SYSTEMATIC DESIGN OF MULTIPLE ANTENNA SYSTEMS USING CHARACTERISTIC MODES

DISSERTATION

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ABSTRACT

Complex multiple antenna systems are emerging today as market pressures to improve link capacity and reliability on complex integrated platforms. Complexity is therefore inherent in both the analysis and design of such systems. Modal analysis of various types has been used in a variety of fields to manage this inherent complexity. For single antenna systems, the Theory of Characteristic Modes (CM) has been used to great success in designing single antennas. In this dissertation, four interconnected topics are discussed, which are crucial to the proper extension of Characteristic Modes to complex antenna systems, especially multiple antenna systems.

First, existing characteristic mode systems are reviewed and their common properties examined. A software architecture is then discussed which enables the computation of characteristic modes in general, requiring only that the particular system definition satisfies those common properties. The software is used to perform all computations, analyses, and designs in this dissertation.

Second, a general, high-performance, wideband mode tracking system is proposed. It is shown to be efficient and robust through several challenging examples. It is the first robust tracker proposed in connection with Characteristic Modes.

Third, a procedure is proposed to compute the number, location, and even voltages of ports on a given antenna or antennas using a CM description of the problem. Its mathematical construction is inspired by results from the emerging field of
Compressed Sensing. Its generality is demonstrated through a number of simulated designs.

Lastly, two new modal systems related to CM are proposed. One modal system, Subsystem Classical Characteristic Modes, produces modes which successively optimize the ratio of stored power to radiated power for an antenna embedded in a multistructure system. The other modal system, Target Coupling Characteristic Modes, produces modes which successively optimize the ratio of induced current intensity on a target structure to the current intensity on a source antenna. The modal systems are related through a projection matrix, which demonstrates the tradeoff between mutual coupling and radiation properties for a given multiple antenna system. The two modal systems are applied to several examples to validate their properties. Then, they are used in a new design procedure to systematically reduce the mutual coupling between two parallel dipoles. It is found that through loading an intermediate structure suggested by the proposed modal systems, the mutual coupling may be reduced, also predicted by the modal coupling analysis. The resulting designs generally improve upon designs available in the literature.

The dissertation concludes with a summary and discussion of future work. The appendices discuss various definitions, including a novel derivation of Classical Characteristic Modes from the starting point of orthogonal eigenpatterns.
To Jesus, His Church, Genevieve, Samuel and Lucia.
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1.1 Background and Motivation

Antenna engineering, like other modern engineering disciplines, involves the application of physical analysis techniques to solving design problems. For a particular antenna design problem, such as designing a simple linear dipole antenna to give a certain radiation pattern and peak gain level over a narrow frequency band, there may be several competing design techniques. One typical approach is to apply specific knowledge of the given antenna geometry parameters (in this case, the dipole length and feed location) determined from experience and antenna theory [2] to achieve the desired result. Usually, this specific knowledge is physical in nature and is derived from an approximate physical and analytical model that is properly parameterized. Deriving this accurate physical, if not entirely analytical, model for an arbitrary antenna is extremely difficult, as evidenced by the lack of literature on this subject. Reporting on antenna design techniques has therefore evolved into reporting on specific antennas with specific parameterizations. A notable exception has been the case of frequency-independent antennas [3, 2], which give general principles for the class of frequency-independent antennas, such as spiral antennas [4], conical antennas [5], and log-periodic antennas [6].

While the literature extensively catalogs existing antenna systems optimized for
various design problems and is occasionally peppered with general antenna design principles [3, 7], it still lacks a design methodology applicable to arbitrary antennas (or at least, an extremely large class of antennas). To attempt to fill this gap is a grand challenge for antenna designers going forward, as conformal and/or integrated antenna design is becoming a significant trend in the industry [8, 9, 10]. To provide a reasonably general design methodology, we first need to have a general analysis methodology.

1.1.1 Engineering Complex Systems

Engineering a system requires a systematic approach, especially in analysis. Analyzing a complex system requires one to decompose the system into simpler components. By simpler, we do not necessarily require the component to be simple to make or even necessarily simple in overall function. Rather, since the overall analysis is directed toward a particular set of engineering goals, we only require the system components to be simple in the context of meeting the overall problem.

For example, a rocket is an enormously complex system. Similarly, a satellite is a complex system. If the engineering problem is to supply television signals to wide geographical areas, one solution is to launch a satellite equipped with television transmission equipment using a rocket. Functionally, breaking the problem (TV signals over large geographical areas) down into three components (rocket, satellite with transceiver capability, and stationary planetary orbit) makes the problem simpler precisely because the role of each component is unique and well-understood. The complexity of each component may be similarly decomposed into a set of functionally simpler elements until each element’s role is so well understood in the context of the overall problem that it may be engineered. Again, the final engineered components
may be complex overall, but its functionally simple role in the overall problem context allowed it to be well-engineered.

1.1.2 Modal Analysis

Managing complexity using modal analysis is a popular technique. It also happens to be quite general. In the field of electromagnetic engineering, it is common practice to engineer waveguides using modal analysis, which generates excitation-independent modal fields based solely upon the cross-section of the proposed waveguide [11, 12]. Another popular analysis experimental electromagnetic modal analysis technique for time domain problems is the Singularity Expansion Method (SEM) [13]. It will not be examined in further detail here because of this work’s focus on frequency-domain methods.

In the fields of aerospace and mechanical engineering, mechanical modal analysis (known simply as modal analysis) has revolutionized the design of components [14]. Using the Finite Element Method (FEM) [15], mechanical modal analysis also generates excitation-independent modal deflections. These modes may be computed from FEM or measured experimentally. From this data, structures are modified to be more earthquake resilient [16, 17], bridges more stable in the presence of wind [18], and aircraft control surfaces more reliable [19].

In all these cases, the modal analysis allows the problem to be decomposed into functionally simpler (i.e. unique roles) modes. The modes are excited by some external forcing function (e.g., a coaxial probe or a gust of wind) and their collective response determines the overall system’s behavior. Most importantly, a useful modal description of a system will involve a relatively small number of modes with significant coupling to the external forcing function. That is, a useful modal system definition is one where the overall behavior of the system may be well approximated by the
excitation of only a few modes. If instead of approximating the system’s behavior using only a few modes, many modes are required, then it is arguable as to whether the original problem’s complexity has been satisfactorily managed, if not expanded.

1.1.3 Characteristic Mode Analysis

The Theory of Characteristic Modes (CM), also known as the Theory of Classical Characteristic Modes (CCM), is a frequency-domain modal analysis system defined for a large class of electromagnetic radiating and scattering systems, such as antennas. It was first proposed by Robert Garbacz [20] in 1965 and later formally associated with the Method of Moments (MoM) [21] by Roger Harrington and Joseph Mautz [22] in 1971. It is an extremely general modal analysis technique, owing its practical generality to MoM. It is defined by a generalized eigenvalue problem parameterized by frequency. Assuming that it is applied to lossless structures [23], CCM provides for each mode: an eigenvalue indicating the ratio of stored power to radiated power of that mode at each frequency; orthogonal eigencurrents, or a surface current density associated with that particular mode; and orthogonal eigenpatterns, or the radiated far-fields produced by an eigencurrent. Because of the orthogonality, both the total surface current density and the total far-field pattern may be decomposed into a weighted summation of modes.

Decomposing the total surface current density or total far-field pattern into a weighted summation of modes is important, not only because each mode has an eigenvalue describing its suitability as a wideband radiator, but also because the weights allow one to approximate the total current density or far-field pattern using a subset of the modes. Usually, the subset is small because the number of modes with relatively large modal weights is also small. This feature is important because it allows for general problem complexity to be managed. Most significantly, the modal
weights describe how well a particular set of antenna feed ports or incident field couple power to each mode, giving the designer both quantitative and qualitative insight into the antenna or scatterer’s physics.

1.1.4 Goals

The overall goal of this work is to provide a systematic approach to designing complex radiator systems, especially those involving more than one antenna. I draw on my experience in designing actual antenna systems for sponsors, usually involving coexistence issues, but sometimes involving direct multi-antenna interaction issues. To solve these problems, I will leverage the benefits of modal analysis to create a new form of characteristic mode analysis for multiple antenna systems, capable of analyzing coexistence and coupling issues. Necessarily, I will need to make a few related contributions to making the analysis of single antennas analyzed and designed using characteristic modes (and related modal systems) treat bandwidth more explicitly.

The organization of the dissertation follows from these general elements.

1.1.5 Key Contributions

In this dissertation, I have made the following key contributions:

- **Wideband Mode Tracking:** Introduced the first high performance wideband mode tracking system in the field of characteristic modes

- **Computer Aided Feed Design:** Introduced the first system to automatically determine the number and placement of feed ports on antennas given a characteristic mode-based description of the antenna’s operation over a given bandwidth, leveraging concepts from Compressed Sensing
• Radiation Modes: Introduced a new type of characteristic mode analysis suited to analyzing the radiation characteristics of a single antenna operating in the presence of other antennas

• Coupling Modes: Introduced a new type of characteristic mode analysis suited to analyzing the coupling between a single antenna and multiple target structures in its vicinity

• Designs to Minimize Mutual Coupling Between Two Antennas: Developed designs to minimize the mutual coupling between two closely spaced co-polarized dipoles using information derived from the radiation modes and coupling modes

1.2 Literature Review

CM theory has been used in the past usually for antenna analysis, and antenna placement on larger support structures, although there are a few examples of using CM for antenna design in the literature. This section takes a historical overview of the literature discussing characteristic modes.

1.2.1 Characteristic Modes: 1960-1980

The theory of Characteristic Modes was first considered by Robert Garbacz in his dissertation [20] and later summarized in [24]. It came from considering the problem

I choose to provide a fuller summary of his work here because it is often glossed over as important only in connection with Harrington’s later work. I speculate that Garbacz’s work, in particular the physical reasoning leading up to the modal method, could enable experimental measurements of characteristic modes using the scattering or perturbation matrix.
of finding scattering eigenfunctions for arbitrary configurations of conducting surfaces. That is, given a scatterer, a scattering operator $S$ may be formed:

$$\vec{E}^{\text{out}} = S(\vec{E}^{\text{in}})$$

where the total far-field $\vec{E}$ is the superposition of the incoming field $\vec{E}^{\text{in}}$ and the outgoing field $\vec{E}^{\text{out}}$: $\vec{E} = \vec{E}^{\text{in}} + \vec{E}^{\text{out}}$.

An associated perturbation operator $P$ may be defined which maps the incoming field $\vec{E}^{\text{in}}$ to the scattered field $\vec{E}^{\text{s}} \equiv \vec{E} - \vec{E}^{\text{in}}$:

$$\vec{E}^{\text{s}} = P(\vec{E}^{\text{in}})$$

Critically, the perturbation operator is normal when the scatterer is lossless, thereby admitting orthogonal eigenfunctions; Garbacz sought these orthogonal eigenfunctions of the perturbation operator. By construction, these modal fields are such that if they are incident upon the scatterer, the scattered field is simply a complex scaled version of themselves.\(^2\) He then proposed a way to numerically compute the modes using a form of MoM with point testing and entire domain basis functions (i.e., Fourier basis) through an ordinary eigenvalue problem:

$$[X(\alpha)]^T[X(\alpha)] \vec{J}_n = \epsilon_n(\alpha) \delta$$

where $[X(\alpha)]$ is the imaginary part of $[Z(\alpha)] = [Z]e^{-j\alpha}$, $\alpha$ is some real constant, and $\delta$ is the imaginary part of $\vec{E}$, the tangential component of the scattered electric field on the surface of the scatterer.

His technique yielded real modal currents and ensured that each associated modal tangential electrical field on the scatterer surface was equiphase with its corresponding

\(^2\)Harrington [22] notes that the modal scattered field is the complex conjugate of the incident modal field
modal current. Garbacz theorized that for electrically small to intermediate structures, the number of modes required to compute the scatterer’s response would be small, justified by several examples involving thin wire structures. For the first time, a general technique was available to compute and analyze the modal response of an arbitrary scatterer.

Harrington and Mautz [22, 23] chose to define the characteristic modes of an arbitrary lossless metallic structure through a generalized eigenvalue problem. It is discussed in much more detail in Chapter 2. Like Garbacz’s technique, it sought real modal currents whose associated incident or impressed modal electric fields tangential to the surface of the lossless scatterer are equiphase. Instead of defining the modes to directly minimize the phase variation of the tangential modal electric field, a generalized eigenvalue problem (GEP) was defined to generate modal currents such that the associated tangential modal electric fields would be equiphase: 

\[
[Z] \vec{J}_n = (1 + j\lambda_n)[R] \vec{J}_n.
\]

The GEP also defined orthogonal modal far-fields. Their technique required the complex MoM generalized impedance matrix \([Z]\) to be symmetric, implying that it should be constructed using the Galerkin method and sub-domain basis/testing functions. Since the GEP directly used the impedance matrix from a particular formulation of MoM to produce modal currents and modal far-fields with the same properties as Garbacz’s technique, it became the de-facto method of constructing the characteristic modes.

Harrington and others went on to apply CM theory to reducing the radar cross section of obstacles using reactive loading [25], pattern control using loading [26], and defining the modes for multiport antennas (network characteristic modes) [27]. He went on to extend characteristic modes (with some limitations) to dielectric and magnetic bodies [28, 29]. A separate work extended CM theory to use the MFIE instead of the EFIE MoM formulation to improve its lower frequency behavior [30].
for certain scatterers. Important for antennas, the modal self-admittance and mutual admittance were defined by Garbacz in [31]. Finally, characteristic modes were shown to be useful for locating a small antenna on a much larger conducting body (airframe) in a systematic fashion [32].

Dramatically, a pair of papers using properties of characteristic modes (in particular, Garbacz’s insight that the modal electric field tangential to the surface of a lossless structure is equiphase) emerged [33, 34], enabling the computation of 3D antenna structures designed to have a dominant mode which radiated a desired pattern. The structures were limited to bodies of revolution. The key idea of the papers was to define a far-to-near field transform using spherical modes and then find equiphase electric field surfaces on which the equivalent equiphase electric current lead by a specified phase. As an example of its generality, the vertically polarized desired and realized far-field patterns are shown in Figure 1.1 and the synthesized characteristic surface (rotationally symmetric about the Z-axis) is shown in Figure 1.2.

Unfortunately, the problem was left open-ended, as it would be difficult to locate appropriate feed points to excite the synthesized surface such that the desired mode was dominant. Still, it could be used to generate rotationally-symmetric scatterers with a specific pattern. More importantly, the concept is not theoretically limited to rotationally-symmetric scatterers, although the computers of the day practically limited their investigation to such scatterers.

Garbacz and Inagaki went on to define a new modal method [35, 36], commonly termed Inagaki characteristic modes. It bore some similarities to Garbacz’s original approach, but used a GEP to optimize the electric field over some target surface to be optimized with respect to the source field. The modal currents were still orthogonal because both matrices in the GEP were Hermitian, but they were complex.

A simplification of Inagaki modes in [37, 38], called generalized characteristic
Figure 1.1: Desired and realized modal far-field patterns

Figure 1.2: Synthesized characteristic surface at 100 MHz
modes, restored some of the benefits of the original characteristic modes, now termed classical characteristic modes, while preserving some of the flexibility of the original Inagaki formulation. Here, the $[R]$ matrix was replaced by another matrix defining regions of far-field orthogonality. The regions were specified by appropriately weighting the field points on the far-field sphere. It could be related back to the classical characteristic modes using a weight of 1 at all far-field points (thus, the term "generalized"). Generalized CM were later extended by Ethier, et al. [39] to allow per-polarization far-field weights to be defined.

Despite these advances, characteristic mode analysis did not catch on in the community (assuming that the literature is an accurate measure of community popularity), probably because MoM could not be used to accurately analyze structures of practical interest, considering the limitations of computers at the time. Mechanical modal analysis, on the other hand, did not suffer from the same problems because of the less demanding requirements of FEM, becoming extremely popular in the mechanical, civil, and related engineering communities.

1.2.2 Characteristic Modes: Recent Work

More recently, a few groups worldwide have taken up characteristic mode theory once again.

CM Theory

There was work defining a special class of characteristic modes to aperture problems [40] and discussing the particular advantages of using characteristic modes over standard MoM [41], [42], [43], [44].

Another group in Italy was using characteristic modes in a somewhat different way. Instead of being used primarily as an analysis tool, they used the modal currents as
entire-domain basis functions to accelerate the MoM solution of array problems [45], [46], [47], made especially clear in their later papers [48], [49]. In [46], they also provided analytical characteristic modes for finite cylinders in order to verify the numerical accuracy of their mode computations.

In the same spirit, a separate group published a technique of rapidly computing RCS using a combination of entire-domain basis functions (i.e. dominant characteristic mode currents) and AWE (asymptotic waveform evaluation) [50].

Their last major contribution was to apply a recent mathematical result to CM, enhancing the accuracy of the GEP computation in classical CM [51]. The basic problem is that the although the $[R]$ matrix in the GEP computation is theoretically positive-definite for lossless structures without internal resonances, the computed $[R]$ matrix is sometimes indefinite for numerical reasons [23]. Distinct from the approach in [23], Angiulli et al. used the Higham-Cheng theorem [52] to define a new pair of matrices $[X']$ and $[R']$ which are a “positive-definite pair.” The modes are computed from the new matrices and transformed back to $[X]$ with a modified positive-definite $[R]$.

Finally, there was a contribution of a group in Spain to explain the slow convergence in input susceptance [53] observed by Garbacz earlier [31] (i.e. many more modes were required to reconstruct the input admittance for a given antenna compared to its far-field pattern). They proposed to accelerate convergence through the introduction of a physically-derived ”source mode,” although it is hypothesized that the susceptance should in fact contain the contributions of several excited, but poorly radiating modes, similar to the concept of evanescent modes in guided waves [11].
CM in Antenna Analysis

Continuing the line of research started by Newman in using characteristic modes for antenna placement [32], Strohschein et al. published a dissertation [54] and a conference paper [55]. Especially interesting is the description of how the overall system modes evolve as the "probe" antenna is placed at various points on the structure.

Significant research has also been conducted into specifically using characteristic modes to understand the qualitative, as well as quantitative, behavior of various antennas by the Spanish since 2000 [56], [57], although one paper was published earlier in 1989 discussing the analysis of log-periodic antennas using characteristic modes [58].

Separately, two papers have been published by a Canadian group in 2010 discussing new computationally simpler metrics for complex antenna analysis [59] and optimization [60].

Lastly, an analysis by Obeidat and Raines on the modal behavior around series and parallel resonance for single port antennas was recently published [61]. It’s primary contribution is the proof that a series resonance is caused primarily by one characteristic mode, while a parallel resonance is caused by the interaction of multiple modes. This analysis enabled Prof. Rojas’s group to experimentally determine whether an antenna prototype’s structure was sufficiently similar to simulation by noting the series resonance frequencies, since the series resonance frequencies are less sensitive to the feed position than parallel resonances.

CM in Antenna Design

Research into using characteristic modes as an aid to design conformal or otherwise non-traditional antennas was first conducted by K. P. Murray [62], [63], [64], [65]. Various design metrics were later proposed by the Spanish group to identify modes
with broader bandwidth impedance and pattern behavior, which made the design process of several commercially-popular antennas more controlled [66], [57, 67, 68, 69, 70]. In [71], they summarized much of their work.

Prof. Roberto Rojas and his students at The Ohio State University have been contributing various design procedures for arbitrary lossless antennas, including lumped reactive loading in order to expand the bandwidth of a single equiphase modal current [72], [73], [74], lumped reactive loading to realize frequency-reconfigurable antennas [75], [76], and techniques for improving the bandwidth and gain of electrically small conformal antennas [77], [78], [8], [79]. Much of the work is summarized in [80].

Another group in Canada has also begun to investigate the application of characteristic modes to antenna design, capitalizing on the orthogonality of modes over a single antenna to improve port isolation in MIMO antenna design [81] and developing an extension to generalized characteristic modes to improve the directional performance of MIMO antennas [39].

1.2.3 Mutual Coupling

The literature concerning mutual coupling and its effects on system performance is split into two main branches. The first documents the mutual coupling between various antennas, ranging from general analysis on two element arrays [82] to the coupling between two adjacent microstrip patch antennas [83, 84]. The second documents how to compensate for the presence of mutual coupling in arrays, mainly through signal processing [85, 86]. Since this work examines a method to reduce mutual coupling physically, the relevant literature on mutual coupling should concern physical modifications to antennas rather than signal processing algorithms. While certain success has been achieved with isolating two antennas using a large plate [87], more general methods have appeared, all of which claim to use resonant parasitic structures to
physically reduce mutual coupling [88], [89], [90]. The most general of the methods is likely the one presented in [90] and I will be examining it in greater detail in Chapter 5. Separately, there has been very promising work to design custom metamaterial structures for mutual coupling reduction [91].

1.3 Organization of the Dissertation

The dissertation is organized in the following manner. First, Chapter 2 discusses the necessary background and design methodology supporting the systematic design of multiple antenna systems using a new form of characteristic modes. The chapter opens with the mathematical background behind all the forms of characteristic mode analysis. It then reviews the formulations of each type of characteristic mode analysis. The chapter concludes with considerations of general modal systems in antenna (and multiple antenna) systematic design.

Chapter 3 discusses a novel algorithm enabling wideband characteristic mode tracking. This algorithm is required for any serious systematic modal design of a wideband antenna system, since each type of characteristic mode analysis defines a frequency-dependent (generalized) eigenvalue problem. The modes computed at one frequency must be automatically related to the modes at another frequency in order for the information to be useful. To the best of my knowledge, it is the first modal tracking method presented in connection with characteristic modes, although similarities to tracking in other fields are discussed. In particular, the method can track a large number of modes using the modal eigenvectors from a generalized eigenvalue problem parameterized by frequency. The method and a naive alternative are detailed following a brief overview of the relevant elements of CM theory. The two techniques are applied to three distinct geometries and the results discussed. Relative
to alternative tracking techniques from other disciplines implemented in MATLAB [92], the proposed method features greatly improved performance.

Chapter 4 discusses another novel algorithm enabling the systematic determination of the number, location, and excitation voltages of feed ports for an arbitrary lossless antenna given the desired complex modal weights over some frequency range. The method defines an underdetermined system of equations and draws upon recent results from the field of Compressed Sensing as well as image processing to obtain a physically useful solution. Several examples are shown to demonstrate the method’s capabilities and generality.

Chapter 5 uses the methods from the previous chapters combined with a new modal system definition specific to multiple antenna systems to explicitly design such systems with reduced mutual coupling. The modal system is explained through physical arguments and defined mathematically. The multiple antenna design method is applied to a practical example and compared with existing work in the literature to illustrate its use and benefits.

Finally, Chapter 6 summarizes the key contributions of this dissertation and suggestions for future work are given.
CHAPTER 2
METHODOLOGY

This chapter will first review some relevant mathematical tools, which shall hopefully prove useful in understanding the machinery of successful modal systems for radiating devices. Then, the major types of Characteristic Mode theory will be reviewed and some generalizations made. Then, we shall present some elements and challenges unique to multiple antenna systems, summarizing the approach taken in this dissertation. Finally, the chapter is rounded out with some high-level discussion on the software required to enable this work and this level of generality.

2.1 Mathematical Tools

The key mathematical concept underlying efficient modal solutions and effective modal system definitions is found in the generalized eigenvalue problem, a classic generalization of the ordinary eigenvalue problem in linear algebra.\footnote{Its solutions, however, should be distinguished from those of the degenerate ordinary eigenvalue problem, in which the algebraic multiplicity or the geometric multiplicity of an eigenvalue is not 1\cite{93}.} First, however, we must establish why we should use the generalized eigenvalue problem in the first place.

One of the most important characteristics of modal analysis of complex phenomena is simplicity. The simplicity arises from two distinct characteristics of a useful
modal decomposition. First, the modes are orthogonal according to some meaningful functional. Orthogonality allows the modes to be considered as separate or uncoupled, greatly simplifying both the mathematics and the concepts. Second, the modes are specific to a given structure’s geometry and its material composition, but are also excitation-independent. This quality allows the modes to satisfy the particular physical constraints of the problem so that attention is directed at understanding their action in determining the system response given a certain system input.

2.1.1 Notation

The mathematical notation used throughout this dissertation is usually standard in finite-dimensional linear algebra, but it is good to define the notation for clarity.

A matrix ”M” is denoted as \([M]\), while a column vector ”x” is denoted as \(\bar{x}\). The transpose operation is denoted as \((\cdot)^T\) (e.g., \([M]^T\) or \(\bar{x}^T\)), while the Hermitian (complex conjugate) transpose is denoted as \((\cdot)^H\). Some inner products have a subscript \(\Sigma\) (c.f. section 2.2.1): this implies a special inner product defined over the infinite far-field sphere and not an inner product over a finite-dimensional vector space. Without subscript, a ”standard” inner product is defined as \(\langle\bar{a}, [M]\bar{b}\rangle \equiv \bar{a}^H [M]\bar{b}\). Complex conjugation is denoted as \((\cdot)^*\) and the imaginary number is \(j = \sqrt{-1}\).

Unless otherwise noted, an \(e^{j\omega t}\) time convention is assumed throughout, where \(\omega = 2\pi f\) denotes the radial frequency (units of radians) and \(f\) the linear frequency (units of Hertz).

2.1.2 Generalized Rayleigh Quotient

To compute modes with the above characteristics for an antenna system, it is useful to have modes which are orthogonal in two domains: the source domain, and the field domain. In particular, this suggests that the modal sources (usually currents) are
orthogonal with respect to an inner product defined over the volume of the antenna, while the modal fields (electric or magnetic or both) are orthogonal with respect to a different inner product defined remotely from the antenna extents.

If the modal orthogonality is determined by two inner products, then the modes may be defined as successive stationary extrema of some (generalized) Rayleigh quotient:

$$\rho(\vec{J}) = \frac{\langle \vec{J}, N\vec{J} \rangle_S}{\langle \vec{J}, D\vec{J} \rangle_S}$$

where $$\langle \vec{J}, A\vec{J} \rangle_S$$ is some inner product defined over the source domain $$S$$ with some associated operator $$A$$. Either $$N$$ or $$D$$, or both operators are related to the field domain. It is extremely advantageous numerically to define the modes using this particular Rayleigh quotient because it is directly related to a generalized eigenvalue problem under certain conditions. The physical meaning of the modes is derived from the physical meaning of the two operators $$N$$ and $$D$$, as well as their ratio.

In this work, the operators $$N$$ and $$D$$ are numerically approximated using finite-dimensional matrices $$[N]$$ and $$[D]$$, usually through a boundary element method. In this case, the Quotient reduces to

$$\rho(\vec{J}) = \frac{\langle \vec{J}, [N]\vec{J} \rangle}{\langle \vec{J}, [D]\vec{J} \rangle}$$ (2.1.1)

where we now use the standard inner products defined in Section 2.1.1.

### 2.1.3 Lagrange Multipliers

The stationary points of Eq. 2.1.1, like in some variational problems, can be computed by maximizing/minimizing the functional $$f(\vec{J}) = \langle \vec{J}, [N]\vec{J} \rangle$$ subject to the constraint that $$\langle \vec{J}, [D]\vec{J} \rangle = C$$. This problem may be stated using the method of Lagrange multipliers:

$$f(\vec{J}) = \langle \vec{J}, [N]\vec{J} \rangle_s - \lambda_n(\langle \vec{J}, [D]\vec{J} \rangle_s - C)$$ (2.1.2)
provided that \([N]\) and \([D]\) are Hermitian matrices and \([D]\) is a positive-definite matrix. These conditions come from two limitations to the method of Lagrange multipliers. First, the functional \(f\) in the method of Lagrange multipliers is must be real. Second, the constraint functional must have a minimum (i.e. the constraint must be bounded from below).

The solutions to this Lagrange multiplier problem can be shown to be equivalent to the generalized eigenvalue problem. The following proof of this connection is provided assuming that \([N]\) and \([D]\) are complex Hermitian matrices and \(\bar{J}\) is real. These assumptions clarify the concepts in the proof, but it can be extended to complex \(\bar{J}\) using the concept of the complex gradient operator [94], but it is an exercise left to the reader.

Before the proof, one definition and one theorem must be established.

**Definition 2.1.1.** Let \(f(\bar{x})\) be a functional operating on an \(N\) dimensional real vector space, where \(\bar{x} \in \mathbb{R}^N\). Then, the gradient of \(f(\bar{x})\) is given as a column vector (in contrast to the closely associated gradient operator, which produces a row vector):

\[
\nabla_{\bar{x}} f(\bar{x}) = \frac{\partial f}{\partial \bar{x}} = \begin{bmatrix}
\frac{\partial f}{\partial x_1} \\
\vdots \\
\frac{\partial f}{\partial x_N}
\end{bmatrix}
\]

**Theorem 2.1.2.** Let \([A] \in \mathbb{R}^{N \times N}\), \(\bar{x} \in \mathbb{R}^{N \times 1}\), and \(f(\bar{x}) = \langle \bar{x}, [A]\bar{x} \rangle\). Then,

\[
\frac{\partial f}{\partial \bar{x}} = (|A| + [A]^T)\bar{x}
\]
Proof. Begin by expanding \( f(\bar{x}) \):
\[
f(\bar{x}) = \sum_{m}^{N} \sum_{n}^{N} x_m x_n A_{mn}
\]
\[
= \sum_{m}^{N} x_m \left( \sum_{n}^{N} x_n A_{mn} \right)
\]
\[
= x_1 \left( \sum_{n}^{N} x_n A_{1n} \right) + x_2 \left( \sum_{n}^{N} x_n A_{2n} \right) + \ldots + x_N \left( \sum_{n}^{N} x_n A_{Nm} \right)
\]
Now compute the gradient with respect to \( \bar{x} \) using Definition 2.1.1:
\[
\frac{\partial f}{\partial \bar{x}} = \begin{bmatrix}
\sum_{n}^{N} x_n A_{1n} \\
\sum_{n}^{N} x_n A_{2n} \\
\vdots \\
\sum_{n}^{N} x_n A_{Nn}
\end{bmatrix}
+ \begin{bmatrix}
x_1 A_{11} + x_2 A_{21} + \ldots + x_N A_{N1} \\
x_1 A_{12} + x_2 A_{22} + \ldots + x_N A_{N2} \\
\vdots \\
x_1 A_{1N} + x_2 A_{2N} + \ldots + x_N A_{NN}
\end{bmatrix}
\]
\[
= [A]\bar{x} + [A]^T \bar{x} 
\]
\[
= ([A] + [A]^T) \bar{x}
\]

Corollary 2.1.3. Let \([A] \in \mathbb{C}^{N \times N}\) and \(\bar{x} \in \mathbb{R}^{N \times 1}\). Furthermore, let \([R], [X] \in \mathbb{R}^{N \times N}\)
where \([A] = [R] + j[X]\). Also, let \(f(\bar{x}) = \langle \bar{x}, [A]\bar{x} \rangle\). Then by Theorem 2.1.2,
\[
\frac{\partial f}{\partial \bar{x}} = ([R] + [R]^T) \bar{x} + j ([X] + [X]^T) \bar{x}
\]
For a Hermitian matrix \([A]\), Corollary 2.1.3 implies that
\[
\frac{\partial f}{\partial \bar{x}} = 2[R] \bar{x}
\]
(2.1.3)
since, by definition, for any Hermitian matrix \([A] = [R] + j[X], [R] = [R]^T ([R] is
symmetric) and \([X] = -[X]^T ([X] is skew-symmetric).

Now, we are finally ready to connect the Lagrange multiplier problem to the
generalized eigenvalue problem (GEP).
Let $\bar{J} \in \mathbb{R}^{N \times 1}$ and $[N], [D] \in \mathbb{C}^{N \times N}$. Furthermore, let $[N]$ and $[D]$ be Hermitian matrices and $[D]$ be a positive definite matrix. The following functional defines the Lagrange multiplier problem, assuming that the scalar $C$ is a real, positive constant:

$$f(\bar{J}) = \langle \bar{J}, [N]\bar{J} \rangle - \lambda \left( \langle \bar{J}, [D]\bar{J} \rangle - C \right)$$

From Equation 2.1.3, the gradient of the functional is

$$\frac{\partial f}{\partial \bar{J}} = 2[N]\bar{J} - 2\lambda[D]\bar{J}$$

At the extrema of $f$, $\frac{\partial f}{\partial \bar{J}} = 0$. Let us denote the minimum or maximum points as $\bar{J}_n$ and the associated multiplier as $\lambda_n$. Evaluating the functional $f$ at an extreme point, we obtain:

$$\frac{\partial f}{\partial \bar{J}} = 0 = 2([N] - \lambda_n[D])\bar{J}_n$$

which simplifies to the generalized eigenvalue problem:

$$[N]\bar{J}_n = \lambda_n[D]\bar{J}_n$$

### 2.1.4 Generalized Eigenvalue Problem

The points $\bar{J}_n$ at the extrema of the Lagrange multiplier problem were shown to reduce to a generalized eigenvalue problem

$$[N]\bar{J}_n = \lambda_n[D]\bar{J}_n$$

under the condition that $[N]$ and $[D]$ are Hermitian matrices and that $[D]$ is furthermore a positive definite matrix and that the extrema coordinates $\bar{J}_n$ (i.e. eigenvectors) are real. Since the method of Lagrange multipliers optimizes a real functional, the multipliers (i.e. eigenvalues) are also real.

Also, by the complex spectral theorem [93, pg. 296], the eigenvectors are both $N$ and $D$ orthogonal. That is, for $m \neq n$:

$$\langle \bar{J}_m, [N]\bar{J}_n \rangle = 0 = \langle \bar{J}_m, [D]\bar{J}_n \rangle$$
Finally, notice then that the eigenvectors $\bar{J}_n$ (extrema in the method of Lagrange multipliers), as used here, are equal to the stationary points of the generalized Rayleigh quotient 2.1.1 by construction. Evaluating the Quotient at these stationary points yields the eigenvalue:

$$\rho(\bar{J}_n) = \frac{\langle \bar{J}_n, [N] \bar{J}_n \rangle}{\langle \bar{J}_n, [D] \bar{J}_n \rangle} = \lambda_n$$

2.1.5 Conceptual Implications

The generalized eigenvalue problem (GEP) has been shown to be related to both a particular Lagrange multiplier problem (under certain conditions), which in turn is connected to the stationary points of a generalized Rayleigh quotient problem. In other words, the GEP computes stationary points of the Rayleigh quotient, which is itself just another functional. If we define the Quotient in a physical way that makes its stationary points interesting for analysis and/or design, then both the eigenvectors and eigenvalues of the GEP gain physical meaning. The theory of characteristic modes has used this result very effectively.

2.2 Characteristic Mode Theory

The theory of characteristic modes, as formulated in [22], and related modal analysis systems are defined by a generalized eigenvalue problem. This section will review the versions of characteristic mode analysis and make some generalizations about what makes a version of characteristic mode analysis useful. There are some useful details omitted here which are noted in the various appendices.
2.2.1 Classical CM

The theory of characteristic modes, also known as the theory of classical characteristic modes (CCM) to distinguish it from later modal systems, is defined by the following Rayleigh quotient for perfectly conducting bodies:

\[
\rho_{\text{CCM}}(\vec{J}) = \frac{\langle \vec{J}, X \vec{J} \rangle_S}{\langle \vec{J}, R \vec{J} \rangle_S} \tag{2.2.1}
\]

where

\[
\vec{E}^i = Z(\vec{J})
\]

which defines \( Z = R + jX \) as the EFIE (electric field integral equation) operator that relates the tangential component of the applied electric field to the conductor surface current. If the \( Z \) operator is discretized using subsectional basis functions and Galerkin’s method in a Method of Moments (MoM) [21] scheme, then, the operator \( Z \) is effectively approximated by the symmetric (not Hermitian) \( N \times N \) matrix \([Z] = [R] + j[X]\), where \([R]\) and \([X]\) are real. The applied electric field tangential to the conductor surfaces is approximated as \( \vec{E}^i \rightarrow \vec{E}^i \), while the surface current density is approximated as \( \vec{J} \rightarrow \vec{J} \), both \( N \) element-long column vectors.

Minimizing this functional using the discrete analogs implies the following generalized eigenvalue problem:

\[
[X] \vec{J}_n = \lambda_n [R] \vec{J}_n \tag{2.2.2}
\]

Each eigenvector (known as an eigencurrent or modal current) \( \vec{J}_n \) represents a modal surface current. Each eigenvalue represents the modal stored reactive power.
relative to the stored radiated power, or $Q/\omega$:

$$P_{\text{stored}} = \frac{1}{2} \langle J_n, [X] J_n \rangle$$

$$P_{\text{rad}} = \frac{1}{2} \langle \tilde{J}_n, [R] \tilde{J}_n \rangle$$

$$\lambda_n = \frac{P_{\text{stored}}}{P_{\text{rad}}} = \frac{\langle \tilde{J}_n, [X] \tilde{J}_n \rangle}{\langle J_n, [R] J_n \rangle}$$

Since both $[R]$ and $[X]$ are real symmetric matrices by construction and $[R]$ is positive-definite\(^2\), both the eigenvalues and eigenvectors are real. More importantly, the eigenvectors simultaneously diagonalize both $[R]$ and $[X]$. Since the eigencurrents are not unique without some normalization, CCM normalizes each eigencurrent to radiate unit power: $\langle \tilde{J}_m, [R] \tilde{J}_n \rangle = \delta_{mn}$. This functional physically represents the real power shared between two eigencurrents $\tilde{J}_m$ and $\tilde{J}_n$ on a lossless radiator, which implies orthogonal modal fields [22] (see Appendix A for a detailed exposition):

$$\langle \tilde{J}_m, [R] \tilde{J}_n \rangle = \frac{1}{\eta_0} \langle \vec{E}_n, \vec{E}_n \rangle_{\Sigma} = \delta_{mn}$$

where $\vec{E}_k(\theta, \phi)$ is the modal far electric field radiated by $\vec{J}_k$, $\eta_0$ is the free-space wave impedance, and $\Sigma$ is the closed spherical surface at infinity. The leftmost equality is guaranteed as a form of conservation of power. More specifically, it is a statement of Parseval’s relation in that the power in the source region is equal to the power in the far-field region, since the far-field transform is unitary for lossless media [37].

With the above modal orthogonalities, it can be shown [22, 23] that any arbitrary surface current density coefficient vector $\vec{J}$ (called a ”current” for simplicity of reference, unless otherwise noted) may be expanded as a weighted sum of modal currents (see Appendix C for more details):

$$\vec{J} = \sum_{n}^{N} \alpha_n \tilde{J}_n = \sum_{n}^{N} \frac{\langle \tilde{J}_n, \vec{E}_i \rangle}{1 + j \lambda_n} \tilde{J}_n$$

\(^2\)Assuming the radiator has no cavity or internal modes, which do not radiate any power [22].
It follows from linearity that any far electric field may also be expanded as a weighted sum of modal electric fields (see Appendix D for more details):

\[ \vec{E} = \sum_{n}^{N} \langle \vec{E}_{n}, \vec{E} \rangle_{\Sigma} = \sum_{n}^{N} \alpha_n \vec{E}_{n} \]

Since both the eigencurrents and eigenvalues are real, classical characteristic mode analysis is uniquely suited to performing a modal decomposition of antenna input admittance [31]. In particular, classical characteristic modes can succinctly describe the phenomena of series and parallel resonance [61].

### 2.2.2 Inagaki CM

Whereas classical characteristic mode theory essentially constructs an \( R \)-orthogonal set of modes in coincident source and field regions (\( \Sigma \) naturally contains \( S \)), Inagaki modes [35], [36] define a family of orthogonal eigenvectors in potentially non-coincident source and field regions. Specifically, Inagaki modes are defined by the following Rayleigh quotient:

\[ \rho_{\text{ICM}} = \frac{\langle s(\vec{J}), s(\vec{J}) \rangle_{S}}{\langle \vec{f}(\vec{J}), \vec{f}(\vec{J}) \rangle_{R}} \]  

(2.2.4)

If we let \( \vec{f} = G(\vec{J}) \), then \( G \) is the operator which maps \( \vec{J} \), a source-dependent variable, in the region \( S \) into the field \( \vec{f} \) in the region \( R \), where \( S \) and \( R \) are not necessarily coincident. Assuming a reciprocal medium enveloping both \( S \) and \( R \), the operator \( G \) is symmetric: \( \vec{f}_A \cdot G(\vec{f}_B) = \vec{f}_B \cdot G(\vec{f}_A) \). In this case, the Quotient simplifies to:

\[ \rho_{\text{ICM}}(\vec{J}) = \frac{\langle \vec{s}(\vec{J}), \vec{s}(\vec{J}) \rangle_{S}}{\langle \vec{G}(\vec{J}), \vec{G}(\vec{J}) \rangle_{R}} = \frac{\langle \vec{s}(\vec{J}), \vec{s}(\vec{J}) \rangle_{S}}{\langle \vec{J}, (G^{\dagger}G)(\vec{J}) \rangle_{S}} \]  

(2.2.5)

where \( G^{\dagger} \) is the adjoint of the operator \( G \).
Significantly, the region $R$ could be in the far-field or the near-field of an antenna. This technique has been used for near-field focusing [37], as well as pattern synthesis [36].

For the purpose of applying this type of characteristic mode analysis to perfectly conducting bodies, the functional may be further simplified by restricting the source of the surface of the antenna using the EFIE:

$$
\vec{s} \rightarrow \vec{E}_{\text{tan}} = Z(\vec{J})
$$

We can ease computation by approximating the operator $Z$ using MoM and subsectional basis functions, thereby transforming $\vec{E}_{\text{tan}} \rightarrow \vec{E}^i$, $\vec{J} \rightarrow \vec{J}$, and $G \rightarrow [G]$. Thus, the Quotient is now:

$$
\rho_{\text{ICM}}(\vec{J}) = \frac{\langle [Z] \vec{J}, [Z] \vec{J} \rangle}{\langle \vec{J}, [G]^H [G] \vec{J} \rangle} = \frac{\langle \vec{J}, [Z]^H[Z] \vec{J} \rangle}{\langle \vec{J}, [G]^H [G] \vec{J} \rangle}
$$

We recognize that the stationary points of the quotient can be computed using the following generalized eigenvalue problem (see Section 2.1):

$$
([Z]^H[Z]) \vec{J}_n = \lambda_n ([G]^H[G]) \vec{J}_n
$$

Since the matrices $[Z]^H[Z]$ and $[H] \equiv [G]^H[G]$ are positive semidefinite Hermitian matrices, the modal surface currents $\vec{J}_n$ are H-orthogonal and the eigenvalues $\lambda_n$ are real.

Thus, the Inagaki modes successively minimize the source field intensity with respect to the field intensity in region $R$.

### 2.2.3 Generalized CM

Generalized characteristic modes [37], [38] are yet another type of modal analysis in the characteristic mode family. They aim to be a more general form of the original
characteristic modes. They are related to Inagaki modes [37] and are defined by the following Rayleigh quotient

$$\rho_{GCM}(\vec{J}) = \frac{\langle \vec{J}, [X] \vec{J} \rangle}{\langle \vec{J}, [G]^H[G] \vec{J} \rangle}$$ \hspace{1cm} (2.2.8)$$

where we have used MoM to approximate the surface current density $\vec{J}$ as $\bar{J}$ and the EFIE operator $Z$ as $[Z] = [R] + j[X]$ using the Galerkin method and subsectional basis functions. By construction, $[X]$ is a real symmetric matrix, while $[G]^H[G]$ is a positive semidefinite Hermitian matrix.

It creates a set of modal fields that are orthogonal over a certain field region $R$ through $[G]$, just like Inagaki modes, while also ensuring that the same modes are $X$-orthogonal over the source region, just like classical characteristic modes. The stationary points of Eq. 2.2.8 are computed through the usual associated generalized eigenvalue problem because of the aforementioned properties of $[X]$ and $[G]^H[G]$:

$$[X] \bar{J}_n = \lambda_n[G]^H[G] \bar{J}_n$$ \hspace{1cm} (2.2.9)$$

Thus, the modes successively minimize the modal reactive power with respect to the field intensity in region $R$. If the region $R$ is set to $\Sigma$, then it can be shown that $[G]^H[G] = \eta_0[R]$, where $\eta_0 = \sqrt{\mu_0 / \epsilon_0}$ is the background wave impedance and $[R]$ is the real part of $[Z]$ [37]. Thus, generalized characteristic modes essentially reduce to classical characteristic modes when the region $R$ is set to the entire sphere at infinity.

A typical use in the literature of generalized characteristic modes [38] is to define the operator $G$ as:

$$W(\theta, \phi) \vec{E}(\theta, \phi) = G(\vec{J})$$

where $W(\theta, \phi)$ is some real weighting function on the far-field pattern $\vec{E}(\theta, \phi)$. 

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2.2.4 General Requirements of Modal Systems

From the previous sections, it may be observed that the modal systems have several common features. First, the modal systems should feature orthogonal modes defined over some source region. The modes are defined mathematically through a generalized Rayleigh quotient, and more specifically, through two matrices. Both matrices must be Hermitian and one matrix (the one in the denominator of the quotient) should be positive definite.

Beyond mathematics, the quotient itself should be a physically meaningful ratio of two distinct quantities. In the case of classical CM, the ratio is the power stored by a mode relative to the power radiated by that mode. In the case of Inagaki CM, the ratio is the field intensity at the source relative to the field intensity at some other remote region.

These properties will be used in developing a new modal system optimized for optimizing the mutual coupling in a multiple antenna system.

2.3 Outline of Approach

The problem considered herein is that there is a single antenna (herein termed the source antenna) under consideration which can be changed. The number and location of feed ports on this source antenna is unknown. The remaining structures and antennas are fixed and are fully known. The antennas are (electrically) near each other. The challenge is to identify the necessary geometric modifications to the source antenna. A pair of modal systems are proposed for multiple antenna systems. The first plays the role of classical CM, but for only one antenna in the system radiating in the presence of the others. We shall refer to this modal system alternately as Subsystem Classical CM (SCCM) or Radiation Modes. The second modal system
shall analyze the power coupled from the source antenna to another target antenna (or antennas) in the multiple antenna system. We shall refer to this second modal system alternately as Target Coupling CM (TCCM) or Coupling Modes.

2.4 UCM Software

Underlying all these computations is necessarily software. Whereas characteristic mode software in the past has apparently focused on implementing a particular type of characteristic mode analysis for a particular numerical electromagnetics solver, the work undertaken here requires considerable generality in both the type of characteristic mode analysis and the numerical solver.

2.4.1 Architecture

Like waveguide modes, characteristic mode-based analysis is physical, but it is usually perceived to be tied to a particular piece of software or electromagnetic simulation technique. While characteristic mode analysis utilizes various numerical routines to enable general analysis of radiating structures, it is not fundamentally numerical in nature. Thus, it is the software implementation of such analysis which can give the analyst the first (incorrect) impression that characteristic modes are numerical in nature.

To overcome this impression and explore new types of characteristic mode analysis, there should be a modern software architecture which presents a unified interface to analyze characteristic modes, regardless of the simulator which provides the raw data for the analysis. It should be flexible with respect to the simulator, so that the simulator becomes almost entirely abstracted away, hopefully leaving the impression that the characteristic modes are physical and defined by a particular generalized eigenvalue problem. Such architecture is difficult to preconceive in the abstract, but
after a few years and two attempts, Figure 2.1 has emerged as a useful software architecture.

![UCM Analysis Software Architecture Diagram](image)

**Figure 2.1: UCM analysis software architecture**

The above architecture has been implemented in MATLAB [92] and should be compatible with versions R2008a and newer. It makes use of the object-oriented facilities provided by MATLAB since R2008a, so as to more easily enforce the above layer relationships and abstractions.

The architecture is broken down in several components, each of which is essentially layered atop each other. At the lowest layer, the simulation data source is a class which implements the abstract class interface (a guaranteed list of services provided by all classes claiming to be valid simulation data sources) specified in

---

3MATLAB R2009b has a bug which makes it more difficult to use layered object-oriented architectures, so it is recommended that the reader use a different version of MATLAB.
UCMSimInterface.m. It is typically completely specific to the particular numerical electromagnetic solver. At the time of this writing, there are four simulation engines supported: ESP5 [95], FEKO 5 [96], Agilent Momentum [97], and a generic engine for performing analysis on multiport Touchstone files (a common file format for storing multiport network parameters).

In Layer 1, the eigenvalues, eigenvectors, eigenpatterns, and even characteristic near-fields are computed. The majority of this work is defined through a simulation-engine agnostic class UCMLayer1Foundation.m, but limited customization is offered through specific Layer 1 classes. All such classes must satisfy the abstract interface class UCMLayer1Interface.m. Examples of specific customizations are offering features and optimizations which are specific to the particular simulation engine. Additionally, certain features in the foundation class are configured here and user-defined preferences set. A particularly important preference is selecting the generalized eigenvalue problem solution method. There are three possible solution methods and they are all discussed in the subsequent section 2.4.2. Also important is code which compensates for numerical inaccuracies in the eigenpattern and characteristic near-field computations. Those compensations are discussed in D. Lastly, the eigenvalue tracking algorithm discussed in Chapter 4 is implemented in the Layer 1 foundation class.

To support a new simulation engine, a new data source class must be created, along with a new Layer 1 class configuring the Layer 1 foundation class to work properly with the new simulation engine.

While Layer 1 supports the computation of the various modal quantities, the modal system itself is defined through the Analysis module layer. The Analysis module defines both matrices involved in the generalized eigenvalue problem, along with all the necessary machinery to compute those matrices if necessary. Since each
Analysis module can communicate the simulation data source, if needed, it is possible for it to be simulator-specific, so it can declare itself to be compatible with a list of Layer 1 classes. As an analyst works on a problem, it may be necessary for him or her to view the problem through the lens of several different types of characteristic mode analysis types, so it is easy to switch between modules. There have been many Analysis modules created at the time of this writing, but they all must obey UCMAnalysisInterface.m, supported by the common functionality found in UCMAnalysisFoundation.m.

The Layer 2 class is responsible for performing all modal analysis functions. It is both simulator-agnostic and Analysis-agnostic. Layer 2 cooperates with Layer 1 and the analysis module to collect all data necessary to compute quantities like modal weights $\alpha_n$ and total current or pattern from a weighted summation of modes. Layer 2 is also responsible for plotting quantities such as eigenvalue magnitude $\lambda_n$, characteristic angle ($\pi - \tan^{-1} \lambda_n$), and eigencurrents in 3D. While all the lower layers feature some ability to be reconfigured depending on the simulator or modal analysis type, there is only one Layer 2 class.

Other classes exist which process the entire stack, such as a diagnostics module designed to evaluate the accuracy of the computed modes UCM_Diagnostics.m, and various Layer 3 classes which implement proposed design techniques. A prominent Layer 3 class implements the computer-aided feed design algorithm featured in Chapter 5.

A notable omission here is that no graphical user interface has been designed or implemented. In my opinion, such an interface can only come after some knowledge of the typical CM analysis workflow is obtained. It is hoped that the UCM software may find wider use so that such an interface may be developed.
2.4.2 Computing the Generalized Eigenvalue Problem

There are some numerical considerations which should be reviewed when computing the GEP. Notice that $[D]$ was constrained to not only be a Hermitian matrix, but also a positive-definite matrix. If $[D] \in \mathbb{C}^{N \times N}$ is positive-definite, then by definition $\langle \tilde{J}, [D] \tilde{J} \rangle > 0 \ \forall \ \tilde{J} \in \mathbb{C}^{N \times 1}$ except when $\tilde{J} = \tilde{0}$ (the null set). If $\tilde{J} = \tilde{0}$, then $\langle \tilde{J}, [D] \tilde{J} \rangle = 0$.

From the definition of a positive-definite matrix and the spectral theorem [93, pg. 296], we may infer that all the eigenvalues of $[D]$ should be positive, if only slightly larger than 0. As mentioned in [23], the small eigenvalues of $[D]$ may become slightly negative due to numerical error. There are two ways to address this problem in the literature: the method discussed in [23], and the method discussed in [51].

In [23], the problem is addressed for a real symmetric matrix $[D]$. This method seeks to compute the eigenvalues and eigenvectors of an associated ordinary eigenvalue problem. This associated problem’s matrix is generated by using the portion of the $[D]$ matrix only eigenvalues of $[D]$ larger than some positive threshold. The eigenvalues and eigenvectors of this associated ordinary eigenvalue problem are then mapped back to $\lambda_n$ and $\tilde{J}_n$. While the original GEP allows up to $N$ modes, this method essentially computes only a subset of those modes, which can sometimes cause problems when performing eigenvalue tracking. In those cases, reducing the number of modes tracked is the only solution.

In [51], the problem is addressed for a complex Hermitian matrix $[D]$. This method depends upon [52] to compute something called the nearest positive definite pair of matrices $[N]$ and $[D]$. That is, the matrix pair $[N]$ and $[D]$ are both modified according to a procedure to force $[D]$ to be a positive definite matrix. The results of the modified $[N']$ and $[D']$ are mapped back to the original problem, pretending as though $[D]$ was a positive-definite matrix. Accuracy is reported to be substantially
improved for certain test cases in [51]. From our testing, this method should be limited to minor adjustments of $[D]$. That is, if only a few eigenvalues of $[D]$ were slightly negative, then this approach should work well. If there are many eigenvalues of $[D]$ that are slightly negative or a few eigenvalues of $[D]$ are quite negative, then Harrington’s method works better, despite restricting the modal analysis to a subset of the total number of modes.

2.4.3 Computing Modal Weighting Coefficients

Computing the modal weighting coefficients $\alpha_n$ in the classical characteristic mode system relies upon the well-known formula in 2.2.3. For modal systems in general, however, a different expression was derived in C:

$$\alpha_n = \frac{\langle \bar{J}_n, [M] \bar{J} \rangle}{\langle \bar{J}_n, [M] \bar{J}_n \rangle}$$

(2.4.1)

where $[M] = [N]$ or $[M] = [D]$ and $[N] \bar{J}_n = \lambda_n [D] \bar{J}_n$. Another expression for computing the modal weighting coefficients from theoretically orthogonal modal far-fields was derived in D, along with some important numerical considerations:

$$\alpha_n = \frac{\langle \vec{E}_n, \vec{E} \rangle_\Sigma}{\langle \vec{E}_n, \vec{E}_n \rangle_\Sigma}$$

(2.4.2)

where

$$\langle \vec{E}_A, \vec{E}_B \rangle_\Sigma = \int_0^{2\pi} \int_0^\pi \vec{E}_A^* \cdot \vec{E}_B \sin \theta d\theta d\phi$$

Again, these expressions are defined so that they are applicable to general modal systems.

2.4.4 Performance Considerations

While the purpose of the UCM software is to provide a general framework to perform characteristic mode analysis on arbitrary geometries using a variety of simulation
engines and a variety of analysis types, its performance characteristics are important, especially considering that it is written in M, in the native language of Matlab.

After the first prototypes of the UCM software were created, several common computations were profiled (i.e. eigenvalue tracking, computing $\alpha_n$ from the total current in classical characteristic modes, etc.). The performance was found to be predominately determined by disk I/O. The CPU performance was improved by redesigning a few data structures.

**Disk Performance**

Most of the computation time for a given problem was spent on disk I/O. Specifically, loading MoM Z matrices from disk and saving eigenvectors and eigenpatterns to disk consumed a significant amount of time. To address this problem, a persistent associative array class interface called *SimpleDatabaseInterface.m* was developed. Two classes implement the interface, *FlatFileDatabase.m*, a 100% native Matlab class, and *MySQLDatabase.m*, a class which utilizes a MEX program to connect to a MySQL database (local or remote). The performance of both classes was tested and it was found that for typical problems, there was only a slight speed advantage to using MySQL. Therefore, the entire UCM system uses *FlatFileDatabase.m* by default, although it can be changed by simply modifying *DatabasePreferences.m*.

The persistent associative array class *FlatFileDatabase.m* trades off disk loading times for memory consumption. By default, it is limited to consuming approximately 100 MB of RAM or the size of a single item, whichever is larger.

**Visualization Performance**

Since a number of routines require visualizing geometry in Matlab instead of in the simulation data engine’s post-processing software, it was also necessary to improve the visualization performance in Matlab. Specifically, the most convenient functions
available to visualize line segments and surfaces in three dimensions have poor performance for arbitrary geometries using default options. For this reason, the class `Geometria.m` was created. It can visualize a fairly large number of line segments and triangle/quadralateral patches (tested for over 1000 elements) using low-level Matlab graphics routines at interactive frame rates, assuming OpenGL hardware acceleration. It can also import and export several different popular graphics formats.
CHAPTER 3
WIDEBAND CHARACTERISTIC MODE TRACKING

3.1 Introduction

The theory of Characteristic Modes (CM) [22] is a frequency-domain modal analysis of radiating systems. Usually, the modes are numerically computed from a matrix. They may be computed from either network Z-parameters [27] or from the Method of Moments (MoM) generalized impedance matrix [23]. For the purposes of this chapter, we shall simply consider the matrix $[Z]$ as referring to the MoM matrix, but the results apply equally well to the network parameters. We require that $[Z(\omega)]$ be constructed such that it is symmetric, which typically implies the Galerkin method applied to subsectional basis functions in MoM [23].

CM theory is built around a defining generalized eigenvalue problem (GEP). Although the results in this chapter apply equally well to GEPs in other types of Characteristic Mode analysis [36, 38], the GEP associated with classical Characteristic Modes [22] shall be assumed:

$$[X(\omega)]\bar{J}_n(\omega) = \lambda_n(\omega)[R(\omega)]\bar{J}_n(\omega)$$

(3.1.1)

where $[Z(\omega)] = [R(\omega)] + j[X(\omega)]$ is the $N\times N$ complex MoM generalized impedance matrix at radial frequency $\omega$, $\bar{J}_n(\omega)$ is the $N\times 1$ eigenvector associated with mode $n$, and $\lambda_n(\omega)$ is the eigenvalue associated with mode $n$. $\bar{J}_n(\omega)$ is also known as the $n^{th}$ eigencurrent. Since $[X(\omega)]$ and $[R(\omega)]$ are real symmetric matrices and $[R(\omega)]$
is positive definite for lossless structures [23], the eigenvalues \( \lambda_n(\omega) \) are real and the eigenvectors \( \vec{J}_n(\omega) \) are \( X \) and \( R \)-orthogonal. Since it is an important concept, the reader is reminded that if \( \langle \vec{x}_1, [M]\vec{x}_2 \rangle = 0 \) for \( \vec{x}_1 \neq \vec{x}_2 \), then the vectors \( \vec{x}_1 \) and \( \vec{x}_2 \) are termed "M-orthogonal." We shall further assume that \( \vec{J}_n(\omega) \) is normalized such that \( \langle \vec{J}_n(\omega), [R(\omega)]\vec{J}_n(\omega) \rangle = 1 \).

Notice that these results are parameterized by the angular frequency \( \omega \). More specifically, the eigenpair \( \lambda_n(\omega), \vec{J}_n(\omega) \) is parameterized by \( \omega \) because \([X(\omega)]\) and \([R(\omega)]\) are parameterized by \( \omega \). Since these two matrices are provided only at discrete frequencies, practical wideband CM analysis requires the modes at some frequency \( \omega_2 \) to be associated with the modes at \( \omega_1 \), independent of the bandwidth separating \( \omega_1 \) and \( \omega_2 \). This process is termed modal tracking and is the topic of this chapter. It is believed that this is the first work to discuss robust wideband modal tracking in the context of Characteristic Modes.

### 3.2 Modal Tracking

#### 3.2.1 Problem Description

To give some context of the problem, the CM spectrum of a 1.2 meter dipole without tracking is shown in Figure 1. The dipole was divided into 33 segments, which translated into a 32 x 32 MoM Z-matrix at each frequency. All 32 eigenpairs were computed every 1 MHz from 50 to 500 MHz and are ordered at each frequency according to ascending eigenvalue magnitude.

By considering the figure, there are a few properties of a successful modal tracking algorithm which become evident. First, it is difficult to determine how many eigenpairs to compute at a given frequency a priori, since the number depends on how
the eigenvalues evolve over the entire frequency band. Naturally, it is easier to estimate this value for narrow bandwidths than wide bandwidths. The most conservative estimate is to compute and store all available eigenpairs at each frequency: in this case, 32. Second, for accuracy considerations, it is likely that the frequency sweep is relatively fine rather than coarse, so that the various observable modal properties may be observed to smoothly vary over the frequency band; therefore, computing many eigenpairs is intrinsically expensive, whether it is performed using standard algorithms (such as those used by the MATLAB [92] function \texttt{eig}), or more recent iterative methods [98]. This computational expense means that the modal tracking algorithm should not add substantial computational complexity to the overall problem. Lastly, it is observed that the dynamic range of the eigenvalue magnitudes is quite large: for this particular problem, they span -20 dB to almost 90 dB (110 dB range). In this chapter, all eigenvalue and eigenvector computations are computed by the MATLAB function \texttt{eig}.

The problem of associating eigenpairs from two related matrices is not a new problem. In a general way, the algorithms fall into three categories: solving a system of differential equations [99]; tracking eigenvalues [100]; and tracking eigenvectors [101]. Of these classes of algorithms, modal tracking based on tracking eigenvalues in Characteristic Modes cannot work, since some antennas have degenerate (or nearly degenerate) modes when their structures feature some symmetry [56].

Between the remaining algorithm classes, it is observed that modal tracking via the solution of differential equations for large, dense Hermitian matrices is computationally expensive. One of the reasons that the problem is computationally expensive is that most methods perform the tracking using numerical integration of a set of differential equations. In the case of MoM, it can be quite expensive to compute many new matrices at arbitrary frequencies, as would be required by any useful quadrature
algorithm. While it is possible to use interpolation methods [102, 103] to generate an MoM matrix at some intermediate frequency between two closely spaced frequencies, we have found experimentally that tracking via differential equations tends to be computationally expensive when applied to moderately-sized matrices (e.g., 1000 x 1000). Another reason for the higher computational complexity is that eigenvalue tracking algorithms usually assume only a few extreme eigenvalues and eigenvectors will be tracked, which is clearly not necessarily the case, as was observed earlier in Figure 3.1.

This chapter introduces a relatively low-complexity algorithm for robust wideband modal tracking via eigenvectors in the context of CM. To our knowledge, it is the first time such an algorithm has been discussed in connection with CM analysis. It is noted that the proposed algorithm has been successfully applied to over one hundred wideband CM analyses, varying from small $[Z]$ (5 x 5) to moderately large $[Z]$ (1000
x 1000), to verify its operation. The physical rationale for tracking eigenvectors is that while the CM eigenvalue magnitudes can vary rapidly versus frequency (in linear units) because they represent the ratio of stored power to radiated power at any given frequency, the eigenvectors represent modal currents on the surface of antennas. It is reasonable to expect that these modal currents should vary slowly versus frequency and we have found this to be empirically true for modes with eigenvalue magnitudes less than 60 or 70 dB, even around modal resonances. Therefore, a modal current \( \bar{J}_n(\omega_2) \) should resemble \( \bar{J}_n(\omega_1) \), provided that \( \omega_1 \) and \( \omega_2 \) are sufficiently close and that \( \bar{J}_n(\omega_1) \) evolves into \( \bar{J}_n(\omega_2) \).

### 3.2.2 Proposed Tracking Algorithm

Let us assume that we have computed the eigenpairs \( \lambda_n, \bar{J}_n \) at two adjacent frequencies at \( \omega_1 \) and \( \omega_2 \). Specifically,

\[
\begin{align*}
[X(\omega_1)] [\Gamma(\omega_1)] &= [R(\omega_1)] [\Lambda(\omega_1)] [\Gamma(\omega_1)] \\
[X(\omega_2)] [\Gamma(\omega_2)] &= [R(\omega_2)] [\Lambda(\omega_2)] [\Gamma(\omega_2)]
\end{align*}
\]

where \([\Gamma(\omega_k)]\) is the matrix whose columns are formed by the eigenvectors \( \bar{J}_n(\omega_k) \), and \([\Lambda(\omega_k)]\) is a diagonal matrix whose entries are the eigenvalues \( \lambda_n(\omega_k) \). Also, we assume that the eigenpairs at \( \omega_1 \) are ordered properly.

The tracking algorithm should track the evolution of \([\Gamma(\omega_1)]\) into \([\Gamma(\omega_2)]\). There are two general stages to the tracking algorithm: the association stage, where eigenvectors at \( \omega_1 \) and \( \omega_2 \) are determined to be related; and the arbitration stage, where ambiguous relationships between eigenvectors are resolved.

**Association Stage**

We begin by assuming that the eigenvectors at \( \omega_1 \) are sorted and the eigenvectors at \( \omega_2 \) are unsorted. Obviously, this requires the first set of computed eigenvectors
to be initially sorted according to some sort of scheme. A simple scheme is to sort according to ascending eigenvalue magnitude (smallest to largest).

We define a correlation matrix $[C]$ which relates the eigenvectors $\omega_1$ to $\omega_2$:

$$[\Gamma(\omega_2)] = [C][\Gamma(\omega_1)]$$  \hfill (3.2.1)

Thus, $[C]$ is given as:

$$[C] = [\Gamma(\omega_2)][\Gamma(\omega_1)]^{-1}$$  \hfill (3.2.2)

Since $[C]$ is potentially complex from the above definition, we define the correlation matrix instead as:

$$[C] = |[\Gamma(\omega_2)][\Gamma(\omega_1)]^{-1}|$$  \hfill (3.2.3)

The goal of the tracking algorithm is to reduce the correlation matrix $[C]$ to a permutation matrix $[P]$. If the correlation between an eigenvector at $\omega_1$ and an eigenvector at $\omega_2$ is suitably high, then the two vectors are regarded as the same mode. The MAC literature [104] seems to recommend a minimum correlation of about 0.9. In our implementation, as long as only one eigenvector at $\omega_2$ is predominately correlated with an eigenvector at $\omega_1$, we do not check for a minimum correlation between eigenvectors at different frequencies, which has been successful in most cases. If more than one eigenvector at $\omega_2$ is predominately correlated with an eigenvector at $\omega_1$, then some arbitration process must occur. Arbitration will be discussed in the next section. Assuming that a unique mapping is discovered from $[C]$, $[P]$ may be constructed by thresholding $[C]$ (e.g., according to the minimum correlation value) and the tracking algorithm is deemed successful in tracking the evolution of $[\Gamma(\omega_1)]$ into $[\Gamma(\omega_2)]$.

**Arbitration Stage**

Arbitration is necessary in cases where the matrix $[C]$ is unable to resolve the relationships among some subset of $[\Gamma(\omega_1)]$ to some subset of $[\Gamma(\omega_2)]$. The ambiguity
may arise because of insufficiently high correlation among some set of eigenvectors between the two frequencies or because of multiple eigenvectors at $\omega_2$ are apparently mapping to a single eigenvector at $\omega_1$ (or vice-versa). All such problems are numerical and not physical in nature. They arise primarily because $\omega_2 - \omega_1$ is too large, but can also occur because of limited accuracy in the eigenvalue solver. The solution to the latter problem is not explored here. While there are several possible arbitration schemes, we use a combination of two schemes.

The first scheme, called *adaptive tracking* for the purposes of this chapter, is based on the reasoning that if there is ambiguity between some set of eigenvectors at $\omega_1$ and at $\omega_2$, it would probably be resolved by making the frequency step, $(\omega_2 - \omega_1)$, smaller. Thus, adaptive tracking simply subdivides each frequency interval into two pieces and applies the first stage algorithm to track the evolution of the eigenvectors over each subinterval. If any given subinterval has some ambiguity, that subinterval is subdivided again. This scheme requires fast matrix interpolation to generate the $[Z]$ matrices between $\omega_1$ and $\omega_2$, but since we already assume that the original frequency step is somewhat small, linear interpolation is used. Lastly, adaptive tracking is well-suited to recursive and parallel implementations.

The second scheme is based on the concept that if the source of ambiguity is that multiple eigenvectors at $\omega_2$ are being associated with one eigenvector at $\omega_1$, the eigenvector at $\omega_2$ with the highest correlation to the eigenvector at $\omega_1$ is deemed to be the same mode as the eigenvector at $\omega_1$. Then, the process is repeated for each remaining unassociated eigenvector at $\omega_2$ and $\omega_1$, with the restriction that only unassociated eigenvectors may be associated. As an example, let us assume that there are three unassociated eigenvectors at $\omega_1$ and three unassociated eigenvectors at $\omega_2$. Furthermore, let us assume that the unassociated eigenvectors at $\omega_1$ are modes 1, 5, and 7, while the unassociated eigenvectors at $\omega_2$ are sorted as 1, 2, and 3. The
portion of the correlation matrix describing the correlation among these eigenvectors between the two frequencies is given below:

\[
[C] = \begin{bmatrix}
0.9 & 0.8 & 0.7 \\
0.3 & 0.6 & 0.65 \\
0.1 & 0.5 & 0.2 \\
\end{bmatrix}
\] (3.2.4)

Recalling that \([C]\) maps the sorted eigenvectors at \(\omega_1\) to the unsorted eigenvectors at \(\omega_2\), this scheme would map \(\vec{J}_1(\omega_1)\) to \(\vec{J}_1(\omega_2)\), since 0.9, the largest value in the first row, is in the first column. To associate \(\vec{J}_5(\omega_1)\) to a particular eigenvector at \(\omega_2\), we note that the correlation matrix of unassociated eigenvectors is now:

\[
[C] = \begin{bmatrix}
0.6 & 0.65 \\
0.5 & 0.2 \\
\end{bmatrix}
\] (3.2.5)

Therefore, this scheme would map \(\vec{J}_5(\omega_1)\) to \(\vec{J}_3(\omega_2)\), since 0.65 > 0.6. By process of elimination, \(\vec{J}_7(\omega_1)\) is mapped to \(\vec{J}_2(\omega_2)\).

In our algorithm, adaptive tracking is used as the arbitration scheme until the frequency subinterval bandwidth is less than a certain value, in which case the second scheme is applied. Specifically, the value is 0.01 MHz, which works well for many geometries and was empirically chosen after much testing. This choice was made by observing that when arbitration is typically required, the correlation among the ambiguous eigenvectors was low (usually less than 0.6). Such low overall correlation indicates that the unassociated eigenvectors at \(\omega_1\) are only loosely correlated with the unassociated eigenvectors at \(\omega_2\), which implies that the frequency step \((\omega_2 - \omega_1)\) is too large. This problem is best resolved by the first arbitration scheme (adaptive tracking) and not the second.
**Complexity**

It is already assumed that the generalized eigenvalue problem is solved at each frequency (and that most, if not all, eigenvalues and eigenvectors are computed), so it will not be counted here. In terms of complexity, the association stage of the proposed modal tracking algorithm is asymptotically $O(N^3)$ [105], where $[Z]$ is $N \times N$, since an inversion of an $N \times N$ matrix is involved and assuming that we are tracking all $N$ eigenvectors. The first arbitration scheme, adaptive tracking, is also $O(N^3)$, since it simply repeats the association stage at progressively finer frequency steps. The second arbitration scheme (usually the rarer of the two) requires a sorting operation for each unassociated eigenvector at $\omega_2$, which varies in complexity depending on the sort algorithm, but is typically $O(N \log(N))$ [105]. Overall, the typical problem will see a complexity of $O(N^3)$, since the typical problem uses the first arbitration scheme most frequently. To reduce the computational complexity, one can choose to track fewer eigenvectors over the band (say $M$ eigenvectors). Then, the overall tracking algorithm complexity will be reduced to $O(M^3)$ [105], assuming that we use a least squares approach to computing $[C]$ (i.e., pseudoinverse), since $[\Gamma(\omega_i)]$ will be a rectangular matrix and cannot be inverted as in equation 3.2.3.

It is arguable that an algorithm with $O(N^3)$ complexity hardly qualifies as absolutely efficient, but in practice, MATLAB implementations of this algorithm and a differential equation-based tracker [100], when applied over many antenna geometries of varying complexity, have demonstrated that the proposed algorithm finishes faster with lower average memory consumption. The main reason for this result is because the differential equation-based tracker had slow convergence in all cases. While it is possible for improved future differential equation-based trackers to have better performance than the proposed algorithm, our experience has been that the proposed tracking algorithm has reasonable performance with minimal resource consumption.
for problems up to 2000 x 2000 in size and tracking at most 200 eigenvectors simultaneously.

### 3.3 Examples

Several geometries were analyzed using Characteristic Modes and the proposed modal tracking algorithm. All antennas were analyzed using FEKO 5.5 [96] and MATLAB R2009a [92] using double precision on a Core 2 Duo 2.5 GHz machine with Windows 7 Professional 64 bit and 4 GB of RAM.

#### 3.3.1 Linear Wire Dipole

A 1.2 meter dipole antenna was simulated from 50 to 500 MHz in 2 MHz steps. It was divided into 33 segments, which yielded a 32 x 32 \([Z]\) matrix at each frequency. Tracking 32 modes across the frequency band took about 10.1 seconds. The tracked spectrum is illustrated in Figure 3.2. Demonstrating the reduced complexity of tracking fewer modes, tracking just 8 modes across the same frequency range took about 9.2 seconds.

From Figure 3.2, the modal tracking allows us to see that there are four modal resonances, or frequencies when a particular mode’s eigenvalue magnitude becomes much less than 1, between 50 and 500 MHz. After each resonant frequency, the eigenvalue magnitude of each mode tends to converge to about 5 dB. Although not easily noticeable in Figure 3.2, the eigenvalue magnitudes are slightly separated (i.e. the modes are not degenerate) and their magnitudes are slowly decreasing with frequency. The same basic trend can be roughly observed with many different geometries, such as the spherical helix discussed next.
3.3.2 Spherical Helix

The spherical helix antenna presented by S. R. Best [1] was analyzed from 50 to 550 MHz in 2 MHz steps. The geometry is shown below in Figure 3.3: it was divided into 240 segments, which translated into a 250 x 250 \([Z]\) matrix at each frequency. Tracking 40 modes across the frequency band took 394 seconds. The tracked spectrum is shown in Figure 3.4. Tracking just 8 of the modes across the same frequency range took less time: 296 seconds.

Figure 3.4 demonstrates the benefit of accurate modal tracking, as it can be clearly observed that there are three primary modal resonances within the band, with three other minor modal resonances (frequencies where the eigenvalue magnitude has a dramatic trough). This spherical helix antenna was designed to operate using the mode which resonates at about 300 MHz. It can be observed that if this mode is excited with a single feed point along with the mode which resonates at about 380
MHz, there may be a parallel resonance in the input impedance [61], reducing the usable impedance and pattern bandwidth.

3.3.3 Rectangular Microstrip Patch

A 38.4 x 29.5 mm rectangular patch on an infinite slab of 2.87 mm thick lossless FR4 dielectric ($\epsilon_r = 4.4$) was analyzed from 1 to 5 GHz in 5 MHz steps. The geometry is shown below in Figure 3.5: it was divided into 50 triangles and one thin wire segment, which resulted in a 68 x 68 [Z] matrix at each frequency. Tracking 10 modes across the frequency band took 11.1 seconds; the computed spectrum is shown below in Figure 3.6.

As can be seen in Figure 3.6, there are a few frequencies where the eigenvalue magnitudes are discontinuous. These are tracking errors. In this particular case, the [Z] matrix provided by FEKO for triangle-type geometries is not exactly symmetric.
Figure 3.4: Spherical helix antenna eigenvalue spectrum with 40 tracked modes

Figure 3.5: Microstrip patch antenna geometry
Figure 3.6: Patch antenna eigenvalue spectrum with 10 tracked modes

theoretically [106]. Moreover, the \([R]\) matrix is not theoretically positive definite (or even semi-positive definite). These two problems significantly contribute to eigenvector errors, which naturally causes modal tracking errors. The matrices are being forced to be symmetric before the GEP is solved, but the \([R]\) matrix problem is not resolved. While it is possible to use techniques discussed in [23] and [51] to repair any numerical errors in \([R]\) which cause it to become an indefinite matrix, they do not precisely apply here, since the \([R]\) matrix is indefinite by construction. Furthermore, as can be observed in 3.6, some of the computed eigenvalues do not vary smoothly versus frequency. While we are then able to compute the Characteristic Modes when we force the theoretically asymmetric \([Z]\) from FEKO to be symmetric, the resulting symmetric \([Z]\) does not correspond exactly to the original problem. Therefore, it is a numerical artifact, rather than a result of the physics of that particular mode.
3.4 Alternative Algorithm

3.4.1 Formulation

The tracking algorithm works well for nearly all simulated geometries with only minor tracking errors. Since the association stage performs most of the work, it is reasonable to consider an alternate method of computing \( [C] \) which replaces the matrix inverse with a simpler operation. It is possible to formulate \( [C] \) using the Cholesky decomposition of \( [R] \) and the well-known Modal Assurance Criterion (MAC) [104] (also known as direction cosines in other literature [107]). Here, we shall use the direction cosines definition for the MAC (Eq. 13 from [107]).

We begin with the generalized eigenvalue problem evaluated at some frequency. Note that since \( [R(\omega_k)] \) is theoretically positive definite, the generalized eigenvalue problem may be transformed into an ordinary eigenvalue problem by using the Cholesky decomposition of \( [R] \):

\[
\bar{L}_n \equiv [A] \bar{J}_n
\]

\[
[B] \equiv [A]^{-1}
\]

\[
[X] \bar{J}_n = \lambda_n [A]^H [A] \bar{J}_n
\]

\[
[X] \bar{J}_n = \lambda_n [A]^H \bar{L}_n
\]

\[
[B]^H [X] [B] \bar{L}_n = \lambda_n \bar{L}_n
\]

Thus, \( \langle \bar{L}_m, \bar{L}_n \rangle = \delta_{mn} \).

Since the \( \bar{L}_n \) are orthonormal, we can use them to form a permutation matrix at a given frequency: \( [P] = [L]^H [L] \). This property seems to be a good alternative to the correlation matrix proposed earlier. We define an alternative correlation matrix using this property:

\[
[C] = ([A(\omega_2)][\Gamma(\omega_2)])^H ([A(\omega_1)][\Gamma(\omega_1)])
\]
where \([R(\omega_k)] = [A(\omega_k)]^H[A(\omega_k)]\) (i.e., \([A]\) is the Cholesky decomposition of \([R]\)) and \([\cdot]^H\) is the Hermitian transpose operator. To clarify the relationship of this expression to the MAC, let \([L(\omega_k)] = [A(\omega_k)][\Gamma(\omega_k)]\). Then, we can rewrite \([C]\) as

\[
[C] = [L(\omega_2)]^H[L(\omega_1)]
\]

Again, to ensure that \([C]\) is composed of positive real numbers, we take the absolute value instead:

\[
[C] = |[L(\omega_2)]^H[L(\omega_1)]|
\]

The above equation is identical to the MAC.

### 3.4.2 Performance

Unfortunately, while this algorithm appears to be more efficient owing to the Cholesky decomposition [108], it has much slower convergence for a large number of tracked modes. For example, in the case of the spherical helix antenna, the proposed algorithm performed the association stage 522 times. This translated into 522 eigenvalue computations. In contrast, the alternate algorithm performed the association stage 622 times, which translated into 1867 eigenvalue computations. The difference in the number of eigenvalue computations between the two algorithms is primarily due to the number of arbitration stages encountered. In this particular example, the alternate algorithm entered arbitration 3.58 times more frequently than the proposed algorithm. As a result, it took about 58 seconds longer to track 40 eigenvectors than the proposed algorithm.

In all tested cases, the alternate algorithm used arbitration much more frequently than the proposed algorithm. Therefore, the alternate algorithm, while promising theoretically, actually performs more work in practice than the proposed algorithm for a variety of Characteristic Mode analyses on practical antenna designs.
3.5 Summary

This chapter discussed the challenges unique to tracking the eigenvalues and eigenvectors in Characteristic Mode analysis over wide frequency bands. It introduced a new modal tracking algorithm which featured relatively low complexity and has been tested on over one hundred CM analysis projects. The algorithm is concerned with tracking the evolution of eigenvectors over the frequency range. To our knowledge, this is the first time such an algorithm has been discussed in the field of Characteristic Modes. Three example antennas were shown and the algorithm performance described in each case. The proposed algorithm was shown to be faster when tracking fewer eigenvectors. An alternative algorithm was described which promised lower complexity and is related to eigenpair tracking efforts in other disciplines, but practically had poorer performance than the proposed algorithm and is therefore not recommended.

For small to medium sized matrices, the proposed algorithm, as currently implemented in MATLAB, has good enough performance on a modern personal computer. For moderately large matrices (on the order of 1000 x 1000), the implementation would need to take advantage of the algorithms inherent parallel structure to provide tracked eigenpairs within a reasonable amount of time.
CHAPTER 4
COMPUTER-AIDED ANTENNA FEED DESIGN USING CHARACTERISTIC MODES

4.1 Introduction

Antenna design is typically composed of three distinct, but related tasks: geometry and materials definition, feed design, and impedance matching (when necessary). By feed design, we mean the determination of the number and location of feed ports. Usually, the first two tasks are lumped together, as optimum feed design for classical antennas such as spiral antennas or others is part of the overall geometrical design. The last task is sometimes considered independent of the antenna design and is well treated by a variety of techniques for the single-port case [109].

For conformal antennas, an increasingly popular technique for both commercial and military applications, the feed design does not necessarily follow from the geometry definition. The geometry definition necessarily depends upon the structure to which the antenna must be conformal, thereby restricting the antenna geometry. Depending on the structure, the antenna geometry may be substantially different than that of any classical antenna. In these cases, the feed design can be challenging, as the transition from guided waves to induced antenna currents may be quite novel. Ideally, the feed design, like the overall antenna geometry should take advantage of the underlying structure.
The theory of Characteristic Modes (CM), as introduced by Garbacz [20] and reformulated for MoM by Harrington [22], has been instrumental in our research group’s efforts to design conformal antennas for challenging platforms [79, 77]. Like modal methods in other disciplines [11, 14], CM defines excitation-independent modes through an (generalized) eigenvalue problem. These modes depend solely upon the antenna geometry and materials. In CM, the modal surface current densities are orthogonal, which means that the total surface current density $\bar{J}$ can be expressed as a weighted summation of the modal surface current densities $\bar{J}_n$. Each weighting coefficient describes how well a feed design excites its associated modal current. Because of these properties and more, CM are quite suitable for designing the geometry and sometimes allow for intuitive feed design. The next section of this chapter reviews CM theory in more detail.

While CM theory may be used to effectively design antennas to take advantage of challenging platforms, the second antenna design task of feed design is not always clear. This chapter proposes a design technique which can readily provide practical feed designs when guided by a human operator. To the best of our knowledge, this is the first time such a technique has been proposed in the literature.

### 4.2 Background

Characteristic Mode theory, as formulated by Harrington and Mautz [22], [23], utilizes the frequency-varying Method of Moments (MoM) generalized impedance matrix $[Z(\omega)] = [R(\omega)] + j[X(\omega)]$ (with the radial frequency $\omega$ hereafter suppressed for notational convenience) to form a generalized eigenvalue problem:

\[
[X] \bar{J}_n = \lambda_n [R] \bar{J}_n \quad \text{(4.2.1)}
\]
While their initial treatment restricted the possible antenna and scatterer geometries to perfectly conducting, non-resonant metallic structures, later developments extended CM analyses to antennas incorporating lossless and lossy dielectrics \[28], \[29]. Provided that \([Z]\) is an \(N\times N\) symmetric matrix and \([R]\) is positive definite, the set of eigenvalues \(\{\lambda_n\}\) is real and indefinite, the set of eigenvectors (i.e. modal currents) \(\{\bar{J}_n\}\) is real and both \(X\) and \(R\)-orthogonal, and the set of electric fields (i.e. modal patterns) radiated by the modal currents \(\{\vec{E}_n(\theta, \phi)\}\) are orthogonal over the infinite sphere. Specifically, the modal current orthogonality is defined by the following inner product:

\[
\langle \bar{J}_m, [M] \bar{J}_n \rangle = 0 \text{ for } [M] = [R] \text{ or } [M] = [X] \text{ and } m \neq n \tag{4.2.2}
\]

where \(\langle a, b \rangle = \bar{a}^H b\) and \((\cdot)^H\) indicates the conjugate transpose (i.e. Hermitian transpose) of a column vector. The modal pattern orthogonality is defined by the following inner product:

\[
\langle \vec{E}_m(\theta, \phi), \vec{E}_n(\theta, \phi) \rangle_{\Sigma} = \\
\int_0^\pi \int_0^{2\pi} \vec{E}_m^*(\theta, \phi) \cdot \vec{E}_n(\theta, \phi) \sin \theta d\phi d\theta = 0 \text{ for } m \neq n \tag{4.2.3}
\]

where \((\cdot)^*\) indicates conjugation.

The eigenvalue \(\lambda_n\) represents how much power is stored relative to radiated for a particular mode:

\[
\lambda_n = \frac{\langle \bar{J}_n, [X] \bar{J}_n \rangle}{\langle \bar{J}_n, [R] \bar{J}_n \rangle} \tag{4.2.4}
\]

Therefore, when \(\lambda_n < 0\), the mode is termed \textit{capacitive}, while an \textit{inductive} mode features \(\lambda_n > 0\).
The total surface current density $\vec{J}$ can be expressed as a weighted summation of modal currents:

$$\vec{J} = \sum_{n}^{N} \alpha_n \vec{J}_n$$

(4.2.5)

where the modal weighting coefficients (also known as modal excitation coefficients) are defined by the following relationship:

$$\alpha_n = \frac{\langle J_n, E^i \rangle}{(1 + j\lambda_n) \langle J_n, [R]J_n \rangle} = \frac{\langle J_n, [R]J \rangle}{\langle J_n, [R]J_n \rangle}$$

(4.2.6)

We define the impressed electric field as $\vec{E}^i = [Z] \vec{J}$. Typically, it is assumed that the modal currents $\vec{J}_n$ are normalized such that $\langle \vec{J}_n, [R]\vec{J}_n \rangle = 1$. For the purposes of this chapter, it shall be assumed that each modal current is so normalized. Additionally, the total pattern may be similarly decomposed into a weighted summation of modal patterns:

$$\vec{E}(\theta, \phi) = \sum_{n}^{N} \alpha_n \vec{E}_n(\theta, \phi)$$

where

$$\alpha_n = \frac{\langle \vec{E}_n, \vec{E} \rangle_{\Sigma}}{\langle \vec{E}_n, \vec{E}_n \rangle_{\Sigma}}$$

(4.2.7)

and

$$\langle \vec{E}_A, \vec{E}_B \rangle_{\Sigma} \equiv \int_{0}^{2\pi} \int_{0}^{\pi} \vec{E}_A^*(\theta, \phi) \cdot \vec{E}_B(\theta, \phi) \sin(\theta) d\theta d\phi$$

where $(\cdot)^*$ indicates complex conjugation.

Notice that $\{\lambda_n\}$, $\{\vec{J}_n\}$, and $\{\vec{E}_n(\theta, \phi)\}$ depend solely on the antenna geometry and materials because they are uniquely defined by the generalized eigenvalue problem 4.2.1, while $\{\alpha_n\}$ depend upon all these quantities and the excitation $\vec{E}^i$. Specifically, since the excitation $\vec{E}^i$ depends upon the feed design, the $\{\alpha_n\}$ describe how much power is coupled from the feed port(s) to each modal current.

For the remainder of this chapter, it shall be assumed that the Characteristic
Modes are computed for perfectly conducting, non-resonant antenna structures using subsectional basis functions and the Galerkin formulation of MoM.

4.3 Theory

After the design of the antenna geometry is complete for a given design iteration, the characteristic modes are fixed, since the sets \( \{\lambda_n\} \), \( \{\bar{J}_n\} \), and \( \{\vec{E}_n(\theta, \phi)\} \) are functions of geometry alone. This implies that the next major challenge in the antenna design is to determine \( \bar{E}_i \). With \( \bar{E}_i \) known, \( \{\alpha_n\} \) are completely determined. \( \bar{E}_i \) is a function of three variables: number of feed ports, feed port locations, and feed port excitations. This triplet is herein referred to as the ”feed arrangement.” If we consider \( \bar{E}_i \) to be unknown, then \( \{\alpha_n\} \) is also unknown. If \( \{\alpha_n\} \) is unknown, then \( \bar{J} \) (the total surface current density) is also unknown. This makes sense, since if \( \bar{J} \) is known exactly, then \( \bar{E}_i \) is determined exactly through \( \bar{E}_i = [Z]\bar{J} \).

Despite the fact that the antenna feed arrangement is unknown in this scenario, there still are some pieces of information that are known. First, if the structure is adequately meshed with subsectional basis functions, then any realistic feed arrangement will result in an \( \bar{E}_i \) consisting of only a few non-zero components; that is, \( \bar{E}_i \) is a sparse vector such that \( 0 < |\bar{E}_i|_0 <\ll N \). Secondly, based on the CM analysis of the structure, a few modal coefficients \( \alpha_n \) can be regarded as known, if only approximately. Alternatively, these coefficients could be considered as the desired modal coefficients of the realized feed arrangement. For example, the designer may identify over a particular frequency range that modes 1 and 3 should be strongly excited, while modes 2 and 4 should be suppressed. Thus, one possible approximation for
some of the $\alpha_n$ is:

$$\alpha_1(\omega) \approx 1 \quad \alpha_2(\omega) \approx 10^{-30}$$
$$\alpha_3(\omega) \approx 1 \quad \alpha_4(\omega) \approx 10^{-30}$$

It is important to recognize that since $\bar{E}_i^i$ is regarded as unknown, $\{\alpha_n(\omega)\}$ is unknown as a whole. That is, the designer can, at best, have only approximate knowledge of $\{\alpha_n(\omega)\}$. In summary, the two pieces of knowledge that the designer has of the desired feed arrangement are that $\bar{E}_i^i$ is a sparse vector (and probably frequency-independent), and some properties of the desired total current $\bar{J}(\omega)$ are approximately known (by means of a few approximately known $\alpha_n(\omega)$).

### 4.3.1 Mathematical Formulation

To formulate the problem in mathematical terms, we begin by specifying the assumptions. Specifically, we assume to approximately know $K \leq N$ modal weighting coefficients at $P$ discrete frequencies. We also assume that there are at most $M$ suitable locations at which feed ports may be placed on the mesh. Note that $M$ may be greater than $N$ for certain meshes, since $N$ is the number of basis function coefficients which are not necessarily associated with every possible feed port location.

From the definition of the modal weighting coefficient 4.2.6, we have at each frequency:

$$\bar{\alpha} = [\bar{J}_n]^H([R] \bar{J})$$

where $\bar{\alpha}$ is the $K \times 1$ column vector of approximately known modal coefficients, and $[\bar{J}_n]$ is the $N \times K$ matrix whose columns are formed from the eigencurrents associated with $\bar{\alpha}$.

The total current $\bar{J}$ may be expressed as:

$$J = [Z]^{-1}E^i = [Y]E^i$$
Let $[F]$ be the $N \times M$ frequency-independent matrix, which maps the $M$-element port voltage vector $\vec{\gamma}$ to $\vec{E}^i$:

$$\vec{E}^i = [F] \vec{\gamma}$$

Thus, at each frequency, we finally have the following system of equations:

$$\vec{\alpha} = [\vec{J}_n]^H [R][Y][F] \vec{\gamma}$$

For clarity, let

$$[A] := [\vec{J}_n]^H [R][Y][F]$$

where $[A]$ is a $K \times M$ matrix. If we assume that the designer has chosen $K < M$, then

$$\vec{\alpha} = [A] \vec{\gamma} \quad (4.3.1)$$

is an underdetermined system of equations. If we assume that the designer has only a vague sense of how many potential feed ports are required to realize the desired $\vec{\alpha}$, it is a good assumption that $0 < |\vec{\gamma}|_0 << M$ (i.e. that $\vec{\gamma}$ is sparse).

Therefore, the ideal solution to solving 4.3.1 is to compute $\vec{\gamma}$ at each frequency such that it is the sparsest possible vector consistent with the observation vector $\vec{\alpha}$. Unfortunately, this is a very difficult problem computationally $[110]$.

Drawing on the literature surrounding compressed sensing $[111]$, it has been shown that in a wide variety of practical cases, one may approximate $|\vec{c}|_0 \approx |\vec{c}|_1$. In other words, instead of solving the problem 4.3.1 using an $\ell_0$ solver, we may approximately solve it using an $\ell_1$ solver. In this chapter, we chose to use a collection of $\ell_1$ numerical solvers packaged as YALL1 developed by Rice University $[112]$. It should be noted that while no optimization techniques have been explicitly employed at this stage in describing the proposed computer aided feed design technique, constrained optimization is often used within the typical $\ell_1$ solver.

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4.3.2 Iterations Required

Note that 4.3.1 is solved at each frequency, so the port voltages $\bar{\gamma}$ will vary with frequency. If we wish to develop a feed arrangement with frequency-independent port voltages, we need to collapse the $P$ port voltage vectors into a single voltage vector. We break this process down into two stages: thresholding, and averaging.

Before examining the two stages, it is important to note that at each frequency the port voltage vector $\gamma$ is normalized according to its largest magnitude voltage. Without this normalization, the iterations can fail to converge upon a single set of frequency-independent port voltages $\bar{\gamma}$.

**Stage 1: Thresholding**

For a particular antenna, the voltage vectors $\bar{\gamma}$ are visualized in Figure 4.1. By examining Figure 4.1, it is apparent that there should be three ports over the frequency range.
range. To make such a determination without manual intervention requires the voltages to surpass some threshold or otherwise have the computer determine horizontal lines in an image. In our implementation, we render the above data as an image (specifically, Figure 4.1) and input it into a Sobel edge detection algorithm [113] to obtain a suitable threshold. Naturally, other approaches can be applied to obtain a suitable voltage magnitude threshold.

Stage 2: Averaging

The voltages shown in Figure 4.1 are averaged over frequency; the result is shown in Figure 4.2. Using the threshold computed from Stage 1, the $Q$ port locations at which there are substantial port voltages are identified. Using these new port locations, a new $[F]$ matrix is formed which maps from the $Q$ port voltages $\gamma'$ to $\bar{E}^i$. Then, $[A]'\gamma' = \bar{\alpha}$ (i.e. Equation 4.3.1 with new $[A]'$ and new $\gamma'$) is solved at each frequency. The process repeats until the port locations have converged.
In summary, the proposed technique computes the number and location of the feed ports, along with a complex, frequency-independent voltage at each port, given some approximate modal properties of the total current over one or more frequency bands of interest. The modal properties are approximate $\alpha_n$ for $M < N$ modes, which may be obtained from some desired far field pattern or a desired total current distribution over the frequency bands of interest.

**Convergence**

It is possible that this process may fail to obtain a frequency-independent set of port voltages $\bar{\gamma}$ because the $\ell_1$ solver fails to compute an answer. We have empirically found this problem to occur when the desired $\bar{\alpha}$ are impossible to realize (i.e. they are self-contradictory as a set). Further studies on convergence will be conducted in future work.

**Considerations on Choice of $\bar{\alpha}$**

It is important to recognize that simply choosing modes whose approximate/desired $\alpha_n$ are large and feeding those values into the proposed procedure can fail to produce the overall desired antenna operation. It is important to specify both the significant modes (i.e. large $|\alpha_n|$) and suppressed modes (i.e. very small $|\alpha_n|$). Otherwise, a realized feed arrangement may indeed excite the specified modes, but may also fail to suppress undesirable modes.

Finally, it is important to not overly constrain the problem 4.3.1 by specifying too many $\alpha_n$. Indeed, as more $\alpha_n$ are specified, the probability increases that the approximate modal weights are inconsistent with any realizable feed arrangement.
4.4 Example: Dipole Antenna

For all examples, the proposed procedure was implemented in MATLAB R2010b [92] and FEKO 5.5 [96] was used to compute the MoM matrices for the given geometry over a frequency range. It should be noted that while all the examples shown here use wires, the method is not intrinsically limited to wire antennas. Antennas meshed using metallic triangles or other surface elements can also be used, but it is more difficult to define a port physically among such mesh elements. For simplicity of presentation, we therefore limit the examples to using wire structures only, but future work will take advantage of more sophisticated structures.

In this particular example, a 1.2 meter z-aligned dipole was meshed into 32 segments and simulated from 50 MHz to 500 MHz. The eigenvalue spectrum for the first 6 modes is shown in Figure 4.3.

![Figure 4.3: 1.2 meter dipole characteristic mode eigenvalue spectrum](image-url)
4.4.1 Design 1

As an initial test, we tried to recover the usual center feed position associated with the dipole’s first mode. The first dipole mode is associated with mode 1 [72] (that is, its eigenpattern is substantially similar to the total pattern at or below the first series resonance of a center-fed dipole). For the desired modal properties, we let:

$$\alpha_1(\omega) \approx 1 \quad \alpha_2(\omega) = \alpha_3(\omega) \approx 10^{-30}$$

from 50 MHz to 150 MHz. After 3 iterations, the algorithm yielded the solution shown in Figure 4.4.

![Figure 4.4: Dipole design 1](image)

This design realizes the modal weight magnitudes shown in Figure 4.5 and produced the radiation patterns seen in Figure 4.6.
Figure 4.5: Dipole design 1 realized modal weight magnitudes

Figure 4.6: Dipole design 1 directive elevation patterns
As is seen directly from Figure 4.6 and indicated by Figure 4.5, the radiation pattern holds according to the desired first dipole mode pattern shape up to about 300 MHz. After that frequency, the pattern degrades as higher order modes are excited by the single feed point.
4.4.2 Design 2

In this design, we explore the question: what if we wanted to excite Mode 1 over more bandwidth? Specifically, let us use the same desired $\alpha_n$ coefficients as in Design 1, but require that they be satisfied from 50 to 400 MHz. In this case, the feed points were computed after 4 iterations and are shown in Figure 4.7. The ports are located at 0.21 meters on either side of the dipole’s midpoint and both are excited with 1 V.

![Dipole design 2](image)

Figure 4.7: Dipole design 2

The realized modal weight magnitudes are shown in Figure 4.8, while the pattern at several frequencies is shown in Figure 4.9.

From Figure 4.9, it is obvious that the pattern maintains its desired single beam at horizon up to 500 MHz, although the beam obvious squints at higher frequencies due to the increased aperture electrical size. Comparing Figure 4.8 with Figure 4.5, it
Figure 4.8: Dipole design 2 realized modal weight magnitudes

Figure 4.9: Dipole design 2 directive elevation patterns
is clear that Design 2 excites primarily Mode 1 up to almost 500 MHz, while Design 1 excites primarily Mode 1 up to only 320 MHz. The tradeoff is that two feed points are required for this extra bandwidth instead of one.

4.4.3 Design 3

In this design, we explore a different question: what if we wanted to excite Mode 2 (which is associated with dipole mode 2)? Specifically, let us define the following desired $\alpha_n$ coefficients from 50 MHz to 400 MHz:

$$\alpha_2(\omega) \approx 1 \quad \alpha_1(\omega) = \alpha_3(\omega) \approx 10^{-30}$$

After 4 iterations, the feed points were computed and are shown in Figure 4.10.

![Figure 4.10: Dipole design 3](image)

The feed points are located 0.3 meters on either side of the dipole’s midpoint. One port is excited with 1V, while the other port is excited with -1V.
Figure 4.11: Dipole design 3 realized modal weight magnitudes

Figure 4.12: Dipole design 3 directive elevation patterns
Examining both the realized modal weights in Figure 4.11 and the pattern cuts in Figure 4.12, it is clear that the second dipole mode has been well-excited by the feed arrangement over the entire frequency range of 50-500 MHz.

4.5 Example: Folded Spherical Helix Antenna

In this example, a more challenging wire geometry designed by S.R. Best from [1] is examined. The four-arm folded spherical helix antenna is shown in Figure 4.13 and its corresponding characteristic mode eigenvalue spectrum is shown in Figure 4.14.

![Four-arm folded spherical helix antenna geometry from [1]](image)

The method proposed in [1] to feed this antenna is to feed at the midpoint of one of the arms (horizontal plane of symmetry). With this single feed point, the input resistance is approximately 50 Ohms and there is very little reactance at 300 MHz,
Figure 4.14: Four-arm folded spherical helix antenna eigenvalue spectrum

while producing an approximately omnidirectional pattern about horizon. The input $S_{11}$ is shown in Figure 4.15, along with the impedance Q (computed using the formula from [114]) shown in Figure 4.16. Additionally, pattern cuts at several frequencies in Figure 4.17. Notice that the pattern shape degraded unacceptably somewhere between 400 and 500 MHz.

4.5.1 Design 1

In this design, we would like to find the feed point(s) to implement the pattern generated by a short dipole, shown in Figure 4.18, between 300 and 500 MHz.

From the desired pattern, equation 4.2.7 is used to compute the $\alpha_n$ for modes 1-4, 7, and 19. These desired $\alpha_n$ are shown in Figure 4.19. This particular list of mode numbers were obtained by both observing the modes with large (1, 7, and 19) and very small magnitudes (2, 3, and 4) in Figure 4.19.
Figure 4.15: Folded spherical helix input reflectance versus frequency

Figure 4.16: Folded spherical helix impedance Q versus frequency
Figure 4.17: Folded spherical helix directive elevation patterns

Figure 4.18: Desired omnidirectional pattern
Figure 4.19: Folded spherical helix design 1 desired modal weight magnitudes

Iteration 1 is shown in Figure 4.20, while iteration 3 is shown in Figure 4.21. The normalized port voltage magnitudes are shown here in relief instead of a two-dimensional representation to emphasize the large voltage magnitudes (the width of each peak is very thin).

After 3 iterations, several ports are located and their respective port excitation voltages computed. The locations are shown in Figure 4.22. The ports at the top of the helix are excited with 1V, while the ports at the bottom of the helix are excited with -1V.

This solution realizes the modal weights shown in Figure 4.23. Various pattern cuts are shown in Figure 4.24. In contrast to the earlier example, note that the pattern does not degrade appreciably until about 550 MHz.

This is certainly a dramatic improvement in pattern bandwidth compared to the original design, but unfortunately, it requires several feed points, which implies a
Figure 4.20: Folded spherical helix port voltages for solution iteration 1

Figure 4.21: Folded spherical helix port voltages for solution iteration 3
Figure 4.22: Folded spherical design 1

Figure 4.23: Folded spherical helix design 1 realized modal weight magnitudes
potentially complex feed network. Examining the overall distribution of feed voltages, however, suggests an improved feed design. Since the ports at the top of the helix are excited out of phase with the bottom ports, it suggests that we could introduce an axial wire connecting the top and bottom of the helix. Then, we could feed the axial wire at its center.

4.5.2 Design 2

An improved design suggested by the feed arrangement of Design 1 is shown below in Figure 4.25.

Using this feed arrangement, we obtain the $S_{11}$ shown in Figure 4.26, the Q shown in Figure 4.27, and the pattern shown in Figure 4.28. The pattern does not degrade significantly until about 780 MHz.

Comparing the Q of the new design in Figure 4.27 with the Q of the original
Figure 4.25: Folded spherical helix design 2 (with axis)

Figure 4.26: Folded spherical helix design 2 input reflectance versus frequency
Figure 4.27: Folded spherical helix design 2 impedance $Q$ versus frequency

Figure 4.28: Folded spherical helix design 2 total elevation gain patterns
design in Figure 4.16 demonstrates that the impedance $Q$ is substantially reduced from 300 to 700 MHz. This generally implies that using one or more matching networks (of useful bandwidth), it may be possible to operate the original design from 300-400 MHz, while the new design can operate from 300 to 600 or 700 MHz. The comparison of the original design with this new axial design is discussed in more detail in separate work [78, 115].

4.6 Summary

In this chapter, a new design technique, which readily provides practical feed designs with minimal guidance from a human operator for arbitrary PEC structures, was introduced. Its theoretical basis relies upon the fact that any practical feed arrangement for an antenna must be sparse relative to the number of basis functions meshing the same antenna. This simple fact combined with some characteristic mode information allows us to construct an underdetermined system of equations and use an $\ell_1$ solver to solve it. From this solution, we can obtain the number, locations, and excitation voltages of one or more feed ports to realize certain modal properties over some specified frequency range, inspired by results from the field of compressed sensing. From many practical designs not shown here, it was verified that using an $\ell_2$ solver in place of the $\ell_1$ solver does not generally converge in a realistic solution.

There are some important properties of this procedure which are fundamental to its function. The fact that this problem comes down to an underdetermined system of equations allows us to use an $\ell_1$ solver is certainly critical to the algorithm’s efficiency, but more interesting is that we have an underdetermined system of equations at all. The system of equations arises because the proposed method uses multiple properties of the desired current in the same domain (i.e. the CM domain). Using modal properties of the desired current allows an almost qualitative description of each
mode’s contribution to the desired current to translate into a practical feed design. This quality of the proposed method is extremely useful in exploring the design space and finalizing designs.
CHAPTER 5
SYSTEMATIC MUTUAL COUPLING REDUCTION THROUGH MODAL ANALYSIS

5.1 Introduction

Multiple antenna systems, such as antenna arrays, (multiple input multiple output) MIMO systems, direction-finding systems, and multiple protocol systems (like a 802.11b antenna coexisting with a Bluetooth antenna) are important applications of antenna technology in today’s marketplace. Especially important are the emerging MIMO systems, since they promise not only improved real-world bandwidth, but also improved signal strength in complex multipath environments [116].

Mutual coupling among these multiple antennas, whether they operate at the same center frequency or different frequencies, is inherent to multiple antenna systems. While some of these systems, especially arrays [117], can be designed to take advantage of mutual coupling, others, such as MIMO or direction-finding systems, usually find mutual coupling to be detrimental. In direction-finding systems, it is possible to process the received signals to compensate for mutual coupling, a topic which has been extensively reported upon in the literature [85], [86]. In MIMO systems, mutual coupling primarily causes a reduction in the overall system efficiency, which reduces the Mean Effective Gain (MEG) and therefore decreases the link capacity [118]. The drop in efficiency is attributable to the power absorbed by one or
more unintended antennas relative to the power absorbed by the intended antenna (i.e. the antenna whose radiation pattern is best suited to the incident wave). Thus, the problems caused by mutual coupling cannot all be removed using post-processing, since some are inherently physical.

As with single antennas, it would be useful to use Characteristic Mode theory to systematically design multiple antenna systems. Since the chief problem unique to multiple antenna systems is mutual coupling (this is in addition to the recognized problems of single antenna systems), whatever proposed methodology should be both general and somehow take into consideration mutual coupling. Computing the Classical Characteristic Modes of these multiple antenna systems is entirely straightforward using the many works referenced in the Introduction chapter, assuming the Method of Moments (MoM) formulation produces a theoretically symmetric generalized impedance matrix \([Z]\), but it fails to explicitly take into consideration the phenomenon of mutual coupling. That is, Classical Characteristic Mode (CCM) theory describes the radiation performance of the entire multiple antenna system, but gives no explicit information on the performance of each constituent antenna. An approach describing the performance of each individual antenna operating in the presence of the other antennas is necessary when operating the system as a collection of independent receivers (as in the MIMO and direction-finding cases) rather than as a single multiport aperture.

In this chapter, two new modal systems are proposed, along with a method to relate them. The first, formally termed Subsystem Classical Characteristic Modes (SCCM) and informally as \(\text{Radiation Modes}\), describes the radiation performance of a single antenna radiating among several other terminated antennas or structures. The second, formally termed Target Coupling Characteristic Modes (TCCM) and informally as \(\text{Coupling modes}\), describes a generalized form of mutual coupling between
a source antenna and one or more target antennas or structures. By decomposing one set of modes in terms of the other, we can understand how an efficient radiation mode may impact mutual coupling, allowing the designer to systematically trade off radiation performance against mutual coupling for general metallic structures.

This chapter is divided into three major sections. First, the theory and properties of each modal system are stated. Next, a few multiple antenna systems are analyzed using the pair of modal systems. Lastly, some important conclusions a particular multiple antenna system described in the literature will be obtained using this new theory, leading to three new designs which minimize the mutual coupling among the member antennas.

5.2 Theory

In systems employing multiple antennas, there are several unique considerations that can otherwise be ignored in single antenna systems: antenna system excitation and mutual coupling.

In a multiple antenna system, all ports on all antennas may be connected to the same transmitting generator or they may all be connected to the same receiver. In this case, when one port is being driven, the remaining ports are also being driven, so using active (or driving point) input impedance to characterize the impedance at each port is necessary. When the antennas are configured in this manner, they are typically being used as an array. When each antenna or port is connected to its own receiver/generator, then when that antenna or port is driven, the other antennas/ports will appear to be terminated with loads. When the antennas are configured in this manner, they are typically being used either independently or collectively for direction finding or MIMO applications.

Mutual coupling is a physical property of both multiple antenna systems and
single antenna systems radiating in the presence of other (parasitic) structures. It fundamentally describes the power transferred from one antenna to one or more other antennas. Physically, one can think of mutual coupling as inducing some current from some source antenna onto another antenna (especially near its ports).

In this chapter, we consider two proposed modal systems: Radiation Modes and Coupling Modes. The radiation modes identify the modal currents on the source structure which radiate efficiently, and the coupling modes identify the modal currents on the source structure which have minimal coupling to the target structure.

To clarify this discussion, a preliminary discussion is necessary. Partitioned matrices are useful to simplify the mathematical and conceptual description of a system of objects. In this section, we are examining the interactions among two structures. In this dissertation, the concept of a structure is not limited simply to a distinct connected set of elements, but can also describe a group of disconnected elements as well. For examine, the term ”structure” could be applied to describe an entire dipole, as well as an Yagi-Uda antenna, which is composed of several disconnected elements. On the other hand, ”structure” may also be used to describe only a portion of a dipole.

5.2.1 Partitioned Matrices: Two Structures

In order to discuss the notion of mutual coupling between a source antenna and other antennas or structures (its environment) using the mathematical framework of MoM, we need to utilize the concept of partitioned matrices.

Without loss of generality, the MoM generalized impedance matrix \([Z]\) may be partitioned into 4 block matrices:

\[
[Z] = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\]
where \( [Z_{11}] \) is the \( P \times P \) generalized impedance matrix of the source antenna alone in free space without neighboring structures, \( [Z_{22}] \) is the \( Q \times Q \) generalized impedance matrix of the other structures alone in free space without the source antenna, and \( [Z_{12}] (P \times Q) \) and \( [Z_{21}] (Q \times P) \) describe the generalized coupling between the source antenna and its environment. Because of reciprocity and the symmetry of \( [Z] \), \( [Z_{21}] = [Z_{12}]^T \).

More generally, we can also partition the excitation coefficients vector \( \vec{E}^i \) and the total surface current coefficient vector \( \vec{J} \):

\[
\vec{E}^i = [Z] \vec{J} = \begin{bmatrix} [Z_{11}] & [Z_{12}] \\ [Z_{21}] & [Z_{22}] \end{bmatrix} \begin{bmatrix} \vec{J}^{(1)} \\ \vec{J}^{(2)} \end{bmatrix}
\]

If we require that the total impressed tangential electric field on the environmental structures is zero (i.e. \( \vec{E}^{i(2)} = \vec{0} \)), we will compute the effective \( [Z] \) matrix for the source antenna in which it is operating in the presence of the environmental structures rather than free space (i.e. \( [Z_{11}] \)). In other words, we will "short" the elements of the environmental structures together, including terminating any ports in the environment with shorts. Thus, we can derive the generalized impedance matrix of the source antenna radiating in the presence of the environmental structures:

\[
\begin{bmatrix} \vec{E}^{i(1)} \\ \vec{0} \end{bmatrix} = \begin{bmatrix} [Z_{11}] & [Z_{12}] \\ [Z_{21}] & [Z_{22}] \end{bmatrix} \begin{bmatrix} \vec{J}^{(1)} \\ \vec{J}^{(2)} \end{bmatrix}
\]

which implies that

\[
\vec{E}^{i(1)} = [Z_{11}]\vec{J}^{(1)} + [Z_{12}]\vec{J}^{(2)} \tag{5.2.1}
\]

\[
\vec{0} = [Z_{21}]\vec{J}^{(1)} + [Z_{22}]\vec{J}^{(2)} \tag{5.2.2}
\]

Solving for \( \vec{J}^{(2)} \) using the second equation 5.2.2, we obtain

\[
\vec{J}^{(2)} = -[Z_{22}]^{-1}[Z_{21}]\vec{J}^{(1)}
\]
Substituting for $\bar{J}^{(2)}$ in the first equation 5.2.1, we obtain

$$\bar{E}^{(1)} = ([Z_{11}] - [Z_{12}][Z_{22}]^{-1}[Z_{21}]) \bar{J}^{(1)}$$

$$= ([Z_{11}] - [Z_{M}^{(1)}]) \bar{J}^{(1)}$$

$$\equiv [Z^{(1)}]\bar{J}^{(1)} \quad (5.2.3)$$

where $[Z^{(1)}]$ and $[Z_{M}^{(1)}]$ are obvious from inspection. Notice that $[Z^{(1)}]$ is composed of two components: $[Z_{11}]$, the source antenna in isolation, and $[Z_{M}^{(1)}]$, which describes the effect of the environment on the source antenna.

Notice that if we wish to include the effect of termination loads in the environment, we need to simply modify $[Z_{22}]$ by some (probably diagonal) load matrix $[Z_{L2}]$ which models the environmental terminations:

$$[Z_{ML}^{(1)}] \equiv [Z_{12}] ([Z_{22}] + [Z_{L2}])^{-1} [Z_{21}] \quad (5.2.4)$$

For the purposes of this dissertation, we shall assume that the environment is terminated with arbitrary loads rather than shorts, so we shall use the following definition of the embedded generalized impedance matrix of the source antenna $[Z^{(1)}]$:

$$[Z^{(1)}] = [Z_{11}] - [Z_{12}] ([Z_{22}] + [Z_{L2}])^{-1} [Z_{21}] \quad (5.2.5)$$

### 5.2.2 Partitioned Matrices: Three Structures

Here, we consider a useful generalization of the partitioned matrix theory described in the previous section: we wish to describe the interaction between two structures in the presence of a third. We begin with a partitioned matrix with 6 submatrices:

$$\bar{E} = \begin{bmatrix} \bar{E}^{(1)} \\ \bar{E}^{(2)} \\ \bar{E}^{(3)} \end{bmatrix} = \begin{bmatrix} [Z_{11}] & [Z_{12}] & [Z_{13}] \\ [Z_{21}] & [Z_{22}] & [Z_{23}] \\ [Z_{31}] & [Z_{32}] & [Z_{33}] \end{bmatrix} \begin{bmatrix} \bar{J}^{(1)} \\ \bar{J}^{(2)} \\ \bar{J}^{(3)} \end{bmatrix}$$
Borrowing the terms *isolated* and *embedded* from antenna array pattern analysis, let 
\([Z_{11}]\) denote the isolated *source* structure generalized impedance matrix, 
\([Z_{22}]\) denote the isolated *target* structure matrix, and 
\([Z_{33}]\) denote the isolated *environmental* structure(s) matrix. We wish to compute the embedded two structure generalized impedance matrix (as opposed to the embedded single structure generalized matrix from the previous section); that is, we wish to characterize the interactions between the source structure and the target structure in the presence of the parasitic environmental structure(s). Similar to the approach taken in the previous section, we shall require that the total impressed tangential electric field on the parasitic environmental structure is zero: \(\bar{E}^i_3 = 0\). Before shorting all elements together all elements on the parasitic environmental structure(s), we shall define 
\([Z_{33L}] = [Z_{33} + Z_{L3}]\), where \([Z_{L3}]\) describes the terminations (if any) on the environmental structures. Thus:

\[
0 = [Z_{31}] \bar{J}^{(1)} + [Z_{32}] \bar{J}^{(2)} + [Z_{33L}] \bar{J}^{(3)}
\]

\[
\bar{J}^{(3)} = -[Z_{33L}]^{-1}[Z_{31}] \bar{J}^{(1)} - [Z_{33L}]^{-1}[Z_{32}] \bar{J}^{(2)}
\]  

(5.2.6)

Substituting the expression for \(\bar{J}^{(3)}\) into the remaining expressions for \(\bar{E}^i(1)\) and \(\bar{E}^i(2)\), we have:

\[
\bar{E}^i(1) = [Z_{11}] \bar{J}^{(1)} + [Z_{12}] \bar{J}^{(2)} - [Z_{13}] ([Z_{33L}]^{-1}[Z_{31}] \bar{J}^{(1)} + [Z_{33L}]^{-1}[Z_{32}] \bar{J}^{(2)})
\]

\[
= ([Z_{11}] - [Z_{13}] [Z_{33L}]^{-1}[Z_{31}]) \bar{J}^{(1)} + ([Z_{12}] - [Z_{13}] [Z_{33L}]^{-1}[Z_{32}]) \bar{J}^{(2)}
\]

\[
= [Z'_{11}] \bar{J}^{(1)} + [Z'_{12}] \bar{J}^{(2)}
\]  

(5.2.7)

and

\[
\bar{E}^i(2) = ([Z_{21}] - [Z_{23}] [Z_{33L}]^{-1}[Z_{31}]) \bar{J}^{(1)} + ([Z_{22}] - [Z_{23}] [Z_{33L}]^{-1}[Z_{32}]) \bar{J}^{(2)}
\]

\[
= [Z'_{21}] \bar{J}^{(1)} + [Z'_{22}] \bar{J}^{(2)}
\]  

(5.2.8)
Thus, we arrive at the desired embedded two structure generalized impedance matrix:

\[
\begin{bmatrix}
\bar{E}^{(1)} \\
\bar{0}
\end{bmatrix} =
\begin{bmatrix}
[Z'_1] & [Z'_2] \\
[Z'_3] & [Z'_4]
\end{bmatrix}
\begin{bmatrix}
\bar{J}^{(1)} \\
\bar{J}^{(2)}
\end{bmatrix}
\]  

(5.2.9)

where

\[
[Z'_1] = [Z_{11}] - [Z_{13}][Z_{33}]^{-1}[Z_{31}]
\]

(5.2.10)

\[
[Z'_2] = [Z_{12}] - [Z_{13}][Z_{33}]^{-1}[Z_{32}]
\]

(5.2.11)

\[
[Z'_3] = [Z_{21}] - [Z_{23}][Z_{33}]^{-1}[Z_{31}]
\]

(5.2.12)

\[
[Z'_4] = [Z_{22}] - [Z_{23}][Z_{33}]^{-1}[Z_{32}]
\]

(5.2.13)

\[
[Z'_5] = [Z_{31}] - [Z_{33}]^{-1}[Z_{33}]
\]

(5.2.14)

5.2.3 Proposed Radiation Mode System

Intuitively, we would expect that the classical characteristic modes of the source antenna in the presence of its environmental structures are different from than the classical characteristic modes of the source antenna in isolation. The embedded single structure generalized impedance matrix formulation in 5.2.5 allows us a way to compute the characteristic modes of the source antenna in the presence of the environmental structures by defining a new modal system closely linked to classical characteristic modes. We begin with the definition of the desired generalized Rayleigh quotient:

\[
\rho_{SCCM}(\bar{J}) = \frac{\langle \bar{J}^{(1)}, [X^{(1)}] \bar{J}^{(1)} \rangle}{\langle \bar{J}^{(1)}, [R^{(1)}] \bar{J}^{(1)} \rangle}
\]

(5.2.15)

Since \([R^{(1)}]\) and \([X^{(1)}]\) are symmetric matrices and \([R^{(1)}]\) is positive definite, the stationary solutions of this quotient are given by the following generalized eigenvalue problem:

\[
[X^{(1)}] \tilde{J}_n^{(1)} = \lambda_n^{(1)} [R^{(1)}] \tilde{J}_n^{(1)}
\]

(5.2.16)
where \( [Z^{(1)}] = [R^{(1)}] + j[X^{(1)}] \). Since \([R^{(1)}]\) and \([X^{(1)}]\) are real and symmetric and \([R^{(1)}]\) should be positive definite, the eigenvalues \( \lambda_n^{(1)} \) and eigenvectors \( \bar{J}_n^{(1)} \) are real. Additionally, the eigenvectors \( \bar{J}_n^{(1)} \) are \( R^{(1)} \) and \( X^{(1)} \) orthogonal. If only the source structure is excited (i.e. \( \bar{E}^{(2)} = 0 \)) and all environmental terminations are lossless, then \( \langle \bar{J}_n^{(1)}, [R^{(1)}] \bar{J}_n^{(1)} \rangle = 2P_{rad} \); therefore, the modal patterns \( \bar{E}^{(1)}_n \) radiated by the modal currents \( \bar{J}_n^{(1)} \) are orthogonal under these conditions (see Appendix A for more information).

This modal system is formally referred to as Subsystem Classical Characteristic Modes (SCCM), or more simply, Radiation Modes.

### 5.2.4 Proposed Coupling Mode System

Since mutual coupling is between two antennas is defined only when the antennas have ports and our problem definition assumes that we do not know the port location(s) on the source antenna, mutual coupling is not strictly defined in our case. Still, we can define a modal system which utilizes the spirit of mutual coupling. Specifically, we wish to define modes which are orthogonal over the source structure and which successively minimize the magnitude of the induced current on some target structure relative to the current magnitude on the source antenna in the presence of other environmental structures. Both the target and environmental structures are assumed to have their ports terminated with their appropriate loads. Practically, the target "structure" should be the areas immediately surrounding each port on the target antenna(s).

Mathematically, this modal system will require the use of 5.2.9 under the additional constraint that the total impressed tangential electric field on the target structure is zero: \( \bar{E}^{(2)} = 0 \).
The target and environmental structures are assumed to have all ports terminated in their appropriate loads, denoted by $[Z_{22L}]$ and $[Z_{33L}]$, respectively:

$$[Z_{22L}] = [Z_{22} + Z_{L2}] \quad (5.2.17)$$
$$[Z_{33L}] = [Z_{33} + Z_{L3}] \quad (5.2.18)$$

The current $\bar{J}_1$ may be computed from 5.2.5 and the remaining current distributions on the target and environmental structures may be obtained from the following matrix problem:

$$\begin{bmatrix} [Z_{22}] & [Z_{23}] \\ [Z_{32}] & [Z_{33}] \end{bmatrix} \begin{bmatrix} J^{(2)} \\ J^{(3)} \end{bmatrix} = \begin{bmatrix} -[Z_{21}]\bar{J}^{(1)} \\ -[Z_{31}]\bar{J}^{(1)} \end{bmatrix} \quad (5.2.19)$$

From this mathematical setup, we may define a generalized Rayleigh quotient to compute the ratio of the induced current distribution magnitude on a target structure relative to the current distribution magnitude on the source structure:

$$\rho_{TCCM}(\bar{J}) = \frac{\langle \bar{J}^{(2)}, \bar{J}^{(2)} \rangle}{\langle \bar{J}^{(1)}, \bar{J}^{(1)} \rangle} = \frac{\langle \bar{J}^{(1)}, [G]\bar{J}^{(1)} \rangle}{\langle \bar{J}^{(1)}, [I]\bar{J}^{(1)} \rangle} \quad (5.2.20)$$

where

$$[G] = [G_{21}]^H[G_{21}] \quad (5.2.21)$$
$$[G_{21}] = -[Z_{22}']^{-1}[Z_{21}'] \quad (5.2.22)$$
$$[I] \text{ is the identity matrix} \quad (5.2.23)$$

and $\bar{J}^{(2)}$ is the current induced on the target antenna/structure(s) by the current on the source antenna $\bar{J}^{(1)}$. $[G_{21}]$ is the matrix which maps the current from the source antenna to the target structure. $[G]$ is the matrix which represents the induced current magnitude on the target structure. From the quotient, we define the coupling modes through the following (ordinary) eigenvalue problem:

$$[G]\bar{J}^{(1)}_n = \lambda^{(1)}_n[I]\bar{J}^{(1)}_n \quad (5.2.24)$$
Since both $[G]$ and $[I]$ are Hermitian matrices and $[I]$ is always positive definite, the eigencurrents $\bar{J}_n^{(1)}$ are both $G$ and $I$-orthogonal. Also, since the denominator of Equation 5.2.20 just uses the identity matrix, the eigenpatterns radiated by $\bar{J}_n^{(1)}$ are not orthogonal. The physical meaning of the eigenvalue $\lambda_n$ is that it is a ratio of the induced current distribution magnitude on the target antenna/structure to the current distribution magnitude of the associated eigencurrent $\bar{J}_n^{(1)}$ on the source antenna in the presence of the target and environmental structures.

5.2.5 Expressing Modes in Different Modal System

Since we have two different modal systems, it is useful to analyze how one modal system maps to another. Here, we decompose an eigencurrent from one modal system (say radiation modes) in terms of the modal currents in another modal system (say coupling modes). Mathematically,

$$\bar{J}_{k,SCCM} = \sum_n^N \alpha_k^* \bar{J}_{n,TCCM}$$  \hspace{1cm} (5.2.25)

where

$$\bar{J}_{k,SCCM} = \begin{bmatrix} \bar{J}_{k,SCCM}^{(1)} \\ \bar{J}_{k,SCCM}^{(2)} \end{bmatrix}$$  \hspace{1cm} (5.2.26)

$$\bar{J}_{n,TCCM} = \begin{bmatrix} \bar{J}_{n,TCCM}^{(1)} \\ \bar{J}_{n,TCCM}^{(2)} \\ \bar{J}_{n,TCCM}^{(3)} \end{bmatrix}$$  \hspace{1cm} (5.2.27)

$$\alpha_k^n = \frac{\langle \bar{J}_{n,TCCM}^{(1)}, \bar{J}_{k,SCCM}^{(1)} \rangle}{\langle \bar{J}_{n,TCCM}^{(1)}, \bar{J}_{n,TCCM}^{(1)} \rangle}$$  \hspace{1cm} (5.2.28)

and $\bar{J}_{k,SCCM}^{(1)}$ is the first partition of the column vector $\bar{J}_{k,SCCM}$, assuming the partitioning specified in 5.2.27 and not 5.2.26. It is crucial to use the right partitioning so that $\bar{J}_{k,SCCM}^{(1)}$ and $\bar{J}_{n,TCCM}^{(1)}$ are the same length so that the inner product
in Eq. 5.2.28 is mathematically defined. Generally, it is recommended that the source structure in both radiation and coupling modes is exactly the same so that $\tilde{J}_{k,SCCM}^{(1)} = \tilde{J}_{k,SCCM}^{(1)}$.

From Eq. 5.2.28, one can build up a ”projection” matrix:

$$\begin{bmatrix}
\alpha_1^1 & \alpha_2^1 & \ldots \\
\alpha_1^2 & \alpha_2^2 & \ldots \\
\vdots & \vdots & \ddots
\end{bmatrix}$$ (5.2.29)

The entries in the $k^{th}$ column are the coefficients in Eq. 5.2.28 corresponding to the decomposition of the $k^{th}$ radiation mode current in terms of coupling mode eigencurrents. A similar matrix may be computed to map from coupling modes to radiation modes. Notice that $[P_{SCCM-TCCM}] \neq [P_{TCCM-SCCM}]^H$.

5.3 Analysis

Before analyzing some example multiple antenna systems with the radiation and coupling modes, it is instructive to point out what is desirable behavior in terms of eigenvalue spectrum, as well as the projection matrix relating the two antennas. Furthermore, we shall always define the source antenna as the same structure for both radiation and coupling modes.

In Figure 5.1, the radiation mode eigenvalue magnitude spectrum is plotted for a particular antenna in a multi-antenna system. The mode whose eigenvalue is denoted by the solid blue line is a good radiator at about 2.45 GHz because it has a small magnitude, implying that it stores relatively little reactive power relative to radiated power. Conversely, the mode whose eigenvalue is denoted by the dashed red line is a poor radiator over the band because it has a relatively large magnitude, implying that it stores a lot of reactive power relative to radiated power at all frequencies over the band.
Figure 5.1: Example of desirable and undesirable radiation modes according to subsystem classical characteristic modes

Figure 5.2: Example of desirable and undesirable coupling modes according to target coupling characteristic modes
In Figure 5.2, the coupling mode eigenvalue magnitude spectrum for the same source antenna in a multi-antenna system is plotted. The mode whose eigenvalue is denoted by the solid blue line couples poorly to the target structure over the band because it has a relatively low average eigenvalue magnitude over the band, which implies that it induces little current on the target structure relative to the current on the source antenna. In contrast, the mode whose eigenvalue is denoted by the dashed red line inherently couples well to the target structure over the band because it has a large average eigenvalue magnitude, especially at 2.40 GHz.

When we use Eq. 5.2.29 to express the first 12 radiation modes in terms of the first 13 coupling modes at 2.4 GHz, we obtain the normalized projection matrix whose entries’ magnitude in decibels is visualized in Figure 5.3. Examining the last column, we see that the highest coupling mode, Mode 13, is primarily composed of radiation
mode 1. Coupling Modes 9 and 11, both modes with a much lower coupling eigenvalue magnitude, are primarily composed of radiation modes 1 and 3.

5.3.1 Example: Dipole Pair

The geometry described in this section is taken from [90]. The basic setup is two 2.45 GHz dipoles spaced 20 mm apart. Each dipole is modeled as a strip 0.5 mm wide and 57.8 mm long. The dipoles are z-aligned and are in the y-z plane. The geometry is shown in Figure 5.4. Each dipole is center-fed. The source antenna, also referred to as Dipole 1, is defined as the metal contained in the blue outline (left dipole) and the target structure is defined as the area immediately around the port (marked as a red dot) on the second dipole, referred to as Dipole 2, outlined in red. Furthermore, the target structure is terminated with 50Ω.

The $S_{11}$ and $S_{12}$ curves are shown below in Figure 5.5. Notice that the minimum
Figure 5.5: $S_{11}$ and $S_{12}$ of the dipole pair

$S_{11}$ occurs at 2.45 GHz with a level of -25 dB, while the maximum $S_{12}$ occurs at 2.44 GHz with a level of -5.97 dB.

**Source Antenna CCM**

The classical characteristic modes of just the source antenna (i.e. alone in free space) were computed and the resulting CCM eigenvalue spectrum is shown in Figure 5.6. The associated modal weighting coefficients are shown in Figure 5.7. As expected, Mode 1 is resonant at 2.46 GHz and is dominant over the entire band of 2-3 GHz.

**System CCM**

The classical characteristic modes for the entire system were also computed and the resulting CCM eigenvalue spectrum is shown in Figure 5.8. Notice that CCM 1 (lowest blue line) is resonant at approximately 2.35 GHz, while the CCM 2 (green
Figure 5.6: Classical characteristic mode eigenvalue spectrum of a single dipole

Figure 5.7: Classical characteristic mode modal weighting coefficients of a single dipole
Figure 5.8: Classical characteristic mode eigenvalue spectrum of system composed of two dipoles

line) is resonant at 2.49 GHz, neither of which correspond to the design frequency of 2.45 GHz.

Radiation Modes

The radiation modes were computed for the source antenna and the eigenvalue spectrum is shown in Figure 5.9, while the associated modal weighting coefficients are shown in Figure 5.10.

In the case of radiation modes, the resonance of Mode 1 (the dominant mode over the entire band) is at 2.44 GHz, which is in far better agreement with the observed minimum in the $S_{11}$ level in Figure 5.5. Also, note that only odd-numbered modes are excited, which agrees well with intuition. From the previous discussion on the desirable behavior of this antenna, we can understand that Radiation Mode 1 is the
Figure 5.9: Radiation mode eigenvalue spectrum of Dipole 1 in the presence of Dipole 2, with port 2 terminated with 50Ω

Figure 5.10: Modal weighting coefficients of radiation modes of Dipole 1
Figure 5.11: Coupling mode eigenvalue spectrum of Dipole 1 (source antenna) coupling to the port of Dipole 2 (target structure)

desirable mode, since it stores the least amount of power relative to the power it radiates into its environment.

Coupling Modes

The coupling modes were computed between the source antenna and target structure. The resulting eigenvalue spectrum is shown in Figure 5.11 and the associated modal weighting coefficients are shown in Figure 5.12. Notice that while Coupling Mode 1 is the most desirable, it is not excited. Instead, Coupling Modes 9 and 13 are primarily excited, along with Mode 2 in a secondary manner. Of Modes 9 and 13, Mode 13 is the most problematic, because it has a substantially higher eigenvalue magnitude than Mode 9, implying that it is contributing the most to the mutual coupling.
Figure 5.12: Modal weighting coefficients of coupling modes
5.3.2 Example: Yagi-Uda

The two geometries described in this section are taken from the design curves in the classic NIST document on Yagi-Uda antennas [119]. The lengths and spacing of Yagi 1 and Yagi 2 were further numerically optimized to obtain better input impedance, so their designs diverge from those published in the NIST document. Both antennas were designed to operate at 400 MHz (i.e. $\lambda_0 = \frac{c}{400 \text{ MHz}}$).

Yagi 1

The first Yagi-Uda antenna we shall consider is a three element Yagi and is shown in Figure 5.13. The driver is center-fed and its port is denoted as a red dot. The blue dashed line denotes the source antenna (the driver), while the red dashed lines enclose the target structures (the reflector and director).
The radiation modes were computed for this structure and the eigenvalue spectrum is shown in Figure 5.14. The associated modal weighting coefficients are shown in Figure 5.15. Notice that the driver excites Mode 1 over the band and is resonant at approximately 400 MHz, as expected.

The coupling modes were also computed for this structure and the resulting eigenvalue spectrum is shown in Figure 5.16, along with the associated modal weighting coefficients in Figure 5.17. The coupling mode eigenvalue spectrum has a peak in Mode 13 around 400 MHz, while the modal weighting coefficients demonstrate that indeed Mode 13 is the dominant coupling mode at 400 MHz and it is strongly excited. From the coupling modes, one correctly predicts that the maximum coupling between the source antenna (driver) and target structures (reflector and director) should occur at 400 MHz. It also clearly predicts that this strongly coupling is narrowband.

The first 13 coupling modes were projected onto the first 9 radiation modes at 400
Figure 5.15: Yagi Antenna 1 radiation modal weighting coefficients

Figure 5.16: Yagi Antenna 1 coupling mode eigenvalue spectrum
MHz and resulted in the normalized projection matrix whose entries are visualized in decibels in Figure 5.18. One may note that the first radiation mode is strongly associated with Coupling Mode 13. Coupling Modes 7, 9, and 11, which feature weaker mutual coupling since their associated eigenvalue magnitudes at 400 MHz are much lower than Mode 13, are all primarily composed of Modes 1 and 3 excited in nearly equal proportion. In this case, mutual coupling is critical to establish the high gain beam associated with a Yagi-Uda antenna.

**Yagi 2**

The second Yagi-Uda antenna we shall consider is a six element Yagi and is shown in Figure 5.19. The radiation modes and coupling modes are defined in similar ways as the last Yagi-Uda example (see Figure 5.19 for more details).

The radiation modes were again computed for this case and the eigenvalue spectrum is shown in Figure 5.20. The associated modal weighting coefficients are shown
in Figure 5.21. As expected, the first radiation mode is dominant over the band and
is approximately resonant at 396 MHz.

The coupling modes were then computed: the coupling eigenvalue spectrum is
shown in Figure 5.22, while the modal weighting coefficients are shown in Figure 5.23.
Notice that Coupling Mode 12, the mode with the highest coupling, is dominant over
the band and is especially strongly excited at 396 MHz. The coupling modes correctly
predict that the peak gain should occur at 396 MHz, and they also predict that it
will be narrowband. The radiation modes predict that the antenna should likely be
well-matched at this frequency, since Radiation Mode 1 is resonant at 396 MHz.
Figure 5.19: Yagi Antenna 2

Figure 5.20: Yagi Antenna 2 radiation mode eigenvalue spectrum
Figure 5.21: Yagi Antenna 2 radiation modal weighting coefficients

Figure 5.22: Yagi Antenna 2 coupling mode eigenvalue spectrum
Figure 5.23: Yagi Antenna 2 coupling modal weighting coefficients
5.3.3 Example: Dipole Beside Microstrip Patch

This section will examine the radiation and coupling modes of a dipole antenna in the presence of a closely spaced patch antenna, all resonant at approximately 2.4 GHz. The geometry is shown in Figure 5.24. The dipole is the blue line beside the microstrip patch antenna on a finite ground plane. The patch is colored green, while the ground plane is colored grey. The microstrip patch antenna is fed using a wire probe, which is offset from the center of the patch antenna by 10 mm, indicated in the figure as a red dot. Both the patch and dipole were designed to resonate at 2.4 GHz when in isolation. They are spaced $\frac{\lambda}{8}$ apart at 2.4 GHz. The feed probe is terminated by 50Ω.

The input reflectance at port 1 (at the dipole’s center) and port 2 (the microstrip patch antenna feed port) is shown in Figure 5.25, while the mutual coupling $S_{12}$ is shown in Figure 5.26.
Figure 5.25: Input reflectance of dipole and patch antenna system

Figure 5.26: Mutual coupling between dipole and patch antenna
Notice that although the patch feed point is physically shielded by the top plate from the dipole and the patch is detuned somewhat by the presence of the dipole, there is still a very significant coupling level of -2.36 dB at 2.4 GHz.

The radiation modes were computed by defining the source antenna as the dipole. The eigenvalue spectrum is shown in Figure 5.27, while the modal weighting coefficients are shown in Figure 5.28.

From the radiation modes, it is evident that Radiation Mode 1 is dominant over the entire band and resonates at 2.4 GHz, as expected. Notice that because the modal resonance in Figure 5.27 is narrowband, the actual dipole resonance in Figure 5.25 is also narrowband.

The coupling modes were also computed by defining the source antenna as the dipole and the target structure as the microstrip patch antenna’s terminated wire.

Figure 5.27: Dipole-patch radiation mode eigenvalue spectrum
feed probe. The resulting eigenvalue spectrum is shown in Figure 5.29, while the associated modal weighting coefficients are shown in Figure 5.30.

Examining these two coupling mode figures, one observes that Mode 4 is the eigencurrent which provides most of the coupling because of its substantially higher eigenvalue magnitude. It is also one of two dominant currents, alongside Mode 2. The peak Mode 4 excitation occurs at 2.4 GHz and is relatively narrowband, which is in agreement with the relatively narrow peak in mutual coupling in Figure 5.26. If one wished to reduce the coupling between the patch antenna and the dipole, one must first analyze the coupling in terms of radiation modes; the projection of coupling modes onto radiation modes is shown below in Figure 5.31.

From the projection, we can observe that Radiation Mode 1 is a strong component of Coupling Mode 4. If we were to reduce the coupling by only exciting Coupling Mode 1 or 2, the matrix reports that those two modes depend significantly upon Radiation
Figure 5.29: Dipole-patch coupling mode eigenvalue spectrum

Figure 5.30: Dipole-patch coupling modal weighting coefficients
Mode 2, which gives a fairly different pattern than the desired Radiation Mode 1. Therefore, if one practically tries to reduce the coupling, one must excite primarily Radiation Mode 3 and Radiation Modes 1 and 2 weakly. Since the Radiation Mode 3 eigenvalue magnitude over the band is much larger than that of Mode 1 (see Figure 5.27), it is highly unlikely that a feed network could be designed to efficiently excite primarily Radiation Mode 3 over Mode 1. Therefore, if the mutual coupling is to be reduced, another method of exciting Radiation Mode 3 must be found, ideally lowering Radiation Mode 3’s eigenvalue on the source antenna.
5.4 Design

In this section, three designs shall be proposed, all which reduce the mutual coupling between a pair of dipoles (the same geometry as in the first analysis example) at 2.45 GHz using the information from radiation and coupling modes. Neither dipole will be modified. They will be compared against a design published in the literature which successfully reduces the mutual coupling between the same pair of dipoles.

Before embarking on the reduced mutual coupling designs, let us summarize the critical observations made of the two dipole system after analysis using radiation and coupling modes. Firstly, the peak mutual coupling level without any treatment (just two closely spaced parallel dipoles) is -5.97 dB at 2.44 GHz. From the coupling modal weighting coefficients in Figure 5.12, we observed that Modes 9 and 13 are primarily excited at the design frequency of 2.45 GHz.

If we did not care at all about the radiation pattern, then Figure 5.32 offers us a reasonably straightforward solution: we could attempt to excite Coupling Mode 12, which is seen to be composed primarily of Radiation Mode 2 (similar to the second dipole mode), and it features a substantially reduced mutual coupling level according to the coupling eigenvalue spectrum relative to Coupling Mode 13. Of course, because the eigenvalue magnitude associated with Radiation Mode 2 is larger than Radiation Mode 1 around 2.45 GHz, one can expect moderate to severe mismatch at the two required feed points. Therefore, primarily exciting Radiation Mode 2 is not practical without geometrical modifications or loading of the source antenna.

When we examine the associated coupling eigenvalue magnitudes of Coupling Modes 9 and 13 in Figure 5.11, we observe that Mode 13 has a substantially higher magnitude than Mode 9, implying that the bulk of the coupling is caused by Mode 13. Thus, we have identified Mode 13 as undesirable and we wish to not excite it so strongly.
To understand how to modify our excitation, let us examine the projection of the coupling modes on the radiation modes in Figure 5.32.

From the 13th column, we observe that Coupling Mode 13 is substantially comprised of Radiation Mode 1, our desired radiation mode. If we now examine the 9th column corresponding the decomposition of Coupling Mode 9 in terms of the radiation modes, we see that Radiation Modes 1 and 3 must be almost equally excited. This stands in contrast to Coupling Mode 13. So, the analysis shows us that in order to reduce the mutual coupling between the dipoles, we must find a way to excite Radiation Modes 1 and 3 more equally on the source antenna (Dipole 1).

**Geometry Modification Through Parasitic Structures**

While one could use lumped reactive loading on the source antenna [74, 75] to lower the Radiation Mode 3 eigenvalue magnitude on the source antenna around 2.45 GHz, it has the unfortunate side-effect of raising the Mode 1 eigenvalue magnitude at the
same frequency. Since the loading technique only enables the control of a single mode at any given frequency, it is clear that loading the source antenna cannot be used to provide equal excitation of Radiation Modes 1 and 3 on it. Simultaneously, it is also clear that it is impossible to design a feed network to directly excite Radiation Modes 1 and 3 with equal magnitudes, since the modal weighting coefficients (see Eq. 2.2.3) are approximately inversely proportional to the Radiation Mode eigenvalue. Therefore, approximately equal excitation of the two modes would require an impressed $\bar{E}^i$ that is impractical.

I propose that the addition of a neighboring parasitic structure, which when illuminated with a Radiation Mode 1 near-field, reflects a Radiation Mode 3 field, thereby indirectly enhancing the excitation of Mode 3 on the source antenna. To ensure that the Radiation Mode 3 eigenvalue on the source antenna is lowered, the parasitic structure must be close to resonance at 2.45 GHz such that its near-field sufficiently interacts with the source antenna (i.e. that there is some coupling between the parasitic structure and the source antenna).

The next section analyzes a proposed design for the parasitic structure (although its design was derived from very different principles), and the subsequent sections show three improvements on the original parasitic structure, based on the previous modal analysis of this dipole pair.

5.4.1 Original Design

In [90], the problem of the reducing the coupling between parallel dipoles was proposed and a solution obtained. A parasitic "top-hat" loaded dipole was inserted between the two dipoles. Both the design discussed in this section and in subsequent sections can be generally described by the parameterized geometry in Figure 5.33. The two dipoles (black) are 0.5 mm wide and 57.8 mm long. The red dot denotes the
terminated port on Dipole 2, while Dipole 1 (the source antenna) lacks a dot. Both
dipoles lie in the yz-plane, while the parasitic structure, herein termed the coupler,
lies in the xz-plane, exactly midway between the two dipoles. The “hats” on the
coupler (blue lines) are $L_1$ mm long and $W_2$ mm wide. The coupler itself (green line)
is $L_2$ mm long and $W_2$ mm wide.

In the original coupler design, the geometrical parameters for the coupler were:
$L_1 = 39$ mm, $W_1 = 4$ mm, $L_2 = 12$ mm, and $W_2 = 2$ mm. With these parameters,
the radiation modes were computed for Dipole 1. The eigenvalue spectrum is shown
in Figure 5.34, and the associated modal weighting coefficients are plotted in Figure
5.35. The black vertical dashed line indicates 2.45 GHz. Comparing Figure 5.35 with
Figure 5.10, one can clearly observe that Radiation Mode 3 is better excited in this
design than the design without a coupler. Furthermore, comparing the two radiation
mode eigenvalue spectrums, one can observe that the eigenvalue for Mode 3 decreased when the coupler was inserted.

In this design, the coupler is nearly resonant at 2.45 GHz, as can be observed by computing the radiation mode eigenvalue spectrum of the coupler, shown in Figure 5.36. The coupler is actually resonant at 2.42 GHz. The addition of the coupler improves the isolation between the two dipoles (Figure 5.38), at the cost of input reflectance (Figure 5.37), although it remains acceptable at -9.06 dB at 2.45 GHz.\(^1\) The isolation improved from 6.00 dB to 12.50 dB.

\(^1\)It should be noted that FEKO computed a higher input reflectance level in the presence of the coupler than was reported by CST and measurements in [90]. The simple nature of the FEKO feed model is likely the cause of this discrepancy. In any case, all comparisons are made with respect to the FEKO version of the problem.
Figure 5.35: Dipole 1 radiation modal weighting coefficients with the original coupler design

Figure 5.36: Original coupler radiation mode eigenvalue spectrum
Figure 5.37: Original coupler design input reflectance

Figure 5.38: Original coupler design mutual coupling
5.4.2 Design Theory

The results from the original coupler design are very interesting, especially since they verify that the concept that as Radiation Mode 3 is made more easily excitable alongside Mode 1 on the source antenna, the mutual coupling is reduced. The coupler design excites Mode 3 on the source antenna because of reflections coming from both the coupler and the terminated dipole, as shown in Figure 5.39.

Still, the coupler is fairly bulky, primarily because of its hat. What if we shortened $L_2$ to make the overall design more compact? If we shorten the hat, then we can use reactive loading to cause the coupler to resonate. The equation for reactive loading used in [74] must be modified slightly, however. Notice that the original coupler design did not resonate at the operating frequency, but slightly below. Thus, the eigenvalue of the dominant mode of the coupler is non-zero (in this case, inductive). We shall use the basic methodology reported in [74], but with a slight modification
of Equation 3, which was used to compute the reactive loads, so that the desired eigenvalue may be non-zero.

Let us assume that we have chosen $N$ ports on the coupler and have identified the desired real current $\bar{I}_d(\omega)$ that we wish to enforce on the coupler using uncoupled reactive loads at those ports. Let the $N$ uncoupled loads be represented by the diagonal matrix $[X_L(\omega)]$. If $\lambda_d(\omega)$ is the corresponding desired eigenvalue for my desired current $\bar{I}_d(\omega)$, then the loaded network characteristic mode generalized eigenvalue problem is:

$$[X(\omega) + X_L(\omega)]\bar{I}_d(\omega) = \lambda_d(\omega)[R(\omega)]\bar{I}_d(\omega) \quad (5.4.1)$$

$$[X_L(\omega)]\bar{I}_d(\omega) = ([\lambda_d(\omega)R(\omega) - X(\omega)])\bar{I}_d(\omega) \quad (5.4.2)$$

Since $[X_L(\omega)]$ is diagonal, the reactance at a port $i$ is:

$$X_{L_i}(\omega) = \frac{1}{\bar{I}_{d_i}(\omega)} ([\lambda_d(\omega)R(\omega) - X(\omega)]\bar{I}_d(\omega))_i \quad (5.4.3)$$

For Designs A-C, we defined 5 ports placed equally along the coupler’s length.

### 5.4.3 Design A

For Design A, we wish to load the coupler to compensate for shorter hats. In this design, $L_1 = 20$ mm (same as the spacing) and all other parameters are the same as in the original coupler design. I chose to enforce a constant current distribution over the coupler to emulate the behavior of the original coupler: $\bar{I}_d = [1 1 1 1 1]^T$. With some experimentation, I chose $\lambda_d = 10^{0.75/10}$. $\lambda_d$ controls the degree of influence that the coupler has on the source antenna. If $\lambda_d$ is too small, then the current near the source antenna’s feed port is affected too much and the input reflectance becomes poor. If $\lambda_d$ is too large, then Radiation Mode 3 won’t be sufficiently excited on the
source antenna, thereby having only a minimal reduction in mutual coupling. For each design, I vary $\lambda_d$ until the input $S_{11}$ level matches that of the original design at 2.45 GHz.

With these settings, the ideal load reactances were computed using Eq. 5.4.3 and are shown in Figure 5.40. While the reactances are obviously non-Foster in nature and are therefore difficult to directly realize, they demonstrate the potential of the design method, always assuming that the loads may be realized over a smaller bandwidth using realistic circuit components and topologies.

The input reflectance and mutual coupling are shown in Figure 5.41 and Figure 5.42, respectively. The isolation at 2.45 GHz improved from 6 dB to 10.89 dB, which is less than the improvement offered by the original coupler design.

To better understand the reason why this design did not work as well as the original coupler design, it is instructive to examine the radiation modes of the source
Figure 5.41: Coupler Design A input reflectance

Figure 5.42: Coupler Design A mutual coupling
antenna. The eigenvalue spectrum is shown in Figure 5.43, and the associated modal weighting coefficients are plotted in Figure 5.44.

The problem in this design is that while Radiation Mode 3’s eigenvalue was lowered, it occurred at a higher frequency band (around 2.5-2.65 GHz) than intended, which is due to the smaller nature of the coupler. That is, reactive loading cannot be used to dramatically increase the influence of an electrically small structure (like the coupler in this case).

### 5.4.4 Design B

In this design, we seek to improve upon Design A by using an electrically larger coupler, but making it nearly planar. In this case, $L_1 = 10$ mm and $L_2 = 57.78$ mm (the dipole’s length).

I chose to enforce a current resembling the Mode 3 eigencurrent, since I want the
coupler to reflect back a Radiation Mode 3 field: $\vec{I}_d = [-0.6 \ 0.6 \ 1 \ 0.6 \ -0.6]^T$. With some experimentation, I chose $\lambda_d = -10^{1.2/10}$. With these settings, the ideal load reactances were computed and are shown in Figure 5.45.

The input reflectance and mutual coupling are shown in Figure 5.46 and Figure 5.47, respectively. The isolation at 2.45 GHz improved from 6 dB to 13.63 dB, which is an improvement over the original coupler design.

### 5.4.5 Design C

In this design, we seek to improve upon Design B by employing a purely planar design. In this case, $L_1 = 0$, and $L_2 = 57.78$ mm (the dipole’s length).

I chose to enforce a current resembling the Mode 3 eigencurrent: $\vec{I}_d = [-0.7 \ 0.5 \ 1 \ 0.5 \ -0.7]^T$. Deciding upon the eigencurrent values does involve some guess-and-check, as this particular eigencurrent gave improved isolation than the eigencurrent in Design B for
**Figure 5.45**: Design B ideal loads

**Figure 5.46**: Coupler Design B input reflectance
this geometry. Obviously, an optimization routine could be used to compute the best possible eigencurrent choice to maximize isolation while minimizing reflectance at the design frequency. With these settings, the ideal load reactances were computed and are shown in Figure 5.48.

The input reflectance and mutual coupling are shown in Figure 5.49 and Figure 5.50, respectively. The isolation at 2.45 GHz improved from 6 dB to 13.06 dB, which is an improvement over the original coupler design, but slightly less than Design B.
Figure 5.48: Design C ideal loads

Figure 5.49: Coupler Design C input reflectance
Figure 5.50: Coupler Design C mutual coupling
5.4.6 Comparison of Designs

The input reflectance of the various coupler designs is compared in Figure 5.51. While at the design frequency, all the couplers feature approximately -9 dB of input reflectance, the no coupler case clearly performs better. This is not surprising, since the coupler is destructively interfering with the current at the feed of the source dipole, thereby producing mismatch. Comparing among coupler designs, however, Design B has the smallest variation in $S_{11}$ level around 2.45 GHz, implying more accessible bandwidth. Of course, loads must be developed to access this potential bandwidth.

The tradeoff between $S_{11}$ level and isolation is apparent when examining the effect of each coupler design on reducing the mutual coupling level near the design frequency. Compared to the no coupler case, every design features improved isolation. Designs B and C offer improved isolation relative to the original coupler design in a significantly
reduced volume. Again, Design B has the smallest frequency variation about 2.45 GHz, so it has the most potential bandwidth.

Another indicator of reduced mutual coupling is the radiation efficiency. If the source antenna is more isolated from the port on the other dipole, then less power should be transferred to that port. Therefore, the radiation efficiency of a single antenna radiating in the presence of the other antenna terminated should be higher with greater isolation. The radiation efficiency of each design is shown in Figure 5.53.

The radiation efficiencies at 2.45 GHz for each design are tabulated in Table 5.1. The source dipole is radiating in the presence of the other dipole, which is terminated with 50Ω.

Perhaps the most interesting result is the embedded radiation patterns at 2.45 GHz, shown in Figure 5.54. While the original design and Design A both feature
Figure 5.53: Comparison of radiation efficiency for all designs (ideal loads)

Table 5.1: Radiation efficiency of a source dipole 2.45 GHz

<table>
<thead>
<tr>
<th>Design Name</th>
<th>Efficiency (percent)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No coupler</td>
<td>74.76</td>
</tr>
<tr>
<td>Original</td>
<td>93.58</td>
</tr>
<tr>
<td>Design A</td>
<td>90.70</td>
</tr>
<tr>
<td>Design B</td>
<td>95.05</td>
</tr>
<tr>
<td>Design C</td>
<td>94.35</td>
</tr>
</tbody>
</table>
Figure 5.54: Comparison of embedded vertical gains at 2.45 GHz at $\theta = 0^\circ$ (azimuthal cuts) for all designs

"beams" which point away from the other dipole, Designs B and C feature fear-field beams which point towards the other dipole.

5.5 Summary

This chapter introduced two new modal systems. One is formally termed Subsystem Classical Characteristic Modes (SCCM), also called Radiation Modes. It is an extension of Classical Characteristic Modes, but applied to antennas or structures embedded among other structures rather than free-space. It was shown to be quite useful in identifying the true modal resonant frequency of a structure operating in the presence of other structures, among other applications. The second new modal system, formally termed Target Coupling Characteristic Modes (TCCM), also called Coupling Modes, is a modal system which can analyze the coupling between structures. It was applied to several antennas and was shown to accurately predict the
qualitative behavior of Yagi-Uda antennas, antennas that depend upon high amounts of coupling to produce high gain. Finally, a projection matrix was defined, which relates the two modal systems. The projection matrix was used to modify a multiple antenna system to reduce the mutual coupling among the antennas by introducing a loaded parasitic. The loads were determined based on the information provided by the coupling modes.

I believe that this pair of modal systems could be used to systematically analyze coupling among structures and possibly control the coupling through the creative application of the modal information.
CHAPTER 6
CONCLUSIONS

6.1 Summary and Conclusions of the Dissertation

The work considered in this dissertation is overall concerned with the systematic analysis and design of complex antenna problems. Although such systems contain a wealth of information, the sheer volume is overwhelming and various specific approaches have been historically applied to extract the relevant physics. This dissertation applies and develops types of characteristic mode analysis through mathematics and software to provide a general description of the physics, making clearer some key antenna design characteristics, such as bandwidth, feed design, and mutual coupling.

6.1.1 Software

In Chapter 2, the mathematical foundation and key properties of successful modal systems were reviewed. Furthermore, the chapter provided an overview of the software developed in connection with this dissertation. In total, over 60,000 lines of Matlab code were written to aid Prof. Roberto Rojas’ research group and are maintained through a custom decentralized software update system. The Unified Characteristic Mode (UCM) system, in particular, has been the foundational system responsible for all characteristic mode analysis, from the computation of the modes using data extracted from various commercial electromagnetic simulation software packages, to
their efficient storage (individual UCM projects can grow to be gigabytes in size), to a plugin system concisely expressing new modal systems, all using an object-oriented approach and in nearly all native Matlab code.

6.1.2 Wideband Mode Tracker

In Chapter 3, a wideband, robust mode tracking system was developed to automatically associate modes at different frequencies in a practically computationally efficient way, the first of its kind reported in the field of characteristic modes. Such a tracker is essential to considering how the characteristic modes (whether they are classical, generalized, Inagaki, or others) evolve over frequency for a given antenna or scatterer. Especially crucial is its automated nature, since in practice, perhaps upwards of 40 or 50 modes must be tracked over hundreds or thousands of frequency points. Comparisons to existing trackers demonstrated that they were either too inaccurate or too computationally expensive for this problem scale. An alternative tracking algorithm was provided as a point of comparison for the computational cost. While not explicitly discussed in Chapter 3, the algorithm is foundational to Chapters 4 and 5, as the developments in both chapters critically depend upon considering the evolution of modes over some frequency band to derive their unique results. The tracker has been under development since 2008 and has been applied to over 100 antennas, enabling the author and his colleagues, Khaled Obeidat and Brandan Strojny, to understand the frequency behavior of the characteristic modes on actual antenna designs. While certain aspects can still be improved, its robustness has been an important contribution to both this work and that of others.
6.1.3 Antenna Feed Design

In Chapter 4, a semi-automated design technique was developed which computes the number, location, and voltages of ports on a given antenna according to realize some desired modal and bandwidth specifications. It uses the desired modal weighting coefficients computed according to some type of characteristic mode analysis to form an underdetermined system of equations over frequency. Using the fact that any realistic port arrangement is sparse relative to the surface area of the antenna and the properties of $\ell_1$ solvers, discussed extensively in the emerging field of Compressed Sensing, the procedure computes the most likely port locations to realize the desired modal weighting coefficients over some frequency range. In fact, the frequency range need not be contiguous, enabling the procedure to potentially compute the optimum feed locations for multiband antennas. The desired modal weighting coefficients $\alpha_n$ are derived from known performance specifications, such as the desired pattern or even potentially the desired input admittance. Thus, the procedure can transform real design criteria, such as the desired pattern over some frequency bandwidth, to the characteristic mode domain, where the number and location of feed points, along with their associated voltages, may be provided. Such a procedure was shown to be extremely general, even suggesting geometrical modifications in the case of a folded spherical helix antenna to enhance bandwidth.

6.1.4 Mutual Coupling Reduction

In Chapter 5, two new modal systems were introduced for the purpose of analyzing and mitigating mutual coupling between a single antenna and one or more neighbors. The first, called Subsystem Classical CM (SCCM), was a generalization of a single isolated antenna’s classical characteristic modes to the situation of the CM of an embedded antenna (i.e. an antenna operating the presence of one or more neighboring
antennas or structures). Using SCCM, several antennas were analyzed to demonstrate its generality and utility.

The second modal system is called Target Coupling CM (TCCM), and it analyzes the coupling between a single antenna \( (\text{source antenna}) \) and one or more neighboring structures \( (\text{target structure}) \). Both SCCM and TCCM are defined to produce orthogonal eigencurrents over the source antenna. Based on Classical CM, only SCCM defines orthogonal eigenpatterns, provided that all antennas/structures involved are lossless, including any port terminations.

A method to map between the two modal systems was also introduced, which enables an analyst to not only perform a modal analysis in either modal system, but also informs the designer on how well certain coupling modes radiate (i.e. mapping TCCM to SCCM). The twin modal systems were applied to the analysis of several example antennas to verify that they operate as desired.

Finally, the modal systems were applied to first understand and then reduce the mutual coupling between two closely spaced parallel dipoles. Three designs were developed with different tradeoffs and compared against the performance of a solution proposed in the literature. The latter two designs, one of which is planar, provided superior performance to that of the reference solution from the literature. Both TCCM and SCCM have promise to not only analyze existing general multi-antenna systems, but also to potentially systematically trade off mutual coupling against pattern and impedance performance, all in a general framework exposing the underlying physics of the problem.
6.2 Suggestions for Future Work

The topics discussed in this dissertation have considerable potential for further development, especially the concepts of the feed design procedure and the systematic analysis and design of minimal/maximal coupled antenna systems.

6.2.1 Software

Although considerable software has been written, further improvements could be made. The most straightforward enhancement would be to develop HFSS, CST and COMSOL simulation engine plugins to extend the compatibility and flexibility of the UCM system. Naturally, all three proposed engines would only support network characteristic mode analysis, since they none support a compatible MoM formulation.¹

The next most useful enhancement would be to develop a fast, in-house Galerkin MoM code supporting structures composed of wires, plates, and dielectric/magnetic volumes. Such software would likely just implement the formulation discussed in a recent dissertation [120] and use commercially available meshing software (or even borrow the mesh from a geometry developed using a commercial electromagnetic simulation software code such as FEKO).

Another enhancement would be design a graphical user interface to the entire UCM system. Such an undertaking would be time-consuming, but it would likely improve the overall ease of use. It is recommended that researchers take cues from interfaces such as Paraview [121], or even developing a graphical language such as Labview [122], since the UCM system supports a considerable degree of flexibility owing to its origin as a research system, rather than a simplified commercial code.

Finally, the most significant change to the UCM system would be to port it to

¹Actually, CST does have a separate MoM engine available for licensing, but current versions do not allow the export of the MoM generalized impedance matrix.
a platform-agnostic high-performance language, such as C++. Such a change would also naturally enable threads or other naturally parallel computational structures to accelerate computations. Data storage (i.e. replacing the FlatFileDatabase code) would likely be handled well by the portable and open-source SQLite [123]. The researcher would likely need to have considerable mastery of several topics in computer science, making this last alteration less likely for the pure electromagnetic engineering researcher.

6.2.2 Wideband Mode Tracking

In the area of wideband mode tracking, the proposed tracker has a fairly low computational cost, but it assumes that all the eigenvalues and eigenvectors have been previously computed. Unfortunately, the practical cost of performing a generalized eigenvalue/eigenvector decomposition of a matrix pair is quite high and repeated evaluations do not scale well for large systems involving several thousand unknowns. While GPU acceleration has shown some promise to alleviate the temporal cost of repeated eigenvalue decompositions, a better theoretical approach could be possible. For future work on the tracking procedure, it is suggested that an underlying state space model be created for each eigenvector and its coefficients computed. Based on the single parameter of frequency, the model would essentially compute a particular mode’s eigenvector at very low cost at arbitrary frequencies. Thus, tracking would be inherent to the model.

To build the state space model, eigenvectors would need to be explicitly computed at several frequencies. Such frequencies would form only a small subset of the total number of frequency points. The technology would behave very much like an adaptive frequency sweep in popular commercial frequency-domain electromagnetic simulation
packages. In this way, the researcher could take advantage of the latest developments in model order reduction techniques [124], which lie at the heart of such techniques.

6.2.3 Antenna Feed Design

While a few examples of the antenna feed design procedure were shown for some antennas Prof. Rojas’ research group has investigated in the past using known design criteria, it would be significant if designs could be developed where the desired impedance behavior and pattern bandwidth were explicitly mapped onto modal weighting coefficients and fed into the algorithm. An outstanding problem characteristic of antennas using multiple feeds requiring further research is how to match at multiple ports over some bandwidth. At the present, the author is not aware of any such general solutions, although there are some recent interesting developments [125].

An unexplored area requiring some development is the application of this procedure to designing MIMO antennas. In particular, it should be possible to develop a single structure with multiple feed points. For simplicity of discussion, let each feed point excite a unique set of classical characteristic modes. Because of orthogonality, the feed points would be highly isolated from each other. The antenna feed design procedure could be applied to compute each port’s location on the structure given the information that each feed port should excite a different set of modes.

Finally, an important line of research would investigate the definition of ports on non-wire geometries. Currently, the procedure can handle such geometries and locate a physical port on the geometry, but these ”ports” are simply thin gaps between mesh triangle edges. In general, an actual feed port cannot be physically constructed in such a way, as it would usually be shorted by neighboring triangular patches. Currently, the author uses the port locations computed by the procedure on such structures.
as suggested places to define ports. The geometry is then modified, introducing sufficient gaps to define a physical port. Refinements are usually necessary to excite the modes according to the desired modal weighting coefficients. A more rigorous approach would clarify and accelerate this process.

6.2.4 Mutual Coupling Reduction

The concepts explored by the computation of the TCCM and SCCM for a given multiple antenna system show great promise, especially in analyzing multiple antenna systems. Still, while the TCCM and SCCM were successfully applied to reduce the mutual coupling between two parallel dipoles using a non-obvious geometry, the most obvious future work is to apply them to reduce the mutual coupling among more diverse multiantenna system geometries and arrangements.

While the antenna feed design procedure was originally developed to excite minimum coupling modes on an arbitrary source antenna, it was found that geometry modification was substantially more appropriate for the design example in Chapter 5. An useful extension of the TCCM/SCCM line of research would be to develop source antenna geometries which feature minimum coupling modes which have low associated SCCM eigenvalue magnitudes. With lower eigenvalue magnitudes, the antenna feed design procedure could be applied to develop multiport source antennas with minimum mutual coupling to a target antenna.

While the two modal systems were used to minimize the mutual coupling between two antennas, it is conceivable that they could also be used for antenna placement on larger fixed structures. In particular, the TCCM could potentially be used to modify a source antenna to have minimum induced current on some neighboring structures, like a ship’s hull or a cellphone enclosure. Such development would be a useful
extension of Newman’s earlier work on antenna port placement using characteristic modes [32].

Finally, an interesting application of the TCCM would be to develop designs to maximize the coupling between two antennas. Such designs would allow for a general treatment of wireless charging technology. In this case, one would seek out modes which would have large TCCM eigenvalues and which would have minimum radiation (i.e. larger SCCM eigenvalues). Of course, to extract a useful high transducer power gain, the mismatch problem would need to be solved on both the source and target antennas, implying the use of potentially small SCCM eigenvalues on both antennas. An exploration of this tradeoff would be another useful development of these modal systems.
Appendix A

ALTERNATIVE DERIVATION FOR CLASSICAL CHARACTERISTIC MODES

This appendix presents an alternative derivation for classical characteristic modes to [22], which begins with the requirement that modal patterns be orthogonal and results in the defining classical characteristic mode generalized eigenvalue problem.

A.1 Derivation

Let us consider the following generalized eigenvalue problem:

\[ [Z] \bar{J}_n = \gamma_n [M] \bar{J}_n \]

where \([Z] = [R] + j[X], [M]\) are complex symmetric matrices, and \(\{\gamma_n = \alpha_n + j\beta_n, \bar{J}_n\}\) is a generalized eigenpair. Then,

\[
\langle \bar{J}_m, [Z] \bar{J}_n \rangle = \langle \bar{J}_m, \gamma_n [M] \bar{J}_n \rangle \\
\langle \bar{J}_m, [R] \bar{J}_n \rangle + j \langle \bar{J}_m, [X] \bar{J}_n \rangle = \alpha_n \langle \bar{J}_m, [M] \bar{J}_n \rangle + j \beta_n \langle \bar{J}_m, [M] \bar{J}_n \rangle
\]

If we restrict \([M]\) to be real,

\[
\langle \bar{J}_m, [M] \bar{J}_n \rangle = \frac{1}{\alpha_n} \langle \bar{J}_m, [R] \bar{J}_n \rangle \\
\langle \bar{J}_m, [M] \bar{J}_n \rangle = \frac{1}{\beta_n} \langle \bar{J}_m, [X] \bar{J}_n \rangle
\]
But what is $\langle \bar{J}_m, [R]\bar{J}_n \rangle$ or $\langle \bar{J}_m, [X]\bar{J}_n \rangle$? These are powers.

Let $(\vec{E}_n, \vec{H}_n)$ be the associated modal fields produced by some impressed source $\bar{J}_n$. A critical property from MoM is that $\bar{J}_n$ is represented by the basis weighting coefficient vector $\bar{J}_n$ and the operator $Z$ is represented by the generalized impedance matrix $[Z]$ such that the following is exactly satisfied:

$$\langle \bar{J}_m, [Z]\bar{J}_n \rangle = \iint_{V'} \bar{J}_m^* \cdot Z(\bar{J}_n) dS$$

We want these fields to be orthogonal in the far-field. $\bar{J}_n$ exists in some closed volume $V'$, which is bounded by some surface $S'$.

From Maxwell’s equations,

$$\nabla \times \vec{E}_n = -j\omega \mu \vec{H}_n \quad (A.1.1)$$
$$\nabla \times \vec{H}_n = j\omega \varepsilon \vec{E}_n + \bar{J}_n \quad (A.1.2)$$

We can derive the complex power from Equations A.1.1 and A.1.2 in a standard way [126]:

$$\vec{E}_m \cdot \nabla \times \vec{H}_n^* - \vec{H}_n^* \cdot \nabla \times \vec{E}_m = -j\omega \varepsilon \vec{E}_m \cdot \vec{E}_n^* + j\omega \mu \vec{H}_n^* \cdot \vec{H}_m + \vec{E}_m \cdot \bar{J}_n^*$$

But

$$\vec{E}_m \cdot \nabla \times \vec{H}_n^* - \vec{H}_n^* \cdot \nabla \times \vec{E}_m = -\nabla \cdot \left( \vec{E}_m \times \vec{H}_n^* \right)$$

Thus,

$$- \iiint_V \nabla \cdot \left( \vec{E}_m \times \vec{H}_n^* \right) dV' = j \omega \iiint_V \left( \mu \vec{H}_m \cdot \vec{H}_n^* - \varepsilon \vec{E}_m \cdot \vec{E}_n^* \right) dV' + \iiint_V \vec{E}_m \cdot \bar{J}_n^* dV' \quad (A.1.3)$$

where $V$ is a volume enclosing $V'$, $S$ is the closed surface of $V$. Let

$$P_{source} = - \iiint_V \vec{E}_m \cdot \bar{J}_n^* dV'$$

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Substituting A.1.3 into the above equation, we obtain:

\[
P_{\text{source}} = \iint_S \left( \vec{E}_m \times \vec{H}_m^* \right) \cdot d\vec{S} + j \omega \iiint_V \left( \mu \vec{H}_m \cdot \vec{H}_n^* - \epsilon \vec{E}_m \cdot \vec{E}_n^* \right) dV' \tag{A.1.4}
\]

In the network domain,

\[
P_{\text{source}} = \langle \bar{J}_n, [Z] \bar{J}_m \rangle = \langle \bar{J}_n, [R] \bar{J}_m \rangle + j \langle \bar{J}_n, [X] \bar{J}_m \rangle
\]

Adding Eq. A.1.4 with its conjugate of the case where \( n \) and \( m \) are interchanged, we obtain

\[
\begin{align*}
&\iint_S \left( \vec{E}_m \times \vec{H}_n^* \right) \cdot d\vec{S} + j \omega \iiint_V \left( \mu \vec{H}_m \cdot \vec{H}_n^* - \epsilon \vec{E}_m \cdot \vec{E}_n^* \right) dV' \\
&+ \iint_S \left( \vec{E}_n^* \times \vec{H}_m^* \right) \cdot d\vec{S} - j \omega \iiint_V \left( \mu \vec{H}_n^* \cdot \vec{H}_m^* - \epsilon \vec{E}_n^* \cdot \vec{E}_m^* \right) dV' = \\
&\langle \bar{J}_n, [R] \bar{J}_m \rangle + j \langle \bar{J}_n, [X] \bar{J}_m \rangle + \langle \bar{J}_m^*, [R] \bar{J}_n^* \rangle - j \langle \bar{J}_m^*, [X] \bar{J}_n^* \rangle
\end{align*}
\]

Since \( \langle b_1, [A] b_2 \rangle = \langle b_2^*, [A] b_1^* \rangle \) for Hermitian \([A]\), the R.H.S. of the above equation is equal to

\[
\begin{align*}
\text{R.H.S.} &= \langle \bar{J}_n, [R] \bar{J}_m \rangle + j \langle \bar{J}_n, [X] \bar{J}_m \rangle + \langle \bar{J}_m^*, [R] \bar{J}_n^* \rangle - j \langle \bar{J}_m^*, [X] \bar{J}_n^* \rangle \\
&= 2 \langle \bar{J}_n, [R] \bar{J}_m \rangle
\end{align*}
\]

Thus,

\[
\iint_S \left( \vec{E}_m \times \vec{H}_n^* + \vec{E}_n^* \times \vec{H}_m^* \right) \cdot d\vec{S} = 2 \langle \bar{J}_n, [R] \bar{J}_m \rangle \tag{A.1.5}
\]

In the far field (i.e. \( S \to \Sigma, V \to V_\infty \)),

\[
\begin{align*}
\vec{E}_n' &= \eta \vec{H}_n \times \hat{r} \\
\vec{H}_n &= \frac{1}{\eta} \hat{r} \times \vec{E}_n
\end{align*}
\]
Under these conditions,

\[
\vec{E}_m \times \vec{H}_n^* = \frac{1}{\eta} \left( \vec{E}_m \times \hat{r} \times \vec{E}_n^* \right) \\
= \frac{1}{\eta} \hat{r} \left( \vec{E}_m \cdot \vec{E}_n^* \right) - \vec{E}_n^* \left( \vec{E}_m \cdot \hat{r} \right) \\
= \frac{1}{\eta} \hat{r} \left( \vec{E}_m \cdot \vec{E}_n^* \right)
\]

Thus, Eq. A.1.5 simplifies in the far-field to the following:

\[
\frac{1}{\eta} \iint_\Sigma \left( \vec{E}_m \cdot \vec{E}_n^* + \vec{E}_m \cdot \vec{E}_n^* \right) \hat{r} \cdot \hat{r} dS' = 2 \langle \vec{J}_n, [R] \vec{J}_m \rangle \\
\frac{1}{\eta} \iint_\Sigma \vec{E}_m \cdot \vec{E}_n^* dS' = \langle \vec{J}_n, [R] \vec{J}_m \rangle
\]

We want \( \vec{E}_n \) and \( \vec{E}_m \) to be orthogonal over the entire sphere in the far-field \( \Sigma \).
Specifically,

\[
\frac{1}{\eta} \iint_\Sigma \vec{E}_m \cdot \vec{E}_n^* dS' = \delta_{mn}
\]

It follows that there is a similar orthogonality between the modal magnetic fields:

\[
\eta \iint_\Sigma \vec{H}_m \cdot \vec{H}_n^* dS' = \delta_{mn}
\]

Therefore, the property of far-field orthogonality requires

\[
\langle \vec{J}_n, [R] \vec{J}_m \rangle = \delta_{mn}
\]

This condition implies

\[
\langle \vec{J}_m, [M] \vec{J}_n \rangle = \frac{1}{\alpha_n} \delta_{mn}
\]

If we let \( \alpha_n = 1 \) and \( \beta_n = \lambda_n \), then

\[
\langle \vec{J}_m, [M] \vec{J}_n \rangle = \langle \vec{J}_m, [R] \vec{J}_n \rangle
\]

The above relationship is obviously satisfied if we let \([M] = [R]\). Then,

\[
[Z] \vec{J}_n = ([R] + j[X]) \vec{J}_n \\
= (1 + j\lambda_n)[R] \vec{J}_n
\]

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Simplifying, we finally obtain the defining generalized eigenvalue problem for classical characteristic modes:

\[
[X] \bar{J}_n = \lambda_n [R] \bar{J}_n
\]

Notice that the above GEP implies far-field orthogonality in addition to orthogonality over the source domain (by virtue of the spectral theorem [93, pg. 296]).
Appendix B

MODAL INPUT POWER IN CLASSICAL CHARACTERISTIC MODES

This appendix discusses the modal decomposition of total input power in classical characteristic mode analysis.

B.1 Derivation

In general, the input power for an antenna in MoM can be described by [23]

\[ P_{in} = \frac{1}{2} \text{Re} \left\{ \langle \bar{J}, [Z] \bar{J} \rangle \right\} \]

To decompose the input power in terms of classical characteristic modes, we begin by decomposing the total current into a weighted sum of modal currents:

\[ \bar{J} = \sum_{n}^{N} \alpha_{n} \bar{J}_{n} \]

By linearity, it follows that

\[ [Z] \bar{J} = \sum_{n}^{N} \alpha_{n} (1 + j \lambda_{n}) [R] \bar{J}_{n} \]

Computing the complex total power:

\[ \langle \bar{J}, [Z] \bar{J} \rangle = \sum_{n}^{N} \alpha_{m}^{*} \left\langle \bar{J}_{m}, \sum_{n}^{N} \alpha_{n} (1 + j \lambda_{n}) [R] \bar{J}_{n} \right\rangle \]

\[ = \sum_{n}^{N} \alpha_{m}^{*} \sum_{n}^{N} \alpha_{n} (1 + j \lambda_{n}) \langle \bar{J}_{m}, [R] \bar{J}_{n} \rangle \]

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Since $\bar{J}_n$ are $[R]$ orthogonal, the complex total power simplifies to:

$$\langle \bar{J}, [Z] \bar{J} \rangle = \sum_n |\alpha_m|^2 (1 + j\lambda_m) \langle \bar{J}_m, [R] \bar{J}_m \rangle$$

Therefore, the total input power for the antenna is given by

$$P_{in} = \frac{1}{2} \sum_n |\alpha_n|^2 \langle \bar{J}_n, [R] \bar{J}_n \rangle$$

If we assume that the modal currents have been normalized such that $\langle \bar{J}_n, [R] \bar{J}_n \rangle$, then the total input power expression simplifies further:

$$P_{in} = \frac{1}{2} \sum_n |\alpha_n|^2$$

(B.1.1)
Appendix C

MODAL WEIGHTING COEFFICIENT DERIVATION
FOR GENERAL MODAL SYSTEMS

This appendix discusses the modal weighting coefficient definition for general modal systems. We begin by discussing an alternate formula for the modal weighting coefficient in classical characteristic modes.

C.1 General Modal Systems

In all characteristic mode related modal systems, we can decompose a total current into a weighted sum of modal vectors (usually modal currents, so we shall denote them by $\bar{J}_n$):

$$\bar{J} = \sum_{n}^{N} \alpha_n \bar{J}_n$$

But what are the $\alpha_n$’s and how are they related to the excitation field vector $E^i$? From MoM, recall that

$$E^i = [Z]J$$
C.2 Classical CM

Classical characteristic modes are defined by the following generalized eigenvalue problem

\[
[X] \vec{J}_n = \lambda_n [R] \vec{J}_n \quad \text{(C.2.1)}
\]

This generalized eigenvalue problem may be equivalently stated as (see A)

\[
[Z] \vec{J}_n = ([R] + j[X]) \vec{J}_n = (1 + j\lambda_n)[R] \vec{J}_n
\]

Then,

\[
\langle \vec{J}_n, \vec{E}_i \rangle = \langle \vec{J}_n, [Z]\vec{J} \rangle \\
= \sum_m \alpha_m \langle \vec{J}_n, [Z]\vec{J}_m \rangle \\
= \sum_m \alpha_m (1 + j\lambda_m) \langle \vec{J}_n, [R]\vec{J}_m \rangle \\
= \alpha_n (1 + j\lambda_n)
\]

The last step uses the fact that \( \vec{J}_n \) are \( R \) orthogonal in classical characteristic modes and that we assume that \( \vec{J}_n \) has been normalized such that \( \langle \vec{J}_n, [R]\vec{J}_n \rangle = 1 \). Finally, we can solve for the modal weighting coefficient \( \alpha_n \):

\[
\alpha_n = \frac{\langle \vec{J}_n, \vec{E}_i \rangle}{1 + j\lambda_n} \quad \text{(C.2.2)}
\]

The above equation is the standard modal weighting coefficient formula for classical characteristic modes found in the literature [22]. If we instead assume that the MoM problem is defined as

\[
\vec{E}_i = -[Z]\vec{J}
\]
or equivalently\(^1\),

\[
\left[ L(\vec{J}) + \vec{E}_i \right]_{\tan} = 0,
\]

\(^1\)We have been writing \( \vec{E}_i \) instead of \( \vec{E}_{i\tan} \) for convenience.
then the modal weighting coefficient formula is instead:

\[ \alpha_n = -\frac{\langle \bar{J}_n, E^i \rangle}{1 + j\lambda_n} \]  

(C.2.3)

Modes from ESP5 [95] use Eq. C.2.2, while modes from FEKO should use Eq. C.2.3. To ensure equivalence, the processing code for FEKO MoM Z matrices includes a negative sign so that modes computed using FEKO in the UCM system also use C.2.2.

### C.2.1 Alternative Definition

There is an alternative way to compute the \( \alpha_n \)'s in classical characteristic modes. We begin with a slightly different inner product involving only eigencurrents \( \bar{J}_n \) and the source current \( \bar{J} \):

\[
\langle \bar{J}_n, [R] \bar{J} \rangle = \sum_m^N \alpha_m \langle \bar{J}_n, [R] \bar{J}_m \rangle
= \sum_m^N \alpha_m \delta_{mn} \langle \bar{J}_n, [R] \bar{J}_n \rangle
= \alpha_n \langle \bar{J}_n, [R] \bar{J}_n \rangle
\]

Thus, the modal weighting coefficient in classical characteristic modes is also:

\[ \alpha_n = \frac{\langle \bar{J}_n, [R] \bar{J} \rangle}{\langle \bar{J}_n, [R] \bar{J}_n \rangle} \]  

(C.2.4)

Since we normally choose the normalize the \( \bar{J}_n \) such that \( \langle \bar{J}_n, [R] \bar{J}_n \rangle = 1 \), then

\[ \alpha_n = \langle \bar{J}_n, [R] \bar{J} \rangle \]

This way of deriving the expression of the modal weighting coefficients \( \alpha_n \) in the source domain is important, since it is more easily related to the generalized eigenvalue problem C.2.1. Since both the \([X]\) and \([R]\) matrices are real symmetric matrices (and more generally, they are complex Hermitian matrices), the eigenvalues \( \lambda_n \) are real and the eigenvectors \( \bar{J}_n \) are orthogonal with respect to \([R]\) and \([X]\).
C.3 Generalized CM

Generalized characteristic modes are defined by the following generalized eigenvalue problem:

\[
[X] \bar{J}_n = \lambda_n [H] \bar{J}_n
\]  \hspace{1cm} (C.3.1)

where

\[
[H] = [G]^H [G]
\]

\[
W(r, \theta, \phi) \bar{F}(r, \theta, \phi) = [G] \bar{J}
\]

and \([G]\) is the matrix operator which maps the meshed current \(\bar{J}\) onto the electric field \(\bar{F}(r, \theta, \phi)\) weighted by some real weighting function \(W(r, \theta, \phi)\).

Since both \([X]\) and \([H]\) are Hermitian (\([X]\) is a real symmetric matrix, while \([H]\) is a complex Hermitian matrix) and \([H]\) is positive definite, the eigenvalues \(\lambda_n\) are real and the eigenvectors \(\bar{J}_n\) are orthogonal with respect to \([H]\) or \([X]\). Specifically,

\[
\langle \bar{J}_m, [H] \bar{J}_n \rangle = 0
\]

\[
\langle \bar{J}_m, [X] \bar{J}_n \rangle = 0
\]

for \(m \neq n\).

To compute the modal weighting coefficients \(\alpha_n\) for generalized characteristic modes, we can use these orthogonality relationships. For \([M] = [H]\) or \([M] = [X]\), we have

\[
\langle \bar{J}_n, [M] \bar{J} \rangle = \sum_n \alpha_m \langle \bar{J}_m, [M] \bar{J}_n \rangle
\]

\[
= \sum_m \alpha_m \delta_{mn} \langle \bar{J}_m, [M] \bar{J}_n \rangle
\]

\[
= \alpha_n \langle \bar{J}_n, [M] \bar{J}_n \rangle
\]
Thus, the modal weighting coefficients $\alpha_n$ for generalized characteristic modes are given by

$$\alpha_n = \frac{\langle \bar{J}_n, [M] \bar{J} \rangle}{\langle \bar{J}_n, [M] \bar{J}_n \rangle} \quad (C.3.2)$$

where $[M] = [H]$ or $[M] = [X]$.

### C.4 Other CM Systems

Other compatible characteristic mode analysis systems may be specified as

$$[N] \bar{J}_n = \lambda_n [D] \bar{J}_n \quad (C.4.1)$$

Assuming that $[N]$ and $[D]$ are complex Hermitian matrices and $[D]$ is a positive definite matrix, the eigenvectors $J_n$ are orthogonal with respect to $[N]$ and $[D]$.

Generalizing from the previous section, if we let $[M] = [N]$ or $[M] = [D]$, we may similarly derive the modal weighting coefficients $\alpha_n$ as

$$\alpha_n = \frac{\langle \bar{J}_n, [M] \bar{J} \rangle}{\langle \bar{J}_n, [M] \bar{J}_n \rangle} \quad (C.4.2)$$

Notice that $[M]$ is not necessarily related to the impedance matrix $[Z]$, which implies that the modal weighting coefficients are not directly related to $\bar{E}^i$. We can explicitly show the modal weighting coefficient dependence on $\bar{E}^i$ (assuming that $\bar{J}_n$ is a modal current) by noting that

$$\bar{J} = [Y] \bar{E}^i$$

Thus, the modal weighting coefficients are restated as

$$\alpha_n = \frac{\langle \bar{J}_n, [M] [Y] \bar{E}^i \rangle}{\langle \bar{J}_n, [M] \bar{J}_n \rangle} \quad (C.4.3)$$

Equation C.4.3 is the most general expression for the modal weighting coefficient that explicitly depends upon the excitation vector $E^i$. 

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Appendix D

MODAL WEIGHTING COEFFICIENT DERIVATION USING MODAL PATTERNS

For modal systems defining far-field modal orthogonality such as CCM, it is possible to decompose a total field into a weighted superposition of modal fields:

\[ \vec{E} = \sum_{n}^{N} \alpha_n \vec{E}_n \]

The computation of the weights \( \{\alpha_n\} \) can be achieved due to the orthogonality of the modal fields

\[ \langle \vec{E}_m, \vec{E}_n \rangle_{\Sigma} = 0 \text{ for } m \neq n \]

where the inner product is defined as

\[ \langle \vec{E}_m, \vec{E}_n \rangle_{\Sigma} \equiv \int_{0}^{2\pi} d\phi \int_{0}^{\pi} \vec{E}_m^* \vec{E}_n \sin \theta d\theta \]  \hspace{1cm} (D.0.1)

The far-field orthogonality is an essential feature of especially CCM theory, enabling us to derive the value of the weights \( \{\alpha_n\} \). A brief derivation of the modal weights is provided below and is followed by important numerical considerations.

D.1 Derivation of Modal Weights

Let some total far-field pattern \( \vec{E} \) be a weighted superposition of modal fields \( \{\vec{E}_n\} \):

\[ \vec{E} = \sum_{n}^{N} \alpha_n \vec{E}_n \]
Now, compute the inner product of this total pattern with some particular modal far-field pattern \( \vec{E}_m \):

\[
\langle \vec{E}_m, \vec{E} \rangle_{\Sigma} = \langle \vec{E}_m, \sum_n^{N} \alpha_n \vec{E}_n \rangle_{\Sigma} = \sum_n^{N} \alpha_n \langle \vec{E}_m, \vec{E}_n \rangle_{\Sigma} = \alpha_m \langle \vec{E}_m, \vec{E}_m \rangle_{\Sigma}
\]

since \( \langle \vec{E}_m, \vec{E}_n \rangle_{\Sigma} = 0 \) for \( m \neq n \).

Thus, the modal weight is simply

\[
\alpha_n = \frac{\langle \vec{E}_n, \vec{E} \rangle_{\Sigma}}{\langle \vec{E}_n, \vec{E}_n \rangle_{\Sigma}} \tag{D.1.1}
\]

### D.2 Numerical Considerations

To reliably use the Equation D.1.1 to compute the modal weights using the modal patterns, it is required that the modal pattern orthogonality holds for all \( N \) modes. If it does not not (due to numerical errors), then Equation D.1.1 is at best approximate.

The formula requires the precise computation of two inner products, both involving Equation D.0.1. The denominator of Equation D.1.1 is concerned with the radiated power of the modal pattern. The error from the double integral underlying this inner product (Equation D.0.1), approximated using the trapezoidal rule, depends only on the magnitude of \( \vec{E}_n \), rather than the magnitude and phase. Thus, the denominator is fairly stable in practice, since the modal pattern magnitude is typically much less susceptible to numerical noise.

The numerator of Equation D.1.1, however, involves a double integral which does depend on both the magnitude and phase of the two modal patterns. In order for modal pattern orthogonality to be satisfied, the phases of \( \vec{E}_n \) and \( \vec{E} \) must be known.
accurately, such that the outcome of the integral represents only the projection of \( \vec{E} \) onto \( \vec{E}_n \). If the pattern phases have some small error, then the result of the integral shall involve some contributions from the other modal patterns. Unfortunately, the actual pattern phases obtained from several numerical EM solvers lack sufficient precision, causing modal orthogonality to degrade, especially for the high-directivity modal patterns radiated by higher order modes.

How do we practically address this serious numerical problem? We return to the above derivation, but assume that the patterns are not orthogonal.

\[
\langle \vec{E}_m, \vec{E} \rangle_\Sigma = \sum_{n=1}^{N} \alpha_n \langle \vec{E}_m, \vec{E}_n \rangle_\Sigma
\]

The above equation actually represents \( N \) linearly independent equations \((m = 1 \text{ to } N)\). Let

\[
\hat{\alpha}_m = \frac{\langle \vec{E}_m, \vec{E} \rangle_\Sigma}{\langle \vec{E}_m, \vec{E}_m \rangle_\Sigma}
\]

Then, we can represent the \( N \) linear equations in matrix form:

\[
\vec{\hat{\alpha}} = [C]\vec{\alpha}
\]

where

\[
C_{mn} = \frac{\langle \vec{E}_m, \vec{E}_n \rangle_\Sigma}{\langle \vec{E}_m, \vec{E}_m \rangle_\Sigma}
\]

\[
\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_2 \end{bmatrix}, \quad \vec{\hat{\alpha}} = \begin{bmatrix} \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_2 \end{bmatrix}
\]

Ideally, \([C]\) is the identity matrix (for orthogonal modal patterns), but any spillover
in the projection of one modal pattern on another is also recorded using this formulation. Therefore, to obtain the actual weighting coefficients \( \{ \alpha_n \} \) from the raw pattern weight coefficients \( \{ \hat{\alpha}_n \} \), we can simply invert the matrix \([C]\):

\[
\hat{\alpha} = [C]^{-1} \bar{\alpha}
\]

The matrix \([C]\) is referred to as the *eigenpattern calibration matrix*. An important property to notice about \([C]\) is that it is *almost* Hermitian:

\[
C_{mn} = \frac{\langle \vec{E}_m, \vec{E}_n \rangle_{\Sigma}}{\langle \vec{E}_m, \vec{E}_m \rangle_{\Sigma}}
\]

\[
C_{nm} = \frac{\langle \vec{E}_n, \vec{E}_m \rangle_{\Sigma}}{\langle \vec{E}_n, \vec{E}_n \rangle_{\Sigma}} = \frac{\langle \vec{E}_m, \vec{E}_n \rangle^*_{\Sigma}}{\langle \vec{E}_n, \vec{E}_n \rangle_{\Sigma}}
\]

The only case in which \(C_{mn} = C_{nm}^*\) is when \(\langle \vec{E}_m, \vec{E}_m \rangle_{\Sigma} = \langle \vec{E}_n, \vec{E}_n \rangle_{\Sigma}\). That is, if each modal field is normalized such that \(\langle \vec{E}_m, \vec{E}_m \rangle_{\Sigma} = K\) (for some real positive \(K\)) for any \(m\), then \([C]\) is a Hermitian matrix. This property can save considerable time in computing \([C]\) at each frequency.
Appendix E

MUTUAL IMPEDANCE DERIVATION

The mutual impedance of a two port system $Z_{21}^P$ is defined as:

$$\left. Z_{21}^P = \frac{V_{oc}^2}{I_1} \right|_{I_1=0}$$

The expression of mutual impedance involving electromagnetic quantities may be derived using the Lorentz reciprocity principle [126, pp. 118-119]. The reciprocity principle is applied to the two problems (a) and (b) shown in Figure E.1. From reciprocity, we have

$$\int_{V_{Ant.~2}} \vec{E}_2^A \cdot \vec{J}_2^B dV = \int_{V_{Ant.~2}} \vec{E}_2^B \cdot \vec{J}_2^A dV$$

Note that

$$\int_{V_{Ant.~2}} \vec{E}_2^A \cdot \vec{J}_2^B dV = \int_{Port~2} \vec{E}_2^A \cdot I_2^B d\vec{l} = -V_2^A I_2^B$$

which implies that the open-circuit voltage at port 2 in Figure E.1(a) is identical to $V_{oc}^2$:

$$V_{oc}^2 = V_2^A = -\frac{1}{I_2^B} \int_{V_{Ant.~2}} \vec{E}_2^A \cdot \vec{J}_2^B dV$$

From the definition of mutual impedance, we have

$$Z_{21} = \frac{V_{oc}^2}{I_1} = \frac{V_2^A}{I_2^B} = -\frac{1}{I_1 I_2^B} \int_{V_{Ant.~2}} \vec{E}_2^A \cdot \vec{J}_2^B dV$$

If we let the terminal currents be the same $I_p = I_1 = I_2^B$, then we arrive at the final expression for mutual impedance:

$$Z_{21} = -\frac{1}{(I_p)^2} \int_{V_{Ant.~2}} \vec{E}_2^A \cdot \vec{J}_2^B dV \quad \text{(E.0.1)}$$
where $\vec{E}_2^A$ is the total induced electric field across port 2 due to the impressed current density $\vec{J}_1^A$ on antenna 1, generated by some impressed current source $I_1$ at port 1, provided that port 2 is open-circuited; and $\vec{J}_2^B$ is the impressed current density on antenna 2, generated by an impressed current source $I_2^B$ at port 2, provided that port 1 is open-circuited.

This expression E.0.1 is the same as Equation (24) from [84].
BIBLIOGRAPHY


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