MODELING WAVES IN LINEAR AND NONLINEAR SOLIDS BY FIRST-ORDER HYPERBOLIC DIFFERENTIAL EQUATIONS

DISSERTATION

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By

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In this dissertation, a new theoretical and numerical approach to model linear and nonlinear stress waves in complex solids has been developed. The model equations are derived based on the conservation laws in the Eulerian frame in conjunction with the constitutive relations of several types of media, including (i) anisotropic elastic solids, (ii) piezoelectric crystals, (iii) hypoelastic solids, (iv) nonlinear elastic solids, (v) soft tissues modeled by viscoelasticity, and (vi) plastic deformation in metals.

For waves in a thin rod, detailed formulations are provided, including a two-equation model, in which Hookean elasticity is assumed, and a three-equation model with formal hypo-elasticity relationship. For piezoelectric solids, the model equations include the equations of motion, a part of the Maxwell equations, and the constitutive relations for anisotropic and piezoelectric solids. For waves in soft tissues, the governing equations include the equation of motion, the viscoelastic constitutive relations, and the equations for internal variables. At the end of the dissertation, a hypo-plasticity relationship coupled with conservation of mass and momentum was developed to model wave propagation in plastic medium. For all these media, the model equations are composed of a set of first-order, linear or nonlinear, coupled hyperbolic partial differential equations (PDEs) with velocity and stress components as the unknowns.

To understand the governing equations and to facilitate numerical solution, various forms of each model have been derived and reported in the dissertation, including
the conservative form, the non-conservative form, the diagonal characteristic form if it can be derived. The eigen systems of the model equations are then analyzed by deriving the eigenvalues and eigenvectors of the Jacobian matrices of the first order PDEs.

The Conservation Element and Solution Element (CESE) method is then used to solve these model equations for time-accurate solutions of propagating waves. For nonlinear elastic waves, I conducted simulations of wave propagating in a thin rod, including a sudden expansion wave, a compression wave, resonant waves, etc. In the linear elastic regime, numerical solution is directly compared to the classical analytical solution. For nonlinear waves, numerical solution shows salient features of nonlinearity, including the appearance of super harmonics of imposed oscillations, limiting cycle, etc.

I also performed numerical calculations of waves in anisotropic solids, including (i) two- and three-dimensional slowness profiles of wave propagation in anisotropic solids of cubic, trigonal, and hexagonal symmetries, (ii) direct calculations of wave propagation in compression and non-compression directions inside anisotropic solids, and (iii) Two-dimensional wave propagation. The calculated group velocity and energy profiles are presented. (iv) The formulation for anisotropic elastic solids is then extended for piezoelectric crystals.

Numerical simulations of viscous-elastic medium focus on ultrasonic waves in soft tissue. The material response is modelled by Fung’s model and a modified Fung’s model developed by Iatridis et al. For numerical simulation, I used parallel connected standard linear solid (SLS) models to discretize Fung’s model and the modified Fung’s model. The resultant relaxation functions formulated in an integral form are then transformed to be differential equations by using the method of internal variables. The constitutive relations formulated in differential equations are then coupled with
the equation of motion. The complete governing equations are a set of hyperbolic PDEs, which are solved by using the CESE method. The approach is validated by simulation of an one-dimensional impact wave. The numerical results compare well with the analytical solutions. I then perform simulation of wave absorption in soft tissues. Contrast to the conventional approach of determining the parameters in the relaxation functions by quasi-steady compression/expansion tests, I used the measured wave absorption coefficients to determine the relaxation functions. As such, the constructed relaxation functions are inherently suitable for wave dynamics. For the last part of this dissertation, nonlinear wave propagating in copper is simulated. For waves with large displacements, copper is treated as a hypo-plastic medium. I simulate elasto-plastic wave propagation in one-dimensional impact problems and study the time history of wave propagation at loading and unloading process. A forced sinusoid wave with the displacement in the nonlinear range is also simulated.

The results in this dissertation present a general theoretical framework of using the first-order, hyperbolic PDEs to model wave motion in complex media. Most of the model equations are presented in three-dimensional space. To demonstrate the capabilities of the model equations, I report numerical solutions of one-dimensional model equations. Nevertheless, it cannot be overemphasized that the theoretical and numerical framework proposed here is general and useful for modeling multi-dimensional, complex wave motion in a wide range of media.
To my parents Zhongzhou Yang and Biyun Cai, my wife Zhongfang Zheng and my brother Libin Yang

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CHAPTER 1
INTRODUCTION

In this research work, I have constructed a new theoretical and numerical approach to model propagating waves in solids. Mechanical waves are of concern. Comprehensive formulation and numerical solutions are sought for disturbances propagating through a deformable medium due to forced motion of a portion of the medium ([1]). To demonstrate that the framework developed here is general, a wide range of media are considered, including nonlinear elastic waves in metals, waves in anisotropic crystals, waves in piezoelectric solids, waves in soft tissues, and waves in plastic solids. For all media, the model equations are formulated based on the conservation laws in the Eulerian frame in conjunction with the constitutive relations. The result is a set of coupled, first-order, nonlinear, hyperbolic equations. The Conservation Element and Solution Element (CESE) method is then used to integrate the hyperbolic equations in the space-time domain for the numerical solutions of the propagating stress waves in solids. To demonstrate the unique approach, I will focus on theoretical and numerical analysis of one-dimensional wave propagation in the present dissertation.

Numerous studies of linear elastic wave propagation in rods exist in the literature, e.g., ([1, 9, 58]). In general, one can derive the one-dimensional models and their solutions for waves in an elastic rod via a direct approach, including longitudinal, flexural, and torsional waves. On the other hand, many nonlinear models have been developed, e.g., ([3, 36, 40, 63, 88, 143, 144]) to examine material behaviour in regimes
where large deformations and/or inherently nonlinear elastic material response are significant. Nonlinear behavior in elastic solids ([60, 61, 88]) can arise due to: (i) kinetic, or convective, effects which arise in the presence of finite deformations which necessitate the retention of the convective portion of the acceleration in the momentum equation; (ii) inherently nonlinear material response shown in the constitutive relations; (iii) geometric/kinematic effects, i.e., a nonlinear strain-displacement relationship; and (iv) nonlinear boundary conditions.

In addition to the above theoretical analyses, numerical studies for nonlinear waves in solids with discontinuities fronts are of interest. The main challenge in employing the above hyperbolic equations is the development of viable numerical methods for time-accurate solutions. Conventional finite-volume method in conjunction with the use of an approximate Riemann solver has been the mainstream approach to solve the hyperbolic system of equations. Most of the development employed upwind methods to solve nonlinear waves in solids. Godunov [57] developed a finite-volume method with a built-in Riemann solver to capture nonlinear waves. Modern upwind schemes extended Godunov’s method by utilizing more efficient approximate Riemann solvers and limiter functions to achieve crisp resolution of jump conditions. Trangenstein [125, 126] and Trangenstein and Pember [128] develop a second-order Godunov method to model impact processes involving multiple materials. LeVeque [84] reported solutions of multi-dimensional waves in elastic solids. Tran and Udaykumar [124] employ an Essentially Non-Oscillating (ENO) scheme and an interface-tracking technique to study wave motion in a tungsten rod impacting a steel plate. Colella [37] and Miller and Colella [98] presented an second-ordered Godunov method for direct calculations of waves in solids. Operator splitting was not applied in the two- and three-dimensional problems. The equations were written in non-conservative form, and deformation tensor was solved. Trangenstein [125, 126], and Trangenstein and
Pember [128] developed another second-order Godunov method for impact problems in solids involving multiple materials. Lin and Ballmann [8] used Zwas method [47] and reported numerical solutions of wave propagation around a crack. They combined the method of bi-characteristics and a finite-difference scheme for a second-order Godunov scheme [8]. They validated their method by applying it to an anti-plane shear problem, a plane-strain problem [8], and an axisymmetric stress wave propagation [8]. They also provided an extensive review of simulating stress waves in solids [8].

Fey [52, 53] proposed the method of transport, which does not rely on the use of a Riemann solver. The method was designed based on the concept of consistency of a set of wave vectors.

In the present research work, the CESE method by Chang [24] will be employed. The CESE method is a novel numerical framework for hyperbolic conservation laws. The tenet of the CESE method is a uniform treatment of space and time for flux conservation. Contrast to the modern upwind method, no Riemann solver or reconstruction step is used as a building block of the CESE method and the logic and operational count of the CESE method are simpler and more efficient. Based on the CESE method, a suite of computer one-, two-, and three-dimensional codes using structured and unstructured meshes have been developed. The two- and three-dimensional codes have been parallelized and can be used for large-scaled simulations. Previously, Yu and coworkers [24, 31, 32, 67, 68, 74, 111, 139, 145, 153], have reported a wide range of highly accurate solutions of hyperbolic systems, including detonations, cavitations, complex shock waves, turbulent flows with embedded dense sprays, dam breaking flows, MHD flows, and aero acoustics. Detailed algebra of the method has been extensively illustrated in the cited references. For conciseness, only the basic ideas of the CESE method in one spatial domain will be illustrated in Chapter 2.

The rest of this dissertation is divided into eleven chapters. A brief description
of each chapter is provided in the following. Chapter 2 illustrates the CESE method, in which a space-time integral form of the governing equations is integrated for time-marching solutions. Space-time flux conservation over Conservation Elements (CEs) is imposed. The integration is aided by the prescribed discretization of the unknowns in each Solution Elements (SE), which in general does not coincide with a CE.

In Chapter 3 and 4, we report theoretical and numerical study of stress wave propagation in anisotropic solid, e.g., hexagonal and trigonal 32 symmetry crystals. We report the eigenstructure of a set of first-order hyperbolic partial differential equations for modelling waves in solids with hexagonal symmetry and trigonal 32 symmetry. The governing equations include the equation of motion and partial differentiation of the elastic constitutive relation with respect to time. The result is a set of nine, first-order, fully coupled, hyperbolic partial differential equations with velocity and stress components as the unknowns. The wave physics are fully described by the eigenvalues and eigenvectors of these matrices: i.e., the nontrivial eigenvalues are the wave speeds, and a part of the corresponding left eigenvectors represents wave polarization. For a wave moving in a certain direction, three wave speeds can be identified by calculating the eigenvalues of the coefficient matrix in a rotated coordinate system. In this process, without using the plane-wave solution, we recover the Christoffel matrix and thus validate the formulation. To demonstrate this approach, two- and three-dimensional slowness profiles of cadmium and quartz are calculated. Wave polarization vectors for wave propagation in several compression directions as well as noncompression directions are discussed. With this established mathematical model, an extension of the space-time Conservation Element and Solution Element (CESE) method was used to simulate stress waves in elastic solids of hexagonal symmetry. To demonstrate this approach, numerical results in these two chapters include one-dimensional expansion waves in a suddenly stopped rod, two-dimensional wave
expansion from a point in a plane, and waves interacting with interfaces separating hexagonal solids with different orientations. All results show salient features of wave propagation in hexagonal solids and the results compared well with the available analytical solutions.

In Chapter 5, theoretical derivations of wave propagation in piezoelectric medium are reported. The research work by Kyame [80], Tiersten [119], and Auld [4] are briefly restated. Then a hyperbolic differential equations coupling Maxwell’s equation and stress wave equation is derived. The eigen-values of the model equations are wave speed in piezoelectric materials. Then a simplified governing equation about quasistatic electro-wave coupling with stress wave equation are studied.

In Chapter 6, we employ the hypo-elasticity constitutive relation to simulate material response. The resultant modeling equations are a set of three first-order, fully coupled, nonlinear hyperbolic partial differential equations. We will present detailed derivation of the equation set and its various forms, including the conservative form, the non-conservative form, and the diagonal form. We also analyze the eigen system of the model equations. The model equations here can simultaneously describe nonlinear waves with large deformation as well as linear elastic waves, which tend to coexist with nonlinear deformations in solids. We study the resonant waves in a thin rod. The imposed oscillations are well in the linear elastic regime. Due to the resonant effect, the amplitude of the wave grows continuously and the waves become nonlinear due to significant material deformation. In the linear regime, classical solution is used to validate the numerical solution.

In Chapter 7, the one-dimensional model equations proposed by Drumheller [9, 45] and Krashanitsa [78] for nonlinear waves are employed and solved. The model employed, with the conservation laws as its foundation, consists of a pair of coupled,
first-order, hyperbolic partial differential equations in the Eulerian frame. For numerical solutions, we recast the system of equations into a conservative form. To support the numerical implementation, model equations in the non-conservative form, the diagonal form, and the characteristic form with its Riemann invariants are also derived. As such, the eigenvalues and eigenvector matrices of the Jacobian matrix of the quasi-linear hyperbolic system are ready to be compared with the analytical solutions of linear wave equations with known wave speeds. For model and code validation, we perform detailed comparison between the numerical solutions and the analytical solution of the classical linear wave equation for elastic waves. To facilitate the comparison, we also present the linearized version of the nonlinear equations employed.

In Chapter 8, Chapter 9, and Chapter 10, we report a theoretical and numerical framework to model wave propagation in soft tissues. Material response of the media is modeled by viscoelastic relations based on Fung’s model [54] and an extended Fung’s model by Iatridis et al. [65]. First, the constitutive relations employed are reconstructed by using parallelly connected Standard Linear Solid (SLS) models. The resultant relations in an integral form are then transformed into differential equations by using internal variables such that modern numerical methods for time-accurate solutions of hyperbolic differential equations could be applied. The governing equations include the equation of motion, the constitutive relations, and the equations for internal variables. Together, these equations are a set of coupled, first-order, hyperbolic partial differential equations with sink terms, in which velocities, stresses, and internal variables are the unknowns. For numerical solutions, we, again, use the space-time Conservation Element and Solution Element (CESE) method. To demonstrate the approach, three cases of waves in soft tissues are reported. To validate the approach, we consider a Maxwellian medium by using one internal variable. Numerical results of an impact wave compare well with the analytical solution. For realistic soft tissues,
we consider rabbit mesentery. The material response is modeled by Fung’s model with eight internal variables. Numerical results show no relaxation effect. Thus, Fung’s model is unsuitable for dynamics problems with short time durations. Finally, we consider the subcalcaneal [82] by using Iatridis’ model with four internal variables. Simulated impact waves show apparent relaxation effect. We further studied ultrasonic absorption of soft tissue. Start with power law theory of absorption coefficient and experiment data, we reconstructed the absorption modulus of Fung’s model and modified Fung’s model. Then a generalized standard linear solid model was used to fit these mathematical models. By using collocation method and internal variable method, we can calculate how ricker’s wavelet damped out because of the absorption effect of soft tissue. The numerical results can be used to compare with original experiments and validate our numerical model. In order to capture the large deformation of soft tissue, linear viscoelastic model coupled with hyperelastic model was used to get a new hypobolic partial differential system. These formula can be used to simulate large amplitude wave propagation in soft tissue.

In Chapter 11, I analyze nonlinear plastic wave in copper, which is treated as a hypo-plastic medium. The infinitesimal plasticity theory have been generalized to hypo-plasticity by using co-rotational derivative, e.g., Jaumann stress rate. The isothermal hyperbolic model of stress wave in elastic-plastic solid does not include energy conservation equation, and equation of state employed relates pressure and density, without considering internal energy. We applied the isothermal model to simulate low speed impact problem, e.g., the impact speed at $u = 30m/s$, plastic loading and unloading problem and ultrasonic forced plastic wave propagation. In Chapter 12, I provide the conclusions, followed by a list of cited references.
2.1 The CESE Method

Conventional finite volume methods are formulated according to a flux balance over a fixed spatial domain. The conservation laws state that the rate of change of the total amount of a substance contained in a fixed spatial domain, i.e., the control volume $V$, is equal to the flux of that substance across the boundary of $V$, denoted as $S(V)$. Consider the differential form of a conservation law as follows:

$$\frac{\partial u}{\partial t} + \nabla \cdot f = 0$$

(2.1)

where $u$ is density of the conserved flow variable, $f$ is the spatial flux vector. By applying Reynolds’ transport theorem to the above equation, one can obtain the integral form as:

$$\frac{\partial}{\partial t} \int_V u dV + \oint_{S(V)} f \cdot dS = 0$$

(2.2)

where $dV$ is a spatial volume element in $V$, $ds = d\sigma n$ with $d\sigma$ and $n$ being the area and the unit outward normal vector of a surface element on $S(V)$ respectively. By
integrating Eq.(2.2), one has

$$\left[ \int_V u dV \right]_{t=t_f} - \left[ \int_V u dV \right]_{t=t_s} + \int_{t_s}^{t_f} \left( \oint_{S(V)} (f \cdot dS) dt \right) = 0 \quad (2.3)$$

The discretization of Eq.(2.3) is the focus of the conventional finite-volume methods. In particular, the calculation of the flux terms in Eq.(2.3) would introduce the upwind methods due to its nonlinearity of the convection terms in the conservation laws.

In the CESE method, we do not use the above formulation based on the Reynolds transport theorem. Instead, the conservation law is formulated by treating space and time in an equal-footing manner. This unified treatment for space and time would allow a consistent integration in calculation space-time and thus ensure local and global flux balance. This chapter only briefly illustrates the CESE method in one-spatial dimension.

### 2.1.1 One-Dimensional CESE Method

To proceed, let time and space be the two orthogonal coordinates of a space-time system, i.e., $x_1 = x$ and $x_2 = t$. They constitute a two-dimensional Euclidean space $E_2$. Define $h \equiv (f, u)$, then by using the Gauss divergence theorem, Eq.(2.1) becomes

$$\int_{\partial \Omega} h \cdot ds = 0 \quad (2.4)$$

Equation (2.4) states that the total space-time flux $h$ leaving the space-time volume through its surface vanishes. Refer to Figure 2.1 for a schematic of Eq.(2.4). To integrate Eq. (2.4) we employed the CESE method [24].

The tenet of the CESE method is the uniform treatment of space and time in calculating flux conservation. Based on the CESE method, a suite of computer one-, two-, and three-dimensional codes using structured and unstructured meshes have
been developed. The two- and three-dimensional codes have been parallelized and can be used to perform large-scaled simulations of nonlinear stress waves in fluids and solids. In the present report, only basic ideas of the CESE method in one spatial domain will be illustrated.

In the CESE method, separated definitions of Solution Element (SE) and Conservation Element (CE) are introduced. In each SE, solutions of unknown variables are assumed continuous and a prescribed function is used to represent the profile. In the present calculation, a linear distribution is used. Over each CE, the space-time flux in the integral form, Eq.(2.4), is imposed. Figure (2.2) shows the space-time mesh and the associated SEs and CEs. Solutions of variables are stored at mesh nodes which are denoted by filled circular dots. Since a staggered mesh is used, solution variables at neighboring SEs leapfrog each other in time-marching calculation. The SE associate with each mesh node is a yellow rhombus. Inside the SE, the solution variables are assumed continuous. Across the interfaces of neighboring SEs, solution discontinuities are allowed. In this arrangement, solution information from on SE to another propagates only in one direction, i.e., toward the future through the oblique interface as denoted by the red arrows. Through this arrangement of space-time
staggered mesh, the classical Riemann problem has been avoided. Figure 2.2(b) illustrates a rectangular CE, over which the space-time flux conservation is imposed. This flux balance provides a relation between the solutions of three mesh nodes: \((j, n)\), \((j - 1/2, n - 1/2)\), and \((j + 1/2, n - 1/2)\). If the solutions at time step \(n - 1/2\) are known, the flux conservation condition would determine the solution at \((j, n)\).

In the present research, many differential equations have source terms. Thus, we consider the one-dimensional equations with source terms:

\[
\frac{\partial u_m}{\partial t} + \frac{\partial f_m}{\partial x} = s_m, \tag{2.5}
\]

where \(m = 1, 2, 3\) and the source term \(s_m\) are functions of the unknowns \(u_m\) and their spatial derivatives. For any \((x, t) \in \text{SE}(j, n)\), \(u_m(x, t)\), \(f_m(x, t)\) and \(h_m(x, t)\), are approximated by \(u^*(x, t; j, n)\), \(f^*(x, t; j, n)\), and \(h^*(x, t; j, n)\). By assuming linear distribution inside an SE, we have

\[
\begin{align*}
u^*_m(x, t; j, n) &= (u_m)_j^n + (u_{mx})_j^n(x - x_j) + (u_{mt})_j^n(t - t^n), \\
f^*_m(x, t; j, n) &= (f_m)_j^n + (f_{mx})_j^n(x - x_j) + (f_{mt})_j^n(t - t^n), \\
h^*_m(x, t; j, n) &= (f^*_m(x, t; j, n), u^*_m(x, t; j, n)).
\end{align*}
\]
where

\[
(u_{mx})^n_j = \left( \frac{\partial u_m}{\partial x} \right)^n_j,
\]

\[
(f_{mx})^n_j = \left( \frac{\partial f_m}{\partial x} \right)^n_j = (f_{m,t})^n_j(u_{tx})^n_j,
\]

\[
(u_{mt})^n_j = \left( \frac{\partial u_m}{\partial t} \right)^n_j = -(f_{mx})^n_j = -(f_{m,t})^n_j(u_{tx})^n_j,
\]

\[
(f_{mt})^n_j = \left( \frac{\partial f_m}{\partial t} \right)^n_j = (f_{m,t})^n_j (u_{tx})^n_j,
\]

and \((f_{m,t})^n_j \equiv (\partial f_m/\partial u)^n_j\) is the Jacobian matrix. Assume that, for any \((x, t) \in SE(j, n), u_m = u^*_m(x, t; j, n)\) and \(f_m = f^*_m(x, t; j, n)\) satisfy Eq. (2.5), i.e.,

\[
\frac{\partial u^*_m(x, t; j, n)}{\partial t} + \frac{\partial f^*_m(x, t; j, n)}{\partial x} = s^*_m(x, t; j, n),
\]

(2.6)

where we assume that \(s^*_m\) is constant within \(SE(j, n)\), i.e., \(s^*_m(x, t; j, n) = (s_m)^n_j\). Eq. (2.6) becomes

\[
(u_{mt})^n_j = -(f_{mx})^n_j + (s_m)^n_j.
\]

(2.7)

Since \((f_{mx})^n_j\) are functions of \((u_m)^n_j\) and \((u_{mx})^n_j\); and \((s_m)^n_j\) are also functions of \((u_m)^n_j\), Eq. (2.7) implies that \((u_{mt})^n_j\) are also functions of \((u_m)^n_j\) and \((u_{mx})^n_j\). Aided by the above equations, we determine that the only unknowns are \((u_m)^n_j\) and \((u_{mx})^n_j\) at each mesh point \((j, n)\).

Next, we impose space-time flux conservation over \(CE(j, n)\) to determine the unknowns \((u_m)^n_j\). Refer to Fig. 2.2(b). Assume that \(u_m\) and \(u_{mx}\) at mesh points \((j - 1/2, n - 1/2)\) and \((j + 1/2, n - 1/2)\) are known and their values are used to
Fig. 2.2: A schematic of the CESE method in one spatial dimension. (a) Zigzagging SEs. (b) Integration over a CE to solve $u_i$ and $(u_x)_i$ at the new time level.
calculate \((u_m)^n\) and \((u_{mx})^n\) at the new time level \(n\). By enforcing the flux balance over \(CE(j,n)\), i.e.,

\[
\int_{S(CE(j,n))} h^*_m \cdot ds = \int_{CE(j,n)} s^*_m d\Omega,
\]

one obtains

\[
(u_m)^n_j - \frac{\Delta t}{4}(s_m)^n_j = \frac{1}{2}\left[(u_m)^{n-1/2}_{j-1/2} + (u_m)^{n-1/2}_{j+1/2}
+ \frac{\Delta t}{4}(s_m)^{n-1/2}_{j-1/2} + \frac{\Delta t}{4}(s_m)^{n-1/2}_{j+1/2}
+ (p_m)^{n-1/2}_{j-1/2} - (p_m)^{n-1/2}_{j+1/2}\right],
\]

(2.8)

where

\[
(p_m)^n_j = \frac{\Delta x}{4}(u_{mx})^n_j + \frac{\Delta t}{\Delta x}(f_m)^n_j + \frac{\Delta t^2}{4\Delta x}(f_{mt})^n_j.
\]

Given the values of the marching variables at the mesh nodes \((j - 1/2, n - 1/2)\) and \((j + 1/2, n - 1/2)\), the right-hand side of Eq. (2.8) can be explicitly calculated. Since \((s_m)^n_j\) on the left hand side of Eq. (2.8) is a function of \((u_m)^n_j\), we use Newton’s method to solve for \((u_m)^n_j\). The initial guess of the Newton iterations is

\[
(\bar{u}_m)^n_j = \frac{1}{2}\left[(u_m)^{n-1/2}_{j-1/2} + (u_m)^{n-1/2}_{j+1/2}
+ \frac{\Delta t}{4}(s_m)^{n-1/2}_{j-1/2} + \frac{\Delta t}{4}(s_m)^{n-1/2}_{j+1/2}
+ (p_m)^{n-1/2}_{j-1/2} - (p_m)^{n-1/2}_{j+1/2}\right],
\]

i.e., the explicit part of the solution of \((u_m)^n_j\).

The solution procedure for \((u_{mx})^n_j\) at node \((j,n)\) follows the standard \(a-\varepsilon\) scheme
[24] with \( \varepsilon = 0.5 \). To proceed, we let

\[
(u_{mx})_j^n = \frac{(u_{mx}^+)_j^n + (u_{mx}^-)_j^n}{2},
\]

(2.9)

where

\[
(u_{mx}^\pm)_j^n = \pm \frac{(u_m)^{n\pm1/2}_j - (u_m)^n_j}{\Delta x/2},
\]

\[
(u_m)^{n\pm1/2}_j = (u_m)^{n-1/2}_j + \frac{\Delta t}{2}(u_{mt})^{n-1/2}_{j\pm1/2}.
\]

For solutions with discontinuities, Eq. (2.9) is replaced by a re-weighting procedure to add artificial damping at the jump

\[
(u_{mx})_j^n = W\left((u_{mx}^-)_j^n, (u_{mx}^+)_j^n, \alpha\right),
\]

where the re-weighting function \( W \) is defined as:

\[
W(x_-, x_+, \alpha) = \frac{|x_+|^\alpha x_- + |x_-|^\alpha x_+}{|x_+|^\alpha + |x_-|^\alpha},
\]

and \( \alpha \) is an adjustable constant. The complete discussion of the one-dimensional CESE method can be found in [24, 28]. The above method with CE and SE defined as in Fig. (2.2) is useful for solving the hyperbolic PDEs with non-stiff source terms.
2.2 Verification of the CESE Method

2.2.1 Validation Example-1

To assess the ability of the CESE method to accurately solve nonlinear hyperbolic problems, we solve Burgers’ equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \quad x > 0, \quad t > 0 \tag{2.10}
\]

subject to the initial condition

\[
u(x, 0) = 1 \tag{2.11}
\]

and the boundary condition

\[
u(0, t) = t + 1 \tag{2.12}
\]

We note the Burgers’ equation and the nonlinear hyperbolic system studied share a similar mathematical structure (cf. Eqs. (7.13) and (2.10)). Using the method of characteristics, we construct analytical solutions of Eqs. (2.10)-(2.12) that satisfy both the Rankine-Hugoniot jump condition

\[
\frac{dx_s}{dt} = \frac{u_l + u_r}{2} \tag{2.13}
\]

and the entropy condition

\[
u_l > u_r \tag{2.14}
\]
at all solution discontinuities, where \( x_s(t) \) is the shock location, and \( u_l \) and \( u_r \) represent the limits of \( u(x, t) \) as the discontinuities are approached from the left and right, respectively. The analytical solution before shock formation at the breaking time \( t = 1 \) is

\[
u(x, t) = \begin{cases} 
(1 + t + \sqrt{(1 + t)^2 - 4x})/2 & \text{if } x < t \\
1 & \text{if } x > t 
\end{cases}
\]  \quad (2.15)

and after shock formation

\[
u(x, t) = \begin{cases} 
(1 + t + \sqrt{(1 + t)^2 - 4x})/2 & \text{if } x < (t + 3)(3t + 1)/16 \\
1 & \text{if } x > (t + 3)(3t + 1)/16 
\end{cases}
\]  \quad (2.16)

We also solve Eqs. (2.10 - 2.12) numerically using the CESE method. An accurate numerical method must be able to capture the correct shock speed with minimal numerical diffusion and dispersion at the sharp solution discontinuity. The results shown in Fig. (2.3) and Fig. (2.4) demonstrate that the CESE method successfully predicts the salient features of nonlinearity modelled by Burgers’ equation, e.g., shock waves and rarefaction waves, without excessive smearing or spurious oscillations at the shock front.

### 2.2.2 Validation Example-2

To validate the CESE method, we investigate a nonlinear initial—value problem involving Burgers’ equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \quad x > 0, \quad t > 0,
\]  \quad (2.17)
Fig. 2.3: Numerical (x symbol) and theoretical (solid line) solutions of Burgers’ equation at (a) t = 0.35, (b) t = 0.875, (c) t = 2.625, and (d) t = 3.5 for $\Delta t = 0.0035$ and $\Delta x = 0.04$.

Fig. 2.4: Numerical (x symbol) and theoretical (solid line) solutions of Burgers’ equation at (a) t = 0.5, (b) t = 0.9, (c) t = 1.5, and (d) t = 2.0 for $\Delta t = 0.001$ and $\Delta x = 0.02$. 

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subject to the initial conditions

\[ u(x,0) = \begin{cases} 
1, & 0 < x < 1.8, \\
0, & 1.8 < x < 3.6, \\
2, & 3.6 < x < 5.4, \\
0, & x > 5.4. 
\end{cases} \]  \tag{2.18}

Using the method of characteristics, we construct a unique weak solution of the initial-value problem (2.17) and (2.18) that satisfies both the Rankine–Hugoniot jump condition and the entropy condition at all solution discontinuities in the x–t domain: for 0 < t < 1.8:

\[ u(x,t) = \begin{cases} 
1, & 0 < x < 1.8 + t/2, \\
0, & 1.8 + t/2 < x < 3.6, \\
(x - 3.6)/t, & 3.6 < x < 3.6 + 2t, \\
2, & 3.6 + 2t < x < 5.4 + t, \\
0, & x > 5.4 + t, 
\end{cases} \]  \tag{2.19}

for 1.8 < t < 3.6:

\[ u(x,t) = \begin{cases} 
1, & 0 < x < 1.8 + t/2, \\
0, & 1.8 + t/2 < x < 3.6, \\
(x - 3.6)/t, & 3.6 < x < 3.6 + \sqrt{7.2t}, \\
2, & 3.6 + \sqrt{7.2t} < x < 3.6 + \sqrt{7.2t}, \\
0, & x > 3.6 + \sqrt{7.2t}, 
\end{cases} \]  \tag{2.20}
for $3.6 < t < 10.8 + 3.6\sqrt{2}$:

\[
\begin{align*}
  u(x,t) &= \begin{cases}
    1, & \text{if } x < 3.6 + t - \sqrt{3.6t}, \\
    0, & \text{if } 3.6 + t - \sqrt{3.6t} < x < 3.6, \\
    (x - 3.6)/t, & \text{if } 3.6 < x < 3.6 + 2t, \\
    2, & \text{if } 3.6 < x < 3.6 + \sqrt{7.2t}, \\
    0, & \text{if } x > 3.6 + \sqrt{7.2t},
  \end{cases}
\end{align*}
\]

(2.21)

for $t > 3.6(1 + \sqrt{2})^2$:

\[
\begin{align*}
  u(x,t) &= \begin{cases}
    1, & \text{if } x < 0.5t + 5.4, \\
    0, & \text{if } x > 0.5t + 5.4.
  \end{cases}
\end{align*}
\]

(2.22)

We also solve the initial-value problem (2.17) and (2.18) numerically using the CESE method. The computational domain $0 < x < 18$ is discretized using nodes equally spaced at $\Delta x = 0.06$. A time step of $\Delta t = 0.02$ is chosen to ensure numerical stability. Excellent agreement between the analytical and numerical solutions (see Fig.(2.5)) demonstrates the ability of the CESE method to accurately solve nonlinear hyperbolic equations and capture salient features of nonlinearity, e.g., sharp solution discontinuities and rarefaction waves.
Fig. 2.5: Analytical (solid line) and numerical (circles) solutions of Burgers’ equation at (a) $t = 1$, (b) $t = 3.8$, (c) $t = 10$, and (d) $t = 22$ with $\Delta t = 0.22$ and $\Delta x = 0.06$
Many references are available on the basic theory of elastic waves in anisotropic solids, e.g., [17], [99], [100], [92], [50], [16], [46], [49], [4], [121], [122], [120], [114]. As a part of general discussions of elasticity, the theory of anisotropic medium was also succinctly summarized in [90] and [81]. In the conventional approach, one would invoke the equation of motion and the elastic relation of the anisotropic medium to model wave propagation in anisotropic mediums. By eliminating the stress tensor from these equations, one derives the second-order acoustic field equations in terms of velocity components. An assumed plane wave solution in the form of \( \mathbf{v} = \mathbf{a} f(\mathbf{k} \cdot \mathbf{x} - t) \) is then employed as the solution of the acoustic field equations. As such, the Christoffel equation is derived. Conventional analyses of wave propagating in anisotropic solids have been largely built upon solving the Christoffel equation for wave speeds and polarization directions. Even when studying inhomogeneous plane waves and energy transmission this approach was commonly used, e.g., [43].

In the present chapter, we will focus instead on the original model equations in the first-order form. The governing equations include the equation of motion in conjunction with partial differentiation of the elastic constitutive equations with respect to time. We treat velocities and stress components as the primary unknowns. The result is a set of nine, first-order, fully-coupled, hyperbolic partial differential
equations, referred to as the velocity-stress equations. We then cast the first-order equations into a vector form with three $9 \times 9$ coefficient (or Jacobian) matrices.

We will show that the physics of wave propagation in anisotropic solids are fully described by the eigen structure of these three $9 \times 9$ coefficient matrices of the first-order governing equations. In particular, the eigenvalues of the Jacobian matrices represent the wave speeds and the corresponding left eigenvectors represent the wave polarization. Without invoking the plane-wave solution for velocities and the Christoffel equations, the two- and three-dimensional slowness profiles can be straightforwardly calculated. As an example, we will show the two- and three-dimensional slowness profiles of a cadmium sulfide crystal by direct numerical calculations of eigenvalues and eigenvectors of the Jacobian matrices in the present chapter.

Next, CESE method was used to numerically solve this first-order equations. The CESE method is a novel numerical method for time accurate solutions of linear and non-linear hyperbolic partial differential equations. The CESE method treats space and time in a unified way in calculating the space-time flux. The method is simple, robust, and its operational count is comparable to that of a second-order central difference scheme.

The backbone of the CESE method is the $a$ scheme [24], which is neutrally stable without artificial dissipation for solving a scalar convection equation with a constant coefficient. For more complex wave equations, the $a$ scheme is extended with added artificial damping, including the $a-\alpha-\epsilon$ scheme [24] for shock capturing and the $c-\tau$ scheme [29] for Courant-Friedrichs-Lewy (CFL) number insensitive calculations. To model multi-dimensional problems, the CESE method uses unstructured meshes [139] with triangles and tetrahedra as the basic elements in two- and three-dimensional spaces. Extensions to use quadrilaterals and hexahedra are also available [153]. In this chapter, we will use the $a-\alpha-\epsilon$ scheme to solve the velocity-stress equations for
waves in solids. For two-dimensional problems, we will use unstructured meshes composed of triangular elements.

In the past, the CESE method has been successfully applied to solve a wide range of various fluid dynamics problems, including aero-acoustics [89], cavitations [111], complex shock waves [75], detonations [136], magnetohydrodynamics (MHD) [150, 152], etc. Moreover, the method has been applied to model electromagnetic waves [140] and nonlinear stress waves in isotropic solids [19, 146, 148].

To model stress waves in hexagonal solids, we solve the equation of motion in conjunction with the elasticity equation of the medium. Although Rue et al. [115] and Liu and Cai [87] investigated diffractions and scattering of elastic waves in cylindrical and spherical systems, it is more convenient for us to use Cartesian coordinate system for numerical calculation. First, we differentiate the constitutive equation with respect to time and treat velocity and stress components as the unknowns. We then cast these first-order partial differential equations into a vector form. The key feature of the first-order equations are three $9 \times 9$ coefficient matrices. The physics of stress waves are described by the eigen structure of the coefficient matrices.

Numerical solutions of the one- and two-dimensional wave equations are calculated by using the CESE method, including wave propagation in a suddenly stopped rod, and expanding wave from a point in a plane. Results of complex wave-interface interactions are also demonstrated in two-dimensional simulations.

In the remainder of this section, we review the basic equations to be used in the following sections. First, we employ the equation of elastodynamics: \( \nabla \cdot T = \rho \ddot{u} \) with \( T \) as the Cauchy stress tensor, \( \rho \) the density, \( b \) the body force, and \( u \) the displacement vector. We assume that the body force is negligible. The independent variables are Cartesian coordinates \( x = (x_1, x_2, x_3) \) and time \( t \). This equation can be rewritten in the index form: \( T_{ij,j} = \rho \ddot{u}_i \), where a subscript following a comma indicates
partial differentiation with respect to the spatial coordinate and dots indicate partial
differentiation with respect to time. For linear elasticity, \( \ddot{u}_i = \dot{v}_i \) with \( \dot{v}_i \) as the
ith component of the velocity vector. The equation of motion is rewritten in the
component form:

\[
\rho \ddot{v}_1 - T_{11,1} - T_{12,2} - T_{13,3} = 0 \\
\rho \ddot{v}_2 - T_{12,1} - T_{22,2} - T_{23,3} = 0 \\
\rho \ddot{v}_3 - T_{13,1} - T_{23,2} - T_{33,3} = 0
\]  

(3.1)

Next, according to the notation convention for treating anisotropic solids, e.g., [4],
we change the double-index notation for stresses to be single indexed:

\[
\mathbf{T} = \begin{bmatrix}
T_{11} & T_{12} & T_{13} \\
T_{12} & T_{22} & T_{23} \\
T_{13} & T_{23} & T_{33}
\end{bmatrix} = \begin{bmatrix}
T_1 & T_6 & T_5 \\
T_6 & T_2 & T_4 \\
T_5 & T_4 & T_3
\end{bmatrix}
\]  

(3.2)

and Eq.(3.1) becomes:

\[
\rho \frac{\partial}{\partial t} \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} = \begin{bmatrix}
\frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2}
0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1}
0 & 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_1}
\end{bmatrix} \begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{bmatrix}
\]  

(3.3)

To proceed, we consider the constitutive relation. Stress is related to strain
through a linear relation \( T_{ij} = c_{ijkl} S_{kl} \), where \( c_{ijkl} \) is a fourth-rank stiffness ten-
sor and \( S_{kl} = (u_{k,l} + u_{l,k})/2 \) is the strain tensor. Both \( T_{ij} \) and \( S_{kl} \) are symmetric and
Fig. 3.1: A schematic of solids of hexagonal symmetry and the Cartesian coordinates each has six independent components. Aided by the index rule shown in Eq. (3.2), the elastic constitutive equation can be recast into the form $T_p = c_{pq}S_q$, $p, q = 1, \ldots, 6$ where stress and strain become single-indexed and the stiffness constants are double indexed. For a lossless medium, we must have $c_{klij} = c_{ijkl}$, or $c_{pq} = c_{qp}$, i.e., the $6 \times 6$ matrix of $c_{pq}$ is symmetric. For the most general anisotropic solids, e.g., a triclinic solid, there are 21 independent stiffness constants.

We consider solids with hexagonal symmetry. As shown in Fig. (3.1), Cartesian coordinates are chosen such that the $x_3$ axis is aligned with the principal axis of hexagonal cylinder of the medium, the $x_2$ axis is aligned with one of crystallographic axes of horizontal plane, and the $x_1$ axis is perpendicular to the $x_3 - x_2$ plane. In this coordinate system, only 5 of 21 stiffness constants are nonzero, and the constitutive relation can be written in the following form:
Differentiate Eq.(3.4) with respect to time and we have:

\[
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2
\end{pmatrix}
\begin{pmatrix}
\partial u_1/\partial x_1 \\
\partial u_2/\partial x_2 \\
\partial u_3/\partial x_3 \\
\partial v_2/\partial x_3 + \partial v_3/\partial x_2 \\
\partial v_1/\partial x_3 + \partial v_3/\partial x_1 \\
\partial v_1/\partial x_2 + \partial v_2/\partial x_1
\end{pmatrix}
\]

(3.4)

where \( u_i, i = 1, 2, 3 \) are the displacements in \( x_i, i = 1, 2, 3 \) directions respectively. Differentiate Eq.(3.4) with respect to time and we have:

\[
\frac{\partial}{\partial t}
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2
\end{pmatrix}
\begin{pmatrix}
\partial v_1/\partial x_1 \\
\partial v_2/\partial x_2 \\
\partial v_3/\partial x_3 \\
\partial v_2/\partial x_3 + \partial v_3/\partial x_2 \\
\partial v_1/\partial x_3 + \partial v_3/\partial x_1 \\
\partial v_1/\partial x_2 + \partial v_2/\partial x_1
\end{pmatrix}
\]

(3.5)

where \( v_i, i = 1, 2, 3 \) are the velocities in \( x_i, i = 1, 2, 3 \) directions respectively. Eqs.(3.3) and (3.5) are the governing equations to be analyzed in the following sections. The rest of the chapter is organized in the following. Section 3.1 reiterates the full governing equations by recasting them into a vector form. Section 3.2 shows the one-dimensional version of the governing equations along the Cartesian coordinate axes. The non-trivial eigenvalues of the Jacobian matrix represent the wave speeds along
the coordinate axes. Section 3.3 illustrates the two-dimensional governing equations in the $x_1 - x_2$ and $x_1 - x_3$ planes. Aided by the rotated coordinates, we analyze wave propagation in two-dimensional plane. With all directions considered, we constructed the slowness curves in the $x_1 - x_2$ plane and in the $x_1 - x_3$ plane. Section 3.4 illustrates three-dimensional equations in arbitrarily rotated coordinates. The eigenvalues of the Jacobian matrix in an arbitrary direction in the three-dimensional space are numerically calculated and plotted to show the three-dimensional slowness surfaces. Wave polarization is calculated by the eigenvectors of the Jacobian matrix. In both two- and three-dimensional slowness profiles, the properties of cadmium sulfide are used in the calculations. At the last part of the chapter, we provide numerical results and discussions of stress wave propagation in hexagonal symmetry crystals. A one-dimensional expansion wave propagation and two-dimensional expanding wave in a plane are considered. Numerical results are directly compared to available analytical solutions. Finally, a conclusions are given.

### 3.1 First-order Equations in a Vector Form

We combine Eqs.(3.3) and (3.5) and recast the equation set into a vector form:

$$
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x_1} + B \frac{\partial U}{\partial x_2} + C \frac{\partial U}{\partial x_3} = 0 \quad \text{(3.6)}
$$

where the unknown vector $U$:

$$
U = (v_1, v_2, v_3, T_1, T_2, T_3, T_4, T_5, T_6)^T, \quad \text{(3.7)}
$$
and the Jacobian matrices are:

$$
A = \begin{bmatrix}
0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
-c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -(c_{11} - c_{12})/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad (3.8)
$$

$$
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \\
0 & -c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-(c_{11} - c_{12})/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad (3.9)
$$
\[
\mathbf{C} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (3.10)
\]

Eqs. (3.8)-(3.10) include nine equations for nine unknowns. As will be shown in the following sections, this equation set is hyperbolic because \(k_1\mathbf{A} + k_2\mathbf{B} + k_3\mathbf{C}\) has real eigenvalues for arbitrary real constants \(k_1, k_2\), and \(k_3\). With suitable initial and boundary conditions, the governing equations can be solved for propagating stress waves in elastic solids with hexagonal symmetry. Moreover, the physics of wave propagation in the medium can be fully described by the eigenstructure of the Jacobian matrices \(\mathbf{A}, \mathbf{B}, \) and \(\mathbf{C}\).

### 3.2 One-dimensional Wave Equations

We assume unknown vector \(\mathbf{U}\) is function of \(x_1\) and \(t\) only. And Eq.(3.6) becomes:

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x_1} = 0. \quad (3.11)
\]

As shown in Eq.(3.8), the entries of the fifth, sixth, and seventh columns of matrix \(\mathbf{A}\) are all zero. This implies that the solutions of \(T_2 = T_{22}, T_3 = T_{33},\) and \(T_4 = T_{23}\)
are decoupled from the rest of the equations. Thus Eq.(3.11) can be reduced to:

$$\frac{\partial U_1}{\partial t} + \tilde{A} \frac{\partial U_1}{\partial x_1} = 0.$$  \hspace{1cm} (3.12)

where

$$U_1 = (v_1, v_2, v_3, T_1, T_5, T_6)^T,$$  \hspace{1cm} (3.13)

and

$$\tilde{A} = \begin{bmatrix}
0 & 0 & 0 & -1/\rho & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & -1/\rho & 0 & 0 \\
-c_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{44} & 0 & 0 & 0 \\
0 & -(c_{11} - c_{12})/2 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  \hspace{1cm} (3.14)

The eigenvalues of matrix $\tilde{A}$ can be readily derived as:

$$\alpha_{1,2} = \pm \sqrt{\frac{c_{11}}{\rho}}, \quad \alpha_{3,4} = \pm \sqrt{\frac{c_{11} - c_{12}}{2\rho}}, \quad \alpha_{5,6} = \pm \sqrt{\frac{c_{44}}{\rho}}.$$  \hspace{1cm} (3.15)

To proceed, we rewrite Eq.(3.12) into six individual equations.

$$\frac{\partial v_1}{\partial t} - \frac{1}{\rho} \frac{\partial T_1}{\partial x_1} = 0, \quad \frac{\partial T_1}{\partial t} - c_{11} \frac{\partial v_1}{\partial x_1} = 0,$$

$$\frac{\partial v_2}{\partial t} - \frac{1}{\rho} \frac{\partial T_6}{\partial x_1} = 0, \quad \frac{\partial T_6}{\partial t} - \frac{c_{11} - c_{12}}{2} \frac{\partial v_2}{\partial x_1} = 0,$$

$$\frac{\partial v_3}{\partial t} - \frac{1}{\rho} \frac{\partial T_5}{\partial x_1} = 0, \quad \frac{\partial T_5}{\partial t} - c_{44} \frac{\partial v_3}{\partial x_1} = 0.$$  \hspace{1cm} (3.16)

The six equations can be divided into three pairs. By combining the two equations
in each row of Eq. (3.16), we have:

\[
\begin{align*}
\frac{\partial^2 v_1}{\partial t^2} &= c_{11} \frac{\partial^2 v_1}{\partial x_1^2} / \rho \quad \frac{\partial^2 v_2}{\partial t^2} = \frac{c_{11} - c_{12}}{2\rho} \frac{\partial^2 v_2}{\partial x_1^2}, \\
\frac{\partial^3 v_1}{\partial t^2} &= \frac{c_{14}}{\rho} \frac{\partial^2 v_3}{\partial x_1^2}.
\end{align*}
\]

(3.17)

Eqs. (3.17) are three second-order partial differential equations, which model pure longitudinal and pure shear waves propagating along the \( x_1 \) axis. The wave speeds are the square root of the coefficients on the right hand sides of the equations. By comparing these wave speeds shown in Eq. (3.17) with the eigenvalues of \( \tilde{A} \) shown in Eq. (3.15), we identify that \(|\alpha_{1,2}|\) is the longitudinal wave speed, \(|\alpha_{3,4}|\) is the shear wave speed with polarization in the \( x_2 \) direction, \(|\alpha_{5,6}|\) is another shear wave speed with polarization in the \( x_3 \) direction.

Next, we consider the one-dimensional wave equation in the \( x_2 \) direction. Similar to the above derivation, we have:

\[
\frac{\partial U_2}{\partial t} + \tilde{B} \frac{\partial U_2}{\partial x_2} = 0.
\]

(3.18)

where

\[
U_1 = (v_1, v_2, v_3, T_2, T_4, T_6)^T,
\]

(3.19)
The eigenvalues of matrix $\tilde{B}$ can be readily derived as:

$$
\beta_{1,2} = \pm \sqrt{\frac{c_{11}}{\rho}}, \quad \beta_{3,4} = \pm \sqrt{\frac{c_{44}}{\rho}}, \quad \beta_{5,6} = \pm \sqrt{\frac{c_{11} - c_{12}}{2\rho}}.
$$

We rewrite Eq.(3.18) in its scalar form.

$$
\begin{align*}
\frac{\partial v_2}{\partial t} - \frac{1}{\rho} \frac{\partial T_2}{\partial x_2} &= 0, \\
\frac{\partial v_3}{\partial t} - \frac{1}{\rho} \frac{\partial T_4}{\partial x_2} &= 0, \\
\frac{\partial v_1}{\partial t} - \frac{1}{\rho} \frac{\partial T_6}{\partial x_2} &= 0,
\end{align*}
$$

The six equations are recast into three second-order equations:

$$
\begin{align*}
\frac{\partial^2 v_2}{\partial t^2} - \frac{c_{11}}{\rho} \frac{\partial^2 v_2}{\partial x_2^2} &= 0, \\
\frac{\partial^2 v_3}{\partial t^2} - \frac{c_{44}}{\rho} \frac{\partial^2 v_3}{\partial x_2^2} &= 0, \\
\frac{\partial^2 v_1}{\partial t^2} - \frac{c_{11} - c_{12}}{2\rho} \frac{\partial^2 v_1}{\partial x_2^2} &= 0.
\end{align*}
$$

By comparison between these three second-order equations with the eigenvalues of $\tilde{B}$ shown in Eq.(3.21), we identify that $|\beta_{1,2}|$ is the longitudinal wave speed, and $|\beta_{3,4}|$
and $|\beta_{5,6}|$ are the shear wave speeds with wave polarization in the $x_3$ and $x_1$ directions, respectively.

To proceed, we consider the one-dimensional wave equations in the $x_3$ direction:

$$\frac{\partial \mathbf{U}_3}{\partial t} + \mathbf{C} \frac{\partial \mathbf{U}_3}{\partial x_3} = 0.$$  \hspace{1cm} (3.24)

where

$$\mathbf{U}_1 = (v_1, v_2, v_3, T_3, T_4, T_5)^T,$$  \hspace{1cm} (3.25)

and

$$\mathbf{C} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & 0 & -c_{33} & 0 & 0 & 0 \\
0 & -c_{44} & 0 & 0 & 0 & 0 \\
-c_{44} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$  \hspace{1cm} (3.26)

The eigenvalues of matrix $\mathbf{C}$ can be readily derived as:

$$\gamma_{1,2} = \pm \sqrt{\frac{c_{33}}{\rho}}, \quad \gamma_{3,4,5,6} = \pm \sqrt{\frac{c_{44}}{\rho}}.$$  \hspace{1cm} (3.27)
We rewrite Eq.(3.24) to be six scalar equations.

\[
\frac{\partial v_3}{\partial t} - \frac{1}{\rho} \frac{\partial T_3}{\partial x_3} = 0, \quad \frac{\partial T_3}{\partial t} - c_{33} \frac{\partial v_3}{\partial x_3} = 0, \\
\frac{\partial v_1}{\partial t} - \frac{1}{\rho} \frac{\partial T_5}{\partial x_3} = 0, \quad \frac{\partial T_5}{\partial t} - c_{44} \frac{\partial v_1}{\partial x_3} = 0, \\
\frac{\partial v_2}{\partial t} - \frac{1}{\rho} \frac{\partial T_4}{\partial x_3} = 0, \quad \frac{\partial T_4}{\partial t} - c_{44} \frac{\partial v_2}{\partial x_3} = 0,
\]

(3.28)

The six equations are recast into three second-order equations:

\[
\frac{\partial^2 v_3}{\partial t^2} = \frac{c_{33}}{\rho} \frac{\partial^2 v_3}{\partial x_3^2}, \\
\frac{\partial^2 v_1}{\partial t^2} = \frac{c_{44}}{\rho} \frac{\partial^2 v_1}{\partial x_3^2}, \\
\frac{\partial^3 v_2}{\partial t^2} = \frac{c_{44}}{\rho} \frac{\partial^3 v_2}{\partial x_3^2}.
\]

(3.29)

By inspecting these second-order equations, we identify the physical meaning of the eigenvalues of C: \(|\gamma_{1,2}|\) is the longitudinal wave speed in the \(x_3\) direction. \(|\gamma_{3,4,5,6}|\) is the shear wave speed in the \(x_3\) direction.

### 3.3 Two-dimensional Wave Equations

First, we assume that the unknowns in \(U\) are functions of \(x_1, x_2\) and \(t\) only. Thus Eq.(3.6) becomes:

\[
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x_1} + B \frac{\partial U}{\partial x_2} = 0.
\]

(3.30)

The entries in the sixth column of both \(A\) and \(B\) are zero. This implies that the equation for \(T_3 = T_{33}\) is decoupled from the rest of the governing equations. Thus
Eq. (3.30) is reduced to be:

\[
\frac{\partial \hat{U}}{\partial t} + \hat{A} \frac{\partial \hat{U}}{\partial x_1} + \hat{B} \frac{\partial \hat{U}}{\partial x_2} = 0,
\]  

(3.31)

where

\[
\hat{U} = (v_1, v_2, v_3, T_1, T_2, T_3, T_4, T_5, T_6)^T,
\]  

(3.32)

\[
\hat{A} = \begin{bmatrix}
0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 \\
-c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{44} & 0 & 0 & 0 & 0 & 0 \\
0 & -(c_{11} - c_{12})/2 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]  

(3.33)

\[
\hat{B} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \\
0 & -c_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{44} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-(c_{11} - c_{12})/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]  

(3.34)
To proceed, we transform Eq. (3.2) into a rotated Cartesian coordinate system \((x_1), (x_2)\). Refer to Fig. (3.2), The rotated coordinates are related to the original coordinates through the direction-cosine tensor \(l_{pq}\):

\[
x_{(p)} = l_{pq}x_q; \quad x_q = l_{qp}x_{(p)}; \quad (p), q = 1, 2,
\]

\[
\frac{\partial x_{(p)}}{\partial x_q} = l_{pq}, \quad \frac{\partial x_q}{\partial x_{(p)}} = l_{qp}; \quad (3.35)
\]

\[
l_{pq} = \cos[\vec{I}_{(p)}, \vec{I}_q], \quad l_{qp} = \cos[\vec{I}_q, \vec{I}_{(p)}],
\]

where \(\vec{I}_{(p)}\) and \(\vec{I}_q\) are the unit normal vectors along the \((p)\) axis in the new coordinate system and the \(q\) axis in the old coordinate system, respectively. In Eq. (3.35), \([\vec{I}_{(p)}, \vec{I}_q]\) represents the angle between \(\vec{I}_{(p)}\) and \(\vec{I}_q\). Aided by Eqs. (3.35), then Eqs. (3.31) can
be transformed into the new coordinates by using the chain rule:

\[
\frac{\partial}{\partial x_q} = l_{(p)q} \frac{\partial}{\partial x_p}.
\] (3.36)

To proceed, we let \( \theta = \theta_l(q) \), where \( 0 \leq \theta \leq 2\pi \) and \( \theta \) is measured in the counterclockwise direction and the direction-cosine tensor is:

\[
[l_{(p)q}] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.
\] (3.37)

Aided by Eqs.(3.36) and Eqs.(3.37), Eq.(3.31) in the new coordinate system becomes:

\[
\frac{\partial \hat{U}}{\partial t} + A^* \frac{\partial \hat{U}}{\partial x_{(1)}} + B^* \frac{\partial \hat{U}}{\partial x_{(2)}} = 0,
\] (3.38)

where the Jacobian matrices are:

\[
A^* = l_{(1)1} \hat{A} + l_{(1)2} \hat{B},
\]

\[
B^* = l_{(2)1} \hat{A} + l_{(2)2} \hat{B}.
\] (3.39)

To find the wave speed in the direction specified by \( \theta \) in the \( x_1 - x_2 \) plane, we consider the one-dimensional version of Eq.(3.38):

\[
\frac{\partial \hat{U}}{\partial t} + A^* \frac{\partial \hat{U}}{\partial x_{(1)}} = 0,
\] (3.40)

where

\[
\begin{pmatrix} 0_3 & A^*_3 \times 5 \\ A^*_5 \times 3 & 0_5 \end{pmatrix}
\] (3.41)
where $\mathbf{0}_3$ is $3 \times 3$ identity zero matrix, $\mathbf{0}_5$ is $5 \times 5$ identity zero matrix,

$$
\mathbf{A}_{3 \times 5}^* = \begin{bmatrix}
-\cos \theta / \rho & 0 & 0 & 0 & -\sin \theta / \rho \\
0 & -\sin \theta / \rho & 0 & 0 & -\cos \theta / \rho \\
0 & 0 & -\sin \theta / \rho & -\cos \theta / \rho & 0
\end{bmatrix},
$$

(3.42)

$$
\mathbf{A}_{5 \times 3}^* = \begin{bmatrix}
-c_{11} \cos \theta & -c_{12} \sin \theta & 0 \\
-c_{12} \cos \theta & -c_{11} \sin \theta & 0 \\
0 & 0 & -c_{44} \sin \theta \\
0 & 0 & -c_{44} \cos \theta \\
-(c_{11} - c_{12}) \sin \theta / 2 & -(c_{11} - c_{12}) \cos \theta / 2 & 0
\end{bmatrix}.
$$

(3.43)

The eigenvalues of $\mathbf{A}^*$ can be readily derived as:

$$
\lambda_{1,2} = 0, \quad \lambda_{3,4} = \pm \sqrt{\frac{c_{11}}{\rho}}, \quad \lambda_{5,6} = \pm \sqrt{\frac{c_{44}}{\rho}}, \quad \lambda_{7,8} = \pm \sqrt{\frac{(c_{11} - c_{12})^2}{2\rho}}.
$$

(3.44)

All eight eigenvalues of $\mathbf{A}^*$ are real. Thus Eq.(3.40) is a hyperbolic wave equation. Moreover, because of the definition of $\mathbf{A}^*$, i.e., Eq.(3.39), all real eigenvalues of $\mathbf{A}^*$ also imply that $k_1 \mathbf{A} + k_2 \mathbf{B}$ has real eigenvalues for all possible real numbers $k_1$ and $k_2$. Therefore, the original two-dimensional governing equation, Eq.(3.38), is a hyperbolic wave equation.

The eigenvalues shown in Eq.(3.44) represent the wave speeds in the two-dimensional plane. Two of the eigenvalues are zero. The rest of them form three plus/minus pairs. The absolute values of nontrivial eigenvalues are identical to that of $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ in the one-dimensional governing equations along the $x_1$ and $x_2$ axes. Refer to Eqs.(3.15) and (3.21). Therefore, the absolute values of the non-trivial eigenvalues are the wave
Fig. 3.3: Two-dimensional slowness curves of cadmium sulfide in $x_1 - x_3$ plane, in which the solid is isotropic

speeds of the corresponding longitudinal and shear waves. Moreover, the eigenvalues of $\mathbf{A}^*$ are independent of $\theta$. Therefore, the medium is isotropic in the $x_1 - x_2$ plane.

As an example, Fig.(3.3) shows the reciprocals of three wave speeds, i.e., Eq.(3.44), as the two-dimensional slowness profiles of cadmium sulfide, a solid with hexagonal symmetry. According to [4], the stiffness constants of cadmium sulfide are $c_{11} = 9.07 \times 10^{10} \text{N/m}^2$, $c_{33} = 9.38 \times 10^{10} \text{N/m}^2$, $c_{44} = 1.504 \times 10^{10} \text{N/m}^2$, $c_{12} = 5.81 \times 10^{10} \text{N/m}^2$, $c_{13} = 5.10 \times 10^{10} \text{N/m}^2$. The density $\rho = 4.82 \times 10^3 \text{kg/m}^3$.

Corresponding to each non-trivial eigenvalue, the eigenvector represents the wave polarization. By definition, we have $\mathbf{A}^*\vec{x}_i = \lambda_i\vec{x}_i$ where $\vec{x}$, a $8 \times 1$ column vector, is the $i$th right eigenvector corresponding to the eigenvalue $\lambda_i$. We put eight column vectors together and form a $8\times8$ square matrix $\mathbf{M} = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_8)$, i.e., the right
eigenvector matrix of $A^*$. Moreover, we have:

$$M^{-1}A^*M = \Lambda = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & \lambda_8
\end{bmatrix} \quad (3.45)$$

where $M^{-1}$ is the inverse of $M$, i.e., the left eigenvector matrix of $A^*$. In Eq.(3.45), $\Lambda$ is a diagonal matrix with eigenvalues of $A^*$ as its entries. Aided by Eq.(3.45), we pre-multiply Eq.(3.45) by $M^{-1}$ and have:

$$M^{-1}\frac{\partial \hat{U}}{\partial t} + M^{-1}A^*MM^{-1} \frac{\partial \hat{U}}{\partial x(1)} = 0,$$

$$M^{-1}\frac{\partial \hat{U}}{\partial t} + \Lambda M^{-1} \frac{\partial \hat{U}}{\partial x(1)} = 0.$$  \quad (3.46)

Note that we insert an identity matrix $I = MM^{-1}$ between $A^*$ and $\partial \hat{U}/\partial x(1)$ in Eq.(3.46). Since $M^{-1}$ is not a function of $x(1)$ or $t$, we let $U' = M^{-1}U = (u'_1, u'_2, \ldots, u'_8)$ and have:

$$\frac{\partial}{\partial t} \left( \begin{bmatrix} U' \end{bmatrix} \right) + \Lambda \frac{\partial}{\partial x(1)} \left( \begin{bmatrix} U' \end{bmatrix} \right) = 0.$$  \quad (3.47)

Eq.(3.47) implies eight separate scalar equations:

$$\frac{\partial \hat{u}'_i}{\partial t} + \lambda_i \frac{\partial \hat{u}'_i}{\partial x(1)} = 0, \quad i = 1, 2, \ldots, 8,$$

$$\hat{u}'_i$$

where $\hat{u}'_i$ is the $i$th characteristic variable of the wave equations Eq.(3.40). According to the d’Alembert solution, the solution profile of $u'_i$ moves at a speed of $\lambda_i$ in the space-time domain. Because $U' = M^{-1}\hat{U}$, the values of the characteristic variables
\( \hat{u}_i \) are determined by the inner product between the \( i \)th row vector in \( M^{-1} \) and the column vector \( \hat{U} \). In other words, eight row vectors contained in \( M^{-1} \) are the bases of an eight-dimensional vector space, where the characteristic unknowns \( U' \) is determined by linear combination of the eight bases by using the values of \( \hat{U} \).

Recall that only three of eight eigenvalues represent positive wave speeds. Refer to Eq. (3.44). We are interested in the left eigenvectors (the row vectors contained in \( M^{-1} \)) associated with these three non-trivial eigenvalues. Recall that the first three entries \( \hat{U} \) are velocity components. Therefore, for any one of three positive eigenvalues, the first three entries of the associated left eigenvector would determine the polarization direction of the velocity components. For linear elasticity, the polarization direction of the velocity components is identical to that of the displacement components. Thus, the first three entries of the left eigenvectors associated with a positive eigenvalue reveal the wave polarization direction for the wave moving at the speed determined by the corresponding the eigenvalue.

Fig.(3.4) shows that wave polarization for waves propagating in the \( x_1 - x_2 \) plane for cadmium sulfide. The plot shows two polarization vectors for each wave propagation direction specified by \( \theta \) with the third one normal to the \( x_1 - x_2 \) plane. Fig.(3.4) also shows that in all directions, waves are always pure longitudinal or pure shear. We arbitrarily plot the polarization for wave propagating in the directions of \( \theta = 30^\circ \) and \( 60^\circ \).

Next, we assume that the unknowns in vector \( U \), Eq.(3.6) becomes:

\[
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x_1} + C \frac{\partial U}{\partial x_3} = 0. \tag{3.49}
\]

The entries in the fifth column of both \( A \) and \( C \) are zero. Refer to Eqs. (3.8) and
Fig. 3.4: The polarization directions of waves propagating in the $x_1 - x_2$ plane in solids of hexagonal symmetry (3.10). Thus Eq. (3.49) is reduced to be:

$$\frac{\partial \hat{U}}{\partial t} + \hat{A} \frac{\partial \hat{U}}{\partial x_1} + \hat{C} \frac{\partial \hat{U}}{\partial x_3} = 0.$$  \hspace{1cm} (3.50)

where

$$\hat{U} = (v_1, v_2, v_3, T_1, T_3, T_4, T_5, T_6)^T.$$  \hspace{1cm} (3.51)
and $A$ and $C$ become $8 \times 8$ matrices and they are:

$$
\hat{A} = \begin{bmatrix}
0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
-c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{44} & 0 & 0 & 0 & 0 & 0 \\
0 & -(c_{11} - c_{12})/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

(3.52)

$$
\hat{C} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 \\
0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{13} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{33} & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

(3.53)

To proceed, we rotate the coordinates with respect to the $x_2$ axis with an angle $-\pi \leq \phi \leq \pi$, measured in the counterclockwise direction from the $x_1$ axis. Aided by the chain rule and the direction-consine tensor similar to that shown in Eqs.(3.36) and (3.37) but formulated in $x_1 - x_3$ plane, Eq.(3.50) is transformed into a new coordinate system:
\[ \frac{\partial \hat{U}}{\partial t} + \hat{A}' \frac{\partial \hat{U}}{\partial x_{(1)}} + \hat{C}' \frac{\partial \hat{U}}{\partial x_{(3)}} = 0. \] (3.54)

where the Jacobian matrices are:

\[ \hat{A}' = l_{(1)1} \hat{A} + l_{(1)3} \hat{C}, \] (3.55)
\[ \hat{C}' = l_{(3)1} \hat{A} + l_{(3)3} \hat{C}, \]

and the entries of direction-cosine tensor are

\[
\begin{bmatrix}
  l_{(1)1} & l_{(1)3} \\
  l_{(3)1} & l_{(3)3}
\end{bmatrix}
= \begin{bmatrix}
  \cos \phi & \sin \phi \\
  -\sin \phi & \cos \phi
\end{bmatrix}. \] (3.56)

To find the wave speed in the direction specified by \( \phi \) in the \( x_1 - x_3 \) plane, we consider the one-dimensional version of Eq.(3.54):

\[ \frac{\partial \hat{U}}{\partial t} + \hat{A}' \frac{\partial \hat{U}}{\partial x_{(1)}} = 0. \] (3.57)

where

\[
\hat{A}' = \begin{bmatrix}
  0_{3\times3} & \hat{A}'_{3\times5} \\
  \hat{A}'_{5\times3} & 0_{5\times5}
\end{bmatrix}. \] (3.58)

\( 0_{3\times3} \) is \( 3 \times 3 \) zero matrix and \( 0_{5\times5} \) is \( 5 \times 5 \) zero matrix and

\[
\hat{A}'_{3\times5} = \begin{bmatrix}
  -\cos \phi/\rho & 0 & 0 & -\sin \phi/\rho & 0 \\
  0 & 0 & -\sin \phi/\rho & 0 & -\cos \phi/\rho \\
  0 & -\sin \phi/\rho & 0 & -\cos \phi/\rho & 0
\end{bmatrix}
\]
\[ \hat{A}'_{5 \times 3} = \begin{bmatrix} -c_{11} \cos \phi & 0 & -c_{13} \sin \phi \\ -c_{12} \cos \phi & 0 & -c_{13} \sin \phi \\ 0 & -c_{44} \sin \phi & 0 \\ -c_{44} \sin \phi & 0 & -c_{44} \cos \phi \\ 0 & -(c_{11} - c_{12}) \cos \phi / 2 & 0 \end{bmatrix} \]

The eigenvalues of \( \hat{A}' \) can be readily derived as:

\[ \eta_{1,2} = 0, \]
\[ \eta_{3,4} = \pm \sqrt{\frac{(c_{11} - c_{12}) \sin^2 \phi}{2} + c_{44} \cos^2 \phi} / \rho, \]
\[ \eta_{5,6} = \pm \sqrt{\left( c_{11} \sin^2 \phi + c_{33} \cos^2 \phi + c_{44} - \chi \right) / (2 \rho)}, \]
\[ \eta_{7,8} = \pm \sqrt{\left( c_{11} \sin^2 \phi + c_{33} \cos^2 \phi + c_{44} + \chi \right) / (2 \rho)}, \]
\[ \chi = \sqrt{\left( (c_{11} - c_{44}) \sin^2 \phi + (c_{44} - c_{33}) \cos^2 \phi \right)^2 + (c_{13} + c_{44})^2 \sin^2 2\phi}. \]

Again, two of the eight eigenvalues are zero. The remainder six eigenvalues form three plus/minus pairs. All eight eigenvalues are real and Eq.(3.57) is hyperbolic. Because of the definition of \( \hat{A}' \), this implies that \( k_1 \hat{A} + k_2 \hat{C} \) has real eigenvalues for all possible real numbers \( k_1 \) and \( k_2 \). As a result, Eq.(3.50) is a hyperbolic wave equation.

The three positive eigenvalues are the wave speeds of three different waves: \( |\eta_{3,4}| \) is the wave speed of a pure shear wave with wave polarization in the \( x_2 \) direction. \( |\eta_{5,6}| \) and \( |\eta_{7,8}| \) are the waves speeds of quasi-longitudinal and quasi-shear waves with propagation in the \( x_1 - x_3 \) plane. For waves propagating in all directions, \( 0 \leq \theta \leq 2\pi \) in the \( x_1 - x_3 \) plane, we plot the reciprocals of these three wave speeds to show the two-dimensional slowness profiles of cadmium sulfide in Fig.(3.5).

Fig.(3.6) shows the polarization directions of waves propagating in the first quadrant of the \( x_1 - x_3 \) plane at angles specified by \( \theta \). Along each wave propagation
direction, two polarization vectors are plotted in the $x_1 - x_3$ plane with the third one is normal to the $x_1 - x_3$ plane. Wave becomes pure-longitudinal and pure-shear when the wave propagation direction is aligned with the original Cartesian axes. In addition, for waves propagating in the direction at $\pi/2 - 52.9^\circ$ about the $x_3$-axis, we also observe pure longitudinal and shear waves. This result will be further illustrated in next section. In all other directions, the wave are quasi-longitudinal and quasi-shear.

3.4 Three-dimensional Wave Equations

We consider the three-dimensional governing equation in a rotated Cartesian coordinate system $(x_1, x_2, x_3)$. Refer to Fig.(3.7). The rotated coordinates are related
Fig. 3.6: The polarization directions of waves propagating in the $x_1 - x_3$ plane of solids of hexagonal symmetry.
to the original coordinates through the tensor of direction-cosine $l_{ij}$:

\[
x_{(i)} = l_{ij} x_q, \quad x_j = l_{j(i)} x_i, \quad (i), j = 1, 2, 3,
\]

\[
\frac{\partial x_{(i)}}{\partial x_j} = l_{ij}, \quad \frac{\partial x_j}{\partial x_{(i)}} = l_{j(i)},
\]

\[
l_{(i)j} = \cos[\vec{I}_{(i)}, \vec{I}_j], \quad l_{j(i)} = \cos[\vec{I}_j, \vec{I}_{(i)}],
\]

(3.60)

where $\vec{I}_{(i)}$ and $\vec{I}_j$ are the unit normal vectors along the $i$th axis in the new coordinate system and the $j$th axis in the old coordinate system, respectively, and $[\vec{I}_{(i)}, \vec{I}_j]$ is the angle between $\vec{I}_{(i)}$ and $\vec{I}_j$. Aided by Eq.(3.60), coordinate transformation can be achieved by using the chain rule:

\[
\frac{\partial}{\partial x_j} = l_{ij} \frac{\partial}{\partial x_{(i)}}, \quad (i), j = 1, 2, 3.
\]

(3.61)

The rotated coordinates are obtained by two consecutive coordinate rotations. First, we rotate the original coordinates with respect to the $x_3$-axis with an angle $0 \leq \theta \leq 2\pi$ measured in the counterclockwise direction. The direction-cosine tensor for this rotation is:

\[
[\lambda_\theta] = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(3.62)

Next, we rotate the coordinates with respect to the new $x_{(2)}$ axis with an angle $0 \leq \phi \leq \pi$ which is measured in the counterclockwise direction. The direction-cosine
Fig. 3.7: Rotation of three-dimensional Cartesian coordinates.

tensor for this rotation is:

\[ \lambda_\phi = \begin{bmatrix} \cos \phi & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{bmatrix} \]  

(3.63)

The direction-cosine tensor for the overall coordinate transformation is the multiplication of the above two tensors:

\[ [l_{ij}] = [\lambda_\phi][\lambda_\theta] = \begin{bmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & \sin \theta \\ -\sin \phi & \cos \phi & 0 \\ -\sin \theta \cos \phi & -\sin \theta \sin \phi & \cos \theta \end{bmatrix} \]  

(3.64)

For completeness, the inverse of the direction-cosines matrix is:

\[ [l_{ij}]^{-1} = [l_{ij}]^T = \begin{bmatrix} \cos \theta \cos \phi & -\sin \phi & -\sin \theta \cos \phi \\ \sin \phi \cos \theta & \cos \phi & -\sin \theta \sin \phi \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \]  

(3.65)
Aided by Eqs.(3.61) and (3.64), Eq. (3.6) can be transformed into the new coordinate system as:

$$\frac{\partial U}{\partial t} + A' \frac{\partial U}{\partial x(1)} + B' \frac{\partial U}{\partial x(2)} + C' \frac{\partial U}{\partial x(3)} = 0,$$

(3.66)

where the Jacobian matrices are:

$$A' = l_{(1)1}A + l_{(1)2}B + l_{(1)3}C,$$

$$B' = l_{(2)1}A + l_{(2)2}B + l_{(2)3}C,$$

$$C' = l_{(3)1}A + l_{(3)2}B + l_{(3)3}C.$$

(3.67)

Next, we consider the one-dimensional version of Eq. (3.66):

$$\frac{\partial U}{\partial t} + A' \frac{\partial U}{\partial x(1)} = 0,$$

(3.68)

where

$$A' = \begin{bmatrix} 0_{3 \times 3} & A'_{3 \times 6} \\ A'_{6 \times 3} & 0_{6 \times 6} \end{bmatrix}$$

(3.69)

In Eqs.(3.69), $0_{3 \times 3}$ is $3 \times 3$ zero matrix and $0_{6 \times 6}$ is $6 \times 6$ zero matrix and

$$A'_{3 \times 6} = \begin{bmatrix} \cos \phi \cos \theta \\ \rho \\ \sin \phi \sin \theta \\ \rho \\ 0 \\ \sin \phi \\ \rho \\ \cos \phi \sin \theta \\ \rho \end{bmatrix}$$
Eq. (3.68) describes wave propagation in the direction determined by the prescribed values of $\theta$ and $\phi$. The non-trivial eigenvalues of $A'$ are the wave speeds for waves propagating in this particular direction. Since $A'$ is a $9 \times 9$ matrix, there are 9 eigenvalues, which can be numerically calculated if the material properties and wave propagation direction are provided. Again, we use cadmium sulfide as an example of solids of hexagonal symmetry and the eigenvalues of $A'$ are calculated by using a simple code of Matlab.

In all directions specified by $\theta$ and $\phi$, three of the nine eigenvalues are always zeros. The other six eigenvalues form three plus/minus pairs. The absolute values of these three pairs represent three distinct wave speeds. In Fig. (3.8), we plot the reciprocals of three wave speeds in all directions to show three slowness surfaces of cadmium sulfide.

As shown in Section (3.3), wave polarization can be calculated as a part of the left eigenvector of $A'$ corresponding to the three positive eigenvalues. We used a simple Matlab code to calculate these left eigenvector, and the first three entries of the eigenvector are taken as the polarization vectors for the associated wave speed.

Borgnis [11] showed the existence of pure longitudinal wave and pure shear wave for waves propagating in certain directions, i.e., the compression direction, in solids of hexagonal symmetry. In these compression directions, wave polarization vectors
Fig. 3.8: Three-dimensional slowness surfaces of cadmium sulfide. (a) The inner surface for the quasi-longitudinal and fastest waves. (b) The intermediate slowness surface for pure-shear waves. (c) The outer slowness surface for quasi-shear waves.
are either aligned or orthogonal to the wave propagation direction. The finding was supported by Brugger [14], Kolodner [76] and Truesdell [130] and later extended by Zuo and Schreyer [155], Ostrosablin [106], and Ting [122]. For solids of hexagonal symmetry, the compression directions include (i) the direction along the $x_3$ axis, (ii) all directions in the $x_1-x_2$ plane as shown in Section (3.3), and (iii) waves propagating in all directions lying on a circular cone, the generating lines of which form an angle of $\pi/2 - \phi$ about the $x_3$-axis. Shown by Borgnis [11], the value of $\phi$ is determined by stiffness constants of the medium:

$$\tan \phi = \left[ \frac{c_{33} - c_{13} - 2c_{44}}{c_{11} - c_{13} - 2c_{44}} \right]^{0.5}$$

(3.70)

For cadmium sulfide, $\tan \phi = 1.322$ and $\phi = 52.9^\circ$.

Fig.(3.9) shows the wave polarization vectors of cadmium sulfide for waves propagating on the surface of such circular cone. By using the Matlab code, each of three wave polarization vectors associated with a wave speed is calculated by taking the first three entries of the three left eigenvectors corresponding to the positive eigenvalues of $A'$. Clearly, along each wave propagation direction, three polarization vectors form a mutually orthogonal triplet. If one of the three polarization vectors is aligned with wave propagation direction, pure longitudinal and pure shear waves exist. The result shown in Fig.(3.8) clearly demonstrated that our approach is validated by the result of Borgnis [11].

If we chose an arbitrary cone angle other than that specified by Eq.(3.70), the polarization vectors are not parallel or orthogonal to the wave propagation direction. Fig.(3.10) shows the result of waves propagating on the surface of a cone of $70^\circ$ with respect to $x_3$ axis. Obviously, none of the polarization vectors is aligned or normal to the wave propagation direction.
Fig. 3.9: Polarization vectors for waves propagating on the surface of a circular cone representing the compression directions. Polarization vectors show one pure longitudinal wave along the wave propagation direction and two pure-shear waves orthogonal to the wave propagation direction.

Fig. 3.10: Polarization vectors for waves propagating on the surface of a circular cone of 70° with respect to $x_3$ axis. Polarization vectors are not aligned or orthogonal to the wave propagation direction.
3.5 Results and Discussions

3.5.1 One-Dimension Expansion Wave

We consider a block of cadmium sulphide, a hexagonal crystal of \( \text{6mm} \) point group. Its material properties are taken from Auld [4]:

- \( c_{11} = 9.07 \times 10^{10} \text{ Pa} \),
- \( c_{33} = 9.38 \times 10^{10} \text{ Pa} \),
- \( c_{44} = 1.504 \times 10^{10} \text{ Pa} \),
- \( c_{12} = 5.81 \times 10^{10} \text{ Pa} \),
- \( c_{13} = 5.10 \times 10^{10} \text{ Pa} \),
- \( \rho = 4820 \text{ kg/m}^3 \).

The length of the computational domain is \( L = 1 \text{ m} \). When \( t < 0 \), the solid translates horizontally at constant velocity \( 1 \text{ m/s} \). Refer to Fig. (3.11). When \( t > 0 \), the left end of the solid, i.e., \( x = 0 \), is suddenly held fixed. For all time, the right end of the solid is kept traction-free. Numerical calculation is performed to model the expansion wave propagating in the one-dimensional block. Two cases are considered: (i) The wave propagation is aligned with the \( x_3 \) axis, which is a compression direction of the hexagonal solid. In this direction, pure longitudinal wave is expected. (ii) The plane wave is not in a compression direction. In this case, we show complex wave interactions.

For Case 1, Fig. (3.11) shows four snapshots of the calculated \( T_3 \) and \( v_3 \) profiles, at \( t = 0, 40, 200, \) and \( 280 \mu \text{s} \). Since we have chosen a compression direction, the normal stress is uncoupled from the shear stress, as shown in Eqn. (A.1). Since the suddenly stopped condition on the left end of the solid only induces normal stress, there is no shear wave in this case. The result of the normal stress is a simple longitudinal wave moving as if in an isotropic solid. For completeness, the analytical solution is given in Appendix (A.1). Fig. (3.11(a)) shows the initial condition, at which velocity \( v_3 \) is constant and there is no stress. Fig. (3.11(b)) and (c) show the wave front move in the \( +\vec{x}_1 \), or \( x_3 \), direction. The numerical solution of the wave front compares well with the analytic solution. In Fig. (3.11(d)), the waves reflected back from the free end of the solid.
For Case 2, we consider expansion wave propagating in the direction: \((\theta, \phi) = (20^\circ, 70^\circ)\), which is not a compression direction of the solid. Figures 3.12, 3.13, and 3.14 show six snapshots of the calculated results at three different times: \(t = 40, 200, \text{and } 280 \mu s\). Figures 3.12(a), 3.13(a), and 3.14(a) show the solutions of the primary unknowns \((v, T)\) while Figs. 3.12(b), 3.13(b), and 3.14(b) show the characteristic variables \((\hat{U})\) with propagating waves, which are obtained by post-processing the solutions by using Eqn.(3.48). The solutions of the characteristic variables are uncoupled and each propagates at its own wave speed. The wave speeds for \(\hat{u}_1, \ldots, \hat{u}_9\) are -4993.43, -3313.21, -2976.47, 0, 0, 0, 2976.47, 3313.21, 4993.43 m/s.

In numerical calculations, the maximum CFL number \(\nu\) is about 0.998.

Initially, the solid is stress-free and at a constant velocity. When \(t > 0\) the left end of the solid is suddenly stopped to give rise to qL and qS waves propagating in the solid. All components of velocities and stresses appear in the solution and the evolving profiles of the unknowns are complex. Figures 3.12(a) and (b) show snapshots of the primary unknowns and the characteristic variables with positive wave speeds at \(t = 40 \mu s\). The waves travel in the \(+\bar{x}_1\) direction. We can identify the three distinct wave speeds in Fig. 3.12(b). Figures 3.13(a) and (b) show the similar results at \(t = 120 \mu s\) before waves hit the free boundary on the right end. Profiles of the calculated characteristic variables shown in Fig. 3.13(b) clearly identify three moving wave fronts. Figures 3.14(a) and (b) show the further evolution of the expansion wave at \(t = 280 \mu s\). The results show that the qL wave is reflected from the right end of the solid and is moving to the left, while two qS waves remain propagating toward the right. The reflected qL wave propagates in characteristic variable \(\hat{u}_1\) shown in Fig. 3.14(b), which has negative wave speed, instead of \(\hat{u}_9\) shown in Figs. 3.12(b) and 3.13(b), which has positive wave speed.
3.5.2 Expanding Wave in an Plane

To proceed, we consider a two-dimensional wave problem in an infinite domain. The initial condition is a two-dimensional step function \( v = a \mathcal{H}(\sigma - |\mathbf{x}|) \), where \( a \) represents a constant initial velocity to initiate the expanding wave from the origin. The solid is beryl, a solid of hexagonal symmetry, and the material properties are taken from [99] as \( \rho = 2.7 \text{ g/cm}^3 \), \( c_{11} = 26.94 \times 10^{11} \text{ dynes/cm}^2 \), \( c_{12} = 9.61 \times 10^{11} \text{ dynes/cm}^2 \), \( c_{13} = 6.61 \times 10^{11} \text{ dynes/cm}^2 \), \( c_{33} = 23.63 \times 10^{11} \text{ dynes/cm}^2 \), and \( c_{44} = 6.53 \times 10^{11} \text{ dynes/cm}^2 \). The computational domain is \(-1 \text{ m} \leq x_1 \leq 1 \text{ m} \) and \(-1 \text{ m} \leq x_2 \leq 1 \text{ m} \). The square domain is divided into 2.2 millions of triangular elements. The maximum length of the element edge is \( 2.68 \times 10^{-3} \text{ m} \). The constant \( \sigma \) in the initial condition is chosen to be \( 5 \times 10^{-3} \text{ m} \).

The two-dimensional code is parallelized by domain decomposition. First, a graph of element connectivity is built based on the unstructured mesh. The connectivity graph is then processed by METIS [71] to partition the domain. As shown in Fig. 3.15, the overall spatial domain is decomposed into 16 sub-domains. Numerical calculations for elements in each sub-domain is distributed to a workstation as a part of a networked cluster. In all cases, \( \Delta t = 65 \times 10^{-9} \text{ s} \) and the maximum CFL number \( \nu \) is about 0.95. Three cases of wave propagation are calculated: (1) calculation of the group velocity profile to assess numerical accuracy, (2) waves propagation in a plane of an arbitrary orientation to assess the non-reflective boundary condition, and (3) waves propagation in a plane composed of three blocks of beryl in different lattice orientations.

For elastic solids, the group velocity is identical to the energy velocity [4]. In Case 1, two-dimensional numerical simulations are performed to calculate the density of total energy, which is the summation of the kinetic energy and the strain energy normalized by area.
In Case 1, we consider wave propagation in 100 \((x_2-x_3)\) or 010 \((x_1-x_3)\) plane of hexagonal solids. The orientation of the solid is set to be \(\theta = 0^\circ\) and \(\phi = 90^\circ\), to have \(x_{(1)}\) aligned to \(x_3\), and \(x_{(2)}\) aligned to \(x_2\). The group velocities have the analytical solution. For completeness, the analytical solution is provided in Appendix A.2. The group velocities of pure-shear (SH), quasi-longitudinal (qL), and quasi-shear (qS) waves for beryl are plotted in Fig. 3.16(a) [99]. The simulation result, shown in Fig. 3.16(b), compares well with the analytical solution.

In Case 2, the orientation of the plane is in the direction of \(\theta = 20^\circ\) and \(\phi = 70^\circ\). Similar to that in Case 1, the initial condition is a velocity source at the origin with \(\sigma = 0.005\) m. Figure 3.17 shows two snapshots of the calculated energy profiles at \(t = 26\) and \(130\) \(\mu s\). With the non-reflecting boundary condition implemented on the four boundaries of the computational domain, waves propagate out of the domain without spurious reflection.

For Case 3, we calculate wave propagation in a domain composed of three blocks of beryl with three different lattice orientations. The partition of the domain is shown in Fig. 3.18. Shown in Fig. 3.19(a), an initial, cylindrical wave is generated at the origin. This is because of the lattice orientation of the central block such that an isotropic wave expansion occurs. This wave expands and interacts with the interfaces separating the neighbouring blocks of beryl with different lattice orientations. Figure 3.19 shows four snapshots of the calculated energy profiles at different times. Due to complex wave/interface interactions, all wave modes, including SH, qL and qS are excited at the interface. As a results, refracted waves at different speeds can be seen in Figs. 3.19(b) and 3.19(c). The original circular wave front is fractured after passing through the interfaces, as shown in Fig. 3.19(d). Complex wave features have been successfully captured by the newly developed CESE code.
Fig. 3.11: Four snapshots of simulated wave propagation along a compression direction in a block of cadmium sulphide, which is suddenly stopped. Calculated solutions are marked with × symbol, while the analytical solutions are plotted as solid lines. The maximum CFL number ($\nu$) is about 0.95. (a) $t = 0 \mu s$. (b) $t = 40 \mu s$. (c) $t = 200 \mu s$. (d) $t = 280 \mu s$. 
Fig. 3.12: The snapshot for simulated wave propagation in a block of cadmium sulphide at $t = 40 \mu s$ with (a) original variables and (b) characteristic variables.
Fig. 3.13: The snapshot for simulated wave propagation in a block of cadmium sulphide at $t = 120 \mu s$ with (a) original variables and (b) characteristic variables.
Fig. 3.14: The snapshot for simulated wave propagation in a block of cadmium sulphide at $t = 280 \mu s$ with (a) original variables and (b) characteristic variables.
Fig. 3.15: Two-dimensional mesh (2.2 million triangular elements). (a) close look at the mesh around origin. (b) decomposed 16 sub-domains.
Fig. 3.16: Comparison between the analytical solutions of the group velocities (in SH, qS, and qL polarization) and the calculated energy profiles for beryl at $t = 91 \, \mu s$, including: (a) The analytical solution calculated from Eqn. (A.13), and (b) The calculated energy density profiles.
Fig. 3.17: Plots for the normalized total energy density \( e / \text{max}(e) \) at two times: (a) \( t = 26 \mu s \), (b) \( t = 130 \mu s \).
Fig. 3.18: The computational domain is divided into three regions. The orientations of the solids in the central, left and right regions are $\theta = 0^\circ$ and $\phi = 0^\circ$, $\theta = 0^\circ$ and $\phi = 60^\circ$, and $\theta = 0^\circ$ and $\phi = 30^\circ$, respectively.
3.6 Conclusions

In this chapter, a new formulation was developed to model wave propagation in solids of hexagonal symmetry. The governing equations include the equation of motion and the first-order derivative of the elastic constitutive relations with respect to time. The result is a set of nine, first-order, hyperbolic partial differential equations with velocity components and stress components as the unknowns. The model equations are cast into a vector form with three $9 \times 9$ Jacobian matrices. We systematically analyzed these three matrices by looking into the one-, two-, three-dimensional versions of the governing equations in the rotated coordinates. Without invoking the plane-wave solution or the Christoffel equation, we showed that the characteristics of wave motions can be fully described by the eigenvalues and the left eigenvectors of the Jacobian matrices of the first-order system of equation.

In a one-dimensional space, the number of equations reduces to be six. These six equations are further grouped into three pairs with each pair combined to form a second-order wave equation in terms of a velocity component. The absolute values of the six eigenvalues of the Jacobian matrix are the wave speeds. In the two-dimensional planes, we derived the eigenvalues of the Jacobian matrix in the rotated coordinates, we derived the eigenvalues of the Jacobian matrix in the rotated coordinates. Waves in both $x_1 - x_2$ and $x_1 - x_3$ planes are considered. The non-trivial eigenvalues of the Jacobian matrix are the wave speeds of wave propagating in an arbitrary direction in the two-dimensional planes. By using cadmium sulfide as an example, we plotted the reciprocals of the wave speeds in all directions in the two-dimensional planes to show the slowness curves of three different waves in the two-dimensional planes. We transferred the three-dimensional equations to a new coordinate system by coordinate rotations. We then analyzed the Jacobian matrices of the first-order wave equations for waves propagating in all directions in the three-dimensional space. The values of
Fig. 3.19: Waves propagation in a heterogeneous domain. The total energy density ($e/\max(e)$) are plotted at different times: (a) $t = 72 \mu s$, (b) $t = 96 \mu s$, (c) $t = 108 \mu s$, (d) $t = 180 \mu s$. $\Delta t = 60 \text{ns}$, and CFL number = 0.95.
the non-trivial eigenvalues of the Jacobian matrix are wave speeds and the first three entries of the associated left eigenvectors form the polarization directions. We plotted the reciprocals of the three wave speeds of cadmium sulfide as the three slowness surfaces. Polarization vectors for wave propagation along compression directions on a surface of a circular cone as well as along arbitrary directions were also illustrated.

Finally the space-time CESE method is employed to solve a set of first-order partial differential equations with velocity and stress components as the unknowns. To demonstrate the present approach, numerical results of one and two-dimensional waves are reported, including one-dimensional expansion waves in a suddenly stopped rod, and two-dimensional wave expansion from a point. In the one-dimensional solution along a compression direction, the calculated wave fronts compare well with the analytical solution. For wave moving in a non-compressions direction, the calculated characteristic variables clearly show three wave fronts with different wave speeds, which compare well with the calculated eigenvalues of the Jacobian matrix. For two-dimensional calculations, a parallelized CESE code based on domain decomposition has been developed to resolve fine wave structures. The calculated energy profiles compare well with the analytical solution. The non-reflective boundary condition treatment shows no spurious reflection. We also simulate wave propagation in a heterogeneous solid composed of three blocks of beryl with different lattice orientations. These three blocks of beryl are pieced together to form the square computational domain with internal interface separating different lattice orientations. Numerical results show complex wave/interface interactions. The results in this chapter demonstrate the potential of the CESE method for modelling waves in complex solids.
CHAPTER 4
THEORETICAL ANALYSIS OF WAVE PROPAGATION IN TRIGONAL SYMMETRY CRYSTAL

4.1 Introduction

The basic theory of elastic waves in anisotropic solids has been extensively illustrated in many references, e.g., Refs. [2, 4, 11, 14, 16, 17, 43, 46, 49, 50, 64, 76, 80, 92, 99, 101, 106, 114, 119–122, 130, 137, 138, 142, 155]. However, to our knowledge, no one has analyzed or solved the governing equations in a first-order hyperbolic vector form, as will be discussed in the present chapter. For waves in anisotropic solids, conventional approach, Refs. [2, 4, 11, 14, 16, 17, 46, 49, 50, 64, 76, 80, 92, 100, 101, 114, 119, 121, 130, 142], employs the equation of motion and the elastic constitutive relations. By eliminating the stresses from the governing equations, a set of second-order wave equations in terms of velocity components is derived. Next, one substitutes a plane-wave solution in the form of \( \vec{v} = a \vec{f}(\vec{k} \cdot \vec{x} - \omega t) \) into the acoustic field equations to derive a set of algebraic equations, i.e., the Christoffel equations. Analyses of wave propagation in anisotropic solids have been based on solving the eigenvalues of Christoffel equations. Even for an inhomogeneous plane wave, this approach was commonly used, e.g., Ref. [43].

In this chapter, we consider an elastic solid with a trigonal 32 symmetry. A suitable Cartesian coordinate system shown in Fig.(4.1) is employed. Accordingly,
the constitutive relations are

\[
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{pmatrix} =
\begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\
c_{12} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & c_{14} \\
0 & 0 & 0 & 0 & c_{14} & \frac{(c_{11} - c_{12})}{2}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial u_1}{\partial x_1} \\
\frac{\partial u_2}{\partial x_2} \\
\frac{\partial u_3}{\partial x_3} + \frac{\partial u_4}{\partial x_2} \\
\frac{\partial u_4}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \\
\frac{\partial u_3}{\partial x_1} + \frac{\partial u_2}{\partial x_1}
\end{pmatrix}
\] (4.1)

where \(c_{ij}\) are the elastic stiffness constants. As an example in the present chapter, quartz without piezoelectric effect, a solid of trigonal 32 symmetry, is considered. The stiffness constants of quartz are \(c_{11} = 8.674 \times 10^{10} N/m^2\), \(c_{33} = 10.72 \times 10^{10} N/m^2\), \(c_{44} = 5.79 \times 10^{10} N/m^2\), \(c_{12} = 0.699 \times 10^{10} N/m^2\), \(c_{13} = 1.191 \times 10^{10} N/m^2\), \(c_{14} = -1.791 \times 10^{10} N/m^2\). The density of quartz is \(\rho = 2.65 \times 10^3 kg/m^3\).

The rest of the chapter is organized as follows. Section 4.2 illustrates the first-order pde’s as the governing equations in a vector form. Section 4.3 discusses the one-dimensional equations along the three Cartesian axes. Section 4.4 illustrates the
two-dimensional equations in three planes normal to the three Cartesian axes. In each
plane, we perform a coordinate rotation and derive a set of two-dimensional equations
by the chain rule. The wave speeds for a wave propagating in a certain direction
in the two-dimensional plane can be determined by the non-trivial eigenvalues of
the Jacobian matrices in the rotated coordinates. By considering all directions in
the plane, the two-dimensional slowness (or the inverse wave velocity) profiles will
be numerically calculated. Section 4.5 illustrates three-dimensional equations in an
arbitrarily rotated coordinate system. Wave speeds in a prescribed direction in the
three-dimensional space are determined by calculating the non-trivial eigenvalues of
the Jacobian matrix of the model equations. At the end of the chapter, we state the
conclusions.

4.2 First-Order Hyperbolic Equations

We apply partial differentiation with respect to time to the classic constitutive equa-
tions for solids of trigonal 32 symmetry, e.g. Eq.(4.1), and we have

\[
\begin{pmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{pmatrix}
= \begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\
c_{12} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
c_{14} & c_{14} & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & c_{14} \\
0 & 0 & 0 & 0 & c_{14} & c_{66}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial v_1}{\partial x_1} \\
\frac{\partial v_2}{\partial x_2} \\
\frac{\partial v_3}{\partial x_3} \\
\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \\
\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \\
\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}
\end{pmatrix}
\]
where \( c_{66} = (c_{11} - c_{12})/2 \) for convenience. Combine Eqs.(4.2) and (3.3) and cast them into a vector form:

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x_1} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial x_2} + \mathbf{C} \frac{\partial \mathbf{U}}{\partial x_3} = 0 \tag{4.3}
\]

where the unknown vector \( \mathbf{U} \):

\[
\mathbf{U} = (v_1, v_2, v_3, T_1, T_2, T_3, T_4, T_5, T_6)^T, \tag{4.4}
\]

and the coefficient matrices are:

\[
\mathbf{A} = \begin{bmatrix}
0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
-c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{14} & -c_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{66} & -c_{14} & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}, \tag{4.5}
\]
\[ B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & -c_{12} & -c_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{14} & -c_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad (4.6) \]

\[ C = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \\
0 & -c_{14} & -c_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{14} & -c_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \quad (4.7) \]

Eq. (4.3) includes nine equations for nine unknowns. As will be shown in the following sections, this equation set is hyperbolic because \( k_1A + k_2B + k_3C \) has real eigenvalues for arbitrary real constants \( k_1, k_2 \) and \( k_3 \). Moreover, wave physics can be fully described by the eigenvalues and eigenvectors of \( k_1A + k_2B + k_3C \).
4.3 One-dimensional Wave Equations

In this section, we consider one-dimensional wave propagation along the three Cartesian axes.

4.3.1 Wave Along the $x_1$ Axis

We assume unknown vector $U$ in Eq.(4.3) depends on $x_1$ and $t$ only, and Eq.(4.3) becomes

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x_1} = 0.$$ (4.8)

As shown in Eq.(4.5), the entries of the fifth, sixth, and seventh columns of matrix $A$ are all zeros, which imply that the solutions of $T_2 = T_{22}$, $T_3 = T_{33}$, and $T_4 = T_{23}$ are decoupled from the rest of the equations. Thus, Eq.(4.8) can be reduced to:

$$\frac{\partial U_1}{\partial t} + A_1 \frac{\partial U_1}{\partial x_1} = 0.$$ (4.9)

where

$$U_1 = (v_1, v_2, v_3, T_1, T_5, T_6)^T,$$ (4.10)
and

\[
A_1 = \begin{bmatrix}
0 & 0 & 0 & -1/\rho & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & 0 & -1/\rho & 0 \\
-c_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{14} & -c_{44} & 0 & 0 & 0 \\
0 & -c_{66} & -c_{14} & 0 & 0 & 0
\end{bmatrix}
\]

(4.11)

The six equations in Eq.(4.11) can be divided into three pairs.

\[
\frac{\partial v_1}{\partial t} - \frac{1}{\rho} \frac{\partial T_1}{\partial x_1} = 0, \quad \frac{\partial T_1}{\partial t} - c_{11} \frac{\partial v_1}{\partial x_1} = 0, \\
\frac{\partial v_2}{\partial t} - \frac{1}{\rho} \frac{\partial T_6}{\partial x_1} = 0, \quad \frac{\partial T_6}{\partial t} - c_{66} \frac{\partial v_2}{\partial x_1} = 0, \\
\frac{\partial v_3}{\partial t} - \frac{1}{\rho} \frac{\partial T_5}{\partial x_1} = 0, \quad \frac{\partial T_5}{\partial t} - c_{14} \frac{\partial v_2}{\partial x_1} - c_{44} \frac{\partial v_3}{\partial x_1} = 0,
\]

(4.12)

Two equations in the first row can be combined to form a second-order wave equation for \(v_1\):

\[
\frac{\partial^2 v_1}{\partial t^2} = \frac{c_{11}}{\rho} \frac{\partial^2 v_1}{\partial x_1^2}
\]

(4.13)

Equations (4.13) is a typical second-order wave equation, which describes the propagation of longitudinal waves along the \(x_1\) axis, and the wave speed is \(\sqrt{c_{11}/\rho}\).
To proceed, the eigenvalues of $A_1$ can be readily derived as

$$\alpha_{1,2} = \pm \sqrt{\frac{c_{11}}{\rho}}$$

$$\alpha_{3,4} = \pm \left( \frac{c_{44} + c_{66} + \sqrt{(c_{66} - c_{44})^2 + 4c_{14}^2}}{2\rho} \right)^{0.5}$$

$$\alpha_{5,6} = \pm \left( \frac{c_{44} + c_{66} - \sqrt{(c_{66} - c_{44})^2 + 4c_{14}^2}}{2\rho} \right)^{0.5}$$

(4.14)

Shown in Eq.(4.14), all six eigenvalues of $A_1$ are real, and Eq.(4.9) is hyperbolic. The six eigenvalues form three $\pm$ pairs. Aided by Eq.(4.13), we recognize that $|\alpha_{1,2}|$ is the wave speed of a pure longitudinal wave moving along the $x_1$ axis. As will be shown in the following, the absolute values of other eigenvalues represent the wave speeds of pure shear waves along the $x_1$ axis.

Corresponding to each eigenvalue, the eigenvector can be derived. By definition, $A_1 \vec{x}_i = \alpha_i \vec{x}_i$, where $\vec{x}_i$, a $6 \times 1$ column vector, is the $i$th eigenvector corresponding to the $i$th eigenvalue $\alpha_i$. Combine six column vectors together, and we form a $6 \times 6$ square matrix $M = (\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_6)$ which is the right eigenvector matrix of $A_1$:

$$M = \begin{bmatrix} M_f & M_l \end{bmatrix}$$

(4.15)

where

$$M_f = \begin{pmatrix} 1 & 0 & 0 & -\sqrt{\rho c_{11}} & 0 & 0 \\ 1 & 0 & 0 & 0 & \sqrt{\rho c_{11}} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$
For conciseness, we use the eigenvalues $\alpha_i, i = 1, \ldots, 6$ in Eq. (4.15). The inverse of $M$, i.e., $M^{-1}$, is the left eigenvector matrix of $A_1$:

$$M^{-1} = \begin{bmatrix} M_f^{-1} & M_l^{-1} \end{bmatrix} \quad (4.16)$$
\[
M_t^{-1} = \begin{pmatrix}
\frac{1}{2\sqrt{\rho c_{11}}} & 0 & 0 \\
\frac{1}{2\sqrt{\rho c_{11}}} & \frac{-c_{14}}{c_{14}} & \frac{-\left(\rho \alpha^2_3 - c_{44}\right)}{\left(\rho \alpha^2_3 - c_{44}\right)} \\
0 & \frac{2(\rho \alpha^2_3 - c_{44} - c_{66})\rho \alpha_3}{c_{14}} & \frac{2(\rho \alpha^2_3 - c_{44} - c_{66})\rho \alpha_3}{\left(\rho \alpha^2_3 - c_{44}\right)} \\
0 & \frac{2(\rho \alpha^2_3 - c_{44} - c_{66})\rho \alpha_3}{c_{14}} & \frac{2(\rho \alpha^2_3 - c_{44} - c_{66})\rho \alpha_3}{\left(\rho \alpha^2_3 - c_{44}\right)} \\
0 & \frac{2(\rho \alpha^2_3 - c_{44} - c_{66})\rho \alpha_5}{c_{14}} & \frac{2(\rho \alpha^2_3 - c_{44} - c_{66})\rho \alpha_5}{\left(\rho \alpha^2_3 - c_{44}\right)} \\
0 & \frac{2(\rho \alpha^2_3 - c_{44} - c_{66})\rho \alpha_5}{c_{14}} & \frac{2(\rho \alpha^2_3 - c_{44} - c_{66})\rho \alpha_5}{\left(\rho \alpha^2_3 - c_{44}\right)} \\
\end{pmatrix}
\]

Aided by the right and left eigenvector matrices, we have

\[
M^{-1}A_1M = \Lambda = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 \\
0 & \alpha_2 & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & \alpha_6 \\
\end{pmatrix}
\]

with \( \Lambda \) as a diagonal matrix with eigenvalues of \( A_1 \) as its entries. Aided by Eq.(4.17), we pre-multiply Eq.(4.8) by \( M^{-1} \) and have

\[
M^{-1}\frac{\partial U_1}{\partial t} + M^{-1}A_1M^{-1}\frac{\partial U_1}{\partial x_1} = 0,
\]

\[
M^{-1}\frac{\partial U_1}{\partial t} + \Lambda M^{-1}\frac{\partial U_1}{\partial x_1} = 0.
\]

Note that we insert an identity matrix \( I = MM^{-1} \) between \( A_1 \) and \( \partial U_1/\partial x_1 \) in Eq.(4.18). Since \( M^{-1} \) is a constant matrix, we let \( \bar{U}_1 = M^{-1}U_1 = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_6)^T \) and have

\[
\frac{\partial \bar{U}_1}{\partial t} + \Lambda \frac{\partial \bar{U}_1}{\partial x_1} = 0
\]

(4.19)
Equation (4.19) implies six decoupled scalar equations:

$$\frac{\partial \tilde{u}_i}{\partial t} + \alpha_i \frac{\partial \tilde{u}_i}{\partial x_1} = 0, \; i = 1, 2, \ldots, 6 \tag{4.20}$$

where \(\tilde{u}_i\) is the \(i\)th characteristic variable, the solution profile of which moves at the speed of \(\alpha_i\) along the \(x_1\) axis. Because \(\tilde{U}_1 = M^{-1}U_1\), the values of the characteristic variable \(\tilde{u}'_i\) are determined by the inner product between the \(i\)th row vector in \(M^{-1}\) and the column vector \(U_1\). The six row vectors of \(M^{-1}\) are the bases of a six-dimensional vector space, in which the characteristic unknowns \(\tilde{U}_1\) are defined by the entries in \(U_1\) as the coordinates of the vector space.

Aided by Eq. (4.16), the first three entries of the first, the third, and the fifth left eigenvectors in \(M^{-1}\) form the following three vectors:

$$v_{11}^p = (0.5, 0, 0)$$

$$v_{12}^p = \left(0, \frac{\rho \alpha_3^2 - c_{44}}{2(2\rho \alpha_3^2 - c_{44} - c_{66})}, \frac{c_{14}}{2(2\rho \alpha_3^2 - c_{44} - c_{66})}\right)$$

$$v_{13}^p = \left(0, \frac{\rho \alpha_3^2 - c_{44}}{2(2\rho \alpha_3^2 - c_{44} - c_{66})}, -\frac{c_{14}}{2(2\rho \alpha_3^2 - c_{44} - c_{66})}\right) \tag{4.21}$$

These three vectors are the bases of a subvector space, in which values of the velocity components, i.e., the first entries of \(U_1\) are the coordinates. The three vectors shown in Eq.(4.21) represent the wave polarization of the three wave speeds at \(\alpha_1\), \(\alpha_3\), and \(\alpha_5\). A simple calculation also show that \(v_{11}^p\), \(v_{12}^p\), and \(v_{13}^p\) are mutually orthogonal. Obviously, waves at the wave speeds of \(\alpha_3\) and \(\alpha_5\), respectively, are polarized in the \(x_2 - x_3\) plane, and they are pure shear waves.
4.3.2 Wave Along the $x_2$ Axis

To proceed, we consider the one-dimensional equations in the $x_2$ direction. Similar to the derivation in Sec 4.3.1, we have

$$\frac{\partial U_2}{\partial t} + B_2 \frac{\partial U_2}{\partial x_2} = 0 \quad (4.22)$$

where

$$U_2 = (v_1, v_2, v_3, T_2, T_4, T_6)^T, \quad (4.23)$$

and

$$B_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & -1/\rho & 0 & 0 \\
0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & -c_{11} & c_{14} & 0 & 0 & 0 \\
c_{14} & -c_{44} & 0 & 0 & 0 & 0 \\
c_{66} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \quad (4.24)$$

Eq.(4.22) are equivalent to the following six equations:

$$\frac{\partial v_1}{\partial t} - \frac{1}{\rho} \frac{\partial T_6}{\partial x_2} = 0, \quad \frac{\partial T_6}{\partial t} - c_{66} \frac{\partial v_1}{\partial x_2} = 0,$$

$$\frac{\partial v_2}{\partial t} - \frac{1}{\rho} \frac{\partial T_2}{\partial x_2} = 0, \quad \frac{\partial T_2}{\partial t} - c_{11} \frac{\partial v_2}{\partial x_2} + c_{14} \frac{\partial v_3}{\partial x_2} = 0,$$

$$\frac{\partial v_3}{\partial t} - \frac{1}{\rho} \frac{\partial T_4}{\partial x_2} = 0, \quad \frac{\partial T_4}{\partial t} + c_{14} \frac{\partial v_2}{\partial x_2} - c_{44} \frac{\partial v_3}{\partial x_2} = 0, \quad (4.25)$$

Two equations in the first row of Eq.(4.25) are combined to form a second-order
wave equation for $v_1$:

$$\frac{\partial^2 v_1}{\partial t^2} = \frac{c_{66}}{\rho} \frac{\partial^2 v_1}{\partial x_2^2} \quad (4.26)$$

Eq.(4.26) governs a pure shear wave, propagating along the $x_2$ axis and at a speed of $\sqrt{c_{66}/\rho}$. To proceed, the eigenvalues of matrix $B_2$ are

$$\beta_{1,2} = \pm \sqrt{\frac{c_{66}}{\rho}}$$

$$\beta_{3,4} = \pm \left( \frac{c_{44} + c_{11} + \sqrt{(c_{44} - c_{11})^2 + 4c_{14}^2}}{2\rho} \right)^{0.5}$$

$$\beta_{5,6} = \pm \left( \frac{c_{44} + c_{11} - \sqrt{(c_{44} - c_{11})^2 + 4c_{14}^2}}{2\rho} \right)^{0.5} \quad (4.27)$$

Shown in Eq.(4.27), all six eigenvalues of $B_2$ are real, and Eq.(4.22) is hyperbolic. These six eigenvalues form three $\pm$ pairs. The absolute values of the eigenvalues represent three wave speeds. Eq. (4.26) confirms that $|\beta_{1,2}|$ is the wave speed of pure shear waves propagating in the $x_2$ direction and polarized in the $x_1$ direction.

To proceed, the right eigenvector matrix of $B_2$ can be readily derived as

$$N = \begin{bmatrix} N_f \ \ N_l \end{bmatrix} \quad (4.28)$$

$$N_f = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\sqrt{\rho c_{66}} \\ 1 & 0 & 0 & 0 & 0 & \sqrt{\rho c_{66}} \end{pmatrix}^T$$

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\[
N_l = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & e_{11} - \rho \beta_3^2 & e_{11} - \rho \beta_3^2 & e_{11} - \rho \beta_5^2 \\
0 & e_{11} - \rho \beta_3^2 & e_{11} - \rho \beta_3^2 & e_{11} - \rho \beta_5^2 \\
\rho \beta_3 & -\rho \beta_3 & -\rho \beta_5 & \rho \beta_5 \\
\rho \beta_3 & -\rho \beta_3 & -\rho \beta_5 & \rho \beta_5 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Its inverse, i.e., the left eigenvector matrix, is

\[
N^{-1} = \begin{pmatrix}
N_f^{-1} & N_l^{-1}
\end{pmatrix}
\]

(4.29)

\[
N_f^{-1} = \begin{pmatrix}
0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2(2\rho \beta_3^2 - c_{11} - c_{44}) & 2(2\rho \beta_3^2 - c_{11} - c_{44}) & c_{14} \\
0 & 2(2\rho \beta_3^2 - c_{11} - c_{44}) & 2(2\rho \beta_3^2 - c_{11} - c_{44}) & c_{14} \\
0 & 2(2\rho \beta_5^2 - c_{11} - c_{44}) & 2(2\rho \beta_5^2 - c_{11} - c_{44}) & c_{14} \\
0 & 2(2\rho \beta_5^2 - c_{11} - c_{44}) & 2(2\rho \beta_5^2 - c_{11} - c_{44}) & c_{14} \\
\end{pmatrix}
\]
\[
N_i^{-1} = \begin{bmatrix}
0 & 0 & -1/2\sqrt{\rho c_{66}} \\
0 & 0 & -1/2\sqrt{\rho c_{66}} \\
-(\rho \beta_3^2 - c_{44}) & c_{14} & 2(\rho \beta_3^2 - c_{11} - c_{44}) \\
2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) & 2(\rho \beta_3^2 - c_{11} - c_{44}) & 2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) \\
2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) & 2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) & 2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) \\
2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) & 2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) & 2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) \\
2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) & 2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) & 2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) \\
2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) & 2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) & 2\rho \beta_3(2\rho \beta_3^2 - c_{11} - c_{44}) \\
\end{bmatrix}
\]

By taking the three entries of the first, third, and fifth row of \( N^{-1} \), we form the following three wave polarization vectors:

\[
v_{p}^{21} = (0.5, 0, 0)
\]

\[
v_{p}^{22} = \left( 0, \frac{\rho \beta_3^2 - c_{44}}{2(\rho \beta_3^2 - c_{11} - c_{44})}, \frac{c_{14}}{2(\rho \beta_3^2 - c_{11} - c_{44})} \right)
\]

\[
v_{p}^{23} = \left( 0, \frac{\rho \beta_3^2 - c_{44}}{2(\rho \beta_3^2 - c_{11} - c_{44})}, \frac{c_{14}}{2(\rho \beta_3^2 - c_{11} - c_{44})} \right)
\]

By examining these three wave polarization vectors, we recognize that \( |\beta_{1,2}| \) is the wave speed of pure shear waves, polarized in the \( x_1 \) direction, and \( |\beta_{3,4}| \) is the wave speed of quasi-longitudinal-shear waves, polarized in the \( x_2 - x_3 \) plane, propagating in the \( x_2 \) direction.

**Wave Along the \( x_3 \) Axis**

We proceed to consider the one-dimensional wave equations in the \( x_3 \) direction:

\[
\frac{\partial U_3}{\partial t} + C_3 \frac{\partial U_3}{\partial x_3} = 0.
\]
where

\[
U_1 = (v_1, v_2, v_3, T_3, T_4, T_5)^T,
\]

and

\[
C_3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & 0 & 0 & -1/\rho & 0 & 0 \\
0 & 0 & -c_{33} & 0 & 0 & 0 \\
0 & -c_{44} & 0 & 0 & 0 & 0 \\
-c_{44} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(4.33)

Eq.(4.31) can be recast into six scalar equations.

\[
\frac{\partial v_3}{\partial t} - \frac{1}{\rho} \frac{\partial T_3}{\partial x_3} = 0, \quad \frac{\partial T_3}{\partial t} - c_{33} \frac{\partial v_3}{\partial x_3} = 0,
\]

\[
\frac{\partial v_1}{\partial t} - \frac{1}{\rho} \frac{\partial T_5}{\partial x_3} = 0, \quad \frac{\partial T_5}{\partial t} - c_{44} \frac{\partial v_1}{\partial x_3} = 0,
\]

\[
\frac{\partial v_2}{\partial t} - \frac{1}{\rho} \frac{\partial T_4}{\partial x_3} = 0, \quad \frac{\partial T_4}{\partial t} - c_{44} \frac{\partial v_2}{\partial x_3} = 0.
\]

(4.34)

These six equations can be combined into three second-order equations:

\[
\frac{\partial^2 v_3}{\partial t^2} = \frac{c_{33}}{\rho} \frac{\partial^2 v_3}{\partial x_3^2},
\]

\[
\frac{\partial^2 v_1}{\partial t^2} = \frac{c_{44}}{\rho} \frac{\partial^2 v_1}{\partial x_3^2},
\]

\[
\frac{\partial^2 v_2}{\partial t^2} = \frac{c_{44}}{\rho} \frac{\partial^2 v_2}{\partial x_3^2}.
\]

(4.35)

Equation (4.35) represents three second-order wave equations for one pure longitudinal wave and two pure shear waves propagating along the \(x_3\) axis. To proceed, the
eigenvalues of \( C_3 \) are

\[
\gamma_{1,2} = \pm \sqrt{\frac{c_{33}}{\rho}}, \quad \gamma_{3,4,5,6} = \pm \sqrt{\frac{c_{44}}{\rho}}. \tag{4.36}
\]

By comparing Eq.(4.36) with Eq.(4.35), we identify that \(|\gamma_{1,2}|\) is the speed of a pure longitudinal wave in the \( x_3 \) direction, and \(|\gamma_{3,4,5,6}|\) is the speed of two pure shear waves, polarized in the \( x_1 \) and \( x_2 \) directions, respectively. For completeness, the right and left eigenvector matrices of \( C_3 \) are

\[
P = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
-\sqrt{\rho c_{33}} & \sqrt{\rho c_{33}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\sqrt{\rho c_{44}} & \sqrt{\rho c_{44}} \\
0 & 0 & -\sqrt{\rho c_{44}} & \sqrt{\rho c_{44}} & 0 & 0
\end{pmatrix} \tag{4.37}
\]

\[
P^{-1} = \begin{pmatrix}
0 & 0 & 0.5 & -1/2\sqrt{\rho c_{33}} & 0 & 0 \\
0 & 0 & 0.5 & 1/2\sqrt{\rho c_{33}} & 0 & 0 \\
0.5 & 0 & 0 & 0 & 0 & -1/2\sqrt{\rho c_{44}} \\
0.5 & 0 & 0 & 0 & 0 & 1/2\sqrt{\rho c_{44}} \\
0 & 0.5 & 0 & 0 & -1/2\sqrt{\rho c_{44}} & 0 \\
0 & 0.5 & 0 & 0 & 1/2\sqrt{\rho c_{44}} & 0
\end{pmatrix} \tag{4.38}
\]
Therefore, the wave polarization vectors are

\[ \vec{v}_{31} = (0, 0, 0.5), \]
\[ \vec{v}_{32} = (0.5, 0, 0), \]
\[ \vec{v}_{33} = (0, 0.5, 0). \]

(4.39)

To summarize this section of one-dimensional wave equations along Cartesian axes, Fig. (4.2) shows the wave polarization vectors for waves propagating along the three Cartesian axes. Aided by Eqs.(4.21),(4.30), and (4.39), the wave polarization vectors for the wave propagating along the \( x_1, x_2, \) and \( x_3 \) axes are plotted. Since the signs of the wave polarization vectors are insignificant, the vectors shown in Fig.(4.2) may be in the opposite direction compared to that shown in Eqs.(4.21),(4.30), and (4.39).

Along each Cartesian axis, three wave polarization vectors form a mutually orthogonal triplet. If one of the wave polarization vectors is aligned with wave propagation direction, pure longitudinal and pure shear waves exist. This direction is referred to as the compression direction. As shown in Fig(4.2) the \( x_1 \) and \( x_3 \) axes are the compression directions.

### 4.4 Two-dimensional Wave Equations

We proceed to analyze wave propagation in three two-dimensional planes normal to the Cartesian axes.

#### 4.4.1 Waves in the \( x_2 - x_3 \) Plane

Consider wave propagation in the plane normal to the \( x_1 \) axis or the \( x_2 - x_3 \) plane. We assume that the unknowns in \( U \) are functions of \( x_2, x_3, \) and \( t \) only, and Eq. (4.3)
becomes

\[ \frac{\partial U}{\partial t} + B \frac{\partial U}{\partial x_2} + C \frac{\partial U}{\partial x_3} = 0. \] (4.40)

Shown in Eqs.(4.6) and (4.7), entries in the fourth column of both \( B \) and \( C \) are null, which implies that the equation for \( T_1 = T_{11} \) is decoupled from the remainder of Eq.(4.40). Thus, Eq. (4.40) is reduced to be

\[ \frac{\partial U_{23}}{\partial t} + B_{23} \frac{\partial U_{23}}{\partial x_2} + C_{23} \frac{\partial U_{23}}{\partial x_3} = 0, \] (4.41)

where

\[ U_{23} = (v_1, v_2, v_3, T_2, T_3, T_4, T_5, T_6)^T \] (4.42)
and

\[
\mathbf{B}_{23} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & -c_{11} & -c_{14} & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c_{14} & -c_{14} & 0 & 0 & 0 & 0 & 0 \\
-e_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]  \hspace{5cm} (4.43)

\[
\mathbf{C}_{23} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \\
0 & -c_{14} & -c_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]  \hspace{5cm} (4.44)

Next, we rotate the coordinates with respect to the \(x_1\) axis to a new two-dimensional coordinate system \((x_{(2)}, x_{(3)})\). Eq. (4.41) will be transformed to be in the new coordinate systems:

\[
\frac{\partial U_{23}}{\partial t} + \mathbf{B}_{23}^* \frac{\partial U_{23}}{\partial x_{(2)}} + \mathbf{C}_{23}^* \frac{\partial U_{23}}{\partial x_{(3)}} = 0,
\]  \hspace{5cm} (4.45)
where the Jacobian matrices are:

\[
B_{23}^* = l_{(2)2}B_{23} + l_{(2)3}C_{23},
\]

\[
C_{23}^* = l_{(3)2}B_{23} + l_{(3)3}C_{23}.
\] \hspace{1cm} (4.46)

To find the wave speed in the direction specified by \( \phi \) in the \( x_2-x_3 \) plane, we consider the one-dimensional version of Eq.(4.45):

\[
\frac{\partial U_{23}}{\partial t} + B_{23}^* \frac{\partial U_{23}}{\partial x(2)} = 0,
\] \hspace{1cm} (4.47)

where

\[
B_{23}^* = \begin{bmatrix}
0_{3\times 3} & B_{23}^{3\times 5*} \\
-5_{5\times 3*} & -B_{23} \\
B_{23} & 0_{5\times 5}
\end{bmatrix}
\] \hspace{1cm} (4.48)

and

\[
B_{23}^{3\times 5*} = \begin{bmatrix}
0 & 0 & 0 & -\sin \phi/\rho & -\cos \phi/\rho \\
-\cos \phi/\rho & 0 & -\sin \phi/\rho & 0 & 0 \\
0 & -\sin \phi/\rho & -\cos \phi/\rho & 0 & 0
\end{bmatrix},
\]

\[
B_{23}^{5\times 3*} = \begin{bmatrix}
0 & -c_{11} \cos \phi + c_{14} \sin \phi & c_{14} \cos \phi - c_{13} \sin \phi \\
0 & -c_{13} \cos \phi & -c_{33} \sin \phi \\
0 & -c_{14} \cos \phi + c_{44} \sin \phi & -c_{44} \cos \phi \\
-c_{14} \cos \phi - c_{44} \sin \phi & 0 & 0 \\
-c_{66} \cos \phi - c_{14} \sin \phi & 0 & 0
\end{bmatrix}.
\]
The eigenvalues of $B_{23}^*$ can be readily derived as:

\[
\begin{align*}
\lambda_{1,2} &= 0, \\
\lambda_{3,4} &= \pm \rho^{-0.5}(c_{66} \sin^2 \phi + c_{44} \cos^2 \phi + c_{14} \sin 2\phi)^{0.5}, \\
\lambda_{5,6} &= \pm (2\rho)^{-1/2}(A - \sqrt{B^2 + C})^{0.5}, \\
\lambda_{7,8} &= \pm (2\rho)^{-1/2}(A + \sqrt{B^2 + C})^{0.5},
\end{align*}
\]

where

\[
\begin{align*}
A &= c_{44} + c_{11} \sin^2 \phi + c_{33} \cos^2 \phi - c_{14} \sin 2\phi, \\
B &= (c_{44} - c_{11}) \sin^2 \phi + (c_{33} - c_{44}) \cos^2 \phi + c_{14} \sin 2\phi, \\
C &= [(c_{13} + c_{44}) \sin 2\phi - 2c_{14} \sin^2 \phi]^2.
\end{align*}
\]

All eight eigenvalues are real, and Eq.(4.47) is hyperbolic. Based on the definition of $B_{23}^*$ shown in Eq.(4.46), all real eigenvalues of $B_{23}^*$ also imply that any linear combination of $B_{23}$ and $C_{23}$, i.e., $k_1B_{23} + k_2C_{23}$ has real eigenvalues for all possible real numbers $k_1$ and $k_2$. Thus, Eq.(4.45) and Eq.(4.40), are hyperbolic. Two of the eight eigenvalues of $B_{23}^*$ are zero, and the remainder six can be grouped into three $\pm$ pairs. The absolute values of non-trivial eigenvalues are the wave speeds.

By considering all angles in the $x_2 - x_3$ plane, we plot the reciprocals of the three positive eigenvalues to show the two dimensional slowness profiles in Fig.(4.3).

The left eigenvectors of $B_{23}^*$ can be numerically calculated. Fig. (4.4) shows wave polarization vectors of quartz, formed by the first three entries of the third, fifth, and seventh left eigenvectors, for waves propagating in five different different directions in the $x_2 - x_3$ plane. Four of them are in the compression directions, and the last one is not in a compression direction. One of four compression directions shown in Fig.(4.4) is the $x_3$ axis, which is consistent with the result shown in Fig.(4.2). The
Fig. 4.3: Two-dimensional slowness curves of quartz for waves propagating in the $x_2 - x_3$ plane.

Fig. 4.4: Wave polarization of quartz in the $x_2 - x_3$ plane. The compression directions on the $x_2 - x_3$ plane are at $\phi = -72$ deg, $-17.8$ deg, $40.8$ deg, and $90$ deg with respect to the $x_2$ axis. In addition, we add $\phi = 0$ deg, a non-compression direction, as a reference.
other three compression directions are obtained by using Borgnis’ formula ([11]). For a wave propagation direction, let \( l_i \) be the cosine direction with respect to the \( x_i \) axis with \( i = 1, 2 \) and \( 3 \). By definition, \( l_1^2 + l_2^2 + l_3^2 = 1 \). Moreover, since all directions of concern are in the \( x_2 - x_3 \) plane, \( l_1 = 0 \). According to Ref.[11], the compression directions in the \( x_2 - x_3 \) plane satisfy the following equation:

\[
(-c_{13} + c_{33} - 2c_{44})(l_3/l_2)^3 + 3c_{14}(l_3/l_2)^2 + (-c_{11} + c_{13} - 2c_{44})(l_3/l_2) - c_{14} = 0
\]  
(4.50)

For quartz, Eq.(4.50) has three real roots with the corresponding angles at \( \phi = -72 \) deg, \( -17.8 \) deg, and \( 40.8 \) deg. The three compression directions are numerically calculated and plotted in Fig.(4.4). In addition, we add \( \phi = 0 \) deg, a non-compression direction, as a reference.

4.4.2 Waves in the \( x_1 - x_3 \) Plane

We assume that the unknowns in \( U \) of Eq.(4.3) are functions of \( x_1, x_3, \) and \( t \) only, and Eq.(4.3) becomes

\[
\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x_1} + C \frac{\partial U}{\partial x_3} = 0
\]  
(4.51)

As shown in Eqs. (4.5) and (4.7), entries of the fifth column of both \( A \) and \( C \) are zeros, which implies that the solution of \( T_2 = T_{22} \) is decoupled from the rest of Eq.(4.51), and Eq.(4.51) can be reduced to be

\[
\frac{\partial U_{13}}{\partial t} + A_{13} \frac{\partial U_{13}}{\partial x_1} + C_{13} \frac{\partial U_{13}}{\partial x_3} = 0
\]  
(4.52)
where

\[
U_{13} = (v_1, v_2, v_3, T_1, T_3, T_4, T_5, T_6)^T
\]  \hspace{1cm} (4.53)

and

\[
A_{13} = \begin{bmatrix}
0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 \\
-c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{14} & -c_{44} & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{66} & -c_{14} & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]  \hspace{1cm} (4.54)

\[
C_{13} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 \\
0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\
0 & -c_{14} & -c_{13} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c_{33} & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]  \hspace{1cm} (4.55)
Aided by the chain rule, the direction-cosine tensor, Eq.(4.52) is transformed into a
new coordinate system:

\[
\frac{\partial U_{13}}{\partial t} + A^{*}_{13} \frac{\partial U_{13}}{\partial x^{(1)}} + C^{*}_{13} \frac{\partial U_{13}}{\partial x^{(3)}} = 0
\]  (4.56)

where the Jacobian matrices are

\[
A^{*}_{13} = l_{(1)1} A_{13} + l_{(1)3} C_{13}
\]

\[
C^{*}_{13} = l_{(3)1} A_{13} + l_{(3)3} C_{13}
\]  (4.57)

To find the wave speed in the direction specified by \( \phi \) in the \( x_1 - x_3 \) plane, we consider
the one-dimensional version of Eq.(4.56):

\[
\frac{\partial U_{13}}{\partial t} + A^{*}_{13} \frac{\partial U_{13}}{\partial x^{(1)}} = 0
\]  (4.58)

where

\[
A^{*}_{13} = \begin{bmatrix}
  0_{3\times3} & A_{13}^{3\times5*} \\
  A_{13}^{5\times3*} & 0_{5\times5}
\end{bmatrix}
\]  (4.59)

\[
A_{13}^{3\times5*} = \begin{pmatrix}
  -\cos \phi / \rho & 0 & 0 & -\sin \phi / \rho & 0 \\
  0 & 0 & -\sin \phi / \rho & 0 & -\cos \phi / \rho \\
  0 & -\sin \phi / \rho & 0 & -\cos \phi / \rho & 0
\end{pmatrix}
\]
Given a rotation angle $\phi$ (measured in the counterclockwise direction with respect to the $x_1$ axis), the eigenvalues of $A^*_{13}$ can be numerically calculated: Two of the eight eigenvalues are always zero, and the remainder six eigenvalues form three $\pm$ pairs. All eight eigenvalues are real, and Eq. (4.58) is hyperbolic. Because of the definition of $A^*_{13}$, this also implies that $k_1 A_{13} + k_2 C_{13}$ has real eigenvalues for all possible real numbers $k_1$ and $k_2$. Therefore, Eq.(4.52) is a hyperbolic wave equation. For waves propagating in all directions, i.e., $0 \leq \phi \leq 2\pi$, in the $x_1 - x_3$ plane, we plot the reciprocals of these three positive eigenvalues to show the slowness profiles of quartz.

We use Matlab to calculate the eigenvalues and the corresponding left eigenvectors of $A^*_{13}$ defined in Eq.(4.59). Wave polarization vectors are obtained by taking the first three entries of left eigenvectors associated with the three positive eigenvalues of the Jacobian matrix $A^*_{13}$. Fig. (4.5) shows the two-dimensional slowness profiles of quartz in $x_1 - x_3$ plane. Fig. (4.6) shows wave polarization vectors of quartz for waves in the $x_1 - x_3$ plane. Three wave propagation directions are considered: $\phi = 0$ deg, 30 deg, and 90 deg. Along each wave propagation direction, three polarization vectors form a mutually orthogonal triplet. The directions along the $x_1$ and $x_3$ axes, i.e., $\phi = 0$ deg and 90 deg, are the compression directions. For $\phi = 30$ deg, a non-compressional direction, all three polarization vectors deviate from the wave propagation direction.
Fig. 4.5: The slowness curves of quartz in the $x_1 - x_3$ plane

Fig. 4.6: Polarization vectors for waves propagating on the $x_1 - x_3$ plane. Polarization vectors show that pure longitudinal wave along $x_1$ and $x_3$ directions. The compression directions are at $\phi = 0$ deg and 90 deg. In addition, we add $\phi = 30$ deg as an arbitrary direction for reference.
4.4.3 Wave in the $x_1-x_2$ Plane

For waves in the $x_1-x_2$ plane, we assume that the unknowns in $U$ are functions of $x_1, x_2,$ and $t$ only and Eq.(4.3) becomes

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x_1} + B \frac{\partial U}{\partial x_2} = 0$$

(4.60)

Shown in Eqs. (4.5) and (4.6), entries in the sixth column of both $A$ and $B$ are zeros. This implies that the equation for $T_3 = T_{33}$ is decoupled from the remainder of Eq.(4.60). Thus, Eq.(4.60) is reduced to be

$$\frac{\partial U_{12}}{\partial t} + A_{12} \frac{\partial U_{12}}{\partial x_1} + B_{12} \frac{\partial U_{12}}{\partial x_2} = 0$$

(4.61)

where

$$U_{12} = (v_1, v_2, v_3, T_1, T_2, T_4, T_5, T_6)^T$$

and

$$A_{12} = \begin{bmatrix} 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho \\ 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho & 0 \\ -c_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_{14} & -c_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_{66} & -c_{14} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
\[
B_{12} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/\rho \\
0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 \\
0 & -c_{12} & -c_{14} & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{11} & -c_{14} & 0 & 0 & 0 & 0 & 0 \\
0 & -c_{14} & -c_{44} & 0 & 0 & 0 & 0 & 0 \\
-c_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-c_{66} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Aided by the chain rule and the direction-cosine tensor for coordinate rotation in the \(x_1 - x_2\) plane, we derive the transformed equation in the new coordinate system:

\[
\frac{\partial U_{12}}{\partial t} + A_{12}^* \frac{\partial U_{12}}{\partial x_{(1)}} + B_{12}^* \frac{\partial U_{12}}{\partial x_{(2)}} = 0 \tag{4.62}
\]

where the Jacobian matrices are

\[
A_{12}^* = l_{(1)1} A_{12} + l_{(1)2} B_{12},
\]

\[
B_{12}^* = l_{(2)1} A_{12} + l_{(2)2} B_{12}. \tag{4.63}
\]

To find the wave speed in the direction specified by \(\theta\) (measured in the counterclockwise direction with respect to the \(x_1\) axis) in the \(x_1 - x_2\) plane, we consider the one-dimensional version of Eq. (4.62):

\[
\frac{\partial U_{12}}{\partial t} + A_{12}^* \frac{\partial U_{12}}{\partial x_{(1)}} = 0 \tag{4.64}
\]
where

\[
A^*_{12} = \begin{bmatrix}
0_{3\times3} & A^{3\times5*}_{12} \\
\hline
A^{5\times3*}_{12} & 0_{5\times5}
\end{bmatrix}
\]

(4.65)

and

\[
A^{3\times5*}_{12} = \begin{bmatrix}
-c_{11} \cos \theta & -c_{12} \sin \theta & -c_{14} \sin \theta \\
-c_{12} \cos \theta & -c_{11} \sin \theta & c_{14} \sin \theta \\
-c_{14} \cos \theta & -c_{14} \sin \theta & -c_{44} \sin \theta \\
-c_{14} \sin \theta & -c_{14} \cos \theta & -c_{44} \cos \theta \\
-c_{66} \sin \theta & -c_{66} \cos \theta & -c_{14} \cos \theta
\end{bmatrix}
\]

\[
A^{5\times3*}_{12} = \begin{bmatrix}
-c_{11} \cos \theta & 0 & 0 & 0 \\
0 & -\sin \theta / \rho & 0 & 0 \\
0 & 0 & -\sin \theta / \rho & 0 \\
0 & 0 & 0 & -\cos \theta / \rho
\end{bmatrix}
\]

\[
0_{3\times3} \text{ and } 0_{5\times5} \text{ are } 3 \times 3 \text{ and } 5 \times 5 \text{ zero matrices respectively. The eigenvalues of } A^*_{12}
\]

are numerically calculated by using Matlab. In all directions specified by \( \theta \), two of
the eight eigenvalues of \( A^*_{12} \) are always zero. The rest of the eigenvalues form three
\( \pm \) pairs. The absolute values of nontrivial eigenvalues form three wave speeds for
waves propagating in the \( x_1 - x_2 \) plane. Since all eight eigenvalues are real, Eq.(4.64)
is hyperbolic. Moreover, real eigenvalues of \( A^*_{12} \) implies that \( k_1 A_{12} + k_2 B_{12} \)
has real eigenvalues for arbitrary real numbers \( k_1 \) and \( k_2 \). Therefore, Eqs.(4.61) and (4.62)
are also hyperbolic. By considering all directions in the \( x_1 - x_2 \) plane, Fig.(4.7) shows the
two-dimensional contour plots of the reciprocals of three wave speeds as the slowness
profiles.
Fig. 4.7: Two-dimensional slowness curves of quartz in the $x_1 - x_2$ plane

Fig.(4.8) shows the polarization vectors in seven wave propagation directions in the $x_1 - x_2$ plane. Due to the trigonal symmetry of the medium, a rotation of the two-dimensional coordinate system about the $x_3$ axis by $\pm 120$ deg results in the same governing equation as Eq.(4.64). In the original coordinate system in the $x_1 - x_2$ plane, the positive $x_1$ direction is a compression direction, and rotations about the $x_3$ axis by $\pm 120$ deg also lead to compression directions. Therefore, three compression directions in the $x_1 - x_2$ plane exist at $\theta = 0$ deg, 120 deg, and 240 deg. Moreover, the reverse directions of the above three directions are also the compression directions. The polarization vectors along these six compression directions are plotted in Fig.(4.8), Moreover, polarization vectors for wave propagation along $\theta = 30$ deg, a non-compression direction, are also plotted for reference.
Fig. 4.8: Polarization vectors for waves propagating on $x_1-x_2$ plane in the directions of $\theta = 0$ deg, 60 deg, 120 deg, 180 deg, 240 deg, and 300 deg. All above directions are compression direction. As a reference, $\theta = 30$ deg, a non-compression direction is also considered.

4.5 Three-dimensional First-Order Wave Equations

This section reports the eigen-structure of the three-dimensional first-order wave equations. In the process of calculating the eigenvalues, we recover the Christoffel matrix, and the result is shown in Sec 4.5.1. The last section illustrates the eigenvectors of non-trivial eigenvalues as the wave polarization vectors.

Slowness Profiles

Aided by Eqs. (3.61) and (3.64), Eq. (4.3) can be transformed into the new coordinate system as

$$\frac{\partial U}{\partial t} + A^* \frac{\partial U}{\partial x^{(1)}} + B^* \frac{\partial U}{\partial x^{(2)}} + C^* \frac{\partial U}{\partial x^{(3)}} = 0$$  \hspace{1cm} (4.66)
where the Jacobian matrices are

\[
A^* = l_{(1)1}A + l_{(1)2}B + l_{(1)3}C
\]

\[
B^* = l_{(2)1}A + l_{(2)2}B + l_{(2)3}C
\]  \hspace{1cm} \text{(4.67)}

\[
C^* = l_{(3)1}A + l_{(3)2}B + l_{(3)3}C
\]

Next, we consider the one-dimensional version of Eq.(4.66):

\[
\frac{\partial U}{\partial t} + A^* \frac{\partial U}{\partial x(1)} = 0
\]  \hspace{1cm} \text{(4.68)}

where

\[
A^* = \begin{pmatrix}
0 & A_v^* \\
A_T^* & 0
\end{pmatrix}
\]  \hspace{1cm} \text{(4.69)}

where

\[
A_v^* = -\frac{1}{\rho} \begin{pmatrix}
\cos \phi \cos \theta & 0 & 0 & 0 & \sin \phi & \cos \phi \sin \theta \\
0 & \cos \phi \sin \theta & 0 & \sin \phi & 0 & \cos \phi \cos \theta \\
0 & 0 & \sin \phi & \cos \phi \sin \theta & \cos \phi \cos \theta & 0
\end{pmatrix}
\]  \hspace{1cm} \text{(4.70)}

is a $3 \times 6$ matrix.
Equation (4.68) models wave propagation in the direction determined by the prescribed $\theta$ and $\phi$. By systematically specifying the values of $\theta$ and $\phi$, we cover all directions in the three-dimensional space.

Given a set of values of $\theta$ and $\phi$, we solve the polynomial $\text{det}(\mathbf{A}^* - \lambda \mathbf{I}) = 0$ numerically by using Matlab. By considering all directions specified by the values of $\theta$ and $\phi$, the eigenvalues are always real: Three eigenvalues are zero, and the remainder six form three $\pm$ pairs. The absolute values of the non-trivial eigenvalues are three wave speeds. We then plot the reciprocals of the three wave speeds to show three slowness surfaces of quartz in Fig.(4.9). Fig. (4.9)(a) shows an inner slowness surface for the fastest wave. Fig.(4.9)(b) shows the other two outer slowness surfaces for slower waves. The two outer slowness profiles are intertwined and are plotted together. To show that there are two eigenvalues in Fig.(4.9)(b), Fig.(4.9)(c) shows
4.5.1 The Christoffel Matrix

Alternatively, we can calculate the eigenvalues of $A^*$ by recognizing that $A^*$ is formed by the null square matrices and two rectangular matrices. As such, the eigenvalue problem can be rewritten as

$$
det(A^* - \lambda I_9) = det \begin{pmatrix} -\lambda I_3 & A_v^* \\ A_T^* & -\lambda I_6 \end{pmatrix} = 0 $$

Fig. 4.9: Three-dimensional outer slowness surfaces of quartz. (a) The inner slowness surface plotted based on the fastest wave speed. (b) The combined outer surface for two slower wave speeds. (c) The combined slowness surfaces cut by using an $x_1 - x_2$ plane. (d) The combined slowness surfaces cut by using an $x_2 - x_3$ plane.

the same three-dimensional profiles, cut open by the $x_1 - x_2$ plane. Fig. (4.9)(d) shows the same plot cut open by the $x_2 - x_3$ plane.

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Eqs. (4.72) can be further simplified to be

\[
\begin{align*}
\det \left( -\lambda \mathbf{I}_3 \quad \mathbf{A}_v^* \right) \\
\mathbf{A}_T^* \quad -\lambda \mathbf{I}_6 
\end{align*}
\]

\[
\det \begin{pmatrix} \mathbf{I}_3 & -\lambda^{-1} \mathbf{A}_v^* \\ 0 & \mathbf{I}_6 \end{pmatrix} \times \det \begin{pmatrix} -\lambda \mathbf{I}_3 + \lambda^{-1} \mathbf{A}_v^* \mathbf{A}_T^* & 0 \\ \mathbf{A}_T^* & -\lambda \mathbf{I}_6 \end{pmatrix}
\]

\[
= \lambda^6 \det [\lambda^{-1} (\mathbf{A}_v^* \mathbf{A}_T^* - \lambda^2 \mathbf{I}_3)] = 0
\]

or simply,

\[
\det (\mathbf{A}_v^* \mathbf{A}_T^* - \lambda^2 \mathbf{I}_3) = 0
\] (4.74)

where

\[
\mathbf{A}_v^* \mathbf{A}_T^* = \frac{1}{\rho} \Gamma = \frac{1}{\rho} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{pmatrix}
\] (4.75)
with

\[ \Gamma_{11} = c_{11} \cos^2 \phi \cos^2 \theta + c_{66} \cos^2 \phi \sin^2 \theta + c_{44} \sin^2 \phi + 2c_{14} \cos \phi \sin \phi \sin \theta \]

\[ \Gamma_{12} = (c_{12} + c_{66}) \cos^2 \phi \cos \theta \sin \theta + 2c_{14} \cos \phi \sin \phi \cos \theta \]

\[ \Gamma_{13} = (c_{13} + c_{44}) \cos \phi \sin \phi \cos \theta + 2c_{14} \cos^2 \phi \cos \theta \sin \theta \]

\[ \Gamma_{22} = c_{66} \cos^2 \phi \cos^2 \theta + c_{11} \cos^2 \phi \sin^2 \theta + c_{44} \sin^2 \phi \]

\[ \Gamma_{23} = (c_{13} + c_{44}) \cos \phi \sin \phi \sin \theta + c_{14} (\cos^2 \phi \cos^2 \theta - \cos^2 \phi \sin^2 \theta) \]

\[ \Gamma_{33} = c_{44} (\cos^2 \phi \cos^2 \theta + \cos^2 \phi \sin^2 \theta) + c_{33} \sin^2 \phi \]

(4.76)

We note that \( \Gamma_{12} = \Gamma_{21}, \Gamma_{23} = \Gamma_{32}, \) and \( \Gamma_{13} = \Gamma_{31}. \) \( \Gamma \) and \( A^*_T A^* \) are symmetric 3 \( \times \) 3 matrices with real entries. Equation (4.74) is a six-order polynomial of \( \lambda, \) and its solutions are three pairs of real eigenvalues \( \pm \lambda_{(1)}, \pm \lambda_{(2)}, \) and \( \pm \lambda_{(3)}, \) which are also the nontrivial eigenvalues of \( A^*. \)

Moreover, the direction cosines between the transformed coordinate \( x_{(1)} \) and the three old Cartesian coordinates are \( l_{(1)1} = \cos \phi \cos \theta, \ l_{(1)2} = \cos \phi \sin \theta, \) and \( l_{(1)3} = \sin \phi. \) Substitute these three equalities into Eq.(4.76) and we recover the original Christoffel matrix for solids of trigonal 32 symmetry, as shown in Auld ([4]).

### 4.5.2 Wave Polarization Vectors

We calculate wave polarization vectors by numerically solving for the left eigenvectors of \( A^*. \) We are interested in the compression directions in solids of trigonal 32 symmetry. According to Borgnis [12], for solids of trigonal 32 symmetry, the compression directions include (i) the \( x_3 \) axis, (ii) three directions in the \( x_1 - x_2 \) plane, as shown in Fig.4.8, and (iii) three directions in the \( x_2 - x_3 \) plane, for each of which there
Table 4.1: Cosine directions of the six compression directions of quartz not in the $x_2 - x_3$ plane

<table>
<thead>
<tr>
<th>Index</th>
<th>$l'_1$</th>
<th>$l'_2$</th>
<th>$l'_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.26</td>
<td>-0.151</td>
<td>-0.9525</td>
</tr>
<tr>
<td>2</td>
<td>-0.26</td>
<td>-0.151</td>
<td>-0.9525</td>
</tr>
<tr>
<td>3</td>
<td>0.655</td>
<td>-0.378</td>
<td>0.6537</td>
</tr>
<tr>
<td>4</td>
<td>-0.655</td>
<td>-0.378</td>
<td>0.6537</td>
</tr>
<tr>
<td>5</td>
<td>0.824</td>
<td>-0.476</td>
<td>0.3039</td>
</tr>
<tr>
<td>6</td>
<td>-0.824</td>
<td>-0.476</td>
<td>0.3039</td>
</tr>
</tbody>
</table>

are two additional compression directions not in the $x_2 - x_3$ plane because of trigonal symmetry. For quartz, these six compression directions are tabulated in Table 4.1.

For solids of trigonal 32 symmetry, there are other compression directions not in the $x_1 - x_2$, $x_2 - x_3$, or $x_1 - x_3$ plane. Based on the compression directions on the $x_2 - x_3$ plane, the additional compression directions can be calculated by using the following relations:

$$l'_1 = \pm \frac{\sqrt{3}}{2} l_2, \quad l'_2 = -\frac{l_2}{2}, \quad l'_3 = l_3$$  \hspace{1cm} (4.77)$$

where $l'_1$, $l'_2$, and $l'_3$ are the cosine directions of a compression direction associated with the compression direction in the $x_2 - x_3$ plane denoted by $l_1$, $l_2$, and $l_3$ shown in Eq.(4.50). For quartz, the direction cosines of the six additional compression directions are tabulated in Table 4.1. Fig.(4.10) shows the polarization vectors of waves propagating along the nine compression directions, specified by Eqs.(4.50) and (4.77). The results are divided into three groups. In each group, one of the three compression directions is calculated by using Eq.(4.50), i.e., in the $x_2 - x_3$ plane. The other two compression directions in the same group are not in the $x_2 - x_3$ plane, and they are calculated by using Eq.(4.77) with the direction cosine tabulated in
Fig. 4.10: Wave polarization vectors of waves propagating in the nine compression directions inside a quartz crystal, a solid of trigonal 32 symmetry.
Table 4.1. As shown in Fig.(4.10), the three compression directions in each group are trigonally symmetric with respect to the $x_3$ axis. The results shown here further verify that the formulation developed in the present chapter can be used to obtain classical solutions of three-dimensional wave polarization vectors.

4.6 Conclusions

In this chapter, we reported the eigen-structure of a first-order velocity-stress formulation for analyzing wave propagation in solids with a trigonal $32$ symmetry. The governing equations include the equation of motion and the differentiation of the elastic constitutive relations with respect to time. The result is a set of nine first-order, hyperbolic partial differential equations with velocities and stress components as the unknowns. The model equations are cast into a vector form with three $9 \times 9$ Jacobian matrices. We showed that wave physics are fully described by the eigenvalues and eigenvectors of the Jacobian matrices of the first-order pde’s. The values of the non-trivial eigenvalues are wave speeds. The first three entries of the associated left eigenvectors form the wave polarization vectors of the wave motions. We calculate wave speeds and wave polarization vectors for wave propagation (i) along the Cartesian axes, (ii) in the planes normal to the Cartesian axes, and (iii) in arbitrary directions in the three-dimensional space. In the process, we recover the Christoffel matrix without invoking the plane-wave solution.
CHAPTER 5
WAVE PROPAGATION IN PIEZOELECTRIC CRYSTALS

5.1 Introduction

Work by Joseph Kyame

The production of an electric polarization by the application of a mechanical stress on certain crystals was firstly demonstrated by Pierre and Jacques Curie. The piezoelectric equations of state are given in different forms. Usually the thermodynamic consideration is adopted to get the state equations. As shown by Mason ([92]), Electric Gibbs function:

\[
G = U - E_m D_m - \Theta \sigma
\]  

(5.1)

can be selected as thermodynamics functions, where \( E \) and \( D \) are electric field and electric displacement respectively, \( \Theta \) is temperature and \( \sigma \) represent entropy. The total energy \( U \) of a body is the sum of all the different types of energy, electric energy, mechanical energy, and thermal energy. The independent variables are respectively the stains, electric field and temperature. It can be expressed as

\[
dU = E_i dD_i + T_{kl} dS_{kl} + \Theta d\sigma
\]

(5.2)
By using the differential form of Eq. (5.1) and Eq. (5.2), the differential form of the Gibbs function:

\[
dG = T_{kl}dS_{kl} - D_i dE_i - \sigma d\Theta
\]

\[
T_{kl} = \frac{\partial G}{\partial S_{kl}}; \quad D_i = -\frac{\partial G}{\partial E_i}; \quad \sigma = -\frac{\partial G}{\partial \Theta}
\]

Since the dependent variables are functions of the independent variables, they can be developed in the form of the partial differential equations

\[
dT_{kl} = \frac{\partial T_{kl}}{\partial S_{ij}}|_{E, \Theta} dS_{ij} + \frac{\partial T_{kl}}{\partial E_m}|_{S_{ij}, \Theta} dE_m + \frac{\partial T_{kl}}{\partial \Theta}|_{E, S_{ij}} d\Theta
\]

\[
DD_n = \frac{\partial D_n}{\partial S_{ij}}|_{E, \Theta} dS_{ij} + \frac{\partial D_n}{\partial E_m}|_{S_{ij}, \Theta} dE_m + \frac{\partial D_n}{\partial \Theta}|_{E, S_{ij}} d\Theta
\]

\[
d\sigma = \frac{\partial \sigma}{\partial S_{ij}}|_{E, \Theta} dS_{ij} + \frac{\partial \sigma}{\partial E_m}|_{S_{ij}, \Theta} dE_m + \frac{\partial \sigma}{\partial \Theta}|_{E, S_{ij}} d\Theta
\]

The terms on the subscript indicate the variables that are held constant during the differentiation. And the partial derivatives have special names as follows by using Eqs.(5.3) and Eqs. (5.4):

\[
\frac{\partial T_{kl}}{\partial S_{ij}}|_{E, \Theta} = c^{E, \Theta}_{ijkl} = \text{Elastic stiffness at constant fields and temperatures}
\]

\[
\frac{\partial T_{ij}}{\partial E_m}|_{S_{ij}, \Theta} = \frac{\partial}{\partial E_m} \frac{\partial G}{\partial S_{ij}} = \frac{\partial}{\partial S_{ij}} \frac{\partial G}{\partial E_m} = -\frac{\partial D_m}{\partial S_{ij}} = -e^{\Theta}_{mij} = \text{Piezoelectric constant}
\]

\[
\frac{\partial T_{kl}}{\partial \Theta} = \frac{\partial}{\partial \Theta} \frac{\partial G}{\partial S_{kl}} = \frac{\partial}{\partial S_{kl}} \frac{\partial G}{\partial \Theta} = -\frac{\partial \sigma}{\partial S_{kl}} = -\lambda^E_{kl} = \text{thermal stress constant}
\]

\[
\frac{\partial D_n}{\partial E_m}|_{S, \Theta} = e^{S, \Theta}_{mn} = \text{dielectric constants at constant strain and temperature}
\]

\[
\frac{\partial D_n}{\partial \Theta} = -\frac{\partial}{\partial \Theta} \frac{\partial G}{\partial E_n} = -\frac{\partial}{\partial E_n} \frac{\partial G}{\partial \Theta} = \frac{\partial \sigma}{\partial \Theta} = p^{S}_{n} = \text{pyroelectric constant}
\]

\[
\frac{\partial \sigma}{\partial \Theta}|_{S, E} = \frac{\rho C^{S,E}}{\Theta} = \text{specific heat per unit volume divided by the absolute temperature}
\]
Since the partial derivatives are regarded as being constants over the range of variables used, the equations can be integrated, with the result that $dT_{kl}$ is replaced by $T_{kl}$, and so on. Introducing these values in Eq.(5.4), these can be written as:

$$T_{ij} = c_{ijkl}^E S_{kl} - e_{mij}^\Theta E_m - \lambda_{ij}^E \delta \Theta$$

$$D_n = e_{nkl}^\Theta S_{kl} + \varepsilon_{ijkl}^S E_m + p_n^S \delta \Theta$$

$$\delta \sigma = \lambda_{kl} S_{kl} + p_m^S E_m + \rho C^{S,E} \delta \Theta$$

(5.5)

The term $c_{ijkl}^E$ represents the elastic constants measured at constant field and temperature, $e_{mij}^\Theta$ is a piezoelectric constant relating a stress to an applied field or the negative of an electric displacement to a strain, $\lambda_{ij}^E$ is the thermal stress constant relating an increase in temperature to a stress at constant strain or field, $p_n^S$ is a pyroelectric constant relating an increase in polarization to an increase in temperature when the strain is constant. By multiplying the last equation through by the absolute temperature $\Theta$, it is seen that $\Theta \lambda_{kl}$ represents the heat of deformation and $\Theta p_m^S$ is the electrocaloric effect at constant strain. For adiabatic conditions, $\delta \Theta$ can be eliminated from these equations by setting $\delta \sigma = 0$, and one finds

$$T_{ij} = c_{ijkl}^{E,\sigma} S_{kl} - e_{mij}^{\sigma} E_m; \quad D_n = e_{nkl}^{\sigma} S_{kl} + \varepsilon_{mn}^{S,\sigma} E_m$$

(5.6)

Where

$$c_{ijkl}^{E,\sigma} = c_{ijkl}^E + \frac{\lambda_{ij}^E \lambda_{kl} \Theta}{\rho C^{E,T}}$$

$$e_{mij}^{\sigma} = e_{mij}^\Theta - \frac{\lambda_{ij}^E p_n^S \Theta}{\rho C^{E,T}}$$

$$\varepsilon_{mn}^{S,\sigma} = \varepsilon_{mn}^S + \frac{p_m^S p_n^S \Theta}{\rho C^{S,E}}$$
The piezoelectric medium is assumed continuous and the polarization charge produced in the medium is assumed to be negligible. Were it appreciable, any polarization charge produced in the crystal would leak away. Introduction of a conductivity would lead to a complex propagation constant and thereby result in a damping factor in the solutions for the wave equations.

Any solution for wave propagation in the piezoelectric medium must satisfy simultaneously Maxwell’s field equations and Newton’s law of force. A 5 by 5 matrix are got and can be used to obtain the wave speed in the crystal. The results are also shown by Hutson and White ([64]). After eliminating some minor errors in the chapter, the theory is briefly restated here. The very similar derivation can also be found in [4].

If we consider $x_1, x_2, x_3$ be orthogonal axes arbitrarily oriented with respect to the crystal axes and consider the propagation of plane waves in the $x_1$ direction. Then let derivatives with respect to $x_2$ and $x_3$ to be zero. The force on a volume element in the $i$ direction in terms of the stress tensor is $\partial T_{1i}/\partial x_1$. The usual definition of the strain tensor in terms of the displacement is

$$S_{ij} = 0.5 \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (5.7)$$

For the plane wave problem, it is convenient to define a strain as

$$S'_{1k} = S_{1k} + S_{k1} = \partial u_k/\partial x_1 \quad (5.8)$$

Under adiabatic conditions, the applicable mechanical and electrical equations of
state can be written

\[ T_{1i} = c_{11k}S'_{1k} - e_{k1i}E_k, \]
\[ D_k = e_{k1i}S'_{1i} + \epsilon_{ik}E_i. \]  

(5.9)

The constitutive equations for the medium are

\[ B_i = \mu_0 H_i \quad \text{and} \quad J_k = \sigma_{ik}E_i. \]  

(5.10)

The electromagnetic field quantities must simultaneously satisfy Matxwell’s equations

\[ \nabla \times H = \frac{\partial D}{\partial t} + J; \]
\[ \nabla \times E = -\frac{\partial B}{\partial t}; \]
\[ \nabla \cdot B = 0; \]
\[ \nabla \cdot D = \rho_e. \]  

(5.11)

For the plane wave condition, \( B_1 \) and \( H_1 \) are constant in space and time and \((\nabla \times H)_1 = 0\) which yield the continuity equation for \( J_1 \) and \( \rho_e \) upon differentiating with respect to \( x_1 \). From the curl equations, one can obtain equations for \( E_2 \) and \( E_3 \) where \( D \) and \( J \) can be eliminated by using (5.4) and (5.5).

\[ \frac{\partial^2 E_p}{\partial x_1^2} = \mu \frac{\partial}{\partial t} \left[ \epsilon_{pi1} \frac{\partial^2 u_i}{\partial x_1 \partial t} + \epsilon \frac{\partial E_i}{\partial t} + \sigma_{ip}E_i \right] \]  

(5.12)

\( i, j, k = 1, 2, 3, \) but \( p, q = 2 \) or 3. Differentiation with respect to the first equation of Eq.(5.6) yields three equations

\[ \frac{\partial T_{1i}}{\partial x_1} = \rho \frac{\partial^2 u_i}{\partial t^2} = c_{11k} \frac{\partial^2 u_k}{\partial x_1^2} - c_{k1i} \frac{\partial E_k}{\partial x_1}. \]  

(5.13)
Eqs. (5.12) and (5.13) are five coupled wave equations in the six variables $u_i$ and $E_i$. Using $(\nabla \times H)_1 = 0$ and Eqs. (5.9) and (5.10), $E_1$ can be expressed as

$$
\left( \sigma_{11} E_1 + \epsilon_{11} \frac{\partial E_1}{\partial t} \right) = -\epsilon_{11} \frac{\partial^2 u_k}{\partial x_1 \partial t} - \left( \sigma_{q1} E_q + \epsilon_{q1} \frac{\partial E_q}{\partial x_1} \right). \tag{5.14}
$$

By assuming plane wave solutions of the form

$$
u_i = u_i^0 e^{j(kx - \omega t)} \quad \text{and} \quad E_i = E_i^0 e^{j(kx - \omega t)} \tag{5.15}
$$

Substitute Eq. (5.15) into Eq. (5.12) - Eq. (5.14), the secular determinant is then

$$
A = \begin{bmatrix}
A_l & A_r
\end{bmatrix} \tag{5.16}
$$

$$
A_l =
\begin{pmatrix}
\left( \frac{c_{111}'}{\rho} - \frac{\omega^2}{k^2} \right) & \frac{c_{112}'}{\rho} & \frac{c_{113}'}{\rho} \\
\frac{c_{121}'}{\rho} & \left( \frac{c_{122}'}{\rho} - \frac{\omega^2}{k^2} \right) & \frac{c_{123}'}{\rho} \\
\frac{c_{131}'}{\rho} & \frac{c_{132}'}{\rho} & \left( \frac{c_{133}'}{\rho} - \frac{\omega^2}{k^2} \right)
\end{pmatrix}
$$

$$
A_r =
\begin{pmatrix}
\frac{v_{211}' \omega}{k} & \frac{v_{212}' \omega}{k} & \frac{v_{213}' \omega}{k} \\
\frac{v_{311}' \omega}{k} & \frac{v_{312}' \omega}{k} & \frac{v_{313}' \omega}{k}
\end{pmatrix}
$$

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and

\[
A_r = \begin{pmatrix}
\frac{\nu'_{211}}{\omega k} & \frac{\nu'_{311}}{\omega k} \\
\frac{\nu'_{212}}{\omega k} & \frac{\nu'_{312}}{\omega k} \\
\frac{\nu'_{213}}{\omega k} & \frac{\nu'_{313}}{\omega k} \\
\frac{\left(\omega^2 \varepsilon'_{22}/k^2 + 1/\varepsilon \mu_0\right)}{\omega^2 \varepsilon'_{32}/k^2} & \frac{\left(\omega^2 \varepsilon'_{33}/k^2 + 1/\varepsilon \mu_0\right)}{\omega^2 \varepsilon'_{32}/k^2}
\end{pmatrix}
\]

where

\[
\nu'_{ijk} = \frac{e'_{ijk}}{(\varepsilon \rho)^{0.5}}
\]

\[
e'_{1i1k} = c_{1i1k} + \frac{e_{11i} e_{11k}}{(\varepsilon_{11} + j \sigma_{11}/\omega)}
\]

\[
e'_{q1i} = e_{q1i} + \frac{e_{11i}(\varepsilon_{q1} + j \sigma_{q1}/\omega)}{(\varepsilon_{11} + j \sigma_{11}/\omega)}
\]

\[
e'_{qp} = (\varepsilon_{qp} + j \sigma_{qp}/\omega) - \frac{(\varepsilon_{q1} + j \sigma_{q1}/\omega)(\varepsilon_{q1} + j \sigma_{q1}/\omega)}{(\varepsilon_{11} + j \sigma_{11}/\omega)}
\]

If the piezoelectric tensor is zero, equation (5.16) splits into the usual third-order acoustic wave determinant and second-order electromagnetic wave determinant.

**Work by H.F. Tiersten [119]**

Instead of using Maxwell’s equation and stress wave equation, Tiersten modelled wave propagation in piezoelectric materials using quasistatic electric equations and stress wave equations. The procedures are summarized as following:
Started with Maxwell’s equations

\[ \nabla \times \mathbf{B} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}; \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}; \]
\[ \nabla \cdot \mathbf{B} = 0; \quad \nabla \cdot \mathbf{D} = \rho_e. \]

These equations can be reformulated in terms of the vector and scalar potentials \( \mathbf{A} \) and \( \phi \) which are

\[ \nabla \cdot \mathbf{B} = 0 \implies \mathbf{B} = \nabla \times \mathbf{A} \implies B_i = e_{ijk}A_{k,j}; \]
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \implies \mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \implies E_k = -\phi_k - \dot{A}_k; \]
\[ \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \implies e_{ijk}H_{k,j} = \dot{D}_i + J_i; \]
\[ \nabla \cdot \mathbf{D} = \rho_e \implies D_{i,i} = \rho_e. \]

When we consider piezoelectricity, which, to a good approximation, gives the same equations, we will consider polarizable dielectrics only. Consequently, \( \rho_e = J_i = 0 \). It should be noted that under these circumstances, Maxwell’s equations take the form

\[ B_i = e_{ijk}A_{k,j}, \]
\[ E_k = -\phi_k - \dot{A}_k, \]
\[ e_{ijk}H_{k,j} = \dot{D}_i, \]  \hspace{1cm} (5.17)
\[ D_{i,i} = 0. \]

The basic simplifying assumption which is used to get the piezoelectric equations is
that each component of $\phi,i$ that enters in a problem satisfies

$$|\dot{A}_i| \ll |\phi,i|$$  \hspace{1cm} (5.18)

As shown by Tiersten, the linear theory of piezoelectricity with wave motion equations can be written as:

Conservation of linear momentum:  $T_{ij,j} + \rho f_i = \rho \ddot{u}_i$,  

The charge equation:  $D_{i,i} = 0$,  

The constitutive relationships:  $T_{ij} = c_{ijkl} S_{kl} - e_{ij} E_k$,  

$D_i = e_{ijk} S_{jk} + \epsilon_{ij} E_j$,  \hspace{1cm} (5.19)  

The strain-displacement relationship:  $S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$,  

And the electric field-potential relationship:  $E_i = -\phi_{,i}$,  

where $\dot{()} \equiv \partial/\partial t$ and $()_{,i} \equiv \partial/\partial x_i$. The primitive unknowns in Eqs.(5.19) are the Cauchy stress $T_{ij}$, the displacement $u_i$, the electric potential $\phi$, the electric displacement $D_i$, the infinitesimal strain $S_{ij}$, and the electric field $E_i$, each a function of position $x$ and $t$. The constant mass density $\rho$, the elasticity tensor $c_{ijkl}$, the piezoelectric tensor $e_{ijk}$, and the dielectric tensor $\epsilon_{ij}$ are specified material constants, while the mechanical body force per unit mass $f_i$.

The system includes 22 equations and 22 unknowns (i.e., 6 stresses, 6 strains, 3 displacements, 3 electric fields, 3 electric displacements and 1 electric potential). After manipulation of these 22 equations, four 2nd order governing equations with
four unknowns can be got:

\[ c_{ijkl}u_{k,li} + e_{kl}j\phi_{j,ki} = \rho \tilde{u}_j \]

\[ e_{ijkl}u_{k,li} - \epsilon_{ik}j\phi_{j,ki} = 0. \]  

(5.20)

These governing equations are composed of 4 unknowns and 4 equations which can be solved together with boundary conditions. A lot of work were based on this formula. Wang and Zhang solved the static state case of Eqs. (5.20). They assume that displacement and potential can be expressed as

\[ u = \frac{\partial \psi}{\partial x} - \frac{\partial \chi}{\partial y}, \quad v = \frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x}, \quad w = k_1 \frac{\partial \psi}{\partial z}, \quad \phi = k_2 \frac{\partial \psi}{\partial z} \]  

(5.21)

where \( \psi(x, y, z) \) and \( \chi(x, y, z) \) are potential functions. \( k_1 \) and \( k_2 \) are unknown constants. After substituting Eq.(5.21) into static state case of Eqs. (5.20), \( \psi(x, y, z) \) and \( \chi(x, y, z) \) both satisfy Laplace equation. Then it can be solved.

**Work by Auld**

Auld also ([4]) worked on wave propagation in piezoelectric materials. The governing equations he gave are

\[ \nabla \cdot c^E : \nabla_s v - \frac{\partial^2 v}{\partial t^2} = -\nabla \cdot \left( e \cdot \frac{\partial \nabla \phi}{\partial t} \right) \]

\[ 0 = -\mu_0 \nabla \cdot \left( \epsilon_s \cdot \frac{\partial^2 \nabla \phi}{\partial t^2} \right) + \mu_0 \nabla \cdot \left( e \cdot \nabla_s \frac{\partial \nabla \phi}{\partial t} \right). \]  

(5.22)

where \( c^E \) is stiffness matrix with electric field is constant. \( v \) is velocity vector \((v_1, v_2, v_3)\). \( \rho \) is density. \( e \) is piezoelectric strain matrix. \( \phi \) is electric potential.
\( \epsilon^S \) is the permittivity matrix with constant strain. And \( \nabla \cdot \) is given by

\[
\nabla \cdot = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\
0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0
\end{bmatrix}
\]  
\text{(5.23)}

\( \nabla_s \) is given by

\[
\nabla_s = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0
\end{bmatrix}
\]  
\text{(5.24)}

When Eqs. (5.22) are compared to Eqs. (5.20), it can be found that Eqs. (5.22) can be got by taking time derivative with respect to Eqs. (5.20). Next, if a plane wave solution is assumed as \( e^{i(\omega t - k \cdot r)} \), where \( \hat{l} \) is a unit vector in the propagation direction, then Eqs. (5.22) can be reduced to

\[
-k^2(l_i e^E_{K Li} l_j) v_j + \rho \omega^2 v_i = i \omega k^2 (l_i e^E_{K Li} l_j) \phi,
\]

\[
\omega^2 k^2 (l_i e^S_{ij} l_j) \phi = -i \omega k^2 (l_i e^S_{ij} l_j) v_j,
\]  
\text{(5.25)}
where \( l_{iK} \) and \( l_{Lj} \) are given by

\[
\begin{bmatrix}
  l_x & 0 & 0 & 0 & l_z & l_y \\
  0 & l_y & 0 & 0 & l_z & 0 \\
  0 & 0 & l_z & 0 & l_x & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  l_x & 0 & 0 \\
  0 & l_y & 0 \\
  0 & 0 & l_z \\
  0 & l_z & l_y \\
  l_z & 0 & l_x \\
  l_y & l_x & 0 \\
\end{bmatrix}
\]

The factor multiplying \( \phi \) on the left hand side of Eqs.(5.25) is a scalar and may be divided out, giving the potential in terms of the particle velocity. That is,

\[
\frac{1}{i\omega} \frac{(l_{i}e_{iL}l_{L})}{l_{i}e_{iL}l_{L}} v_j.
\]

(5.26)

After substitution into the first equation of Eqs.(5.25) and re-arrangement of terms, one has

\[
k^2 \left( l_{iK} \left[ e_{KL}^E + \frac{[e_{Kj}l_{j}][l_{i}e_{iL}]}{l_{i}e_{iL}l_{j}} \right] l_{Lj} \right) v_j = \rho \omega^2 v_i.
\]

(5.27)

This has exactly the same form as the Christoffel equation

\[
k^2(l_{iK}e_{KL}^E l_{Lj})v_j = \rho \omega^2 v_i,
\]

(5.28)
with $c_{KL}^E$ replaced by the expression

$$c_{KL}^E + \frac{[e_{Kj}][I_i e_{iL}]}{I_i e_{ij} I_j},$$

(5.29)

which is called a piezoelectrically stiffened elastic constant. There are three uniform plane wave solutions for each propagation direction $\hat{l}$. Once $v$ has been found from the stiffened Christoffel equation, the electric potential is easily calculated from Eq.(5.26).

### 5.2 Modeling Wave Propagation in Piezoelectric Materials by Hyperbolic System Equations

First, we will consider the governing equations which are composed of Maxwell’s equation and stress wave equation. In order to simplify the derivation, we use hexagonal crystal as an example, that is, wave propagation along the $x_1$ axis of a hexagonal(6mm) crystal. Here we will also use contracted indices $11 \rightarrow 1, 22 \rightarrow 2, 33 \rightarrow 3, 23 \rightarrow 4, 31 \rightarrow 5$ and $12 \rightarrow 6$. From our discussion before, the conservation of linear momentum:

$$\rho \frac{\partial v_1}{\partial t} = \frac{\partial T_1}{\partial x_1} + \frac{\partial T_6}{\partial x_2} + \frac{\partial T_5}{\partial x_3},$$

$$\rho \frac{\partial v_2}{\partial t} = \frac{\partial T_6}{\partial x_1} + \frac{\partial T_2}{\partial x_2} + \frac{\partial T_4}{\partial x_3},$$

$$\rho \frac{\partial v_3}{\partial t} = \frac{\partial T_5}{\partial x_1} + \frac{\partial T_4}{\partial x_2} + \frac{\partial T_3}{\partial x_3}.$$  

(5.30)

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The piezoelectric stress matrices for hexagonal crystal which is 3rd order tensor can be shown as

\[
e_{i,j} = \begin{bmatrix}
0 & 0 & 0 & 0 & e_{x5} & 0 \\
0 & 0 & 0 & e_{x5} & 0 & 0 \\
e_{z1} & e_{z1} & e_{z3} & 0 & 0 & 0
\end{bmatrix}.
\] (5.31)

And the permittivity matrix for hexagonal crystal is

\[
\varepsilon^{S}_{ij} = \begin{bmatrix}
\varepsilon^{S}_{11} & 0 & 0 \\
0 & \varepsilon^{S}_{11} & 0 \\
0 & 0 & \varepsilon^{S}_{33}
\end{bmatrix}
\] (5.32)

and stiffness matrix for hexagonal crystal is

\[
C^{E}_{ij} = \begin{bmatrix}
C^{E}_{11} & C^{E}_{12} & C^{E}_{13} & 0 & 0 & 0 \\
C^{E}_{12} & C^{E}_{11} & C^{E}_{13} & 0 & 0 & 0 \\
C^{E}_{13} & C^{E}_{13} & C^{E}_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C^{E}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C^{E}_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & C^{E}_{66}
\end{bmatrix}.
\] (5.33)
If we take a time derivative of constitutive relationships, e.g., Eqs.(5.19), we obtain

\[
\frac{\partial}{\partial t} \begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6 \\
\end{bmatrix} = \begin{bmatrix}
C_{11}^E & C_{12}^E & C_{13}^E & 0 & 0 & 0 \\
C_{12}^E & C_{11}^E & C_{13}^E & 0 & 0 & 0 \\
C_{13}^E & C_{13}^E & C_{33}^E & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44}^E & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44}^E & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}^E \\
\end{bmatrix}
\begin{bmatrix}
\frac{\partial v_1}{\partial x_1} \\
\frac{\partial v_2}{\partial v_2} \\
\frac{\partial v_3}{\partial x_3} \\
\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\
\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\
\frac{\partial v_1}{\partial x_3} + \frac{\partial v_2}{\partial x_1} \\
\end{bmatrix}
\]

(5.34)
and

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 & e_{x5} & 0 \\ 0 & 0 & 0 & e_{x5} & 0 & 0 \\ e_{z1} & e_{z1} & 0 & 0 & e_{z3} & 0 \\ 0 & 0 & e_{z1} & e_{z1} & e_{z3} & 0 \\ 0 & 0 & 0 & e_{z1} & e_{z1} & e_{z3} \end{bmatrix}
\end{align*}
\]

\[
+ \begin{bmatrix} \varepsilon_{11}^S & 0 & 0 \\ 0 & \varepsilon_{11}^S & 0 \\ 0 & 0 & \varepsilon_{33}^S \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \]

(5.35)
Substituting Eqs. (5.35) into Eqs. (5.34) and using Maxwell's equation, we have

\[
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6
\end{bmatrix}
= 
\begin{bmatrix}
C_{11}^E + \frac{e_{z1}^2}{\varepsilon_{33}^s} & C_{12}^E + \frac{e_{z1}^2}{\varepsilon_{33}^s} & C_{13}^E + \frac{e_{z1}e_{z3}}{\varepsilon_{33}^s} & 0 & 0 & 0 \\
C_{12}^E + \frac{e_{z1}^2}{\varepsilon_{33}^s} & C_{11}^E + \frac{e_{z1}^2}{\varepsilon_{33}^s} & C_{13}^E + \frac{e_{z1}e_{z3}}{\varepsilon_{33}^s} & 0 & 0 & 0 \\
C_{13}^E + \frac{e_{z1}e_{z3}}{\varepsilon_{33}^s} & C_{13}^E + \frac{e_{z1}e_{z3}}{\varepsilon_{33}^s} & C_{33}^E + \frac{e_{z3}^2}{\varepsilon_{33}^s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44}^E + \frac{e_{z5}^2}{\varepsilon_{11}^s} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}^E + \frac{e_{z5}^2}{\varepsilon_{11}^s} \\
0 & 0 & 0 & 0 & 0 & 0 & C_{66}^E
\end{bmatrix}
\times
\begin{bmatrix}
\frac{\partial v_1}{\partial x_1} \\
\frac{\partial v_2}{\partial x_2} \\
\frac{\partial v_3}{\partial x_3} + \frac{\partial v_3}{\partial x_1} + \frac{\partial v_3}{\partial v_1} + \frac{\partial v_3}{\partial v_2} + \frac{\partial v_3}{\partial x_2} + \frac{\partial v_3}{\partial x_1} \\
\frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_2} + \frac{\partial v_3}{\partial v_2} + \frac{\partial v_3}{\partial v_1} + \frac{\partial v_3}{\partial x_1} + \frac{\partial v_3}{\partial x_2} + \frac{\partial v_3}{\partial x_1} \\
\frac{\partial v_1}{\partial x_1} + \frac{\partial v_3}{\partial x_1} + \frac{\partial v_3}{\partial v_1} + \frac{\partial v_3}{\partial v_2} + \frac{\partial v_3}{\partial x_1} + \frac{\partial v_3}{\partial x_2}
\end{bmatrix}
- 
\begin{bmatrix}
0 & 0 & \frac{e_{z1}}{\varepsilon_{33}^s} & 0 & 0 & 0 \\
0 & 0 & \frac{e_{z1}}{\varepsilon_{33}^s} & \frac{e_{z3}}{\varepsilon_{33}^s} & 0 & 0 \\
0 & 0 & \frac{e_{z3}}{\varepsilon_{33}^s} & \frac{e_{z3}}{\varepsilon_{33}^s} & 0 & 0 \\
0 & 0 & 0 & \frac{e_{z5}}{\varepsilon_{11}^s} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{e_{z5}}{\varepsilon_{11}^s} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{e_{z5}}{\varepsilon_{11}^s}
\end{bmatrix}
\times
\begin{bmatrix}
\frac{\partial H_2}{\partial x_2} + \frac{\partial H_3}{\partial x_1} \\
\frac{\partial H_2}{\partial x_3} - \frac{\partial H_3}{\partial x_1} \\
\frac{\partial H_2}{\partial x_3} - \frac{\partial H_3}{\partial x_1} \\
\frac{\partial H_1}{\partial x_2} + \frac{\partial H_2}{\partial x_1} \\
\frac{\partial H_1}{\partial x_3} - \frac{\partial H_2}{\partial x_1}
\end{bmatrix}
.
\]

(5.36)
We re-write Maxwell’s equations as

\[
\frac{\partial}{\partial t} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = - \begin{bmatrix} 0 & -\frac{\partial}{\partial x_3} & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{5.37}
\]

and

\[
\frac{\partial}{\partial t} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\varepsilon_{11}} & 0 & 0 \\ 0 & \frac{1}{\varepsilon_{11}} & 0 \\ 0 & 0 & \frac{1}{\varepsilon_{33}} \end{bmatrix} \begin{bmatrix} -\frac{\partial H_2}{\partial x_3} + \frac{\partial H_3}{\partial x_2} \\ -\frac{\partial H_2}{\partial x_3} - \frac{\partial H_1}{\partial x_2} - \frac{\partial H_3}{\partial x_1} \\ -\frac{\partial H_2}{\partial x_3} - \frac{\partial H_1}{\partial x_2} + \frac{\partial H_3}{\partial x_1} \end{bmatrix} \tag{5.38}
\]

This hexagonal crystal is assumed to be non-magnetic material where

\[
\mathbf{B} = \mu_0 \mathbf{H}
\]

If we combined Eqs.(5.36),(5.37) and (5.38) with Eqs.(5.30), a hyperbolic system can be obtained

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x_1} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial x_2} + \mathbf{C} \frac{\partial \mathbf{U}}{\partial x_3} = 0, \tag{5.39}
\]

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where

\[ U = (v_1, v_2, v_3, T_1, T_2, T_3, T_4, T_5, T_6, B_1, B_2, B_3, E_1, E_2, E_3)^T \]

where matrix \( \mathbf{A} \) can be expressed as

\[
\mathbf{A} = \begin{bmatrix} \mathbf{A}_t & \mathbf{A}_r \end{bmatrix}
\]

(5.40)

\[
\mathbf{A}_t = \begin{bmatrix}
0 & 0 & 0 & 1/\rho & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{C_{11}^E + \frac{e_{21}^2}{\varepsilon_{33}^1}}{\varepsilon_{33}^1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{C_{12}^E + \frac{e_{21}^2}{\varepsilon_{33}^1}}{\varepsilon_{33}^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{C_{13}^E + \frac{e_{21}^2}{\varepsilon_{33}^1}}{\varepsilon_{33}^3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{C_{44}^E + \frac{e_{25}^2}{\varepsilon_{33}^4}}{\varepsilon_{33}^5} & 0 & 0 & 0 & 0 \\
0 & \frac{C_{66}^E}{\varepsilon_{33}^6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{e_{25}^2}{\varepsilon_{11}^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e_{21}}{\varepsilon_{33}^1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{e_{21}}{\varepsilon_{33}^3} & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[ A_r = - \begin{bmatrix} 
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/\rho & 0 & 0 & 0 & 0 & 0 & 0 \\
1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{e_{z1}}{\mu_0 \varepsilon_{33}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{e_{z1}}{\mu_0 \varepsilon_{33}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{e_{z1}}{\mu_0 \varepsilon_{33}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{e_{z5}}{\mu_0 \varepsilon_{11}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\mu_0 \varepsilon_{11}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\mu_0 \varepsilon_{11}} & 0 & 0 & 0 & 0 
\end{bmatrix} \]
The eigenvalues of matrix $A$ can be got. There are 5 pairs of eigenvalues

$$
\lambda_{1,2} = \pm \sqrt{\frac{C_{44}^E + e_{25}^2/\varepsilon_{11}^S}{\rho}},
$$

$$
\lambda_{3,4} = \pm \sqrt{\frac{C_{66}^E}{\rho}},
$$

$$
\lambda_{5,6} = \pm \sqrt{1/\mu_0\varepsilon_{11}^S},
$$

$$
\lambda_{7,8} = \pm \sqrt[2]{\frac{C_{11}^E + e_{21}^2/\varepsilon_{33}^S}{\rho} + \frac{1}{\mu_0\varepsilon_{33}^S} + \sqrt{\left(\frac{C_{11}^E + e_{21}^2/\varepsilon_{33}^S}{\rho} - \frac{1}{\mu_0\varepsilon_{33}^S}\right)^2 + \frac{4e_{21}^2}{\rho\mu_0(\varepsilon_{33}^S)^2}}},
$$

$$
\lambda_{9,10} = \pm \sqrt[2]{\frac{C_{11}^E + e_{21}^2/\varepsilon_{33}^S}{\rho} + \frac{1}{\mu_0\varepsilon_{33}^S} - \sqrt{\left(\frac{C_{11}^E + e_{21}^2/\varepsilon_{33}^S}{\rho} - \frac{1}{\mu_0\varepsilon_{33}^S}\right)^2 + \frac{4e_{21}^2}{\rho\mu_0(\varepsilon_{33}^S)^2}}},
$$

(5.41)

For wave propagation in $x_1$ direction, there are 5 wave speeds. $\lambda_{1,2}$ is stiffened acoustic wave speed. $\lambda_{3,4}$ is the acoustic wave speeds. And $\lambda_{5,6}$ is one of speed of light in hexagonal medium. $\lambda_{7,8}$ is another speed of light after interacting with acoustic wave. $\lambda_{9,10}$ is an acoustic wave speed interacting with light speed. Since hexagonal crystal is isotropic in $x_1 - x_2$ plane, the wave speeds in $x_2$ are the same as those in $x_1$ direction. In other words, the eigenvalues of matrix $B$ are the same as those of matrix $A$. Next, we will discuss wave speed in $x_3$ direction. Since the wave speed in $x_3$ is the eigenvalues of matrix of $C$, The $C$ can be shown as

$$
C = \begin{bmatrix} C_l & C_r \end{bmatrix}
$$

(5.42)
where

\[
C_l = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/\rho \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{C_{13}^E}{\rho} & \frac{e_x e_z}{\varepsilon_{33}} & 0 & 0 & 0 \\
0 & 0 & \frac{C_{13}^E}{\rho} & \frac{e_x e_z}{\varepsilon_{33}} & 0 & 0 & 0 \\
0 & 0 & \frac{C_{33}^E}{\rho} & \frac{e_x e_z}{\varepsilon_{33}} & 0 & 0 & 0 \\
0 & 0 & \frac{C_{44}^E}{\rho} & \frac{e_x^2}{\varepsilon_{11}} & 0 & 0 & 0 \\
0 & 0 & \frac{C_{44}^E}{\rho} & \frac{e_x^2}{\varepsilon_{33}} & 0 & 0 & 0 \\
0 & 0 & \frac{-e_x^2}{\varepsilon_{11}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-e_x^2}{\varepsilon_{11}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-e_x^2}{\varepsilon_{33}} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
C_r = \begin{bmatrix}
1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\varepsilon_{z5} \frac{S}{\mu_0 \varepsilon_{11}^S} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\varepsilon_{z5} \frac{S}{\mu_0 \varepsilon_{11}^S} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \frac{S}{\mu_0 \varepsilon_{11}^S} & 0 & 0 & 0 \\
0 & 0 & 1 \frac{S}{\mu_0 \varepsilon_{33}^S} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
The eigenvalues of matrix $C$ can be got. There are 5 pairs of eigenvalues

- $\lambda_{1,2} = \pm \sqrt{\frac{C_{33}^E + e_{23}^2 / \varepsilon_{33}^S}{\rho}}$,
- $\lambda_{3,4} = \pm \frac{\sqrt{C}}{2} \left[ \frac{C_{44}^E + e_{25}^2 / \varepsilon_{11}^S}{\rho} + \frac{1}{\mu_0 \varepsilon_{11}^S} + \sqrt{\left( \frac{C_{44}^E + e_{25}^2 / \varepsilon_{11}^S}{\rho} - \frac{1}{\mu_0 \varepsilon_{11}^S} \right)^2 + \frac{4e_{25}^2}{\rho \mu_0 (\varepsilon_{11}^S)^2}} \right]$,
- $\lambda_{5,6} = \pm \frac{\sqrt{C}}{2} \left[ \frac{C_{44}^E + e_{25}^2 / \varepsilon_{33}^S}{\rho} + \frac{1}{\mu_0 \varepsilon_{33}^S} - \sqrt{\left( \frac{C_{44}^E + e_{25}^2 / \varepsilon_{33}^S}{\rho} - \frac{1}{\mu_0 \varepsilon_{33}^S} \right)^2 + \frac{4e_{25}^2}{\rho \mu_0 (\varepsilon_{33}^S)^2}} \right]$,
- $\lambda_{7,8} = \pm \frac{\sqrt{C}}{2} \left[ \frac{C_{44}^E + e_{25}^2 / \varepsilon_{11}^S}{\rho} + \frac{1}{\mu_0 \varepsilon_{11}^S} + \sqrt{\left( \frac{C_{44}^E + e_{25}^2 / \varepsilon_{11}^S}{\rho} - \frac{1}{\mu_0 \varepsilon_{11}^S} \right)^2 + \frac{4e_{25}^2}{\rho \mu_0 (\varepsilon_{11}^S)^2}} \right]$,
- $\lambda_{9,10} = \pm \frac{\sqrt{C}}{2} \left[ \frac{C_{44}^E + e_{25}^2 / \varepsilon_{11}^S}{\rho} + \frac{1}{\mu_0 \varepsilon_{33}^S} - \sqrt{\left( \frac{C_{44}^E + e_{25}^2 / \varepsilon_{33}^S}{\rho} - \frac{1}{\mu_0 \varepsilon_{33}^S} \right)^2 + \frac{4e_{25}^2}{\rho \mu_0 (\varepsilon_{33}^S)^2}} \right]$.

(5.43)

5.3 Modelling Wave Propagation in Piezoelectric Materials by Quasi-static Assumptions

It is well known that the electromagnetic wave speed is 5th order higher than the speeds of acoustic wave. As shown by Hutson ([64]), the effect to speed of acoustic wave due to the coupling to the electromagnetic waves is very small, which magnitude of the correction $\delta^2$ to $v^2$ (acoustic wave speed) is about $2(v/c)^2$ ($c$ is the average velocity of electromagnetic waves), which is very small.

Based on this consideration, quasi-static electromagnetic field coupling acoustic wave are considered here. Instead of using the popular formula by Tiersten, we start to model wave in piezoelectric material using hyperbolic equations. The quasi-static means an electromagnetic field changes very slowly with time. The criterion of "sufficient slowness" ([96]) of the change in the field may be expressed as: The change in the electromagnetic field is so slow that the time variation of displacement current...
can be ignored. In very low conducting media, from the continuity equation

$$\text{div} J = -\frac{\partial \rho_e}{\partial t}$$

and Gauss’ theorem

$$\text{div} D = \rho_e.$$  

It can be seen that the varying current density due to the piezoelectric fields is zero since $J$ in piezoelectric medium is zero. From here, we deduce that divergence of the electric displacement $D$ is zero in the piezoelectric fields. Furthermore, we make an assumption that $D$ also changes very slow with time. This lead to

$$\frac{\partial D}{\partial t} = 0. \quad (5.44)$$

With this assumption, Eq.(5.34) can be simplified to be

$$\frac{\partial}{\partial t} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = H \begin{bmatrix} \frac{\partial v_1}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} \\ \frac{\partial v_2}{\partial x_2} \\ \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_3} \end{bmatrix} + \begin{bmatrix} \frac{\partial v_1}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} \\ \frac{\partial v_2}{\partial x_2} \\ \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_3} \end{bmatrix} \quad (5.45)$$
where

\[
H = \begin{bmatrix}
C_{11}^E + \frac{e_{z1}^2}{\varepsilon_{33}} & C_{12}^E + \frac{e_{z1}^2}{\varepsilon_{33}} & C_{13}^E + \frac{e_{z1}e_{z3}}{\varepsilon_{33}} & 0 & 0 & 0 \\
C_{12}^E + \frac{e_{z1}^2}{\varepsilon_{33}} & C_{11}^E + \frac{e_{z1}^2}{\varepsilon_{33}} & C_{13}^E + \frac{e_{z1}e_{z3}}{\varepsilon_{33}} & 0 & 0 & 0 \\
C_{13}^E + \frac{e_{z1}e_{z3}}{\varepsilon_{33}} & C_{13}^E + \frac{e_{z1}e_{z3}}{\varepsilon_{33}} & C_{33}^E + \frac{e_{z3}}{\varepsilon_{33}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44}^E + \frac{e_{z5}^2}{\varepsilon_{11}} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{44}^E + \frac{e_{z5}^2}{\varepsilon_{11}} \\
0 & 0 & 0 & 0 & 0 & 0 & C_{66}^E
\end{bmatrix}
\]

Eq. (5.45) can be coupled with conservation of linear momentum equation to form a close hyperbolic differential system. The coupled system of governing equations is recast in the following vector form:

\[
\frac{\partial \hat{U}}{\partial t} + \hat{A} \frac{\partial \hat{U}}{\partial x_1} + \hat{B} \frac{\partial \hat{U}}{\partial x_2} + \hat{C} \frac{\partial \hat{U}}{\partial x_3} = 0,
\]

(5.46)

with \(\hat{U}\) the flux vector; \(\hat{A}\), \(\hat{B}\), and \(\hat{C}\) the Jacobian matrices; \(\hat{U}, \hat{A}, \hat{B},\) and \(\hat{C}\) are defined as below;

\[
\hat{U} = (v_1, v_2, v_3, T_1, T_2, T_3, T_4, T_5, T_6)^T.
\]
\[
\hat{A} = - \begin{bmatrix}
0 & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\rho \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\rho & 0 \\
\frac{\varepsilon_{11}^2}{\varepsilon_{33}} + C_{11}^E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\varepsilon_{12}^2}{\varepsilon_{33}} + C_{12}^E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\varepsilon_{13}^2}{\varepsilon_{33}} + C_{13}^E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\varepsilon_{25}^2}{\varepsilon_{11}} + C_{44}^E & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{66}^E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\hat{B} = - \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\rho \\
0 & 0 & 0 & 1/\rho & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/\rho & 0 \\
0 & \frac{\varepsilon_{11}^2}{\varepsilon_{33}} + C_{12}^E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\varepsilon_{11}^2}{\varepsilon_{33}} + C_{11}^E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\varepsilon_{11}^2}{\varepsilon_{33}} + C_{11}^E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\varepsilon_{25}^2}{\varepsilon_{11}} + C_{44}^E & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{66}^E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
When we consider one-dimensional wave propagation ($x_1$ direction), this system can be simplified to be:

$$\dot{\mathbf{U}} + \mathbf{\hat{A}} \frac{\partial \mathbf{U}}{\partial x_1} = 0,$$

(5.47)

$\mathbf{\hat{A}}$ is a $9 \times 9$ matrix which is singular. After crossing out the zero columns and corresponding rows, a $6 \times 6$ matrix can be got:

$$- \begin{bmatrix} 0 & 0 & 0 & 0 & 1/\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/\rho \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{11}^E & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{33}^E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(5.48)
The eigenvalues can be got as

\[
\lambda_{1,2} = \pm \sqrt{\frac{\varepsilon_{11}^2 + C_{11}^E}{\rho}}, \quad \lambda_{3,4} = \pm \sqrt{\frac{\varepsilon_{11}^2 + C_{44}^E}{\rho}}, \quad \lambda_{5,6} = \pm \sqrt{\frac{C_{66}^E}{\rho}}. \tag{5.49}
\]

For comparison, we also list the three wave speeds got by Auld ([4]) as

\[
\lambda_{1,2} = \pm \sqrt{\frac{C_{11}^E}{\rho}}, \quad \lambda_{3,4} = \pm \sqrt{\frac{\varepsilon_{11}^2 + C_{44}^E}{\rho}}, \quad \lambda_{5,6} = \pm \sqrt{\frac{C_{66}^E}{\rho}}. \tag{5.50}
\]
This chapter is directly taken from the published paper "Numerical simulation of linear and nonlinear waves in hypoelastic solids by the CESE method." [148]

6.1 Introduction

In this chapter, we report a novel nonlinear mathematical model and its numerical solution for elastic waves in slender rods. The model equations are based on the Eulerian form of the conservation laws and a hypoelastic constitutive equation ([129]). This set of first-order, fully-coupled, nonlinear hyperbolic partial differential equations (PDEs) can simultaneously model nonlinear elastic waves in the regime of large deformations as well as linear elastic waves in the regime of small deformations. Traditionally, problems in time-dependent elasticity have been solved numerically using the finite-difference method [110],[6], the finite-element method [154], or variations thereof [149], [73]. Spectral methods [77], the finite-volume method [44],[85], [69], and other numerical schemes such as the elastodynamic finite integration technique [117],[116], [51],[132] have also been employed. In this chapter, we use the Conservation Element and Solution Element (CESE) method [28],[24], an
explicit space-time finite-volume scheme, to solve our system of nonlinear elastodynamic model equations. It has been formerly used to solve dynamics and combustion problems, including detonations, cavitations, flows with complex shock structures, turbulent flows with embedded dense sprays, and magnetohydrodynamics; see, for instance, Refs. [136], [75], [150], [151]. In this chapter, the CESE method is employed to solve problems in dynamic non-linear elasticity for the first time.

To demonstrate our approach for modeling linear and nonlinear elastic waves in slender rods, we numerically simulate: (i) resonant standing waves and (ii) propagating compression waves. For problem (i), a time-harmonic axial load, driven at a resonant frequency, is applied to the right boundary of the rod. We observe a linear-to-nonlinear evolution of the resulting resonant vibrations, the emergence of super-harmonics of the forcing frequency, and the distribution of wave energy among multiple modes. For problem (ii), propagating compression waves are generated by a bi-material collinear impact. The CESE method successfully captures the sharp propagating wavefronts, wave reflection and transmission, and the different wave propagation speeds in each material. The remainder of the chapter is organized as follows: In Section 6.2, we derive the nonlinear model equations. We present conservative, non-conservative, and diagonal formulations; the former is solved numerically using the CESE method, while the latter two are used to study the eigenvalues and eigenvectors of the equation set. The nonlinear model equations are then linearized to recover the classical second-order wave equation in displacement. In Section 6.3, we demonstrate the numerical simulation of two initial-value/boundary-value problems. At the end of the chapter, we provide the conclusions.
6.2 Model Equations

6.2.1 Conservative Form

The three-dimensional, time-dependent field equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

(6.1)

for conservation of mass, and, in the absence of body forces,

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} - \mathbf{T}) = 0,$$

(6.2)

for conservation of linear momentum. In Eqs.(6.1) and (6.2), $\rho$ is the density, $\mathbf{v}$ is the velocity, and $\mathbf{T}$ is the Cauchy stress, all functions of position $\mathbf{x}$ and time $t$. A homogeneous isotropic hypoelastic solid of grade zero is chosen as the elastic constitutive response:

$$\frac{D \mathbf{T}}{Dt} = \lambda (tr \mathbf{D}) \mathbf{I} + 2\mu \mathbf{D},$$

(6.3)

where

$$\mathbf{D} = \frac{1}{2} [\nabla \mathbf{v} + (\nabla \mathbf{v})^T],$$

(6.4)

$$\frac{D(\mathbf{\cdot})}{Dt} = \frac{\partial (\mathbf{\cdot})}{\partial t} + \mathbf{v} \cdot \nabla (\mathbf{\cdot}) + (\mathbf{\cdot}) \mathbf{W} - \mathbf{W}(\mathbf{\cdot}) - a[(\mathbf{\cdot}) \mathbf{D} + \mathbf{D}(\mathbf{\cdot})],$$

(6.5)

and

$$\mathbf{W} = \frac{1}{2} (\nabla \mathbf{v} - (\nabla \mathbf{v})^T)^T.$$  

(6.6)
In Eqs. (6.3)–(6.6), \( \mathbf{I} \) is the identity tensor. \( \mathbf{D} \) and \( \mathbf{W} \) are the symmetric and skew parts of the velocity gradient, and \( \lambda \) and \( \mu \) are Lame’s first and second parameters. Several particular choices of the slip parameter \( a \) in Eq. (6.5) result in familiar forms of the rate operator \( \mathbf{D}/\mathbf{D}t \). For instance, when \( a = 1, a = 0, \) and \( a = -1 \), Eq. (6.5) represents the upper-convected rate, the Jaumann corotational rate, and the lower-convected rate, respectively. In this chapter, we let \( a = 0 \), a common choice in the literature [79], [18].

We choose Cartesian coordinates and express Eqs. (6.1) and (6.2) as four scalar equations:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_1}{\partial x_1} + \frac{\partial \rho v_2}{\partial x_2} + \frac{\partial \rho v_3}{\partial x_3} = 0, \tag{6.7}
\]

\[
\frac{\partial \rho v_1}{\partial t} + \frac{\partial (\rho v_1^2 - T_{11})}{\partial x_1} + \frac{\partial (\rho v_1 v_2 - T_{12})}{\partial x_2} + \frac{\partial (\rho v_1 v_3 - T_{13})}{\partial x_3} = 0, \tag{6.8}
\]

\[
\frac{\partial \rho v_2}{\partial t} + \frac{\partial (\rho v_2 v_1 - T_{21})}{\partial x_1} + \frac{\partial (\rho v_2^2 - T_{22})}{\partial x_2} + \frac{\partial (\rho v_2 v_3 - T_{23})}{\partial x_3} = 0, \tag{6.9}
\]

\[
\frac{\partial \rho v_3}{\partial t} + \frac{\partial (\rho v_3 v_1 - T_{31})}{\partial x_1} + \frac{\partial (\rho v_3 v_2 - T_{32})}{\partial x_2} + \frac{\partial (\rho v_3^2 - T_{33})}{\partial x_3} = 0, \tag{6.10}
\]

where \( v_1, v_2 \) and \( v_3 \) are the Cartesian components of \( \mathbf{v} \) and \( T_{11}, T_{12} \) and \( T_{13} \), etc. are the components of \( \mathbf{T} \). To proceed, we consider a slender circular-cylindrical rod, with \( x_1 \) the axial coordinate. We assume that the rod has traction-free lateral boundaries.
and is sufficiently slender so that the Cauchy stress is approximately uni-axial:

\[
\mathbf{T} = \begin{bmatrix}
T_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]  

(6.11)

The constitutive equation, Eq. (6.3), can be expressed as six scalar transport equations for each of the Cauchy stress components. Aided by Eq. (6.11), the transport equations for the vanishing stress components \(T_{22}, T_{33}, T_{13}, T_{23}\) yield the following strain-rate relationships:

\[
\begin{align*}
\frac{\partial v_2}{\partial x_2} &= -\frac{\lambda}{2(\lambda + \mu)} \frac{\partial v_1}{\partial x_1}, & \frac{\partial v_3}{\partial x_3} &= -\frac{\lambda}{2(\lambda + \mu)} \frac{\partial v_1}{\partial x_1}, \\
\frac{\partial v_1}{\partial x_2} &= -\frac{\partial v_2}{\partial x_1}, & \frac{\partial v_1}{\partial x_3} &= -\frac{\partial v_3}{\partial x_1}, & \frac{\partial v_2}{\partial x_3} &= -\frac{\partial v_3}{\partial x_2}.
\end{align*}
\]  

(6.12)

Using Eqs. (6.11) and (6.12), and noting \(W_{11} = 0\), the transport equation for \(T_{11}\) becomes

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1} \left[ \rho v_1 \left(\frac{\mu}{\lambda + \mu} \frac{\partial T_{11}}{\partial x_1} - T_{11}\right) \right] = -\frac{\lambda}{\lambda + \mu} \frac{\partial \rho}{\partial x_1} - \frac{\mu}{\lambda + \mu} \frac{\partial \rho v_1}{\partial x_1}.
\]  

(6.13)

We now postulate that the density \(q(x_1, x_2, x_3, t)\), axial velocity component \(v_1(x_1, x_2, x_3, t)\), and axial normal stress component \(T_{11}(x_1, x_2, x_3, t)\) are functions of axial coordinate \(x_1\) and time \(t\) only, a customary assumption in the analysis of slender cylindrical bodies e.g. [72]. Thus, Eqs. (6.7), (6.8) and (6.13) become

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1} \left[ \rho v_1 \left(\frac{\mu}{\lambda + \mu} \frac{\partial v_1}{\partial x_1} - \frac{\mu}{\lambda + \mu} T_{11}\right) \right] &= -\frac{\lambda}{\lambda + \mu} \frac{\partial \rho}{\partial x_1}, \\
\frac{\partial \rho v_1}{\partial t} + \frac{\partial}{\partial x_1} \left[ \rho v_1 \left(\frac{\mu}{\lambda + \mu} \frac{\partial v_1}{\partial x_1} - T_{11}\right) \right] &= -\frac{\lambda}{\lambda + \mu} \frac{\partial \rho v_1}{\partial x_1}, \\
\frac{\partial T_{11}}{\partial t} + \frac{\partial}{\partial x_1} \left[ \rho v_1 \left(\frac{\mu}{\lambda + \mu} \frac{\partial v_1}{\partial x_1} - \frac{\mu}{\lambda + \mu} T_{11}\right) \right] &= E \frac{\partial v_1}{\partial x_1} - \frac{\lambda}{\lambda + \mu} \frac{\partial T_{11}}{\partial x_1},
\end{align*}
\]  

(6.14, 6.15, 6.16)
where we have used the relationship $E = \mu(3\lambda+2\mu)/(\lambda+\mu)$ between Young’s modulus $E$ and Lame’s parameters. Aided by Eq.(6.14), Eq.(6.16) is rewritten in a conservative form:

$$
\frac{\partial \rho T_{11}}{\partial t} + \frac{\partial (\frac{\mu}{\lambda+\mu}) \rho v_1 T_{11}}{\partial x_1} = E \rho \frac{\partial v_1}{\partial x_1} - \frac{\lambda v_1 \partial T_{11}}{\lambda + \mu},
$$

(6.17)

Eqs.(6.14),(6.15), and (6.17), a system of three coupled nonlinear conservative partial differential equations in three primitive unknowns $\rho(x_1,t), v_1(x_1,t)$, and $T_{11}(x_1,t)$, are cast into the vector form

$$
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \mathbf{S},
$$

(6.18)

where the vector of conserved variables, the vector of fluxes, and the vector of source terms are:

$$
\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho v_1 \\ \rho T_{11} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \frac{\mu}{\lambda+\mu} \rho v_1 \\ \frac{\mu}{\lambda+\mu} \rho v_1^2 - T_{11} \\ -\frac{\mu}{\lambda+\mu} v_1 \frac{\partial \rho}{\partial x_1} \\ -\frac{\mu}{\lambda+\mu} v_1 \frac{\partial (\rho v_1)}{\partial x_1} \\ E \rho \frac{\partial v_1}{\partial x_1} - \frac{\mu}{\lambda+\mu} v_1 \frac{\partial (\rho T_{11})}{\partial x_1} \end{bmatrix},
$$

(6.19)

Using the chain rules, Eq.(6.18) can be expressed as a quasi-linear system:

$$
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial x} = \mathbf{S},
$$

(6.20)

where the Jacobian of the flux vector is

$$
\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}} = \begin{bmatrix} 0 & \frac{\mu}{\lambda+\mu} & 0 \\ \frac{\mu v_1^2}{\lambda+\mu} + \frac{T_{11}}{\rho} & \frac{\lambda}{\lambda+\mu} & -\frac{T_{11}}{\rho} \\ -\frac{\mu v_1 T_{11}}{\lambda+\mu} & \frac{\mu T_{11}}{\lambda+\mu} & \frac{\mu v_1}{\lambda+\mu} \end{bmatrix},
$$

(6.21)
with eigenvalues
\[
\lambda_{1,2,3} = \frac{\mu}{\lambda + \mu} v_1. 
\]  
(6.22)

Generally, the eigenvalues represent the characteristic speeds of the hyperbolic system. However, in this case, due to the influence of the source term \( S \), these eigenvalues are not the true wave speeds. Thus, we examine a non-conservative form of the equations to deduce the physically-correct wave speeds.

### 6.2.2 Non-Conservative Form

We recast Eq. (6.20) into a non-conservative form, with \( \tilde{U} = (\rho, u, T_{11})^T \) the vector of unknowns. This is achieved by premultiplying Eq.(6.20) by a transformation matrix \( M^{-1} \):

\[
M^{-1} \frac{\partial \tilde{U}}{\partial t} + \tilde{M}^{-1} \tilde{A} M \frac{\partial \tilde{U}}{\partial x_1} = \tilde{M}^{-1} S, 
\]  
(6.23)

to produce

\[
\frac{\partial \tilde{U}}{\partial t} + \tilde{A} \frac{\partial \tilde{U}}{\partial x_1} = \tilde{S}, 
\]  
(6.24)

where

\[
M^{-1} = \frac{\partial \tilde{U}}{\partial \tilde{U}} = \begin{bmatrix} 1 & 0 & 0 \\ -v_1/\rho & 1/\rho & 0 \\ -T_{11}/\rho & 0 & 1/\rho \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 \\ v_1 & 1/\rho & 0 \\ T_{11} & 0 & \rho \end{bmatrix}, 
\]  
(6.25)
and

\[ \tilde{A} = M^{-1}AM = \begin{bmatrix} \frac{\mu v_1}{\lambda + \mu} & \frac{\mu \rho}{\lambda + \mu} & 0 \\ \frac{\mu v_1}{\lambda + \mu} & -1/\rho \\ 0 & \frac{\mu v_1}{\lambda + \mu} \end{bmatrix}, \]

\[ \tilde{S} = M^{-1}S = \begin{bmatrix} \frac{\mu v_1}{\lambda + \mu} \frac{\partial \rho}{\partial x_1} \\ \frac{\mu v_1}{\lambda + \mu} \frac{\partial v_1}{\partial x_1} \\ E \frac{\partial v_1}{\partial x_1} - \frac{\mu v_1}{\lambda + \mu} \frac{\partial T_{11}}{\partial x_1} \end{bmatrix}. \]

Equation (6.26)

Moving the source term \( \tilde{S} \) to the left-hand side of Eq. (6.24), we obtain

\[ \frac{\partial \tilde{U}}{\partial t} + \tilde{A} \frac{\partial \tilde{U}}{\partial x_1} = 0, \]

Equation (6.27)

where

\[ \tilde{A} = \begin{bmatrix} v_1 & \frac{\mu \rho}{\lambda + \mu} & 0 \\ 0 & v_1 & -1/\rho \\ 0 & -E & v_1 \end{bmatrix}. \]

Equation (6.28)

The eigenvalues of \( \tilde{A} \) are

\[ \lambda_1 = v_1, \quad \lambda_{2,3} = v_1 \pm \tau = v_1 \pm \sqrt{E/\rho}, \]

Equation (6.29)

where \( \tau = \sqrt{E/\rho} \) is the local longitudinal propagation speed of sound. The eigenvectors of \( \tilde{A} \) are arranged column-wise to form the right eigenvector matrix

\[ \tilde{M} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{\lambda + \mu}{\mu} \frac{1}{\rho} \sqrt{\frac{E}{\rho}} & -\frac{\lambda + \mu}{\mu} \frac{1}{\rho} \sqrt{\frac{E}{\rho}} \\ 0 & \frac{\lambda + \mu}{\mu} \frac{E}{\rho} & -\frac{\lambda + \mu}{\mu} \frac{E}{\rho} \end{bmatrix}. \]

Equation (6.30)
Its inverse is the left eigenvector matrix

\[
\tilde{M}^{-1} = \begin{bmatrix}
1 & 0 & \frac{\mu - \rho}{\lambda + \mu E} & \frac{\mu - \rho}{\lambda + \mu 2E} \\
0 & \frac{\mu - \rho}{\lambda + \mu 2\sqrt{E/\rho}} & -\frac{\mu - \rho}{\lambda + \mu 2E} & -\frac{\mu - \rho}{\lambda + \mu 2E} \\
0 & -\frac{\mu - \rho}{\lambda + \mu 2\sqrt{E/\rho}} & -\frac{\mu - \rho}{\lambda + \mu 2E} & -\frac{\mu - \rho}{\lambda + \mu 2E}
\end{bmatrix}.
\]

(6.31)

We pre-multiply Eq.(6.27) by \(\tilde{M}^{-1}\):

\[
\tilde{M}^{-1}\frac{\partial \tilde{U}}{\partial t} + \tilde{M}^{-1}\tilde{A}\tilde{M}^{-1}\frac{\partial \tilde{U}}{\partial x_1} = 0,
\]

(6.32)

to produce

\[
\frac{\partial \hat{U}}{\partial t} + \Lambda \frac{\partial \hat{U}}{\partial x_1} = 0,
\]

(6.33)

where \(\partial \hat{U} = \tilde{M}^{-1}\partial \tilde{U}\) and

\[
\Lambda = \tilde{M}^{-1}\tilde{A}\tilde{M} = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix} = \begin{bmatrix}
v_1 & 0 & 0 \\
0 & v_1 + \sqrt{E/\rho} & 0 \\
0 & 0 & v_1 - \sqrt{E/\rho}
\end{bmatrix}.
\]

(6.34)

The physically-correct, real, distinct characteristic speeds in Eq.(6.34) indicate that our system of nonlinear governing Eqs. (6.14),(6.15), and (6.17) is strictly hyperbolic.

### 6.2.3 Linearized Model Equations

In this section, we linearize the nonlinear model Eqs.(6.14), (6.15), and (6.17) to recover the classical second-order linear wave equation in displacement. We posit the
following expansions for the unknown variables:

\[ \rho(x_1, t) = \rho^0(x_1, t) + \epsilon \rho^1(x_1, t) + \cdots, \]
\[ v_1(x_1, t) = v_1^0(x_1, t) + \epsilon v_1^1(x_1, t) + \cdots, \]  
\[ T_{11}(x_1, t) = T_{11}^0(x_1, t) + \epsilon T_{11}^1(x_1, t) + \cdots, \]  

where \( \rho^0(x_1, t), v_1^0(x_1, t), \) and \( T_{11}^0(x_1, t) \) are the rest states that we perturb about, \( \rho^1(x_1, t), v_1^1(x_1, t), \) and \( T_{11}^1(x_1, t) \) are the first corrections, and \( \epsilon \ll 1 \) is a small scalar slenderness parameter (e.g., the ratio of a characteristic transverse length scale to a characteristic axial length scale). Inserting the expansions, Eq.(6.35), into Eqs.(6.14), (6.15), and (6.17), we obtain the following equations from the order—\( \epsilon \) problem:

\[ \partial \rho^1 \partial t + v_1^0 \partial \rho_1 \partial x_1 + v_1^1 \partial \rho_0 \partial x_1 + \frac{\mu}{\lambda + \mu} v_1^0 \partial v_1^1 \partial x_1 + \mu \frac{v_1^0}{\lambda + \mu} \partial v_1^0 \partial x_1 = 0, \]  
\[ \frac{\partial}{\partial t} (\rho v_1^0 + \rho^1 v_1^0) + \frac{\lambda}{\lambda + \mu} \left[ v_1^0 \frac{\partial}{\partial x_1} (\rho v_1^0 + \rho^1 v_1^0) + v_1^1 \frac{\partial}{\partial x_1} (\rho v_1^0) \right] \]
\[ + \frac{\partial}{\partial x_1} \left\{ \frac{\mu}{\lambda + \mu} [2\rho v_1^0 v_1^1 + \rho^1 (v_1^0)^2] - T_{11}^1 \right\} = 0, \]  
\[ \partial T_{11}^1 \partial t + v_1^0 \partial T_{11}^1 \partial x_1 + v_1^1 \partial T_{11}^0 \partial x_1 - E \partial v_1^1 \partial x_1 = 0. \]  

For the rest states, we select

\[ \rho^0(x_1, t) = \rho_0, v_1^0(x_1, t) = 0, v_1^1(x_1, t) = 0, T_{11}^0(x_1, t) = 0, \]  

with \( \rho_0 \) the reference density. Our choices for the rest density, rest velocity, and rest stress necessarily satisfy Eqs.(6.14),(6.15), and (6.17). Inserting the rest values (6.39)
into Eqs.(6.36)−(6.38), we obtain a set of first-order linear PDEs:

\[
\frac{\partial \rho^1}{\partial t} + (1 - 2\nu)\rho_0 \frac{\partial v^1_1}{\partial x_1} = 0, \quad (6.40)
\]

\[
\frac{\partial v^1_1}{\partial t} - \frac{1}{\rho_0} \frac{\partial T^1_{11}}{\partial x_1} = 0, \quad (6.41)
\]

\[
\frac{\partial T^1_{11}}{\partial t} - E \frac{\partial v^1_1}{\partial x_1} = 0, \quad (6.42)
\]

where we have utilized the relationship \(\mu/(\lambda + \mu) = 1 - 2\nu\) between Lame’s parameters and Poisson’s ratio \(\nu\). Eqs. (6.41) and (6.42), which are decoupled from Eq.(6.40), can be manipulated to yield linear wave equations for the velocity and stress corrections:

\[
\frac{\partial^2 v^1_1}{\partial t^2} = c^2 \frac{\partial^2 v^1_1}{\partial x_1^2}, \quad \frac{\partial^2 T^1_{11}}{\partial t^2} = c^2 \frac{\partial^2 T^1_{11}}{\partial x_1^2}, \quad (6.43)
\]

with \(c = \sqrt{E/\rho_0}\) the constant longitudinal wave propagation speed. If we integrate Eq.(6.43)\(_1\) with respect to time and set the arbitrary function of \(x_1\) which arises to be zero, we recover:

\[
\frac{\partial^2 u_1}{\partial t^2} = c^2 \frac{\partial^2 u_1}{\partial x_1^2}, \quad (6.44)
\]

with \(u_1(x_1, t)\) the axial displacement component. Eq.(6.44) is identically the classical second-order linear wave equation in displacement.

We express the linearized Eqs.(6.40)-(6.42) in vector form:

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x_1} = 0, \quad (6.45)
\]
with the vector of conserved variables $\mathbf{U}$ and the flux Jacobian matrix $\mathbf{A}$ given by:

$$
\mathbf{U} = \begin{bmatrix} \rho_1 \\ v_1 \\ T_{11} \end{bmatrix},
\mathbf{A} = \begin{bmatrix} 0 & (1 - 2\nu)\rho_0 & 0 \\ 0 & 0 & -1/\rho_0 \\ 0 & -E & 0 \end{bmatrix}.
$$

(6.46)

The Jacobian matrix $\mathbf{A}$ has three real, distinct eigenvalues:

$$
\lambda_1 = 0, \lambda_{2,3} = \pm c = \pm \sqrt{E/\rho_0},
$$

(6.47)

and three linearly-independent eigenvectors. Thus, the system of linear Eqs.(6.40 -6.42) is strictly hyperbolic. Further, since $\lambda_1 = 0$, variations in $\rho_1(x_1,t)$ have no effect on wave propagation in the $x_1$ (axial) direction.

### 6.3 Results and Discussions

In this section, we demonstrate the numerical simulation of two initial-value/boundary-value problems: (i) resonant standing waves arising from the application of a time-harmonic external axial load and (ii) propagating compression waves arising from a bi-material collinear impact. Numerical results of the nonlinear problems and analytical solutions of the linearized problems are illustrated. In the linear regime, the numerical results are directly compared with the analytical solutions.

#### 6.3.1 Resonant Standing Waves

We consider a slender aluminium (Young’s modulus $E = 70$ GPa, reference density $\rho_0 = 2700\text{kg/m}^3$) rod of length $L = 20\text{m}$. The left boundary of the rod is fixed, and
a time-harmonic force

\[ F(t) = T_{11}(L, t)A = f \sin(\Omega t) \]  

(6.48)

is applied to the right boundary. In Eq.(6.48), the amplitude \( f = 10 \) MN and the cross-sectional area \( A = 180\text{cm}^2 \). To induce resonant vibrations, we set the driving frequency \( \Omega = 5198\text{s}^{-1} \), the seventh natural frequency of the rod.

**Analytical Solution of the Linearized Problem**

The linearized problem consists of

\[ \frac{\partial^2 u_1}{\partial t^2} = c^2 \frac{\partial^2 u_1}{\partial x_1^2}, \quad 0 < x_1 < L, t > 0; \quad c = \sqrt{E/\rho_0}, \]  

(6.49)

together with the initial conditions

\[ u_1(x_1, 0) = 0, \quad v_1(x_1, 0) = \frac{\partial u_1(x_1, t)}{\partial t}|_{t=0} = 0, \]  

(6.50)

and the boundary conditions

\[ u_1(x_1, t) = 0, \quad T_{11}(L, t) = E \frac{\partial u_1(x_1, t)}{\partial t}|_{x_1=L} = \frac{f \sin(\Omega t)}{A}. \]  

(6.51)

The solution of Eqs.(6.49)- (6.51) is

\[ u_1(x_1, t) = \frac{1}{L} \sin(\alpha_r x_1) \frac{(-1)^r f \omega_r \cos(\omega_r t) - \sin(\omega_r t)}{\rho A \omega_r^2} + \sum_{m=1, m \neq r}^{\infty} \frac{2}{L} \sin(\alpha_m x_1) \frac{(-1)^{m-1} f}{\rho A \omega_m^2 - \Omega^2} \left( \sin \Omega t - \frac{\Omega}{\omega_m} \sin \omega_m t \right). \]  

(6.52)
where

\[ \alpha_m = \frac{(2m - 1)\pi}{2L}, \quad \omega_m = \frac{\alpha_m c}{2L}, \quad m \neq r = 1, 2, \ldots, \infty, \]

and the positive integer \( r \) corresponds to the resonant mode. The axial stress is

\[
T_{11}(x_1, t) = E \frac{\partial u_1}{\partial x_1} = E \frac{\alpha_r}{L} \cos(\alpha_r x_1) \left( -1 \right)^r \left( \frac{L}{\rho A} \omega_r \cos(\omega_r t) - \sin(\omega_r t) \right) \]

\[+ \sum_{m=1, m \neq r}^{\infty} \frac{2E\alpha_m}{L} \cos(\alpha_m x_1) \left( -1 \right)^{m-1} \left( \frac{f}{\rho A} \omega_m^2 - \Omega^2 \right) \left( \sin \Omega t - \frac{\Omega}{\omega_m} \sin \omega_m t \right).\]

(6.53)

Numerical Solution of the Nonlinear Problem

The nonlinear problem consists of the conservative form governing Eq.(6.14), Eq.(6.15) and Eq.(6.17), together with the initial conditions (6.50) and the boundary conditions (6.51). We solve these equations numerically using the CESE method. The rod is discretized using nodes equally spaced at \( \Delta x = 10 \text{cm} \), and the time step is \( \Delta t = 10 \mu s \). The CFL number \( \approx c\Delta t/\Delta x \) is about 0.76. To numerically implement the fixed boundary condition, Eq.(6.51), on the left end of the rod, we add ghost cells. For time \( t = n\Delta t \), with \( n = 0, 1, 2, \ldots \), we take the values of density \( \rho \), the reverse velocity \( -v_1 \), and the reverse stress \( -T_{11} \) from Node 1 ( the first node to the right of the ghost cell) to be the solution at the ghost cell. Moreover, the solution profile at the ghost cell is a mirror image of that at Node 1. Thus, the spatial derivatives of the unknowns at the ghost cell are determined:

\[
\rho_{gc} = \rho_1, \quad (\rho v_1)_{gc} = -(\rho v_1)_1, \quad (\rho T_{11})_{gc} = -(\rho T_{11})_1, \quad (\rho_x)_{gc} = -(\rho_x)_1, \quad \left[(\rho v_1)_x\right]_{gc} = \left[ (\rho v_1)_x \right]_1, \quad \left[ (\rho T_{11})_x \right]_{gc} = -\left[ (\rho T_{11})_x \right]_1.
\]

(6.54)
where the subscript $gc$ denotes the ghost cell. In this arrangement, the fixed boundary is imposed at the midpoint between the ghost point and Node 1 on the $t$ axis.

At $t = n \Delta t, n = 1/2, 3/2, \cdots$, the solution at the node located at the fixed boundary can be directly calculated without using any boundary condition. With the complete solutions at the ghost cell $1/2\Delta t$ earlier, the node on the fixed boundary for time $t = n\Delta t, n = 1/2, 3/2, \cdots$, becomes an interior node, and its solution can be readily calculated based on the CESE method.

On the right end of the rod, we numerically implement the forced boundary condition, Eq.(6.51). At $t = n\Delta t$ with $n = 1, 2, \cdots$, we let

$$
\rho^n_N = \rho^{n-1/2}_{N-1/2}, \quad (\rho v_1)^n_N = (\rho v_1)^{n-1/2}_{N-1/2}, \quad (\rho T_{11})^n_N = (\rho)^{n-1/2}_{N-1/2} f \sin(\Omega t)/A,
$$

(6.55)

where the subscript $N$ denotes the last mesh node on the right end, where the forced vibrations are imposed. Essentially, we let the values of $\rho$ and $\rho v_1$ be taken from the immediate interior node from the previous half time step, and the boundary condition for $T_{11}$ is determined by the imposed forced vibrations, Eq. (6.48). For the spatial derivative of the unknowns, we let

$$
(\rho_x)^n_N = (\rho_x)^{n-1/2}_{N-1/2},
$$

$$
[(\rho v_1)_x]^n_N = [(\rho v_1)_x]^{n-1/2}_{N-1/2},
$$

$$
[(\rho T_{11})_x]^n_N = [(\rho)_x]^{n-1/2}_{N-1/2} f \sin(\Omega t)/A.
$$

(6.56)

In the setting of the CESE method, the above boundary conditions, Eqs. (6.55) and (6.56), are strictly based on space-time flux conservation.
Comparison of the Analytical and Numerical Solutions

Fig.(6.1) illustrates a travelling wave saturating into a standing wave profile over the course of four snapshots of time: \( t = 3.15, 3.75, 11.55, 12.15 \) ms. Snapshots at \( t = 3.15 \) and \( 3.75 \) ms illustrate the stress waves initiated by the time-harmonic load propagating leftward toward the fixed boundary. For snapshots at \( t = 11.55 \) and \( 12.15 \) ms, the wave-front has reflected from both boundaries over the course of several cycles and interfered with the leftward-travelling waves generated by the time-harmonic load, producing a standing wave profile. About three and a quarter periods of resonant vibrations span the length of the rod. After more than seven hundred time steps, the response is still within the linear regime as the numerical solution compares well with the analytical solution in both phase speed and amplitude as shown in Fig.(6.1). Fig. (6.2) shows four snapshots of the stress profile at \( t = 49.35, 50.55, 78, \) and \( 79.2 \) ms. The resonant vibrations are fully non-linear, as evidenced by the discrepancy between the numerical solution of the non-linear problem and the analytical solution of the linearized problem. At these later times, super-harmonics of the forcing frequency are clearly discerned. Fig illustrates the evolution of dimensionless stress at \( x = 9.6 \) m, an anti-node in the standing wave profile. Up to \( t/t_0 = 7.63 \), in the linear regime, the numerical solution compares well with the analytical solution. Because of the effects of nonlinearity, for \( t/t_0 > 7.63 \), wave energy is channelled into modes other than that of the imposed frequency. Thus, the numerically-obtained wave amplitude levels off as shown in Fig.(6.3). Conversely, for the analytical solution, all wave energy is locked into the mode of the forcing frequency, and amplitude of the standing waves grows continuously.

Fig.(6.4) shows a comparison of the analytical and numerical power spectra. In Fig. (6.4)a, the CESE result shows the imposed frequency at 827 Hz and its first and
Fig. 6.1: Four snapshots of the analytical (solid line) and numerical (diamonds) dimensionless resonant axial stress wave profiles during the initial stages of wave development in the linear regime at (a) $t = 3.15$ ms, (b) $t = 3.75$ ms, (c) $t = 11.55$ ms, and (d) $t = 12.15$ ms. Axial stress is normalized with respect to Young’s modulus, and axial position is normalized with respect to the length of the rod.

Fig. 6.2: Four snapshots of the analytical (solid line) and numerical (diamonds) dimensionless resonant axial stress wave profiles in the nonlinear regime at (a) $t = 49.35$ ms, (b) $t = 50.55$ ms, (c) $t = 78$ ms, and (d) $t = 79.2$ ms.
second super-harmonics at 1654 Hz. In Fig.(6.4)b, the analytical solution has only one mode: the frequency of the imposed oscillations.

6.3.2 Propagating compression waves

We consider the collinear impact of an aluminium impact rod (denoted by the subscript \( l \)) translating rightward at \( V_0 = 10 \) m/s and a stationary copper target rod (denoted by the subscript \( r \)), which is shown in Fig.(6.5) For aluminium, Young’s modulus \( E_l = 70\) GPa and reference density \( \rho_l = 2.7 \times 10^3 \) kg/m\(^3\). For copper, \( E_r = 130 \) GPa and \( \rho_r = 8.23 \times 10^3 \) kg/m\(^3\). The length of the impact and target rods is \( L_l = 2 \) m and \( L_r = 6 \) m, respectively; both have identical constant cross-sectional areas. The
Fig. 6.4: Power spectra of the calculated resonant waves. (a) The numerical solution of the nonlinear problem predicts a dominant frequency of 827 Hz, with first and second super-harmonics at 1654 Hz and 2481 Hz. (b) The analytical solution of the linearized problem has only one wave mode: the frequency of the imposed oscillations.

origin of a Cartesian coordinate system is affixed to the left end of the impact rod. The impact is assumed to be fully elastic, i.e., no rebound.

**Analytical solution of the linearized problem**

The linearized equations governing the longitudinal wave motion in the impact and target rods are

\[
\frac{\partial^2 u_l}{\partial t^2} = c_l^2 \frac{\partial^2 u_l}{\partial x_1^2}, \quad 0 < x_1 < L_l, \; t > 0; \quad c_l = \sqrt{E_l/\rho_l}, \\
\frac{\partial^2 u_r}{\partial t^2} = c_r^2 \frac{\partial^2 u_r}{\partial x_1^2}, \quad L_l < x_1 < L_l + L_r, \; t > 0; \quad c_r = \sqrt{E_r/\rho_r},
\]

(6.57)

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where \( u_l(x_1, t) \) and \( u_r(x_1, t) \) are the axial displacements of the impact and target rods. The initial conditions are

\[
\begin{align*}
u_l(x_1, 0) &= 0, \quad v_l(x_1, 0) = \frac{\partial u_l(x_1, t)}{\partial t}|_{t=0} = V_0, \quad 0 < x_1 < L_l, \quad (6.58) \\
u_r(x_1, 0) &= 0, \quad v_r(x_1, 0) = \frac{\partial u_r(x_1, t)}{\partial t}|_{t=0} = 0, \quad L_l < x_1 < L_l + L_r, \quad (6.59)
\end{align*}
\]

The boundary conditions include the traction-free left end of the impact rod:

\[
T_{11}(0, t) = E \frac{\partial u_l(x_1, t)}{\partial x_1}|_{x_1=0} = 0, \quad (6.60)
\]

the traction-free right end of the target rod:

\[
T_{11}(L_l + L_r, t) = E \frac{\partial u_r(x_1, t)}{\partial x_1}|_{x_1=L_l+L_r} = 0, \quad (6.61)
\]

and continuity of axial stress and axial velocity at the interface:

\[
\sigma_I(L_l, t) + \sigma_R(L_l, t) = \sigma_T(L_l, t), \quad v_I(L_l, t) + v_R(L_l, t) = v_T(L_l, t), \quad (6.62)
\]

where the subscripts \( I, R, \) and \( T \) denote incident, reflected, and transmitted longitudinal waves, respectively. Based on the d’Alembert solution of the second-order linear wave equation, the analytical solution of Eqs.(6.57)-(6.62) is constructed for five time periods: (i) \( 0 < t < L_l/c_l \), (ii) \( L_l/c_l < t < 2L_l/c_l \), (iii) \( 2L_l/c_l < t < 3L_l/c_l \), (iv) \( 3L_l/c_l < t < L_r/c_r \), (v) \( L_r/c_r < t < 4L_r/c_r \). Time period (i) is the initial stage of the impact process. The solution is focused on the interaction between the initial condition and the rod interface in forming the left-running (reflected) wave and the right-running (transmitted) wave. In time period (ii), the left-running wave is reflected from the left end of the impact rod. In (iii), the reflected wave from the left
Fig. 6.5: A schematic of the problem of collinear impact between two bars of different materials.

end interacts with the rod interface to form a secondary wave reflection and transmission. In (iv), the secondary waves are reflected from the left end of the impact rod. In (v), the initial transmitted right-running wave is reflected from the right end of the target rod.

**Numerical solution of the non-linear problem**

We numerically solve the non-linear problem consisting of the conservative-form Eqs. (6.14),(6.15), and (6.17), the initial conditions (6.59), and the boundary conditions (6.60)-(6.62) using the CESE method. The 2 m impact rod and the 6 m target rod are together divided into equally-spaced elements of length $\Delta x = 1$ cm, and the time step is $\Delta t = 1.5 \nu$ s. The CFL number is about 0.76 for the impact rod and about 0.59 for the target rod. We numerically implement the traction-free boundary (6.61) on the right end of the target rod as follows. For time $t = n \Delta t$ with $n = 1, 2, \cdots$, we let

$$
\rho_N^n = \rho_r, \ (\rho v_1)_N^n = \rho_r (v_1)_{N-1/2}^{n-1/2}, \ (\rho T_{11})_N^n = \rho_r (T_{11})_{N-1/2}^{n-1/2}, \quad (6.63)
$$
where the subscript \( N \) denotes the last mesh node on the right end. Essentially, we let \( \rho = \rho_r \) and \( v_1 \) and \( T_{11} \) be taken from the immediate interior node from the previous half time step. For the spatial derivatives of the primary unknowns, we let

\[
(\rho_x)^n_N = (\rho_x)^{n-1/2}_N, \quad [(\rho v_1)_x]^n_N = [(\rho v_1)_x]^{n-1/2}_N, \quad [(\rho T_{11})_x]^n_N = [(\rho T_{11})_x]^{n-1/2}_N,
\]

(6.64)

A similar treatment is employed for the traction-free boundary condition (6.60) at the left end of the impact rod.

**Comparison of the analytical and numerical solutions**

Figure (6.6) shows eight snapshots of velocity profiles at \( t = 0.15, 0.543, 0.9, 1.33, 1.65, 2.02, 2.5 \) and 2.92 ms. The eight snapshots are taken in Time period (i) to (iv) corresponding to the analytical solution. The plots show that the initial right-running and left-running waves, emanating from the rod interface due to the initial impact, travel towards the ends of the two rods and rebound back to the interface. The CESE method accurately capture the left-running wave speed in the impact (aluminum) bar at 5091 m/s and the right-running wave speed in the target (copper) bar at 3959 m/s. The CESE method also accurately capture the wave reflection and transmission at the bar interface as well as wave reflection from the free ends of the two rods. For \( t = 1.65 \) ms, the snapshots are taken within Time period (v), when the right-running wave in the copper rod reflects from the right end of the rod. In both impact rod and target rod, there are left and right-running waves. The calculated wave fronts in the two rods propagate with correct wave speeds in aluminum and copper rods. Figures 6.7 show the corresponding eight stress profiles at the same time steps as that in Figs. 6.6. As shown in the figures, the calculated longitudinal stress changes sign at the free end during wave reflection. A compression wave becomes a tension wave and vice
versa. This result is consistent with the analytical solution. The numerical treatment of the boundary conditions is successful. Although not shown, the numerical results here also satisfy the energy conservation. Initially, the kinetic energy carried by the impact rod is transmitted into the target rod through the transmission of a right-running wave into the target bar. Stress profiles of the two rod are directly related to the strain energy and the velocity profiles of the two rods are related to the kinematic energy. Since there is no damping effect in the model equations, the total energy must be conserved. In all above solutions, numerical results calculated by the CESE method compares well with the analytical solutions. The CESE method also crisply captures the reflected and transmitted wave fronts at the material interface, moving at different speeds in different materials.

6.4 Conclusions

In this chapter, we report a comprehensive framework for numerically simulating linear and nonlinear elastic waves. Our approach is a synergy of: (i) a set of first-order, coupled, nonlinear hyperbolic partial differential equations based on the conservation laws and a hypoelastic constitutive equation and (ii) the CESE method, a high-fidelity explicit space-time finite-volume scheme for solving nonlinear hyperbolic systems. Various forms of the governing equations are presented, including the conservative form, which we solve numerically, and the non-conservative and diagonal forms, which we use to explore the eigen-structure of the equation set. We present numerical simulations of: (i) resonant standing waves arising from a time-harmonic external axial load and (ii) travelling compression waves arising from a bi-material collinear impact. Our numerical solution of the resonant vibrations problem shows that energy is channelled into additional modes (super-harmonics of the forcing frequency) when the resonant waves become nonlinear, preventing unbounded amplitude growth. For
Fig. 6.6: Analytical and numerical solutions of velocity profiles at (a) $t = 0.15$ ms, (b) $t = 0.543$ ms, (c) $t = 0.9$ ms, (d) $t = 1.33$ ms, (e) $t = 1.65$ ms, (f) $t = 2.02$ ms, (g) $t = 2.5$ ms, and (h) $t = 2.92$ ms.
Fig. 6.7: Analytical and numerical solutions of stress profiles at (a) $t = 0.15$ ms, (b) $t = 0.543$ ms, (c) $t = 0.9$ ms, (d) $t = 1.33$ ms, (e) $t = 1.65$ ms, (f) $t = 2.02$ ms, (g) $t = 2.5$ ms, and (h) $t = 2.92$ ms.
the bi-material impact problem, the CESE method accurately captures the sharp propagating wavefronts, the different wave propagation speeds in each material, wave reflection at the traction-free boundaries, and wave reflection/transmission at the interface.
CHAPTER 7  
FIRST-ORDER HYPERBOLIC PARTIAL  
DIFFERENTIAL EQUATIONS FOR NONLINEAR  
ELASTICITY  

This chapter is directly taken from the published paper ”Numerical solution by the CESE method of a first-order hyperbolic form of the equations of dynamic nonlinear elasticity.” [146]

7.1 Introduction

The present chapter reports the application of the space-time CESE method to the numerical solution of nonlinear waves in elastic solids. Our model is a synergy of an open theoretical framework for nonlinear elastic waves based on first principles and a highly accurate numerical method for solving a class of technically important wave propagation problems. Unlike traditional semi-inverse approaches in the solid mechanics literature ([143]), our direct approach does not a priori postulate a particular motion, deformation, and /or solution structure. Furthermore, our flexible theoretical framework can be straightforwardly generalized to incorporate 2D or 3D effects, different constitutive models, complex geometries, additional physics, etc.

To demonstrate this novel approach, we focus on one-dimensional longitudinal waves in slender elastic rods. The mathematical model, based on the Eulerian form of
the conservation laws, consists of two coupled first-order nonlinear hyperbolic partial differential equations (PDEs). Solving these model equations for complex nonlinear waves with multiple wave speeds and frequencies with limited numerical diffusion and dispersion is challenging. To this end, we employ the CESE method [28], [23], [24], [27], [31], [32], [25], [29], [26], [139], [136], [111], [75], [67], [66], [150], [151], [105], [107], [70], [13], [123]. In what follows, we present a set of first-principles-based model equations for one-dimensional nonlinear waves in slender elastic rods. We obtain conservative and diagonal forms of these equations, both of which are necessary for the implementation of our numerical method. Numerical simulations of two benchmark elastic wave problems are demonstrated using the CESE method: one involving linear propagating extensional waves, the other involving nonlinear resonant standing waves. For the extensional wave problem, the CESE method accurately captures the sharp propagating wave front without excessive numerical diffusion or spurious oscillations, and predicts correct reflection characteristics at the boundaries. For the resonant standing wave problem, the CESE method captures the linear-to-nonlinear evolution of the resonant vibrations and the distribution of the wave energy among multiple modes in the nonlinear regime.

The present chapter is organized as follows: Sec. 7.2 presents the nonlinear model equations in conservative, non-conservative, and diagonal forms. We also derive the Riemann invariants of the hyperbolic system and linearize the model equations. Section 7.3 presents numerical simulations of two benchmark elastic wave problems. We offer our concluding remarks at the end of this chapter.
7.2 Nonlinear Model Equations

In this section, we recall from Refs.([9],[45] [78]) a nonlinear model for one-dimensional longitudinal waves in slender homogeneous isotropic elastic rods of constant cross-sectional area. The one-dimensional time-dependent field equations are:

conservation of mass:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0, \tag{7.1}
\]

and balance of linear momentum:

\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 - \sigma)}{\partial x} = 0 \tag{7.2}
\]

where \(\rho(x,t)\) is the density, \(v(x,t)\) is the axial velocity component, and \(\sigma(x,t)\) is the axial normal stress component, which are all functions of the axial coordinate \(x\) (with the origin at the left boundary of the rod) and time \(t\). The elastic constitutive assumption is that \(\sigma = \sigma(\rho)\). Expanding this form in the constant coefficient power series \(\rho = a_0 + a_1 \epsilon_v + a_2 \epsilon_v^2 + \cdots\), where \(\epsilon_v = \rho_0/\rho - 1\) is the volumetric strain, and selecting \(a_0 = a_2 = a_3 = \cdots = 0, a_1 = E\) yields the constitutive equation [78]

\[
\sigma = E \left( \frac{\rho_0}{\rho} - 1 \right). \tag{7.3}
\]

Young’s modulus \(E\) and reference density \(\rho_0\) are specified material properties. Equations (7.2) and (7.3) are combined to obtain

\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial \left[ \rho v^2 - E \left( \frac{\rho_0}{\rho} - 1 \right) \right]}{\partial x} = 0 \tag{7.4}
\]
Equations (7.1) and (7.4), a pair of one-dimensional coupled first-order nonlinear PDEs, govern the longitudinal wave motion.

### 7.2.1 Conservative Form

Equations (7.1) and (7.4) are cast into a vector form

\[
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0 \tag{7.5}
\]

where the vector of conserved quantities \( \mathbf{U} \) and the vector of fluxes \( \mathbf{F} \) are

\[
\mathbf{U} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \rho v \\ \rho v^2 - E \left( \frac{\rho_0}{\rho} - 1 \right) \end{pmatrix}. \tag{7.6}
\]

Equation (7.5) is referred to as the conservative form. The governing equations (Eqs. (7.1) and (7.4)) can be manipulated to produce several other vector forms.

### 7.2.2 Diagonal Form

To diagonalize Eq. (7.5), we first apply the chain rule to the flux term

\[
\frac{\partial \mathbf{F}}{\partial x} = \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}} \frac{\partial \mathbf{U}}{\partial x} = \mathbf{B}(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial x} \tag{7.7}
\]

and express Eq. (7.5) as a quasi-linear system

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{B}(\mathbf{U}) \frac{\partial \mathbf{U}}{\partial x} = 0 \tag{7.8}
\]
where the Jacobian of the flux vector is

\[ B(U) = \frac{\partial F}{\partial U} = \begin{bmatrix} 0 & 1 \\ \frac{E\rho_0}{\rho^2} - v^2 & 2v \end{bmatrix}. \]  

(7.9)

We multiply Eq. (7.8) by the left eigenvector matrix of \( B \):

\[ \mathbf{N}^{-1} = \begin{bmatrix} \frac{1}{2} - \frac{v}{2\sqrt{E\rho_0/\rho^2}} & \frac{1}{2\sqrt{E\rho_0/\rho^2}} \\ \frac{1}{2} + \frac{v}{2\sqrt{E\rho_0/\rho^2}} & -\frac{1}{2\sqrt{E\rho_0/\rho^2}} \end{bmatrix}, \]

(7.10)

to produce

\[ \mathbf{N}^{-1}\frac{\partial \mathbf{U}}{\partial t} + \mathbf{N}^{-1}\mathbf{B}\mathbf{N}^{-1}\frac{\partial \mathbf{U}}{\partial x} = 0 \]

(7.11)

where \( \mathbf{NN}^{-1} = \mathbf{I} \) and the right eigenvector matrix of \( B \) is

\[ \mathbf{N} = \begin{bmatrix} 1 & 1 \\ v + \sqrt{E\rho_0/\rho^2} & v - \sqrt{E\rho_0/\rho^2} \end{bmatrix}. \]  

(7.12)

Using \( \partial \tilde{\mathbf{U}} = \mathbf{N}^{-1}\partial \mathbf{U} \), Eq. (7.11) becomes the diagonal form:

\[ \frac{\partial \tilde{\mathbf{U}}}{\partial t} + \tilde{\mathbf{A}} \frac{\partial \tilde{\mathbf{U}}}{\partial x} = 0 \]  

(7.13)

with \( \tilde{\mathbf{A}} \) the matrix of the eigenvalues of \( B \)

\[ \tilde{\mathbf{A}} = \mathbf{N}^{-1}\mathbf{B}\mathbf{N} = \begin{bmatrix} \tilde{\lambda}_1 & 0 \\ 0 & \tilde{\lambda}_2 \end{bmatrix} = \begin{bmatrix} v + \tilde{c} & 0 \\ 0 & v - \tilde{c} \end{bmatrix}. \]

(7.14)
where \( \tilde{c} = \sqrt{\frac{E\rho_0}{\rho^2}} \) is the local longitudinal propagation speed of sound. Equations (7.1) and (7.4) are hyperbolic since the eigenvalues \( \tilde{\lambda}_1 \) and \( \tilde{\lambda}_2 \) of the Jacobian matrix \( \mathbf{B} \) are real and distinct.

### 7.2.3 Nonconservative Form and Riemann Invariants

Equations (7.1) and (7.4) have much more structure than what is necessary to employ our numerical method. In this section, we present the nonconservative form of the model equations, which is manipulated to explicitly deduce the Riemann invariants of the hyperbolic system. Equation (7.1) and (7.4) are expanded as

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + E\rho_0 \rho \frac{\partial \rho}{\partial x} + v \frac{\partial v}{\partial x} &= 0
\end{align*}
\]  

and expressed as a quasi-linear system

\[
\frac{\partial \hat{\mathbf{U}}}{\partial t} + \hat{\mathbf{B}}(\hat{\mathbf{U}}) \frac{\partial \hat{\mathbf{U}}}{\partial x} = 0
\]

where

\[
\hat{\mathbf{U}} = \begin{bmatrix} \rho \\ v \end{bmatrix}, \quad \hat{\mathbf{B}}(\hat{\mathbf{U}}) = \begin{bmatrix} v & \rho \\ \frac{E\rho_0}{\rho^3} & v \end{bmatrix}
\]

Equation (7.16) is referred to as the nonconservative form. In diagonal form, we have

\[
\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \tilde{\Lambda} \frac{\partial \tilde{\mathbf{U}}}{\partial x} = 0
\]
where \( \partial \tilde{U} = \hat{N}^{-1} \partial \hat{U} \),

\[
\tilde{\Lambda} = \hat{N}^{-1} \hat{B} \hat{N} = \begin{bmatrix}
\tilde{\lambda}_1 & 0 \\
0 & \tilde{\lambda}_2
\end{bmatrix} = \begin{bmatrix}
v + \tilde{c} & 0 \\
0 & v - \tilde{c}
\end{bmatrix}
\]  

(7.19)

and

\[
\hat{N}^{-1} = \begin{bmatrix}
-\frac{1}{2\rho^2} & -\frac{1}{2\sqrt{E\rho_0}} \\
\frac{1}{2\rho^2} & \frac{1}{2\sqrt{E\rho_0}}
\end{bmatrix}, \hat{N} = \begin{bmatrix}
-\rho^2 & -\rho^2 \\
-\sqrt{E\rho_0} & \sqrt{E\rho_0}
\end{bmatrix}
\]  

(7.20)

Note that \( \hat{N}^{-1} \) and \( \hat{N} \) are the left and right eigenvector matrices of \( \hat{B}(\hat{U}) \). We now have identical diagonal representations Eqs. (7.13) and (7.18)

\[
\frac{\partial \tilde{u}_1}{\partial t} + \tilde{\lambda}_1 \frac{\partial \tilde{u}_1}{\partial x} = 0,
\frac{\partial \tilde{u}_2}{\partial t} + \tilde{\lambda}_2 \frac{\partial \tilde{u}_2}{\partial x} = 0.
\]  

(7.21)

We use the method of characteristics to deduce that \( \tilde{u}_1 = \) constant along the right-running characteristics with slope \( dx/dt = \tilde{\lambda}_1 = v + \tilde{c} \), while \( \tilde{u}_2 = \) constant along the left-running characteristics with slope \( dx/dt = \tilde{\lambda}_2 = v - \tilde{c} \), where \( \tilde{c} = \sqrt{E\rho_0/\rho^2} \).

Deriving the Riemann invariants \( \tilde{u}_1 \) and \( \tilde{u}_2 \) from the relationship \( \partial \tilde{U} = \hat{N}^{-1} \partial \hat{U} \) is generally cumbersome, while the relationship \( \partial \tilde{U} = \hat{N}^{-1} \partial \hat{U} \) is usually more straightforward to work with. Thus, the left eigenvector matrix \( \hat{N}^{-1} \) is carefully scaled (see Eq.(7.20)) so that it can be absorbed into the differential in \( \partial \tilde{U} = \hat{N}^{-1} \partial \hat{U} \). Upon integrating this expression in its component form, we obtain

\[
\tilde{u}_1(x,t) = \frac{1}{2\rho(x,t)} - \frac{v(x,t)}{2\sqrt{E\rho_0}} = \text{constant along } \frac{dx}{dt} = \tilde{\lambda}_1
\]  

(7.22)

\[
\tilde{u}_2(x,t) = \frac{1}{2\rho(x,t)} + \frac{v(x,t)}{2\sqrt{E\rho_0}} = \text{constant along } \frac{dx}{dt} = \tilde{\lambda}_2
\]  

(7.23)
where the arbitrary constant of integration is taken to be zero without loss of generality.

7.2.4 Linearized Model Equations

In this section, we linearize Eqs. (7.1) and (7.4) and compare the result with the classical second-order linear wave equation. We expand \( \rho(x,t) \) and \( v(x,t) \) in \( \epsilon \) about the rest state \( \rho^0, v^0 \)

\[
\rho(x,t) = \rho^0 + \epsilon \rho^1(x,t) + \epsilon^2 \rho^2(x,t) + \cdots
\]

\[
v(x,t) = \rho^0 + \epsilon v^1(x,t) + \epsilon^2 v^2(x,t) + \cdots
\]

(7.24)

where \( \rho^i(x,t) \) and \( v^i(x,t) (i = 1, 2, \cdots) \) are the \( i \)th corrections of the density and velocity, respectively, and \( \epsilon \) is a small positive dimensionless quantity, taken to be the slenderness parameter for the rod

\[
\epsilon = \frac{a}{L} \ll 1
\]

(7.25)

where \( a \) is the rod radius and \( L \) is the rod length. Inserting the expansions, e.g., Eq. (7.24) into Eqs (7.1) and (7.4), and selecting \( \rho^0 = \rho_0, v^0 = 0 \) as the rest state, the following linear equations are obtained from the order-\( \epsilon \) problem:

\[
\frac{\partial \rho^1}{\partial t} + \rho_0 \frac{\partial v^1}{\partial x} = 0
\]

\[
\frac{\partial v^1}{\partial t} + \frac{E}{\rho_0} \frac{\partial \rho^1}{\partial x} = 0
\]

(7.26)
These equations are combined to recover second-order linear wave equations in the velocity and density

\[ \frac{\partial^2 v^1}{\partial t^2} = c^2 \frac{\partial^2 v^1}{\partial x^2} \]

\[ \frac{\partial^2 \rho^1}{\partial t^2} = c^2 \frac{\partial^2 \rho^1}{\partial x^2} \]  

(7.27)

where \( c = \sqrt{E/\rho_0} \) is the constant longitudinal wave propagation speed. If we integrate Eq. (7.27) with respect to time and set the arbitrary function of \( x \) which arises to be zero, we recover

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]  

(7.28)

with \( u(x, t) \) the axial displacement component. Equation (7.28) is identically the second-order linear wave equation, which governs the longitudinal motion of slender elastic rods in the regime of small deformations.

### 7.3 Results and Discussion

In general, problems in dynamic nonlinear elasticity involve waves that excite a wide spectrum of wavenumbers and frequencies, with magnitudes that range from small-amplitude linear waves to strong shocks. As time evolves, the small-amplitude linear waves undergo complex interactions with the large-amplitude nonlinear waves. Thus, a high-fidelity model for time-dependent nonlinear elasticity must accurately capture both linear and nonlinear waves, and correctly predict their interactions.

For this reason, it is essential to ensure that our model renders accurate numerical solutions not only in the nonlinear regime, but also in the linear regime. In the
Fig. 7.1: Schematic illustrating our verification process for numerical solutions in the linear regime. The so-called nonlinear problem consists of the nonlinear governing equations along with appropriate initial and boundary conditions. The linearized problem consists of the linearized governing equations together with the same initial and boundary conditions.

Benchmark elastic wave problems to follow, we accomplish the latter using the verification process illustrated in Fig (7.1). Note that the solution in the linear regime is obtained (i) analytically from the linearized problem and (ii) numerically as an asymptotic limit of the nonlinear problem when the imposed deformations are small and the effects of nonlinearity are not significant.

### 7.3.1 Linear Propagating Extensional Waves

A slender elastic rod of length $L$ and constant cross-sectional area $A$ rigidly translates rightward with uniform velocity $V_0$ when $t < 0$. At $t = 0$, the left end of the rod is abruptly stopped and held fixed thereafter. Refer to Table 7.1 for our particular choices of material properties (e.g., Young’s modulus and reference density), geometric parameters (e.g., rod length), and initial conditions (e.g., initial velocity). We impose fixed and stress-free boundary conditions on the left ($x = 0$) and right
Table 7.1: Material properties, geometric parameters, and initial conditions for extensional waves

<table>
<thead>
<tr>
<th>E</th>
<th>Young’s modulus</th>
<th>70 GPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_0$</td>
<td>Reference density</td>
<td>2700 kg/m$^2$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c = \sqrt{E/\rho_0}$</td>
<td>5092 m/s</td>
</tr>
<tr>
<td>$L$</td>
<td>Length</td>
<td>6 m</td>
</tr>
<tr>
<td>$V_0$</td>
<td>Initial velocity</td>
<td>10 m/s</td>
</tr>
</tbody>
</table>

\[(x = L)\) ends of the rod, respectively

\[
u(0, t) = 0, \quad \sigma(L, t) = E \frac{\partial u(x, t)}{\partial x}|_{x=L} = 0, \quad t > 0 \quad (7.29)\]

along with initial conditions on the displacement and velocity

\[
u(x, 0) = 0
\]

\[
v(x, 0) = \left. \frac{\partial u(x, t)}{\partial t} \right|_{t=0} = V_0, \quad 0 < x < L \quad (7.30)
\]

For the material properties and initial conditions considered in this simulation (see Table 7.1), only small deformations arise. Hence, the effect of the nonlinear terms in the governing equations (Eqs. (7.1) and (7.4)) is small. In this problem, we thus demonstrate that the numerical solution of the small amplitude nonlinear problem coincides with the analytical solution of the linearized problem.

**Analytical Solution of the Linearized Problem**

Based on the method of separation of variables, we obtain a general solution to the linearized problem given by Eqs. (7.28), (7.29) and (7.30). A detailed derivation is provided as following.
The solution of Eq. (7.28) is multiplicatively decomposed via separation of variables

\[ u(x,t) = X(x)T(t) \] (7.31)

Substituting Eq.(7.31) into Eq.(7.28) yields

\[ \frac{X''(x)}{X(x)} = \frac{1}{c^2} \frac{T''(t)}{T(t)} \] (7.32)

The left-hand side of Eq.(7.32) is a function of \( x \) only, and the right-hand side is a function of \( t \) only. Both sides, then, are equal to a constant, e.g., \(-\xi\)

\[ \frac{X''(x)}{X(x)} = \frac{1}{c^2} \frac{T''(t)}{T(t)} = -\xi \] (7.33)

Equation (7.33) is recast as two ordinary differential equation (ODEs) in \( x \) and \( t \), respectively

\[ X''(x) + \xi X(x) = 0 \] (7.34)

\[ T''(x) + c^2 \xi T(t) = 0 \] (7.35)

and the boundary condition (7.29) is re-expressed in terms of the separable solution (7.31)

\[ X(0) = X'(L) = 0 \] (7.36)

The only non-trivial solution of Eq. (7.34) that can satisfy the boundary condition
(7.36) is

\[ X(x) = A \sin(\alpha x) + B \cos(\alpha x) \]  

(7.37)

where \( \xi = \alpha^2 > 0 \), and \( A \) and \( B \) are arbitrary constants. Application of Eq.(7.36) to Eq.(7.37) yields

\[ B = 0 \]  

(7.38)

and

\[ \alpha_m = \frac{(2m - 1)\pi}{2L}, m = 1, 2, 3, \ldots, \infty \]  

(7.39)

where \( m \) is a positive integer index and \( \alpha_m \) is the square root of the eigenvalue \( \xi_m \) corresponding to the \( m \)th vibrational mode. Thus, the eigenfunctions (mode shapes) are

\[ X_m(x) = A_m \sin(\alpha_m x) \]  

(7.40)

Similarly, the solution of Eq.(7.35) is

\[ T_m(t) = C_m \sin(\omega_m t) + D_m \cos(\omega_m t) \]  

(7.41)

where \( C_m \) and \( D_m \) are arbitrary constants, and \( \omega_m = \alpha_m c \) is the angular frequency of the \( m \)th vibrational mode. The general solution of Eq.(7.28), which satisfies the
boundary condition (7.29), is the superposition of all modes (particular solutions)

\[
    u(x, t) = \sum_{m=1}^{\infty} u_m(x, t) = \sum_{m=1}^{\infty} X_m(x)T_m(t)
\]

\[
    \sum_{m=1}^{\infty} \sin(\alpha_m x)[C_m \sin(\omega_m t) + D_m \cos(\omega_m t)]
\]

where \( C_m = A_m C_m \) and \( D_m = A_m D_m \). The initial condition on displacement Eq.(7.30) leads to \( D_m = 0 \), while the initial condition on velocity Eq.(7.30) and orthogonality of the sine function determine \( C_m \)

\[
    C_m = \frac{2V_0}{\omega_m L} \int_0^L \sin(\alpha_m x) dx = \frac{2V_0}{L \omega_m \alpha_m} = \frac{8V_0 L}{\pi^2 (2m - 1)^2 c}
\]

Upon substituting Eq.(7.43) into Eq.(7.42), we obtain the general solutions for the displacement, stress, and velocity

\[
    u(x, t) = \frac{8V_0 L}{\pi^2 c} \sum_{m=1}^{\infty} \frac{\sin(\alpha_m x) \sin(\omega_m t)}{(2m - 1)^2}
\]

\[
    \sigma(x, t) = E \frac{\partial u}{\partial x} = \frac{8V_0 L}{\pi^2 c} \sum_{m=1}^{\infty} \frac{\cos(\alpha_m x) \sin(\omega_m t)}{2m - 1}
\]

\[
    v(x, t) = \frac{\partial u}{\partial t} = \frac{4V_0 L}{\pi} \sum_{m=1}^{\infty} \frac{\sin(\alpha_m x) \cos(\omega_m t)}{2m - 1}
\]

**Numerical Solution Of The Non-linear Problem**

We numerically solve the non-linear problem given by Eqs.(7.1),(7.4),(7.29), and (7.30) using the CESE method. To start the numerical calculation, we impose the initial conditions

\[
    \rho = \rho_0, \quad v = V_0, \quad \rho_x = (\rho v)_x = 0
\]
Fig. 7.2: Numerical implementation of the boundary conditions in problem 7.3.1. For the fixed boundary on the left end of the rod, ghost cells are added to the left of the computational domain.

at all mesh nodes at $t = 0$ except for the first node on the left-hand side, where the rod is held fixed and $v = 0$.

The fixed boundary condition on the left end of the rod is achieved by using the ghost cells to the first node of the space-time mesh, as shown in Fig.(7.2). The unknowns at the ghost cell at time $t = n\Delta t$, where $n = 1, 2, 3, \ldots$, are the mirror images of the solutions and their spatial derivatives at the first interior node at the same time level (hereafter referred to as node 1)

$$
\rho_{gc} = \rho_1, \quad (\rho v)_{gc} = -(\rho v)_1
$$

$$
(\rho x)_{gc} = -(\rho x)_1, \quad [(\rho v)_x]_{gc} = [(\rho v)_x]_1
$$

(7.48)

where the subscripts $gc$ and 1 denote quantities corresponding to the ghost cell and node 1, respectively. This treatment imposes a fixed boundary at the midpoint between the ghost cell and node 1 at time $t = n\Delta t$, where $n = 1, 2, 3, \ldots$ For time $t = n\Delta t$, with $n = 1/2, 3/2, 5/2, \ldots$, the first (leftmost) mesh mode is located directly on the fixed boundary. With the complete solutions at the ghost cell $\Delta t/2$ earlier,
the node on the fixed boundary becomes an interior node. As such, its solutions and spatial derivatives are calculated based on the procedure discussed in Sec 6.3.1.

On the right-hand side of the rod, we numerically implement the free-end boundary condition. As shown in Fig.7.3, for time $t = n\Delta t$, where $n = 1/2, 3/2, 5/2, \cdots$, the last (rightmost) mesh node in the computational domain is an interior node, and no special treatment is needed. For time $t = n\Delta t (n = 0, 1, 2, \cdots)$ we impose the free-end condition on the last mesh node $N$ by letting

$$
\rho_N^n = \rho_0, (\rho v)_N^n = -\rho_0 v_{N-1/2}^{n-1/2} \tag{7.49}
$$

$$
(\rho_x)_N^n = (\rho_x)_{N-1/2}^{n-1/2}, [(\rho v)_x]_N^n = [(\rho v)_x]_{N-1/2}^{n-1/2} \tag{7.50}
$$

where the density at node $N$ is set to be the reference density, and the other unknowns $v$, $\rho_x$, and $(\rho v)_x$ are taken from the immediate interior node $\Delta t/2$ earlier. Additional details about boundary condition treatments in the setting of the CESE method can be found in Refs.[30], [33].

The computational domain (6 m in length; see Table 7.1) is discretized into 100
Comparison of the Numerical and Analytical Solutions

Fig. (7.4)(a) - (c) show the numerical and analytical stress profiles (amplitude versus position) at three different snapshots of time: $t = 0.171$ ms, $0.891$ ms, and $1.791$ ms. The stress wave progresses rightward, as shown in Figs (7.4)(a) and (7.4)(b), and induces tensile stresses behind its wavefront. In Fig. (7.4)(c), the stress wave has reflected as a compression wave from the traction-free right end and is now propagating leftward toward the fixed end, rendering the rod stress free behind its wavefront. Figures (7.4)(d) and (7.4)(f) illustrate the three corresponding numerical
Fig. 7.5: Time histories of the analytical (solid line) and numerical (x symbol) solutions of stress and velocity at $x = 2$ m for problem 7.3.1 respectively.

and analytical velocity profiles, each in phase with its corresponding numerical and analytical velocity profiles, each in phase with its corresponding stress profile. In Figs. (7.4)(d) - (7.4)(e), the velocity wave travels rightward, halting the material particles behind the wave-front. Fig. (7.4)(f) shows that upon reflection from the traction-free right end, the velocity wave propagates leftward the fixed end, inducing a leftward velocity upon the previously stationary particles behind its wave front as it progresses. We note that each individual mode in the analytical solution (see Eqs.(7.46)) of the initial-value/boundary-value problem specified by Eqs.(7.28)(7.29), and(7.30) represent a standing wave, i.e., its shape does not travel. However, an infinite series of these standing waves of mode $m = 1, 2, \cdots, \infty$ is a travelling-wave solution, as demonstrated in Fig.(7.4).

Fig. (7.5) shows the periodic stress and velocity profiles at $x = 2$ m. Within 5 ms, the waves have travelled a full cycle. Fig. (7.5) demonstrates that our stress and velocity snapshots at $t = 0.171$ ms, 0.891 ms, and 1.791 ms, as shown in Fig.7.4, all occur within the first cycle prior to $t = 5$ ms, and that the stress and velocity waves are time harmonic. In Figs.7.4 and 7.5, we find good agreement in the waveforms,
Table 7.2: Material properties, geometric parameters, and initial conditions for resonant standing waves

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>Young’s modulus</td>
<td>$70GP_n$</td>
</tr>
<tr>
<td>$\rho_0$</td>
<td>Reference density</td>
<td>$2700 \text{kg/m}^2$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c = \sqrt{E/\rho_0}$</td>
<td>$5092 \text{m/s}$</td>
</tr>
<tr>
<td>$L$</td>
<td>Length</td>
<td>$6 \text{m}$</td>
</tr>
<tr>
<td>$A$</td>
<td>Cross-sectional area</td>
<td>$180 \text{cm}^2$</td>
</tr>
<tr>
<td>$F$</td>
<td>Driving force amplitude</td>
<td>$100 \text{kN}$</td>
</tr>
<tr>
<td>$r$</td>
<td>Resonant mode</td>
<td>$10$</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Driving frequency</td>
<td>$7598 \text{1/s}$</td>
</tr>
</tbody>
</table>

wave amplitudes, and wave propagation speeds between the numerical solution of the small amplitude nonlinear problem and the analytical solution of the small linearized problem. In contrast to the presence of the Gibbs phenomenon at the wavefronts of the analytical stress and velocity profiles, the CESE method effectively captures the sharp propagating wavefronts without spurious oscillations.

7.3.2 Nonlinear Resonant Standing Waves

For $t > 0$, a slender elastic rod of length $L$ and constant cross-sectional area $A$ is fixed on its left end and driven by a time-harmonic external axial force

$$f(t) = F \sin(\Omega t)$$

on its right end. In Eq.(7.51), $F$ is the constant amplitude of the applied force and $\Omega$ is the driving frequency. To induce resonant vibrations, the imposed frequency $\Omega$ is taken to be one of the rod’s natural frequencies. Refer to Table 7.2 for our particular choices of material properties, geometric parameters, and boundary conditions.

We impose fixed and forced boundary conditions on the left ($x = 0$) and right ($x = 185$)
ends of the rod, respectively

\[ u(0, t) = 0, \quad \sigma(L, t) = E\frac{\partial u(x, t)}{\partial x}\bigg|_{x=L} = \frac{F\sin(\Omega t)}{A}, \quad t > 0 \]  

(7.52)

and initial conditions on the displacement and velocity

\[ u(x, 0) = 0 \]
\[ v(x, 0) = \frac{\partial u(x, t)}{\partial t}\bigg|_{t=0} = 0, \quad 0 < x < L \]  

(7.53)

**Analytical Solution of the Linearized Problem**

The solution of the linearized problem given by Eqs.(7.28), (7.52), and (7.53) obtained using an eigenfunction expansion technique and L’Hospital’s rule. A detail derivation is provided as following:

The eigenfunctions of the corresponding linearized problem with homogeneous boundary conditions are

\[ X_m(x) = \sin(\alpha_m x), \quad \alpha_m = \frac{(2m - 1)\pi}{2L}, m = 1, 2, 3, \ldots, \infty \]  

(7.54)

We expand the solution \( u(x, t) \) of the linearized problem with inhomogeneous boundary conditions Eqs. (7.28), (7.52), and (7.53) using the eigenfunctions (Eq.(7.54)) over \( 0 < x < L \)

\[ u(x, t) = \sum_{m=1}^{\infty} u_m(x, t) = \sum_{m=1}^{\infty} T_m(t)X_m(x) = \sum_{m=1}^{\infty} T_m(t)\sin(\alpha_m x) \]  

(7.55)

Multiplying Eq.(7.55) by \( X_n(x) = \sin(\alpha_n x) \), where \( n = 1, 2, 3, \ldots, \infty \), integrating
over $0 \leq x \leq L$, and invoking orthogonality of the sine function, we obtain

$$T_m(t) = \frac{2}{L} \int_0^L u(x, t) \sin(\alpha_m x) dx$$  \hspace{1cm} (7.56)$$

We also expand the continuous partial derivatives $\partial^2 u / \partial t^2$ and $\partial^2 u / \partial x^2$ in the Fourier sine series over $0 < x < L$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{m=1}^{\infty} w_m(t) \sin(\alpha_m x)$$  \hspace{1cm} (7.57)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{m=1}^{\infty} y_m(t) \sin(\alpha_m x)$$  \hspace{1cm} (7.58)$$

where the time-dependent coefficients are given by

$$w_m(t) = \frac{2}{L} \int_0^L \frac{\partial^2 u}{\partial t^2} \sin(\alpha_m x) dx = \frac{d^2 T_m(t)}{dt^2}$$  \hspace{1cm} (7.59)$$

$$y_m(t) = \frac{2}{L} \int_0^L \frac{\partial^2 u}{\partial x^2} \sin(\alpha_m x) dx$$  \hspace{1cm} (7.60)$$

We note that fixed bounds and a continuous integrand allow us to commute differentiation and integration in Eqs. (7.59) and (7.60). Using Green’s second identity, we rewrite Eq. (7.60) as

$$y_m(t) = -\frac{2}{L} \int_0^L \alpha_m^2 u(x, t) \sin(\alpha_m x) dx - \frac{2}{L} \alpha_m u(x, t) \cos(\alpha_m x) \bigg|_0^L$$

$$+ \frac{2}{L} \sin(\alpha_m x) \frac{\partial u(x, t)}{\partial x} \bigg|_0^L$$  \hspace{1cm} (7.61)$$
Invoking the boundary condition (7.52) and simplifying, we obtain

\[ y_m(t) = -\alpha_m^2 T_m(t) + \frac{2}{L}(-1)^{m-1} \frac{F \sin(\Omega t)}{AE} \]  \hspace{1cm} (7.62)

Aided by Eqs.(7.59) and (7.60), the governing PDE Eq.(7.28) demands

\[ y_m(t) - \frac{1}{c^2}w_m(t) = \frac{2}{L} \int_0^L \left( \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \right) \sin(\alpha_m x) dx = 0 \]  \hspace{1cm} (7.63)

Upon substituting Eqs.(7.59) and (7.62) into Eq.(7.63), we obtain the following second-order, linear, inhomogeneous ordinary differential equation:

\[ \frac{d^2 T_m(t)}{dt^2} + \omega_m^2 T_m(t) = \frac{2}{L}(-1)^{m-1} \frac{F \sin(\Omega t)}{A\rho_0} \]  \hspace{1cm} (7.64)

where \( \omega_m = \alpha_m c \) is the angular frequency of the \( m \)th vibration mode. We additively decompose the solution of Eq.(7.11) into the solution of the homogeneous problem

\[ T_m^h(t) = A_m \cos(\omega_m t) + B_m \sin(\omega_m t) \]  \hspace{1cm} (7.65)

and a particular solution

\[ T_m^p(t) = \frac{2}{L}(-1)^{m-1} \frac{F}{A\rho_0} \frac{\sin(\Omega t)}{\omega_m^2 - \Omega^2} \]  \hspace{1cm} (7.66)

Thus

\[ T_m(t) = T_m^h(t) + T_m^p(t) = A_m \cos(\omega_m t) + B_m \sin(\omega_m t) + \frac{2}{L}(-1)^{m-1} \frac{F}{A\rho_0} \frac{\sin(\Omega t)}{\omega_m^2 - \Omega^2} \]  \hspace{1cm} (7.67)
We obtain the coefficients $A_m$ and $B_m$ by applying the homogeneous initial condition Eq.(7.53) directly to the time-dependent part of the general solution. The initial condition on displacement Eq.(7.53) leads to $A_m = 0$, while the initial condition on velocity Eq.(7.53) determines $B_m$

$$B_m = -\frac{2}{L}(-1)^{m-1} \frac{F}{A \rho_0} \frac{\Omega}{\omega_m^2 - \Omega^2}$$ \hspace{1cm} (7.68)

Therefore

$$T_m(t) = \frac{2}{L}(-1)^{m-1} \frac{F}{A \rho_0} \frac{1}{\omega_m^2 - \Omega^2} \left( \sin(\Omega t) - \frac{\Omega}{\omega_m} \sin(\omega_m t) \right)$$ \hspace{1cm} (7.69)

and

$$u(t) = \sum_{m=1}^{\infty} \frac{2F}{AL \rho_0} (-1)^{m-1} \sin(\alpha_m x) \frac{1}{\omega_m^2 - \Omega^2} \left( \sin(\Omega t) - \frac{\Omega}{\omega_m} \sin(\omega_m t) \right)$$ \hspace{1cm} (7.70)

To induce resonant vibrations in the rod, we let the driving frequency $\Omega$ approach one of the natural frequencies $\omega_m$, where $m = 1, 2, 3, \cdots, \infty$. We denote this resonant frequency as $\omega_r$, where $r$ is a positive integer. For the resonant case, the $r$th mode is singular, and we employ l’hospital’s rule to deduce the solutions for the axial displacement, axial velocity, and axial stress

$$u(x, t) = \sin(\alpha_r x) \frac{(-1)^{r} F t_\omega}{AL \rho_0} \frac{\omega_r}{\omega_r^2} \cos(\omega_r t) - \sin(\omega_r t)$$

$$+ \sum_{m=1}^{\infty} \frac{2F}{AL \rho_0} (-1)^{m-1} \sin(\alpha_m x) \frac{1}{\omega_m^2 - \Omega^2} \left( \sin(\Omega t) - \frac{\Omega}{\omega_m} \sin(\omega_m t) \right)$$ \hspace{1cm} (7.71)
\[
v(x, t) = \frac{\partial u}{\partial t} = \sin(\alpha, x)\frac{(-1)^{r+1}F}{AL\rho_0} t \sin(\omega, t) + \sum_{m=1}^{\infty} 2F\Omega AL\rho_0(-1)^{m-1} \sin(\alpha, x) \frac{1}{\omega_m^2 - \Omega^2} \left(\cos(\Omega t) - \cos(\omega_m t)\right)
\]

\[
\sigma(x, t) = E\frac{\partial u}{\partial x} = E\alpha_x \cos(\alpha, x)\frac{(-1)^{r}F t\omega_r \cos(\omega, t) - \sin(\omega, t)}{\omega_r^2} + \sum_{m=1}^{\infty} 2E\alpha_m AL\rho_0(-1)^{m-1} \sin(\alpha, x) \frac{F}{\omega_m^2 - \Omega^2} \left(\sin(\Omega t) - \frac{\Omega}{\omega_m} \sin(\omega_m t)\right)
\]

### Numerical Solution of the Non-linear Problem

We use the CESE method to numerically solve the non-linear problem given by Eq.(7.1),(7.4),(7.52), and (7.53). The treatment for the fixed boundary condition on the left end is identical to that discussed in section 7.3.1 (refer to Fig.(7.2)). For the right end, where the time-harmonic force Eq.(7.51) in conjunction with the constitutive relation Eq.(7.3), and the boundary conditions on \(\rho_x, \rho v\), and \((\rho v)_x\) are obtained from the immediate interior node \(\Delta t/2\) earlier. For the free-end boundary conditions, the treatment is similar to the procedure discussed in Sec 7.3.1. Please refer to Eq.(7.50) and Fig. (7.3). The computational domain (20 m in length; see Table 7.2) of the rod is discretized into 200 equally spaced elements of length \(\Delta x = 100\) mm. The time-marching step is chosen to be \(\Delta t = 10\mu s\) so that the CFL number (about 0.76 here) falls within the stability bound (CFL < 1) of the CESE method.

### Comparison of the Numerical and Analytical Solutions

Fig. (7.6) shows three snapshots of the stress and velocity profiles (amplitude versus position) at \(t = 2.94\) ms, 5.68 ms, and 10.05 ms. These three snapshots show the early stages of resonant wave development and are well within the linear regime. In
Fig. 7.6: Analytical (solid line) and numerical (x symbol) axial stress and axial velocity profiles for problems 7.3.2, respectively, at $t = 2.92$ ms ((a) and (d)), $t = 5.68$ ms ((b) and (e)), and $t = 10.05$ ms ((c) and (f)). The analytical solution is truncated at 2500 modes.

Fig. 7.7: Analytical (solid line) and numerical (x symbol) axial stress and axial velocity profiles for problems 7.3.2, respectively, at $t = 239.20$ ms ((a) and (d)), $t = 245.20$ ms ((b) and (e)), and $t = 249.95$ ms ((c) and (f)). The analytical solution is truncated at 2500 modes.
Figs.(7.6) (a) and (7.6)(d), the stress and velocity waves initiated on the right end of the rod by the external driving force travel leftward until they reach the fixed left end of the rod. Upon reflection, they interact with incoming waves generated at the force boundary, as shown in Figs.(7.6)(b) and (7.6)(e). Fig.(7.6)(c) and (7.6)(f) illustrate the saturation of the travelling wave profiles into stationary wave profiles with fixed nodes. In each case, excellent agreement in the waveforms, wave amplitudes, and wave speeds is observed between the numerical solution of the nonlinear problem and the analytical solution of the linearized problem.

Fig. (7.7) shows three snapshots of the stress and velocity profiles (amplitude versus position) in the nonlinear regime at \( t = 239.20 \text{ ms}, 245.20 \text{ ms}, \text{ and } 249.95 \text{ ms} \). The effects of nonlinearity are now significant, and the numerical solutions of the nonlinear problem diverge appreciably from the analytical solutions of the linearized problem. This discrepancy is further illustrated in Fig.(7.8), which shows the evolution of the axial stress and axial velocity at \( x = 3.1 \text{ m} \), which is an antinode in the resonant wave profile. Up to \( t = 50 \text{ ms} \) in the linear regime, the numerical solution of the nonlinear problem compare well with the analytical solution of the linearized problem. However, for \( t > 50 \text{ ms} \), nonlinear effects dominate, resulting in the distribution of vibrational energy among multiple nodes (cf. Figs.(7.9)(a) and (7.9)(b)) and a diminished amplitude growth rate for the solutions of the nonlinear problem.

### 7.4 Conclusions

In this dissertation, we employ the CESE method, which is an explicit space-time finite-volume scheme, to numerically simulate nonlinear waves in elastic solids. Previously, the CESE method has been employed to accurately solve a wide range of
Fig. 7.8: Time histories of analytical (filled circles) and numerical (symbol x) stress and velocity at $x = 3.1$ m for problems 7.3.2, respectively. Up to about $t = 50$ ms, the resonant waves are in the linear regime. For $t > 50$ ms, the effects of nonlinearity become significant, and the numerical and analytical solutions diverge.

Fig. 7.9: Power spectra generated from (a) the numerical obtained stress data for the non-linear problem 7.3.2 (left figure) and (b) the analytically obtained stress data for the linearized problem 7.3.2, both at $t = 249.95$ ms, well into the non-linear regime. Unlike the linear problem, where the wave energy is confined as a single mode, the wave energy is distributed among multiple superharmonics of the forcing frequency for the non-linear problem.
highly nonlinear multiphysics fluid dynamics and combustion problems. In this dissertation, we illustrate its success with time-dependent nonlinear elasticity problems. To demonstrate our novel approach, we investigate longitudinal waves in slender elastic rods. The mathematical model, which is based on first principles and formulated in the Eulerian frame, consists of two coupled first-order nonlinear hyperbolic PDEs. Three different forms of the model equations are derived: conservative, nonconservative, and diagonal. The CESE method is used to numerically solve the conservative form.

Unlike traditional semi-inverse approaches in the solid mechanics literature, our direct approach does not a priori postulate a particular motion, deformation, and/or solution structure. We demonstrate this by numerically simulating linear propagating extensional wave problem, our particular choices of the materials properties and initial conditions are such that only small deformations arise and the effects of nonlinearity are small. Good agreement is found between our numerical solution of the small amplitude nonlinear problem and our analytical solution of the linearized problem. In particular, the CESE method accurately captures the sharp propagating wavefront with minimal smearing and spurious oscillations, and correctly predicts wave reflection at the boundaries. For the resonance problem, our numerical solution illustrates the linear-to-nonlinear evolution of the resonant vibrations and the appearance of salient features of nonlinearity, including the emergence of superharmonics of the imposed forcing frequency and the distribution of wave energy among multiple modes.
8.1 Introduction

In this chapter, a theoretical and numerical framework is developed to model wave propagation in soft tissues. The material response is modelled by well-established viscoelasticity relations, which are formulated in the conventional integral forms. In order to employ a modern numerical method for time-accurate solutions of waves in a time domain, the constitutive relations are transformed into Partial Differential Equations (PDEs) by using parallelly connected Standard Linear Solid (SLS) models in conjunction with the memory variables. As such, the constitutive relation is an open framework and can be readily extended to model complex soft tissues. A generic viscoelasticity relation is factorized into time- and strain-dependent terms. The strain-dependent part is formulated for elastic-like behaviour of the medium. On the other hand, hereditary integration is employed to model time-dependent/relaxation effects. Typical models for soft tissues include Fung’s model [54] and many extended Fung’s models. Extensive experiments on various soft tissues have been conducted to determine the viscoelasticity relation formulated in this two-part functional form. Essentially, experimental data were used to determine the relaxation functions in the constitutive relation. However, such constitutive models are formulated in integrals,
which are cumbersome to be coupled with the equation of motion for numerical solutions. Modern numerical methods for solving wave equations were designed to solve coupled, first-order, hyperbolic PDEs. To circumvent this difficulty, we recast the constitutive relations into PDEs by the following two steps:

(i) The constitutive relations are reconstructed by using parallelly connected SLS models. By adding and tuning SLS modules, the composite SLS model can be readily adapted to various soft tissues.

(ii) The composite SLS model in an integral form is transformed into PDEs by using the method of internal variables [20, 22, 113].

The transformed constitutive relations in the form of PDEs can be directly coupled with the equation of motion. As will be shown, the complete governing equations are a set of fully coupled, first-order, hyperbolic PDEs with source terms. The eigenvalues of the Jacobian matrices of the PDEs are real and they represent wave speeds. Thus, the governing equations can be readily solved by using a modern numerical methods for time-accurate solutions.

To solve the governing equations, we employ the space-time Conservation Element and Solution Element (CESE) method [24]. In the past, similar attempts have been made in applying modern numerical methods to solve first-order PDEs for dynamics in soft tissues. For examples, Banks et al. [5] modeled wave propagation originated from coronary stenoses. The wave magnitudes are significant and the wave motions are modeled by non-linear, first-order PDEs. Ramakrishnan et al. [112] studied sound propagation in a human thoracic cavity. Bastard et al.[7] used the Maxwell model in axis-symmetric coordinates to simulate waves in a heterogeneous soft tissue. In their model, stress is divided into elastic and viscous parts. The elastic stress is modeled by Hooke’s law and the viscous stress is modeled by the Voigt model. They compared the numerical solutions with the analytical solutions.
The rest of this chapter is organized as follows. Section 8.2 illustrates the constitutive equations for viscoelastic media. General theories of viscoelasticity is briefly reviewed. The viscoelasticity relations are reconstructed by composite SLS models. Next, by using the method of internal variables [20, 22, 113], we derive the constitutive relations in PDEs. Section 8.3 presents the complete governing equations in a vector-matrix form with analyses about eigenvalues of the Jacobian matrices. Section 8.4 illustrates Fung’s model and the model by Iatridis et al. [65]. The collocation method is used to determine the parameters in the internal variable equations. Section 8.5 shows the numerical results. Three cases are reported. For validation, we simulate an impact wave in an Maxwellian medium. The numerical results compare well with the analytical solution. We then consider an impact wave in rabbit mesentery, modelled by Fung’s model. Eight internal variables are used to reconstruct Fung’s model. Next, we consider wave in a cow subcalcaneal, modeled by Iatridis’ model. In this case, four internal variables are used. We offer conclusions in Section 8.6, followed by the list of cited references. For completeness, the internal variable equations are derived in an appendix.

8.2 Viscoelasticity

Stresses in a viscoelastic medium are functions of the material deformation at the present time as well as the history of the deformation. The most general form of constitutive relations for viscoelastic media can be expressed by the Weierstrass approximation theorem [59], in which Pipkin and Rogers’ integral series [109] are used:

\[ S(t) = \sum_{n=1}^{\infty} P_n(t), \]
where $S$ is the Piola-Kirchhoff stress and \[ P_n(t) = \frac{1}{n!} \int_{\mathbf{E}(-\infty)}^{\mathbf{E}(t)} \cdots \int_{\mathbf{E}(-\infty)}^{\mathbf{E}(t)} \frac{\partial^n \mathbf{R}_n[\mathbf{E}(\tau_1), t - \tau_1; \cdots; \mathbf{E}(\tau_n), t - \tau_n]}{\partial \mathbf{E}(\tau_1) \cdots \partial \mathbf{E}(\tau_n)} \, \mathbf{E}(\tau_1) \cdots \mathbf{E}(\tau_n) \, dE(\tau_1) \cdots dE(\tau_n). \]

In the above equation, $\mathbf{E}$ is the Green-St. Venant strain tensor, $\mathbf{R}_n$ is the $n$th relaxation function, and $\tau_n$ is the $n$th discrete time between 0 and $t$. The Pipkin-Rogers model is valid for non-aging materials.

To proceed, we assume the material is simple in the sense that a single relaxation function $\mathbf{R}_1$ is adequate to model the material response. As such, the Pipkin-Rogers model becomes

\[ S(t) = P_1(t) = \int_{\mathbf{E}(-\infty)}^{\mathbf{E}(t)} \frac{\partial \mathbf{R}_1[\mathbf{E}(\tau_1), t - \tau_1]}{\partial \mathbf{E}(\tau_1)} \, dE(\tau_1). \quad (8.1) \]

Next, we illustrate the functional form of the relaxation function $\mathbf{R}_1$ in Eq. (8.1). Based on experimental observations, when $\mathbf{R}_1$ is plotted against time $t$, or an normal strain $\mathbf{E}$, in a logarithm plot, one finds straight lines parallel to each other. This implies that the stress relaxation modulus can be factorized into time- and strain-dependent terms, i.e.,

\[ \log \mathbf{R}_1(\mathbf{E}, t) = A \log t + B \log \mathbf{E} = \log (t^A E^B), \]

where $A$ and $B$ are constants to be determined by experimental data. This type of material response is commonplace for a wide range of viscoelastic media [91], including soft tissues. As such, the relaxation equation $\mathbf{R}_1$ can be generalized to be

\[ \mathbf{R}_1(\mathbf{E}(\tau), t - \tau) = \mathbf{G}(t - \tau) S^\alpha[\mathbf{E}(\tau)]. \quad (8.2) \]
where $G$ is a function of $t$ only and $S^e$ is a function of $E$ only. By substituting Eq. (8.2) into Eq. (8.1) with $\tau_1$ replaced by $\tau$, we obtain

$$S(t) = \int_{-\infty}^{t} G(t - \tau) \frac{\partial S^e(E(\tau))}{\partial E(\tau)} d\tau.$$  \hspace{1cm} (8.3)$$

To proceed, we assume $E = 0$ when $\tau < 0$, and at $\tau = 0$, $E$ experiences a jump start. Thus, Eq. (8.3) can be written as

$$S(t) = S^e(0)G(t) + \int_0^{t} G(t - \tau) \frac{\partial S^e(E(\tau))}{\partial E(\tau)} \frac{\partial E(\tau)}{\partial \tau} d\tau,$$ \hspace{1cm} (8.4)$$

where $S^e(0)G(t)$ is obtained by integrating $S(t)$ from $\tau = -\infty$ to $\tau = 0$.

To proceed, we recast Eq. (8.4) into the index form by using a Cartesian coordinate system:

$$S_{ij}(t) = S^e_{kl}(0)G_{ijkl}(t) + \int_0^{t} G_{ijkl}(t - \tau) \frac{\partial S^e_{kl}}{\partial E_{mn}} \frac{\partial E_{mn}}{\partial \tau} d\tau,$$ \hspace{1cm} (8.5)$$

where $S_{ij}$ is the $(i,j)$th component of the second Piola-Kirchhoff stress tensor, $E_{mn}$ is the $(m,n)$th component of Green’s strain tensor, and $G_{ijkl}$ is the component of the fourth-order relaxation function tensor. Equation (8.5) is a general form of the classical Fung’s model for modeling material response of soft tissues.

Next, we assume small deformation and adapt the constitutive relation for linear viscoelasticity. Thus, the second Piola-Kirchhoff stress tensor becomes the Cauchy stress, $S_{ij} = \sigma_{ij}$, and the Green’s strain tensor becomes the infinitesimal strain tensor, $E_{mn} = \epsilon_{mn}$. As such, Eq. (8.5) is changed to

$$\sigma_{ij}(t) = \sigma^e_{kl}(0)G_{ijkl}(t) + \int_0^{t} G_{ijkl}(t - \tau) \frac{\partial \epsilon_{kl}}{\partial \tau} d\tau,$$ \hspace{1cm} (8.6)$$

which is equivalent to a general form of Fung’s model for linear deformations in soft
tissues. Moreover, we assume the stress-free initial condition, i.e., \( \sigma_{kl}^e(0) = 0 \). The constitutive model Eq. (8.6) can be simplified to

\[
\sigma_{ij}(t) = \int_0^t G_{ijkl}(t - \tau) \frac{\partial \epsilon_{kl}}{\partial \tau} d\tau. \tag{8.7}
\]

Next, the medium is assumed isotropic. The fourth-order relaxation tensor \( G_{ijkl} \) can be written as:

\[
G_{ijkl}(t - \tau) = \lambda(t - \tau) \delta_{ij} \delta_{kl} + \mu(t - \tau)(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \tag{8.8}
\]

where \( \lambda(t) \) and \( \mu(t) \) are the two viscoelastic variables. Aided by Eq. (8.8), the constitutive relation Eq. (8.6) can be simplified to

\[
\sigma_{ij}(t) = \int_0^t \left( \lambda(t - \tau) \frac{\partial \epsilon_{kk}}{\partial \tau} \delta_{ij} + 2 \mu(t - \tau) \frac{\partial \epsilon_{ij}}{\partial \tau} \right) d\tau
\]

\[
= \lambda \ast \frac{\partial \epsilon_{kk}}{\partial t} \delta_{ij} + 2 \mu \ast \frac{\partial \epsilon_{ij}}{\partial t}, \tag{8.9}
\]

where the symbol * denotes the convolution integral and the overdot denotes the differentiation in time. Because of small deformation, the time rate of the strain becomes

\[
\frac{\partial \epsilon_{ij}}{\partial t} = \frac{1}{2} \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right). \tag{8.10}
\]

Aided by Eq. (8.10), Eq. (8.9) becomes

\[
\sigma_{ij}(t) = \lambda(t) \ast \frac{\partial v_k}{\partial x_k} \delta_{ij} + \mu(t) \ast \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right). \tag{8.11}
\]

Equation (8.11) is the general constitutive relationship between stress and strain for small deformation in a linear viscoelastic medium.
8.2.1 Standard Linear Solid Models

In this section, we proceed to determine the two relaxation parameters $\lambda(t)$ and $\mu(t)$ in the above constitutive relation Eq. (8.11) for soft tissues. To proceed, we approximate the general viscoelasticity relation by a series of parallelly connected SLS models. Each SLS model includes two springs and one dash-pot. The relaxation function of the parallelly connected SLS models is a linear combination of a series of exponential functions. It is very easy to integrate or differentiate exponential functions. Thus, the functional form of the relaxation parameters $\lambda$ and $\mu$ can be readily derived. In this process, we consider one-dimensional, uni-axial loading only. Nevertheless, the resultant formula of the two relaxation parameters can be substituted back to the general constitutive relation Eq. (8.11), which in turn can be applied to model waves in two and three spatial dimensions.

Recall Eq. (8.7), which can be rewritten as $\sigma_{ij} = G_{ijkl} \ast \partial \epsilon_{kl}/\partial t$. Aided by the Voigt notation, the Cauchy stress tensor can be written as a 6-component vector as

$$
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}
= \sigma,
$$

where $\sigma$ is the 6-component Cauchy stress vector. Similarly, the rate of the Cauchy
strain tensor can be written as a 6-component vector as

\[
\begin{bmatrix}
\frac{\partial \epsilon_{11}}{\partial t} \\
\frac{\partial \epsilon_{22}}{\partial t} \\
\frac{\partial \epsilon_{33}}{\partial t} \\
\frac{\partial \epsilon_{23}}{\partial t} \\
\frac{\partial \epsilon_{13}}{\partial t} \\
\frac{\partial \epsilon_{12}}{\partial t}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\frac{\partial \epsilon_1}{\partial t} \\
\frac{\partial \epsilon_2}{\partial t} \\
\frac{\partial \epsilon_3}{\partial t} \\
2\frac{\partial \epsilon_4}{\partial t} \\
2\frac{\partial \epsilon_5}{\partial t} \\
2\frac{\partial \epsilon_6}{\partial t}
\end{bmatrix}
= \frac{\partial \mathbf{\epsilon}'}{\partial t},
\]

where \( \mathbf{\epsilon}' \) is the 6-component vector of strain. As such, Eq. (8.7) can be rewritten as the following vector-matrix equation:

\[
\sigma = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\star \frac{\partial \mathbf{\epsilon}}{\partial t},
\]  (8.12)

where \( \lambda \) and \( \mu \) are functions of time.

To determine the viscoelastic parameters, \( \lambda \) and \( \mu \), we consider two stress-strain relations: (i) an uni-axial compression/expansion case, and (ii) a simple shear case. To proceed, we consider the case of compression/expansion along the \( x_1 \) direction. We assume \( \sigma_1 \neq 0 \) and \( \epsilon_1 \neq 0 \). All of other stress and strain are null. Equation (8.12) becomes \( \sigma_1 = (\lambda(t) + 2\mu(t)) \star \frac{\partial \epsilon_1}{\partial t} \). For convenience, we let \( \Pi(t) = \lambda(t) + 2\mu(t) \).
By using the parallelly connected SLS models, $\Pi(t)$ can be shown as

$$\Pi(t) = \pi \left( 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{pl}}{\tau_{\sigma l}} \right) e^{-t/\tau_{pl}} \right) H(t)$$

(8.13)

where $H(t)$ is the Heaviside function, and $L$ is the total number of the SLS models involved. In Eq. (8.13), $\pi$, $\tau_{\sigma l}$, and $\tau_{pl}$ are the tension or compression modulus and relaxation parameters, respectively. The values of these parameters must be determined by the compression/tension tests. If the test data are presented in the form of Fung’s model for soft tissues, we use the collocation method to deduce the values of these parameters. Section 8.4 illustrates this procedure. Once the values of these parameters are provided, $\Pi$ is determined by Eq. (8.13).

Next, we consider a case of simple shear under the condition of $\sigma_4 \neq 0$ and $\epsilon_4 \neq 0$. All other stress and strain components are null. Equation (8.12) becomes, for pure shear waves in a soft tissue, $\sigma_4 = \mu(t) \times \partial \epsilon_4 / \partial t$. For this pure shear condition, the relaxation function of the parallelly connected SLS models [10] can be formulated as

$$\mu(t) = \nu \left( 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}}{\tau_{\sigma l}} \right) e^{-t/\tau_{\sigma l}} \right) H(t),$$

(8.14)

where $\nu$, $\tau_{\sigma l}$ and $\tau_{sl}$ are the shear modulus and the two relaxation parameters of viscoelastic material. The values of these parameters need to be determined by experiments. If needed, the collocation method is used to deduce the data. Once $\mu$ is determined, $\lambda(t)$ can be calculated by $\lambda(t) = \Pi(t) - 2\mu(t)$. To recapitulate, the parameters $\lambda$ and $\mu$ in the general three-dimensional isotropic viscoelasticity model, Eq. (8.12), can be obtained by (i) approximating the relaxation function by a series of parallelly connected SLS models, and (ii) two simple tests of a pure compression/tension test and a pure shear test.

To proceed, we substitute the derived functions for the viscoelasticity parameters
Eq. (8.14) and Eq. (8.13) into Eq. (8.11). We then differentiate Eq. (8.11) by time. To present the resultant equations, we separate the diagonal terms and the off-diagonal terms [12, pp. 118]. The diagonal elements \((i \neq j)\) of \(\partial \sigma_{ij}/\partial t\) can be expressed as:

\[
\frac{\partial \sigma_{ij}}{\partial t} = \left( \frac{\partial \Pi}{\partial t} - 2 \frac{\partial \mu}{\partial t} \right) \ast \frac{\partial v_k}{\partial x_k} + 2 \frac{\partial \mu}{\partial t} \ast \frac{\partial v_j}{\partial x_i} .
\]

(8.15)

The off-diagonal element \((i \neq j)\) are

\[
\frac{\partial \sigma_{ij}}{\partial t} = \frac{\partial \mu}{\partial t} \ast \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) .
\]

(8.16)

### 8.2.2 Internal Variables

In this section, we transform the constitutive relation in the integral form derived in Section 8.2.1 into a differential form by using the internal variables [21, 113]. For completeness, details of applying the internal variable method to Eqs. (8.15) and (8.16) are given in Appendix A.3. After the derivation, Eq. (8.15) becomes:

\[
\frac{\partial \sigma_{ij}}{\partial t} = \left\{ \pi \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{p}^{l}}{\tau_{\sigma l}} \right) \right] - 2 \nu \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{s}^{l}}{\tau_{\sigma l}} \right) \right] \right\} \frac{\partial v_k}{\partial x_k} + 2 \nu \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{s}^{l}}{\tau_{\sigma l}} \right) \right] \frac{\partial v_j}{\partial x_i} + \sum_{l=1}^{L} \gamma_{ij}^{l} .
\]

(8.17)

Equation (8.16) becomes

\[
\frac{\partial \sigma_{ij}}{\partial t} = \nu \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{p}^{l}}{\tau_{\sigma l}} \right) \right] \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) + \sum_{l=1}^{L} \gamma_{ij}^{l} ,
\]

(8.18)

where \(\gamma_{ij}^{l}\) are the internal variables. Shown in Appendix A.3, we have the following governing equations of the internal variables. For the diagonal terms, i.e., \(\gamma_{ij}^{l}\) with
\[ i = j, \text{ the equations of the internal variables are} \]
\[ \frac{\partial \gamma_{ij}^l}{\partial t} = -\frac{1}{\tau_{\sigma l}} \left[ \gamma_{ij}^l + \pi \left( \frac{\tau_{\sigma l}^p}{\tau_{\sigma l}} - 1 \right) \frac{\partial v_k}{\partial x_k} \right. \]
\[ \left. -2\nu \left( \frac{\tau_{\sigma l}^s}{\tau_{\sigma l}} - 1 \right) \frac{\partial v_k}{\partial x_k} + 2\nu \left( \frac{\tau_{\sigma l}^s}{\tau_{\sigma l}} - 1 \right) \frac{\partial v_j}{\partial x_i} \right]. \] (8.19)

For the off-diagonal terms \((i \neq j)\), the equations are
\[ \frac{\partial \gamma_{ij}^l}{\partial t} = -\frac{1}{\tau_{\sigma l}} \left[ \gamma_{ij}^l + \nu \left( \frac{\tau_{\sigma l}^s}{\tau_{\sigma l}} - 1 \right) \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \right]. \] (8.20)

### 8.3 The Governing Equations

In this section, we discuss the complete governing equations for waves in viscoelastic media. The equations include the equation of motion and the constitutive relations, Eqs. (8.17)–(8.20), derived in Section 8.2.2. To proceed, we consider the equation of motion \( \nabla \cdot \sigma = \rho \frac{\partial^2 u}{\partial t^2} \), where \( \sigma \) is the Cauchy stress tensor, \( \rho \) is density, and \( u \) is the displacement vector. We assume that the body force is negligible. The independent variables are the position \( x = (x_1, x_2, x_3) \) and time \( t \). By using the index notation, the equation of motion is rewritten as \( \sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2} \), where a subscript following a comma denotes partial differentiation with respect to the spatial coordinate. The velocity components \( v_i = \frac{\partial u_i}{\partial t} \) are used as the unknowns instead of the displacement. The equation of motion is then coupled with the constitutive equations, Eqs. (8.17) and (8.18), and the equations for the internal variables, Eqs. (8.19) and (8.20), to form the complete set of the governing equations. In what follows, we
list the governing equations with one internal variable:

\[
\begin{align*}
\frac{\partial v_1}{\partial t} &= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}, \\
\frac{\partial v_2}{\partial t} &= \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3}, \\
\frac{\partial v_3}{\partial t} &= \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}, \\
\frac{\partial \sigma_{11}}{\partial t} &= \frac{\tau^p}{\tau_\sigma} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \\
&\quad - 2\nu \frac{\tau^s}{\tau_\sigma} \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_3}{\partial x_2} \right) + \gamma_{11}, \\
\frac{\partial \sigma_{22}}{\partial t} &= \frac{\tau^p}{\tau_\sigma} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \\
&\quad - 2\nu \frac{\tau^s}{\tau_\sigma} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_3}{\partial x_3} \right) + \gamma_{22}, \\
\frac{\partial \sigma_{33}}{\partial t} &= \frac{\tau^p}{\tau_\sigma} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \\
&\quad - 2\nu \frac{\tau^s}{\tau_\sigma} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) + \gamma_{33}, \\
\frac{\partial \sigma_{12}}{\partial t} &= \nu \frac{\tau^s}{\tau_\sigma} \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_3}{\partial x_1} \right) + \gamma_{12}, \\
\frac{\partial \sigma_{13}}{\partial t} &= \nu \frac{\tau^s}{\tau_\sigma} \left( \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_1} \right) + \gamma_{13}, \\
\frac{\partial \sigma_{23}}{\partial t} &= \nu \frac{\tau^s}{\tau_\sigma} \left( \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_2} \right) + \gamma_{23},
\end{align*}
\]

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\[
\begin{align*}
\frac{\partial \gamma_{11}}{\partial t} &= -\frac{1}{\tau_\sigma} \left\{ \gamma_{11} + \pi \left( \frac{\tau_p}{\tau_\sigma} - 1 \right) \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \right. \\
&\quad \left. -2\nu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \right\}, \\
\frac{\partial \gamma_{22}}{\partial t} &= -\frac{1}{\tau_\sigma} \left\{ \gamma_{22} + \pi \left( \frac{\tau_p}{\tau_\sigma} - 1 \right) \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \right. \\
&\quad \left. -2\nu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_3}{\partial x_3} \right) \right\}, \\
\frac{\partial \gamma_{33}}{\partial t} &= -\frac{1}{\tau_\sigma} \left\{ \gamma_{33} + \pi \left( \frac{\tau_p}{\tau_\sigma} - 1 \right) \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \right. \\
&\quad \left. -2\nu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) \right\}, \\
\frac{\partial \gamma_{12}}{\partial t} &= -\frac{1}{\tau_\sigma} \left\{ \gamma_{12} + \nu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \right\}, \\
\frac{\partial \gamma_{13}}{\partial t} &= -\frac{1}{\tau_\sigma} \left\{ \gamma_{13} + \nu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) \right\}, \\
\frac{\partial \gamma_{23}}{\partial t} &= -\frac{1}{\tau_\sigma} \left\{ \gamma_{23} + \nu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) \right\}.
\end{align*}
\]

where \(\pi = \lambda + 2\mu\) is the relaxation modulus for compression/expansion waves, and \(\nu\) is the relaxation modulus for shear waves. \(\tau_p \) and \(\tau_s \) are the strain relaxation times for the compression/expansion and shear waves, respectively. \(\tau_p \) and \(\tau_s \) are the stress relaxation time for the compression/expansion and shear waves, respectively [10]. \(v_1, v_2\) and \(v_3\) are the velocity components. \(\sigma_{ij}, i, j = 1, 2, 3\) are the stress components. \(\gamma_{ij}, i, j = 1, 2, 3\) are the internal variables.

The above first-order PDEs can be recast into a matrix-vector form:

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}_1 \frac{\partial \mathbf{U}}{\partial x_1} + \mathbf{A}_2 \frac{\partial \mathbf{U}}{\partial x_2} + \mathbf{A}_3 \frac{\partial \mathbf{U}}{\partial x_3} = \mathbf{S}. \tag{8.21}
\]
where the unknown vector \( \mathbf{U} \) is defined as

\[
\mathbf{U} = [v_1, v_2, v_3, \sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23},
\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{12}, \gamma_{13}, \gamma_{23}]^T,
\]

the source term vector \( \mathbf{S} \) is

\[
\mathbf{S} = [0, 0, 0, \gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{12}, \gamma_{13}, \gamma_{23},
-\gamma_{11}/\tau_{\sigma}, -\gamma_{22}/\tau_{\sigma}, -\gamma_{33}/\tau_{\sigma},
-\gamma_{12}/\tau_{\sigma}, -\gamma_{13}/\tau_{\sigma}, -\gamma_{23}/\tau_{\sigma}]^T,
\]

and \( \mathbf{A}_1, \mathbf{A}_2, \) and \( \mathbf{A}_3 \) are the Jacobian matrices.

The Jacobian matrix \( \mathbf{A}_1 \) is written as:

\[
\mathbf{A}_1 = \begin{bmatrix}
0_3 & \mathbf{A}_{1v} \\
\mathbf{A}_{1\sigma v} & 0_{12}
\end{bmatrix},
\]

where

\[
\mathbf{A}_{1v} = \begin{bmatrix}
-1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
\[ A_{1\sigma v} = \]
\[
\begin{bmatrix}
-\frac{\pi \tau_p}{\tau_\sigma} & 0 & 0 \\
-\frac{\pi \tau_p}{\tau_\sigma} + 2\nu \frac{\tau_s}{\tau_\sigma} & 0 & 0 \\
0 & -\nu \frac{\tau_s}{\tau_\sigma} & 0 \\
0 & 0 & -\nu \frac{\tau_s}{\tau_\sigma} \\
0 & 0 & 0 \\
\pi \frac{\tau^p}{\tau_\sigma} - 1 & 0 & 0 \\
\frac{\pi \tau_s}{\tau_\sigma} - 1 + 2\nu \frac{\tau^s}{\tau_\sigma} & 0 & 0 \\
0 & 0 & \nu \frac{\tau^s}{\tau_\sigma} - 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix},
\]

and \(0_3\) and \(0_{12}\) denote \(3 \times 3\) and \(12 \times 12\) null matrices. Based on the Schur complement [97], the non-trivial eigenvalues of \(A_1\) can be obtained by solving the following equation:

\[ \det(A_{1v}A_{1\sigma v} - \beta^2 I_3) = 0. \]

where \(\beta\) is the eigenvalues of \(A_1\) and

\[
A_{1v}A_{1\sigma v} = \begin{bmatrix}
\pi \frac{\tau_p}{\tau_\sigma} & 0 & 0 \\
\rho \frac{\tau_s}{\tau_\sigma} & 0 & \nu \frac{\tau^s}{\tau_\sigma} \\
0 & \rho \frac{\tau^s}{\tau_\sigma} & 0 \\
0 & 0 & \nu \frac{\tau^s}{\tau_\sigma}
\end{bmatrix}.\]
$A_1$ has 15 eigenvalues: 9 of them are null while the remaining 6 are

$$\beta_{1,2} = \pm \sqrt{\frac{\pi \tau^p}{\rho \tau_\sigma}}, \quad \beta_{3,4} = \beta_{5,6} = \pm \sqrt{\frac{\nu \tau^s}{\rho \tau_\sigma}}. \quad (8.22)$$

If the relaxation parameters $\pi$ and $\nu$ are positive, all 6 non-zero eigenvalues $A_1$ are real. Similar expressions can be obtained for the eigenvalues of $A_2$ and $A_3$.

### 8.3.1 One-Dimensional Equation with One Internal Variable

We let $\partial / \partial x_2 = 0$ and $\partial / \partial x_3 = 0$ and Eq. (8.21) is reduced to the one-dimensional governing equations for waves propagation along the $x_1$ axis. The resultant equations can be divided into four groups. The first group includes the following three equations:

$$\rho \frac{\partial v_1}{\partial t} = \frac{\partial \sigma_{11}}{\partial x_1},$$

$$\frac{\partial \sigma_{11}}{\partial t} = \pi \frac{\tau^p}{\tau_\sigma} \frac{\partial v_1}{\partial x_1} + \gamma_{11},$$

$$\frac{\partial \gamma_{11}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ \gamma_{11} + \pi \left( \frac{\tau^p}{\tau_\sigma} - 1 \right) \frac{\partial v_1}{\partial x_1} \right\}. \quad (8.23)$$

These three equations are self-sufficient and they can be solved for solutions of longitudinal waves propagating in the $x_1$ direction. The second and the third groups include three equations each for shear waves moving along the $x_1$ axis:

$$\rho \frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{1i}}{\partial x_1},$$

$$\frac{\partial \sigma_{1i}}{\partial t} = \nu \frac{\tau^s}{\tau_\sigma} \frac{\partial v_i}{\partial x_1} + \gamma_{1i},$$

$$\frac{\partial \gamma_{1i}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ \gamma_{1i} + \nu \left( \frac{\tau^s}{\tau_\sigma} - 1 \right) \frac{\partial v_i}{\partial x_1} \right\}, \quad (8.24)$$

where $i = 2, 3$. The above three equations are self-sufficient and they can be solved for shear wave propagation along the $x_1$ axis. The fourth set includes the following
six equations:

\[
\begin{align*}
\frac{\partial \sigma_{22}}{\partial t} &= \tau \frac{\tau_p}{\tau \sigma} \frac{\partial v_1}{\partial x_1} - 2\nu \tau \frac{\tau_s}{\tau \sigma} \frac{\partial v_1}{\partial x_1} + \gamma_{22}, \\
\frac{\partial \sigma_{33}}{\partial t} &= \tau \frac{\tau_p}{\tau \sigma} \frac{\partial v_1}{\partial x_1} - 2\nu \tau \frac{\tau_s}{\tau \sigma} \frac{\partial v_1}{\partial x_1} + \gamma_{33}, \\
\frac{\partial \sigma_{23}}{\partial t} &= \gamma_{23}, \\
\frac{\partial \gamma_{22}}{\partial t} &= -\frac{1}{\tau \sigma} \left\{ \gamma_{22} + \pi \left( \frac{\tau_p}{\tau \sigma} - 1 \right) \frac{\partial v_1}{\partial x_1} \\
&\quad - 2\nu \left( \frac{\tau_p}{\tau \sigma} - 1 \right) \frac{\partial v_1}{\partial x_1} \right\}, \\
\frac{\partial \gamma_{33}}{\partial t} &= -\frac{1}{\tau \sigma} \left\{ \gamma_{33} + \pi \left( \frac{\tau_p}{\tau \sigma} - 1 \right) \frac{\partial v_1}{\partial x_1} \\
&\quad - 2\nu \left( \frac{\tau_p}{\tau \sigma} - 1 \right) \frac{\partial v_1}{\partial x_1} \right\}, \\
\frac{\partial \gamma_{23}}{\partial t} &= -\frac{1}{\tau \sigma} \gamma_{23}.
\end{align*}
\] (8.25)

The solutions of these six equations depend on the solutions of Eqs. (8.23) and (8.24). When the pure longitudinal waves and pure shear waves are solved, the six equations in Eq. (8.25) become ordinary differential equations and can be readily integrated.

For the rest of this chapter, we will focus on the numerical solutions of longitudinal waves modeled by Eq. (8.23). To proceed, we rewrite Eq. (8.23) into a matrix-vector form:

\[
\frac{\partial \hat{U}}{\partial t} + \hat{A}_1 \frac{\partial \hat{U}}{\partial x_1} = \hat{S}. \tag{8.26}
\]
\[
\hat{U} = [v_1, \sigma_{11}, \gamma_{11}]^T, \quad \hat{S} = \left[ 0, \gamma_{11}, -\frac{\gamma_{11}}{\tau}\right]^T, \quad \text{and}
\]
\[
\hat{A}_1 = \begin{bmatrix}
0 & -\frac{1}{\rho} & 0 \\
\rho & 0 & 0 \\
-\frac{\pi \tau}{\tau_{\sigma}} & 0 & 0 \\
\frac{\pi}{\tau_{\sigma}} \left( \frac{\tau_{p}}{\tau_{\sigma}} - 1 \right) & 0 & 0 \\
\end{bmatrix}.
\]

\(\hat{A}_1\) has three eigenvalues: one is null and the other two are \(\beta_{1,2}\) shown in Eq. (8.22).

For numerical solution by using the CESE method, Eq. (8.26) is cast into a conservative form:

\[
\frac{\partial \hat{U}}{\partial t} + \frac{\partial \hat{F}}{\partial x_1} = \hat{S}, \quad \text{(8.27)}
\]

where \(\hat{F} = \hat{A}_1 \hat{U}\). Note that \(\hat{A}_1\) is a constant matrix.

### 8.3.2 Governing Equations with Eight Internal Variables

For complex media, Eq. (8.26) can be extended to include more internal variables:

\[
\frac{\rho}{\partial t} \frac{\partial v_1}{\partial t} = \frac{\partial \sigma_{11}}{\partial x_1},
\]

\[
\frac{\partial \sigma_{11}}{\partial t} = \left\{ \pi \left[ 1 - \sum_{l=1}^{8} \left( 1 - \frac{\tau_{p}}{\tau_{\sigma}} \right) \right] \frac{\partial v_1}{\partial x_1} \right\} + \sum_{l=1}^{8} \gamma_{11}^l,
\]

\[
\frac{\partial \gamma_{11}^l}{\partial t} = -\frac{1}{\tau_{\sigma}} \left[ \gamma_{11}^l + \pi \left( \frac{\tau_{p}}{\tau_{\sigma}} - 1 \right) \frac{\partial v_1}{\partial x_1} \right].
\]

Without losing generality, we arbitrarily use 8 internal variables and \(l = 1, \ldots, 8\). To proceed, we rewrite the equations into a matrix-vector form:

\[
\frac{\partial \mathbf{V}}{\partial t} + \hat{A}_1 \frac{\partial \mathbf{V}}{\partial t} = \mathbf{Q},
\]

\[\text{(8.29)}\]
where the unknown vector \( \mathbf{V} = [v_1, \sigma_{11}, \gamma_{11}^1, \gamma_{11}^2, \cdots, \gamma_{11}^8]^T \), the source term
\[
\mathbf{Q} = \left[ 0, \sum_{l=1}^{8} \gamma_{11}^l, -\gamma_{11}^1/\tau_{\sigma 1}, \cdots, -\gamma_{11}^8/\tau_{\sigma 8} \right]^T,
\]
and the Jacobian matrix
\[
\overline{\mathbf{A}}_1 = \begin{bmatrix}
\mathbf{0}_3 & \overline{\mathbf{A}}_{1v} \\
\overline{\mathbf{A}}_{1T} & \mathbf{0}_7
\end{bmatrix},
\]
where
\[
\overline{\mathbf{A}}_{1v} = \begin{bmatrix}
0 & -\frac{1}{\rho} & 0 \\
-\pi \left[ 1 - \sum_{l=1}^{8} \left( 1 - \frac{\tau_{p l}}{\tau_{\sigma l}} \right) \right] & 0 & 0 \\
\frac{\pi}{\tau_{\sigma 1}} \left( \frac{\tau_{p 1}}{\tau_{\sigma 1}} - 1 \right) & 0 & 0
\end{bmatrix},
\]
and
\[
\overline{\mathbf{A}}_{1T} = \begin{bmatrix}
\frac{\pi}{\tau_{\sigma 2}} \left( \frac{\tau_{p 2}}{\tau_{\sigma 2}} - 1 \right) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma 3}} \left( \frac{\tau_{p 3}}{\tau_{\sigma 3}} - 1 \right) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma 4}} \left( \frac{\tau_{p 4}}{\tau_{\sigma 4}} - 1 \right) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma 5}} \left( \frac{\tau_{p 5}}{\tau_{\sigma 5}} - 1 \right) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma 6}} \left( \frac{\tau_{p 6}}{\tau_{\sigma 6}} - 1 \right) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma 7}} \left( \frac{\tau_{p 7}}{\tau_{\sigma 7}} - 1 \right) & 0 & 0 \\
\frac{\pi}{\tau_{\sigma 8}} \left( \frac{\tau_{p 8}}{\tau_{\sigma 8}} - 1 \right) & 0 & 0
\end{bmatrix}.
\]
Simple derivation shows that the nonzero eigenvalues of $A_1$ are

$$\tilde{\beta}_{1,2} = \pm \sqrt{\frac{\pi}{\rho} \left[ 1 - \sum_{l=1}^{8} \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) \right]}.$$ 

The other 8 eigenvalues are null.

### 8.4 Fung’s Model

In this section, we determine the values of $\pi$, $\tau^p_l$, and $\tau_{sl}$, where $l = 1, \ldots, L$, in Eq. (8.28) for modeling longitudinal waves in soft tissues. To this end, we relate the relaxation function $G(t)$ in Fung’s model to the general constitutive relation Eq. (8.12) for parallelly connected SLS models. To proceed, we recall that the relaxation function $G(t)$ in Fung’s model [54] is

$$G(t) = \frac{1 + c [\eta(t/\tau_2) - \eta(t/\tau_1)]}{1 + c \ln(\tau_2/\tau_1)},$$

(8.30)

where $\eta(z) = \int_{z}^{\infty} e^{-t/tdt}$, $c$ is a dimensionless constant, and $\tau_1$ and $\tau_2$ are two constants for slow and fast relaxation times, respectively. In Fung’s model, the functional form of $G(t)$ represents the relaxation effect, which is insensitive to the frequency $\omega$ of the wave, i.e., the relaxation effect remains constant for $2\pi/\tau_2 < \omega < 2\pi/\tau_1$. In Fung’s original model [54] for rabbit mesentery, $\tau_1 = 1.735 \times 10^{-5}s$, $\tau_2 = 1.869 \times 10^{4}s$, and $c = 0.02657$. These values were obtained by experiments.

Next, we will apply Carson Transform to Eq.(8.30).

Carson Transform is defined as

$$F(s) = s \int_{0}^{\infty} f(t)e^{-st}dt.$$  

(8.31)
If Carson Transform is taken with respect to Eq.(8.30) and let \( s = i\omega \), The dimensionless imaginary part of Fung’s model can be got as

\[
G''(\omega) = \frac{c[\tan^{-1}(\omega \tau_2) - \tan^{-1}(\omega \tau_1)]}{1 + c \ln \frac{\tau_2}{\tau_1}}.
\] (8.32)

To proceed, we let the relaxation function of the parallelly connected SLS models, Eq. (8.13), be equal to that of Fung’s model, Eq. (8.30):

\[
G(t) = \left( G_e + \sum_{l=1}^{L} G_l e^{-t/\tau_{\sigma l}} \right) H(t)
= 1 + c[\eta(t/\tau_2) - \eta(t/\tau_1)]
\]

\[
= \frac{1 + c \ln(\tau_2/\tau_1)}{1 + c \ln(\tau_2/\tau_1)}.
\] (8.33)

where \( G_e = \pi \) and \( G_l = -\pi(1 - \tau''_d/\tau_{\sigma l}) \). As such, \( \pi, \tau''_d \), and \( \tau_{\sigma l} \) in Eq. (8.28) can be determined. The curve-fitting process is done by using the collocation method [131].

For completeness, a brief description of the method is provided in the following. To proceed, we substitute a series of collocation times into Eq. (8.33) and have

\[
G(t_m) = \left( G_e + \sum_{l=1}^{L} G_l e^{-t_m/\tau_{\sigma l}} \right), m = 1, 2, \ldots,
\] (8.34)

where \( t \) is replaced by \( t_m \) for each discrete collocation time. Numerical values of the left hand side of Eq. (8.34), i.e., \( G(t_m) \), are given by Fung’s model with the values of \( c, \tau_1, \) and \( \tau_2 \) experimentally determined. The task here is to determine the parameters \( G_e, G_l \) and \( \tau_{\sigma l} \) at the right hand side of Eq. (8.34) such that \( \pi, \tau''_d \), and \( \tau_{\sigma l} \) in the governing equations can be determined.

The idea of collocation method is to minimize the error in the non-linear curve fitting process. We first let \( t_m \) to be very large, approaching \( \infty \). By using Eq. (8.34), we can determine \( G_e \) by letting \( G_e = G(\infty) \). We then let the discrete collocation times \( t_m \) with \( m = 1, 2, \ldots \), be separated with orders of magnitudes. For example, we could let \( t_1 = 0.001 \) s, \( t_2 = 0.01 \) s, \( t_3 = 0.1 \) s, et cetera. We then choose each \( \tau_{\sigma l} \) so that its value is in the same order of magnitude as that of \( t_m \). For example, let
\( \tau_{\alpha 1} = 2 \times 0.001 \text{ s} \quad \tau_{\alpha 2} = 2 \times 0.01 \text{ s}, \) and so on. We write down the linear algebraic equations for all collocation times:

\[
\begin{bmatrix}
A_{11} & A_{12} & \cdots \\
A_{21} & A_{22} & \cdots \\
\vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
G_1 \\
G_2 \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
G(t_1) - G_e \\
G(t_2) - G_e \\
\vdots
\end{bmatrix},
\tag{8.35}
\]

where \( A_{ij} = \exp(-t_{m}/\tau_{\alpha l}) \). Equation (8.35) can be readily solved by inverting matrix \( A \) to obtain \( G_1, G_2, \ldots \) Finally, we substitute \( G_e, G_l, \) and \( \tau_{\alpha l} \) with \( l = 1, \ldots, L \) back to Eq. (8.34) to obtain the values of the constants \( \pi, \tau_{\alpha l}^p, \) and \( \tau_{\alpha l} \) for \( l = 1, \ldots, L \). By using the collocation method, the generalized SLS model with \( L = 1, 3, 5, \) and 8 to match Fung’s model are obtained as:

\[
G_1(t) = 0.64 + 0.28e^{-\frac{t}{1.18}},
\]

\[
G_3(t) = 0.64 + 0.0825e^{-\frac{t}{0.025}}
+ 0.165e^{-\frac{t}{0.025}} + 0.025e^{-\frac{t}{0.0025}},
\]

\[
G_5(t) = 0.64 + 0.078e^{-\frac{t}{0.025}}
+ 0.082e^{-\frac{t}{0.025}} + 0.076e^{-\frac{t}{0.025}} + 0.07e^{-\frac{t}{0.0025}},
\tag{8.36}
\]

\[
G_8(t) = 0.64 + 0.037e^{-\frac{t}{0.025}}
+ 0.0482e^{-\frac{t}{0.025}} + 0.024e^{-\frac{t}{0.025}} + 0.054e^{-\frac{t}{0.025}}
+ 0.03e^{-\frac{t}{0.025}} + 0.044e^{-\frac{t}{0.0025}} + 0.0355e^{-\frac{t}{0.0025}}
+ 0.0243e^{-\frac{t}{0.0025}}.
\]

For completeness, the parameters in the generalized SLS model are tabulated in Table 8.1.

Fig. 8.1 shows the reconstructed relaxation function as compared to that in the original Fung’s model. We consider four cases by using 1, 3, 5, and 8 internal variables,
Table 8.1: Parameters of the reconstructed Fung’s model by using 1, 3, 5, and 8 SLS modules.

<table>
<thead>
<tr>
<th>Relaxation Functions</th>
<th>$G_1$</th>
<th>$G_3$</th>
<th>$G_5$</th>
<th>$G_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi$</td>
<td>0.64</td>
<td>0.64</td>
<td>0.64</td>
<td>0.64</td>
</tr>
<tr>
<td>$\tau_{\sigma 1}$</td>
<td>11.78</td>
<td>0.02</td>
<td>0.002</td>
<td>0.002</td>
</tr>
<tr>
<td>$\tau_{e 1}$</td>
<td>16.9</td>
<td>0.023</td>
<td>0.0022</td>
<td>0.0021</td>
</tr>
<tr>
<td>$\tau_{\sigma 2}$</td>
<td>-</td>
<td>20</td>
<td>0.2</td>
<td>0.02</td>
</tr>
<tr>
<td>$\tau_{e 2}$</td>
<td>-</td>
<td>25.16</td>
<td>0.226</td>
<td>0.0215</td>
</tr>
<tr>
<td>$\tau_{\sigma 3}$</td>
<td>-</td>
<td>20000</td>
<td>20</td>
<td>0.2</td>
</tr>
<tr>
<td>$\tau_{e 3}$</td>
<td>-</td>
<td>20781</td>
<td>22.37</td>
<td>0.2075</td>
</tr>
<tr>
<td>$\tau_{\sigma 4}$</td>
<td>-</td>
<td>-</td>
<td>2000</td>
<td>2</td>
</tr>
<tr>
<td>$\tau_{e 4}$</td>
<td>-</td>
<td>-</td>
<td>2219</td>
<td>2.17</td>
</tr>
<tr>
<td>$\tau_{\sigma 5}$</td>
<td>-</td>
<td>-</td>
<td>20000</td>
<td>20</td>
</tr>
<tr>
<td>$\tau_{e 5}$</td>
<td>-</td>
<td>-</td>
<td>20000</td>
<td>20.9</td>
</tr>
<tr>
<td>$\tau_{\sigma 6}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>200</td>
</tr>
<tr>
<td>$\tau_{e 6}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>213.75</td>
</tr>
<tr>
<td>$\tau_{\sigma 7}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2000</td>
</tr>
<tr>
<td>$\tau_{e 7}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>2111</td>
</tr>
<tr>
<td>$\tau_{\sigma 8}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>20000</td>
</tr>
<tr>
<td>$\tau_{e 8}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>20759</td>
</tr>
</tbody>
</table>

Fig. 8.1: The relaxation function of the reconstructed Fung’s model for rabbit mesentery by using 1, 3, 5 and 8 SLS modules in the time domain.
which correspond to the use of 1, 3, 5, and 8 parallelly connected SLS models. Shown in Fig. 8.1, by using 8 paralleled connected SLS models, the original Fung model has been reconstructed. Fig. 8.2 shows the relaxation effect of the reconstructed Fung’s model using 8 internal variables. The relaxation effect is modelled by the imaginary part of the relaxation functions.

To proceed, we illustrate a modified Fung’s model developed by Iatridis et al. [65]. Fig. 8.1 shows the relaxation function of the original Fung’s model, i.e., Eq. (8.30), plotted against the logarithm function of time. A straight line in Fig. 8.1 indicates that Fung’s model has a constant relaxation effect for dynamics problems at all frequencies. However, for dynamics problems with short durations or at high frequencies, experimental data [82] shows that the relaxation function plotted against the logarithmic function of time is nonlinear. Experimental data show frequency-sensitive relaxation effects. Specifically, the relaxation effect in the dynamics problems for short time durations or high-frequency is significantly higher than that in that with
long time durations or at low frequencies. To capture high frequency dynamical behaviours, Iatridis et al. [65] developed a modified Fung’s model with the following relaxation function:

$$G(t) = \frac{1 + \int_0^\infty S(\tau) \exp(-t/\tau) d\tau}{1 + \int_0^\infty S(\tau) d\tau}$$  \hspace{1cm} (8.37)

where \(S(\tau)\) is the relaxation spectrum:

$$S(\tau) = \begin{cases} 
  \frac{c_1}{\tau} + \frac{c_2}{\tau^2}, & \text{for } \tau_1 \leq \tau \leq \tau_2, \\
  0, & \text{for } \tau < \tau_1, \tau > \tau_2.
\end{cases}$$  \hspace{1cm} (8.38)

In the above definition, \(c_1\) is the amplitude of the relaxation effect and \(c_2\) is a constant representing linear increase in the relaxation effect with respect to the frequency. \(\tau_1\) and \(\tau_2\) are two times corresponding to the frequency limits of the relaxation function.

To proceed, we substitute Eq. (8.38) into Eq. (8.37) and obtain

$$G(t) = \left[1 + c_1 \ln \left(\frac{\tau_2}{\tau_1}\right) + c_2 \left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)\right]^{-1} \left\{1 + c_1 \left[\eta \left(\frac{t}{\tau_2}\right) - \eta \left(\frac{t}{\tau_1}\right)\right] + c_2 \frac{t}{\tau_2} [\exp(-t/\tau_2) - \exp(-t/\tau_1)]\right\}$$  \hspace{1cm} (8.39)

where \(\eta\) has the same definition as that in Fung’s model. To proceed, we let \(c_1 = 0.314\) and \(c_2 = 0.165\) according Ledoux et al. [82] for the cow subcalcaneal. \(\tau_1\) and \(\tau_2\) can be determined by considering the frequency range of interest. With these values determined, Eq. (8.39) can be plotted as Fig. 8.3.

To proceed, we use the parallelly connected SLS models to reconstruct the Ledoux’s model for the cow subcalcaneal. Again, the collocation method is used to determine the parameters in the reconstructed model. In this case, the use of 4 internal variables (or 4 SLS modules) is enough to reconstruct an satisfactory relaxation function for the modified Fung’s model. Fig. 8.3 shows the relaxation function plotted against
the logarithm function of time. The plotted curve in the range of short time periods, e.g., $10^{-4}$ to $10^{-1}$ has a negative and very steep slope. This indicates that the relaxation effect is more pronounced in the range of small time periods or at high frequencies. The line becomes flat in the time range beyond $10^{-1}$s. The relaxation effect is minimum in this range of longer times or at low frequencies.

The reconstructed relaxation function by using 4 SLS modules is

$$G_4 = 0.0001 + 0.8035 \times \exp\left(-\frac{t}{0.0002}\right) + 0.138 \times \exp\left(-\frac{t}{0.02}\right) + 0.0114 \times \exp\left(-\frac{t}{0.02}\right) + 3.78 \times 10^{-5} \times \exp\left(-\frac{t}{0.02}\right). \quad (8.40)$$

For completeness, the parameters in Eq. (8.40) for $G_4$ are provided in Table 8.2.
Table 8.2: Parameters in the reconstructed Ledoux model [82] by using 4 SLS modules

<table>
<thead>
<tr>
<th>Relaxation Function</th>
<th>$\tau_{\sigma 1}$</th>
<th>$\tau_{\sigma 2}$</th>
<th>$\tau_{\sigma 3}$</th>
<th>$\tau_{\sigma 4}$</th>
<th>$\tau_{\epsilon 1}$</th>
<th>$\tau_{\epsilon 2}$</th>
<th>$\tau_{\epsilon 3}$</th>
<th>$\tau_{\epsilon 4}$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_4$</td>
<td>0.0002</td>
<td>0.002</td>
<td>0.02</td>
<td>0.2</td>
<td>1.607</td>
<td>2.76</td>
<td>2.3</td>
<td>0.28</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

8.5 Results and Discussions

In this section, we report numerical results of one-dimensional longitudinal waves in a semi-infinite medium in the range $x > 0$. Initially, the medium is unstrained and at rest. When $t = 0$, the left end of the block, i.e., $x = 0$, is suddenly subjected to an impact at a constant velocity. The one-dimensional governing equations Eq. (8.26) are solved by using the CESE method. Solutions of three problems for impact wave propagation are reported. First, we consider wave propagation in a Maxwellian medium, which is a special case of the SLS model. Numerical solutions of wave in the medium is used to assess numerical accuracy by comparing the numerical results with the derived analytical solution. In what follows, we proceed to derive the analytical solution of the impact wave propagating in the Maxwellian medium.

The relation between the normal stress $\sigma$ and the normal strain $\epsilon$ of a Maxwellian medium is $\partial \epsilon / \partial t = \partial v / \partial x = (1/E) \partial \sigma / \partial t + \mu \sigma$, where $v$ is the velocity in the uni-axial direction, $E$ and $\mu$ are two constants of the Maxwellian medium. The equation of motion is $\partial \sigma / \partial x = \rho \partial v / \partial t$, where $\rho$ is the density of the unstrained material. By combining the two equations, we have the governing equation for the normal stress:

$$\frac{\partial^2 \sigma}{\partial x^2} - \frac{1}{c^2} \left( \frac{\partial^2 \sigma}{\partial t^2} + \frac{1}{\tau_0} \frac{\partial \sigma}{\partial t} \right) = 0,$$

where $c = \sqrt{E/\rho}$ is the elastic wave velocity, and $\tau_0 = 1/(E\mu)$ is the relaxation time. For the boundary conditions, at $x = 0$, $v = VH(t)$, where $H(t)$ is the Heaviside
function, and at $x = \infty, \sigma = 0$. Equation (8.41) is a particular form of the telegraph equation. To proceed, we perform the Laplace transform to Eq. (8.41) and obtain

$$\frac{\partial^2 \bar{\sigma}}{\partial x^2} - \frac{1}{c^2} \left( s^2 + \frac{s}{\tau_0} \right) \bar{\sigma} = 0,$$  \hspace{1cm} (8.42)

where the over-bar denotes the transformed unknown and $s$ is the Laplace parameter. Equation (8.42) is a second-order ordinary differential equation. To solve the equation, two boundary conditions on the two end of the spatial domain are needed. These boundary conditions can be obtained by performing the Laplace transformation to the original boundary conditions, i.e., at $x = 0$, $\bar{v} = V/s$ and at $x = \infty$, $\bar{\sigma} = 0$. The boundary condition at $x = 0$ is for $\bar{v}$, which can be changed to be a condition for $\bar{\sigma}$ by performing the Laplace transformation to the equation of motion: $\partial \bar{\sigma} / \partial x = \rho s \bar{v}$.

The analytical solution of $\sigma$ can be obtained by performing the inverse Laplace transformation and was provided by Lee [83]:

$$\sigma = -pcVe^{-\frac{x}{c\tau}}I_0 \left( \frac{\sqrt{t^2 - x^2/c^2}}{2\tau_0} \right) H(t - \frac{x}{c}),$$

where $I_0$ is the zeroth-order Bessel function of the first kind. The dimensionless form of the analytical solution is

$$-\frac{\sigma}{pcV} = e^{-\frac{\tau}{\tau_0}}I_0 \left( \frac{\sqrt{\tau^2 - \xi^2}}{2} \right) H(\tau - \xi).$$

where dimensionless time $\tau = t/\tau_0$ and the dimensionless distance $\xi = x/c\tau_0$.

For numerical solutions, we use the CESE method to solve Eq. (8.26), i.e., the one-dimensional governing equations with one internal variable for longitudinal waves. As a typical medium, we let $\rho = 800 \text{kg/m}^3$ and $E = 1 \times 10^9 \text{ Pa}$. The wave speed is $c = \sqrt{E/\rho} = 1118 \text{m/s}$. In Eq. (8.26), the wave speed is the eigenvalue of the
Jacobian matrix $\tilde{A}_1$. Shown in Eq. (8.22), the wave speed $c = \sqrt{\frac{\pi \tau_c \rho}{\eta \tau}}$. The values of three constants in Eq. (8.27) are chosen to be $\pi = 10 \times 10^7 \text{N/m}^2$, $\tau_p = 0.1 \text{s}$, and $\tau_\sigma = 0.001 \text{s}$. These values are chosen so that the wave speed is consistent with the wave speed of the analytical solution. In the calculations, the length of the one-dimensional domain is 20 m, $\Delta x = 0.025 \text{m}$ and $\Delta t = 2 \times 10^{-5} \text{s}$. The impact velocity is $V = 1 \text{m/s}$.

Figure 8.4 shows the numerical solution and the analytical solution of the moving impact wave in the Maxwellian medium. Three snapshots of dimensionless stress profiles at three different times are presented. The analytical solutions are presented by lines. The numerical solutions are presented by symbols. The solution jump at the wave fronts is resolved by about 3 mesh nodes. The result shows the calculated wave speed is a constant, which matches that of the analytical solution. The amplitude of the wave decays continuously due to the relaxation effect. The calculated wave amplitudes are slightly higher than that of the analytical solution. Overall, numerical solutions compare well with the analytical solutions.

Next, we consider wave propagation in the rabbit mesentery. The material response is modelled by using Fung’s original model [54] for the rabbit mesentery. The CESE code is used to solve Eq. (8.29), i.e., the governing equations with eight internal variables for longitudinal wave. For the rabbit mesentery, Young’s modulus is $2.5 \times 10^9 \text{Pa}$ and the density is $1200 \text{kg/m}^3$. The parameters in the governing equations Eq. (8.29) are determined by using the collocation method. The constants are tabulated in Table 8.1. The parameters in Eq. (8.36) are made dimensionless by divided by Young’s modulus.

Numerical results of moving waves are shown in Fig. 8.5. Three snapshots of stress profiles at three different times are presented. The amplitude of the wave front remains the same as the wave propagates to the right. Apparently, within
Fig. 8.4: Comparison between the analytical solution and the numerical solution of the normal stress at $\tau = 1, 5$ and $8$. 
the short period of time, the relaxation effect modelled by the original Fung model is negligible. However, experiments shows that bio-tissues exhibit large absorption coefficients when ultrasonic wave propagate through it. This implies that the use of Fung’s model for wave propagation during short period times or at high frequencies is unsuitable.

To proceed, we consider impact wave propagation in the subcalcaneal, a cow bone material. The material response is modelled by using a modified Fung’s model, which was developed by Iatridis et al. [65] for modelling material response of the nucleus pulposus as a part of the human intervertebral disc. Ledoux et al. [82] extended Iatridis’ constitutive relation to model the material response of the cow subcalcaneal. We employ Ledoux’s model for this case. As shown in previous discussions, the modified Fung’s model was first reconstructed by using parallelly connected SLS models. The resultant model formulated in the integral form is then transformed
to be differential equations by using the method of internal variables. Four internal variables are used for this case. The dimensionless parameters used in the model equations are tabulated in Table 8.2. In numerical calculation, we let $dx = 0.005 \text{ m}$, $dt = 0.000001 \text{ s}$. Numerical results are shown in Fig. 8.6. Three snapshots of stress profiles at three different times are shown. The amplitude of the wave front decays continuously as the wave propagates to the right. Numerical results in this case show apparent relaxation effect by the material.

### 8.6 Conclusions

In this chapter, a new theoretical and numerical framework has been developed to calculate waves propagating in soft tissues. A generalized constitutive model based on paralleling connected SLS models is employed to represent the original Fung’s model and a modified Fung’s model for material response of soft tissues. To be coupled
with the equation of motion, the constitutive relation formulated in an integral form is transformed into a differential form by using the method of internal variables. The values of additional parameters in the governing equations for the internal variables are determined by comparing to Fung’s model. The curve-fitting process is done by using the collocation method. The complete governing equations, including the equation of motion, the constitutive relation between stress and the rate of strain, and the equations of internal variables. The governing equations can be cast into a set of first-order, fully coupled, hyperbolic PDEs with sink terms.

To demonstrate the capabilities of the present approach, the one-dimensional equations are solved by the CESE method for time-accurate solutions of propagating longitudinal waves in soft tissues. Results of three cases are reported. In the first case, a simple Maxwellian medium is used and its material response is modelled by using one internal variable. The accuracy of the numerical results is validated by favourable comparison between the numerical solutions and the analytical solutions in terms of the calculated wave speed and the relaxation effect. In the second case, we employ the original Fung’s model for the rabbit mesentery. The governing equations include eight internal variables. The results show no relaxation effect by the original Fung’s model due to the short time duration of the traveling impact wave. Since Fung’s model did not capture the relaxation effect for the dynamics problem with the short time duration, we proceed to employ a modified Fung’s model developed by Iatridis et al. [65]. An extension of Iatridis’ model by Ledoux et al. [82] for the cow subcalcaneal was used. The numerical results show apparent relaxation effect in the material response. In all three cases, wave fronts are accurately captured by the CESE method.
CHAPTER 9
WAVE ABSORPTION BY SOFT TISSUES MODELLED
BY FIRST-ORDER VELOCITY STRESS EQUATIONS

9.1 Introduction

A wide range of soft tissues have acoustic impedances resemble that of water. As a
common practice, the governing equations employed to model waves in soft tissues
have been by and large based on extensions of the second-order acoustics equations,
originally developed for lossless waves in liquids. To model the observed wave ab-
sorption effect, additional terms have been added to the wave equations to model
relaxation and damping. Oftentimes the added terms were nonlinear functions of un-
knowns. Hamilton and Blackstock [61] provided a comprehensive review on nonlinear
acoustics equations for biomedical applications.

Many have used finite-element, finite-volume, or finite-difference methods to solve
the nonlinear acoustics equations and reported numerical solutions of waves in soft
tissues. Most of reported simulations were performed in the time domain because,
in general, material response and geometries of the media were too complex to ren-
der normal mode analysis useful. Mast et al. [93–95] reported simulations of waves
in cross sections of anatomical tissues, which were modeled as a fluid with spatially
dependent sound speeds and densities. They used the MacCormack method, a finite-volume method, and the $k$-space method, a spectral method, for two- and three-dimensional simulations of waves in tissue-mimicking cylinders, walls, and spheres. Tabei [118] reported simulations of waves in abdominal wall and breast tissue. The time-shift compensation method was used. Varslot et al. [133–135] employed an operator splitting method to solve the Khokhlov-Zabotskaya-Kuznetsov (KZK) equation, a nonlinear acoustics equation, to simulate waves in soft tissues. Their results captured wave diffraction and attenuation. To develop therapeutic transducers, Ginter et al. [56] solved the nonlinear acoustics equation, which was formulated in a conservative form, by using an explicit high-order Finite-Difference Time-Domain (FDTD) method. Norton [102] applied the FDTD method to simulate waves passing through a heterogeneous medium. He incorporated a convolution operator into the acoustics equation for the viscoelastic effect. Pinton et al. [108] used the FDTD method to solve a nonlinear wave equation to simulate diagnostic ultrasound pulses propagating through a human abdominal wall. The medium was treated with spatial variations of wave speed and density. The results showed wave attenuation. In above works, linear and nonlinear acoustics equations were solved numerically. Spatial variations in density and wave speed were incorporated in the model equations. Wave attenuation and dissipation were modeled by the added damping terms to the acoustics equations.

It is generally recognized that soft tissues are viscoelastic media, in which the material response to the applied stress is dictated by the memory or relaxation effect. This key attribute of viscoelastic media demands hereditary integration in the constitutive relation when simulating wave dynamics so that the observed wave dispersion and dissipation due to relaxation of the medium can be correctly calculated. Banks et al. [5] formally incorporated viscoelasticity relations into the governing equations.
for modeling dynamic problems. A system of non-linear, first-order, hyperbolic PDEs were developed to simulate waves from coronary stenoses.

In this chapter, we use Fung’s model [54] and an extended Fung’s model by Iatridis et al. [65] to simulate material response of soft tissues. It is well known that the functional form of the relaxation functions in these models are particularly useful for modeling material response of soft tissues. Conventionally, parameters in the relaxation function employed have been determined by quasi-steady compression/tension and/or shear tests. Such approach, however, may not be useful for dynamic problems characterized by very short time durations and cyclical loadings at high frequencies. In this chapter, instead of using quasi-steady testing data, we will use the measured wave absorption coefficients, which are widely available for many soft tissues, to determine the relaxation functions in Fung’s model [54] and Iatridis’ model [65]. As such, the resultant constitutive models will be inherently suitable for wave dynamics with correct wave absorption effect.

The second objective of this chapter is to transform the hereditary integration in the constitutive relation into PDEs with the same mathematical form as that in the equation of motion. As such, the complete model equations become amenable to numerical solutions by using a finite-volume or finite-difference method without additional complexity of coupling the numerical method employed for PDEs with a specialized ODE integrator for hereditary integration. The numerical method employed can be uniformly applied to all equations for controllable numerical stability and accuracy. Without such treatment, the hereditary integration in its original form demands integration in time from the onset of the process. Such integration requires separate treatment and the intermediate results must be coupled with intermediate solutions solved by the finite-volume or finite-difference methods employed.

Two steps are involved to transform the viscoelasticity relation into PDEs. First,
we discretize the relaxation function by using parallel-connected SLS models. We then differentiate the constitutive relation in time and introduce internal variables [20, 22, 113]. The resultant PDEs for constitutive relations can be readily coupled with the equation of motion and the complete governing equations are a set of first-order, hyperbolic PDEs with source terms. Cast into a vector-matrix form, we then show that the eigenvalues of the Jacobian matrix in the model equations are real and they represent the wave speeds of the medium.

To demonstrate the approach, we use the space-time Conservation Element and Solution Element (CESE) method [24] to solve the governing equations for time-accurate solutions of longitudinal waves in a pig muscle. The CESE method is a space-time finite-volume method for solving generic hyperbolic PDEs and conservation laws. The method has been applied to many wave problems in solids, including linear and nonlinear stress waves in hypoelastic medium [19, 146, 148] and anisotropic elastic solids [34, 147]. The material response is modeled by Fung’s model and Iatridis’ model. The relaxation functions in both models will be determined by the measured absorption coefficients. Each model is then discretized by using 6 parallel connected SLS modules. Propagating Ricker waves are calculated. The simulated absorption effect are obtained by post processing the transient solutions. For Iatridis’ model, numerical results show that the simulated wave absorption effect compares well with the measured data.

The rest of this chapter is organized as follows. Section 9.2 illustrates viscoelasticity relations and the process of determining the relaxation functions by using measured wave absorption coefficients. Section 9.3 presents the complete governing equations as a set of first-order, hyperbolic PDEs with source terms. We show that the governing equations can be cast into a vector matrix form such that the eigenvalues of the Jacobian matrix can be derived as the wave speed of the medium. Section 9.4
reports numerical solutions of propagating Ricker wave in a pig skeletal muscle by using the CESE method. By using Iatridis’ model, the simulated wave absorption effect in the pig muscle compares well with the measured data. Section 9.5 provides concluding remarks, followed by a list of cited references. For completeness, appendices are attached to provide detailed derivation of the Carson transformation of the relaxation functions of Fung’s model and Iatridis’ model.

9.2 Wave Absorption and Viscoelasticity

In this section, we illustrate the use of Fung’s model and Iatridis’ model for modeling wave motion in soft tissues. There are three subsections. The first subsection shows that the measured wave absorption coefficients can be connected with the Carson transform of a generic relaxation function of a viscoelastic relation. In the second subsection, Fung’s and Iatridis’ relaxation functions are transformed to the frequency domain to be connected to the measured wave absorption coefficients. The third subsection illustrate the parallel connected SLS models for numerical calculation.

9.2.1 Wave Absorption Effect

Based on observation, wave amplitudes decay as waves propagate in soft tissues. The wave absorption effect has been usually represented by an exponential decay function:

\[ A(x) = A_0 \exp(-\mu_A x), \]  

(9.1)

where \( A(x) \) is the magnitude of the propagating stress wave at location \( x \) after a certain period of time, \( A_0 \) is the initial wave amplitude, and \( \mu_A \) is the wave absorption coefficient and it is positive. In experiments, \( A_0 \) is given and \( A(x) \) are recorded. The
measured absorption effect were commonly presented as decrease in decibels (dB), i.e., $20 \log_{10}(A(x)/A_0)$. The wave absorption coefficient $\alpha$ is then defined as

$$\alpha = 10 \log_{10}(A(x)/A_0)^2/x = 20 \log_{10}(e)\mu_A \approx 8.7\mu_A.$$  

(9.2)

For soft tissues, the absorption coefficients, i.e., $\alpha$ and $\mu_A$, strongly depends on the wave frequency. The experimental results were often presented in the form of a power law:

$$\alpha = a\hat{f}^b,$$  

(9.3)

where $\hat{f}$ is the frequency measured in MHz, and $a$ and $b$ are experimentally determined constants, depending on the condition of the medium, e.g., temperature, humidity, PH value, etc. For most of soft tissues, $b \geq 1$. One can find measured data of $\alpha$ presented in the form of Eq. (9.3) for many soft tissues in the literature.

In this chapter, the classical viscoelasticity relations, i.e., Fung’s model [54] and an extended Fung’s model by Iatridis et al. [65] are employed to model material response of soft tissues. The key contribution of the present chapter is the use of measured absorption coefficients $\alpha$ to determine the relaxation functions in the constitutive relations. The constitutive models so obtained are tuned up to model the wave absorption in soft tissues. As will be shown in the following section, the essential derivation is to relate the measured $\alpha$ to the imaginary part of the transformed relaxation function by the Fourier transformation in the frequency domain. The relation in the frequency domain can fully determine the parameters in the original relaxation function employed.

To proceed, we consider a generic viscoelastic constitutive relation with a stress-free initial condition:

$$\sigma_{ij}(t) = \int_0^t G_{ijkl}(t - \tau) \frac{\partial \epsilon_{kl}}{\partial \tau} d\tau.$$  

(9.4)
where $\sigma_{ij}$ is the stress tensor, $\epsilon_{kl}$ is the strain tensor, and $G_{ijkl}$ is the relaxation function, which in general is a fourth-order tensor. For one-dimensional longitudinal waves in a homogeneous medium, Eq. (9.4) is simplified to be

$$\sigma = G(t) * \frac{\partial \epsilon}{\partial t}$$  \quad (9.5)$$

where $\sigma = \sigma_{11}$ is the normal stress on the $x_1$ surface, $G$ is the relaxation function, $\epsilon = \epsilon_{11} = \partial u_1 / \partial x_1 = \partial u / \partial x$ is the normal strain with $u$ as the displacement, and $*$ represents the convolution integral. Aided by the convolution theorem, the constitutive relation Eq. (9.5) is equivalent to

$$\tilde{\sigma} = \tilde{G} \tilde{\epsilon} = \tilde{G} \tilde{\epsilon},$$  \quad (9.6)$$

where the symbol dot on top of a variable denotes differentiation by time, and the last equality in Eq. (9.6) is obtained with integration by part. The symbol tilde denotes the transformed variables by the Fourier transformation in time:

$$\tilde{G} = \int_{-\infty}^{\infty} \frac{\partial G(t)}{\partial t} e^{i\omega t} dt = G' + iG'',$$

$$\tilde{\epsilon} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\omega t} dt = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} u e^{i\omega t} dt = \frac{\partial \tilde{u}}{\partial x}.$$  \quad (9.8)$$

where $\omega = 2\pi f$ is the angular frequency with the unit rad/s. The transform relaxation function $\tilde{G}(\omega)$ is a function of $\omega$. It is complex with its real and imaginary parts denoted by $G'$ and $G''$, respectively. To proceed, we consider the one-dimensional momentum equation governing the longitudinal waves in the $x$ direction:

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}.$$  \quad (9.7)$$

By applying the Fourier transform in time to Eq. (9.7) we obtain

$$\frac{\partial \tilde{\sigma}}{\partial x} = \rho (i\omega)^2 \tilde{u}.$$  \quad (9.8)$$

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Next, we apply the Fourier transform to Eq. (9.8) in space. Aided by Eq. (9.6), we obtain

\[
\int_{-\infty}^{\infty} \frac{\partial}{\partial x} \left( \tilde{G}(\omega) \tilde{\epsilon} \right) e^{ikx} dx = \int_{-\infty}^{\infty} \rho(i\omega)^2 \tilde{\epsilon} e^{ikx} dx,
\]

\[
\tilde{G}(\omega)(ik)^2 \int_{-\infty}^{\infty} \tilde{\epsilon} e^{ikx} dx = \rho(i\omega)^2 \int_{-\infty}^{\infty} \tilde{\epsilon} e^{ikx} dx.
\]

After cancelling the equivalent terms on both sides, we obtain the following relation:

\[
\tilde{G}(\omega)k^2 = \rho\omega^2.
\] (9.9)

To proceed, we consider the solution of a plane wave with damping: \( A = A_0 \exp[i(\omega t - kx)] \), where \( A \) is the magnitude of the wave and the wave number

\[ k = \kappa - i\mu_A \]

is a complex with the damping effect represented by \( \mu_A \). The phase speed of the plane wave \( v_p = \omega/\kappa \). Aided by complex \( k \), simple manipulation shows that the imaginary part of Eq. (9.9) becomes

\[
\frac{2\omega^2 \kappa \mu_A}{(\kappa^2 + \mu_A^2)^2} = \frac{G''}{\rho}.
\] (9.10)

If \( \kappa \gg \mu_A \), i.e., relatively small damping, Eq. (9.10) becomes \( \mu_A = (\pi G'' f)/ (\rho v_p^3) \), or \( \alpha = (8.7 \pi G'' f)/ (\rho v_p^3) \approx a f^b \). Finally, the dispersion relation can be related to \( G'' \) as

\[
4.35 \frac{G'' \omega}{\rho v_p^3} = a \left( \frac{10^6 \omega}{2\pi} \right)^b.
\] (9.11)

We note that the frequency \( f = \frac{\omega}{2\pi} \) in Hz is equal to \( 10^6 \hat{f} \), where \( \hat{f} \) is the frequency in MHz. In general, the phase velocity \( v_p \) is constant within one type of soft tissue. This assumption has been extensively verified by experiments [141]. Within one soft tissue, density \( \rho \) is also constant. Thus Eq. (9.11) shows that \( G'' \) is a function of frequency \( \omega \).
9.2.2 Transformed Relaxation Functions

In this subsection, we proceed to derive $G''$ of Fung’s model and Iatridis’ model such that the derived In Fung’s model [54], the relaxation function $G(t)$ is defined as

$$G(t) = \frac{1 + c[\eta(t/\tau_2) - \eta(t/\tau_1)]}{1 + c \ln(\tau_2/\tau_1)},$$  \hspace{1cm} (9.12)

where

$$\eta(z) = \int_0^\infty \frac{e^{-t} e^{-t}}{t} dt,$$  \hspace{1cm} (9.13)

c is a dimensionless constant, and $\tau_1$ and $\tau_2$ are the slow and fast relaxation times, respectively. To proceed, we apply the Fourier transformation to the time derivative of the Fung’s relaxation function $\dot{G}$, where $G$ is defined by Eq. (9.12). This transformation is equivalent to the Carson transformation of the relaxation function $G$ itself. We note that for a generic function $f(t)$, its Carson transform [42] is $F(s) = s \int_0^\infty f(t) e^{-st} dt$. The Carson transform is related to the Laplace transform: $F(s) = s \overline{f(s)} - f(0) = \overline{\dot{f}} - f(0)$, where $\overline{f(s)}$ and $\overline{\dot{f}(s)}$ are the Laplace transform of $f(t)$ and $\dot{f}(t)$, respectively. Moreover, the Laplace transform is related to the Fourier transform in time by letting $s = -i\omega$. For completeness, the derivation of $\tilde{G}$ for Fung’s model is provided in Appendix. The transformed $\tilde{G}$ is also available in [54] and it is

$$\tilde{G}(\omega) = G'(\omega) + iG''(\omega)$$

$$= \frac{1 + \frac{c}{2}[\ln(1 + \omega^2 \tau_2^2) - \ln(1 + \omega^2 \tau_1^2)]}{1 + c \ln(\tau_2/\tau_1)} + \frac{i c[\tan^{-1}(\omega \tau_2) - \tan^{-1}(\omega \tau_1)]}{1 + c \ln(\tau_2/\tau_1)}$$  \hspace{1cm} (9.14)

We then substitute the derived $G''$ into the dispersive relation Eq. (9.11). With measured $a$ and $b$ in Eq. (9.11), we can determine $c$, $\tau_1$, and $\tau_2$ in $G''$, which in turn
would completely determine the relaxation function $G$ itself. We remark that there are only three parameters $c$, $\tau_1$, and $\tau_2$ in Fung’s model. The slow time $\tau_1$ and the fast time $\tau_2$ must be used to determine the frequency range, i.e., $1/\tau_1 \leq 2\pi f \leq 1/\tau_2$. This leave $c$ as the only parameter to determine the damping effect. As a result, the function form of Fung’s model is such that $G''$ is a linear function of $\omega$ and the damping would be insensitive to wave frequencies within the range decided by $\tau_1$ and $\tau_2$. In other words, if one uses Fung’s model, one is committed to let $b = 1$ for the wave frequencies of interest.

Next, we consider the relaxation function of Iatridis’ model

$$G(t) = \frac{1 + c_1 \left[ \eta \left( \frac{t}{\tau_2} \right) - \eta \left( \frac{t}{\tau_1} \right) \right] + \frac{c_2}{\tau_2} \left[ e^{-t/\tau_2} - e^{-t/\tau_1} \right]}{1 + c_1 \ln \left( \frac{\tau_2}{\tau_1} \right) + c_2 \left( \frac{1}{\tau_1} - \frac{1}{\tau_2} \right)}.$$  

(9.15)

where $\eta$ is identical to that in Fung’s model, i.e., Eq. (9.13). $c_1$ and $c_2$ are two constants. We apply the Carson transformation to the above relaxation function to yield

$$\tilde{G}(\omega) = G' + iG''$$

$$= 1 + \frac{c_1}{\tau_2} \left[ \ln(1 + \omega^2 \tau_2^2) - \ln(1 + \omega^2 \tau_1^2) \right] + \frac{c_2\omega}{\tau_2} \left[ \tan^{-1}(\omega \tau_2) - \tan^{-1}(\omega \tau_1) \right]$$

$$+ \frac{c_1}{\tau_2} \left[ \tan^{-1}(\omega \tau_2) - \tan^{-1}(\omega \tau_1) \right] + \frac{c_2\omega}{\tau_2} \left[ \ln \frac{\tau_2}{1 + \omega^2 \tau_2^2} - \ln \frac{\tau_1}{1 + \omega^2 \tau_1^2} \right]$$

$$1 + c_1 \ln \left( \frac{\tau_2}{\tau_1} \right) + c_2 \left( \frac{1}{\tau_1} - \frac{1}{\tau_2} \right)$$

(9.16)

For completeness, details of the derivation for the transformed $\tilde{G}$ can be found in Appendix. Again, the derived expression of $G''$ can be substitute into the wave dispersion relation Eq. (9.11) to determine the parameters in Iatridis’ relaxation function. $\tau_1$
and \( \tau_2 \) are chosen such that the frequency range of interest is included. \( c_1 \) and \( c_2 \) can then be determined by matching the measured damping effect.

### 9.2.3 Standard Linear Solid Models

To proceed, we approximate the relaxation functions of Fung’s model and Iatridis’ model by parallel connected SLS models. We first discretize the relaxation function \( G(t) \) by using parallel connected SLS models. Next, by comparing the relaxation function of the parallel connected SLS models to Fung’s model and Iatridis’ model, we can determine the parameters in the SLS models. For a series of \( L \) parallel connected SLS models, the relaxation function \( G(t) \) is:

\[
G(t) = G_e \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{pl}}{\tau_{sl}} \right) e^{-t/\tau_{sl}} \right] H(t),
\]

where \( G_e, \tau_{pl}, \tau_{sl}, \) with \( l = 1, \ldots, L \) are parameters representing springs and dashpots in SLS models. \( H(t) \) is the Heaviside function.

Based on the derivation shown in the last section, the relaxation functions \( G(t) \) of Fung’s model and Iatridis’ model can be determined by the measured wave absorption coefficients. Here, we use the known \( G(t) \) on the left hand side of Eq. (9.17) to determine the parameters \( G_e, \tau_{pl}, \tau_{sl}, \) with \( l = 1, \ldots, L \) for the parallel connected SLS models. This can be done by the well known collocation method. Examples of using this procedure to develop the SLS models for a pig skeletal muscle will be demonstrated in Section 9.4 for result and discussion.

### 9.3 Governing Equations of Waves in Soft Tissues

This section discusses the complete governing equations amenable for numerical solution by a finite-volume or finite-difference method. The first subsection illustrates
the use of internal variables to transform the hereditary integration needed in the relaxation function into a set of PDEs. The second subsection presents the complete governing equations in a vector-matrix form.

### 9.3.1 Internal Variables

To proceed, the relaxation function Eq. (9.17) of parallel connected SLS models is substituted into the constitutive relation Eq.(9.5), which is then differentiated by time, i.e., $\dot{\sigma} = \dot{G} * \partial\varepsilon/\partial t$. And we have

$$\frac{\partial \sigma}{\partial t} = \int_0^t G_e \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau^p_{cl}}{\tau_{sl}} \right) \exp \left( -\frac{t - \tau}{\tau_{sl}} \right) \right] \delta(t - \tau) \frac{\partial v}{\partial x} d\tau$$

$$+ \int_0^t \frac{G_e}{\tau_{sl}} \left[ \sum_{l=1}^L \left( 1 - \frac{\tau^p_{cl}}{\tau_{sl}} \right) \exp \left( -\frac{t - \tau}{\tau_{sl}} \right) \right] H(t - \tau) \frac{\partial v}{\partial x} d\tau,$$

where the strain rate $\partial\varepsilon/\partial t$ has been replace by $\partial v/\partial x$ with $v$ as the velocity. Aided by the properties of the delta function, i.e., $\delta(-\tau) = \delta(\tau)$ and for a generic function $f(t)$, $\int_{-\infty}^{\infty} f(\tau)\delta(\tau - t)d\tau = f(t)$, and the definition of the internal variables $\gamma^l$ with $l = 1, 2, \ldots, L$, Eq. (9.18) becomes

$$\frac{\partial \sigma}{\partial t} = G_e \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau^p_{cl}}{\tau_{sl}} \right) \right] \frac{\partial v}{\partial x} + \sum_{l=1}^L \gamma^l,$$

where

$$\gamma^l = \int_0^t \frac{G_e}{\tau_{sl}} \left[ \left( 1 - \frac{\tau^p_{cl}}{\tau_{sl}} \right) \exp \left( -\frac{t - \tau}{\tau_{sl}} \right) \right] H(t - \tau) \frac{\partial v}{\partial x} d\tau.$$
To proceed, Eq. (9.20) is differentiated by time to yield

\[
\frac{\partial \gamma_l}{\partial t} = -\frac{1}{\tau_{sl}} \int_0^t G_e \left[ \left( 1 - \frac{\tau_p}{\tau_{sl}} \right) \exp \left( -\frac{t-\tau}{\tau_{sl}} \right) \right] H(t-\tau) \frac{\partial v}{\partial x} d\tau + G_e \left( 1 - \frac{\tau_p}{\tau_{sl}} \right) \int_0^t \exp \left( -\frac{t-\tau}{\tau_{sl}} \right) \delta(t-\tau) \frac{\partial v}{\partial x} d\tau.
\]

(9.21)

According to the definition of \( \gamma_l \) Eq. (9.20), we recognize that the first term on the right hand side of Eq. (9.21) is \(-\gamma_l/\tau_{sl}\). Moreover, the second term on the right hand side can be simplified by using the above-mentioned property of the delta function. As a result, Eq. (9.21) becomes

\[
\frac{\partial \gamma_l}{\partial t} = -\frac{1}{\tau_{sl}} \left[ \gamma_l + G_e \left( \frac{\tau_p}{\tau_{sl}} - 1 \right) \frac{\partial v}{\partial x} \right], \quad l = 1, \ldots, L
\]

(9.22)

Equations (9.19) and (9.22) are the new constitutive relation, in which the hereditary integration has been replaced by a set of PDEs for the internal variables \( \gamma_l \). As such, the constitutive relation is ready to be coupled with the equation of motion for numerical solution.

### 9.3.2 Governing Equations in a Vector-Matrix Form

The complete governing equations for longitudinal waves in a viscoelastic medium modeled by using \( L \) parallel connected SLS models include the equation of motion Eqs. (9.7) and the constitutive relation Eqs. (9.19, 9.22). These equations can be cast into a vector-matrix form:

\[
\frac{\partial V}{\partial t} + A \frac{\partial V}{\partial t} = Q
\]

(9.23)

where \( V \) is a column vector of the unknowns:

\[
V = [v, \sigma, \gamma^1, \gamma^2, \ldots, \gamma^L]^t
\]

(9.24)
where the superscript $t$ denotes transpose. The equation of motion has been formulated in term of velocity instead of displacement. The Jacobian matrix $A$ is defined as

$$A = \begin{bmatrix} A_v & 0_2 \\ \cdots & \cdots \\ A_t & 0_L \end{bmatrix},$$

where $0_n$ denotes an $n \times n$ null matrix. The sub-matrices are defined as

$$A_v = \begin{bmatrix} 0 \\ -G_e \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{\sigma l}} \right) \right] \\ 0 \end{bmatrix}, \quad A_t = \begin{bmatrix} \frac{G_e}{\tau_{\sigma 1}} (\frac{\tau_{p1}}{\tau_{\sigma 1}} - 1) & 0 \\ \frac{G_e}{\tau_{\sigma 2}} (\frac{\tau_{p2}}{\tau_{\sigma 2}} - 1) & 0 \\ \vdots & \vdots \\ \frac{G_e}{\tau_{\sigma L}} (\frac{\tau_{pL}}{\tau_{\sigma L}} - 1) & 0 \end{bmatrix}$$

(9.25)

and $Q$ is the vector of the source terms:

$$Q = \left[ 0, \sum_{l=1}^L \gamma^l, -\frac{\gamma^1}{\tau_{\sigma 1}}, -\frac{\gamma^2}{\tau_{\sigma 2}}, \ldots, -\frac{\gamma^L}{\tau_{\sigma L}} \right]^t.$$

(9.26)

Simple derivation shows that the Jacobian matrix $A$ has two nontrivial eigenvalues:

$$\beta_{1,2} = \pm \sqrt{\frac{G_e \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau_{pl}}{\tau_{\sigma l}} \right) \right]}{\rho}}.$$

(9.27)

They are also the eigenvalues of the sub-matrix $A_v$. The system of PDEs have real eigenvalues. Without considering the source terms, the PDEs are hyperbolic and the absolute value of the eigenvalues is the wave speed. With the source term, the mathematical property is less clear. Nevertheless, the eigenvalues here will be used to calculate the CFL number, which in term would determine the time increment for stable numerical calculation.
Table 9.1: $G''$ of Fung’s model and Iatridis’ model.

<table>
<thead>
<tr>
<th>Frequency (MHz)</th>
<th>Fung’s model (Pa) ($\times 10^6$)</th>
<th>Iatridis’ model (Pa) ($\times 10^6$)</th>
</tr>
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<tbody>
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<td>1</td>
<td>8.229</td>
<td>8.229</td>
</tr>
<tr>
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<td>8.229</td>
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<td>8.229</td>
<td>16.3</td>
</tr>
<tr>
<td>9</td>
<td>8.229</td>
<td>16.9</td>
</tr>
<tr>
<td>10</td>
<td>8.229</td>
<td>17.6</td>
</tr>
</tbody>
</table>

9.4 Result and Discussion

In this section, the theoretical results presented in previous section are used to model wave motion in a pig skeletal muscle. El-Brawany et al. [48] provided the measured absorption coefficient:

$$\alpha = 0.54 \hat{f}^{1.33},$$  \hspace{1cm} (9.28)

where $\hat{f}$ is in MHz. The frequency range of interest is 1 to 10 MHz. Both Fung’s model and Iatridis’ model are used in the calculation. However, for Fung’s model, $\alpha = 0.54 \hat{f}$ is used because the damping effect in Fung’s model is insensitive to the wave frequency. According to El Brawany et al. [48], the density of the pig skeletal muscle $\rho = 1.06 \times 10^3$ kg/m$^3$ and the wave speed $v_p=1578$ m/s. Aided by the dispersion relation Eq. (9.11), Table 1 lists the values of $G''$ for Fung’s model and Istridis’ model. In what follows, we proceed to determine the parameters in parallel connected SLS models for the approximated Fung’s and Iatridis’ models. Conventionally, the relaxation functions are usually normalized by a constant. In this chapter we let the constant $\Gamma = \rho v_p^2$. Compared to the wave speed for an elastic medium, $\Gamma$ can be
perceived as an equivalent Young’s modulus for viscoelastic medium. With the given
\( \rho \) and \( v_p \) of the pig muscle, \( \Gamma = 2.63 \times 10^9 \text{Pa} \).

The rest of this section is divided into 3 subsections. The first two subsections
report the construction of Fung’s model and Iatridis’ model for modeling the pig
skeletal muscle. The last subsection reports the calculated Ricker wavelets by using
the developed viscoelastic models. The numerical method employed for solving the
governing equations is the CESE method.

9.4.1 Fung’s Model

To construct Fung’s model, three parameters \( \tau_1, \tau_2 \), and \( c \) in the relaxation function
are to be determined by matching \( G'' \) with the dispersion relation Eq. (9.11). First,
\( \tau_1 \) and \( \tau_2 \) are chosen based on the frequency range of interest. In this case, 1MHz
\( \leq \hat{f} \leq 10\text{MHz} \) and we let \( \tau_1 = 10^{-4}s \) and \( \tau_2 = 10^{-9}s \). The value of \( c \) is then calculated
based on the measured wave absorption effect, and we have \( c = 0.0021 \).

Next, we use 6 parallel connected SLS models to approximate Fung’s relaxation
function. The parameters in SLS models are calculated by using the collocation
method. In the following equations, the relaxation function \( G \) is normalized by the
equivalent Young’s modulus \( \Gamma \).

\[
\frac{G(t)}{\Gamma} = \frac{1}{\Gamma} \left( G_e + \sum_{l=1}^{6} G_l e^{-t/\tau_l} \right) H(t) \quad (9.29)
\]

where \( G_l = -G_e (1 - \tau_p^l/\tau_{ol}) \). The parameters in the 6 parallel connected SLS models
are listed in Table 2 and \( G_e/\Gamma = 0.9768 \). The corresponding polynomial for \( G \) is

\[
G(t)/\Gamma = 0.9768 + 0.0041e^{-t/2\times10^{-9}} + 0.0043e^{-t/2\times10^{-8}} + 0.0000789e^{-t/2\times10^{-7.5}}
+ 0.0051e^{-t/2\times10^{-7}} + 0.0046e^{-t/2\times10^{-6}} + 0.0052e^{-t/2\times10^{-5}}.
(9.30)
\]
Table 9.2: Parameters in 6 parallel connected SLS models for Fung’s model.

<table>
<thead>
<tr>
<th>$l$</th>
<th>$\tau_{\sigma l}$</th>
<th>$\tau_{c l}$</th>
<th>$G_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2 \times 10^{-3}$</td>
<td>$2.0083 \times 10^{-9}$</td>
<td>$4.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>2</td>
<td>$2 \times 10^{-8}$</td>
<td>$2.0085 \times 10^{-8}$</td>
<td>$4.3 \times 10^{-3}$</td>
</tr>
<tr>
<td>3</td>
<td>$2 \times 10^{-7.5}$</td>
<td>$2.0001 \times 10^{-7.5}$</td>
<td>$7.89 \times 10^{-5}$</td>
</tr>
<tr>
<td>4</td>
<td>$2 \times 10^{-7}$</td>
<td>$2.0101 \times 10^{-7}$</td>
<td>$5.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>5</td>
<td>$2 \times 10^{-6}$</td>
<td>$2.0092 \times 10^{-6}$</td>
<td>$4.6 \times 10^{-3}$</td>
</tr>
<tr>
<td>6</td>
<td>$2 \times 10^{-5}$</td>
<td>$2.0104 \times 10^{-5}$</td>
<td>$5.2 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Fig. 9.1: $G(t)$ of Fung’s model developed by measured wave absorption coefficients and approximated by 6 parallel connected SLS models.

Figure 9.1 shows the relaxation function $G(t)$ of Fung’s model in the time domain. The solid line represents $G(t)$ of Fung’s model with $\tau_1$, $\tau_2$, and $c$ determined by using measured wave absorption coefficient, i.e., Eq. (9.11). The symbols represent the approximated Fung’s model by 6 parallel connected SLS models. To proceed, we
apply the Carson transform to Eq.(9.29), i.e., the relaxation function of 6 parallel connected SLS models, and the imaginary part \( G'' \) can be derived as

\[
\frac{G''}{\Gamma} = \frac{1}{\Gamma} \sum_{l=1}^{6} \left( \frac{\omega \tau_{\sigma_l} G_l}{1 + \omega^2 \tau_{\sigma_l}^2} \right),
\]  

(9.31)

Figure 9.2 shows \( G''(\omega) \) of Fung’s model. Again, the solid line represents Fung’s model plotted by using \( \tau_1, \tau_2, \) and \( c \), and symbols are the 6 parallel connected SLS models. The profile of \( G''(\omega) \) shows a flat plateau for the frequencies of interest. This implies that the damping effect of the soft tissue modeled by Fung’s model is insensitive to wave frequencies, or a constant damping to waves in the ultrasonic range.

9.4.2 Iatridis’ Model

The same procedure is repeated for Iatridis’ model. Unlike Fung’s model, however, Iatridis’ model is capable to represent frequency-sensitive damping as shown in Table 9.1. The Iatridis’ model is first constructed by matching \( G''(\omega) \) to the dispersion
relation Eq. (9.11) in the frequencies ranging from 1 to 10 Hz to determine 4 parameters \( \tau_1, \tau_2, c_1 \) and \( c_2 \) in the relaxation function. Similarly to that in Fung’s model, \( \tau_1 = 10^{-5} \text{s} \) and \( \tau_2 = 10^{-8.5} \text{s} \) are chosen to cover the frequency range. \( c_1 \) and \( c_2 \) can then be determined by matching the \( G'' \) profile with the dispersion relation Eq. (9.11), and we have \( c_1 = 0.000627 \) and \( c_2 = 6 \times 10^{-11} \).

Figure 9.3 shows the relaxation function \( G''(\omega) \) of Iatridis’ model constructed by using the measured wave absorption coefficient. The solid line is the Iatridis model and symbols are the power law relation Eq. (9.11) based on the measured wave absorption coefficient. For Iatridis’ model, the damping effect represented by \( G'' \) increases with higher frequencies. To proceed, the constructed Iatridis’ model is approximated by using 6 parallel connected SLS models. Again, the collocation method is used to determine parameters in the SLS models. The additional parameter
Table 9.3: Parameters in Iatridis’ model approximated by 6 parallel connected SLS models.

<table>
<thead>
<tr>
<th>l</th>
<th>(\tau_{cl})</th>
<th>(\sigma_{l})</th>
<th>(\tau_{l})</th>
<th>(\epsilon_{l})</th>
<th>(G_{l})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2 \times 10^{-8.5})</td>
<td>2.0303 (\times 10^{-8.5})</td>
<td>1.51 (\times 10^{-2})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(2 \times 10^{-8})</td>
<td>2.0021 (\times 10^{-8})</td>
<td>1.1 (\times 10^{-3})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(2 \times 10^{-7.5})</td>
<td>2.00678 (\times 10^{-7.5})</td>
<td>3.4 (\times 10^{-3})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(2 \times 10^{-7})</td>
<td>2.0022 (\times 10^{-7})</td>
<td>1.1 (\times 10^{-3})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(2 \times 10^{-6})</td>
<td>2.0029 (\times 10^{-6})</td>
<td>1.5 (\times 10^{-3})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(2 \times 10^{-5})</td>
<td>2.0006 (\times 10^{-5})</td>
<td>3.1 (\times 10^{-4})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(G_{e}/\Gamma = 0.9765\) and the polynomial for the relaxation function \(G(t)\) is

\[
\frac{G}{\Gamma} = 0.9765 + 0.0151e^{-t/2\times10^{-8.5}} + 0.0011e^{-t/2\times10^{-8}} + 0.0034e^{-t/2\times10^{-7.5}} \\
+ 0.0011e^{-t/2\times10^{-7}} + 0.0015e^{-t/2\times10^{-6}} + 0.00031e^{-t/2\times10^{-5}}. \tag{9.32}
\]

Figure 9.4 shows the \(G\) profile of Iatridis’ model. The solid line is Iatridis’ model and symbols are the SLS models. To proceed, we apply the Carson transform to the relaxation function of parallel connected SLS models and obtain \(G''(\omega)\) profile in the frequency domain, i.e., Eq. (9.31). The numerical result of \(G''(\omega)\) is plotted in Fig. 9.5. The solid line represents Iatridis’ model and symbols are the SLS models.

### 9.4.3 Ricker’s Wavelets

In this section, we report numerical solutions of propagating Ricker’s wavelet in the pig skeleton muscle. We apply the CESE method [24] to solve the governing equations, Eq. (9.23). The space-time CESE method is a novel numerical method for time-accurate solution of conservation laws and hyperbolic PDEs. The method was originally developed for solving the Euler equations for compressible flows with shock capturing capabilities. The method has been extended to solve a wide range of problems, including plasma, combustion, cavitations, compressible flows with complex...
Fig. 9.4: $G(t)$ of Iatridis’ model approximated by 6 parallel connected SLS models.

Fig. 9.5: $G''(\omega)$ of Iatridis’ model, approximated by 6 parallel connected SLS models.
shocks, two-phase flows, and waves in anisotropic elastic solids. A suite of one-, two-, and three-dimensional solver based on the CESE method has been developed.

To use the CESE method, it is imperative to cast the model equations into a vector-matrix form as shown in Eq. (9.23). As such, all equations including the constitutive relation can be solved in a uniform manner by the CESE method. The eigenvalues of the Jacobian matrix are used to calculate the CFL number, defined as $|\beta_{1,2}|\Delta t/\Delta x$, for controlling numerical stability. For the CESE method, the Courant number constraint demands that $\text{CFL} \leq 1$.

The CESE method uses a unique space-time integral equation. Aided by Gauss’ theorem, the CESE method changes the Eq. (9.23) to a space-time integral equation:

$$\int_{\partial \Omega} h_m \cdot ds = \int_{\Omega} q_m d\Omega, \quad m = 1, 2, \ldots, L + 2$$

(9.33)

where $\Omega$ is a space-time domain, $\partial \Omega$ is the surface of $\Omega$. The space-time flux vector $h_m = (f_m, v_m)^t$, $m = 1, 2, \ldots, L + 2$, where $f_m$ and $v_m$ are the $m$th components of $F = AV$ and $V$, respectively. $q_m$ is the $m$th components of the source term vector $Q$ in Eq. (9.23). The CESE method performs space-time integration and enforces space-time flux conservation over each Conservation Element (CE). The integration is facilitated by the prescribed discretization in each Solution Element (SE). In general CEs does not coincide with SEs. For details of the CESE method, please refer to the cited references.

To proceed, we consider the Ricker wavelet, which can be expressed as

$$\sigma(t) = \sigma_0 (1 - 2\pi^2 \omega^2 t^2) \exp(-\pi^2 f^2 t^2),$$

(9.34)

where $\sigma_0$ is the wave amplitude and $f$ is the central frequency. Ricker’s wavelet has a very narrow frequency band. Two sets of solutions are calculated. The first set is based on the use of Fung’s model and the second set Iatidis’ model. For each model,
we consider 5 cases with the central frequencies of Ricker’s wavelets equal to 1 to 9 MHz spaced by 2 MHz. For all cases \( \sigma_0 = 1 \times 10^5 \text{Pa} \). Since there is only one central frequency in Ricker’s wavelet, the profile of the wavelet will remain in the same shape when wave propagate from left to right. The amplitude of the wavelets will decay because of the attenuation effect of the soft tissue.

In all calculations, the computational domain is 0.02 m. \( \Delta x = 2 \times 10^{-5} \text{m} \), \( \Delta t = 1.2 \times 10^{-8} \text{s} \), and CFL = 0.95. On the left side of the computational domain, the boundary condition is a time varying stress profile based on Ricker’s wavelet. Figures 9.6 show snapshots of numerical solutions of stress profiles for 10 cases using Fung’s model. Dash lines are stress profiles at \( t = 3 \times 10^{-6} \text{s} \) and solid lines are stress profiles at \( t = 1.17 \times 10^{-5} \text{s} \). In the figure, stresses are normalized by \( \rho V_p^2 \).

The damping effect of the calculated wavelets is calculated by post processing the transient wave solutions based on

\[ \alpha = \frac{20 \log_{10}(A(x_1)/A(x_2))}{x_2 - x_1}. \]

Figure 9.7 shows the calculated wave dispersion relation. Because the damping effect by Fung’s model is insensitive to frequency, a linear relation, \( \alpha = 0.54f \), is used to accommodate the property of Fung’s model. Shown in Fig.(9.7), numerical results compare well with the linear dispersion relation, and Fung’s model successfully captures the damping effect for ultrasonic waves.

The numerical results slightly diverge from the straight line. This may come from the use of SLS models in fitting Fung’s model. Shown in Fig. 9.2, the parallel connected SLS models is not constant in the frequency domain which might be the reason of difference between the numerical results and the damping effect described by Fung’s model.
Fig. 9.6: Calculated Ricker’s wavelet with central frequencies. Fung’s model is used for material response. (a) 1 MHz, (b) 3 MHz, (c) 5 MHz, (d) 7 MHz, and (e) 9 MHz.

Figure 9.8 shows snapshots of propagating wavelets at 10 different central frequencies by using Iatridis’ model. Dash lines are stress profiles at $t = 3 \times 10^{-6}$ s and solid lines at $t = 1.17 \times 10^{-5}$ s. Figure 9.9) show the calculated wave absorption effect as compared to the experimental data. The calculated results compare well with experiment data in the ultrasonic range.

9.5 Conclusion

In this chapter, we presented a novel framework to simulate wave motion in soft tissues, which were formally treated as viscoelastic media. The classical Fung’s model and an extended Fung’s model developed by Iatridis et al. were employed to model
Fig. 9.7: Calculated wave absorption effect by using Fung’s model and its comparison with the linear power law.

material response. The key contribution of the present chapter is the use of measured absorption coefficients to construct the viscoelastic relations. We showed that in the frequency domain the transformed relaxation function by the Carson transformation is directly related to the measured wave absorption coefficients. As such, the constructed relaxation function of the soft tissue would provide accurate wave attenuation effect and would be suitable for modeling wave motion.

For numerical solution, we transformed the constructed relaxation functions, which include hereditary integration, into PDEs, which can be readily coupled with the equation of motion for numerical calculation. The transformation involved two steps: (i) approximation of the relaxation function by parallel connected SLS models, and (ii) introduction of the internal variables. The complete governing equations are a set of first-order, fully coupled PDEs with source terms. The primary unknowns are
Fig. 9.8: Calculated Ricker’s wavelets with central frequencies. Iatridis’ model is used for material response. (a) 1 MHz, (b) 3 MHz, (c) 5 MHz, (d) 7 MHz, and (e) 9 MHz.
velocity, stress, and internal variables. The model equations were solved by using the space-time CESE method for simulation of Ricker’s wavelets propagating in a pig skeletal muscle. El-Brawany et al. [48] provided the measured properties of the tissue and the empirical relation of the wave absorption coefficient. Numerical results were obtained by using Fung’s model and Iatridis’ model. For each model, 5 cases were calculated for wavelets with frequencies ranging from 1 to 9 MHz. The transient solutions were processed to obtain the calculated wave absorption effect. For solutions using Fung’s model, the result compared well with the linear dispersion relation. For Iatridis’ model, the calculated wave absorption effect compared well with the measured data by El-Brawany et al. [48].
CHAPTER 10
WAVES IN SOFT TISSUES BY VISCO-ELASTICITY
AND HYPER-ELASTICITY

10.1 Viscoelasticity

The generalized viscoelastic relations are given by Eq. (8.4) To proceed, we recast Eq. (8.4) into the index form by using a Cartesian coordinate system:

\[
S_{ij}(t) = S_{kl}^e(0)G_{ijkl}(t) + \int_0^t G_{ijkl}(t - \tau) \frac{\partial S_{kl}^e}{\partial E_{mn}} \frac{\partial E_{mn}}{\partial \tau} d\tau,
\]

where \( S_{ij} \) is the \((i,j)\)th component of the second Piola-Kirchhoff stress tensor, \( E_{mn} \) is the \((m,n)\)th component of Green’s strain tensor, and \( G_{ijkl} \) is the component of the fourth-order relaxation function tensor.

Next, we assume small deformation and adapt the constitutive relation for linear viscoelasticity. Thus, the second Piola-Kirchhoff stress tensor becomes the Cauchy stress, \( S_{ij} = \sigma_{ij} \), and the Green’s strain tensor becomes the infinitesimal strain tensor, \( E_{mn} = \epsilon_{mn} \).

As such, Eq. (10.1) is changed to

\[
\sigma_{ij}(t) = \sigma_{kl}^e(0)G_{ijkl}(t) \frac{\partial \sigma_{kl}^e}{\partial \epsilon_{mn}} \frac{\partial \epsilon_{kl}}{\partial \tau} d\tau,
\]

which is equivalent to a general form of Fung’s model for linear deformations in soft tissues.
tissues. Moreover, we assume the stress-free initial condition, i.e., $\sigma_{kl}(0) = 0$. The constitutive model Eq. (10.2) can be simplified to

$$\sigma_{ij}(t) = \int_0^t G_{ijkl}(t - \tau) \frac{\partial \sigma_{kl}^e}{\partial \epsilon_{mn}} \frac{\partial \epsilon_{kl}}{\partial \tau} d\tau.$$  \hspace{1cm} (10.3)

The relationship of $S^e$ and $\epsilon$ are assumed to non-linear function

$$\sigma_{kl}^e = e^{\alpha_{klmn}} \epsilon_{mn} + \beta_{kl}.$$  \hspace{1cm} (10.4)

Next, the medium is assumed isotropic. The fourth-order relaxation tensor $G_{ijkl}$ and $\alpha_{klmn}$ can be written as:

$$G_{ijkl}(t - \tau) = \lambda(t - \tau) \delta_{ij} \delta_{kl} + \mu(t - \tau)(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

$$\alpha_{klmn} = a \delta_{kl} \delta_{mn} + b(\delta_{km} \delta_{ln} + \delta_{kn} \delta_{lm})$$ \hspace{1cm} (10.5)

where $\lambda(t)$ and $\mu(t)$ are the two viscoelastic variables, $a$ and $b$ are elastic constants.

Aided by Eq. (10.5), the constitutive relation Eq. (10.4) can be simplified to

$$\sigma^e = \begin{bmatrix}
\sigma_{11}
\sigma_{22}
\sigma_{33}
\sigma_{23}
\sigma_{13}
\sigma_{12}
\end{bmatrix} = \begin{bmatrix}
 e^{a(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2b\epsilon_{11}} + \beta_{11}
 e^{a(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2b\epsilon_{22}} + \beta_{22}
 e^{a(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2b\epsilon_{33}} + \beta_{33}
 e^{2b\epsilon_{23}} + \beta_{23}
 e^{2b\epsilon_{13}} + \beta_{13}
 e^{2b\epsilon_{12}} + \beta_{12}
\end{bmatrix}.$$  \hspace{1cm} (10.6)

Aided by the Voigt notation, we proceed to let the indices $11 \rightarrow 1$, $22 \rightarrow 2$, $33 \rightarrow 3$, $23 \rightarrow 4$, $13 \rightarrow 5$, and $12 \rightarrow 6$. As such, the Cauchy stress tensor can be
written as a 6-component vector as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix}
= \boldsymbol{\sigma},
\]

where \( \boldsymbol{\sigma} \) is the 6-component Cauchy stress vector. Similarly, the rate of the Cauchy strain tensor can be written as a 6-component vector as

\[
\begin{bmatrix}
\partial \varepsilon_{11}/\partial t \\
\partial \varepsilon_{22}/\partial t \\
\partial \varepsilon_{33}/\partial t \\
\partial \varepsilon_{23}/\partial t \\
\partial \varepsilon_{13}/\partial t \\
\partial \varepsilon_{12}/\partial t
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\partial \varepsilon_1/\partial t \\
\partial \varepsilon_2/\partial t \\
\partial \varepsilon_3/\partial t \\
2\partial \varepsilon_4/\partial t \\
2\partial \varepsilon_5/\partial t \\
2\partial \varepsilon_6/\partial t
\end{bmatrix}
= \partial \varepsilon^e / \partial \tau,
\]

where \( \varepsilon^e \) is the 6-component vector of strain.

Then

\[
\frac{\partial \sigma^e_i}{\partial \varepsilon_j} \frac{\partial \varepsilon_j}{\partial \tau} =
\begin{bmatrix}
(a + 2b)(\sigma^e_1 - \beta_1) \frac{\partial \varepsilon_1}{\partial \tau} + a(\sigma^e_1 - \beta_1) \frac{\partial \varepsilon_2}{\partial \tau} + a(\sigma^e_1 - \beta_1) \frac{\partial \varepsilon_3}{\partial \tau} \\
\quad a(\sigma^e_2 - \beta_2) \frac{\partial \varepsilon_1}{\partial \tau} + (a + 2b)(\sigma^e_2 - \beta_2) \frac{\partial \varepsilon_2}{\partial \tau} + a(\sigma^e_2 - \beta_2) \frac{\partial \varepsilon_3}{\partial \tau} \\
\quad a(\sigma^e_3 - \beta_3) \frac{\partial \varepsilon_1}{\partial \tau} + a(\sigma^e_3 - \beta_3) \frac{\partial \varepsilon_2}{\partial \tau} + (a + 2b)(\sigma^e_3 - \beta_3) \frac{\partial \varepsilon_3}{\partial \tau} \\
\quad b(\sigma^e_4 - \beta_4) \frac{\partial \varepsilon_4}{\partial \tau} \\
\quad b(\sigma^e_5 - \beta_5) \frac{\partial \varepsilon_5}{\partial \tau} \\
\quad b(\sigma^e_6 - \beta_6) \frac{\partial \varepsilon_6}{\partial \tau}
\end{bmatrix}, \quad (10.7)
\]
where $i$ and $j$ are from 1 to 6. Similarly, $G_{ijkl}$ turns to be $G_{mn}$ which was shown in chapter 8 will be rewritten as

\[
\begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\] (10.8)

Substitute Eq.(10.7) and Eq.(10.8) into Eq.(10.3), we obtained

\[
\begin{align*}
\sigma_1(t) &= \int_0^t \{ A_1(\lambda + 2\mu) + A_2\lambda + A_3\lambda \} d\tau \\
\sigma_2(t) &= \int_0^t \{ A_1\lambda + A_2(\lambda + 2\mu) + A_3\lambda \} d\tau \\
\sigma_3(t) &= \int_0^t \{ A_1\lambda + A_2\lambda + A_3(\lambda + 2\mu) \} d\tau \\
\sigma_4(t) &= \int_0^t \{ (\sigma_4^e - \beta_4) \frac{\partial\epsilon_1}{\partial t} b\mu \} d\tau \\
\sigma_5(t) &= \int_0^t \{ (\sigma_5^e - \beta_5) \frac{\partial\epsilon_5}{\partial t} b\mu \} d\tau \\
\sigma_6(t) &= \int_0^t \{ (\sigma_6^e - \beta_6) \frac{\partial\epsilon_6}{\partial t} b\mu \} d\tau.
\end{align*}
\] (10.9)

where

\[
\begin{align*}
A_1 &= (a + 2b)(\sigma_1^e - \beta_1) \frac{\partial\epsilon_1}{\partial t} + a(\sigma_1^e - \beta_1) \frac{\partial\epsilon_2}{\partial t} + a(\sigma_1^e - \beta_1) \frac{\partial\epsilon_3}{\partial t} \\
A_2 &= a(\sigma_2^e - \beta_2) \frac{\partial\epsilon_1}{\partial t} + (a + 2b)(\sigma_2^e - \beta_2) \frac{\partial\epsilon_2}{\partial t} + a(\sigma_2^e - \beta_2) \frac{\partial\epsilon_3}{\partial t} \\
A_3 &= a(\sigma_3^e - \beta_3) \frac{\partial\epsilon_1}{\partial t} + a(\sigma_3^e - \beta_3) \frac{\partial\epsilon_2}{\partial t} + (a + 2b)(\sigma_3^e - \beta_3) \frac{\partial\epsilon_3}{\partial t}.
\end{align*}
\]

Equation (10.9) is the general constitutive relationship between stress and strain for
nonlinear deformation in a linear viscoelastic medium. The time rate of the strain becomes

\[
\begin{align*}
\frac{\partial \epsilon_1}{\partial t} &= \frac{\partial v_1}{\partial x_1}, \\
\frac{\partial \epsilon_2}{\partial t} &= \frac{\partial v_2}{\partial x_2}, \\
\frac{\partial \epsilon_3}{\partial t} &= \frac{\partial v_3}{\partial x_3}, \\
\frac{\partial \epsilon_4}{\partial t} &= \frac{1}{4} \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right), \\
\frac{\partial \epsilon_5}{\partial t} &= \frac{1}{4} \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right), \\
\frac{\partial \epsilon_6}{\partial t} &= \frac{1}{4} \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right),
\end{align*}
\]

(10.10)

where \( v_1, v_2 \) and \( v_3 \) are velocities of particles.

Aided by Eqs.(10.10), we then differentiate Eq. (10.9) with respect to time. We obtain

\[
\begin{align*}
\frac{\partial \sigma_1}{\partial t} &= \int_0^t \left\{ B_1 \frac{\partial (\lambda + 2\mu)}{\partial t} + B_2 \frac{\partial \lambda}{\partial t} + B_3 \frac{\partial \lambda}{\partial t} \right\} d\tau \\
\frac{\partial \sigma_2}{\partial t} &= \int_0^t \left\{ B_1 \frac{\partial \lambda}{\partial t} + B_2 \frac{\partial (\lambda + 2\mu)}{\partial t} + B_3 \frac{\partial \lambda}{\partial t} \right\} d\tau \\
\frac{\partial \sigma_3}{\partial t} &= \int_0^t \left\{ B_1 \frac{\partial \lambda}{\partial t} + B_2 \frac{\partial \lambda}{\partial t} + B_3 \frac{\partial (\lambda + 2\mu)}{\partial t} \right\} d\tau \\
\frac{\partial \sigma_4}{\partial t} &= \frac{1}{4} \int_0^t \left\{ (\sigma_4^e - \beta_4) \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) b \frac{\partial \mu}{\partial t} \right\} d\tau \\
\frac{\partial \sigma_5}{\partial t} &= \frac{1}{4} \int_0^t \left\{ (\sigma_5^e - \beta_5) \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) b \frac{\partial \mu}{\partial t} \right\} d\tau \\
\frac{\partial \sigma_6}{\partial t} &= \frac{1}{4} \int_0^t \left\{ (\sigma_6^e - \beta_6) \left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) b \frac{\partial \mu}{\partial t} \right\} d\tau.
\end{align*}
\]

(10.11)
where

\[
B_1 = (a + 2b)(\sigma_1^e - \beta_1)\frac{\partial v_1}{\partial x_1} + a(\sigma_1^e - \beta_1)\frac{\partial v_2}{\partial x_2} + a(\sigma_1^e - \beta_1)\frac{\partial v_3}{\partial x_3}
\]

\[
B_2 = a(\sigma_2^e - \beta_2)\frac{\partial v_1}{\partial x_1} + (a + 2b)(\sigma_2^e - \beta_2)\frac{\partial v_2}{\partial x_2} + a(\sigma_2^e - \beta_2)\frac{\partial v_3}{\partial x_3}
\]

\[
B_3 = a(\sigma_3^e - \beta_3)\frac{\partial v_1}{\partial x_1} + a(\sigma_3^e - \beta_3)\frac{\partial v_2}{\partial x_2} + (a + 2b)(\sigma_3^e - \beta_3)\frac{\partial v_3}{\partial x_3}.
\]

### 10.1.1 Internal Variable Method

The values of coefficients of this model such as \(a\), \(b\), \(\lambda\) and \(\mu\) can be got by following procedures. In this section, we transform the constitutive relation in the integral form into a differential form by using the internal variables [21, 113]. For completeness, details of applying the internal variable method to Eqs. (10.11) are given in Appendix A.3. After the derivation, Eqs. (10.11) becomes:

\[
\frac{\partial \sigma_1}{\partial t} = \left\{ \frac{\pi}{2} \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}^p}{\tau_{sl}} \right) \right] \right\} B_1 + \nu \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}^s}{\tau_{sl}} \right) \right] (B_2 + B_3) + \sum_{l=1}^{L} \gamma_l^1,
\]

\[
\frac{\partial \sigma_2}{\partial t} = \left\{ \frac{\pi}{2} \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}^p}{\tau_{sl}} \right) \right] \right\} B_2 + \nu \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}^s}{\tau_{sl}} \right) \right] (B_1 + B_3) + \sum_{l=1}^{L} \gamma_l^2,
\]

\[
\frac{\partial \sigma_3}{\partial t} = \left\{ \frac{\pi}{2} \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}^p}{\tau_{sl}} \right) \right] \right\} B_3 + \nu \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}^s}{\tau_{sl}} \right) \right] (B_1 + B_2) + \sum_{l=1}^{L} \gamma_l^3,
\]

\[
\frac{\partial \sigma_4}{\partial t} = \nu \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}^p}{\tau_{sl}} \right) \right] (\sigma_4^e - \beta_4) b \left[ \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right] + \sum_{l=1}^{L} \gamma_l^4.
\]

\[
\frac{\partial \sigma_5}{\partial t} = \nu \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}^p}{\tau_{sl}} \right) \right] (\sigma_5^e - \beta_5) b \left[ \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right] + \sum_{l=1}^{L} \gamma_l^5.
\]

\[
\frac{\partial \sigma_6}{\partial t} = \nu \left[ 1 - \sum_{l=1}^{L} \left( 1 - \frac{\tau_{sl}^p}{\tau_{sl}} \right) \right] (\sigma_6^e - \beta_6) b \left[ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right] + \sum_{l=1}^{L} \gamma_l^6.
\]

\[(10.12)\]
where $\gamma^l_i$ are given by

\[
\frac{\partial \gamma^l_1}{\partial t} = -\frac{1}{\tau_{\sigma l}} \left[ \gamma^l_1 + \pi \left( \frac{\tau^p_{\sigma l}}{\tau_{\sigma l}} - 1 \right) B_1 + \nu \left( \frac{\tau^p_{\sigma l}}{\tau_{\sigma l}} - 1 \right) (B_2 + B_3) \right],
\]

\[
\frac{\partial \gamma^l_2}{\partial t} = -\frac{1}{\tau_{\sigma l}} \left[ \gamma^l_2 + \pi \left( \frac{\tau^p_{\sigma l}}{\tau_{\sigma l}} - 1 \right) B_2 + \nu \left( \frac{\tau^p_{\sigma l}}{\tau_{\sigma l}} - 1 \right) (B_1 + B_3) \right],
\]

\[
\frac{\partial \gamma^l_3}{\partial t} = -\frac{1}{\tau_{\sigma l}} \left[ \gamma^l_3 + \pi \left( \frac{\tau^p_{\sigma l}}{\tau_{\sigma l}} - 1 \right) B_3 + \nu \left( \frac{\tau^p_{\sigma l}}{\tau_{\sigma l}} - 1 \right) (B_1 + B_2) \right],
\]

\[
\frac{\partial \gamma^l_4}{\partial t} = -\frac{1}{\tau_{\sigma l}} \left[ \gamma^l_4 + \nu \left( \frac{\tau^p_{\sigma l}}{\tau_{\sigma l}} - 1 \right) (\sigma^{e}_4 - \beta_4) b \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) \right],
\]

\[
\frac{\partial \gamma^l_5}{\partial t} = -\frac{1}{\tau_{\sigma l}} \left[ \gamma^l_5 + \nu \left( \frac{\tau^p_{\sigma l}}{\tau_{\sigma l}} - 1 \right) (\sigma^{e}_5 - \beta_5) b \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) \right],
\]

\[
\frac{\partial \gamma^l_6}{\partial t} = -\frac{1}{\tau_{\sigma l}} \left[ \gamma^l_6 + \nu \left( \frac{\tau^p_{\sigma l}}{\tau_{\sigma l}} - 1 \right) (\sigma^{e}_6 - \beta_6) b \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \right].
\]

(10.13)

10.2 The Governing Equations

In this section, we discuss the complete governing equations for waves in viscoelastic media. To proceed, we consider the equation of motion $\nabla \cdot \sigma = \rho \partial^2 \mathbf{u} / \partial t^2$, where $\sigma$ is the Cauchy stress tensor, $\rho$ is density, and $\mathbf{u}$ is the displacement vector. We assume that the body force is negligible. The independent variables are the position $\mathbf{x} = (x_1, x_2, x_3)$ and time $t$. By using the index notation, the equation of motion is rewritten as $\sigma_{i,j} = \rho \partial^2 u_i / \partial t^2$, where a subscript following a comma denotes partial differentiation with respect to the spatial coordinate. The velocity components $v_i = \partial u_i / \partial t$ are used as the unknowns instead of the displacement. The equation of motion is then coupled with the constitutive equations, Eqs. (10.12), and the equations for the internal variables, Eqs. (10.13), to form the complete set of the governing equations.
In what follows, we list the governing equations:

\[
\begin{align*}
\rho \frac{\partial v_1}{\partial t} &= \frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \sigma_6}{\partial x_2} + \frac{\partial \sigma_5}{\partial x_3}, \\
\rho \frac{\partial v_2}{\partial t} &= \frac{\partial \sigma_6}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \sigma_4}{\partial x_3}, \\
\rho \frac{\partial v_3}{\partial t} &= \frac{\partial \sigma_5}{\partial x_1} + \frac{\partial \sigma_4}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3},
\end{align*}
\]

(10.14)

The above first-order PDEs, e.g., Eqs. (10.12), Eqs. (10.13) and Eqs. (10.14) with one internal variable can be recast into a matrix-vector form:

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}_1 \frac{\partial \mathbf{U}}{\partial x_1} + \mathbf{A}_2 \frac{\partial \mathbf{U}}{\partial x_2} + \mathbf{A}_3 \frac{\partial \mathbf{U}}{\partial x_3} = \mathbf{S},
\]

(10.15)

where the unknown vector \( \mathbf{U} \) is defined as

\[
\mathbf{U} = [v_1, v_2, v_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \\
\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6]^T,
\]

the source term vector \( \mathbf{S} \) is

\[
\mathbf{S} = [0, 0, 0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \\
- \gamma_1/\tau_\sigma, - \gamma_2/\tau_\sigma, - \gamma_3/\tau_\sigma, \\
- \gamma_4/\tau_\sigma, - \gamma_5/\tau_\sigma, - \gamma_6/\tau_\sigma]^T,
\]

and \( \mathbf{A}_1, \mathbf{A}_2, \) and \( \mathbf{A}_3 \) are the Jacobian matrices.

The Jacobian matrix \( \mathbf{A}_1 \) is written as:

\[
\mathbf{A}_1 = \begin{bmatrix}
0_3 & \mathbf{A}_{1v} \\
\mathbf{A}_{1\sigma v} & 0_{12}
\end{bmatrix},
\]

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where

\[
A_{1v} = \begin{bmatrix}
-1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

\[
A_{1\sigma v} = \begin{bmatrix}
A_{1\sigma v1} & A_{1\sigma v2}
\end{bmatrix}
\]

where

\[
A_{1\sigma v1} = \\
\begin{bmatrix}
-\pi \frac{\tau^p}{\tau^\sigma}(a + 2b)(\sigma_1 - \beta_1) - \nu \frac{\tau^s}{\tau^\sigma}a(\sigma_2 + \sigma_3 - \beta_2 - \beta_3) \\
-\pi \frac{\tau^p}{\tau^\sigma}a(\sigma_2 - \beta_2) - \nu \frac{\tau^s}{\tau^\sigma}[(a + 2b)(\sigma_1 - \beta_1) + a(\sigma_3 - \beta_3)] \\
-\pi \frac{\tau^p}{\tau^\sigma}a(\sigma_3 - \beta_3) - \nu \frac{\tau^s}{\tau^\sigma}[(a + 2b)(\sigma_1 - \beta_1) + a(\sigma_2 - \beta_2)] \\
0 \\
0 \\
0 \\
\pi \frac{\tau^p}{\tau^\sigma}(\frac{\tau^p}{\tau^\sigma} - 1)(a + 2b)(\sigma_1 - \beta_1) + \nu \frac{\tau^s}{\tau^\sigma}(\frac{\tau^s}{\tau^\sigma} - 1)a(\sigma_2 + \sigma_3 - \beta_2 - \beta_3) \\
\pi \frac{\tau^p}{\tau^\sigma}(\frac{\tau^p}{\tau^\sigma} - 1)a(\sigma_2 - \beta_2) + \nu \frac{\tau^s}{\tau^\sigma}(\frac{\tau^s}{\tau^\sigma} - 1)[(a + 2b)(\sigma_1 - \beta_1) + a(\sigma_3 - \beta_3)] \\
\pi \frac{\tau^p}{\tau^\sigma}(\frac{\tau^p}{\tau^\sigma} - 1)a(\sigma_3 - \beta_3) + \nu \frac{\tau^s}{\tau^\sigma}(\frac{\tau^s}{\tau^\sigma} - 1)[(a + 2b)(\sigma_1 - \beta_1) + a(\sigma_2 - \beta_2)] \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]
and

\[ A_{1\sigma v} = \]

\[
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & \frac{-\nu b \tau_s}{4} (\sigma_5 - \beta_5) \\
0 & \frac{-b \tau_s}{4 \tau_\sigma} (\sigma_6 - \beta_6) \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & \frac{\nu}{4 \tau_\sigma} (\frac{\tau_s}{\tau_\sigma} - 1) b (\sigma_5 - \beta_5) \\
\frac{\nu}{4 \tau_\sigma} (\frac{\tau_s}{\tau_\sigma} - 1) b (\sigma_6 - \beta_6) & 0
\end{bmatrix}
\]

and \( 0_3 \) and \( 0_{12} \) denote \( 3 \times 3 \) and \( 12 \times 12 \) null matrices. Based on the Schur complement [97], the non-trivial eigenvalues of \( A_1 \) can be obtained by solving the following equation:

\[ \det(A_{1\tau} A_{1\sigma} - \beta^2 I_3) = 0. \]
where \( \beta \) is the eigenvalues of \( A_1 \) and

\[
A_{1v}A_{1\sigma v} = \begin{bmatrix}
\pi \frac{\tau_p}{\rho \tau_\sigma} (a + 2b)(\sigma_1 - \beta_1) & 0 & 0 \\
\nu \frac{\tau_\sigma}{\rho \tau_\sigma} a(\sigma_2 + \sigma_3 - \beta_2 - \beta_3) & \frac{\nu}{4\rho} \frac{\tau_\sigma}{\tau_\sigma} (\sigma_5 - \beta_5) & 0 \\
0 & 0 & \frac{\nu}{4\rho} \frac{\tau_\sigma}{\tau_\sigma} (\sigma_6 - \beta_6)
\end{bmatrix}.
\]

\( A_1 \) has 15 eigenvalues: 9 of them are null while the remaining 6 are

\[
\beta_{1,2} = \pm \sqrt{\pi \frac{\tau_p}{\rho \tau_\sigma} (a + 2b)(\sigma_1 - \beta_1) + \nu \frac{\tau_\sigma}{\rho \tau_\sigma} a(\sigma_2 + \sigma_3 - \beta_2 - \beta_3)}, \tag{10.16}
\]

\[
\beta_{3,4} = \pm \sqrt{\frac{\nu}{4\rho} \frac{\tau_\sigma}{\tau_\sigma} (\sigma_5 - \beta_5)}, \quad \beta_{5,6} = \pm \sqrt{\frac{\nu}{4\rho} \frac{\tau_\sigma}{\tau_\sigma} (\sigma_6 - \beta_6)}. \tag{10.17}
\]

Similar expressions can be obtained for the eigenvalues of \( A_2 \) and \( A_3 \).

### 10.2.1 One-Dimensional Equation with Five Internal Variables

We let \( \partial/\partial x_2 = 0 \) and \( \partial/\partial x_3 = 0 \) and Eq. (10.15) is reduced to the one-dimensional governing equations for waves propagation along the \( x_1 \) axis. The resultant equations can be divided into four groups. The first group includes the following three equations:
\[
\frac{\partial v_1}{\partial t} = \frac{\partial \sigma_1}{\partial x_1},
\]

\[
\frac{\partial \sigma_1}{\partial t} = \left\{ \pi \left[ 1 - \sum_{i=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{pl}} \right) \right] \right\} B_1 + \nu \left[ 1 - \sum_{i=1}^{5} \left( 1 - \frac{\tau_{sl}}{\tau_{sl}} \right) \right] (B_2 + B_3) + \sum_{i=1}^{L} \gamma_i^1,
\]

\[
\frac{\partial \sigma_2}{\partial t} = \left\{ \pi \left[ 1 - \sum_{i=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{pl}} \right) \right] \right\} B_2 + \nu \left[ 1 - \sum_{i=1}^{5} \left( 1 - \frac{\tau_{sl}}{\tau_{sl}} \right) \right] (B_1 + B_3) + \sum_{i=1}^{L} \gamma_i^2,
\]

\[
\frac{\partial \sigma_3}{\partial t} = \left\{ \pi \left[ 1 - \sum_{i=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{pl}} \right) \right] \right\} B_3 + \nu \left[ 1 - \sum_{i=1}^{5} \left( 1 - \frac{\tau_{sl}}{\tau_{sl}} \right) \right] (B_1 + B_2) + \sum_{i=1}^{L} \gamma_i^3,
\]

\[
\frac{\partial \gamma_{l1}}{\partial t} = -\frac{1}{\tau_{sl}} \left[ \gamma_{l1} + \pi \left( \frac{\tau_{pl}}{\tau_{pl}} - 1 \right) B_1 + \nu \left( \frac{\tau_{sl}}{\tau_{sl}} - 1 \right) (B_2 + B_3) \right],
\]

\[
\frac{\partial \gamma_{l2}}{\partial t} = -\frac{1}{\tau_{sl}} \left[ \gamma_{l2} + \pi \left( \frac{\tau_{pl}}{\tau_{pl}} - 1 \right) B_2 + \nu \left( \frac{\tau_{sl}}{\tau_{sl}} - 1 \right) (B_1 + B_3) \right],
\]

\[
\frac{\partial \gamma_{l3}}{\partial t} = -\frac{1}{\tau_{sl}} \left[ \gamma_{l3} + \pi \left( \frac{\tau_{pl}}{\tau_{pl}} - 1 \right) B_3 + \nu \left( \frac{\tau_{sl}}{\tau_{sl}} - 1 \right) (B_1 + B_2) \right], l = 1, 2, 3, 4, 5.
\]

(10.18)

To proceed, we rewrite the equations into a matrix-vector form:

\[
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}_t \frac{\partial \mathbf{V}}{\partial t} = \mathbf{Q}.
\]

(10.19)

where the unknown vector \( \mathbf{V} = [v_1, \sigma_1, \sigma_2, \sigma_3, \gamma_{l1}, \cdots, \gamma_{l5}^1, \cdots, \gamma_{l2}, \cdots, \gamma_{l5}^2, \cdots, \gamma_{l3}, \cdots, \gamma_{l5}^3, \cdots, \gamma_{l5}^5]^T \).

There are 19 equations and 19 unknowns. The source term

\[
\mathbf{Q} = [0, \sum_{l=1}^{5} \gamma_{l1}, \sum_{l=1}^{5} \gamma_{l2}, \sum_{l=1}^{5} \gamma_{l3}, -\gamma_{l1} / \tau_{sl}, \cdots, -\gamma_{l5} / \tau_{sl},
\]

\[
-\gamma_{l1} / \tau_{sl}, \cdots, -\gamma_{l5} / \tau_{sl}, -\gamma_{l3} / \tau_{sl}, \cdots, -\gamma_{l5} / \tau_{sl},]^T,
\]

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and the Jacobian matrix

\[
\mathbf{A}_1 = \begin{bmatrix}
\mathbf{A}_{1v} & \mathbf{0}_{3 \times 16} \\
\mathbf{A}_{1T} & \mathbf{0}_{16 \times 16}
\end{bmatrix},
\]

where

\[
\mathbf{A}_{1v} = \begin{bmatrix}
0 & -1/ho & 0 \\
\left\{ \pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{ol}} \right) \right] \right\} (a + 2b)(\sigma_1 - \beta_1) & 0 & 0 \\
+ \nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{ol}} \right) \right] a(\sigma_2 + \sigma_3 - \beta_2 - \beta_3) & 0 & 0 \\
\left\{ \pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{ol}} \right) \right] \right\} a(\sigma_2 - \beta_2) & 0 & 0 \\
+ \nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau_{pl}}{\tau_{ol}} \right) \right] [(a + 2b)(\sigma_1 - \beta_1) + a(\sigma_3 - \beta_3)] & 0 & 0
\end{bmatrix}.
\]
\[
\bar{A}_{1T} = \\
\begin{bmatrix}
\left\{ \begin{array}{c}
\pi \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^p_l}{\tau_{\sigma l}} \right) \right] \end{array} \right\} a(\sigma_3 - \beta_3) \\
+ \nu \left[ 1 - \sum_{l=1}^{5} \left( 1 - \frac{\tau^p_l}{\tau_{\sigma l}} \right) \right] [(a + 2b)(\sigma_1 - \beta_1) + a(\sigma_2 - \beta_2)] \\
\frac{1}{\tau_{\sigma 1}} \left[ \pi \left( \frac{\tau^p_{\sigma 1}}{\tau_{\sigma 1}} - 1 \right) (a + 2b)(\sigma_1 - \beta_1) \\
+ \nu \left( \frac{\tau^p_{\sigma 1}}{\tau_{\sigma 1}} - 1 \right) a(\sigma_2 + \sigma_3 - \beta_2 - \beta_3) \right] \\
\vdots \\
\frac{1}{\tau_{\sigma 1}} \left[ \pi \left( \frac{\tau^p_{\sigma 1}}{\tau_{\sigma 1}} - 1 \right) a(\sigma_2 - \beta_2) \\
+ \nu \left( \frac{\tau^p_{\sigma 1}}{\tau_{\sigma 1}} - 1 \right) [(a + 2b)(\sigma_1 - \beta_1) + a(\sigma_3 - \beta_3)] \right] \\
\vdots \\
\frac{1}{\tau_{\sigma 1}} \left[ \pi \left( \frac{\tau^p_{\sigma 1}}{\tau_{\sigma 1}} - 1 \right) a(\sigma_3 - \beta_3) \\
+ \nu \left( \frac{\tau^p_{\sigma 1}}{\tau_{\sigma 1}} - 1 \right) [(a + 2b)(\sigma_1 - \beta_1) + a(\sigma_3 - \beta_3)] \right] \\
\vdots 
\end{bmatrix}
\]
CHAPTER 11
FIRST-ORDER EULERIAN EQUATIONS FOR
HYPO-PLASTICITY

11.1 Introduction

General theories of elastic-plastic waves has been treated extensively and reviewed from various points of view in the literature, e.g., Nowacki [103], Clifton [35], Cristescu [39], Craggs [38] Herrmann ([104]). Numerical simulations of wave motion in elastic-plastic media were reported by Buchar et al ([15]) with aid of finite element analysis.

Trangenstein and Collella [127] studied finite deformation in elastic-plastic solids by using hyper-elastic constitutive and kinematic evolution equation. By using higher-order Godunov scheme, they mainly studied wave propagation in hyperelastic solids for one dimensional simulation. Miller and Collella [98] extend this work in hyperelasticity and visco-plasticity to multiple dimension. Hill et al. [62] used a hybrid of the weighted essentially non-oscillatory schemes combined with explicit centered difference to solve the equations of motion expressed in an Eulerian formulation. This formulation allows for a wide range of constitutive relations. Giese [55] studied elastic-plastic wave in three space dimensions. Since the governing equations are composed of two part. He solve the flux equation by using method of transport and integrate the stress-strain relationship in time with a high order ODE solver. In order to understand the formation of the plastic zone at the crack tip, Lin and Ballmann
[86] used a characteristic-based difference method to simulate elastic-plastic wave propagation in two-dimensional anisotropic plane strain problems. Their study focuses on small plastic deformation problems. Tran and Udaykumar [124] developed an Eulerian, sharp interface, Cartesian grid method to simulate impact and denotation. Since energy equations are considered, the Mie-Gruneisen equation of state is used to obtain pressure. The Essentially non-oscillatory scheme is employed to capture shocks and sharp immersed boundaries are captured by using a hybrid particle level set technique. In this paper, we will employ the Conservation Element and Solution Element (CESE) method [24], an explicit space-time finite-volume scheme, to solve our system of nonlinear elasto-dynamic model equations. It has been formerly used to solve dynamics and combustion problems, including detonations, cavitations, flows with complex shock structures [136, 151]. Recently, it also employed to solve problems in solids structures, see, for instance. Refs. [34, 146, 147]. Since the application we are interested in is acoustical ultrasonic welding process, the thermo-effect is ignored which was demonstrated by experiment. So in this paper, we report a novel theoretical and numerical approach to model elastic-plastic wave motion in solids without energy equation included. We extend the isothermal model for modeling stress wave propagation in elastic-plastic media. by using a suitable constitutive equation to model the material response of elastic-plastic material.

The rest of the present paper is organized as follows. Section 11.2 illustrates the basic formulation of constitutive relation for plasticity of solids. Section 11.3 summarizes the model equations, including the continuity, momentum, and constitutive relations. The model equations are cast into a vector-matrix form. The Jacobian matrix of the One-dimensional equations are analyzed to show the eigenvalues, which represent the wave speeds. We then offer the concluding remarks.

11.2 Hypo-Elastic-Plastic Solids

The constitutive relation of Hypo-elastic-plastic media is developed in this section. We first develop the elastic-plastic constitutive equation based on the infinitesimal theory. We then generalize the model for problems with finite deformations by using Jaumann rate. The medium of interest is assumed isotropic, homogeneous, non-porous, and metallic. For the constitutive relation, we adopt the customary assumptions of incompressibility of the plastic strain, yield insensitivity to the spherical/hydrostatic part of the stress, and, for the sake of simplicity, strain-rate-independent response. To proceed, based on observation, infinitesimal plasticity admits an additive decomposition of the infinitesimal strain tensor

\[ \varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}, \]  

(11.1)

where \( \varepsilon_{ij} \) is the strain with \( \varepsilon^e_{ij} \) and \( \varepsilon^p_{ij} \) as the elastic and plastic contributions, respectively. Equivalently, Eq. (11.1) can be expressed in an incremental form:

\[ d\varepsilon_{ij} = d\varepsilon^e_{ij} + d\varepsilon^p_{ij}, \]  

(11.2)

where \( d\varepsilon_{ij}, d\varepsilon^e_{ij}, \) and \( d\varepsilon^p_{ij} \) are the cumulative, elastic, and plastic strain increments, respectively. Aided by the above definitions, the constitutive equation for a homogeneous isotropic material in the linear elastic regime is

\[ \varepsilon^e_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij}, \]  

(11.3)

which can be expressed in an incremental form as

\[ d\varepsilon^e_{ij} = \frac{1 + \nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij}. \]  

(11.4)

In the above equation, \( \sigma_{ij} \) is the stress tensor, \( d\sigma_{ij} \) is the stress increment, and Young’s modulus \( E \) and Poisson’s ratio \( \nu \) are material-dependent constants. To proceed, we
consider the equation of the yield surface of a material which undergoes isotropic strain-hardening:

\[ F(\sigma_{ij}, \varepsilon_{ij}^p) = 0, \]  

(11.5)

where \( F(\sigma_{ij}, \varepsilon_{ij}^p) \) is a scalar-valued yield function, whose form is made explicit by the chosen yield criterion, e.g., von Mises or Tresca. Essentially, the yield surface is the union of all points in the stress space that satisfy Eq. (11.5). The associated flow rule is

\[ d\varepsilon_{ij}^p = d\lambda \frac{\partial F}{\partial \sigma_{ij}}, \]  

(11.6)

which implies normality of the plastic strain increment with respect to the yield surface defined in the stress space. In Eq. (11.6), \( d\lambda \) is a scalar function that represents the magnitude of the plastic strain increment. \( d\lambda \) will be made explicit in the following. To proceed, the loading criteria are

\[
\begin{align*}
F < 0 & \quad \text{elastic deformation} \\
F = 0, \quad \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} > 0 & \quad \text{plastic loading} \\
F = 0, \quad \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} = 0 & \quad \text{neutral loading} \\
F = 0, \quad \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} < 0 & \quad \text{elastic unloading}
\end{align*}
\]

Plastic strain only occurs during plastic loading. Otherwise, the plastic strain increment vanishes. Thus, to deduce \( d\lambda \) in Eq. (11.6), we adopt the plastic loading criteria \( F = 0, \) and \( (\partial F/\partial \sigma_{ij}) d\sigma_{ij} > 0 \) in the following discussions.

Based on observation, strain-rate-insensitive material would harden during plastic deformation. And points on the original yield surface remain on all subsequent yield
surfaces. This observation, together with Eq. (11.5) and aided by the chain rule, imply the consistency condition:

\[ dF = \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p = 0. \quad (11.7) \]

Aide by Eqs. (11.6) and (11.7), we have

\[ d\lambda = -\frac{\partial F}{\partial \sigma_{ij}} \frac{d\sigma_{ij}}{d\sigma_{kl}}. \quad (11.8) \]

We employ the von Mises yield criterion, i.e., the $J_2$ flow theory, which specifies the yield function $F(\sigma_{ij}, \varepsilon_{ij}^p)$ in Eq. (11.5) as

\[ F(S_{ij}, \varepsilon_{ij}^p) = \frac{1}{2} S_{ij} S_{ij} - \frac{1}{3} (\sigma^y(\varepsilon_{ij}^p))^2 = 0, \quad (11.9) \]

where $J_2 = S_{ij} S_{ij}/2$ is the second invariant of the deviatoric stress tensor $S_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$, and $\sigma^y(\varepsilon_{ij}^p)$ is the yield stress in uniaxial tension, which evolves with plastic strain as the material hardens during plastic deformation. Using

\[ \frac{\partial F}{\partial \sigma_{ij}} = S_{ij}, \quad S_{ij} d\sigma_{ij} = \frac{2}{3} \sigma^y d\sigma^y, \quad \frac{\partial F}{\partial \varepsilon_{kl}^p} = -\frac{2}{3} \sigma^y \frac{d\sigma^y}{d\varepsilon_{kl}^p} \quad (11.10) \]

in Eq. (11.8) yields

\[ d\lambda = \frac{d\sigma^y}{d\varepsilon_{kl}^p} S_{kl}. \quad (11.11) \]

For the purpose of characterizing $d\sigma^y/d\varepsilon_{kl}^p$ in Eq. (11.11), we introduce the effective stress $\bar{\sigma}$ and the effective plastic strain $\varepsilon^p$, defined by

\[ \bar{\sigma} = \sqrt{\frac{3}{2} S_{ij} S_{ij}, \quad \varepsilon^p = \sqrt{\frac{2}{3} \varepsilon_{ij}^p \varepsilon_{ij}^p}. \quad (11.12) \]

Equations (11.12) imply that the effective stress, or, more precisely in this case, the effective stress encapsulating a fully three-dimensional state of stress that satisfies the yield criterion, is equivalent to the yield stress in uniaxial tension, i.e.,

\[ \bar{\sigma}(\varepsilon_{ij}^p) = \sigma^y(\varepsilon_{ij}^p), \quad (11.13) \]
from which
\[ F(S_{ij}, \varepsilon^p_{ij}) = \frac{1}{2} S_{ij} S_{ij} - \frac{1}{3} (\bar{\sigma}(\varepsilon^p_{ij}))^2 = 0 \] (11.14)

and
\[ d\lambda = \frac{d\bar{\sigma}}{d\bar{\varepsilon}_{kl} S_{kl}} \] (11.15)

follow. For strain-hardening materials, the tensile yield stress evolves with plastic deformation through the effective plastic strain. Thus, use of the chain rule gives
\[ \frac{d\bar{\sigma}}{d\bar{\varepsilon}_{kl}} S_{kl} = \frac{d\bar{\sigma}}{d\bar{\varepsilon}_{kl} S_{kl}} d\bar{\varepsilon}_{kl}, \quad dW^p = \bar{\sigma} d\bar{\varepsilon}_p, \] (11.16)

where we have used the results
\[ dW^p = S_{ij} d\varepsilon^p_{ij}, \quad dW^p = \bar{\sigma} d\bar{\varepsilon}_p, \] (11.17)

where \( dW^p \) is the plastic work increment. Substitution of Eq. (11.16) into Eq. (11.15), with the aid of Eq. (11.14), yields
\[ d\lambda = \frac{3}{2} \frac{d\bar{\sigma}}{\bar{\sigma} d\bar{\varepsilon}_p} S_{ij}. \] (11.18)

which, in conjunction with the flow rule, gives the plastic strain increment
\[ d\varepsilon^p_{ij} = \frac{3}{2} \frac{d\bar{\sigma}}{\bar{\sigma} d\bar{\varepsilon}_p} S_{ij}. \] (11.19)

For a linear strain-hardening material, the tensile yield stress increases linearly with the effective plastic strain, i.e.,
\[ \sigma^y(\bar{\varepsilon}^p) = \sigma^y_o + B_{SH} \bar{\varepsilon}^p, \] (11.20)

where the initial tensile yield stress \( \sigma^y_o \) and the strength coefficient \( B_{SH} \) are material-dependent constants. It follows from Eqs. (11.13) and (11.20) that
\[ \frac{d\bar{\sigma}}{d\bar{\varepsilon}_p} = B_{SH} \] (11.21)
and thus

\[ d\varepsilon_{ij}^p = \frac{3}{2} \frac{d\bar{\sigma}}{B_{SH} \bar{\sigma}} S_{ij}. \]  

(11.22)

As infinitesimal plasticity admits an additive decomposition of the elastic and plastic strain increments, Eqs. (11.3) and (11.22) are linearly combined to obtain

\[ d\varepsilon_{ij} = \frac{1 + \nu}{E} d\sigma_{ij} + \nu \frac{d\sigma_{kk} \delta_{ij}}{E} + \frac{3}{2} \frac{d\bar{\sigma}}{B_{SH} \bar{\sigma}} S_{ij}, \]  

(11.23)

which can be rewritten as

\[ \frac{1 + \nu}{E} d\sigma_{ij} = d\varepsilon_{ij} + \nu \frac{d\sigma_{kk} \delta_{ij}}{E} - \frac{3}{2} \frac{d\bar{\sigma}}{B_{SH} \bar{\sigma}} S_{ij}. \]  

(11.24)

Taking the inner product of the deviatoric stress \( S_{ij} \) and Eq. (11.23), and subsequently using Eqs. (11.10) and (11.14), allows us to solve for the effective stress increment

\[ d\bar{\sigma} = \frac{S_{ij} d\varepsilon_{ij}}{\left( \frac{2}{3} \frac{1 + \nu}{E} + \frac{1}{B_{SH}} \right) \bar{\sigma}}. \]  

(11.25)

Taking the trace of Eq. (11.3) and assuming plastic incompressibility, i.e., the plastic strain increment \( d\varepsilon_{ij}^p \) is traceless, yields the following relationship between the mean stress increment and the volumetric strain increment:

\[ d\sigma_{ii} = \frac{E}{1 - 2\nu} d\varepsilon_{ii}. \]  

(11.26)

Substitution of Eqs. (11.25) and (11.26) into Eq. (11.24) gives the deviatoric stress increment

\[ dS_{ij} = 2\mu d\varepsilon_{ij} - \frac{2}{3} \mu d\varepsilon_{tt} \delta_{ij} - 3\mu \frac{S_{kl} d\varepsilon_{kl}}{\left( \frac{B_{SH}}{2\mu} \right) \left( \frac{3}{2} \right) S_{mn} S_{mn}} S_{ij}, \]  

(11.27)

where we have used Eq. (11.14); the relationship \( \mu = \frac{E}{2(1 + \nu)} \) between the shear modulus \( \mu \), Young’s modulus \( E \), and Poisson’s ratio \( \nu \); and the definition \( dS_{ij} = d\sigma_{ij} - \frac{1}{3} d\sigma_{kk} \delta_{ij} \). Equivalently, Eq. (11.27) may be expressed in rate form as

\[ \dot{S}_{ij} = 2\mu \dot{\varepsilon}_{ij} - \frac{2}{3} \mu \dot{\varepsilon}_{tt} \delta_{ij} - 3\mu \frac{S_{kl} \dot{\varepsilon}_{kl}}{\left( \frac{B_{SH}}{2\mu} \right) \left( \frac{3}{2} \right) S_{mn} S_{mn}} S_{ij}. \]  

(11.28)
To proceed, we generalize the infinitesimal elastic-plastic constitutive equation, Eq. (11.28), to finite deformations by (i) replacing the infinitesimal strain rate $\dot{\varepsilon}_{ij}$ with its finite Eulerian analog $D_{ij}$, the rate of deformation, (ii) replacing the stress tensor $\sigma_{ij}$ with its finite Eulerian analog $T_{ij}$, the Cauchy stress, and (iii) employing an objective rate $D/Dt$ to the stress. The objective rate would ensure that the constitutive equation is invariant under an arbitrary superposed rigid body motion. One common choice is the Jaumann rate:

$$\frac{DS_{ij}}{Dt} = \frac{\partial S_{ij}}{\partial t} + v_k \frac{\partial S_{ij}}{\partial x_k} - W_{ik} S_{kj} + S_{ik} W_{kj},$$

(11.29)

where $W_{ij} = (\partial v_i/\partial x_j - \partial v_j/\partial x_i)$ is the skew part of the velocity gradient. The resultant constitutive equation becomes

$$\frac{D}{Dt} S_{ij} = 2\mu D_{ij} - 2\frac{\mu}{3} D_{kk} \delta_{ij} - 3\mu \left( \frac{B_{ij}}{2\mu} + \frac{3}{2} \right) S_{mn} S_{mn} S_{ij},$$

(11.30)

where $D_{ij} = 1/2 (L_{ij} + L_{ji})$ is the symmetric part of the Eulerian velocity gradient $L_{ij} = \partial v_i/\partial x_j$, $v_i$ is the velocity, and the first two terms in Eq. (11.30) reflect elastic contributions to the deviatoric stress, while the final term represents the plastic contribution.

### 11.3 Governing Equations

Based on the above constitutive relation, the three-dimensional governing equations for elastic-plastic wave motion formulated in the Eulerian frame are *Conservation of mass:*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho v_i) = 0,$$

(11.31)

*Conservation of linear momentum:*

$$\frac{\partial}{\partial t} (\rho v_i) + \frac{\partial}{\partial x_j} (\rho v_i v_j + p - S_{ij}) = 0$$

(11.32)
and Cauchy stress components and pressure are $T_{ij} = -p + S_{ij}$, $p = -\sum_{i=1}^{3} T_{ii}/3$. 

Elastic-plastic constitutive relation:

$$\frac{D}{Dt}S_{ij} = 2\mu D_{ij} - \frac{2}{3}\mu D_{kk}\delta_{ij} - \beta(s)S_{kl}D_{kl}S_{ij}, \quad (11.33)$$

where $s = S_{mn}S_{mn}$ and

$$\beta(s) = \begin{cases} 
0 & \text{if } F < 0 \\
0 & \text{if } F = 0 \text{ and } \frac{\partial F}{\partial \sigma_{ij}}d\sigma_{ij} < 0 \\
0 & \text{if } F = 0 \text{ and } \frac{\partial F}{\partial \sigma_{ij}}d\sigma_{ij} = 0 \\
\frac{6\mu^2}{(3\mu + B_{SH})s} & \text{if } F = 0 \text{ and } \frac{\partial F}{\partial \sigma_{ij}}d\sigma_{ij} > 0
\end{cases} \quad (11.34)$$

The four rows of Eq. (11.37) represent elastic deformation, elastic unloading, neutral loading, and plastic loading, respectively. In Eqs. (11.31)-(11.37), $\rho$ is the density, $v_i$ is the velocity, $\bar{p} = -T_{kk}/3$ is the mean pressure, and $D_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$ is the symmetric part of the velocity gradient, all functions of time $t$ and the position $\mathbf{x} = \{x_1, x_2, x_3\}$ of a typical material particle in the current configuration referred to a fixed Cartesian basis. The shear modulus $\mu$ is a prescribed material constant. If $J_2$ flow plastic theory is adopted, we have

$$\frac{\partial F}{\partial \sigma_{ij}}d\sigma_{ij} = d(\frac{1}{2}S_{ij}S_{ij} - k^2), \quad (11.35)$$

where $k$ is a constant for perfect plastic materials and in general it is

$$k^2 = \frac{1}{3}(\bar{\sigma}(\varepsilon_p^{eq}))^2.$$ 

Then Eq.(11.36) and Eq.(11.37) can be written as

$$\frac{D}{Dt}S_{ij} = 2\mu D_{ij} - \frac{2}{3}\mu D_{kk}\delta_{ij} - \beta(s)S_{kl}D_{kl}S_{ij}, \quad (11.36)$$

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where $s = S_{mn}S_{mn}$ and

$$
\beta(s) = \begin{cases} 
0 & \text{if } F < 0 \\
0 & \text{if } F = 0 \text{ and } d\left(\frac{1}{2}S_{ij}S_{ij} - k^2\right) \leq 0 \\
\frac{6\mu^2}{(3\mu + B_{SH}) s} & \text{if } F = 0 \text{ and } d\left(\frac{1}{2}S_{ij}S_{ij} - k^2\right) > 0 
\end{cases}
$$

(11.37)

1-D Formulation

The modeling equations of the elastic-plastic stress waves include the mass, momentum conservation equation and constitutive equation. For one-dimensional cases (ignore $\partial/\partial x_2, \partial/\partial x_3$), the mass and momentum equations are

$$
\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_1}{\partial x_1} = 0, 
$$

(11.38)

$$
\frac{\partial \rho v_1}{\partial t} + \frac{\partial}{\partial x_1} (\rho v_1 v_1 + p - S_{11}) = 0, 
$$

(11.39)

where $p$ is mean stress and $S_{11}$ is deviatoric stress. Moreover, for an elastic-plastic constitutive equation in one spatial dimension strain problem, the nonzero deviatoric stress, e.g., $S_{ij}$ are $S_{11}, S_{22}, S_{33}$ and nonzero term in $D_{ij}$ is $D_{11}$. After assume $\partial/\partial x_2$ and $\partial/\partial x_3$ is zero, the 3-d elastic-plastic constitutive equation can be simplified to be

$$
\frac{\partial \rho S_{11}}{\partial t} + \frac{\partial \rho v_1 S_{11}}{\partial x_1} = \frac{4}{3}\mu \rho \left(1 - \frac{\beta}{1 + \frac{2\mu}{3\rho}}\right) \frac{\partial v_1}{\partial x_1},
$$

(11.40)

where

$$
\beta = \begin{cases} 
0 & \text{if } F < 0 \\
0 & \text{if } F = 0 \text{ and } d\left(\frac{1}{2}S_{ij}S_{ij} - k^2\right) \leq 0 \\
1 & \text{if } F = 0 \text{ and } d\left(\frac{1}{2}S_{ij}S_{ij} - k^2\right) > 0 
\end{cases}
$$

(11.41)

where $J_2$ yielding criterion has been used for this one dimensional stress case. And $B_{SH}$ is the modulus for the elastic-linear hardening plastic material.

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Based on the previous analysis, Eqs. (11.38), (11.39) and (11.40) could generate a hyperbolic system, which could be written in vector form as:

\[
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x_1} = \mathbf{H} \tag{11.42}
\]

where

\[
\mathbf{U} = (\rho, \rho v_1, \rho S_{11})^T, \\
\mathbf{E} = (\rho v_1, \rho v_1 v_1 + p - S_{11}, \rho S_{11} v_1)^T, \\
\mathbf{H} = \left(0, 0, \frac{4}{3} \mu \rho \left(1 - \frac{\beta}{1 + B_{SH}/3 \mu} \right) \frac{\partial v_1}{\partial x_1} \right)^T.
\]

To close the system of equations, we employ the following equation of state to relate pressure to density:

\[
p = k \ln \frac{\rho}{\rho_o} + p_0. \tag{11.43}
\]

By analyzing the eigen-structure of this hyperbolic system, we directly calculate the speed of sound in the solid with plastic deformation. In order for numerical calculation, Eq. (11.42) can be recast to be

\[
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x_1} = \mathbf{H} \tag{11.44}
\]

where

\[
\mathbf{A} = \begin{pmatrix}
0 & S_{11} & 1 & 0 \\
-v_1^2 + \frac{k}{\rho} + \frac{S_{11}}{\rho} & 2v_1 & -1 & \rho \\
-v_1 S_{11} & S_{11} & v_1 \\
\end{pmatrix} \tag{11.45}
\]

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and

\[
H = \left[ 0, 0, \frac{4}{3} \mu \rho \left( 1 - \frac{\beta}{1 + B_{SH}/3\mu} \right) \frac{\partial v_1}{\partial x} \right]^T
\]  

Alternatively, we rewrite the above hyperbolic system by using the non-conservative variables vector

\[
\tilde{U} = (\rho, v_1, S_{11})^T.
\]

We have non-conservative form as:

\[
\frac{\partial \tilde{U}}{\partial t} + \tilde{A} \frac{\partial \tilde{E}}{\partial x_1} = \tilde{H}
\]

\[
\tilde{A} = \text{MAM}^{-1} = \begin{pmatrix}
    v_1 & \rho & 0 \\
    k & v_1 & \frac{1}{\rho} \\
    0 & 0 & v_1
\end{pmatrix}
\]

and

\[
\tilde{H} = \left[ 0, 0, \frac{4}{3} \mu \left( 1 - \frac{\beta}{1 + B_{SH}/3\mu} \right) \frac{\partial v_1}{\partial x} \right]^T
\]

By moving the source term from right side to the left side, we could transform Eq.(11.48) into the form as

\[
\frac{\partial \tilde{U}}{\partial t} + \tilde{A} \frac{\partial \tilde{U}}{\partial x_1} = 0
\]
where

\[
\begin{pmatrix}
\begin{bmatrix}
\frac{v_1}{\rho^2} & \frac{1}{\rho} & 0 \\
\frac{k}{\rho} & v_1 & \frac{1}{\rho} \\
0 & -4\mu & \frac{\beta}{1 + \frac{B_{SH}}{3\mu}} \frac{\partial v_1}{\partial x} & v_1
\end{bmatrix}
\end{pmatrix}
\]  

(11.52)

The eigenvalues of matrix \( \mathbf{A} \) can be readily derived and they are

\[
\lambda_1 = v_1, \lambda_{2,3} = v_1 \pm c = v_1 \pm \sqrt{\frac{K + 4\mu(1 - \frac{\beta}{1 + \frac{B_{SH}}{3\mu}})}{\rho}},
\]

(11.53)

where \( K \) is bulk modulus and \( B_{SH} \) is strength coefficient given by Eq.(11.21). \( \mu \) is shear modulus. It can be seen that wave speed is \( \sqrt{(k + \frac{4\mu}{3})/\rho} \) by letting \( \beta = 0 \). It can be seen that this plastic wave speed is slower than elastic wave speed in the bulk material. In particular, for the elastic-perfect materials, i.e. \( \beta = 1 \) and \( B_{SH} = 0 \), the plastic wave speed is given by

\[
c = \sqrt{\frac{k}{\rho}}.
\]

(11.54)

In the rest of the present chapter, the above formulation will be numerically solved by the CESE method. The computational conditions and numerical results and remarks will be illustrated in the following sections.

11.4 Linearization of nonlinear plastic wave and wave speed illustration

For the purpose of comparison to the small-amplitude numerical solution of the nonlinear, we obtain the analytical solution of the linearized problem. We linearize
the nonlinear governing equations (11.38),(11.39) and (11.40) as follows. We expand \( \rho(x_1,t), v(x_1,t) \), and \( S_{11} \) in \( \epsilon \) about the rest state \( \rho^0, v^0 \) and \( S_{11}^0 \):

\[
\begin{align*}
\rho(x_1,t) &= \rho^0 + \epsilon \rho^1(x_1,t) + \epsilon^2 \rho^2(x_1,t) + \ldots, \\
v(x_1,t) &= v^0 + \epsilon v^1(x_1,t) + \epsilon^2 v^2(x_1,t) + \ldots, \\
S_{11}(x_1,t) &= S_{11}^0 + \epsilon S_{11}^1(x_1,t) + \epsilon^2 S_{11}^2(x_1,t) + \ldots,
\end{align*}
\]

(11.55)

where \( \rho^i(x_1,t), v^i(x_1,t) \) and \( S_{11}^i(x_1,t), i = 1,2,... \) are the i-th corrections of density, velocity and deviatoric stress, respectively, and \( \epsilon \) is a small, dimensionless, positive, scalar quantity. Inserting the expansions, e.g. Eq.(11.55) into Eq.(11.38), Eq. (11.39) and Eq.(11.40). By selecting \( \rho^0 = \rho_0, v^0 = 0 \) and \( S_{11}^0 = 0 \) as the rest state, the following linear equations are obtained from the order-\( \epsilon \) problem:

\[
\begin{align*}
\partial \rho^1 \partial t + \rho_0 \partial v^1 \partial x_1 &= 0, \\
\rho_0 \frac{\partial v^1}{\partial t} + K \frac{\partial \rho^1}{\rho_0} \frac{\partial v^1}{\partial x} - \frac{\partial S_{11}^1}{\partial x} &= 0, \\
\frac{\partial S_{11}^1}{\partial t} &= \frac{4}{3} \mu (1 - \frac{\beta}{1 + \frac{B_{SH}}{\mu}}) \frac{\partial v^1}{\partial x}.
\end{align*}
\]

(11.56)

Eqs.(11.56) are combined to recover second order linear linear wave equations in velocity:

\[
\frac{\partial^2 v^1}{\partial t^2} = c^2 \frac{\partial^2 v^1}{\partial x^2}.
\]

(11.57)

where

\[
c = \sqrt{\frac{K + \frac{4}{3} \mu (1 - \frac{\beta}{1 + \frac{B_{SH}}{\mu}})}{\rho_0}}
\]
is the plastic wave propagation speed. If we integrate Eq.(11.57) with respect to time and set the arbitrary function of \( x \) which arise to be zero, we recover:

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}
\]  

(11.58)

with \( u(x, t) \) the axial displacement component. It can be seen in Eq.(11.58) that plastic wave speed depends on plastic modulus \( B_{SH} \) where is constant in linear hardening plastic materials. Generally, it is not a constant which is shown in Eq. (11.21). So the variation of the velocity of plastic wave propagation, as a function of \( \epsilon \), is governing by the slope of the plastic stress-strain curve. For instance, for most metals the stress-strain curve takes the form illustrated in Fig.(11.1(a)), i.e. where \( \sigma d^2 \sigma / d \epsilon^2 < 0 \). For such case, the wave speed \( c(\epsilon) \) decreases when the stress increase. In the impact case, since the stress will increase at the impact end, the waves generated by the impact will propagate with continually decreasing velocities. This also means that the wave fronts will become wide during their propagation. Therefore, an expansion wave will be formed. However, for some rubbers, soils and certain metals, the constitutive equation takes the form shown in Fig. (11.1(b)), i.e. for which the slope increases continuously \( (\sigma d^2 \sigma / d \epsilon^2 > 0 \text{ for any } \epsilon) \). The speed of wave propagation will increase when the stress increases. In this case the distance between the wave fronts becomes short during propagation and there is a tendency to form shock waves. Detail analysis of infinitesimal plastic wave speed was shown by Cristescu [39]. In the following section, some numerical examples, e.g., impact plastic wave, unloading plastic wave and sinusoidal plastic wave will be discussed.

11.5 Numerical Results
Fig. 11.1: A schematic of one dimensional stress-strain curve (a) A stress-strain curve for a work-hardening material. (b) A stress-strain curve concave towards the stress axis.

The configuration can be viewed as impact of a flat plate whose diameter is large compared to its thickness in a direction perpendicular to its surface. Before the arrival of release wave from the edge of the plate, the center portion is in a confined state of one-dimensional or uniaxial strain.

11.5.1 Semi-infinite Domain Impact Analysis

We consider a one-dimensional copper plate with an initial speed $u = 30\text{m/s}$ hitting a stationary copper plate. Both of these copper plants are considered to be semi-infinite. The plants are assumed to be elastic and isotropic hardening plastic materials. We consider 3 cases in this example. Case 1: material has one effective yielding stress, e.g., $6.0 \times 10^7 \text{ Pa}$ In other words, the material is elastic, e.g., elastic young’s modulus $(122 \times 10^9 \text{ Pa})$, shear modulus $(45 \times 10^9 \text{ Pa})$ and bulk modulus $(140 \times 10^9 \text{ Pa})$. After equivalent stress surpasses effective yielding stress, the material properties will change for once only which is decided by the slop of effective stress and strain curve; Case 2: The material will have two effective yielding stresses, e.g., $6.0 \times 10^7 \text{ Pa}$ and $8.0 \times 10^7 \text{ Pa}$. If stress is larger than yielding stress, the material properties will change for
the first time. This is the same as Case 1. However, if the stress in the materials keep increasing to the second yielding stress, the material properties will change for the second time. Case 3: The material will have 4 effective yielding stresses, e.g., $6.0 \times 10^7$ Pa, $8.0 \times 10^7$ Pa, $1.0 \times 10^8$ Pa, $1.2 \times 10^8$ Pa. With the materials’ properties defined, we use CESE method to calculate plastic propagation. In the computation, $\Delta x = 0.66$ mm and $\Delta t = 1.3 \times 10^{-7}$ s. The initial pressures $p$ and deviatoric stress component $S_{11}$ in both copper bulks are null.

Figure 11.2 show density and stress profiles for 3 cases respectively. Wave propagate to the right direction. It was demonstrated there are two wave fronts for case 1. There is one elastic wave speed and one plastic wave speed. From the discussion above, we know elastic wave is faster than plastic wave front. In case 2, there 3 wave front, again, there is one elastic wave which is fastest wave in Figure 11.2. And there are two plastic wave fronts which is different from case 1. The wave speed will depend on the slop of effective stress and strain curve. Similarly, there are five wave fronts, e.g., 4 plastic wave fronts and 1 elastic wave front. It should be mentioned that the wave fronts could become infinite if the effective stress and strain curve becomes smooth. As the result, an expansion wave will be formed.

11.5.2 Unloading of Plastic Wave

In this numerical example, we test the unloading properties of plastic wave. It is well known that material will have a residue stress if a load is unloaded from plastic zone in static state. In this calculation, we found it has the same characters in dynamical loading. We set the numerical example to be the same as the former impact analysis. However, the traverse direction of impact plate is very short. The left side of the plate is stress free. We defined a free boundary as we used in section 7.2. Since we know compression wave will become tension wave at the free boundary
Fig. 11.2: Density and stress profiles of plastic wave propagation in bulk materials for 3 cases: (top) material has one isotropic hardening effective yielding stress; (middle) material has two isotropic hardening effective yielding stress; (bottom) material has four isotropic hardening effective yielding stress.
Fig. 11.3: Density and stress profiles of stress wave propagation in bulk elastic-hardening plastic materials. The unloading stress wave propagate with elastic wave speed.
condition, it can be used in the case to demonstrate the unloading of plastic wave propagation.

Fig.(11.3) shows a short copper plate hit a semi-infinite copper plate. The wave starts from the contact section. There is left running wave and right running wave. The left running wave will be reflected from the left end and become right-running wave and join the other right running wave. In Fig. (11.3), all the elastic and plastic waves propagate to the right-hand direction. There are one elastic wavefront which is the fastest wavefront and 2 plastic wavefronts. But there is only only elastic wave front in the unloading zone. It demonstrates that unloading path will be parallel to the elastic line in the stress-strain curve.

11.5.3 Sinusoidal Ultrasonic Forced Plastic Wave

In this setting, we consider a copper plate whose left end is forced by a sinusoidal ultrasonic force which is

$$F = 2 \times 10^8 \sin(10^5 t) P_a.$$  \hspace{1cm} (11.59)

For the simplicity, we set the initial conditions are zero and area of plate we considered is unit. Since the external force will lead to copper yielding, elastic and plastic wave in the plate will be generated and propagate to the other end. In Fig.(11.4) and Fig.(11.5), stress wave and velocity wave profiles are plotted. When wave propagate to the right hand side, wave shapes have been changed. It can be noticed that exist elastic tensional and compressional waves and plastic tensional and compressional waves. The elastic waves propagate in one speed which plastic wave speeds are slow. It also can be noticed that the original symmetrical external sinusoidal wave profiles have been changed to unsymmetrical elasto-plastic wave profiles.
Fig. 11.4: Density profiles of stress wave propagation in bulk elastic-hardening plastic materials. The left end is applied with a sinusoidal force in 10 kHz frequency. The amplitude of the force is larger than yielding stress of copper.

The author of this dissertation deduced that it could be accused by the unsymmetrical effective stress-strain curve. In other word, In the process that a copper material is loaded to a yielding compression stress and unloaded to zero stress. Then a tensional loading will be added to this copper to a yielding tensile stress and unloaded to zero stress. The strain and stress state is not symmetric in our mathematical model even though we have not considered Bauschinger effect.

11.6 Conclusion

The isothermal hyperbolic model of stress wave in elastic-plastic solid does not include energy conservation equation, and equation of state employed relates pressure and density, without considering internal energy. We applied the isothermal model to
Fig. 11.5: Stress profiles of stress wave propagation in bulk elastic-hardening plastic materials. The left end is applied with a sinusoidal force in 10 kHz frequency. The amplitude of the force is larger than yielding stress of copper.
simulate low speed impact problem, e.g., the impact speed at $u = 30\text{m/s}$. The numerical results were validated by comparing to the analytical solution, which was derived by using a more comprehensive equation state with the thermal effect.

The above results show that the isothermal model developed in the present chapter could correctly predict the elastic-plastic wave propagation. Thus, if temperature is not of concern in a low-impact-speed problem, process which might be able to assumed as a isothermal problem due to slight temperature change and low material particle speed, one may use the isothermal model to simulate process instead of complete model including the thermal effect.
CHAPTER 12
CONCLUSIONS AND FUTURE WORKS

In this dissertation, I have reported a general framework to model stress waves in a wide range of complex media, including linear and nonlinear elastic solids, anisotropic elastic solids, piezoelectric crystals, viscoelastic soft tissues, and plastic deformation of metals. In the following I will summarize my contribution in Subsection Conclusions, followed by recommended future works.

12.1 Conclusions

The theoretical and numerical approach developed and reported in the present dissertation consists of a novel theoretical framework for modeling stress wave propagation in solids based on the use of a set of coupled, first-order, hyperbolic partial differential equations. A wide range of different physical models for various material responses of complex media can all be cast into the same framework of the first-order hyperbolic partial differential equations. Then, the CESE method was used to solve the couple differential equations to demonstrate the approach. The wave characteristics can be assessed by analyzing the eigenvalues and eigenvectors of the Jacobian matrices of the model equations. To recapitulate, unique contributions by the present research includes

2. I have provided detailed analysis of the eigen-structure of each model. The nature of the wave propagation can be determined by the eigenvalues, eigenvectors of Jacobian matrices. The corresponding conservative form, non-conservative form, characteristic form, and the Riemann invariants have been derived for many cases. In many cases, the numerical and theoretical results compared well with the available analytical solutions.

3. A unique formulation for modeling wave motion in piezoelectric solids is derived and reported. Two sets of governing equations are obtained, including (i) the equation of motion is coupled with the full Maxwell equation for complete electromagnetic effects, and (ii) quasi-static electric wave equations are coupled with the equation of motion for electrical effect only.

4. Wave propagation in soft-tissue has been studied. Linear and nonlinear viscoelastic model was used to capture the material responses. The model equations were solved by the CESE method for an impact problem. The numerical results compared well with the analytical solution, and the model equations and the computer code are validated. The validated code was then used to calculate impact problems in real soft tissues. Moreover, I also developed a new approach to use the measured absorption coefficients of wave propagation in soft tissues to determine the relaxation function in the viscoelasticity relations for soft tissues.
5. I have further developed the CESE method for simulation of wave propagation. In particular, source term treatment was developed to solve the first-order wave equations with stiff source terms in model equations for waves in soft tissues.

12.2 Future Works

Based on my current results and experience, I would make the following suggestions for the future works:

1. For wave motion in anisotropic solids, the use of new open-sourced code SOLVCON for large scale two- and three-dimensional simulations could be very fruitful. The medium could include heterogeneous materials. The model equations could include nonlinear effect of the materials. The phase shift caused by waves with large amplitude could be a subject for further study. The application of this field will be seismic wave in the earth.

2. Further development for model equations for wave motion in piezoelectric media could focus on surface wave propagation because of ample applications of Surface Acoustic Wave (SAW) devices in industry. Theoretical and numerical analyses of surface wave propagation related to SAW devices will improve the design.

3. Wave propagation in soft tissue is a wide open subject. The future work may include new mathematical models, e.g., poro-elastic or bio-phase model. The simulation of current model should be extend to three-dimensional computation. The numerical simulation work must find clinic applications and cooperation with medical research group. In particular, wave propagation in brain tissue could be an interesting subject, in which the new CESE solver SOLVCON could be used for very large scale simulations.
4. For wave propagation in hypo-elastic solid, future works may extend to two- and three-dimensional simulations. In general, the hyper-elasticity may be included in the model equations to capture nonlinear material response. The geometric nonlinearity will be captured by Jaumann time derivative. Nonlinear dynamics analysis should be used to explain why and how sub- and super-harmonics phenomena occur in nonlinear wave propagation.

5. For wave propagation in hypo-plastic solid, this research work can be directly connected to the ultrasonic welding processes. Future works can have three-dimensional dynamical analysis of the contact plane of the two materials to be joined. The energy flow, longitudinal and shear wave interaction, phase transfer, and dislocation of crystal will be important research issue. The model equations could also be used to study structural changes when nonlinear waves interact with a crack for fracture mechanics.
Appendix A

A.1 Analytical Solution of Expansion Wave in a Suddenly Stopped Solid

This appendix provides the analytical solution of longitudinal wave propagation along a compression direction in a hexagonal solid.

Recall the governing equation for wave dynamics in solids of hexagonal symmetry, Eqn. (3.68). Let $\phi = \pi/2$ and $\theta$ be arbitrary. Moreover, we let $\partial/\partial x(2)$ and $\partial/\partial x(3)$ be null. The resultant one-dimensional equations in the component form are:

\[
\begin{align*}
\frac{\partial v_1}{\partial t} - \frac{1}{\rho} \frac{\partial T_5}{\partial x(1)} &= 0, \\
\frac{\partial v_2}{\partial t} - \frac{1}{\rho} \frac{\partial T_4}{\partial x(1)} &= 0, \\
\frac{\partial v_3}{\partial t} - \frac{1}{\rho} \frac{\partial T_3}{\partial x(1)} &= 0, \\
\frac{\partial T_1}{\partial t} - c_{13} \frac{\partial v_3}{\partial x(1)} &= 0, \\
\frac{\partial T_2}{\partial t} - c_{13} \frac{\partial v_3}{\partial x(1)} &= 0, \\
\frac{\partial T_6}{\partial t} &= 0.
\end{align*}
\]

(A.1)

The first 6 equations in Eqn. (A.1) can be grouped into 3 pairs, which in turn form 3 second-order wave equations:

\[
\begin{align*}
\frac{\partial^2 v_1}{\partial t^2} &= \frac{c_{44}}{\rho} \frac{\partial^2 v_1}{\partial x^2(1)}, \\
\frac{\partial^2 v_2}{\partial t^2} &= \frac{c_{44}}{\rho} \frac{\partial^2 v_2}{\partial x^2(1)}, \\
\frac{\partial^2 v_3}{\partial t^2} &= c_{33} \frac{\partial^2 v_3}{\partial x^2(1)}.
\end{align*}
\]

(A.2)
According to the chosen rotational angles, we let \( x = x_3 = x_{(1)} \). Equation (A.2) the longitudinal wave speed is \( \sqrt{c_{33}/\rho} \) and two shear with wave speeds are \( \sqrt{c_{44}/\rho} \).

Differentiate Eqn. (A.2) by time, and we have the following second-order wave equations for the displacement \( u_3 \):

\[
\frac{\partial^2 u_3}{\partial t^2} = \frac{c_{33}}{\rho} \frac{\partial^2 u_3}{\partial x^2}.
\] (A.3)

To proceed, we derive the analytical solution of Eqn. A.3. First, we impose the fixed and stress-free boundary conditions on the left end and the right end of the solid, respectively:

\[
u_3(0, t) = 0, \quad \frac{\partial u_3(L, t)}{\partial x} = 0, \quad \text{for } t > 0.
\] (A.4)

The initial conditions are

\[
u_3(x, 0) = 0, \quad \frac{\partial u_3(x, 0)}{\partial t} = V_0, \quad \text{for } 0 \leq x \leq L.
\] (A.5)

We use the method of separation of variables to obtain the solution of Eqn. (A.3) as:

\[
u_3(x, t) = \frac{8V_0L}{\pi^2 c} \sum_{m=1}^{\infty} \frac{\sin(\alpha_m x) \sin(\omega_m t)}{(2m - 1)^2}
\] (A.6)

where \( \alpha_m = (2m - 1)\pi/2L \), \( \omega_m = \alpha_m c \), and the speed of the wave \( c = \sqrt{c_{33}/\rho} \), for \( m = 1, 2, \cdots, \infty \).

Based on the solution of the displacement, the solution of stress and velocity can
be readily obtained as:

\[
\sigma(x,t) = c_{33} \frac{\partial u(x,t)}{\partial x} = \frac{4V_0 c_{33}}{\pi c} \sum_{m=1}^{\infty} \frac{\cos(\alpha_m x) \sin(\omega_m t)}{2m - 1},
\]

\[
v_3(x,t) = \frac{\partial u_3(x,t)}{\partial t} = \frac{4V_0}{\pi} \sum_{m=1}^{\infty} \frac{\sin(\alpha_m x) \cos(\omega_m t)}{2m - 1},
\]

(A.7)

A.2 Group Velocity in the 100 and 010 Planes of a Hexagonal Solid

This appendix provides the analytical solution of the group velocity in a hexagonal solid. We will use the classical solution based on the Christoffel equations for the derivation. For waves propagate in the \(x_1-x_3\) plane, the Christoffel equation becomes:

\[
\begin{bmatrix}
\Gamma_{11} - \rho v^2 & 0 & \Gamma_{13} \\
0 & \Gamma_{22} - \rho v^2 & 0 \\
\Gamma_{13} & 0 & \Gamma_{33} - \rho v^2
\end{bmatrix}
\begin{bmatrix}
A_1 \\
A_2 \\
A_3
\end{bmatrix}
= 0,
\]

(A.8)

where \(\Gamma_{11} = c_{11}n_1^2 + c_{55}n_3^2\), \(\Gamma_{33} = c_{55}n_1^2 + c_{33}n_3^2\), \(\Gamma_{22} = c_{66}n_1^2 + c_{44}n_3^2\), \(\Gamma_{13} = (c_{13} + c_{55})n_1n_3\), and \(\mathbf{n} = (n_1, n_2, n_3)\) is the unit vector of the wave propagation direction.

Phase velocities can be obtained by calculating the eigenvalues of the Christoffel matrix [41] as:

\[
v_2 = \sqrt{\frac{c_{66}n_1^2 + c_{44}n_3^2}{\rho}},
\]

\[
\rho v_1^2 = \frac{1}{2} \left[ c_{11}n_1^2 + c_{33}n_3^2 + c_{55} + \sqrt{D} \right],
\]

\[
\rho v_3^2 = \frac{1}{2} \left[ c_{11}n_1^2 + c_{33}n_3^2 + c_{55} - \sqrt{D} \right],
\]

(A.9)
where

\[ D = [(c_{11} - c_{55})n_1^2 + (c_{55} - c_{33})n_3^2]^2 + 4(c_{13} + c_{55})^2 n_1^2 n_3^2. \]

In the above equation, \( v_2 \) is the phase velocity of the SH wave. \( v_1 \) and \( v_3 \) are the phase velocities of the qL and the qS waves, respectively.

The group velocity \( \mathbf{v}_g = (v_{g1}, v_{g2}, v_{g3}) \) is given by [4]:

\[ v_{gi} = -\frac{\partial \Omega}{\partial k_i} \frac{\partial \Omega}{\partial \omega}, \quad i = 1, 2, 3, \tag{A.10} \]

where \( \Omega \) is the determinant of the Christoffel matrix, \( \omega \) is the frequency, and \( \mathbf{k} = (k_1, k_2, k_3) \) is the wave number vector. Because \( v_i = \omega/k_i, i = 1, 2, 3 \), \( \Omega \) is a function of \( \omega \) and \( k_i \). Since the wave have three polarization directions, similar to the phase velocities, there are three group velocities.

For a pure shear wave polarized in the \( x_2 \) direction and propagating in the \( x_1 - x_3 \) plane, the group velocity can be calculated by let

\[ \Omega^{SH} (\omega, k_1, k_3) = (c_{66}k_1^2 + c_{44}k_3^2) - \rho \omega^2 = 0. \tag{A.11} \]

and the group velocity of the SH wave is:

\[ v_{g1} = \frac{n_1c_{66}}{\rho v_2}, \quad v_{g3} = \frac{n_3c_{44}}{\rho v_2}. \tag{A.12} \]
For qL and qS waves propagating in the $x_1$-$x_3$ plane, the group velocities are

$$v_{g1} = -\frac{\partial \Omega / \partial k_1}{\partial \Omega / \partial \omega} = n_1 \left[c_{11}\{\rho v^2 - (c_{35} n_1^2 + c_{33} n_3^2)\} + c_{35}\{\rho v^2 - (c_{11} n_1^2 + c_{55} n_3^2)\}\right] + \left[c_{13} + c_{55}\right]^2 n_3^2 \rho v \left[2\rho v^2 - (c_{35} n_1^2 + c_{33} n_3^2 + c_{55})\right].$$

$$v_{g3} = -\frac{\partial \Omega / \partial k_3}{\partial \Omega / \partial \omega} = n_3 \left[c_{33}\{\rho v^2 - (c_{11} n_1^2 + c_{35} n_3^2)\} + c_{35}\{\rho v^2 - (c_{35} n_1^2 + c_{33} n_3^2)\}\right] + \left[c_{13} + c_{55}\right]^2 n_1^2 \rho v \left[2\rho v^2 - (c_{35} n_1^2 + c_{33} n_3^2 + c_{55})\right].$$

(A.13)

where $v$ is the phase velocity. For the qL wave, $v = v_1$. For the qS wave, $v = v_3$. The values of $v_1$ and $v_3$ can be determined by using Eqn. (A.9).

A.3 The Method of Internal Variable

The derivation of the governing equations for longitudinal wave in a soft tissue are illustrated in this appendix. The diagonal element ($i = j$) in Eq. (8.15) can be recast.
\[
\frac{\partial \sigma_{ij}}{\partial t} = \int_0^t \pi \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau_{il}^p}{\tau_{ol}} \right) e\left(\frac{-t - \tau_{il}}{\tau_{ol}}\right) \right] \delta(t - \tau) \frac{\partial v_k(t)}{\partial x_k} d\tau
\]

\[
+ \int_0^t \pi \left[ \left( \frac{1}{\tau_{ol}} \right) \sum_{l=1}^L \left( 1 - \frac{\tau_{il}^s}{\tau_{ol}} \right) e\left(\frac{-t - \tau_{il}}{\tau_{ol}}\right) \right] H(t - \tau) \frac{\partial v_k(t)}{\partial x_k} d\tau
\]

\[
- 2 \int_0^t \nu \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau_{il}^s}{\tau_{ol}} \right) e\left(\frac{-t - \tau_{il}}{\tau_{ol}}\right) \right] \delta(t - \tau) \frac{\partial v_k(t)}{\partial x_k} d\tau
\]

\[
- 2 \int_0^t \nu \left[ \left( \frac{1}{\tau_{ol}} \right) \sum_{l=1}^L \left( 1 - \frac{\tau_{il}^s}{\tau_{ol}} \right) e\left(\frac{-t - \tau_{il}}{\tau_{ol}}\right) \right] H(t - \tau) \frac{\partial v_j(t)}{\partial x_i} d\tau
\]

\[
+ 2 \int_0^t \nu \left[ \left( \frac{1}{\tau_{ol}} \right) \sum_{l=1}^L \left( 1 - \frac{\tau_{il}^s}{\tau_{ol}} \right) e\left(\frac{-t - \tau_{il}}{\tau_{ol}}\right) \right] H(t - \tau) \frac{\partial v_j(t)}{\partial x_i} d\tau.
\]

Recall the properties of the delta function \( \delta(-\tau) = \delta\tau \) and \( \int_{-\infty}^{\infty} f(\tau)\delta(\tau-t)dt = f(t) \).

Thus Eq. (A.14) becomes

\[
\frac{\partial \sigma_{ij}}{\partial t} = \left\{ \pi \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau_{il}^p}{\tau_{ol}} \right) \right] - 2\nu \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau_{il}^s}{\tau_{ol}} \right) \right] \right\} \frac{\partial v_k}{\partial x_k} + 2\nu \left[ 1 - \sum_{l=1}^L \left( 1 - \frac{\tau_{il}^s}{\tau_{ol}} \right) \right] \frac{\partial v_j}{\partial x_i} + \sum_{l=1}^L \gamma_{ij}.
\]

The governing equations of the internal variables, are

\[
\gamma_{ij}^l = \int_0^t \pi \left[ \left( \frac{1}{\tau_{ol}} \right) \left( 1 - \frac{\tau_{il}^p}{\tau_{ol}} \right) e\left(\frac{-t - \tau_{il}}{\tau_{ol}}\right) \right] H(t - \tau) \frac{\partial v_k(t)}{\partial x_k} d\tau
\]

\[
- 2 \int_0^t \nu \left[ \left( \frac{1}{\tau_{ol}} \right) \left( 1 - \frac{\tau_{il}^s}{\tau_{ol}} \right) e\left(\frac{-t - \tau_{il}}{\tau_{ol}}\right) \right] H(t - \tau) \frac{\partial v_k(t)}{\partial x_k} d\tau
\]

\[
+ 2 \int_0^t \nu \left[ \left( \frac{1}{\tau_{ol}} \right) \left( 1 - \frac{\tau_{il}^s}{\tau_{ol}} \right) e\left(\frac{-t - \tau_{il}}{\tau_{ol}}\right) \right] H(t - \tau) \frac{\partial v_j(t)}{\partial x_i} d\tau.
\]
To proceed, we differentiate Eq. (A.15) by time to obtain

$$\frac{\partial \gamma^l_{ij}}{\partial t} = \left( \frac{1}{\tau_{\alpha l}} \right) \int_0^t \left[ \frac{1}{\tau_{\alpha l}} \left( 1 - \frac{\tau^p_{\alpha l}}{\tau_{\alpha l}} \right) e^{-\frac{t-\tau}{\tau_{\alpha l}}} \right] H(t-\tau) \frac{\partial v_k(t)}{\partial x_k} d\tau$$

$$+ \frac{2}{\tau_{\alpha l}} \int_0^t \nu \left[ \frac{1}{\tau_{\alpha l}} \left( 1 - \frac{\tau^s_{\alpha l}}{\tau_{\alpha l}} \right) e^{-\frac{t-\tau}{\tau_{\alpha l}}} \right] H(t-\tau) \frac{\partial v_j(t)}{\partial x_i} d\tau$$

$$- 2 \int_0^t \left[ \frac{1}{\tau_{\alpha l}} \left( 1 - \frac{\tau^s_{\alpha l}}{\tau_{\alpha l}} \right) e^{-\frac{t-\tau}{\tau_{\alpha l}}} \right] \delta(t-\tau) \frac{\partial v_k(t)}{\partial x_k} d\tau$$

$$+ 2 \int_0^t \left[ \frac{1}{\tau_{\alpha l}} \left( 1 - \frac{\tau^s_{\alpha l}}{\tau_{\alpha l}} \right) e^{-\frac{t-\tau}{\tau_{\alpha l}}} \right] \delta(t-\tau) \frac{\partial v_j(t)}{\partial x_i} d\tau.$$

(A.16)

Recall the definition of $\gamma^l_{ij}$ and recognize that the first three terms in the right hand side of Eq. (A.16) is $-1/\tau_{\alpha l}$ multiplied by $\gamma^l_{ij}$. Next, we apply the equality

$$\int_{-\infty}^{\infty} f(\tau) \delta(\tau - t) d\tau = f(t)$$

to the remainder three terms in Eq. (A.16), and the equation becomes

$$\frac{\partial \gamma^l_{ij}}{\partial t} = -\frac{1}{\tau_{\alpha l}} \left[ \gamma^l_{ij} + \pi \left( \frac{\tau^p_{\alpha l}}{\tau_{\alpha l}} - 1 \right) \frac{\partial v_k}{\partial x_k} - 2\nu \left( \frac{\tau^s_{\alpha l}}{\tau_{\alpha l}} - 1 \right) \frac{\partial v_j}{\partial x_i} \right]$$

To proceed, we consider the equations of the internal variables for the off-diagonal
elements \((i \neq j)\) i.e., Eq. (8.16):

\[
\frac{\partial \sigma_{ij}}{\partial t} = \int_{0}^{t} \nu \left[ 1 - \sum_{i=1}^{L} \left( 1 - \frac{\tau_{s}^{\sigma}}{\tau_{\sigma l}} \right) e^{-\frac{t - \tau}{\tau_{\sigma l}}} \right] \delta(t - \tau) \left[ \frac{\partial v_{j}(t)}{\partial x_{i}} + \frac{\partial v_{i}(t)}{\partial x_{j}} \right] d\tau + \int_{0}^{t} \nu \left[ \frac{1}{\tau_{\sigma l}} \sum_{i=1}^{L} \left( 1 - \frac{\tau_{s}^{\sigma}}{\tau_{\sigma l}} \right) e^{-\frac{t - \tau}{\tau_{\sigma l}}} \right] \left[ \frac{\partial v_{j}(t)}{\partial x_{i}} - \frac{\partial v_{i}(t)}{\partial x_{j}} \right] d\tau.
\]

(A.17)

Aided by the \(\delta\) function, Eq. (A.17) is rewritten to be

\[
\frac{\partial \sigma_{ij}}{\partial t} = \nu \left[ 1 - \sum_{i=1}^{L} \left( 1 - \frac{\tau_{s}^{\sigma}}{\tau_{\sigma l}} \right) \right] \left[ \frac{\partial v_{j}}{\partial x_{i}} + \frac{\partial v_{i}}{\partial x_{j}} \right] + \sum_{i=1}^{L} \gamma_{ij}^{l},
\]

where the eight internal variables in the governing equations, for longitudinal wave in a soft tissue are

\[
\gamma_{ij}^{l} = \int_{0}^{t} \nu \left[ \frac{1}{\tau_{\sigma l}} \sum_{i=1}^{L} \left( 1 - \frac{\tau_{s}^{\sigma}}{\tau_{\sigma l}} \right) e^{-\frac{t - \tau}{\tau_{\sigma l}}} \right] \left[ \frac{\partial v_{j}(t)}{\partial x_{i}} - \frac{\partial v_{i}(t)}{\partial x_{j}} \right] d\tau.
\]

(A.18)

Next, we differentiate Eq. (A.18) by time to obtain the governing equations of the
eight internal variables for longitudinal wave in a soft tissue,

\[
\frac{\partial \gamma_{ij}^l}{\partial t} = \frac{1}{\tau_{sl}} \int_0^t \nu \left[ \left( \frac{1}{\tau_{sl}} \right) \sum_{i=1}^L \left( 1 - \frac{\tau_{sl}}{\tau_{sl}} \right) e^{-\frac{t-\tau}{\tau_{sl}}} \right] \nabla^{ij} H(t-\tau) \left[ \frac{\partial v_j(t)}{\partial x_i} + \frac{\partial v_i(t)}{\partial x_j} \right] d\tau \\
+ \int_0^t \nu \left[ \left( \frac{1}{\tau_{sl}} \right) \sum_{i=1}^L \left( 1 - \frac{\tau_{sl}}{\tau_{sl}} \right) e^{-\frac{t-\tau}{\tau_{sl}}} \right] \delta(t-\tau) \left[ \frac{\partial v_j(t)}{\partial x_i} + \frac{\partial v_i(t)}{\partial x_j} \right] d\tau.
\]

(A.19)

Again, the first term of right hand side of Eq. (A.19) is the internal variable \( \gamma_{ij}^l \). Aided by the property of the \( \delta \) function, the second term in the right hand side Eq. (A.19) can be simplified. After manipulation, Eq. (A.19) becomes

\[
\frac{\partial \gamma_{ij}^l}{\partial t} = -\frac{1}{\tau_{sl}} \left[ \gamma_{ij}^l + \nu \left( \frac{\tau_{sl}}{\tau_{sl}} - 1 \right) \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \right]
\]

The resultant constitutive equations and the equations for the internal variables are listed in Section 8.2.2.

A.4 Derivation of Fung’s and Modified Fung’s Model in Time and Frequency Domain

Fung’s Model in Time Domain

Start with Fung’s model which is shown as

\[
G(t) = \frac{1 + \int_0^\infty S(\tau) e^{-\frac{t}{\tau} d\tau}}{1 + \int_0^\infty S(\tau) d\tau}
\]

\[
S(\tau) = \begin{cases} 
\frac{c}{\tau} & \text{for } \tau_1 \leq \tau \leq \tau_2 \\
0 & \text{for } \tau < \tau_1, \tau > \tau_2
\end{cases}
\]

(A.20)
Substitute $S(\tau)$ into $G(t)$ in Eq.(A.20) and let $t/\tau = u$, we have

$$G(t) = \frac{1 + c \int_{t/\tau}^{t/\tau_2} (-e^{-u})du}{1 + c \ln(\frac{\tau_2}{\tau_1})}$$

(A.21)

It can be easily viewed that Eq.(A.21) is Eq.(9.12).

**Fung’s Model in Frequency Domain**

We take Carson’s Transform with respect to Eq.(A.20) and use $s = i\omega$, then

$$G(\omega) = i\omega \int_0^\infty G(t)e^{-i\omega t} dt$$

$$= i\omega \int_0^\infty \frac{[1 + \int_{\tau_1}^{\tau_2} e^{-i/\tau} d\tau]}{1 + \int_{\tau_1}^{\tau_2} \frac{1}{\tau} d\tau} e^{-i\omega t} dt$$

$$= \frac{i\omega}{1 + c \ln(\frac{\tau_2}{\tau_1})} \left[ \int_0^\infty e^{-i\omega t} dt + c \int_{\tau_1}^{\tau_2} \int_0^\infty \frac{e^{(-1/\tau - i\omega)t}}{\tau} dtd\tau \right]$$

$$= \frac{1}{1 + c \ln(\frac{\tau_2}{\tau_1})} \left[ 1 + \frac{1}{i\omega} \int_{\tau_1}^{\tau_2} \frac{1}{1 + i\omega \tau} d\tau \right]$$

$$= \frac{1}{1 + c \ln(\frac{\tau_2}{\tau_1})} \left[ 1 + ic \int_{\tau_1}^{\tau_2} \frac{1}{(1 + i\omega \tau)(1 - i\omega \tau)} d\omega \tau \right]$$

$$= \frac{1}{1 + c \ln(\frac{\tau_2}{\tau_1})} \left[ 1 + \int_{\tau_1}^{\tau_2} \frac{\omega \tau}{1 + \omega^2 \tau^2} d\omega \tau + ic \int_{\tau_1}^{\tau_2} \frac{1}{(1 + \omega^2 \tau^2)^2} d\omega \tau \right]$$

(A.22)

After using integration table, Eq.(9.14), e.g., Fung’s Model in Frequency Domain, can be got.
Modified Fung’s Model in Time Domain

Start with Modified Fung’s model which is shown as

\[ G(t) = \frac{[1 + \int_0^\infty S(\tau)e^{-t/\tau}d\tau]}{[1 + \int_0^\infty S(\tau)d\tau]} \]

\[ S(\tau) = \begin{cases} 
\frac{c_1}{\tau} + \frac{c_2}{\tau^2} & \text{for } \tau_1 \leq \tau \leq \tau_2 \\
0 & \text{for } \tau < \tau_1, \tau > \tau_2 
\end{cases} \]  \hfill (A.23)

Substitute \( S(\tau) \) into \( G(t) \) in Eq.(A.23), we have

\[ G(t) = \left[ 1 + \int_{t/\tau_1}^{t/\tau_2} \frac{c_1}{\tau}e^{-t/\tau}d\tau + \int_{t/\tau_1}^{t/\tau_2} \frac{c_2}{\tau^2}e^{-t/\tau}d\tau \right] 
\left[ 1 + \int_{t/\tau_1}^{t/\tau_2} \frac{c_1}{\tau}d\tau + \int_{t/\tau_1}^{t/\tau_2} \frac{c_2}{\tau^2}d\tau \right]^{-1} \]  \hfill (A.24)

with using Fung’s model, e.g., Eq.(9.12), Modified Fung’s model in time domain, Eq.(9.15) will be obtained.

Modified Fung’s Model in Frequency Domain

Carson’s Transform is taken with respect to Eq.(A.23), we obtain

\[ G(\omega) = i\omega \int_0^\infty \frac{1 + \int_{t/\tau_1}^{t/\tau_2} \frac{c_1}{\tau}e^{-t/\tau}d\tau + \int_{t/\tau_1}^{t/\tau_2} \frac{c_2}{\tau^2}e^{-t/\tau}d\tau}{1 + \int_{t/\tau_1}^{t/\tau_2} \frac{c_1}{\tau} + \frac{c_2}{\tau^2}d\tau} e^{-i\omega t}dt 
= i\omega \int_0^\infty \frac{1 + \int_{t/\tau_1}^{t/\tau_2} \frac{c_1}{\tau}e^{-t/\tau}d\tau}{1 + c_1 \ln(\tau_2/\tau_1) + c_2(1/\tau_1 - 1/\tau_2)} e^{-i\omega t}dt 
+ i\omega \int_0^\infty \frac{1 + \int_{t/\tau_1}^{t/\tau_2} \frac{c_2}{\tau^2}e^{-t/\tau}d\tau}{1 + c_1 \ln(\tau_2/\tau_1) + c_2(1/\tau_1 - 1/\tau_2)} e^{-i\omega t}dt \]  \hfill (A.25)
With using results of Fung’s model, the first part of Eq.(A.25) can be easily got. Then Eq.(A.25) becomes

\[ G(\omega) = A + \frac{i\omega c_2 \int_{\tau_1}^{\tau_2} \int_0^\infty \frac{1}{\tau} e^{-\tau/\tau_1} \tau e^{-i\omega\tau} d\tau dt}{1 + c_1 \ln \tau_2/\tau_1 + c_2 (1/\tau_1 - 1/\tau_2)} \]

where

\[ A = \frac{1 + \frac{c_2}{2} [\ln(1 + \omega^2 \tau_2^2) - \ln(1 + \omega^2 \tau_1^2)] + ic_1 [\tan^{-1}(\omega \tau_2) - \tan^{-1}(\omega \tau_1)]}{1 + c_1 \ln \tau_2/\tau_1 + c_2 (1/\tau_1 - 1/\tau_2)} \]

(A.26)

To proceed,

\[ G(\omega) = A + \frac{i\omega c_2}{1 + c_1 \ln \tau_2/\tau_1 + c_2 (1/\tau_1 - 1/\tau_2)} \int_{\tau_1}^{\tau_2} \frac{1}{\tau + i \omega \tau^2} d\tau \]
\[ = A + \frac{i\omega c_2}{1 + c_1 \ln \tau_2/\tau_1 + c_2 (1/\tau_1 - 1/\tau_2)} \left[ \int_{\tau_1}^{\tau_2} \frac{\omega}{\omega \tau (1 + \omega^2 \tau^2)} d\tau - i \int_{\tau_1}^{\tau_2} \frac{d\omega \tau}{1 + \omega^2 \tau^2} \right] \]
\[ = A + \frac{\omega c_2 [\tan^{-1}(\omega \tau_2) - \tan^{-1}(\omega \tau_1)]}{1 + c_1 \ln \tau_2/\tau_1 + c_2 (1/\tau_1 - 1/\tau_2)} + \frac{\omega c_2}{2} \frac{[\ln(\tau_2^2) - \ln(\tau_1^2)]}{1 + \omega^2 \tau^2} \]
\[ \frac{1 + c_1 \ln \tau_2/\tau_1 + c_2 (1/\tau_1 - 1/\tau_2)} \]

(A.27)

where integration table

\[ \int \frac{dx}{x(a^2 + x^2)} = \frac{1}{2a^2} \ln \left( \frac{x^2}{a^2 + x^2} \right) \]

(A.28)

is used. It can be noticed that Eq. (A.27) is the same as Eq.(9.16).

A.5 One Dimension Plastic Strain Wave Analysis

The conventional uniaxial stress-strain curve, as depicted by the idealized models of Figure (A.1), does not adequately represent the state of stress and strain to which a material is subjected under shock loading.
If we consider the case of plane waves propagating through a material where dimensions and constraints are such that the lateral strains are zero, the stress and strain curve takes on a different form. This situation is commonly referred to as uniaxial strain. To understand why these changes occur, consider the stress-strain relationship for one-dimensional deformation. In the general case the three principal strains can be divided into an elastic and a plastic part:

\[ \epsilon_1 = \epsilon_1^e + \epsilon_1^p, \epsilon_2 = \epsilon_2^e + \epsilon_2^p, \epsilon_3 = \epsilon_3^e + \epsilon_3^p, \]  
(A.29)

where the superscripts \(e\) and \(p\) refer to elastic and plastic, respectively, and the subscripts are the three principal directions. In one-dimensional deformation

\[ \epsilon_2 = \epsilon_3 = 0, \epsilon_2^p = -\epsilon_2^e, \epsilon_3^p = -\epsilon_3^e, \]  
(A.30)

The plastic portion of the strain is taken to be incompressible, so that

\[ \epsilon_1^p + \epsilon_2^p + \epsilon_3^p = 0 \]  
(A.31)

which gives

\[ \epsilon_1^p = -\epsilon_2^p - \epsilon_3^p = -2\epsilon_2^p \]  
(A.32)
since $\epsilon_2^p = \epsilon_3^p$ due to symmetry. From Eq.(A.30) we have that

$$\epsilon_1^p = 2\epsilon_2^e. \tag{A.33}$$

And the total strain $\epsilon_1$ may be written as

$$\epsilon_1 = \epsilon_1^e + \epsilon_1^p = \epsilon_1^e + 2\epsilon_2^e. \tag{A.34}$$

The elastic strain in terms of the stresses and elastic constants is given by

$$\epsilon_1^e = \frac{\sigma_1}{E} - \frac{\nu}{E}(\sigma_2 + \sigma_3) = \frac{\sigma_1}{E} - \frac{2\nu}{E}\sigma_2, \tag{A.35}$$

$$\epsilon_2^e = \frac{\sigma_2}{E} - \frac{\nu}{E}(\sigma_1 + \sigma_3) = \frac{1 - \nu}{E}\sigma_2 - \frac{\nu}{E}\sigma_1, \tag{A.36}$$

$$\epsilon_3^e = \frac{\sigma_3}{E} - \frac{\nu}{E}(\sigma_1 + \sigma_2) = \frac{1 - \nu}{E}\sigma_3 - \frac{\nu}{E}\sigma_1. \tag{A.37}$$

Combining Eq.(A.34) and Eq.(A.37), we get

$$\epsilon_1 = \frac{\sigma_1(1 - 2\nu)}{E} + \frac{2\sigma_2(1 - 2\nu)}{E}. \tag{A.38}$$

And we know that the Von Mises plasticity yielding condition for this case is

$$\sigma_1 - \sigma_2 = Y_0, \tag{A.39}$$

where $Y_0$ is the static yield strength. If Eq.(A.38) and Eq.(A.39) are combined to remove $\sigma_2$, we get

$$\sigma_1 = \frac{E}{3(1 - 2\nu)}\epsilon_1 + \frac{2}{3}Y_0 = K\epsilon_1 + \frac{2Y_0}{3}, \tag{A.40}$$

where $K = E/3(1 - 2\nu)$ is called the bulk modulus. Eq.(A.40) is valid only above yield. The uniaxial-strain elastic stress-strain relation can be easily got by considering first two equations of Eq. (A.37) with $\epsilon_2^e = 0$

$$\sigma_1 = \frac{E(1 - \nu)}{1 + \nu(1 - 2\nu)}\epsilon_1 = (K + \frac{4\mu}{3})\epsilon_1. \tag{A.41}$$
The uniaxial stress-strain curve for elastic and plastic material is shown in Fig. (A.5). Linear momentum equation and general one-dimensional stress-strain relation may be combined to be

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho_0} \frac{\partial \sigma_1}{\partial x} = \frac{1}{\rho_0} \frac{\partial \sigma_1}{\partial \epsilon_1} \frac{\partial^2 u}{\partial x^2},$$  \hspace{1cm} (A.42)

where relation $\epsilon_1 = \partial u/\partial x$ have been used. So the wave speed of a strain increment is given by

$$c = \left( \frac{1}{\rho_0} \frac{\partial \sigma_1}{\partial \epsilon_1} \right)^{0.5}.$$  \hspace{1cm} (A.43)

Combine Eq. (A.41) and Eq. (A.43), an elastic wave in uniaxial strain configuration is

$$c_e = \sqrt{\frac{K + 4\mu/3}{\rho_0}}.$$  \hspace{1cm} (A.44)
Combine Eq. (A.40) and Eq. (A.43), an plastic wave in uniaxial strain configuration is

\[ c_p = \sqrt{\frac{K}{\rho_0}}. \]  

(A.45)

The wave speeds shown in Eq. (A.44) and Eq. (A.45) are consistent with the eigenvalues shown in Eq. (11.53) where \( \beta = 0 \) correspond to elastic wave speed and \( \beta = 1, B_{SH} = 0 \) correspond to plastic wave speed.
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