LIPSCHITZ PROPERTIES OF HARMONIC AND HOLOMORPHIC FUNCTIONS

DISSERTATION

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ABSTRACT

We prove two results in this dissertation, one concerning Lipschitz harmonic functions and the other concerning Lipschitz holomorphic functions.

Let $B$ be a regular majorant. We show that a harmonic function, in a smoothly bounded domain $\Omega$ in $\mathbb{R}^n$, that is Lipschitz-$B$ along a family of curves transversal to $\partial \Omega$ is Lipschitz-$B$ in $\Omega$ (i.e., Lipschitz-$B$ in all directions in $\Omega$).

Let $\Omega$ be a smoothly bounded domain in $\mathbb{C}^n (n > 1)$. Let $P \in \partial \Omega$ and let $\nu_P$ be the outward unit normal to $\partial \Omega$ at $P$. Fix $P \in \partial \Omega$ and a unit vector $\vec{v} \in \mathbb{C}^n$. For $\delta > 0$, we define $R(P_{\delta}; \vec{v})$, where $P_{\delta} = P - \delta \nu_P$, to be the radius of a complex disc centred at $P_{\delta}$ in the $\vec{v}$ direction that fits inside $\Omega$ satisfying some additional properties. We show that a Lipschitz-$B$ holomorphic function in $\Omega$ has a Lipschitz gain along complex discs centred at $P_{\delta}$ in the $\vec{v}$ direction. This gain is given by the inverse of $R(P_t; \vec{v})$ as function of $t$. Some examples including an application to convex domains of finite type in $\mathbb{C}^n$ are discussed.
To Amma, Appa, Thatha, and Ammayi
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CHAPTER 1

INTRODUCTION

The central theme of this dissertation is to investigate, somewhat loosely speaking, the “gain” in Lipschitz continuity for some special classes of functions. We focus on harmonic and holomorphic functions defined on smoothly bounded domains in $\mathbb{R}^n$ and $\mathbb{C}^n$ ($n > 1$), respectively. For harmonic functions, we show that transversal Lipschitz regularity transfers to all other directions. This is primarily a consequence of harmonicity. In the case of holomorphic functions, we show that there is an increase in Lipschitz regularity in complex tangential directions and this increase is determined by the relative size of complex discs that can be fit in these directions. In contrast to the former case, this emerges from the interplay between the geometry of the boundary of the domain and the regularity of a holomorphic function which is a truly several complex variables phenomenon.

We begin by generalizing the class of Lip-$\alpha$, $0 < \alpha \leq 1$, (also known as Lipschitz-$\alpha$ or Hölder-$\alpha$) functions by introducing the notion of a majorant. Chapter 2 is devoted to understanding the properties of majorants and the Hardy-Littlewood Theorem (Theorem 2.11) which gives a sufficient condition on a function to be in this generalized Lipschitz class in terms of the rate of blow-up of its derivative.
A majorant function is a certain generalization of the power functions $t^\alpha$. To describe this generalization, we fix some notation. At the risk of abusing terminology, we say increasing to mean non-decreasing and similarly, decreasing to mean non-increasing. $A \subset\subset B$ will mean that $A \subset B$ and has compact closure in $B$. Also, we use $a \lesssim b$ or $b \gtrsim a$ to mean $a \leq Cb$ for some constant $C > 0$ which is independent of certain parameters. It will be mentioned, or clear from the context, what these parameters are. We use $a \approx b$ to mean $a \lesssim b$ and $b \lesssim a$. We call a function or the boundary of a domain smooth if it is $C^\infty$-smooth. A function $B : [0, \infty) \to [0, \infty)$ is called a majorant if $B(0) = 0$, $B$ is increasing, and $B(t)/t$ is decreasing. A function $f$ defined on a smoothly bounded domain $\Omega$ is called Lipschitz-$B$ if $\exists C_f > 0$ such that

$$|f(x) - f(y)| \leq C_f \cdot B(|x - y|) \quad \forall x, y \in \Omega.$$ 

Let $\Lambda_B(\Omega)$ denote the set of all Lipschitz-$B$ functions on $\Omega$. For the majorant function $B(t) = t^\alpha$, we will denote $\Lambda_\alpha(\Omega)$ by $\text{Lip}_\alpha(\Omega)$.

Let us recall what is already known about transverse regularity of a harmonic function transferring to all directions. Détraz [Dét81] showed that the $L^p$-behaviour of a transverse derivative of a harmonic function transfers to its full gradient. More precisely, let $u$ be a harmonic function on a bounded domain $\Omega$ with $C^1$-boundary and let $L$ be a continuous unit vector field in a neighbourhood of $b\Omega$ and transverse to $b\Omega$. Then, $Lu \in L^p_a(\Omega) \implies \nabla u \in L^p_a(\Omega)$, for $p > 0$ and $a > -1$. Here,

$$L^p_a(\Omega) = \left\{ f \text{ measurable on } \Omega : \int_{\Omega} |f(x)|^p \delta(x)^a \, dx < \infty \right\}$$

where $\delta(x)$ is the Euclidean distance of $x$ to $b\Omega$. 

2
Pavlović [Pav07] showed that the following conditions \((a) - (f)\) are equivalent for a function \(u\) harmonic on the unit ball \(B \subset \mathbb{R}^n\) and continuous on \(\overline{B}\). Let \(B\) be a regular majorant (see Definition 2.5).

\[
\begin{align*}
(a) & \quad u \in \Lambda_B(B) \\
(b) & \quad |u| \in \Lambda_B(B) \\
(c) & \quad u \in \Lambda_B(bB) \\
(d) & \quad |u| \in \Lambda_B(bB) \\
(e) & \quad \exists C > 0 \text{ such that } |u(\zeta) - u(r\zeta)| \leq C \cdot B(1 - r), \text{ for } \zeta \in bB, 0 < r < 1 \\
(f) & \quad \exists C > 0 \text{ such that } ||u(\zeta)| - |u(r\zeta)|| \leq C \cdot B(1 - r), \text{ for } \zeta \in bB, 0 < r < 1
\end{align*}
\]

The methods used in proving this hinge on estimating the derivative of a harmonic function on a ball by the values of the function on a larger ball. We use this technique significantly in showing our result too. Our result is a generalization of the equivalence \((a) \iff (e)\) above. We show in Theorem 3.8 that if \(u\) is harmonic in \(B\) and Lipschitz-\(B\) along a family of curves transversal to \(bB\) (see Definition 3.1), then \(u \in \Lambda_B(B)\). We use a scaling argument via \(u_\lambda(x) = u(\lambda x)\). We exploit the fact that \(u_\lambda\) is harmonic in \(B\), \(u_\lambda \in C^\infty(B)\), and \(u_\lambda\) is Lipschitz-\(B\) along a perturbation of the family of transversal curves. Since the constants in our estimates are independent of the scaling parameter \(\lambda\), we let \(\lambda \to 1\) to get the result. We use this to extend the result to a smoothly bounded domain obtaining the following main theorem of that chapter.

**Theorem 3.4.** Let \(\Omega\) be a smoothly bounded domain in \(\mathbb{R}^n\). Let \(\Gamma\) be a family of curves transversal to \(b\Omega\). If \(u\) is harmonic in \(\Omega\) and Lipschitz-\(B\) along \(\Gamma\), then \(u \in \Lambda_B(\Omega)\).
The gain in Lipschitz regularity in complex tangential directions for a Lipschitz
holomorphic function was first observed by Stein [Ste73]. He noticed that a Lip-
\( \alpha \) holomorphic function is Lip-(2\( \alpha \)) along complex tangential curves. To state this
more precisely, we need the following definitions. Let \( \Omega \subset \subset \mathbb{C}^n (n > 1) \) have smooth
boundary and let \( \mathcal{O}(\Omega) \) denote the set of holomorphic functions in \( \Omega \). Let \( \pi : \Omega \rightarrow b\Omega \)
be a smooth mapping such that for \( z \in \Omega \) close to \( b\Omega \), \( \pi(z) \) is the normal projection
of \( z \) onto \( b\Omega \). For \( P \in b\Omega \), let \( \nu_P \) denote the outward unit normal to \( b\Omega \) at \( P \) and
let \( C_{\nu_P}(b\Omega) \) be the orthogonal complement of \( C_{\nu_P} \). For \( 0 < \alpha \leq \beta < 1 \), we say that
\( u \in \Gamma_{\alpha,\beta}(\Omega) \) if \( u \in \text{Lip}_\alpha(\Omega) \) and \( \sup \| u \circ \gamma \|_{\text{Lip}_\beta([0,1])} < \infty^* \)
where the supremum is taken over curves \( \gamma : [0,1] \rightarrow \Omega \) such that \( \gamma'(t) \in C_{\pi(\gamma(t))}(b\Omega) \)
and \( |\gamma'(t)| \leq 1 \). Such curves are called normalized complex tangential curves. Stein’s result is as follows.

**Theorem** (Stein). Let \( f \in \mathcal{O}(\Omega) \). Then, for \( 0 < \alpha < \frac{1}{2} \),

\[
f \in \text{Lip}_\alpha(\Omega) \iff f \in \Gamma_{\alpha,2\alpha}(\Omega).
\]

For a proof of the above theorem see [GS77, pp. 182-185] or [Kra01, pp. 367-370]. Rudin [Rud78] showed a similar gain for holomorphic functions in the unit ball
\( \mathbb{B} \subset \mathbb{C}^n \) that satisfy weaker hypothesis . Let \( f \) be defined on \( \overline{\mathbb{B}} \) and holomorphic in \( \mathbb{B} \).
For \( w \in b\mathbb{B} \), define \( f_w(\lambda) = f(\lambda w) \) where \( \lambda \) is in the closed unit disc in \( \mathbb{C} \). He proved
that, for \( 0 < \alpha < 1/2 \), if \( \{f_w\} \) is a bounded subset of \( \text{Lip}_\alpha(S^1) \), then \( f \) is Lip-(2\( \alpha \))
along normalized (with \( |\gamma'(t)| = 1 \)) complex tangential curves in \( \mathbb{B} \). His elegant proof
heavily utilizes the symmetry of the ball.

Krantz [Kra80] extended this result to a smoothly bounded domain \( \Omega \subset \mathbb{C}^n \).
A curve \( \gamma : [0,1] \rightarrow b\Omega \) is called special complex normal if \( \gamma'(t) \in C_{\nu_{\pi(\gamma(t))}} \).
Let

\[
^* \|g\|_{\text{Lip}_\alpha([0,1])} = \|g\|_\infty + \sup_{x,y \in [0,1], x \neq y} \frac{|g(x) - g(y)|}{|x - y|^\beta}.
\]
\( f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega}) \) and \( 0 < \alpha < 1/2 \). Suppose that for each complex normal curve \( \gamma \) with \( ||\gamma'||_{\infty} \leq 1 \) we have \( ||f|_{b\Omega} \circ \gamma||_{\text{Lip}_\alpha([0,1])} \leq C \). Then, he showed that \( f \in \Gamma_{\alpha,2\alpha}(\Omega) \).

A version of these results hold for \( \alpha \geq 1/2 \), but require modification of the definition of the space of Lip-\( \alpha \) functions. We shall restrict the range of \( \alpha \) in this dissertation to avoid introducing these technicalities.

The key geometric fact behind these results is the following. Let \( \delta_{b\Omega}(z) \) be the Euclidean distance of \( z \) to \( b\Omega \). For a point \( z \in \Omega \) with \( \delta_{b\Omega}(z) = \delta \), one can fit a complex disc of radius \( \approx \sqrt{\delta} \) in the complex tangential directions inside \( \Omega \) whereas only a complex disc of radius \( \delta \) can be fit inside \( \Omega \) in the normal direction. It follows almost immediately that this gain from \( \alpha \) to \( 2\alpha \) is sharp only for strongly pseudo-convex domains. For weakly pseudo-convex domains, we expect more gain near Levi-flat points. In fact, if the domain is pseudo-concave near a point, we know that a holomorphic function extends holomorphically across the boundary near this point and hence is in Lip-\( \alpha \), for all \( \alpha > 0 \).

This is our starting point. Let \( \vec{v} \) be a unit vector in \( \mathbb{C}^n \) and \( B \) be a regular majorant. Let \( P \in b\Omega \). For small \( t > 0 \), let \( P_t = P - t\nu_P \). We introduce the function \( R(P_t; \vec{v}) \) which measures the radius of the largest complex disc centred at \( P_t \) that can be fit inside \( \Omega \) in the \( \vec{v} \) direction. This essentially captures the geometry of the boundary near \( P \) in the \( \vec{v} \) direction. We aim to describe the Lipschitz gain in the \( \vec{v} \) direction using the inverse of \( R \). So, we require \( R(P_t; \vec{v}) \) to be increasing in \( t \). We also impose some mild conditions on \( R \) and its interaction with \( B \). For these to be compatible, we relax the definition of \( R \), so that it only satisfies

\[
0 < R(P_t; \vec{v}) \leq \sup \left\{ r \geq 0 : \{ P_t + \zeta \vec{v} : \zeta \in \mathbb{C}, |\zeta| \leq r \} \subset \subset \Omega \right\}.
\]
The precise definition is given in Definition 4.1. Let $S(t)$ invert $R(P_t; \vec{v})$ (as a function of $t$), i.e., $S(R(P_t; \vec{v})) = t$ and $R(P_{S(t)}; \vec{v}) = t$. Let $U$ be a neighbourhood of $b\Omega$ satisfying (4.2.1) and (4.2.2). We are now ready to state our theorem.

**Theorem 4.5.** Let $f \in \mathcal{O}(\Omega) \cap \Lambda_B(\Omega)$. Let $P \in b\Omega$ and $z \in U \cap \Omega$. Suppose $\exists \delta > 0$ such that $P_\delta - z$ is $\mathbb{R}$-parallel to $\vec{v}$ and $|P_\delta - z| < R(P_\delta; \vec{v})$.

Then,

$$|f(P_\delta) - f(z)| \lesssim B(S(|P_\delta - z|)).$$

The only dependence of the constant in the above inequality on $f$ is on its Lipschitz-$B$ constant. It follows that the most gain in Lipschitz order is obtained for $v \in CT_P(b\Omega)$. In contrast to the results of Stein, Rudin, and Krantz mentioned above, where the gain was along complex tangential curves, the gain obtained here is in terms of points that lie on a complex disc.

In $\mathbb{C}^2$, for $m > 1$, let $\Omega_m$ be the ellipsoid $\{(z, w) : |z|^2 + |w|^{2m} < 1\}$ and $P$ be a point of the form $(e^{i\theta}, 0) \in b\Omega_m$. We show that a Lip-$\alpha$ ($\alpha < 1/2m$) holomorphic function is Lip-$(2m\alpha)$ on complex tangential discs in $\Omega_m$ centred at $P_\delta$. If $b\Omega$ is pseudo-concave at $P$ or a hyperplane near $P$, then we get Lip-$(1 - \epsilon)$, for any $0 < \epsilon < 1$, estimates on a complex tangential disc. In Section 4B, we explore these examples in greater detail and also discuss an application to convex domains of finite type in $\mathbb{C}^n$.

Krantz [Kra90] formulates this Lipschitz gain phenomenon using analytic discs and the Kobayashi metric. The statement and proof of this result is very technical, a reflection of the difficulty in computing invariant metrics on arbitrary domains. Also, the resulting gain is still along normalized complex tangential curves. In the case of finite type domains in $\mathbb{C}^2$ (with type $k$), using the estimates of invariant metrics
obtained by Catlin [Cat89], this result states that a Lip-$\alpha$ ($\alpha < 1/k$) holomorphic function is Lip-$k\alpha$ along normalized complex tangential curves. This is dealt with in greater detail in Chang-Krantz [CK91].

Grellier [Gre91] casts this tangential gain in terms of point-wise and integral estimates of complex tangential derivatives of a holomorphic function. The results provide estimates of these derivatives by the average of relevant functions on a product of a ball in the complex tangential direction and a complex disc in the normal direction. All complex tangential directions are treated equally and only the radius of the largest ball in the complex tangential directions enter these estimates. The proofs of these results and some converse results are presented in Grellier [Gre92].
CHAPTER 2

LIPSCHITZ FUNCTIONS: DEFINITION AND BASIC PROPERTIES

Definition 2.1. Let $\Omega \subset \subset \mathbb{R}^n$ and $0 < \alpha \leq 1$. A function $f$ defined on $\Omega$ is said to be Lip-$\alpha$ (also known as Lipschitz-$\alpha$ or Hölder-$\alpha$) if there exists $C_f > 0$, such that

$$|f(x) - f(y)| \leq C_f \cdot |x - y|^\alpha, \quad x, y \in \Omega.$$ 

For $0 < \alpha \leq 1$, let $\text{Lip}_\alpha(\Omega)$ denote the Lip-$\alpha$ functions defined on $\Omega$. This definition suggests that the properties of these functions are linked to the behaviour of the function $t^\alpha$. In Section 2A, we recall the definition of a majorant function and study a more general class of Lipschitz functions defined using these majorant functions. The notion of a majorant captures the key properties of the function $t^\alpha$ essential for the analysis of this generalized class of Lipschitz functions. Readers not interested in this generalization may skip Section 2A and replace every occurrence of the majorant function $B(t)$ with the function $t^\alpha$ in what follows. In Section 2B, we recall a theorem of Hardy and Littlewood, generalized to our setting of Lipschitz functions with respect to a majorant, and provide a proof for the reader’s convenience. This theorem gives a sufficient condition for a function to be Lipschitz-$B$ in terms of the rate of blow-up of its derivative.
2A Majorants

Majorant functions and their regularity appear in the work of Dyakonov [Dya97] and go back at least to the work of Havin [Hav71], and Zygmund [Zyg59], if not any farther.

**Definition 2.2.** A continuous function $B : [0, \infty) \to [0, \infty)$ is called a majorant function if

$$B(0) = 0, \ B \text{ is increasing, and } \frac{B(t)}{t} \text{ is decreasing}.$$ 

We will discuss some examples of majorants shortly. For a majorant function $B$ we define Lipschitz-$B$ functions analogously to Definition 2.1 as follows.

**Definition 2.3.** Let $B$ be a majorant. A function $f$ defined on $\Omega$ is said to be Lipschitz-$B$ if there exists $C_f > 0$, such that

$$|f(x) - f(y)| \leq C_f \cdot B(|x - y|), \ x, y \in \Omega.$$ 

Let $\Lambda_B(\Omega)$ denote the Lipschitz-$B$ functions defined on $\Omega$. For the majorant function $B(t) = t^\alpha$, we will denote $\Lambda_{t^\alpha}(\Omega)$ by Lip$\alpha(\Omega)$. We now show that the above notion of being Lipschitz is a local condition.

**Lemma 2.4.** Suppose $\exists C, \delta > 0$ such that

$$|f(x) - f(y)| \leq C \cdot B(|x - y|)$$

for all $x, y \in \Omega$ with $|x - y| < \delta$. Then, $f \in \Lambda_B(\Omega)$.

**Proof.** Let $k$ be an integer such that $\overline{\Omega}$ is covered by $k$ balls of radius $\delta/3$ centered at points in $\Omega$. Let $x_0, y_0 \in \Omega$ such that $|x_0 - y_0| \geq \delta$. Then, $\exists x_1, \ldots, x_k \in \Omega$ such that

$$|x_j - x_{j+1}| < \delta, \ j = 0, \ldots, k$$
where $x_{k+1} = y_0$. So,

$$|f(x_0) - f(y_0)| \leq \sum_{j=0}^{k} |f(x_j) - f(x_{j+1})| \leq (k + 1)C \cdot B(\delta)$$

$$\leq (k + 1)C \cdot B(|x_0 - y_0|).$$

Since we have shown that the condition on a function being Lipschitz is local, let us focus on the behaviour of majorant functions near 0. This also suggests that the more a majorant function behaves like $t^\alpha$ near 0 the more we can expect Lipschitz-$B$ functions to ‘behave like’ Lipschitz-$\alpha$ functions. As it will become evident later, the following integral estimate on the majorant function ensures that Lipschitz-$B$ functions ‘behave like’ Lipschitz-$\alpha$ functions.

**Definition 2.5.** A majorant function $B$ is called **regular** if $\exists C > 0$, $\forall \delta > 0$ sufficiently small,

$$\int_0^\delta \frac{B(t)}{t^\alpha} dt + \delta \int_\delta^\infty \frac{B(t)}{t^{2+\alpha}} dt \leq C \cdot B(\delta).$$

Let us now look at some examples of majorant functions and discuss their regularity.

**Example 2.6.** As we would expect, the functions $B(t) = t^\alpha$ ($0 < \alpha \leq 1$) are majorant functions. The following computation also shows that $t^\alpha$ is a regular majorant when $0 < \alpha < 1$.

$$\int_0^\delta \frac{B(t)}{t^\alpha} dt + \delta \int_\delta^\infty \frac{B(t)}{t^{2+\alpha}} dt = \int_0^\delta t^{-1+\alpha}dt + \delta \int_\delta^\infty t^{-2+\alpha} dt$$

$$= \left( \frac{1}{\alpha} + \frac{1}{1-\alpha} \right) \cdot \delta^\alpha \quad \text{for } 0 < \alpha < 1.$$

$B(t) = t$ is not a regular majorant even if we redefine it to be a constant for $t \geq t_0$. 


Example 2.7. Fix $0 < \alpha < 1$.

$$B(t) = \begin{cases} 
0, & \text{if } t = 0 \\
-t^\alpha \ln t, & \text{if } 0 < t \leq 1/e \\
e^{-\alpha}, & \text{if } t > 1/e 
\end{cases}$$

is a majorant function. The following shows that this is a regular majorant.

$$\delta \int_0^\infty \frac{B(t)}{t} \, dt = \delta \int_0^{1/e} -t^{\alpha-1} \ln t \, dt = -\frac{\delta^\alpha \ln \delta}{\alpha} + \frac{\delta^\alpha}{\alpha^2} \leq C (-\delta^\alpha \ln \delta).$$

$$\delta \int_0^\infty \frac{B(t)}{t^2} \, dt = \delta \int_0^{1/e} -t^{\alpha-2} \ln t \, dt + \delta \int_0^{1/e} \frac{e^{-\alpha}}{t^2} \, dt = -\frac{\delta^\alpha \ln \delta}{1-\alpha} + \delta \left( \frac{e^{1-\alpha}}{(1-\alpha)^2} - \frac{e^{1-\alpha}}{1-\alpha} + e^{1-\alpha} \right) - \frac{\delta^\alpha}{(1-\alpha)^2} \leq C (-\delta^\alpha \ln \delta).$$

Example 2.8. Letting $\alpha = 1$ in the above example, we get the following majorant;

$$B(t) = \begin{cases} 
0, & \text{if } t = 0 \\
-t \ln t, & \text{if } 0 < t \leq 1/e \\
1/e, & \text{if } t > 1/e. 
\end{cases}$$

But this majorant fails to be regular, since, for small $\delta > 0$

$$\delta \int_0^\infty \frac{B(t)}{t^2} \, dt = \delta \int_0^{1/e} -\frac{\ln t}{t} \, dt + \delta \int_0^{1/e} \frac{1}{et^2} \, dt = \frac{\delta}{2} \left( (\ln \delta)^2 - 1 \right) + \delta$$

$$= \frac{\delta (\ln \delta)^2}{2} + \frac{\delta}{2} \not\leq C \cdot (-\delta \ln \delta).$$

Example 2.9.

$$B(t) = \begin{cases} 
0, & \text{if } t = 0 \\
1/(\ln t)^2, & \text{if } 0 < t \leq 1/e \\
1, & \text{if } t > 1/e. 
\end{cases}$$
is a majorant function. For small \( \delta > 0 \),

\[
\int_0^\delta \frac{B(t)}{t} \, dt = \int_0^\delta \frac{1}{t (\ln t)^2} \, dt = \frac{1}{-\ln \delta} \not\leq C \cdot \frac{1}{(\ln \delta)^2}.
\]

Hence, this majorant is not regular.

Notice that the last two examples fail to be regular by violating two distinct parts of the inequality (2.5.1) that defines regularity. In fact, for a majorant \( B \),

\[
B \text{ is regular} \iff \exists C > 0, \text{ such that } \\
\int_0^\delta \frac{B(t)}{t} \, dt \leq C \cdot B(\delta) \text{ and } \int_\delta^\infty \frac{B(t)}{t^2} \, dt \leq C \cdot \frac{B(\delta)}{\delta}.
\]

A majorant \( B \) is called \textbf{fast} if \( \exists C > 0 \), such that \( \int_0^\delta \frac{B(t)}{t} \, dt \leq C \cdot B(\delta) \) and it is called \textbf{slow} if \( \exists C > 0 \), such that \( \int_\delta^\infty \frac{B(t)}{t^2} \, dt \leq C \cdot \frac{B(\delta)}{\delta} \). So, \( B \) is regular iff it is both fast and slow. Notice that for any majorant \( B \) we have the inequalities

\[
\int_0^\delta \frac{B(t)}{t} \, dt \geq B(\delta) \quad \text{and} \quad \int_\delta^\infty \frac{B(t)}{t^2} \, dt \geq \frac{B(\delta)}{\delta}.
\]

Now, let us proceed to characterize each of the conditions fast and slow in terms of properties of \( B \) that are easier to identify.

Let us recall the following definition introduced by Bernšteǐn [Ber49]. A non-negative function \( f \), of a real variable, is said to be \textbf{almost increasing} (or \textbf{almost decreasing}) if \( \exists C > 0 \), for \( x \leq y \), we have \( f(x) \leq C \cdot f(y) \) (or \( f(y) \leq C \cdot f(x) \)).

**Lemma 2.10.** Let \( B \) be a majorant.

\[
B \text{ is fast} \iff \frac{B(t)}{t^\alpha} \text{ is almost increasing for some } 0 < \alpha \leq 1. \quad (2.10.1)
\]

\[
B \text{ is slow} \iff \frac{B(t)}{t^\beta} \text{ is almost decreasing for some } 0 \leq \beta < 1. \quad (2.10.2)
\]
Proof. Let us prove (2.10.1). Let \( F(t) = \int_0^t \frac{B(s)}{s} \, ds \). Then,

\[
B \text{ is fast } \iff F(t) \leq C \cdot tF'(t) \quad \text{for some } C \geq 1
\]

\[
\iff \alpha \cdot \frac{1}{t} \leq \frac{F'(t)}{F(t)} \quad \text{(where } \alpha = 1/C)\]

\[
\iff \alpha \cdot \int_a^b \frac{1}{t} \, dt \leq \int_a^b \frac{F'(t)}{F(t)} \, dt \quad \text{for } 0 < a \leq b
\]

\[
\iff \alpha (\ln b - \ln a) \leq \ln F(b) - \ln F(a)
\]

\[
\iff \frac{F(a)}{a^\alpha} \leq \frac{F(b)}{b^\alpha}.
\]

Now, for \( 0 < a \leq b \)

\[
B \text{ is fast } \implies \frac{B(a)}{a^\alpha} \leq \frac{1}{a^\alpha} \int_0^a \frac{B(s)}{s} \, ds \leq \frac{1}{b^\alpha} \int_0^b \frac{B(s)}{s} \, ds \text{ fast } \leq C \cdot \frac{B(b)}{b^\alpha}
\]

The first inequality follows from \( B(t) \leq \int_0^t \frac{B(s)}{s} \, dt \) and the middle inequality follows from the equivalence above.

\[
\frac{B(t)}{t^\alpha} \text{ almost increasing } \implies \exists C > 0, \frac{B(a)}{a^\alpha} \leq \frac{B(b)}{b^\alpha}
\]

\[
\implies \int_0^t \frac{B(s)}{s} \, ds = \int_0^t \frac{B(s)}{s^\alpha} \cdot \frac{1}{s^{1-\alpha}} \, ds \leq C \cdot \frac{B(t)}{t^\alpha} \cdot \frac{t^\alpha}{\alpha}.
\]

This proves (2.10.1) and (2.10.2) follows similarly. \( \square \)

Now, one can easily check that the majorant from Example 2.9 has the property that \( B(t)/t^\alpha \) is not almost increasing for any \( 0 < \alpha \leq 1 \) and the majorant from Example 2.8 has the property that \( B(t)/t^\beta \) is not almost decreasing for any \( 0 \leq \beta < 1 \).
2B Hardy-Littlewood Theorem

Theorem 2.11 (Hardy-Littlewood). Let $\Omega \subset \subset \mathbb{R}^n$ have smooth boundary and let $B$ be a fast majorant. Let $U$ be a neighbourhood of $b\Omega$. If $f \in C^1(\Omega) \cap L^\infty(\Omega)$ satisfies

$$|\nabla f(x)| \lesssim \frac{B(\delta(x))}{\delta(x)}, \quad x \in U \cap \Omega,$$

then $f \in \Lambda_B(\Omega)$.

(Here $\delta(x)$ is the Euclidean distance of $x$ to $b\Omega$.)

Proof. Notice that it is enough to prove this near $b\Omega$ since on any compact subset of $\Omega$, $f$ is $C^1$ and hence is in $\Lambda_1 \subset \Lambda_B$. More precisely, for any compact set $K \subset \subset \Omega$, $\exists \delta_0 > 0$ such that $\text{dist}(K, b\Omega) > 3\delta_0$. Let $C_{\delta_0} := \sup \{|\nabla f(x)| : \delta(x) > \delta_0\}$. Let $x, y \in K$. If $|x - y| \geq \delta_0$, then

$$|f(x) - f(y)| \leq \frac{2||f||_\infty}{\delta_0} \cdot \delta_0 \leq \frac{2||f||_\infty}{\delta_0} \cdot |x - y| \cdot \frac{B(|x - y|)}{B(|x - y|)} \leq \frac{2||f||_\infty}{\delta_0} \cdot \frac{\text{diam } K}{B(\text{diam } K)} \cdot B(|x - y|).$$

If $|x - y| < \delta_0$, then

$$|f(x) - f(y)| = \frac{|f(x) - f(y)|}{|x - y|} \cdot \frac{|x - y|}{B(|x - y|)} \cdot B(|x - y|).$$

Since $\delta((1 - s)x + sy) > \delta_0$, we have

$$|f(x) - f(y)| \leq C_{\delta_0} \cdot \frac{\text{diam } K}{B(\text{diam } K)} \cdot B(|x - y|).$$

This shows that $f \in \Lambda_B(K)$.

Fix $0 < \delta_0 < 1$ so that $V := \{x \in \Omega : \delta(x) < 3\delta_0\} \subset U \cap \Omega$. Let $K := \{x \in \Omega : \delta(x) \geq \delta_0\}$. Since we know that being Lipschitz-$B$ is a local condition,
it suffices to consider $T, S \in \Omega$ such that $|T - S| < \delta_0$. Then, either $T, S \in K$ or $T, S \in V$. To show that $f \in \Lambda_B(\Omega)$, it suffices to show

$$|f(T) - f(S)| \lesssim B(|T - S|)$$

for $T, S \in U$ such that $|T - S| < \delta_0$.

The estimate on $\nabla f$ is in terms of the distance to the boundary. To show that $f$ is in $\Lambda_B$ we need to compare the function values at two distinct points in $V$. We achieve this by pushing these points inside $\Omega$ by a fixed $\epsilon$ so that we can use the estimate on $\nabla f$. We then choose $\epsilon$ effectively to achieve the result.

Let $r$ be the signed distance to $b\Omega$, i.e.,

$$r(x) = \begin{cases} 
-\delta(x), & \text{if } x \in \Omega \\
\delta(x), & \text{if } x \notin \Omega.
\end{cases}$$

Figure 2.12: Box Argument - Hardy-Littlewood
We know that \( r \) is smooth near \( b\Omega \) and \( r \) is a defining function for \( \Omega \), i.e., \( \Omega = \{ r < 0 \} \), \( b\Omega = \{ r = 0 \} \), and \( |\nabla r| \neq 0 \) on \( b\Omega \). Decrease \( \delta_0 \), if necessary, so that \( r \) is smooth in \( V \) and \( |\nabla r| \neq 0 \) in \( V \). Hence we may assume, without loss of generality, that \( \partial r / \partial x_n \neq 0 \) near \( T \) and \( S \), i.e., in a neighbourhood of the box below. Now, we can consider \( r \) to be a coordinate in the normal direction on \( V \), i.e., \((x_1, \ldots, x_{n-1}, r)\) are coordinates on \( V \). Let \( T = (t_1, \ldots, t_{n-1}, t_n), S = (s_1, \ldots, s_{n-1}, s_n) \in V \). For \( 0 < \epsilon \leq \delta_0 \), let \( T' = (t_1, \ldots, t_{n-1}, t_n - \epsilon) \) and \( S' = (s_1, \ldots, s_{n-1}, s_n - \epsilon) \). Since \( r \) is a coordinate in the normal direction, we know that \( T', S' \in \Omega \). In fact for any \( P \) in the line \( L' \), in the \((x_1, \ldots, x_{n-1}, r)\) coordinate system, joining \( T' \) and \( S' \), \( \delta(P) > \epsilon \).

\[
|f(T') - f(S')| \leq |\nabla f(P)||T' - S'| \quad \text{(for some } P \in L')
\]
\[
\lesssim \frac{B(\delta(P))}{\delta(P)} \cdot |T - S| \lesssim \frac{B(\epsilon)}{\epsilon} \cdot |T - S| \quad \text{(since } B(x)/x \text{ is decreasing)}.
\]

Choosing \( \epsilon = |T - S| \), we get \( |f(T') - f(S')| \lesssim B(|T - S|) \). Now, let us estimate \( |f(T) - f(T')| \).

\[
|f(T) - f(T')| = |f(t_1, \ldots, t_{n-1}, t_n) - f(t_1, \ldots, t_{n-1}, t_n - \epsilon)|
\]
\[
= \left| \int_0^\epsilon \frac{\partial f}{\partial r}(t_1, \ldots, t_{n-1}, t_n - x) \, dx \right|
\]
\[
\leq \int_0^\epsilon \frac{B(-t_n + x)}{-t_n + x} \, dx \leq \int_0^\epsilon \frac{B(x)}{x} \, dx \lesssim B(|T - S|) \quad \text{(since } B \text{ is fast)}.
\]

Similarly, one estimates \( |f(S) - f(S')| \).

For harmonic functions the converse of the above theorem is also true.

\(^1\)For more on the distance to the boundary function, see Gilbarg-Trudinger [GT83, pp. 354-357] and Herbig-McNeal [HM].
Lemma 2.13. Let $u$ be a harmonic function on a smoothly bounded domain $\Omega$ in $\mathbb{R}^n$. If $u \in \Lambda_B(\Omega)$, then

$$|\nabla u(x)| \lesssim \frac{B(\delta(x))}{\delta(x)}, \quad x \in \Omega.$$ 

Proof. Fix $x_0 \in \Omega$. Let $\epsilon > 0$ such that $B(x_0, \epsilon) \subset \subset \Omega$. Now, by the Poisson integral formula, for $x \in B(x_0, \epsilon)$

$$\nabla u(x) = \frac{1}{\omega_{n-1}} \int_{|\xi|=\epsilon} u(x_0 + \xi) \nabla_x \left( \frac{\epsilon^2 - |x - x_0|^2}{|x - x_0 - \xi|^n} \right) d\sigma(\xi)$$

$$= \frac{1}{\omega_{n-1}} \int_{|\xi|=\epsilon} (u(x_0 + \xi) - u(x_0)) \nabla_x \left( \frac{\epsilon^2 - |x - x_0|^2}{|x - x_0 - \xi|^n} \right) d\sigma(\xi).$$

Calculating $\nabla_x$ inside the integral, setting $x = x_0$, and estimating we get

$$|\nabla u(x_0)| \leq \frac{n}{\epsilon} \cdot \sup_{|\xi|=\epsilon} |u(x_0 + \xi) - u(x_0)| \lesssim \frac{B(\epsilon)}{\epsilon}.$$ 

Now, let $\epsilon = \delta(x_0)/2$ to get

$$|\nabla u(x_0)| \lesssim \frac{B(\delta(x_0)/2)}{\delta(x_0)} \leq \frac{B(\delta(x_0))}{\delta(x_0)} \quad \text{(since $B$ is increasing).}$$

The following example shows us that the assumption of majorant being fast in the Hardy-Littlewood Theorem cannot be dropped.

Example 2.14. Recall that $B$ is fast $\iff \int_0^\delta \frac{B(t)}{t} dt \lesssim B(\delta) \iff B(t)/t^\alpha$ is almost increasing for some $0 < \alpha \leq 1$. As we have seen in Example 2.9 the majorant $B(t) = 1/(\ln t)^2$ is not fast. Let $\chi : \mathbb{R} \to [0,1]$ be a smooth function such that $\chi \equiv 0$ on $(-\infty, \frac{1}{4}]$, $\chi \equiv 1$ on $[\frac{1}{2}, \infty)$, and $0 \leq \chi'(t) \leq 5$ for all $t$. Define $u$ on the unit disc $D \subset \mathbb{C}$ as follows:

$$u(0) = 0 \quad \text{and} \quad u(re^{i\theta}) = \frac{\chi(r)}{\ln(1-r)}, \quad \text{for } 0 < r < 1 \text{ and } \theta \in \mathbb{R}.$$
Clearly \( u \in C^1(D) \cap L^\infty(D) \) and

\[
\left| \nabla u (re^{i\theta}) \right| = \left| \frac{\partial u}{\partial r} (re^{i\theta}) \right| = \left| \frac{\chi'(r) \ln(1-r) + \chi(r)}{(\ln(1-r))^2} \right|
\]

\[
\lesssim \frac{1}{(1-r)(\ln(1-r))^2} = \frac{B(1-r)}{1-r}.
\]

This shows that \( u \) satisfies the hypothesis of Theorem 2.11. Let us now show that \( u \notin \Lambda_B(D) \). For \( \frac{1}{2} < r < s < 1 \),

\[
\frac{\left| u (se^{i\theta}) - u (re^{i\omega}) \right|}{B(s-r)} = (\ln(s-r))^2 \cdot \frac{1}{\ln(1-s)} - \frac{1}{\ln(1-r)}
\]

\[
\to -\ln(1-r) \quad \text{as} \quad s \to 1.
\]

Since \(- \ln(1-r)\) is unbounded as \( r \to 1 \), \( u \notin \Lambda_B(D) \).
CHAPTER 3

TRANSVERSALLY LIPSCHITZ HARMONIC FUNCTIONS

In this chapter we show that a transversally Lipschitz harmonic function is Lipschitz (i.e. Lipschitz everywhere). We begin by defining the necessary notions and state Theorem 3.4, the main theorem of this chapter. We then present the key tools used in the proof of Theorem 3.4 in Lemmas 3.5, 3.6, and 3.7. In Section 3A we prove the special case of Theorem 3.4 corresponding to the unit ball $B$. The proof in this special case captures all the key ideas behind the result. In Section 3B, we prove Theorem 3.4 by attaching ($\mathbb{R}^n$-)sectors of balls to $b\Omega$ and then using the result for $B$.

Let us now define the terms family of transversal curves and a function being transversally Lipschitz-\(B\) with respect to such a family. Let $\Omega \subset \subset \mathbb{R}^n$ have smooth boundary. Recall that for $p \in b\Omega$, $\nu_p$ is the outward unit normal to $b\Omega$ at $p$.

Definition 3.1. Let $U$ be a neighbourhood of $b\Omega$ and $\Gamma : b\Omega \times (-a, a) \to U$ be a $C^2$ map (for some $a > 0$). For $p \in b\Omega$ and $t \in (-a, a)$, let $\gamma_p(t) := \Gamma(p, t)$. $\Gamma$ is called a family of curves transversal to $b\Omega$ if the following hold;

(a) $\gamma_p(0) = p$, for $p \in b\Omega$, and
(b) \( \exists c > 0 \) such that

\[
\gamma'_p(t) \cdot \nu_p \leq -c < 0, \text{ for } p \in b\Omega \text{ and } t \in (-a, a).
\]

Transversality, for us, means \( \gamma'_p(t) \cdot \nu_p \neq 0 \) for \( p \in b\Omega \) and \( t \in (-a, a) \). Using compactness of \( b\Omega \), the continuity of \( \gamma'_p \), and restricting \( t \) to a closed sub-interval around 0 we get that this inner product is uniformly bounded away from 0. By making the negative choice for sign we get condition (b) above.

We now show that \( \Gamma \) is a \( C^1 \) bijection near \( b\Omega \).

**Lemma 3.2.** Let \( U \) be a neighbourhood of \( b\Omega \) and \( \Gamma : b\Omega \times (-a, a) \rightarrow U \) be a family of curves transversal to \( b\Omega \) that is \( C^2 \). Then \( \exists a' > 0 \), a neighbourhood \( U' \) of \( b\Omega \) such that \( \Gamma : b\Omega \times (-a', a') \rightarrow U' \) is a \( C^1 \) bijection.

**Proof.** Let \( \{e_1, \ldots, e_n\} \) denote the standard basis in \( \mathbb{R}^n \). Fix \( p \in b\Omega \). By a translation and rotation, we may assume, \( p = 0 \in \mathbb{R}^n \) and \( \nu_0 = e_n \). Furthermore, by the implicit function theorem, \( \exists V \subset \mathbb{R}^{n-1}, \) a neighbourhood of the origin, and \( f : V \rightarrow \mathbb{R} \) such that \( f(0) = 0, \nabla_{n-1}f(0) = 0, \{x_n = f(x') : x' \in V\} \) is \( b\Omega \) near \( p \), and \( \{(x', x_n - f(x')) : x' \in V, |x_n| \text{ small}\} \) is a neighbourhood of \( p \). Notice that, near \( p \), \( \Gamma \) can now be thought of as a mapping from \( V \times (-a, a) \) to \( U \) satisfying

\[
\Gamma(x', 0) = (x', f(x')) \quad \text{and} \quad \frac{\partial \Gamma}{\partial t}(x', t) \cdot \left(-\nabla_{n-1}f(x')\right) \leq -c.
\]

Now, computing the Jacobian of \( \Gamma \) at \( p \), we get,

\[
\text{Jac } \Gamma(p) = \begin{pmatrix}
Id_{n-1} & * & \vdots \\
& & * \\
* & \ldots & * & \downarrow
\end{pmatrix}, \quad \text{where } \downarrow \leq -c.
\]

Hence, \( \det (\text{Jac } \Gamma) \neq 0 \) on \( b\Omega \). The conclusion follows from the inverse function theorem. \( \square \)
Let us replace $a$ and $U$ from 3.1 with $a'$ and $U'$ from the above lemma. We decrease $a$ and correspondingly shrink $U$, further if necessary, to assume that the following holds;

$$\gamma_p((0,a]) \subset U \cap \Omega \quad \text{and} \quad \gamma_p([-a,0)) \subset U \cap \Omega^c,$$ for $p \in b\Omega$.

**Definition 3.3.** A function $f$ defined on $\Omega$ is said to be **transversally Lipschitz-B** along $\Gamma$ if there exists $C_f > 0$, for all $p \in b\Omega$ and $s,t$ sufficiently small,

$$|f(\gamma_p(s)) - f(\gamma_p(t))| \leq C_f \cdot B(|s-t|).$$

Now, we state the main theorem of this chapter.

**Theorem 3.4.** Let $\Omega$ be a smoothly bounded domain in $\mathbb{R}^n$. Let $\Gamma$ be a family of curves transversal to $b\Omega$. If $u$ is harmonic in $\Omega$ and Lipschitz-B along $\Gamma$, then $u \in \Lambda_{B}(\Omega)$.

In the following section we will prove the main theorem in the case of the unit ball $\mathbb{B}$. Using this result, we will prove the above theorem in Section 3B.

We now present the key tools used in the proof of Theorem 3.4 in the following lemmas. Let us begin by showing that the values of the derivative of a harmonic function on a ball can be estimated by the values of the function on a larger concentric ball.

**Lemma 3.5.** Let $u$ be a harmonic function on $\mathbb{B}$ and let $0 < r < R < 1$. Then,

$$\sup_{|x|=r} |\nabla u(x)| \leq \frac{n}{R - r} \cdot \sup_{|x|=R} |u(x)|.$$

**Proof.** Fix $x_0 \in \mathbb{B}$ such that $|x_0| = r$. Let $\epsilon = R - r$. By Poisson integral formula, for $x \in B(x_0; \epsilon)$,

$$\nabla u(x) = \frac{1}{\omega_{n-1} \epsilon} \int_{|\xi-x_0| = \epsilon} u(\xi) \cdot \nabla_x \left( \frac{\epsilon^2 - |x-x_0|^2}{|\xi-x|^n} \right) d\sigma(\xi).$$
Calculating $\nabla_x$ inside the integral and setting $x = x_0$, we get

$$\nabla_x \left( \frac{\epsilon^2 - |x - x_0|^2}{|\xi - x|^n} \right)_{x=x_0} = -\frac{n}{\epsilon} \cdot (\xi - x_0)$$

and hence

$$|\nabla u(x_0)| \leq \frac{1}{\omega_{n-1} \epsilon} \cdot \sup_{|\xi - x_0| = \epsilon} |u(\xi)| \cdot \frac{n}{\epsilon} \cdot \omega_{n-1} \epsilon^{n-1} \leq \frac{n}{\epsilon} \cdot \sup_{|\xi| \leq R} |u(\xi)|.$$  

By the maximum principle for harmonic functions, we know that

$$\sup_{|\xi| \leq R} |u(\xi)| = \sup_{|\xi| = R} |u(\xi)|.$$  

So,

$$|\nabla u(x_0)| \leq \frac{n}{\epsilon} \cdot \sup_{|\xi| = R} |u(\xi)|.$$  

Taking supremum over $|x_0| = r$ gives the result.  

We now show the following distance estimate. Recall that $\delta(x)$ is the Euclidean distance of $x$ to $b\Omega$.

**Lemma 3.6.** Let $\vec{v}$ be a unit vector that is transverse to $b\Omega$ at $p$. i.e. $\exists c > 0$ such that $\vec{v} \cdot \nu_p \leq -c$. Then, for $0 < a < 1$, $\exists S_a > 0$ so that

$$acs \leq \delta(p + s\vec{v}) \leq s, \text{ for } 0 < s \leq S_a.$$  

**Proof.** Let $\{e_1, \ldots, e_n\}$ denote the standard basis for $\mathbb{R}^n$. Fix $p \in b\Omega$. By a rotation of $\Omega$ we may assume that $\nu_p = e_n$. Let $\vec{v} = (v_1, \ldots, v_n)$. Then, $v_n \leq -c < 0$. Let $r$ be the smooth defining function for $\Omega$ given by the signed distance to the boundary function. Corollary 5.3 from Herbig-McNeal [HM] implies that $\nabla r(p) = \nu_p = e_n$.

It is clear that $\delta(p + s\vec{v}) \leq s$. Now, let us show the other inequality. For small $s > 0$, we have $p + s\vec{v} \in \Omega$. Let $C$ be maximum of all the second derivatives of $r$ in a
neighbourhood of \( b\Omega \). Then, by Taylor’s Theorem \( \exists w \in \Omega \), near \( b\Omega \), such that \( p - w \) is parallel to \( \vec{v} \) and
\[
 r(p + s\vec{v}) = r(p) + s \sum_{j=1}^{n} \frac{\partial r}{\partial e_j}(p)v_j + \frac{s^2}{2} \sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial e_j \partial e_k}(w)v_j v_k.
\]

Recall that we are using the signed distance to the boundary function as the defining function \( r \). So, \( |r(x)| = \delta(x) \). Since \( p \in b\Omega \), we have \( r(p) = 0 \). Also, since \( \nabla r(p) = e_n \), we have
\[
 \sum_{j=1}^{n} \frac{\partial r}{\partial e_j}(p)v_j = \nabla r(p) \cdot \vec{v} = v_n.
\]
Hence,
\[
 |r(p + s\vec{v})| \geq s|v_n| - C s^2 \geq s(c - Cs) = cs \left( 1 - \frac{C}{c} s \right).
\]

Choose \( S_a > 0 \) so that \( 1 - \frac{C}{c} S_a \geq a \). Hence, for \( 0 < s < S_a \),
\[
 |r(p + s\vec{v})| \geq a cs. \quad \square
\]

To show the conclusion of the Theorem 3.4, it suffices by the Hardy-Littlewood Theorem to show
\[
 |\nabla u(x)| \lesssim \frac{B(\delta(x))}{\delta(x)}, \text{ for } x \in U \cap \Omega,
\]
where \( U \) is a neighbourhood of \( b\Omega \), which is equivalent to showing
\[
 \sup_{U \cap \Omega} |\nabla u(x)| \frac{\delta(x)}{B(\delta(x))} < \infty.
\]

For \( 0 < \delta_0 < \delta_1 \), let \( U_0 := \{ x \in \Omega : \delta(x) < \delta_0 \} \) and \( U_1 := \{ x \in \Omega : \delta(x) < \delta_1 \} \). The following lemma allows us to compare the supremum above computed on \( U_0 \) and \( U_1 \).

This is a consequence of the maximum principle for sub-harmonic functions.

**Lemma 3.7.** For \( 0 < \delta_0 < \delta_1 \), let
\[
 U_0 := \{ x \in \Omega : \delta(x) < \delta_0 \} \quad \text{and} \quad U_1 := \{ x \in \Omega : \delta(x) < \delta_1 \}.
\]
If \( u \) is a harmonic function in \( \Omega \) satisfying

\[
\sup_{u_0} |\nabla u(x)| \frac{\delta(x)}{B(\delta(x))} = A_0 < \infty,
\]

then

\[
\sup_{u_1} |\nabla u(x)| \frac{\delta(x)}{B(\delta(x))} \leq \frac{\delta_1}{\delta_0} \cdot A_0.
\]

Proof. Let \( w \in U_1 \setminus U_0 \). By the maximum principle for the sub-harmonic function

\[
|\nabla u|^2 = u_{x_1}^2 + \cdots + u_{x_n}^2,
\]

\( \exists x_w \in \Omega \) with \( \delta(x_w) = \delta_0 \) such that \( |\nabla u(w)| \leq |\nabla u(x_w)| \). By continuity, we have

\[
|\nabla u(x_w)| \frac{\delta(x_w)}{B(\delta(x_w))} \leq A_0.
\]

Hence,

\[
|\nabla u(w)| \frac{\delta(w)}{B(\delta(w))} \leq |\nabla u(x_w)| \cdot \frac{\delta_1}{B(\delta_1)} \cdot \frac{\delta(x_w)}{B(\delta(x_w))} \cdot \frac{B(\delta(x_w))}{\delta(x_w)} \quad \text{(since } t/B(t) \text{ is increasing)}
\]

\[
= \frac{\delta_1}{B(\delta_1)} \cdot |\nabla u(x_w)| \cdot \frac{\delta(x_w)}{B(\delta(x_w))} \cdot \frac{B(\delta(x_w))}{\delta(x_w)} \cdot \frac{\delta_1}{\delta_0} \cdot A_0 \quad \text{(since } B \text{ is increasing)}.
\]

\[\square\]

3A Main Theorem for \( B \)

In this section we prove the special case of Theorem 3.4 corresponding to the unit ball \( B \). Let us sketch the strategy of the proof before we state and prove the theorem. Let \( \Gamma \) be a family of curves transversal to \( bB \). Let \( u \) be a harmonic function in \( B \) that is Lipschitz-\( B \) along \( \Gamma \).
First we approximate $u$ by scaling, $u_\lambda(x) := u(\lambda x)$ for $1/2 < \lambda < 1$. $u_\lambda$ is harmonic in $\mathbb{B}$, smooth up to the boundary and transversally Lipschitz-$B$ with respect to a perturbation of $\Gamma$, called $\Gamma_\lambda$. As $\lambda \to 1$, $u_\lambda$ and its derivatives converge point-wise to $u$ and its derivatives respectively, while $\Gamma_\lambda$ and its derivatives converge uniformly to $\Gamma$ and its derivatives respectively.

Let $M$ be the vector field given by differentiation along curves of $\Gamma_\lambda$. We show that $Mu_\lambda$ blows up no faster than the rate prescribed by the Hardy-Littlewood theorem, modulo an error term involving a small constant times $\nabla u_\lambda$. We accomplish this by using Taylor’s theorem on $u_\lambda$ up to second order along curves of $\Gamma$. Then, we use a constant coefficient approximation $M_0$ of the vector field $M$. Since $M_0u_\lambda$ is harmonic, we estimate the second derivative terms using Lemma 3.5. We then show that $N_0u_\lambda$ has the same rate of growth as $M_0u_\lambda$, where $N_0$ is a constant coefficient vector field orthonormal to $M_0$. Combining these estimates, we show that $\nabla u_\lambda$ has the rate of growth no worse than that prescribed by the Hardy-Littlewood theorem modulo an error term involving a small constant times $\nabla u_\lambda$. We absorb the small constant times $\nabla u_\lambda$ into the $\nabla u_\lambda$ term on the left to show that $\nabla u_\lambda$ grows no worse than the rate prescribed by the Hardy-Littlewood theorem. Since the constants in our estimates are independent of $\lambda$, we let $\lambda \to 1$ to finish the proof.

**Theorem 3.8.** Let $\Gamma$ be a family of curves transversal to $b\mathbb{B}$. If $u$ is harmonic in $\mathbb{B}$ and Lipschitz-$B$ along $\Gamma$, then $u \in \Lambda_B(\mathbb{B})$.

**Proof.** Let us first re-parametrize $\Gamma$ to get $\left| \frac{d}{dt} (\gamma_p(t)) \right| = 1$. To achieve this consider the parametrization given by the arc-length starting from $p$, i.e., define $l_p$ on $(-a, a)$.
by
\[ l_p(t) = \int_0^t \left| \frac{d}{d\tau} (\gamma_p(\tau)) \right| d\tau \]
and consider the family of curves \( \tilde{\Gamma}(p, s) = \Gamma(p, l_p^{-1}(s)) \). Notice that the curve parameter \( s \) takes values in intervals about 0. By decreasing \( a \) if necessary, we get \( \tilde{\Gamma} : \mathbb{B} \times (-a, a) \to U \). It is easy to verify that \( \tilde{\Gamma} \) is a family of transversal curves, \( \left| \frac{d}{dt} (\tilde{\gamma}_p(t)) \right| = 1 \), and \( u \) is Lipschitz-\( B \) along \( \tilde{\Gamma} \). So, we replace the given \( \Gamma \) by \( \tilde{\Gamma} \).

To show the conclusion of the theorem it suffices, by Hardy-Littlewood Theorem, to show
\[
\sup_{U \cap \mathbb{B}} |\nabla u(x)| \frac{\delta(x)}{B(\delta(x))} \leq C < \infty, \quad (3.8.1)
\]
where \( U \) is a neighbourhood of \( b\mathbb{B} \). Let
\[
C_u := \sup_{U \cap \mathbb{B}} |\nabla u(x)| \frac{\delta(x)}{B(\delta(x))}.
\]
Notice that if \( u \) is in \( C^1(\mathbb{B}) \), then the condition \( (3.8.1) \) is automatically satisfied with \( C \) depending on \( u \). This is our starting point. For \( \frac{1}{2} < \lambda < 1 \), define
\[
u_{\lambda}(x) := u(\lambda x).
\]
Note that \( u_{\lambda} \in C^{\infty}(\mathbb{B}) \) and harmonic in \( \mathbb{B} \). Since \( t/B(t) \) is increasing,
\[
C_{u, \lambda} := \sup_{U \cap \mathbb{B}} |\nabla u_{\lambda}(x)| \frac{\delta(x)}{B(\delta(x))} \leq ||| \nabla u_{\lambda} |||_{\infty} \cdot \frac{\text{diam} \mathbb{B}}{B(\text{diam} \mathbb{B})} < \infty.
\]
We show that \( u_{\lambda} \) is Lipschitz-\( B \) along a transversal family of curves \( \Gamma_{\lambda} \) which is related to \( \Gamma \). Using this we then show that \( C_{u, \lambda} \) can, in fact, be dominated by a constant independent of \( \lambda \). We conclude that \( u \) satisfies \( (3.8.1) \) by letting \( \lambda \to 1 \).

Since \( \Gamma \) gives a foliation of \( U \) by curves, we get a projection \( \pi_{\Gamma} : U \cap \mathbb{B} \to b\mathbb{B} \) along \( \Gamma \), i.e., for \( x \in U \cap \mathbb{B} \), \( \exists! \pi_{\Gamma}(x) \in b\mathbb{B} \) and \( 0 < T_x < a \), such that \( \Gamma(\pi_{\Gamma}(x), T_x) = x \). For
simplicity of notation let us drop the subscript $\Gamma$ in $\pi_{\Gamma}$ and simply call it $\pi$. Define $\Gamma_\lambda$ by,

$$\Gamma_\lambda(p, t) = \frac{1}{\lambda} \cdot \Gamma(\pi(\lambda p), t + T_{\lambda p}), \quad p \in bB \text{ and } |t| \text{ small.}$$

We restrict $\lambda$ sufficiently close to 1 to make $\Gamma_\lambda$ well-defined near $bB$. In all of the analysis that follows we will be working in a small neighbourhood of such a boundary point. We will exploit this localization when we generalize the result to a general smoothly bounded domain. First, let us verify that $\Gamma_\lambda$ is a family of curves transversal to $bB$. Let $p, \lambda p$ and $q$ be as in Figure 3.9. Clearly,

$$\Gamma_\lambda(p, 0) = \frac{1}{\lambda} \cdot \Gamma(\pi(\lambda p), T_{\lambda p}) = \frac{1}{\lambda} \cdot \lambda p = p.$$

Now, let us check the transversality. Since, $\gamma_q(T_{\lambda p}) = \lambda p$, we have

$$\lambda p - q = \gamma_q(T_{\lambda p}) - \gamma_q(0) = T_{\lambda p}\gamma_q'(t^*), \text{ for some } 0 < t^* < T_{\lambda p}.$$

So, $p = \frac{1}{\lambda} \left( q + T_{\lambda p}\gamma_q'(t^*) \right)$. Hence,

$$\frac{\partial \Gamma_\lambda}{\partial t}(p, t) \cdot \nu_p = \frac{\partial \Gamma_\lambda}{\partial t}(p, t) \cdot p = \frac{1}{\lambda} \cdot \frac{\partial \Gamma}{\partial t}(q, t + T_{\lambda p}) \cdot \frac{1}{\lambda} \left( q + T_{\lambda p}\gamma_q'(t^*) \right)$$

$$= \frac{1}{\lambda^2} \left( \gamma_q'(t + T_{\lambda p}) \cdot q + T_{\lambda p}\gamma_q'(t + T_{\lambda p}) \cdot \gamma_q'(t^*) \right)$$

$$\leq \frac{1}{\lambda^2} (-c + T_{\lambda p}).$$

As $\lambda \to 1$, $T_{\lambda p} \to 0$. So, choose $\lambda \geq (1/2)$ close enough to 1 so that $T_{\lambda p} \leq (c/2)$ for all $p \in bB$. Then, we have

$$\frac{\partial \Gamma_\lambda}{\partial t}(p, t) \cdot \nu_p \leq -\frac{c}{2}. $$
Without loss of generality let us suppose that the Lipschitz-B constant of \( u \) along \( \Gamma \) is 1. Then, we also have

\[
|u_\lambda (\Gamma_\lambda (p, s)) - u_\lambda (\Gamma_\lambda (p, t))| = |u (\Gamma (\pi (\lambda p), s + T_{\lambda p})) - u (\Gamma (\pi (\lambda p), t + T_{\lambda p}))|
\]

\[
= |u (\gamma_q (s + T_{\lambda p})) - u (\gamma_q (t + T_{\lambda p}))|
\]

\[
\leq B \left| s - t \right|.
\]

This shows that \( u_\lambda \) is Lipschitz-\( B \) along \( \Gamma_\lambda \). Also, notice that

\[
\frac{\partial \Gamma_\lambda}{\partial t} (p, t) = \frac{1}{\lambda} \frac{\partial \Gamma}{\partial t} (q, t + T_{\lambda p}) \implies \left| \frac{\partial \Gamma_\lambda}{\partial t} (p, t) \right| = \frac{1}{\lambda}.
\]

Let us denote \( u_\lambda \) by \( v \) and \( \Gamma_\lambda \) by \( \hat{\Gamma} \). Let \( \hat{\gamma} \) denote the curves of \( \hat{\Gamma} \) and for \( x \in U \), let \( \hat{T}_x \) be such that \( \hat{\Gamma} (\hat{\pi}(x), \hat{T}_x) = x \).

Let \( M \) be the vector field of unit length given by the curves of \( \hat{\Gamma} \), i.e.,

\[
Mf(x) = \lambda \nabla f(x) \cdot \frac{\partial}{\partial t} \left( \hat{\Gamma} (\hat{\pi}(x), t) \right) \bigg|_{t=\hat{T}_x} = \lambda \nabla f(x) \cdot \hat{\gamma}'_{\hat{\pi}(x)}(\hat{T}_x).
\]

Let us now choose two neighbourhoods of \( bB \) to work in. Let

\[
0 < \epsilon < \min \left\{ \frac{1}{6}, \frac{c^2}{6400n(n-1)C_2} \right\},
\]

where \( C_2 \) is the constant from the inequality (2.5.1). By the uniform continuity of the tangent vectors to the curves of \( \Gamma \), \( \exists \delta_0 > 0 \) such that

\[
x, y \in B, \text{ and } |x - y| < \delta_0 \implies \left| \hat{\gamma}'_{\hat{\pi}(x)}(\hat{T}_x) - \hat{\gamma}'_{\hat{\pi}(y)}(\hat{T}_y) \right| < \epsilon.
\]

Decrease \( \delta_0 \), if necessary, so that \( \delta_0 < \epsilon \) and \( \{ x \in \mathbb{R}^n : \delta(x) < 2\delta_0 \} \) is contained in the neighbourhood corresponding to the bijection given by \( \hat{\Gamma} \), and in the part of the neighbourhood coming from Lemma 3.6 that lies in \( \Omega \). Let

\[
U := \{ x \in B : \delta(x) < \delta_0 \} \quad \text{and} \quad U' := \left\{ x \in B : \delta(x) < \frac{\delta_0}{4} \right\}.
\]
Let us recall some of the constants introduced above and modify their notation for clarity;

\[ C_u^U := \sup_U |\nabla u(x)| \frac{\delta(x)}{B(\delta(x))} \quad \text{and} \quad C_{u,\lambda}^U := \sup_U |\nabla u_\lambda(x)| \frac{\delta(x)}{B(\delta(x))}. \]

\( C_u' \) and \( C_{u,\lambda}' \) are defined similarly.

Fix \( x_0 \in U' \). Now we begin estimating \( Mv(x_0) \). Let \( p = \hat{\pi}(x_0) \). For \( s > \hat{T}_{x_0} \), we have

\[ v(\hat{\gamma}_p(s)) - v(\hat{\gamma}_p(\hat{T}_{x_0})) = (s - \hat{T}_{x_0}) \frac{Mv(x_0)}{\lambda} + \frac{(s - \hat{T}_{x_0})^2}{2} \cdot \frac{d^2}{dt^2} (v(\hat{\gamma}_p(t))), \]

for some \( \hat{T}_{x_0} < t < s \). Since \( v \) is Lipschitz-B along \( \hat{\gamma}_p \), we have

\[ |Mv(x_0)| \leq \frac{B(s - \hat{T}_{x_0})}{s - \hat{T}_{x_0}} + \frac{s - \hat{T}_{x_0}}{2} \cdot \left| \frac{d^2}{dt^2} (v(\hat{\gamma}_p(t))) \right|. \]  (3.8.2)

In what follows, we will estimate any dependence on \( \lambda \) using \( 1/2 < \lambda < 1 \) so that the inequalities we obtain are independent of \( \lambda \). On many occasions this step may not be explicitly shown.

Let \( \bar{M}_0 = \lambda \hat{\gamma}'(\hat{T}_{x_0}) \) and \( M_0f(x) = \lambda \nabla f(x) \cdot \bar{M}_0 \). Let

\[ s = \hat{T}_{x_0} + \frac{\kappa \delta(x_0)}{4} \min \left\{ \frac{c^2}{1024n}, \frac{c}{4C_1} \right\}, \]  (3.8.3)

where \( 0 < \kappa < 1 \) is to be chosen later and \( C_1 \) is the maximum of the length of the second derivatives of the curves of \( \hat{\Gamma} \). So,

\[ |\hat{\gamma}_p(t) - x_0| = |\hat{\gamma}_p(t) - \hat{\gamma}_p(\hat{T}_{x_0})| \leq 2 |t - \hat{T}_{x_0}| \leq \frac{\delta(x_0)}{2} < \frac{\delta_0}{8}. \]

Hence, \( \hat{\gamma}_p(t) \in U \), and \( |Mv(\hat{\gamma}_p(t)) - M_0v(\hat{\gamma}_p(t))| < \epsilon |\nabla v(\hat{\gamma}_p(t))| \). Using this, we compute the last term in (3.8.2) to get
\[
\left| \frac{d^2}{dt^2} (v(\hat{\gamma}_p(t))) \right| \leq \frac{1}{\lambda^2} |\nabla (M_0v)(\hat{\gamma}_p(t))| + C_1 |\nabla v(\hat{\gamma}_p(t))| \\
+ 2\epsilon \max \{|v_{xx}|, |v_{xy}|, |v_{yy}|\} (\hat{\gamma}_p(t)). \tag{3.8.4}
\]

Now, we want to use Lemma 3.5 to estimate the first and last term. To do this we need to estimate \( \delta(\hat{\gamma}_p(t)) \). We now show that
\[
\delta(\hat{\gamma}_p(t)) \geq c\delta(x_0)/8;
\]

\[
\hat{\gamma}_p(t) - p = \hat{\gamma}_p(t) - \hat{\gamma}_p(0) = t \hat{\gamma}_p'(t^*) \quad \text{for some } 0 < t^* < t.
\]

Hence, by Lemma 3.6 with \( a = 1/2 \), we have
\[
\delta(\hat{\gamma}_p(t)) = \delta(p + t \hat{\gamma}_p(t^*)) \geq \frac{c}{4} t \cdot |\hat{\gamma}_p'(t^*)| = \frac{ct}{4\lambda} \geq \frac{cT_{x_0}}{4}.
\]

Similarly,
\[
\delta(x_0) = \delta(\hat{\gamma}_p(T_{x_0})) = \delta(p + T_{x_0} \cdot \hat{\gamma}_p(t^{**})) \leq T_{x_0} \cdot |\hat{\gamma}_p'(t^{**})| = \frac{T_{x_0}}{\lambda} \leq 2T_{x_0}.
\]

Combining this with the previous inequality, we get
\[
\delta(\hat{\gamma}_p(t)) \geq \frac{c\delta(x_0)}{8}.
\]

Notice in (3.8.4) that \( v_x \) and \( v_y \) are harmonic in \( \mathbb{B} \). Since \( M_0 \) is a constant coefficient vector field, \( M_0v \) is harmonic too. By Lemma 3.5 we have the following;
\[
|\nabla (M_0v)(\hat{\gamma}_p(t))| \leq \frac{16n}{c\delta(x_0)} \sup \left\{ |M_0v(y)| : |y - \hat{\gamma}_p(t)| = \frac{c\delta(x_0)}{16} \right\},
\]

and
\[
\max \{|v_{xx}|, |v_{xy}|, |v_{yy}|\} (\hat{\gamma}_p(t)) \leq \frac{16n}{c\delta(x_0)} \sup \left\{ |\nabla v(y)| : |y - \hat{\gamma}_p(t)| = \frac{c\delta(x_0)}{16} \right\}.
\]

For \( |y - \hat{\gamma}_p(t)| = c\delta(x_0)/16 \), we have \( \delta(y) \geq c\delta(x_0)/16 \) and hence
\begin{align*}
|\mathcal{M}_0 v(y)| &= |\mathcal{M}_0 v(y)| \cdot \frac{\delta(y)}{B(\delta(y))} \cdot \frac{\mathcal{B}(\delta(y))}{\delta(y)} \leq |\mathcal{M}_0 v(y)| \cdot \frac{\delta(y)}{B(\delta(y))} \cdot \mathcal{B} \left( \frac{c \delta(x_0)}{16} \right) \\
&\leq \frac{16}{c} \cdot \frac{\mathcal{B} (\delta(x_0))}{\delta(x_0)} \cdot |\mathcal{M}_0 v(y)| \cdot \frac{\delta(y)}{B(\delta(y))} \leq \frac{16}{c} \cdot \frac{\mathcal{B} (\delta(x_0))}{\delta(x_0)} \cdot C_{u, \lambda}.
\end{align*}

The last inequality follows since \( \delta(y) < \delta_0 \). So,

\[
|\nabla (\mathcal{M}_0 v)(\hat{\gamma}_p(t))| \leq \frac{256n}{c^2 \delta(x_0)} \cdot \frac{\mathcal{B} (\delta(x_0))}{\delta(x_0)} \cdot C_{u, \lambda}.
\]

A similar calculation yields,

\[
\max \left\{ |v_{xx}|, |v_{xy}|, |v_{yy}| \right\} (\hat{\gamma}_p(t)) \leq \frac{256n}{c^2 \delta(x_0)} \cdot \frac{\mathcal{B} (\delta(x_0))}{\delta(x_0)} \cdot C_{u, \lambda}.
\]

Let us now estimate the only remaining term (the middle term) in (3.8.4);

\[
|\nabla v(\hat{\gamma}_p(t))| = |\nabla v(\hat{\gamma}_p(t))| \cdot \frac{\delta(\hat{\gamma}_p(t))}{B(\delta(\hat{\gamma}_p(t)))} \cdot \frac{B(\delta(\hat{\gamma}_p(t)))}{\delta(\hat{\gamma}_p(t))} \\
\leq \frac{8}{c} \cdot \frac{\mathcal{B} (\delta(x_0))}{\delta(x_0)} \cdot C_{u, \lambda}.
\]

Hence (3.8.4) becomes,

\[
\left| \frac{d^2}{dt^2} (v(\hat{\gamma}_p(t))) \right| \leq 4 |\nabla (\mathcal{M}_0 v)(\hat{\gamma}_p(t))| + C_1 |\nabla v(\hat{\gamma}_p(t))| \\
+ 2\epsilon \max \left\{ |v_{xx}|, |v_{xy}|, |v_{yy}| \right\} (\hat{\gamma}_p(t)) \\
\leq \frac{\mathcal{B} (\delta(x_0))}{\delta(x_0)} \left( \frac{1024n}{c^2 \delta(x_0)} (1 + \epsilon) + \frac{8}{c} \cdot C_1 \right) C_{u, \lambda}.
\]

Using this in (3.8.2) we get,

\[
|\mathcal{M} v(x_0)| \leq \frac{\mathcal{B} (s - \hat{T}_x)}{s - \hat{T}_x} + \frac{s - \hat{T}_x}{2} \cdot \frac{\mathcal{B} (\delta(x_0))}{\delta(x_0)} \left( \frac{1024n}{c^2 \delta(x_0)} (1 + \epsilon) + \frac{8}{c} \cdot C_1 \right) C_{u, \lambda}.
\]

Using the choice of \( s \) from (3.8.3), we have a positive constant \( C = C(\kappa) = O(1/\kappa) \).
\[ |Mv(x_0)| \leq C \cdot \frac{B(\delta(x_0))}{\delta(x_0)} + \kappa \frac{B(\delta(x_0))}{\delta(x_0)} \cdot \left(\frac{1 + \epsilon}{8} + \frac{\delta(x_0)}{4}\right) C_{u,\lambda}^{U'}. \]

Later, we will choose \( \kappa \) small to make the coefficient in front of \( C_{u,\lambda}^{U'} \) small. This choice will make \( C \) large, but for our purposes it does not matter. Since \( \delta(x_0) < \delta_0 < \epsilon \),

\[ |Mv(x_0)| \frac{\delta(x_0)}{B(\delta(x_0))} \leq C + \kappa \left(\frac{1 + 3\epsilon}{8}\right) C_{u,\lambda}^{U'}. \]  

(3.8.5)

Note that the above estimate holds for any \( x_0 \in U' \). In what follows, let us restrict \( x_0 \) further close to \( b\mathbb{B} \), i.e., \( x_0 \in U'' \), where

\[ U'' := \left\{ x \in \mathbb{B} : \delta(x) < \frac{\delta_0}{10} \right\}. \]

Let us now estimate the derivatives of \( v \) in directions orthogonal to \( \vec{M}_0 \). Let \( \vec{N}_0 \) be a unit vector orthogonal to \( \vec{M}_0 \). Let \( N_0f = \nabla f \cdot \vec{N}_0 \). To estimate \( N_0v(x_0) \), we use the fundamental theorem of calculus in the \( \vec{M}_0 \) direction. Since \( M_0 \) and \( N_0 \) are constant coefficient, they commute and also preserve harmonic functions. Then, we proceed to use the estimate on \( M_0 \) by applying Lemma 3.5 to \( |M_0N_0v| = |N_0M_0v| \leq |\nabla M_0v| \).

Let \( y_0 = x_0 + (\delta_0/10)\vec{M}_0 \). So,

\[ N_0v(x_0) = N_0v(y_0) - \int_0^{\delta_0/10} M_0N_0v(x_0 + s\vec{M}_0) \, ds, \text{ and hence} \]

\[ |N_0v(x_0)| \leq |N_0v(y_0)| + \int_0^{\delta_0/10} \left| \nabla (M_0v)(x_0 + s\vec{M}_0) \right| \, ds. \]

Since \( \exists P_{x_0}^{M_0} \in \partial \mathbb{B} \) such that \( x_0 - P_{x_0}^{M_0} \) is parallel to \( M_0 \), by Lemma 3.6, we have

\[ \frac{c}{4} (\delta(x_0) + s) \leq \delta(x_0 + s\vec{M}_0) \leq \delta(x_0) + s. \]
Hence,

\[ |N_0v(x_0)| \leq |N_0v(y_0)| + \frac{8n}{c} \int_0^{\delta_0} \frac{1}{(\delta(x_0) + s)} \sup \left\{ |M_0v(\xi)| : |\xi - (x_0 + sM_0)| = \frac{c}{8} (\delta(x_0) + s) \right\} \, ds. \tag{3.8.6} \]

For \( |\xi - (x_0 + sM_0)| = \frac{c}{8} (\delta(x_0) + s) \), by Lemma 3.6, we have

\[ \frac{c}{8} (\delta(x_0) + s) \leq \delta(\xi) \leq \left(1 + \frac{c}{8}\right) (\delta(x_0) + s), \]

and hence

\[ \frac{B(\delta(\xi))}{\delta(\xi)} \leq \frac{B\left(\frac{c}{8} (\delta(x_0) + s)\right)}{\frac{c}{8} (\delta(x_0) + s)}. \]

Since \( |\xi - x_0| < \delta_0/8, |M_0v(\xi)| \leq |Mv(\xi)| + \epsilon |\nabla v(\xi)| \). Also, since \( \delta(\xi) < \delta_0/4 \), we can use (3.8.5) to estimate \( |Mv(\xi)| \) to obtain

\[ |M_0v(\xi)| \leq |M_0v(\xi)| \cdot \frac{\delta(\xi)}{B(\delta(\xi))} \cdot \frac{B\left(\frac{c}{8} (\delta(x_0) + s)\right)}{\frac{c}{8} (\delta(x_0) + s)} \leq B\left(\frac{c}{8} (\delta(x_0) + s)\right) \left(C + \frac{\kappa(1 + 3\epsilon)}{8} C_{u,\lambda} + \epsilon C_{u,\lambda} \right). \]

Since \( B \) is increasing and regular,

\[ \frac{64n}{c^2} \int_0^{\delta_0} B\left(\frac{c}{8} (\delta(x_0) + s)\right) \left(B\left(\frac{c}{8} (\delta(x_0) + s)\right) \left(C + \frac{\kappa(1 + 3\epsilon)}{8} C_{u,\lambda} + \epsilon C_{u,\lambda} \right) \cdot \frac{64nC_2}{c^2} \right) \leq \frac{64nC_2}{c^2} \frac{B(\delta(x_0))}{\delta(x_0)}, \]

where \( C_2 \) is the constant from the inequality (2.5.1). Using this in (3.8.6), we get

\[ |N_0v(x_0)| \leq |N_0v(y_0)| + \frac{64nC_2}{c^2} \left(C + \frac{\kappa(1 + 3\epsilon) + 8\epsilon}{8} C_{u,\lambda} \right) \frac{B(\delta(x_0))}{\delta(x_0)}. \]

Let

\[ C_3 := \sup \left\{ |\nabla u(x)| : \delta(x) \geq \frac{c\delta_0}{40} \right\}. \]
Since
\[ \delta(y_0) \geq \frac{c}{4} \left( \delta(x_0) + \frac{\delta_0}{10} \right) \geq \frac{c\delta_0}{40}, \]
we have
\[ |N_0v(x_0)| \frac{\delta(x_0)}{B(\delta(x_0))} \leq C_3 \frac{\delta(x_0)}{B(\delta(x_0))} + \frac{64nC_2}{c^2} \left( C + \frac{\kappa(1 + 3\epsilon) + 8\epsilon}{8} \cdot C_{u,\lambda}^U \right). \]
Now, since \( \delta(x_0) < (\delta_0/10) < \delta_0 \), and \( t/B(t) \) is increasing,
\[ |N_0v(x_0)| \frac{\delta(x_0)}{B(\delta(x_0))} \leq C_3 \frac{\delta_0}{B(\delta_0)} + \frac{64nC_2}{c^2} \left( C + \frac{\kappa(1 + 3\epsilon) + 8\epsilon}{8} \cdot C_{u,\lambda}^U \right). \]
Combining (3.8.5) with the above estimate applied to the \((n-1)\) orthonormal directions in \( \tilde{M}_0 \), we get
\[ |\nabla v(x_0)| \frac{\delta(x_0)}{B(\delta(x_0))} \leq C_4 + \left\{ \frac{\kappa(1 + 3\epsilon)}{8} + \frac{64n(n-1)C_2}{c^2} \left( \frac{\kappa(1 + 3\epsilon) + 8\epsilon}{8} \right) \right\} C_{u,\lambda}^U \]
\[ = C_4 + \left\{ \left( 1 + \frac{64n(n-1)C_2}{c^2} \right) \frac{\kappa(1 + 3\epsilon)}{8} + \frac{64n(n-1)C_2}{c^2} \cdot \epsilon \right\} C_{u,\lambda}^U. \]
for some \( C_4 > 0 \). Choose
\[ \kappa = \left( 10 + \frac{640n(n-1)C_2}{c^2} \right)^{-1}. \]
So, for \( x_0 \in U'' \), the choices of \( \epsilon \) and \( \kappa \) give us
\[ |\nabla v(x_0)| \frac{\delta(x_0)}{B(\delta(x_0))} \leq C_4 + \frac{3}{100} C_{u,\lambda}^U. \]
Taking supremum over \( U'' \),
\[ C_{u,\lambda}^{U''} \leq C_4 + \frac{3}{10} C_{u,\lambda}^U. \]
By Lemma 3.7,
\[ C_{u,\lambda}^U \leq \frac{\delta_0}{(\delta_0/10)} \cdot C_{u,\lambda}^{U''} \leq 10C_4 + \frac{3}{10} C_{u,\lambda}^U, \]
and hence

\[ C_{u,\lambda}^U \lesssim C_4. \]

Since the constants involved in the inequalities are independent of \( \lambda \), let \( \lambda \to 1 \), to get \( C_u^U < \infty \). Now by the Hardy-Littlewood Theorem, \( u \in \Lambda_B(\mathbb{B}) \).

\[ \square \]

### 3B Main Theorem

As alluded to earlier, notice that all the analysis so far was local, centred around a point near \( b\mathbb{B} \). None of this depends on the behaviour of \( u \) elsewhere in \( \mathbb{B} \) or on the fact that the radius of this ball was 1. So, we have the following corollary to Theorem 3.8.

**Corollary 3.10.** Let \( \mathbb{B}_R = \{ |x| < R \} \) for some \( R > 0 \). Let \( \Gamma \) be a family of transversal curves to \( b\mathbb{B}_R \) (with transversality constant \( c > 0 \)). Let \( S \) be an open \( \mathbb{R}^n \)-sector in \( \mathbb{B}_R \) and \( u \) be harmonic in \( \mathbb{B}_R \). If \( u \) is Lipschitz-B along \( \Gamma \) near \( b\mathbb{B}_R \) in \( S \), then there exists a \( \mathbb{R}^n \)-sub-sector \( \tilde{S} \) of \( S \) in \( \mathbb{B}_R \) such that \( u \in \Lambda_B(\tilde{S} \cap U) \), where \( U \) is a neighbourhood of \( b\mathbb{B}_R \).

**Proof.** Let \( U \) be a neighbourhood of \( b\mathbb{B}_R \) such that \( \Gamma \) defines a \( C^1 \) bijection onto \( U \). Restrict \( U \), if necessary, so that it satisfies the requirements of the neighbourhood in the proof of the above theorem and also so that there exists a \( \mathbb{R}^n \)-sub-sector of \( S \), call it \( \tilde{S} \), such that

\[ \tilde{S} \cap U \subset \{ x \in S \cap U : B(x, \delta_{b\mathbb{B}_R}(x)) \subset S \} \]

\[ \square \]

We use this in our generalization of the above theorem to the case of a smoothly bounded domain.
Theorem 3.4. Let $\Omega$ be a smoothly bounded domain in $\mathbb{R}^n$. Let $\Gamma$ be a family of curves transversal to $b\Omega$. If $u$ is harmonic in $\Omega$ and Lipschitz-$B$ along $\Gamma$, then $u \in \Lambda_B(\Omega)$.

Proof. Let $c > 0$ be the transversality constant of $\Gamma$. There exists a neighbourhood $U$ of $b\Omega$ such that the restriction of curves of $\Gamma$ to $U$ defines a $C^1$ bijection onto $U$ and Lemma 3.6 holds in $U \cap \Omega$. \( \exists s_0 > 0 \) such that $p - s_0 \nu_p \in U \cap \Omega$ for all $p \in b\Omega$.

Fix a $p \in b\Omega$. Let $B_p = B(p - s_0 \nu_p, s_0)$. Then, $bB_p \cap b\Omega = \{p\}$.

Let $S_p$ be a $\mathbb{R}^n$-sector of $B_p$ such that $\Gamma$ is a family of curves transversal to the $\mathbb{R}^n$-spherical boundary of $S_p$ with a transversality constant of at least $c/2$. Since $u$ is harmonic in $S_p$ and Lipschitz-$B$ along $\Gamma$, by Corollary 3.10 we have a sub-sector $\tilde{S}_p$ of $S_p$ and a neighbourhood $V$ of $b\Omega$ such that $u \in \Lambda_B(\tilde{S}_p \cap V)$. It is evident that $x \in V \cap \Omega$ belongs to $\tilde{S}_p$ for some $p \in b\Omega$. This shows that $u \in \Lambda_B(\Omega)$. \(\square\)
In this chapter, we show that a Lipschitz-B holomorphic function has a Lipschitz gain near the boundary in certain directions. This gain, or the lack thereof, in a fixed direction is determined by the relative size of complex discs (relative to the distance of its centre to the boundary of the domain) that can be fit inside the domain in this direction. For instance, it will be clear that there is gain in the complex tangential directions but no gain in the normal direction. We discuss examples and an application to convex domains of finite type in Section 4B.

Let us begin with some definitions and lemmas. Let $\Omega \subset \subset \mathbb{C}^n (n > 1)$ have smooth boundary. Let $\mathcal{O}(\Omega)$ denote the holomorphic functions in $\Omega$. Let $U$ be a neighbourhood of $b\Omega$ and $\pi : U \to b\Omega$ be the normal projection onto the boundary. Let $B$ be a regular majorant. Recall that for $P \in b\Omega$, $\nu_P$ denotes the outward unit normal to $b\Omega$ at $P$.

**Definition 4.1.** Fix $P \in b\Omega$ and a unit vector $\vec{v} \in \mathbb{C}^n$. For $t > 0$, let $P_t = P - t\nu_P$. Let $R(P_t; \vec{v})$ be a real valued function satisfying the following properties;

(a) $0 < R(P_t; \vec{v}) \leq \sup \{ r > 0 : \{ P_t + \zeta \vec{v} : \zeta \in \mathbb{C} \text{ and } |\zeta| \leq r \} \subset \subset \Omega \}$,

(b) $R(P_t; \vec{v})$ is $C^1$ and strictly increasing in $t$ for $0 \leq t \leq 2\delta_0$ (for some $\delta_0 > 0$),
(c) \( \exists c > 0, \) such that

\[
R(P_{2t}; \vec{v}) \geq (1 + c) R(P_t; \vec{v}), \quad 0 \leq t \leq 2\delta_0.
\]

(d) \( \exists 0 < \beta < 1, \) such that \( t^\beta \cdot \frac{d}{dt} (R(P_t; \vec{v})) \) is almost increasing,

(e) \( \frac{R(P_t; \vec{v})}{B(t)} \) is a majorant, and

(f) \( \int_{h}^{2\delta_0} \frac{B(t)}{t R(P_t; \vec{v})} dt \lesssim \frac{B(h)}{R(P_h; \vec{v})} \) for \( 0 < h \leq \delta_0. \)

Let us now make a few remarks about this definition. Since we describe the Lipschitz gain in Theorem 4.5 in terms of the inverse of \( R, \) 4.1(b) is natural. For the same reason, it would be ideal to choose \( R \) as large as possible, namely we would like \( R \) to equal the supremum in 4.1(a). But this choice may not satisfy the other conditions in Definition 4.1, so we relax \( R \) to be less than or equal to the supremum in 4.1(a). 4.1(c) and (d) are rather mild concavity conditions on \( R. \) We make use of this in Lemma 4.7 which is utilized in the proof of Theorem 4.5. 4.1(e) and (f)
are restrictions on how \( R \) interacts with \( B \). This is akin to the condition \( \alpha < 1/2 \), for theorems referenced in the introduction which give us the gain from \( \text{Lip-\( \alpha \)} \) in the domain to \( \text{Lip-(2\( \alpha \)} \) along complex tangential curves. We want to note that conditions 4.1(c), (d), and (f) may not be optimal. We did not investigate this aspect further.

Notice that, for \( k \in \mathbb{N} \), \( \alpha < 1/k \), \( R(P_t ; \vec{v}) \approx t^{1/k} \) and \( B(t) = t^\alpha \) satisfy 4.1(b)-(f). This corresponds to \( P \) being finite type-\( k \) and \( \vec{v} \) being complex tangential to \( b\Omega \) at \( P \). We will discuss this in greater detail in Section 4B.

Since \( b\Omega \) is smooth, we know that \( \nu_P \), for \( P \in b\Omega \), is transversal to \( b\Omega \) at points in \( b\Omega \) close to \( P \). We restrict \( U \) so that

\[
U \cap \Omega \subset \{ z \in \Omega : \delta(z) \leq \delta_0 \},
\]

(4.2.1)

where \( \delta_0 \) comes from Definition 4.1 and also so that, if \( z - w \) is \( \mathbb{R} \)-parallel to \( \vec{v} \) and \(|z - w| < R(z; \vec{v})\), for \( z, w \in U \cap \Omega \), then

\[
\nu_{\pi(z)} \cdot \nu_{\pi((1-s)z+sw)} > 0, \text{ for } 0 \leq s \leq 1.
\]

(4.2.2)

In what follows, \( U \) will always denote this neighbourhood. The following lemma will let us estimate the directional derivatives of Lipschitz-\( B \) holomorphic functions by the radius of discs that can be fit in those directions.

**Lemma 4.3.** Let \( z \in U \cap \Omega \). Let \( \vec{n} \) and \( \vec{v} \) be unit vectors in \( \mathbb{C}^n \). Suppose \( f \in \mathcal{O}(\Omega) \cap \Lambda_B(\Omega) \) (with Lipschitz constant 1). Then,

(a) \( \{ z + \zeta \vec{n} : \zeta \in \mathbb{C} \text{ and } |\zeta| \leq \delta \} \subset \Omega \) \( \Rightarrow \) \( \left| \frac{\partial f}{\partial \vec{n}} (z) \right| \leq \frac{B(\delta)}{\delta} \),

and

(b) \( \{ z + \zeta \vec{n} + \tau \vec{v} : \zeta, \tau \in \mathbb{C}, |\zeta| \leq \delta, \text{ and } |\tau| \leq r \} \subset \Omega \)

\[
\Rightarrow \left| \frac{\partial^2 f}{\partial \vec{n} \partial \vec{v}} (z) \right| \leq \frac{B(\delta)}{\delta r}.
\]

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Proof. These are direct consequences of Cauchy’s integral formula.

\[ \frac{\partial f}{\partial \vec{n}} (z) = \frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{f(z + \zeta \vec{n})}{\zeta^2} d\zeta = \frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{f(z + \zeta \vec{n}) - f(z)}{\zeta^2} d\zeta \]

\[ \left| \frac{\partial f}{\partial \vec{n}} (z) \right| \leq \frac{1}{2\pi} \cdot \frac{B(\delta)}{\delta^2} \cdot 2\pi \delta = \frac{B(\delta)}{\delta}. \]

\[ \frac{\partial^2 f}{\partial \vec{n} \partial \vec{v}} (z) = \frac{1}{2\pi i} \int_{|\tau|=r} \frac{\frac{\partial f}{\partial \vec{n}} (z + \tau \vec{v})}{\tau^2} d\tau. \]

For $|\tau| = r$, we can fit a disc of radius $\delta$ in the $\vec{n}$ direction centred at $z + \tau \vec{v}$ contained in $\Omega$. Hence, by a),

\[ \left| \frac{\partial f}{\partial \vec{n}} (z + \tau \vec{v}) \right| \leq \frac{B(\delta)}{\delta}. \]

So,

\[ \left| \frac{\partial^2 f}{\partial \vec{n} \partial \vec{v}} (z) \right| \leq \frac{1}{2\pi} \cdot \frac{B(\delta)}{\delta r^2} \cdot 2\pi r = \frac{B(\delta)}{\delta r}. \]

\[ \Box \]

4A Main Theorem

Definition 4.4. Fix $P \in b\Omega$, and a unit vector $\vec{v} \in \mathbb{C}^n$. Since $R$ satisfies 4.1(b), let $S(t)$ invert $R(P_t; \vec{v})$ (as a function of $t$), i.e.,

\[ S \left( R(P_t; \vec{v}) \right) = t \quad \text{and} \quad R \left( P_{S(t)}; \vec{v} \right) = t. \]

Note that $S$ depends on $P$ and $\vec{v}$ even though we do not make this notationally explicit. Let $U$ be a neighbourhood of $b\Omega$ satisfying (4.2.1) and (4.2.2). Now, we are ready to state and prove the main theorem of this chapter.
Theorem 4.5. Let $f \in \mathcal{O}(\Omega) \cap \Lambda_B(\Omega)$. Let $P \in b\Omega$ and $z \in U \cap \Omega$. Suppose $\exists \delta > 0$ such that

\[ P_\delta - z \text{ is } \mathbb{R}-\text{parallel to } \vec{v} \text{ and } |P_\delta - z| < R(P_\delta; \vec{v}). \]

Then,

\[ |f(P_\delta) - f(z)| \lesssim B(S(|P_\delta - z|)). \]

Proof. Without loss of generality, let us assume that the Lipschitz-B constant of $f$ is 1. We use a box argument similar to the one used in the proof of the Hardy-Littlewood theorem. For $t > 0$, let $z_t = z - t \nu_P$. Let $h = S(|P_\delta - z|)$. We now estimate $|f(P_\delta) - f(z)|$ by estimating the change of $f$ in each side of this box, i.e.,

\[ |f(P_\delta) - f(z)| \leq |f(P_\delta) - f(P_{\delta+h})| + |f(P_{\delta+h}) - f(z_h)| + |f(z_h) - f(z)|. \]

For $I$, by Lemma 4.3, we have

\[ \left| \frac{\partial f}{\partial \nu_P}(P_{\delta+s}) \right| \leq \frac{B(\delta + s)}{\delta + s} \leq \frac{B(s)}{s}. \]
Hence,

\[ I = |f(P_\delta) - f(P_{\delta+h})| \leq \int_0^h \left| \frac{\partial f}{\partial \nu_P}(P_{\delta+s}) \right| \, ds \leq \int_0^h \frac{B(s)}{s} \, ds \]

\[ \lesssim B(h) = B(S(|P_\delta - z|)). \]

For III, since \( \nu_P \) is transversal near \( \pi(z) \), by Lemma 3.6 we have \( \delta(z_s) \gtrsim s \). So,

\[ \left| \frac{\partial f}{\partial \nu_P}(z_s) \right| \lesssim \frac{B(s)}{s}. \]

The constant in the above inequality depends only on the geometry of \( \partial \Omega \) near \( P \).

Estimating III similarly to I, we have III = \( |f(z_h) - f(z)| \lesssim B(S(|P_\delta - z|)) \).

To estimate II, we need to estimate \( \partial f / \partial \vec{v} \) at points in the lower edge of the box. It turns out that we get a better estimate if we first estimate \( \partial^2 f / (\partial \nu_P \partial \vec{v}) \) using the size of the radii of discs that fit in the \( \nu_P \) and \( \vec{v} \) directions and then integrating this in the \( \nu_P \) direction. Now, let us estimate the size of the discs that can be fit in these directions at points on the dotted line, given by \( (1-s)P_{\delta+h+t} + sz_{h+t}, \) in Figure 4.6.

So,

\[ R((1-s)P_{\delta+h+t} + sz_{h+t}; \vec{v}) \geq R(z_{h+t}; \vec{v}) \geq R(P_{\delta+h+t}; \vec{v}) - |P_\delta - z|. \]

Now, we consider two cases.

Case 1: \( |P_\delta - z| \leq \frac{1}{1+c} R(P_\delta; \vec{v}) \). Here \( c \) is the constant appearing in the doubling condition (4.1(c)) that \( R \) satisfies. Then,

\[ R(P_{\delta+h+t}; \vec{v}) - |P_\delta - z| \geq R(P_{\delta+h+t}; \vec{v}) - \frac{1}{1+c} R(P_\delta; \vec{v}) \]

\[ = \frac{1}{1+c} (R(P_{\delta+h+t}; \vec{v}) - R(P_\delta; \vec{v})) + \frac{c}{1+c} R(P_{\delta+h+t}; \vec{v}) \]

\[ \geq \frac{c}{1+c} R(P_{\delta+h+t}; \vec{v}). \]
Case 2: \(|P_\delta - z| \geq \frac{1}{1 + c} R(P_\delta ; \vec{v})\). Since \(|P_\delta - z| < R(P_\delta ; \vec{v})\), we have

\[ R(P_{\delta+h+t} ; \vec{v}) - |P_\delta - z| \geq R(P_{\delta+h+t} ; \vec{v}) - R(P_\delta ; \vec{v}) . \]

Also, since \(h = S(|P_\delta - z|)\), we have

\[ \frac{\delta}{2} \leq S \left( \frac{1}{1 + c} R(P_\delta ; \vec{v}) \right) \leq h \leq S (R(P_\delta ; \vec{v})) = \delta . \]

The conditions 4.1(c) and (d) that \(R\) satisfies imply that, \(\exists \epsilon > 0\) such that

\[ R(P_{\delta+h+t} ; \vec{v}) - R(P_\delta ; \vec{v}) \geq \epsilon R(P_{h+t} ; \vec{v}), \quad 0 \leq t \leq \delta_0 . \]

We detail the proof of this fact in Lemma 4.7 following this proof. So,

\[ R(P_{\delta+h+t} ; \vec{v}) - |P_\delta - z| \geq \epsilon R(P_{h+t} ; \vec{v}) . \]

So far, we’ve shown that for \(z\) as in the hypothesis, \(0 \leq s \leq 1\), and \(0 \leq t \leq \delta_0\),

\[ R((1 - s)P_{\delta+h+t} + sz_{h+t} ; \vec{v}) \geq R(z_{h+t} ; \vec{v}) \gtrsim R(P_{h+t} ; \vec{v}) . \]

Since \(\nu_P\) is transversal to \(b\Omega\) at \(\pi((1 - s)P_\delta + sz)\),

\[ \delta ((1 - s)P_{\delta+h+t} + sz_{h+t}) \gtrsim h + t . \]

Let us point out that the constant in the above inequality depends only on \(R\). By Lemma 4.3,

\[ \left| \frac{\partial^2 f}{\partial \nu_P \partial \nu} ((1 - s)P_{\delta+h+t} + sz_{h+t}) \right| \lesssim \frac{B(h + t)}{(h + t) R(P_{h+t} ; \vec{v})} . \]
Let $M := \sup \{ |\partial f(w)| : w \in \Omega \text{ and } \delta(w) \geq \delta_0 \}$. Now, we integrate the above estimate in the $\nu_P$ direction to get an estimate on $\partial f/\partial \vec{v}$;

\[
\left| \frac{\partial f}{\partial \vec{v}}((1-s)P_{\delta+h} + sz) \right| \leq \left| \frac{\partial f}{\partial \vec{v}}((1-s)P_{\delta+h+\delta_0} + sz_{\delta+\delta_0}) \right| \\
+ \int_0^{\delta_0} \left| \frac{\partial^2 f}{\partial \nu_P \partial \vec{v}}((1-s)P_{\delta+h+t} + sz_{\delta+t}) \right| dt
\]

\[
\leq M + \int_0^{\delta_0} \frac{B(h+t)}{(h+t)R(P_{h+t};\vec{v})} dt
\]

\[
\lesssim M + \frac{B(h)}{R(P_h;\vec{v})}.
\]

By 4.1(e), $B(h)/R(P_h;\vec{v}) \to \infty$ as $h \to 0$. So, we can restrict $\delta_0$ further if necessary to get

\[
\left| \frac{\partial f}{\partial \vec{v}}((1-s)P_{\delta+h} + sz) \right| \lesssim \frac{B(h)}{R(P_h;\vec{v})}.
\]

By Lemma 4.3, we have $M \lesssim B(\delta_0)/\delta_0$. So, the only dependence of the constant in the above inequality on $f$ is on the Lipschitz-B constant of $f$, which, we have assumed here to be 1. Now, $II$ is estimated as follows;

\[
II = |f(P_{\delta+h}) - f(z_h)| \leq \sup_{0 \leq s \leq 1} \left| \frac{\partial f}{\partial \vec{v}}((1-s)P_{\delta+h} + sz) \right| \cdot |P_{\delta+h} - z_h|
\]

\[
\lesssim \frac{B(h)}{R(P_h;\vec{v})} \cdot |P_{\delta} - z| = \frac{B(S(|P_{\delta} - z|))}{|P_{\delta} - z|} \cdot |P_{\delta} - z|
\]

\[
= B(S(|P_{\delta} - z|)).
\]

We now present the lemma that we used in the proof above.

**Lemma 4.7.** Let $u : [0, 2] \to [0, \infty)$ be $C^1$ and increasing with $u(0) = 0$. Suppose

(i) $\exists c > 0$ and $0 < \delta_0 < 1$ such that $u(2t) \geq (1+c)u(t)$, for $0 \leq t \leq \delta_0$. 

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and

(ii) \( \exists 0 < \beta < 1 \) such that \( x^\beta u'(x) \) is almost increasing (with constant \( C \)).

Then, \( \exists \epsilon > 0 \), \( \forall 0 \leq a \leq \delta_0 \),

\[
    u(a + x) - u(a) \geq \epsilon u(x), \text{ for } \frac{a}{2} \leq x \leq \delta_0.
\]

Proof. Let \( 0 \leq a \leq \delta_0 \) and \( a/2 \leq x \leq \delta_0 \). Then,

\[
    u(a + x) - u(a) = \int_0^x u'(a + t) \, dt = \int_0^x (a + t)^\beta u'(a + t) \cdot \frac{1}{(a + t)^\beta} \, dt
\]

\[
    \geq \frac{1}{C} \int_0^x t^\beta u'(t) \cdot \frac{1}{(a + t)^\beta} \, dt \geq \frac{1}{C} \int_{a/4}^x \frac{t^\beta}{(a + t)^\beta} \cdot u'(t) \, dt
\]

\[
    \geq \frac{1}{5^\beta C} (u(x) - u(a/4)).
\]

(i) \( \implies u(a/2) \geq (1 + c)u(a/4) \). So,

\[
    u(a + x) - u(a) \geq \frac{1}{5^\beta C} \left( u(x) - \frac{1}{1+c} u \left( \frac{a}{2} \right) \right)
\]

\[
    \geq \frac{1}{5^\beta C} \left( u(x) - \frac{1}{1+c} u(x) \right) = \frac{1}{5^\beta C} \cdot \frac{c}{1+c} \cdot u(x).
\]

\[\square\]

4B Examples

Let us now consider some examples and see what Theorem 4.5 says in each of these cases. The first example recovers Stein’s original observation in the setting of Lipschitz gain along complex tangential discs.
Example 4.8. Let \( \Omega \) be a smoothly bounded domain in \( \mathbb{C}^n (n > 1) \). Let \( P \in \partial \Omega \) and \( \vec{v} \in \mathbb{C}T_P(\partial \Omega) \). Then, by using Taylor’s theorem we see that we may choose

\[
R(P_\delta; \vec{v}) \approx \sqrt{\delta}, \quad \text{and hence } S(t) \approx t^2.
\]

It is easily verified that this choice of \( R \) along with \( B(t) = t^\alpha \), for \( 0 < \alpha < 1/2 \), satisfies the conditions of definition 4.1. By Theorem 4.5 we have the following; if \( f \in \mathcal{O}(\Omega) \cap \text{Lip}_\alpha(\Omega) \), for \( 0 < \alpha < 1/2 \), then

\[
|f(P_\delta) - f(z)| \lesssim |P_\delta - z|^{2\alpha},
\]

for \( z \in \Omega \) near \( \partial \Omega \) such that \( P_\delta - z \in \mathbb{C}T_P(\partial \Omega) \) and \( |P_\delta - z| < R(P_\delta; \vec{v}) \).

Example 4.9. For \( m > 1 \), let \( \Omega_m = \{ (z, w) \in \mathbb{C}^2 : |z|^2 + |w|^{2m} < 1 \} \). Let \( P = (e^{i\theta}, 0) \in \partial \Omega_m \), for some \( \theta \in \mathbb{R} \). Then, \( \mathbb{C}T_P(\partial \Omega_m) = \{ (0, \tau) : \tau \in \mathbb{C} \} \). Now, we can choose

\[
R(P_\delta; \vec{v}) \approx \delta^{1/2m}, \quad \text{for } \vec{v} = (0, \tau) \in \mathbb{C}T_P(\partial \Omega_m),
\]

and hence,

\[
S(t) \approx t^{2m}.
\]

As in the previous example, this choice of \( R \) and \( B(t) = t^\alpha \), for \( 0 < \alpha < 1/(2m) \), satisfies definition 4.1. So, by Theorem 4.5, if \( f \in \mathcal{O}(\Omega_m) \cap \text{Lip}_\alpha(\Omega_m) \), for \( 0 < \alpha < 1/(2m) \), then

\[
|f(P_\delta) - f(z)| \lesssim |P_\delta - z|^{2m\alpha},
\]

for \( z \in \Omega_m \) near \( \partial \Omega_m \) such that \( P_\delta - z = (0, \tau) \), for some \( \tau \in \mathbb{C} \), and \( |\tau| < R(P_\delta; \vec{v}) \).

In the above example \( P \) is a point of type \( 2m \). Since the domain is in \( \mathbb{C}^2 \), the different notions of finite type agree. The concept of finite type introduced by D'Angelo
[D’A82], is the (normalized) maximal order of contact of (possibly singular) complex varieties with \( b\Omega \) at \( P \in b\Omega \). For a detailed discussion see D’Angelo [D’A93, Chapter 4]. These concepts have a deep connection to the \( \overline{\partial} \)-Neumann problem as seen in the work of Catlin [Cat83, Cat87].

Now, we recall the definitions of finite variety type and finite line type for domains in \( \mathbb{C}^n \). These are genuinely different when \( n \geq 3 \). Let \( \Omega \subset \subset \mathbb{C}^n \) be a domain with smooth boundary. Fix \( P \in b\Omega \) and a neighbourhood \( U \) of \( P \). Let \( r \) be a smooth defining function for \( \Omega \) in \( U \), i.e., \( \Omega \cap U = \{ z \in U : r(z) < 0 \} \), \( b\Omega \cap U = \{ z \in U : r(z) = 0 \} \), and \( |\nabla r| \neq 0 \) on \( b\Omega \cap U \).

If \( f \) is a smooth, complex valued function, defined near the origin in \( \mathbb{C} \), let \( \nu(f) \) denote the order of vanishing of \( f - f(0) \) at the origin. For a vector valued function \( F = (f_1, \ldots, f_n) \), let \( \nu(F) = \min_j \nu(f_j) \).

**Definition 4.10.** \( P \) is a point of **finite (one-dimensional) variety type** if

\[
\Delta_1(P) = \sup_F \frac{\nu(r \circ F)}{\nu(F)} < \infty,
\]

for \( F \) a holomorphic parametrization of a complex analytic subvariety of \( \mathbb{C}^n \) with \( F(0) = P \). \( \Delta_1(P) \) is called the **one-dimensional variety type** of \( P \).

A complex line in \( \mathbb{C}^n \) is a set of points of the form \( \{ \zeta \vec{a} + \vec{b} : \zeta \in \mathbb{C} \} \) for fixed \( \vec{a}, \vec{b} \in \mathbb{C}^n \). The line type of \( P \) measures the order of contact of such complex lines to \( b\Omega \) at \( P \).

**Definition 4.11.** \( P \) is a point of **finite line type** if

\[
L(P) = \sup_l \nu(r \circ l) < \infty,
\]

for \( l \) a parametrization of a complex line with \( l(0) = P \). \( L(P) \) is called the **line type** of \( P \).
McNeal [McN92] showed that if $\Omega \cap U$ is convex and has finite line type $L(P)$ at $P$, then $P$ is a point of finite one-dimensional variety type at $P$ and $\Delta_1(P) = L(P)$. Boas-Straube [BS92] gave a simpler and more geometric proof of this fact. The following example illustrates Lipschitz gain for $f \in \mathcal{O}(\Omega) \cap \text{Lip}_\alpha(\Omega)$ in a complex tangential disc near $P \in \partial \Omega$ where $\Omega$ is convex near $P$ and has finite line type at $P$.

**Example 4.12.** Fix $P \in \partial \Omega$ and a neighbourhood $U$ of $P$. Let $r$ be the defining function for $\Omega$ in $U$ given by the signed distance to $\partial \Omega$. Suppose $\Omega \cap U$ is convex and $P$ is a point of finite line type with $L(P) = k$ for some $k \geq 2$. Let $\vec{n}_P$ be the outward unit normal to $\partial \Omega$ at $P$. By a translation, we may assume that $P = 0 \in \partial \Omega$. Then, there is a complex line $l$, with $l(\zeta) = \zeta \vec{a}$ for some $\vec{a} \in \mathbb{C}^n \setminus \{0\}$, passing through 0 with $\nu(r \circ l) = k$. This means,

$$(r \circ l)^{(j)}(0) = 0, \quad \text{for } 0 \leq j \leq k - 1 \quad \text{and} \quad (r \circ l)^{(k)}(0) \neq 0.$$ 

Hence, $r(\zeta \vec{a}) = O(|\zeta|^k)$. Also, $(r \circ l)'(0) = 0$ means that $a \in \mathbb{C}T_0(\partial \Omega)$. Let $\delta > 0$ be sufficiently small. Since $\vec{n}_0$ is transversal to $\partial \Omega$ near 0, by Lemma 3.6 we have

$$r(-\delta \vec{n}_0 + \zeta \vec{a}) < 0, \quad \text{for } |\zeta| \lesssim \delta^{1/k}.$$ 

By choosing $R(0_\delta; \vec{a}) \approx \delta^{1/k}$, we get $S(t) \approx t^k$, and hence we obtain the following proposition as a consequence of Theorem 4.5.

**Proposition 4.13.** Let $\Omega \subset \subset \mathbb{C}^n(n > 1)$ have smooth boundary. Suppose $\Omega$ is convex near $P \in \partial \Omega$, a point of (finite) one-dimensional variety type $k$. If $f \in \mathcal{O}(\Omega) \cap \text{Lip}_\alpha(\Omega)$, for $\alpha < 1/k$, then $\exists \vec{a} \in \mathbb{C}T_P(\partial \Omega)$, for $\delta > 0$ sufficiently small,

$$|f(P_\delta) - f(z)| \lesssim |P_\delta - z|^{k\alpha},$$

for $z$ such that $P_\delta - z$ is $\mathbb{R}$-parallel to $\vec{a}$, and $|P_\delta - z| \lesssim \delta^{1/k}$. 

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Now, let us consider an example where the boundary is a hyperplane or pseudo-concave near a point in the boundary.

**Example 4.14.** Let $\Omega$ be a smoothly bounded domain in $\mathbb{C}^n (n > 1)$. Let $P \in b\Omega$ and $\vec{v} \in CT_P(b\Omega)$. Suppose that $b\Omega$ near $P$ is a hyperplane or pseudo-concave. Let $f \in \mathcal{O}(\Omega) \cap \text{Lip}_\alpha(\Omega)$, for $0 < \alpha < 1$. For $0 < \epsilon < (1 - \alpha)$ and $\vec{v} \in CT_P(b\Omega)$, we may choose

$$R(P_\delta; \vec{v}) \approx t^{\alpha + \epsilon}, \text{ and hence } S(t) \approx t^{1/(\alpha + \epsilon)}.$$

This choice of $R$ and $B(t) = t^\alpha$, for $0 < \alpha < 1$, satisfies definition 4.1. So, by Theorem 4.5, we have

$$|f(P_\delta) - f(z)| \lesssim |P_\delta - z|^{\alpha/(\alpha + \epsilon)}$$

for $z \in \Omega$ near $b\Omega$ such that $P_\delta - z \in CT_P(b\Omega)$ and $|P_\delta - z| < R(P_\delta; \vec{v})$. The constant in the above inequality depends on $\epsilon$. In particular, it may be unbounded as $\epsilon \to 0$. Hence, $f$ is Lipschitz continuous of order $1 - \epsilon'$, for all sufficiently small $\epsilon' > 0$, along complex tangential discs centred at $P_\delta$. 

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BIBLIOGRAPHY


