numerical torque impulse cases. Bearing models with fully-populated and diagonal stiffness matrices are compared as shown in Figure 5.15. Differences in resonant frequencies and amplitudes are evident between these two bearing models. This stresses the significance of the cross-coupling stiffnesses.

5.3.4 Shaft Modeling and Validation

The long shafts in the gearbox introduce system compliance, could cause misalignment, and eventually affect transmission error. The accuracy of the shaft models are important.

The shaft bending and torsional deformations are compared against classical beam
The shaft bending model is considered as the elastic beam with a concentrated load applied at the gear location. The beam bending boundary conditions are chosen as simply-supported at each end of the shaft. The transverse deflection $y$ along the shaft is calculated as

$$y = \frac{Pb[x^3 - (L^2 - b^2)x^{3/2}]}{6EIL}, \quad x < a$$

where the parameter $P$ is the applied concentrated force. The parameters $a, b$ are the distances between one shaft end and the location where $P$ is applied. The quantities $E, I, L$ denote the Young’s modulus, moment of inertia, and shaft length. The quantities $R_{out}, R_{in}$ denote the shaft outer and inner radii.

The analytical model to calculate the shaft torsional deflection has the clamped-free boundary condition. A torque $T$ is applied at the free end of the shaft. The
Figure 5.13: (a) Radial and (b) tilting stiffness of the cylindrical bearing over a ball pass period. The bottom figures show the number of rolling elements in contact over a ball pass period. The applied load and moment are 1000 N and 1 Nm, respectively.

torsional deflection $\theta$ is calculated as

$$\theta = \frac{T_x}{GJ}$$

$$G = \frac{E}{2(1 + \nu)}$$

(5.3)

$$J = \frac{\pi}{2} \left( R_{out}^4 - R_{in}^4 \right)$$

where the quantities $\nu$, $J$ denote the Possion’s ratio and the second moment of inertia.

Figures 5.16(a) and 5.16(b) show the shaft bending and torsional deformations calculated by analytical solutions (solid line) and finite element results (square marker) at various torques/forces. The finite element results of shaft bending and torsional deformations agree with the analytical predictions.
Figure 5.14: (a) Radial and (b) tilting stiffness of the ball bearing over a ball pass period. The bottom figures show the number of rolling elements in contact over a ball pass period. The applied load and moment are 1000 N and 1 Nm, respectively.

In addition, as shown in Figure 5.3.4, the bending deformation is the same order of magnitude of transmission error near the operating torque. Shaft deformation is significant. Thus, including shafts would increase the overall accuracy of gearbox modeling.

5.3.5 Gearbox Dynamic Analysis and Correlation with Experiments

Modal analysis is performed numerically on the gearbox model by applying a torque impulse at the input shaft and measuring the dynamic response. Numerical impulse test provides natural frequencies and mode shapes of the gearbox. Results of the impulse test are correlated with experimental measurements conducted by NASA personnel [133] and the Ohio State research group.

The frequency spectrum of dynamic transmission error and shaft displacement for
Figure 5.15: Numerical torque impulse response of gear dynamic transmission error with fully-populated (−) and diagonal (−−) stiffness matrices of the rolling element bearings mounted in the examined gearbox based on [8]. The input torque equals $84.74 \, Nm$.

The speed range from 0 to 8000 $Hz$ are shown in Figures 5.18(a) and 5.18(b). Multiple resonances are present, which are the natural frequencies of the gearbox. The computed natural frequencies agree with experiments as compared in Table 5.3. NASA accelerometers identify natural frequencies near 3000 to 4500 $Hz$, but the coherence is relatively low at these higher frequencies. Measurements show several modes near 6500 to 7500 $Hz$ that are difficult to resolve, while simulations predict only one natural frequency at 6856 $Hz$. The mode shapes of these natural frequencies include mesh deflection modes, shaft modes, housing modes, and coupled modes.
<table>
<thead>
<tr>
<th></th>
<th>Calyx</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>759</td>
<td>750</td>
<td></td>
</tr>
<tr>
<td>1942</td>
<td>2000</td>
<td></td>
</tr>
<tr>
<td>2195, 2283</td>
<td>2500</td>
<td></td>
</tr>
<tr>
<td>3803, 4449, 4654</td>
<td>3000-4500</td>
<td></td>
</tr>
<tr>
<td>6856</td>
<td>6500-7500</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.3: Natural Frequencies Predicted by Numerical Impact Tests and Measurements (Hz)

5.4 Mathematical Gearbox Dynamic Modeling and Analysis

Mathematical modeling of the gearbox provides fast dynamic analysis of the gearbox. Dynamic analysis, especially at multiple operating speeds, requires much computational effort using the finite element model. Mathematical model gives the natural frequencies and mode shapes, and dynamic response at various operating speeds.

This mathematical model employs crucial information on transmission error, mesh stiffness, bearing stiffness, and housing compliance from the finite element analysis.

5.4.1 The Gear Pair

The gear pair model is constructed following the literature [134, 135, 15, 136, 137, 138, 139]. Detail discussions into the validity of the modeling can be found in the literature. The model consists of two gears mounted on shafts. Each gear body is combined with its supporting shaft into a single rigid body. These gear-shaft bodies are each mounted on up two bearings placed at arbitrary axial locations.
Figure 5.19 shows the gear model and the bases. A fixed, right-handed, orthonormal basis \( \{ \mathbf{E} \} = \{ \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3 \} \) is oriented such that \( \mathbf{E}_1 \) is parallel to the line of action of the gear mesh. The origin is on the rotation axis of the pinion body, midway in the active facewidth. This dimensioning allows arbitrary axial positioning of meshing gears with different facewidth because any inactive facewidth is treated as part of the shaft. The translational \( (x_p, y_p, z_p) \) and angular \( (\phi_p, \theta_p, \beta_p) \) coordinates of the pinion body are assigned to translations along and rotations about \( \mathbf{E}_1, \mathbf{E}_2, \) and \( \mathbf{E}_3, \) respectively. The translational and angular coordinates of the gear body follow similarly with subscript \( g \). Body-fixed bases \( \{ \mathbf{e}^p \} = \{ \mathbf{e}^p_1, \mathbf{e}^p_2, \mathbf{e}^p_3 \} \) and \( \{ \mathbf{e}^g \} = \{ \mathbf{e}^g_1, \mathbf{e}^g_2, \mathbf{e}^g_3 \} \) for the pinion and gear are adopted. Positive axial quantities are measured along \( \mathbf{E}_3 \) from the dashed line in Figure 5.19.

The pinion translational and angular velocity vectors are

\[
\dot{\mathbf{r}}_p = \dot{x}_p \mathbf{E}_1 + \dot{y}_p \mathbf{E}_2 + \dot{z}_p \mathbf{E}_3,
\]

\[
\mathbf{\omega}_p = \left[ \dot{\phi}_p - \theta_p \left( \dot{\beta}_p + \Omega_p \right) \right] \mathbf{e}^p_1 + \left[ \dot{\theta}_p + \phi_p \left( \dot{\beta}_p + \Omega_p \right) \right] \mathbf{e}^p_2 + \left[ \dot{\beta}_p + \Omega_p - \phi_p \dot{\theta}_p \right] \mathbf{e}^p_3,
\]

where \( \Omega_p \) is the constant angular rotational speed of the pinion. The velocity vectors for the gear are identical except with components for the gear.

The pinion body is supported by two bearings at points \( A_p \) and \( B_p \). The axial positions of these bearings measured along \( \mathbf{E}_3 \) are \( L_p^A \) and \( L_p^B \). The pinion bearing deflection vectors at point \( A_p \) and \( B_p \) are the relative deflections of points \( A_p \) and \( B_p \) with respect to ground, giving

\[
\mathbf{d}^A_p = \left[ \theta_p \left( L_p^A - e_p \right) + x_p \right] \mathbf{E}_1 + \left[ \phi_p \left( e_p - L_p^A \right) + y_p \right] \mathbf{E}_2 + z_p \mathbf{E}_3,
\]

\[
\mathbf{d}^B_p = \left[ \theta_p \left( L_p^B - e_p \right) + x_p \right] \mathbf{E}_1 + \left[ \phi_p \left( e_p - L_p^B \right) + y_p \right] \mathbf{E}_2 + z_p \mathbf{E}_3.
\]

The bearing deflections for the gear follow similarly. The bearings resist tilting as well. The angular deflection of the pinion body bearing at \( A_p \) is

\[
\mathbf{\Gamma}^A_p = \dot{\phi}_p \mathbf{E}_1 + \theta_p \mathbf{E}_2 + \beta_p \mathbf{E}_3.
\]
The angular bearing deflection at point $B_p$ is identical to Eq. (5.6) for rigid shafts. The bearings are isotropic in the $E_1 - E_2$ plane. At point $A_p$, the bearing stiffness matrix for translation is $K^A_{A_p} = \text{diag}[k^A_{A_p}, k^A_{A_p}, k^A_{A_p}]$, where the equality of stiffness in the two translation directions is evident. The bearing stiffness matrix for rotation is $\chi^A_{A_p} = \text{diag}[\kappa^A_{A_p}, \kappa^A_{A_p}, \kappa^A_{A_p}]$. Similar definitions follow for point $B_p$ and for the gear body.

The gear mesh interface is modeled by a series of springs along the nominal lines of contact for no mesh deflection. These lines change as the gears rotate. Each spring acts at a point denoted by $C_i$. When the gear bodies deflect, the contact points on the pinion separate or compress against the contact points on the gear. The difference between the position vectors of the contact points on the pinion and gear gives the relative mesh deflection vector at $C_i$. The projection of the relative mesh deflection vector on the tooth surface normal gives the relative compressive deflection at the $i$th contact point. The relative compressive deflection is

$$
\delta_i(q, t) = \left\{ [e_p - c_i(t)] \theta_p + [c_i(t) - e_g] \theta_g - x_p + x_g + h_i + \beta_p r_p + \beta_g r_g \right\} \cos \psi \\
- \left\{ [b_i(t) + h_i] \theta_p + [(r_p + r_g) \theta_g \tan \Phi - b_i(t)] + z_p - z_g + \phi_p r_p + \phi_g r_g \right\} \sin \psi,
$$

where $r_p$ and $r_g$ are the base radii, $\Phi$ is the transverse operating pressure angle, and $\psi$ is the base helix angle. The vector $q$ comprises generalized coordinates described in the upcoming development. The axial position of a contact point is $c_i(t)$ measured from the origin along $E_3$, and the transverse position of a contact point is $b_i(t)$ measured from the origin along $-E_1$. They are known functions of time determined by the contact line progressions as the gears rotate. Micron-level deviations of the tooth surface from an involute at any contact point $i$, such as from gear tooth surface modifications and manufacturing errors, are denoted by $h_i$. Figures 5.19 and 5.20(b) depict these quantities.
The kinetic and potential energies are

\[ T = \frac{1}{2} (\omega_p^T J_p \omega_p + \omega_g^T J_g \omega_g + \dot{r}_p^T m_p \dot{r}_p + \dot{r}_g^T m_g \dot{r}_g) , \]

\[ V = \frac{1}{2} \left[ d_p^A T_p^A d_p^A + d_p^B T_p^B d_p^B + \Gamma_p^A \chi_p^A + \Gamma_p^B \chi_p^B \right] + \frac{1}{2} \sum_{i=1}^{n(t)} k_i(q(t)) \delta_i(q(t))^2 , \]

where \( k_i(q, t) \) is the \( i \)th contact stiffness, and \( n(t) \) is the number of contact segments at an instant \( t \). These quantities change as the gears rotate, hence the time dependence.

The inertia tensor of the axisymmetric pinion body is \( J_p = \text{diag}[J_{px}, J_{py}, J_{pz}] \) with similar definition for the axisymmetric gear body.

Lagrange’s equations of motion for unconstrained generalized coordinates follow after substitution of equations Eqs. (5.4) through (5.7) into the energy expressions Eq. (5.8). In matrix form they are

\[ M \ddot{q} + D \dot{q} + \Omega_p G \dot{q} + [K(q, t) - \Omega_p^2 C] q = f(q, t) , \]

\[ q = \left( \phi_p, \theta_p, \beta_p, x_p, y_p, z_p, \phi_g, \theta_g, \beta_g, x_g, y_g, z_g \right). \]

The vector \( f \) includes external loading; the driving and absorbing torques and tooth surface modifications \( h_i \) appear here. The matrix \( K \) represents the system elasticity with losses contained in the modal damping matrix \( D \). Tooth surface modifications \( h_i \) are neglected in \( K \) because \( h_i \ll b_i(t) \), and the \( h_i \) appear as additions to \( b_i(t) \). The terms that arise from the constant rotation speed are contained in the gyroscopic matrix \( G \) and the centripetal acceleration matrix \( C \). Individual elements of \( M, K, G, C \), and \( f \) are given in the appendix.

Following [140, 138] the nominal contact lines are discretized into \( n(t) \) segments of equal length \( l(t) \), as shown in Figure 5.20(a). Each contact point \( C_i \) is positioned at the center of its segment. As the contact lines progress with gear rotation, the
total number of segments \( n(t) \) and the length of a segment \( l(t) \) change. Each contact line has a specified number of segments. This discretization is based on the nominal lines of contact with no gear deflections.

Each contact spring is attached to its contact point \( C_i \). The stiffness \( k_i(q, t) \) of contact springs are obtained by considering two separate categories of tooth deflection: local (\( \epsilon_i \)) and bulk (\( \delta_b \)). Discussion of this categorization can be found in [140, 137, 141]. The local deflection represents the Hertz contact deflections. The associated local stiffness is \( k_c(t) \), where the constant \( k_c \) is the local stiffness per unit contact length. The bulk deflection represents all deflections except local deflection, and those include gear blank deflection, tooth bending, shear, etc. Because the Hertz contact deflections are localized and far enough from the bulk deflections, the bulk deflection is assumed to be the same for all contact segments. The bulk stiffness \( k_b \) is assumed constant. The bulk spring is in series with the local springs, so the total deflection at the \( i \)th contact point \( C_i \) is

\[
\delta_i = \epsilon_i + \delta_b \tag{5.11}
\]

The mesh force \( F \) equals the sum of all forces carried by the local springs and also the force carried by the bulk spring due to the series connection. The mesh force is

\[
F = \sum_{i=1}^{n} F_i = k_c l(t) \sum_{i=1}^{n} \epsilon_i H(\epsilon_i) = k_b \delta_b, \tag{5.12}
\]

where

\[
H(\epsilon_i) = \begin{cases} 
1; & \epsilon_i \geq 0 \\
0; & \epsilon_i < 0 
\end{cases} \tag{5.13}
\]

is the Heaviside function that represents the contact condition at each contact spring. Use of Eqs. (5.11) and (5.12) reduce the network of local and bulk springs into \( n(t) \)
contact springs \((k_i, i = 1, 2, \ldots, n)\) in parallel, as shown in Figure 5.20(b). The \(i\)th contact stiffness is given by

\[
k_i(q, t) = \frac{k_b k_c l(t) H(\epsilon_i)}{k_b + k_c l(t) \sum_{i=1}^{n(t)} H(\epsilon_i)}.
\]

Note that even though the local stiffness per unit contact length \(k_c\) and the bulk stiffness \(k_b\) are constants, the stiffness of each contact segment \(k_i(q, t)\) changes with partial contact loss and contact line length. This is a mathematical result of the spring network that Eq. (5.14) gives, and it is more realistic than a constant stiffness per contact length [138, 142]. One can then calculate the \(i\)th potential energy \(\frac{k_i \delta_i^2}{2}\) stored in the gear mesh in Eq. (5.8) from the contact force \(F_i = k_i \delta_i\).

The local stiffness per unit length \(k_c\) and the bulk stiffness \(k_b\) are parameters of the gear pair determined by the contact mechanics and elasticity of the gears. These constants can be approximated analytically [140] or semi-analytically [138] by assigning certain types of stiffnesses such as Hertz contact, tooth bending, shear, etc. to \(k_c\) and \(k_b\). A different approach is used in this work, where \(k_c\) and \(k_b\) are solved for from the deflections obtained from an external analysis tool for computational static analyses. In this case, \(k_c\) and \(k_b\) are numerical values that best fit the deflection obtained from finite element contact analysis of gears.

The following stipulations simplify the algebra to find \(k_c\) and \(k_b\). 1) The tooth surface is perfectly involute, that is, \(h_i = 0\) for all \(i\); 2) All degrees of freedom are constrained to be zero except the pinion rotation \(\beta_p\); and 3) A specified moment about \(E_3\) is applied to the pinion. With these stipulations, the deflections at all contact points are identical, that is, \(\delta_i = \delta\) for all \(i\) in Eq. (5.7). Consequently, all points are in contact; \(H_i = 1\) for all \(i\). The subscript \(i\) of \(k_i\) is unnecessary because when all segments are in contact, \(k_1 = k_2 = \ldots = k_n\). Use of static equilibrium, Eq. (5.11),
and Eq. (5.14) gives
\[ \delta(t) = F \left[ \frac{1}{k_b} + \frac{1}{k_c L(t)} \right], \tag{5.15} \]
where \( \delta(t) \) is the static transmission error, \( L(t) = n(t) l(t) \) is the total contact line length at an instant \( t \), and \( F \) is the constant mesh force obtained from the known applied torque. The two unknowns \( (k_c, k_b) \) are solved using the data from finite element analysis results at two instances \( \{ \delta(t_1), L(t_1) \} \) and \( \{ \delta(t_2), L(t_2) \} \) within a mesh cycle. To increase accuracy, values at these two instances are calculated from averages of the four points where transmission error is highest (giving the values for the first instance) and the four points where transmission error is lowest (giving the values for the second instance).

The gear mesh model is validated by finite element analysis. Figure 5.21 shows transmission error comparison between the finite element and the analytical model for a helical and a spur gear pair.

5.4.2 Housing

Connecting the Housing to the Gear & Shaft System

The equations of motion for gears, shafts, and bearings are given in matrix form by
\[ M_s \ddot{q}_s + K_s q_s = F_s \tag{5.16} \]
where \( M_s, K_s, \) and \( f_s \) are the mass and stiffness matrices and the force vector of the system as found in the foregoing section. The displacements of the degrees of freedom are cast in the vector \( q_s \). The equations of motion for the housing is given by
\[ M_H \ddot{q}_H + K_H q_H = F_H \tag{5.17} \]
where \( M_H \) and \( K_H \) are the mass and the stiffness matrices with degrees of freedom \( q_H \) and the force vector \( f_H \). The purpose in what follows is to connect the housing to
the bearings, which in (5.16) are attached to the ground. Figure 5.22 helps explain the approach.

The system equations of motion (5.16) are expanded by the addition of bearing degrees of freedom $q_b$. The new equations of motion are given by

$$
\begin{bmatrix}
M_s & 0 \\
0 & M_b
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_s \\
\ddot{q}_b
\end{bmatrix}
+ 
\begin{bmatrix}
K_s & K_{sb} \\
K_{sb}^T & K_b
\end{bmatrix}
\begin{bmatrix}
q_s \\
q_b
\end{bmatrix}
= 
\begin{bmatrix}
f_s \\
f_b
\end{bmatrix}
$$

(5.18)

where, the matrix $K_{sb}$ includes the coupling between the bearing degrees of freedom and the system. The bearing ends are massless, $M_b = 0$.

Partitioning of (5.17) to separate the degrees of freedom where the bearings are attached from the rest of the housing gives

$$
\begin{bmatrix}
M_c & 0 \\
0 & M_h
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_c \\
\ddot{q}_h
\end{bmatrix}
+ 
\begin{bmatrix}
K_c & K_{ch} \\
K_{ch}^T & K_h
\end{bmatrix}
\begin{bmatrix}
q_c \\
q_h
\end{bmatrix}
= 
\begin{bmatrix}
f_c \\
f_h
\end{bmatrix}
$$

(5.19)

where $M_c$ and $K_c$ are the mass and stiffness matrices, $q_c$ is the degrees of freedom, and $f_c$ is the force at the connection points. The rest of the housing mass, stiffness, degrees of freedom and forcing are contained in $M_h$, $K_h$, $q_h$, and $f_h$.

The four matrix equations from (5.18) (5.19) are

$$
M_s\ddot{q}_s + K_s q_s + K_{sb}q_{sb} = f_s
$$

(5.20)

$$
K_{sb}^T q_s + K_b^T q_b = f_b
$$

(5.21)

$$
M_c\ddot{q}_c + K_c q_c + K_{ch}q_{ch} = f_c
$$

(5.22)

$$
M_h\ddot{q}_h + K_h q_h + K_{ch}^T q_c = f_h
$$

(5.23)

The connection between the bearings and the housing requires that at the interface
displacements are equal, \( q_b = q_c \), and the forces are transmitted \( f_b = -f_c \). Eliminating \( q_c \) using these connection requirements from (5.21) and (5.22) gives

\[
\begin{bmatrix}
M_s & 0 & 0 \\
0 & M_b & 0 \\
0 & 0 & M_h
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_s \\
\ddot{q}_b \\
\ddot{q}_h
\end{bmatrix}
+ \begin{bmatrix}
K_s & K_{sb} & 0 \\
K_{sb}^T & K_c + K_b & K_{ch} \\
0 & K_{ch}^T & K_h
\end{bmatrix}
\begin{bmatrix}
q_s \\
q_b \\
q_h
\end{bmatrix}
= \begin{bmatrix}
f_s \\
f_b \\
f_h
\end{bmatrix}
\quad (5.24)
\]

This is the general form that connects the gears, shafts, and bearings to the full housing model which includes the flexibility as well as the inertia. If the housing inertia is neglected \( M_h = M_b = 0 \) and there are no external forces on the housing \( f_h = 0 \), the last row of (5.24) becomes \( q_h = -K_h^{-1}K_{ch}^T q_b \). Substitution of \( q_h \) into the second row of (5.24) gives \( q_b = \left[ K_b + K_c - K_{ch}K_h^{-1}K_{ch}^T \right]^{-1}K_{sb}q_s \). Substitution of \( q_b \) into the first row of (5.24) gives

\[
M_s \ddot{q}_s + \left\{ K_s - K_{sb}\left[ K_b + K_c - K_{ch}K_h^{-1}K_{ch}^T \right]^{-1}K_{sb} \right\} q_s = 0
\quad (5.25)
\]

This equation incorporates the housing flexibility, without added degrees of freedom and the housing inertia, into the equations of motion of the gears, shafts, and bearings. To this end, the calculation of \( K_c, K_{ch}, \) and \( K_h \) separately is unnecessary. A shorter path can be taken by evaluating the housing stiffness \( K_f \) at the connection interface using influence coefficients of the housing (5.17). This is discussed in the following subsection.

**Including the Housing by Using Influence Coefficients**

The influence coefficient matrix \( C_{f} \) can be solved from (5.17). It is given by

\[
C_{f}f_e = q_c
\quad (5.26)
\]

Solving for \( q_h \) from the second row of (5.19) and substituting it into the first row gives

\[
C_{f}^{-1} = K_f = K_c - K_{ch}K_h^{-1}K_{ch}^T
\quad (5.27)
\]
which allows use of the influence coefficient matrix in (5.25).

The procedure to obtain elements of $C_{f_i}^{n \times n}$ consists of applying a unit force to each degree of freedom at the connection points $f_c$ one at a time, and measuring the deflection of all the degrees of freedom at the connection points $q_c$. Explicitly, elements in the $i^{th}$ column, $C_{f_i}^{<i><j>}$, are given by the displacements $q_c^{<j>}$, calculated by (5.17) when there is a unit force at $f_c^{<i>} = 1$, where $j = 1, \ldots, n$.

Mathematical modeling of the gearbox is completed by Dr. Eritenel.

### 5.4.3 Mathematical Model Dynamic Analysis

Table 5.4.3 shows the natural frequencies of four systems: 1) analysis with cylindrical/ball bearings with housing flexibility, 2) analysis with wave bearings with housing flexibility, 3) analysis with cylindrical/ball bearings without housing flexibility, and 4) analysis with wave bearings without housing flexibility. Figure 5.23 shows the mesh deflection mode from analysis with cylindrical/ball bearings with housing flexibility. The mesh deflection mode is an important mode because the tooth pass excitation excites primarily the mesh deflection mode. The mesh deflection mode persists when bearing types is changed, and when the housing flexibility is included.

Linear dynamic analysis is performed using the mean system stiffness matrix obtained by averaging the quasi-static time-dependent system stiffness over a few mesh periods. The harmonic excitation is approximated from the first ten harmonics of the quasi-static displacement vector. Figure 5.24 shows the dynamic peak-to-peak transmission error from the four systems mentioned above. Because the the gear mesh parametric excitation excites the mesh deflection mode, all systems have similar dynamic transmission error. Figure 5.25 shows the dynamic bearing reactions of the four systems. Wave bearings seem to produce higher dynamic bearing loads.
Use of housing flexibility moves the natural frequencies and thus the resonant peaks, but does not alter the peak amplitudes in a significant manner.

<table>
<thead>
<tr>
<th>Cylindrical/ball bearings</th>
<th>Wave bearings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Housing</td>
<td>No Housing</td>
</tr>
<tr>
<td>576.6</td>
<td>975.5</td>
</tr>
<tr>
<td>954.6</td>
<td>1000</td>
</tr>
<tr>
<td>987.7</td>
<td>1597</td>
</tr>
<tr>
<td>1515</td>
<td>1711</td>
</tr>
<tr>
<td>1599</td>
<td>1868</td>
</tr>
<tr>
<td>1793</td>
<td>1977</td>
</tr>
<tr>
<td>1915</td>
<td>2736</td>
</tr>
<tr>
<td>2075</td>
<td>2848</td>
</tr>
<tr>
<td>2356</td>
<td>3910</td>
</tr>
<tr>
<td>2722</td>
<td>-</td>
</tr>
<tr>
<td>3893</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 5.4: Natural frequencies of the NASA gearbox with cylindrical/ball bearings, wave bearings, with housing flexibility and without housing flexibility. Red boxes indicate the mesh deflection mode natural frequencies.

5.5 Acoustic Radiation Gearbox Modeling and Analysis

The boundary element model of the gearbox housing is shown in Figure 5.26. This model includes 50,401 tetrahedron elements. Each element has nearly uniform length in order to have good aspect ratio. This model employs the fast multipole method (detailed in [126]) to calculate the gearbox sound radiation. This method provides fast acoustic calculation with a realistic system modeling: it can analyze well over 500,000 degrees of freedom efficiently. Given the complexity of housings, the great
increase in the number of degrees of freedom admits solutions for realistic models not possible with conventional boundary element software. This method is implemented in Coustyx [122].

5.5.1 Model Validation and Mesh Convergence Study

This boundary element model of the gearbox has been validated against theoretical solutions of the sound field with a monopole. A monopole is a source which radiates sound equally well in all directions. The simplest example of a monopole source would be a sphere whose radius alternately expands and contracts sinusoidally. The far-field pressure radiated by a monopole may be written as

\[ p(r, \theta, t) = \frac{iQ \rho c k}{4\pi r} e^{i(\omega t - kr)} \]  

(5.28)

where \( \rho \) is the fluid density, \( c \) is the speed of sound, \( k \) is the wave number, \( \omega \) is the frequency, and \( r \) is the local distance from source. \( Q \) is the complex source strength, which is a constant number defined in [143].

The sound field excited by a monopole is not affected by the gearbox geometry, which allows simulation results to be compared against the theoretical solution of a monopole for an arbitrary geometry. In other words, the noise radiation from the gearbox surface equals to the theoretical solution of a monopole in the free space (shown in Figure 5.27). The velocity field caused by the point source (monopole) in free space at gearbox surface is calculated theoretically and used as the boundary condition for the actual housing model.

Optimal element length has been carefully chosen through mesh convergence study. The relative error of the computed pressure compared to the theoretical value in (5.28) with various element length is shown in Figure 5.28. Result accuracy is improved when element length is small. However, when element length is smaller
than 8.636 mm, the result accuracy is disrupted as round off error becomes significant. The optimal element length is between 8.890 mm and 10.16 mm with less than 2% relative error. The mesh convergence study is important as the element length affects the solution accuracy.

Sound pressure of the radiated noise calculated by Coustyx and the theoretical solution is compared in Figure 5.29 for the speed range from 500 to 3500 Hz using 9.000 mm element length. Excellent agreement is present.

5.5.2 Sound Pressure and Power Computation Using Transfer Functions

A network of transfer functions from unit bearing load to sound pressure are used to predict sound pressure of the radiated noise. Transfer functions describe the noise radiation property of the gearbox housing. The computation of these transfer functions is entirely independent of the vibration analysis of the bearing reaction forces. Thus, these transfer functions accept different bearings or gears. The transfer function method is exceptionally useful to preform the gearbox optimization study to reduce noise.

The procedure to calculate transfer functions consists of applying a unit dynamic force to each degree of freedom \((x, y, z, \theta_x, \theta_y, \theta_z)\) at every bearing one at a time in the finite element model in ANSYS, measuring the velocity at the gearbox surface for various speeds in ANSYS, and computing the sound pressure of the radiated noise at specified locations in the gearbox environment using the computed velocity as the boundary condition in Coustyx. Figure 5.30 shows a few calculated transfer functions at four different bearings. Sound pressure at arbitrary locations outside the gearbox is determined by the superposition of these transfer functions and multiplying the fluctuating amplitudes of dynamic bearing forces at each degree of freedom of individual bearings.
Sound power is a better measure of the gearbox noise than sound pressure. Sound power measurements in experiments, however, require much effort. This study calculates the sound power using transfer functions, which are used to estimate the sound pressure with arbitrary bearing forces in the aforementioned study.

To obtain accurate estimate of sound power, many numerical “microphones” are mounted outside the gearbox. Those microphones are mounted on a measurement surface enveloping the gearbox, in this study, a sphere. According to the international standard ISO 3745, 20 microphones are “mounted”. The sound power is computed by integrating the sound pressure at these “microphones” in the sphere.

The sound power level, $L_w$ is computed as:

$$L_w = L_p + 10 \log_{10}\left(\frac{s}{s_0}\right)\text{(in dB)} + c_1 + c_2$$

$$c_1 = -10 \log_{10}\left[\frac{B}{B_0} \left(\frac{313.15}{273.15 + \theta}\right)^{0.5}\right]\text{(in dB)}$$

$$c_2 = -15 \log_{10}\left[\frac{B}{B_0} \left(\frac{296.15}{273.15 + \theta}\right)^{0.5}\right]\text{(in dB)}$$

(5.29)

where $s$ is the area of the measurement surface and $s_0 = 1 \text{ m}^2$. $B$ is the barometric pressure during measurements, in Pascals; $B_0$ is the reference barometric pressure. $\theta$ is the air temperature during measurement. $L_p$ is the weighed surface sound pressure level over the measurement surface in decibels.

$$L_p = 10 \log_{10} \left(\frac{1}{N} \sum_i 10^{0.1(L_{p,i} + W)}\right)\text{in dB}$$

(5.30)

$$L_{p,i} = 20 \log_{10} \left(\frac{p_i}{p_0}\right)$$

where $N$ is the number of microphones positions. $W$ is the weighting function applied by the filter at the frequency of analysis. $p_i$ is the root mean square pressure (Pascals) and $p_0 = 2 \times 10^{-5} Pa$. 
5.5.3 Radiated Noise Correlation with Measurements

The simulated noise is compared against the measurements taken at NASA GRC gearbox. Two microphones are mounted to measure sound pressure in experiments. They are mounted about 79 cm directly above the gearbox, and separated horizontally by about 18 cm as shown in Figure 5.31.

Here, only the noise transmitted through rolling element bearings are compared against measurements because experimental data on sound pressure with wave bearings is limited. Figure 5.32 shows the time averages of the mean squared sound pressure 

\[ (p)_{av}^2 = \sum_b (p_b)_{av}^2 \]

where \((p_b)_{av}^2\) is the additive measure of sound pressure associated with the frequencies within the band \(b\) from 0 to 8000 Hz. For each mesh frequency, two different experimental results are shown: the minimum sound pressure including only mesh frequency harmonics; and the maximum sound pressure including mesh frequency harmonics and other frequency components. These frequencies are the sidebands caused by the shaft rotation frequency, belt slip frequency, etc. The simulated sound pressure is within the experimental data range.

Figure 5.33 shows the frequency spectrum of the measured and simulated sound pressure when mesh frequency equals 2000 Hz. Good agreement is evident between the measurements and simulations at mesh frequency harmonics. Sidebands around the mesh frequency harmonics are present in the measured data. These sidebands are not included in the computational model because the causes are not known. Including these sidebands in the vibration/acoustics model would provide better agreement with the measurements.
Figure 5.16: (a) Input shaft bending deformation calculated by analytical beam theory (solid line) and finite element method (square marker) with simply-supported boundary conditions under various input torques; (b) Input shaft torsional deformation calculated by analytical beam theory (solid line) and finite element method (square marker) with clamped-free boundary conditions under various input torques. The shaft has uniform outer diameter (30.23 mm).
Figure 5.17: Shaft bending deformation under various input torques.
Figure 5.18: Calyx impulse test results of (a) dynamic transmission error and (b) the input shaft horizontal displacement of the gear-bearing-housing system within speed range from 0 Hz to 7000 Hz. The applied torque is 79.09 Nm.
Figure 5.19: Gear pair model. The parameters are defined in [9]. The dashed line is at the center of the active facewidth.

Figure 5.20: (a) Distributed spring network over a contact line with the local and bulk stiffnesses, $k_c l(t)$ and $k_b$. (b) Local $k_c l(t)$ and bulk $k_b$ stiffnesses are combined into contact stiffness $k_i(q, t)$ by Eq. (5.14).
Figure 5.21: Static transmission error from the analytical (solid line) and finite element (circles) model. (a) A helical gear pair. Quadratic tip relief starting at $\alpha = 28$ deg and root relief at $\alpha = 27$ deg. Tip relief, root relief, and circular lead crown are $10 \ \mu m$. The applied torque is 200 N-m. (b) Spur gear pair in [10]. Linear tip relief starting at $\alpha = 23.6$ deg with amplitude $10 \ \mu m$. Circular lead crown is $5 \ \mu m$. The applied torque is 340 N-m.

Figure 5.22: Description of the connection between the gears, shafts and bearings to the housing model.
Figure 5.23: Graphical representation of the 12th mode at 3893 Hz (mesh deflection mode) from the system with ball/cylindrical bearings with housing.

\[ \omega_{12} = 3893 \text{ Hz} \]

Figure 5.24: Peak-to-peak dynamic transmission error of four systems. Analysis with roller element bearings are marked by B, analysis with wave bearings are marked by W, analysis including the housing flexibility is marked by H, analysis without the housing is marked by nH.
Figure 5.25: Dynamic bearing forces. Analysis with roller element bearings are marked by B, analysis with wave bearings are marked by W, analysis including the housing flexibility is marked by H, analysis without the housing is marked by nH.

Figure 5.26: Boundary element model of the gearbox established in Coustyx.
Figure 5.27: Theoretical solutions used to validate the boundary element housing model. Noise radiated from the housing with monopole velocity field at the gearbox surface as the boundary condition equals to that with monopole in the free space.

Figure 5.28: Effects of boundary element length on the relative error of calculated sound pressure compared to the theoretical solution.
Figure 5.29: Sound pressure at the NASA microphone 1 location calculated by Coustyx (□) and theoretical models (−).

Figure 5.30: Sound pressure transfer functions when unit dynamic loads are applied at bearings. Six transfer functions are generated per each bearing along \(x, y, z, \theta_x, \theta_y, \theta_z\) directions. The input torque is 79.09 Nm.
Figure 5.31: Microphones (1 and 2) mounted above the NASA GRC gearbox.

Figure 5.32: The time average of the experimental (−−) and simulated (−) mean squared sound pressure at the microphone 1 location within mesh frequency range from 500 to 3000 Hz. The input torque equals 79.09 Nm.
5.5.4 Noise Radiation Properties with Different Bearings

To understand the roles of rolling element and fluid film wave bearings on the vibro-acoustic propagation path, the sound pressure and power of the radiated noise are compared between these bearings.

Sound pressure at the microphone 1 location excited by the first (left) and second (right) mesh frequency harmonics is compared between rolling element and wave bearings. As shown in Figure 5.34, the relative amplitude of sound pressure between rolling element and wave bearings changes with speed. When the second mesh frequency harmonic is near 3803 Hz (dominant mesh deflection mode), sound pressure with rolling element bearings is similar in shape and amplitude to that with wave bearings. The noise radiation is independent of bearing types at this mesh deflection mode.
Figure 5.34: The frequency spectrum of the measured (left) and simulated (right) sound pressure at the microphone 1 location when mesh frequency is between 1000 Hz and 2500 Hz. The input torque equals 79.09 Nm.

The sound power radiated from the gearbox is also compared between wave and rolling element bearings in Figure 5.35. The sound power level with finite element and wave bearings is also speed-dependent. The radiated gearbox sound power with rolling element bearings is lower than that with fluid film wave bearings in general within the speed range of interest. There is no obvious benefit to replace rolling element bearings with wave bearings to reduce the gearbox noise at these speeds. This conclusion could change when an advanced wave bearing model is used (The current model does not include the tilting degree of freedom.).

5.6 Summary and Conclusions

A multi-physics multi-dynamics model of the examined gearbox is established to address the vibro-acoustic propagation of the gear dynamics. It includes: 1) full fidelity finite element gearbox model in Calyx; 2) lumped-parameter gearbox model
Figure 5.35: Sound power of radiated gearbox noise excited by certain mesh frequency harmonics of bearing forces with rolling element (−) and fluid film wave bearings (−−) at 79.09 Nm input torque. The 1st, 2nd, 3rd, 4th, 5th and 6th mesh frequency harmonics of bearing forces are considered during the computation. A weighing filter (ISO standard) is used to adjust sound pressure levels.

in Matlab; 3) finite element housing model in ANSYS; and 4). boundary element housing model in Coustyx. The procedure of the noise radiation analysis consists of: a). dynamic bearing force computation using the finite element and lumped-parameter gearbox models; b). gearbox surface velocity estimate with excitations from dynamic bearing forces obtained from a); and c). acoustic analysis of the gearbox with the boundary condition of the surface velocity calculated from b).

The full fidelity finite element/contact mechanics model of the NASA gearbox has been validated. Gear transmission error is compared among commercial software
and matches the design objective. The rolling element and gear contact properties have been validated against experiments in the literature. Modal analysis has been performed on the entire gearbox model and results agree with measurements.

Cross-coupling stiffnesses are significant as they affect the gearbox dynamics. A fully-populated stiffness matrix is needed to fully represent rolling element bearings. Bearing stiffnesses fluctuate periodically as rolling elements enter and leave the region of loaded rollers. These stiffness variations could excite gearbox vibration.

The simplified lumped-parameter model of the gearbox is suitable for parametric studies on the dynamic response. The analytical model has been validated against the finite element analysis.

This realistic noise radiation housing model has been validated by comparing against theoretical solutions and measurements taken at the NASA GRC gearbox. The transfer function method determines sound pressure and power for arbitrary bearings, which is exceptional suitable for the gearbox optimization study to reduce noise. Result indicates that the noise radiation property with fluid film wave bearings is speed-dependent, and wave bearings are not superior to rolling element ones in reducing gearbox noise.

An one-way vibro-acoustic pass of the gearbox is present at the mesh deflection mode. Gear dynamics significantly affects the housing vibration and noise radiation. Housing and bearings, however, have limited effects on the gear vibration, leading to similar noise radiation properties at this mode.
CHAPTER 6
SUMMARY AND CONCLUDING REMARKS

This body of work investigates the gearbox vibration considering tooth wedging, bearing clearance, and noise propagation. Tooth wedging is a potential cause for gearbox fatigue failure by elevating planet bearing forces and disrupting load sharing. Bearing clearance causes complicated nonlinear dynamic behaviors and unexpected vibration in planetary gears. Reliable $5 \times 5$ stiffness matrices of rolling element bearings are obtained using a novel numerical algorithm. The full fidelity rolling element bearing model improves the modeling accuracy by capturing time-varying stiffnesses, cross-coupling stiffnesses, and bearing micro-geometry. Multi-body computational tools are established to address the vibro-acoustic propagation of the gear/bearing/housing system.

6.1 Dynamic Modeling and Analysis of Planetary Gears Involving Tooth Wedging

A lumped-parameter model including tooth wedging, tooth separation, bearing clearance, and internal/external excitations is developed to examine the planetary gear dynamics. This analytical model is validated by finite element analysis.

Field investigation of a failed wind turbine gearbox indicates potential tooth wedging and premature planet bearing failures. Tooth wedging elevates planet bearing
forces significantly and disrupts load sharing. It is a potential source for gearbox fatigue failures.

Gravity is the dominant cause of tooth wedging compared to aerodynamic loads and dynamic forces from the meshing teeth. Gravity disturbs the planetary gear symmetry and thus breaks the mesh phasing rules that can reduce planetary gear vibration. Gravity could be the primary excitation source for planetary gears with heavy components.

6.2 Dynamic Analysis of Planetary Gears with Bearing Clearances

The harmonic balance method with arc-length continuation is formulated and applied to the analytical planetary gear model to determine the dynamic response. Floquet multipliers are used to analyze the solution stability. The harmonic balance formulation is verified by numerical integration and finite element analysis.

Bearing clearance changes system dynamics, elevates vibration, and causes instability. Bearing clearance in the central gears and carrier significantly decreases the natural frequencies of translational modes and changes the mode shapes. Rich nonlinear behaviors, various bifurcations, coexisting periodic orbits, and chaos are often present in the dynamic response within a wide speed range. Planet bearing clearance breaks the system symmetry and further softens the system when tooth separation is present.

Input torque can potentially eliminate some of the nonlinear behaviors.
6.3 Stiffness matrix calculation of rolling element bearings using a finite element/contact mechanics model

Reliable stiffness estimate of rolling element bearings is not available.

A finite element model for rolling element bearings including bearing microgeometry is established. A unique contact algorithm is employed to analyze the rolling element contact.

A stiffness determination method based on finite difference analysis is developed to provide accurate stiffness matrices for rolling element bearings. The accuracy order of this method depends on the finite element/contact mechanics analysis and the step size selected for the finite difference formulae. This stiffness determination method is validated against published experiments.

A fully-populated stiffness matrix is required to fully describe rolling element bearings. These cross-coupling terms change the dynamics of the examined NASA gearbox.

All elements in the stiffness matrix fluctuate periodically about the ball pass frequency. These fluctuations can excite vibration of the connected structures.

6.4 Vibration Propagation of Gear Dynamics in a Gear-Bearing-Housing System

Limited work has linked the gearbox vibration to noise radiation.

A computational multi-body model of the examined gearbox is established to address the structural-borne noise propagation of dynamic forces at the meshing teeth to the surrounding environment. This multi-physics model couples the finite element/contact mechanics model of the gear/bearing/housing systems and acoustic
model of the gearbox housing. The detailed finite element gearbox model provides accurate estimates of bearing stiffness, bearing force, mesh stiffness, mesh force, and transmission error, all of which are crucial for the gearbox noise computation. The acoustic model uses a fast multipole method, which provides calculation with a realistic system modeling.

This model has been validated by theoretical solutions and experiments including modal testing on the gearbox and sound pressure measurements provided by NASA Glenn research center.

The noise radiation property with rolling element and wave bearings is speed-dependent.

An one-way vibro-acoustic pass of the gearbox is evident at the mesh deflection mode. Gear dynamics significantly affects the housing vibration and noise radiation. Bearings, however, have limited effects on the gear vibration.
CHAPTER 7
FUTURE RECOMMENDATION

7.1 Effects of Bearing Microgeometry on Rolling Element Bearing Stiffness

The microgeometry dimensions of rolling element bearings include internal clearance, crowning on the rolling elements and races, raceway dimensions, etc. Existing analytical bearing models in the literature, however, are not capable of including these dimensions. Thus, the investigation on effects of microgeometry on bearing stiffness is limited.

The finite element/contact mechanics model used in Chapter 4 captures all the important design details of rolling element bearing including bearing microgeometry. Figure 7.1 shows the effects of radial clearance on the tilting stiffness of the cylindrical bearing C and the effects of race crown curvature of ball bearing D. Internal clearance and race crown curvature significantly change bearing stiffness: bearing stiffness decreases with internal clearance and increases with race crown curvature. Microgeometry of rolling element bearings affects bearing stiffness. This proposed study aims to fully understand the effects of bearing microgeometry on the overall stiffness using the detailed finite element/contact mechanics model.
Figure 7.1: Tilting stiffness of cylindrical bearing C (− − □ − −) and radial ball bearing D (−□−) vs. applied moments on x direction calculated by the approach in Chapter 4. Bearings C and D are described in Tables 4.3 and 4.4 in Chapter 4. Results are computed by the finite element analysis in Chapter 4.

### 7.2 Time-Varying Property of Rolling Element Bearing Stiffness and Its Effects on Gear Dynamics

Bearing stiffness is periodic about its ball pass period. This time-varying bearing stiffness is a secondary excitation source for the gearbox compared to mesh stiffness variation. Traditional gearbox models consider the bearing stiffness as constant. This assumption, however, might not be appropriate for some applications. For instance, when rolling element bearings have fewer rolling elements, the peak to peak value of bearing stiffness in the ball pass period is significant as shown in Figure 7.2. This proposed study would investigate the conditions when the bearing stiffness variation is significant.

Further research could investigate the effects of time varying bearing stiffness on gear dynamics, which has not been studied in the past. Prior studies treat gears
and bearings as separate research topics. In experiments, side bands about mesh
frequency harmonics are often present. These side bands could be related to the
shaft rotation frequency, belt slip frequency, bearing stiffness variation, etc. This
proposed study would study the contribution of bearings to the gearbox vibration
and dynamics.

7.3 Symmetry Breaking Induced by Planet Bearing Clearance in Planetary Gears

Symmetry breaking describes a physical phenomenon that small fluctuation acting
on the system could lead to different solutions when it is passing a critical point
[144]. When symmetry breaking happens, the system may experience more coexisting
solutions, various bifurcations, and chaos [145, 146]. Symmetry breaking phenomena
and bifurcation have been well-discussed theoretically and observed in mechanical
oscillators. Symmetry breaking has, however, not been investigated in planetary
gears.

Planetary gears are highly symmetric systems because of the nearly equally-spaced
planet mounting. Planet bearing clearance can break the system symmetry and cause
unexpected vibration. For instance, with an equally-spaced planet in-phase design,
the central gears and carrier have zero translational vibration (To have in-phase de-
sign, the numbers of gear teeth defined in Chapter 3 are tuned slightly to 28, 36, 100
for the sun, planet, and ring with equal planet spacing,). With planet bearing clear-
ance, the translational vibration of the central members is present as shown in Figure
7.3. Bearing clearance changes the phasing relations among planets from in-phase to
out-of-phase by destroying the symmetry of tooth loads. Translational vibration of
the central gears, therefore, appears and translational modes are excited. This proposed work would focus on the symmetry breaking caused by planet bearing clearance.
Figure 7.2: The radial stiffness of the cylindrical bearing C defined in Chapter 4 with (a) even and (b) odd numbers of rolling elements in a ball pass period. Results are calculated by the finite element analysis in Chapter 4.
Figure 7.3: The rms (mean removed) of the sun translational displacement for varying speeds with $\rho_p = 0$ and $\rho_p = 3$ in in-phase planetary gear. The system parameters are defined in Chapter 3. $\rho_p$ is the nondimensional planet bearing clearance defined in Chapter 3.
APPENDIX

Appendix A: Operators Used in Harmonic Balance

In the time domain, one period of the excitation frequency is discretized into \( n \) equally-spaced time intervals.

\[
L = \begin{pmatrix}
  L_e & \ldots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \ldots & L_e
\end{pmatrix}
\]

\[\text{(1)}\]

\[
L_e = \begin{pmatrix}
  1 & \cos \Omega t_1 & \sin \Omega t_1 & \cos 2\Omega t_1 & \sin 2\Omega t_1 & \cdots & \sin R\Omega t_1 \\
  1 & \cos \Omega t_2 & \sin \Omega t_2 & \cos 2\Omega t_2 & \sin 2\Omega t_2 & \cdots & \sin R\Omega t_2 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  1 & \cos \Omega t_n & \sin \Omega t_n & \cos 2\Omega t_n & \sin 2\Omega t_n & \cdots & \sin R\Omega t_n
\end{pmatrix}
\]

\[\text{(2)}\]

\[
H = \begin{pmatrix}
  H_e & \ldots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \ldots & H_e
\end{pmatrix}
\]

\[\text{(3)}\]
\[
H_e = \frac{1}{n} \begin{pmatrix}
& 1 & 1 & \ldots & 1 \\
2 \cos(\Omega t_1) & 2 \cos(\Omega t_2) & \ldots & 2 \cos(\Omega t_n) \\
2 \sin(\Omega t_1) & 2 \sin(\Omega t_2) & \ldots & 2 \sin(\Omega t_n) \\
2 \cos(2\Omega t_1) & 2 \cos(2\Omega t_2) & \ldots & 2 \cos(2\Omega t_n) \\
2 \sin(2\Omega t_1) & 2 \sin(2\Omega t_2) & \ldots & 2 \sin(2\Omega t_n) \\
& \vdots & \ddots & \vdots \\
2 \sin(R\Omega t_1) & 2 \sin(R\Omega t_2) & \ldots & 2 \sin(R\Omega t_n)
\end{pmatrix}
\]

(4)

\[
E = \begin{pmatrix}
E_e & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & E_e
\end{pmatrix}
\]

(5)

\[
E_e = \text{diag}(0, 1^2, 1^2, 2^2, 2^2, \ldots, R^2, R^2)
\]

(6)

\[
D_e = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
-1 & 0 & 0 & \ldots & 0 \\
0 & 2 & 0 & \ldots & 0 \\
-2 & 0 & 0 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
0 & R & 0 & \ldots & 0 \\
-R & 0 & -R & \ldots & 0
\end{pmatrix}
\]

(8)
Appendix B: Jacobian Matrix Used in Harmonic Balance

\[
J = K + K_{TL} + H \frac{\partial g_0}{\partial q} L + H \frac{\partial g_m}{\partial q} L
\]  
(9)

\[
K_{TL} = \text{diag}(0, 0, \Pi_c, 0, 0, \Pi_r, 0, 0, \Pi_s, 0, \ldots, 0)
\]  
(10)

\[
\Pi_l = \text{diag}(k_{l,u}, \ldots, k_{l,u})
\]

Jacobian Matrix Associated with Discontinuous Tooth Loads

\[
J_m = H \frac{\partial g_m}{\partial q} L = \begin{pmatrix}
0 & \cdots & \sum \Lambda_{r1}^{i,m} & \Lambda_{r2}^{1,m} & \cdots & \Lambda_{r2}^{n,m} \\
& \cdots & \sum \Lambda_{s1}^{i,m} & \Lambda_{s2}^{1,m} & \cdots & \Lambda_{s2}^{n,m} \\
& & \Lambda_{p}^{1,m} & \cdots & \Lambda_{p}^{n,m}
\end{pmatrix}
\]  
(11)

By defining \( \Xi(\cdot) = H \text{diag}(\cdot) L \),

\[
\Lambda_{r1,i}^{m} = \begin{pmatrix}
-\Xi(X_{ri}) \sin^2 \psi_{ri} & \Xi(X_{ri}) \sin \psi_{ri} \cos \psi_{ri} & \Xi(X_{ri}) \sin \psi_{ri} \\
\Xi(X_{ri}) \sin \psi_{ri} \cos \psi_{ri} & -\Xi(X_{ri}) \cos^2 \psi_{ri} & -\Xi(X_{ri}) \cos \psi_{ri} \\
\Xi(X_{ri}) \sin \psi_{ri} & -\Xi(X_{ri}) \cos \psi_{ri} & -\Xi(X_{ri})
\end{pmatrix}
\]  
(12)

\[
\Lambda_{r2,i}^{m} = \begin{pmatrix}
\Xi(X_{ri}) \sin \psi_{ri} \sin \alpha_r & -\Xi(X_{ri}) \sin \psi_{ri} \cos \alpha_r & -\Xi(X_{ri}) \sin \psi_{ri} \\
-\Xi(X_{ri}) \cos \psi_{ri} \sin \alpha_r & \Xi(X_{ri}) \cos \psi_{ri} \cos \alpha_r & \Xi(X_{ri}) \cos \psi_{ri} \\
-\Xi(X_{ri}) \sin \alpha_r & \Xi(X_{ri}) \cos \alpha_r & \Xi(X_{ri})
\end{pmatrix}
\]  
(13)

\[
\Lambda_{s1,i}^{m} = \begin{pmatrix}
-\Xi(X_{si}) \sin^2 \psi_{si} & \Xi(X_{si}) \sin \psi_{si} \cos \psi_{si} & \Xi(X_{si}) \sin \psi_{si} \\
\Xi(X_{si}) \sin \psi_{si} \cos \psi_{si} & -\Xi(X_{si}) \cos^2 \psi_{si} & -\Xi(X_{si}) \cos \psi_{si} \\
\Xi(X_{si}) \sin \psi_{si} & -\Xi(X_{si}) \cos \psi_{si} & -\Xi(X_{si})
\end{pmatrix}
\]  
(14)

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\[ \Lambda_{s2,i}^m = \begin{pmatrix} -\Xi(\chi_{si}) \sin \psi_{si} \sin \alpha_s & -\Xi(\chi_{si}) \sin \psi_{si} \cos \alpha_s & -\Xi(\chi_{si}) \sin \psi_{si} \\ \Xi(\chi_{si}) \cos \psi_{si} \sin \alpha_s & \Xi(\chi_{si}) \cos \psi_{si} \cos \alpha_s & -\Xi(\chi_{si}) \cos \psi_{si} \\ \Xi(\chi_{si}) \sin \alpha_s & \Xi(\chi_{si}) \cos \alpha_s & -\Xi(\chi_{si}) \end{pmatrix} \]  

\[ \Lambda_{p,i}^m = \begin{pmatrix} -\Xi(\chi_{ri}) \sin^2 \alpha_r & \Xi(\chi_{ri}) \sin \alpha_r \cos \alpha_r & -\Xi(\chi_{ri}) \sin \alpha_r \\ \Xi(\chi_{ri}) \sin \alpha_r \cos \alpha_r & -\Xi(\chi_{ri}) \cos^2 \alpha_r & -\Xi(\chi_{ri}) \cos \alpha_r \\ -\Xi(\chi_{ri}) \sin \alpha_r & \Xi(\chi_{ri}) \cos \alpha_r & -\Xi(\chi_{ri}) \end{pmatrix} + \begin{pmatrix} -\Xi(\chi_{si}) \sin^2 \alpha_s & -\Xi(\chi_{si}) \sin \alpha_s \cos \alpha_s & \Xi(\chi_{si}) \sin \alpha_s \\ -\Xi(\chi_{si}) \sin \alpha_s \cos \alpha_s & \Xi(\chi_{si}) \cos^2 \alpha_s & \Xi(\chi_{si}) \cos \alpha_s \\ -\Xi(\chi_{si}) \sin \alpha_s & \Xi(\chi_{si}) \cos \alpha_s & -\Xi(\chi_{si}) \end{pmatrix} \]  

\[ \chi_{ri} = -\frac{1}{2} k_{ri}(t) \left( \sigma sech^2(\sigma \delta_{ri}) \delta_{ri} + 1 + \tanh(\sigma \delta_{ri}) \right) \]  

\[ \chi_{si} = -\frac{1}{2} k_{si}(t) \left( \sigma sech^2(\sigma \delta_{si}) \delta_{si} + 1 + \tanh(\sigma \delta_{si}) \right) \]  

\[ \delta_{si} = L(q_{iy} \cos \psi_{si} - q_{ix} \sin \psi_{si} - q_{ix} \sin \alpha_s - q_{iy} \cos \alpha_s + q_{ui} + q_{u_x}) \]  

\[ \delta_{ri} = L(q_{iy} \cos \psi_{ri} - q_{ix} \sin \psi_{ri} + q_{ix} \sin \alpha_r - q_{iy} \cos \alpha_r - q_{ui} + q_{u_r}) \]  

\[ \psi_{si} = \psi_i - \alpha_s, \quad \psi_{ri} = \psi_i + \alpha_r \]  

**Jacobian Matrix Associated with Discontinuous Bearing Forces**

\[ J_b = H \frac{\partial g_b}{\partial q} L = \begin{pmatrix} \sum \Lambda_{c1,i}^b + \Lambda_{c1}^b & \Lambda_{c2,1}^b & \cdots & \Lambda_{c2,N}^b \\ \Lambda_{c2,1}^b & \cdots & \Lambda_{c2,N}^b \\ \Lambda_{c1}^b & \cdots & \Lambda_{c2,N}^b \end{pmatrix} \]  

\[ Symmetric \]  

\[ \Lambda_{p,1}^b \cdots 0 \]  

\[ \Lambda_{p,N}^b \]
\[
\begin{align*}
\Lambda^b_i &= \begin{pmatrix}
\frac{\text{Lq}_{i2} \gamma^b_i \cos(\vartheta^b_i)}{b^2_i} + \frac{\text{Lq}_{i0} \nu^b_i \sin(\vartheta^b_i)}{b^2_i} & \frac{\text{Lq}_{i3} \gamma^b_i \cos(\vartheta^b_i)}{b^2_i} + \frac{\text{Lq}_{i1} \nu^b_i \sin(\vartheta^b_i)}{b^2_i} & 0 \\
\frac{\text{Lq}_{i2} \gamma^b_i \sin(\vartheta^b_i)}{b^2_i} & \frac{\text{Lq}_{i3} \gamma^b_i \sin(\vartheta^b_i)}{b^2_i} & 0 \\
0 & 0 & 0
\end{pmatrix} \\
\end{align*}
\]

\(\delta^b_i = (\text{Lq}_{i2} + \text{Lq}_{i0})^{\frac{1}{2}}\)  
\(\vartheta^b_i = \tan^{-1}\left(\frac{\text{Lq}_{i3}}{\text{Lq}_{i1}}\right), \quad l = c, r, s\)

\[
\gamma^b_i = -\frac{1}{2} k_l \left(\text{sech}^2(\sigma(\delta^b_i - \Delta_l))(\delta^b_i - \Delta_l) + 1\right) + \tanh(\sigma(\delta^b_i - \Delta_l)), \quad l = c, r, s
\]

\[
\nu^b_i = -\frac{1}{2} k_l \left(1 + \tanh(\sigma(\delta^b_i - \Delta_l))\right)(\delta^b_i - \Delta_l), \quad l = c, r, s
\]

\[
\begin{align*}
\Lambda^b_{P,i} &= \begin{pmatrix}
\kappa^i_{1,0} \gamma^b_{P,i} \cos(\vartheta^b_{P,i} - \psi_i) - \kappa^i_{1,0} \nu^b_{P,i} \sin(\vartheta^b_{P,i} - \psi_i) & \kappa^i_{1,0} \gamma^b_{P,i} \cos(\vartheta^b_{P,i} - \psi_i) - \kappa^i_{2,0} \nu^b_{P,i} \sin(\vartheta^b_{P,i} - \psi_i) & 0 \\
\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} - \psi_i) + \kappa^i_{1,0} \nu^b_{P,i} \cos(\vartheta^b_{P,i} - \psi_i) & \kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} - \psi_i) + \kappa^i_{2,0} \nu^b_{P,i} \cos(\vartheta^b_{P,i} - \psi_i) & 0 \\
0 & 0 & 0
\end{pmatrix} \\
\end{align*}
\]

\[
\begin{align*}
\Lambda^b_{c1,i} &= \begin{pmatrix}
-\kappa^i_{1,0} \gamma^b_{P,i} \cos(\vartheta^b_{P,i} - \psi_i) & -\kappa^i_{1,0} \gamma^b_{P,i} \cos(\vartheta^b_{P,i} - \psi_i) & -\kappa^i_{1,0} \gamma^b_{P,i} \cos(\vartheta^b_{P,i} + \psi_i) \\
-\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i) & -\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i) & -\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i) \\
-\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i) & -\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i) & -\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i)
\end{pmatrix}
\] 
\]

\[
\begin{align*}
\Lambda^b_{c2,i} &= \begin{pmatrix}
-\kappa^i_{1,0} \gamma^b_{P,i} \cos(\vartheta^b_{P,i} + \psi_i) & -\kappa^i_{1,0} \gamma^b_{P,i} \cos(\vartheta^b_{P,i} + \psi_i) & 0 \\
-\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i) & -\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i) & 0 \\
-\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i) & -\kappa^i_{1,0} \gamma^b_{P,i} \sin(\vartheta^b_{P,i} + \psi_i) & 0 \\
\end{pmatrix}
\] 
\]

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\[ \kappa_{1,x}^i = \frac{\delta^b_{\xi,i} \cos(\psi_i) - \delta^b_{\eta,i} \sin(\psi_i)}{\delta^b_{P,i}} \]

\[ \kappa_{1,y}^i = \frac{\delta^b_{\xi,i} \sin(\psi_i) + \delta^b_{\eta,i} \cos(\psi_i)}{\delta^b_{P,i}} \]

\[ \kappa_{1,\zeta}^i = -\frac{\delta^b_{\xi,i}}{\delta^b_{P,i}} \]

\[ \kappa_{1,\eta}^i = -\frac{\delta^b_{\eta,i}}{\delta^b_{P,i}} \]

\[ \kappa_{1,u}^i = \frac{\delta^b_{\eta,i}}{\delta^b_{P,i}} \]

\[ \kappa_{2,x}^i = -\frac{\delta^b_{\xi,i} \sin(\psi_i) - \delta^b_{\eta,i} \cos(\psi_i)}{(\delta^b_{P,i})^2} \]

\[ \kappa_{2,y}^i = \frac{\delta^b_{\xi,i} \cos(\psi_i) - \delta^b_{\eta,i} \sin(\psi_i)}{(\delta^b_{P,i})^2} \]

\[ \kappa_{2,\zeta}^i = \frac{\delta^b_{\eta,i}}{(\delta^b_{P,i})^2} \]

\[ \kappa_{2,\eta}^i = -\frac{\delta^b_{\xi,i}}{(\delta^b_{P,i})^2} \]

\[ \kappa_{2,u}^i = \frac{\delta^b_{\eta,i}}{(\delta^b_{P,i})^2} \]

\[ \delta^b_{\xi,i} = L (q_{x_i} \cos \psi_i + q_{y_i} \sin \psi_i - q_{\zeta}) \]

\[ \delta^b_{\eta,i} = L (-q_{x_i} \sin \psi_i + q_{y_i} \cos \psi_i + q_{u_i} - q_{\eta}) \]

\[ \delta^b_{P,i} = \left[ \left( \delta^b_{\xi,i} \right)^2 + \left( \delta^b_{\eta,i} \right)^2 \right]^{1/2} \]

\[ \vartheta^b_{P,i} = \tan^{-1} \left( \frac{\delta^b_{\eta,i}}{\delta^b_{\xi,i}} \right) \]

\[ \gamma^b_{P,i} = -\frac{1}{2} k_P \left( \sigma \operatorname{sech}^2(\varepsilon(\delta^b_{P,i} - \Delta P_i)) (\delta^b_{P,i} - \Delta P_i) + 1 + \tanh(\sigma(\delta^b_{P,i} - \Delta P_i)) \right) \]

\[ \nu^b_{P,i} = -\frac{1}{2} k_P \left( 1 + \tanh(\sigma(\delta^b_{P,i} - \Delta P_i)) \right) (\delta^b_{P,i} - \Delta P_i) \]
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