Homogenization of Heterogeneous Composites by Using Effective Electromagnetic Properties

THESIS

Presented in Partial Fulfillment of the Requirements for the Degree Master of Science in the Graduate School of The Ohio State University

By

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Nowadays, multi-scale or multi-physics modeling plays a very important role in many important problems. In the past few years, there have been increasingly growing research activities aiming at developing novel multi-scale computational methods. In this thesis, we will focus on one computational electromagnetics method (CEM) which is based on the homogenization theory and considers the microscopically heterogeneous system to be a macroscopically homogeneous system by averaging the local electromagnetic fields and current distributions. The averaging process adopted here is to replace the original inhomogeneous structure by a homogeneous material with effective anisotropic permittivity and permeability tensors.
Dedicated to my parents for all their support over the years . . .
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CHAPTER 1

INTRODUCTION

The study on the electromagnetic wave propagation and scattering of heterogeneous materials has found its obvious importance in the practical applications, such as the design and performance test of structural material coating which is applied on the real life air platform to reduce the radar signature. In order to simplify the calculation on such a multi-scale electromagnetic problem with heterogeneous composites, analytic treatments using homogenization theory are usually preferred. They have been studied for many years and is still an active field of such problems [1].

The idea of homogenization theory is developed in the early 1970’s, primarily by a group of French mathematicians [2], originally refer to the mechanics theory specifically designed to analyze micro-structured materials [3], and then it is widely used in the applied mechanics. A few of the published results are applicable to the electromagnetic problems: [4] and [5] talked about wedge or pyramidal electromagnetic absorbers; [6] and [7] introduced a generalized impedance-type equivalent boundary condition by using the homogenization method for rough surfaces; [8] and [9] proposed an equivalent method by using effective material properties. They also extensively investigated the quantitative interrelationship between the coating materials, coating layers, coating thickness and the shape design in the electromagnetic scattering
problems. Even though the homogenization technique assumes that the period of the structure is small compared to one wavelength, results given in [5] indicate that the homogenization models can be accurate for periodicity as large as half to one free space wavelength, and possibly even larger for lossy structures.

The advantages of considering a homogeneous system by averaging of the local electromagnetic fields and current distributions are two folds: first, it reduces considerably the cost of numerical simulation, second, it provides the understanding of the macro dynamics of considered problem. However, the influences of the small scales of such systems at a macro-scale are usually non-trivial. Finding the correct averaging process plays a central role in homogenization theory. In this thesis, the electromagnetic properties of heterogeneous composite will be represented by a homogeneous material with effective permittivity and permeability tensors. The electromagnetic properties of a heterogeneous composite can be completely determined by solving Maxwell’s equations inside. However, if the particular details of the inhomogeneous structure are complicated but unimportant to our interest, the local electromagnetic properties such as field and current can be averaged as long as they share the same scattering characteristics with the original ones. Therefore, a heterogeneous structure can be replaced by a continuous and homogeneous material.

1.1 Object of Thesis

The homogenization methodology adopted in this thesis is to derive a the constitutive parameters of such materials in terms of anisotropic tensors. The work is motivated by an equivalence scheme for a inhomogeneous anisotropic coating proposed in [10]. Maxwell’s equations are cast into a $4 \times 4$ matrix formulation. Scattering
Solution based upon both eigenfunction expansions and the hybrid finite element and boundary integral formulation [11] are considered. By setting them to be equal, we can derive the equivalent material property by the inversion of the scattering solution.

1.2 Organization of Thesis

The following chapters of the thesis are organized as follows: Chapter 2 will focus on the introduction to the hybrid finite element and boundary integral method (infinite FE-BE method), on the derivation of boundary value problems (BVP) in a unit periodic cell, and on the interior penalty formulation. Chapter 3 will discuss the analytical formulation for the computation of scattering by general anisotropic material, which employs a first-order state-vector differential equation representation of Maxwell’s equations whose solution is given in terms of a $4 \times 4$ transition matrix relating the tangential field components at input and output planes of the anisotropic regions. The scattering properties we obtain in Chapter 2 will be inversed numerically from the analytical formulation to get the equivalent material property.

In Chapter 4, we will examine the model of our homogenization scheme by applying it on the composite honeycomb structure. We will compare the electromagnetic properties of the original honeycomb structure with the homogenized ones. Finally, Chapter 5 provides conclusions for the whole thesis.
CHAPTER 2

GENERAL INTRODUCTION TO HYBRID
FINITE/BOUNDARY ELEMENT METHOD FOR
PERIODIC STRUCTURES

In this chapter, we would like to introduce the hybrid Finite Element Method/Boundary Element Method (FEM-BEM) for periodic structures. FEM-BEM is one of the most appealing approaches to analyze unbounded electromagnetic radiation and scattering from heterogeneous structures. The method has the FEM’s versatility to model geometrically complex structures and materials, and at the same time, it enforces the "exact" boundary conditions prescribed at the truncation boundary through the BEM [12][13]. For periodic structures, the electromagnetic properties are analyzed by solving for a single unit cell utilizing Floquet’s theorem. [14] and [15] can analyze periodic structure without the constraint of periodicity in meshes. A FEM-BEM based on non-periodic meshes is presented in [11]. To account for non-periodic meshes, the interior penalty approach is utilized in this publication to enforce proper periodic boundary conditions across non-matching grids.
2.1 Formulation

2.1.1 Boundary value problem for infinite periodic structure

Fig. 2.2 shows the problem of an infinite periodic structure layer we are interested. An infinite layer of composite honeycomb structure is adopted here for the demonstration. As we can see from Fig. 2.2, the entire geometry is divided into two regions, the interior region $\Omega$ and the exterior region $\Omega^{(1)}_{ex}$ and $\Omega^{(2)}_{ex}$. $D_x$ and $D_y$ denote the periods in $x$ and $y$ directions.

![Diagram of Infinite Periodic Structure](image)

Figure 2.1: Demo. of Infinite Periodic Structure

In order to solve both electromagnetic wave radiation and scattering with infinite periodic structures, we should state our boundary value problem (BVP) as:

\[
\nabla \times \frac{1}{\mu_r} \nabla \times E - k_0^2 z \cdot E = -j k_0 \eta J^{imp}, \text{ in } \Omega
\]

\[
\nabla \times \nabla \times E - k_0^2 E = 0, \text{ in } \Omega^{(i)}_{ex} i = 1, 2
\]

\[
[E]_\tau = \pi_\tau (E^{inc}), \text{ on } \Gamma_i i = 1, 2
\]

\[
\left[ \frac{1}{\mu_r} \nabla \times E \right]_\gamma = \gamma_\tau (\nabla \times E^{inc}), \text{ on } \Gamma_i i = 1, 2,
\]

\[
\lim_{z \to \infty} \nabla \times E \times z - j k_0 z E = 0
\]
where $\varepsilon_r$ and $\mu_r$ are the relative permittivity and permeability, respectively, $k_0$ is the wavenumber in free space, $\eta$ is the intrinsic impedance, $J^{imp}$ is the impressed current, and $E^{inc}$ represents the incident electric field. The tangential surface trace $\pi_r$ and twisted tangential surface trace $\gamma_r$ are defined as:

$$\pi_r(v_i) := n_i \times (v_i \times n_i) \quad (2.6)$$

$$\gamma_r(v_i) := n_i \times v_i \quad (2.7)$$

where $n_i$ is the outgoing normal on the surface $\Gamma_i$. The associated jump operators are given by:

$$[v]_\gamma := \gamma_r(v^-) - \gamma_r(v^+) \quad (2.8)$$

$$[v]_\pi := \pi_r(v^-) - \pi_r(v^+) \quad (2.9)$$

We want to have a BVP to be formulated for an infinite periodic problem, so for this purpose, we need to enforce that material properties and excitations to be periodic functions of the form $f(r + mD_x x + nD_y y) = \alpha_x^m \alpha_y^n f(r)$, with $\alpha_x = e^{-jk_0 \sin \theta \cos \phi D_x}$ and $\alpha_y = e^{-jk_0 \sin \theta \cos \phi D_y}$, where $m$ and $n$ are any integers. By the introduction of the electric, magnetic currents, $J_i := \gamma_r(H_i)$ and $M_i := -\gamma_\pi(E_i)$, the scattered electric field, $E_i$, and scattered magnetic field, $H_i$, in the exterior region, can be expressed through the representation formula [11]:

$$E_i(r) = -jk_0\eta A(J_i)(r) - \frac{j\eta}{k_0} \nabla \phi (J_i)(r) + C(M_i)(r), \ r \in \Omega_i \quad (2.10)$$

$$\nabla \times E_i(r) = -k_0^2 A(M_i)(r) - \nabla \phi(M_i)(r) + jk_0\eta C(J_i)(r), \ r \in \Omega_i \quad (2.11)$$
where the single-layered vector and scalar potential $A$ and $\Phi$, the double-layered potential $C$ are defined by [16]:

$$A(v_i)(r) := \int_{\Gamma_i^+} G(r, r') v_i(r') \, dS', \quad r \in \Omega_{ex}^{(i)} \quad i = 1, 2 \tag{2.12}$$

$$\Phi(v_i)(r) := \int_{\Gamma_i^+} G(r, r') \nabla' \cdot v_i(r') \, dS', \quad r \in \Omega_{ex}^{(i)} \quad i = 1, 2 \tag{2.13}$$

$$C(v_i)(r) := \int_{\Gamma_i^+} \nabla' G(r, r') \times v_i(r') \, dS', \quad r \in \Omega_{ex}^{(i)} \quad i = 1, 2, \tag{2.14}$$

and $G(r, r') = \frac{\exp(-jk_0|\bar{r} - r'|)}{4\pi|\bar{r} - r'|}$. $\Gamma_i^+$ and $\Gamma_i^-$ represent the exterior and interior sides of $\Gamma$, respectively. As for the total field, we can have similar result just by substituting $E$ by $E^t$ where the notation $t$ denotes the total field. Obviously, $E_i^t = E_i + E_i^{inc}$, $H_i^t = H_i + H_i^{inc}$, $J_i^t = J_i + J_i^{inc}$, $M_i^t = M_i + M_i^{inc}$.

To derive the surface integral equations, we can take the representation formula (2.10) and (2.11), take the limit of $r \left( \in \Omega_{ex}^{(i)} \right) \rightarrow \bar{r} \left( \in \Gamma_i \right)$, and apply the limit of

$$\lim_{r \left( \in \Omega_{ex}^{(i)} \right) \rightarrow r \left( \in \Gamma_i \right)} A(v_i)(r) = A(v_i)(\bar{r}) = \int_{\Gamma_i^+} G(\bar{r}, r') v_i(r') \, dS', \tag{2.15}$$

$$\lim_{r \left( \in \Omega_{ex}^{(i)} \right) \rightarrow r \left( \in \Gamma_i \right)} \Phi(v_i)(r) = \Phi(v_i)(\bar{r}) = \int_{\Gamma_i^+} G(\bar{r}, r') \nabla' \cdot v_i(r') \, dS', \tag{2.16}$$

$$\lim_{r \left( \in \Omega_{ex}^{(i)} \right) \rightarrow r \left( \in \Gamma_i \right)} C(v_i)(r) = -\frac{1}{2} v_i \times n_i + pv \int_{\Gamma_i^+} \nabla' G(\bar{r}, r') \times v_i(r') \, dS' = -\frac{1}{2} v_i \times n_i + C(v_i)(\bar{r}), \tag{2.17}$$

to get

$$\frac{1}{2} M_i^t(\bar{r}) = jk_0 \eta \gamma_\tau \left( A(J_i^t)(\bar{r}) \right) + \frac{j}{k_0} \gamma_\tau \left( \nabla' \Phi(J_i^t)(\bar{r}) \right) - \gamma_\tau \left( C(M_i^t)(\bar{r}) \right), \quad \bar{r} \in \Gamma_i. \tag{2.18}$$

$$\frac{1}{2} J_i^t(\bar{r}) = -\frac{j}{\eta} \gamma_\tau \left( A(M_i^t)(\bar{r}) \right) - \frac{j}{k_0 \eta} \gamma_\tau \left( \nabla' \Phi(M_i^t)(\bar{r}) \right) - \gamma_\tau \left( C(J_i^t)(\bar{r}) \right), \quad \bar{r} \in \Gamma_i. \tag{2.19}$$

In (2.17), $pv$ indicates integration in principal value.
Since \( \mathbf{J}_i^+ = \mathbf{J}_i^- - \mathbf{J}_i^{inc} \), \( \mathbf{M}_i^+ = \mathbf{M}_i^- - \mathbf{M}_i^{inc} \), and \( \gamma_\tau (\mathbf{C}(\mathbf{v}_i)(\mathbf{r})) = 0 \) for planar \( \Gamma_i \), we can form the needed Dirichlet to Neumann (DtN) condition from (2.19):

\[
\frac{1}{2} \mathbf{J}_i^-(\mathbf{\bar{r}}) = -\frac{j k_0}{\eta} \gamma_\tau (\mathbf{A}(\mathbf{M}_i^-)(\mathbf{\bar{r}})) - \frac{j}{k_0 \eta} \gamma_\tau (\nabla_\tau \Phi(\mathbf{M}_i^-)(\mathbf{\bar{r}})) + \mathbf{J}_i^{inc} \tag{2.20}
\]

Finally, we can constrain our problem domain inside the the interior region \( \Omega \):

\[
\nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{E} - k_0^2 \varepsilon_r \mathbf{E} = -j k_0 \eta \mathbf{J}_\text{imp}, \quad \text{in } \Omega \tag{2.21}
\]

\[
\gamma_\tau (\mathbf{E}) = 0, \quad \text{on } \Gamma_{PEC} \tag{2.22}
\]

\[
\frac{1}{2} \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{E}|_{\Gamma_i} \right) = -k_0^2 \gamma_\tau (\mathbf{A}(\mathbf{M}_i)) - \gamma_\tau (\nabla \Phi(\mathbf{M}_i)) + \gamma_\tau (\nabla \times \mathbf{E}^{inc}|_{\Gamma_i}), \quad \text{on } \Gamma_i \ i = 1, 2. \tag{2.23}
\]

### 2.1.2 Boundary value problem for a unit cell

Fig. 2.2 shows the unit cell of an infinite periodic structure. The interior domain becomes \( \tilde{\Omega} \). \( \Gamma_m \) and \( \Gamma_s \) are for master and slave boundaries, respectively.

![Figure 2.2: Unit Cell of a Periodic Honeycomb Structure](image)

In the periodic case, we should utilize the periodic Green’s function \( G_p \) which can represent the repetition of \( \tilde{\Gamma}_i \) to replace the original Green’s function in equation (2.12), (2.13) and (2.14). We should also establish the boundary conditions on \( \Gamma_m \) and \( \Gamma_s \).
In the following, let’s talk more about the periodic Green’s function $G_p$. Assume that the boundary element surface $\overline{\Gamma}_i$ is flat and coincide with $x - y$ plane. According to [17], we employ the Ewald transformation for the fast evaluation of the periodic Green’s function:

$$G_p(\mathbf{r}, \mathbf{r}') = G_{p1}(\mathbf{r}, \mathbf{r}') + G_{p2}(\mathbf{r}, \mathbf{r}') ,$$

(2.24)

where

$$G_{p1}(\mathbf{r}, \mathbf{r}') = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-jk_{tnm}(\mathbf{r}, \mathbf{r}')} \frac{erfc\left(\frac{jk_{zmnm}}{2E_{opt}}\right)}{2jAk_{zmnm}},$$

(2.25)

$$G_{p2}(\mathbf{r}, \mathbf{r}') = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-jk_{rpmn}R_{mn}} \left[ e^{-jk_0R_{mn}}erfc\left(\frac{R_{mn}E_{opt} - \frac{jk_0}{2E_{opt}}}{2jAk_{rpmn}}\right) + e^{-jk_0R_{mn}}erfc\left(\frac{R_{mn}E_{opt} + \frac{jk_0}{2E_{opt}}}{2jAk_{rpmn}}\right) \right],$$

(2.26)

In (2.25) and (2.26), $A$ is the area of $\overline{\Gamma}_i$, and

$$\rho_{mn} = mD_x x + nD_y y$$

(2.27)

$$R_{mn} = |\mathbf{r} - \mathbf{r}' - \rho_{mn}|$$

(2.28)

$$k_{tnm} = k_t + \frac{2\pi m}{D_x} x + \frac{2\pi n}{D_y} y$$

(2.29)

$$k_{zmnm} = \sqrt{k_0^2 - k_t \cdot k_t}$$

(2.30)

$$E_{opt} = \sqrt{\frac{2\pi}{A}}$$

(2.31)

The complementary error function $erfc$ is defined as

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du$$

(2.32)

According to Floquet theorem, the relationship between $E_m = \pi_{\tau}(E|_{r_m})$ and $E_m = \pi_{\tau}(E|_{r_s})$ can be described as

$$E_m = \alpha E_s$$

(2.33)
where $\alpha$ is the phase shift:

$$\alpha = e^{-jkr_0 \rho}$$  \hspace{1cm} (2.34)

and

$$k_t = k_0 \sin \theta \cos \phi \hat{x} + k_0 \sin \theta \sin \phi \hat{y}$$  \hspace{1cm} (2.35)

The Dirichlet and Neumann boundary conditions on $\Gamma_m$ and $\Gamma_s$ are

$$\pi_{\tau} (E_m) = \alpha \pi_{\tau} (E_s)$$  \hspace{1cm} (2.36a)

$$\gamma_{\tau} \left( \frac{1}{\mu_r} \nabla \times E_m \right) = -\alpha \gamma_{\tau} \left( \frac{1}{\mu_r} \nabla \times E_s \right).$$  \hspace{1cm} (2.36b)

In summary, the BVP for a unit periodic cell is

$$\nabla \times \frac{1}{\mu_r} \nabla \times E - k_0^2 \varepsilon_r E = -jk_0 \eta J^{imp}, \text{ in } \tilde{\Omega}$$  \hspace{1cm} (2.37)

$$\gamma_{\tau} (E) = 0, \text{ on } \Gamma_{PEC}$$  \hspace{1cm} (2.38)

$$\pi_{\tau} (E_m) = \alpha \pi_{\tau} (E_s), \text{ on } \Gamma_s \Gamma_m$$  \hspace{1cm} (2.39)

$$\gamma_{\tau} \left( \frac{1}{\mu_r} \nabla \times E_m \right) = -\alpha \gamma_{\tau} \left( \frac{1}{\mu_r} \nabla \times E_s \right), \text{ on } \Gamma_s \Gamma_m$$  \hspace{1cm} (2.40)

$$\frac{1}{2} \gamma_{\tau} \left( \frac{1}{\mu_r} \nabla \times E \big|_{\Gamma_i} \right) = -k_0^2 \gamma_{\tau} (A_p (M_\Gamma)) - \gamma_{\tau} (\nabla \Phi_p (M_\Gamma)) + \gamma_{\tau} (\nabla \times E^{inc} \big|_{\Gamma_i}), \text{ on } \Gamma_i \ i = 1, 2.$$  \hspace{1cm} (2.41)

### 2.1.3 Interior Penalty Formulation

1. **Galerkin weak statement**

The weak statement of the above BVP equations is obtained by testing each equation with the appropriate set of vector functions. The spirit of duality pairing [13] should be adopted. It would be beneficial to review it here. We may categorize the electromagnetic quantities into four forms as [18]:

\[\]
Table 2.1: Electromagnetic forms and residing function spaces

<table>
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<tr>
<th>Form</th>
<th>EM Quantities</th>
<th>Residing Function Space</th>
</tr>
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<tbody>
<tr>
<td>0-form (0D)</td>
<td>$\phi^e, \phi^m$</td>
<td>$H^1(\Omega)$</td>
</tr>
<tr>
<td>1-form (1D)</td>
<td>$E, H$</td>
<td>$H(curl, \Omega)$</td>
</tr>
<tr>
<td>2-form (2D)</td>
<td>$D, B, J, M$</td>
<td>$H(div, \Omega)$</td>
</tr>
<tr>
<td>3-form (3D)</td>
<td>$\rho^e, \rho^m$</td>
<td>$L^2(\Omega)$</td>
</tr>
</tbody>
</table>

In a physical point of view, a proper dual pairing results in an energy density. Notice that a $p$ form pairs with a $3 - p$ form in the energy density. We will follow this rule in the testing procedure.

The solution $E$ resides in the $H(curl)$ function space where $H(curl, \tilde{\Omega}) = \{v|v, \nabla \times v \in (L^2(\tilde{\Omega}))\}$. Based on the BVP shown in equation (2.37) to (2.41), we shall have four residuals associated with any given trial electric field $u \in H(curl, \tilde{\Omega})$, based on the BVP. They are:

$$R^{(1)}_{\tilde{\Omega}}(u) := \nabla \times \frac{1}{\mu_r} \nabla \times u - k_0^2 \varepsilon_r u + jk_0\eta J^{imp}$$

(2.42)

$$R^{(2)}_{\Gamma_p}(u) := \pi_r(u_m) - \alpha \pi_r(u_s)$$

(2.43)

$$R^{(3)}_{\Gamma_p}(u) := \gamma_r \left( \frac{1}{\mu_r} \nabla \times u_m \right) + \alpha \gamma_r \left( \frac{1}{\mu_r} \nabla \times u_s \right)$$

(2.44)

$$R^{(4)}_{\Gamma_i}(u) := \frac{1}{2} \gamma_r \left( \frac{1}{\mu_r} \nabla \times u_i \right) + k_0^2 \gamma_r (A_p(M_i^u)) + \gamma_r (\nabla \Phi_p(M_i^u)) - \gamma_r (\nabla \times E^{inc}|_{\Gamma_i})$$

(2.45)

where we use $\Gamma_p$ to represent both $\Gamma_m$ and $\Gamma_s$. As is shown above, the residual $R^{(1)}_{\tilde{\Omega}}(u)$ is the volume error current to support the difference between trial electric field $u$ and the exact solution $E$; $R^{(2)}_{\Gamma_p}(u)$ is the surface error magnetic current to maintain the discontinuity between surface electric field $\pi_r(u_m)$ and $\alpha \pi_r(u_s)$; and $R^{(3)}_{\Gamma_p}(u)$ is the surface electric current error to maintain the discontinuities between surface magnetic fields $\gamma_r(\nabla \times u|_{\Gamma_m})$ and $-\alpha \gamma_r(\nabla \times u|_{\Gamma_s})$ on the periodic boundary $\Gamma_p$. Similarly, the
residual $R_i^{(4)}(u)$ on the truncation boundary $\Gamma_i$ has the physical meaning of error surface current needed to support the discontinuity between $\gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_i \right)$ and the exterior trace $\gamma_\tau \left( \nabla \times u \big|_{\Gamma_i} \right)$ [11], respectively. Therefore, we can pair $R_i^{(1)}(u)$ with test function $\forall v \in H \left( \text{curl}, \tilde{\Omega} \right)$, $R_i^{(2)}(u)$ with the averaged test surface electric current $\frac{1}{2} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_m \right) - \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_s \right) \right)$, $R_i^{(3)}(u)$ with the averaged test surface electric field $\frac{1}{2} \left( \pi_\tau (v_m) + \alpha \pi_\tau (v_s) \right)$, and $R_i^{(4)}(u)$ with the test surface electric field $\pi_\tau (v_i)$ to construct inner products, according to the power density dual pairing. Moreover, different scalar coefficients will be weighted to form our Galerkin weak formulation:

Find $u \in H \left( \text{curl}, \tilde{\Omega} \right)$ such that

$$\left( v, R_i^{(1)}(u) \right)_{\tilde{\Omega}} + c_1 \left( \frac{1}{2} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_m \right) - \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_s \right) \right), R_i^{(2)}(u) \right)_{\Gamma_p} + c_2 \left( \frac{1}{2} \left( \pi_\tau (v_m) + \alpha \pi_\tau (v_s) \right), R_i^{(3)}(u) \right)_{\Gamma_p} + \sum_{i=1}^{2} c_3 \left( \pi_\tau (v_i), R_i^{(4)}(u) \right)_{\Gamma_i} = 0$$

(2.46)

The justification of the interior penalty formulation can be found in [19].

Moreover, additional inner products regarding the jumps of electric and magnetic traces on the periodic boundary $\Gamma_p$ are added in the interior penalty formulation. The inner products are

$$\left( \pi_\tau (v_m) - \alpha \pi_\tau (v_s), \frac{1}{Z_p} \left( \pi_\tau (u_m) - \alpha \pi_\tau (u_s) \right) \right)_{\Gamma_p}$$

(2.47)

and

$$\left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_m \right) + \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_s \right), Z_p \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_m \right) + \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_s \right) \right) \right)_{\Gamma_p}$$

(2.48)

with $Z_p = \sqrt{\frac{\mu_r}{\varepsilon_{r,p}}}$, which indicates an average relative impedance for $\Gamma_p$. These two inner products have the physical meaning of power density dissipation. Adding (2.47) and (2.48) into (2.46), we have a more general Galerkin weak formulation:
Next we will discuss how to choose coefficients $c_1$, $c_2$, $c_3$, $p$ and $q$. We can do the following expansion according to the Green’s identities,

\[
\begin{align*}
\langle v, R^{(1)}_{\tilde{\Omega}} (u) \rangle_{\tilde{\Omega}} &= \left( \nabla \times v, \frac{1}{\mu_r} \nabla \times u \right)_{\tilde{\Omega}} - k_0^2 \langle \nabla \times \varepsilon_r u, \nabla \times \varepsilon_{\text{imp}} \phi \rangle_{\tilde{\Omega}} + jk_0 \eta \langle \nabla \times u, J^{\text{imp}} \rangle_{\tilde{\Omega}} \\
&+ \left( \pi_\tau (v_m), \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_m \right) \right)_{\Gamma_m} + \left( \pi_\tau (v_s), \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_s \right) \right)_{\Gamma_s} \\
&+ \sum_{i=1}^{2} \left( \pi_\tau (v_i), \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_i \right) \right)_{\tilde{\Gamma}_i} \\
\langle \pi_\tau (v_i), R^{(4)}_{\Gamma_i} (u) \rangle_{\tilde{\Gamma}_i} &= \left( \pi_\tau (v_i), \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u \right) \right)_{\tilde{\Gamma}_i} + \left( \pi_\tau (v_i), 2k_0^2 \gamma_\tau (A_p (M^u)) \right)_{\tilde{\Gamma}_i} \\
&+ \left( \pi_\tau (v_i), 2\gamma_\tau (\nabla \times E^{\text{inc}}) \right)_{\tilde{\Gamma}_i} - \left( \pi_\tau (v_i), 2\gamma_\tau (\nabla \times E^{\text{inc}}) \right)_{\tilde{\Gamma}_i} \\
&= 0
\end{align*}
\]

(2.49)

2. Finding coefficients

It can be seen from (2.50) and (2.51) obviously that choosing $c_3 = -1$ can remove the inner product term $\left( \pi_\tau (v_i), \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_i \right) \right)_{\tilde{\Gamma}_i}$. 

13
The choice of $c_1$ and $c_2$ is due to the symmetric matrix equation, where $\alpha$ needs to be 1. Let’s do expansion on the second and the third inner product terms in (2.49)

$$\frac{1}{2} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_m \right) - \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_s \right) \right) , R^{(2)}_{\Gamma_p} (\mathbf{u}) ,_{\Gamma_p}$$

$$= \frac{1}{2} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_m \right) - \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_s \right) \right) , \pi_\tau (\mathbf{u}_m) - \pi_\tau (\mathbf{u}_s) ,_{\Gamma_p}$$

$$= \frac{1}{2} \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_m \right) , \pi_\tau (\mathbf{u}_m) - \frac{1}{2} \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_m \right) , \pi_\tau (\mathbf{u}_s) ,_{\Gamma_p}$$

$$- \frac{1}{2} \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_s \right) , \pi_\tau (\mathbf{u}_m) + \frac{1}{2} \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_s \right) , \pi_\tau (\mathbf{u}_s) ,_{\Gamma_s} . \tag{2.52}$$

$$\frac{1}{2} \left( \pi_\tau (\mathbf{v}_m) + \alpha \pi_\tau (\mathbf{v}_s) \right) , R^{(3)}_{\Gamma_p} (\mathbf{u}) ,_{\Gamma_p}$$

$$= \frac{1}{2} \pi_\tau (\mathbf{v}_m) , \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) + \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) ,_{\Gamma_p}$$

$$= \frac{1}{2} \pi_\tau (\mathbf{u}_m) , \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) + \frac{1}{2} \pi_\tau (\mathbf{u}_m) , \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) ,_{\Gamma_p}$$

$$+ \frac{1}{2} \pi_\tau (\mathbf{u}_s) , \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) + \frac{1}{2} \pi_\tau (\mathbf{u}_s) , \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) ,_{\Gamma_s} . \tag{2.53}$$

Setting $c_1 = 1$ and $c_2 = -1$, we can get the following inner products on the periodic boundary $\Gamma_p$

$$\frac{1}{2} \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_m \right) , \pi_\tau (\mathbf{u}_m) - \frac{1}{2} \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_m \right) , \pi_\tau (\mathbf{u}_s) ,_{\Gamma_p}$$

$$- \frac{1}{2} \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_s \right) , \pi_\tau (\mathbf{u}_m) + \frac{1}{2} \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_s \right) , \pi_\tau (\mathbf{u}_s) ,_{\Gamma_s}$$

$$+ \frac{1}{2} \pi_\tau (\mathbf{v}_m) , \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) - \frac{1}{2} \pi_\tau (\mathbf{v}_m) , \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) ,_{\Gamma_p}$$

$$- \frac{1}{2} \pi_\tau (\mathbf{v}_s) , \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) + \frac{1}{2} \pi_\tau (\mathbf{v}_s) , \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) ,_{\Gamma_s} . \tag{2.54}$$

Finally, the parameters $p$ and $q$ are set to rearrange Dirichlet and Neumann boundary conditions to Robin transmission conditions on $\Gamma_p$. Use chosen coefficients
\( c_1 = 1, \ c_2 = -1, \ \text{and} \ c_3 = -1: \)

\[
\left\{ \begin{array}{l}
\gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_m \right), (\pi_r (\mathbf{u}_m) - \alpha \pi_r (\mathbf{u}_s)) + 2qZ_p \left( \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) + \alpha \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) \right)_{\Gamma_m} = 0 \\
\gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{v}_s \right), (\pi_r (\mathbf{u}_m) - \alpha \pi_r (\mathbf{u}_s)) - 2qZ_p \left( \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) + \alpha \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) \right)_{\Gamma_s} = 0 \\
\left\{ \begin{array}{l}
\pi_r (\mathbf{v}_m), \frac{2p}{Z_p} (\pi_r (\mathbf{u}_m) - \alpha \pi_r (\mathbf{u}_s)) - 2qZ_p \left( \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) + \alpha \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) \right)_{\Gamma_m} = 0 \\
\pi_r (\mathbf{v}_s), \frac{2p}{Z_p} (\pi_r (\mathbf{u}_m) - \alpha \pi_r (\mathbf{u}_s)) + 2qZ_p \left( \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) + \alpha \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) \right)_{\Gamma_s} = 0 
\end{array} \right.
\]

and enforce Robin transmission conditions used in [20]:

\[
\begin{align*}
\gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) - \frac{j k_0}{Z_p} \pi_r (\mathbf{u}_m) &= -\alpha \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) - \frac{j k_0}{Z_p} \pi_r (\mathbf{u}_s) \\
\gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) - \frac{j k_0}{Z_p} \pi_r (\mathbf{u}_s) &= -\alpha \gamma_r \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) - \frac{j k_0}{Z_p} \pi_r (\mathbf{u}_m)
\end{align*}
\]

we can obtain \( p = \frac{i k_0}{2} \), and \( q = \frac{1}{2 j k_0} \). Finally, we can rewrite (2.49) by using all the coefficients we choose as:

Find \( \mathbf{u} \in \mathbf{H} (\text{curl}, \bar{\Omega}) \) such that

\[
a (\mathbf{v}, \mathbf{u}) = f (\mathbf{v}), \ \forall \mathbf{v} \in \mathbf{H} (\text{curl}, \bar{\Omega})
\]
with
$$a(v, u) = \left( \nabla \times v, \frac{1}{\mu_r} \nabla \times u \right)_\Omega - k_0^2 \left( v, \varepsilon_r u \right)_\Omega$$
$$+ \frac{1}{2} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_m \right) - \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_s \right), \pi_\tau (u_m) - \alpha \pi_\tau (u_s) \right)_{\Gamma_p}$$
$$+ \frac{1}{2} \left( \pi_\tau (v_m) - \alpha \pi_\tau (v_s), \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_m \right) - \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_s \right) \right)_{\Gamma_p}$$
$$+ \frac{j k_0}{2} \left( \pi_\tau (v_m) - \alpha \pi_\tau (v_s), \frac{1}{Z_p} \left( \pi_\tau (u_m) - \alpha \pi_\tau (u_s) \right) \right)_{\Gamma_p}$$
$$- \frac{1}{2 j k_0} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_m \right) + \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_s \right), Z_p \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_m \right) + \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_s \right) \right) \right)_{\Gamma_p}$$
$$+ 2 \sum_{i=1}^2 \left( \pi_\tau (v_i), \gamma_\tau \left( k_0^2 A_p (M_i^u) + \nabla \Phi_p (M_i^u) \right) \right)_{\Gamma_i} \tag{2.62}$$

and
$$f(v) = - j k_0 \eta \left( v, J^{imp} \right)_{\Omega} + 2 \sum_{i=1}^2 \left( \pi_\tau (v_i) \right) \gamma_\tau \left( \nabla \times E^{inc} \right) \tag{2.63}$$

3. Equivalence of weak solution and classic solution

We can do integration by part on equation (2.61) if the weak solution $u$ is smooth enough:

$$\left( v \nabla \times \frac{1}{\mu_r} \nabla \times u - k_0^2 \varepsilon_r u + j k_0 \eta J^{imp} \right)$$
$$+ \frac{1}{2} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_m \right), (\pi_\tau (u_m) - \alpha \pi_\tau (u_s)) \right) - \frac{Z_p}{j k_0} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_m \right) + \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_s \right) \right)_{\Gamma_p}$$
$$- \frac{\alpha^s}{2} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times v_s \right), (\pi_\tau (u_m) - \alpha \pi_\tau (u_s)) \right) + \frac{Z_p}{j k_0} \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_m \right) + \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_s \right) \right)_{\Gamma_p}$$
$$+ \frac{1}{2} \left( \pi_\tau (v_m), - \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_m \right) + \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_s \right) \right) + \frac{j k_0}{Z_p} \left( \pi_\tau (u_m) - \alpha \pi_\tau (u_s) \right) \right)_{\Gamma_p}$$
$$+ \frac{\alpha^s}{2} \left( \pi_\tau (v_s), - \left( \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_m \right) + \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times u_s \right) \right) - \frac{j k_0}{Z_p} \left( \pi_\tau (u_m) - \alpha \pi_\tau (u_s) \right) \right)_{\Gamma_p}$$
$$+ 2 \sum_{i=1}^2 \left( \pi_\tau (v_i), \gamma_\tau \left( \nabla \times u_i \right) + 2 \gamma_\tau \left( k_0^2 A_p (M_i) + \nabla \Phi_p (M_i) - \nabla \times E^{inc} \right) \right)_{\Gamma_i} = 0 \tag{2.64}$$
We can rewrite the Dirichlet and Neumann type boundary conditions in the Robin type as:

\[
\gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) + \frac{j k_0}{Z_p} \pi_\tau (\mathbf{u}_m) = - \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) + \frac{j k_0}{Z_p} \pi_\tau (\mathbf{u}_s), \quad \text{on } \Gamma_p \quad (2.65)
\]

\[
\gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_m \right) - \frac{j k_0}{Z_p} \pi_\tau (\mathbf{u}_m) = - \alpha \gamma_\tau \left( \frac{1}{\mu_r} \nabla \times \mathbf{u}_s \right) - \frac{j k_0}{Z_p} \pi_\tau (\mathbf{u}_s), \quad \text{on } \Gamma_p, \quad (2.66)
\]

and since equation (2.64) holds true for all \( \mathbf{v} \in H \left( \text{curl}, \tilde{\Omega} \right) \), we can easily get

\[
\nabla \times \frac{1}{\mu_r} \nabla \times \mathbf{u} - k_0^2 \varepsilon \mathbf{u} = - j k_0 \eta \mathbf{J}^{imp}, \quad \text{in } \tilde{\Omega} \quad (2.67)
\]

The Galerkin statement in equation (2.61) also needs to be modified slightly to take the constraints on the corner edges in to consideration [11]:

Find \( \mathbf{u} \in X \in H \left( \text{curl}, \tilde{\Omega} \right) \) such that

\[
a (\mathbf{v}, \mathbf{u}) = f (\mathbf{v}), \quad \forall \mathbf{v} \in X
\]

with the function space \( X \) defined as:

\[
X := \{ \mathbf{v} \in H \left( \text{curl}, \tilde{\Omega} \right), \pi_\tau (\mathbf{v}_m)|_{\Gamma_i \cap \Gamma_m} = \alpha \pi_\tau (\mathbf{v}_s)|_{\Gamma_i \cap \Gamma_s} \}
\]

### 2.2 Finite dimensional implementation

In order to implement the weak statement shown above, we need first partition the unit cell domain \( \tilde{\Omega} \) into a finite element mesh \( \tilde{\Omega}^h \), which is formed by tetrahedral elements \( K \). Assume the tetrahedral mesh is regular and with the largest diameter of \( h \). As to the basis functions, we use the most popular curl-conforming basis functions, the first type of Nédélec elements. Following is our discrete Galerkin formulation:

Find \( \mathbf{u}^h \in X^h \subset V^h \) such that \( a (\mathbf{v}^h, \mathbf{u}^h) = f (\mathbf{v}^h), \quad \forall \mathbf{v} \in X^h \). where

\[
V^h = \left\{ \mathbf{v}^h \in H \left( \text{curl}, \tilde{\Omega} \right), \mathbf{v}^h|_K, \nabla \times \mathbf{v}^h|_K \in \left( P^1 (K) \right)^3 \right\}
\]
\[
X^h = \left\{ v^h \in V^h, \int_{\Gamma \cap \Gamma_m} (w^h_{i,m} \cdot \hat{t}) ((v_{i,m} - \alpha v_{i,s}) \cdot \hat{t}) \, dl = 0, \forall w \in V^h \right\} \tag{2.70}
\]

with \( P^1(K) \) is the set of linear polynomials defined in the element \( K \).

We will apply the \( p \)-type multiplicative Schwarz preconditioner (pMUS) \([21]\) in the solving process. The order of basis function is \( p = 2 \), we can partition the the matrix \( A \) resulting from (2.61) in terms of the basis order as:

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I_1 & 0 \\ A_{21}A_{11}^{-1}I_2 \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \begin{bmatrix} I_1 & A_{11}^{-1}A_{12} \\ 0 & I_2 \end{bmatrix}, \tag{2.71}
\]

Its inverse is simply:

\[
A^{-1} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ -A_{21}A_{11}^{-1} & I_2 \end{bmatrix}, \tag{2.72}
\]

For a fast and efficient computation, we first approximate the Schur complement

\[ S = A_{22} - A_{21}A_{11}^{-1}A_{12} \]

by \( S \approx A_{22} \), and then perform incomplete \( LU \) factorization of \( A_{11} \) and \( A_{22} \).

\[
A_{11} = \tilde{L}_1\tilde{U}_1 + E_1, \quad \| E_1 \| \leq 10^{-6}, \tag{2.73}
\]

\[
A_{22} = \tilde{L}_2\tilde{U}_2 + E_2, \quad \| E_2 \| \leq 10^{-3}. \tag{2.74}
\]

Finally, the pMUS preconditioner \( M^{-1} \) can be constructed as

\[
M^{-1} = \begin{bmatrix} I_1 & 0 \\ -\tilde{L}_1\tilde{U}_1^{-1} & I_2 \end{bmatrix} \begin{bmatrix} (\tilde{L}_1\tilde{U}_1)^{-1} & 0 \\ 0 & (\tilde{L}_2\tilde{U}_2)^{-1} \end{bmatrix} \begin{bmatrix} I_1 & 0 \\ -A_{21}(\tilde{L}_1\tilde{U}_1)^{-1} & I_2 \end{bmatrix}. \tag{2.75}
\]

### 2.3 Reflection coefficient retrieval

For each scanning direction, the normalized reflection coefficient \( R(\theta, \phi) \) is computed through:

\[
R(\theta, \phi) = \frac{Z_{in}(\theta, \phi) - Z_{in}(0,0)}{Z_{in}(\theta, \phi) + Z_{in}(0,0)} \tag{2.76}
\]

which is based on a conventional planewave transmission line model of the problem domain. In equation (2.76), \( Z_{in}(\theta, \phi) \) is the input impedance at location \( (\theta, \phi) \).
CHAPTER 3

EFFECTIVE MATERIAL PROPERTY RETRIEVAL FROM EIGENFUNCTION EXPANSION

In this chapter, we will outline the procedures needed for the retrieval of material parameters from reflection coefficients ($S$ parameters) for homogeneous materials. Maxwell’s equations are first cast into a first-order state-vector differential equation. Then, the eigenfunction expansions of the transverse field components inside the anisotropic material are used to solve for the tangential ones on the surface of the anisotropic material through the boundary conditions on the metal-based surface and the interface at material. Finally, an analytical expression is obtained for the relation between the material property of the anisotropic material and the reflection coefficients on the anisotropic material backed by a metal surface. The relation expression is degenerated into the common form for the simple material property case.

3.1 General Formulation

In this section, we want to derive the reflection coefficient based on the geometry shown in Fig. 3.1. An infinite layer of homogeneous and anisotropic absorbing material with thickness $d$ is coated on an infinite metal plane. Region 0 is the free space with dielectric permittivity $\varepsilon_0$ and magnetic permeability $\mu_0$. Region 1 is the
Figure 3.1: Wave Scattering from an Anisotropic Material Backed by Metal Surface

anisotropic layer. The electromagnetic properties of the anisotropic media can be entirely specified by the permittivity tensor $\varepsilon$ and permeability tensor $\mu$:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad (3.1)$$

$$\mu = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \quad (3.2)$$

As is shown in Fig. 3.1, the x-y axis is chosen to coincide with the material-air interface, and the z axis is set to be perpendicular to that surface. A monochromatic plane wave is illuminated obliquely at the interface with the arriving angle $\theta^i$. 
In region 1, using an $e^{j\omega t}$ time convention, the complex phasor fields can be written in the separate product form as:

$$\vec{E}(x, z) = \vec{E}(z) e^{-jk_x x} \quad (3.3a)$$
$$\vec{H}(x, z) = \vec{H}(z) e^{-jk_x x} \quad (3.3b)$$

where $k_x = k_0 \sin \theta$ is the $x$-component of the incident wave vector $\hat{k}$, $k_0 = \omega \sqrt{\mu_0 \varepsilon_0}$ is the free space wave number, and

$$\vec{E}(z) = E_{x1} \hat{x} + E_{y1} \hat{y} + E_{z1} \hat{z} \quad (3.4a)$$
$$\vec{H}(z) = H_{x1} \hat{x} + H_{y1} \hat{y} + H_{z1} \hat{z} \quad (3.4b)$$

We should notice here that we assume the incident wave is inside $xoz$ plane for simplicity. The final material property result should also be valid since we focus the problem in an infinite plane. The variables $E_{x1}$, $E_{y1}$, $E_{z1}$, $H_{x1}$, $H_{y1}$ and $H_{z1}$ are only the functions of $z$ after we have done the variable separation.

In region 1, the Maxwell equations are expressed as:

$$\nabla \times \vec{E}(x, z) = -j\omega \vec{\mu} \cdot \vec{H}(x, z) \quad (3.5a)$$
$$\nabla \times \vec{H}(x, z) = j\omega \vec{\varepsilon} \cdot \vec{E}(x, z). \quad (3.5b)$$

Substitute Equation (3.3) into Equation (3.5), notice that the $\frac{\partial}{\partial y}$ values of both electromagnetic fields are 0 because our incident wave is illuminating inside $xoz$ plane,
and cancel out the common exponential factor $e^{jk_x x}$, we have:

\[
\frac{\partial E_{y1}}{\partial z} = j\omega \left( \mu_{xx} H_{x1} + \mu_{xy} H_{y1} + \mu_{xz} H_{z1} \right) (3.6a)
\]

\[
-j k_x E_{z1} - \frac{\partial E_{x1}}{\partial z} = j\omega \left( \mu_{yx} H_{x1} + \mu_{yy} H_{y1} + \mu_{yz} H_{z1} \right) (3.6b)
\]

\[
j k_x E_{y1} = j\omega \left( \mu_{xx} H_{x1} + \mu_{zy} H_{y1} + \mu_{zz} H_{z1} \right) (3.6c)
\]

\[
-\frac{\partial H_{y1}}{\partial z} = j\omega \left( \varepsilon_{xx} H_{x1} + \varepsilon_{xy} H_{y1} + \varepsilon_{xz} H_{z1} \right) (3.6d)
\]

\[
\frac{\partial H_{x1}}{\partial z} + j k_x H_{z1} = j\omega \left( \varepsilon_{yx} H_{x1} + \varepsilon_{yy} H_{y1} + \varepsilon_{yz} H_{z1} \right) (3.6e)
\]

\[
- j k_x H_{y1} = j\omega \left( \varepsilon_{xx} H_{x1} + \varepsilon_{zy} H_{y1} + \varepsilon_{zz} H_{z1} \right). (3.6f)
\]

The resulting six scalar component equations in (3.6) can be further reduced to four independent equations through the algebraic elimination of the $z$-components of the electric and magnetic field intensities. From the (3.6c) and (3.6f), we can represent $E_{z1}$ and $H_{z1}$ in terms of $E_{x1}$, $E_{y1}$, $H_{x1}$, $H_{y1}$ by

\[
E_{z1} = \frac{1}{\varepsilon_{zz}} \left( -\frac{k_x H_{y1}}{\omega} - \varepsilon_{xx} E_{x1} - \varepsilon_{xy} E_{y1} \right) (3.7a)
\]

\[
H_{z1} = \frac{1}{\mu_{zz}} \left( \frac{k_x E_{y1}}{\omega} - \mu_{xx} H_{x1} - \mu_{yx} H_{y1} \right). (3.7b)
\]

Using Equation (3.7) and the rest equations in (3.6), we can easily obtain the following independent partial differential equations:

\[
\frac{\partial E_{x1}}{\partial z} = j k_x \frac{\varepsilon_{xx}}{\varepsilon_{zz}} E_{x1} + j k_x \left( \frac{\varepsilon_{xy}}{\varepsilon_{zz}} - \frac{\mu_{yz}}{\mu_{zz}} \right) E_{y1} + j\omega \left( \frac{\mu_{yx}}{\mu_{zz}} \frac{\mu_{xx}}{\mu_{zz}} - \mu_{xy} \right) H_{x1} + j \left[ \frac{k_x^2}{\omega \varepsilon_{zz}} + \omega \left( \frac{\mu_{yx} \mu_{zx}}{\mu_{zz}} - \mu_{yy} \right) \right] H_{y1} (3.8a)
\]

\[
\frac{\partial E_{y1}}{\partial z} = j k_x \frac{\mu_{xx}}{\mu_{zz}} E_{y1} + j\omega \left( \mu_{xx} - \frac{\mu_{xz}}{\mu_{zz}} \right) H_{x1} + j\omega \left( \mu_{xy} - \frac{\mu_{zz}}{\mu_{zz}} \right) H_{y1} (3.8b)
\]

\[
\frac{\partial H_{x1}}{\partial z} = j\omega \left( \varepsilon_{yx} - \frac{\varepsilon_{yz} \varepsilon_{xx}}{\varepsilon_{zz}} \right) E_{x1} + j \left[ \omega \left( \frac{\varepsilon_{yy} - \varepsilon_{yz} \varepsilon_{xx}}{\varepsilon_{zz}} \right) + \frac{k_x^2}{\omega \mu_{zz}} \right] E_{y1} + j k_x \frac{\mu_{xx}}{\mu_{zz}} H_{x1} + j k_x \left( \frac{\mu_{yy}}{\mu_{zz}} - \frac{\varepsilon_{yz}}{\varepsilon_{zz}} \right) H_{y1} (3.8c)
\]

\[
\frac{\partial H_{y1}}{\partial z} = j\omega \left( \frac{\varepsilon_{xx} \varepsilon_{zz} - \varepsilon_{xx}}{\varepsilon_{zz}} \right) E_{x1} + j\omega \left( \frac{\varepsilon_{zy} \varepsilon_{yy} - \varepsilon_{zy}}{\varepsilon_{zz}} \right) E_{y1} + j k_x \frac{\varepsilon_{zz}}{\varepsilon_{zz}} H_{y1}. (3.8d)
\]
The four coupled linear first-order ordinary differential equations in (3.8) can be further expressed in matrix notation:

\[
\frac{d}{dz} \Phi(z) = \Gamma \Phi(z),
\]

(3.9)

where the state vector \( \Phi \) contains the transverse field components of \( \vec{E} \) and \( \vec{H} \) in (3.4):

\[
\Phi(z) = \begin{bmatrix} E_{x1} \\ E_{y1} \\ H_{x1} \\ H_{y1} \end{bmatrix},
\]

(3.10)

and

\[
\Gamma = \begin{bmatrix}
jk_x \left( \frac{\varepsilon_{xx}}{\varepsilon_{zz}} - \frac{\mu_y}{\mu_z} \right) & jk_x \left( \frac{\varepsilon_{yy}}{\varepsilon_{zz}} - \frac{\mu_x}{\mu_z} \right) & j \omega \left( \frac{\mu_x \mu_z}{\mu_z} - \mu_{yy} \right) & j \left[ k_x^2 + \omega^2 \left( \frac{\mu_x \mu_z}{\mu_z} - \mu_{yy} \right) \right] \\
0 & jk_x \left( \frac{\mu_x}{\mu_z} \right) & j \omega \left( \frac{\mu_x \mu_z}{\mu_z} \right) & jk_x \left( \frac{\mu_y \mu_z}{\mu_z} \right) \\
\left( \frac{\varepsilon_{xx}}{\varepsilon_{zz}} - \frac{\varepsilon_{yy}}{\varepsilon_{zz}} \right) & j \left( \frac{\varepsilon_{xx}}{\varepsilon_{zz}} - \frac{\varepsilon_{yy}}{\varepsilon_{zz}} \right) & jk_x \left( \frac{\mu_x}{\mu_z} \right) & j \omega \left( \frac{\mu_x \mu_z}{\mu_z} \right) \\
j \omega \left( \frac{\varepsilon_{xx}}{\varepsilon_{zz}} - \frac{\varepsilon_{yy}}{\varepsilon_{zz}} \right) & j \left( \frac{\varepsilon_{xx}}{\varepsilon_{zz}} - \frac{\varepsilon_{yy}}{\varepsilon_{zz}} \right) & jk_x \left( \frac{\mu_x}{\mu_z} \right) & \frac{\mu_{xx}}{\mu_z} - \frac{\varepsilon_{yy}}{\varepsilon_{zz}} \\
\end{bmatrix}.
\]

(3.11)

Solving for Equation (3.9), we find:

\[
\Phi(z) = \sum_{m=1}^{4} k_m x_m e^{\lambda_m z},
\]

(3.12)

where \( \lambda_m \) represents four eigenvalues of \( \Gamma \), and \( x_m \) is a set of corresponding eigenvectors. \( k_m \) is the coefficient we want to determine by boundary conditions.

In region 0, the transverse fields in Fig.1 can be found as follows:

\[
E_{x0}(z) = E_{x0}^i e^{-jk_z z} + E_{x0}^r e^{jk_z z}
\]

(3.13a)

\[
E_{y0}(z) = E_{y0}^i e^{-jk_z z} + E_{y0}^r e^{jk_z z}
\]

(3.13b)

\[
H_{x0}(z) = p \left( -E_{x0}^i e^{-jk_z z} + E_{y0}^r e^{jk_z z} \right)
\]

(3.13c)

\[
H_{y0}(z) = q \left( E_{x0}^i e^{-jk_z z} + E_{x0}^r e^{jk_z z} \right),
\]

(3.13d)

where \( (E_{x0}^i, E_{y0}^i) \) are the \( x \)- and \( y \)-components of the known incident electric field intensity respectively, and \( (E_{x0}^r, E_{y0}^r) \) are the \( x \)- and \( y \)-components of the unknown
reflected electric field intensity respectively; \( k_z = k_0 \cos \theta \) is the \( z \)-component of the incident field wave vector; \( p = \frac{\cos \theta_i}{\eta_0} \), \( q = \frac{1}{\eta_0 \cos \theta_i} \), and \( \eta_0 = \sqrt{\mu_0 / \varepsilon_0} \approx 377 \Omega \) is the free space wave impedance.

By enforcing the boundary condition, which is that the tangential electric fields should be vanished at the metal-backed surface (\( z = d \)), and the tangential electromagnetic fields must be continuous across the interface between free space and anisotropic material (\( z = 0 \)):

\[
E_{x1}(d) = 0 \quad (3.14a)
\]
\[
E_{y1}(d) = 0 \quad (3.14b)
\]
\[
E_{x1}(0) = E_{x0}^i + E_{x0}^r \quad (3.14c)
\]
\[
E_{y1}(0) = E_{y0}^i + E_{y0}^r \quad (3.14d)
\]
\[
H_{x1}(0) = p \left( E_{y0}^r - E_{y0}^i \right) \quad (3.14e)
\]
\[
H_{y1}(0) = q \left( E_{x0}^i - E_{x0}^r \right) \quad (3.14f)
\]

we can obtain six linear equations about \( k_i \) (\( i = 1, 2, 3, 4 \)), \( E_{x0}^r \), and \( E_{y0}^r \). The matrix notation is as follows:

\[
ZU = V \quad (3.15)
\]

where

\[
Z = \begin{bmatrix}
x_1^{(1)} e^{\lambda_1 d} & x_2^{(1)} e^{\lambda_2 d} & x_3^{(1)} e^{\lambda_3 d} & x_4^{(1)} e^{\lambda_4 d} & 0 & 0 \\
x_1^{(2)} e^{\lambda_1 d} & x_2^{(2)} e^{\lambda_2 d} & x_3^{(2)} e^{\lambda_3 d} & x_4^{(2)} e^{\lambda_4 d} & 0 & 0 \\
x_1^{(3)} e^{\lambda_1 d} & x_2^{(3)} e^{\lambda_2 d} & x_3^{(3)} e^{\lambda_3 d} & x_4^{(3)} e^{\lambda_4 d} & 0 & 0 \\
x_1^{(4)} e^{\lambda_1 d} & x_2^{(4)} e^{\lambda_2 d} & x_3^{(4)} e^{\lambda_3 d} & x_4^{(4)} e^{\lambda_4 d} & 0 & 0 \\
x_1 \quad x_2 \quad x_3 \quad x_4 & 1 & 0 & -1 & 0 & -p & 0 & p & q & 0 \\
\end{bmatrix}, \quad (3.16)
\]
the right hand side

\[
V = \begin{bmatrix}
0 \\
0 \\
E_{x0}^i \\
E_{y0}^i \\
-pE_{y0}^i \\
qE_{x0}^i
\end{bmatrix},
\tag{3.17}
\]

and the unknown vectors

\[
U = \begin{bmatrix}
k_1 \\
k_2 \\
k_3 \\
k_4 \\
E_{x0}^r \\
E_{y0}^r
\end{bmatrix}.
\tag{3.18}
\]

Our purpose is to retrieve the material property from the reflection coefficient, hence, it is crucial for us to observe the relations between \((E_{x0}^i, E_{y0}^i)\) and \((E_{x0}^r, E_{y0}^r)\), which, in this case, is the reflection coefficient on the interface between free space and anisotropic material. For this purpose, \((E_{x0}^r, E_{y0}^r)\) can be expressed in terms of \((E_{x0}^i, E_{y0}^i)\) as:

\[
\begin{bmatrix}
E_{x0}^r \\
E_{y0}^r
\end{bmatrix} = S \begin{bmatrix}
E_{x0}^i \\
E_{y0}^i
\end{bmatrix},
\tag{3.19}
\]

with

\[
S = \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\tag{3.20}
\]

According to Cramer’s rule, \(S_{11}, S_{12}, S_{21}\) and \(S_{22}\) can be solved as:

\[
S_{11} = \frac{q_1 - qq_4}{|Z|},
\tag{3.21a}
\]

\[
S_{22} = \frac{q_6 + pq_7}{|Z|},
\tag{3.21b}
\]

\[
S_{12} = -\frac{q_2 - pq_3}{|Z|},
\tag{3.21c}
\]

\[
S_{21} = \frac{qq_8 - qq_5}{|Z|},
\tag{3.21d}
\]
where $q_1 \sim q_8$ can be found in Appendix A, and they are only dependent on the
the constitutive parameters of the anisotropic material $\varepsilon_{xx} \sim \varepsilon_{zz}$, $\mu_{xx} \sim \mu_{zz}$ and the
incident wave angle $\theta_i$.

### 3.2 Inversion of Reflection Coefficient for Effective Material Properties

In Chapter 2, the reflection coefficients can be calculated through the numerical
approach, namely the hybrid FE/BE method for modeling infinite periodic struc-
tures. At the same time, the last section provides us a way to analytically derive the
expressions for the scattering parameters. We can combine these two results together
to solve for the constitutive parameters for the equivalent material. Our object in
this section is to give an idea of how to to solve for this system of nonlinear equations.

Given the incident wave one incident angle $\theta_i$ with both parallel and vertical
polarization, we can have four nonlinear angle $\theta_i$ with both parallel and vertical
polarization, we can have four nonlinear angle $\theta_i$ with both parallel and vertical
polarization, we can have four nonlinear angle $\theta_i$ with both parallel and vertical
polarization, we can have four nonlinear angle $\theta_i$ with both parallel and vertical
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polarization, we can have four nonlinear angle $\theta_i$ with both parallel and vertical
polarization, we can have four nonlinear angle $\theta_i$ with both parallel and vertical
polarization, we can have four nonlinear equation $S$. By changing incident wave angles, we are able to obtain sufficient
equations to solve for the unknown $\varepsilon$ and $\mu$.

We can use Newton’s method for solving this set of nonlinear equations. Suppose
we have a set of $n$ nonlinear equations as:

$$
\begin{align*}
  f_1(x_1, x_2, \ldots, x_n) &= 0 \\
  f_2(x_1, x_2, \ldots, x_n) &= 0 \\
  \vdots \\
  f_n(x_1, x_2, \ldots, x_n) &= 0.
\end{align*}
$$

(3.22)
We can write it in vector notation by defining

\[
\begin{bmatrix}
    f_1(x_1, x_2, \ldots, x_n) \\
    f_2(x_1, x_2, \ldots, x_n) \\
    \vdots \\
    f_n(x_1, x_2, \ldots, x_n)
\end{bmatrix},
\]

so Equation (3.22) can be written as:

\[
f(x) = 0,
\]

with \( x \in \mathbb{C}^n \) and \( f : \mathbb{C}^n \to \mathbb{C}^n \). The Jacobian matrix of \( f(x) \) evaluated at \( \hat{x} \) is:

\[
Df(\hat{x}) = \begin{bmatrix}
    \frac{\partial f_1(\hat{x})}{\partial x_1} & \frac{\partial f_1(\hat{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\hat{x})}{\partial x_n} \\
    \frac{\partial f_2(\hat{x})}{\partial x_1} & \frac{\partial f_2(\hat{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\hat{x})}{\partial x_n} \\
    \vdots & \vdots & \ddots & \vdots \\
    \frac{\partial f_n(\hat{x})}{\partial x_1} & \frac{\partial f_n(\hat{x})}{\partial x_2} & \cdots & \frac{\partial f_n(\hat{x})}{\partial x_n}
\end{bmatrix}.
\]

If \( f \) is differentiable at \( \hat{x} \in \mathbb{C}^n \), and \( x \) is near \( \hat{x} \), then

\[
f(x) \approx f(\hat{x}) + Df(\hat{x})(x - \hat{x}).
\]

Therefore, Newton iteration method can be summarized as following [22] by assuming \( Df(x) \) is nonsingular:

given an initial \( x \), and tolerance \( \epsilon > 0 \) repeat

1. Evaluate \( g = f(x) \) and \( H = Df(x) \).
2. If \( |g| \leq \epsilon \), return \( x \).
3. Solve \( Hv = -g \).
4. \( x := x + v \) until maximum number of iterations is exceeded.

Let \( x^* \) denotes the true solution, and \( x^{(n)} \) is the numerical solution after \( nth \) iterations. If \( Df(x^*) \) is nonsingular and the initial \( x^{(0)} \) is sufficiently close to \( x^* \), then Newton’s
method converges and there exists a $c > 0$ such that

$$|x^{(k+1)} - x^*| \leq c|x^{(k)} - x^*|^2. \quad (3.27)$$

From the convergence result, we can see that Newton’s method is quadratically converged. If we start near a solution, the convergence will be very fast. As to how to choose initial guess $x^{(0)}$, we may have some comments.

In our case, we can choose the initial values of the material property in between with the material properties of each heterogeneous component*. For example, assume we have two homogeneous and isotropic materials with relative permittivity $\varepsilon_r^1$ and $\varepsilon_r^2$. We can plot them in the real and imaginary axis. According to Figure 3.2, the initial values of each element inside $\varepsilon^r$ can be chosen inside the cross hatched region.

![Figure 3.2: Demo. of Choosing Initial Values](image)

*The choice of the initial condition is subject to the nature of the problem. The initial value we choose here is only compatible with the structure of our problem and Newton algorithm.
3.3 Validity Test

In this section, we will test the validity of our material property retrieval model. As is shown in Chapter 2, the reflection coefficient could be obtained numerically from the hybrid finite element and boundary integral formulation, then Equation (3.21) is inverted numerically to get the material properties $\varepsilon$ and $\mu$. To this end, a C++ code, which is based on the previous research, is implemented to retrieve the reflection coefficient, and a Mathematica code is written to numerically invert the reflection coefficient to get the material property.

We would like to simplify our material property model by focusing on the non-magnetic material with permittivity tensor as $2 \times 2 \times 1$ matrix, which means:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}$$  \hspace{1cm} (3.28)

and

$$\mu = \mu_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \hspace{1cm} (3.29)$$

Suppose we have an infinite layer with depth $d = 3mm$, and permittivity of

$$\varepsilon = \varepsilon_0 \begin{bmatrix} \varepsilon_{xx}^r & \varepsilon_{xy}^r & 0 \\ \varepsilon_{yx}^r & \varepsilon_{yy}^r & 0 \\ 0 & 0 & \varepsilon_{zz}^r \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 4.0 - j2.0 & 1.5 - j1.0 & 0 \\ 1.0 - j0.5 & 3.0 - j1.0 & 0 \\ 0 & 0 & 2.0 - j1.5 \end{bmatrix}, \hspace{1cm} (3.30)$$

we can build up a homogeneous unit cell geometry. We need to mention that at this time we don’t know the material property of this layer, so we want to find it out and compare it with the exact material property provided above. The reflection coefficients on the interface between the homogeneous material and the free space are calculated using infinite FEM/BEM formulation which introduced in Chapter 2. Finally we can obtain the material property by the inversion of the reflection
coefficient as:

\[
\varepsilon = \varepsilon_0 \begin{bmatrix}
3.99752 - j1.97727 & 1.49221 - j0.99203 & 0 \\
0.99717 - j0.49480 & 2.97779 - j1.02251 & 0 \\
0 & 0 & 1.99804 - j1.50566
\end{bmatrix},
\]  

(3.31)

Define the error function as:

\[
Error = \frac{|\varepsilon^{cal} - \varepsilon^{exa}|}{|\varepsilon^{exa}|},
\]  

(3.32)

we can find out the error of the material we calculated. Summarize what we have got:

<table>
<thead>
<tr>
<th>Table 3.1: Material Property for Validity Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculated</td>
</tr>
<tr>
<td>(\varepsilon_{xx})</td>
</tr>
<tr>
<td>(\varepsilon_{xy})</td>
</tr>
<tr>
<td>(\varepsilon_{yx})</td>
</tr>
<tr>
<td>(\varepsilon_{yy})</td>
</tr>
<tr>
<td>(\varepsilon_{zz})</td>
</tr>
</tbody>
</table>

From Table 3.1, we can see the agreement is good, so that the material property retrieval model is valid.
In this chapter, we will invest the electromagnetic properties of honeycomb composite materials. Since the electromagnetic behavior of a honeycomb composite is governed by effective permittivity and permeability of the material, we can utilize the material property retrieval method presented in the previous chapters to characterize the electromagnetic behavior of honeycomb structure.

4.1 Introduction to Honeycomb Composite Materials

Honeycomb materials are a class of cellular material with both advantageous mechanical properties and appealing electromagnetic characteristics. They have high strength per density but are lightweight, and also the damage on the structure is local. For electromagnetic properties [23], honeycombs can be used to provide protection for sensors (radomes) provided that the permittivity is loss free. At relatively low frequencies, at which the radome’s thickness is less than one-quarter wavelength, it is necessary to minimize the honeycomb’s permittivity to reduce the impact of the window on the electromagnetic signal. However at higher frequencies, the window’s permittivity and thickness can be chosen to exploit resonance phenomena that cause
the window to be highly transparent. Honeycomb materials can also be used for wave absorbing applications, in which the permittivity will become complex.

Fig. 4.1 shows the most commonly use honeycomb composite materials. In Fig. 4.1(b), the geometry shown is called honeycomb sandwich structure, which is widely used in many real life air platform to obtain certain kind (mechanical or electromagnetic) properties. Fig. 4.2 demonstrates a cross-section of a typical two-dimensional dielectric honeycomb material. The red dash line covers the smallest unit cell which can represent the periodic nature of the honeycomb structure. The quantity $r$ is the perimeter of this regular hexagon, and $d$ is the thickness of the honeycomb’s walls. The host material of the wall is assumed to be homogeneous and isotropic with relative permeability $\mu_r = 1 - j0$. We also assume that the rest part of honeycomb structure is just free space.
In the following section, we will discuss the electromagnetic behavior of this kind of structure in detail.

4.2 EM Behavior of Honeycomb Composite Material

In this section, our goal is to use homogeneous honeycomb model to examine the EM manner of the real honeycomb structure. We will begin with the verification of feasibility of the presented approach on honeycomb structure, and then find out what will happen if we increase the honeycomb layer depth. Moreover, a typical sandwich honeycomb layer will be studied by using homogeneous model, and finally, the back scattered waveform of a finite honeycomb layer will be compared between the real structure and the homogenized model. Suppose all the following examples work at X band.
4.2.1 Homogenization on Honeycomb Composite Material

In Chapter 3, we have introduced a homogenization model based on the inversion of reflection coefficient, and test the feasibility of the model by using a homogeneous material whose material properties is known in Chapter 3.2. Now we will use this model to find out the effective material properties of the honeycomb structure.

The configuration of the honeycomb structure we are interested is shown in Fig. 4.3(a). It is the cross-section of a typical two-dimensional dielectric honeycomb material. The radius for one hexagon cell is \( r = 2.5 \text{mm} \), and the depth of the honeycomb layer is \( d = 2.5 \text{mm} \). The gray part is the honeycomb wall with width \( t = 1 \text{mm} \), and homogeneous material of \( \varepsilon_r = 2 - j0.5 \) and \( \mu_r = 1 - j0 \).
The geometric and material properties of a honeycomb cause the material to be uniaxially anisotropic at microwave frequencies, since it is only anisotropic in the $X - Y$ plane, the permittivity tensor of such a sheet should be in the form of (3.28).

By taking a look at the cross polarized reflection coefficient $S_{21}$ and $S_{12}$:

$$S_{21} = -0.551 \times 10^{-4} - j6.480 \times 10^{-4} \quad (4.1)$$

$$S_{12} = -4.724 \times 10^{-5} - j4.321 \times 10^{-4} \quad (4.2)$$

which are very small, we can reduce the dielectric constant tensor to a diagonal tensor, with non-magnetic material properties. We may write:

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} & 0 & 0 \\ 0 & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}. \quad (4.3)$$

Thus, equation (3.21) can be reduced to:

$$S_{11} = \frac{(Z_\parallel - \cos \theta i)}{(Z_\parallel + \cos \theta i)} \quad (4.4a)$$

$$S_{22} = \frac{(Z_\perp \cos \theta i - 1)}{(Z_\perp \cos \theta i + 1)} \quad (4.4b)$$

$$S_{12} = 0 \quad (4.4c)$$

$$S_{21} = 0 \quad (4.4d)$$

where

$$Z_\parallel = j \tan \left( k_0 d \sqrt{\sqrt{\varepsilon_r^x - \frac{\varepsilon_r^x \sin^2 \theta_i}{\varepsilon_r^z}} \sqrt{\varepsilon_r^z - \sin^2 \theta_i} / \sqrt{\varepsilon_r^x \varepsilon_r^z} } \right) \quad (4.5a)$$

$$Z_\perp = j \tan \left( k_0 d \sqrt{\sqrt{\varepsilon_r^y - \sin^2 \theta_i} / \sqrt{\varepsilon_r^y - \sin^2 \theta_i} } \right) \quad (4.5b)$$

In the next step, we will follow the procedure of homogenization introduced in Chapter 2 and Chapter 3, which we may reiterate here in a flowchart.
According to the flowchart, we can obtain the equivalent material property for this structure as

\[ \varepsilon_{xx}^r = 1.284 - j0.105 \]
\[ \varepsilon_{yy}^r = 1.279 - j0.160 \quad (4.6) \]
\[ \varepsilon_{zz}^r = 1.332 - j0.164. \]

Till now we have obtained the effective material properties for the honeycomb structure, the next step is to test whether this homogeneous material share the same scattering properties with the original honeycomb structure. We want to compare the near field with the incident wave illuminating from different angles. An error function is also defined here to denote the accuracy of the homogenization model.
Define the error as:

\[
\text{Error } S = \frac{|S^{\text{Homogenization}} - S^{\text{Honeycomb}}|}{|S^{\text{Honeycomb}}|} \times 100\% \quad (4.7)
\]

Scanning over the entire half plane, we may plot the error as:

From Fig. 4.5, we can observe good agreement in reflection coefficient between the original honeycomb structure and the equivalent homogeneous material.
4.2.2 Different Layer Height

In the real life honeycomb applications, the height of the honeycomb layer is flexible according to its specific purpose. Sometimes the honeycomb layer will be made to 1 cm or several centimeters, which can achieve one wavelength or even thicker. Therefore, it is of great importance to examine how homogenization model works in such cases. We did several experiments for this purpose. The performances are shown in Fig. 4.7 to Fig. 4.9.

Figure 4.7: $S$ Error for $d = 2.5 mm = 0.07\lambda$

Figure 4.8: $S$ Error for $d = 20 mm = 0.53\lambda$
Figures above show that the accuracy of the homogenization model will decrease when the depth of the layer is increasing in the horizontal polarized case. The same phenomenon can be observed in the vertical polarized case. However, the error is acceptable because the maximum value is less than 9%.

4.2.3 Sandwich Honeycomb

Another honeycomb related structure which enjoys wide application in the real life is the sandwich structured honeycomb composite, which is fabricated by attaching
two skins to the lightweight honeycomb core. Radar Absorbing Material (RAM) is coated on both side of honeycomb structure to obtain better electromagnetic scattering reduction; laminates of glass or carbon fiber reinforced thermoplastics or mainly thermoset polymers are widely used as skin materials for mechanical purpose: the composite layer is far lighter than a metal plate of comparable thickness and has greater resistance to bending.

The RAMs are coated on the both sides of honeycomb layer, with the properties shown in Table. 4.2.3

<table>
<thead>
<tr>
<th>RAM Layer</th>
<th>Thickness $d$ (mm)</th>
<th>Material Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>RAM 1</td>
<td>2.5</td>
<td>5.0-j0.8</td>
</tr>
<tr>
<td>Honeycomb Structure</td>
<td>2.5</td>
<td>2.0-j0.5 (wall)</td>
</tr>
<tr>
<td>RAM 2</td>
<td>2.5</td>
<td>3.0-j0.6</td>
</tr>
</tbody>
</table>

After the homogenization process, we can compare the error of the near field described in (4.7) in Fig. 4.11:

![Horizontal Polarized Error](image1)

(a) Horizontal Polarized

![Vertical Polarized Error](image2)

(b) Vertical Polarized

Figure 4.11: Error for Sandwich Honeycomb
The big error around $\theta = 75^\circ$ in the vertical polarized comparison comes from the small reflection coefficients at those incident angles. One conjecture for this phenomenon is the fields cancel out with each other at the measurement point for reflection coefficient. The other conjecture is that the fields become resonant in this specific structure at certain incident angles.

**4.2.4 Honeycomb Structure Coating on Areofoil**

Here we would like to put honeycomb structure on the real wing of the aircraft. We want to check out the feasibility of the homogenization model in the real finite dimension. For this purpose, we will have two problems need to be done: Problem A is the numerical computation of the microscopically heterogeneous system with all fine details, and Problem B is the numerical computation of homogenization of material with permittivity and permeability tensors. A typical areofoil with a thin coating is shown in Fig. 4.12

![Figure 4.12: Areofoil with Thin Coating](image-url)
Problem A: Honeycomb Structure

Fig. 4.13 shows the basic meshing scheme we adopt in this work. We use UV-mapping based tessellation of NURBS surface, then do analytical surface meshing on the quadrilateral following the shape of honeycomb cell. Then we will grow prism mesh from the surface triangular mesh, and finally grow tetrahedral mesh from the prism mesh.

![Mesh Generation](image)

Figure 4.13: Mesh Generation

Problem B: Homogenized Equivalent Material

By using the homogenization scheme mentioned in Chapter 3, we obtained the effective material property for such structure as mentioned in (4.6). Therefore, the
structure on the wing is replaced by a homogeneous material with constitutive parameters as (4.6). By using mesh generation toolkit CUBIT [24], the geometry for this problem can be easily prepared.

**Numerical Computation**

The scattering problem we are studying is demonstrated in Fig. 4.2.4.

As to the numerical computation, we choose finite element (FE) method to account for the complex geometrical features and spatially varying material properties. Surface integral equation (SIE) method is used to truncate the unbounded computational domain. The numerical solutions of the electromagnetic wave scattering from such a composite aerofoil with the real honeycomb structures will be compared against
the solutions of the corresponding simplified homogenized aerofoil with the effective material properties.

Fig. 4.15 compares the electric current on the aerofoil surfaces of both problems. Fig. 4.16 compares the far field patterns of both cases.

Figure 4.15: Comparison of Current Distributions

(a) top view  (b) bottom view
Reasonable agreement between the near field current and mono-static far field patterns is observed.
CHAPTER 5

CONCLUSIONS

In this thesis, we have proposed a homogenization methodology based on the effective constitutive parameters. In Chapter 2, we discussed the basic BVP of a unit periodic cell and interior penalty formulation for the hybrid FE/BE method for periodic structures. In Chapter 3, we have derived the analytical formulation for the scattering parameters, and numerically invert it to obtain the effective material property. Finally, some homogenization experiments related to the honeycomb structure were studied in Chapter 4.
APPENDIX A

$q_1 \sim q_8$ found in equation (3.21) are determined as follows:

\[
q_1 = \begin{bmatrix}
    x_1^{(1)} e^{\lambda_1 d} & x_2^{(1)} e^{\lambda_2 d} & x_3^{(1)} e^{\lambda_3 d} & x_4^{(1)} e^{\lambda_4 d} & 0 \\
    x_1^{(2)} e^{\lambda_1 d} & x_2^{(2)} e^{\lambda_2 d} & x_3^{(2)} e^{\lambda_3 d} & x_4^{(2)} e^{\lambda_4 d} & 0 \\
    x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & x_4^{(2)} & -1 \\
    x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & x_4^{(3)} & -p \\
    x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & 0 
\end{bmatrix}, \quad (A.1)
\]

\[
q_2 = \begin{bmatrix}
    x_1^{(1)} e^{\lambda_1 d} & x_2^{(1)} e^{\lambda_2 d} & x_3^{(1)} e^{\lambda_3 d} & x_4^{(1)} e^{\lambda_4 d} & 0 \\
    x_1^{(2)} e^{\lambda_1 d} & x_2^{(2)} e^{\lambda_2 d} & x_3^{(2)} e^{\lambda_3 d} & x_4^{(2)} e^{\lambda_4 d} & 0 \\
    x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & x_4^{(1)} & 0 \\
    x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & x_4^{(3)} & -p \\
    x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & 0 
\end{bmatrix}, \quad (A.2)
\]

\[
q_3 = \begin{bmatrix}
    x_1^{(1)} e^{\lambda_1 d} & x_2^{(1)} e^{\lambda_2 d} & x_3^{(1)} e^{\lambda_3 d} & x_4^{(1)} e^{\lambda_4 d} & 0 \\
    x_1^{(2)} e^{\lambda_1 d} & x_2^{(2)} e^{\lambda_2 d} & x_3^{(2)} e^{\lambda_3 d} & x_4^{(2)} e^{\lambda_4 d} & 0 \\
    x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & x_4^{(1)} & 0 \\
    x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & x_4^{(2)} & -1 \\
    x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & 0 
\end{bmatrix}, \quad (A.3)
\]

\[
q_4 = \begin{bmatrix}
    x_1^{(1)} e^{\lambda_1 d} & x_2^{(1)} e^{\lambda_2 d} & x_3^{(1)} e^{\lambda_3 d} & x_4^{(1)} e^{\lambda_4 d} & 0 \\
    x_1^{(2)} e^{\lambda_1 d} & x_2^{(2)} e^{\lambda_2 d} & x_3^{(2)} e^{\lambda_3 d} & x_4^{(2)} e^{\lambda_4 d} & 0 \\
    x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & x_4^{(1)} & 0 \\
    x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & x_4^{(2)} & -1 \\
    x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & x_4^{(3)} & -p 
\end{bmatrix}, \quad (A.4)
\]
$$\begin{align*}
q_5 &= \begin{pmatrix}
  x_1^{(1)} e^{\lambda_1 d} & x_2^{(1)} e^{\lambda_2 d} & x_3^{(1)} e^{\lambda_3 d} & x_4^{(1)} e^{\lambda_4 d} & 0 \\
x_1^{(2)} e^{\lambda_1 d} & x_2^{(2)} e^{\lambda_2 d} & x_3^{(2)} e^{\lambda_3 d} & x_4^{(2)} e^{\lambda_4 d} & 0 \\
x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & x_4^{(3)} & 0 \\
x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & q
\end{pmatrix}, \quad (A.5) \\
q_6 &= \begin{pmatrix}
x_1^{(1)} e^{\lambda_1 d} & x_2^{(1)} e^{\lambda_2 d} & x_3^{(1)} e^{\lambda_3 d} & x_4^{(1)} e^{\lambda_4 d} & 0 \\
x_1^{(2)} e^{\lambda_1 d} & x_2^{(2)} e^{\lambda_2 d} & x_3^{(2)} e^{\lambda_3 d} & x_4^{(2)} e^{\lambda_4 d} & 0 \\
x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & x_4^{(3)} & 0 \\
x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & q
\end{pmatrix} -1, \quad (A.6) \\
q_7 &= \begin{pmatrix}
x_1^{(1)} e^{\lambda_1 d} & x_2^{(1)} e^{\lambda_2 d} & x_3^{(1)} e^{\lambda_3 d} & x_4^{(1)} e^{\lambda_4 d} & 0 \\
x_1^{(2)} e^{\lambda_1 d} & x_2^{(2)} e^{\lambda_2 d} & x_3^{(2)} e^{\lambda_3 d} & x_4^{(2)} e^{\lambda_4 d} & 0 \\
x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & x_4^{(3)} & 0 \\
x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & q
\end{pmatrix} -1, \quad (A.7) \\
q_8 &= \begin{pmatrix}
x_1^{(1)} e^{\lambda_1 d} & x_2^{(1)} e^{\lambda_2 d} & x_3^{(1)} e^{\lambda_3 d} & x_4^{(1)} e^{\lambda_4 d} & 0 \\
x_1^{(2)} e^{\lambda_1 d} & x_2^{(2)} e^{\lambda_2 d} & x_3^{(2)} e^{\lambda_3 d} & x_4^{(2)} e^{\lambda_4 d} & 0 \\
x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & x_4^{(3)} & 0 \\
x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & q
\end{pmatrix} -1. \quad (A.8)
\end{align*}$$
BIBLIOGRAPHY


[22] ECE103 13-1 lecture notes from UCLA.
