A STUDY OF SERIES CONVERGENCE

A Thesis

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by

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CHAPTER I

I. Introduction

Since the Eighteenth Century, great mathematicians such as Gauss, Cauchy, Remann, Able and many others have been studying the structure of the infinite series.

One aspect of the study of infinite series is the study of convergence. Thus, and because of the usefulness of infinite series, many tests of convergence of infinite series have been discovered.

For Cauchy one test is not enough, as usual, so he discovered the so called integral test and Cauchy test. When Gauss was working with the hypergeometric series he needed then a stronger test. He did not ask for one, he discovered one.

In this paper I shall study first the real series and its tests such as the integral test, Erma Koof's test, and Kummer's test, which Dr. Hyslop in his book "Infinite Series" considered as a very general test.

At the end of this chapter I shall give some examples showing the relative strength of such tests.

In Chapter 3 I shall study the complex power series. The most important example for our purposes is the study of Lusin's series and its related series, which I accomplish in Chapter 4.
II. Definitions and Basic Theorems

Suppose that \(u_0, u_1, u_2, \ldots, u_n, \ldots\) is an infinite sequence of numbers. The expression

\[(l-1) \quad u_0 + u_1 + u_2 + \ldots + u_n + \ldots\]

is called an infinite series and the numbers \(u_0, u_1, \ldots\) are called the terms of the series. Now consider the sequences of numbers \(s_0, s_1, \ldots, s_n, \ldots\) where

\[s_i = u_0 + u_1 + u_2 + \ldots + u_i; \quad i = 0, \ldots, n.\]

This sequence is called the sequence of partial sums. If the sequence \(s_n\) has a limit as \(n \to \infty\), we say the series \((l-1)\) is convergent, and we say that \(s\) is the sum of the series, and we write

\[s = u_0 + u_1 + \ldots + u_n + \ldots\]

If \((x_n)\) and \((y_n)\) are two sequences such that a number \(n_0\) exists such that \((x_n/y_n) < K\), whenever \(n > n_0\), where \(K\) is independent of \(n\), we say that \(x_n\) is "of order of" \(y_n\) and we write:

\[x_n = o(y_n).\]

If \(\lim (x_n/y_n) = 0\), we write:

\[x_n = o(y_n).\]

Suppose \(f(x)\) is bounded for all \(x \geq x_0\), let \(M(x)\) and \(m(x)\) be the upper and the lower bound of \(f(x)\) for \(x \leq t \leq x_0\). Then \(M(t)\) and \(m(t)\) are respectively monotonic decreasing and monotonic increasing bounded functions of \(t\); i.e., as \(x\) increases
in a certain interval \((a, b)\), \(f(x)\) does not decrease (respectively, increase).

**Lemma 1-1:** If \(f(x)\) is a monotonic increasing (decreasing) function of \(x\) for \(x > a\), then \(x \to -\infty\) as \(f(x) \to a\) (finite) or to \(+\infty\), according as \(f(x)\) bounded above (below) or not.

**Proof:** Suppose \(f(x)\) is bounded, with least upper bound \(K\). Then i) \(f(x) \leq K, x > a\), ii) given \(\varepsilon > 0\), there is an \(x_0 > a\) such that \(f(x_0) < K - \varepsilon\). \(f(x)\) is monotonic increasing, so \(k - \varepsilon < f(x) \leq K < K + \varepsilon\) implies \(f(x) \to K\) as \(x \to -\infty\).

2. \(f(x)\) is not bounded. Then given \(L\), there is an \(x_1\) such that \(f(x_1) > L\) implies \(f(x) > L\) for all \(x \geq x_1\). A similar proof can be obtained for a function which is bounded below.

\(M(x), m(x)\) tend to finite limits as \(t \to -\infty\). These limits are called the upper limit and the lower limit of \(f\).

\[
\lim_{x \to -\infty} f(x) = \lim_{t \to -\infty} M(t) \\
\lim_{x \to -\infty} f(x) = \lim_{t \to -\infty} m(t)
\]

**Definition.** A sequence \(\{a_n\}_{n=1}^{\infty}\) is said to be convergent to the sum \(\alpha\), if \(\lim_{n \to \infty} a_n = \alpha\); i.e., if \(\varepsilon > 0\) implies there exists an \(N(\varepsilon)\) such that \(n \geq N\) implies \(|\alpha - a_n| < \varepsilon\); otherwise it is divergent.

**Theorem 1-2:** A necessary and sufficient condition for
\( \{a_n\}_n \) to be convergent is that: given \( \varepsilon > 0 \) there is an \( N(\varepsilon) \) such that \( |a_{n+p} - a_n| < \varepsilon \), for every positive integer \( p \).

**PROOF:** If \( \{a_n\}_n \) is convergent, then there exists \( a \) such that \( a_n \to a \), given \( \varepsilon \) we can find \( N(\varepsilon) \), such that \( |a_n - a| < \frac{\varepsilon}{2} \) for all \( n > N \), thus for any positive \( p \).

\[
|a_{n+p} - a_n| \leq |a_{n+p} - a| + |a_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

If \( |a_{n+p} - a_n| < \varepsilon \) for \( n > N(\varepsilon) \) then \( n \geq N + 1 \Rightarrow |a_n - a_{n+1}| < \varepsilon = a_{n+1} - a_n < a_{n+1} + \varepsilon = a_n \) is bounded. Thus \( \lim a_n \) and \( \lim a_n \) are finite (Lemma 1.1). \( a_{N+1} - \varepsilon \leq \lim a_n \leq \lim a_n \leq a_{N+1} + \varepsilon \), so \( 0 \leq \lim a_n - \lim a_n \leq 2\varepsilon \). But \( \varepsilon \) is an arbitrarily small, so

\[
\lim a_n = \lim a_n, \text{ and } \lim a_n = a \text{ (finite), i.e.} \]

\( \{a_n\}_n \) converges.

The next theorem is a consequence of the above.

**Theorem 1.3:** Fundamental theorem of convergence.

A necessary and sufficient condition for the convergence of the series \( \sum a_n \) is that: given \( \varepsilon > 0 \), there is a number \( n_0 = n_0(\varepsilon) \) such that for every \( n > n_0 \) and every \( k \geq 1 \), we have

\[
|\delta_{n+k} - \delta_n| < \varepsilon, \text{ that is to say, } |a_{n+1} + a_{n+2} + \ldots + a_{n+k}| < \varepsilon.
\]

\(^1\text{Proof on Knopp's "Theory and Application of Infinite Series", 1928, page 126.}\)
\[ \sum a_n b_n \text{ converges, if } \sum (a_n - b_n) \text{ converges absolutely and } \sum a_n \text{ converges conditionally at least. The proof is postponed because it is a special case of Abel's test.} \]

**Definition.** A series \( \sum a_n \) converges absolutely if \( \sum |a_n| \) converges.

**Definition.** A power series with center \( x_0 \) and coefficients \( a_n \) is a series of the form \( \sum_{n=0}^{\infty} a_n (x - x_0)^n \).

**Theorem 1-4:** If \( \sum a_n x^n \) is any power series which does not merely converge everywhere or nowhere, then a definite positive number \( r \) exists such that \( \sum_{n=0}^{\infty} a_n x^n \) converges for \( |x| < r \) (absolutely), but diverges for \( |x| > r \).

The number \( r \) is called the radius of convergence, and \((-r, r)\) is the interval of convergence of \( \sum a_n x^n \). In case of series of complex terms \( |x| < r \) is called the disk of convergence.
CHAPTER II

TESTS FOR CONVERGENCE OF REAL SERIES

I. The Tests

A. The Integral Test. This test is applicable only in the case of for which \( \sum a_n \) is a monotonic decreasing function.

Theorem 2-1: If for \( x \geq 1 \), \( f(x) \) is a non-negative, monotonic decreasing, integrable function such that \( f(n) = a_n \) for all positive integral values of \( n \), then:

\[
\lim_{n \to \infty} (A_n - \int_1^n f(x) \, dx) \text{ exists and satisfies the inequality:}
\]

\[0 \leq \lim_{n \to \infty} (A_n - \int_1^n f(x) \, dx) \leq a_1, \quad \text{where } A_n = \sum_{v=1}^{n} a_v.
\]

PROOF: \( \int_r^{r+1} f(r) \, dx \geq \int_r^{r+1} f(x) \, dx \geq \int_r^{r+1} f(r + 1) \, dx, \)

i.e. \( a_r \geq \int_r^{r+1} f(x) \, dx \geq a_{r+1} \). Adding, we have \( A_n - a_1 \geq \int_1^n f(x) \, dx \geq A_n - a_1 \) and \( 0 \leq a_n \leq A_n - \int_1^n f(x) \, dx \leq a_1 \).

Now \( [A_n - \int_1^n f(x) \, dx] - [A_{n+1} - \int_1^{n+1} f(x) \, dx] = \)

\[^2\text{Hyslop, 1942, page 38.}\]
= \int_{n}^{n+1} f(x) \, dx - a_{n-1} \geq 0, \text{ so by Lemma 1-1, } [A_n - \int_{1}^{n} f(x) \, dx]\n tends to a limit which satisfies the inequality.

**Cor. 2-2:** If for \( x \leq 1 \), \( f(x) \) is a non-negative, monotonic decreasing, integrable function such that \( f(n) = a_n \) for all \( n \n then \( \sum a_n \) and \( \int_{1}^{\infty} f(x) \, dx \) converge and diverge together.

**PROOF:** \( A_n = \int_{1}^{n} f(x) \, dx + \{A_n - \int_{p}^{n} f(x) \, dx\}. \) So if \( \int_{1}^{n} f(x) \, dx \) converges, so does \( A_n \), because by Theorem 1-1, \( 0 \leq \lim_{n \to \infty} \{A_n - \int_{1}^{n} f(x) \, dx \} \leq a_1 \).

**B. Ermakoff's Test:** If \( f(x) \) satisfies the same conditions as is Cor. 2-2, then:

\[ \sum a_n = \sum f(n) \left\{ \begin{array}{ll} \text{converges} & \text{if } \frac{e^x f(x)}{f(x)} \leq r < 1 \\ \text{diverges} & \text{if } r \geq 1 \end{array} \right. \n for large \( x \).

**PROOF:** Suppose the first of these inequalities is satisfied
for \( x \geq x_0 \), then for these \( x \)'s \( \int_{e}^{x} f(t) \, dt = \sum_{x_0}^{x} e^t f(e^t) \, dt \leq \theta \int_{x_0}^{x} f(t) \, dt \leq \theta \int_{x_0}^{x} f(t) \, dt - \int_{e}^{x} f(t) \, dt \leq \theta \int_{x_0}^{x} f(t) \, dt - \int_{e}^{x} f(t) \, dt \leq \theta \int_{x_0}^{x} f(t) \, dt = \text{fixed number}. \) Then the integral on the left, and hence

---

\( \int_{x_0}^{x} f(t) \, dt \), are less than a certain number for \( x > x_0 \). Thus by the integral test, the series must converge.

On the other hand, if we assume the second inequality is satisfied for \( x > x_1 \), we have:

\[
\int_{x_1}^{e^x} f(t) \, dt = \int_{x_1}^{e} e^t f(e^t) \, dt \geq \int_{x_1}^{x} f(t) \, dt.
\]

Thus by comparing the first integral and the third, we have:

\[
\int_{x}^{e^x} f(t) \, dt \geq \int_{x_1}^{x_1} f(t) \, dt.
\]

On the right hand of this inequality we have a fixed quantity \( \gamma > 0 \), and for every \( n > x_1 \), we can find \( k_n \), so that \( (J_n = \int_{1}^{n} f(t) \, dt): \)

\[
J_{n+k_n} - J_n = \int_{n}^{n+k_n} f(t) \, dt \geq \gamma > 0.
\]

Thus, \( J_n \) cannot be bounded and \( \sum_{n=0}^{\infty} a_n \) is then divergent.

\[ G. \text{ The General Test, "Kummer's Test"} \]

\[ \text{Theorem 2-3: Let } a_n > 0 \text{ and } \sum b_n \text{ be a divergent series} \]

with \( b_n > 0 \).

Let \( \lim_{n \to \infty} \left( \frac{1}{b_n} - \frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}} \right) = r. \)

Then, \( \sum a_n \) converges if \( r > 0 \), and diverges if \( r \leq 0 \).
First Proof: If \( r > 0 \), there exists \( N \) such that whenever \( n \geq N \):

\[
\frac{1}{b_n} \cdot \frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}} > \frac{1}{2} r, \text{ i.e.}
\]

\[
a_{n+1} < \frac{2}{r} \left( \frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}} \right).
\]

Then

\[
\sum_{i=N+1}^{n+1} a_i < \frac{2}{r} \left( \frac{a_N}{b_N} - \frac{a_{n+1}}{b_{n+1}} \right) < \frac{2}{r} \cdot \frac{a_N}{b_N}
\]

and for \( \sum a_n \), \( a_n \) is bounded. Thus \( \sum a_n \) converges.

2. If \( r = 0 \), the series \( \sum a_n \) clearly diverges, for

\[
\frac{a_n}{a_{n+1}} = \frac{b_n}{b_{n+1}}
\]

If \( r < 0 \), we can find \( N \) such that, whenever \( n \geq N \),

\[
\frac{1}{b_n} \cdot \frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}} < 0. \text{ Then}
\]

\[
a_{n+1} > \frac{a_n}{b_n} \cdot b_{n+1}
\]

\[
a_{N+1} > \frac{a_N}{b_N} \cdot b_{N+1}
\]

\[
a_{N+2} > \frac{a_{N+1}}{b_{N+1}} \cdot b_{N+2} > \frac{a_N}{b_N} \cdot \frac{b_{N+1}}{b_{N+1}} \cdot b_{N+2} = \frac{a_N}{b_N} \cdot b_{N+2}
\]

\[
\ldots\ldots\ldots
\]

\[
a_{N+m} > \frac{a_N}{b_N} \cdot b_{N+m}, \text{ adding}
\]
\[
\sum_{N+1}^{N+m} a_i > \frac{a_N}{b_N} \cdot \sum_{N+1}^{N+m} b_i.
\]

But \( \Sigma b_n \) diverges, hence \( \Sigma a_n \) diverges.

**Second Proof:** The convergence test requires in the first instance, that for every sufficient large \( n \)

\[
\frac{c_{n+1}}{c_n} - \frac{a_{n+1}}{a_n} \geq 0 \quad \text{or} \quad \frac{1}{c_n} - \frac{a_{n+1}}{a_n} \cdot \frac{1}{c_{n+1}} \geq 0, \quad \text{where} \quad \Sigma c_n \text{ is a convergent series.}
\]

Let \( c_n = \frac{p_n - p_{n-1}}{p_n p_{n-1}} \). Then \( \frac{p_{n+1} - p_{n-1}}{p_{n+1} p_n} \cdot \frac{p_{n} p_{n-1}}{p_n - p_{n-1}} - \frac{a_{n+1}}{a_n} \geq 0 \Rightarrow d_{n+1} \left( \frac{1}{d_n} - 1 \right) - \frac{a_{n+1}}{a_n} \geq 0, \quad \text{where} \quad d_{n+1} = \frac{p_{n+1} - p_n}{p_{n+1}} \).

\[
\frac{d_{n+1}}{d_n} - \frac{a_{n+1}}{a_n} \geq d_{n+1}. \quad \text{But} \quad \Sigma r_d \text{ } (r > 0) \text{ is a divergent series with}
\]

\[
\Sigma d_n, \text{ so} \quad \frac{1}{d_n} - \frac{a_{n+1}}{a_n} \cdot \frac{1}{d_{n+1}} \geq r > 0 \Rightarrow \Sigma a_n \text{ is convergent} \Rightarrow
\]

\[
b_n - \frac{a_{n+1}}{a_n} \cdot b_{n+1} \geq r > 0
\]

\[
\frac{1}{d_n} - \frac{a_{n+1}}{a_n} \cdot \frac{1}{d_{n+1}} \left\{ \begin{array}{ll} \geq r > 0 & \text{\( \Sigma a_n \) converges} \\ \leq 0 & \text{\( \Sigma a_n \) diverges} \end{array} \right.
\]

**Third Proof:** (Stalz's Proof):\(^2\)

The criterion is that from some stage on \( a_n b_n - a_{n+1} b_{n+1} \geq \)

\(^1\)Knopp, page 311.

\(^2\)Knopp, page 311.
It follows in particular, that the product $a_n b_n$ diminishes monotonely and therefore tends to a definite limit $t > 0$. Then

$$\sum_{r} \frac{1}{r} (a_n b_n - a_{n+1} b_{n+1})$$

is a convergent series of positive terms and as its terms are not less than the corresponding terms of the series $\sum a_n$ then $\sum a_n$ is convergent.

This extremely general criterion is due to E. Kummer, 1835. All the convergence tests fall as corollaries of this theorem except the Integral Test and Eramakoff's.

**Cor. 3-4: The Ratio Test (d'Alembert Test)**

If $a_n > 0$, $\lim_{n \to \infty} \frac{a_n + 1}{a_n} = u$, then $\sum a_n$ is convergent, if $u < 1$, and divergent if $u > 1$.

**Proof:** Let $b_n = 1$; then $\sum b_n = 1 + 1 + 1 + \ldots$ is a divergent series. Then for all $n$

$$\lim_{n \to \infty} \left( \frac{1}{b_n} \cdot \frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}} \right) = r \Rightarrow$$

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = r + 1 = v = \frac{1}{u}.$$

1) If $r > 0$, then $v > 1$, then $u < 1$ and the series converges.

2) If $r < 0$, then $v < 1$, then $u > 1$ and the series diverges.
Cor. 3-5: The Comparison Test

Let \( a_n \geq 0, b_n \geq 0, k > 0 \) independent of \( n \), and an integer \( N > 0 \), such that:

1) \( a_n < k b_n \), then \( \Sigma a_n \) converges if \( \Sigma b_n \) converges.

2) \( a_n > k b_n \), then \( \Sigma a_n \) diverges if \( \Sigma b_n \) diverges.

**Proof:** (2) \( a_n > k b_n \), \( \Sigma b_n \) is divergent, to show \( \Sigma a_n \) diverges.

\[
\frac{1}{b_n} \cdot \frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}} > \frac{k b_n}{b_n a_{n+1}} - \frac{1}{b_{n+1}} < \frac{k}{kb_{n+1}} - \frac{1}{b_{n+1}} < 0.
\]

Then by the Theorem 2-3, \( \Sigma a_n \) diverges.

1) \( a_n < k b_n \), and \( \Sigma b_n \) converges, to show that \( \Sigma a_n \) converges.

Let \( \Sigma d_n \) be a divergent series, to show

\[
\frac{1}{d_n} \cdot \frac{a_n}{a_{n+1}} - \frac{1}{d_{n+1}} = r > 0 \Rightarrow \frac{1}{d_n} \cdot \frac{k b_n}{a_{n+1}} - \frac{1}{d_{n+1}} < r
\]

\[
\frac{d b_n}{d_n} < \frac{a_{n+1}}{d_{n+1}} + r a_{n+1} < \frac{kb_{n+1}}{d_{n+1}} + rkb_{n+1} \Rightarrow
\]

\[
\frac{b_n}{d_n} - \frac{b_{n+1}}{d_{n+1}} < r b_{n+1}, \text{ but } \Sigma b_n \text{ converges} \Rightarrow \frac{b_n}{d_n} - \frac{b_{n+1}}{d_{n+1}} > 0 \Rightarrow
\]

\( r b_{n+1} > 0 \Rightarrow r > 0 \Rightarrow \Sigma a_n \) converges.
Cor. 3-6: Cauchy Test

I shall prove Cauchy Test by proving the following lemma:

Lemma 3-7: If \( a_n > 0 \), \( \frac{a_{n+1}}{a_n} \rightarrow l \) then \( \frac{\sqrt{n}}{a_n} \rightarrow l \).

Proof: \( \frac{a_{n+1}}{a_n} \rightarrow l \neq 0 \)

\[
\log a_{n+1} - \log a_n \rightarrow \log l \Rightarrow \log l - \varepsilon < \left( \log a_{n+1} - \log a_n \right) < \\
< \log l + \varepsilon
\]

\[
\log l - \varepsilon < \left( \log a_{N+1} - \log a_N \right) < \log l + \varepsilon
\]

\[
\ldots
\]

\[
\log l - \varepsilon < \left( \log a_{N+m} - \log a_{N+m-1} \right) < \log l + \varepsilon
\]

Adding

\[
m(\log l - \varepsilon) < \left( \log a_{N+m} - \log a_N \right) < m(\log l + \varepsilon)
\]

\[
\Rightarrow \log l - \varepsilon < \frac{1}{m} \left( \log a_{N+m} - \log a_N \right) < \log l + \varepsilon
\]

\[
m \rightarrow \infty \Rightarrow \log l - \varepsilon \leq \lim_{m \rightarrow \infty} \frac{1}{m} \left( \log a_{N+m} \right) \leq \lim_{m \rightarrow \infty} \left( \frac{1}{m} \log a_{N+m} \right) \leq \\
\leq \log l + \varepsilon. \text{ But } \varepsilon \text{ is arbitrary, so}
\]

\[
\log l = \lim_{m \rightarrow \infty} \frac{1}{m} \log a_{N+m}
\]

Let \( n = N + m \), then

\[
\log l = \lim_{n \rightarrow \infty} \frac{1}{n} \log a_n \Rightarrow \\
\log l = \lim_{n \rightarrow \infty} \log \frac{\sqrt{n}}{a_n} = \log \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{a_n} \Rightarrow \frac{\sqrt{n}}{a_n} \rightarrow l.
\]

Cor. 3-8: Raabe's Test

If \( a_n > 0 \), \( n \rightarrow \infty \), \( \frac{a_n}{a_{n+1}} = 1 + \frac{b}{n} + o\left( \frac{1}{n} \right) \), then:
if \( h > 1 \), then \( \sum a_n \) is convergent

if \( h < 1 \), then \( \sum a_n \) is divergent.

**Proof:** \( \frac{a_n}{a_{n+1}} = 1 + \frac{h}{n} + o\left(\frac{1}{n}\right) \)

\[
\begin{align*}
    n \frac{a_n}{a_{n+1}} - n &= h + n o\left(\frac{1}{n}\right) \\
    n \frac{a_n}{a_{n+1}} - n - 1 &= h + n o\left(\frac{1}{n}\right) - 1 \\
\end{align*}
\]

\[
\Rightarrow \lim_{n \to \infty} (n \frac{a_n}{a_{n+1}} - (n + 1)) = h - 1, \text{ let } b_n = \frac{1}{n}, \text{ } h - 1 = r
\]

\[
\Rightarrow \sum b_n \text{ diverges } \Rightarrow \lim_{n \to \infty} \left( \frac{\frac{a_n}{b_n}}{a_{n+1}} - \frac{1}{b_{n+1}} \right) = r
\]

If \( r > 0 \Rightarrow h > 1 \) and \( \sum a_n \) is convergent.

If \( r < 0 \Rightarrow h < 1 \) and \( \sum a_n \) is divergent.

**Cor. 3-9: Gauss' Test**

1. If \( a_n > 0 \) and \( \frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{h_n}{n \log n} \) then:

   1) \( \sum a_n \) converges, if \( h_n > 1 \)

   2) \( \sum a_n \) diverges, if \( h_n < 1 \).

**Proof:** Let \( b_n = \frac{1}{n \log n} \), \( n = 2, 3, \ldots \). Then

\[
\frac{1}{b_n} \frac{a_n}{a_{n+1}} - \frac{1}{b_{n+1}} = n \log n \frac{a_n}{a_{n+1}} - (n + 1) \log (n + 1) = w_n - 1, \text{ say.}
\]

Then \( \frac{a_n}{a_{n+1}} = \frac{n + 1}{n} \frac{\log (n + 1)}{\log n} + \frac{w_n - 1}{n \log n} = \)
\[(1 + \frac{1}{n}) \{1 + \frac{\log (1 + \frac{1}{n})}{\log n} \} + \frac{w_n - 1}{n \log n} = (1 + \frac{1}{n})(1 + \frac{1}{n \log n}) + \]

\[+ O\left(\frac{1}{n^2 \log n}\right) + \frac{w_n - 1}{n \log n} = (1 + \frac{1}{n}) + \frac{w_n}{n \log n} + O\left(\frac{1}{n^2 \log n}\right) =\]

\[= 1 + \frac{1}{n} + \frac{w_n}{n \log n} + o(\frac{1}{n \log n}).\]

Hence:

If \(w_n > 1\) then \(\sum a_n\) converges.

If \(w_n < 1\) then \(\sum a_n\) diverges.

**Other forms of Gauss' Test:**

1. Let \(\lambda > 1\) and \((r_n)\) is bounded; then

\[\sum a_n \text{ converges if } h > 1, \text{ and diverges if } h \leq 1.\]

2. Gauss expressed his test as

"If \[
\frac{a_{n+1}}{a_n} = 1 - \frac{h}{n} - \frac{r_n}{n^\lambda}\]

\(\lambda > 1\) and \((r_n)\) is bounded; then \(\sum a_n\) converges when \(b_1 - c_1 < -1\) and diverges if \(b_1 - c_1 \geq -1.\)"

2. **Examples**

1. Consider the series \(\sum \frac{1}{2 n (\log n)^\lambda};\) this series

converges if \(\lambda > 1,\) and diverges if \(\lambda \leq 1.\)

**Proof:** Using the Integral Test:

\[\text{Knopp, page 288.}\]
(1) \( \lambda > 1 \). Consider
\[ J = \int_1^\infty \frac{dx}{x(\log x)^\lambda} \] Let \( x = e^y \). Then
\[ J = \int_\log 2^{\infty} \frac{dy}{y^\lambda} = \left[ \frac{1}{1 - \lambda} \right]_1^\infty \log 2 \cos \left( \frac{1}{(1 - \lambda)y^{1 - \lambda}} \right) \log 2 = \lim_{b \to -\infty} \left( \frac{1}{b^{\lambda - 1}} - \frac{1}{(\log 2)^{\lambda - 1}} \right) = \frac{1}{(1 - \lambda)(\log 2)^{\lambda - 1}}, \text{ and the series converges.} \]

(2) If \( \lambda < 1 \), then
\[ J = \int_2^\infty \frac{dx}{x \log x} = \log |\log x| \bigg|_2^\infty, \text{ and as } y \to \infty, J \to \infty, \]
which means that the series diverges.

(3) \( \lambda = 1 \). Then
\[ J = \int_1^\infty \frac{dx}{x \log x} = \log |\log x| \bigg|_2^\infty, \]
and the series is divergent.

But \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n(\log n)^\lambda}{(n + 1)(\log(n + 1))^\lambda} = 1 \), so the Ratio Test does not apply. Thus, in such examples, it is better to use the Integral Test than other tests.

II. Consider the series \( \sum_{n=0}^\infty a_n \) where \( a_n = 2^{-n}(-1)^n \).
\[ \sqrt[n]{a_n} = 2^{-1}(-1)^n/n^{1/2}, \text{ so by the Root Test, the series converges.} \]

But \( \frac{a_{n+1}}{a_n} = 2^{-1+(-1)^n}(-1)^{n+1} \). Then \( \frac{a_{n+1}}{a_n} \to \left\{ \begin{array}{ll} 2 & \text{if } n \text{ is even} \\ \frac{1}{8} & \text{if } n \text{ is odd} \end{array} \right. \)
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 2, \quad \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}
\]
which means that this series cannot be tested by the Ratio Test because the limit does not exist.

### III.

Consider \(\sum_{1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{1}{n}\).

\[
\frac{a_n}{a_{n+1}} = \frac{(2n + 2)(n + 1)}{(2n + 1)n} = \frac{2n^2 + 4n + 2}{2n^2 + n} = \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = 1; \quad \text{no information by Ratio Test.}
\]

But by Raabe's Test:

\[
\frac{a_n}{a_{n+1}} = 1 + \frac{\frac{3}{2n} + O\left(\frac{1}{n^2}\right)}{2n}, \quad \text{so the series converges.}
\]

### IV.

Consider the hypergeometric series at \(x = 1\):

\[
1 + \frac{\alpha \beta}{1\gamma} + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{2!\,\gamma(\gamma + 1)} + \frac{\alpha(\alpha + 1)(\alpha + 2)\beta(\beta + 1)(\beta + 2)}{3!\,\gamma(\gamma + 1)(\gamma + 2)} + \ldots =
\]

\[
= \sum_{0}^{\infty} \frac{\alpha(\alpha + 1)\cdots(\alpha + n - 1)\beta(\beta + 1)\cdots(\beta + n - 1)}{n!\,\gamma(\gamma + 1)\cdots(\gamma + n - 1)}
\]

\[
\frac{a_n}{a_{n+1}} = \frac{(n + 1)(\gamma + n)}{(\alpha + n)(\beta + n)} = \frac{n^2 + (\gamma + 1)n + \gamma}{n^2 + (\alpha + \beta)n + \alpha\beta}
\]

\[= 1 + \frac{\gamma + 1 - \alpha - \beta}{n} + O\left(\frac{1}{n^2}\right), \quad \text{then:}
\]

By Raabe's Test or by Gauss' test:

\(\sum a_n\) converges if \(\gamma > \alpha + \beta\) and diverges if \(\gamma < \alpha + \beta\). But if \(\gamma = \alpha + \beta\) then \[
\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right) \quad \text{and here we cannot apply Raabe's Test. But by Gauss' test, (Form 3):}
\]

\[
\frac{a_{n+1}}{a_n} = \frac{n^2 + (\alpha + \beta)n + \alpha \beta}{n^2 + (\gamma + \lambda)n + \gamma}, \quad b_1 = \alpha + \beta \text{ and } c_1 = \gamma + 1.
\]

If \( b_1 - c_1 = \alpha + \beta - \gamma - 1 < -1 \), i.e. \( \alpha + \beta < \gamma \), the series converges.

If \( b_1 - c_1 = \alpha + \beta - \gamma - 1 \geq -1 \), i.e. \( \alpha + \beta \geq \gamma \) the series diverges. Then when \( \alpha + \beta = \gamma \) the series diverges.

It is obvious in this example, testing it by the Integral Test or by others is too complicated, if not impossible.

V. To test \( \sum \frac{1}{n \log n \cdots \log_{p-1} n (\log_p n)^{\alpha}} \) by using test

B, consider \( f(x) = \frac{1}{x \log x \cdots \log_{p-1} x (\log_p x)^{\alpha}} \), and

\[
e^{x} f(e^{x}) = \frac{e^{x}}{e^{x} \log x \cdots \log_{p-2} x (\log_{p-1} x)^{\alpha}}.
\]

Then

\[
R = \frac{e^{x} f(e^{x})}{f(x)} = \frac{e^{x} \log x \cdots \log_{p-1} x (\log_p x)^{\alpha}}{e^{x} \log x \cdots \log_{p-2} x (\log_{p-1} x)^{\alpha}} = \frac{\log_{p-1} x (\log_p x)^{\alpha}}{(\log_{p-1} x)^{\alpha-1}} = \frac{(\log_p x)^{\alpha}}{(\log_{p-1} x)^{\alpha-1}}.
\]

If \( \alpha > 1 \), then \( R \to 0 \), and the series converges.

If \( \alpha \leq 1 \), then \( R \to +\infty \), and the series diverges.
CHAPTER 3

COMPLEX SERIES

I. General Remarks

Lemma 3-1: The sequence $(z_n) = (x_n + i y_n)$ converges to $w = (u + iv)$ if, and only if $(x_n) \to u$ and $(y_n) \to v$.

Proof: a. If $x_n \to u$ and $y_n \to v$, then $(x_n - u)$ and $(y_n - v)$ are null sequences, i.e., given $\varepsilon > 0$ we can find $n_0$, then $\sum_{n_0+1}^{\infty} |x_n - u|$ and $\sum_{n_0+1}^{\infty} |y_n - v|$ are less than $\varepsilon$ which means $(x_n - u) + i(y_n - v) = (z_n - w)$ is also a null sequence. Then $z_n \to w$ and so $(z_n)$ is a convergent sequence.

b. $z_n \to w$ then $(z_n - w)$ is a null sequence, but $|x_n - u| \leq |z_n - w|$ and so does $|y_n - v|$; then $(x_n - u)$ and $(y_n - v)$ are null sequences; i.e., $(x_n)$ converges to $u$ and $(y_n)$ converges to $v$.

Theorem 3-2: A series $\sum a_n$ of complex terms is convergent if and only if the series $\sum R(a_n)$ of the real parts and $\sum I(a_n)$ of the imaginary parts converge separately. And if these two...
series converge to \( s' \) and \( s'' \), then \( \sum a_n \) converges to \( s \) where \( s = s' + is'' \).

**Proof**: By Lemma 2-1.

**Theorem 2-2**: \( \sum a_n \) is absolutely convergent if and only if both \( \sum R(a_n) \) and \( \sum I(a_n) \) are absolutely convergent.

**Proof**: a) \( |x| \leq |z| \leq |x| + |y| \) and \( |y| \leq |z| \leq |x| + |y| \), so if \( \sum |x_n| \) and \( \sum |y_n| \) are convergent, then \( \sum |x_n| + \sum |y_n| \) is a finite number, then \( \sum |z_n| \) is convergent.

b) If \( \sum |z_n| \) is convergent, then \( \sum |x_n| \leq \sum |z_n| \) and then \( \sum |x_n| \) is convergent, and so does \( \sum |y_n| \).

**II. The Power Series**

We recall the definition and basic properties of power series from Chapter I. If a complex power series has radius of convergence \( r \) and center \( 0 \), the circle of convergence is given by \( |z| = r \).

**Theorem 3-2**: If \( \sum a_n z^n \) converges for \( z = z_0 \) (\( z_0 \neq 0 \)), or even if the sequence \((a_n z_0^n)\) is only known to be bounded, then \( \sum a_n z^n \) is absolutely convergent for \( z = z_1 \) where \( |z_1| < |z_0| \).

**Theorem 3-4**: Given \( \sum a_n z^n \), let \( L \) denote the upper limit of the sequence \(|a_1|, \sqrt{|a_2|}, \sqrt[3]{|a_3|}, ..., \sqrt[n]{|a_n|}, ...; \ i.e., \)**
\[ L = \lim_{n \to \infty} \sqrt[n]{a_n}. \]

a) If \( L = 0 \) then \( \sum_{n} a_n z^n \) converges everywhere.

b) If \( L = \infty \) then \( \sum_{n} a_n z^n \) converges nowhere.

c) If \( 0 < L < \infty \) then \( \sum_{n} a_n z^n \)

i. Converges absolutely for any \( |Z| > \frac{1}{L} \)

ii. Diverges for every \( |Z| > \frac{1}{L} \)

iii. Thus, we can find \( r \) such that \( r = \frac{1}{L} = \frac{1}{\lim \sqrt[n]{|a_n|}} \) as a radius of convergence.

**Proof:**

a. If \( Z_0 \neq 0 \) then \( \frac{1}{2|Z_0|} > 0 \), and \( \sqrt[n]{|a_n|} < \frac{1}{2|Z_0|} \) or \( |a_n z^n| < \frac{1}{2^n} \), which means that \( \sum_{n} a_n z^n \) converges also for \( |Z| < |Z_0| \).

b. If \( \sum_{n} a_n z^n \) converges absolutely for \( z = z_1 \neq 0 \) then \( (a_n z_1^n)^\infty \) and \( (\sqrt[n]{|a_n z_1^n|})^\infty_0 \) are bounded and \( < K_1 \), say for every \( n \).

Then \( \sqrt[n]{|a_n|} \frac{K_1}{|z_1|} = K \), but \( \lim \sqrt[n]{|a_n|} = \infty \), so \( \sum_{n} a_n z^n \) cannot converge.

If \( Z' \) is any complex number such that \( |Z'| < \frac{1}{L} \), then \( \text{let } h > 0 \text{ be such that } |Z'| < h < \frac{1}{L} \text{ so } \frac{1}{h} > L \). Then by the

---

1Knopp, p. 153.
definition of \( L \) there exists \( n_0 \) such that for \( n > n_0 \), \( \sqrt[n]{|a_n|} < \frac{1}{h} \).

Then \( \sqrt[n]{a_n Z^n} < \frac{|Z^n|}{h} < 1 \) and therefore, \( \Sigma a_n Z^n \) is absolutely convergent. If \( |Z^n| > \frac{1}{L} \) then \( \frac{1}{|Z^n|} < L \) and we can find infinitely many \( n \)'s such that \( \sqrt[n]{|a_n|} > \frac{1}{|Z^n|} \) or \( \sqrt[n]{|a_n Z^n|} > 1 \). Therefore, the series is divergent for \( |Z^n| > \frac{1}{L} \).

**Theorem 3-5: "Weierstrass Criterion"**

If \( \frac{a_{n+1}}{a_n} = 1 - \alpha \cdot \frac{A_n}{n} \), where \( \alpha \) is an arbitrary complex number, \( \lambda > 1 \), and \( (A_n) \) is bounded, then the power series \( \Sigma a_n Z^n \) is absolutely convergent for \( |Z| < 1 \), divergent for \( |Z| > 1 \) and, for the points of the circle of convergence \( |Z| = 1 \) the power series will:

a. Converge absolutely if \( R(\alpha) > 1 \),

b. Be conditionally convergent, if \( 0 < R(\alpha) \leq 1 \), except perhaps for \( Z = +1 \),

c. Diverge, if \( R(\alpha) \leq 0 \).

**PROOF:**

\[
\frac{a_{n+1} Z^{n+1}}{a_n Z^n} \rightarrow |Z|,
\]

so the statements where \( |Z| < 1 \) and \( |Z| > 1 \) are verified from the theorem above. Suppose now \( |Z| = 1 \). Let \( \alpha = \beta + i \gamma \), and suppose \( |A_n| < K \) for all \( n \).

a. \( R(\alpha) = \beta > 1 \).
b. $0 < R(\alpha) \leq 1$.

First we have to show that $\Sigma |a_n - a_{n+1}|$ is convergent.

Now as $n \to \infty$, $\frac{|a_{n+1}|}{a_n} < 1 - \frac{\beta'}{n}$ with $0 < \beta' < \beta$.

So $|a_n|$ diminishes monotonically from some stage on and therefore tends to a definite limit $\geq 0$.

Then $\Sigma (|a_n| - |a_{n+1}|)$ is convergent and all its terms are positive, for sufficiently large $n$

$$\frac{|a_n - a_{n+1}|}{|a_n| - |a_{n+1}|} = \frac{|1 - \frac{a_{n+1}}{a_n}|}{1 - \frac{|a_{n+1}|}{a_n}} \leq \frac{|a_n + A_n|}{|a_n - A_n|} \to \frac{|a|}{\beta'}$$

this means that $\Sigma |a_n - a_{n+1}|$ converges with $\Sigma (|a_n| - |a_{n+1}|)$.

So far $0 < R(\alpha) = \beta \leq 1$ and for $Z \neq +1$, the convergence of

$\Sigma a_n Z^n$ follows from Dedekind's Test, by putting $a_n = b_n$.

$\Sigma |a_n - a_{n+1}|$ converges and $a_n \to 0$ which means that the partial sums of $\Sigma Z^n$ are bounded for every fixed $Z \neq +1$ on $|Z| = 1$.

But for $Z = +1$ suppose $R(\alpha) = \beta \leq 1$. Then for $A \leq K$ we will have

$$\frac{|a_{n+1}|}{a_n} \leq 1 - \frac{\beta}{n} - \frac{K}{n},$$

and by Gauss' Test, $\Sigma |a_n|$ diverges. Then

$$\frac{|a_{n+1}|}{a_n} \leq |1 - \frac{\beta + \gamma i}{n}| + \frac{K}{n\lambda}.$$ Hence $\frac{|a_{n+1}|}{a_n} < 1 - \frac{\beta'}{n}$ for all $\beta'$ such that $1 < \beta' < \beta$ and sufficiently large $n$, and by Raabe's test, $\Sigma a_n$ converges.
c. \( R(a) \leq 0 \). In this case \(|a_n|\) remains greater than a certain positive number for every sufficient large \( n \).

For example:

1. \( \sum \frac{Z^n}{n^2} \) converges absolutely for all points of \(|Z| = 1\).

2. \( \sum \frac{Z^n}{n} \) converges conditionally at all points of \(|Z| = 1\), except \( Z = \pm 1 \).

3. \( \sum \frac{Z^{2n}}{2n} \) diverges at \( Z = 1 \) and converges conditionally at all other points of \(|Z| = 1\).

4. \( \sum \frac{Z^{4n}}{4n} \) converges everywhere on \(|Z| = 1\), except \( Z = \pm 1, Z = \pm i \).

**Theorem 3-6: Abel's Test for Complex Series**

If \( \sum a_n \) is convergent, and if \( \sum (v_n - v_{n+1}) \) is absolutely convergent, then \( \sum a_n v_n \) is convergent.

**Proof:** \[ \left| \sum_{m=1}^{m+p} a_n v_n \right| < H_m V_m, \] where

\[ H_m = \lim \{ |a_{m+1}|, \ldots, |a_{m+1} + a_{m+2} + \ldots + a_{m+p}| \} \]

and

\[ V_m - \lambda = \sum_{i=1}^{\infty} |V_{m+i} - V_{m+i+1}| \]

where \( \lambda = \lim |V_m| \). But \( \sum a_n \) converges, so \( H_m \leq \varepsilon \) and

\[ \sum_{m=1}^{m+p} a_n v_n < \varepsilon V. \] Hence \( \sum a_n v_n \) is convergent.
CHAPTER 4

BEHAVIOR OF POWER SERIES ON OR NEAR

THE CIRCLE OF CONVERGENCE

4-1: Abel's Theorem.

If \( \Sigma a_n z^n \), of radius 1, remains convergent at \( Z = 1 \) and if

\[ \Sigma a_n = s, \text{ then } \lim_{Z \to 1} (\Sigma a_n Z^n) = s, \]

if the mode of approach of \( Z \) to 1 is restricted only in that \( Z \) must be on a Stolz curve, i.e., \( Z \) should remain within \( |Z| = 1 \), and in the angle between two arbitrary rays which penetrate into the interior of the unite circle starting from the point +1. (See figure.)

PROOF: This theorem is also a special case of Picard's extension of Abel's theorem.

4-2: Picard's Theorem.

If \( \Sigma a_n z^n \), of radius 1, remains convergent at \( Z = +1 \) and if \( \Sigma a_n = s \), then

\[ \lim_{Z \to +1} \Sigma a_n Z^n = s \]

along any regular curve which cuts the circle at a positive angle.
PROOF: Let \( v_n = z^n \); then
\[
|v_n - v_{n+1}| = |1 - z||z^n|,
\]
then
\[
V = \sum_{n=0}^{\infty} |v_n - v_{n+1}| = \frac{|1 - z|}{1 - |z|} \geq 1,
\]
so \( \sum (v_n - v_{n+1}) \)
is absolutely convergent in any area for which \(|1 - z| \leq K (1 - |Z|)\), where \( K > 1 \) and \(|Z| < 1\) from the figure below.

\(|1 - Z| = j, |Z| = r, j \leq K(1 - r) \) or \((K - j)^2 \geq K^2 r^2,\)

\(1 - Z = j(Cos \varphi + i Sin \varphi), K^2 - 2Kj + j^2 \geq K^2(1 - 2j Cos \varphi + j^2)\)
or \((K^2 - 1) j \geq 2(K^2 Cos \theta - K)\) where \(-\frac{\pi}{2} < \varphi < \frac{\pi}{2}\)

For large \( K \), the limaçon approaches nearly to \(|Z| = 1\).

In any regular curve is drawn from a point inside the circle to the point \( Z = +1 \), then provided the curve cuts the circle at a positive angle, we can draw one of these limacons to enclose the whole of the curve.

Hence: \( \lim_{Z \to +1} \sum a_n Z^n = \sum a_n \), where \( Z \to +1 \) along any regular curve which cuts the circle at a positive angle.

The Converse of Abel's Theorem

The converse of Abel's Theorem is not always true, i.e. if
$\lim_{n \to 1} \sum a_n z^n$ exists, it is not possible to conclude that $\sum a_n$ is convergent without some restrictions on the coefficients.

1. Pringsheim made this inference by his example,

$$\sum a_n z^n = e^{(z-1)^{-1}} \to 0 \quad \text{as } z \to 1.$$  
He then assumed the restriction that the coefficients $a_n$ are all positive after a certain stage. Then $z \to 1$ by real values we can infer from the existence of $\lim \sum a_n z^n$ that $\sum a_n$ cannot diverge and cannot oscillate, hence $\sum a_n$ converges to $A$ by Abel's theorem.

2. Tauber gave this condition:

$$\lim n a_n = 0 \quad \text{when } z \to 1 \text{ by any path within the limacon}$$
(of the proof of Picard's theorem). In this example we can write:

$$n |a_n| = c_n, \text{ then we find } \sum_{0}^{k-1} \sum a_n (1 - Z^n) | <$$

$$\sum_{0}^{k-1} \sum a_n (1 - Z^n) | < \sum_{0}^{k-1} \sum C_n \text{ since } \frac{1 - Z^n}{1 - Z} = 1 + Z + \ldots +$$

$$+ Z^{n-1} | \leq n. \text{ Also, if } H_i, \text{ is the upper limit of } C_i,$$

$$C_{i+1}, \ldots, \text{ we will have}$$

$$\left| \sum_{n=v}^{\infty} a_n z^n \right| < H_v \frac{|z|^v}{v} (1 + |z| + \ldots + |z|^v + \ldots) <$$

$$< H_v \frac{|z|^v}{v(1 - |z|)}. \text{ If we take } z \text{ on the given path such}$$
that $|Z| = 1 - \frac{1}{i}$, we will have $|1 - Z| \leq K(1 - |Z|) = \frac{K}{v}$

then $|\Sigma_{n=0}^{v-1} a_n - \Sigma_{n=0}^{v-1} a_n Z^n| < \frac{K}{v} \Sigma_{n=0}^{v-1} c_n + H_1$

then

$\lim_{v \to \infty} \Sigma_{n=0}^{v-1} a_n = \lim_{v \to \infty} \Sigma_{n=0}^{v-1} a_n Z^n = A$. If $a_n$ has no limit then

the convergence of $\Sigma a_n$ follows from the existence of

$\lim_{n \to \infty} \Sigma a_n Z^n$ and from $\lim_{n \to \infty} \frac{1}{n}(a_1 + \ldots + a_n) = 0$.

**THE BINOMIAL SERIES**

The binomial series $\Sigma_{n=0}^{\infty} \frac{(\alpha)}{n} Z^n$ is a power series of radius $\alpha \leq 0$

defined by $\frac{(\alpha)}{n} = \frac{\alpha(\alpha - 1)\ldots(\alpha - n + 1)}{n!}$.

**Theorem 4-3**: For real integral values of $\alpha \geq 0$ the binomial series $\Sigma_{n=0}^{\infty} \frac{\alpha}{n} Z^n$ is reduced to finite sum, i.e., $(1 + Z)^\alpha$, if $\alpha = 0$, it has the value 1.

If $\alpha$ does not have those values then, the series converge absolutely for $|Z| < 1$, diverges for $|Z| > 1$ and its behavior on the circumference $|Z| = 1$ is:

a. If $R(\alpha) > 1$, it converges absolutely at all points of the circumference.

b. If $R(\alpha) \leq -1$, it converges nowhere.

c. If $-1 < R(\alpha) \leq 0$, it diverges at $Z = -1$, and converges conditionally at every other point of the circumference.

The sum of the series when it converges is $(1 + Z)^\alpha$ for $Z \neq -1$,
and 0 for \( Z = -1 \).

**Proof:** Let \((-1)^{n}(\frac{\alpha}{n}) = a_{n+1}\), then

\[
\frac{a_{n+1}}{a_{n}} = \frac{-\alpha}{n(n-1)} = \frac{\alpha(\alpha - 1) \ldots (\alpha - n + 1)}{(n-1)!} = 1 - \frac{\alpha + 1}{n}
\]

then by Weistrass' criterion, the statements a, b, c follow immediately.

At \( Z = -1 \), \( \Sigma (-1)^{n}(\frac{\alpha}{n}) \) needs special care:

\[
1 - \binom{\alpha}{1} + \binom{\alpha}{2} = 1 - \alpha + \frac{\alpha(\alpha - 1)}{2!} = (1 - \alpha)(1 - \frac{\alpha}{2})
\]

\[
1 - \binom{\alpha}{1} + \binom{\alpha}{2} - \binom{\alpha}{3} = (1 - \alpha)(1 - \frac{\alpha}{2})(1 - \frac{\alpha}{3})
\]

\[
\ldots
\]

\[
1 - \binom{\alpha}{1} + \binom{\alpha}{2} - \ldots + (-1)^{n}(\frac{\alpha}{n}) = (1 - \alpha) \ldots (1 - \frac{\alpha}{n}).
\]

In general

\[
\Sigma (-1)^{n}(\frac{\alpha}{n}) = \prod_{1}^{\infty}(1 - \frac{\alpha}{n}).
\]

1. If \( R(\alpha) = \beta > 0 \), let \( 0 < \beta' < \beta \). For all \( n \geq m \)

\[
|(1 - \frac{\alpha}{m})(1 - \frac{\alpha}{m+1}) \ldots (1 - \frac{\alpha}{n})| < (1 - \frac{\beta'}{m}) \ldots (1 - \frac{\beta'}{n}) \rightarrow 0.
\]

Then \( \Sigma (-1)^{n}(\frac{\alpha}{n}) \rightarrow 0 \), the series converges to 0.

2. \( R(\alpha) = -\beta < 0 \). Then

\[
|1 - \frac{\alpha}{n}| > 1 + \frac{\beta}{n} \quad \text{and} \quad |(1 - \frac{\alpha}{1}) \ldots (1 - \frac{\alpha}{n})| >
\]

\[
(1 + \frac{\beta}{1}) \ldots (1 + \frac{\beta}{n}). \quad \text{The left hand side} \rightarrow \infty, \text{so the}
\]

series diverges in this case.
3. \( R(\alpha) = 0, \alpha = -i\gamma, \gamma \neq 0 \). The \( n \)th partial sum of our series is \((1 + i\gamma) \cdots (1 + \frac{i\gamma}{n})\). This tends to no limit as \( n \to \infty \).

\((1 + \frac{i\gamma}{1}) \cdots (1 + \frac{i\gamma}{n}) \sim e^{i\gamma(1+1/2+\cdots+1/n)}\), the right hand side tends to no limit. On the contrary, the points which it represents for successive values of \( n \) circulate incessantly around the circumference of the unit circle in a constant sense. The interval between two successive points becomes smaller and smaller at each term.

Hence, \( \Sigma (-1)^{n} \left( \frac{1}{n} \right) \) also diverges at \( R(\alpha) = 0 \).

SPECIAL EXAMPLES

1. **Lusin Example**

There is a power series \( f(z) = \sum_{n=0}^{\infty} a_{n} z^{n} \) with radius of convergence one and with \( a_{n} \to 0 \), that is divergent on the whole circumference of the unit circle.

**PROOF:** For every \( m = 1, 2, \ldots, \) let:

\[
g_{m}(Z) = 1 + Z + \cdots + Z^{m-1} = \frac{1 - Z^{m}}{1 - Z}.
\]

Then for \( Z_{0} = e^{\phi i} \neq 1 \),

\[
g_{m}(Z_{0}) = \frac{1 - e^{m\phi i}}{1 - e^{\phi i}} = \frac{1 - e^{m\phi i}}{1 - e^{\phi i}} \times \frac{e^{-(\pi/2)\phi i}}{e^{-(\pi/2)\phi i}} =
\]

\[\text{Landau, E., page 62.}\]
\[
e^{-\frac{(m/2)\varphi i}{\sin (\varphi/2)}} = \frac{e^{-(m/2)\varphi i} - e^{-(\varphi/2)i}}{e^{-(\varphi/2)i} - e^{(\varphi/2)i}} = \sin \frac{m\varphi}{2} \sin \frac{\varphi}{2}.
\]

Now for \( \varphi \leq \frac{\pi}{2} \),

\[
\frac{2}{m} \varphi \leq \varphi \leq \varphi \leq \frac{\pi}{2}, \quad \text{so for } \frac{|m\varphi|}{\varphi} \leq \frac{\pi}{2}, \quad |g_m(z_0)| \geq 0.
\]

\[
\frac{2}{m} \varphi \leq \frac{|m\varphi|}{\varphi} = \frac{2m}{\varphi}. \quad \text{For } \varphi = 0 \quad \text{we have } g_m(1) = m > \frac{2}{m} \varphi, \quad \text{so}
\]

(1) \( |g_m(e^{i\varphi})| \geq \frac{2m}{\varphi} \) for \( \varphi \leq \frac{\pi}{m} \).

Now given any \( \varphi \in [0, 2\pi) \) there is an integer \( k_1 \), \( 0 \leq k_1 < m \) such that \( \varphi - \frac{2\pi k_1}{m} \in [0, 2\pi) \) and (1) would apply to

\( e^{-(2\pi k/m) i \varphi} \). Hence:

(2) \( \text{Max } |g_m(e^{-(2\pi k/m) i \varphi})| \geq \frac{2m}{\varphi} \) for any \( Z_0 = e^{\varphi i} \), \( 0 \leq k < m \). Now let:

\( h_m(z) = g_m(z) + z^m g_m(e^{-(2\pi i/m) z}) + \ldots + z^{m(m-1)} g_m(e^{-(2\pi (m-1)/m) i z}) \)

Then \( z^{km} g_m(e^{-(2\pi ki/m) z}) \) contributes terms of degree between \( km \)
and \((k + 1)(m - 1)\). Thus, no term of the form \( C_n z^n \) appears more
than once, and each \( C_n \) that does appear is of the form \( e^{ia} \)
and hence, it is of length one.

Finally, let \( \Sigma \frac{1}{\sqrt{m}} z^{1^2 + 2^2 + \ldots + (m-1)^2} h_m(z) \)

The last term of \( z^{1^2 + \ldots + (m-1)^2} h_m(z) \) is of degree \( d_m = 1^2 + \)
+ \( \ldots + (m - 1)^2 + m(m - 1) + m - 1 \) and the first term of
\[ \lim_{m \to \infty} Z_m^2 (z) \]

is of degree \( l^2 + \ldots + (m - l)^2 + m^2 = d_m + 1 \).

Thus, again, no term of the form \( C_n Z^n \) appears more than once,
and each \( C_n \) that does appear is of length \( \frac{1}{\sqrt{k_m}} \) where \( k_m \to \infty \)
as \( n \to \infty \). Hence, \( a_n \to 0 \) as desired, and the power series must
have radius of convergence at least one.

Now if the series converged for some \( Z_0 = e^{\varphi i} \) we would have
to have \( \lim_{n \to \infty} |a_n Z^n| = 0 \) and in particular
\[
\lim_{m \to \infty} \frac{1}{\sqrt{m}} \left| Z_0 \right|^{l^2 + \ldots + (m-1)^2} \max_{0 \leq k \leq m} g_m(e^{-2\pi ki/m} Z_0) = 0.
\]

But according to (2), the terms of this last sequence are
bounded below by \( \frac{1}{\sqrt{m}} \frac{2m}{\pi} = \frac{2\sqrt{m}}{\pi} \), which \( \to \infty \) as \( m \to \infty \). Thus the
radius of convergence is one and the series diverges everywhere
on the unit circle.

II. Sierpinski Example

There is a power series \( g(z) = \sum b_n z^n \) which diverges at
every point on the unit circle except at \( z = +1 \) where it
converges.

**Proof:** By using Lusia example; let
\[
g(z) = a_0 - a_0 z + a_1 z^2 - a_1 z^3 + a_2 z^4 - a_2 z^5 + \ldots \quad \text{This series}
\]
converges at \( z = +1 \), because \( a_n \to 0 \). For \( Z_0 = e^{\varphi i} \neq 1 \) the
series diverges for if the series converged, so likewise would
the series \( a_0 + a_1z^2 + a_2z^4 + \ldots \) (for \( g(z) = a_0(1 - z) + \\
+ a_1z^2(1 - z) + \ldots \), which is divergent by Lusin's example.

III. Let \( g(z) = a_0 - a_0z^2 + a_1z^4 - a_1z^6 + a_2z^8 - a_2z^{10} + \ldots \)
at \( z = 1 \) the series converges.
At \( z = -1 \), \( g(-1) = a_0 - a_0 + a_1 - a_1 + \ldots \) which converges by
Sierpenski example. Then \( g(z) = \sum_{k=0}^{\infty} a_k(z^{4k} - z^{4k+2}) \) converges
only at two points.

IV. The series \( g(z) = a_0 - a_0z^4 + a_1z^8 - a_1z^{12} + \ldots \)
converges at \( z = 1 \) and \( z = -1 \) (as in Example III). At \( z = i \),
\( g(i) = a_0 - a_0 + a_1 - a_1 + a_2 - a_2 + \ldots \) which converges as in
Example III.
At \( z = -i \), \( g(-i) = a_0 - a_0 + a_1 - a_1 + a_2 - a_2 + \ldots \)
which converges as before.

So the series \( g(z) = \sum_{k=0}^{\infty} a_k(z^{4k} - z^{4k+4}) \) converges at \( z = 1, -1, \\
i, -i \) and diverges everywhere on the unit circle.

V. Let \( g(z) = a_0 - a_0z^3 + a_1z^6 - a_1z^9 + \ldots \) This series
converges at \( z_1 = 1 \) and it is also convergent at \( z_2 = \\
= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \) and \( z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \) for \( z_{1n} = z_{2n} = z_{3n}^3 \)
\( n = 1, 2, \ldots \) and diverges everywhere.

In general, if we take Sierpinski's example and put for every \( z = \zeta^n \) then there are \( n \)-distinct values for \( \zeta \) such that
\[
\zeta_i^n (i = 1, 2, \ldots, n) = z.
\]
Now the series \( g(z) = a_0 - a_0 z + a_1 z^2 - a_1 z^3 + \ldots \) converges at \( z = 1 \) only. Then,
\[
g(\zeta) = a_0 - a_0 \zeta^n + a_1 \zeta^{2n} - a_1 \zeta^{3n} + \ldots \quad \text{converges at} \quad n \text{values of} \quad \zeta \quad \text{on the unit circle, but diverges everywhere else.}
\]

Moreover, these \( n \)-values of \( \zeta \) at which the series converge, are distributed on the unit circle in an even way; i.e., the distances between each two successor are equal.

VI. A power series whose radius of convergence is 1 and which diverges at \( p \) pre-assigned points of the unit circle.

If \( |Z_i| = 1 \), then the power series
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{Z}{Z_i} \right)^n
\]
it converges at \( Z \neq Z_i \), but it diverges at \( Z = Z_i \); for we will have in this case the well-known harmonic series \( \sum \frac{1}{n} \). Let the \( p \) pre-assigned
points be \( z_1, z_2, \ldots, z_p \). The series
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{Z}{z_1} \right)^n + \frac{1}{n} \left( \frac{Z}{z_2} \right)^n + \ldots + \frac{1}{n} \left( \frac{Z}{z_p} \right)^n
\]
possesses the required property. For if \( Z = Z_i \)
i = 1, 2, \ldots, p, then:
\[ * = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{z_i}{z_1} \right)^n + \ldots + \frac{z_i}{z_i} + \ldots + \frac{z_i}{z_p} \] = \\
= \sum_{l=1}^{\infty} \frac{1}{n} \left( \frac{z_i}{z_1} \right)^n + \ldots + \frac{z_i}{z_{i-1}} + \frac{z_i}{z_{i+1}} + \ldots + \frac{z_i}{z_p} \] + \sum_{l=1}^{\infty} \frac{1}{n}

But \( \sum \frac{1}{n} \) is divergent.

Then \( * \) is divergent. On the other hand if \( z \neq z_i, 1 \leq i \leq p \), then each \( \sum \frac{1}{n} \left( \frac{z_i}{z_1} \right)^n \) converges, and consequently, so does the series.
BIBLIOGRAPHY


