WIDEBAND LUMPED CIRCUIT MODELS
FOR INTEGRATED SPIRAL INDUCTORS
AND CAPACITORS

A Thesis
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ABSTRACT

Integrated circuits have shown a dramatic increase in the frequency of operation in recent years, up to tens of GHz. At GHz level frequencies, integrated spiral inductors and interdigitated capacitors suffer severe parasitic effects, making the accurate modeling of these devices more difficult. To aid in their design, the most common tool available for accurate modeling is numerical EM solvers. However, their use is often very time consuming and computationally very expensive.

This thesis proposes a lumped pi-model representation of the spiral inductor and interdigitated capacitor that is accurate up to and including the first resonant frequency of the device. Based upon the Partial Element Equivalent Circuit (PEEC) distributed model, approximations are made to simplify the distributed model down to a more useful pi-lumped circuit model that enables the designer to visualize the parasitic impedances that each device experiences. Other lumped circuit models are outlined in the literature, but the component values are determined using empirically derived or approximated formulas. The method in this thesis is more rigorous and physically derived, using the partial capacitance matrix and partial inductance matrix, along with a resistance matrix that takes into account the skin effect and, in the case of the spiral inductor, the eddy current effect.
The thesis begins with a review of the PEEC technique. An improved resistance matrix is derived that works for a wide range of inductor geometries by more accurately accounting for the eddy current effect and skin effect. A resistance matrix that is dependent only upon the skin effect is derived for interdigitated capacitors. The approximations made to simplify the PEEC model to a lumped pi-model are reviewed for both the spiral inductor and interdigitated capacitor. In addition, a general technique for the de-embedding of RF passive devices is reviewed. There are many de-embedding techniques outlined in the literature for smaller active devices, but these techniques are often erroneously applied to larger passive devices. The parasitics that typically need to be de-embedded in an RF passive device test structure are discussed, and a few techniques for accurately dealing with these parasitics are introduced. The thesis concludes with a comparison of the 2-port S-parameters, inductor $Q$, and effective inductance arrived with our model to that arrived with ADS RF Momentum for a set of five spiral inductors. We also compare our model to de-embedded measured data for a 1 nH spiral inductor. In the case of the interdigitated capacitor, we compare the 2-port S-parameters, 2-port differential resistance, and effective capacitance calculated with our model to that calculated with ADS RF Momentum.
This thesis is dedicated to the memory of my dad.
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CHAPTER 1

INTRODUCTION

As the speed of transistors has risen in recent years, acquiring accurate models for the passive devices used in integrated circuits has become a greater challenge. Current device speeds are in the tens of GHz, and at these frequencies the parasitic effects of inductors and capacitors can become difficult to predict. At GHz level frequencies, monolithic inductors and interdigitated capacitors suffer from severe parasitic effects. The relatively large size of the inductor and capacitor (hundreds of microns), and the high dielectric constant of the substrate, cause the inductor to experience significant capacitive coupling. The magnetic coupling of the parallel fingers in the interdigitated capacitor causes a parasitic series inductance, resulting in a lowered resonant frequency. The skin effect, along with the fields coupling into a lossy substrate, causes the structures to dissipate a significant amount of power. In the case of the spiral inductor, the effect of eddy currents must be understood as they also result in power loss. These parasitic effects result in a limited peak $Q$. Since many RF circuits such as LNA’s and VCO’s require inductors and capacitors with a high $Q$, it is critical for the circuit designer to accurately model these parasitics.
At present, the best way to model the parasitic effects of a monolithic inductor is to apply data fitting techniques to a lumped circuit model, or to use a numerical EM solver. Data fitting techniques require the inductor to be fabricated multiple times which is both time consuming and expensive. EM solvers such as FEM, MoM, and FDTD can be both time consuming and numerically unstable when simulating electrically small structures (inductors are much smaller than \( \lambda / 20 \)). Although new schemes have been developed to handle electrically small structures, numerical techniques are still time consuming.

There are techniques in the literature, such as in [19], that use microstrip transmission line approximations to develop general expressions for the inductance and \( Q \) of spiral inductors. Other techniques, like that of [1], [6], [8], [15], and [16], involve the development of approximated empirical or semi-analytical formulas. Unfortunately, as mentioned by [11], these formulas can sometimes give results with an error on the order of 20%.

Capacitor models have been developed that use conformal mapping techniques such as in [18], but the use of conformal mapping is limited to simple geometries and a lumped model that accounts for losses is not produced. Also, models have been developed that use basic microstrip theory such as [19] and [20]. Again, these models will be limited in the geometries that they can solve and do not apply to other geometries such as quasi-fractals shown in Figure 7.16 which do not exhibit the properties of coupled microstrip lines. As with inductors, EM solvers such as FEM, MoM, and FDTD can be used but their use presents the same disadvantages as with inductors.
In this thesis, we begin in Chapter 2 with a summary of the Partial Element Equivalent Circuit (PEEC) distributed circuit model for a spiral inductor. We apply the same technique to develop a distributed model for an interdigitated capacitor. Chapter 3 will deal with the derivation of the frequency dependent resistance model that accounts for skin effect and, in the case of the spiral inductor, eddy currents. In Chapter 4, we review the approximations made to simplify the Partial Element Equivalent Circuit (PEEC) model into a simple lumped pi-model for the spiral inductor. Chapter 5 will deal with a similar derivation for the interdigitated capacitor. Chapter 6 will discuss a technique for the de-embedding of measured data for inductors and capacitors at GHz level frequencies. In Chapter 7, we will see that the lumped models can be evaluated fairly quickly and, as will be demonstrated, with great accuracy. We will summarize our results in Chapter 8.
CHAPTER 2

SUMMARY OF THE PEEC DISTRIBUTED CIRCUIT

The PEEC technique has been thoroughly developed by [4] as a tool by which Maxwell’s equations are used to construct a distributed circuit to model interconnects in digital and microwave circuits. As the construction of the PEEC distributed circuit for inductors is discussed in great detail in [1],[4]-[6], and [9], it is only briefly summarized here, with reference to [11]. A flowchart showing the progression from Maxwell’s equations (with a quasistatic approximation) to a distributed model is given in Figure 2.1 below.
Review of EM Equations

Using Faraday's law in point form for the time periodic case concurrently with the constitutive relations for linear isotropic materials, one can arrive at the following expression for the electric field.

\[ \vec{E} = -j\omega \vec{A} - \nabla \phi \]  

Taking the divergence of both sides of eq.(2.1) and applying the Lorenz gauge to the result will give
\[ \nabla^2 \phi + \omega^2 \mu \varepsilon \phi = -\frac{\rho}{\varepsilon} \]  

(2.2)

where \( \phi \) is the scalar electric potential.

In a similar manner beginning from Ampere’s law, we arrive at the magnetic equivalent to eq.(2.2)

\[ \nabla^2 \bar{A} + \omega^2 \mu \varepsilon \bar{A} = -\mu \bar{J}, \]

(2.3)

where \( \bar{A} \) is the vector magnetic potential. Since the devices we are studying are electrically very small, we can apply a quasi-static \( (\omega \approx 0) \) assumption to eq.(2.2) and eq.(2.3) which gives

\[ \nabla^2 \phi = -\frac{\rho}{\varepsilon} \]

(2.4)

\[ \nabla^2 \bar{A} = -\mu \bar{J} \]

(2.5)

Eq. (2.5) is based upon the assumption that the substrate has very low loss. If we want to take into account losses in the substrate, we can use \( \varepsilon_s = \varepsilon_0 \varepsilon_{rs} - j \frac{\sigma_s}{\omega} \) in eq.(2.3) to get (within the substrate)

\[ \nabla^2 \bar{A} + \omega^2 \mu_0 \left( \varepsilon_0 \varepsilon_{rs} - j \frac{\sigma_s}{\omega} \right) \bar{A} = \nabla^2 \bar{A} - j \omega \sigma_s \mu_0 \bar{A} = -\mu \bar{J}. \]

(2.6)

where the \( \omega^2 \) term is ignored because we are using a quasi-static approximation. The solution to eq. (2.5) and eq.(2.6) with the appropriate boundary conditions is expressed in terms of the magnetostatic \( (G_m) \) Green’s function.

\[ \bar{A} = \int G_m \bar{J} d\nu \]

(2.7)
The solution for $G_m$ for the lossless case (eq.(2.5) ) is simple because it does not involve the permittivity. Since the permeability of each layer of our substrate is the same as free space, $G_m$ is simply the free space Green’s function without the $e^{jBR}$ term, namely,

$$G_m(R) = \frac{\mu_0}{4\pi R},$$  \hspace{1cm} (2.8)

where $R$ is the distance between the observation and source points. The solution to eq.(2.6) (lossy case) is more involved and requires the application of appropriate boundary conditions to solve for a multi-layer $G_m$. That case will not be considered in this thesis. The solution to eq.(2.4), with the appropriate boundary conditions, can be expressed in terms of the electrostatic ($G_e$) Green’s function,

$$\phi = \int G_e \rho dv.$$  \hspace{1cm} (2.9)

As mentioned previously, the inductors we are studying are electrically very small, making the use of static Green’s functions appropriate. Following the method outlined in [5] and [10], $G_e$ for the multi-layer substrate shown in Figure 2.2 is solved directly.
As will be discussed, eq.(2.7) and eq.(2.9) form the foundation for the system of equations that will be used to construct the PEEC distributed circuit.

**Discretizing the Inductor and Capacitor**

In general, the current and charge on a structure can be discretized any arbitrary way. In other words, the charge and current can be expressed in terms of a set of basis functions that is most appropriate for the geometry. However, for simplicity, as done in [11]-[13], the monolithic inductors are discretized into a set of series rectangles as illustrated in Figure 2.3, while the discretization of the interdigitated capacitor is illustrated in Figure 2.5. The coordinate system we use is shown in Figure 2.4.
Figure 2.3 Discretization for 1.5 turn inductor

Figure 2.4 Coordinate system used: (a) and (b): top view of inductor and capacitor, (c) side view of substrate
The current and charge is then decomposed into a summation of pulse basis functions as done in [11]-[13]. The volume current density is given as

$$\bar{J}_i = \frac{I_i b(x, y, z)_i \hat{J}_i}{a_i},$$  \hspace{1cm} (2.10)

where $b(x, y, z)_i$ is a pulse function with a magnitude of 1 when the point $(x, y, z)$ is inside rectangle $i$ and is equal to zero otherwise. $\hat{J}_i$ is a unit vector that gives the direction of current flow and is always perpendicular to the cross-section into which it flows, whose area is denoted by $a_i$. $I_i$ is a constant and represents the magnitude of the current.

Similarly, the volume charge density in each segment is given as

$$\rho_i = \frac{Q_i b(x, y, z)_i}{v_i},$$  \hspace{1cm} (2.11)

where $b(x, y, z)_i$ is the same as in eq.(2.10), $v_i$ is the volume of rectangle $i$, and $Q_i$ is a constant that represents the total charge in the volume $v_i$. The total current and charge in the inductor is a summation of the individual basis functions. This gives the total current and charge density as

$$\bar{J} = \sum_{i=1}^{n} \frac{I_i b(x, y, z)_i \hat{J}_i}{a_i},$$

$$\rho = \sum_{i=1}^{n} \frac{Q_i b(x, y, z)_i}{v_i},$$
\[ \bar{J} = \sum_{i=1}^{N} \bar{J}_i \]  
\[ \rho = \sum_{i=1}^{N} \rho_i \]  

(2.12)  
(2.13)

**Formulation of the PEEC System of Equations**

It is now possible to construct the PEEC distributed circuit with the equations we have derived up to this point. Plugging eq.(2.7) into eq.(2.1) and putting all terms to the right side will give

\[ 0 = \bar{E} + j \omega \int \bar{G}_m \bar{J} dv + \nabla \phi \]  
\[ (2.14) \]

We can replace \( \bar{J} \) in eq.(2.14) with eq.(2.10) and eq.(2.12) to arrive at

\[ 0 = \bar{E} + j \omega \sum_{j=1}^{N} \frac{I_j}{a_j} \int \bar{G}_m b(x,y,z) \hat{J}_j dv + \nabla \phi . \]  
\[ (2.15) \]

By defining the testing function to be the same as the basis function from eq.(2.10), we can apply Galerkin’s method and take the inner product of both sides of eq.(2.15) with the testing function to arrive at

\[ 0 = \int_{\Omega} \frac{\bar{E} \cdot \hat{J}}{a} dv_j + j \omega \sum_{j=1}^{N} \frac{I_j}{a_j} \int_{\Omega} \bar{G}_m \left( \hat{J}_i \cdot \hat{J}_j \right) dv_j + \int_{\Omega} \frac{\bar{J}_i \cdot \nabla \phi}{a_i} dv_i . \]  
\[ (2.16) \]

From the boundary condition on a good conductor, it can be shown that the first term on the right side of eq.(2.16) represents the resistance \( (I/\sigma) \) of the inductor.

\[ \bar{E} \cdot \dot{\bar{J}}_i = \frac{\left| \bar{J}_i \right|}{\sigma} . \]  
\[ (2.17) \]
We call the double integral in the second term of eq.(2.16) $M_{ij}$, the partial inductance matrix, namely,

$$
M_{ij} = \frac{1}{\alpha_i \alpha_j} \int_{V_i} \int_{V_j} G_m(\mathbf{j}_i, \mathbf{j}_j) \, dv_i \, dv_j,
$$

(2.18)

where $G_m$ was defined in eq.(2.8). Plugging eq.(2.8) into eq.(2.18), we get

$$
M_{ij} = \frac{\mu_0}{4\pi \alpha_i \alpha_j} \int_{V_i} \int_{V_j} \frac{\mathbf{dl}_i \cdot \mathbf{dl}_j}{|r_i - r_j|} \, da_i \, da_j.
$$

(2.19)

The analytical solution to eq.(2.19) can be very complicated. The techniques outlined in [2],[6], [7] and summarized in Appendix C offer approximate solutions to eq.(2.19) that give reasonable accuracy.

Finally, a finite difference approximation can be applied to the third term of eq.(2.16) and simplify the equation to

$$
0 = \frac{1}{a_i} \int_{V_i} R_i \mathbf{j}_i \, dv_i + j\omega \sum_{j=1}^{N} I_j M_{ij} - \Delta V_i.
$$

(2.20)

By definition, $\Delta V_i$ is the change in voltage from one end of rectangle $i$ to the other end.

As in [11]-[13], we define $\Delta V_i$ as

$$
\Delta V_i = \phi(x_i^+, y_i^+, z_i^+) - \phi(x_i^-, y_i^-, z_i^-),
$$

(2.21)

where $\phi(x_i^+, y_i^+, z_i^+)$ denotes the potential at the boundary between rectangle $i$ and rectangle $i-1$. Similarly, $\phi(x_i^-, y_i^-, z_i^-)$ denotes the potential at the boundary between rectangle $i$ and rectangle $i+1$. Expressing eq.(2.20) in matrix form gives

$$
[\Delta V] = ([R] + j\omega [M])[I].
$$

(2.22)
 Whereas the system of equations expressed in eq.(2.22) is based on eq.(2.1) and eq.(2.7), a second system of equations is generated from eq.(2.9) and eq.(2.23), the continuity equation.

$$\nabla \cdot \bar{J} = -j\omega \rho.$$  (2.23)

Similar to the application of Galerkin’s method used in arriving at eq.(2.16), we obtain eq.(2.24).

$$V_i = \frac{1}{v_i} \int_{V_i} \phi \, dv_i = \sum_{j=1}^{N} \int_{V_j} \int_{V_i} \frac{Q_j}{v_i v_j} G_{i,j} \, dv_j \, dv_i.$$  (2.24)

Here, $V_i$ is defined as the average potential on rectangle $i$. If we assume that the potential decreases linearly along the length of the rectangle, then $V_i$ is approximately the potential at the center of the rectangle.

In using this technique for an $N$ rectangle structure, there are two variables that are not assigned to rectangles, namely $V_0$ and $V_{N+1}$. However, the input and output ports of the inductor and capacitor are located at the input boundary of rectangle 1 and the output boundary of rectangle $N$ respectively. Therefore, $V_0$ and $V_{N+1}$ can be assigned to the applied port voltage sources.

$Q_j$ is the total charge present on rectangle $j$ and can be found by integrating the charge density over the volume of rectangle $j$.

$$Q_j = \int_{V_j} \rho \, dv_j.$$  (2.25)

By solving eq.(2.23) for $\rho$ and inserting the result into eq.(2.25), substituting the current $\bar{J}$ with a summation of pulse basis functions, we get
\[ Q_j = \frac{1}{j\omega} \int \sum_{i=1}^{N} \frac{(\nabla \cdot I_j \hat{v}_i) b(x, y, z)}{a_i} \, dv_j. \]  

(2.26)

The divergence is taken on the volume current density in rectangle \( j \). The result is a derivative with respect to the direction in which current is flowing. Again, a finite difference approximation can be made giving

\[ Q_j = \frac{\Delta I_j}{j\omega} \]  

(2.27)

Here \( \Delta I_j \) is the difference between the currents at the input and output boundaries of rectangle \( j \). The current at the boundaries is assumed to be the average of the currents on the two rectangles sharing this boundary. Here we are going to let \( I_0 = I_1 \) and \( I_{N+1} = I_N \). By substituting eq.(2.27) into eq.(2.24), we obtain

\[ V_j = \frac{1}{j\omega} \sum_{i=1}^{N} \frac{\Delta I_j}{V_j} \int v_i G_{e j} \, dv_j. \]  

(2.28)

Writing eq.(2.28) in matrix form gives

\[ [V] = \frac{1}{j\omega} [P][\Delta I]. \]  

(2.29)

Here, \([P]\) is the well-known coefficient of potential matrix whose elements are

\[ p_{ij} = \frac{1}{V_j V_i} \int v_i G_{e j} \, dv_j. \]  

(2.30)

We then calculate eq.(2.30) using a discrete cosine transform, a technique outlined in detail in [5] and [9] and summarized in Appendix B. Solving eq.(2.29) for \([\Delta I]\) gives the matrix equation,

\[ [\Delta I] = j\omega [P]^{-1}[V] = j\omega[C][V]. \]  

(2.31)
Creating the Distributed Circuit from the System of Equations

Eq.(2.22) and eq.(2.31) are in the form of Kirchoff's voltage law (KVL) and Kirchoff's current law (KCL), respectively. From eq.(2.31) we construct the circuit shown in Figure 2.6 using KCL with some algebraic manipulation as in [11]-[13] so that the topology of the circuit agrees with the properties of the capacitance matrix outlined in [14]. The capacitance of rectangle \( i \) to ground \( C_u \) can be written as

\[ C_u = \sum_{j=1}^{N} c_{ij} \tag{2.32} \]

where \( c_{ij} \) are the individual terms of the \([C]\) matrix for an \( N \) rectangle system.

![Figure 2.6 Equivalent KCL for three series rectangles](image)

Also, the direct capacitance between two rectangles \( i \) and \( j \) can be written as

\[ C_{ij} = -c_{ij} \tag{2.33} \]

This leads us to construct the circuit so that the current to ground from rectangle \( i \) is written as \( j \omega C_i V_i \), and the current from rectangle \( i \) to \( j \) is written as \( j \omega C_{ij} (V_i - V_j) \). An equivalent circuit of Figure 2.6 is re-drawn with capacitors in Figure 2.7.
Figure 2.7 Equivalent KCL with capacitors

To complete the derivation of the PEEC distributed circuit, we use eq.(2.22) to model the ohmic loss and magnetic coupling within the metal of the inductor and capacitor. Utilizing the KVL form of eq.(2.22), the circuit in Figure 2.8 is developed.

Figure 2.8 KVL circuit for one rectangle

We combine the circuit of Figure 2.8, which represents the magnetic coupling and power loss in the conductor, with the circuit of Figure 2.7, which models the electrostatic coupling between different conductive segments and between the conductor and ground. The resulting distributed model is formed in Figure 2.9.
Figure 2.9 PEEC distributed circuit for three series rectangles

The capacitors in Figure 2.7 are complex. This is related to the fact that $G_e$ is complex and accounts for loss in the substrate. As a result, each partial capacitance is represented by an ideal capacitor in parallel with a resistor. The ideal capacitance is simply the real part of the partial capacitance. The parallel resistance is simply given by

$$R_y = \frac{-1}{\omega \text{imag}(C_y)}. \tag{2.34}$$

From [5] we know that the imaginary part of $C_y$ is approximately inversely proportional to frequency, which tells us that $R_y$ is approximately constant with respect to frequency. The terms $R_y$ above are not to be confused with the terms of $[R]$, which is discussed in Chapter 3 and deals with the frequency dependent resistance.

The application of the distributed model to the inductor is simple. It is observed in Figure 2.3 that the rectangles comprising the inductor are in series from port 1 to port 2, allowing us to add the distributed models in series. Thus, for an inductor of twelve rectangles, we would have four the circuits from Figure 2.9 in series. The application to
the interdigitated capacitor is slightly different. From Figure 2.5, we see that some of the rectangles are in parallel, while some others are in series. For example, since each finger of the interdigitated capacitor is divided into three rectangles in series, the circuit of Figure 2.9 is used to represent each finger. All fingers from the same side are in parallel. As can be seen from Figure 5.2, the fingers on the port 1 side are not in parallel with the fingers on the port 2 side, but rather they are in series and separated by the capacitance $C_0$, which we will discuss in Chapter 5.
CHAPTER 3

CALCULATING THE RESISTANCE MATRIX

In Chapter 2, the current distribution is assumed to be uniform inside the volume of the rectangle. However, the current will be non-uniform due to the skin effect and, in the case of the spiral inductor, eddy currents. While a uniform current approximation is reasonably adequate for the calculation of the partial inductance and partial capacitance matrices, it is not adequate to calculate the resistance matrix.

Eddy Currents in a Spiral Inductor

We begin with a discussion of the derivation of $[R]$ for the spiral inductor, a slightly more complicated topic compared to its calculation for the interdigitated capacitor due to the eddy current effect. The resistance matrix, $[R]$, for the inductor is a diagonal matrix whose elements represent two important effects: the amount of power dissipated in each rectangle due to skin effect and eddy currents, as well as the effects due to the internal inductance of each rectangle. To calculate the total power dissipated in each rectangle, the total current in each rectangle is first calculated. As an approximation, the total current in a rectangle is expressed as a sum of an excitation
current and an eddy current. The excitation current is uniform across the length and
width of the rectangle. The eddy current describes how the current is disturbed by the
presence of a magnetic field.

\[ \mathbf{J}_{\text{tot}} = \mathbf{J}_{\text{ex}} + \mathbf{J}_{\text{eddy}} \]  

(3.1)

Applying eq.(3.1) to the case of uniform excitation current and an eddy current density
that is linear in \( x \) results in the illustration of Figure 3.1. The (+) sign indicates current is
flowing into the paper and a (-) sign indicates current is flowing out of the paper.

![Figure 3.1 Cross-section of a trace illustrating the total current distribution](image)

The coordinate system for a rectangle is illustrated by Figure 3.2. The origin is
located at the center of the rectangle.

![Figure 3.2 Coordinate system for the top view of a rectangle](image)

The author in [8] gives the relationship between eddy current density and the \( z \-
component of the magnetic flux, using Maxwell’s equations, as
\[
\vec{J}_{\text{eddy}} = \sigma \vec{E}_y = j \omega \sigma \hat{y} \int_0^L B_z(x') dx'
\] (3.2)

Note that the x-component of the eddy current is neglected since the length of the rectangle is assumed to be much larger than the width.

Although approximating \( \vec{J}_{\text{eddy}} \) to be linear in \( x \) as in eq.(3.2) is a reasonable approximation, a more accurate formulation would be to allow \( \vec{J}_{\text{eddy}} \) to vary as \( x^\alpha \), where \( \alpha \) is a constant, determined empirically, that is dependent on the technology being used. This allows eq.(3.2) to be re-written in a more general form as

\[
\vec{J}_{\text{eddy}} = j \omega \sigma B_z x^\alpha \hat{y}.
\] (3.3)

The author in [8] uses an empirical approximation for \( B_z \). However, the magnetic flux acting on a given rectangle can be approximated using the partial inductance matrix found from eq.(2.18). The total flux acting on rectangle \( i \) is

\[
\psi_i = \sum_{j=1}^N M_{ij} I_j.
\] (3.4)

By assuming uniform flux distribution over the area of rectangle \( i \), the magnitude of the magnetic flux density can be approximated as

\[
B_{zr} \approx \frac{\sum_{j=1}^N M_{ij} I_j}{W_i}.
\] (3.5)

The components of the partial inductance matrix that will contribute to the \( \hat{z} \) component of \( B \), \( M_{zr} \), can be determined by the right hand rule. Orthogonal rectangles have zero mutual inductance. We neglect the self-inductance term, because the magnetic field
generated by rectangle \( i \) will have a zero average \( z \) component inside its volume. From [11], a general expression for \( M_{zi} \) is given as

\[
M_{zi} = \sum_{n=1, n \neq i}^{N} (-1)^{\alpha_{in}} M_{in},
\]

where \( \alpha_{in} \) is determined by the orientation of rectangle \( i \) relative to rectangle \( n \) and is equal to zero or one. Our expression for \( B_z \) on rectangle \( i \) is

\[
B_{zi} = \frac{M_{zi} I_{ex}}{W_i I_i}.
\]

**The Skin Effect**

In general, a conductor with a rectangular cross section will have current crowding at all of the surfaces as illustrated by Figure 3.3. The current crowding at high frequencies is due to the internal inductance being higher in the center of the conductor as a result of the internal magnetic field being largest at the center. The current tends to flow through the path of least impedance, which is the path of least inductance at high frequencies.

![Figure 3.3 Cross-section of current distribution](image)

An exact calculation, in terms of number of skin depths \( \Delta (1 \leq \Delta \leq 2) \), of the thickness that the current in the conductor penetrates is not a trivial task and its result will
vary from technology to technology, depending greatly on the dielectric properties of the substrate being used. Using $\Delta$ as a variable for the number of skin depths the current takes up, the volume current density in the rectangle can be expressed as

$$\bar{J}(x, y, z) = \Delta \ast J(x, y) e^{\frac{-z}{\delta}} \hat{y}.$$

(3.8)

Here, $z_i$ is the distance from the top of the substrate to rectangle $i$. $z$ is equal to zero at the top of the substrate and is negative inside the substrate. Integrating eq.(3.8) over the thickness of the conductor gives

$$\int_{z_i}^{0} \bar{J}(x, y, z) dz = \Delta \ast \hat{y} J(x, y) \left( \delta - \delta e^{\frac{-z_i}{\delta}} \right).$$

(3.9)

The effective thickness, $t_{eff}$, is then defined as

$$t_{eff} = \Delta \ast \delta \left( 1 - e^{\frac{-z_i}{\delta}} \right).$$

(3.10)

As we will see, $t_{eff}$ plays a big role in determining the frequency dependent resistance and therefore the peak $Q$ of the device. This allows $\Delta$ to be treated as a degree of freedom that will be determined empirically for each technology. Using $Q$ obtained by another EM solver or from measured data as a benchmark, $\Delta$ is adjusted so that the peak $Q$ obtained by this model matches that of the EM solver or measured data. For the technology used in this model, a value of $\Delta = 1.1$ is used.

**Calculating the Frequency Dependent Resistance**

The total power dissipated in the rectangle can be expressed in terms of the total volume current density in the rectangle.
\[ P_{dxx} = \int \frac{1}{2\sigma} |J_{ax}|^2 \, dv \]  

(3.11)

We first substitute eq.(3.7) into eq.(3.3) for \( B_z \), plugging that result into eq.(3.1) and eq.(3.11). Eq.(3.11) is then integrated which gives the total power dissipated as

\[ P_{dxx} = \frac{l_{eff}^2}{2\sigma} |I_{cr}|^2 \left\{ W_i + \frac{K}{2\alpha + 1} \left[ \left( \frac{W_i}{2} \right)^{2\alpha + 1} - \left( -\frac{W_i}{2} \right)^{2\alpha + 1} \right] \right\}. \]  

(3.12)

The variable \( K \) is defined as

\[ K = \left( \frac{\omega \sigma M_{cr, eff}}{I_t} \right)^2. \]  

(3.13)

The effective resistance of the rectangle relates the amount of power dissipated in the rectangle to the amount of current applied to the rectangle.

\[ R_{eff} = \frac{2P_{dxx}}{|I_{cr}|^2} = \frac{2P_{dxx}}{\left\{ \int_{S_{cr, eff}} (\vec{J}_{cr} \cdot dS_{cr, eff}) \right\}^2}. \]  

(3.14)

Evaluating the integral gives

\[ \int_{z_i}^{z_f} \int \frac{w}{2} \vec{J}_{cr} \cdot \vec{y} dxdz = J_{cr} W_{l_{eff}} \]  

(3.15)

The result of eq.(3.15) and eq.(3.12) is then combined into eq.(3.14), which results in

\[ R_{eff, ind} \]  

for a given rectangle \( i \) of an inductor being

\[ R_{eff, ind} = \frac{l_{eff}}{\sigma W_i^2 I_{eff}} \left\{ W_i + \frac{K}{2\alpha + 1} \left[ \left( \frac{W_i}{2} \right)^{2\alpha + 1} - \left( -\frac{W_i}{2} \right)^{2\alpha + 1} \right] \right\}. \]  

(3.16)

As with \( \Delta \), \( \alpha \) is also determined empirically by the comparison of \( Q \) between this model and another reliable benchmark. The procedure for determining \( \alpha \) is to start off
with $\alpha = 1$. The value for $\Delta$ is varied first, as its value will affect $Q$ the most. The value for $\Delta$ will always be between one and two. If adjusting $\Delta$ does not give a correct value for peak $Q$, $\Delta$ will be set to the value that gives the best $Q$. Next, the value for $\alpha$, is adjusted. Its value should not be too far from 1, as the linear approximation for the eddy current distribution is fairly accurate. For this model, the linear approximation for the eddy current distribution, $\alpha = 1$, is used.

As mentioned previously, interdigitated capacitors do not experience a significant effect due to eddy currents. This is due to the fact that the magnetic flux density perpendicular to the surface of the interdigitated capacitor is negligible. This can be handled with our model by setting $M_{\sigma} = 0$ in eq. (3.13). Thus we can modify eq.(3.16) for the interdigitated capacitor to only account for the skin effect. This gives $R_{\text{eff,cap}}$ for a given rectangle $i$ of a capacitor as

$$R_{\text{eff,cap}} = \frac{1}{\sigma W_{\text{eff}}}$$

The rectangles of the capacitor and inductor will experience an internal inductance because of the magnetic flux internal to the conductor. Paul in [3] gives the per-unit-length internal inductance of a round wire as

$$L_{\text{int}} = \frac{2\delta}{r_w} \left( \frac{\mu_0}{8\pi} \right)$$

for high frequencies ($r_w \gg \delta$) and

$$L_{\text{int}} = \frac{\mu_0}{8\pi}$$
for low frequencies \( r_w << \delta \). Here, \( r_w \) is the radius of the wire. \( 2\delta \) is recognized as the effective thickness of the wire. As the conductor is rectangular, it is unknown what should be substituted into \( r_w \). However, if we assume that at DC, the per-unit-length internal inductance should be eq.(3.19), then it is reasonable to approximate the per-unit-length internal inductance for all frequencies as

\[
L^{\text{int}} = \frac{t_{\text{eff}}}{t} \left( \frac{\mu_0}{8\pi} \right),
\]  

(3.20)

because at low frequencies, \( t_{\text{eff}} \) is approximately equal to \( t \). The total reactance of the rectangle can be calculated as

\[
X_{\text{eff}} = \omega L^{\text{int}} l.
\]  

(3.21)
CHAPTER 4

DERIVING THE INDUCTOR LUMPED PI-MODEL

The techniques of Chapters 2 can be used to construct a PEEC distributed model for any planar monolithic inductor. Shown below is a flowchart outlining the progression from Maxwell’s equations to an inductor lumped pi-model.

[Diagram of flowchart]

Figure 4.1 Inductor lumped pi-model derivation flowchart

In this chapter, we describe a method to simplify the distributed model down to a lumped pi-model for a spiral inductor. A lumped pi-model makes it easy for the designer
to visualize the parasitics inherent to the spiral inductor. The inductor lumped pi-model we use is commonly used for integrated inductors and is shown below in Figure 4.2. The loss in the substrate due to the electrostatic coupling between metal and ground is captured by $R_{s1}$ and $R_{s2}$. The capacitance caused by this electrostatic coupling through the substrate is captured by $C_{s1}$ and $C_{s2}$. The capacitive coupling due to the lossless oxide is represented by $C_{ox1}$ and $C_{ox2}$. The lumped model used by [11]-[13] does not account for this oxide capacitance separately from $C_{s1}$, $R_{s1}$, $C_{s2}$, and $R_{s2}$. By taking into account the effect due to $C_{ox1}$ and $C_{ox2}$ separately from the rest of the substrate admittance, we are able to accurately model the device at lower frequencies. If $C_{ox1}$ and $C_{ox2}$ are absent, we effectively have a low impedance to ground through $R_{s1}$ and $R_{s2}$ at low frequencies, which if physically not accurate since the lossless oxide would make the impedance very large.

As mentioned in Chapter 1, there are approximate empirical and semi-analytical methods ([1],[6],[8],[15],[16]) used to calculate the component values of the lumped pi-model. As illustrated in the flowchart in Figure 4.1, we use two low frequency approximations to the distributed model similar to that of [11]-[13] to calculate $R_s$, $L$, $C_{s1,2}$ and $R_{s1,2}$. We use a high frequency approximation that uses the overlap capacitance of the inductor bridge to approximate a value for $C_0$ and $R_0$. 

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First Low Frequency Approximation

The first low frequency approximation involves recognizing that at low frequencies, the complex capacitors are approximately open circuits. By applying this approximation to the PEEC circuit of Figure 2.9, the circuit of Figure 4.3 results.

For a circuit with $N$ series rectangles, it is easily observed that the total series impedance of this circuit is

$$Z_{\text{series}} = \sum_{j=1}^{N} R_j + s \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij}$$

(4.1)
The form of eq.(4.1) is
\[ Z_{\text{series}} = R_s + sL \]  \hspace{1cm} (4.2)
which is the same form as the branch of the pi-model shown in Figure 4.4.

\[ \begin{array}{c}
  R_s \\
  L
\end{array} \hspace{1cm} \begin{array}{c}
  \text{Figure 4.4 Pi-circuit neglecting capacitances}
\end{array} \]

Thus the components of Figure 4.3 can be found as
\[ R_s = \sum_{j=1}^{N} R_j \]  \hspace{1cm} (4.3)
\[ L = \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij} \]  \hspace{1cm} (4.4)

**Second Low Frequency Approximation**

The second low frequency approximation does not neglect all of the capacitors. Because the impedance of an inductor is much smaller than the impedance of a capacitor at low frequencies, the complex capacitances in parallel with a series RL combination are neglected. All of the complex capacitances connected to ground remain. When this approximation is applied to the PEEC circuit of Figure 2.9, the result is the circuit of Figure 4.5. The currents through each of the rectangles of the inductor are assumed to be approximately equal because of the negligible current in the substrate. This allows us to lump the mutual inductances with the self-inductances.
Figure 4.5 PEEC circuit after second low frequency approximation

\[ I_y = \sum_{j=1}^{N} M_{yj} \]  

(4.5)

The next step is to short port 2 and calculate the input admittance \( Y_{11} \). We ignore terms with \( s \) to the power of three or greater since we are using a low frequency approximation. This gives

\[ Y_{11} \approx s \left( \frac{C_{ox1} C_{s1}'}{C_{ox1} + C_{s1}'} + \alpha_2 \frac{C_{ox2} C_{s2}'}{C_{ox2} + C_{s2}'} + \alpha_3 \frac{C_{ox3} C_{s3}'}{C_{ox3} + C_{s3}'} \right) + \frac{1}{Z_{series}} \]  

(4.6)

where \( Z_{series} \) is found from eq.(4.1) and

\[ \alpha_i = \frac{s \sum_{j=1}^{N} L_j}{Z_{series}} \approx \frac{s \sum_{j=2}^{N} L_j}{L} \]  

(4.7)

if we assume \( sL_j \gg R_j \) as in [11] to make the substrate impedance independent of frequency. The form of eq.(4.6) matches the form for the input admittance looking into the circuit of Figure 4.6.
Equating eq.(4.6) to eq.(4.8) for $N$ rectangles gives the substrate admittance $Y'_{s1}$. Since $s=j\omega$ and keeping in mind that $C'_{si}$ is complex,

$$Y'_{s1} = \sum_{i=1}^{N} \alpha_i Y'_{si} = \sum_{i=1}^{N} \alpha_i \frac{C_{ox} C'_{si}}{C_{ox} + C'_{si}} = j\omega \left\{ \text{Re}\left(\tilde{Y}'_{s1}\right) \right\} + j\omega \left\{ j \cdot \text{Im}\left(\tilde{Y}'_{s1}\right) \right\}$$

$$= j\omega \left\{ \text{Re}\left(\tilde{Y}'_{s1}\right) \right\} - \omega \left\{ \text{Im}\left(\tilde{Y}'_{s1}\right) \right\}$$

where $Y'_{s1}$ is the total complex admittance to ground at port 1 as illustrated in Figure 4.6.

$Y'_{si}$ is the total complex capacitance to ground of rectangle $i$ and is found from the capacitance matrix as

$$Y'_{si} = \sum_{j=1}^{N} c_{ij},$$

where $c_{ij}$ represents the individual terms of $[C]$.

In a manner similar to [15] and [16], we use a parallel plate approximation for the oxide capacitance. This gives the oxide capacitance in terms of the area, thickness, and permittivity of rectangle $i$ as
\[ C_{ox} = A_i \frac{\varepsilon_{ox}}{t_{ox}} \]  

(4.11)

where \( A_i \) is the surface area (neglecting thickness) of rectangle \( i \). We make the assumption that the oxide capacitance is equal at both ports. Therefore we can take the sum of the oxide capacitance due to all the rectangles and divide it by two to calculate the oxide capacitance at port 1 and port 2.

\[ C_{ox,p1} = C_{ox,p2} = \frac{1}{2} \sum_{i=1}^{N} C_{ox} \]  

(4.12)

We can express \( Y''_{s1} \) as the series combination of the admittances due to \( C'_{sub1} \) and \( C'_{ax,p1} \), which are defined in Figure 4.6.

\[ Y''_{s1} = j\omega \left\{ \frac{C_{ax,p1}C_{R,1}(C_{R,1} + C_{ax}) + C_{ax,p1}C_{I,1}^2}{(C_{R,1} + C_{ax,p1})^2 + C_{I,1}^2} \right\} - \omega \left\{ \frac{C_{ax,p1}^2C_{I,1}}{(C_{R,1} + C_{ax,p1})^2 + C_{I,1}^2} \right\} \]  

(4.13)

where \( C_{I,1} = \text{Im}\{C'_{sub1}\} \) and \( C_{R,1} = \text{Re}\{C'_{sub1}\} \). Equating the real and imaginary parts of eq.(4.9) with the real and imaginary parts of eq.(4.13) gives us two equations to solve for the two unknowns, \( C_{I,1} \) and \( C_{R,1} \). From \( C_{I,1} \) and \( C_{R,1} \) we use the complex capacitance technique to extract the component values, \( C_{sub1} \) and \( R_{sub1} \).

\[ C_{sub1} = C_{R,1} \]  

(4.14)

\[ R_{sub1} = -\frac{1}{\omega C_{I,1}} \]  

(4.15)

This technique will work for any \( \omega > 0 \). A similar procedure is applied to find the component values at port 2.
\[ Y_{i2} = \sum_{i=1}^{N} \beta_i Y'_{i} = \sum_{i=1}^{N} \beta_i \frac{C_{ao} C'_{n}}{C_{ao} + C'_{n}} = j\omega \left\{ \text{Re} \left( \tilde{Y}_{i2} \right) \right\} + j\omega \left\{ j \ast \text{Im} \left( \tilde{Y}_{i2} \right) \right\} \]

(4.16)

\[ \beta_i = \frac{\sum_{j=1}^{i} L_j}{L} \]

(4.17)

\[ C_{sub2} = C_{R,2} \]

(4.18)

\[ R_{sub2} = -\frac{1}{\omega C_{l,2}} \]

(4.19)

**High Frequency Approximation**

For an accurate calculation of the self-resonant frequency, the most important component of the equivalent pi-model apart from the inductance is the interwinding capacitance \( C'_o \), the parallel combination of \( R_o \) and \( C_o \) in Figure 4.2. While the frequency dependent resistance plays an important role in determining the value of the peak \( Q \) of the inductor, the interwinding capacitance determines when the inductor \( Q \) becomes zero and the inductor behaves as a capacitor rather than an inductor. We replace the complete distributed model for three consecutive rectangles with the simplified version shown in Figure 4.7 for 4 rectangles, where the inductors are treated as open circuits. Note we also replace the admittances to ground from each rectangle with the lumped values at the ports found with the second low frequency approximation.
Our method is based on a common technique used in other analytical pi-models, such as [15] and [16], which use the overlap capacitance of the bridge, illustrated in Figure 4.8, to determine the interwinding capacitance.
The methods in [15] and [16] use a parallel plate approximation, whereas we use the [C] matrix, which is more accurate in that it accounts for fringing. In order to justify the method of using the overlap capacitance to determine the total interwinding capacitance, we must show that the portion of the inductor that overlaps the bridge is close to the port 1 voltage and that the bridge is close to the port 2 voltage at resonance. Once this is accomplished, we can ignore all of the turn-to-turn interwinding capacitances because most of the turns are at the same potential and the overlap capacitance will dominate. After inspecting the voltage along the inductor for various geometries by solving the above equations, we consistently find that the voltage along the first half of the inductor rectangles remained fairly constant and that most of the voltage drop occurred along the inner-most turns of the inductor after the last overlap of the bridge. The plot of the voltage along the inductor at resonance is shown below for a 1.5 turn and 3.5 turn inductor in Figure 4.9 and Figure 4.10, respectively, and it shows that the overlap rectangles are close to the same voltage as port 1. Similar results were found for other N-turn inductors.
Figure 4.9 Inductor voltage vs. Inductor rectangle for 1.5 turn inductor

Figure 4.10 Inductor voltage vs. Inductor rectangle for 3.5 turn inductor

As a result of these findings, we can express the total interwinding capacitance as the parallel combination of each bridge overlap capacitance. A general expression for the interwinding capacitance using the discretization scheme shown in Figure 2.3 is given as
\[ C'_0 = \sum_{i=6}^{r-9} \{ c(i, \gamma) + c(i+1, \gamma) \} \]  
(4.20)

\[ \gamma = 8N + 3 \]  
(4.21)

\[ c(i, j) = -C(i, j) \]  
(4.22)

with \( N \) being the number of turns of the inductor.

We use the complex capacitance technique to extract the real capacitance and resistance values from the complex capacitance.

\[ C_0 = \text{real}(C'_0) \]  
(4.23)

\[ R_0 = \frac{1}{\omega(\text{imag}(C'_0))} \]  
(4.24)
CHAPTER 5

DERIVING THE CAPACITOR LUMPED PI-MODEL

For the interdigitated capacitor, the distributed model is actually a combination of two distributed models, one for port 1 (left-side) and one for port 2 (right-side). \( C' \), the parallel combination of \( C_0 \) and \( R_0 \) in Figure 5.2, separates the two distributed models and its calculation will be explained shortly.

Figure 5.1 Capacitor lumped pi-model derivation flowchart
Low Frequency Approximation

In a manner similar to that used for the inductor, we use a low frequency approximation for both ports of the capacitor to find the impedance $Z_{l.f}=R_s+j\omega L_s$, the branch of the circuit in Figure 5.2 comprised of $R_s$ and $L_s$. Little current will flow through the capacitors at low frequency, leading to the distributed model for one side of the capacitor shown in Figure 5.3.

Figure 5.2 Capacitor lumped pi-model

Figure 5.3 Capacitor distributed circuit for one side after low frequency approximation
The total series resistance of each finger with \((k-j)\) rectangles is the sum of the \(R_i = R_{ii}\) found from \([R]\).

\[
R_{sf} = \sum_{i=j}^{k} R_i
\]  

(5.1)

Because the capacitors are treated as opens, we can approximate the current in each rectangle as being equal. We further approximate that each finger has the same current. Making this assumption, the mutual inductances are lumped with the self inductances as with the inductor model, and the inductance due to each rectangle \(L_i\) is found as

\[
L_i = \sum_{j=1}^{N} M_{ij}
\]  

(5.2)

where \(N\) is the total number of rectangles comprising one side of the capacitor. The total series inductance due to each finger with \((k-j)\) rectangles is the sum of the \(L_i\).

\[
L_{sf} = \sum_{i=j}^{k} L_i
\]  

(5.3)

The total series impedance due to finger \(n\) is

\[
Z_{sf,n} = R_{sf} + j\omega L_{sf}
\]  

(5.4)

For the feed rectangles, from which the fingers branch out, the total impedance is found the same way. Since they have the same geometry, we treat the impedance of the feed rectangles to be the same and call it \(Z_A\). The series combination of each finger impedance \(Z_{sf,n}\) with its adjacent feed rectangle impedance \(Z_A\) is then added in parallel with the next finger of the capacitor. This pattern continues until we have resolved the circuit below to one impedance, \(Z_{s,tot} = R_s + j\omega L_s\). A generalized expression which captures this impedance for a capacitor with \(M\) fingers is
\[ Z_{s,\text{tot}}^{-1} = \sum_{n=1}^{M-1} Z_{s,n}^{-1} \]  \hspace{1cm} (5.5)

where

\[ Z_{s,n}^{-1} = \frac{1}{Z_A + Z_{s,n-1} + \frac{1}{Z_{sf,n+1}}} \]  \hspace{1cm} (5.6)

for \( n = 2 \) to \( n = M - 1 \). For \( n = 1 \), we have

\[ Z_{s,1}^{-1} = \frac{1}{Z_A + Z_{sf,3} + \frac{1}{Z_{sf,2}}} \]  \hspace{1cm} (5.7)

Finally, we find the total series impedance due to the fingers and feed rectangles of one side of the capacitor as

\[ Z_{s,\text{tot}} = \frac{1}{Z_{s,\text{tot}}^{-1}}. \]  \hspace{1cm} (5.8)

The method above is applied to both sides of the capacitor and \( Z_{s,\text{tot}} \) found for both sides can be simply added together since they are in series.

**High Frequency Approximation**

To determine the value for \( C'_0 \) and the substrate admittances \( Y'_1 \) and \( Y'_2 \) (the parallel combination of \( C_{s1,2} \) and \( R_{s1,2} \) in series with \( C_{ox,1,2} \)), a high frequency approximation is used. At high frequencies, we assume negligible current in the conductor due to the high impedance of the inductance of each rectangle. From this, we can conclude that the rectangles on either side of the interdigitated capacitor are at the same potential, allowing us to ignore capacitances between rectangles on the same side of the capacitor as they are shorted out. Additionally, this equipotential approximation
allows us to add the capacitance to ground of each rectangle in parallel. From [14], we know that from \( [C] \) the capacitance of rectangle \( i \) to ground, \( \tilde{C}_i \), can be written as

\[
\tilde{C}_i = \sum_{j=1}^{N} c_{ij},
\]

where \( c_{ij} \) are the individual terms of the \( [C] \) matrix for an \( N \) rectangle system. If we discretize the interdigitated capacitor into \( N/2 \) rectangles on either side, for a total of \( N \) rectangles making up the entire structure as in Figure 5.4, we can solve for the complex admittances \( Y'_{s1} \) and \( Y''_{s2} \).

![Figure 5.4 Discretization for a 6 finger interdigitated capacitor](image)

We number the rectangles such that rectangles 1 through \( N/2 \) make up the port 1 side, leading to \( \tilde{C}_{s1} \).

\[
\tilde{C}_{s1} = \sum_{i=1}^{N/2} \tilde{C}_i
\]

(5.10)

\( \tilde{C}_{s2} \) is found from rectangles \( N/2+1 \) to \( N \).

\[
\tilde{C}_{s2} = \sum_{i=N/2+1}^{N} \tilde{C}_i
\]

(5.11)
Similar to calculating the bridge overlap capacitance for the inductor, we use a parallel plate approximation for the oxide capacitance as done in [15]. This gives the oxide capacitance in terms of the area of the rectangle \( A \), oxide thickness \( t_{ox} \), and permittivity \( \varepsilon_{ox} \) as

\[
C_{ox} = A \frac{\varepsilon_{ox}}{t_{ox}}.
\]  

(5.12)

We again make the assumption that the oxide capacitance is equal at both ports. Therefore we can take the sum of the oxide capacitance due to all the rectangles and divide it by two to calculate the oxide capacitance at port 1 and port 2.

\[
C_{ox,p1} = C_{ox,p2} = \frac{1}{2} \sum_{j=1}^{N} C_{oxj}
\]  

(5.13)

Since \( Y'_{sl} \) is a complex admittance with a real part, we can express it as

\[
Y'_{sl} = j\omega \left\{ \text{real} \left( \tilde{C}_{sl} \right) \right\} + j\omega \left\{ j \text{ * imag } \left( \tilde{C}_{sl} \right) \right\}
\]

\[
= j\omega \left\{ \text{real} \left( \tilde{C}_{sl} \right) \right\} - \omega \left\{ \text{imag} \left( \tilde{C}_{sl} \right) \right\}
\]  

(5.14)

where \( \tilde{C}_{sl} \) is the total complex capacitance to ground at port 1 defined in eq.(5.10). We can also express \( Y'_{sl} \) as the series combination of the admittances due to \( C'_{sl} \) and \( C_{sl} \).

\[
Y'_{sl} = j\omega \left\{ \frac{C_{sl} C_{R,l} \left( C_{R,l} + C_{sl} \right) + C_{sl}^2 C_{l,l}}{(C_{R,l} + C_{sl})^2 + C_{l,l}^2} \right\} - \omega \left\{ \frac{C_{sl}^2 C_{l,l}}{(C_{R,l} + C_{sl})^2 + C_{l,l}^2} \right\}
\]

(5.15)

where \( C_{l,l} = \text{imag} \left\{ C'_{sl} \right\} \) and \( C_{R,l} = \text{real} \left\{ C'_{sl} \right\} \). Equating the real and imaginary parts of eq.(5.14) with the real and imaginary parts of eq.(5.15) gives us two equations to solve for the two unknowns, \( C_{l,l} \) and \( C_{R,l} \). From \( C_{l,l} \) and \( C_{R,l} \) we use the complex capacitance technique to extract the component values, \( C_{sub1} \) and \( R_{sub1} \).
\[ C_{s1} = C_{R,1} \quad (5.16) \]

\[ R_{s1} = \frac{1}{\omega C_{i,1}} \quad (5.17) \]

This technique will work for any \( \omega > 0 \). A similar procedure is applied to find the component values at port 2.

The calculation of \( C'_0 \) in Figure 5.2 is relatively simple compared to the inductor pi-model. Using the concepts outlined in [14] relating charge, potential and the capacitance matrix of a system of rectangles, and applying to the interdigitated capacitor, we can derive a rigorous formulation for \( C'_0 \). To begin the derivation, Figure 5.5 will be used, and one of the ports of the capacitor model will be shorted, which effectively sets the voltage of rectangle \( i, V_i \), to zero.

![Diagram of Pi-circuit with High Frequency Approx.](image)

Figure 5.5 Pi-circuit with High Frequency Approx.

We short port 1, which allows us to write the total charge of the system as

\[ Q_{tot} = Q_{c_{s1}} + Q_{c_{s2}} = \sum_{i=1}^{N} \left( \tilde{C}_i V_i + \sum_{j=1}^{N} C_{ij} v_{ij} \right), \quad (5.18) \]

keeping in mind that port 1 consists of rectangles \( i = 1 \) to \( N/2 \) and port 2 consists of rectangles \( i = (N/2 + 1) \) to \( N \). In eq.(5.18) above, \( v_{ij} \) is the change in voltage from \( i \) to \( j \), while \( C_{ij} = -C_{ji} \), the individual terms of \([C]\).
However, since port 1 voltage is set to zero, \( V_i = 0 \) for \( i = 1 \) to \( N/2 \). Also, since all rectangles on either port of the capacitor are at the same voltage, we can write \( v_{ij} = 0 \) for \( i,j = 1 \) to \( N/2 \) and \( v_{ij} = 0 \) for \( i,j = N/2 \) to \( N \). This allows the total charge \( Q_{tot} \) to be rewritten as

\[
Q_{tot} = \sum_{i=1}^{N/2} \sum_{j=N/2+1}^{N} C_{ij} v_{ij} + \sum_{i=N/2+1}^{N} \left\{ \tilde{C}_i V_i + \sum_{j=1}^{N/2} C_{ij} v_{ij} \right\}
\]

(5.19)

We can substitute \( v_{ij} \) to be \( V_{p2} \), the port 2 voltage, for all \( i \) and \( j \) used in eq.(5.19). We also set \( V_i = V_{p2} \) for \( i = (N/2 + 1) \) to \( N \). Applying these substitutions, eq.(5.19) is then rewritten as

\[
Q_{tot} = V_{p2} \left\{ \sum_{i=1}^{N/2} \sum_{j=N/2+1}^{N} C_{ij} + \sum_{i=N/2+1}^{N} \sum_{j=1}^{N/2} C_{ij} \right\} + V_{p2} \left\{ \sum_{i=N/2+1}^{N} \tilde{C}_i \right\}
\]

(5.20)

\[
= V_{p2} C_0' + V_{p2} \tilde{C}_{s2}
\]

From eq.(5.20) we can easily see that for an \( N \) rectangle system (\( N/2 \) rectangles per side) \( C_0' \) is found as

\[
C_0' = \sum_{i=1}^{N/2} \sum_{j=N/2+1}^{N} C_{ij} + \sum_{i=N/2+1}^{N} \sum_{j=1}^{N/2} C_{ij} .
\]

(5.2)

As with the inductor, we use the complex capacitance technique to extract the real capacitance and resistance values from the complex capacitance.

\[
C_0 = \text{real}(C_0')
\]

(5.22)

\[
R_0 = -\frac{1}{\omega \text{imag}(C_0')}
\]

(5.23)
CHAPTER 6

DE-EMBEDDING TECHNIQUE FOR RF PASSIVES

There are many published methods that deal with the de-embedding of active devices at higher frequencies, similar to that of [21]. However, locating de-embedding techniques that deal with larger RF passive devices such as inductors and capacitors is difficult. The techniques for de-embedding smaller active devices cannot be blindly applied to RF inductors and capacitors. In the case of inductors, the length of a thru de-embedding structure would add too much inductance and cause error in the de-embedded result. For capacitors, especially MIM capacitors, the ohmic loss is so small that it is easy to over-estimate the loss of the test structure, resulting in a negative real part in the 2-port differential impedance. It is sometimes better to neglect de-embedding this loss, as it is so small to begin with that the de-embedded result is usually no more accurate than the raw measured value.

We begin with a diagram in Figure 6.1 showing the typical parasitics that need to be de-embedded from an RF passive device measurement made on a ground-signal-
ground test structure such as the one shown in Figure 6.2. It is observed from the figure that there are only four parasitics that need to be de-embedded. The coupling from port to port can be neglected due to the large size of the test structure.

Figure 6.1 RF passive device testbed parasitics

The test structures fabricated and measured and simulated in this chapter use a BiCMOS process with seven available metal layers. The design and measurement was performed by the Mixed Signal Design Team at the AFRL at Wright Patterson Air Force Base. Differing from the FDSOI process discussed previously in this thesis, the silicon bulk is lossy with a resistivity of 13.5 $\Omega \cdot cm$. For the inductors, a non-conductive deep trench is used beneath for the ground-plane. For the capacitors, a conductive ground-plane is used.
Capacitor Test Structure

We first discuss a technique for calculating the parasitics for a capacitor test structure shown in Figure 6.2. To de-embed the coupling from the ground pads to the middle signal pad, we use the open test structure shown below in Figure 6.3. The parasitics of the open structure are given in Figure 6.4.

Figure 6.2 Capacitor test structure
From the open test structure, we obtain the Y-parameters and the value of the parasitics $Y_{p1}$ and $Y_{p2}$ are found as

$$Y_{p1} = Y_{11,open} + Y_{12,open} + Y_{11,open} \quad (6.1)$$

$$Y_{p2} = Y_{22,open} + Y_{12,open} + Y_{22,open} \quad (6.2)$$

where $Y_{12,open}$ is neglected due to the negligible coupling between the ports. The next step is to determine the values of the impedance from the DUT feed-lines $Z_1$ and $Z_2$. Due to
the capacitors small size, we have two options: a thru or a short test structure. First we will discuss the thru method. The thru test structure used for this fabrication is given below in Figure 6.5. As will be discussed, the design is not correct, due to the fact that it is a transmission line test structure and not the actual thru designed for this fabrication. Since the thru test structure that was designed was erroneous and could not be used, a transmission line was used in its place to obtain measured data for the thru

![Diagram of Capacitor Thru Test Structure]

Figure 6.5 Capacitor thru

As can be observed in Figure 6.5, the layout of the thru line being used is not identical to the DUT interconnect line of Figure 6.2. However, we will see from the results that this still gives reasonably accurate de-embedding. The thru test structure should be identical to the open shown in Figure 6.3 with the exception that the transmission line continues through the open area where the DUT would be. A schematic showing the parasitics of the thru is given below in Figure 6.6.
To determine the value for $Z_1$ and $Z_2$ using the thru, we measure the $Y$-parameters of the device and obtain $Y_{12}$. The impedance of $Z_1$ in series with $Z_2$ and the thru is equal to $-1/Y_{12}$. We can then get $Z_1$ and $Z_2$ as

$$Z_1 = Z_2 = \alpha \left(\frac{1}{2}\right) \left(-\frac{1}{Y_{12,\text{thru}}}\right)$$

(6.3)

The variable $\alpha$ in eq.(6.3) accounts for the length of the thru. We find $\alpha$ as

$$\alpha = \frac{\text{length}(Z_1 + Z_2)}{\text{length}(Z_1 + Z_2) + \text{length}(\text{thru})}.$$  

(6.4)

The above formula is exact when the width of the lines for the thru and the impedances $Z_1$ and $Z_2$ are the same as shown in Figure 6.5. Otherwise, it is an approximate formula.

The short test structure for the capacitor is given below in Figure 6.7. It is important to keep in mind where the reference plane for the device is. This is not too important for the open structure, but for the short structure we must insert the short at this reference plane. If we insert it too close to the signal pad (away from the device), we
include too much of the DUT feed-line in our de-embedded measurement, resulting in added inductance and loss to our de-embedded measurement.

![Figure 6.7 Capacitor short](image)

The squares at the end of the feed-lines represent vias to ground, as noted on the figure. The parasitics of the short structure are given below in Figure 6.8. There are some losses and inductance ($Z_3$) in the vias as well as the return path from the vias to the ground pads. However, for the capacitor, they are ignored ($Z_3=0$)
We calculate $Z_1$ and $Z_2$ from the measured port 1 and port 2 input impedances, $Z_{11}$ and $Z_{22}$, and the previously calculated $Y_{p1}$ and $Y_{p2}$.

$$Z_{11,\text{short}} = Z_1 \parallel \frac{1}{Y_{p1}} \Rightarrow Z_1 = \frac{\left( Z_{11,\text{short}} / Y_{p1} \right)}{\left( 1 / Y_{p1} - Z_{11,\text{short}} \right)}$$ \hspace{1cm} (6.5)

$$Z_{22,\text{short}} = Z_2 \parallel \frac{1}{Y_{p2}} \Rightarrow Z_2 = \frac{\left( Z_{22,\text{short}} / Y_{p2} \right)}{\left( 1 / Y_{p2} - Z_{22,\text{short}} \right)}$$ \hspace{1cm} (6.6)

**Inductor Test Structure**

Next we will apply the open/short method to an inductor test structure. The test structures as laid out are not correct, but they still give reasonable results. Corrected test structures will be shown as well, with simulated results to support their design. One obstacle with the inductors fabricated is that there is no metal ground plane (lossless deep
trench used instead) beneath the inductor to put our short at the proper reference plane. This causes an error in the de-embedded inductance due to a shift in the reference plane towards the signal pads. The inductor test structure is given below.

![Inductor test structure](image)

**Figure 6.9 Inductor test structure**

The parasitics of the test structure are the same as for the capacitor, shown in Figure 6.1. The open test structure is given below in Figure 6.10. The parasitics are the same as that shown in Figure 6.4. The parasitics $Y_{p1}$ and $Y_{p2}$ are calculated from the open in the same manner as for the capacitor:

\[
Y_{p1} = Y_{11,open} + Y_{12,open} - Y_{11,open}
\]  

(6.7)

\[
Y_{p2} = Y_{22,open} + Y_{12,open} - Y_{22,open}
\]  

(6.8)

The next step is to determine the values for $Z_1$ and $Z_2$. For the inductor, we use the short test structure as the thru is too long to be used accurately. The calculation for
these parasitics is the same as for the capacitor. The inductor short is shown below in Figure 6.11.

Figure 6.10 Inductor open

Figure 6.11 Inductor short

As with the capacitor, we calculate $Z_1$ and $Z_2$ from the measured port 1 and port 2 input impedances, $Z_{11}$ and $Z_{22}$, and the previously calculated $Y_{p1}$ and $Y_{p2}$.
\[ Z_{11,\text{short}} = Z_1 \parallel \frac{1}{Y_{p1}} \Rightarrow Z_1 = \left( \frac{Z_{11,\text{short}}}{Y_{p1}} \right) \left( \frac{1}{Y_{p1} - Z_{11,\text{short}}} \right) \]

\[ Z_{22,\text{short}} = Z_2 \parallel \frac{1}{Y_{p2}} \Rightarrow Z_2 = \left( \frac{Z_{22,\text{short}}}{Y_{p2}} \right) \left( \frac{1}{Y_{p2} - Z_{22,\text{short}}} \right) \]

(6.9)  

(6.10)

If the technology allows, it would be better to use vias to short to ground instead of the bar across the pads shown in Figure 6.11. The length of the bar, and the current loop caused by it, introduces too much inductance and causes an error in our de-embedding. Also, if a conductive ground plane beneath the inductor is present, the vias should be placed at the DUT so that a proper reference plane is provided. If vias cannot be used, then a wider bar, inserted right after the pads is preferred as it introduces less impedance.

**De-embedding the Parasitics**

Now that we have determined the parasitics, the next step is to subtract their values from the raw measurement. The method is a simple two-step procedure and is the same for inductors and capacitors.
Step 1 de-embeds the parasitic pad coupling. Its calculation is given as

\[ Y_A = Y_{\text{meas}} - \begin{bmatrix} Y_{p1} & 0 \\ 0 & Y_{p2} \end{bmatrix} \rightarrow Z_A \quad (6.1) \]

Step 2 de-embeds the DUT feed-lines and gives the final de-embedded measurement, \( Z_{\text{DUT}} \). Its calculation is given as

\[ Z_B = Z_A - \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} \rightarrow Z_B = Z_{\text{DUT}} \quad (6.12) \]

Note that for the capacitor, a device that generally has small losses, the additional losses in \( Z_1/Z_2 \) cause an error in the de-embedding. For this reason, we only subtract the imaginary part, the reactance, of \( Z_1 \) and \( Z_2 \) in eq.(6.12). The results for de-embedding the capacitor are given below in Figure 6.2 using the open/short method. The effective capacitance and capacitor \( Q \) are shown first. We compare our de-embedded measurement (in blue) with the values generated by the design kit (in black).
As mentioned previously, the loss de-embedding is very difficult interdigitated capacitors due to the inherently low loss. Any error in the loss is magnified due to its
small value. There is a slight improvement in $Q$ visible in Figure 6.14, but it is small. The de-embedded S-parameters are given below. Good agreement between the de-embedded measurement and the design kit is obtained.

![S11 & S12](image1)

![S22](image2)

Figure 6.15 S-parameters – capacitor open/short

Using the open/thru method for the capacitor provides slightly better results for the $Q$ due to the lack of error introduced by the return path of the short. However, the effective capacitance is not as good as with the open/short, and the S-parameters are basically the same.
Figure 6.16 Effective capacitance – capacitor open/thru

Figure 6.17 Capacitor Q – capacitor open/thru
Figure 6.18 S-parameters – capacitor open/thru

Our de-embedding of the inductor measured data did not turn out as well. This is due to the error in the layout of the short test structure, as well as the shift in reference plane due to the location of the short. For this type of inductor, there is little that can be done about the shift in reference plane due to the lack of a conductive ground plane beneath the inductor to form a short. The results are given below.

Figure 6.19 Effective inductance - inductor open/short

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**Improved Inductor Test Structures for Open/Short Technique**

A suggested set of test structures for better inductor de-embedding will now be given. The only change is in the short test structures. As mentioned in previously, the vias provide a better short because a current loop is avoided, thus introducing a smaller
impedance. If the technology does not allow vias as in Figure 6.24, then a line across the ground-signal-ground test pads can be used. As mentioned previously, the line should be as wide as possible to introduce the smallest impedance possible. Like the test structures used previously in this chapter, the reference plane will still be shifted because of the deep trench ground plane, however the ideal values we will compare to will be arrived at from an ADS RF Momentum simulation that also has its reference plane shifted. This could have been done with the ideal values given above using the design kit if we used a model for a length of transmission line equal to the shift in reference plane. The modified test structures are given below in Figure 6.22 - Figure 6.24.

![Figure 6.22 Improved inductor test structure](image)
Figure 6.23 Improved inductor open

Figure 6.24 Improved inductor short
Applying the de-embedding procedure, we get very good simulated results. The effective inductance is better compared to that shown previously, but this is only due to the shift in reference plane being accounted for in the comparison. A large improvement in de-embedding the loss is achieved. This is because the vias in the short test structure of Figure 6.24 introduce much less impedance than the short method used in Figure 6.11.

![Effective Inductance](image)

Figure 6.25 Effective inductance - improved inductor open/short
Figure 6.26 Inductor Q- improved inductor open/short

Figure 6.27 S-parameters - improved inductor open/short

Another option for accurate de-embedding is the use of coplanar waveguide tapers from the pads to the DUT interconnects. This results in the parasitic impedance of the device feed-lines being negligible, making only the use of open de-embedding necessary. Some processes do not allow traces at arbitrary angles, making this option
process-dependent. The de-embedding of the measurement for inductor #1 (defined in Table 7.1 in Chapter 7) is based upon this type of de-embedding.
CHAPTER 7

COMPARISON WITH MOMENTUM AND
MEASURED DATA

Inductor Results

A set of five spiral inductors was designed, with the 1.5 turn design being done by Idstein [11], with turns varying from 1.5 to 5.5 turns as described by Table 7.1. The substrate used is based on a 0.18 um FDSOI CMOS technology, a process utilizing a high resistivity (2000 Ω-cm.) bulk silicon substrate. A description of the layer stack and its properties is given in Table 7.2 and Table 7.3.

<table>
<thead>
<tr>
<th>Inductor #</th>
<th>Inductance (nH)</th>
<th>Number of turns</th>
<th>Outer diameter (um)</th>
<th>Trace width (um)</th>
<th>Trace spacing (um)</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1.5</td>
<td>248</td>
<td>28</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
<td>2.5</td>
<td>248</td>
<td>14</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>1.6</td>
<td>3.5</td>
<td>146</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>4.5</td>
<td>190</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>4.9</td>
<td>5.5</td>
<td>210</td>
<td>10</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 7.1 Inductor geometry summary
<table>
<thead>
<tr>
<th>Material</th>
<th>Thickness (um)</th>
<th>Dielectric Constant</th>
<th>Conductivity (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oxide</td>
<td>7.75</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Silicon</td>
<td>675</td>
<td>11.7</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 7.2 Substrate dielectric properties

<table>
<thead>
<tr>
<th>Metal Layer</th>
<th>Thickness (um)</th>
<th>Distance from metal center to top(um)</th>
<th>Conductivity (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Metal 1</td>
<td>.6</td>
<td>6.15</td>
<td>2.08E+07</td>
</tr>
<tr>
<td>Metal 2</td>
<td>.6</td>
<td>4.55</td>
<td>2.08E+07</td>
</tr>
<tr>
<td>Metal 3</td>
<td>2</td>
<td>1.75</td>
<td>2.64E+07</td>
</tr>
</tbody>
</table>

Table 7.3 Substrate metal properties

Using the concepts discussed in this thesis, the component values of the lumped pi-circuit of Figure 4.2 were calculated. The lumped pi-circuit was then simulated using the Agilent ADS schematic tool. The ADS schematic showing the pi-circuit and its component values for inductor #1 is given in Figure 7.1. The box in the middle of the schematic titled “SNP1” contain the 2-port S-parameters of the frequency dependent impedance. The 2-port S-Parameters were computed from 100 MHz to the first resonant frequency of each inductor. The inductors were also simulated using ADS RF Momentum, which is a Method of Moments (MoM) EM simulator. Measured data was obtained for inductor #1, the 1.5 turn spiral. The layout of the test structure is illustrated by Figure 7.2. The pads are arranged in a ground-signal-ground configuration and 2-port S-Parameter measurements from 1-20 GHz were performed using on-chip cascade probes.
**Figure 7.1** ADS schematic for inductor #1

- **Figure 7.2** 1.5 turn inductor (a) and open (b) test structure

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The measured data was de-embedded by subtracting the $Y$-Parameters ($Y_{11} + Y_{12}$, $Y_{22} + Y_{12}$) of the de-embed open (Figure 7.2b) from the measured $Y$-Parameters. This effectively removes the parasitic pad capacitance from the measurements. The open/short method described in Chapter 6 was not necessary because the taper in the interconnect between the test pad and DUT was designed to match the impedance. Also, the loss introduced by the taper is minimal and can be ignored. For the other inductors, the 2-port $S$-Parameters generated from the Momentum simulation were used as the benchmark to compare our model to.

The differential impedance, which describes the series impedance connecting ports 1 and 2, is defined using the $Z$-Parameters of the 2-port inductor.

\[ Z_{\text{diff}} = Z_{11} + Z_{22} - Z_{12} - Z_{21} \]  

\( (7.1) \)

Using the differential impedance, the effective inductance of the inductor can be computed at frequency $f$ as

\[ L_{\text{eff}} = \frac{\text{imag}(Z_{\text{diff}})}{2\pi f}. \]  

\( (7.2) \)

The inductor $Q$ can also be computed using the differential impedance as

\[ Q = \frac{\text{imag}(Z_{\text{diff}})}{\text{real}(Z_{\text{diff}})}. \]  

\( (7.3) \)

The values for $Q$ and $L_{\text{eff}}$, as well as the 2-port $S$-Parameters generated by Momentum and our model were then compared. Again, we were able to compare to measured data for inductor #1. As can be seen from Figure 7.3 - Figure 7.7, reasonable agreement between Momentum and our model is achieved. Additionally, our model agrees well with the de-embedded measured data in predicting the frequency and value of
the peak $Q$, as well as accurately modeling effective inductance and S-parameters, as can be seen in Figure 7.3.
Figure 7.4 Simulated results inductor #2

Figure 7.5 Simulated results inductor #3
Figure 7.6 Simulated results inductor #4

Figure 7.7 Simulated results inductor #5
The biggest discrepancy observed is in the prediction of $Q$ at higher frequencies. It is typical for $Q$ to vary slightly from one model to the next, as the methods of capturing the ohmic loss in the metal are usually different. The most important aspect of $Q$ is the magnitude and frequency of its maximum, since this is typically in the frequency range that the device will be used. In comparing our model with Momentum, the peak $Q$ is reasonably close, and the peak $Q$ frequencies are nearly identical which leads to a high degree of confidence in both models. Slight deviations in the effective inductance values are most likely due to approximations made in the modeling of internal inductance. The slight over-estimation in the resonant frequency is most likely due to the method in determining the interwinding capacitance, which neglects some of the turn-to-turn capacitances and relies only on the bridge overlap capacitance for its value.
Capacitor Results

Interdigitated Capacitors

To verify the accuracy of the capacitor model, five interdigitated capacitors, as well as one fractal capacitor, were designed and simulated with our model and ADS RF Momentum. The substrate used is based on the same 0.18 um FDSOI CMOS technology as the inductors. The fractal capacitor layout is given in Figure 7.16. A summary of the five interdigitated capacitors, using the parameters indicated in the layout in Figure 7.9 is given below in Table 7.4. The ADS schematic showing the lumped pi-model and its component values for capacitor #1 is given in Figure 7.10.
<table>
<thead>
<tr>
<th>Capacitor #</th>
<th>Number of Fingers</th>
<th>g(μm)</th>
<th>d(μm)</th>
<th>L1(μm)</th>
<th>L2(μm)</th>
<th>W(μm)</th>
<th>Lin(μm)</th>
<th>Win(μm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>8</td>
<td>50</td>
<td>258</td>
<td>272</td>
<td>20</td>
<td>15</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>8</td>
<td>50</td>
<td>258</td>
<td>348</td>
<td>20</td>
<td>15</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>8</td>
<td>50</td>
<td>258</td>
<td>496</td>
<td>20</td>
<td>15</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>10</td>
<td>50</td>
<td>258</td>
<td>410</td>
<td>20</td>
<td>15</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>12</td>
<td>50</td>
<td>258</td>
<td>384</td>
<td>20</td>
<td>15</td>
<td>40</td>
</tr>
</tbody>
</table>

Table 7.4 Capacitor geometry summary

Figure 7.10 ADS schematic for capacitor #1

Whereas the differential impedance is used for inductors, the differential admittance $Y_{diff}$ is used for capacitors. The differential admittance, which describes the series admittance connecting ports 1 and 2, is defined using the Y-Parameters of the 2-port capacitor.

$$Y_{diff} = Y_{11} + Y_{22} - Y_{12} - Y_{21}$$

Using the differential admittance, the effective capacitance of the capacitor can be computed at frequency $f$ as
The capacitor $Q$ can also be computed using the differential admittance as

$$Q = \frac{\text{imag}(Y_{\text{diff}})}{\text{real}(Y_{\text{diff}})}. \quad (7.6)$$

As will be demonstrated, the loss inherent to the capacitors is very small. Thus, an error in calculating this value translates to a large error in $Q$. For this reason, we will compare the 2-port resistance, or loss, rather than the $Q$ found from ADS RF Momentum and our model. The 2-port loss is defined as

$$R_{2-\text{port}} = \text{Re}\left\{ \frac{1}{Y_{\text{diff}}} \right\} = \text{Re}\{Z_{\text{diff}}\}. \quad (7.7)$$

We proceed with a comparison of the 2-port S-parameters, the 2-port resistance, and the effective capacitance found with ADS RF Momentum and our model. A trend appears in accuracy versus the number of fingers and the spacing between fingers. As the number of fingers increases, the accuracy also increases. This is most likely due to the smaller parasitic inductance due to the increase in width of the capacitor. The larger the length $L_2$ is relative to length $L_1$, the smaller the parasitic inductance will be. This parasitic inductance is difficult to predict exactly. This trend is demonstrated by the results in Figure 7.11 - Figure 7.13 as the number of fingers is increased for the same spacing in capacitors #1, #2, and #3.
Figure 7.11 Simulated results for capacitor #1

Figure 7.12 Simulated results for capacitor #2
Although all the results are very good for all three capacitors, it is apparent that the accuracy gets better as the number of capacitor fingers increases. For capacitor #3, with 18 fingers, the effective capacitance is almost identical for ADS RF Momentum and our model. A similar trend will be demonstrated showing an increase in accuracy as the spacing between the fingers increases. This increase in accuracy with an increase in finger spacing is similar to the increase in accuracy as the number of fingers increases. The parasitic inductance will be smaller as the spacing increases, and as mentioned previously, the largest inaccuracy with this model is with the calculation of the parasitic inductance due to the difficulty in calculating it exactly. The results for capacitors #4 and #5 are given in Figure 7.14 and Figure 7.15 below. If we compare capacitor #2 in Figure 7.12 to capacitors #4 and #5 below, we see the increase in accuracy as the spacing
increases. The results from Momentum and our model in Figure 7.15 agree almost exactly, and that is the capacitor with the largest finger spacing.

Figure 7.14 Simulated results for capacitor #4
Fractal Capacitors

Outside of numerical EM solvers such as ADS RF Momentum, there are few, if any, techniques in the literature available to model the complicated geometry of a fractal. The current state of the art cannot manufacture reliably a true fractal, but quasi-fractal geometries such as the one shown in Figure 7.16, based on the Hilbert-curve after two iterations, can still exhibit some of the fractal’s useful properties.
One of these properties is the randomization of the current direction, leading to a smaller parasitic series inductance relative to standard interdigitated capacitors. Another advantage of fractal capacitors, according to [17], is that fractals have a large number of rough edges that can accumulate electrostatic charge more efficiently compared to interdigitated capacitors, leading to a high capacitance density. Next we will demonstrate the capability to accurately model the quasi-fractal capacitor shown in Figure 7.16.

The results below in Figure 7.17 show a comparison of the effective capacitance, 2-port resistance, and S-parameters obtained with ADS RF Momentum and our model. Excellent agreement up to first resonance is observed. In comparison to most of the results for the interdigitated capacitors, the accuracy is better. This could be related to the smaller parasitic series inductance. This agrees with the earlier finding that as the
number of fingers and/or finger spacing increased, and hence a smaller series inductance, 
the accuracy also increased.

Figure 7.17 Simulated results for the quasi-fractal capacitor
CHAPTER 8

CONCLUSION

In this paper, the technique proposed in [11]-[13] for modeling spiral inductors
with a simple lumped pi-circuit was improved upon and applied to interdigitated and
fractal capacitors as well. The modeling of a wide range capacitor geometries and
inductors with different number of turns was done with good accuracy. In addition, the
bandwidth of this model goes up to the first resonant frequency, a significant
improvement in the bandwidth achieved previously.

As in [11]-[13], the PEEC technique was used to construct the distributed circuit,
upon which the lumped pi-circuits were based. For the inductor, a low frequency
approximation similar to that in [11]-[13] was applied to the distributed circuit to arrive at
values for the substrate impedance and series impedance of the inductor. The method in
this work differs slightly in that it accounts for the lossless oxide capacitance. At higher
frequencies, a different method was used to determine the interwinding capacitance. This
method has also been used in [15] and [16], but we justify its application by inspecting
the voltage along the inductor near resonance. As previously mentioned, the
interwinding capacitance must be calculated accurately in order to correctly predict the resonant frequency.

The approximations made to the capacitor distributed circuit involved a high frequency approximation to determine the complex nominal capacitance $C'_0$, as well as the values for the substrate impedance. At high frequencies, we assume the rectangles comprising the same side of the capacitor are at the same voltage, essentially shorting out the coupling between them and allowing the substrate admittance of each rectangle to be added in parallel. A low frequency approximation is used to determine the series impedance resulting from the magnetic coupling and ohmic loss within the capacitor metal.

A rigorous method for de-embedding RF inductors and capacitors was presented. The typical parasitic impedances present in an RF test structure were discussed. Basic knowledge of these parasitics allows the designer to effectively adjust the test structures to account for any limitations imposed by the technology being used. Some technologies allow traces to be etched at arbitrary angles, making it possible for a coplanar waveguide interconnect to be used, thus minimizing the interconnect impedance.

In calculating the partial capacitance matrix, [11]-[13] used numerical integration which proved to be very time-consuming. A method employing FFTs as discussed by [5] was used in this work to speed up the calculation of [C].

As demonstrated in Chapter 7, good agreement in $Q$, $L_{eff}$, and 2-port S-parameters up to the first resonance is obtained with the inductor model presented in this work and ADS RF Momentum. We also verify the accuracy of the model by comparing to the de-
embedded measurements for a 1.5 turn 1 nH inductor. For the capacitor model, we compare the 2-port resistance, $C_{eff}$, and 2-port S-parameters predicted by our model to the values predicted by ADS RF Momentum, and excellent agreement is achieved. We also demonstrated the ability to accurately model a complex quasi-fractal capacitor.

A big improvement that could be made to this model would be to account for eddy current losses in the substrate when calculating the partial inductance matrix $[M]$. To accomplish this, a multi-layer $G_m$ would need to be calculated, as mentioned in Chapter 2. The technique outlined in [23], using image theory, could be used as a guideline for calculating $G_m$. One potential problem with using a $G_m$ that accounts for loss is that the matrix $[M]$ will become frequency dependent, which leads to a more complicated model, which deviates from the goal of this work.

Another improvement to the inductor model would be to allow for more general geometries, such as octagonal spirals or circular spirals. As the shape of a spiral inductor approaches circular, the efficiency in terms of inductance per-unit-area increases, leading to more efficient use of real-estate in the integrated circuit design.

The model discussed in this thesis is based on rigorously derived equations that are true to the physics of the device. The S-parameter data generated in the ADS schematic can be incorporated into other circuit simulators such as Cadence and used in the design of complex integrated circuits such as LNAs, matching networks, LC tanks for oscillators, etc. Since this model can be run in a very short period of time, the design cycle time can be greatly reduced compared to the time involved with other numerical EM solvers.
APPENDIX A

A GENERAL APPROACH TO SOLVING THE
ELECTROSTATIC GREEN’S FUNCTION FOR A
MULTILAYERED SUBSTRATE

![Diagram of L-layered substrate]

Figure A.1 Profile of L-layered substrate

With reference to [5], [10], and [22], we consider the derivation of the multilayer
electrostatic Green’s function, $G_e$, for the L-layered structure in Figure A.1. The
dimensions of the structure shown in Figure A.1 are $a$ in the x-direction and $b$ in the y-direction. $G_e(r;r')$ is found by inverting Poisson’s equation and gives the potential at the observation point $r$ due to a point charge source placed at a point $r'$. In our case, all metal layers are in the top layer, which means both the source and observation points will be in the top layer. However, for generality we will perform the derivation assuming the source location can be in any layer.

Poisson’s equation in eq.(A.1) can be written in the Green’s function form of eq.(A.2).

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon}$$  \hspace{1cm} (A.1)

$$\nabla^2 G_e(r;r') = -\frac{\delta(x-x')\delta(y-y')\delta(z-z')}{\varepsilon_i}$$  \hspace{1cm} (A.2)

where the subscript $i$ indicates the source layer. The form of $G_e$ is

$$G_e = X(x;x')Y(y;y')Z'(z;z').$$  \hspace{1cm} (A.3)

We substitute $X$, $Y$, and $Z'$ into eq.(A.2) which gives

$$YZ' \frac{d^2}{dx^2} X + Z'X \frac{d^2}{dy^2} Y + XY \frac{d^2}{dz^2} Z' = -\frac{\delta(x-x')\delta(y-y')\delta(z-z')}{\varepsilon_i}$$  \hspace{1cm} (A.4)

The boundary condition on the side-walls of the structure in Figure A.1 is $\hat{n} \cdot \vec{E} = 0$

(perfect magnetic walls). To satisfy this, we let $X = \cos\left(\frac{m\pi}{a} x\right)$ and $Y = \cos\left(\frac{n\pi}{b} y\right)$.

Plugging in the functions for $X$ and $Y$ we re-write eq.(A.4) as
\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) \times \left\{ \frac{d^2}{dz^2} Z' - \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] Z' \right\} = \frac{\delta(x-x')\delta(y-y')\delta(z-z')}{\varepsilon_i} \tag{A.5}
\]

We multiply both sides by \( \cos \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) \) and integrate both sides over \( x \) and \( y \) from 0 to \( a \) and 0 to \( b \), respectively, and this gives

\[
\frac{ab}{4} \left\{ \frac{d^2}{dz^2} Z' - \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] Z' \right\} = -\frac{\delta(z-z')}{\varepsilon_i} \cos \left( \frac{m\pi}{a} x' \right) \cos \left( \frac{n\pi}{b} y' \right) \tag{A.6}
\]

We substitute \( Z' = Z_{mn}(z;z') \cos \left( \frac{m\pi}{a} x' \right) \cos \left( \frac{n\pi}{b} y' \right) \) into eq.(A.6) to simplify to

\[
\frac{ab}{4} \left( \frac{d^2}{dz^2} - t_{mn}^2 \right) Z_{mn}(z;z') = -\frac{\delta(z-z')}{\varepsilon_i} \tag{A.7}
\]

where \( t_{mn}^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \). From eq.(A.7) we can solve for \( Z_{mn}(z;z') \) using the method of solving a one-dimensional Green’s functions outlined in [22]. The method is summarized below.

A solution to eq.(A.7) valid for \( z \neq z' \) has the form

\[
Z_{mn}(z;z') = C_e e^{i\omega z} + D_e e^{-i\omega z} \tag{A.8}
\]

in layer \( k \). A recursive relation is obtained between the coefficients in adjacent layers using the boundary conditions. At the boundary between each interface, the voltage and \( \hat{n} \times \vec{D} \) (where \( \hat{n} \) is normal to the interface) is continuous. In addition, the voltage at the bottom is zero due to the PEC ground plane. At the top, we have zero normal \( \vec{E} \) field due
to the magnetic wall condition. This recursive relation can be represented by a matrix 

\[
\begin{bmatrix}
C_{k+1} \\
D_{k+1}
\end{bmatrix} = [A] \begin{bmatrix}
C_k \\
D_k
\end{bmatrix}
\]  \hspace{1cm} (A.9)

where \([A]_k\) can be found from the boundary conditions as

\[
[A]_k = \begin{bmatrix}
\frac{1}{2} \left(1 + \frac{\varepsilon_k}{\varepsilon_{k+1}}\right) & \frac{1}{2} \left(1 - \frac{\varepsilon_k}{\varepsilon_{k+1}}\right) e^{-2l_{\text{wd}k}} \\
\frac{1}{2} \left(1 - \frac{\varepsilon_k}{\varepsilon_{k+1}}\right) e^{2l_{\text{wd}k}} & \frac{1}{2} \left(1 + \frac{\varepsilon_k}{\varepsilon_{k+1}}\right)
\end{bmatrix} \hspace{1cm} (A.10)
\]

Next, we introduce the self-adjoint operator \(L\).

\[
L = \frac{d}{dz} \left[ p(z) \frac{d}{dx} \right] - q(x) \hspace{1cm} (A.11)
\]

We want a Green’s function that satisfies an equation of the form

\[
[L + \lambda w(z)] Z_{mn}(z; z') = -\delta(z - z') \hspace{1cm} (A.12)
\]

together with the boundary conditions

\[
\alpha_{\alpha, \beta} p(z) Z'_{mn}(z; z') + \beta_{\alpha, \beta} Z_{mn}(z; z') \bigg|_{z=\alpha, \beta} = 0, \hspace{1cm} (A.13)
\]

where \(L\) is the self-adjoint operator of eq. (A.11). The solution to eq. (A.12) is divided into two intervals.

\[
Z_{mn}(z; z') = \begin{cases} 
\frac{U(z) T(z')}{C} & z \leq z' \\
\frac{U(z') T(z)}{C} & z \geq z'
\end{cases} \hspace{1cm} (A.14)
\]

where \(C\) is a constant and \(U\) and \(T\) are two independent solutions of

\[
[L + \lambda w(z)] \begin{bmatrix} U(z) \\
T(z) \end{bmatrix} = 0 \hspace{1cm} (A.15)
\]
with
\[ \alpha_u p(z)U'(z) + \beta_o U(z) = 0 \bigg|_{z=a} \]
\[ \alpha_s p(z)T'(z) + \beta_s T(z) = 0 \bigg|_{z=b} \]  
(A.16)

and
\[ C = p(z') [T(z')U'(z') - T'(z')U(z')] \]  
(A.17)

Eq.(A.16) is necessary to ensure that eq.(A.13) is satisfied. To use this method for eq.(A.7), we re-write it as
\[ \frac{a b \varepsilon_i}{\Delta_{mn}} \left( \frac{d^2}{dz^2} - t_{mn}^2 \right) Z_{mn}(z; z') = -\delta(z - z') \]  
(A.18)

where
\[ \Delta_{mn} = \begin{cases} 
4 & m, n \neq 0 \\
2 & (m = 0, n \neq 0) \text{ or } (m \neq 0, n = 0) \\
1 & m = n = 0 
\end{cases} \]  
(A.19)

To put eq.(A.18) into the same form as eq.(A.12), we recognize that \( p(z) = \frac{a b \varepsilon_i}{\Delta_{nn}} \), \( w(z) = 1 \), \( q(z) = 0 \), and \( \lambda = -t_{nn}^2 \). In addition, \( \alpha_o = 0 \), \( \beta_o = 1 \), \( \beta_h = 0 \), and \( \alpha_h = \frac{\Delta_{nn}}{a b \varepsilon_i} \) to satisfy the boundary conditions of eq.(A.16).

We proceed with solving for \( T(z) \) and \( U(z') \) for \( z \geq z' \). Using eq.(A.8) in matrix form, we solve for \( T(z) \) as
\[ T(z) = \begin{bmatrix} e^{t_{mn}z} & e^{-t_{mn}z} \end{bmatrix} \begin{bmatrix} C_k \\ D_k \end{bmatrix} \]  
(A.20)
where \( k \geq i \). We can find \( C_k \) and \( D_k \) using the boundary condition at \( z = d_L \) in conjunction with the recursive relation, applied from the top down towards the \( k^{th} \) layer. From the boundary condition at \( d_L \) we find

\[
D_L = C_L e^{2\nu_w d_L}. \tag{A.21}
\]

We can set \( D_L \) to 1 since it will drop out in a ratio with \( D_L \) later on. This allows us to find

\[
\begin{bmatrix}
C_k \\
D_k
\end{bmatrix}
\]

as

\[
\begin{bmatrix}
C_k \\
D_k
\end{bmatrix} = [A_k]^\dagger [A_{k+1}]^\dagger \ldots [A_{L-2}]^\dagger [A_{L-1}]^\dagger \begin{bmatrix}
1 \\
e^{2\nu_w d_L}
\end{bmatrix} C_L, \tag{A.22}
\]

We plug the result for eq.(A.22) into eq.(A.20) to find \( T(z) \). Next we need to find \( U(z') \) for \( z \geq z' \).

\[
U(z') = \begin{bmatrix}
e^{i\nu_w z'} & e^{-i\nu_w z'}
\end{bmatrix} \begin{bmatrix}
C'_i \\
D'_i
\end{bmatrix} \tag{A.23}
\]

The superscript \( l \) in the coefficients \( C \) and \( D \) indicates that the source location \( z' \) is below the observation point \( z \). The coefficients \( C'_i \) and \( D'_i \) for \( U(z') \) are found in a manner similar to how the coefficients \( C_k \) and \( D_k \) were found for \( T(z) \) except we start from the bottom and apply the recursive relation towards the top. From the boundary condition at \( z = 0 \) where we have a ground plane, we can relate the coefficients \( C_l \) and \( D_l \) by plugging \( z = 0 \) into eq.(A.8) and setting \( Z_{nn}(0, z') = 0 \). From this we get

\[
C_l = -D_l. \tag{A.24}
\]

We can set \( D_l = 1 \) for the same reason we can set \( D_L = 1 \) previously. This gives us
\[
\begin{bmatrix}
C'_{i'}

D'_{i'}
\end{bmatrix}
= [A_{i-1}] [A_{i-2}] \ldots [A_1] \begin{bmatrix}
1

-1
\end{bmatrix} C_i.
\]

(A.25)

The values for the coefficients found in eq.(A.25) are then plugged into eq.(A.23).

Up to this point, half of the solution to eq.(A.14) has been found. The next step is to solve for \( T(z') \) and \( U(z) \) for the case \( z \leq z' \). For this case, layer \( k \leq \) layer \( i \). \( U(z) \) is found as

\[
U(z) = \begin{bmatrix}
e^{i\omega z'}

-e^{-i\omega z'}
\end{bmatrix} \begin{bmatrix}
C_k

D_k
\end{bmatrix}.
\]

(A.26)

The coefficients \( C_k \) and \( D_k \) are found using the recursive relation applied from the bottom layer to the \( k^{th} \) layer, with the aid of the boundary condition \( Z_{nm}(0; z') = 0 \).

\[
\begin{bmatrix}
C_k

D_k
\end{bmatrix}
= [A_{k-1}] [A_{k-2}] \ldots [A_1] [A_i] \begin{bmatrix}
1

-1
\end{bmatrix} C_i
\]

(A.27)

Again, we plug the result of eq.(A.27) into eq.(A.26). \( T(z') \) is found for \( z \leq z' \) as

\[
T(z') = \begin{bmatrix}
e^{i\omega z'}

-e^{-i\omega z'}
\end{bmatrix} \begin{bmatrix}
C''_{i'}

D''_{i'}
\end{bmatrix}
\]

(A.28)

The superscript \( u \) in the coefficients \( C \) and \( D \) indicates that the source location \( z' \) is above the observation point \( z \). The coefficients \( C''_{i'} \) and \( D''_{i'} \) are found by applying the recursive relation with \([A]\), starting again from the \( z = d_i \), in conjunction with the boundary condition at \( z = d_i \) and working downwards towards layer \( i \). This gives

\[
\begin{bmatrix}
C''_{i'}

D''_{i'}
\end{bmatrix}
= [A_i^{-1}] [A_{i+1}]^{-1} \ldots [A_{i-2}]^{-1} [A_{i-1}]^{-1} \begin{bmatrix}
1

e^{2i\omega d_i}
\end{bmatrix} C_{i'}.
\]

(A.29)

Again, we plug the result of eq.(A.29) into eq.(A.28).
An expression for $C$ in eq.(A.14) is found using eq.(A.18). However, the expression found for $C$ would be broke into two intervals, one for $z \leq z'$ and one for $z \geq z'$. We introduce a variable $\tilde{C}$, as well as coefficients $C_\times, D_\times, C_\circ, D_\circ$ to allow us to write $Z_{mn}(z;z')$ in one expression.

$$Z_{mn}(z;z') = \frac{1}{\tilde{C}} \begin{bmatrix} e^{l_m z_0} & e^{-l_m z_0} \\ e^{l_n z_0} & e^{-l_n z_0} \end{bmatrix} \begin{bmatrix} C_\times \\ D_\times \end{bmatrix} \begin{bmatrix} e^{l_m z_0} & e^{-l_m z_0} \\ e^{l_n z_0} & e^{-l_n z_0} \end{bmatrix} \begin{bmatrix} C_\circ \\ D_\circ \end{bmatrix} = G_C(z;z')$$

(A.30)

where $z_\times$ clearly means the use of $z$ if $z \leq z'$ and the use of $z'$ if $z > z'$. We also have

$$C_\times = \begin{cases} C_k & k \leq i, z \leq z' \\ C_i & k \geq i, z \geq z' \end{cases}$$

(A.31)

$$D_\times = \begin{cases} D_k & k \leq i, z \leq z' \\ D_i & k \geq i, z \geq z' \end{cases}$$

(A.32)

$$C_\circ = \begin{cases} C_k & k \geq i, z \geq z' \\ C_i & k \leq i, z \leq z' \end{cases}$$

(A.33)

$$D_\circ = \begin{cases} D_k & k \geq i, z \geq z' \\ D_i & k \leq i, z \leq z' \end{cases}$$

(A.34)

The variable $\tilde{C}$ is defined as

$$\tilde{C} = \frac{2a_k b_k}{\Delta_{mn}} t_{mn} \begin{bmatrix} 1 & -1 \end{bmatrix} [A]_0^T [A]_2^T \ldots [A]_{-2}^T [A]_{-1}^T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [A]_1^T [A]_{-1}^T \ldots$$

(A.35)

$$\begin{bmatrix} 1 & e^{l_n z_0} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [A]_{-1}^T [A]_{-1}^T \ldots$$

We tested this method by applying it to the substrate under consideration in this paper. For this, we set $i=j=2$ and $L=2$ since we have a two-layer substrate with both metal layers (source and observation points) in the top layer. Without getting into the details, we find $Z_{mn}(z;z')$ as follows. For $z \leq z'$,
\[ Z_{mn}(z; z') = \frac{(C^l_2e^{r_{mn}} + D^l_2e^{-r_{mn}})(e^{r_{mn}} + e^{2i\omega t_{ij}}e^{-r_{mn}})}{2ab\epsilon_i^t \Delta_{mn} (C_2^l e^{2i\omega t_{ij}} - D_2^l)}. \]  \hspace{1cm} (A.36)

Eq. (A.36) matches the result found in [10] found for the case \( z \leq z' \). For \( z \geq z' \),

\[ Z_{mn}(z; z') = \frac{(C^l_2e^{r_{mn}} + D^l_2e^{-r_{mn}})(e^{r_{mn}} + e^{2i\omega t_{ij}}e^{-r_{mn}})}{2ab\epsilon_i^t \Delta_{mn} (C_2^l e^{2i\omega t_{ij}} - D_2^l)}. \]  \hspace{1cm} (A.37)

This also matches the result found in [10] for the case \( z \geq z' \).

The above derivation works for all values of \( m \) and \( n \) except for \( m = n = 0 \). We will study that case in detail in Appendix B.
APPENDIX B

FAST COMPUTATION OF THE COEFFICIENT OF

POTENTIAL MATRIX

The following summary of the method for obtaining the coefficient of potential matrix is done with reference to [5] and [9].

Recall the coefficient of potential matrix has elements $p_{ij}$, which gives the potential $V_i$ on rectangle $i$ due to a charge $Q_j$ placed on rectangle $j$, is defined as

$$p_{ij} = \frac{1}{v_i v_j} \int_{v_i} \int_{v_j} G_{ij} dv_i dv_j = \frac{V_i}{Q_j}.$$  \hspace{1cm} (B.1)

Because the thickness is very small relative to the surface area, we ignore the thickness of rectangles $i$ and $j$ and integrate over the surface, which gives

$$p_{ij} = \frac{1}{s_i s_j} \int_{s_i} \int_{s_j} G_{ij} ds_i ds_j.$$  \hspace{1cm} (B.2)

From Appendix A we know that $G_{ij}$ is of the form $G_{ij} = X(x; x')Y(y; y')Z(z; z')$.

Plugging in $X = \cos\left(\frac{m\pi}{a} x\right)$, $Y = \cos\left(\frac{n\pi}{b} y\right)$ and $Z = Z(z; z')\cos\left(\frac{m\pi}{a} x'\right)\cos\left(\frac{n\pi}{b} y'\right)$,

we can re-write $G_{ij}$ as

$$G_{ij} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \cos\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) \cos\left(\frac{m\pi}{a} x'\right) \cos\left(\frac{n\pi}{b} y'\right) \right\} Z_{mn}(z; z').$$  \hspace{1cm} (B.3)
where $Z_{mn}(z;z')$ is found from eq.(A.30). The set of rectangles $i$ and $j$ and their coordinates is given below.

![Coordinates for rectangles i and j](image)

Figure B.1 Coordinates for rectangles $i$ and $j$ (based on [5])

Performing the integration from eq.(B.2) on the set of contacts for rectangles $i$ and $j$ shown in Figure B.1, ignoring the $m = n = 0$ case, gives

$$
p_{ij} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin \left( m \pi \frac{a_2}{a} \right) - \sin \left( m \pi \frac{a_1}{a} \right)}{a_3 - a_1} \frac{\sin \left( m \pi \frac{a_4}{a} \right) - \sin \left( m \pi \frac{a_5}{a} \right)}{a_4 - a_3} \times \frac{\sin \left( n \pi \frac{b_2}{b} \right) - \sin \left( n \pi \frac{b_1}{b} \right)}{b_3 - b_1} \frac{\sin \left( n \pi \frac{b_4}{b} \right) - \sin \left( n \pi \frac{b_3}{b} \right)}{b_4 - b_3} \times Z_{mn}
$$

(B.4)

The sinusoidal terms in eq.(B.4) are then multiplied out and we use the identity shown in eq.(B.5) to put eq.(B.4) into the form of a discrete cosine transform (DCT).
\[\sin\left( m\pi \frac{a_i}{a} \right) \sin\left( m\pi \frac{a_f}{a} \right) = \frac{1}{2} \left( \cos\left( m\pi \frac{a_i - a_f}{a} \right) - \cos\left( m\pi \frac{a_i + a_f}{a} \right) \right) \quad (B.5)\]

As in [5] and [9], this allows us to re-cast eq.(B.4) into a sum of 64 terms of the form
\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \tilde{Z}_{mn} \cos\left( m\pi \frac{a_{1,i} \pm a_{1,4}}{a} \right) \cos\left( n\pi \frac{b_{3,2} \pm b_{3,4}}{b} \right) \quad (B.6)
\]

This is of the same form as a 2-D DCT. By inspection of eq.(B.6), it is apparent that for the case \( m = 0, n \neq 0 \) or \( m \neq 0, n = 0 \), the form of eq.(B.6) becomes the form of a 1-D DCT.

Eq.(B.6) is not solvable due to the infinite upper bound on the DCT. To make eq.(B.6) solvable with a DCT, we need to discretize the substrate in the \( x \) and \( y \) directions and represent the coordinates of the contacts as ratios of integers as in [5] and [9].

\[
\frac{a_k}{a} = \frac{p_k}{P}, \quad \frac{b_k}{b} = \frac{Q_k}{Q} \quad (B.7)
\]

This allows eq.(B.6) to be re-written as
\[
\sum_{m=0}^{P-1} \sum_{n=0}^{Q-1} \tilde{Z}_{mn} \cos\left( m\pi \frac{p_{1,i} \pm p_{3,4}}{P} \right) \cos\left( n\pi \frac{q_{1,2} \pm q_{3,4}}{Q} \right). \quad (B.8)
\]

The variables \( a \) and \( b \) in eq.(B.7) are the \( x \) and \( y \) dimensions of the substrate. They do not need to be the exact value, just large enough for convergence. The inductors we studied are on the order of a few hundred microns in diameter and \( a=b=1024 \) um is adequate for convergence. The values of \( P \) and \( Q \) determine the number of iterations over which the infinite summations are made, or in other words, the size of the DCT.

The capacitors we studied are typically larger, some approaching 1000 um by 1000 um in size. This lead us to use \( a=b=2048 \) um and \( P=Q=1024 \) to ensure convergence.
Eq.(B.8) is not exactly a DCT because of the ± terms in the numerator which have an upper limit up to 2P and 2Q. To truly be a DCT, the indices in the numerator of the cosine arguments in the DCT must not be greater than the upper limits, P-1 and Q-1 in this case. The following properties of symmetry of the DCT allow us to get around this. We define $Z_{pq}$ as

$$Z_{pq} = \sum_{m=0}^{p-1} \sum_{n=0}^{q-1} \tilde{Z}_{mn} \cos \left( m \pi \frac{p}{P} \right) \cos \left( n \pi \frac{q}{Q} \right)$$

(B.9)

$$Z_{p,q} = Z_{2p-p,q} = Z_{p,2Q-q} = Z_{2p-p,2Q-q}$$

(B.10)

Eq.(B.9) is in the exact form of a P by Q DCT. Thus, if the terms $p_{1,2} \pm p_{3,4}$ and $q_{1,2} \pm q_{3,4}$ are greater than $P-1$ or $Q-1$, respectively, we can flip the values across $P$ and $Q$ using the identities in eq.(B.10).

For the case $m = n = 0$, the portion of $G_e$ that is a function of $x$ and $y$ drops out because it is equal to 1 for $m = n = 0$. Thus, eq.(B.2) becomes

$$p_{y}^{00} = \frac{1}{s_j s_j} \int \int_{s_j} Z_{00} ds_j ds_j.$$  

(B.11)

If we perform this integration, we find that $p_{y}^{00}$ is simply the Green’s function for $m = n = 0$ since the integration over the contacts is in $x$ and $y$ while $Z_{00}$ is a function of $z$ only.

$$p_{y}^{00} = G_e^{00}(z; z') = Z_{00}(z; z')$$

(B.12)

To find $Z_{00}$, we inspect eq.(A.18) with $t_{nn} = 0$. The solution to $Z_{00}$ is of the form

$$Z_{00} = K(C_r^{\prime} z + D_r^{\prime}).$$

(B.13)

where the superscript $u$ indicates $z' \geq z$ as in Appendix A. For $z' \leq z$, we have

$$Z_{00} = K(C_r^{\prime} z' + D_r^{\prime}).$$

(B.14)
As in [10], we find $K$ by integrating eq.(A.18) for $z = z'$ and applying the boundary condition that the potential be continuous. Doing this gives

$$K = \frac{1}{ab\varepsilon_i C_{i,j}},$$ \hspace{1cm} (B.15)

where $\varepsilon_i$ is the dielectric constant of the source layer. We choose the superscript $u$ or $l$ in eq.(B.15) depending upon the location of $z$ with respect to $z'$ as done previously.

To find the values for $C_{i,j}^u$ and $D_{i,j}^u$ we use a matrix $[A]$ as in eq.(A.9), but $[A]$ is defined differently for the case $m = n = 0$. If we apply the boundary conditions of continuous potential and continuous $\hat{n} \times \vec{D}$ at the interface, as well as $Z_{oo}(0; z') = 0$ due to the ground plane and $\hat{n} \cdot \vec{E} = 0$ at the top of the substrate, we find $[A]$ as

$$[A] = \begin{bmatrix}
\frac{\varepsilon_k}{\varepsilon_{k+1}} & 0 \\
1 - \frac{\varepsilon_k}{\varepsilon_{k+1}} & 1
\end{bmatrix}$$ \hspace{1cm} (B.16)

We are free to set $C_1 = 1$ and we must set $D_1 = 0$ because of the ground plane at $z = 0$.

Thus far, we have defined $p_g$ for all values of $m$ and $n$. To summarize the procedure from [5] and [9], the steps for fast computation of $[P]$ is given below.

1. Input the substrate data. This includes the thickness and dielectric constants of the dielectric layers of the substrate.

2. Compute $Z_{mn}$ as outlined in Appendix A and $Z_{oo}$ as outlined in Appendix B.

3. Compute the DCT of $Z_{mn}$ for all cases of $m$ and $n$ except for $m=n=0$. 

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4. Read in the $x$ and $y$ coordinates of the corners of the rectangles comprising the current geometry and compute the equivalent integer representation as given by eq.(B.7).

5. Compute eq.(B.4) using eq.(B.8). Recall that for $m=n=0$, use

$$p_{ij}^{00} = G_{ij}^{00}(z; z') = Z_{00}(z; z').$$

6. Generate $[P]$ by computing all possible combinations of rectangles. For example, if we have an inductor of 20 rectangles, $[P]$ will be $20 \times 20$.

7. Steps 1-3 only need to be done for a given substrate. If $[P]$ needs to be recomputed for a different geometry, go to step 4.
APPENDIX C

MUTUAL INDUCTANCE CALCULATION

The magnetostatic Green’s function, $G_m$, is used to calculate the partial inductance matrix $[M]$. We make the assumption that the dielectric substrate under consideration has low loss, which leads to, from eq.(2.3)

$$\nabla^2 \vec{A} = -\mu \vec{J}.$$  

(C.1)

By inverting eq.(C.1) and setting $\vec{J}$ to be a dirac delta function, we get $G_m$. Since $\mu = \mu_0$ for all layers, $G_m$ is

$$G_m = \frac{\mu_0}{4\pi R}.$$  

(C.2)

As discussed in Chapter 2, we find $M_{ij}$ from $G_m$ as

$$M_{ij} = \frac{\mu_0}{4\pi a_i a_j} \int \int \frac{d\vec{l}_i \bullet d\vec{l}_j}{|\vec{r}_i - \vec{r}_j|} da_i da_j.$$  

(C.3)

In this appendix, we will review the techniques of [2],[6], and [7] to solve eq.(C.3).

We begin with the diagonal terms of $[M]$, also known as the self inductance terms. For a rectangle of thickness $t$, width $W$, and length $L$, we get $M_{ii}$ to be

$$M_{ii}(nH) = 2 \times 10^{-4} \times L(\mu m) \times \left( \ln \left( \frac{2L(\mu m)}{W(\mu m) + t(\mu m)} \right) + 0.5 \right).$$  

(C.4)
The calculation of the off-diagonal terms of $[M]$, also called the mutual inductance terms, is more complicated. The calculation of the mutual inductance has two parts: the 4 dimensional integral and the dot product.

The dot product is quite simple. If the rectangles are oriented in such a way that the lengths are perpendicular, then $\vec{d}_i \cdot \vec{d}_j$ is zero. If the lengths are in parallel or in series, then $\vec{d}_i \cdot \vec{d}_j = 1$ for current in the same direction. The dot product $\vec{d}_i \cdot \vec{d}_j = -1$ if the current is in the opposite direction.

The calculation of the 4 dimensional integral is done in one of two ways, depending upon the relation of the center-to-center distance $D$ and length of the rectangles $L$. First, we treat the rectangles as filaments with no width and the same length $L$ and separated by a distance $d$. The solution to the integral of eq.(C.3) for this case is

$$M' = 2 \times 10^{-4} \times L \times \left[ \ln \left( \sqrt{1 + \left( \frac{L}{d} \right)^2 + \frac{L}{d}} \right) - \ln \left( \sqrt{1 + \left( \frac{d}{L} \right)^2 + \frac{d}{L}} \right) \right].$$

(C.5)

where $L$ and $d$ is in $\mu m$ and $M'$ is in $nH$. If $L >> d$, we can simplify eq.(C.5) to

$$M' = 2 \times 10^{-4} \times L \times \left( \frac{d}{L} - \ln(d) + \ln(2L) - 1 \right).$$

(C.6)

If the width of the rectangles is not negligible, then we use the arithmetic mean distance (AMD) for the center to center distance $d$ and the geometric mean distance (GMD) for $\ln(d)$. The AMD defined as

$$AMD = \frac{1}{6} \left[ \left( d + \frac{W_2 - W_1}{2} \right)^3 - \left( d + \frac{W_2 + W_1}{2} \right)^3 - \left( d - \frac{W_2 + W_1}{2} \right)^3 + \left( d - \frac{W_2 - W_1}{2} \right)^3 \right]$$

(C.7)

and the GMD is
GMD = \frac{-1}{2} \left( d + \frac{W_2 - W_1}{2} \right)^2 \ln \left( d + \frac{W_2 - W_1}{2} \right) + \frac{3}{4} \left( d + \frac{W_2 - W_1}{2} \right)^2 + \\
\frac{1}{2} \left( d + \frac{W_2 + W_1}{2} \right)^2 \ln \left( d + \frac{W_2 + W_1}{2} \right) - \frac{3}{4} \left( d + \frac{W_2 + W_1}{2} \right)^2 + \\
\frac{1}{2} \left( d - \frac{W_2 + W_1}{2} \right)^2 \ln \left( d - \frac{W_2 + W_1}{2} \right) - \frac{3}{4} \left( d - \frac{W_2 + W_1}{2} \right)^2 + \\
- \frac{1}{2} \left( d - \frac{W_2 - W_1}{2} \right)^3 \ln \left( d - \frac{W_2 - W_1}{2} \right) + \frac{3}{4} \left( d - \frac{W_2 - W_1}{2} \right)^2 \right) \quad \text{(C.8)}

W_2 \text{ and } W_1 \text{ are the widths of the rectangles. Eq. (C.6) becomes}

\begin{align*}
M' &= 2 \times 10^{-4} \times L \times \left( \frac{AMD}{L} - \text{GMD} + \ln(2L) - 1 \right). 
\end{align*} \quad \text{(C.9)}

The criteria for deciding when to use the filamental approximation of eq.(C.5) or the GMD/AMD approximation of eq.(C.9) is based upon the size of \( d \) in relation to the length of the rectangles. If \( d < 0.5L \), the widths of the rectangles cannot be ignored and we use the GMD/AMD approximation. If \( d > 0.5L \), we use the filamental approximation.

In general, the two rectangles will not be the same size and will sometimes have an offset. \[2\] gives a method for calculating the mutual inductance of two rectangles that are offset and not equal in length. There are four different possible orientations for two rectangles of arbitrary length and offset. The first is given in Figure C.1

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{unequal_parallel_filaments_with_offset.png}
\caption{Unequal parallel filaments with offset}
\end{figure}

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The mutual inductance $M'$ of the rectangles with this configuration is given as

$$M' = \frac{1}{2} \left( M_{l+m+\delta} + M_{\delta} \right) - \frac{1}{2} \left( M_{l+\delta} + M_{w+\delta} \right),$$

where the subscripts are the values for $L$ to be used in eq.(C.5) and (C.9). Eq.(C.10) is for the case when the offset is positive, i.e. the filaments do not overlap. If the filaments do overlap, then we have

$$M' = \frac{1}{2} \left( M_{l+m-\delta} + M_{\delta} \right) - \frac{1}{2} \left( M_{l-\delta} + M_{w-\delta} \right).$$

If $d$ and $\delta$ go to zero in Figure C.1, we get two rectangles in series. For the case when $d$ and $\delta$ go to zero we use

$$M' = \frac{1}{2} \left( M_{l+m} - M_{l} \right) - \frac{1}{2} \left( M_{w} \right).$$

A variation of Figure C.1 is when there is a full overlap of the rectangles as shown in Figure C.2.

![Diagram](image)

Figure C.2 Unequal parallel filaments with full overlap

The mutual inductance for the case of Figure C.2 is
\[ M' = \frac{1}{2} \left( M_{m+p} + M_{m+q} \right) - \frac{1}{2} \left( M_p + M_q \right). \]  

(C.13)

Lastly, there is the case when both filaments are unequal in length but have no offset as shown in Figure C.3.

![Figure C.3 Unequal parallel filaments with no offset](image)

The mutual inductance for this case is found as

\[ M = \frac{1}{2} \left( M_{l} + M_{m} - M_{l-m} \right). \]  

(C.14)
BIBLIOGRAPHY


