DEFLECTION SURFACE EQUATIONS FOR A
THIN, UNIFORMLY LOADED CIRCULAR PLATE
SUPPORTED BY DISCRETE POINT SUPPORTS

A Thesis

Presented in Partial Fulfillment of the Requirements
for the Degree Master of Science

by

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1963

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ACKNOWLEDGMENTS

The author wishes to express his appreciation to Professor A. W. Leissa, Department of Engineering Mechanics, for his guidance and suggestions in the preparation of this thesis.
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INTRODUCTION

The problem of a thin uniformly loaded circular plate supported at a number of points along its boundary was solved first by Nadai (Reference 1) in 1922. Since this time many authors have solved this or similar problems, most always requiring that the supports be located on the periphery of the plate. Bassali (Reference 2) in 1953 first attempted a solution to the problem of a thin circular plate supported at points interior to its boundary. He used the complex variable approach developed by Muskhelishvili for his solution. No numerical results were obtained except for special limiting cases.

It is, therefore, the purpose of this thesis to obtain the solution for the deflection at any point of a thin uniformly loaded circular plate supported at symmetrically located interior points, or boundary points. The approach will involve the use of Fourier series and the notation, as adopted by Timoshenko (Reference 3), will be used. Graphs, for the solution of the plate with four point supports, will be drawn illustrating the deflection surface along various radial lines and for various locations of the supports.
DISCUSSION OF PERTINENT REFERENCES

A general method for analyzing circular plates symmetrically loaded with respect to the center and supported at several points along the boundary was first given by Nadai (Reference 1), and quoted by Timoshenko (Reference 3). De Beer (Reference 4) used a similar method for the analysis of a thin circular plate, symmetrically loaded and supported at a number of points regularly distributed along the boundary. Chankretadze (Reference 5) applied Muskhelishvili's complex variable method to investigate the thin circular plate normally and uniformly loaded over the whole plate and supported at a number of points on its periphery. Lourye (Reference 6) also applied the complex variable method to the discussion of several problems of the bending of circular plates. Bassali (Reference 2) obtained the deflection at any point of a thin circular plate supported at several symmetrical points along the boundary and subjected to certain loadings along the circumference or over the area of an eccentric circle. Bassali (Reference 7), using the complex variable method developed by Kolossoff and Muskhelishvili, obtained the solution for the thin circular plate supported at several interior or boundary points and normally loaded over the area of an eccentric circle, the
boundary of the plate being free. Dawoud and Bassali (Reference 8) used the complex variable method to find the deflection, bending and twisting moments, and shearing forces at any point of a thin circular plate normally loaded over a sector and supported at its edge under a general boundary condition including the clamped and simply supported boundaries.
PLATE THEORY

Under the assumptions that:

1) Points originally on the normal to the undeformed middle surface, which is the plane midway between the faces, remain approximately on the normal to the surface in the deformed state

2) There is no deformation in the middle surface of the plate

3) The normal stress perpendicular to the faces of the plate is negligible in comparison to the other stresses

4) The deflection of the middle surface is small in comparison to the thickness of the plate

the general equation for the deflection of a thin plate can be derived.

This equation, the biharmonic in polar coordinates is

$$\nabla^4 w = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^4 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) = \frac{q}{D}$$

where

$$D = \frac{Eh^3}{12 (1 - \nu^2)}$$, w = deflection at any point, and q = load on the plate.

Further, h is the thickness of the plate, and E and ν are the material constants, Young's modulus and Poisson's ratio.
If \( w \) is known, the moments and transverse shears can be found from the following relationships:

\[
M_r = -D \left[ \frac{\partial^2 w}{\partial r^2} + \nu \left( \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right]
\]

\[
M_\theta = -D \left[ \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \nu \frac{\partial^2 w}{\partial r^2} \right]
\]

\[
M_{r\theta} = (1 - \nu)D \left( \frac{1}{r} \frac{\partial^2 w}{\partial \theta \partial \theta} - \frac{1}{r} \frac{\partial w}{\partial \theta} \right)
\]

\[
V_r = Q_r - \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta}, \quad V_\theta = Q_\theta - \frac{\partial M_{r\theta}}{\partial \theta}
\]

where

\[
Q_r = -D \frac{\partial}{\partial r} (\nabla^2 w), \quad Q_\theta = -D \frac{\partial}{r} (\frac{\nabla^2 w}{\partial \theta})
\]

The boundary conditions for a free edge are:

1) \( M_r \bigg|_{\text{boundary}} = 0 \)

2) \( V_r \bigg|_{\text{boundary}} = 0 \)
PROPOSED METHOD

The proposed method of solution for a thin, circular, uniformly loaded plate with symmetrically located interior point supports is to find a singular solution around the point supports which will satisfy \( \nabla^4 w = 0 \) and will also satisfy the required deflection, slope, moment, and shear conditions at these points. The boundary conditions for the plate will then be applied to the total solution which is the sum of the complementary, particular, and singular solutions. The terms resulting will be expressed as a Fourier series so that the necessary constants can be found.

The notation to be used for this thesis is the same as Timoshenko's (Reference 3). Special cases will include the thin circular plate supported by three point supports along a diameter, by a central point support, and by two, four, and eight symmetrical supports.

The solution for the four point support case will be programmed for the computer and numerical values for the deflection obtained. Graphs, illustrating the deflection surface of the plate along different radial lines and for various locations of the supports will be included.
SOLUTION OF THE PLATE WITH TWO POINT SUPPORTS

The first problem to be considered is a thin, uniformly loaded circular plate with two point supports where $r, r_1, r_2, b,$ and $\theta$ are measured as shown in Figure 1.

![Figure 1](image)

The fundamental equation to be solved is

$$\nabla^4 w = \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right] \left[ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right] = \frac{q}{D} \quad (1)$$

For the solution to the homogeneous equation

$$\nabla^4 w = 0$$

a solution $w = \sum_{n} f_n (r) \cos \theta$ is assumed. \quad (2)
Assuming \( f_n(r) = r^P \) and substituting the assumed \( w \) and \( f_n(r) \) into equation (1), it is found that

\[ P = n, \ n + 2, \ -n, \ -n + 2 \]

from which

\[ w = \sum_{n=2}^{\infty} \left[ A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2} \right] \cos n \theta \quad (3) \]

The complete solution of (1) would involve a similar term \( \sum g_n(r) \sin(n \theta) \). However, when the problem is restricted to one having at least one axis of symmetry, \( w(\theta) \) must equal \( w(-\theta) \) and for this reason the second part of the solution can be eliminated. In addition to this, two terms of the form \( k_1 r^2 \theta \) and \( k_2 \theta \) appear in the complete solution. These are discarded because they are multivalued functions.

It is necessary to check the values of \( n \) for which there are not distinct roots. These are \( n = 0 \) and \( n = 1 \). If \( n = 0 \) then it can be shown that

\[ f_0(r) = A_0 + B_0 r^2 \ln r + C_0 r^2 + D_0 \ln r \]

and if \( n = 1 \), then

\[ f_1(r) = A_1 r + B_1 r \ln r + C_1 r^3 + D_1 r^{-1} \]

The complete solution for equation (1) is then

\[ w = A_0 + B_0 r^2 \ln r + C_0 r^2 + D_0 \ln r + A_1 r + B_1 r \ln r + C_1 r^3 + D_1 r^{-1} \]

\[ + \sum_{n=2}^{\infty} \left[ A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2} \right] \cos n \theta \quad (4) \]
In order to have a finite deflection at \( r = 0 \),

\[
B_n = D_1 = D_0 = 0
\]  \hspace{1cm} (a)

In order to have finite moment at \( r = 0 \),

\[
D_n = B_0 = B_1 = 0
\]  \hspace{1cm} (b)

Because \( w(\theta) \) must equal \( w(\theta + \pi) \),

\[
\cos n \theta = \cos n \pi \cos n \theta
\]

from which

\[
n = 0, \ 2, \ 4, \ 6, \ \cdots
\]  \hspace{1cm} (c)

Substituting (a) and (b) into equation (4),

\[
w = A_0 + C_0 r^2 + A_1 r + C_1 r^3 + \sum_{n=2}^{\infty} [A_n r^n + C_n r^{n+2}] \cos n \theta
\]

or which is equivalent,

\[
w = \sum_{n=0, 2, 4, \cdots}^{\infty} [A_n r^n + C_n r^{n+2}] \cos n \theta
\]  \hspace{1cm} (d)

Applying the fact that \( n \) must be even, equation (d) can be written as

\[
w = \sum_{n=0}^{\infty} [A_{2n} r^{2n} + C_{2n} r^{2(n+1)}] \cos 2n \theta
\]  \hspace{1cm} (5)

In addition to the homogeneous solution, a particular solution \( w_p \) associated with the uniform load and a singular solution \( w_s \) associated with the concentrated forces due to the point supports must also be found.
The particular solution can be shown to be

$$w_p = \frac{gr^4}{64D}$$  \hspace{1cm} (6)

where \( q \) = constant load per unit area on the plate.

Define \( r_i \) as the distance from the \( i \)th support to any point on the plate. The singular solution must satisfy the following conditions around a point support:

1) \( w \bigg|_{r_i=0} = 0 \)

2) \( \frac{\partial w}{\partial r_i} \bigg|_{r_i=0} = 0 \)

3) \( \frac{\partial^2 w}{\partial r_i^2} \bigg|_{r_i=0} = \infty \) and \( \frac{1}{r_i} \frac{\partial w}{\partial r} \bigg|_{r_i=0} = \infty \) which together imply that \( M_r \) and \( M_\theta \) evaluated at \( r_i=0 \) are infinitely large.

4) \( \frac{\partial^3 w}{\partial r_i^3} \bigg|_{r_i=0} = \infty \), \( \frac{\partial^2 w}{\partial r^2} \bigg|_{r_i=0} = \infty \) and \( \frac{1}{r_i^2} \frac{\partial w}{\partial r} \bigg|_{r_i=0} = \infty \) which together imply that \( V_r \) and \( V_\theta \) evaluated at \( r_i=0 \) are infinitely large.

The function which satisfies \( \nabla^4 w = 0 \) and the above conditions is

\( w = r_i^2 \ln r_i \).

The singular solution is then

$$w_s = \sum_{i=1}^{j} \frac{F_i}{2} \frac{r_i^2 \ln r_i^2}{r_i^2}$$  \hspace{1cm} (7)
where \( F_i \) are some constants to be determined by the magnitudes of the forces and \( j \) is the number of supports.

If \( r_1 \) and \( r_2 \) are each expressed in terms of \( r \) and \( \theta \) by the cosine law, then \( w_s \) for the present problem can be rewritten as

\[
w_s = \frac{F}{2} (r^2 + b^2 - 2rb \cos \theta) \ln (r^2 + b^2 - 2rb \cos \theta) + \frac{F}{2} (r^2 + b^2 + 2rb \cos \theta) \ln (r^2 + b^2 + 2rb \cos \theta) \tag{8}
\]

If the homogeneous solution is defined as \( w_1 \), then the total solution becomes

\[
w = w_1 + w_p + w_s
\]

where \( w_1 \) is equation (5), \( w_p \) is equation (6), and \( w_s \) is equation (8).

The boundary conditions which must be applied to this solution are

1) moment at the free edge = 0, and
2) shear at the free edge = 0.

\[
M_r \bigg|_{r=a} = -D \left[ \frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r} \frac{\partial w}{\partial r} + \frac{\nu}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right]_{r=a} = 0 \tag{9}
\]

\[
V_r \bigg|_{r=a} = \left[ Q_r - \frac{\partial M_r \theta}{\partial \theta} \right]_{r=a} = 0 \tag{10}
\]

where

\[
Q_r = -D \frac{\partial}{\partial r} (\nabla^2 w)
\]

and

\[
M_r \theta = (1 - \nu) D \left[ \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r} \frac{\partial w}{\partial \theta} \right]
\]
After expansion, the second boundary term becomes

\[
\frac{V_r}{r=a} = \left[ Q_r \frac{\partial M_r}{\partial \theta} \right]_{r=a} = -D \left[ \frac{\partial^3 w}{\partial r^3} + \frac{1}{r} \frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2} \frac{\partial w}{\partial r} \right]_{r=a} + D \left[ \frac{2}{r^2} \frac{\partial^3 w}{\partial r^2 \partial \theta} + \frac{3}{r^3} \frac{\partial^2 w}{\partial \theta^2} \right]_{r=a} + Dv \left[ \frac{1}{r^2} \frac{\partial^3 w}{\partial r \partial \theta^2} - \frac{1}{r^3} \frac{\partial^2 w}{\partial \theta^2} \right]_{r=a} = 0
\]

The boundary terms which \( w_s \) yields will be expressed as Fourier series. This will facilitate the determination of the arbitrary constants.

**Development of the Fourier Series for Boundary Terms Contributed by \( w_s \)**

The first boundary condition \( M_r \bigg|_{r=a} = 0 \) applied to \( w_s \) yields an expression \( M_r \bigg|_{r=a} = -\frac{FD}{2} f(\theta) \) (See Appendix II). Representing \( f(\theta) \) by

\[
\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n \theta
\]

where

\[
A_n = \frac{2}{\pi} \int_0^\pi f(\theta) \cos n \theta \, d\theta
\]
it is found that

\[- \frac{\text{FD}}{2} f(\theta) = - \frac{\text{FD}}{2} \left[ (A_0/2)' + \sum_{n=1}^{\infty} (A_{2n})' \cos 2n \theta \right] \]

where the prime (') indicates that the constant is associated with the first boundary condition. The constants are evaluated to be:

\[(A_0/2)' = 4(1 + \nu) + 4(1 + \nu) \ln a^2 + \frac{4a}{a^2 - b^2} \frac{2}{b} (1 - \nu) + \frac{4b}{a^2 - b^2} (-3 + \nu) + \frac{8a}{a^2 - b^2} \]

and

\[(A_{2n})' = \frac{8a}{a^2 - b^2} \frac{2(n+1)}{b} (-1 + \nu) + \frac{4a}{a^2 - b^2} \frac{2(-n+1)}{b} \frac{2n}{a} (1 - \nu) + \frac{4a}{a^2 - b^2} \frac{2(n+1)}{b} \frac{2(n+2)}{a} (1 - \nu) (e) \]

\[+ \frac{4}{n} (b/a)^{2n} (-1 - \nu) \]

The second boundary condition applied to \(w_s\) yields

\[V_r \bigg|_{r=a} = - \frac{\text{FD}}{2} g(\theta) \mid_{w=w_s} \]

(See Appendix II). It is found that by representing \(g(\theta)\) by its Fourier series and by defining \(\Psi = \frac{a^2 - b^2}{a^2}\) and \(\Phi = \left[\frac{a^2 - b^2}{a^2}\right]^3\) that

\[V_r \bigg|_{r=a} = - \frac{\text{FD}}{2} g(\theta) = - \frac{\text{FD}}{2} \left[ (A_0/2)'' + \sum_{n=1}^{\infty} (A_{2n})'' \cos 2n \theta \right] \]

(f)
where the double prime (") indicates that the constant is associated with the second boundary condition and where

\[
(A_{\Phi}/2)^n = \frac{24a - 1}{\Psi} \frac{b^2}{\Phi} + \frac{8a - 3}{\Psi} \frac{b^4}{\Phi} - \frac{32a}{\Psi} + \frac{16a - 1}{\Phi} \\
- \frac{40a - 3}{\Phi} \frac{b^2}{\Phi} + \frac{24a - 5}{\Phi} \frac{b^4}{\Phi} + \frac{8a - 7}{\Phi} \frac{b^6}{\Phi} - \frac{8a - 9}{\Phi} \frac{b^8}{\Phi} \\
+ \nu \left\{ \frac{8a - 1}{\Psi} \frac{b^2}{\Phi} - \frac{8a - 3}{\Phi} \frac{b^4}{\Phi} - \frac{8a - 5}{\Phi} \frac{b^6}{\Phi} + \frac{24a - 5}{\Phi} \frac{b^4}{\Phi} \\
- \frac{24a - 7}{\Phi} \frac{b^6}{\Phi} + \frac{8a - 9}{\Phi} \frac{b^8}{\Phi} \right\}
\]

and

\[
(A_{\Phi}^n)^n = -\frac{32a}{\Psi} \frac{-2n + 1}{b} \frac{2n}{\Phi} + \frac{24a}{\Psi} \frac{-2n - 1}{b} \frac{2(n + 1)}{\Phi} + \frac{8a - 2n - 3}{b} \frac{2(n + 2)}{\Phi} \\
+ \frac{a}{\Phi} \frac{-2n - 1}{b} \frac{2n}{\Phi} \left[ -8n + 16 \right] + \frac{a}{\Phi} \frac{-2n - 3}{b} \frac{2(n + 1)}{\Phi} \left[ 32n - 40 \right] + \frac{a}{\Phi} \frac{-2n - 5}{b} \frac{2(n + 2)}{\Phi} \\
\left[ -48n + 24 \right] + \frac{a}{\Phi} \frac{-2n - 7}{b} \frac{2(n + 3)}{\Phi} \left[ 32n + 8 \right] + \frac{a}{\Phi} \frac{-2n - 9}{b} \frac{2(n + 4)}{\Phi} \\
\left[ -8n - 8 \right] + \nu \left\{ \frac{8a - 2n - 1}{\Psi} \frac{b^2}{\Phi} \frac{2(n + 1)}{\Phi} - \frac{8a}{\Phi} \frac{-2n - 3}{b} \frac{2(n + 2)}{\Phi} \\
+ \frac{a}{\Phi} \frac{-2n - 1}{b} \frac{2n}{\Phi} \left[ 8n \right] + \frac{a}{\Phi} \frac{-2n - 3}{b} \frac{2(n + 1)}{\Phi} \left[ -32n - 8 \right] \\
+ \frac{a}{\Phi} \frac{-2n - 5}{b} \frac{2(n + 2)}{\Phi} \left[ 48n + 24 \right] + \frac{a}{\Phi} \frac{-2n - 7}{b} \frac{2(n + 3)}{\Phi} \left[ -32n - 24 \right] \\
+ \frac{a}{\Phi} \frac{-2n - 9}{b} \frac{2(n + 4)}{\Phi} \left[ 8n + 8 \right] \right\}
\]
Applications of the Boundary Conditions
to the Complementary and Particular Solutions

Particular Solution

The first boundary condition \( M_r \bigg|_{r=a} = 0 \) applied to \( w_p = \frac{q a^4}{64 D} \) yields

\[
-D \left( \frac{\partial^2 w_p}{\partial r^2} + \frac{\nu}{r} \frac{\partial w_p}{\partial r} + \frac{\nu}{r^2} \frac{\partial^2 w_p}{\partial \theta^2} \right) \bigg|_{r=a} = \frac{q a^2}{16} (3 + \nu)
\]  

\[\text{(g)}\]

The second boundary condition \( V_r \bigg|_{r=a} = 0 \) applied to \( w_p \) yields

\[
V_r \bigg|_{r=a} = -\frac{q a}{2}
\]
\[
\bigg|_{w=w_p}
\]

\[\text{(h)}\]

Complementary Solution

The first boundary condition \( M_r \bigg|_{r=a} = 0 \) applied to \( w_1 = A_0 r^2 \)

\[
+ \sum_{n=1}^{\infty} \left[ A_{2n} r^{2n} + C_{2n} r^{2(n+1)} \right] \cos 2n \theta \text{ yields}
\]

\[
M_r \bigg|_{r=a} = -2 D C_0 (1 + \nu)
\]

\[\text{- (i)}\]
The second boundary term \( V_r |_{r=a} \) contributed by the complementary solution is

\[
V_r |_{r=a} = \left. \right|_{w=w_1} \sum_{n=1}^{\infty} D \left( -4n^2 + 4vn^2 + 8n^3 - 8vn^3 \right) A_{2n} a^{2n-3} \cos 2n\theta
\]

\[
+ \sum_{n=1}^{\infty} D \left( 8n^3 - 12n^2 - 8n - 8vn^3 - 4vn^2 \right) C_{2n} a^{2n-1} \cos 2n\theta \quad (j)
\]

Define:

\[
S = 4n^2 - 4vn^2 - 2n + 2vn
\]

\[
T = 4n^2 - 4vn^2 + 6n + 2vn + 2v + 2
\]

\[
U = -4n^2 + 4vn^2 + 8n^3 - 8vn^3
\]

\[
W = 8n^3 - 12n^2 - 8n - 8vn^3 - 4vn^2
\]

The first boundary condition \( M_r |_{r=a} = 0 \) for the general solution \( w = w_1 + w_p + w_s \) is found by adding equations (e), (g), and (i). This is

\[
M_r |_{r=a} = \sum_{n=1}^{\infty} \left( 2DC_0 (1 + \nu) - D \sum_{n=1}^{\infty} (SA_{2n} a^{2n-2} + TC_{2n} a^{2n}) \cos 2n\theta \right)
\]

\[
- \frac{qa^2}{16} (3 + \nu) - \frac{FD}{2} \left[ (A_0/2)^t + \sum_{n=1}^{\infty} (A_{2n})^t \cos 2n\theta \right] = 0 \quad (9)
\]
and the second boundary condition \( V_r \bigg|_{r=a} = 0 \) for the general solution if found by adding equations (f), (h), and (j). This is

\[
V_r \bigg|_{r=a} =
\]

\[
\sum_{n=1}^{\infty} \left[ (U a^{2n-3} + W C a^{2n-1}) \cos 2n \theta \right] - \frac{qa}{2} + \frac{FD}{2} \left[ (A_0/2)'' + \sum_{n=1}^{\infty} (A_{2n})'' \cos 2n \theta \right] = 0
\]

(10)

Setting the \( n = 0 \) terms equal to zero yields the following two equations:

\[
-2 DC_0 (1 + \nu) - \frac{qa^2}{16} (3 + \nu) - \frac{FD}{2} (A_0/2)' = 0
\]

(11)

and

\[
- \frac{qa}{2} + \frac{FD}{2} (A_0/2)'' = 0
\]

(12)

From equations (11) and (12) it is found that

\[
F = \frac{qa}{D(A_0/2)''}
\]

and

\[
C_0 = -\frac{qa}{4D(1 + \nu)} \left[ \frac{(A_0/2)'}{(A_0/2)''} \right] - \frac{qa^2}{32D} \frac{(3 + \nu)}{(1 + \nu)}
\]
COMPLETE SOLUTION FOR TWO POINT SUPPORTS

The complete general solution for two supports is then

\[ w = A_0 + C_0 r^2 + \sum_{n=1}^{\infty} \left[ A_{2n} r^{2n} + C_{2n} r^{2n+2} \right] \cos 2n \theta \]

\[ + \frac{qr^4}{64D} + \frac{F}{2} \left( r^2 + b^2 + 2rb \cos \theta \right) \ln \left( r^2 + b^2 + 2rb \cos \theta \right) \]

\[ + \frac{F}{2} \left( r^2 + b^2 - 2rb \cos \theta \right) \ln \left( r^2 + b^2 - 2rb \cos \theta \right) \]

where:

\[ F = \frac{qa}{D(A_0/2)^n} \]

\[ C_0 = -\frac{qa}{4D(1+\nu)} \frac{(A_0/2)^l}{(A_0/2)^m} - \frac{qa^2}{32D} \frac{(3+\nu)}{(1+\nu)} \]

\[ A_{2n} = \frac{F}{2a} \frac{(A_{2n})}{2n-3} \left[ \frac{T(A_{2n})}{SW - UT} \right] \]

\[ C_{2n} = -\frac{SA_{2n} a^{-2}}{T} - \frac{F}{2} \frac{(A_{2n})}{Ta^{2n}} \]

and

\[ A_0 = -C_0 b^2 - \frac{qb^4}{64D} - 2F b^2 \ln (4b^2) - \sum_{n=1}^{\infty} (A_{2n} b^{2n} + C_{2n} b^{2n+2}) \]

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Likewise, the \( n \geq 1 \) coefficients yield

\[
-S_n a^{2n-2} - T_n a^{2n} - \frac{F}{2} \left( A_{2n} \right)' = 0
\]  
(13)

and

\[
U_n a^{2n-3} + W_n a^{2n-1} + \frac{F}{2} \left( A_{2n} \right)'' = 0
\]  
(14)

Solving for \( A_{2n} \) and \( C_{2n} \) from the equations (13) and (14) it is found that

\[
A_{2n} = \frac{F}{2a} \left[ \frac{T(A_{2n})'' - Wa^{-1}(A_{2n})'}{SW - UT} \right]
\]

and

\[
C_{2n} = -\frac{SA_n a^{-2}}{T} - \frac{F(A_{2n})'}{Ta^{2n}}
\]

All of the constants of the general solution are now determined except \( A_0 \). \( A_0 \) is found by applying the condition that at

\[ \theta = 0 \text{ and } r = b, \; w = 0. \]

This condition applied to the general solution \( w = w_1 + \frac{w}{p} + w_s \) gives

\[
A_0 = -C_0 b^2 - \frac{gb^4}{64D} - 2Fb^2 \ln(4b^2)
\]

\[
- \sum_{n=1}^{\infty} \left( A_{2n} b^{2n} + C_{2n} b^{2n+2} \right).
\]
\[ S = 4n^2 - 4vn^2 - 2n + 2n\nu \]
\[ T = 4n^2 - 4vn^2 + 6n + 2nv + 2\nu + 2 \]
\[ U = -4n^2 + 4vn^2 + 8n^3 - 8\nu n^3 \]
\[ W = 8n^3 - 12n^2 - 8n - 8nv^3 - 4vn^2 \]

Define \((A_0/2)^* = (A_0/2)' - 4(1+\nu)\ln a^2\)

and
\[ \beta = \frac{Ta(A_{2n})^n - W(A_{2n})'}{SW - UT} \]

Then, with the parameters expressed as ratios (i.e., \(b/a\) and \(r/a\)) and the Fourier series constants reduced to their simplest form, the general solution for two point supports is found to be

\[ \frac{wD}{qa} = \frac{(b/a)^2(A_0/2)^*}{-64(1+\nu)} + \frac{(b/a)^2}{32} \frac{[3 + \nu]}{1+\nu} - \frac{(b/a)^4}{64} \]

\[ - \frac{1}{32} \sum_{n=1}^{\infty} \left\{ \frac{S}{T} (b/a)^{2n+2} - (b/a)^{2n} \right\} \beta + (b/a)^{2n+2} \frac{(A_{2n})'}{T} \}

\[ + \frac{(r/a)^2(A_0/2)^*}{64(1+\nu)} - \frac{(r/a)^2}{32} \frac{[3 + \nu]}{1+\nu} + \frac{(r/a)^4}{64} \]

\[ + \frac{1}{32} \sum_{n=1}^{\infty} \left\{ \frac{S}{T} (r/a)^{2n+2} - (r/a)^{2n} \right\} \beta + \frac{(A_{2n})'}{T} (r/a)^{2n+2} \cos 2n \theta \]
\[- \frac{1}{32} \left[ (r/a)^2 + (b/a)^2 + 2 \frac{r}{a} \frac{b}{a} \cos \theta \right] \left[ \ln \left( (r/a)^2 + (b/a)^2 + 2 \frac{r}{a} \frac{b}{a} \cos \theta \right) \right] \]

\[- \frac{1}{32} \left[ (r/a)^2 + (b/a)^2 - 2 \frac{r}{a} \frac{b}{a} \cos \theta \right] \left[ \ln \left( (r/a)^2 + (b/a)^2 - 2 \frac{r}{a} \frac{b}{a} \cos \theta \right) \right] \]

\[+ \frac{(b/a)^2}{4} \ln \left( 2 \frac{b}{a} \right) \]

(15)

where

\[(A_0/2)^* = 4 (b/a)^2 \left( \nu - 1 \right) + 4 (\nu + 3)\]

\[a(A_0/2)^n = - 16\]

\[(A_{2n})^k = - 4 (b/a)^{2n} \left[ \frac{\nu + 1}{n} + \left( 1 - \frac{b^2}{a^2} \right) (\nu - 1) \right]\]

\[a(A_{2n})^n = 8 (b/a)^{2n} \left\{ - 2 + n \left[ 1 - \frac{b^2}{a^2} \right] (\nu - 1) \right\}\]

\[S = 4n^2 - 4 \nu n^2 + 2n + 2 \nu \]

\[T = 4n^2 - 4 \nu n^2 + 6n + 2 \nu n + 2 \nu + 2\]

\[U = - 4n^2 + 4 \nu n^2 + 8n^3 - 8 \nu n^3\]

\[W = 8n^3 - 12n^2 - 8n - 8 \nu n^3 - 4 \nu n^2\]

\[\beta = \frac{Ta(A_{2n})^n - W(A_{2n})^n}{SW - UT}\]
DEVELOPMENT OF A FOUR POINT SOLUTION

A solution for a uniformly loaded plate supported by four point supports is found by applying the superposition principle to the two point solution. The four point solution should be a sum of two, two point solutions, one with q replaced by q/2, and one with q replaced by q/2 and θ replaced by θ - π/2. However, when these two solutions are added, it is found that a deflection exists at two of the support points; r = b and θ = π/2 and 3π/2. In order to get zero deflection at these points, it is necessary to add an adjustment to the solution. This required deflection is found by evaluating the sum of the two, two point solutions at θ = π/2 and r = b. This solution for the deflection is found from equation (15) to be

\[
\frac{wD}{qa} \left| \begin{array}{c}
  r=b \\
  \theta=\pi/2
\end{array} \right| = \frac{1}{32} \sum_{n=1,3,5}^{\infty} \left\{ \frac{S}{T} (b/a)^{2n+2} - (b/a)^{2n} \right\} \beta \\
+ \frac{(A_{2n})'}{T} \left( b/a \right)^{2n+2} \right\} - \frac{1}{16} \left( b/a \right)^{2} \ln 2
\]

If

\[
B_n = \left[ \frac{S}{T} (b/a)^{2n+2} - (b/a)^{2n} \right] \beta + \frac{(A_{2n})'}{T} \left( b/a \right)^{2n+2}
\]
\[
\frac{wD}{qa} \bigg|_{r=b, \theta=\pi/2} = \frac{1}{32} \sum_{n=1, 3, 5\ldots}^{\infty} B_n - \frac{1}{16} (b/a)^2 \ln 2
\]

If
\[
\beta_n = \left[ \frac{S}{T} (r/a)^{2n} + 2 - (r/a)^{2n} \right] + \frac{(A_2n')}{T} (b/a)^{2n} + 2
\]

then the total solution for four point supports becomes
\[
\frac{wD}{qa} = \frac{(b/a)^2 (A_{0/2})'}{-64 (1 + \nu)} + \frac{(b/a)^2}{32} \left[ \frac{3 + \nu}{1 + \nu} \right] - \frac{(b/a)^4}{64}
\]

\[
- \frac{1}{32} \sum_{n=1}^{\infty} B_n + \frac{(r/a)^2 (A_{0/2})'}{64 (1 + \nu)} - \frac{\nu (r/a)^2}{32} \left[ \frac{3 + \nu}{1 + \nu} \right] + \frac{(r/a)^4}{64}
\]

\[
+ \frac{1}{64} \sum_{n=1}^{\infty} R_n \cos 2n\theta - \frac{1}{64} \left[ (r/a)^2 + (b/a)^2 + 2 \frac{r}{a} \frac{b}{a} \cos \theta \right]
\]

\[
\ln \left[ (r/a)^2 + (b/a)^2 + 2 \frac{r}{a} \frac{b}{a} \cos \theta \right]
\]

\[
- \frac{1}{64} \left[ (r/a)^2 + (b/a)^2 - 2 \frac{r}{a} \frac{b}{a} \right] \ln \left[ (r/a)^2 + (b/a)^2 + 2 \frac{r}{a} \frac{b}{a} \cos \theta \right]
\]

\[
+ \frac{1}{4} \ln (2b/a) + \frac{1}{64} \sum_{n=1}^{\infty} R_n \cos n(\pi - 2\theta)
\]
\[-\frac{1}{6^4} [(r/a)^2 + (b/a)^2 + 2 \frac{r}{a} \frac{b}{a} \sin \theta][\ln \{(r/a)^2 + (b/a)^2 + 2 \frac{r}{a} \frac{b}{a} \sin \theta\}]
\]

\[-\frac{1}{6^4} [(r/a)^2 + (b/a)^2 - 2 \frac{r}{a} \frac{b}{a} \sin \theta][\ln \{(r/a)^2 + (b/a)^2 - 2 \frac{r}{a} \frac{b}{a} \sin \theta\}]
\]

\[+ \frac{1}{32} \sum_{n=1, 3, 5} B_n \frac{1}{n} (b/a)^2 \ln 2 \]

All the constants are defined as they are in the two point solution.

The above solution for four point supports was programmed for the IBM 1620 computer. Numerical results are shown on Graphs 1 through 4 for \( \nu = 0.3 \). In these particular graphs \( (b/a) \) is shown in increments of 0.1. Graphs 5 through 8 show the deflection surface for \( (b/a) = .6, .63, .65, .66, \) and .7. As can be observed from Graphs 5 through 8, the location of the supports at \( (b/a) = .63 \) gives almost no deflection of the plate along any of the radial lines up to and including \( (r/a) = .25 \).

The radial lines chosen for the graphs were \( \theta = 0, \theta = \pi/12, \theta = \pi/6, \) and \( \theta = \pi/4 \). This affords a full picture of the deflection surface of the plate as there is symmetry about the lines \( \theta = \pm \pi/4 \).
DEVELOPMENT OF THE EIGHT POINT SUPPORT SOLUTION

A solution for a uniformly, loaded plate supported by eight point supports is found by applying superposition to the four point solution. This is done by adding two, four point solutions; one with q replaced by q/2 and one with and one with θ replaced by θ - π/4 and with q replaced by q/2. Again a deflection occurs at the points θ = ±π/4, ±3π/4, and r = b. It is, therefore necessary to add an adjustment deflection to insure a zero deflection at these points. The required deflection is found by evaluating the sum of the two, four point solutions at θ = π/4 and r = b, which is

\[ \frac{wD}{4qa} \bigg|_{\theta=\pi/4, \ r=b} = -0.0044 \left( \frac{b}{a} \right)^2 + \frac{1}{32} \sum_{n=2}^{\infty} B_n \]
The complete solution for eight point supports becomes

\[
\frac{wD}{qa} = \frac{(b/a)^2 (A_0/2)^*}{-64 (1 + \nu)} + \frac{(b/a)^2}{32} \left[ \frac{3 + \nu}{1 + \nu} \right] - \frac{(b/a)^4}{64}
\]

\[
- \frac{1}{32} \sum_{n=1}^{\infty} B_n + \frac{(r/a)^2 (A_0/2)^*}{64 (1 + \nu)} - \frac{(r/a)^2}{32} \left[ \frac{3 + \nu}{1 + \nu} \right] + \frac{(r/a)^4}{64}
\]

\[
+ \frac{(b/a)^2}{16} \left[ 4 \ln (b/a) + 3 \ln 2 \right] + \frac{1}{128} \sum_{n=1}^{\infty} R_n \cos 2n \theta
\]

\[
+ \frac{1}{128} \sum_{n=1}^{\infty} \cos n (\pi - 2 \theta) + \frac{1}{32} \sum_{n=1, 3, 5} B_n
\]

\[
+ \frac{1}{128} \sum_{n=1}^{\infty} R_n \cos n (2 \theta - \pi/2) + \frac{1}{128} \sum_{n=1}^{\infty} R_n \cos (3\pi/2 - 2 \theta)
\]

\[
- \frac{1}{128} \left[ (r/a)^2 + (b/a)^2 + 2 \frac{r b}{a a} \cos \theta \right] \ln \left\{ (r/a)^2 + (b/a)^2 + 2 \frac{r b}{a a} \cos \theta \right\}
\]

\[
- \frac{1}{128} \left[ (r/a)^2 + (b/a)^2 - 2 \frac{r b}{a a} \cos \theta \right] \ln \left\{ (r/a)^2 + (b/a)^2 - 2 \frac{r b}{a a} \cos \theta \right\}
\]

\[
- \frac{1}{128} \left[ (r/a)^2 + (b/a)^2 + 2 \frac{r b}{a a} \sin \theta \right] \ln \left\{ (r/a)^2 + (b/a)^2 + 2 \frac{r b}{a a} \sin \theta \right\}
\]

\[
- \frac{1}{128} \left[ (r/a)^2 + (b/a)^2 - 2 \frac{r b}{a a} \sin \theta \right] \ln \left\{ (r/a)^2 + (b/a)^2 - 2 \frac{r b}{a a} \sin \theta \right\}
\]

\[
- \frac{1}{128} \left[ (r/a)^2 + (b/a)^2 + 2 \frac{r b}{a a} \cos (\theta - \pi/4) \right] \ln \left\{ (r/a)^2 + (b/a)^2 + 2 \frac{r b}{a a} \cos (\theta - \pi/4) \right\}
\]

\[
- \frac{1}{128} \left[ (r/a)^2 + (b/a)^2 - 2 \frac{r b}{a a} \cos (\theta - \pi/4) \right] \ln \left\{ (r/a)^2 + (b/a)^2 - 2 \frac{r b}{a a} \cos (\theta - \pi/4) \right\}
\]
\[-\frac{1}{128} \left[(\frac{r}{a})^2 + (\frac{b}{a})^2 + 2 \frac{r}{a} \frac{b}{a} \sin(\theta - \frac{\pi}{4})\right] \ln\left((\frac{r}{a})^2 + (\frac{b}{a})^2 + 2 \frac{r}{a} \frac{b}{a} \sin(\theta - \frac{\pi}{4})\right) \]

\[-\frac{1}{128} \left[(\frac{r}{a})^2 + (\frac{b}{a})^2 - 2 \frac{r}{a} \frac{b}{a} \sin(\theta - \frac{\pi}{4})\right] \ln\left((\frac{r}{a})^2 + (\frac{b}{a})^2 - 2 \frac{r}{a} \frac{b}{a} \sin(\theta - \frac{\pi}{4})\right) \]

\[-0.0044 \left(\frac{b}{a}\right)^2 + \frac{1}{32} \sum_{n=2,6,10}^{\infty} B_n \]

The expression for the deflection at the center of the plate (i.e., \(r/a = R_n = 0\)) with eight point supports is

\[
\frac{wD}{qa} \bigg|_{r/a=0} = \frac{(b/a)^2 (A_0/2)\ast}{\frac{q}{4}} + \frac{(b/a)^2}{32} \left[ \frac{3 + \nu}{1 + \nu} \right] - \frac{(b/a)^4}{64} \]

\[-\frac{1}{32} \sum_{n=1}^{\infty} B_n + \frac{(b/a)^2}{16} [4 \ln (b/a) + 3 \ln 2] \]

\[+ \frac{1}{32} \sum_{n=1,3,5}^{\infty} B_n - 0.0044 \left(\frac{b}{a}\right)^2 + \frac{1}{32} \sum_{n=2,6,10}^{\infty} B_n \]
SOLUTIONS WITH SUPPORTS LOCATED ON THE PERIPHERY

It is desirable to know the solutions for the various cases when the supports are located on the periphery because most of the literature found on this subject treats only this case.

Various ratios that must be evaluated at \( b/a = 1 \) or as \( b/a \to 1 \) are found in equation (15). These ratios are found to be:

\[
\lim_{b/a \to 1} \frac{a(A_{2n})^n}{a(A_0/2)^n} = 1
\]

\[
\lim_{b/a \to 1} \frac{(b/a)^2 \ln (b/a)}{a(A_0/2)^n} = 0
\]

\[
\lim_{b/a \to 1} \frac{(A_{2n})^{n'}}{a(A_0/2)^n} = \frac{1 + \nu}{4n}
\]

\[
\lim_{b/a \to 1} \frac{(A_0/2)^*}{a(A_0/2)^n} = -\frac{(\nu + 1)}{2}
\]

\[
\lim_{b/a \to 1} \frac{[1 + (b/a)^2] \ln [1 + (b/a)^2]}{a(A_0/2)^n} = \frac{\ln (2)}{8}
\]

It is now possible to use the above values applied to the general two, four, and eight support cases to find solutions with the supports located on the periphery.
These solutions for the center deflection (i.e., \( r/a = 0 \)) are:

1) **Two Supports on Periphery**

\[
\frac{wD}{qa} \bigg |_{r/a=0} = - \frac{9}{64} + \frac{3 + \nu}{32 (1 + \nu)} - \frac{\ln (2)}{4} - \frac{5n + n\nu (2 + \nu) + 2 (\nu + 1)}{(SW - UT)} + \frac{\ln (2)}{4}
\]

If \( \Phi_n \) is defined as

\[
\Phi_n = \frac{5n + n\nu (2 + \nu) + 2 (\nu + 1)}{(SW - UT)}
\]

then

\[
\frac{wD}{qa} \bigg |_{r/a=0} = - \frac{9}{64} + \frac{3 + \nu}{32 (1 + \nu)} + \frac{\ln (2)}{4} - \frac{\ln (2)}{16} - \sum_{n=1}^{\infty} \Phi_n
\]

2) **Four Supports on Periphery**

\[
\frac{wD}{qa} \bigg |_{r/a=0} = - \frac{9}{64} + \frac{3 + \nu}{32 (1 + \nu)} + \frac{3}{16} \ln (2)
\]

\[
+ \sum_{n=1, 3, 5}^{\infty} \frac{1}{2} \Phi_n + \sum_{n=1}^{\infty} \frac{1}{4} \Phi_n
\]

3) **Eight Supports on Periphery**

\[
\frac{wD}{qa} \bigg |_{r/a=0} = - \frac{9}{64} + \frac{3 + \nu}{32 (1 + \nu)} + \frac{3}{16} \ln (2) - 0.0044
\]

\[
- \sum_{n=1}^{\infty} \frac{1}{2} \Phi_n + \sum_{n=1, 3, 5}^{\infty} \frac{1}{2} \Phi_n + \sum_{n=2, 6, 10}^{\infty} \frac{1}{2} \Phi_n
\]

...
DEVELOPMENT OF THE SOLUTION
FOR THREE POINT SUPPORTS
LOCATED ON A DIAMETER

The problem to be considered is that of a circular, uniformly
loaded plate that is supported by three point supports as shown below.

This solution can be found by adding the solution for one point support
to the solution for the two point support case with the supports on the
outer edge (i.e., b/a = 1).

The one point support case can be found by setting b = 0 in the two
point case. When b = 0 the two point solution becomes

$$\frac{wD}{qa^4} = \frac{(r/a)^2}{32} \frac{(3 + v)}{(1 + v)} + \frac{(r/a)^4}{64} - \frac{(r/a)^2}{16} \ln (r/a)^2$$  \hspace{1cm} (16)
The two point solution for \( \frac{b}{a} = 1 \) and any general \( r \) is found by evaluating equation (15) at \( \frac{b}{a} = 1 \) and is

\[
\frac{wD}{4qa} \bigg|_{\frac{b}{a}=1} = -\frac{9}{64} + \frac{3 + \nu}{32(1 + \nu)} - \sum_{n=1}^{\infty} \Phi_n \\
+ \ln \frac{2}{4} + \frac{(r/a)^2}{8} - \frac{(r/a)^2}{32} \left[ \frac{3 + \nu}{1 + \nu} \right] + \frac{1}{32} \sum_{n=1}^{\infty} R_n \cos 2n \theta + \frac{(r/a)^4}{64} \\
- \frac{1}{32} \left[ (r/a)^2 + 1 + 2 \frac{r}{a} \cos \theta \right] \ln \left( (r/a)^2 + 1 + 2 \frac{r}{a} \cos \theta \right) \\
- \frac{1}{32} \left[ (r/a)^2 + 1 - 2 \frac{r}{a} \cos \theta \right] \ln \left( (r/a)^2 + 1 - 2 \frac{r}{a} \cos \theta \right) 
\] (17)

Each support in the two point case is supporting a \( q/2 \) load. In the three point case each is supporting \( q/3 \) load. Consequently, the support is carrying \( 2/3 \) of the load it was in the two point case. Therefore, \( q \) in the two point case should be replaced by \( 2/3q \) when superposition is applied. Likewise, \( q \) in the one point case should be replaced by \( q/3 \).

When \( q \) is replaced by \( 2/3q \) in equation (17), and \( q \) is replaced by \( q/3 \) in equation (16) and these are added, a deflection is found to exist at \( r/a = 0 \). This is the deflection that must be subtracted from the sum of the two in order to find the solution for the plate supported by three point supports. This deflection (with the sign already changed) is

\[
\frac{wD}{4qa} = \frac{1}{12} - \frac{3\nu + 1}{48(1 + \nu)} - \frac{1}{48} \frac{(3 + \nu)}{(1 + \nu)} 
\] (18)
When equations (16) and (17) with the altered q's are added to equation (18), the solution of the plate with three point supports is found to be

\[
\frac{wD}{qa} = - \frac{(r/a)^2}{24} \left[ \frac{3 + \nu}{1 + \nu} \right] + \frac{5}{192} (r/a)^4 - \frac{(r/a)^2}{16} \ln (r/a)^2 + \frac{1}{48} \Sigma R_n
\]

\[
- \frac{1}{48} \left[ (r/a)^2 + 1 + 2 \frac{r}{a} \cos \theta \right] \ln \left[ \frac{r}{a} \right] \left[ 1 + 2 \frac{r}{a} \cos \theta \right]
\]

\[
- \frac{1}{48} \left[ (r/a)^2 + 1 - 2 \frac{r}{a} \cos \theta \right] \ln \left[ \frac{r}{a} \right] \left[ 1 + 2 \frac{r}{a} \cos \theta \right]
\]

This three point solution should approximate the solution for a semicircular plate with the straight edge clamped, because along the line of supports, \( w = 0, \frac{\partial w}{\partial r} = 0 \), and a moment exists.

With \( \nu = 0.25 \) the maximum deflection \( w \), at \( r/a = 1 \) and \( \theta = \pi/2 \) for three point supports becomes

\[
w = 0.1784 \frac{qa^4}{D}
\]

A more accurate approximation for the clamped semicircular plate could be obtained by adding other solutions, e.g., \( b/a = 0.5 \) for two point supports, to the three point solution. In fact, approximations for various circular sectors of plates clamped along the radial edges could be obtained by superposition of other solutions given above.
LIMITING CASES AND COMPARISON OF RESULTS WITH KNOWN SOLUTIONS

All of the following results are solutions with the supports located on the periphery and with $\nu = .25$.

Two Point Supports

Evaluating $w$ at the center $(r/a = 0)$ Timoshenko, (Reference 3) quoting Nadai (Reference 1) has

$$w = .269 \frac{qa^4}{D}$$

Bassali (Reference 7) shows

$$w = .270 \frac{qa^4}{D}$$

and the solution presented here yields

$$w = .268 \frac{qa^4}{D}.$$

Evaluating $w$ at $r/l = 1$ and $\theta = \pi/2$ Nadai shows

$$w = .371 \frac{qa^4}{D}$$

and the solution shown above yields

$$w = .369 \frac{qa^4}{D}. $$
Four Point Supports

Evaluating \( w \) at \( r/a = 0 \), Bassali (Reference 7) has

\[
w = 0.084 \frac{qa^4}{D}
\]

Nadai shows

\[
w = 0.083 \frac{qa^4}{D}
\]

and the above solution yields

\[
w = 0.084 \frac{qa^4}{D}
\]

Eight Point Supports

Obviously with eight supports the deflection should be approaching the deflection of a simply supported, uniformly loaded plate.

Timoshenko (Reference 3, page 66) gives the solution for a simply supported, uniformly loaded plate as

\[
w | r = 0 = \frac{qa^4}{16D} \left[ \frac{3 + \nu}{1 + \nu} - \frac{7 + 3\nu}{4(1 + \nu)} \right]
\]

When this is evaluated for \( \nu = 0.25 \), it becomes

\[
w | r = 0 = 0.0656 \frac{qa^4}{D}.
\]

The author's solution for the center deflection of a plate with eight point supports at the circumference is

\[
w = 0.0663 \frac{qa^4}{D}.
\]
The above results show very good agreement between known solutions and limiting cases of the author's solutions.

There are obviously other plate problems which are variations of the ones given above which could be solved by similar methods or by sums of the solutions presented here.

It is, however, a very laborious job to express the boundary terms as Fourier series. It might, therefore, be advisable to attempt a solution with an approximate method such as point matching if a solution were desired that could not be obtained directly from the above solutions.
APPENDIX I

Evaluation of Integrals

By repeated use of trigonometric identities and the following integrals, it is possible to evaluate all integrals which arise from expressing the boundary terms of \( w_s \) in a Fourier series. These integrals were evaluated from Integraltafel and Nouvelle Table d'Integrale (References 14 and 15, respectively).

\[
\begin{align*}
\int_0^\pi \frac{dx}{1 - 2P\cos x + P^2} &= \frac{\pi}{1 - P^2} [P^2 < 1] \\
&= \frac{\pi}{P^2 - 1} [P^2 > 1]. \quad (1)
\end{align*}
\]

\[
\begin{align*}
\int_0^\pi \frac{\cos x \, dx}{1 - 2P\cos x + P^2} &= \frac{P\pi}{1 - P^2} [P^2 < 1] \\
&= \frac{\pi}{P(P^2 - 1)} [P^2 > 1]. \quad (2)
\end{align*}
\]

\[
\begin{align*}
\int_0^\pi \frac{\cos ax \, dx}{1 - 2P\cos x + P^2} &= \frac{\pi P^a}{1 - P^2} [P^2 < 1] \\
&= \frac{\pi P^{-a}}{P^2 - 1} [P^2 > 1]. \quad (3)
\end{align*}
\]
\[ \int_{0}^{\pi} \frac{\cos ax \cos x \, dx}{1 - 2P \cos x + P^2} = \frac{\pi}{2} \left[ \frac{1 + P^2}{1 - P^2} \right]^{\frac{a}{2} - 1} \quad [P^2 < 1] \]

\[ = \frac{\pi}{2P^a + 1} \left[ \frac{P^2 + 1}{P^2 - 1} \right] \quad [P^2 > 1] \quad (4) \]

\[ \int_{0}^{\pi} \ln (1 - 2P \cos x + P^2) \, dx = 0 \quad [P^2 < 1] \quad (5) \]

\[ \int_{0}^{\pi} \ln (1 - 2P \cos x + P^2) \cos ax \, dx = -\frac{\pi}{a} \cdot P^a \quad (6) \]

\[ \int_{0}^{\pi} \ln (1 - 2P \cos x + P^2) \cos ax \cos x \, dx = -\frac{\pi}{2} \left( \frac{P^a + 1}{a + 1} + \frac{P^a - 1}{a - 1} \right) \quad (7) \]

\[ \int_{0}^{\pi} \frac{dx}{(1 - 2P \cos x + P^2)^{a + 1}} = \frac{\pi}{(1 - P^2)^2a + 1} \sum_{n=0}^{a} \left( \frac{a}{n} \right)^2 P^{2n} \]

where

\[ \left( \frac{a}{n} \right) = \frac{a!}{n! (a - n)!} \quad [P^2 < 1] \quad (8) \]

\[ \int_{0}^{\pi} \frac{\cos bx \, dx}{(1 - 2P \cos x + P^2)^2} = \frac{\pi P^b}{(1 - P^2)^3} \left[ \frac{b + 1}{(b + 1) P^2} \right] \quad [P^2 < 1] \quad (9) \]
APPENDIX II

Development of Fourier Series for Boundary Terms Contributed by $w_s$

First Boundary Term Contributed by $w_s$

The first boundary term is

$$
M_r \left|_{r=a}^{w=w_s} = -D \left[ \frac{\partial^2 w_s}{\partial r^2} + \frac{\nu}{r} \frac{\partial w_s}{\partial r} + \frac{\nu}{r^2} \frac{\partial^2 w_s}{\partial \theta^2} \right] \right|_{r=a}
$$

$$
= -\frac{FD}{2} \left\{ 4 \left( \frac{2a + 2b \cos \theta}{a^2 + b^2 + 2ab \cos \theta} \right)^2 + 2 \ln \left( \frac{2a + 2b \cos \theta}{a^2 + b^2 + 2ab \cos \theta} \right) + \frac{(2a - 2b \cos \theta)^2}{a^2 + b^2 - 2ab \cos \theta} + 2 \ln \left( \frac{2a - 2b \cos \theta}{a^2 + b^2 - 2ab \cos \theta} \right) + \frac{\nu}{a} (2a + 2b \cos \theta) \left( 1 + \ln \left[ \frac{a^2 + b^2 + 2ab \cos \theta}{a^2 + b^2 - 2ab \cos \theta} \right] \right) + \frac{\nu}{a} (2a - 2b \cos \theta) \left( 1 + \ln \left[ \frac{a^2 + b^2 - 2ab \cos \theta}{a^2 + b^2 + 2ab \cos \theta} \right] \right) + \frac{\nu}{a^2} \left[ \frac{4ab \sin^2 \theta}{a^2 + b^2 + 2ab \cos \theta} + \frac{4ab \sin^2 \theta}{a^2 + b^2 - 2ab \cos \theta} - 2ab \cos \theta \ln \left( \frac{a^2 + b^2 + 2ab \cos \theta}{a^2 + b^2 - 2ab \cos \theta} \right) + 2ab \cos \theta \ln \left( \frac{a^2 + b^2 + 2ab \cos \theta}{a^2 + b^2 - 2ab \cos \theta} \right) \right] \right\} = -\frac{FD}{2} f(\theta)
$$

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The second boundary term is

\[ V_r \bigg|_{r=a} = -D \left[ \frac{\partial^3 w_s}{\partial r^3} + \frac{1}{r} \frac{\partial^2 w_s}{\partial r^2} - \frac{1}{r} \frac{\partial w_s}{\partial r} \right]_{r=a} \]

+ \frac{D}{2} \left[ \frac{-2}{r^2} \frac{\partial^3 w_s}{\partial r \partial \theta^2} + \frac{3}{r^3} \frac{\partial^2 w_s}{\partial \theta^2} \right]_{r=a}

+ \nu D \left[ \frac{1}{r^2} \frac{\partial^3 w_s}{\partial r \partial \theta^2} - \frac{1}{r^3} \frac{\partial^2 w_s}{\partial \theta^2} \right]_{r=a}

= - \frac{FD}{2} \left\{ \frac{12(a + b \cos \theta)}{a^2 + b^2 + 2ab \cos \theta} + \frac{12(a - b \cos \theta)}{a^2 + b^2 - 2ab \cos \theta} \right. 

- \frac{(2a + 2b \cos \theta)^3}{(a^2 + b^2 + 2ab \cos \theta)^2} - \frac{(2a - 2b \cos \theta)^3}{(a^2 + b^2 - 2ab \cos \theta)^2}

+ \frac{1}{a} \left[ \frac{(2a + 2b \cos \theta)^2}{a^2 + b^2 + 2ab \cos \theta} \right] + \frac{1}{a} \left[ \frac{(2a - 2b \cos \theta)^2}{a^2 + b^2 - 2ab \cos \theta} \right]

+ \frac{2b \cos \theta}{2} \ln \left( a^2 + b^2 - 2ab \cos \theta \right)

\left. - \frac{2b \cos \theta}{a^2} \ln \left( a^2 + b^2 + 2ab \cos \theta \right) \right\}

+ \frac{FD}{2} \left\{ \frac{-16b^2 \sin^2 \theta}{a(a^2 + b^2 + 2ab \cos \theta)} - \frac{16b^2 \sin^2 \theta}{a(a^2 + b^2 - 2ab \cos \theta)} \right.

+ \frac{8b^2 \sin^2 \theta (2a + 2b \cos \theta)}{(a^2 + b^2 + 2ab \cos \theta)^2} + \frac{8b^2 \sin^2 \theta (2a - 2b \cos \theta)}{(a^2 + b^2 - 2ab \cos \theta)^2} \right\}
\[
\frac{4b \cos \theta (2a + 2b \cos \theta)}{a(a^2 + b^2 + 2ab \cos \theta)} - \frac{4b \cos \theta (2a - 2b \cos \theta)}{a(a^2 + b^2 - 2ab \cos \theta)} \\
+ \frac{4b^2 \cos \theta \ln (a^2 + b^2 + 2ab \cos \theta)}{a^2} \\
- \frac{4b^2 \cos \theta \ln (a^2 + b^2 + 2ab \cos \theta)}{a^2} \\
+ \frac{12b^2 \sin^2 \theta}{a^2 + b^2 + 2ab \cos \theta} + \frac{12b^2 \sin^2 \theta}{a^2 + b^2 - 2ab \cos \theta} \\
- \frac{6b \cos \theta \ln (a^2 + b^2 + 2ab \cos \theta)}{a} \\
+ \frac{6b \cos \theta \ln (a^2 + b^2 - 2ab \cos \theta)}{a} \right) \\
+ \frac{F D v}{2} \left\{ \frac{8b^2 \sin^2 \theta}{a(a^2 + b^2 + 2ab \cos \theta)} + \frac{8b^2 \sin^2 \theta}{a(a^2 + b^2 - 2ab \cos \theta)} \\
- \frac{4b^2 \sin^2 \theta (2a + 2b \cos \theta)}{(a^2 + b^2 + 2ab \cos \theta)^2} - \frac{2b \cos \theta (2a + 2b \cos \theta)}{a(a^2 + b^2 + 2ab \cos \theta)} \\
+ \frac{2b \cos \theta (2a - 2b \cos \theta)}{a(a^2 + b^2 - 2ab \cos \theta)} - \frac{4b^2 \sin^2 \theta (2a - 2b \cos \theta)}{(a^2 + b^2 - 2ab \cos \theta)^2} \\
- \frac{4b^2 \sin^2 \theta}{a^2 + b^2 + 2ab \cos \theta} - \frac{4b^2 \sin^2 \theta}{a^2 + b^2 - 2ab \cos \theta} \right\} = -\frac{F D}{2} g(\theta)
\]
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