TWO PROBLEMS CONCERNING THE INTERNAL STRUCTURE
OF THE STARS

DISSERTATION

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HERMAN MOE ROTH B. S. E., M. S.

The Ohio State University
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Approved by:

L K. Thomas
Adviser.
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APPRECIATION

I wish to express my appreciation to my adviser, Professor J. H. Thomas, for the suggestion of the problems and also for considerable assistance throughout all phases of the work.
III
Notation.

A  Mean atomic weight
a  Stefan's constant = $7.64 \times 10^{-15}$
c  velocity of light = $3 \times 10^{10}$ cm. per sec.
C_v  specific heat at constant volume
D  $L/4$
f'/f  logarithmic rate of contraction
G  constant of gravitation = $6.66 \times 10^{-8}$
H  energy flow in ergs per cm.$^2$ per sec. through a shell at a distance $r$ from center
k  opacity of the material to radiation
L  luminosity in ergs per sec.
m  mass contained within a radius $r$
M  total mass of star = $8.3 \times 10^{33}$ gm. in Part I
p  gas pressure in dynes/ cm.$^2$ at a distance $r$ from center
$p_r$  radiation pressure in dynes/ cm.$^2$ at a distance $r$ from center
P  total pressure in dynes per cm.$^2$ at a distance $r$ from center
r  distance from center in cm.
R  universal gas constant = $8.26 \times 10^7$
$R_0$  radius of the star = $9.55 \times 10^{11}$ cm. in Part I
T  absolute temperature at a distance $r$ from center
Z  mean atomic weight
$p/p_r$  mean molecular weight
density of the matter at a distance $r$ from the center
TWO PROBLEMS CONCERNING THE INTERNAL STRUCTURE OF THE STARS

Introduction -

Astronomical observation gives us little direct information about the interiors of the stars. No experiments can be performed to tell us the characteristics of the material which we are considering. The relevant data that are obtainable consist of the radii, the total masses, and the absolute luminosities or the total rates of outflow of energy, of a few stars selected because of their proximity to the Sun, their large size, or their membership of some special group. Russell has plotted the absolute magnitudes of these stars for which they are known against their effective temperatures and has found that they fall into three groups - the 'main sequence', the 'giants', and the 'white dwarfs'. Stars of the main sequence show an approximate empirical relation - the luminosity varies as the cube of the mass, while the giants have higher luminosities and the dwarfs lesser luminosities. Any theory of the internal structure of the stars must account for the simultaneous existence of these three types and the energy they emit by assuming a state of the matter inside which agrees with known physical laws.

The physical conditions in stellar interiors must be very different from those at the earth's surface. The
enormous outflow of energy can be explained only by assuming temperatures of the order of a million degrees. The transfer of this energy must be accomplished by a radiative process, and not by conduction or convection; for conduction is too slow and large convection currents would lead to instability. Hence the opacity of the matter to radiation plays an important role.

The stars are assumed to be spheres in which the gravitational attraction towards the center is just balanced by the pressure exerted by the material. The pressure is usually assumed to be that exerted by a perfect gas together with that due to radiation; although Jeans has attempted to show by his liquid star model that a small variation from the perfect gas laws can be very important. Various sources of energy have been assumed: Contraction, as in this work; radioactive disintegration of elements of higher atomic weight than those known on the earth; the building up of heavy elements from hydrogen; and the annihilation of matter. In the point source model no assumption is made concerning the nature of the source of energy, but its position is limited to being in the neighborhood of the center of the star. No single source has been able to account convincingly for all the energy emitted by a single star during its supposed life time.

Kramers' formula for the opacity has been found valid from X-Ray measurements at the earth's surface but
values of the mean atomic weight, the mean atomic number, and the mean molecular weight are needed to make it definite. In this paper, Kramers' formula with a mean atomic weight (A) of one hundred, a mean atomic number (Z) of fifty and a mean molecular weight (\(\mu\)) of two is assumed. These values are approximately those suggested by Eddington, but lower than those suggested by Jeans. It has been pointed out recently (since this work has been done) that if \(Z^2/A\mu\) were increased tenfold, the difficulties pointed out in Part I might be removed. Eddington has assumed that the product of the rate of generation of energy and the opacity is constant throughout the star. There was no physical reason for this assumption; it merely made the mathematics of Eddington's point source model simpler.

In Part I of this paper, the star is assumed to be a sphere of perfect gas with a point source of energy at the center. The magnitude of the radiation pressure was investigated and found to be small, and therefore this was neglected. In Part II, we assume a fluid sphere slowly contracting or expanding without internal generation of energy. Here the radiation pressure was comparable with the gas pressure and therefore was included in the calculations.

Our equations are valid for all constant values of the specific heat at constant volume of the material,
unless $C_v \approx R$ when the contraction or expansion becomes very rapid. Our equations lead to such a homologous solution that we can vary the dependent variables—the pressure, the temperature, the luminosity, and the radius,—in certain proportions without changing the independent variable, the mass of the star. The model was found to be dynamically stable for all constant values of the specific heat at constant volume except $C_v \approx R$, when our equations are no longer valid.

On this basis, we proceed to consider the problem of a point source model.
Part I - Point Source Model.

Previous writers\textsuperscript{12} on point source models of gas spheres have usually attempted to obtain solutions of the equilibrium equations by choosing the values of the pressure and temperature at the center in such a manner that after integrating outward, the boundary conditions at the surface, viz., \( p = 0, T = 0 \), were satisfied. Eddington\textsuperscript{12} has succeeded in obtaining a solution of such a model which leads to 'astronomical' values of the opacity. In this work the equilibrium equations were integrated inward. Kramers' formula\textsuperscript{7} for the opacity was used and for the boundary conditions at the surface, the mass, radius, and luminosity of Capella as given by Eddington\textsuperscript{13} were adopted.

For the first step in the integration of our equations, the values of \( \rho, p, T, \) and \( r \), at a depth corresponding to one tenth the mass were determined by solving the equation\textsuperscript{14}

\[
0.1 M = \int_{r_0}^{r} 4\pi r^2 \rho \, dr
\]

for \( r \). This was accomplished by expressing \( r \) and \( \rho \) as functions of \( T \), the new limits being zero and \( T \), and solving for \( T, \rho, r \) and \( p \), are then determined since they are functions of \( T \). The remainder of the solution was obtained by numerical integration.
A. First Layer.

The equations of equilibrium for a gas sphere with a point source of energy at the center, are:

\[
\frac{dP}{dr} = -\frac{\rho G m}{r^2}
\]

\[
H = \frac{D}{\lambda^2} = -\frac{c}{3\kappa c} \frac{d}{dr} (aT^4)
\]  

(1.01)

(1.02)

where

\[
\frac{dm}{dr} = 4\pi \lambda^2 \rho
\]

(1.03)

\[
P = p + p_r = \frac{e^{RT}}{\mu} + \frac{1}{3} aT^4
\]

(1.04)

\[
\kappa = 4.46 \times 10^{-3} \frac{25}{\lambda \mu} \rho T^{-3.5}
\]

(1.05)

(1.05) is Kramers' formula for the opacity as given by Jeans. In order to get the values of \( p, T, \) and \( r, \) at a depth corresponding to one tenth of the mass, we proceed in a manner similar to Jeans. 15

Let \( \lambda = \frac{p}{p_r}, \)

then

\[
P = p_r (1 + \lambda) = \frac{1}{3} \lambda aT^4 (1 + \lambda)
\]

and

\[
\frac{\lambda aT^4}{\rho} = \frac{e^{RT}}{\mu}
\]

or

\[
\rho = \frac{\lambda \mu aT^3}{3\kappa}
\]

(1.06)

Hence

\[
\frac{dp}{dT} = \frac{dp}{d\rho_r} \frac{d\rho_r}{dT}
\]

\[
\frac{dp}{dr} = (p_r \frac{d\lambda}{d\rho_r} + 1 + \lambda)(\frac{2}{3} aT^3) \frac{d\tau}{dr} = -\frac{\rho G m}{\lambda^2}
\]

(1.07)
(1.02) may be written
\[
\frac{Dx^F}{cY^2} = -\frac{4}{3} \alpha T^3 \frac{dT}{dY}
\]  

(1.08)

Dividing (1.07) by (1.08), we obtain
\[
\frac{cGm}{Dx} = \left[ p_y \frac{d\lambda}{dp_y} + 1 + \lambda \right]
\]  

(1.09)

Substituting for \(k\) and \(p\) from (1.05) and (1.06) respectively
\[
\frac{3RcGm}{\mu a N D} \frac{1}{T_0^2} \lambda \left[ p_y \frac{d\lambda}{dp_y} + 1 + \lambda \right]
\]  

(1.10)

where
\[
N = 4.46 \times 10^{25} \frac{A_2}{\Lambda \mu}
\]

Let
\[
\chi = \sqrt{\frac{3RcGm}{\mu a N D}} T_0^2
\]  

(1.11)

and since
\[
p_y = \frac{1}{3} \alpha T^2
\]

\[
\chi = \sqrt{\frac{3RcGm}{\mu a N D}} \left( \frac{3p_y}{a} \right)^{\frac{1}{4}}
\]  

(1.12)

and assuming \(m\) constant in the first layer
\[
\frac{d\chi}{d\rho_y} = \frac{1}{4} \frac{\rho_y}{\chi} \rho_y
\]  

(1.13)

Writing (1.10) as
\[
\chi^2 = \lambda \left[ p_y \frac{d\lambda}{dx} + 1 + \lambda \right]
\]  

(1.14)

and substituting for \(\frac{d\chi}{d\rho_y}\), we obtain
\[
\lambda(\lambda + 1) = \chi^2 = \frac{1}{16} \lambda \frac{d\lambda}{d\chi}
\]

An approximate solution is
\[
\lambda(\lambda + 1) = \chi^2
\]

which for \(\chi\) large gives
\[
\lambda = \chi
\]
The assumption of $\lambda$ large was investigated and found valid \textit{a posteriori} by computing its value at various points in the solution.

Upon inserting the value of the physical constants in (1.11) we obtain

$$\lambda = a = 0.2415 \sqrt{\frac{m}{d}} T^2$$

(1.15)

With $\lambda$ and the physical constants (1.06) gives

$$\rho = 0.1489 \times 10^{-22} \sqrt{\frac{m}{d}} T^{3.25}$$

(1.16)

Using $k$ as given by (1.05), with $g^2/4\mu$ equal to 12.5 and $\rho$ from (1.16), we obtain from (1.01)

$$-\frac{d\gamma}{\gamma^2} = \frac{2.473 \times 10^{15}}{m} dT$$

(1.17)

Then if $m$ is assumed constant in the first layer,

$$\frac{1}{\gamma} = \frac{2.473 \times 10^{15}}{m} T + u$$

(1.18)

where

$M =$ total mass of the star

$u = 1/R_0$, $R_0$ being the radius of the star and

$T$ the temperature at a distance $r$ from the center.

Substituting the values of $\rho$ and $r$ as given by (1.16) and (1.18) into

$$0.1M = \int R_0^r \rho d\gamma$$

(1.19)
we obtain an integral in $T$ alone.

\[ 2.161 \times 10^{5} \, \nu^{4} \, \nu^{1/2} \, M^{2} = \int_{0}^{T_{0}} \frac{T^{3.25}}{\left[ \frac{2.473 \times 10^{15}}{\nu^{3}} + T + 1 \right]^{4}} \, dT \]

(1.20)

The left hand member becomes $0.3839 \times 10^{26}$ if Eddington's values for the mass, luminosity and radius of Capella, viz;

\[ L = 4 \pi D = 4.8 \times 10^{35} \text{ ergs per second} \]
\[ M = 8.3 \times 10^{33} \text{ grams} \]
\[ R_{0} = 9.55 \times 10^{11} \text{ cm.} \]

are used. Upon integration of the right hand member the value of $T_{0}$ that satisfies (1.20) is found to be $3.048 \times 10^{6}$ degrees. This leads to

\[ \varphi = 0.00822 \text{ gms. per c.c.} \]
\[ p = 1.0335 \times 10^{12} \text{ dynes per cm.}^{2} \]
\[ \frac{r}{R_{0}} = 0.508 \]

for a depth corresponding to $1/10$ of the mass.
B. Numerical Integration

The values of $p$, $T$, and $r$ at a depth corresponding to one tenth of the mass determined in Part A are used to compute the differential coefficients for the first step in the numerical integration. The assumption and subsequent proof that $\lambda$ is large allows us to neglect the radiation pressure. Since $m$ varies more slowly than $r$, we change the independent variable from $r$ to $m$ in (1.01) and (1.02) and using $\rho = \frac{p\mu}{RT}$, we find after putting in the values of the physical constants:

$$\frac{dp}{dm} = -5.30 \times 10^9 \frac{m}{r^4} \quad (2.01)$$

$$\frac{dT}{dm} = -2.793 \times 10^{20} \frac{Lp}{\lambda^4 T^{1.5}} \quad (2.02)$$

$$\frac{dy}{dm} = 3.287 \times 10^6 \frac{T}{r^2 \rho} \quad (2.03)$$

Adopting convenient units, viz:

$p$, $10^{12}$ dynes/cm$^2$; $T$, $10^6$ degrees absolute;
m, $10^{33}$ grams; $r$, $10^{11}$ cm.; and Eddington's value for the luminosity of Capella$^{13}$, we find

$$\frac{dp}{dm} = -53.00 \frac{m}{r^4} \quad (2.04)$$

$$\frac{dT}{dm} = -13.408 \times 10^5 \frac{p}{r^4 T^{7.5}} \quad (2.05)$$
\[
\frac{d\gamma}{dm} = 3.287 \frac{T}{\gamma^2 \rho} \tag{2.06}
\]

The procedure for the numerical integration is as follows: For any step, we know the initial values of \(m_0, p_0, T_0,\) and \(r_0.\) We choose an arbitrarily small mass interval \(\Delta m.\) Then

\[
\bar{m}_i = m_0 + \frac{1}{2} \Delta m
\]

and

\[
m_i = m_0 + \Delta m
\]

The values of \(p, T,\) and \(r,\) at the halfway and end points of an interval are given by:

\[
\bar{p}_i = p_0 + \left( \frac{dp}{dm} \right)_0 \frac{\Delta m}{2} \quad \text{and} \quad \bar{p}_i = p_0 + \left( \frac{dp}{dm} \right)_0 \Delta m
\]

\[
\bar{T}_i = T_0 + \left( \frac{dT}{dm} \right)_0 \frac{\Delta m}{2} \quad \text{and} \quad \bar{T}_i = T_0 + \left( \frac{dT}{dm} \right)_0 \Delta m
\]

\[
\bar{r}_i = r_0 + \left( \frac{dr}{dm} \right)_0 \frac{\Delta m}{2} \quad \text{and} \quad \bar{r}_i = r_0 + \left( \frac{dr}{dm} \right)_0 \Delta m
\]

where

\[
\left( \frac{dp}{dm} \right)_0 = -330.00 \frac{\bar{m}_i}{\gamma^2 \rho_0}
\]

\[
\left( \frac{dT}{dm} \right)_0 = -13.408 \times 10^4 \frac{\rho_0}{\gamma^2 \rho_0} \frac{r_0}{T_0} 7.5
\]

\[
\left( \frac{dr}{dm} \right)_0 = 3.287 \left( \frac{T_0}{\gamma^2 \rho_0} \right)
\]

As a check, we recompute \(p_0, T_0,\) and \(r_0,\) using differential coefficients determined by \(p_1, T_1,\) and \(r_1.\) The mass interval was chosen so that this check would be within one percent. The size of the intervals varied from 0.05 (for the first step) to 0.5 (for the last step) of
the mass remaining within the radius \( r \). Below, is the first step as a sample calculation.

\[
\begin{align*}
    m_o &= 0.7 M \\
    m_i &= 0.85 M \\
    \Delta m &= 0.15 M = 0.4150 \\
    \overline{m} &= 7.2625 \\
    \Delta \overline{m} &= 0.2075 \\
    \varphi_o &= 1.0335 \\
    T_o &= 3.0481 \\
    \tau_o &= 4.8514
\end{align*}
\]

\[
\begin{align*}
    \left( \frac{dp}{dm_o} \right)_o &= -53.00 \left( \frac{7.2625}{4.8514^4} \right) = -0.69486 \\
    \left( \frac{dT}{dm_o} \right)_o &= -13.408 \times 10^{-5} \left( \frac{1.0335}{(4.8514^4)(3.0481)^3} \right)^2 = -0.58620 \\
    \left( \frac{d\tau}{dm_o} \right)_o &= 3.287 \left( \frac{3.0481}{(4.8514^2)(1.0335)} \right) = 0.37078
\end{align*}
\]

Hence:

\[
\begin{align*}
    \overline{p}_1 &= 1.0335 + 0.1442 = 1.1777 \\
    p_1 &= 1.1777 + 0.1442 = 1.3219 \\
    \overline{T}_1 &= 3.0481 + 0.1216 = 3.1697 \\
    T_1 &= 3.1697 + 0.1216 = 3.2913 \\
    \overline{\tau}_1 &= 4.8514 - 0.07695 = 4.7745 \\
    \tau_1 &= 4.7745 - 0.97695 = 4.6976
\end{align*}
\]

At each step the density \( \rho \) at the point and the mean density \( \overline{\rho} \) of the remaining matter were computed by the following:

\[
\begin{align*}
    \rho &= \frac{M \rho}{RT} \\
    \overline{\rho} &= \frac{m}{\frac{4}{3} \pi \gamma^3}
\end{align*}
\]

The results of the calculations are shown in Table I. It should be noted that the mass becomes zero before the radius.
Thus, on the basis of a point source model and Kramers' formula for the opacity, Capella would have a hollow center. The end of the last mass interval occurred at \( r = 2.60 \times 10^{11} \text{ cm.} \) or approximately 0.26 \( R_\odot \).

We interpret this result to mean that the source of Capella's energy cannot be located within a distance 0.26 \( R_\odot \) of its center.

In the following part, an extended source of energy is taken. For definiteness we suppose it to be the energy liberated by the contraction of the matter.
Part II - A Slowly Contracting or Expanding Fluid Sphere.

Previous investigations\textsuperscript{16} of the stability of stellar models have usually been made by considering a varied motion constrained to have one or two degrees of freedom, e.g. a uniform dilatation. If the constrained motion is unstable, the model is unstable; while if the constrained motion is stable, the model may or may not be stable. Thomas\textsuperscript{17} has pointed out that if a solution of the equilibrium equations has been obtained, the complete conditions for dynamical stability can be applied readily.

With this in view, a solution was obtained of the equilibrium equations for a fluid sphere, which contracts or expands homologously without internal generation of energy, and for which the opacity is given by Kramers' formula.\textsuperscript{7} The conditions for dynamical stability were then applied and this model was found to be dynamically stable.

A. Solution of Equations.

A fluid sphere in radiative equilibrium contracting or expanding homologously without internal generation of energy has, besides the equations of equilibrium as given by (1.01), (1.02), (1.03), (1.04), and (1.05), the additional relation\textsuperscript{17}

\[(3R - C_v) \rho T \frac{f'}{f} = \frac{1}{4\pi r^2} \frac{dL}{dr}\]

(3.01)

\textit{Here it is found necessary to consider the radiation pres-}
sure. Adopting the same units as in Part I with the additional one for $L, 10^{35}$ ergs/sec, we find

$$\frac{dT}{dm} = -2.793 \times 10^5 \frac{L \rho}{r^3 T^{7.5}}$$  \hspace{1cm} (3.02)$$

$$\frac{dp}{dm} = -53.00 \frac{m}{r^3} - 1.019 \times 10^{-2} T^3 \frac{dT}{dm}$$  \hspace{1cm} (3.03)$$

$$\frac{dY}{dm} = 3.287 \frac{T}{\rho^3}$$  \hspace{1cm} (3.04)$$

$$\frac{dL}{dm} = \alpha T$$  \hspace{1cm} (3.05)$$

where $P$ has been replaced by $P + 1/3 \alpha T^4$ and

$$C_V = \frac{3}{2} R$$

$$\alpha = 0.6195 \times 10^{12} \frac{f^l}{f}$$

It is to be noted that a change in $C_V$ will alter $\alpha$ but will not change its order of magnitude, unless $C_V \approx R$ when the contraction or expansion becomes very rapid and our equations break down.

With central temperature and total pressure equal to $5 \times 10^6$ degrees absolute and $8.5917 \times 10^{12}$ dynes per cm.$^2$ (gas pressure $7 \times 10^{12}$), we computed two solutions corresponding to $\alpha \approx 0.17$ and 0.175, ($f^l/f \approx 2.74 \times 10^{-13}$ or a past life of $10^5$ years). In the first of these solutions, the pressure approaches zero and the temperature remains finite, and in the second the temper-
ature approaches zero and the pressure remains finite as $m$ increases. By interpolating between these two solutions\textsuperscript{18}, a solution for which both temperature and pressure become zero within one mass interval is found. Thus the boundary conditions at the surface, viz: \( P = 0 \), and \( T = 0 \), are fulfilled. These solutions are shown in Table II. The work was done by numerical integration.

A solution for which \( p \) tends to zero while \( T \) remains finite has no physical meaning because it approaches the isothermal solution at large distances from the center, and thus has infinite mass. The other solution for which \( T \) tends to zero and \( p \) remains finite, has a finite mass. As the surface is approached, the above equations break down as soon as the outward flow of radiation is comparable to the total radiation present. We believe that it is possible to fit an atmosphere in adiabatic equilibrium to such an interior, provided that the pressure remaining is not too large.

All the solutions for stars of this mass contracting homologously can be obtained by altering the internal pressure, temperature, and radius, for a given included mass, in proportions given by Lense's law, viz:

\[ T \propto r^{-1} \quad p \propto r^{-4} \]

Further:\[ L \propto r^{-\frac{1}{2}} \quad \rho \propto r^{-\frac{1}{2}} \quad T_e \propto r^{-\frac{5}{8}} \]

where \( l \) is the past life, and \( T_e \) is the effective temperature.

The mass of this star is about that of \( \delta \) Aurigae A or Sirius A.\(^{19}\) The luminosity per gram corresponding to the radius of Sirius would be 1.9 times that of the Sun. As in Eddington's model, this single source is too small. Having obtained a particular solution, we wish to test the dynamical stability.

\section*{B. Dynamical Stability.}

Besides the condition of dynamical equilibrium, there are three conditions of stability - the Euler, the Legendre, and the Jacobi. The first two of these relate to the matter at each point in the solution, while the last relates to the solution as a whole. If the matter at each point is stable by itself, the first two hold, the Jacobi condition demands that for any adiabatic displacement of the solution there shall be no point \( m \) in the solution for which the value of \( r \) remains unchanged. The last condition is applied to the solution below.

The entropy of a perfect gas with radiation is given by:

\[
S = \frac{4}{3} \alpha T^3 \sigma + \int \frac{C_v}{T} dT + R \log \sigma
\]

\hspace{1cm} (4.01)

The total pressure is

\[
P = \frac{1}{3} \alpha T^4 + \frac{RT}{\sigma}
\]

\hspace{1cm} (4.02)
where \( \sigma = \frac{1}{\rho} \) is volume per unit mass.

By taking \( \int s = 0 \) in (4.01),

\[
\int_T = -\frac{\left(\frac{4}{3}aT^3 + \frac{R}{\sigma} \right) \int \sigma}{\left(4aT^2\sigma + \frac{C_v}{T} \right)}
\]  

(4.03)

From (4.02),

\[
\int P = \left(\frac{4}{3}aT^3 + \frac{R}{\sigma} \right) \int T - \frac{RT}{\sigma^2} \int \sigma \]

(4.04)

Eliminating \( \int T \) between (4.03) and (4.04) we have,

\[
\int \sigma = -\frac{\int P}{\frac{RT}{\sigma^2} + \left(\frac{4}{3}aT^3 + \frac{R}{\sigma} \right) \left(4aT^2\sigma + \frac{C_v}{T} \right)}
\]

(4.05)

If we change the independent variable in (1.01)
from \( r \) to \( m \) and substitute \( \sigma \) for \( 1/\rho \) in (1.03),

\[
\frac{dP}{dm} = -\frac{Gm}{4\pi \rho^2}
\]

(4.06)

\[
\frac{dY}{dm} = \frac{G}{4\pi \rho^2}
\]

(4.07)

Taking the variation of (4.06) and (4.07), with \( m \) constant, we obtain

\[
\frac{d}{dm}(\delta P) = \frac{Gm \delta Y}{\pi \rho^2} - \frac{2 \sigma \delta Y}{4\pi \rho^2}
\]

\[
\frac{d}{dm}(\delta Y) = \frac{\delta \sigma}{4\pi \rho^2} - \frac{\delta Y}{4\pi \rho^2}
\]
Making the substitution \( v = 4/3 \pi r^3 \) and \( \delta v = 4 \pi r^2 \delta r \),

\[
\frac{d}{dm} (\delta P) = \frac{G m \delta v}{4 \pi^2 r^2 \gamma^2}
\]  
(4.08)

\[
\frac{d}{dm} (\delta v) = -\frac{\sigma \delta P}{P \gamma}
\]  
(4.09)

where

\[
\gamma = \frac{\sigma}{P} \left[ \frac{RT}{\sigma^2} + \frac{\left( \frac{\mu_0}{3} \pi^{-3} + \frac{R}{\sigma}\right)^2}{\left(4 \pi^{-3} \sigma^2 + \frac{c_v}{T}\right)} \right]
\]  
(4.10)

This is the same \( \gamma \) as that given by Eddington.20

Starting with \( \delta P = 1 \) and \( \delta v = 0 \) for \( m = 0 \),

\( \delta v \) was computed at each point in the solution of the equilibrium equations. The Jacobi condition demands that there be no other point for which \( \delta v \) is zero, \( \delta v = 4 \pi^{-3} \gamma \delta P \).

The diagrams show how \( \delta P \) and \( \delta v \) vary with \( m \) for stable (1) and unstable (2) solutions.

**Fig. 1**

**Fig. 2**

For \( c_v = 3R/2 \), \( \gamma = 5/3 \), the test was applied exactly. For the limiting cases \( c_v = 3R \), \( \gamma = 4/3 \), and \( c_v = \infty \), \( \gamma = 1 \); \( \gamma \) is constant throughout the star but
in the former case, our tests are only approximate on account of the rapid expansion or contraction.

We obtain curves such as (1) and conclude that the star is dynamically stable in all these cases.
TABLE I

Solution of Equations for Point Source Model

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<th>( P \times 10^{-12} )</th>
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Table II.
Solutions of Equations for Slowly Contracting Sphere.

\[ \alpha = 0.17 \]

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Interpolated Solution

\[ \alpha = 0.172 \]
Conclusions and Results

Part I. Assuming the star to be a gas sphere with a point source of energy at the center and Kramers' opacity formula, a density distribution within the star Capella was obtained by integrating the equilibrium equations inward rather than outward as had been done by previous writers. It was pointed out that with values assumed for the constants in Kramers' formula, this star would have a hollow center. This adds more weight to Jeans' suggestion of attributing higher mean atomic weights and numbers to the matter within the stars. J. C. P. Miller has recently pointed out that by increasing $Z^2/A\mu$ tenfold, this difficulty can be removed from point source models.

Part II. A homologous solution of the equilibrium equations for a slowly contracting or expanding fluid sphere without internal generation of energy was obtained. This model was shown to be dynamically stable for all values of the specific heat at constant volume of the material.
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20. Eddington, Reference 8, page 191
Autobiography

I, Herman Moe Roth, was born in Richmond, Virginia, December 5th, 1906. I received my elementary and secondary school education in the public and high schools of that city, graduating from John Marshall High School in 1925. I then matriculated at the University of Virginia from which I obtained the degrees of Bachelor of Science in Engineering (1927) and Master of Science in Physics (1928).

During my last year at the University of Virginia I served as a Teaching Fellow in Physics. In 1928 I received an appointment from The Ohio State University as an assistant in the Department of Physics, which position I have held while completing the requirements for the degree of Doctor of Philosophy.