TERMS AND SYMBOLS IN MATHEMATICS
WITH HISTORY AND APPLICATIONS

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by

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Introduction

It is interesting to find that mathematics is at the root of all learning. The Greek word manthano means to learn and another word philomath means one who loves learning, a scholar. It is with a love of learning and a desire for scholarship that this thesis has been prepared.

The purpose of this thesis is to prepare a reference book of the terms and symbols used in mathematics. No work of the kind could be found on the campus of the Ohio State University, except a Mathematical Dictionary and Cyclopedia of Mathematical Science. Definitions of all the terms employed in mathematics, an analysis of each branch and of the whole, as forming a single science, Charles Davies and William G. Peck, New York: A. S. Barnes and Company, (1855). An autographed copy was presented by the author to Dr. Orton in 1856. "All the terms employed in mathematics" are not given, however, e.g., envelope, which would have no mathematical significance to anyone not interested in mathematics; but the dictionary contains much valuable information, difficult to obtain, however, because of the extremely small type in which it is printed. Again there are terms not defined mathematically in Webster's New International Dictionary, second edition, unabridged, G. and C. Merriam Company, Publishers, Springfield, Mass., (1941), such as alibi and alibi.

This thesis is an alphabetized list of mathematical terms and symbols defined and explained with some history
and applications. It is composed of terms used in courses up to and including the 700 courses in the mathematics department of the university. It is far from complete, but an attempt has been made to present the fundamental concepts common to all courses in mathematics and to generalize the more elementary ones so that they would be applicable in higher courses.

Various phases of the work might be much more complete. Each concept might be worked out through the system of n-spaces; or equations might be given in terms of the various systems of coordinates and axes of reference; the ideas might be developed from the standpoint of the various geometries. The work might be revised and extended almost indefinitely. No attempt has been made to include statistical or technical terms although both of these branches of science are dependent on mathematical foundations. Two very excellent works providing this information are: Statistical Dictionary, Albert K. Kurtz and Harold A. Edgerton. New York: John Wiley and Sons, Inc., (1939) and Chamber's Technical Dictionary, Edited by C. F. Tveney and L. H. C. Hughes. New York: The Macmillan Company, (1940).

No claim is made to originality in most of the definitions since they have been taken from texts mentioned in the bibliography or condensed from several pages of explanation and theory. There are probably some unintentional mistakes and incomplete and incorrect statements
from the standpoint of higher mathematics.

The plan used generally was to define a term in the position taken by the beginning letter of the first word since some words cannot be separated and maintain their mathematical meaning, e.g., one-to-one correspondence. Cross references are given, however, so that the word can be conveniently located, as conformal mapping, same as isogonal representation. This term is defined under conformal and reference given to it under isogonal representation, "see conformal mapping".

Words of general meaning, such as increase and decrease, have not been included. Mathematics is based on certain fundamental concepts called undefinables. "A line, the path of a moving point" may not satisfy all the requirements of a good definition but it gives a reasonable explanation of the term. Point is listed, "a point is just a position". All definitions therefore include these undefinables, words of general meaning, or terms defined somewhere in the body of the work.

Aristotle said we should never use negatives and avoid opposites in definitions, but some we may use acceptably, such as, "divergent series, not convergent". Someone has said that if a definition need never be used, it need not be given but assumed. It is very difficult to define some terms in simpler words.

It is hoped that this thesis may afford a convenient means of reference to terms which may be hard to find in a
standard dictionary or whose mathematical significance it may be difficult to separate from all other shades of meaning given.
A.

(ABCD), symbol for the cross ratio of four points on a line; (abcd), symbol for the cross ratio of a pencil of lines.

Abelian functions, name given to higher transcendental functions of multiple periodicity.

Abelian group, a group for which each pair of elements is commutative. All cyclic groups are Abelian.

abscissa, the distance of a point from the y-axis, measured on a line parallel to the x-axis in the system of rectangular coordinates.

abscissa axis, same as the x-axis, the horizontal line of reference in the system of rectangular coordinates.

absolute, the infinite region. See circular points.

absolute coordinates, the coordinates of a point referred to the pair of circular rays through the Cartesian origin. Same as circular coordinates.

absolute continuity, see continuity.

absolute convergence, see convergence.

absolute value, a real positive quantity that expresses the measure of some function; also called modulus.

Symbol, \(|a|\).

acceleration, rate of change of velocity. \(a = \frac{dv}{dt} = \frac{d^2s}{dt^2}\).

acute angle, see angle.

acyclic, pertaining to a region such that any closed curve, C, within it may be shrunk up to nothing without passing through any point and without breaking its continuity, that is, a region every closed curve of which is reducible. 

addition, the operation of combining two classes to form a new class, the cardinal number of the new class being called the sum of the cardinal numbers of the first two classes. Symbol, \( a + b = c \).

addition laws, (1) The sum of two numbers is a unique number; (2) addition is commutative; (3) addition is associative; (4) there is a number zero such that the sum of this number and any other number gives the second number; (5) every number has an inverse with respect to addition such that the sum of that number and its inverse is equal to zero.

adherence, the set of all adherents.

adherent, any isolated point of a set. These terms were used by Cantor.

adjacent angles, see angle.

adjoint circles, circles passing through two vertices of a triangle and tangent to a side passing through one of them. The three adjoint circles of a triangle have a point in common. There are two groups of adjoint circles to every triangle. The two common points are called the Brocard points.
adjoint equation, the equation of the integrating factor, the multiplication by which makes a differential equation exact.

adjoint of a determinant, the determinant of which the element in the \( i \)th row and \( j \)th column \((i, j = 1, \ldots, n)\) is equal to the value of the cofactor \( A_{i j} \) of the element \( a_{i j} \) of the determinant \(|a_{i j}|\), \((i, j = 1, \ldots, n)\).

adjoint of a matrix, a matrix of the same order in which the element in the \( i \)th row and \( j \)th is the cofactor of the element in the \( j \)th row and \( i \)th column of the given matrix.

adjunction, the process by which one field is derived from another. In the Galois theory of equations, we employ a field, \( R(k_1, \ldots, k_m) \), composed of all rational functions with rational coefficients of \( k_1, \ldots, k_m \) either all of which are constants, or certain of which are constants and the others are given functions of specified variables not necessarily independent. \( F \) contains \( \overline{q} = 1, \overline{l + 1} = 2, \) etc., and therefore contains all rational numbers. In other words, the field, \( R \), of all rational numbers is a sub-field of every field, \( F \). The foregoing field, \( R(k_1, \ldots, k_m) \), is said to be derived from \( R \) by the adjunction of \( k_1, \ldots, k_m \) to \( R \). If \( n < m \), \( R(k_1, \ldots, k_m) \) is derived from \( R(k_1, \ldots, k_n) \) by adjunction of \( k_{n+1}, \ldots, k_m \).

admissible values, values of the direction cosines of a line for which the sum of the squares of the direction
cosines is equal to one. \( \lambda^2 + \mu^2 + \nu^2 = 1 \).

Affine coordinates in a plane, the projective coordinates obtained by taking the line at infinity as the side, \( x_3 = 0 \), of the triangle of reference. \( x = \frac{x_1}{x_3}, y = \frac{x_2}{x_3} \). Because of the cross ratios when the fourth point is at infinity, the coordinates reduce to the oblique Cartesian coordinates.

Affine coordinates in space, the tetrahedron of reference of the projective coordinates in space has the plane of infinity as its fourth face, \( x_4 = 0 \).

Affine geometry, the geometry of the plane at infinity together with the ideal points and ideal lines. This geometry may be considered as a generalization of metric geometry or a sub-geometry of projective geometry.

Affine transformation, a transformation by which all finite points go into finite points and the line at infinity into itself.

Affix, same as representative point, the point, referred to rectangular axes, which represents a complex number. Conversely, the number is spoken of as the affix of the point.

Aggregate of real numbers, totality of all numbers, positive and negative, rational and irrational, integral and fractional.

Algebra, "algebra" means the restoration and refers to the fact that the same magnitude may be added to or subtracted from both sides of the equation. The unknown quantity was
formerly referred to as the "thing" or "the root of a plant", from which phrase we get the use of the word root as applied to the solution of an equation. 
algebraic expression, an expression composed of algebraic terms separated by plus and minus signs. An expression may contain only one term. 
algebraic fractions, fractions whose terms are algebraic expressions. 
algebraic functions, rational functions of $x$ whose relation to $y$ is shown by the equation of the form 
$$y^n + f_1(x)y^{n-1} + f_2(x)y^{n-2} + \ldots + f_n(x) = 0,$$
where $y$ is said to be an algebraic function of $x$. Functions which are not algebraic are called transcendental functions. 
algebraic laws, the associative, commutative, and distributive laws, q. v. 
algebraic number, a number that satisfies an algebraic equation whose coefficients are all rational numbers. 
algebraic scale, the representation of the signed numbers as points on a line. 
algebraic sum, the sum of algebraic numbers according to the algebraic rules for addition. 
algebraic term, a quantity expressed by numbers and letters together. 
algorism, the Arabic or decimal system of numeration, hence arithmetic; also, any method of computation.
alias, the renaming of points of a range under projection.
alibi, changing the place of points of a range under pro-
jection. Standard name, collineation, q. v.
aliquant, contained in another number but with a remainder.
aliquot, contained in another number without a remainder.
alternating group, of degree n, the group composed of all
even permutations on n letters.
alternating sequence, a sequence which oscillates instead of
approaching a limit or becoming infinite.
alternating series, a series whose terms are alternately
positive and negative.
albedo, the height of a figure, measured by the perpen-
dicular to the base.
amplitude, same as vectorial angle.
analysis, the separation of anything into its constituent
elements; a method of proving propositions by assuming
the conclusion and reasoning back to data or to already
established principles; the investigation of a problem
by the methods of algebra.
analysis-situs, same as topology, treats of those properties
of geometrical forms common to all forms which can be
transformed into each other by stretching and bending
without tearing.
analytic, pertaining to analysis.
analytic function, a function $f(x)$ is analytic at $a$ if it
can be expanded in a Taylor's series in powers of \( x-a \)
which is valid near \( a \). A function of two or more variables,
say \( f(x,y,z) \), is analytic at a point \((a,b,c)\) if it can be
expanded in a Taylor's series in powers of \((x-a), (y-b),
(z-c)\) valid about the point.

Analytic geometry, attributed to Rene Descartes; sometimes
called Cartesian geometry; the analytic treatment of
geometry whereby geometric figures are represented
algebraically.

Anchor ring, same as torus, the path of a circle revolving
about a straight line which lies in the plane of the circle
but does not cut it.

Angle, a figure formed by two intersecting (straight) lines
or planes. Symbol \( \angle \).

1(a). Angles formed by straight lines or curves.

Plane angle, an angle formed by two intersecting
straight lines in a plane.

Polar angle, same as vectorial angle; one of the
coordinates in the system of polar coordinates.

Spherical angle, the angle formed on the surface of a
sphere by two great circle arcs.

Vectorial angle, the angle formed by the initial line
and the radius vector from the origin to the point in
the system of polar coordinates. Also called ampli-
tude, anomalie, arcus, argument, declination. The
principal value of the amplitude of a complex number
is that one of its values which satisfies the conditions: $-\pi < \phi \leq \pi$, $\phi$ representing the amplitude or vectorial angle.

1(b). angles formed by planes.

dihedral angle, the angle formed by the intersection of two planes.
polyhedral angle, the angle formed by three or more planes intersecting in pairs.
triangular angle, the angle formed by the intersection in pairs of three planes.

2. functions of an angle, certain ratios between the sides of a right triangle are called the trigonometric functions of the angle.
cosecant, the ratio of the hypotenuse to the side opposite the angle.
cosine, the ratio of the side adjacent to the angle to the hypotenuse.
cotangent, the ratio of the side adjacent to the angle to the side opposite the angle.
secant, the ratio of the hypotenuse to the side adjacent to the angle.
sine, the ratio of the side opposite the angle to the hypotenuse.
tangent, the ratio of the side opposite the angle to the side adjacent to the angle.

3. parts of an angle,
bisection of an angle, the line or plane which divides the angle into two equal parts.

degrees of an angle, the unit of measure, one-ninety-sixth of a right angle.

depth of an angle, the intersection of the two planes forming an angle.

faces of an angle, the planes forming an angle.

sides of an angle, or arms, the lines which form the angle.

   initial side, the first position of the rotating line which generates the angle.

   terminal side, the final position of the rotating line which generates the angle.

vertex of the angle, the point of intersection of the two lines forming the angle.

4. angles according to position,

Brocard angle, an angle formed by the lines from the Brocard point of a triangle to one vertex and the adjacent side.

central angle, an angle whose vertex is at the center of a circle and whose sides are radii.

exterior angle, an angle of a polygon formed by one side of the figure and another side produced.

included angle, an angle is included by two sides of a triangle if the sides form the angle.

inscribed angle, an angle is inscribed in a segment of a circle if its vertex is on the circle and its
sides pass through the ends of the arc of the segment. **interior angle**, an angle of a polygon formed by the sides of the polygon.

5. **angles classified as to purpose,**

**angle of depression**, the angle formed by the horizontal and the line to the object from the eye of the observer, who is above the object.

**angle of elevation**, the angle formed by the horizontal and the line to the object from the eye of the observer, who is below the object.

**angle of incidence**, the angle made by a ray of light and the reflecting surface.

**angle of reflection**, the angle made by a line to the image from the eye of the observer and the reflecting surface.

6. **angles of relation,**

**adjacent angles**, two angles which have a common vertex and a common side between them; in space, two angles with a common edge and a common plane between them.

**alternate-exterior angles of parallels**, two angles outside the parallels and on opposite sides of a transversal.

**alternate-interior angles of parallels**, two angles between the parallels on opposite sides of a transversal.

**complementary angles**, two angles whose sum is equal to
11.

a right angle. Each angle is called a complement of the other.

*conjugate angles*, two angles whose sum is equal to a perigon.

*corresponding angles*,

(a) *of parallels*, two angles on the same side of a transversal and in corresponding position with respect to the parallels.

(b) *of congruent or similar figures*, the angles that are opposite equal sides.

*equal angles*, angles containing the same number of degrees or having the same amount of rotation.

*non-adjacent angles*, angles that are not adjacent.

*supplementary angles*, two angles whose sum is equal to a straight angle. Each angle is called a supplement of the other.

*vertical angles*, two angles with a common vertex and the sides of one are the prolongations of the sides of the other.

7. *angles classified as to size*: the size of an angle depends on the amount of rotation and not upon the length of the sides.

*acute angle*, an angle less than a right angle.

*oblique angle*, an angle that is neither a right nor a straight angle.

*obtuse angle*, an angle greater than a right angle but
less than a straight angle.

reflex angle, an angle greater than a straight angle but less than a perigon.

right angle, an angle formed by perpendicular lines.

solid angle, an angle subtended at the vertex of a cone.

straight angle, an angle whose sides extend in opposite directions from the vertex.

angle naming, an angle is named by placing capital letters at the vertex and on the sides of the angle, the vertex letter being between the other two; a small letter may also be used and placed within the angle close to the vertex. If the angle is named in a clockwise direction it is negative; if counter-clockwise, positive.

angle sum, a right angle contains 90°; the sum of all the successive adjacent angles around a point on one side of a straight line is 180°; the sum of all the successive adjacent angles around a point in a plane is 360°. The sum of the interior angles of a polygon is \((n-2)180°\) where \(n\) is the number of sides; the sum of the exterior angles of a convex polygon is 360° when the sides are extended in succession.

anharmonic ratio, same as cross ratio or double ratio.

anomalie, same as vectorial angle.

antecedent, in vector analysis, the first vector in a dyad;

in a proportion, the first terms in the equal ratios.

antiderivative, of a function, is the function whose
derivative is the given function.

antihomologous points, see homologous points.

antilogarithm, the number corresponding to a given logarithm.

antiparallel, two opposite sides of an inscriptable quadrilateral are antiparallel with respect to the other two sides.

antipodal geometry, same as spherical geometry.

apex, see cone.

apolar forms, two forms such that the polar of one with respect to the other vanishes identically, i.e., when every coefficient is zero.

apolar points, same as harmonic points, four points on a line whose cross ratio is -1.

apolar quadratic, unique to a given cubic, (compare apolar forms) same as the Hessian, or the canonizant of the cubic. The apolar quadratic is of fundamental importance in the projective theory of the cubic. See Hessian.

apothem, of a regular polygon, the radius of the incircle.

applicable surfaces, two surfaces represented in such a way that small lengths as well as angles are equal; the representation is called isometric.

Appolonian circle, the locus of a point the ratio of whose distances from two fixed points is constant.

approximation curve, the graphical method for approximation of real roots of a real equation.

Arabic notation and numerals, the decimal system of numera-
tion employing the nine digits and zero.
arbitrary constant, a quantity which has a definite value for any given discussion; this value may be chosen arbitrarily, however the value of some constants never change, for example, π and the cardinal numbers.

arc, a part of a circle between two points on the circle.

arc of a curve, a connected and closed set of points which contains no inner points but only boundary points. (This definition includes the last one.)

Archimedes' axiom, Let A₁ be any point on a straight line between the arbitrarily chosen points, A and B. Take points A₂, A₃, A₄, ..., so that A₁ lies between A and A₂ and A₂ lies between A₁ and A₃, ... and segments AA₁, A₁A₂, A₂A₃, ..., are equal to one another. Then, among this series of points, there always exists a certain point Aₙ such that B lies between A and Aₙ.

arcus, same as vectorial angle.

area, the number of square units of measure contained in a given surface.

Argand diagram, a geometrical representation of complex numbers by the points of the plane.

argument, term used for the independent variable; also used for the vectorial angle.

arithmetic, the science of numbers and computation.

arithmetic mean, any term between any two terms in an arithmetic progression; more specifically, the one mean between two numbers, called the average; hence the arithmetic
mean between two numbers equals half their sum.

**arithmetic progression**, a series in which the difference between any term and the preceding term is the same for all terms. This difference is called the **common difference**. The formula for the **general term** is: \( t = a + (n-1)d \) where \( a \) is the first term, \( n \) is the number of the term and \( d \) is the common difference. The formula for the **sum of \( n \) terms** is: \( S = \frac{1}{2} n \left[ 2a + (n-1)d \right] \).

**arithmetical scale**, a system of arithmetical notation in which the successive places determine the value of figures, as the decimal system.

**arithmetic sequence**, same as arithmetic progression.

**arrangements**, \( n \) objects taken out of a total number of \( m \) objects to form groups, attention being paid to the order of objects in each group.

**assemblages**, same as set.

**associative law**, one of the fundamental laws of algebra, which states that of three elements of the group any two may be combined in any operation and then acted upon by the third to give the same result as though the last two were combined first in the same operation and then acted upon by the first.

**assumption**, the temporarily accepted, conditional part of a statement upon which a proof is based.

**assymetry**, lack of symmetry.

**asymptote**, of any curve, the tangent line whose point of
contact is on the line at infinity.

asymptotic cone, the cone of which every generator is an asymptote of a surface.

asymptotic series, a series of the form \( a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \)

(which need not converge for any value of \( x \)), an expansion of a function, \( F(x) \), which is defined for every sufficiently large positive value of \( x \), if, for every (fixed) \( n=0,1,2,\ldots, \)

\[
\left[ F(x) - \left( a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n} \right) \right] \xrightarrow{x \to \infty} 0
\]

Symbolically: \( F(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots \). This series decreases and reaches a minimum and then increases.

augmented matrix, see matrix.

automorphic function, a function, \( w=f(z) \), which remains unaltered if a definite linear function of \( z \) is substituted for \( z \).

symmetric automorphic function, an automorphic function which takes on conjugate values of the function at conjugate points.

automorphic rational function, called a finite discontinuous group, an automorphic function which remains unchanged under only a finite number of transformations.

autopolar, self-polar.

auxiliary lines, extra lines introduced in a figure to help establish the proof.
average,

arithmetic, see arithmetic mean.

geometric, see geometric average.

harmonic, the reciprocal of the arithmetic average, the harmonic mean in a harmonic progression.

axes, three lines meeting to form a system of reference for points.

coordinate, the x-axis or the horizontal line of reference, and the y-axis or vertical line of reference, and in space, the z-axis perpendicular to the plane of the other two, for the system of rectangular coordinates. Compare oblique system of coordinates.

axiom, a statement about quantities in general which is accepted as true without proof.

Archimedes' axiom, q. v.

Cantor-Dedekind axiom, or the axiom of continuity, q. v.

Paasch's axiom, q. v.

axis, a single line of reference.

abscissa, same as x-axis; also axis of real numbers in complex system.

conjugate, an axis that does not meet the surface in real points. See hyperboloid and hyperbola.

Lemoine, of a triangle, see Lemoine axis.

major, see ellipse.

minor, see ellipse.

ordinate, same as y-axis; also the axis of pure
imaginary numbers.
polar, see polar axis.
radical, see radical axis.
semi-, the half axis.
transverse, see hyperbola.
\textbf{x-axis}, see axes.
\textbf{v-axis}, see axes.
\textbf{z-axis}, see axes.

\textbf{axis coordinates}, the homogeneous coordinates of a line which is determined by two intersecting planes, \( u \) and \( v \), are the six numbers \( q_{12}, q_{13}, q_{14}, q_{34}, q_{42}, q_{23} \), where \( q_{ij} = u_i v_j - u_j v_i \) (\( i,j = 1,2,3,4 \)). They are connected by the relation \( q_{12} q_{34} + q_{13} q_{42} + q_{14} q_{23} = 0 \).

\textbf{axis of conics}, see circle (in Euclidean and hyperbolic geometry), the parabola.
\textbf{axis of cylinder}, of revolution, also of circular helix, q.v.
\textbf{axis of homology}, see homology.
\textbf{axis of perspectivity}, see perspectivity.
\textbf{axis of quadrics}, see quadrics.
\textbf{axis of range of points}, see range.
\textbf{axis of reflection}, see reflection.
\textbf{axis of revolution}, see revolution.
\textbf{axis of similitude}, same as \textbf{homothetic axis}, see similitude.
\textbf{axis of symmetry}, see symmetry.
Barrow's triangle, the differential triangle formed by the tangent to the curve at the given point and the lines parallel to the axes, having the sides \(dx, dy, ds\). Barycentric coordinates, the masses situated at the vertices of a triangle or tetrahedron which locate the center of gravity.

Bend point, a point of the curve at which the tangent is horizontal and all adjacent points lie below the tangent or all above the tangent. In other words, \(f(x)\) has a relative maximum or relative minimum at the abscissa of a bend point, on the graph of \(y=f(x)\).

Bendixson's test for uniform convergence, see convergence.

Bernoulli criterion, two contingent events are considered as equally probable if, after taking into consideration all relevant evidence, one of them can not be expected in preference to the other.

Bernoulli inequality, \((1+x)^n \geq 1 + nx\) where \(x\) is any real number such that \(0 < |x| < 1\) and \(n\) is any integer greater than unity.

Bernoulli numbers, numbers which play a prominent part in the theory of infinite series; these numbers are obtained from calculating the reciprocals of the series \(\sum_{n=1}^{\infty} \frac{x^n}{n!}\).

The numbers: \(B_1 = -1/2, B_2 = 1/2, B_3 = 0, B_4 = -1/20, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/720, \ldots\)
Bernoulli's Theorem, when the number of trials is increased indefinitely, the probability that the discrepancy shall remain numerically less than any given number and the probability that the relative discrepancy shall remain numerically greater than any given number, will both approach 0 as a limit. This theorem is one of the most important theorems in the theory of probability.

Fertrand's curves, curves that have the same principal normals. A necessary and sufficient condition is that a linear relation exist between the first and second curvatures: the distance between corresponding points of the two curves is constant, the osculating planes at these points cut under a constant angle and the torsions of the two curves have the same sign.

Bessel's differential equation, a hypergeometric equation, important for certain physical problems,

\[ x^2 y'' + xy' + (x^2 - n^2) y = 0. \]

Then this equation is solved in series form we have what is called the Bessel function.

Bessel function, \( J_n(x) = \frac{x^n}{2^{n+1} n!} \left[ 1 - \frac{x^2}{2(n+1)} + \frac{x^4}{2^2 \cdot 3!(n+1)(n+2)} - \frac{x^6}{2^3 \cdot 3!(n+1)(n+2)(n+3)} + \ldots \right]. \)

\( J_n(x) \) is defined for all values of \( n \) except negative integers by putting \( \Gamma(n+1) \) in place of \( n! \).

Beta function, \( \beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx. \)

Bezout's Method of Elimination, a method of solution of simultaneous equations of the same degree or different
degrees in $x$ whereby the eliminating of $x$ leads to a relation $F = 0$, where $F$ is a polynomial in the coefficients and has as a factor the true resultant (or eliminant) of the equations. This method reduces the determinant of a similar method by Sylvester from order $2n$ (when $m=n$) to one of order $n$. A necessary and sufficient condition that the equations have a common root is that the resultant is equal to zero.


**biangular**, having two angles.

**biflenode**, a double point both of whose tangents are flex tangents.

**bilinear**, or linear-linear equation, a linear transformation expressed by the equation: $x' = \frac{ax+b}{cx+d}$, $ad - bc \neq 0$.

**bilinear form**, a polynomial in $2n$ variables with each of its terms of the first degree in each variable.

**binary cubic**, a cubic equation in two variables. The complete system of the binary cubic is (1) $f$, the cubic itself;
(2) $H$, the Hessian; (3) $J$, the Jacobian, called the cubic covariant; (4) $\Delta$ the discriminant connected by the following relation: $\Delta f_2 - J_2 \equiv 4H^3$. Such a relation between invariants is called a syzygy. Syzygies occur in the complete systems of all forms except the quadratic.

**binary quadratic**, a form of two variables in the second degree.

**binary scale of notation**, the representation of a number as
a polynomial in 2 with integral coefficients, e.g.,
237 = 1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 1.
Symbol 11101101.

binomial, an algebraic expression of two terms.
binomial expansion, same as binomial series.
binomial theorem, the following group of laws, by means of
which any power of a binomial may be written down: e.g., the expansion of \((a + x)^m\).

1. Terms-The number of terms in the expansion is \(m+1\).
The first term contains only \(a\) and the last term contains
only \(x\). The other terms contain both \(a\) and \(x\).

2. Exponents-The exponent of \(a\) in the first term is \(m\) and
decreases by 1 in each succeeding term until it becomes 0.
The letter \(x\) has a zero exponent in the first term and
increases by 1 in each succeeding term until it becomes \(m\)
in the last term.

3. Coefficients-The coefficient of the first term is 1;
of the second term is \(m\). If the coefficient of any term is
multiplied by the exponent of \(1\) in that term and the result
divided by one more than the exponent of \(x\) in that term, the
coefficient of the succeeding term is obtained.

4. Signs-In the expansion \((a + x)^m\), all terms are positive.
In the expansion \((a - x)^m\), all odd-numbered terms are
positive and all even-numbered terms are negative.

The ancient Hindus were familiar with special cases of
the theorem and used it to extract square roots and cube roots. Sir Isaac Newton first recognized the general truth of the theorem and used it in making some of his greatest mathematical discoveries. Napier was led by the use of the binomial theorem to the discovery of logarithms. 

**binormal**, the normal to the curve which is perpendicular to the osculating plane at the point of tangency.

**bipartite**, having all its real points compressed in two distinct continuous series of points.

**biquadrental versor**, a dyadic which turns the points of space about the axis through two right angles.

**biquadrate**, the 4th power.

**biquadratic**, of the fourth degree.

**biquadratic equation**, an equation of the fourth degree.

**birectangular**, having two right angles.

**biscalars**, the ordinary imaginary scalars.

**biset**, to divide into two equal parts.

**bisector**, that which divides into two equal parts.

**bisegment**, one of the two equal segments of a line.

**bitement**, same as double line or double tangent, a tangent to the curve at two discrete points.

**biunique correspondence**, a one-to-one correspondence between sets whereby one and only one individual mates each individual in the other set.

**bivector**, the sum of two real vectors, of which one has
been multiplied by the imaginary scalar, \( \sqrt{-1} \); also called \textit{imaginary vector}.

\textbf{bordered determinants}, see determinants.

\textit{Borel's method of summability of series}, \( F(x) = e^{-x} \sum_{n=0}^{\infty} s_n \frac{X^n}{n!} \).

\textit{boundary curve}, same as \textit{horocycle}.

\textit{boundary point of a set}, a point, such that in every neighborhood of the point, there are points of the set and also at least one point which does not belong to the set.

\textit{bounded function}, a positive number, \( M \), such that \( |f(x)| < M \) for all values of \( x \) on the range of the definition.

\textit{bounded sequence}, a sequence such that the absolute values of the elements of the sequence do not exceed a definite value, called the \textit{upper bound}, and are never less than another definite value, called the \textit{lower bound}.

\textit{bounded set}, a linear set with all its points in a finite interval.

\textit{branch point}, a point about which a complete circuit is being made by another point in its plane returning to its original position, the value of the function always being changed.

\textit{Brianchon's hexagon}, see Brianchon's theorem.

\textit{Brianchon's point}, see Brianchon's theorem.

\textit{Brianchon's theorem}, The opposite vertices of a hexagon circumscribing a conic lie on lines through a point, called the \textit{Brianchon point}. Brianchon's point and Pascal's line are pole and polar with respect to the conic. Compare Pascal's line.
Brocard angle, see angle.

Brocard circle, the circle described on the line joining the circumcenter of a triangle to the symmedian point as a diameter.

Brocard point, see adjoint circles.

Brocard triangle, see triangle.

Broken line, a line composed of straight-line segments, which do not all lie in the same straight line.

Budan's theorem. Let $a$ and $b$ be real numbers, $a \neq b$, neither a root of the function, $f(x) = 0$, an equation of degree $n$ with real coefficients. Let $V_a$ denote the number of variations of sign of $f(x), f'(x), f''(x), \ldots, f^{(n)}(x)$ for $x = a$, after vanishing terms have been deleted. Then $V_a - V_b$ is either the number of real roots of $f(x) = 0$ between $a$ and $b$ or exceeds the number of those roots by a positive even integer. A root of multiplicity $m$ is here counted as $m$ roots.

Bundle, or star, all the lines and planes in three dimensional space passing through a given point.
C.

calculus, same as infinitesimal calculus, the rules, processes and principles by means of which one may deal mathematically with continuously varying magnitudes.

differential calculus, the processes, rules and theorems relating to differentiation and derivatives.

integral calculus, the method of finding antiderivatives.

of variations, the study of the change in a derived function in consequence of a change in the form of its primitive.

canonical form, the simplest form to which the general collineation can be reduced. The equation: \( x' = kx \), where \( x \) and \( x' \) are corresponding points.

canonization, same as apolar quadratic.

Cantor-Dedekind Axiom, the proposition by which the "continuity of the straight line" is expressly postulated: To each point on the line there corresponds one and only one real number and, conversely, to each real number there corresponds one and only one point on the line.

Cardan's formulas for the roots of a reduced cubic equation, \( y^3 + py + q = 0 \): \( y_1 = A + B \), \( y_2 = \omega A + \omega^2 B \), \( y_3 = \omega^2 A + \omega B \),

where \( A = \sqrt[3]{-\frac{q}{2} + \sqrt{R}} \), \( B = \sqrt[3]{-\frac{q}{2} - \sqrt{R}} \), \( R = \left( \frac{q}{3} \right)^2 + \frac{q}{3} \) and \( \omega \) and \( \omega^2 \) are the two imaginary cube roots of unity.

cardinal numbers, a class of symbols, each of which
represents only one of a set and recognizes two classes as belonging to the same set.

cardiod, a higher plane curve, the locus of the equation \( r = a(1- \cos \theta) \).

Cartesian coordinates, the distances of a point from the axes of reference, measured along lines parallel to the other axis in the same plane, or along the edges of a parallelepiped, formed by the planes determined by the axes in pairs, if the system of reference is in space. The distance from the x-axis is called the ordinate of the point; the distance from the y-axis is called the abscissa of the point; the distance from the xy plane measured along the edge of the parallelepiped which is parallel to the z-axis is the third coordinate and has no name. The system of reference in space may be rectangular or oblique, according as the axes are at right or oblique angles with each other. Both of these systems are called non-homogeneous or regular. Another system of Cartesian coordinates referred to the same axes is the set of homogeneous coordinates, composed of three numbers \( x_1, x_2, x_3 \) for a point in a plane defined by the equations: \( \frac{x_1}{x_3} = x, \frac{x_2}{x_3} = y \). In space the coordinates are the four numbers, \( x_1, x_2, x_3, x_4 \) for which \( \frac{x_1}{x_4} = x, \frac{x_2}{x_4} = y, \frac{x_3}{x_4} = z \), e.g. the point \((3/4, -1/2)\) in homogeneous coordinates is \((3, -2, 4)\) or any
triple of numbers having the same ratio provided the last number is never equal to 0, the triple with 0 in the third place being reserved for the point at infinity. In the general equation of a straight line: \( u_1 x + u_2 y + u_3 = 0 \), the coefficients and every triple similar to \((u_1, u_2, u_3)\) is a set of homogeneous coordinates for the line; the triple \((0,0,u_3)\) is a set of homogeneous coordinates for the ideal line.

cessus irreducibilis, the irreducible case, when the roots of a real cubic equation are all real and distinct, the discriminant, \( \Delta \), is positive and \( R = -\frac{\Delta}{108} \) is negative, so that Cardan's formulas present the values of the roots in a form involving cube roots of imaginaries. This is called the irreducible case since it may be shown that a cube root of a general complex number cannot be expressed in the form \( a + bi \), where \( a \) and \( b \) involve only real radicals. While we cannot find these cube roots algebraically, we have learned to find them trigonometrically.

catalecticant, a determinant, the vanishing of which expresses the condition that the quartic have an apolar quadratic.

The determinant: \[
\begin{vmatrix}
  a_0 & a_1 & a_2 \\
  a_1 & a_2 & a_3 \\
  a_2 & a_3 & a_4
\end{vmatrix}
\]

is 0.

categoricalness, the property of an axiomatic system which states that only one set of elements obeys the axioms essentially.
category, the totality of all bilinear forms which have the same characteristic.
catenary, the curve formed by a perfectly flexible, inextensible, infinitely slender cord suspended by its ends.
catenoid, the surface of revolution whose equations are
\[ x = u \cos \nu, \quad y = u \sin \nu, \quad z = a \log(u + \sqrt{u^2 - a^2}), \]
generated by the rotation of a catenary about its axis. It is the only minimal surface of revolution.
Cauchy criterion for convergence of sequences, see convergence.
Cauchy double series theorem, see convergence.
Cauchy-Lipschitz method of solving differential equations, consists of a broken line approximation to the solution. Choosing an interval, \( h \), we construct a straight line through \( (a, b) \) with the requisite slope, \( f(a, b) \), and extend to meet the line, \( x = a + h \). This gives the approximation, \( y(a + h) = b + f(a, b)h \). Through the point, \( [a + h, y(a + h)] \), we draw a line with the slope of the lineal element there, viz., \( f[a + h, y(a + h)] \), and extend to the line \( x = a + 2h \), and so on. This process yields the recursion formula,
\[ y(a + nh) = y[a + (n-1)h] + f[a + (n-1)h, y[a + (n-1)h]] h. \]
Cauchy-Riemann equations, an expression of the form \( u + iv \), in which \( u, v \), are rational functions of \( x \) and \( y \), can only be put in the form of a rational function of \( z = x + iy \) when \( u \) and \( v \) satisfy the partial differential equations:
\[ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}. \]
Cauchy-Schwarz inequality, If \( f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \)
converges for \( |z-z_0| < r \), then \( |a_p| \leq \frac{M}{a^p} \) \((p=0, 1, 2, \ldots)\),
if \( 0 < C < r \) and \( M \) is a number which \( |f(z)| \) never exceeds
along the circumference \( |z-z_0| = C \).

Cauchy's fundamental integral theorem in the theory of
functions, if a function, \( f(z) \), of a complex argument is
regular in a simply connected domain \( E \), then \( \int_C f(z)dz = 0 \)
for every closed curve which lies entirely inside of \( E \).

Cayleyan, a Jacobian of apolar nets of conics.

Cell, Peaucellier's, see Peaucellier's cell.

Center(s), the midpoint or fixed point of reference, as
the origin of a coordinate system.

Center of curvature, see osculating circle.

Center of gravity, see centroid.

Center of homology, see homology.

Center of inversion, see inverse points.

Center of involution, see involution.

Center of perspectivity, see perspectivity.

Center of projection, see projection.

Center of reflection, see reflection.

Center of similitude, the point in which lines joining
corresponding points of homothetic figures are concurrent.
Also called the homothetic center.

Center of symmetry, see symmetry. The centers of certain
geometric figures are centers of symmetry, e.g., the circle, sphere, ellipse, parabola and hyperbola.

**centers of triangles,**

- **circumcenter**, the point in which the perpendicular bisectors meet; the center of the circumscribed circle.
- **equicenter**, see excenter.
- **excenter**, the point in which the external bisectors of two angles and the internal bisector of the third are concurrent. The three excenters together with the incenter are the equicenters of the triangle.
- **incenter**, the point in which the bisectors of the angles meet; the center of the inscribed circle.
- **nine-point center**, see nine-point circle.
- **orthocenter**, the point in which the altitudes of the triangle meet.

**radical center**, see radical axis.

**central angle**, see angle.

**central projection**, see projection.

**central quadric**, a quadric surface with a single proper center and not tangent to the plane at infinity.

**centroid**, of a triangle, the point in which the medians intersect; the center of gravity of any mass.

**Cesaro's method of summability of series,**

\[
\sum_{n=0}^{\infty} s^{(k)} x^n = \frac{1}{(1-x)^k} \sum_{n=0}^{\infty} s_n x^n = \frac{1}{(1-x)^{k+1}} \sum_{n=0}^{\infty} a_n x^n
\]

for every integral \( k \geq 0 \).
Cesaro moving trihedral, see moving trihedral.

Ceva's Theorem, The lines joining the vertices of a given triangle to a given point determine on the sides of the triangle six segments such that the product of three of these segments having no common end is equal to the product of the remaining three segments.

chance variables, variable quantities with a definite range of values each one of which, depending on chance, can be attained with a definite probability.

characteristic curve, consider a one-parameter family of surfaces: $(x,y,z,a) = 0$. The equations: $\Psi(x,y,z,a) = 0$, $\frac{\partial \Psi(x,y,z,a)}{\partial a} = 0$ define the characteristic curve of the family.

characteristic determinant, see determinant.

characteristic of a logarithm, see logarithm.

characteristic matrix, Let $A$ be an $n$-rowed square matrix whose elements are independent of the variable $\lambda$. Let $I$ be the $n$-rowed identity matrix. The matrix $A - \lambda I$ is called the characteristic matrix of $A$; it may be obtained by subtracting $\lambda$ from each diagonal element of $A$. Its equation is: $\phi(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0$; $a_n = (-1)^n$.

characteristic strip, a curve endowed with a planar element at each point, provided the planar element passes through the tangent to the curve and the orientation of the planar
element varies continuously as we proceed along the curve. A strip is a characteristic strip of \( f(x,y,z,p,q) = 0 \), a partial differential equation and \( z = g(x,y) \), some integral surface, if at each point the curve is tangent to a ruling of the conical element, and the planar element is tangent to the conical element along this ruling.


chord, a straight line joining two points on the circle.

anti-homologous chords, chords joining anti-homologous points of two circles.

common chord, a chord which is a chord of two circles.

circle, a closed curve every point of which is equidistant from a point within called the center; the length of the closed curve is called the circumference of the circle; the distance from the point to the circle is called the radius. In terms of locus it may be defined as the path of a point moving so that its distance from a fixed point is constant. Circle equation: \( x^2 + y^2 = k \) where \( k \) is a constant equal to the radius squared.

diameter, a line segment drawn through the center of the circle and terminated by the circle.

circle of inversion, the circle having the square root of the positive constant of inversion as its radius and all
points of the circle are their own inverse points. The constant is called the radius of inversion. See inverse points.

circular cone, a cone whose base is a circle.
circular coordinates, see absolute coordinates.
circular cylinder, a cylinder whose bases are circles.
circular elements, portions of the osculating circle at the points of the direction field, showing the direction and curvature of the integral curve at that point.
circular functions, functions involving the trigonometric functions.
circular helix, the path of a point moving on the surface of a right circular cylinder so as to intersect its elements at a constant acute angle. The axis of the cylinder is the axis of the helix.
circular measure, the measurement of angles in degrees or radians. A radian is equal to 57°17'45" approximately, being the angle subtended at the center of the circle by an arc equal in length to the radius of the circle.
circular points, the two conjugate imaginary points at infinity; taken together they constitute the absolute.
circular rays, the two conjugate imaginary lines which connect a point with the circular points.
circular transformation, the point transformations of the plane of inversion which are in a one-to-one correspondence and carry a circle always into a circle.
circumcenter, see center.
circumcircle, the circle circumscribed about a polygon and passing through the vertices of the polygon.
circumdiameter, the diameter of the circumcircle.
circumference, see circle.
circumradius, the radius of the circumcircle.
circumscribed, drawn around so that the vertices of the polygon within lie on the circle.
cissoid, a higher plane curve, the graph of the equation: \( x^3 + xy^2 - 2ay^2 = 0 \). This curve is used to solve the problem of the duplication of the cube.
Clairaut's equation, the most general differential equation of a non-parallel one-parameter family of straight lines:
\[ y = x \frac{dy}{dx} + f \left( \frac{dy}{dx} \right). \]
class, a set of elements satisfying a certain condition.
continuous class, a dense class which satisfies Dedekind's axiom, e.g. all points on a straight line.
dense class, see set.
denumerable class, an infinite class which can be put into a one-to-one correspondence with a progression or, in particular, with the set of all positive integers, e.g., the set of all even integers, the set of all rational numbers.
finite class, a class which is not infinite.
infinite class, a class which contains a part which can be put into one-to-one correspondence with the class. 

ordered class, a class satisfying the conditions for linear order.

classical equations, the hypergeometric differential equation: 
\[ x(1-x)y'' + \left[c - (a+b+1)x\right]y' - aby = 0. \]
The hypergeometric equation has the interesting property that the derivative of a solution satisfies an associated hypergeometric equation.

hypergeometric series, a series developed in the solution of the hypergeometric differential equation: 
\[ P(a,b,c,x) = 1 + \frac{ab}{c} x + \frac{a(a+1)b(b+1)}{2!c(c+1)} x^2 \]
\[ + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)} x^3 + \ldots . \]

The Legendre differential equation, a hypergeometrical equation, important for certain physical problems: 
\[ (1-x^2)y'' - 2xy' + n(n+1)y = 0. \]

Legendre polynomials, a particular solution of the Legendre differential equation when n is a positive integer or zero, resulting from substitution in the hypergeometric series. Following are the first few Legendre polynomials: 
\[ P_0(x) = 1, \quad P_1(x) = x, \]
\[ P_2(x) = 1/2(3x^2 - 1), \quad P_3(x) = 1/2(5x^3 - 3x), \]
\[ P_4(x) = 1/8(35x^4 - 50x^2 + 3). \] In developing these polynomials, if \( x = 1 \), we have 
\[ P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} (x^2 - 1)^n. \]
This is called Rodrigues' formula. 

clockwise, in the direction of the clock. 

closed interval, see interval. 

closed set, see set. 

closure, the property of a group, which states that if two elements of the group operate on each other, another element of the group is formed. 

coaxal circles, a system of circles such that the radical axis of any two circles which belong to the system is the same as the radical axis of any other pair of circles of the system. 

coefficient, see polynomial. 

cofactor, of an element of a determinant, the determinant which is its minor or the negative of its minor according as the sum of the row and column indices of the element is even or odd. 

cofunction, of an angle, the function of its complement, e.g., the cosine, cotangent and cosecant are the cofunctions of the sine, tangent and secant respectively. 

coredident variables, sets of variables which are subjected to the same transformation. 

coherece, the set of all coherents. 

coherecent, any limiting point of a set. These terms were used by Cantor. 

coincidence, the state of having all points in common.
collinear, lying on one line.
collineation, a projective transformation which carries points into points and lines into lines and preserves incidence relations.
cologarithm, the negative of a logarithm.
combinant, invariants, the determinants of bilinear forms, which depend on several sets of variables.
combination, a set of things without reference to the order of the things in the set.
commensurable quantities, any two quantities which contain the same suitable unit of measure an integral number of times.
common chord, see chord.
common difference, see arithmetic progression.
common fraction, see fraction.
common logarithm, see logarithm.
common tangent, see tangent.
commutative law, one of the fundamental laws of algebra which states that the order of the elements in a given operation is interchangeable, e.g., \( ab = ba \).
commutative field, see field.
compasses, an instrument for drawing and dividing circles and transferring measurements.
complement, see angle, complementary angles.
complementary set, the set of points obtained from a given set by omitting from the continuum the points of the given set.
completing the square, a method for solving a quadratic equation by adding to both sides of the equation the number which, together with the terms containing the variable, form a perfect square trinomial. The solution is completed by extracting the square root of both sides of the equation and solving the resulting equation for the unknown.

complete quadrangle, a figure of four points in a plane, no three of which are collinear, and the six lines joining them. Any two sides containing, between them, all four of the given points are called opposite sides and the three points of intersection of the pairs of opposite sides are called the diagonal points of the figure.

complete quadrilateral, a figure of four lines in a plane, no three of which are concurrent, and the six points in which they intersect.

complex argument, the argument of a complex number.

complex fraction, see fraction.

complex number, same as complex variable or number pair, the most general type of number, including real and imaginary numbers. Symbol, a + bi and its conjugate, a - bi; in polar form, r(cos θ + i sin θ). The modulus or absolute value of a complex number is the positive square root of the sum of the squares of the real numbers which make up the complex number, e.g., the modulus of a + bi is \(\sqrt{a^2 + b^2}\). If a = 0, the number is a pure imaginary.
complex series, series composed of complex numbers.

component, the parts of a quantity or the simple quantities of which a complex quantity is composed.

velocity component, the velocity of the x-point is called the x-component; the velocity of the y-point is called the y-component of the velocity of the point, \( P \). Symbols, \( v_x \) and \( v_y \).

composite function, a function of a function. Symbol:

\[
\frac{dF}{dx} = \frac{dF}{du} \frac{du}{dx} \quad \text{or} \quad \frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}
\]

where \( z \) is a function of \( y \) and \( y \) is a function of \( x \).

compound fraction, see fraction.

concave downward, a curve is said to be concave downward at a point if an arc of a curve containing the point in its interior lies below the tangent at that point.

concave upward, a curve is said to be concave upward at a point if an arc of a curve containing the point in its interior lies above the tangent at that point.

concentrated set, same as dense set.

concentric circles, circles with the same center.

conchoid, a higher plane curve, the locus of the equation: \( r = a \csc \theta \pm b \). This conchoid of Nicomedes is used to trisect an angle.

conclusion, that part of a conditional statement which can be established as a result of the hypothesis.
concomitant, same as invariant.
concurrent, passing through the same point.
conyclic, lying on the same circle.
condensation, see point of condensation.
condition,
linear, any condition which involves a linear relation among the coefficients of an equation.
independent, those which give rise to independent linear equations.
non-, a condition not linear.
necessary, a condition without which a thing cannot be true; if a relation inevitably follows a certain hypothesis or event, that relation is called a necessary condition for that event.
quadratic, any condition which involves a quadratic relation among the coefficients of an equation.
sufficient, the condition that is a satisfactory reason for saying a thing is true; if an event is the unavoidable consequence of a relation, the relation is called a sufficient condition for the event. Together, these two form the necessary and sufficient condition. To show that a condition is necessary and sufficient entails the proof of a proposition and its converse.
conditional equation, an equation which is true for only certain values of the variable.
cone, a developable surface in which rectilinear elements intersect in a point, called the apex. The length of an element of a cone of revolution is called the slant height; the perpendicular distance from the vertex to the base is the altitude of the cone; axis, see quadrics.

asymptotic cone, q.v.
circular cone, q.v.
frustum of, that portion of a cone which lies between the base and a plane parallel to the base.
of revolution, a circular cone whose axis is perpendicular to the base.
configuration, a figure of related points and lines.
conformal mapping or representation, same as isogonal representation, a transformation, under which the angle between any two curves is equal to the angle between the corresponding curves.
congruences, if the difference of two integers, a and b, is divisible by an integer, m, a and b are said to be congruent to each other with respect to modulus, m.
Symbol: $a \equiv b \mod m$.
congruent figures, figures that can be made to coincide in all their parts.
conic, the curve formed by a section of a right circular cone; same as conic section.

central, a conic with a single proper center, as the
circle or ellipse.

degenerate, a conic whose equation can be factored into linear factors, e.g. the axes and the origin.

focal, the conic generated with reference to a fixed point, called the focus, or points, focii, and a fixed line called the directrix or lines, directrices.

line, the envelope of lines joining corresponding points of two projective, nonperspective ranges of points.

non-degenerate, see conic section.

point, a conic generated by the intersections of corresponding rays of two coplanar projective pencils, which are not perspective.

proper, same as non-degenerate.

conic equation, general equation of the second degree,

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \]

conic section, same as conic, but defined in projective terms, the locus of points of intersection of the corresponding lines of two projective pencils. If the projectivity is not a perspectivity, the locus is a non-degenerate conic; if a perspectivity, a degenerate conic.

The non-degenerate conics are as follows:

ellipse, the conic with no ideal points, or analytically defined as the locus of a point in a plane that moves so that the sum of its undirected distances from two fixed points is equal to a constant. The circle is a special case when the axes are equal. The two fixed points are
called the foci; the line determined by the foci is called the major axis and the perpendicular to the major axis at its midpoint is called the minor axis. The equation of the ellipse: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \), where \( a \) and \( b \) are the squares of the \( x \) and \( y \) intercepts respectively.

Parabola, the conic with one ideal point, or the locus of a point that moves in a plane so that its distance from a given point is always equal to the distance from a given straight line. The given point is called the focus; the given line the directrix; the line through the focus perpendicular to the directrix is the axis. Equation of the parabola: \( y^2 = 4px \), where \( p \) is the distance from the focus to the directrix.

Hyperbola, the conic with two ideal points, or the locus of a point that moves in a plane so that the difference of the distances from two fixed points is constant. The given points are the foci; the line, determined by the foci, is the transverse axis, and the line perpendicular to the transverse axis, at its midpoint, is called the indefinite conjugate axis; the midpoint is called the center. The equation of the hyperbola: \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \), where \( a \) is half the distance between the curves, and \( b^2 = c^2 - a^2 \), \( c \) being half the distance between the foci.

Conicoid, see quadric surface. Conjugate, joined in pairs, reciprocally related, as
conjugate angles or conjugate axes, q. v.

**Conjugate diameters**, two axes of an ellipse or hyperbola, each bisecting the chords of the ellipse or hyperbola which are parallel to the other.

**Conjugate expressions**, two binomials which are identical except for the sign of the second term, e.g. \(3 + \sqrt{4}\) and \(3 - \sqrt{4}\); \(a + bi\) and \(a - bi\).

**Conjugate hyperbolas**, two hyperbolas so related that the transverse axis of each coincides with the conjugate axis of the other.

**Conjugate lines**, two lines so related that each contains the pole of the other with respect to a conic.

**Conjugate points**, two points so related that each is on the polar of the other with respect to a conic. Compare isolated point of a curve.

**Conoid**, the path of a straight line which, as it moves, touches two other lines, one straight and the other curved, and remains parallel to a given plane. When the generatrix of a right helicoid touches the axis, the helicoid is also a conoid.

**Consecutive**, following in uninterrupted succession.

**Consequent**, in vector analysis, the second term in a dyad; in proportion, the second terms in the equal ratios.

**Consistency**, the property of an axiomatic system which states that no contradiction must arise from any combinations of the axioms.
constant, see arbitrary constant.

constant of integration, the parameter which completes the indefinite integral.

collection, in elementary geometry, the drawing of figures that can be made by the compasses and straight edge alone. The criterion for such construction follows: a proposed construction is possible by straight edge and compasses if, and only if, the numbers which define analytically the desired geometric elements can be derived from those defining the given elements by a finite number of rational operations and extractions of real square roots.

The three problems of antiquity, the solutions of which, by ruler and compasses alone, have been attempted for so many years, are the duplication of the cube, the trisection of an angle and the quadrature of the circle. The duplication of the cube, stated algebraically, is to find \( x \) so that \( x^3 = 2a^3 \), \( a \) being the edge of the given cube. A rational root of \( x^3 = 2 \) would be integer which is an exact divisor of 2, but the cubes of \( \pm 1 \) and \( \pm 2 \) are distinct from 2. Hence, there is no rational root and it is impossible to duplicate the cube with straight edge and compasses alone. By means of curves drawn in space, Archytas, about 400 B.C., solved the problem. Plato proved that, if two right triangles, CAE and DAB, having one side, AB in common and their other sides, AD and BC, parallel and their hypotenuses, AC and BD, at right
angles intersecting at P, so that PD = 2PC, then
PC:PB = PB:PA = PA:PD and PB is the required line. It is
to make an instrument by which the triangles can
be constructed mechanically. Appolonius solved the problem
by constructing a rectangle whose length was twice its width
and, by means of shifting lines until three points were
collinear, found the required line. Menaeachmus used conics
to give a solution and Eratosthenes gave a description of
an instrument whereby the cube could be duplicated.

The second problem, the trisection of an angle, can be
shown to be impossible of construction by straight edge and
compasses alone by the use of a trigonometric identity and
the theorem that it is impossible to construct by straight
dge and compasses alone a line whose length is a root of
a cubic equation with rational coefficients having no
rational root. Hippias invented a curve, called the quad-
tratrix, by which any angle can be trisected. Pappus, with
a certain construction and shifting of lines, was able to
trisect any angle. The conchoid of Nicomedes can also be
used.

The quadratrix can also be used to solve the third
problem, the squaring of the circle. Hippocrates studied
this question, which means to find the side of a square
whose area is equal to that of a circle. Many others gave
their time to the study of these problems.

The theory of numbers shows us, too, that only a
polygon of $n$ sides where $n = 2^d p_1 p_2 \ldots p_k$ and $d$ is any integer, $p$ a prime of the form $p = 2^d + 1$ can be constructed with straight edge and compasses alone.

**content of point sets**, the notion, as originally defined by Hankel and Cantor, has been largely replaced by the notion of measure. The term is now used to mean Jordan measure, when the inner and outer content are the same.

- **inner content**, the limit of the sum of the lengths of the sub-intervals of a given interval, which contain only the inner points of a given set which lies in the interval.
- **outer content**, the limit of the sum of the lengths of the sub-intervals of a given interval, which contain both points of the given set and of the complementary set.

**continuity**, Aristotle's definition: a thing is continuous when of any two successive parts the limits at which they touch are one and the same and are, as the word implies, held together. The definition of continuity is involved in the definition of continuous function, q.v.

**uniform continuity**, if a function is continuous in a closed interval, then it has uniform continuity.

**continuous function**, a function is continuous along a curve, $C$, if it is continuous at every point of $C$. By **continuity** at a point $(x_0, y_0)$ of the curve, $C$, we mean that

$$\lim_{x \to x_0} F(x, y) = F(x_0, y_0)$$

when the point $(x, y)$ approaches $x_0, y_0$ along $C$, that is,

$$|F(x, y) - F(x_0, y_0)| < \epsilon$$

for all points
of the curve, \( C \), that satisfy the inequalities, 
\[ |x - x_0| < \delta \quad \text{and} \quad |y - y_0| < \delta \]
where \( \delta \) is a positive number.

**continuum**, any set which is everywhere dense and perfect.

**linear continuum**, same as **real continuum**, the type of order of points on a line and the type of all real numbers.

If an ordered class contains a denumerable sub-class, such that between any two elements of the ordered class there is an element of the sub-class, the ordered class is said to be **linearly continuous**.

**convergence**, (1) for sequence: if \((x_n)\) is a given sequence and there exists a limit, \( L \), such that the sequence \((x_n - L)\) is a null sequence, then the given sequence is said to be convergent and the elements \( x_n \) of the sequence are said to approach the limit, \( L \).

(2) for series: an infinite series \( \sum_{n=1}^{\infty} u_n \) is said to converge if the sequence of partial sums \( \{s_n\} \) is convergent.

**absolute convergence**, the series is said to converge absolutely if the series of absolute values converges.

**conditional convergence**, if a convergent series does not converge absolutely, it is said to converge conditionally.

**uniform convergence**, the series, \( \sum_{n=1}^{\infty} u_n(x) \), defined in the interval \((a, b)\), is uniformly convergent in that interval if, for any \( \epsilon > 0 \), there exists a number, \( N \), independent of \( x \) in \((a, b)\), such that 
\[ |S(x) - s_n(x)| < \epsilon \]
for all values of \( n \geq N \).

**fundamental principle of convergence**, Cauchy's theorem,
A necessary and sufficient condition for the existence of the limit of the sequence \( \{x_n\} \) is that for any \( \epsilon > 0 \) one can find a positive integer, \( N \), such that for any pair of indices \( m \) and \( n \), both greater than or equal to \( N \),

\[ |x_m - x_n| < \epsilon. \]

tests for convergence of sequences,

1. the Cauchy criterion, a necessary and sufficient condition for the convergence of the sequence \( x_1, x_2, \ldots, x_n, \ldots \) is that for any \( \epsilon > 0 \), one can find a positive integer \( N \) such that \( |x_{n+k} - x_n| < \epsilon \) when \( n \geq N \), and for every positive integer \( k \).

2. a monotone sequence which is bounded is convergent.

3. a necessary and sufficient condition for the convergence of a bounded sequence is that the upper limit, \( L \), be equal to the lower limit, \( l \).

tests for convergence of infinite series, of positive terms, 1-4.

1. comparison test, let \( \sum_{n=1}^{\infty} u_n \) and \( \sum_{n=1}^{\infty} v_n \) be two series of positive terms, the first of which is known to be convergent and the second divergent.

(a) if the terms of a given series \( \sum_{n=1}^{\infty} a_n \) of positive terms are such that \( a_n \leq u_n \), for every \( n \geq m \), then the series \( \sum_{n=1}^{\infty} a_n \) is convergent.

(b) if the terms of a given series \( \sum_{n=1}^{\infty} a_n \) of positive terms are such that \( a_n \geq v_n \), for every \( n \geq m \), then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.
2. **Abel's test**, if $\sum_{n=1}^{\infty} u_n$ is a convergent series of positive terms, and $\{a_n\}$ is any bounded sequence of positive numbers, then the series $\sum_{n=1}^{\infty} a_n u_n$ is convergent.

3. **d'Alembert's ratio test**, if the terms of a series $\sum_{n=1}^{\infty} a_n$ of positive terms satisfy, from some value of $n$ onward, the inequality $\frac{a_{n+1}}{a_n} \leq r < 1$ where $r$ is independent of $n$, then the given series is convergent. If, however, from and after some value of $n$, $\frac{a_{n+1}}{a_n} \geq 1$, then the given series diverges. The corollary to this theorem is easier to apply and therefore important. Cor., if the given series $\sum_{n=1}^{\infty} a_n$ is such that $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = r < 1$, then the given series converges. If $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} \geq 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

4. The **$p$-series**, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, converges for every $p > 1$ and diverges whenever $0 < p \leq 1$. If $p = 1$, we have the important harmonic series which always diverges.

5. **For alternating series**, a series whose terms are alternately positive and negative and such that their absolute values form a monotone null sequence is convergent. This leads to tests for absolute convergence.

6. A series $\sum_{n=1}^{\infty} a_n$ is convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

7. If the terms of a series are numerically greater than the corresponding terms of a known convergent series of positive terms, the series converges.
8. the ratio test and others apply to absolute convergence.

9. Cauchy's integral test, let \( \sum_{n=1}^{\infty} a_n \) be a series of decreasing positive terms. If there exists a positive monotone-decreasing function \( f(x) \) for \( x > 1 \), such that \( f(n) = a_n \), then the given series converges if the integral \( \int_1^{\infty} f(x) \, dx \) exists; the given series diverges if the integral does not exist.

tests for absolute and uniform convergence of series of functions,

1. Weierstrass M-test, let \( u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \) be a series of functions such that each \( u_1(x) \) is a bounded function of \( x \) in \( (a,b) \). If there exists a convergent series of positive constants \( M_1 + M_2 + \ldots + M_n + \ldots \) such that \( |u_1(x)| \leq M_1 \) for all values of \( x \) in \( (a,b) \), then the series \( u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \) is uniformly and absolutely convergent in \( (a,b) \).

2. Abel's test for power series, if a power series \( \sum_{n=0}^{\infty} a_n x^n \) is convergent for \( x = x_0 \), then it is absolutely convergent for every value of \( x \) such that \( |x| < |x_0| \).

On the other hand, if the series is divergent for \( x = x_0 \), then it is divergent for every \( x \) such that \( |x| > |x_0| \).

There are also tests for certain trigonometric series and the theorems carry over to the field of complex variables. The necessary and sufficient condition for an infinite series
of complex quantities is, that the series of real parts and the series of imaginary parts respectively converge. Absolute convergence enables one to use algebraic manipulation and uniform convergence justifies the use of calculus. Conversely, the statement formed by interchanging the hypothesis and conclusion in a given theorem.

convex, curving outward.

convolute, the path of a straight line which, as it moves, is always tangent to a curve of double curvature, such as the helix.

helical convolute, same as developable helicoid, a convolute when the directrix is a helix.

coordinate axes, see axes.

coordinates, see axis.

coplanar, situated in the same plane.

corollary, a statement easily derived from a previously proved statement.

correlation, the projective transformation which carries points into planes and planes into points and lines into lines with incidence preserved.

corresponding lines, those lines which have the same relative position in two similar figures.

corresponding vertices, those vertices which have the same relative position in two similar figures.

cos, symbol for cosine.

cosec, symbol for cosecant.
cosecant, see angle, functions.
cosine, see angle, functions.
cot, symbol for cotangent.
cotangent, see angle, functions.
coterminal, having the same terminal sides.
counterclockwise, opposite to the direction of the clock.
covariant, a function of the coefficients of a system of a number of polynomials and a number of sets of variables cogredient to the variables of the polynomials unchanged by any transformation of a certain set.
cramer's rule, if the coefficient determinant of a system of n linear equations in n variables has a value $A$, different from zero, then the system has a single solution, consisting of one value for each of the variables; these are equal to fractions whose denominators are all equal to $A$, and whose numerators are the values of the determinants obtained from the coefficient determinant by replacing the coefficient of each variable in turn by the known terms as they appear on the right-hand side of the equations.
criterion, a standard by which to determine the correctness of a conclusion.
critical values, a value for $x$ for which $f'(x)$ has the value 0 or at which $f'(x)$ is discontinuous.
cross ratio, same as double ratio or anharmonic ratio, a ratio of four points on a line or four lines on a point, having the relation $17/23:14/24$. It is the only non-trivial
metric property of four points which is invariant under projection. The cross ratio of four lines in a pencil is the ratio of the four points obtained by intersection with any transversal.

cross section, the intersection of the given figure and a plane passed through it.

cruciform curve, a higher plane curve, the locus of the equation: $x^2y^2 - a^2x^2 = 0$ where $a$ is a constant measured from origin on both axes.

csc, symbol for cosecant.

cube, a regular polyhedron of six faces, all squares, whose planes are each at right angles with four of the adjacent planes.

cube roots of unity, one, $-1/2 \pm 1/2 \sqrt{3}i$. Symbol for the last two, $\omega$ and $\omega'$.

cubic equation, an equation of the third degree; bipartite cubic curve, the locus of the equation: $y^2 = x(x-a)(x-b)$, $0 < a < b$, which is composed of two entirely separate branches.

cubic form, a homogeneous polynomial of the third degree.

cubic covariant, see binary cubic.

curvature, the limiting value of the ratio of the angle between the tangents to two points on a curve and the length of the arc between these points as one point approaches the other, measures the rate of change of the direction of the tangent and is called first curvature. Its reciprocal is the radius of first curvature.
center of curvature, see osculating circle. Cf. evolute.
mean curvature, the sum of the principal curvatures at the point.
total curvature, the product of the principal curvatures at the point.
curve, a line, no part of which is straight; "a continuous line" -Jordan; "a Jordan curve is a plane set of points which can be brought into continuous (1,1) correspondence with the points of a closed segment of a straight line" - Young; the path of a point which moves so as to change its direction continually.
simple curve, one which has no double points, that is, one on which there are always distinct points corresponding to different values of the parameter.
the parametric representation of a curve, any set of points defined by the parametric equations, which are two continuous functions of a third variable, that can be chosen in many ways so that all points of this arc of the curve (a connected and closed set of points which contains no inner points) and only these are obtained when we put \( x = \) first function and \( y = \) second, and allow the variable to take on the values on a given interval. When all points of the curve lie in the same plane, it is called a line of single curvature or a plane curve; when all points do not lie in the same plane it is called a line of double curvature or a space curve or a twisted curve.
general curve equation, any equation, \( f^n(x_1, x_2, x_3) = 0 \), homogeneous and of degree \( n \) in projective point coordinates represents a curve of order \( n \).

Plane curves, the conics: circle, ellipse, parabola, hyperbola, q.v.; also called point and line curves, according to whether they are the loci of points or lines.

Higher plane curves,

algebraic, see algebraic functions; bipartite cubic, cruciform, folium, hypocycloid, pilaster, quartic, serpentine, strophoid, trisectrix, witch and others, q.v.

transcendental, see algebraic functions; the curves of trigonometric functions, catenary, conchoid, cycloid, epicycloid, exponential, lemniscate, limacon, logarithmic, logarithmic spiral, rose, spiral of Archimedes and others, q.v. The graphs of the trigonometric functions and the inverse trigonometric functions are the same with a reversal of the axes, as are also the exponential and logarithmic.

Space curves, cylindroid, helix, loxodromic, minimal, spherical and others, q.v.

Integral curve, a curve whose equation is a particular solution of a differential equation of the first order obtained by giving different values to the constant in the general solution.

Norm curve, a curve of the \( n \)th degree in \( n \) dimensions.
rational curve, the locus of points whose ternary coordinates can be expressed as rational integral functions of a single parameter; also called unicursal, since the point describes the complete curve in a single circuit.

roulette, the locus of a point on a curve which is rolled along on a fixed curve without slipping. When the rolling curve is a straight line, it is called an involute, q.v.
cusp, same as stationary point of a curve, a point of a curve at which two branches meet and stop and have a common tangent; first species, the branches are on opposites of the tangent; second species, the branches are on the same side of the tangent.
cut, same as partition, a separation of all rational numbers into two sets in such a manner that every number in the first set is less than every number in the second set.
cyclic, pertaining to or moving in a circle. Cf. acyclic.
cyclic group, a group, all of whose operations can be formed by repetition of a definite one of them; a cyclic group of transformations is a transformation with all of its powers, positive and negative.
cyclic permutation, a permutation where the elements are kept in sequence as points on a circle.
cycloid, the locus of a fixed point on a circle as the circle rolls along a straight line.
cyclotomic equation, an equation, \( f(x) = \frac{x^n - 1}{x^k - 1} \).
\[ x^{t(p-1)} + x^{t(p-2)} + \ldots + x^t + 1 = 0, \quad t = p^s - 1, \] whose roots are all the primitive \( p \)th roots of unity, namely, roots of \( x^{p^s} = 1 \) which do not satisfy \( x^t = 1 \).

cyclotomic, a dyadic which combines properties of the cyclic dyadic and the tonic.

cyliner, a developable surface in which the rectilinear elements are parallel.

cylindrical coordinates, a system of coordinates in which the polar coordinates of the \( xy \) plane are combined with the \( z \) axis.

cylindroid, the path of a straight line which, as it moves, touches two curved lines and remains parallel to a given plane. This surface is used in architecture for joining the arched ceilings in two parallel corridors of different levels.
D.

decagon, a polygon of ten sides.
decimal fraction, see fraction.
decimal scale of notation, the representation of a number as a polynomial in 10 with integral coefficients, e.g., \(237 = 2 \cdot 10^2 + 3 \cdot 10 + 7\). Symbol \(\$37\).
deciliation, see angle, vectorial.
Dedekind, see Cantor-Dedekind axiom.
deductive reasoning, the method of reasoning which starts from a general principle or truth already established by definition, by assumption, or by previous reasoning, and draws the necessary conclusions.
defect, the difference between the angle sum of a triangle and two right angles in hyperbolic geometry.
definite integral, the integral defined in the following way: let \(f(x)\) be a continuous function defined in the interval \(a \leq x \leq b\). Let the interval \((a,b)\) be divided into \(n\) subintervals by inserting the points of subdivision \(x_i\) in such a way that \(a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\). Let \(\xi_i\) be any point in the interval of length \(\Delta x_i = x_i - x_{i-1}\). The limit of the sum \(\sum_{i=1}^{n} f(\xi_i) \Delta x_i\), as \(n \to \infty\) in such a way that each \(\Delta x_i \to 0\), is called the Riemann definite integral of \(f(x)\) between the limits \(a\) and \(b\). It is denoted by the symbol \(\int_{a}^{b} f(x) \, dx\).
definition of surfaces, stretching and bending.
degree, see polynomial.
delta, symbol Δ, attached to the variable indicates the increment, q.v.
deltoid, same as hypocycloid.

DeMoivre's theorem, if n is any positive whole number,
(cos θ + i sin θ)^n = cos nθ + i sin nθ.
denominator, see fraction.
dependent variable, the variable whose value is determined when the value of the independent variable is assigned.
derivative, same as differential coefficient or derived function, the limit of the ratio of the increment of the function to the increment of the independent variable when the initial value of that variable is held fixed and its increment varies and approaches 0 in any way. Symbols:
\( \frac{dy}{dx}, \frac{df}{dx}, Dxy, Dxf(x), y', f'(x). \)

left and right hand derivative, the derivatives of a bounded function, f(x), which is integrable in the interval (a,b) but has points of discontinuity.
nth derivative, the result of differentiating n times. Symbol: \( d^ny = f^n(x)(dx)^n. \)

partial derivative, the derivative of f(x,y) with respect to x, for y = \( y_0 \) when u = f(x,y) is a single-valued function of the independent variables x and y and is defined at some point \( (x_0, y_0) \) and for all values of \( (x, y) \) in some region, R, about the point \( (x_0, y_0) \). If y is set equal to \( y_0 \), u becomes a function of the single
variable \( x \), namely, \( u = f(x, y_0) \). In like manner, if a constant value \( x_0 \) is assigned to \( x \), the derivative with respect to \( y \) of the resulting function \( f(x_0, y) \) is called the partial derivative of \( f(x, y) \) with respect to \( y \), for \( x = x_0 \). Symbols: \( \frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, u_x, f_x(x, y) \).

The total derivative, if \( u = f(x, y) \) is a function of the variables \( x \) and \( y \) which in turn are functions of some independent variable \( t \), and if \( u = f(x, y) \) is continuous together with its partial derivatives, then

\[
\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}.
\]

Derived sets, a set consisting of all the limiting points of a given set.

Desargue's theorem on perspective triangles, two triangles (in the plane or in space) which are perspective from a point are also perspective from a line, and conversely.

Descartes' rule of signs, the number of positive real roots of an equation with real coefficients is either equal to the number of its variations of sign or is less than that number by a positive even integer. A root of multiplicity \( m \) is here counted as \( m \) roots.

Descriptive geometry, the geometry that treats of the exact representation of geometrical forms upon planes and of graphical solution on planes of problems involving these forms in space.

determinant, a square array of numbers to which a single
number, called the value of the determinant, is attached. The value is the algebraic sum of all possible products obtainable by taking one and only one factor from each row and from each column, preceded by the plus or minus signs, according as the number of inversions of the column indices of the factors of a product are even or odd, when the row indices are in the natural order 1, 2, 3, etc.

bordered determinant, a determinant of the nth order to which we add one or more rows and the same number of columns of n quantities each and fill in the vacant corners with zero.
catalecticant, a determinant the vanishing of which expresses the condition that the quartic have an apolar quadratic.
Cayleyan, q.v.
cofactor of an element of a determinant, see cofactor.
Hessian, q.v.
Jacobian, q.v.
minor of a determinant,
minor of an element of a determinant, see minor.
rank of a determinant, if a determinant of order n is not zero, it is said to be of rank n. If an r-rowed minor is formed by erasing from the determinant all but r rows and all but r columns and if for 0<r<n, the r-rowed minor is not zero, while every (r+1)-rowed minor is zero, the determinant is said to be of rank r.
**symmetric**, the determinant of a square matrix, whose pairs of terms situated symmetrically with respect to the principal diagonal are equal.

**skew-symmetric**, a determinant whose negative is equal to its adjoint.

**developable surface**, see surface.

**development of a curved surface**, a plane area of such form and size that it can be rolled or folded to form again the original surface; development is obtained by rolling the surface upon a tangent plane until each part of the surface comes in contact with the plane. The part of the plane thus covered is the development of the surface. Theoretically double curved surfaces can not be developed, but approximate developments are made and used in practice.

**dextrorsum**, when a point moves along a curve in a positive direction, it passes from the negative to the positive side of the osculating plane if the torsion is negative and the curve is said to be dextrorsum.

**diagonal**, a line joining opposite points.

**major diagonal of a matrix**, same as **principal diagonal**, the diagonal from the upper left-hand corner to the lower right-hand corner.

**diagonal points**, see complete quadrangle.

**diameter**, see circle.

**conjugate**, q.v.

**diometral complex numbers**, two complex numbers having the
relation: \( x' = \frac{-x}{x^2 + y^2} \), \( y' = \frac{-y}{x^2 + y^2} \); \( x \) and \( x' \), \( y \) and \( y' \) are **diametral points**.

diametral points, see diametral complex numbers.

difference, the result obtained by subtracting one number from another.

**common difference**, see arithmetic progression.

**finite difference**, q.v.

difference equation, an equation used in solving problems in probability.

differential, of an independent variable is any assigned increment of the variable; the differential of the dependent variable, in a functional relation with the independent variable, is the product of the derivative of the function by the differential of the independent variable. Symbol: \( dy = f'(x) \, dx \).

**exact differential**, a differential equation which has the exact form of the differential of a function of the variables involved.

differential calculus, see calculus.

differential coefficient, see derivative.

differential equation, an equation involving a function and certain of its derivatives.

**partial differential equations**, a differential equation of two or more independent variables with the derivatives of the dependent variables with respect to the independent
variables; these are partial derivatives.

differential triangle, see Barrow's triangle.

differentiable, possessing derivatives.

differentiable functions, functions possessing derivatives.

differentiate, to find the derivative.

differentiation, the process by which the ratio of change of
one variable to the change of the other variable on which
it depends is determined.

partial, the process of finding partial derivatives.

successive, the process of differentiating the derivative
and thus finding the higher derivatives or successive
derivatives.

total, the process of finding total derivatives.

digit, any one of the Arabic numerals.

dihedral angle, the angle formed by two intersecting planes.

dihedron, the limiting form of a regular polyhedron, consist-
ing of but two faces which are coincident regular polygons
of n vertices.

dimension, any measurable extent, as length, breadth and
thickness.

diminish, to lessen.

direction angles, the angles between $-180^\circ$ and $+180^\circ$ which
the directed line makes with the positive directions of
the coordinate axes.

direction cosines, the cosines of the direction angles of
a directed line.
direction field, an array of lineal elements showing the
directions of the integral curves.
directrix, the fixed line or lines in the generation of a
conic which together with the focus or foci determine
its shape. Cf. eccentricity.
Dirichlet's summation of Fourier series: if \( f(x) = a_0 + \sum (a_n \cos nx + b_n \sin nx), 0 < x < 2\pi \), the formulas for
the coefficients in the expansion are as follows:
\[
a_0 = \frac{1}{2\pi} \int_{0}^{2\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos nx \, dx,
\]
\[
b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin nx \, dx.
\]

discontinuity, the state of a function which is discontinuous
at a point.

finite or ordinary, when the function is bounded in the
vicinity of the point.
infinite, when the function becomes infinite as the
variable approaches the point.

discontinuous functions, functions which fail to be
continuous at a point.
discrete set, see set.
discriminant, of any equation in which the coefficient of
the highest power of the unknown is unity, the product of
the squares of the differences of the roots.

of the general quadratic equation in one unknown:
\[ ax^2 + bx + c = 0, \] the discriminant is \( b^2 - 4ac \).
of a quadric surface, the determinant of the symmetric square matrix, e.g., \( \Delta = (a_{ij}) \), \( i,j = 1,2,3,4 \),
\( a_{ij} = a_{ji} \) for the equation: \( Q(x,y,z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{13}xz + 2a_{12}xy + 2a_{41}x + 2a_{24}y + 2a_{34}z + a_{44} = 0 \).

distance, length of separation in space, or, by extension, in time.
distributive law, one of the fundamental laws of algebra, expressed by the equation: \( a(b + c) = ab + ac \).
divergence, the failure to satisfy the conditions for convergence.
divergent sequence, a sequence which is not convergent. Cf. convergence.
divergent series, a series which is not convergent. Cf. convergence.
divergence theorem, same as Green's theorem, an important theorem of advanced calculus that establishes the connection between the integral over the volume and the integral over the surface enclosing the volume. (See Advanced Calculus, I. S. Sokolnikoff, pp. 167-169.)
dividend, a number to be divided.
division, the operation of finding how often one quantity is contained in another.
divisor, the number by which the dividend is divided.
dodecagon, a polygon of twelve sides.
dodecahedron, a polyhedron of twelve faces.
domain of rationality, see surface.
double curved surface, see surface.
double-flex tangent, a line which is tangent at each of
two distinct points of inflection.
double integral, defined in an analogous manner to the
definite integral, the region of integration corresponding
to the interval of integration and the term refers to the
dimensionality of the region. Symbol: \[ \int_a^b \int_c^d f(x,y) \, dx \, dy \] The integral is evaluated by considering
\( f(x,y) \) as a function of \( x \) alone but containing \( y \) as a
parameter, and integrating it between \( x = a \) and \( x = b \) and
then integrating the resulting function of \( y \) between \( y = c \)
and \( y = d \). The right hand member is known as an iterated
integral.
double ratio, see cross ratio.
double series, see series.
duality, the principle arising from the algebraic relation
between two variables considered as point or line coordi-
nates, whereby the elements, point and line are inter-
changeable. Since the algebra of dual theorems is identical,
the proof of either establishes the validity of both.
Dupin, cyclides or, the envelopes of two one-parameter
families of spheres.

indicatrix of, the conic sections, which are loci of
certain moving points in the study of curvatures.

duplication of the cube, see construction.
dyad, a matrix of rank one; an expression ab formed by
the juxtaposition of two vectors without the intervention
of a dot or a cross. The first vector in a dyad is called
the antecedent, and the second, the consequent.
dyalitic method of elimination, see Bezout's method of
elimination.
E.

$e$, the base of the natural system of logarithms; the
limit of the expansion of $(1 + x)^{1/x}$ as $x \to 0$ gives the approximate value 2.71828, ... .

**eccentricity**, the constant ratio of the undirected dis-
tance of a point from a fixed point, called the focus,
to its undirected distance from a fixed line, the corre-
sponding directrix, in the generation of a conic.

**edge**, the intersection of two planes with relation to the
figure so formed.

**edge of regression**, see tangent surface of a curve.

**eliminant**, same as resultant. The solution of simultaneous
equations depends upon the resultant being equal to zero.

Bezout, Euler, and Sylvester all had methods of elimina-

**ellipse**, see conic section.

**ellipsoid**, a quadric surface, the locus of the equation:

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \quad \text{if } c = b \text{ it is called a prolate spheroid;}
$$

if $c = a$, an oblate spheroid.

**elliptic functions**, a class of functions with the following
properties: if $f(z)$ be any function of the class, then
$f(z)$ is a one-valued function of $z$ with no singularities
other than poles in the finite part of the $z$ plane; $f(z)$
satisfies, for all values of $z$, the equations $f(z + 2w_1) =
$ $f(z)$ and $f(z + 2w_2) = f(z)$ where $w_1$ and $w_2$ are two quantities
independent of $z$. Functions $f(z)$ are said to admit $2w_1$ and $2w_2$ as periods and are called doubly periodic functions.

elliptic geometry, see spherical geometry.

end point, a point of a curve at which a single branch of the curve stops.

enumerable set, see set.

equation, the equation $f(x,y,a) = 0$ represents a curve if $a$ is constant. If, however, $a$ is allowed to assume a series of values, the equation represents a series of curves called a family and if all the curves are tangent to another curve, the curve to which each curve of the family is tangent is called the envelope of the family.

epicycle, a circle that rolls upon the circumference of another circle.

epicycloid, the locus of a point on the circumference of a circle that rolls tangent externally to a fixed circle.

equation, a statement of equality between two equal expressions. The expressions are called the members of the equation.

algebraic, equations written in algebraic form; these may be in terms of the variables alone or in addition to the variables may have the derivatives or integrals of the variables, when they are called differential or integral equations, q.v.

transcendental, equations which involve other than algebraic functions, e.g., trigonometric functions.
degree, of the equations, according to the term of highest degree: linear, quadratic (quadric, in surfaces), cubic, quartic or biquadratic, quintic, ..., n-ic, q.v. If the terms are all of the same degree it is called a homogeneous equation.

form of the equation, exponential, in which the variable appears as an exponent; radical, in which the variable appears under a radical sign or with a fractional exponent; equations may be given in different systems of coordinates, homogeneous, polar or rectangular.

locus, equations which represent curves or surfaces, q.v. relation, of the variables in the equation or of equations in a system. A function is the expression of the relation of two variables, see function.

conditional, q.v.

consistent, same as simultaneous, a system of two or more equations with a common solution, e.g., \( x+y=8 \); \( x-y=2 \).

derived, equations so related that one is transformed into the other by one or more of the fundamental operations, e.g., \( x-2=0 \); \( x^2 - 5x+6 = 0 \) by multiplying the first equation by \( x-3 \).

equivalent, equations which have the same roots, e.g., \( x-2=0 \); \( 7x = 14 \).

identical, same as identities, equations which are true for all values of the letters or symbols for which the two numbers are defined, e.g., \( 3a + 5a = 8a \).
inconsistent, a system of equations with no solution, e.g., \( x + y = 5; \ x + y = 7 \).

independent, equations which are formed from certain conditions which involve relations among the coefficients of the equation, e.g., the coefficients of the general equation of a line, \( ax + by + c = 0 \), must satisfy the equation \( a + b + c = 0 \), if the line is to go through the point \( (1,1) \).

linearly independent, equations which do not belong to a linear system.

indeterminate, an equation which has an unlimited number of solutions, e.g., \( x - 3y = 5 \).

reciprocal, an equation such that the reciprocal of each root is itself a root of the same multiplicity.

variables, equations are named according to the number of variables involved in the corresponding form, as binary, ternary, quaternary, ..., p-ary.

Certain equations have become famous and are often identified by the name of the one who developed or discovered them, e.g., Clairaut's equation and Laplace's equation, q.v.

equiangular, having equal angles.

equianharmonic, if \( \lambda \), the cross ratio of four numbers, is equal to \( \omega \), the complex cube root of unity, the six ratios reduce to \( \omega \) or \( \omega^2 \) and the four points are called equianharmonic.

equicenter, see centers of triangles.
equiangular geometry, the geometry which preserves shape only, namely the geometry of similar figures, whose characteristic instruments are the straight edge and the pantograph.
equilateral, having equal sides.
equipotential surface, a surface of constant potential.
equivalent equations, see equation, relation.
esential constants, the necessary arbitrary constants in the general equation of a curve or surface.
Euclid's algorithm, Euclid's method of finding the highest common factor: using the polynomial of lesser degree as divisor and in turn using it as dividend and the remainder of the first division as divisor, continuing this process until the remainder is zero.
Euclidean geometry, the geometry based on Euclid's parallel axiom; the geometry of size and shape, whose characteristic instruments are the straight edge and the compasses.
Euler's formulas, \( \cos x + i \sin x = e^{ix} \) and \( \cos x - i \sin x = e^{-ix} \).
Euler function, \( \phi(n) = n \left(1 - \frac{1}{a}\right) \left(1 - \frac{1}{b}\right) \cdots \left(1 - \frac{1}{k}\right) \), for finding the number of positive integers greater than \( n \) and prime with \( n \), where \( a, b, \text{ etc.} \) are prime factors of \( n \).
Euler's theorem on the relation of the edges, faces and vertices of any polyhedron, given by the following formula: \( e + f = v + 2 \).
evolute, see involute.
exact differential, a form $M dx + N dy$ frequently occurs, where $M$ and $N$ are functions of $x$ and $y$. If a function $z = f(x,y)$ exists, such that $dz = M dx + N dy$, then $M dx + N dy$ is called an exact differential and $M dx + N dy = 0$ is called an exact equation.

equation, see exact differential.

excenter, see centers of triangles.

circle, the circle drawn from the excenter as center, tangent to the side of the triangle nearest to it.

explicit function, a function expressed directly in terms of the independent variable.

exponential curve, the graph of the equation: $y = a^x$, $a > 1$.

exponential equation, see equation, form.

exterior angle, see angles, according to position.

exterior point of a set, see set.

extraneous root, a root which satisfies the derived equation but not the original equation.

extreme, the first and last terms of a proportion.

extreme and mean ratio, the division of a segment in such a way that the longer part is the mean proportional between the whole segment and the shorter part. It is also called the golden section or the divine proportion.
factor, the expressions whose product gives a new expression are called factors of the new expression.

Factor Theorem: if \( f(c) \) is zero, the polynomial \( f(x) \) has the factor \( x - c \). In other words, if \( c \) is a root of \( f(x) = 0 \), \( x - c \) is a factor of \( f(x) \).

Factorial notation, symbol \( n! \), denotes the product of all the positive integers from 1 to \( n \) inclusive. \( n! = n(n-1)(n-2) \ldots 3\cdot2\cdot1 \).

Family, see envelope.

Fejer's theorem on Fourier series, a theorem applying an earlier one, due to Frobenius, to the Fourier series, stating that the sum of the series is equal to the sum found by taking the arithmetic mean.

Fermat's Theorem: if a function \( f(x) \) assumes a maximum or a minimum value at an interior point \( x_1 \) of the interval \( (a,b) \), and if \( f(x) \) is differentiable at \( x = x_1 \) then \( f'(x_1) = 0 \).

Fermat's Last Theorem: no whole numbers or fractions exist such that \( x^2 + y^2 = a^2 \) or \( x^4 + y^4 = a^4 \) or, generally, such that \( x^n + y^n = a^n \) if \( n \) is a whole number greater than 2.

Field,

commutative, a set or class of numbers which satisfy the eleven laws of addition and multiplication.

finite, a commutative field which has the following properties: the elements form an Abelian group with respect to an operation called addition. The identical
element in this group is designated by 0. The elements, with the exception of 0, form an Abelian group with respect to an operation called multiplication. The identical element in this group is designated by 1. The distributive law holds for any four elements, that is,

$$(a + b) \cdot (c + d) = ac + ad + bc + bd.$$  

Carnot, q.v.

tangent, q.v.

finite, that which is within the realm of the senses.

finite discontinuous group, see automorphic function.

finite integral, same as proper integral, an integral whose function is bounded in the given interval and whose limits of integration are both finite.

finite series, series with a finite limit.

finite set, see set.

finitesimal, denoted by the ordinal of a finite number.

flecnodal, a double point formed by the tracing point passing through a point of inflection.

flex, see point of inflection.

focus, the fixed point or points in the generation of a conic which, with the corresponding directrix or directrices, determine its shape. Cf. eccentricity.

folium, of Descartes, the locus of the equation $x^2 + y^2 - 3xy = 0$.

follows, symbol $>$; also, greater than.
form, a rational, integral, homogeneous, algebraic
function in any number of variables.
formula, a rule expressed in algebraic symbols.
four-group, a group of order four.
four-step rule for differentiation, in the function \( y(x) \),
replace \( x \) by \( x + \Delta x \) and \( y \) by the corresponding value
\( y + \Delta y \). Find value of \( \Delta y \) (increment of the function) by
subtracting the given value of the function from the new
value. Divide the value of \( \Delta y \) by \( \Delta x \). Find the limit of
the quotient \( \frac{\Delta y}{\Delta x} \) thus found when \( x \) is held fixed and \( \Delta x \) is
allowed to vary and approach 0 as a limit. This limit is
the derivative of \( y \) with respect to \( x \).
Fourier series, the expansion of any function of a variable
in a series of sines and cosines of multiple angles,
\[ f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \]
fourth-dimensional geometry, an extension of the geometry
of three-space by the addition of certain axioms to the
system.
fourth proportional, the fourth term in a proportion.
fraction, the expressed quotient of two integers; the term
of the fraction which expresses the number of equal parts
into which the unit is divided is called the denominator;
the term of the fraction which denotes how many of the
equal parts of the unit (expressed by the denominator)
are taken is called the numerator.
common, a fraction whose numerator and denominator are both expressed.

complex, a fraction, one or both of whose terms are mixed numbers.

compound, a fraction, one or both of whose terms are themselves fractions.

decimal, a fraction in decimal notation without the expression of the denominator.

improper, a fraction whose numerator is equal to or greater than, or of higher degree than the denominator.

partial, the representation of a proper fraction as the sum of several fractions.

proper, a fraction whose numerator is less than, or of lower degree than the denominator.

simple, a fraction whose numerator and denominator are both whole numbers.

Frenet-Serret formulas, formulas fundamental in the theory of twisted curves: \( a' = \frac{1}{\rho}, \quad \beta' = \frac{m}{\rho}, \quad \gamma' = \frac{n}{\rho} \)
\[ \mathcal{L}' = - \left( \frac{a'}{\rho} + \frac{\lambda}{\tau} \right), \quad m' = -\left( \frac{\beta'}{\rho} + \frac{\mu}{\tau} \right), \quad n' = -\left( \frac{\gamma'}{\rho} + \frac{\nu}{\tau} \right) \]
\[ \lambda' = \frac{\mathcal{L}}{\tau}, \quad \mu' = \frac{m}{\tau}, \quad \nu' = \frac{n}{\tau} \]

where \( \tau = \) radius of torsion, \( \lambda, \mu, \nu \) are coordinates of the representative curve on the unit sphere formed by taking the locus of the end points of radii parallel to positive binormals of the curve. \( a, \beta, \gamma; \mathcal{L}, m, n; \lambda, \mu, \nu \) are the direction cosines; \( a', \beta', \gamma'; \)
\( l', m', n' \); \( \lambda', \mu', \gamma' \), their derivatives; \( \rho \) is the radius of first curvature.

**Frustum,**

*of a cone*, see cone.

*of a pyramid*, see pyramid.

**Function,** the relation between two variables such that to every value of one of them (taken at will in a certain range) there corresponds a determined value of the other; the second variable is said to be a function of the first. Functions may be classified in various ways: algebraic and transcendental, see algebraic functions; continuous and discontinuous, e.g., \( y = x^2 - 3x^2 + 3x \) is a continuous function, while the function, \( y = \tan x \) is a discontinuous function; the first of these functions is algebraic and the second is transcendental; functions may be explicit or implicit, q.v.

**Function notation,** function of \( x \): \( y = f(x) \), \( y = f'(x) \), \( y = F(x) \), \( y = \phi(x) \) or in the case of functions of more than one variable \( z = f(x,y) \) etc.

**Fundamental region of a complex variable,** a region in which a single-valued function of a complex variable takes on all of its values once and only once.

**Fundamental theorem of algebra,** every algebraic equation of the \( m \)th degree has exactly \( m \) roots in the field of complex numbers of the form \( a + bi \), where multiple roots are counted according to their order of multiplicity.
fundamental theorem of integral calculus, if \( f(x) \) is continuous in the interval \( a \leq x \leq b \) and \( G(x) \) is a function such that \( \frac{dG}{dx} = f(x) \) for all values of \( x \) in this interval, then \( \int_{a}^{b} f(x) \, dx = G(b) - G(a) \).

fundamental theorem of projective geometry, the cross ratio of four points on a line is invariant under projection.

fundamental theorem of the theory of functions, if a function \( f(z) \) of a complex argument is regular in a simply connected domain \( E \), then \( \oint_{\gamma} f(z) \, dz = 0 \) for every closed curve which lies entirely inside of \( E \).
**Gallilean region,** a finite region where, with respect to a suitably chosen space of reference, material particles move freely without acceleration and in which the laws of the special theory of relativity hold with remarkable accuracy.

**Galois field,** a finite field of order $p^n$ where $p$ indicates a prime, made up of the $p^n$ classes of residues, mod $p$, $P(x)$ obtained from the elements of a finite field of order $p^n$.

**Gamma function,** the improper integral

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx,$$

where $n > 0$.

**gauche,** left-handed, same as skew, applied to two straight lines which do not lie in the same plane.

**generatrix,** see surface.

**geodesic curvature,** the curvature of a normal section, q.v.

symbol: $\delta s/s = 0$.

**geometric average,** same as geometric mean.

**geometric mean,** any term between any two terms in a geometric progression; more specifically, the one mean between two numbers, called the average or mean; hence the geometric mean between two numbers equals the square root of their product. It is also called the mean proportional.

**geometric progression,** the sequence of numbers so related that there is a constant ratio between each term and the following term. The formula for the general term is:

$$L = a r^{n-1}$$

where $a$ is the first term, $n$ is the number of the term and $r$ is the ratio. The formula for
the sum of \( n \) terms is: \( S = \frac{a-r^n}{1-r} \); the formula for the sum of an infinite number of terms is: \( S = \frac{a}{1-r} \).

**geometric series**, see partial sum sequence.

**geometry**, the study of figures with their properties and relationships.

**Gergonne point**, the point in which lines from the vertices of a triangle to the points of contact of the incircle are concurrent.

**golden section**, see extreme and mean ratio.

**gradient**, see operator.

**Gramian**, a determinant, the vanishing of which is both a necessary and sufficient reason for the setting up of linear dependence for linear differential equations of the \( \mu \)th order.

**graph**, an analytical representation of algebraic equations referred to a set of axes for reference.

**great circle**, the section of a sphere made by a plane passing through the center of the sphere.

**Green's theorem**, see divergence theorem.

**ground form**, a rational function of the coefficients of a form or system of forms, which, when these forms are subjected to any non-singular linear transformation, is merely multiplied by the \( \mu^\tau \) power (\( \mu \) an integer) of the determinant of the transformation is called a relative invariant of weight \( \mu \) of the form or system of forms. The forms themselves are called ground forms.

**group**, a class of elements having one law of combination and satisfying the following properties: closure, associative, inverse and the identity, e.g., the set of real
numbers; the set of rational numbers; the set of all integers with respect to the operation of addition.
half-open interval, see interval.

harmonic average, same as harmonic mean.

harmonic conjugates, in an harmonic range each pair of points
separated by the other pair.

harmonic mean, any term between any two terms in an harmonic
progression; more specifically, the one mean between two
numbers. The arithmetic mean must first be found between
the reciprocals of the given numbers. The reciprocals of
these arithmetic means are the required harmonic means.

harmonic motion, every rectilinear motion under the influence
of a force always directed towards the origin and in mag-
nitude proportional to the distance from the origin is ex-
pressible in the form: \( \chi(t) = A \sin \omega(t - t_0) \). This function be-
haves like the sin function and so, such motions are some-
times called sinusoidal motions. Any state of the motion
recurs after a time \( T = \frac{2\pi}{\omega} \) and is therefore called periodic
with period \( T \) or frequency \( \frac{1}{T} \) with maximum displacement
which is called the amplitude of the motion.

harmonic pencil, a pencil of four lines which pass through
four harmonic points of a line.

harmonic perspectivity, same as reflection.

harmonic points, four distinct points on a line whose cross
ratio is equal to \(-1\).

harmonic progression, a progression formed by the recip-
rocals of an arithmetic progression.

harmonic range, a range of harmonic points.

harmonic sequence, same as harmonic progression. Symbol: \( \left\{ \frac{1}{n} \right\} \).
harmonic series, the series formed from a harmonic progression. Symbol: \( \sum_{n=1}^{\infty} \frac{1}{n} \).

height, the distance measured on a perpendicular from the vertex of a figure to its base.

slant height, of a cone, the length of an element of a right circular cone.

of a regular pyramid, the altitude of a lateral face.

Heine-Borel theorem, a fundamental theorem of point sets: given any closed set of points on a straight line and a set of intervals so that every point of the closed set of points is an internal point of at least one of the intervals, then there exists a finite number of the given intervals having the same property.

helical convolute, see convolute. Cf. helicoid.

helicoid, the surface formed by a straight line moving about an axis in such a manner that each point of the moving line traces a helix. All the helices must have the same axis and the same pitch (angle). The generating line may or may not intersect the axis of the surface but cannot be parallel to it. If the generatrix forms an acute angle with the axis, an oblique helicoid is formed as in a triangular screw thread; when the generatrix is tangent to a helix traced by one of its points, the surface generated is a developable helicoid or helical convolute. The ordinary straight line helicoid is sometimes called a screw surface.

helix, same as circular helix.

hemisphere, a half sphere.
heptagon, a polygon of seven sides.

heptahedron, a polyhedron of seven faces.

Hermite's polynomials, same as Hermitian forms, let $S$ be any field containing a number $\gamma$ which is not the square of any number of $S$. Then all the numbers $\chi = a + b\gamma^{\frac{1}{2}}$, in which $a$ and $b$ belong to $S$, form a larger field $F$. Call $\overline{\chi} = a - b\gamma^{\frac{1}{2}}$, the conjugate of $\chi$, e.g., let $S$ be the field of all real numbers, and $\gamma = -1$. Then $F$ is the field of all complex numbers and $\chi$ and $\overline{\chi}$ are conjugate imaginaries. The bilinear form, with coefficients in $F$, $\sum_{i,j} a_{ij} \chi_i \chi_j$, $a_{ij} = \overline{a_{ji}}$ is called an Hermitian bilinear form in $F$.

Hessian, same as canonizant of the cubic, the determinant whose elements are second partial derivatives of a function of two variables. Symbol:

$$H = \begin{vmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{vmatrix}$$

hexagon, a polygon of six sides.

Brianchon's, see Brianchon's theorem.

Pascal's, see Pascal's theorem.

hexahedron, a polyhedron of six faces; a regular hexahedron is the cube.

hodograph, the locus of the terminus of vector velocity.
holomorphic functions, single-valued, regular (monogenic) functions, continuous in the given domain.

homogeneous, having all of its terms of the same degree.

homogeneous coordinates, see Cartesian coordinates.

homogeneous equation, see equation.

homogeneous form, see form.

homogeneous strain, see strain.

homologous points, the corresponding points on two circles formed by the intersections of lines drawn through the center of similitude. Of two pairs of homologous points, the first point of the first pair and the second point of the second pair or the second point of the first pair and the first point of the second pair are called anti-homologous points.

homology, figures which are in plane perspective are said to be in homology or to be homological. The center of perspectivity is called the center of homology, and the axis of perspectivity is called the axis of homology.

homothetic axis, same as axis of similitude, q.v.

dhomothetic center, same as center of similitude, q.v.

dhorizontal, parallel to the horizon.

Horner's method, for computing a real root of a real equation to any required degree of accuracy. A continued process of synthetic division whereby the interval containing the root may be narrowed down by noting the change of sign of the solution of the equation formed by the last two terms ignoring terms of the second degree and beyond, but guided by them in determining the choice of the next divisor. In
each division the resulting numbers give the coefficients of the transformed equation. The number of transformations made by synthetic division should be about half the number of significant figures desired for the root. This method applies only to polynomial equations.

horocycle, a conic having a four point contact with the absolute; the limit of the circle as the center recedes to infinity; a term in hyperbolic geometry.

hyperbola, see conic section.

hyperbolic functions, certain functions involving the exponential functions, the study of which arose in connection with the hyperbola.

hyperbolic geometry, the geometry built on the hyperbolic axiom of parallels, which states that these lines are not equidistant. There are non-intersectors as well as parallels.

hyperbolic paraboloid, the locus of a straight line which, as it moves, touches two straight lines which are not in one plane and remains parallel to another given plane; the moving line is the generatrix, the fixed lines, the directrices, the given plane, the plane director. Any position of the generatrix is called an element of the surface. This surface has a second rectilinear generation in which any two rectilinear elements of the first generation may be taken as directrices and a plane parallel to the first directrices as the plane director. It follows that through any point of the hyperbolic paraboloid, two rectilinear elements can always be drawn. The pilot or "cow catcher" of an American locomotive is usually of the form of the
hyperbolic paraboloid.

**hyperboloid of revolution,**

of one sheet, the locus of a straight line, the generatrix revolving about another straight line as an axis, the generatrix and axis not being in the same plane. This is the only warped surface of revolution.

of two sheets, the locus of an hyperbola revolving about the transverse axis.

**hypercomplex number,** an extension of the notion of the complex number, e.g., matrices, where each element is a matrix of order two and quaternions based on four units 1, i, j, k with the use of real numbers as coefficients to represent the quaternion, $\mathbf{\chi} = \chi_0 + \chi_1 \mathbf{i} + \chi_2 \mathbf{j} + \chi_3 \mathbf{k}$.

**hypergeometric differential equations and series,** see classical equations.

**hyperspace,** the space of more than three dimensions.

**hypersphere,** the locus of the equation of the type:

$$\chi_1^2 + \chi_2^2 + \chi_3^2 + \chi_4^2 + \ldots + \chi_n^2 = R^2$$

where each letter with a subscript represents an independent variable quantity; the analogue in $n$-dimensional space of the three-dimensional sphere.

**hypocycloid,** a higher plane curve, the locus of a point fixed on the circumference of a circle that rolls, tangent internally to a fixed circle. The three cusped hypocycloid is called the **deltoid.**

**hypotenuse,** the side opposite the right angle in a right triangle.

**hypothesis,** that part of a statement which contains the assumption.
i, the imaginary unit; a quantity which satisfies the fundamental laws of algebra and has for its square the negative number \(-1\).

Ischnography, the art of drawing by means of compasses and rule, or of tracing planes.

Icosahedron, a polyhedron of twenty faces.

Ideal line, in two dimensions, the class of all ideal points; in space, the class of all parallel planes. Symbol: \(l\).

Ideal plane, the complete class of all ideal lines.

Ideal point, the class of parallel lines; an improper pencil; a point at infinity; symbols: \(\nearrow\) denoting direction of parallel lines of the improper pencil or the ideal point on a proper line; \(\nearrow\).

Idem factor, the unit matrix; the dyadic which applied as a prefactor or a postfactor to any vector always yields that vector. Symbol I.

Identical equation, see equation, relation.

Identical polynomials, polynomials in which the coefficients of like powers of the unknown are the same.

Identity, that property of a group whereby a certain element when acting upon another element of the group makes no change in that element, in other words the result is identical with the element chosen. Symbol \(\equiv\).

Imaginary axis, see axis.

Imaginary number, see i.

Imaginary root, root containing imaginary numbers.
**implicit function**, a function defined by means of a relation connecting it with the independent variable, but not giving an explicit expression for it.

**improper fraction**, see fraction.

**improper integral**, same as **infinite integral**, if \( f(x) \) becomes infinite in the interval \( a \leq x \leq b \), or if the limits of integration become infinite, then the symbol \( \int_a^b f(x) \, dx \) is called an improper integral. Improper integrals are defined as the limits of certain functions which arise from consideration of ordinary definite integrals.

**improper pencil of lines**, see ideal point.

**incenter**, see centers of triangles.

**incidence**, the fundamental geometric property of projective geometry, the only relations preserved by projective transformations which carry points into points or points into lines, and lines into points or lines into lines.

**incircle**, the circle inscribed in a polygon.

**inclination**, the angle which a line makes with the \( x \) axis measured in a counterclockwise direction.

**incommensurable quantities**, quantities which cannot be measured an integral number of times by some suitable unit of measure.

**inconsistent equations**, see equations, relation.

**increment**, the difference found by subtracting the first or initial value of the variable from the second as the variable changes from one value to another. Symbol: \( \Delta x \). The **increment of the function** is the increment of \( f \), the de-
dependent variable, if \( y = f(x) \) Symbol, \( \Delta y \).

indefinite conjugate axis, see hyperbola.

indefinite integral, any function \( F(x) \) whose derivative is equal to \( f(x) \) is called a primitive or an indefinite integral of \( f(x) \).

independence, a fundamental property of a set of assumptions, states that none of the assumptions may be derived as a formal logical consequence from the others.

independent, linearly, see equation, relation.

independent variable, the variable to which values may be assigned at will; also called argument.

indeterminate equation, see equation, relation.

indeterminate form, the fraction resulting from the numerator and denominator of a fraction simultaneously approaching zero; also if the numerator and denominator approach infinity or increase indefinitely. Symbol: \( \frac{\alpha}{\alpha} \) and \( \infty \).

index of a root, the number which indicates the root to be taken.

indicial equation, an equation whose roots determine the possible exponents in the trial series solution of a reduced differential equation, which cannot be expanded in a Taylor’s series because it becomes infinite at an ordinary point, \( \alpha \). The function must be multiplied by a suitable integral power of \( \lambda - \alpha \).

indirect measurement, measurement which is impossible by direct measurement, but possible through mathematical principles.
indirect proof, the method of proof, which shows that if one of the possible relationships is impossible, the other must be true.

inductive reasoning, the method of reasoning from a great many specific cases to a generalization.

inequality, a statement that one of two quantities is greater or less than another; defined by nests of intervals, if \( S = (x_n \mid y_n) \) and \( S' = (x'_n \mid y'_n) \) then \( s < s' \) if \( x_n < y'_n \) for every \( n \) but not \( x_n \leq y'_n \) for every \( n \), that is \( y'_n < x'_n \) for at least one \( n \).

Bernoulli inequality, q.v.

Cauchy-Schwartz inequality, q.v.

infinite, that which is beyond the realm of the finite.

infinite integral, see improper integral.

infinite series, the indicated sum of an infinite sequence of numbers. Symbol: \( \sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \) when \( u_1, u_2, u_3, \ldots, u_n, \ldots \) is any sequence of quantities.

infinite set, see set.

infinitesimal, a variable which approaches 0.

infinitesimal calculus, see calculus.

infinity, analytic definition: in addition to the points of the plane at finite distances from the origin let us assign to the plane an infinitely distant point which may be regarded as the one corresponding to the origin in the transformation by reciprocal radii; the totality of all things infinite; the inverse of 0 or the result of
dividing by zero. Symbol: $\infty$.

double infinity, when there are two essential constants in an equation, there are infinitely many choices for these constants, and we say there are a double infinity equations of that form or a two parameter family.

single infinity, when there is one essential constant in an equation, there are infinitely many choices for this constant and we say there are single infinity of equations of that form, or a one-parameter family.

inflection point, the point at which the curve crosses the tangent, one of the singular points of the curve.

initial line, the half line from which the vectorial angle is measured in the system of polar coordinates.

initial side of an angle, see angle.

inner product of two functions, $g_m(x)$ and $g_n(x)$, is the number $\langle g_m, g_n \rangle = \int_a^b g_m(x)g_n(x)\,dx$.

innumerable, that which cannot be counted.

inscribe, to draw within.

inscriptible quadrilateral, a quadrilateral which can be inscribed in a circle.

integer, whole number, see cardinal numbers.

integrable function, any bounded function whose upper and lower integrals are equal.

integral, the limit of the summation of infinitely small rectangles under a curve between definite limits. Symbol: $\int$. The function represented by the curve is the integrand. See definite integral.
integral calculus, see calculus.
integral curve, see curve.
integral equation, an equation in which an unknown function appears under the integral sign.
integral function, the antiderivative of a function; also called an indefinite integral since an arbitrary constant must be added to the definite integral.
integrating factor, see adjoint equation.
integrating operator, see operator.
integration, a reversal of the process of differentiation; a method of finding antiderivatives; a summation.
  double integration, see double integral.
  successive integration, integrating the indefinite integral repeatedly.

intercept
  \( \chi \)-intercept, the abscissa of the point whose ordinate is 0.
  \( \gamma \)-intercept, the ordinate of the point whose abscissa is zero.

interior angle, see angle, position.
interpolation, the process of approximating values not given in the tables.
intersection, the crossing of two lines or surfaces: the common points of two figures.
interval, the segment of a straight line lying between two points.
  closed interval, an interval in which both end points
are included.

half open, an interval in which only one of the end points is included.

open interval, an interval in which at least one of the end points is not included.

intrinsic equations of a curve, equations of a curve in terms of the expressions for the radii of first and second curvature in terms of the arc.

invariable, constant.

invariant, the algebraic expression and the relation between figures which remain unchanged under the transformation.

inverse points, two points with respect to a circle, the product of whose distances from the center of the circle is equal to the radius squared. The circle is called the circle of inversion; its center the center of inversion.

inverse functions, a function may be solved for the independent variable in terms of the dependent variable, which gives rise to another function known as the inverse of the first. The derivative of the inverse function and that of the direct function are reciprocals of each other.

inverse trigonometric functions, the functions giving the angle in terms of the sine, cosine, etc.

inverse variation, see variation.

inversion, an involutory transformation which carries each of two arbitrary points which are mutually inverse in a circle into the other.

center of, see inverse points.

circle of, see inverse points.
plane of, q.v.
inversive geometry, the geometry of a property which is invariant with respect to the group of direct circular transformations.
invert, to turn over.
involute, the curve to which tangents to another curve are normal; the latter curve is called an evolute of the former.
involutary transformation, a transformation which is its own inverse. See reflection.
involution, a collineation which is an involutory projective transformation. The double points of an involution are the points harmonic to two given pairs.

elliptic, hyperbolic, and parabolic, an involution in which all pairs of collinear points the product of whose distances from a fixed point is constant, are conjugate pairs. The fixed point is the center of involution. If the constant is positive, the involution is hyperbolic; if negative, elliptic. If the double points coincide the involution is parabolic.
harmonic, an involution in which all pairs of harmonic points are double points to two given points and conversely.

irrational number, a number that cannot be expressed as an integer or the quotient of two integers; not rational.
irrational roots, roots containing irrational numbers.
irrational vector function, a vector function the curl of
which vanishes at every point of space.

**isobaric polynomial**, a polynomial all the terms of which are of the same weight.

**isodynamic points of a triangle**, the two points common to the three Apollonian circles of the triangle.

**isogon**, a polygon whose angles are all equal.

**isogonal conjugates**, two lines passing through the same vertex of a given angle and making equal angles with the bisector of this angle, e.g., the altitude and circum-diameter issuing from the same vertex of a triangle, the median and symmedian issuing from the same vertex of a triangle.

**isogonal representation**, see conformal mapping.

**isolated point of a curve**, same as **conjugate point**, a point which satisfies the equation of a curve, while no other point in its neighborhood satisfies the equation.

**isolated point of a set**, a point which is not a limit point of the set.

**isometric representation**, see applicable surfaces.

**isomorphic**, a property of any two systems of elements which can be put into reciprocal one-to-one correspondence with preservation of properties as to a set of assumptions.

**isosceles**, having two equal sides.

**isothermal surface**, same as **isometric**, a surface of constant temperature.

**isotropic coordinates**, the coordinates \( z, \bar{z} \) for a real point are conjugate complex numbers, \( z = x + iy \)

and \( \bar{z} = x - iy \). In homogeneous coordinates \( z = \frac{z_1}{z_2} \)

and \( \bar{z} = \frac{\bar{z}_1}{\bar{z}_2} \).
101.

isotropic lines, same as circular rays, lines whose slope is \( i \) or \(-i\).

iterated integral, same as double integral.

iterated limit, the limit of two variables of a function which approach the limit by varying separately.
Jacobian, the functional determinant formed by the $n^2$ partial differential coefficients of the first order of $n$ given functions of $n$ independent variables.

Symbol: \[ J(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \]

where $u$ and $v$ are binary forms.

K.

Kirkman's points, see Pascal line.

Kronecker delta, symbol: \( \delta^i_j \) = \( \frac{\partial^i \partial^j}{\partial x^i \partial y^j} \)

where \( \delta^i_j = 1 \) for $i = j$

$0$ for $i \neq j$.
Lagrangian form of the remainder after \( n \) terms, the last term in the Taylor formula:

\[
R_n = \frac{(x-a)^n}{n!} \frac{d^n}{dx^n} f(\xi).
\]

Laplace's development of a determinant, the value of a determinant is equal to the algebraic sum of the products obtained by multiplying each of the \( k \)-rowed minors that can be formed from any \( k \) rows (columns) of the determinant by their algebraic complements.

Laplace's equation,

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.
\]

Laplacian integral, see operator.

latus rectum, the chord through either focus perpendicular to the transverse axis of the hyperbola or the major axis of the ellipse or, in the parabola, through the focus parallel to the directrix and terminated by the curve in each case.

laws of equality, (1) if \( a > b \), then \( b = a \); (2) if \( a = b \) and \( b = c \), then \( a = c \); (3) if \( a = a' \) and \( b = b' \), then \( a + b = a' + b' \) and \( ab = a'b' \).

law of the mean, the mean value theorem, for differential calculus, if a function \( f(x) \) is continuous in the interval \( a \leq x \leq b \) and possesses a derivative at every interior point of \((a, b)\), then there exists a point \( \xi \) such that \( \frac{f(b) - f(a)}{b - a} = f'(\xi) \).
If the values of the function are equal at the end points of the interval, then this formula is equal to 0 and \( f(b) = f(a) \). Cf. Rolle's Theorem. The extended law of the mean leads to Taylor's theorem. Cf. Taylor's Theorem. When written in the form
\[
\int_a^b f(x) \, dx = f(a) + (x-a) \frac{df}{dx} (\xi)
\]
where \( b = x \) so that \( f(x) \) may be expressed as a polynomial of degree \( n \) in \( x-a \).

For integral calculus, let \( f(x) \) and \( \phi(x) \) be two functions which are continuous in the interval \((a, b)\) and suppose that \( \phi(x) \) does not change sign in this interval. Then there exists at least one value \( \xi \), where \( a \leq \xi \leq b \) such that
\[
\int_a^b f(x) \phi(x) \, dx = f(\xi) \int_a^b \phi(x) \, dx.
\]
In case \( \phi(x) = 1 \) this reduces to
\[
\int_a^b f(x) \, dx = f(\xi) (b-a).
\]

Legendre differential equation and polynomials, see classical equations.

Lemma, a subsidiary proposition employed as an auxiliary in demonstrating another one.

Lemniscate, the locus of the equation,

Lemoine axis, same as Lemoine line, the line determined by intersections of the tangents to the circumcircle of a triangle with the corresponding opposite sides.

Lemoine circle, first, the circle of the points of intersection of lines parallel to the sides of a triangle through the Lemoine point, (the intersection of the symmedians of a triangle) with the other two sides; the center of the
circle is the midpoint of the line joining the circum-
center to the Lemoine point; second, the circle, with the
symmedian point as center, passing through the intersections
of the lines antiparallel to the sides with the correspond-
ing sides.

**length**, the greatest dimension of a body.

**less than**, precedes, symbol: <

**limacon**, the locus of the equation, \( r = a - b \cos \theta \).

**limit**, if a variable takes successively a sequence of values
that approach more and more nearly to a constant value in
such a way that the numerical (absolute) value of the dif-
fERENCE between the variable and the constant becomes and
remains less than any assigned constant however small, then
the first constant is called the limit of the variable.

**iterated limit**, q.v.

**simultaneous limit**, q.v.

**limiting point of a set**, a point of the set, in the neigh-
borhood (arbitrarily small) of which there are always other
points.

**line**, the path of a moving point.

**straight line**, the path of a point, which moves always
in the same direction.

**line equation**, general form: \( ax + by + c = 0 \)

**line integral**, the limit of the sum of the scalar products
of a finite number of monotonic arcs into which a continuous
curve is divided and the value of a continuous function at
some point of the arc.

**lineal element**, the small piece of straight line tangent
to a curve and having the slope of the integral curve at
the point of tangency.

**linear equation**, an equation of a straight line; one of the
first degree.

**linear form**, any homogeneous linear function; a form of the
first degree.

**linear fractional function**, general, \( Z = \frac{ax + b}{cy + d} \).

**linear set**, see set.

**linear vector function**, a vector whose components along three
non-coplanar vectors are expressible linearly with scalar
coefficients in terms of the components of a given vector
along those same vectors.

**linear dependent functions**, a set of functions \( u_1(x), u_2(x), \ldots, u_n(x) \) in an interval \((a, b)\) of the \( x \)-axis with
existing constants \( c_1, c_2, \ldots, c_n \), not all zero, such that
in the interval, \( c_1 u_1 + c_2 u_2 + \cdots + c_n u_n \equiv 0 \).

If no such identity holds the functions are **linearly independent**.

**linkage**, a hinged mechanism with certain fixed points used
to draw certain loci.

**literal equations**, equations containing two or more un-
knowns, each of which can be found in terms of the others,
thus giving a formula for that letter.

**locus**, the path of a point moving in accordance with a
given condition.

**logarithm**, of a number, the exponent to which a given base
must be raised to produce the number.

characteristic of, the integral part of the logarithm.
common, a logarithm with base 10.

mantissa of, the decimal part of the logarithm.

Naperian, the natural system of logarithms with the base, \( e \approx 2.71828 \) approximately.

logarithmic curve, the graph of the equation, \( y = \log_a x, a > 1 \).

logarithmic spiral, the locus of the equation, \( \log_e r = a \Theta \) or \( r = e^{a\Theta} \).

loxodromic curve, on a surface of revolution, the curve which cuts the meridians (the generating curve in different positions) under a constant angle.

lune, a figure on a sphere which is bounded by two great semicircles.
Maclaurin's series, a special case of Taylor's series in which $a = 0$.

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^{(n)}(0)x^n}{n!} + \cdots$$

magnitude, a number assigned to a quantity according to some stated rule by means of which the quantity may be compared with other members of the same class.

major axis, see ellipse in conic sections.

manifold, same as family; see envelope.

manifoldness, degree of, the number of parameters necessary.

mantissa, see logarithm.

map, a representation by means of a transformation of geometric figures on a plane.

mapping, conformal, q.v.; mercator, q.v.

mathematics, the science of drawing necessary conclusions.

"Mathematics is the study of all possible axiomatic systems and all possible axiomatic consequences derivable therefrom." - Dr. C. F. Wylie.

"Mathematics is that science in which we do not know what we are talking about, nor whether what we say is true." - Bertrand Russell.

"Mathematics is the class of all propositions of the form, $P$ implies $Q$." - Bertrand Russell.

matrix, a rectangular array of numbers enclosed by parentheses; when the array is square it is called a square
matrix. A matrix defines a new number not included in the system of complex numbers. When the value of a determinant formed from the same array of numbers is equal to zero, the matrix is called singular, otherwise, non-singular. The unit matrix, symbol: \( u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \).

Augmented matrix, the matrix formed by annexing the column of known terms in a system of non-homogeneous equations.

Characteristic matrix, see characteristic determinant.

Rank of, a positive integer or zero, \( r \), such that it is possible to form a determinant of order \( r \) whose value is different from zero and whose rows and columns are obtained from the rows and columns of the matrix, whereas it is not possible to form a determinant of order \( r + 1 \) which satisfies the same condition.

Scalar matrix, a matrix having a scalar for the elements of the principal diagonals but zeros in all other places.

Skew-symmetric matrix, see determinant.

Symmetric matrix, see determinant.

Unimodular matrix, a sub-group of matrices whose determinants have the value, \( \pm 1 \), the rule of combination being multiplication.

Maximum, a function \( f(x) \) is said to have a maximum for \( x = a \) if \( f(x) \) is continuous for \( x = a \) and if \( f(a) \) is greater than the values of \( f(x) \) for \( x \) just preceding and just following \( x = a \).
absolute maximum, the largest value of the function in the range of the definition.
relative maximum, the value of the function for a given $x$ is greater than all of its values for $x$ near $x$.
Maxwellian integral, see operator.
mean, a middle term.
  arithmetic mean, q.v.
  geometric mean, q.v.
  harmonic mean, q.v.
mean proportion, a proportion in which the means are identical.
mean proportional, the middle terms of a mean proportion.
Cf. geometric mean.
mean value theorem, cf. law of the mean.
measure, the number of times any common unit is contained in a given quantity.
  circular, q.v.
medial triangle, the triangle formed by joining the midpoints of the sides of a triangle.
median, the line joining the vertex of a triangle to the midpoint of the opposite side.
meets, points of intersection.
member of an equation, see equation.
Menelaus' Theorem, the six segments determined by a transversal on the sides of a triangle are such that the product of three non-consecutive segments is equal to the product of the three others.
Mercator mapping, the representation of a unit sphere upon a plane; used in making maps of the earth for mariners. meridian, the generating curve of a surface of revolution in its various positions.

metric geometry, the geometry based on measures. The measure of distances in Euclidean plane geometry is parabolic, since the point at infinity remains fixed in the measuring operation. The measure of angles in Euclidean geometry is elliptic, since the circular rays are fixed under a rotation of the plane. The measure of distance is hyperbolic, parabolic or elliptic according as the fundamental points are real and distinct, real and coincident or conjugate imaginary.

midpoint of a segment, the point which divides the segment into two equal parts.

minimal curves, or minimal straight lines, imaginary straight lines through a point, the generators of an imaginary cone with the point as vertex.

minimum, a function \( f(x) \) is said to have a minimum for \( x = a \) if \( f(x) \) is continuous for \( x = a \) and if \( f(a) \) is less than the values of \( f(x) \) for \( x \) just preceding and just following \( x = a \).

absolute minimum, the smallest value of the function in range of the definition.

relative minimum, the value of the function for a given \( x \) is less than all of its values for \( x \) near \( x \).

minor axis, see ellipse in conic sections.
minor of a determinant, any determinant whose rows and columns are formed from the rows and columns of the given determinant.

complementary minor, the determinant formed from the rows and columns not used in forming the minor.

minor of an element of a determinant of order n is the determinant of order n - 1 obtained by deleting the row and column in which this element stands.

minuend, the number from which another number is to be subtracted.

modulus, see absolute value.

moments, the limit of the sum of the product of the distance from a line or plane of reference and the increment of an element of mass.

monomial, an expression of one term.

monotone sequence, decreasing, if for every value of n, $\left| x_n \right| \geq x_{n+1}$; increasing, if for every value of n, $\left| x_n \right| \leq x_{n+1}$.

moving trihedral, the equations of a curve are given with reference to tangent, principal normal and binormal at a certain point as axes. As the point moves along the curve, the configuration formed by these three mutually perpendicular lines is referred to as a moving trihedral; known as the Cesaro moving trihedral.

multiple, the result of multiplying a quantity by whole numbers.
multiple roots, a root occurring more than once in an equation.
multiple-valued function, see single-valued function.
multiplicand, the number to be multiplied.
multiplication, if a class is formed by taking an element from each of two other classes to form pairs, the cardinal number of this new class is called the product of the numbers. Symbol $a \times b = c$.
multiplication laws, (1) the product of two numbers is a unique number; (2) multiplication is commutative; (3) multiplication is associative; (4) there is a number $u$ such that for every $a$, $a \cdot u = a$; this is postulating the unit number; (5) every number except zero has an inverse with respect to multiplication such that the product of a number and its inverse gives unity; (6) multiplication is distributive with respect to addition.
multiplier, the number showing how many times the multiplicand is to be taken.
\( n \to \infty \) or \( n = \infty \), symbol for \( n \) becomes infinite or \( n \) becomes infinite or \( n \) increases without limit.

\( n! \), (\( n \) factorial = \( n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1 \)), see factorial notation.

\( n \)-dimensional, having any number of dimensions.

\( n \)-con, a polygon of any number of sides.

\( n \)-ic, a form of degree \( n \).

Hagel point, the point common to the three perpendiculars dropped from the excenters of a triangle upon the corresponding sides of the triangle.

nappes, the two parts of a conic surface which are separated by the vertex of the surface.

natural logarithm, see logarithm.

natural number, a positive integer.

necessary and sufficient condition, see condition.

negative, that which is less than zero.

neighborhood of a point, all of the points whose distance from a given point is less than a given number.

nest of intervals, a given monotone increasing sequence \( (x_n) \) and a given monotone decreasing sequence \( (y_n) \) whose terms for every \( n \) satisfy the condition \( x_n \leq y_n \) and for which the differences, \( d_n = y_n - x_n \), form a null sequence. Symbol \( (x_n \mid y_n) \). A real number is defined by a nest of intervals, e.g. \( (\sqrt{2} \) to \( n \) decimals \( \mid \sqrt{2} \) to \( n \) decimals \( + \frac{1}{10^n} \)) defines \( \sqrt{2} \).

Newton's formulas, for computing an isolated real root of a
real equation is available not only for polynomial equations but applicable also to logarithms, trigonometric and other equations. Given an approximate value \( a \) of a real root, we can usually find a closer approximation \( a + h \) to the root by neglecting the powers \( h^2, h^3, \ldots \) of the small number \( h \) in Taylor's Formula, \( f(a+h) = f(a) + f'(a)h + f''(a)\frac{h^2}{2} + \ldots \) and hence by taking \( f(a) + f'(a)h = 0 \), \( h = \frac{-f(a)}{f'(a)} \). Repeating the process with \( a_1 = a + h \) in place of the former \( a \).

**Nine-point circle**, the circle on which lie the midpoints of the sides of the triangle, the feet of the altitudes, and the midpoints of the segments joining the orthocenter to the vertices of the triangle. The center of this circle lies midway between the circumcenter and the orthocenter. **Node**, same as **double point**, see singular points. **Non-adjacent**, not adjacent. **Non-collinear**, not collinear. **Non-concurrent**, not concurrent. **Non-convergent**, see divergent. **Non-coplanar**, not coplanar. **Non-enumerable set**, see set. **Non-Euclidean geometry**, geometry in which Euclid's parallel axiom is not assumed. **Non-linear**, not linear. **Non-linear set**, see set. **Norm**, of the function or vector \( g(x) \) or the sum of the squares
of its components, the number \( N(f) = \int_a^b \left[ f(x) \right]^2 \, dx \).

**norm curve**, see curve.

**normal**, a line drawn perpendicular to the tangent at the point of tangency in the osculating plane.

**normal section**, the orthogonal projection of a curve on a surface upon the tangent plane at a point of the curve.

**notation**, any system of signs, figures, or abbreviations employed for convenience in a discussion.

**null circle**, same as a pair of circular rays, q.v., the circle with zero radius.

**null sequence**, a sequence such that if subsequent to the choice of a positive number \( \varepsilon \), however small, one can find a positive integer \( p \) such that \( |x_n| < \varepsilon \) for all values of \( n > p \).

**null system**, an involutory correlation with a skew-symmetric determinant.

**number**, pure number implies nothing but magnitude. Numbers may be classified as follows: in the complex number system we have real and imaginary numbers, q.v.; real numbers may be subdivided into rational and irrational numbers, q.v.; under rational we have the integers or positive natural numbers together with the negative numbers, fractions and zero; the irrational numbers include non-repeating decimals, \( \pi \), \( \sqrt{2} \), e, and other numbers than those arrived at by extracting roots. There is also a class beyond the field of complex numbers, known as hypercomplex numbers, such as
matrices and quaternions, q.v.; the imaginary numbers may be pure imaginaries, having no real number term or \( a = 0 \), see number pair.

**number pair**, the ordinary complex number, symbol: \((a, b) = a + bi\).

**number system**, a class in which two undefined operations, which we denote by \( + \) and \( \times \), exist and operate subject to the following assumptions: (1) a group with respect to operation \( + \), (2) a group with respect to operation \( \times \), except that no inverse of zero is required; (3) the distributive law holds.

**numerator**, see fraction.
oblateral spheroid, see hyperboloid.

oblique coordinates, see Cartesian coordinates.

oblique lines, lines that do not form right angles.

oblique projection, see projection.

obtuse angle, see angle.

octagon, a polygon of eight sides.

octahedron, a polyhedron of eight faces.

octants, the eight parts into which space is divided by the three rectangular coordinate planes.

one, there is only real number w, different from zero, such that w x w = w. We call it one. Symbol: 1.

one-to-one correspondence, if each element of one class corresponds to one and only one element of another class. Symbol: (1, 1).

open interval, an interval in which at least one of the end points is not included.

open set, see set.

operation, the result which exists when, corresponding to certain elements of a class there is a certain order of those elements.

operator, differentiating, a vector $\nabla V$ is obtained from a scalar function $V$ of position in space by differentiating. The symbol $\nabla$ indicates the operator; $\nabla V$ is called the derivative of $V$ and $V$ the primitive of $\nabla V$; the terms gradient and slope of $V$ are also used for $\nabla V$. $\nabla V$ is
read \( \Delta \cdot V \) called divergence, represents the diminution of density; \textit{integrating,} \( \mathbf{P} \) \textit{at} \( V \), symbol, is a vector obtained by integrating the length of a vector between two fixed points with respect to the density at the first fixed point and is called the potential at the second point due to the body whose density at the first fixed point is known. This integral defined as the potential is a solution of Poisson's equation, q.v. The three integrals obtained from the potential by the three differentiating operators are called the Newtonian, Laplacian, and Maxwellian. These are also called integrating operators.

\textbf{order laws}, 1. If \( a, b, c \), etc. are distinct numbers, then \( a < b \) and \( b < c \) are incompatible. This is called asymmetry relation.

2. If \( a < b \) and \( b < c \) then \( a < c \). This is called the transitive relation.

\textbf{order of a differential equation}, the order of the highest derivative which appears.

\textbf{order of the group}, the number of transformations contained in the group.

\textbf{ordered set}, see set.

\textbf{ordinal number}, the number assigned to the last element of a finite set in the process of counting.

\textbf{ordinate}, see axis.

\textbf{ordinate axis}, see axis.

\textbf{origin}, the point of reference from which axes or other lines
of reference are drawn as the initial point.

**Orthic triangle**, same as **Orthocentric triangle**, or **pedal triangle**, see triangle.

**Orthocenter**, see centers of triangles.

**Orthogonal circles**, circles which intersect at right angles, that is, the tangents at one point of intersection are perpendicular.

**Orthogonal trajectories**, each of two systems of curves which cut each other at right angles.

**Orthogonal projection**, see projection.

**Orthonormal set**, the set of mutually perpendicular unit vectors obtained by normalizing a set of mutually perpendicular vectors.

**Oscillating function**, a function which fails to approach a limit, but alternates between positive and negative values, e.g., \( \sin \frac{1}{x^2} \).

**Osculant**, a curve associated with rational curves. If the equation of a rational curve,

\[
x_i = f_i(t, \tau), \ i = 1, 2, 3,
\]

where \( f_i \) are binary forms of order \( n \) in the homogeneous parameter \( t/\tau \), then \( x_i = (t \frac{\partial}{\partial t} + \tau \frac{\partial}{\partial \tau}) f_i \) obtained by taking the first polars of \( f_i \) with respect to \( (t, \tau) \) represent the first osculant at the point \( t/\tau \).

**Osculating circle**, same as circle of curvature, the limiting position of a circle in the osculating plane that has the same tangent as the curve at the given point and passes
through the point of the curve which determines the osculating plane. The center of this circle is called the center of first curvature.

**Osculating plane**, the limiting position of a plane determined by a tangent to a curve at a point and another point of the curve, as this point approaches the first point.

**Osculating sphere**, a moving sphere whose center is on the normal to the osculating plane at the center of curvature, that is, on the polar line; and whose radius is the square root of the sum of the squares of the radius of first curvature and the product of the radius of torsion and the first derivative of the radius of first curvature. The sphere cuts the osculating plane in the osculating circle.

**Osculatory**, having the property of coincident curvature.
$p$-series, $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$

Pasche's axiom, let $A$, $B$, $C$ be three points not lying on the same straight line and let $a$ be a straight line lying on the plane $ABC$ and not passing through any of the points $A$, $B$, $C$, then, if the straight line $a$ passes through a $AB$ it will also pass through either a point of the segment $BC$ or a point of the segment $AC$.

Pantograph, an instrument used for enlarging or reducing designs and maps.

Pappus' theorems, (1) if two sets each of three collinear points are joined crosswise, the points of intersection are collinear. (2) the volume generated by the revolution of a curve about an axis is equal to the product of the area of the curve and the length of the path described by its centroid; the surface is equal to the product of perimeter of the curve and the length of the path described by its centroid.

Parabola, see conic sections.

Paraboloid of revolution,

elliptic, the surface formed by a parabola revolving about its axis.

hyperbolic paraboloid, q.v.

Parallel lines, planes and surfaces, lines, planes and surfaces which do not intersect in finite points, lines or
planes no matter how far produced. They are said to meet at infinity. Two lines in a plane are parallel if they are cut by a transversal at the same angle. **parallel projection**, see projection.

**parallelopiped**, a prism whose bases are parallelograms.

**parallelogram**, a quadrilateral with the opposite sides parallel.

**parameter**, an arbitrary constant characterizing by each of its particular values some particular member of a system of expressions, curves, surfaces, functions, etc.

**parametric equation**, see equation.

**parentheses**, a sign of grouping or aggregation, e.g. 

\[(a, b) \equiv interval.\]

**partial derivatives**, see derivatives.

**partial differential equation**, see differential equation.

**partial differentiation**, see differentiation.

**partial fractions**, see fraction.

**partial sum sequence**, the sequence formed by adding the terms of the geometric progression in which the first term is unity and the common ratio is one-half; the general term: 

\[s_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots\]

This is called the **geometric series**.

**partition**, see cut.

**Pascal's hexagon**, see Pascal's theorem.

**Pascal's line**, see Pascal's theorem.
Pascal's theorem, the opposite sides of a hexagon inscribed in a conic meet in three collinear points and conversely; the line containing these three points is called Pascal's line. By connecting the six vertices of the hexagon in all the different orders possible, sixty simple hexagons can be formed; each of these gives rise to a Pascal line, all distinct. The fifteen lines joining the six points intersect at forty-five other points and through each of these points pass four of the sixty Pascal lines. The Pascal lines also pass by threes through twenty other points, called Steiner's points, one on each line; they also pass by threes through sixty other points called Kirkman points, three to a line. The twenty Steiner points lie by fours on fifteen other lines and the sixty Kirkman points lie by threes on twenty other lines.

Pascal's triangle, a triangular arrangement of numbers, which show the coefficients of the expansion of a binomial; it is also used to show combinations of m things taken n at a time. Symbol: \( \binom{m}{n} = \frac{m(m-1)(m-2) \ldots (m-n+1)}{n!} \).

Path, see surface.

Peaucellier's cell, a linkage formed by four rigid bars of equal length hinged together to form a rhombus; two opposite vertices are connected by equal rigid bars to a fixed point. Then the other pair of opposite vertices are inverse points with respect to a circle whose center is the fixed point, so that they describe inverse curves. This transforms circular
motion into rectilinear motion.

pedal line, same as Simpson line, the line determined by the feet of the three perpendiculars dropped upon the sides of a triangle from any point of its circumcircle; the point is called the pole of the line with respect to the triangle.

pedal triangle, same as orthocentric triangle, see triangle.

pencil of lines, triangle.

pencil of lines, the figure of all lines incident to a point; the point is called the vertex of the pencil.

improper pencil, when the point is ideal.

proper pencil, when the point is a proper point.

syzygetic pencil, a pencil of syzygies.

pentadecagon, a polygon of fifteen sides.

pentagon, a polygon of five sides.

pericon, the whole angular magnitude about a point.

perimeter, the sum of the sides of any polygon.

periodic function, a function such that $f(x) = f(x+a)$; the constant $a$ is called the period of the function; a primitive period, a period no aliquot part of which is a period, e.g. $2\pi$ is a primitive period of the sine and cosine; $2\pi$ is a primitive period of the exponential function. A periodic function is a special kind of automorphic function, q.v.

periodic transformation, a transformation with period $n$ if the identical transformation is obtained after applying
the transformation $n$ (but not less than $n$) times.

**permutation**, an ordered arrangement of a number of things.
Symbol: $P_m = 1 \cdot 2 \cdot 3 \ldots m = m!$

**cyclic permutation**, q.v.

**perpendicular lines**, lines that intersect so that the adjacent angles are equal.

**perspectivity**, a correspondence between figures when all lines joining mated points pass through the same point and all intersections of mated lines are collinear; the point is called the **center of perspectivity** and the line containing the intersections of mated lines, the **axis of perspectivity**.
Symbol: $\overline{\Lambda}$

**harmonic perspectivity**, the perspectivity whose center and axis are ideal elements. See reflection.

**perversion**, a transformation that replaces each figure by a symmetrical figure.

**perversor**, the dyadic which gives a perversion.

**Pfaffian**, the total differential equation in three variables,

$$P(x,y,z) + Q(x,y,z)\frac{dy}{dx} + R(x,y,z)\frac{dz}{dx} = 0.$$ 

**pi**, the constant ratio of the circumference of a circle to its diameter; equal to $3.1416$ approximately.

**piercing point**, the point at which a line pierces a plane.

**nilaster**, a higher plane curve, the locus of the equation:

$$xy^2 - a^2y - b^2x = 0.$$ 

**planar element**, the small piece of a plane perpendicular to the normal to the surface at the given point.
plane coordinates, see point coordinates.

plane curve, see curve.

plane geometry, the geometry of figures which lie in a plane surface.

plane surface, a surface such that if any point in the surface is joined to any point on a straight line in the surface, the line so drawn lies in the surface.

plane of inversion, the real finite plane, with the ideal point at infinity adjoined, every inversion is without exception. The ideal point corresponds to the center of inversion, see inverse points, and inversive geometry.

plane of projection, the plane upon which the projection is made.

planisphere, a plane projection of the sphere.

Platonic solids, the five regular polyhedrons, so called because of the emphasis Plato placed on their study.

Plücker line coordinates, the homogeneous coordinates for a line expressed in two ways:

axis coordinates, q.v., the line determined by the intersection of two planes.

ray coordinates, q.v., the line determined by two points.

point, an undefinable; a point is just a position.

point curve, same as conic.

point equation in line coordinates:

\[ x_1 u_1 + x_2 u_2 + x_3 u_3 = 0. \]

point of condensation, a point in every neighborhood of
which there are a non-enumerably infinite number of points of the given set. Every point of condensation is a limiting point but the converse is not true.

point of inflection, same as flex or stationary line, the point at which the inclination of a curve is continuous and which separates an arc concave upward from an arc concave downward.

point row, see range of points.

point set, see set of points.

Poisson's equation, let \( V \) be any function in space such that the potential \( \text{Pot} \ V \) has in general a definite value. Then

\[
\nabla \cdot \nabla \text{Pot} \ V = -4\pi V + \frac{\partial^2 \text{Pot} \ V}{\partial x_1^2} + \frac{\partial^2 \text{Pot} \ V}{\partial y_1^2} + \frac{\partial^2 \text{Pot} \ V}{\partial z_1^2} = -4\pi V.
\]

polar angle, see angle.

polar axis, the initial line in the system of polar coordinates.

polar coordinates, the coordinates of a point referred to the radius vector and the vectorial angle. The polar coordinates may be obtained from the rectangular coordinates with the same origin by the transformation \( x = r \cos \theta \) and \( y = r \sin \theta \), where \( r \) is the radius vector and \( \theta \) the vectorial angle. These equations are referred to as polar equations.

polar line, the line normal to the osculating plane at the center of curvature; the line determined by the intersections of pairs of opposite sides and the pairs of tangents.
at opposite vertices of an inscribed quadrangle. These four points of intersection form an harmonic range. 

pole, the diagonal point of the quadrangle opposite the polar line; the origin in the system of polar coordinates; in solid geometry, the ends of the axis perpendicular to the plane of the circle considered.

polygon, a closed figure composed of straight lines.

concave polygon, a polygon which has at least one angle greater than a straight angle.

convex polygon, a polygon each of whose interior angles is less than a straight angle.

equilangular polygon, a polygon whose angles are all equal.
equilateral polygon, a polygon whose sides are all equal.

regular polygon, a polygon which is both equiangular and equilateral.

apothem, the radius of the inscribed circle.

center, the center of the inscribed or circumscribed circle: they are identical.

radius, the radius of the circumscribed circle.

polyhedral angle, see angle.

polyhedron, a closed figure composed of intersecting planes.

polynomial, an integral rational function of degree of one or more variables expressed in the following form:

\[ f(x) = c_1 x^{a_1} y^{b_1} + c_2 x^{a_2} y^{b_2} + \ldots + c_k x^{a_k} y^{b_k} \]

where the a's are positive integers or zero, while the c's are constants
real or imaginary. The expressions $c_i x^{a_i}$ are called terms, $c_i$'s the coefficients, $a_i$'s the exponents indicating the degree of the variable. The degree of the term is the sum of the degrees of its variables.

dyadic polynomial, see dyad.

homogeneous polynomials, see form.

Legendre polynomials, q.v.

Tchebichef polynomials, q.v.

positive integers, see cardinal numbers.

post factor, when a vector is multiple in a dyadic, the dyadic is said to be post factor to the vector.

postulate, a geometric intuitive statement accepted without proof.

potential, see operator.

power, "we say two classes which can be put into one-to-one reciprocal correspondence have the same power" - Cantor.

power series, the series in the form:

$$\sum_{n=0}^{\infty} c_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$$

in which the coefficients of the powers of $x$ are independent of $x$.

power of a point, with respect to a circle, the difference between the square of the distance from the point to the center of the circle and the square of the radius of the circle.

precede, see less than.

prefactor, when a dyadic is multiplied into a vector, the
dyadic is called a prefactor to the vector. Either a prefactor or a postfactor determines a linear vector function. The two thus obtained are called conjugate linear vector functions.

prime number, a number that has no other factors except itself and unity.

primitive, see operator.

primitive function, see indefinite integral.

principal axis, same as major axis.

principal diagonal, see determinant.

prism, a polyhedron, two of whose faces are congruent polygons in parallel planes, while the other faces are parallelograms which intersect in parallel lines.

prismatoid, a polyhedron bounded by two polygons in parallel planes, and by other faces which are triangles, parallelograms or trapezoids having one side common with one of the bases and the opposite vertex or side common with the other base.

prismoidal formula, a formula for the area under a curve, \( y = ax^3 + bx^2 + cx + d \), between the ordinates \( y_1 \) and \( y_3 \) with \( y_2 \) midway between them:

\[
A = \frac{h}{3} \left( y_1 + 4y_2 + y_3 \right)
\]

probability, the mathematical measure of our state of expectancy.

probability integral, \( \frac{2}{\sqrt{\pi}} \int e^{-x^2} \, dx \).

produce, to extend.

product, the result of multiplication.
progression, any discrete sequence which has a first element and no last. See arithmetic, geometric and harmonic progression.

projection, more specifically central projection,

in a plane, the intersections of any line with lines drawn from a given point to other points in the plane; the segment between points. The given point is called the center of projection.

in space, the intersections of any plane with lines drawn from a given point without the plane.

oblique projection, the projecting lines are oblique to the plane of projection.

orthogonal, same as orthographic projection, when the projecting lines are perpendicular to the line or plane of projection. In both oblique and orthogonal projection, the center is infinite and the projecting lines are parallel; this is parallel projection.

stereographic projection, the representation of a sphere on a plane so that the great circle determined by the plane corresponds with itself point by point. Symbol: \( \vec{A} \).

projective coordinate, the coordinates of any point on a range of points are the four numbers which define the cross ratio of that point. For points on a plane of points, a triangle of reference is used, in space a tetrahedron of reference. The projective coordinates of a finite point interpreted metrically are proportional to multiples of
directed distances from the sides of the triangle of reference to the point. Considered from this point of view, they are known as the \textit{trilinear coordinates} of the point.

\textbf{Projective geometry}, the geometry based on incidence relations and the transformations which preserve these relations, whose characteristic instrument is the straight edge only. Sometimes called \textit{synthetic geometry} or the geometry of position.

\textbf{Projectivity}, the result of projection; a series of perspectivities.

\textbf{Prolate spheroid}, see ellipsoid.

\textbf{Proof}, the procedure in establishing the correctness of a conclusion.

\textbf{Analytic}, see analysis.

\textbf{Deductive}, see deductive reasoning.

\textbf{Indirect}, q.v.

\textbf{Inductive}, see inductive reasoning.

\textbf{Synthetic}, q.v.

\textbf{Proper fraction}, see fraction.

\textbf{Proper integral}, see finite integral.

\textbf{Proportion}: the expression of equality between two equal ratios. For the \textbf{terms} of a proportion see antecedent, consequent, extreme, fourth proportional, mean, third proportional.

\textbf{Proposition}, same as \textbf{theorem}, a conditional statement to
be proved.

**protractor**, an instrument for measuring angles.

**pseudo-tangent lines**, lines which touch the quadric surface at the vertex only.

**pure imaginary**, see number system.

**pyramid**, a polyhedron whose base is a polygon of any number of sides and whose other faces are triangles having a common vertex.

**Pythagorean theorem**, the famous theorem, concerning the relation of the sides of a right triangle to the hypotenuse expressed by the formula: \( a^2 + b^2 = c^2 \) where \( a \) and \( b \) are the sides and \( c \) is the hypotenuse.
quadrangle, a figure of four points.

complete, q.v.

quadrant, a quarter of a circle.

quadratic equation, an equation of the second degree,
general form \( ax^2 + bx + c = 0 \), \( a \neq 0 \); if \( b = 0 \), \( c \neq 0 \),
an incomplete quadratic or pure quadratic.

quadratic form, a form of the second degree.

apolar and binary quadratic forms, q.v.

quadatrix, a transcendental curve used in squaring the
circle and trisecting an angle.

quadrature, the operation of finding an expression for the
area of a figure bounded wholly or in part by a curved line.

quadrature of the circle, finding the side of a square equal
in area to the circle.

quadric equation, an equation of the second degree in three
unknowns, the equation of a quadric surface; the general
equation:

\[
Q(x, y, z) \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{31}yz + 2a_{51}zx \\
+ 2a_{12}xy + 2a_{14}x + 2a_{24}y + 2a_{34}z + a_{44} = 0 .
\]

quadric surface, same as conicoid, a surface, the locus of
a quadric equation.

central quadric, q.v.

quadrilateral, a polygon of four sides.

complete, q.v.

Saccheri's, q.v.
skew, a quadrilateral, one of whose vertices is outside the plane of the other three.

quadricular, a polynomial of four terms.

quadripole, a group of four points.

quantic, same as form.

quantity, that which has size; an amount.

quartic, an equation or form of the fourth degree.

Lame's special quartic, \( x^4 + y^4 = a^4 \).

quaternion, a hypercomplex number based on four arguments, \( l, i, j, k \) with real number coefficients. Symbol:

\[ x = x_1 + x_2 i + x_3 j + x_4 k . \]

quatrefoil, a figure made of four equal circles, each one tangent to two others and with their centers at the vertices of a square whose side is equal to the diameter of a circle.

quintic, an equation or form of the fifth degree.

quotient, the result of division.
R.

radian, see circular measure.

radical, symbol: \( \sqrt[n]{a} \); \( n \) is called the index of the radical and \( a \) is the radicand.

radical axis, the line, the locus of all points whose powers with respect to two non-intersecting circles are equal; this line is perpendicular to the line of centers. The axes of three circles, taken two at a time are concurrent in a point called the radical center.

radical circle, the circle orthogonal to three given circles with its center the radical center.

radical equation, see equation, form.

radical sign, see radical.

radicand, see radical.

radius, the segment drawn from the center of the circle (sphere) to any point on the circle (sphere).

radius of first curvature, see curvature.

radius of second curvature, same as radius of torsion, see torsion of a curve.

radius of normal curvature, the radius of the normal section, q.v.

radius of torsion, same as radius of second curvature, see torsion of a curve.

radius vector, the half line or terminal side of the vectorial angle in the system of polar coordinates.

range of points, same as point row, a set of all points
range of the variable, the set of all values which the variable may assume in the interval.

rank of a determinant (matrix), see matrix.

rate of change, the varying speed of a moving object.

average rate of change, the ratio of distance to time of a moving object distributed evenly over a given interval.

exact rate of change, the derivative of the average rate of change.

ratio, the quotient of two numbers.

anharmonic, cross ratio and double ratio, see cross ratio.

extreme and mean ratio, q.v.

rational curve, see curve.

rational function, a rational fraction, the quotient of two polynomials; a complex variable is called a rational function of another variable when it is possible to deduce it from the variable by a finite number of additions, subtractions, multiplications and divisions. If there are no quotients in this function, it is a rational integral function.

rational number, any number, positive or negative, that is an integer or can be expressed as the quotient of two integers; zero is included.

rational roots, roots which are rational numbers.

ray, same as half-line, the part of a straight line that extends indefinitely in one direction from a fixed point on it.
circular ray, q.v.

real axis, see axis.

real number, all positive and negative, rational and irrational, integral and fractional numbers.

reciprocal, corresponding to every real number $a$, except zero, there is a real number $a'$ and only one such that $a \times a' = 1$. We call it the reciprocal of $a$ and denote it by the symbol $1/a$.

reciprocal equations, see equation, relation.

reciprocal system of vectors, the system of three vectors found by dividing the three vector products of three non-coplanar vectors by a scalar product.

reciprocity, a transformation which carries points into lines and lines into points and preserves incidence relations.

rectangle, a parallelogram with one right angle.

rectangular coordinates, see Cartesian coordinates.

rectilinear, composed of straight lines.

reduced equation, an equation formed by transforming the original equation of degree $n$ by a substitution which will result in an equation lacking the term of degree $n-1$. This equation may be solved by earlier methods of quadratic equations. In differential equations, the reduced equation is formed by letting the arbitrary function of $x = 0$.

reducibility, a polynomial is said to be reducible if it is identically equal to the product of two polynomials neither of which is a constant.
reflection, the abstract projective view of physical reflection; an involutory transformation; also called a harmonic perspectivity; a special case of homology.

region, a connected and closed set of points which contains inner points.

fundamental, q.v.

Gallilean, q.v.

regression, any discrete sequence which has a last element and no first.

edge of, see tangent surface of a curve.

regula falsi, the method of interpolation in Newton's process of computing the real root of a real equation.

regular function of a complex argument, a function of a complex argument \( z \) in a given domain, \( w = f(z) \), only when the limit, \( \lim_{\zeta \to 0} \frac{f(z + \zeta) - f(z)}{\zeta} \) exists for every point \( z \) of this domain, so that \( \lim f(z) = \lim_{n \to \infty} f(z_n) \), independent of the direction.

regular polygon, see polygon.

regular sequence, see sequence.

relative maximum, see maximum.

relative minimum, see minimum.

remainder, that which is left after division by a number which is not a factor of the dividend.

remainder theorem, if a polynomial \( f(x) \) be divided by \( x - c \) until remainder independent of \( x \) is obtained, this remainder is equal to \( f(c) \), which is the value of \( f(x) \) when \( x = c \);
symbol: \( f(x) \equiv (x-c)q(x) + r \).

remainder in Taylor's theorem, see Lagrange form of the remainder.
repeated integral, see iterated integral.
representation, see conformal mapping.
representative point, see affix.
residue, the remainder obtained when any arbitrary polynomial \( F(x) \) having integral coefficients is divided by an irreducible polynomial of degree \( n \), \( P(x) \), mod \( p \) and the quotient \( Q(x) \) is obtained. The residue \( f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} \), each \( a_i \) being one of the numbers \( 0, 1, 2, \ldots, p-1 \). The coefficient of \(-1^{st}\) power in the expansion of a function about a pole of the function.

resolvent cubic equation, a cubic equation which results from methods of completing the square in a quartic equation. This equation can then be solved by Tartaglia's method for the solution of the general cubic equation, q.v. Ferrari thus obtained the method for the solution of the general quartic equation.

resultant, see Bezout's method of elimination.
revolve, a figure is said to revolve about a straight line as an axis when each of its points moves in the circumference of a circle whose center is in the axis and whose plane is perpendicular to the axis.
rhombus, a parallelogram with adjacent sides equal.
rhumb line, see loxodromic curve.
Riccati equation, a generalized differential equation, the
solution of which gives the angle between the tangents at two points on a curve in terms of functions of arc length between these points: \( \frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2 \)

where \( P, Q \) and \( R \) are functions of \( x \).

**Riemann definite integral**, see definite integral.

**Riemannian geometry**, non-Euclidean geometry of the spherical or antipodal type.

**right angle**, see angle.

**right section**, of cylinder or prism, a section at right angles to the elements.

**Rodríguez's formula**, see classical equations.

**Rolle's theorem**, if a function \( f(x) \) is continuous in the interval \( a \leq x \leq b \) and has a derivative at every interior point of \( (a,b) \), then there exists at least one point \( x = \xi \), \( (a < \xi < b) \), such that \( f'(\xi) = 0 \); whence by the mean value theorem, \( \frac{f(b) - f(a)}{b - a} = 0 \) or \( f(a) = f(b) \) and we can state the theorem: if \( a \) and \( b \) are two roots of the equation \( f(x) = 0 \), then \( f'(x) = 0 \) has at least one root between \( a \) and \( b \), provided that \( f(x) \) is continuous in the interval \( a \leq x \leq b \) and has a derivative at every interior point of \( (a,b) \).

**root**, the number which used as a factor as many times as indicated gives the original number is a root of that number; the number which when substituted in a given equation, satisfies the equation.

See extraneous, imaginary, irrational, rational roots.
square root, one of two equal factors of a number.

rose curves, higher plane curves, the loci of the equations,
\[ r = a \cos n\theta \quad \text{and} \quad r = a \sin n\theta . \]

rotation, a transformation where no change of shape is affected.

roulette, see curve.

rounding off numbers, the process of finding the closest approximation to the value of a number which cannot be found accurately. These rules should be followed: find at least one more decimal place than is called for; do not round off until the final result; if the digit of the next decimal place is less than five we drop it; if it is five we increase the digit to the left one if it is an odd number, but do not change it if it is even; or if the digit is more than five we add one to the digit on the left.

row of points, see range.

rule of three, a rule for finding any term of a proportion, the three others being given.

ruled surface, a surface covered with straight lines.

ruling of a quadric surface, a line every point of which is a point of the quadric.
Sacherry's quadrilateral, a configuration which plays an important role in hyperbolic non-Euclidean geometry.

salient point, the point of a curve at which two branches of a curve meet and stop but do not have a common tangent.

scalar, a quantity which is considered as possessing magnitude but no direction.

imaginary scalar, see biscalar.

scales of notation, a system of arithmetic notation in which the successive places determine the value of the figures. See binary, decimal, septimal scales.

scalene, see triangle.

screw surface, see helicoid.

sec, symbol for secant.

secant, a line which cuts a curve in two points; the trigonometric function, see angle, functions.

second degree equation, see quadratic equation.

second order equation, the differential equation with a second derivative.

section, a cutting or intersection of a figure by a line or plane. See cross section, golden section, right section.

sector, a part of a circle formed by two radii and the intercepted arc.

segment, a definite part of a line determined by two points. self-conjugate, same as self polar, having its own pole and polar, with respect to the conic.
self-corresponding, the common element, if any, of two perspective form is self-corresponding.

self-dual, its own dual, e.g., a line in space or spaces when n is odd.

self-projective, invariant under a collineation group.

semi-axis, see axis.

semi-circle, a half circle.

semi-circumference, a half circumference.

semi-diameter, same as radius.

seminvariant, an homogeneous, isobaric polynomial in the coefficients of a binary form f, an invariant of f with respect to all transformations of the type,

$$T_k : x = \xi + k \gamma ; \quad y = \gamma.$$  

septimal scale, the representation of a number, as a polynomial in 7 with integral coefficients, e.g.,

$$337 = 4 \cdot 7^2 + 5 \cdot 7 + 6.$$  

Symbol: 456.

sequence, an ordered infinite aggregate of real numbers having a one-to-one correspondence with the positive integers in their natural order. Symbol: \((x_n)\) or \((x_1, x_2, \ldots)\).

arithmetic sequence, see arithmetic progression.

bounded sequence, q.v.

finite, both a first and last element.

convergent, see convergence.

discrete, see progression.

harmonic, see harmonic progression.

monotone, q.v.

null, q.v.

partial sum, q.v.
series, a set in order, same as partial sum sequence.
Symbol: \( \sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots \).

**alternating series**, q.v.

**arithmetic**, see arithmetic progression.

**asymptotic**, q.v.

**binomial**, see binomial theorem.

**complex**, q.v.

**convergent**, see convergence.

**divergent**, see convergence.

**double**, the product of two series. Symbol: \( \sum_{i,j=1}^{\infty} a_{ij} \).

**exponential**, expansion of exponential functions.

**finite**, q.v.

**infinite**, q.v.

Laurent, \( \mathcal{L}(\xi) = \sum_{n=\infty}^{\infty} a_n \xi^n \).

**logarithmic**, expansion of logarithmic functions.

**Maclaurin**, see Taylor's series.

**non-convergent**, q.v.

**normal Bernoullian**, see Bernoullian numbers.

**p-series**, q.v.

**power series**, q.v.

**Taylor's**, q.v.

**terminating**, q.v. same as finite.

**serpentine**, a higher plane curve, the locus of the equation
\( \alpha^2 y + b^2 y - a^2 x = 0 \).

**set**, A collection of elements without reference to arrangement; if the set be arranged according to any law it becomes a one-rowed matrix in which any given order of the elements may be taken as normal.
set(s) of points, of a line, plane or space such that
their corresponding numbers form a finite or infinite class.
bounded, q.v.
closed, a set of points which includes all of its
boundary points.
connected set, a set such that given any two points
A, B of the set and a number \( \varepsilon \) (arbitrarily small)
we can always select a finite number of other points
of the set so that each of the distances \( AA_1, A, A_2, \ldots A_{n-1}, A_n, A_n B \) is smaller than \( \varepsilon \).
dense, a set of points which contains no isolated points
and fulfills the condition that corresponding to any
two distinct numbers \( a, b \) of the set for which \( b \) is
less than \( a \), there is at least one number \( x \) of the
set such that \( b < x < a \), e.g., real numbers and
rational numbers.
derived, q.v.
discrete, a point set with content zero.
enumerable, an infinite set whose elements may be
put into one-to-one correspondence with the positive
integers.
exterior point of, a point which can be inclosed in
an interval containing no points of the given set.
inner point of a set, a point such that the neighbor-
hood of the point belongs entirely to the set.
limiting point of a set, q.v.
linear, a set of points limited to a line.
non-enumerable, a set which it is not possible to
put into one-to-one correspondence with the positive
integers.

**non-linear**, a set of points not limited to a line.

**open**, a set every point of which is an inner point.

**ordered**, a set of distinct points which satisfy the order laws and the laws of equality.

**sieve of Erathosthenes**, a scheme for picking out prime numbers: write down all the numbers from one upward; then every second number from two is a multiple of two and may be cancelled; every third number from three is a multiple of three and may be cancelled; and so on.

**sigma functions**, same as **symmetric functions**, functions of several variables which are unaltered by the interchange of any two of the variables. Symbol: \( \sigma \) function.

**sign**, a symbol; a character indicating the relation of quantities or an operation to be performed on them.

**Descartes' rule of signs**, q.v.

**significant figures**, any of the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, 9; a zero may or may not be significant, according as it represents the exact figure which should occur or merely locates the decimal point.

**similar**, alike in shape.

**similitude**, the relation of identity between two figures irrespective of magnitude.

**axes of**, same as **homothetic axes**, the line on which the six centers of similitude of three circles taken in pairs lie.

**center of**, a point with respect to which figures are are directly or inversely similar or **homothetic**.

**simply connected domain**, a domain in which any closed curve
can be contracted to a point by continuous deformation without in so doing, going outside of the domain, e.g., the surface of a circle of square is simply connected, but not the surface between two concentric circles, since a circle on this surface concentric to the two bounding circles cannot be contracted to a point without going outside of the surface.

Simpson's rule for area under a curve,

\[ A = \left( y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n \right) \frac{\Delta x}{3} \]

where the \( y \)'s are the ordinates under the curve.

Simpson line, see pedal line.

Simultaneous equations, see equations, consistent.

Simultaneous limit, the limit which two variables of a function approach simultaneously.

\( \sin \), symbol for sine.

\( \sin \), see angle functions.

\( \sin \) curve, same as sinusoidal wave, the graph of the sine function.

Single-valued function, a function such that to every value of the independent variable there corresponds one and only one value of the dependent variable; otherwise the function is called multiple valued.

Singular points of curves, certain points at which a plane curve has peculiarities. See cusp, end point, isolated point, multiple point, node, salient point, tacnode.

\( \text{sinistrosum} \), when a point moves along a curve in the positive direction it passes from the positive to the negative
side of the osculating plane if the torsion is positive
and the curve is said to be sinistrorsum.
sinusoidal wave, see sine curve.
skew involutions, a point transformation of space which
leaves every point on each of two skew lines fixed and car-
rries every other point into its harmonid conjugate with
respect to the two points on the skew lines which are
collinear with the arbitrary general point. The lines are
called the axes of the transformation.
skew quadrilateral, see quadrilateral.
skew symmetrical determinant, see determinant.
slant height, see height.
slope of a curve, the slope of the tangent line to the curve
at that point.
slope of a line, the ratio of the ordinate to the abscissa
of any point on the line. Symbol: m.
solenoidal vector function, a vector function the divergence
of which vanishes at every point of space.
solid angle, see angle.
solid geometry, the geometry of three dimensional space.
space,

three space, a space of three dimension. Symbol: \( S_3 \),
four space, a space of four dimensions. Symbol: \( S_4 \),
n-space, a space of \( n \) dimension. Symbol: \( S_n \).
space curve, see line.
species, a triangle given in species, its angles given.
sphere, the locus of a point in space that moves so that its
distance from a fixed point called the center is always
constant. The constant distance from the center is the
radius of the sphere; the path of a circle revolving about its diameter.

spherical coordinates, a system of coordinates using the rectangular axes but measured by the angle which a line from the point to the origin makes with the z-axis, the length of this line and the angle which a line from the projection of the point to the origin makes with the x-axis. The spherical coordinates may be obtained from the rectangular coordinates by the transformation, \( x = r \cos \phi \cos \theta \), \( y = r \cos \phi \sin \theta \) and \( z = rsin \theta \), where \( r \) is the radius vector, \( \phi \) is the vectorial angle and \( \theta \) is the angle which the line from the point to the origin makes with the xy plane.

spherical curve, a curve which lies upon a sphere.

spherical geometry, a geometry in which there are no real parallel lines, but lines meet without exception at a finite distance. This geometry is analogous to Euclidean geometry of an unbounded hemisphere, where the great circle arc of Euclidean geometry plays the role of a straight line in spherical geometry. Same as antipodal geometry or elliptic geometry.

spherical indicatrix, the locus of the extremities of radii of the unit sphere which are parallel to the positive directions of the tangents of a curve.

spheroid, see ellipsoid.

spiral of Archimedes, a higher plane curve, the locus of the equation \( r = a \theta \).

logarithmic, q.v.

square, the result of multiplying a number by itself; the geometric representation of this is a figure of four equal lines joined at right angles.
square matrix, see matrix.
square root, see root.
standard position of an angle; the vertex at the origin,
the initial side coinciding with the x-axis.
star, the class of all lines and planes in space incident
with a given fixed point.
stationary line, see point of inflexion.
stationary point, see cusp.
steiner points, see pascal's line.
stereographic projection, see projection.
stirling's formula, if the expression \( n! \) be replaced by
the expression \( n^n e^{-n} \sqrt{2\pi n} \) the true value will have
been divided by a number lying between 1 and \( 1 + \frac{1}{10n} \).

stochastic variables, chance variables.
stokes' theorem, the surface integral of the curl of a vector
function is equal to the line integral of the function taken
along the closed curve which bounds the surface.
straight angle, see angle.
straight line, see line.

strain, an affine transformation of importance in mechanics.

homogeneous, a transformation of points at the termi-
minus of one vector to the point at the terminus of another
vector drawn from the same origin; this also takes
straight lines into straight lines and lines parallel
to the same line go into lines parallel to the same line;
planes also go into planes and the quality of parallel-
ism is invariant.

stretch, the collineation in which the fixed points are the
origin and the point at infinity on the line. It has the effect of multiplying the distance of any point from the origin by a constant.

**stroke**, see translation.

**strophoid**, a higher plane curve, the locus of the equation,

$$x^3 + xy^2 + ax^2 - ay^2 = 0.$$  

**Sturm's functions**, see Sturm's theorem.

**Sturm's theorem**, let \( f(x) = 0 \) be an equation with real coefficients and without multiple roots. Modify the usual process of seeking the greatest common divisor of \( f'(x) \) and its first derivative \( f''(x) \) by exhibiting each remainder as the negative of a polynomial \( f^i \),

\[
\begin{align*}
f &= q_1 f_1 - f_2, \quad f_1 = q_2 f_2 - f_3, \\
f_2 &= q_3 f_3 - f_4, \quad \cdots, \quad f_{n-2} = q_{n-1} f_{n-1} - f_n,
\end{align*}
\]

where \( f_n \) is a constant \( \neq 0 \). If \( a \) and \( b \) are real numbers \( a < b \), neither a root of \( f(x) = 0 \) the number of real roots of \( f(x) = 0 \) between \( a \) and \( b \) is equal to the excess of the number of variations of sign of \( f(x) \),

\[
\begin{align*}
f(x), \quad f_{2}(x), \quad \cdots, \quad f_{n-1}(x), \quad f_{n}(x)
\end{align*}
\]

for \( x = a \) over the number of variations of sign for \( x = b \). Terms which vanish are to be dropped out before counting the variations of sign.

The functions in (1) are known as **Sturm's functions**.

**sub-class**, a class such that every element in it is an element of a larger class of which it is a part.

**sub-group**, a group such that every element in it is an element of a larger group of which it is a part.

**sub-normal**, the line subtended by the angle formed by the normal to the curve and the perpendicular to the \( x \) axis.

**substitutions on \( n \) letters**, the operation which replace \( x \) by \( x_a, x_2, \ldots, x_n \) by \( x_\ell \) where \( a, b, \ldots, \ell \)
form a permutation of $1, 2, \ldots, n$.

sub-tangent, the line subtended by the angle formed by the tangent to the curve and the perpendicular to the $x$-axis.

subtraction, the subtraction of $a$ from $b$ is the operation of finding $d$ when $a + d = b$.

subtrahend, the number to be subtracted.

successive approximations, a method, due to Picard, for solving differential equations. See Differential Equations, L. R. Ford, pp. 82-86.

successive differentiation, see differentiation.

successive integration, see integration.

sufficient condition, see necessary and sufficient condition.

sum, see addition.

summation, the fundamental process of integral calculus.

supplementary angles, see angle.

suré, same as irrational number.

surface(s), a connected and closed set of points containing inner points and whose boundary points form one or a finite number of simple curves not intersecting in pairs. If the functions are continuous and partitively monotonic the curve is called a path; and a surface bounded by a path is called a domain. A further extension of these functions gives an improper path and an improper domain.

of revolution, the path of a line, moving except in the direction of its length. The moving line is the generatrix and its different positions are the elements of the surface.

applicable, q.v.

deformation of surfaces, q.v.
developable surfaces, a ruled surface in which two straight line elements coming one right after the other on a surface lie in a plane, e.g., cylinder, cone. Such surfaces can be rolled into a plane without undergoing distortion.

double curved, one which can only be generated by motion of curved line; they may be double convex as a sphere or ellipsoid, concave outward as a surface of a pulley, or concave convex as the torus or anchor ring.

equipotential, q.v.,
isometric, q.v.
isothermal, q.v.
plane, see planes, q.v.
quadric, q.v.
ruled, one which can be generated by the motion of a straight line. Ruled surfaces are of three kinds, plane, developable and warped.
screw, q.v.
warped, a ruled surface in which any two straight line elements coming one right after the other on a surface do not lie in a plane, e.g., surface of a screw thread. The surface cannot be developed without undergoing distortion.

surface integral, a definition comparable to the definition for a line integral with the surface divided into infinitesimal elements.

symbol, that which stands for or suggests something else. A character used instead of a word or words.
symmedian, the symmetric of the median with respect to the
internal bisector issued from the same vertex.

**symmetric function**, see Sigma function.

**symmetric automorphic function**, see automorphic function.

**symmetric determinant**, see determinant.

**symmetric group**, a group composed of all permutations of n letters.

**symmetry**, the balancing of points of a figure with respect to a fixed point called the center of symmetry, or a fixed line called the axis of symmetry. The centers of a circle, sphere, and certain quadrics, and the axes of certain conics and quadrics are centers and axes of symmetry.

**synthetic division**, the process of division of a polynomial by a binomial shortened by omitting the variables and using only the coefficients in one line, and by changing the sign of the divisor; addition instead of subtraction may be used for the partial products.

**synthetic proof**, same as deductive reasoning.

**system**, a set of elements with its associated rules of combination.

**system of equations**, two or more equations containing the same unknown numbers.

**system of vectors**, see reciprocal system of vectors.

**syzygetic pencil**, see pencil.

**syzygy**, see system of the cubic under binary cubic.
tangent, a point of a curve at which two branches of a curve meet, and have a common tangent but do not stop.

tan, symbol for tangent.

tangent, the limiting position of the secant as one point approaches the other along the curve; see angle functions.

common, a tangent common to two curves or surfaces.

double-flex, q.v.

pseudo-tangent, q.v.

tangent circles, two or more circles tangent to the same line at the same point.

externally, if the circles are on opposite sides of the line.

internally, if the circles are on the same side of the line.

tangent planes, the totality of all the lines tangent to the surface at one point.

tangent surface of a curve, the totality of all the points on the tangents to a twisted curve; this consists of two sheets, corresponding respectively to the positive and negative values of the distance of any point on the surface to the point of tangency, which are tangent to one another along the curve, and thus form a sharp edge; the curve is called the edge of regression of the surface.

tangential component of acceleration, the value of the component of the acceleration along the tangent defined by the
\[
\frac{d\nu}{dt} = \frac{\nu_x \frac{dx}{dt} + \nu_y \frac{dy}{dt}}{\sqrt{\nu_x^2 + \nu_y^2}} = a_t.
\]

**Taylor's Expansion**, same as **Taylor's series**. If a function has an expansion in the form:
\[
 f(x) = C_0 + C_1 (x-a) + C_2 (x-a)^2 + C_3 (x-a)^3 + \ldots
\]
the series may be differentiated repeatedly term by term and the coefficients evaluated in terms of the functions and its derivatives; we may replace \( a \) by \( x \) in the original expression and substitute these values for the coefficients and get the resulting series called **Taylor's series**,
\[
f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots
\]
when \( a = 0 \) it reduces to **Maclaurin's series**, 
\[
f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots
\]

**Tchebichef polynomials**, 
\[
T_n(x) = \frac{1}{2^{n-1}} \cos \left( n \arccos x \right)
\]
\[
T_0(x) = 1 \quad (n = 1, 2, \ldots)
\]

term, see polynomial.

**terminal side of an angle**, see parts of an angle.

**terminating series**, a series in which after a certain finite number of terms all the terms have a value 0.

**terminus**, the head or final point of a stroke.

**tests for convergence**, see convergence.

**tetracyclic coordinates**, the ordered number set of homogeneous coordinates of points on a sphere used as homogeneous co-
ordinates of the corresponding points of the inversive plane.

**tetrahedral groups**, the rotations of a regular tetrahedron into itself. Other groups that correspond to this one are the **octahedral group**, formed by the rotation of a cube and the **icosahedral group**, formed by the rotation of an icosahedron.

**tetrahedron**, a regular polyhedron of four faces; a triangular pyramid whose faces are all equilateral triangles.

**tetrahedron of reference**, see projective coordinates.

**theorem**, see proposition.

**third proportional**, the third term in a mean proportion.

**three-eighths rule of** Cotes,

\[ \int_a^{a+\ell} f(x) \, dx = \frac{3\ell}{8} \left[ f(a) + 3f(a+\ell) + 3f(a+2\ell) + f(a+3\ell) \right]. \]

**tonic**, a dyadic that may be reduced to the form

\[ \phi = aa' + bbb' + ccc' \]

where \( a, b, c \) are scalars and \( a', b', c' \) and \( a', b', c' \) reciprocal systems.

**topology**, see analysis situs.

**torsion of a curve**, the limit of the ratio of the angle between the binormals at two points of the curve and their curvilinear distance; also called **second curvature**; its inverse is called the **radius of second curvature** or the **radius of torsion**.

**torus**, see anchor ring.

**total**, the whole or aggregate of elements considered.

**total derivatives**, see derivative.

**total differentiation**, see differentiation.

**totality**, the entire number, sum, mass or quantity.

**trace**, the line in which a given plane intersects the
coordinate planes.

**trajectory**, a curve or surface intersecting a system of curves or surfaces at a constant angle.

**transcendental curve**, a curve of a transcendental function.

**transcendental functions**, see algebraic functions.

**transcendental numbers**, numbers which are not rational and do not satisfy algebraic equations with rational coefficients, e.g., π and e.

**transfinite numbers**, Cantor assumed that the sequence of all positive integers defined a number. He called this the first transfinite ordinal number.

**transform**, to change by a certain substitution.

**transformation**, any rule which associates in a unique way the elements of one figure with the elements of another. Symbol: \( T \).

**affine**, q.v.

**circular**, q.v.

**equiform**, \( x' = ax - by; y' = bx + ay \).

**involutory**, q.v. and see reflection.

**linear**, a change of variables in homogeneously linear polynomials.

**orthogonal**, a linear transformation which carries over the variables \((x_1, \ldots, x_n)\) into the variables \((x'_1, \ldots, x'_n)\) in such a way that

\[
\chi_1^2 + \chi_2^2 + \ldots + \chi_n^2 = \chi'_1 + \chi'_2 + \ldots + \chi'_n.
\]

**periodic**, q.v.

**transformation of coordinates**, see polar equations.

**transformed equations**, equations resulting from a substitution.

**transfinite points**, points beyond the infinite points and
represented by their polar lines.

transitive relation, a relation such that if \( a < b \) and \( b < c \), then \( a < c \).

translation, a displacement or shifting of space without rotation; the geometric representation of a translation is an arrow whose magnitude and direction are equal to those of the translation; this arrow is called a stroke.

translation of axes, a translation such that the coordinates \( x', y', z' \) of an arbitrary point \( P \) with reference to a Cartesian frame of reference \( O' - x'y'z' \) and the coordinates \( x, y, z \) of the same point with reference to a parallel Cartesian frame of reference \( O - xyz \) whose origin has the coordinates \( a, b, c \) with respect to \( O' - x'y'z' \) satisfy the relations, \( x' = x - a \), \( y' = y - b \), \( z' = z - c \).

transposition, a substitution which interchanges two letters and leaves unaltered the further letters. Every substitution is a product of transpositions.

transversal, a line which cuts or meets another line.

transverse axis, see hyperbola.

trapezium, a general quadrilateral.

trapezoid, a quadrilateral with one pair of parallel sides.

trapezoidal rule for area under a curve,

\[
A = \frac{1}{2}(y_0 + 2y_1 + \cdots + 2y_{n-1} + y_n) \Delta x
\]

where the \( y \)'s are the ordinates under the curve.

trefoil, the figure formed by circles drawn with centers at vertices of an equilateral triangle half of whose side is the radius of the circle.
triad, the expression formed when three vectors are placed side by side with no sign of multiplication uniting them; the sum of such an expression is called triadic; tetrads and tetrads are formed from four vectors; polyads and polyads from any number of vectors.

triangle, a figure of six elements, three non-current lines and three non-collinear points, so arranged that two lines pass through each point and two points are on each line.

according to angles

acute, all angles are acute.
equiangular, if all angles are equal.
obtuse, one angle is obtuse.
right, one angle is a right angle.
birectangular, on the surface of a sphere two angles are right angles.
trirectangular, three angles are right angles.

relation

congruent, they can be made to coincide in all their parts.
equal or equivalent, they have the same area.
polar, on the surface of a sphere, the vertex of one is the pole of the opposite side of the other.
similar, their corresponding angles are equal and their corresponding sides proportional.
spherical, on the surface of a sphere.

according to sides

equilateral, all sides equal.
isosceles, two sides equal.
scalene, no sides equal.
special triangles.

Brocard, first, the triangle having for its vertices the points of intersection of the Brocard circle with the perpendicular bisectors of the sides of the given triangle; second, the triangle having for its vertices the points of intersection of the diameter of the Brocard circle with the symmedians of the given triangle.

differential, see Barrow’s triangle.

medial, q.v.

orthocentric, same as pedal, the triangle determined by the feet of the altitudes of the triangle.

Pascal’s, q.v.

see pedal, orthocentric.

of reference, see projective coordinates.

tangential, the triangle formed by the tangents to the circumcircle at the vertices of the given triangle.

trigonometry, the branch of mathematics concerned with the measurement of the triangle.

trigonometric equations, see equation, transcendental.

trigonometric functions, see angle functions.

triangular angle, see angle.

triangular, Cesaro moving, see moving triangular.

trilinear coordinates, the sides of the triangle of reference.

trinomial, a polynomial of three terms.

perfect square, a trinomial such that the first and last terms are perfect squares and positive and the middle term is twice the product of the square roots of the other two terms.
triple of numbers, three real numbers in an ordered arrangement.

similar triples, proportional triples.

singular triple, \((0, 0, 0)\).

trisect, to divide into three equal parts.

trisection of an angle, see construction.

trisectrix of Maclaurin, a higher plane curve, the locus of the equation, \(\chi^3 + \chi y^2 + ay^2 - 3ax^2 = 0\).

trochoid, a higher plane curve, the locus of the equations
\[
\chi = a \phi - b \sin \phi \quad \text{and} \quad y = a - b \cos \phi,
\]
the tracing point on a fixed radius of a rolling circle at a distance \(b \neq a\) from the center; prolate cycloid if \(b > a\) and a curtate cycloid if \(b < a\).

truncated prism, that part of the prism between the base and a section not parallel to the base.

Tschirnhaus transformation, \(y = g(x)/h(x)\).

where \(g\) and \(h\) are polynomials such that \(h(x)\) vanishes for no root of \(g(x) = 0\).

twisted curve, see line.
ultra-infinite points, same as ideal points.
unblic, a point on the quadric surface at which the tangent
plane is parallel to a plane through the origin which cuts
the surface in a non-degenerate circle.
undecagon, a polygon of eleven sides.
undetermined coefficients, method of the procedure of develop-
ing a function of a complex argument in a power series which
is the result of two series in powers of a complex variable
with positive integral exponents having the same value along
an indefinitely small arc of a curve through the origin; so
that the two series coincide term for term, by writing the
series with undetermined coefficients and then determining
these coefficients by any suitable relation which we know
the function satisfies. This method is quite restricted in
its scope; it is applicable to certain combinations of poly-
nomials, exponentials, sines and cosines.
unicursal curve. a rational curve by which a point describes
the complete curve in a single circuit.
uniform, alike at all points.
unimodular matrix, see matrix.
uniqueness, the result of an operation is determined as soon
as the elements are determined.
unit, the element of a certain kind used to measure a quantity.
unity, one.
unknown, the variable.
valid, based on evidence that can be supported.

value, the worth of a quantity, measured in terms of a suitable unit. See absolute value, admissible values, critical values.

vanish identically, if each of the homogeneous constants in an equation is equal to zero, the equation is said to vanish identically.

variable, a quantity which changes in value throughout a discussion.

cogradient, q.v.

dependent, q.v.

independent, q.v.

range of, q.v.

variation, a change of one variable in proportions to the change in another.

direct, when the change is directly proportional.

inverse, when the change is inversely proportional.

joint, when the change involves more than two variables.

vector, a quantity which is considered as possessing direction as well as magnitude. A typical vector is the displacement of translation in space.

vector analysis, the study of vectors; also called reflexive geometry.

vector field, a field composed of each point P of a region R is associated with a vector V(P),

vector functions, see solenoidal, irrational and linear
vector functions.

vectorial angle, see angle, polar.

vector semi-tangent of version, a vector drawn in the direction of the axis of rotation, whose magnitude is equal to the tangent of one-half the angle of version.

velocity, rate of change of position.

versor, a dyadic which represents a rotation; a special case of a cyclic dyadic.

biquadrantal versor, q.v.

vertex, a point of meeting, e.g., the vertex of an angle is the point at which the sides meet.

vertical, perpendicular to the plane of the horizon.

vertical angles, see angles.

vinculum, a sign of aggregation, e.g., $\overline{ab}$.

volume, the number of cubic units of measure.
warped surface, see surface.

weight, a rational function of the coefficients of a form or system of forms which when these forms are subjected to any non-singular linear transformation, is merely multiplied by the \( \mu \) power of the determinant of the transformation is called a relative invariant of weight \( \mu \) of the form or system of forms.

Weierstrass \( \mathcal{M} \)-test, see convergence tests.

whole number, see integer.

width, the smaller of the two dimensions measured in the plane.

Wronskian, the determinate relating two or more linearly dependent functions,

\[
W(\mu_1, \ldots, \mu_\eta) = \begin{vmatrix}
\mu_1, & \mu_2, & \ldots, & \mu_\eta \\
\mu'_1, & \mu'_2, & \ldots, & \mu'_\eta \\
\mu''_1, & \mu''_2, & \ldots, & \mu''_\eta \\
\vdots & \ddots & \vdots & \vdots \\
\mu^{(\eta-1)}_1, & \mu^{(\eta-1)}_2, & \ldots, & \mu^{(\eta-1)}_\eta \\
\mu^{(\eta)}_1, & \mu^{(\eta)}_2, & \ldots, & \mu^{(\eta)}_\eta
\end{vmatrix}
\]

where \( \mu_1, \ldots, \mu_\eta \) are \( \eta \) functions of \( \chi \), linearly dependent in an interval \( a < \chi < b \), if there exist constants \( c_1, \ldots, c_\eta \) not all zero such that

\[
c_1 \mu_1(x) + \cdots + c_\eta \mu_\eta(x) = 0
\]

holds identically in the interval.
x-axis, see axis.
x-coordinate, see axes.
x-distance, same as x coordinate.
x-intercept, the distance measured on the x axis, perpendicular to the other planes.

y

y-axis, see axis.
y-coordinate, see axes.
y-distance, same as y coordinate.
y-intercept, the distance measured on the y axis, perpendicular to the other planes.
z-axis, see axis.

z-coordinate, see axes.

z-distance, same as z coordinate.

z-intercept, the distance measured on the z-axis, perpendicular to the other planes.

zero, the negative of number, yet included in the series of real numbers, but uniquely different, not having all the properties of the other numbers. There is only one real number z such that \( z + z = z \). Symbol: 0.

\[ \zeta (s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \].

zone, that portion of a sphere which lies between parallel planes.
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