Multi-Physics Interactions for Coupled Thermo-Electro-Magneto-Mechanical Effects

THESIS

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ABSTRACT

Smart or active materials have advantageous properties that allow for material behavior control. This has expanded the capabilities of smart materials for engineering applications, ranging from biomedical devices to defense technology. Optimization of designing and controlling these devices demands robust and reliable experimental characterization. The study of smart materials requires careful consideration of the interactions between coupled physical effects. Smart materials propose a challenge because of the number of physical effects and experiments involved in order to fully characterize coupled material behavior. To clarify these interactions, schematics were developed in earlier literature to convey the couplings of reversible processes. In this thesis, schematics, coined Multi-Physics Interaction Diagrams (MPID), illustrate the potential coupling of thermal, electrical, magnetic, and mechanical effects exhibited by smart materials. By extending and adapting previous diagrams, a thorough evaluation of all reversible and irreversible phenomena can allow for minimizing the potential number of experiments involved to fully characterize material behavior. The MPIDs provided are a starting point for depicting the involved effects and can be extended to other physical effects.

This thesis focuses on visually illustrating the coupled behavior of materials to aid constitutive modeling and illuminate the energetic relations. In recent work, the
mathematical characterization of the fully-coupled, nonlinear, three-dimensional behavior of these materials has been completed through different Thermo-Electro-Magneto-Mechanical (TEMM) processes. One of the energetic functions from these processes was selected as an example and applied to illustrate the use of MPIDs with regards to thermoelastic constitutive modeling. The schematics aid in developing the constitutive relations based on assumptions for particular applications. MPIDs were also shown to relate the energetic functions within thermomechanical processes. A special case of magneto-thermoelastic conditions is used to develop governing equations and proposed for a boundary value problem. With this example, potential directions of the MPIDs can further aid in the development of constitutive modeling for more complex coupled materials.
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I dedicate this thesis to my parents and close friends, who have relentlessly supported me throughout the years. I do not know what I would do without you all.
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CHAPTER 1: Multi-Physics Interactions

1.1 Introduction

Smart materials are used in complex devices that exhibit the coupling of multiple physical effects. To optimize the design of these devices, the study of smart materials requires careful consideration of these interactions. This complex coupling of effects can be clarified through Multi-Physics Interaction Diagrams (MPIDs) to illuminate the processes, material constants, and experiments involved for constitutive modeling, as well as illustrate the energetic functions used to describe the interactions. For the development of these MPIDs, only thermal, electrical, magnetic, and mechanical physical effects are considered for the scope of this thesis. MPIDs are shown in earlier literature for the development of constitutive modeling and thermodynamic potentials. A description of the previous schematics and background information to create MPIDs are provided in the proceeding sections.

1.1.1 Schematics Description

MPIDs were first shown in a tetrahedron diagram, Figure 1 by Cady [2] in the 1940s. Although the primary focus of Cady’s [2] work was piezoelectricity, he formulated the theory in thermodynamic diagrams, involving elastic, dielectric, thermal,
and magnetic effects, and proposed the relationships between these linear physical effects.

Figure 1: Tetrahedron diagram of elastic, dielectric, thermal, and magnetic effects [2]

Using Figure 1, [2] developed thermodynamic potentials for piezoelectric systems and highlighted the relations between thermal, elastic, and electric phenomena. This visual representation was adapted by Nye [3], but narrowed his focus to the study of crystals with only electrical, mechanical, and thermal effects. The work of [3] only presented the material constants with respect to these relations and the associated constitutive relations. The illustration in [3] has been adapted and modified by Smith [4] to include a different conjugate pair for the electric property. Similar to the work of [3], Smith [4] provided
constitutive equations and material constants, but also included a more applicable measurable quantity, polarization, as opposed to electric displacement. The work of [4] also illustrates the use of schematics for developing thermodynamic potentials. Hartley’s [5] adaptation of the diagrams, which includes the magnetic consideration, is much closer to the interaction diagram of [2], but in more general terms. The work of [5] provided only an in depth look into the thermomechanical processes and associated constitutive relations. All of the MPIDs listed are only applied to linear, reversible processes.

With these schematics in mind, the reversible MPID has been extended to included magnetism, similar to [2], and more physically meaningful quantities (polarization and magnetization), similar to [4]. In addition, the MPIDs presented in Chapter 2 include a mechanical conjugate pair that accommodates three-dimensional, time-dependent, finite deformations and stresses: the first Piola-Kirchhoff stress and deformation gradient tensor. This thesis also expands on previous schematics by thoroughly evaluating and uncovering all potential linear processes.

Since the MPIDs are adapted and applied in a more general sense, the fully coupled relationships within thermal, electrical, magnetic, and mechanical effects were extended to irreversible phenomena. The only set of MPIDs for nonequilibrium thermodynamics are given in [6]. By expanding the MPIDs to include irreversible processes, new physical effects can be illuminated by presenting this big picture model; a deeper discussion of this is provided in Section 2.3

The framework of [6] for irreversible processes was extended to include the thermodynamic force for magnetism to highlight the interactions with magnetic field, but
also dismissed a mechanical generalized flow term. Along with transport processes, other irreversible processes, such as magnetostriction were considered as part of the irreversible phenomena for the MPIDs.

1.1.2 Overview

The coupling between at least two different physical effects is exhibited in smart materials. Specifically focusing on the thermal, electrical, magnetic, and mechanical effects, the background information required to construct a Multi-Physics Interaction Diagram (MPID) is explored in Chapter 1. MPIDs are devices used to illustrate complex coupling of physical effects in smart materials and structures. Illustrations of such reversible and irreversible processes are presented in Chapter 2. An overview of a general Thermo-Electro-Magneto-Mechanical (TEMM) process, an application of MPIDs to a thermoelastic constitutive relation, and the energetic relations in thermomechanical processes are shown in Chapter 3. The start of a magneto-thermoelastic example is presented in Chapter 4. An overview of potential applications and future directions are presented in Chapter 5.

1.2 Background

The development of the MPIDs is a multi-disciplinary exercise: concepts from continuum mechanics, elasticity, chemistry, thermodynamics, electricity, and magnetism must be taken into consideration to completely describe smart material behavior. This section discusses the thermodynamic frameworks considered, the reasons for the MPID
parameter selections, and the physical characteristics of materials that impact the effects explained in this thesis.

1.2.1 Thermodynamics

For the construction of the MPIDs, equilibrium and nonequilibrium thermodynamics must be considered. Reversible processes fall into the category of equilibrium thermodynamics, while irreversible phenomena are described through the framework of nonequilibrium thermodynamics. The essential difference is illuminated by the second law of thermodynamics: reversible systems have zero entropy production and are conservative systems, i.e. the system only depends on the initial and final states [5]. Nonequilibrium thermodynamics considers processes when entropy is produced within a system and is no longer in the same initial and final states [5]. Irreversible processes are evaluated within nonequilibrium thermodynamics due to the dissipative material properties and hysteresis. Conventionally, irreversible processes are evaluated at near-equilibrium conditions in linear form [24].

1.2.2 Phenomenological Equations & Linearity Laws

Nonequilibrium thermodynamics consists of cross-effects that produce irreversible processes, which are described by phenomenological equations (1) [13], where $J$ represents a generalized thermodynamic flow and $X$ are the thermodynamic forces.

$$J_a = L_{ab}X_b$$

(1)
The terms, $L_{ab}$ in (1) are the phenomenological coefficients, or proportionality factors, that couple the relationships between thermodynamic forces and flows [10]. Phenomenological equations are applied to the processes to evaluate the irreversible phenomena at near-equilibrium states.

Onsager’s relations are based on equation 1, where diffusion flow of matter and the concentration gradient can be linearly related in a generic set of variables [11]. Onsager’s reciprocal relations treat the phenomenological coefficients as symmetric terms by taking into consideration not only the influence of spatial symmetry, but also the property of time-reversal invariance [11]. The application of Onsager’s principles are to make assumptions that the processes evaluated are only small deviations from equilibrium, and therefore, can be assumed with linear relationships.

1.2.3 Parameter Selection

State functions, which depend on the state variables, fully describe the thermodynamic state of the system. These state variables can be categorized as intensive quantities, i.e. independent of the system size, or extensive quantities, i.e. dependent on the system size. These parameters can also be considered as independent and dependent variables, where the extensive fields can be considered independent variables through the Legendre transformation. The thermodynamic intensive and extensive quantities are used to describe the parameters selected for reversible and irreversible processes. In Chapter 3, the parameters are described as independent and dependent variables to be consistent with the constitutive relations and energetic functions of [1].
In the first and second law of thermodynamics, the energy contributions from the physical effects, i.e. thermal, electrical, magnetic, and mechanical, are described by conjugate pairs of variables, each consisting of an intensive and extensive parameter. For the selection of parameters in the reversible and irreversible processes, the Clausius-Duhem inequality is used to establish the conjugate pairs. The inequality, presented in (2), highlights the terms that are relevant to the reversible and irreversible processes, where \( \dot{e} \) is the material derivative of the internal energy of the system.

\[
-\rho \dot{e} + \frac{1}{f} \rho \dot{F} + \rho \theta \eta + \rho e \cdot \left( \frac{\dot{P}}{\rho} \right) + \rho \mu \delta \cdot \left( \frac{\dot{m}}{\rho} \right) + j \cdot e - \frac{1}{\theta} q \cdot \text{grad} \theta \geq 0
\]  

(2)

For reversible thermodynamics, the convention is to introduce the conjugate pairs as a thermodynamic force (intensive variable) and a generalized displacement term (extensive variable), highlighted in blue in (2). These conjugate pairs are provided in Table 1, where the bold lowercase variables denote vector quantities and the bold uppercase variables denote tensor quantities.

<table>
<thead>
<tr>
<th>Property</th>
<th>Intensive Parameter</th>
<th>Extensive Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal</td>
<td>Absolute Temperature, ( \theta )</td>
<td>Entropy, ( \eta )</td>
</tr>
<tr>
<td>Electric</td>
<td>Electric Field, ( e ) or ( e_i )</td>
<td>Polarization, ( p ) or ( p_i )</td>
</tr>
<tr>
<td>Magnetic</td>
<td>Magnetic Field, ( h ) or ( h_i )</td>
<td>Magnetization, ( m ) or ( m_i )</td>
</tr>
<tr>
<td>Mechanical</td>
<td>First Piola-Kirchhoff Stress Tensor, ( P ) or ( P_{iA} )</td>
<td>Deformation Tensor, ( F ) or ( F_{iA} )</td>
</tr>
</tbody>
</table>

Table 1: Conjugate pairs for reversible processes
Conventionally, electric displacement and magnetic induction terms are used to describe the material, as opposed to the selected parameters, polarization and magnetization. Polarization and magnetization were selected similar to the work in [1].

Mechanical parameters were also selected to be consistent with [1], which employ the deformation gradient tensor, $F$ instead of the infinitesimal strain tensor, $\varepsilon_{ij}$. For small displacement assumptions, the Piola-Kirchhoff stress tensor, $P_{iA}$ can be generalized as an elastic stress tensor, $T_{ij}$. Likewise, the deformation gradient tensor, $F_{iA}$ can first be related to the Eulerian strain tensor, illustrated in (3) [29]. With small displacement and linear elastic assumptions, the Eulerian strain tensor can be expanded in terms of the deformation tensor, $u$, and converted to the infinitesimal strain tensor, described in (4) [29]. These equations are consistent with Einstein’s summation convention, and the tensor, $F_{iA}^{-1}$ is the inverse of the deformation gradient tensor, $F_{iA}$.

$$\varepsilon_{ij} = \frac{1}{2} (\delta_{ij} - F_{Al}^{-1} F_{Aj}^{-1})$$  \hspace{1cm} (3)

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$  \hspace{1cm} (4)

For irreversible phenomena within nonequilibrium thermodynamics, the generalized displacement terms change to generalized flow parameters. Similar to the reversible case, generalized thermodynamic forces are the “causes” of irreversible processes, and generalized flows are the “effects” of thermodynamic forces [13]. The convention of classifying generalized flow terms as “strengths” are for scalar parameters and “fluxes” for vector terms [13]. These different sets of conjugate pairs are highlighted in red in (2) and summarized in Table 2.
<table>
<thead>
<tr>
<th>Property</th>
<th>Intensive Parameter</th>
<th>Extensive Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal</td>
<td>Temperature Gradient, (\nabla \theta) or (\theta_i)</td>
<td>Heat Flow, (q) or (q_i)</td>
</tr>
<tr>
<td>Electric</td>
<td>Electric Field, (e) or (e_i)</td>
<td>Current Density, (j) or (j_i)</td>
</tr>
<tr>
<td>Magnetic</td>
<td>Magnetic Field, (h) or (h_i)</td>
<td>Nonexistent thermodynamic flow term</td>
</tr>
<tr>
<td>Mechanical</td>
<td>First Piola-Kirchhoff Stress Tensor, (P) or (P_{iA})</td>
<td>Nonexistent thermodynamic flow term</td>
</tr>
</tbody>
</table>

Table 2: Conjugate pairs for irreversible processes

For irreversible thermodynamics, both magnetic thermodynamic force and flow terms do not exist. Irreversible thermodynamics is a framework that describes transport processes, which consist of conjugate pairs consisting of thermodynamic forces and generalized flows [5]. This generalized flow describes “the movement of matter or energy from one part of the thermodynamic system to another” [5]. This places constraints on the selection of intensive and extensive parameters, which are not only rate dependent terms, but also dependent on the individual particles of matter. Unlike electrical charges, an equivalent magnetic monopole does not exist, i.e. the magnetic poles are always found in pairs of opposite signs [27]. The absence of magnetic monopoles prevents a possible definition for the thermodynamic force (intensive variable), and this also means that the magnetic spins are not conserved, preventing a definition for the generalized flow term (magnetic extensive variable) [6].

The magnetic field is kept as the intensive parameter, only to show the irreversible relationships with respect to this field term. Similarly, the mechanical stress
The product of the intensive and extensive parameters provides a work and power term for the reversible and irreversible processes, respectively. Since the reversible conjugate pairs provide a work term, the product of the parameters is integratible. This becomes important for the consideration of defining the energy function. In the work of [1], the reversible conjugate pairs highlighted in (2) are the only parameters that can be used to postulate the constitutive relations. Since the nonlinear processes exhibit irreversibility and are non-integratable, the irreversible conjugate terms in (2) become the residual inequality [1].

1.2.4 Physics of Magnetic Materials

This section consists of a brief overview of magnetic material properties to provide a better understanding of the relations described in Chapter 2. Magnetic materials, specifically ferromagnetic, are composed of randomly arranged domains, with each domain consisting of a dipole with positive and negative ends [7]. The range of temperature of the magnetic material has a large impact on the responses of the material. For ferromagnetics below the Curie temperature (conventionally denoted as $T_c$ or $\theta_c$), the response of the material under any applied magnetic field is non-linear and usually exhibits hysteresis [27]. Ferromagnetics exhibit spontaneous magnetism in the absence of a magnetic field, have a high susceptibility, and a strong tendency for the dipoles to line up with the magnetic field lines [7].
CHAPTER 2: Reversible and Irreversible Processes

2.1 Introduction

In this chapter, schematics are illustrated for a comprehensive and detailed description of the reversible and irreversible interactions between thermal, electrical, magnetic, and mechanical physical effects. By specifying these relationships, the material constants and constitutive equations are considered alongside of the cross-effects and experiments involved. An explanation of the elements within MPIDs is first approached with reversible processes, where uncoupled and coupled effects are taken into account. The MPIDs are then extended into irreversible processes.

2.2 Linear, Reversible Processes

For ease of understanding, MPIDs are first evaluated within linear, reversible phenomena restricted to equilibrium thermodynamics. An extension of the schematics specified in Chapter 1, namely [2] and [3], is presented in complete form (Figure 2). The intensive parameters, \( P, \theta, e, \) and \( h \) are clearly depicted on the outside perimeter of Figure 2, and the corresponding conjugate extensive parameters, \( F, \eta, p, \) and \( m \) are highlighted on the inside perimeter.
All of the processes involved between each parameter are illustrated in Figure 2 to fully define the reversible interactions within thermal, electrical, magnetic, and mechanical physical effects.

The presentation of this MPID allows for the processes to be categorized as “primary” and “secondary” effects [2]. A single-step process between two parameters or a primary effect is represented by the thin, solid lines in Figure 2. In instances where the primary effect is more difficult to experiment, the corresponding secondary effect may be evaluated. For example, the process of $P \rightarrow \theta$ can be alternatively described by going through the two-step process of $P \rightarrow F \rightarrow \theta$, which involves the use of secondary effects. Although the number of possible processes to describe the interactions of physical effects
is increased, the alternative paths can potentially minimize the number of experiments conventionally used to characterize the material.

The increased number of potential effects is better interpreted by creating subset diagrams of Figure 2. A modified depiction is presented in Figure 3 to illustrate the possible subsets, or “panels” of MPIDs.

All of the physical processes describing the coupled physical effects can be divided into these six panels shown in Figure 3. These subsets describe the coupling interactions between two physical effects and restrict the material behavior to specific conditions. For example, the purely thermomechanical process, i.e. when electromagnetic effects are negligible, is highlighted as a subset of the reversible MPID in Figure 4.
Figure 4: An example of the isolated subset for coupled interactions

Figure 5 represents the isolation of a coupled set of conjugate pairs, corresponding to a subset of the reversible MPID for the thermomechanical coupled conjugate pairs, as well as a detailed explanation of the corresponding framework. The bold red lines connected by the conjugate pairs in Figure 5 represent the uncoupled effects. Specification of the intensive and extensive parameters for reversible processes is provided in Table 1.

Figure 5: General framework for subset of relationships between conjugate pairs
Within each of these six coupled subsets, Figure 5 also contains four solid, black lines, representing reversible processes that relate the extensive and intensive parameters. The top and bottom solid, black lines are processes that can be found through secondary effects. Specifically, the effects that relate one intensive and one extensive parameter are known as cross-effects [5]. Cross-effects are the primary effects and are usually well-defined within each subset due to the “cause” and “effect” nature in the parameters (described in Chapter 1). Since the intensive variables are the thermodynamic forces, the cross-effects are generally determined with the extensive parameter as the independent variable. For the description of the cross-effects in Section 2.2.2, each coupled subset illustrates the direction of the cross-effects due to the experimental and physical meaning. This does not necessarily imply that the extensive variable is always the independent variable. The intensive variable can also be mathematically described as the independent variable through use of the Legendre transformation. However, to be consistent with thermodynamics, the directions of these cross-effects are used to provide physical meaning to the reversible processes. With each process, a brief description of the associated coupled equations from thermodynamic modeling and general experiments to uncover the material constants are provided.

2.2.1 Uncoupled Constitutive Relationships

For the linear, reversible relations, the uncoupled constitutive relationships within each conjugate pair are highlighted in Figure 6. A description of each uncoupled process, as well as the corresponding equations are provided in the proceeding sections.
Specific Heat Capacity

The thermal capacity or heat capacity, C relates the amount of heat, $Q$ required to change a body’s temperature, as shown in (5), where $\Delta \theta$ is the change in temperature.

$$Q = C \Delta \theta$$

Incorporating the second law of thermodynamics for the reversible system, entropy can be related to the change in temperature in (6).

$$c_P = \theta \left( \frac{\partial s}{\partial \theta} \right)_P$$

For this constitutive equation, it is assumed that the material is homogeneous, i.e. the properties are uniform throughout the specimen. This changes the constant relating thermal capacity to the specific heat capacity at constant pressure, $c_P$. The specific heat can also be held at constant strain ($c_e$), constant stress ($c_T$), or at constant volume ($c_V$) and constant pressure ($c_P$) in thermodynamic systems.
Electric Susceptibility

For dielectric media subjected to an electric field, the electrical charges microscopically redistribute to form dipoles, or a macroscopic polarization [32], which is the conjugate displacement term for the reversible MPID. The relationship between electric field and polarization in a dielectric medium is the electric susceptibility, $\chi^e_{ij}$.

This is indicated in (7) [3], where $\varepsilon_o^e$ is the scalar permittivity constant in vacuum.

$$\mathbf{p}_i = \varepsilon_o^e \chi^e_{ij} \mathbf{e}_i$$

(7)

This measurable quantity is also related to the permittivity of the medium, (8) [3].

$$\varepsilon^e_{ij} = \varepsilon_o^e (1 + \chi^e_{ij})$$

(8)

To be consistent with conjugate pairs demonstrated in literature, the electric field can be related to another measurable quantity, the electric displacement, $\mathbf{d}$ in (9) [3].

$$d_i = \varepsilon^e_{ij} \mathbf{e}_j$$

(9)

The superscript ‘$e$’ on the material constants in (7) through (9) is to denote the coefficients and the relationship to electrical properties. Similarly, the superscript ‘$m$’ is provided in the proceeding section to designate relations to magnetic properties.

Magnetic Susceptibility

The generalized magnetic displacement, magnetization is analogous to the electrical polarization. Relating the magnetic field, $\mathbf{h}$ and intensity of magnetization, $\mathbf{m}$ of the material is the magnetic susceptibility tensor, $\chi^m$ presented in (10) [3], where $\mu_o^m$ is the scalar permeability constant in vacuum.

$$m_i = \mu_o^m \chi^m_{ij} \mathbf{n}_j$$

(10)
The magnetic permeability, $\mu^m$, is related to the susceptibility in (11) [3] and is also the material’s ability to absorb magnetic induction (or flux), $b$, described in (12) [3].

$$\mu^m_{ij} = \mu_0^m (1 + \chi^m_{ij})$$

(11)

$$b_i = \mu^m_{ij} h_j$$

(12)

For a scalar magnetic susceptibility, the constant is positive for paramagnetic and ferromagnetic materials and negative for a diamagnetic material. The focus of the processes within the magnetic physical effects will be on ferromagnetic materials, which has the highest magnetic susceptibility.

**Elasticity**

Elasticity describes the material’s ability to return to its initial shape after an applied stress. Hooke’s law is the governing constitutive relation for relating the mechanical stress and strain tensors. For the linear, isotropic, elastic materials, Hooke’s law is reduced to (13) [29], where $\lambda$ and $\mu$ are the Lamè constants.

$$T_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

(13)

Through these assumptions, the Piola-Kirchhoff stress tensor, $P$ is changed to the general stress tensor, $T$, and the deformation tensor, $F$ is converted to the infinitesimal strain tensor, $\varepsilon$ through (3) and (4). A simple tension test measures the elasticity of a material through uniaxial loading of a bar.
2.2.2 Generalized Linear, Reversible Processes

In this section, the MPID of the fully coupled reversible processes in Figure 2 are isolated and evaluated for the cross-effects within two physical effects. The reversible MPID is separated into six different coupled subsets: electromechanical, thermomechanical, thermomagnetic, electromagnetic, electrothermal, and magnetomechanical processes. A brief description of each process is accompanied by corresponding coupled equations, material constants, and experiments that complement the constitutive relationships.

Electromechanical Processes

The electromechanical reversible effects, shown in Figure 7, are primarily described by piezoelectricity. The direct piezoelectric effect (effect 2 in Figure 7) is the application of mechanical stress to a dielectric material, which causes a change in polarization [32]. When an electric field is applied in a piezoelectric material, the crystal deforms; this process is known as the converse piezoelectric effect (effect 3 in Figure 7) and is sometimes referred to as the “reciprocal” or “inverse” piezoelectric effect. Effects 1 and 4 can be generalized to piezoelectricity through manipulation of the parameters by using the uncoupled constitutive relations.
The coupled constitutive equations are dependent on the selection of independent variables. Smith [4] provides an extensive development of the thermodynamic energy functions and corresponding constitutive equations, using both electric field terms of the electric displacement, $d$ and polarization, $p$. For constant electric field and constant stress, the coupled equations reduce to (14) and (15).

\begin{align}
\varepsilon_{ij} &= s_{ijkl}^T T_{kl} + d_{ijk}^T e_k \\
p_i &= d_{ijk}^T T_{jk} + \chi_{ij}^T e_j
\end{align}

(14)  
(15)

The cross-effect coefficients, $d_{ijk}$ (sometimes referred to as the dielectric constant) in the coupled equations of (14) and (15) are numerically equal [3]. Since the deformation of the body is directly proportional to the electric field, many piezoelectric materials exhibit anisotropic conditions and possess a degree of asymmetry [2].

The piezoelectric constant can be found by a uniaxial tensile stress applied along one of the diad axes of a quartz crystal [3]. By inducing a current in a sample, the
converse piezoelectric coefficient can be found through measuring the resulting strain in the material.

Piezoelectricity is normally accompanied by electrostriction. However, electrostriction is a nonlinear, irreversible processes and falls outside of the scope of reversible phenomena. Therefore, the effect will be discussed in further detail in the irreversible section.

Thermomechanical Processes

The thermomechanical relationships are defined in four generalized processes. Each arrow in Figure 8 demonstrates the biased direction of the process relating the extensive and intensive parameters.

Figure 8: Linear, reversible thermomechanical relationships
The first effect, involving stress and temperature is the thermal pressure of a material. Thermal pressure is the change in temperature that results in stress for a constrained body. However, a stress applied to a body does not necessarily cause a linear change in temperature. In one-dimensional form, the stress tensor is characterized as a linear function of temperature through the linear coefficient of thermal stress, $\beta$, shown in (16).

Note that $\beta$ a constant for the one-dimensional form.

$$T = \beta \Delta \theta$$

(16)

The piezocaloric effect is the linear form of measuring the heat, or increase in entropy in a material caused by an isothermal application of pressure. The change in entropy can be described by the addition of the piezocaloric effect and thermal capacity at constant temperature and constant pressure in (17) [3].

$$\Delta \eta = \alpha_{ij} T_{ij} + \left( \frac{c}{\beta} \right) \Delta \theta$$

(17)

For an isotropic and homogenous material, where material symmetry is taken into consideration, the compliance tensor, $s$ can be greatly reduced. Thermal expansion can be shown by applying a known temperature change to a uniform rod, where one end of the rod is fixed and the Young’s modulus is known. This would provide a change in the overall length and the coefficient of linear expansion, $\alpha_{ij}$ can be developed for the material. The strain tensor is described in (18), where the first coefficient represents elasticity, and the second is the coefficient of thermal expansion.

$$\epsilon_{ij} = s_{ijkl}^\theta T_{kl} + \alpha_{ij} \Delta \theta$$

(18)
The cross-effect coefficients in (17) and (18) are also numerically equal [3]. Heat of deformation (effect 4 in Figure 8), or sometimes referred to as strain power or compression heat, is an isothermal process that relates the change in deformation to heat.

**Thermomagnetic Processes**

Thermomagnetic relationships, shown in Figure 9 are slightly less well-defined compared to the thermomehcanical relationships. The magnetocaloric effect consists of two different processes caused by an applied magnetic field: the isothermal process that involves the change in entropy (effect 2 in Figure 9) and the adiabatic change of temperature (effect 1 in Figure 9) [25]. However, adiabatic demagnetization and adiabatic magnetization are subsets of the magnetocaloric effect.

![Figure 9: Linear, reversible thermomagnetic relationships](image)
Adiabatic demagnetization involves an applied magnetic field that causes a change in temperature. When the magnetic field is increased, the disorder of magnetic moments is decreased, causing the material to increase temperature (at constant entropy); this effect is known as adiabatic magnetization [7]. The converse, where the material is cooling, is known as adiabatic demagnetization [7]. Both adiabatic effects are achieved at very low temperatures and usually applied in refrigeration applications [25].

The pyromagnetic effect is the linear process that relates the temperature to the magnetization vector. The fourth effect in Figure 9 can be generalized as a secondary isothermal effect.

**Electromagnetic Processes**

For the visual depiction for electromagnetism provided in Figure 10, the electromagnetic coupling is not as clearly distinguished as the linear effects for other reversible processes.

Figure 10: Linear, reversible electromagnetic relations
Electrical and magnetic physical effects are closely related, and electromagnetism is governed by Maxwell’s equation, (19_{a,b,c,d}), where $\rho_e$ is the charge density, provided in vector form [20].

\[
(\nabla \times h)_t = j_t + \frac{\partial d_t}{\partial t}
\]

(19_{a,b,c,d})

\[
(\nabla \cdot b) = 0
\]

\[
(\nabla \times e)_t = -\frac{\partial b_t}{\partial t}
\]

\[
(\nabla \cdot d) = \rho_e
\]

Maxwell’s equations can be used to relate all of the electromagnetic effects in Figure 10. Conventionally, electric displacement and magnetic induction are used in Maxwell’s equations and can easily be converted to the polarization and magnetization vectors through the uncoupled relations in (7) through (12).

*Electrothermal Processes*

The reversible electrothermal processes consist of pyroelectricity and electrocaloric effects. Pyroelectricity, effects 1 and 3 in Figure 11, relates polarization and temperature, where crystals develop an electric polarization through a uniform temperature change [3].
The polarization is related to temperature through the pyroelectric coefficients, $L_i$, as shown in (20) [3].

$$\Delta p_i = L_i \Delta \theta$$

(20)

The first effect in Figure 11 can be defined as piezoelectricity with the use of the uncoupled electric parameters to find the electric field in terms of polarization. At constant electric field, the pyroelectric effect can be observed through two different experiments, both that involve uniform heating of a crystal and observing the change in polarization [3].

The second effect is defined as the electrocaloric effect, which relates the change of entropy due to an applied electric field. The linear coefficients in the pyroelectric effect are the same for the electrocaloric effect [3]. Heat of polarization, effect 4 in Figure 11, relates the electric field and the change of heat (entropy) in an analogous manner to heat of deformation in the thermomechanical relations [3].
Magnetomechanical Processes

The reversible magnetomechanical processes are analogous to those found in the electromechanical relationships. Magnetomechanical relationships, shown in Figure 12, are generalized by piezomagnetic effects.

Truly defined piezomagnetic processes are the linear cross-effects in Figure 12, relating stress and magnetization (effect 2) strain and magnetic field (effect 3). Piezomagnetism only exists in some antiferromangetic materials [7]. However, literature seems to refer piezomagnetism to the linear form of magnetostriction for ferromagnetic materials.

Similar to the electromechanical processes, the linear piezomagnetic coupled constitutive equations can be re-arranged depending on the selection of independent variables; these equations are applicable to ferromagnetic materials. Constant magnetic field and constant stress were selected to show the constitutive relations in (21) and (22).
[4], where $d_{ijk}$ is the coupling coefficient, $d^*$ is the transpose of $d_{ijk}$, and $\chi_{ij}$ in this case is the magnetic susceptibility. The superscripts in equations below are meant to represent at constant magnetic field, $h$ or constant elastic stress, $T$.

$$\varepsilon_{ij} = s^h_{ijkl} T_{kl} + d^*_{ijk} h_k$$

$$m_i = d_{ijk} T_{jk} + \chi^T_{ij} h_j$$

(21)  (22)

The linear piezomagnetic effect is inherently of “no practical significance because of the lack of available pizeomagnetic crystals with a reasonable level of coupling” [8]. The process of piezomagnetism is the linear form of magnetostriction, or the Joule Effect. Magnetostriction, a nonlinear, irreversible process is applicable to ferromagnetic materials, which will be discussed in the proceeding section.

### 2.3 Nonlinear, Irreversible Processes

Expanding the scope of applicability, MPIDs are extended to describe the irreversible phenomena in nonequilibrium thermodynamics. Compared to the linear, reversible phenomena illustrated earlier, the generalized MPID for nonlinear, irreversible processes are much more complicated to describe. The original intent was to organize and categorize all of the irreversible phenomena, reflecting the work of [6], into a similar visual framework presented for the reversible MPID. A few conflictions arose while researching nonequilibrium thermodynamics and the associated irreversible processes.

The first confliction involves the classical nonequilibrium thermodynamic framework, which is specifically used to describe transport processes of closed systems [10]. As explained in Section 1.2.3, this prevents a definition for the mechanical and
magnetic thermodynamic flow terms within nonequilibrium thermodynamics. This limits any irreversible effect that is not considered a transport process, such as magnetostriction. In an attempt to resolve this confliction, other frameworks of thermodynamics, namely rational and extended thermodynamics, were explored to characterize the irreversible processes. Since both rational and extended thermodynamics provide similar results and the parameter selection of the physical effects are the same, nonequilibrium thermodynamics are primarily presented for the analysis of transport processes.

The second confliction involves the inability to neglect the electrical effects with the inclusion of magnetic field. For the thermodynamic conjugate pairs of generalized force and flow, there are no magnetic field variables that are sufficient under quantum mechanics, as mentioned in Chapter 1. Therefore, in these MPIDs, the addition of magnetic field as generalized forces is only to provide a better understanding of the relationships between the physical effects. For the transport processes described in the proceeding sections, coupling of magnetic field terms also involves electrical conductivity.

Regardless of these conflictions, the irreversible MPID, shown in Figure 13, is completed for a preliminary assessment of the coupled nonlinear effects within TEMM interactions. Alongside of the schematics, brief descriptions of the transport and other nonlinear effects are provided in the proceeding sections.
2.3.1 Uncoupled Constitutive Relationships

The uncoupled constitutive relations are highlighted in Figure 14, where only three of the four physical effects are considered for the nonequilibrium thermodynamic systems.
**Thermal Conductivity**

Thermal conductivity, $k$ can be described through Fourier’s law of heat conduction in (23) [2], where $k$ denotes the tensor of thermal conductivity.

$$q_i = -k_{ij}	heta_j$$  \hspace{1cm} (23)

If the material is assumed homogenous and the thermal conductivity is independent of temperature, the thermal conductivity, $k$ is assumed a constant; this provides a linear relationship between heat flow and temperature gradient.

**Electrical Conductivity**

Electrical conductivity, $\sigma_{ij}$ is the proportionality constant between current density and electric field and is described by Ohm’s Law, (24) [2].
The conductivity is also a function of direction in a material, depending on the crystal symmetry; it is also isotropic in the absence of a magnetic field [7].

Ohm’s law can change based on the magnetic and mechanical considerations: the electric current density can change based on the magnetic induction and deformation [18]. Since magnetic induction and deformation are not thermodynamic flow terms, the relationship is out of the scope in the irreversible coupled equations described below.

2.3.2 Generalized Nonlinear, Irreversible Processes

Due to the selection of conjugate pairs, the transport processes and other nonlinear effects are described separately in the following sections. Transport processes conventionally involve thermoelectric relations and are well-developed in literature; however, transport processes also involve the coupling of three physical effects, versus the traditional two couplings that have been shown. A brief discussion of the thermoelectric effects is provided, along with the thermomagnetic and galvanomagnetic effects that are dependent on the direction of applied heat flow and electric current.

Thermoelectric Processes

The first effect within thermoelectric phenomena in Figure 15 is related by the Seebeck coefficient. The cross-effects, 2 and 3 in Figure 15, are traditionally related by Onsager relations. The fourth effect, relating electric current and heat flow, is described
by the Peltier effect. This effect involves an isothermal junction of two materials, where heat evolves with the flow of an electric current [12].

![Figure 15: Irreversible thermoelectric relations](image)

The directionality of the cross-effects in Figure 15 is not shown since the intensive and extensive parameters are related by mathematical relations, not a particular experimental effect. However, the thermoelectric relations are not just restricted to relating two parameters: another effect to consider is the Thomson effect, which refers to “the evolution of heat as an electric current traverses a temperature gradient in a material” [12].

**Thermomagnetic and Galvanomagnetic Processes**

Similar to reversible phenomena, the material characterization changes depending on the selection of material. For example, the thermomagnetic behavior for conductors is
vastly different from semiconductors. Consideration for thermomagnetic and galvanomagnetic effects, which is usually applied to semiconductors, is coupled between three physical effects, illustrated in Figure 16.

Characteristic of these effects are the dependencies of the direction of the applied heat flow and uniform magnetic field, shown in Figure 17.
Since thermomagnetic and galvanomagnetic effects involve electrical, magnetic, and thermal physical effects, the distinction between the two is based on the applied field. Thermomagnetic effects are employed by the flow of heat, whereas galvanomagnetic effects are evoked by an electric current [13]. Both thermomagnetic and galvanomagnetic effects exhibit differences in transverse and longitudinal measurements. Details of these irreversible conductive effects is outside of the scope of this thesis; however, a summary of the effects, modified from [30] and [31], is presented in Table 3.
### Thermomagnetic (employed by heat)
- **Thermo-Electric** (potential difference)
- **Thermo-Thermal** (temperature difference)

### Galvanomagnetic (employed by electrical current)
- **Electro-Electric** (potential difference)
- **Electro-Thermal** (temperature difference)

<table>
<thead>
<tr>
<th>Transverse Effect</th>
<th>Longitudinal Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transverse Nernst-Ettingshausen Effect</td>
<td>Change in Electrical Resistance</td>
</tr>
<tr>
<td>Righi-Leduc Effect</td>
<td>Difference of Temperature</td>
</tr>
<tr>
<td>Hall Effect</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Summary of thermomagnetic and galvanomagnetic effects

In addition to the effect listed in Table 3, observations of new effects in semiconductors are arising in recent research. In particular, the spin-Seebeck effect has been observed by [34] in ferromagnetic semiconductors. The work of [34] uncovered a spatial-dependent effect, where the magnetization vector reverses direction under similar conditions of the thermomagnetic transport phenomena. If the irreversible MPID in Figure 13 is remodeled and extended to include thermodynamic displacement terms, such as magnetization and deformation, the interactions of these physical effects can illuminate newly observed effects. The spin-Seebeck effect can be taken as a prototype model for the potential of MPIDs, i.e. by uncovering the material properties and related processes, experiments and material coefficients can be uncovered to better describe the amterial behavior. This leads to the discussion of other nonlinear processes in the proceeding section that should be noted in irreversible systems.
Other Nonlinear Processes

Of interest are the nonlinear processes involved in the electromechanical and magnetomechanical coupling. Little research is complete on the characterization of these effects within irreversible thermodynamics. For better descriptions of these nonlinear processes, the mechanical force term remains to illustrate interactions with electrical and magnetic physical effects. The subset MPIDs for irreversible electromechanical and magnetomechanical are not shown since only one process is described for each subset coupling.

The electromechanical relations, illustrated in Error! Reference source not found., are mainly focused on electrostriction. The linear piezoelectric effect actually supersedes the nonlinear process of electrostriction [2]. Electrostriction is where the deformation is proportional to the square of the electric field. Unlike piezoelectricity, electrostriction is therefore independent of direction.

Similar to the electromechanical relations, magnetomechanical relations focus on one particular effect: magnetostriction. A biased magnetostriction, although it involves nonlinear coupling effects, is present in isotropic bodies [8]. Unlike electrostriction, the nonlinear effect of magnetostriction is dominant over the linear piezomagnetic effect [28]. Magnetostriction is an anisotropic feature in the behavior of ferromagnetic due to hysteresis. The uncoupled equations in Chapter 1 for magnetic materials are behavior laws in linear form, where the magnetic permeability can be assumed to be constant [15]. Nonlinear isotropic models can be taken where permeability of the material is a scalar
function of magnetic field; for nonlinear anisotropic models, the magnetic permeability becomes a tensor function of magnetic field [15].
CHAPTER 3: Applications

3.1 Introduction

The Multi-Physics Interaction Diagrams presented in Chapter 2 accompany the development of constitutive relations that involve the coupling of more physical effects. These schematics were purposefully created to illustrate different fully coupled TEMM processes presented in [1]. The most general form of governing equations that fully describe the coupling of thermal, electrical, magnetic, and mechanical processes are provided in [1]. A TEMM process begins with the selection of an arbitrary set of independent variables. Governing equations are used to place restrictions on a set of dependent variables. Along with balancing terms to satisfy all of the conservation laws, Maxwell’s equations, and employing the second law of thermodynamics, the dependent variables are deduced in the form of constitutive equations, similar to the provided set of equations (25) [1].

\[
\begin{align*}
P &= \rho_o \frac{\partial E^{F\theta e h}}{\partial F}, \\
\eta &= -\frac{\partial E^{F\theta e h}}{\partial \theta} \\
\end{align*}
\]

Based on the selection of independent variables, the fully coupled TEMM model is described through seventy-two energetic functions, presented in [1]. These energetic functions describe the material behavior and are denoted by \( E^{(a)(b)(c)(d)} \), where each
superscript represents a mechanical, thermal, electrical, and magnetic independent variable, respectively.

One of the TEMM processes is employed with the energetic function of $E^{F_{\theta}eh}$ to illustrate the use of MPIDs in constitutive modeling. Constitutive modeling provides mathematical descriptions of material responses due to various loading conditions [14]. This particular energetic function is of importance because of the physical meaning in the selected independent variables. Strain, temperature, electric field, and magnetic field are easier to control compared to the corresponding conjugate parameter; the conjugate extensive parameters are easily measurable quantities.

Since each coupled processes involves different sets of independent variables, the energetic function that describes the system changes for each relation. This also presents a different set of constitutive relations, which describe the material behavior. Therefore, the following sections apply MPIDs to uncover the thermoelastic constitutive relations and the energetic potentials that are involved in thermomechanical processes.

3.2. Constitutive Modeling

The interactions presented in the MPIDs are considered for the development of the constitutive relations. To provide physical insight into constitutive modeling, the complete combination of reversible and irreversible processes involved in TEMM interactions must first be illustrated. The distinguished set of parameters to convey these interactions are shown in Figure 18.
The thermodynamic force term for the thermal physical effects is different for reversible and irreversible phenomena. Therefore, $\theta$ presented in Figure 18 is used to represent both the scalar absolute temperature and temperature gradient.

By using the fully coupled model, the constitutive relations in (25) are modified based on a set of specific assumptions to eliminate any unrelated process. The MPIDs are not used for the derivation of constitutive equations; instead, the schematics are used to illuminate the interactions for the desired relations. By using the special case of thermoelastic coupling, the interactions can be compared to well-developed constitutive relations in literature. This allows for a clear and concise visual representation of the procedure to eliminate interactions based on the associated assumptions. For the
following sections, the fully coupled model in Figure 18 is developed and used for a purely thermoelastic case and extended to a magneto-thermoelastic example.

### 3.2.1 Thermoelastic Constitutive Relation

By applying assumptions to the fully coupled MPID, processes and parameters can be eliminated from the consideration for constitutive relations. This is shown through successive steps for the thermoelastic case. For each assumption, correlating processes and parameters are eliminated; this is represented by grey lines or grey texts. For the thermoelastic case, the first assumption is to restrict the material to a perfectly elastic body, which removes any irreversible process from the MPID. This eliminates the top portion of Figure 18 and becomes Figure 19.

![Figure 19: Thermoelastic application, elastic body assumption](image)
Since only reversible processes are considered, the set of constitutive relations in (25) are valid for this example. The restriction to the energetic function in (25), $E_{\text{Feh}}$ provides the independent variables highlighted in Figure 20.

![Figure 20: Thermoelastic constitutive modeling, independent variables](image)

By restricting the coupling to only thermal and mechanical physical effects, the set of relations in (25) reduces to the set in (26).

$$P = \rho_o \frac{\partial E_{\text{F0}}}{\partial F}, \quad \eta = -\frac{\partial E_{\text{F0}}}{\partial \theta}$$

By neglecting the electromagnetic physical effects, the focus for constitutive modeling is restricted to thermomechanical subset (Figure 21) with the independent variables highlighted in red.
A constitutive equation for stress can arise from the general constitutive relations in (26). Based on this visual depiction of Figure 21, the stress must be a function of the independent variables, deformation gradient tensor and the absolute temperature. These processes correlate with the thermomechanical subset diagram of Figure 8, i.e. thermal pressure and elasticity are considered to develop the constitutive equation. Similar to the procedure presented in [29], thermoelastic assumptions illuminate the well-established Duhamel-Neumann constitutive equation in (27): invariance requirements, material symmetry considerations (isotropy), small displacements, and neglecting inertial forces. The last two assumptions listed change the first Piola-Kirchhoff stress tensor to a general stress tensor, denoted by $T_{ij}$.

$$T_{ij} = 2\mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij} + \beta \theta \delta_{ij}$$  \hfill (27)
In (27), \( \mu \) and \( \lambda \) are the Lamé constants, and \( \beta \) is the linear coefficient of thermal pressure.

### 3.2.2 Magneto-Thermoelastic MPID

For the magneto-thermoelasticity case, the prototype problem is assumed to be dependent on the coupling of electromagnetic forces through the governing equations, not the constitutive relations for stress and deformation. Therefore, the elastic stress is independent of electromagnetic considerations. However, the MPIDs reveal that Ohm’s law and Fourier’s law of heat conduction are modified based on the dependencies of magnetism and elasticity.

Figure 22 illustrates that the uncoupled relations – elasticity, magnetic susceptibility, and thermal conductivity – are taken into account for the magneto-thermoelastic case.

---

Figure 22: MPID of magneto-thermoelastic application
The visual description presented in Figure 22 highlights the focus on the piezomagnetic effect combined with the interactions from the relationships of current density and heat flux terms. Reversible processes in thermomagnetic relations, such as the magnetocaloric effects were not considered because of the material and application selection. Typically, magnetocaloric effects are considered for refrigeration applications [25], and since the prototype problem in the proceeding chapter are well outside of this temperature range, it is not necessary to include these effects. Although entropy arises in the prototype problems through the energy function, the thermal property is excluded in Figure 22 to simplify the number of effects.

With the consideration of the current density, \( j \), this broadens the application to include Ohm’s law, which has been modified in (28) [18], to be dependent on the electric field, magnetic field, deformation, and thermal conductivity.

\[
j_i = \sigma_{ij} \left( e_j + \frac{\partial u_j}{\partial t} \times b_j \right) + \rho_e \frac{\partial u_i}{\partial t} \tag{28}\]

Fourier’s law of heat conduction with consideration of current density is provided in (29) [18], where the comma subscript represents a partial derivative, \( Q \) is the heat source, \( c_v \) is the specific heat at constant volume, \( T_o \) is the reference temperature over which the perturbed temperature is \( T \), \( \beta \) and \( \pi_o \) are constants to relate the electrical terms.

\[
k \theta_{ij} + Q = \rho c_v \frac{\partial \theta}{\partial t} + T_o \beta \frac{\partial e_i}{\partial t} + \pi_o (\nabla \cdot j_i) \tag{29}\]

The uncoupled electric constitutive relations for polarization and electric displacement are provided in (7) and (9), respectively from Chapter 2. Magnetic constitutive relations for magnetization and magnetic induction are presented in (10) and (12), respectively.
3.3 Energetic Functions

By employing a TEMM process, the selection of independent variables determines not only the constitutive relations, but relies heavily on the energetic function, which describes the material behavior. Since seventy-two energetic potentials have been described through [1], an evaluation of those energetic functions can be shown through the reversible MPID. As explained in Section 1.2.3, the constitutive relations and energetic function arise from the reversible conjugate pairs in the Clausius-Duhem inequality. By using Legendre transformations, the energetic potential can be redefined through different sets of independent variables. Since energetic potentials are well-defined in the thermomechanical processes, this special subset will be used to illuminate the different energetic potentials and constitutive relations associated with these effects.

The cross-effects shown in Chapter 2 focused on describing the physical sense of the processes involving intensive and extensive parameters in a thermodynamics sense. The parameters presented in this chapter are mathematically termed as independent or dependent variables. This is to accommodate for the language of the constitutive modeling theory of [1].

3.3.1 Thermomechanical Energetic Relations

The primary focus of this work is to demonstrate that each of the MPID subsets is accompanied by an energetic function. Since energetic potential functions for the thermomechanical processes have been well-defined in literature, this provides the most concise relation of energetic potentials to MPIDs. Depending on the selection of the
independent variables, there are four possible pairs of independent variables. The sets of independent variables change the directionality of processes described in Chapter 2, i.e. reversible processes are integrable and therefore, arise for the development of constitutive relations. The intensive and extensive variables can interchangeably become the independent or dependent variable. This provides the set of independent variables and the corresponding energy functions, summarized in Table 4.

<table>
<thead>
<tr>
<th>Independent Variables</th>
<th>MPID Thermomechanical Subset</th>
<th>Energetic Relation</th>
<th>Constitutive Relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( { F, \theta } )</td>
<td></td>
<td>( E^{\text{Fn}} = \psi(F, \theta) )</td>
<td>( P = \rho_o \frac{\partial \psi}{\partial F} ) ( \eta = - \frac{\partial \psi}{\partial \theta} )</td>
</tr>
<tr>
<td>( { F, \eta } )</td>
<td></td>
<td>( E^{\text{Fn}} = \epsilon(F, \eta) )</td>
<td>( P = \rho_o \frac{\partial \epsilon}{\partial F} ) ( \theta = \frac{\partial \epsilon}{\partial \eta} )</td>
</tr>
<tr>
<td>( { P, \theta } )</td>
<td></td>
<td>( E^{\text{Fn}} = \phi(P, \theta) )</td>
<td>( F = -\rho_o \frac{\partial \phi}{\partial P} ) ( \eta = - \frac{\partial \phi}{\partial \theta} )</td>
</tr>
<tr>
<td>( { P, \eta } )</td>
<td></td>
<td>( E^{\text{Fn}} = \chi(P, \eta) )</td>
<td>( F = -\rho_o \frac{\partial \chi}{\partial P} ) ( \theta = \frac{\partial \chi}{\partial \eta} )</td>
</tr>
</tbody>
</table>

Table 4: Summary of thermoelastic energetic potentials
The modification of independent variables presents different energetic potential functions through Legendre transformations. The thermoelastic constitutive relation shown earlier can be related to the Helmholtz energetic potential. For each thermomechanical process, Table 4 accentuates the energetic potential necessary for constitutive modeling.

Mathematically, the energetic potential can become the independent thermal variables, and the thermal parameter is treated as an energetic potential [33]. This creates another four sets of independent variables and constitutive relations to consider. More details with regards to the Legendre transformations, entropic potentials, and these other sets of independent variables are discussed in [1] and [33].
CHAPTER 4: Example Problem

4.1 Introduction

The focus of the present work is on the linear and nonlinear interactions of smart materials with thermal, electrical, magnetic, and mechanical physical effects. Since the material would have to be sampled through isolated experiments to develop the material constants, this chapter focuses on a constrained rod example, proposing an analytical model to these experiments. The axisymmetric example provides motivation for the use of the developed energy functions and constitutive equations in interesting problems regarding energy comparison and beams under buckling loads.

The magneto-thermoelastic equations and assumptions are developed and applied to ferromagnetic materials. The focus will be on ferromagnetics due to its high susceptibility compared to diamagnetics and paramagnetics. However, the properties of ferromagnetic materials will not be very strict, i.e. some nonlinear relations are excluded to simplify the example. The ferromagnetic rod is subject to a uniform magnetic field, caused by an electrical current; this results material behavior affected by magnetic considerations. Effects caused by an electric field are negligible.
4.2 Governing Equations

In addition to the equations of motion in classical thermoelasticity, Maxwell’s equations will also be incorporated for electromagnetic effects. The following equations are applicable for an elastic body under electromagnetic and thermal loads.

The equations of motion are defined in (30) from [1], where the electromagnetically-induced body force, $\mathbf{f}^e$ is defined in (31).

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \rho(\mathbf{f}_l^{ext} + \mathbf{f}_l^e) + (\nabla \cdot \mathbf{T}_{ij}) \quad (30)$$

$$\rho \mathbf{f}_l^e = (\mathbf{j} \times \mathbf{b})_l \quad (31)$$

The electromagnetic body force is reduced from the defined electromagnetically-induced Lorentz force [1] based on a set of assumptions. The total stress of the system is the addition of the elastic and electromagnetic stress tensors. For the application of the present work, the total stress will only be defined as the elastic, or Cauchy stress tensor, $\mathbf{T}$. Similar to evaluation of electrostatics in a deformable dielectric, the Maxwell stress is experimentally indistinguishable from the Cauchy stress [19]. Due to the disregard of higher order effects, the Maxwell stress tensors will be dismissed for the application of magnetostatics in a deformable ferromagnetic material.

With the consideration of electromagnetic field terms, Maxwell’s equations, presented in (19$_{a,b,c,d}$) from Chapter 2 must be evaluated. For simplification, the magnetostatic conditions [20] limit and reduce the Maxwell’s equations to the set of equations in (32), where the electric displacement, $\mathbf{d}$ and charge density, $\rho_e$ are dismissed.

$$(\nabla \times \mathbf{h})_l = \mathbf{j}_l \quad , \quad (32_{a,b,c})$$

$$(\nabla \cdot \mathbf{b}) = 0 \quad ,$$
The set of Maxwell’s equations in (32a,b,c) assume that the magnetic flux lines are closed and provide applicability to boundary value problems within solenoids [20].

Because the Duhamel-Neumann constitutive equation will be used for the magneto-thermoelastic theory, the application is restricted to isotropic materials. For this to be valid, the assumption of small deformations on the body and neglecting inertial must be made, i.e. the rate terms disappear in the governing equations. To formulate the equations of motion in terms of the deformation tensor, the strain-displacement compatibility equation, (4) is used. Combining (4) with the Duhamel-Neumann constitutive relation, presented in (27), the equations of motion reduce to the differential equations with unknowns of \( u \) and \( h \), where the magnetic permeability, \( \mu^m \) is assumed to be a constant.

\[
\begin{align*}
 u_{i,jj} + (\lambda + \mu)u_{j,ji} + \beta \theta_{ji} + \mu^m(j \times h)_i &= \rho \frac{\partial^2 u_i}{\partial t^2} \\
\end{align*}
\]

(33)

The governing equation is also converted to the axisymmetric case, which can be applied to applications in Chapter 4.

### 4.3 Magnetic Considerations

For magnetic considerations under electrical currents, the magnetic vector potential, \( \mathbf{A} \) must be considered. The magnetic vector potential is related to the magnetic induction in (34) [20]. Using the uncoupled constitutive relation of (12), the magnetic field and magnetic potential relation is defined in (35).

\[
b_i = (\nabla \times \mathbf{A})_i
\]

(34)
\[ h_i = \frac{1}{\mu_m} (\nabla \times A)_i \]  

(35)

The compatibility condition of the Coulomb gauge transformation, (36) is necessary to completely define the magnetic potential field [21].

\[ (\nabla \cdot A)_i = 0 \]  

(36)

This relationship can further define (37) based on the formulations in reference [20], where \( \Delta \) denotes the Laplace operator.

\[ (\Delta A)_i = (\nabla \cdot \nabla A)_i = -\mu_m j_i \]  

(37)

4.3.1 Magnetic Boundary Conditions

The boundary conditions for magnetostatics are not comparable to electrostatics. For electrostatics, the boundary conditions can be assumed under perfectly conducting surfaces for electric fields. Magnetostatics must take into consideration the free space [22]. Therefore, for boundary value problems, a large enough boundary surrounds the material, where \( A = 0 \). This can be applied to the two-dimensional problems.

4.3.2 Analytic Solution

The equations of motion provided above are in terms of the magnetic field, \( h \) and the deformation field, \( u \). To solve the equations of motion for the deformation field, analytical solutions of \( h \) can be found to provide the differential equations in the useable one- or two-dimensional forms.
4.4 Constrained Rod Example

A ferromagnetic rod constrained on both ends is to be modeled for under a uniform magnetic field, applied through a solenoid wrapped around the sample. The length of the rod, $L$ is assumed to be much larger than the radius of the rod, $r$. The rod is constrained at both ends; the mechanical boundary conditions are provided in (38), where the $x_3$ direction corresponds to the Cartesian $z$-direction.

$$u_i = \bar{u}_i(\theta, r, x_3)$$

$$\bar{u}(0, r, 0) = 0, \quad \bar{u}(0, r, L) = 0$$ (38)

The thermal boundary conditions, provided in (39), are considered adiabatic walls at the ends of the rod.

$$\frac{\partial \bar{u}}{\partial x_3}(0, r, 0) = 0, \quad \frac{\partial \bar{u}}{\partial x_3}(0, r, L) = 0$$ (39)

With these conditions applied to a ferromagnetic rod, the axisymmetric form of the governing equations can be solved for the deformation.
CHAPTER 5: Conclusion

Coupled theories in smart materials are complex in their properties and performance. With more devices making use of composites and layered composite materials, this complexity makes the potential of smart materials powerful, but also unavoidable in explaining and modeling the material behavior. The present work provides the general framework by making use of visual depictions and creates the starting point of coupled theories in material behavior. Specifically, the creation of MPIDs provides insight of the coupling phenomena within thermal, electrical, magnetic, and mechanical physical effects.

The MPIDs should be modified in the future to distinguish the specific classes of materials that make use of the associated effects. An example of the need for this accommodation is the piezoelectric effects; the piezoelectric phenomenon is specific to a particular class of dielectrics, whereas the process of electrostriction applies to all crystal symmetries. A careful consideration of the material can illuminate new processes by extending the irreversible MPID to include a different set of conjugate pairs.

MPIDs can also be extended to other physical effects, such as chemical and optical processes. With growing research in thin-film solar cells, there is recent need for
coupled effects in magnetism and optics. This would broaden the scope of MPIDs to smart materials in renewable energy applications.

The magneto-thermoelastic example of MPIDs can also be evaluated in complete form and referenced to create prototype problems in magnetoelectroelastic models, which piezoelectric-piezomagnetic composites are currently studied. Boundary value problems that extend to bending examples of these composite models would be of interest. By applying MPIDs, a better understanding of the associated effects for constitutive modeling assumptions can motivate the advancement of smart materials.
REFERENCES


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