FACTORs INFLUENCINg THE FATIGUE CHARACTERISTICS
OF RUBBER-TEXTILE MACHINE ELEMENTS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
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* * * * * *

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SUMMARY

The following is a brief summary of the important equations and methods presented in detail throughout this paper.

Force analysis. V-belt drives can be divided into three classes depending on the method used to maintain a desired tension in the belt. The three classes are (1) constant tension drives in which a known constant force is applied to each pulley in the drive, (2) locked center drives in which the location of the axis of each pulley in the drive is fixed in space, and (3) spring-loaded drives in which one pulley in the drive has movable sides which are held together by an axial spring.

In each drive type, two pure tensions exist in the belt at any point — one due to centrifugal force and another due to forces exerted on the belt by the pulley. The centrifugal tension can always be treated as separate from the pulley tension, and is shown in Appendix B to be

\[ T_c = \rho v^2 / g \]

For a constant tension drive in which the torque and pulley reactions can be measured, the force analysis is shown in Chapter I for a drive with two pulleys of equal diameter. The two expressions from which the tight and slack side tensions can be obtained are

\[ T_1 + T_2 = F_p \]

and

\[ T_1 - T_2 = J/r \]
A more general case in which the straight segments of belt entering and leaving a pulley form the angle \( \gamma \) is treated by Marco, Starkey, and Hornung. \(^3\) For this situation, the sum of the entrance and exit tensions, \( T_I \) and \( T_E \), are given by

\[
(T_I + T_E) \cos (\theta/2) = F_{PF}
\]

where \( F_{PF} \) is the component of the pulley reaction in the direction along the bisector of the angle \( \gamma \). The tension difference is obtained as before from

\[
T_I - T_E = J/r
\]

where the torque \( J \) is taken as plus in the direction of rotation.

The analysis of the forces in a locked center drive results in a graphical procedure for determining the running tensions in a drive in which the static no-load tension is known. An example showing the application of this method is shown in Chapter I.

A complete analysis of the forces and motions involved in the passage of a V-belt over a V-pulley is performed in Appendix A. Typical chart solutions to the resulting equations as obtained on an analog computer are given, and the application of the analysis to a spring-loaded drive is shown in Chapter I. Spring forces for known belt tensions are computed and show good agreement with measured values.

**Stress analysis.** A stress factor is defined which represents a measure of the severity of a given load on a given belt. Three stress

---

factors are involved since the pulley forces, centrifugal force, and bending each impose a load on the belt. Each stress factor consists of a corrected nominal stress expressed as a percentage of the ultimate strength of the belt. Correction of the nominal stresses is necessary because of the failure of the belt to conform to simplifying assumptions involved in the stress analysis. The three stress factors are

\[ Q_{hp} = \frac{k_{hp} T}{q k_e G_u} \]

\[ Q_c = \frac{\rho V^2}{g q k_e G_u} \]

\[ Q_b = f(Q_{hp} + Q_c) \frac{ye^{3}}{k_e G_u} \]

where the \( k \) values are the correction coefficients and must be obtained empirically.

The total stress factor is then given by

\[ Q = Q_{hp} + Q_c + Q_b \]

**Failure theory.** A plot of the stress factor, \( Q \), versus the number of stress cycles at failure, \( N \), for a large number of belt life tests shows these quantities to be related by

\[ Q = b + p \log N \]

where \( b \) and \( p \) are constants. If there are \( i \) pulleys in a drive and the maximum stress factor at each pulley is denoted by \( Q_1, Q_2, Q_3, \ldots, Q_i \), then a cycle life can be computed for each \( Q \), giving the values \( N_1, N_2, N_3, \ldots, N_i \). These individual cycle lives are assumed to be related to
the total number of belt cycles at failure, $n$, by

$$n = \frac{1}{\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} + \ldots + \frac{1}{N_i}}$$

Once $n$ has been established, the expected life of the belt, $H$, may be obtained from

$$H = \frac{knL}{V}$$

where $k$ is a constant, $L$ is the belt length, and $V$ the belt velocity.
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DEFINITION OF SYMBOLS

A  area, square inches
a  acceleration vector, feet per second per second
C  coefficient
D  diameter, inches
E  elastic modulus in pounds per square inch/inch per inch
E'  strain or unit elongation, inches per inch
F  force, pounds
F'  corrected force, pounds
G  stress, pounds per cord
G'  modified stress, pounds per cord
G'  gravitational constant, feet per second per second
H  moment, inch-pounds
HP  horsepower
J  torque, inch-pounds
K  belt bending constant ( = EI ), pound-inches^2
k  a constant
k_e  correction for variations in endurance strength as percentage of ultimate strength
k_{hp}  correction for stress concentration
L_b  belt half-length, inches
L_c  pulley center distance, inches
M  elastic modulus, pounds per cord/inch per inch
M'  predicted elastic modulus, pounds per cord/inch per inch
m  mass, slugs
N  number of stress cycles at failure
n  number of stress cycles
P  distributed force, pounds per tangential inch
Q  stress factor, dimensionless
q  number of cords
R  radius of curvature, inches
R  radius vector, inches
r  radius, inches
r  radius to centroid, inches
S  stress, pounds per square inch
S_t  slip, or tangential component of belt velocity relative to pulley divided by the pulley velocity
s  length of curve, inches
T  tension, pounds
T  tension vector, pounds
T_s  tension in a no-load drive, pounds
u  radial displacement coordinate, inches
V  velocity, feet per second
V  velocity vector, feet per second
v  tangential displacement coordinate, inches
W  belt width at neutral axis, inches
x, y, z  variables
g  belt or pulley angle, degrees
ρ  belt weight per unit length, pounds per foot
φ  angle, radians
$\phi$ friction angle, degrees

$\tau$ time, seconds

$\omega$ angular position in space, radians

$\omega_g$ total contact angle, radians

Subscripts

a axial

B belt

b bending

C cords

c centrifugal

e exit

f friction

hp horsepower

i inlet or initial

k key

L lower half of belt

M rubber matrix

N normal

p pulley

R relative

r radial

SC single cord

s constant tension

sp spring

t tangential
U  upper half of belt
v  active arc
W  cover
1  tight side
2  slack side
3  all rubber belt
4  rubber-and-cover belt
5  rubber-and-cord belt
6  whole belt
INTRODUCTION

Of the non-metallic machine elements employed in modern machine design, those composed of rubber-textile combinations form an important sub-group. Such elements as shock-absorbing mountings, diaphragms, tires, and belts make use of the strength and flexibility properties of a composite structure of rubber reinforced with either woven or non-woven textile fibers. The methods used in the design of these elements, however, have not been refined as have the design methods for metal parts, but have remained more art than science. As a result they are often over- or under-designed, resulting in unnecessarily high initial cost or perhaps equally costly premature failures.

One of the most important reasons for this lack of technically sound design procedures is the extreme difficulty encountered in attempting to analyze the stresses in these composite structures. Materials of this type are anisotropic, non-homogeneous, have physical properties which are age and temperature sensitive, and usually suffer strains which are large rather than small. Any one of these deviations from the ideal case can by itself cause an analysis of stresses and strains in a structure to become extremely complex. It would appear, then, that a general analysis applicable to all rubber-textile machine members would be too complex to be practical.

When generality cannot be achieved, the only alternative is to consider specific cases. To aid in the development of rational design procedures for rubber-textile machine members and to improve the general
understanding of their behavior under load, a study of a specific rubber-textile machine element, the V-belt, was conducted in the Mechanical Engineering Department of The Ohio State University.

The term V-belt includes all power-transmitting belts of trapezoidal cross-section. Such a belt is most commonly constructed as an endless loop with a core of strong tension-carrying cords surrounded by a rubber or rubber-fabric matrix. A belt of this type is intended to run in a pulley with a V-groove, the angle of which is approximately the same as the included angle of the belt trapezoid. This results in a wedging action of the belt in the groove and thus provides a very high normal force between the pulley and belt for a given tension in the belt. The V-belt is therefore capable of transmitting high torque without large tensions and correspondingly high bearing loads.

At the beginning of this research a thorough study of the literature was made to establish the present state of V-belt technology. With the exception of a limited number of papers the literature dealing with V-belts was found to be largely descriptive, showing typical drive design procedures based mainly on the author's experience. The works of several authors will be discussed at appropriate points throughout the text.

Among the articles in which an attempt is made to analyze various phases of the V-belt problem in a theoretical manner, there appears to be little conformity in the approach, the assumptions, and the results obtained. This, however, is not surprising but simply indicates that the various phenomena involved in the problem are not completely understood and that the relative importance of their effects have not been properly evaluated. Examples of these phenomena are centrifugal force, cord
bending, and frictional resistance to motion in a V groove. As is often the case, the many complicated phenomena involved in the process of energy transfer through a belt drive are so interrelated that their separation is impossible. This means that an analysis can be checked only by experimental results which reflect the combined influence of many effects. If these results then prove the analysis to be in error, the researcher becomes faced with the almost impossible task of discovering which of his basic assumptions, affecting a specific basic phenomenon, should be modified.

In view of these facts, it has been the policy on this research project to refrain from either accepting or rejecting the works of other authors in the field of V-belt research and simply to read these works as background material. Any analysis which was needed throughout the course of the project was performed independently starting from accepted fundamentals and, where assumptions were necessary, making those assumptions which appeared to be most reasonable.
CHAPTER I

FORCE AND MOTION ANALYSIS

Introduction

In a power transmission system consisting of a V-belt, a driving pulley and a driven pulley, the only external forces applied to the belt are the frictional and normal forces exerted on the belt by the pulleys. The body forces acting on the belt are centrifugal force and gravity, but in the usual case the force of gravity is completely negligible in comparison to the other forces in the system. The action of the pulley and centrifugal forces on the belt results in the build-up of an internal resisting stress system within the belt. The variation of this stress system due to the forces exerted on the belt by the pulleys is analyzed in Appendix A. As pointed out in this analysis, the belt moves radially as well as tangentially with respect to the pulley so that only a part of the total frictional force acts tangentially on the belt to cause a tension change. It is this radial motion and the consequent radial frictional force which causes the analysis of forces and motions in a V-belt to be less straightforward than that for flat belts.

From the analysis given in Appendix B, the tension in the belt due to centrifugal force is constant throughout the belt. This tension is shown to be independent of the shape of the belt and so conversely, the shape of the belt will not be influenced by the centrifugal tension. That is, if a certain portion of a belt hangs in the shape of a catenary under static conditions the same section of belt will not tend to assume the shape of a circle arc when the belt is driven at high speed, but
will continue to hang in the shape of a catenary. The belt section will, of course, be slightly longer in the dynamic state than in the static state since the belt is an elastic member and will be elongated by the centrifugal tension. If, however, the ends of the belt section are moved slightly to make up for this increased length, the two curves would be identical.

Nearly all V-belt drives may be placed in one of three categories depending on the method used to control the magnitude of the external force applied to the belt by the pulleys. The three methods are characterized by drives in which (1) both pulleys are rigid members, and a known constant force is applied to the shafts on which the pulleys are mounted, (2) both pulleys are rigid and a fixed constant distance is maintained between them, and (3) one pulley is rigid but the other consists of two separate disks held together by an axial spring. Each of the drive types presents a unique system of forces and must be treated separately.

**Constant Tension Drive**

Drives of the first type in which a constant known force is applied to the pulley shaft are commonly called "constant tension drives." This name, however, is not strictly descriptive of the drive, and should be used with caution. Two ways of achieving a constant shaft force are shown in Figure 1-1. If either of these drives is cut between the pulleys by a plane, the free body diagram of the right half of the drive would be as shown in Figure 1-2. For the sake of simplicity Figure 1-2 has been drawn for a drive with equal diameter pulleys.
Figure 1-1. Constant tension drives.
Figure 1-2. Forces in a constant tension drive.

Here the forces $F_1$ and $F_2$ represent the total tensile forces in the tight and slack sides of the belt respectively at the points where the belt is cut. These forces may be considered as being made up of two components, one whose magnitude is that of the centrifugal tension $\rho v^2/g$ as obtained in Appendix B, and the other, $T_1$ or $T_2$ representing the remainder of the tension in the tight or slack strand of the belt respectively.

Because the belt travels in a curved path around the pulley, a centrifugal force of $2(\rho v^2)/g$ acts to lift the belt away from the pulley. This centrifugal force, however, is exactly balanced by the centrifugal tension $\rho v^2/g$ acting to the left in each of the belt strands. As the belt is driven at different speeds, the centrifugal force acting to the right and the centrifugal tension acting to the left both take on
different values, but each always balances the other. The constant force $F_p$ then is always equal to the sum of the tension components $T_1$ and $T_2$. The tensions $T_1$ and $T_2$ may now be defined as the total tensions $F_1$ and $F_2$ on the tight and slack sides of the belt respectively, each reduced by the amount of the centrifugal tension. As the total tension varies with speed, the length of the belt also varies, but the motor is free to move to accommodate these length changes while keeping the shaft force $F_p$ constant.

By use of a torque measuring device the torque exerted on the pulley by the belt may be determined. If the pulley radius is $r$, the torque $J$ will be related to $r$ and the total tensions according to

$$ J = r(F_1 - F_2) \quad \text{(1-1)} $$

or

$$ J = r \left( T_1 + \frac{\rho v^2}{g} \right) - \left( T_2 + \frac{\rho v^2}{g} \right) = r(T_1 - T_2) \quad \text{(1-2)} $$

For a known force $F_p$, where

$$ F_p = T_1 + T_2 \quad \text{(1-3)} $$

and a known torque and pulley radius, the tensions $T_1$ and $T_2$ may be obtained from Equations 1-2 and 1-3.

"Constant tension drives" are used extensively in laboratory tests where complete knowledge of the force system is necessary. Such drives also make up a significant portion of all V-belt drives in use in industry.
Locked Center Drive

The second important class of V-belt drives is made up of those commonly called "locked center" drives. The name is an appropriate one, since in this arrangement each pulley is mounted on a shaft whose position in space is fixed. This will also be seen to be true for the spring loaded pulley system, but in the present case the pulleys are both rigid, so that the axial force exerted by the pulley on the belt can vary depending on the location of the belt in the groove.

In setting up a drive of this type, the belt is placed under some initial tension and the pulleys are then fixed to prevent further change in center distance. Under these conditions, the tensions in the two belt strands are equal, and the force of the shaft on the pulley is equal to their sum. The drive is then used to transmit torque from one pulley to the other at some operating speed. As the torque is applied, the tensions in the belt strands become unequal, and when the belt is placed in motion a tension due to centrifugal force builds up and causes the belt to stretch. The tension in each strand is now determined by the complex spring system created by the belt and pulleys. It is the object of this analysis to define completely the forces in such a drive under the action of torque and centrifugal forces. The response of the drive to an applied torque will be examined first, and the result will also show the effect of centrifugal force on the system. To eliminate the effect of centrifugal force the drives considered here will be assumed to be stationary or moving very slowly.

A drive of this type is shown diagrammatically in Figure 1-3. The symbols $T_s$, $L_c$, $F_p$, and $D$ represent the initial static tension in
Figure 1-3. Locked center drive with no load.

Figure 1-4. Locked center drive under load.
the belt, the center distance between pulleys, the force exerted on the pulley by the shaft and the pulley diameter respectively.

For the sake of simplicity, the pulleys shown in Figure 1-3 and those treated in the analysis of forces in the locked center drive will be equal in diameter. The extension of any analysis given here to drives involving pulleys of unequal diameters has not been attempted, but should be possible.

If a torque \( J \) is now applied to the drive as shown in Figure 1-4, the tensions in the strands become \( T_1 \) and \( T_2 \) on the tight and slack sides respectively. The torque \( J \) is given by

\[
J = (T_1 - T_2)(D/2) \tag{1-4}
\]

so that if the torque is known, the difference between \( T_1 \) and \( T_2 \) can be found. Also, the pulley force \( F_p \) can be obtained from a horizontal force summation as

\[
F_p = T_1 + T_2 \tag{1-5}
\]

Now if \( F_p \) were known, Equations 1-4 and 1-5 would provide values for \( T_1 \) and \( T_2 \).

In early analyses of belt drives the assumption was made that the pulley force remained constant as torque was applied or

\[
T_1 + T_2 = 2T_S \tag{1-6}
\]

This assumption was justified in the following manner. Consider the two strands of the belt to be straight elastic rods and examine the strains taking place in these rods when torque is applied to a drive. In Figure 1-5 the torque \( J \) causes the lower rod to elongate while the upper rod contracts, resulting in a rotation of each pulley by a small
angle $\theta$. As a result, for small $\gamma$, the strain (change in length per unit length) increases in the lower rod and decreases in the upper rod by approximately equal amounts. If the elastic modulus of the rods is assumed to be constant, then the increase in tension in the lower rod would be equal to the decrease in tension in the upper rod. The sum of the two tensions would then be a constant and would be defined by Equation 1-6.

Measurements, however, showed the sum of the tensions to increase as the torque on a drive was increased. This led to the conclusion that the modulus of the belt strands is not a constant, but increases with increasing tension. This would explain the measurements, since the tight side tension would then be increased more than the slack side is decreased for the same strains due to an applied torque.
This explanation was satisfactory as long as cotton cords were used as the tension-carrying member in V-belts since these cords did indeed have an elastic modulus which increased with increased tension. This fact is shown as the result of measurements made by C. A. Norman.¹ However, modern V-belts have tension-carrying members made of rayon or dacron cords for which the moduli are nearly constant or may even decrease slightly with increased tension. Examples of these moduli are shown in Figures 2-3 and 2-8. Since the sum of the tight and slack side tensions also increases with increased torque for rayon or dacron cord belts, it appears that something must be wrong with the original analysis or with some of the assumptions made.

At the beginning of the analysis it was assumed that the two strands of the belt could be treated as straight elastic rods. Norman² has pointed out two good reasons why the treatment of the belt as a straight rod might be inaccurate: (1) The belt has weight and therefore will tend to hang in the shape of a catenary and (2) the belt resists being bent to the radius of the pulley, and will thus stand away from the pulley in the shape of a bent beam. Norman has attempted to analyse these two effects and was successful to the point of showing the bending effect to be significant but the weight effect to be small. The effect of the belt weight would, of course, be dependent on the length of belt between pulleys, but for V-belt drives the center distance is usually comparatively small, so the assumption of no weight effect would appear reasonable.

¹C. A. Norman, High Speed Belt Drives. Bulletin No. 83, Engineering Experiment Station, The Ohio State University, 1934, p. 15.
²Ibid., pp. 1-12.
If the curvature of the belt between the pulleys is taken into account, when a torque is applied and the pulleys rotate through a small angle $\Theta$ as in Figure 1-5, the resultant increase in the distance between the end points of the lower strand will be due partly to an elastic increase of the belt length and partly to a straightening out of the belt curve. A further complication arises if one considers the fact that as the tension increases in the lower strand, the belt wedges deeper into the pulley groove, and thus tends to offset part of the tension increase. On the slack side of the belt each of these phenomena are present, but their effects are reversed. Figure 1-6 shows schematically the type of curves the belt would take on under the influence of bending and wedging. The exaggerated belt curves are shown for the drive before and after torque is applied.

Figure 1-6. Belt shape in locked center drive.
The equation of the curve assumed by the belt between the pulleys has been derived in Appendix D, and the curve of the belt in the pulley groove can be obtained graphically from the results shown in Appendix A. By using these curves along with the elastic modulus of the belt in the direction of its length, it would appear possible to predict analytically the values of the tight and slack side tensions for a known torque and initial tension applied to a given drive. However, such an analysis would be very complex, and would involve the calculation of small differences in belt length which would require a very accurate knowledge of the physical properties of the belt. To avoid these difficulties, the somewhat more direct approach outlined below has been used to analyze the forces in the belt-pulley system of a locked center drive.

Consider the drive shown in Figure 1-5. When a torque $J$ is applied the belt takes on some shape such as that shown by the solid line in Figure 1-6. However, the shape of the upper strand in Figure 1-6 is the same general shape that would be obtained if the center distance between the pulleys had been reduced rather than the pulleys turned. Also, the general shape of the lower strand could be obtained by increasing the center distance of the drive. In Figure 1-7 the drive has been cut horizontally by an imaginary plane so that the center distances for the upper and lower halves of the drive can be varied independently. These center distances have been shifted to give the upper and lower strands approximately the same shape as the upper and lower strands in Figure 1-6. It is necessary now to compare the tight and slack side belt tensions for the two cases: (1) where the pulleys are turned through an angle $\gamma$ as in Figure 1-5, and (2) where the center distances of the
lower and upper halves of the drive have been increased and decreased respectively by the same amount as shown in Figure 1-7.

![Diagram of a drive mechanism](image)

Figure 1-7. Locked center drive under torque approximated by imaginary split drive.

In either of the above cases the change in tension in a belt strand due to a given pulley rotation or a corresponding change in center distance comes as the result of three separate mechanisms: (1) the resistance of the belt to elongation, (2) the resistance of the belt to bending, and (3) the resistance of the belt to wedging in the groove. The two cases in question will be compared with respect to each of these mechanisms separately.

If the belt is considered to be perfectly flexible (i.e., no moment required to cause bending) and perfectly rigid in the axial direction so that the radial position of the belt in the groove is independent of the tension, then the belt strands between the pulleys will assume the shape of straight rods as discussed earlier in this section.
In this case, the change in the distance between the end points of a belt strand can come only as the result of elastic deformation of the belt along its length. The modulus of elasticity of the belt in the tangential direction, $M_t$, will be defined as the ratio of the tension in pounds to the strain in inches per inch caused by that tension, and will be assumed constant.

Figure 1-8. Belt drive showing effect of $M_t$ alone.

The idealized belt is shown on the drive in Figure 1-8. Here a torque $J$ has been applied to the pulleys causing them to rotate through the angle $\theta$. The midpoints of the belt $A$ and $D$ are thus moved to $A'$ and $D'$ respectively. The resultant tensions in the belt are $T_1$ on the tight side and $T_2$ on the slack side. The total elongation of the lower half of the belt, $ACD$, is $2r\theta$ and is made up of the elongation occurring between the pulleys plus that occurring over the pulleys. If the
tension throughout the belt was \( T_s \) before the torque was applied then
the elongation between the pulleys is given by

\[
\Delta L_c = L_c \frac{T_1 - T_2}{M_t}
\]  

(1-7)

The tension over the pulleys varies from \( T_1 \) at the bottom to \( T_2 \) at the

top so the strain varies correspondingly from \( e_1 \) to \( e_2 \). The total

elongation occurring around the pulleys is therefore the integrated ef-
fact of a variable strain. As a first approximation, let it be assumed

that the variation from \( T_1 \) to \( T_2 \) and thus from \( e_1 \) to \( e_2 \) is linear. The

elongation over the lower half of the two pulleys is then given by

\[
\Delta L_T = (\pi r)(\Delta \varepsilon_{\text{average}})
\]

(1-8)

where \( \pi r \) is the length of belt in contact with the pulleys and \( \Delta \varepsilon_{\text{ave}} \)
is the average increase in strain caused by the application of the

torque. The change in strain at the midpoints A and D is

\[
\Delta \varepsilon_A = \frac{(T_A - T_S)}{M_t}
\]

(1-9)

where \( T_A \) is the tension at A.

At the point of first contact at each pulley the change in

strain is

\[
\Delta \varepsilon_1 = \frac{(T_1 - T_S)}{M_t}
\]

(1-10)

The average change in strain then is

\[
\Delta \varepsilon_{\text{ave}} = \frac{T_1 + T_A - 2T_S}{2M_t}
\]

(1-11)

and from Equation 1-8,

\[
\Delta L_T = \left( \frac{\pi r}{2M_t} \right) (T_1 + T_A - 2T_S)
\]

(1-12)

The total change in length of the lower half of the belt is given by
the sum of Equations 1-7 and 1-12

\[ \Delta L_L = \frac{\pi r}{2M_t} (T_1 + T_A - 2T_s) + \frac{L_c}{M_t} (T_1 - T_s) \]  

(1-13)

For the upper half of the belt, a reduction in length will occur which can be shown in a similar manner to be

\[ \Delta L_U = \frac{\pi r}{2M_t} (2T_s - T_2 - T_A) + \frac{L_c}{M_t} (T_s - T_2) \]  

(1-14)

The total increase in length of the lower half must be the same as the decrease in length of the upper half, so Equations 1-13 and 1-14 can be equated and rearranged to give

\[ \frac{\pi r}{2M_t} (T_1 + T_2 + 2T_A - hT_s) + \frac{L_c}{M_t} (T_1 + T_2 - 2T_s) = 0 \]  

(1-15)

But since the change from \( T_1 \) to \( T_2 \) around the pulleys has been assumed to be linear,

\[ T_A = (T_1 - T_2)/2 \]  

(1-16)

or

\[ 2T_A = T_1 + T_2 \]  

(1-17)

so Equation 1-15 becomes

\[ \frac{2mr}{M_t} (T_A - T_s) + \frac{2L_c}{M_t} (T_A - T_s) = 0 \]  

(1-18)

Then

\[ (T_A - T_s)\left(\frac{2mr}{M_t} + \frac{2L_c}{M_t}\right) = 0 \]  

(1-19)

but the second term can never be zero, so

\[ (T_A - T_s) = 0 \]  

(1-20)

or

\[ T_A = T_s \]  

(1-21)

This means that the applied torque causes a tension increase in the
lower half of the belt which is equal to the tension decrease in the upper half. This is exactly the same tension changes which would take place if the center distances of the upper and lower halves of the drive were decreased and increased respectively by equal amounts as in Figure 1-7. In this case, no difficulty arises over the pulleys, since the tension and hence the strain are constant throughout each half of the belt.

The result obtained above rests on the assumption of a linear change in tension around the pulleys of a drive subjected to a torque. The results of Appendix A show this assumption to be realistic for the driver pulley but not for the driven, since the belt tension on the driven pulley changes more rapidly near the tight side than near the slack side. It should be noted, however, that for the usual case the length of belt in contact with the pulleys will be less than half of the total belt length; thus the error introduced by the non-linearity of the tension change on the pulleys should be fairly small.

A second idealized belt will be examined to show the effect of the bending resistance of the belts between the pulleys for the two cases of (1) turned pulleys and (2) changed center distances. This belt will be assumed to have very large moduli in the tangential and axial directions and to have a normal bending modulus. The belt, therefore, will experience negligible elongation under tension change and will remain at approximately a constant radial position in the pulley groove.

A belt of this type is shown mounted on a drive in Figure 1-9. Here the dimension $L_b$ denotes the length of belt between points A and B which are fixed in space on the centerline of the drive. This length is
Figure 1-9. Locked center drive under no load.

Figure 1-10. Locked center drive under load showing effect of bending resistance.
shown by Equation C-54 in Appendix C to be related to the center distance \( L_c \), the pulley diameters \( D \), and the bending constant for the belt \( K \) in the following manner.

\[
L_b = L_c = \frac{\pi}{2} D = 2 \left( \frac{K}{T} \right)^2 + \frac{2}{D^2} \left( \frac{K}{\pi} \right)^2 = \frac{1}{D^4} \left( \frac{K}{\pi} \right)^5 = D \tan^{-1} \left[ \frac{2 \left( \frac{K}{\pi} \right)^2}{D \left( \frac{K}{\pi} \right)^5} \right]
\]  

(1-22)

If a torque \( J \) is applied to the drive of Figure 1-9 and the pulleys rotate through the angle \( \gamma \) as shown in Figure 1-10, the length of belt included between the points A and B is increased by \( D \gamma \) over the upper half of the drive and is decreased by the same amount over the lower half of the drive. These changes in \( L_b \) result in changes in the belt tensions as defined by Equation 1-22.

Now according to Equation 1-22, the same tension changes would occur if the center distances of the upper and lower halves of the drive were decreased and increased respectively by the same amount, namely \( \gamma D \), while the belt length \( L_b \) remained constant for each half. This indicates that the tension changes due to the resistance of the belt to bending in a drive under torque can be exactly duplicated by a fictitious drive in which the center distances of the upper and lower halves have been decreased and increased by the same amount.

The third phenomenon influencing the tension changes occurring with turned pulleys or changed center distance is that of the belt squashing in the pulley groove. Since the manner in which the belt moves in the groove is governed by the friction angle \( \theta \) as discussed in Appendix A, it appears that no simple method is available to compare the effects of turned pulleys and changed center distance with regard to this phenomenon.
Figures 1-11 and 1-12 show the general manner in which the belt would move if only the squashing effect were present. When the pulleys are turned as in Figure 1-11 the tension in the lower strand would surely increase and that in the upper would certainly decrease. This means that in order to duplicate these tension changes by differential variation of the center distances of the upper and lower halves as shown in Figure 1-12 would require an increase in center distance in the lower half and a decrease in the upper half. The required changes in center distance would not in general be equal, but at least the changes are in the right direction. As was mentioned in the discussion of the effect of tangential modulus, the importance of the phenomenon acting only throughout the contact angle is reduced by virtue of the fact that only a portion of the belt is involved.

From the above discussion it would appear that the tensions arising from the application of torque to a locked center drive might be capable of approximation in a fictitious drive by increasing and decreasing the center distances of the two halves of the drive by equal amounts. The following simple experiment has been used to show the validity of this approximation and also to demonstrate its value.

A C-section belt 68 inches long was mounted on two 9.5 inch pulleys. The pulleys were arranged in such a manner that the center distance between them could be varied and the resulting total tension in the belt could be measured. The measurements were made under no torque load, so that the tensions in the two belt strands were equal. Between measurements the belt was rotated several times to assure complete seat-
Figure 1-11. Locked center drive showing effect of $M_a$ alone.

Figure 1-12. Split drive approximation showing effect of $M_a$ alone.
ing in the pulley grooves. Before each measurement was taken, the belt and pulleys were returned to a predetermined orientation in space.

The resulting plot of center distance versus the tension in each strand of the belt is shown in Figure 1-13. The straight line portion of the curve represents the condition in which the belt has straightened out between the pulleys, and the relationship between tension and center distance is controlled almost entirely by the tangential modulus of the belt and the tendency of the belt to squash in the groove. At lower tensions the non-linear bending effect discussed earlier becomes important and causes the observed curvature. If the belt were a rigid circle in the stress-free state, then the tension should go to zero at the point where the center distance becomes equal to the diameter of the stress-free belt minus the sum of the pulley radii. The belt, however, is not a very rigid member so that the belt tension becomes essentially zero at a much greater center distance than would be so indicated.

The curve in Figure 1-13 was used to predict tension variations due to torque application on a locked center drive. For example, the drive was placed under a no-torque tension of 30 pounds per strand and the pulley center distance was fixed. A torque was then applied such that the tension difference $T_1 - T_2$ became 80 pounds. As shown on the curve, if the tight and slack sides of the drive could be increased and decreased respectively by the same amount until the tension difference were 80 pounds, the resulting tensions would be

\[
T_1 = 89 \text{ pounds} \\
T_2 = 9 \text{ pounds}
\]

By measuring the total tension on the belt and the torque in the system,
Figure 1-13. T versus $L_C$ for a C-belt 68" long over 9.0" pulleys.
the belt tensions were found to be

\[ T_1 = 90 \text{ pounds} \]
\[ T_2 = 10 \text{ pounds} \]

For the same value of no-torque tension two other tests gave the following results:

<table>
<thead>
<tr>
<th>Tension Difference</th>
<th>Predicted</th>
<th>Measured</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_1 - T_2 )</td>
<td>( T_1 )</td>
<td>( T_2 )</td>
</tr>
<tr>
<td>100</td>
<td>106.5</td>
<td>110.6</td>
</tr>
<tr>
<td></td>
<td>6.5</td>
<td>10.6</td>
</tr>
<tr>
<td>120</td>
<td>125</td>
<td>130.5</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>10.5</td>
</tr>
</tbody>
</table>

The accuracy of these predictions is well within the limits of accuracy of the test apparatus, and suggest that the approximations involved are good ones. The method used to obtain the predictions of \( T_1 \) and \( T_2 \) provides a means of specifying the tension that must be placed on a drive before it is locked in order that the tensions under load be those desired.

The empirical curve of Figure 1-13 can also be obtained analytically from Equation 1-22. Since the curve represents a drive under no load, the difficulties involved in defining the effects of tangential modulus and squashing on the center distance-tension relationship are eliminated. An increase in tension acts throughout the entire belt to cause an elongation by elastic action, while the same change in tension causes a change in the diameter at which the belt rides in the pulleys.

Rewriting Equation 1-22 as

\[ L_c = L_b \frac{nD}{2} - 2 \left( \frac{K}{T} \right)^{\frac{1}{2}} - \frac{2}{D^2} \left( \frac{K}{T} \right)^{\frac{3}{2}} + \frac{1}{D^4} \left( \frac{K}{T} \right)^{\frac{5}{2}} + D \tan^{-1} \left[ \frac{(2/D)(K/T)^{1/2}}{1} \right] \]  (1-23)
and referring to Figure 1–9, it can be observed that a tension change affects each term on the right-hand side of the equation. For a tension change $\Delta T$ the change in center distance is, from Equation 1–23,

$$
\Delta L_c = \Delta L_b - \frac{\pi}{2} \Delta D = 2 \Delta \left( \frac{K}{T} \right)^2 - 2 \Delta \left[ \frac{1}{\beta^2} \left( \frac{K}{T} \right)^{\frac{3}{2}} \right] \\
+ \Delta \left[ \frac{1}{5} \left( \frac{K}{T} \right)^{\frac{5}{2}} \right] + \Delta \left[ D \tan^{-1} \frac{2}{D} \left( \frac{K}{T} \right)^{\frac{1}{2}} \right]
$$

(1–24)

The change in belt length due to the tangential modulus results in a change in $L_b$ and is given simply by

$$
\Delta L_b = L_b (\Delta T/M_t)
$$

(1–25)

where $\Delta T$ is the tension change, $M_t$ the tangential modulus, and, since the total change in belt length will be small compared with the length, $L_b$ may be taken safely as one-half the length of the unstressed belt.

The expression for the change in the belt diameter over the pulleys can be obtained by assuming that the friction angle discussed in Appendix A is $90^\circ$, or, more explicitly, by assuming that the frictional force of the pulley on the belt is directed radially outward and that all available frictional force is developed. This assumption is reasonable, since with no tension change over the pulleys, there is no tendency for the belt to move in the tangential direction while during the seating process the belt must move inward radially until the internal forces in the belt are in equilibrium with those of the pulley on the belt surface.

Based on this assumption the forces acting on a segment of belt anywhere throughout the angle of contact will be the same as those act-
ing at the inlet and can be obtained from the analysis of Appendix A.

From Equation A-56 with \( \Phi = 90^\circ \) and dropping the subscript \( i \),

\[
P_a = \frac{T\left(\cos \frac{a}{2} - C_f \sin \frac{a}{2}\right)}{D\left(\sin \frac{a}{2} + C_f \cos \frac{a}{2}\right)} \tag{1-26}
\]

where \( P_a \) is the intensity of the axial force on the belt in pounds per tangential inch, \( T \) is the belt tension, \( D \) is the belt diameter, \( a \) is the groove angle, and \( C_f \) is the coefficient of friction between the belt and pulley. For a change in tension \( \Delta T \),

\[
\Delta P_a = \frac{\Delta T\left(\cos \frac{a}{2} - C_f \sin \frac{a}{2}\right)}{D_{ave}\left(\sin \frac{a}{2} + C_f \cos \frac{a}{2}\right)} \tag{1-27}
\]

where \( D_{ave} \) represents some diameter between \( D \) and \( D + \Delta D \). However, if it is assumed that the changes in diameter will be small compared to the diameter, then \( D_{ave} \) may be taken as the nominal diameter of the belt in the pulley. From Equation A-22

\[
\Delta P_a = \frac{2 M_a \tan (a/2)}{W} \Delta u \tag{1-28}
\]

where \( M_a \) is the axial modulus of the belt in pounds per tangential inch/inch per inch, \( u \) is the coordinate used to locate the belt radially in the groove, and \( W \) is the nominal width of the belt. But from the definition of \( u \)

\[
2\Delta u = -\Delta D \tag{1-29}
\]

so Equation 1-28 becomes

\[
\Delta P_a = \frac{M_a \tan (a/2)}{W} \Delta D \tag{1-30}
\]
Combining Equations 1-27 and 1-30 with \( D = D_{ave} \) and solving for \( \Delta D \) gives

\[
\Delta D = \frac{W \left( \cos \frac{a}{2} - C_f \sin \frac{a}{2} \right)}{(D \tan \frac{a}{2}) \left( \sin \frac{a}{2} + C_f \cos \frac{a}{2} \right)} \Delta T \quad (1-31)
\]

Equation 1-31 may be used to define the diameter change in the second term on the right side of Equation 1-24. Since the changes in \( D \) will always be small in practice, the nominal belt diameter may be used in all remaining terms where \( D \) appears.

The change in center distance \( \Delta L_c \) resulting from a tension change \( \Delta T \) is now defined by the combination of Equations 1-24, 1-25, and 1-31. All of the physical constants involved in these equations are known for the drive represented in the curve of Figure 1-13 except the bending constant \( K \). This constant can be obtained from the data shown in Figure 1-13 by determining the difference in center distance corresponding to two arbitrarily chosen values of tension and then varying \( K \) in Equation 1-24 until this difference is obtained when the chosen values of tension are used in the equation.

For the drive in question the physical constants are:

- \( D = 9.5 \) inches
- \( L_b = 3 \) inches
- \( M_t = 45,500 \) pounds/inch per inch
- \( W = 0.757 \) inch
- \( M_a = 1000 \) pounds per tangential inch/inch per inch
- \( a/2 = 18^\circ \)
- \( C_f = 0.43 \)
Using these values in Equation 1-24 and matching the equation with the curve at tensions of 10 and 40 pounds gives a K value for this C-section belt of 11.9. The curve described by the equation with $K = 11.9$ is shown in Figure 1-14 along with the original curve from the data. The very close agreement between the two curves suggests that the equation represents an excellent approximation of the true physical situation.

**Centrifugal Force**

The above analysis of the locked center drive under an applied torque can also be used to explain the effect of centrifugal force on such a drive. As pointed out in Appendix B, the only effect of centrifugal force on a belt is to cause an elongation due to the increased tension within the belt. This elongation can be expressed in terms of the tangential modulus as

$$\Delta L_b = \left(\frac{L_b}{M_t}\right)\Delta T_c$$  \hspace{1cm} (1-32)

where $L_b$ is the belt half-length, $\Delta T_c$ is a change in tension due to centrifugal force, and $M_t$ is the tangential modulus of the belt. However, from Equation 1-24 a change in belt length can be replaced by a corresponding change in center distance without affecting the resultant change in belt tension. Centrifugal force therefore has the effect of shifting the operating points of a drive to the right or left along the $T$ versus $L_c$ curve of Figure 1-13. It should be pointed out here that the tension involved in Equation 1-24 does not include the tension due to centrifugal force, but only that tension which defines the shape of the belt. A change in tension due to centrifugal force affects the shape-defining tension only through the change it causes in the $L_b$ term.
Figure 1-14. Computed and measured curves of $T$ versus $L_c$ for a C-belt 68" long over 9" pulleys.
of Equation 1-24. Another way to visualize the effect of centrifugal force is to consider a belt under its action as a new belt of slightly different length from the original belt. This new belt would have a $T$ versus $L_C$ curve almost exactly the same as that for the original belt but shifted to the right or left by the amount $\Delta L_b$ as defined by Equation l-32.

A numerical example will show how the preceding analysis can be used in practice. Suppose that it is desired that the belt described by Figure 1-13 operate with a tension ratio of 5 while transmitting 12 horsepower at a belt speed of 5000 feet per minute. The problem is to determine the required no-load static tension to yield these operating conditions. For this case,

$$T_1 - T_2 = \frac{33,000 \text{ HP}}{V} = \frac{33,000 \times 12}{5000}$$

$$T_1 - T_2 = 80 \text{ lbs}$$

But

$$\frac{T_1}{T_2} = 5$$

so

$$5T_2 - T_2 = 80 \text{ lbs}$$

Then

$$T_2 = 20 \text{ lbs}$$

and

$$T_1 = 100 \text{ lbs}$$

From Equation B-9 in Appendix B,

$$T_C = \frac{(\rho V^2)}{g}$$

where $T_C$ is the centrifugal tension, $\rho$ is the belt weight per foot of length, $V$ is the belt speed in feet per second, and $g$ is the gravitational constant 32.2 feet per second per second. With $\rho = 0.22$ pounds per foot.
\[ T_c = \frac{0.22 \left( \frac{5000}{60} \right)^2}{32.2} = 47.4 \text{ lbs} \]

From Equation 1=32 this centrifugal tension causes the belt to elongate by the amount

\[ 2 \Delta L_b = \frac{2L_b}{M_c} \Delta T_c = \frac{68}{45,500} = 47.4 \]

\[ 2 \Delta L_b = 0.0709 \text{ in.} \]

This corresponds to a center distance change of

\[ \Delta L_c = 0.0355 \text{ in.} \]

The curve of Figure 1-13 is shown again in Figure 1-15. Here A and B represent the desired operating points for the belt. Now if the torque is removed from the drive while the belt speed is held constant the previous analysis shows that the new tensions can be obtained by shifting equal distances on the center distance scale of Figure 1-15. As shown in the figure, this results in a tension in each belt strand of \( T_s = 49 \) pounds. When the belt is stopped the centrifugal force ceases to act and allows the belt to decrease in length, which can be represented in Figure 1-15 as a shift to the right along the curve. The amount of the shift has been computed to be \( \Delta L_c = 0.0355 \) and results in a required static initial tension of \( T_s = 80 \) pounds per strand as shown in the figure.
Figure 1-15. Example of tension prediction using $T$ versus $L_c$ curve.
Spring-Loaded Drive

Description. The third drive type is often called a spring-loaded drive and is shown schematically in Figure 1-16. In this arrangement the center distance between the pulleys is fixed at all times, and the tension in the belt is maintained by the spring force acting on the moveable disk of the spring-loaded pulley.

The usual application for this system is in the variable speed drive shown in Figure 1-16. Here the spring-loaded pulley is used in conjunction with an adjustable pulley, the latter having a moveable disk which can be positioned manually by means of a handwheel. A change in speed ratio is achieved by turning the handwheel, thus causing the belt to ride at a new radial position in the adjustable pulley. Since the belt length remains nearly constant, this means that the belt must shift radially in the spring-loaded pulley in the direction opposite to that of the motion in the adjustable pulley. This change in the pitch radii of both pulleys results in a change in the speed ratio for the drive.

The force system acting on the belt in a drive of this type is exactly the same as that acting in a constant tension or locked center drive of the same geometry, belt speed, and belt tensions. There exists a popular misconception that the entire mechanics of power transmission is in some way different when a spring-loaded pulley is installed in a system. Such, of course, is not the case, since as far as the belt is concerned there is no difference between a pulley force exerted due to the action of a spring and a force exerted due to elastic action in a solid pulley.

The great problem involved in the force analysis of the spring-
Figure 1-16. Spring loaded variable speed drive.
loaded drive is that of defining the belt tensions for a given spring force, or, from the designer's point of view, the specification of a spring force if known tensions are to be obtained. The only solution to this problem lies in an analysis of the interface forces acting between the belt and pulley since it is the integrated effect of these forces which results in a certain required axial force for given inlet and outlet tensions. Such an analysis has been performed in Appendix A, and will be discussed in connection with a specific drive. The results of spring force investigations reported by other authors will be discussed later in this chapter.

**Centrifugal force.** Before beginning the discussion of the pulley forces in an actual drive, a few words should be said about the effect of centrifugal force on a spring-loaded drive. As was the case for the constant tension and the locked center drives, the only result of centrifugal force is the build-up of a balancing tension inside the belt. This centrifugal tension is simply superposed on the tensions caused by pulley forces, and, as discussed in Appendix B, has no effect on the belt shape. The slight increase in length due to the centrifugal tension results in a slight shift in the belt radius on the spring-loaded pulley, but the force picture remains essentially unchanged since a spring with a fairly low spring constant is usually involved.

**Test data.** To check the validity of the analysis given in Appendix A as applied to a spring-loaded drive, a Reeves No. 8000 Moto-drive similar to that shown in Figure 1-16 was mounted in such a manner that the belt tensions could be measured while the drive operated under
various loads. Three different springs were used to obtain a variation in spring force.

Typical results of measurements on this apparatus are shown in Figure 1-17. Here the tight and slack side tensions have been plotted versus their difference for three different spring forces at a constant speed and constant spring-loaded pulley radius as indicated. Curves have been drawn to approximate the data, and have been extrapolated to zero load at which point no data were taken. For the case shown, the driver and driven pulley radii were approximately equal.

Analytical results. The physical constants corresponding to the test drive were measured, and these constants were then used in conjunction with Equations A-30, A-39, A-51, and A-59 from Appendix A to obtain chart solutions for the drive variables similar to those shown in that appendix. Only a few solutions for the spring-loaded drive had been plotted when, as mentioned in Appendix A, further operating time on the computer became very difficult to obtain. More curves will be obtained in the future, but it is believed that those shown here are sufficient to encourage such further study.

Since the test drive could be loaded only at the spring-loaded pulley, the equations of Appendix A were solved only for the driven pulley, i.e., the computer was wired to give a tension increase with increasing θ. The constants used in obtaining the curves were:

\[ W_a = 214,000 \text{ pounds per belt} \]

\[ α = 36 \text{ degrees} \]

\[ C_f = 0.327 \]
Figure 1-17. Tight and slack side tensions as measured on spring-loaded drive.
\[ M_a = 7500 \text{ pounds per tangential inch} \]
\[ W = 3.64 \text{ inches} \]

The initial conditions used were
\[ \left( \frac{du}{d\theta} \right)_i = 0 \]
\[ u_i = 0 \]
\[ \phi_i = 90^\circ \text{ or as large as possible} \]
\[ r_i = 5.8 \text{ inches} \]

The curves presented in Figures 1-18 through 1-21 show the variations of each of the drive variables with the angular position coordinate, \( \theta \). The variables shown are the sine of the friction angle, \( \sin \phi \); the belt tension, \( T \); the total axial force acting on the belt, \( F_a \); and the tangential component of the belt velocity relative to the pulley divided by the pulley velocity (or the slip), \( S_t \).

Due to poor scaling of the problem for the computer, the inlet value of the friction angle, \( \phi_i \), could not be held near \( 90^\circ \) without causing the machine to overload. The set of curves obtained under these conditions are marked with unprimed numbers. To check the seriousness of this situation the scaling was changed and the curves with primed numbers were obtained. With the new scale factors it was possible to plot only a short portion of the required curves. However, by comparing corresponding primed and unprimed curves it can be seen that the axial force required to allow a given tension change over the pulley is approximately the same for either curve. The unprimed curves were therefore used in the following comparisons of the computed curves with measured data.
TABLE 1-1
Curve Identification for Figures 1-18 Through 1-21

\[ M_b = 214,000 \text{ pounds per belt} \]
\[ \alpha = 36^\circ \]
\[ C_f = 0.327 \]
\[ M_a = 7500 \text{ pounds per inch} \]
\[ W = 3.64 \text{ inches} \]

<table>
<thead>
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<th>Curve</th>
<th>( \sin \varphi_i )</th>
<th>( T_i ) (lbs)</th>
<th>( S_{ti} )</th>
<th>( r_i ) (in.)</th>
</tr>
</thead>
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<td>10</td>
<td>0</td>
<td>5.8</td>
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<td>0</td>
<td>5.8</td>
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<tr>
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<td>100</td>
<td>0</td>
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<td>150</td>
<td>0</td>
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<td>4'</td>
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<td>100</td>
<td>0</td>
<td>5.8</td>
</tr>
</tbody>
</table>
Figure 1-18. Sin $\phi$ versus $\Theta$ for spring-loaded driven pulley. For curve identification see Table 1-1.
Figure 1-19. T versus θ for spring-loaded driven pulley. For curve identification see Table 1-1.
Figure 1-20. $S_t$ versus $\Theta$ for spring-loaded driven pulley.
For curve identification see Table 1-1.
Figure 1-21. \( F_a \) versus \( \theta \) for spring-loaded driven pulley. For curve identification see Table 1-1.
Comparison of analysis and test data. At first glance, it would appear that the analysis could be checked by simply comparing the computed axial force on the belt, \( F_a \), for given inlet and exit tensions with the spring force required in the actual drive to cause the same tensions at inlet and exit. This comparison immediately shows the computed value to be far greater than the actual one, and suggests a closer look at the force system in the actual pulley. Two possible sources of trouble are at once apparent: (1) Because the belt touches only one-half of the pulley, the hub of the sliding disk is forced to be "cocked" on the shaft, and may tend to move along the shaft as rotation occurs. (2) The key used to transmit torque from the sliding disk to the shaft is subject to a high normal force, and thus could potentially exert a large axial frictional force on the sliding disk. Either of these effects could be fairly significant and could account for the axial force on the belt being different from the axial spring force.

The relative importance of the two effects mentioned above can be estimated by considering the fact that item (1) is a function of the total tension acting in the belt while item (2) is a function of the difference between these tensions.

Consider first the \( T_1 \) and \( T_2 \) curves in Figure 1-17 for a drive under no torque and with a spring force of 468 pounds. For such a drive the no-load tensions are seen to be

\[
T_s = T_1 = T_2 = 227 \text{ pounds},
\]

where the subscript \( s \) indicates a constant tension. Since no tension change takes place in this case, the friction angle can be assumed to be 90° throughout the arc of contact according to the discussion in the
General Remarks of Appendix A. Using $\phi = 90^\circ$, solving Equation A-51 for $P_N$ and substituting this expression into the left-hand side of Equation A-53 gives

$$P_{as} = \frac{T_S (\cos \frac{\alpha}{2} - C_f \sin \frac{\alpha}{2})}{2(r_i - u)(\sin \frac{\alpha}{2} + C_f \cos \frac{\alpha}{2})}$$  \hspace{1cm} (1-33)

For the drive under consideration,

$\alpha = 36$ degrees

$u = 0$

$C_f = 0.327$

Using these values in Equation 1-33 gives

$$P_{as} = 0.685 \left( \frac{T_S}{r_i} \right)$$  \hspace{1cm} (1-34)

Since $P_{as}$ is the distributed force per tangential inch, multiplying Equation 1-34 by the incremental length $r_i d\theta$ and integrating over the arc of contact should give the value of the total axial force on the belt, or

$$F_{as} = \int_0^\theta 0.685 T_S \, d\theta = 0.685 T_S \theta$$  \hspace{1cm} (1-35)

Since the belt is probably not in contact with the pulley throughout a full $180^\circ$, it is advisable to assume an active angle of something less than $n$ radians, say $3.0$ radians.

For the present case with $\phi = 3.0$ and $T_S = 227$ pounds, Equation 1-35 yields

$$F_a = (0.685)(227)(3.0) = 466 \text{ pounds}$$

This is in excellent agreement with the actual spring force of $468$ pounds. Using Equation 1-35 to compute the axial force corresponding
to the no-load case for the remaining two pairs of curves in Figure 1-17
results in computed forces of 354 and 510 pounds as compared with actual
spring forces of 346 and 518 pounds, respectively.

Now again consider the two tension curves in Figure 1-17 for a
spring force of 468 pounds. For these curves, a pair of corresponding
inlet and exit tensions for a drive under torque are seen to be 150 and
450 pounds. These same tensions can be obtained in the analysis by fol-
lowing curve 5 in Figure 1-19 from $\Theta = 0$ to $\Theta = 2.81$ radians. Evident-
ly, then, in this case only part of the contact arc is involved in the
tension change. The axial force required for the active arc is obtained
from curve 5 in Figure 1-21 by reading the value of $F_a$ at $\Theta = 2.81$.
This gives

$$F_{av} = (0.236)(2500) = 590 \text{ pounds}$$

where the subscript $v$ refers to the active arc.

Throughout the idle or inactive arc the tension on the driven
pulley will be that at the pulley inlet, or $T_2$. If the contact angle
is again assumed to be 3.0 radians, then the idle arc is

$$\Theta_s = 3.0 - \Theta_v$$  \hspace{1cm} (1-36)

where the subscript $s$ indicates constant tension. The active arc $\Theta_v$
has been evaluated as 2.81 radians, so Equation 1-36 gives

$$\Theta_s = 3.0 - 2.81 = 0.19 \text{ radians}$$

Using this value of $\Theta_s$ and $T_s = T_2 = 150 \text{ pounds}$ in Equation 1-35 re-
results in the axial force required for the idle arc, which is

$$F_{as} = (0.685)(150)(0.19) = 19 \text{ pounds}$$

The total axial force is then

$$F_a = F_{av} + F_{as}$$  \hspace{1cm} (1-37)
or, for this case

\[ F_a = 590 + 19 = 609 \text{ pounds} \]

This is considerably above the actual spring force of 468 pounds and suggests that one or both of the phenomena mentioned under (1) and (2) above are affecting the system.

These two results indicate that the calculated axial force is approximately equal to the actual spring force when the tension difference \( T_1 - T_2 = 0 \), but that poor agreement is obtained between these numbers with a tension difference of \( T_1 - T_2 = 300 \) pounds. Since the sum of the belt tensions is a large number for either example (454 and 600 pounds), it would appear that the frictional force affecting the system must be only that one which depends on the tension difference. Based on this reasoning, the tendency of the pulley hub to move along the shaft due to cocking may be dismissed as a probable source of discrepancy between computed and actual results. Attention will then be centered on the axial frictional force which might be exerted due to a normal force acting on the key.

In the drive on which the tests were conducted the torque force is exerted on the key at a mean radius of 1.72 inches, while the tensions \( T_1 \) and \( T_2 \) act at a radius of 5.8 inches. Assuming that one-half of the total torque is carried by the sliding disk, the normal force on the key is given by

\[ F_k = \frac{5.8}{2(1.72)} (T_1 - T_2) \quad (1-38) \]

and the corresponding available frictional force is

\[ F_{fk} = 1.69 C_f (T_1 - T_2) \quad (1-39) \]
However, the coefficient of friction $C_f$ for two metal surfaces is difficult to predict accurately without actual measurement. If the force from Equation 1-39 is assumed to be the difference between the actual and computed values of spring force, a value of $C_f$ can be obtained.

For the previous case of $T_1 = 450$ and $T_2 = 150$ pounds the actual and computed values of spring force were 468 and 609 pounds, respectively, giving a difference of 141 pounds. Using these numbers and equating the frictional force to the spring force error gives, from Equation 1-39

$$141 = 1.69 C_f (450 - 150)$$

or

$$C_f = 0.278.$$ 

This value of $C_f$ is a very reasonable one for steel on steel where lubrication may be poor. With this value of $C_f$, Equation 1-39 becomes

$$F_{fk} = 0.469(T_1 - T_2) \quad (1-40)$$

Three other cases will now be checked using Equation 1-40. Curve 4 in Figure 1-19 can be used to represent a drive having $T_2 = 100$ and $T_1 = 360$ pounds, for which the actual spring force is seen from Figure 1-17 to be 346 pounds. On curve 4 the tension change from 100 to 360 pounds occurs in an arc of 3.13 radians. Since the curve indicates a required active angle of 3.13 radians with no slip at the pulley inlet and the available arc has been assumed to be 3.0 radians, a slight slip at the inlet to the drive is indicated. A drive having a slight inlet slip, an active arc of 3.0 radians, an exit tension of 360 pounds, and an inlet tension of about 102 pounds is represented by curve 4 between $\Theta = 0.13$ and $\Theta = 3.13$, and so this segment should give an excellent ap-
proximation of the drive in question. The corresponding spring force can now be obtained from curve 4 in Figure 1-21. The total axial force is given by subtracting the value of $F_a$ at $\theta = 0.13$ from its value at $\theta = 3.13$. This gives

$$F_a = (2500)(0.200 - 0.004) = 490 \text{ pounds} \quad (1-41)$$

Substituting for $T_1$ and $T_2$ in Equation 1-40 yields

$$F_{fk} = (0.469)(360 - 100) = 122 \text{ pounds}$$

Subtracting the key force from the axial force gives the computed spring force

$$F_{sp} = 490 - 122 = 368 \text{ pounds}$$

as compared with $346$ pounds on the drive. The same approach can be used for drives with spring forces of $346$ and $518$ pounds, slack side tensions of $150$ pounds each, and tight side tensions of $235$ and $510$ pounds, respectively. For the drive with an actual spring force of $346$ pounds, the computed force is $334$ pounds, while for the case where the spring force is $518$ pounds, the computed value is $531$ pounds.

It is believed that these computed values are as close to the actual values as can be expected with the present level of perfection in the analog solutions. The assumptions that the key exerts a frictional force on the sliding disk and that it acts in the same direction as the spring force are, of course, open to argument. Unfortunately, the test set-up on which the variable speed drive data were taken has been dismantled, or the phenomenon of key friction could be studied experimentally. The close agreement between computed and measured spring forces is, however, strong evidence in support of these assumptions.
Discussion of literature. The spring force problem as described in the preceding paragraphs has been the object of considerable analytical scrutiny by other researchers. The first apparent mention of an expression for axial spring force is made by Kimmich and Roesler. The equation given is

\[
F_a = \frac{(T_1 + T_2) \theta_g}{4 \tan \alpha/2}
\]

(1-42)

where \( F_a \) is the total axial force, \( T_1 \) and \( T_2 \) are the tight and slack side tensions, respectively, \( \theta_g \) is the angle of contact between the belt and pulley, and \( \alpha \) is the pulley groove angle. No derivation is presented along with the equation, but the same equation is derived by Morgan. The derivation is based on two primary assumptions, (1) that the tension around the pulley can be taken as \( (T_1 + T_2)/2 \), and (2) that no friction acts between the belt and pulley in the radial direction.

As pointed out by Morgan, these assumptions are rather far from reality, and might be expected to contribute considerable error to the computation. Morgan next presents a parallel derivation in which all available friction is assumed to act outward in the radial direction. The resulting expression is, after conversion of symbols,

\[
F_a = \frac{(T_1 + T_2) \left[ 1 - C_f \tan (\alpha/2) \right] \theta_g}{4(C_f + \tan \alpha/2)}
\]

(1-43)

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5Ibid., pp. 6-7.
It would appear that the assumption that all friction acts outward is little better than the assumption of no friction at all, since, if a tension change is occurring over the pulley, a significant portion of the available frictional force must act in the tangential direction leaving only a part of the total frictional force to act in the radial direction. This assumption could be especially erroneous for the driver pulley, in which case there is no assurance that the radial component of friction does not act inward on the belt rather than outward.

Worley⁶ mentions three different spring force equations, which are

\[ F_a = \frac{T_1 - T_2}{2k} \frac{1 - C_f \sin \phi \tan \alpha/2}{C_f \sin \phi + \tan \alpha/2} \]  \hspace{1cm} (1-44)

\[ F_a = \frac{T_1 - T_2}{2k \tan \alpha} \]  \hspace{1cm} (1-45)

and

\[ F_a = \frac{T_1 Q_g}{2} \frac{1 - C_f \tan \alpha/2}{C_f + \tan \alpha/2} \]  \hspace{1cm} (1-46)

where

\[ k = \left[ \ln \left( \frac{T_1}{T_2} \right) \right] / Q_g \]  \hspace{1cm} (1-47)

\( \phi \) is the friction angle, and the other symbols are defined as before.

A separate equation given to define the friction angle \( \phi \) is

\[ C_f \cos \phi = k (\sin \frac{\alpha}{2} + C_f \sin \phi \cos \frac{\alpha}{2}) \]  \hspace{1cm} (1-48)

According to Worley, Equation 1-44 is to be used for a driven pulley and Equations 1-45 and 1-46 are valid for a driver pulley. These equa-

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tions, and those defining \( k \) and \( \phi \) are given without derivation, so their true origin is a matter of conjecture. It is highly probable, however, that the derivations of all of these equations follow, at least roughly, along the same lines as derivations shown by Morgan\(^7\) which lead to the same results.

Equation 1-46 can be seen to be the same as Equation 1-43 with \( (T_1 + T_2)/2 \) replaced by \( T_1 \). This means that the same derivation was probably used, but that \( T_1 \) was found to give better agreement with measured data than did \( (T_1 + T_2)/2 \). The statements made concerning Equation 1-43 thus also hold for Equation 1-46.

The derivations of Equations 1-44 and 1-45 are given by Morgan.\(^8\) The basic assumption on which the derivation depends is that the friction angle \( \phi \) remains constant throughout the contact angle, and consequently that the tension changes gradually throughout the entire angle of contact. Now in general, this assumption would not be expected to be true, and the analogue results shown in this chapter and in Appendix A confirm this expectation. The assumption of a constant friction angle is therefore not an approximation of the actual physical situation, but is equivalent to the assumption of a new and fictitious drive in which only a constant friction angle can exist. This fictitious drive can now be analyzed quite simply using only the three force summations as described in Appendix A. This analysis results in equations for the tensions on the drive in terms of the angle of contact and the friction angle, and for the axial force on the belt in terms of the friction

\(^7\)Morgan, op. cit., pp. 6-15.

\(^8\)Ibid., pp. 9-15.
angle and the tensions. The next step is to make the fictitious drive similar to the actual drive by requiring that the inlet and exit tensions in each system be identical. For this condition, the required constant value of the friction angle in the fictitious drive can be found and then used to compute the axial force on the belt in this drive. This approach results in Equation 1-44. Equations 1-47 and 1-48 can be solved simultaneously to provide the value of the constant friction angle.

The important assumption is now evidently made that the axial force in the fictitious drive is approximately equal to the axial force in the actual drive. Again, this would not, in general, be expected to be true. The tensions and the axial force are each integrated functions of $\phi$, and these functions are different, so that the integrated mean value of $\phi$ for one of these functions will generally be different from that for the other. That this difference exists is amply demonstrated by considering Equation 1-44 as applied to a driver pulley. Worley\(^9\) states that the equation is good only for the driven pulley, but nothing in Morgan's derivation would make this restriction. On the driver pulley $T_1$ is the inlet tension and $T_2$ the exit tension, so Equation 1-47 will yield a negative number equal in magnitude to that which would be obtained for the driven pulley. Also, on the driver pulley tangential friction acts in the direction of rotation so $\cos \phi$ will always be a negative number. The two negatives will cancel in Equation 1-48 and this equation will yield the same $\phi$ for the driver as for the driven.

\(^{9}\) Worley, op. cit., p. 332.
same magnitude as for the driven pulley but will be negative in sign and will cancel the negative value of k in the denominator. This means that Equation 1-44 will give the same spring force for the driver pulley as for the driven. While Worley's data\textsuperscript{10} are difficult to interpret since only ranges of spring force are given, it would appear that the axial force on the driver pulley in this data is about 1.4 times the axial force on the driven pulley. Morgan's data\textsuperscript{11} show a ratio of driver pulley axial force to driven pulley axial force of as high as 2.77 for identical conditions at each pulley. From this type of data, then, it is understandable that Equation 1-44 has been restricted by Worley to use on a driven pulley. However, from a theoretical point of view there is no more reason for Equation 1-45 to be correct for the driven pulley than for the driver pulley. The degree of approximation involved can only be ascertained by reference to measured data, and this will be done in the next section.

As for Equation 1-45 which is one of the equations advanced by Worley for use on the driver pulley, the derivation of this expression as described by Morgan\textsuperscript{12} is quoted directly below with converted nomenclature given in brackets.

\text{... By starting with a diagram essentially the same as Figure 4. [This is a sketch showing the normal, frictional and tension forces acting on a belt element.] and assuming that the pulley is driving and, thus, that the tension is decreasing as the belt element moves around the arc of contact, they have invariably come to an impasse. An analysis of the forces}

\textsuperscript{10} W. S. Worley, Discussion of paper by Kimmich and Roesler. \textit{Agricultural Engineering}, Vol. 31, No. 7 (July, 1950), 337-40.

\textsuperscript{11} Morgan, \textit{op. cit.}, p. 40.

\textsuperscript{12} \textit{Ibid.}, pp. 14-15.
in equilibrium does not permit the radial component of friction to act inward toward the axis of rotation because if it did, the pulley reaction would be in the same direction as the belt tension components and there would be no equilibrium since the radial components of all the forces would be acting in the same direction.

Therefore they arbitrarily assume that \( u_R [C_f \sin \theta] \) is equal to \( \tan \alpha \left[ \tan \alpha/2 \right] \) and make this substitution in equation \( 41 \) [Equation 1-44] which results in the following equation for the driver pulley:

\[
F_x = \frac{T_1 - T_2}{2k \tan 2\alpha} \tag{42}
\]

\[
F_a = \frac{T_1 - T_2}{2k \tan \alpha} \tag{1-45}
\]

It should be noted that this equation does not allow for variation in the coefficient of friction independently of groove angle and therefore it should be viewed with suspicion.

First of all, the statement by Morgan that the radial component of friction cannot act inward is entirely erroneous, since the normal force on the side of the belt has a component in the outward radial direction. This outward component might be in equilibrium with an inward frictional force and the inward components of tension.

Secondly, it is true that the component of frictional force acting inward radially does reach a limit as to its magnitude when it becomes equal to the outward component of the normal force. Under this condition, the value of \( C_f \sin \theta \) must be \(-\tan \alpha/2\), where the negative sign is necessary since in the derivation of Equation 1-44, forces in the outward radial direction have been taken as positive. It is interesting to consider that this might have been the reason for the "arbitrary" assumption that \( C_f \sin \theta = \pm \tan \alpha/2 \) with the plus sign being an error. If \(-\tan \alpha/2\) is substituted for \( C_f \sin \theta \) in Equation 1-44, the
denominator becomes zero, but since the limiting value of the inward radial friction force can occur only when the belt tension is zero, the numerator is also zero and the equation becomes meaningless. If the reason advanced above for the choice of $C_f \sin \varphi = \tan \alpha/2$ is incorrect, then the choice is indeed a very arbitrary one and has no theoretical basis.

Comparison of measured axial forces with those computed by various equations. As mentioned previously, the final evaluation of a spring force equation can only be made by comparing numbers computed from the equation with measured data. In Table 1-2 the results of such a comparison are given. Unfortunately, the only spring force data presently available for use in this publication are those taken on the driven

<table>
<thead>
<tr>
<th>No.</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_1-T_2$</th>
<th>$F_{sp}$</th>
<th>$F_{fk}$</th>
<th>$F_a$</th>
<th>$F_a$</th>
<th>$F_a$</th>
<th>$F_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Eq. 1-42</td>
<td>Eq. 1-43</td>
<td>Eq. 1-44</td>
<td>Analog</td>
</tr>
<tr>
<td>1</td>
<td>360</td>
<td>100</td>
<td>260</td>
<td>346</td>
<td>122</td>
<td>468</td>
<td>1062</td>
<td>473</td>
<td>504</td>
</tr>
<tr>
<td>2</td>
<td>235</td>
<td>150</td>
<td>85</td>
<td>346</td>
<td>40</td>
<td>386</td>
<td>889</td>
<td>396</td>
<td>399</td>
</tr>
<tr>
<td>3</td>
<td>450</td>
<td>150</td>
<td>300</td>
<td>468</td>
<td>141</td>
<td>609</td>
<td>1387</td>
<td>617</td>
<td>645</td>
</tr>
<tr>
<td>4</td>
<td>295</td>
<td>210</td>
<td>85</td>
<td>468</td>
<td>40</td>
<td>508</td>
<td>1167</td>
<td>519</td>
<td>525</td>
</tr>
<tr>
<td>5</td>
<td>510</td>
<td>150</td>
<td>360</td>
<td>518</td>
<td>169</td>
<td>687</td>
<td>1525</td>
<td>678</td>
<td>720</td>
</tr>
<tr>
<td>6</td>
<td>315</td>
<td>230</td>
<td>85</td>
<td>518</td>
<td>40</td>
<td>558</td>
<td>1260</td>
<td>560</td>
<td>568</td>
</tr>
</tbody>
</table>
pulley and presented previously in Figure 1-17. This means that direct comparison of the effectiveness of the various methods of obtaining spring force for the driver pulley will not be possible.

The first four columns of Table 1-2 give the test identification and tension conditions. The fifth and sixth columns represent, respectively, measured values of spring force as given in Figure 1-17 and values of the key force as obtained from Equation 1-39. The significance of the key force has been discussed earlier along with the derivation of Equation 1-39, and for reasons made clear in that discussion, this force has been added to the measured spring forces. The result is a computed axial force on the belt and is given in column seven of Table 1-2. The remaining columns show values of the axial force on the belt as computed by four different methods.

According to these results, it appears that three of the four computational methods yield very satisfactory results. Only the values computed from Equation 1-42 are completely out of the range of usefulness. Equations 1-43 and 1-44 and the analog charts all give values of belt axial force which are as close to the measured values as could be expected for a system involving such a complex interaction of frictional forces.

**Conclusions.** While Equations 1-43 and 1-44 are very much simpler to apply than are the analog charts, it should be remembered that, according to the previous discussion, neither of these equations has a really sound theoretical basis. As such they become subject to the same limitations as are imposed on strictly empirical relationships. One can never be certain when such a relationship might fail to show
the effect of extreme values of the variables. For example, neither of these equations takes into account the physical properties of the belt or the pulley radius. This may be perfectly safe so long as these quantities are within a range characteristic of present-day variable speed V-belt drives, but if important changes are made in belt materials or drive geometry there is no assurance that the equations will continue to give reliable results.

The failure of either Equation 1-4.3 or 1-4.4 to provide accurate values of the spring force for a driver pulley is evidence of their lack of generality. It is to be emphasized, however, that while the analog charts yield excellent results for a driven pulley, no experimental evidence is presented here to indicate their validity for the driver pulley. It is believed, nevertheless, that the analog charts are based on sound reasoning and realistic assumptions, and thus should be sufficiently general to apply to the driver pulley.
CHAPTER II

PHYSICAL PROPERTIES

Introduction

In order to define the stress pattern throughout a V-belt which has been deformed in a known manner, a knowledge of the physical properties of all of the materials used in the construction of the belt must be available. However, to obtain very accurate and complete values for the properties of all belt components under various conditions of temperature, aging, strain rate, etc., would be a great task, and such an extensive testing program was not conducted throughout the present V-belt study. It is further somewhat fruitless to spend a great amount of time testing a particular component, since the compounds or materials used in the construction of the components are constantly and rapidly being changed. The following is a presentation of those data which were deemed essential to the investigation.

The group of tests involves a rather intensive study of the tensile properties of the components of a belt known commonly throughout the industry as a B-section belt. This belt was chosen as typical, and was examined carefully for two primary purposes, (a) to evaluate generally the approximate importance of each component as far as load-carrying capacity is concerned, and (b) to check the accuracy of the physical properties data which are available from the belt manufacturer. Later tests were conducted on other belts to provide the specific data required to complete or to check various phases of the analysis, but these
were conducted in the same general manner as the tests described below.

Tests were conducted on whole belts and also on belt components. The three primary components tested were (1) the solid rubber body, (2) the rayon tension-carrying cords, and (3) the rubber-impregnated cotton-fabric cover.

In the whole belt, the physical properties of one component are not independent of those of the other components. The cotton cover, for instance, is wrapped on the bias, so that the compressibility of the material inside the cover is important in determining the tensile properties of the cover in the direction along the length of the belt. For this reason, the physical properties of each of the belt components were obtained by using specially constructed belts in which one or more of the component parts were omitted. The effect of a given component, then, could be determined by comparing the properties of a belt with that component included to one in which the component had been omitted.

Four types of special belts were constructed and tested: (1) all rubber, (2) rubber and cords, (3) rubber and wrap, and (4) complete belts with rubber, cords, and wrap.

On the specially constructed belts, the physical properties tested were the tensile and compressive moduli in the direction of the belt length and the compressive modulus in the direction of the belt width. Additional tests were performed on single tensile cords to establish their tensile modulus. The effect of cyclic stressing on single cords and on belts containing cords was investigated in an attempt to obtain realistic values for the tensile moduli. The tensile data were taken
only at values of strain up to 6 percent, the maximum strain encountered in normal V-belt applications.

All of the properties studied were measured under static conditions, while in an actual V-belt application all of the loads are constantly changing. It is believed, however, that these static data will at least give the approximate proportional effect of each of the belt components in a dynamically loaded system.

**Apparatus**

All of the test work was performed on standard Olsen tension-testing machines modified to hold belt and cord specimens. Belt specimens were mounted over 5.1 inch pitch diameter B pulleys, and were loaded in tension by increasing the pulley center distance.

Single cords were held at the ends by placing several wraps of the cord on a small circular cylinder. In each case, the gage length used in determining elongations was ten inches in length, and was located approximately midway between the ends of the specimen. The loads were measured either by the beam balance on the testing machine or by a spring scale. The weighing devices were calibrated to insure accurate results. Compression specimens were loaded directly on a platform scales, with deformations being measured by a dial indicator.

The elongation of a tensile specimen was measured by the use of two machinist's scales and two small microscopes. The scales were attached to the specimen at points ten inches apart, and the change in location of each scale due to the application of a given load was read to the nearest 0.002 inch with the aid of a microscope. The difference be-
between the changes in location of the two scales then gave the elongation experienced by the gage length due to the applied load.

Results

Effect of cyclic loading on belt component moduli. Since a V-belt is subjected to cyclic loading throughout its life, it is necessary to include the effect of this type of loading in any measurement of the belt’s physical properties. To determine this effect, data were taken of the belt components before and after cyclic loading. The results showed no measurable difference in the properties of the rubber or cover due to cyclic loading, but indicated a definite effect in the tensile cords.

Figures 2-1, 2-2, and 2-3 are typical examples of data taken on rubber and cord belts, whole belts, and single cords, respectively, with various degrees of preworking. In every case, the slope, and thus the modulus, is seen to increase and become more nearly constant as the number of load cycles is increased. The greatest change takes place during the first few cycles, so that after about five cycles, the modulus holds fairly constant.

As shown by the curves, the effect of cyclic loading is much more pronounced in single cords and whole belts than in belts with no wrap. This apparent discrepancy is evidently due to uncontrolled variables such as manufacturing processes.
Figure 2-1. Effect of cyclic loading on the tensile modulus of a rubber and cord belt.
Figure 2-2. Effect of cyclic loading on the tensile modulus of a whole belt.
Figure 2-3. Effect of cyclic loading on the tensile modulus of a single rayon cord.
Moduli of specially constructed belts. To obtain representative values for the moduli of the specially constructed belts, several belts of each type were tested. Figures 2-4 through 2-7 show the results of these tests. Each figure is a plot of the data taken on a particular type belt, and is composed of three curves.

The middle curve in each case represents the arithmetic mean of all data taken on the belt sample. There is, of course, a certain amount of scatter of the data about this mean. The curves above and below the mean are an indication of this scatter. If it be assumed that the physical properties of all belts of a particular type have a normal distribution and that the sample belts tested were randomly chosen, then the curves on either side of the mean define tolerance limits for the physical properties of all belts of this type. According to statistical analysis, the area between the particular tolerance limits shown will cover the stress-strain curves for 95 percent of all belts of this type 95 percent of the time. The rather large area between the tolerance limits points out the necessity of a large number of tests when scatter is significant.

The true mean for any particular belt is much closer to the mean value curve shown than are the tolerance limits, so the tensile moduli obtained from these data were assumed to be representative and were used in later analytical studies. The moduli for the various belt types are:
Figure 2-4. Stress-strain data for an all rubber belt in tension showing mean and 95% - 95% tolerance limits.
Figure 2-5. Stress-strain data for a rubber and cover belt in tension showing mean and 95% - 95% tolerance limits.
Figure 2-6. Stress-strain data for a rubber and cord belt in tension showing mean and 95% - 95% tolerance limits.
Figure 2-7. Stress-strain data for a whole belt in tension showing mean and 95% - 95% tolerance limits.
<table>
<thead>
<tr>
<th>Belt Type</th>
<th>Tensile Modulus (pounds per belt inch per inch)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solid rubber</td>
<td>242</td>
</tr>
<tr>
<td>Rubber and cover</td>
<td>510</td>
</tr>
<tr>
<td>Rubber and cords</td>
<td>17,670</td>
</tr>
<tr>
<td>Whole belt</td>
<td>13,530</td>
</tr>
</tbody>
</table>

**Single cord modulus.** Figure 2-8 shows a plot of data obtained from three different samples of a typical rayon cord. The three sets of points obtained from these data fit a single curve, indicating that any scatter in the data was probably due to experimental error. The tensile modulus indicated by these data is 2385 pounds per cord/inch per inch. This value agrees very well with the tensile modulus obtained by the manufacturer for this same cord. Throughout the remainder of this study, therefore, the cord tensile moduli supplied by the manufacturer have been used wherever needed.

**Compressive moduli.** The compressive moduli of most interest in the analysis of V-belt phenomena are those of the tension-carrying cords in the direction of the belt length and of the whole belt in the direction of the belt width. To obtain order-of-magnitude values for these moduli, three tests were performed, (1) on solid rubber in lengthwise compression, (2) on rubber and tension-carrying cords in lengthwise compression, and (3) on a whole belt in crosswise compression.

The results of these tests are shown in Figures 2-9, 2-10, and 2-11. The approximate compressive moduli obtained from these data are 192.5 pounds per belt for solid rubber and 2260 pounds per belt for
Figure 2-8. Tensile data on single rayon cords in tension showing mean.
Figure 2-9. Stress-strain data for an all rubber belt in lengthwise compression (1 test).
Figure 2-10. Stress-strain data for a rubber and cord belt in lengthwise compression (1 test).
Figure 2-11. Stress-strain data on a whole belt in axial compression.
rucker and cords in the direction of the belt length and 750 pounds per inch of length for a whole belt in the direction of its width.

**Discussion of composite results.** In order to compare the load-carrying capacities of the various components, the data presented must be combined in such a way as to single out the effect of each of the three belt components on the physical properties of the whole belt. This can be accomplished quite simply if it is remembered that the modulus of a particular component of a belt in pounds per belt/inch per inch is directly proportional to the cross-sectional area of the component present in the belt. For example, if the cross-sectional area of the rubber present in a whole belt were one-half of the total belt area, then the part of the whole belt modulus which may be attributed to the rubber would be equal to one-half of the modulus for an all rubber belt.

The following sample calculation shows the means by which the data may be used to define the effect of the cords on the whole-belt tensile modulus.

From the data on solid rubber belts, the average tensile modulus for this type belt is

\[ M_{32} = 242 \text{ pounds per belt/inch per inch.} \]

For belts composed of rubber and cords, the average tensile modulus is

\[ M_5 = 17,670 \text{ pounds per belt/inch per inch.} \]

The area of the all rubber belt is 0.21 square inches, while the area of rubber in the rubber and cord belt is 0.1876 square inches. Then the part of the tensile modulus of the rubber and cord belt due to the rubber is
\[ M_{5M}^* = \left( \frac{A_{5M}}{A_{3M}} \right) (M_{3M}) = (0.1876/0.21)(242) \]

= 216 pounds per belt/\text{inch per inch}\\

where \(^*\) indicates a computed value, and the part due to the cords is

\[ M_{5C}^* = M_5 - M_{5M}^* \]

= 17,670 = 216

= 17,454 pounds per belt/\text{inch per inch}\\

From measurements on the belts, the ratio of cord area in a whole belt to that in a rubber and cord belt is

\[ \frac{A_{6C}}{A_{5C}} = 0.01632/0.0224 = 0.728 \]

The part of the whole belt tensile modulus due to the cords is then computed to be

\[ M_{6C}^* = M_{5C}^* \left( \frac{A_{6C}}{A_{5C}} \right) = 17,454 \times 0.728 \]

= 12,710 pounds per belt/\text{inch per inch}\\

In the same manner, the average tensile modulus of the cover or wrap of a whole belt can be shown from the data to be

\[ M_{6W}^* = 360 \text{ pounds per belt/\text{inch per inch}} \]

For the rubber in a whole belt the tensile modulus indicated by the data is

\[ M_{6R}^* = 130 \text{ pounds per belt/\text{inch per inch}} \]

If the tensile moduli computed for each of the three components or a belt are added together, the result should be equal to the tensile modulus of a whole belt, and should agree with whole belt data. The sum of the three moduli are

\[ M_6^* = M_{6M}^* + M_{6C}^* + M_{6W}^* \]

= 13,200 pounds per belt/\text{inch per inch}
where $M^*_b$ is the predicted modulus of a whole belt. From the data taken on whole belts, the actual average modulus is

$$M^*_c = 13,530 \text{ pounds per belt/inch per inch}$$

Figure 2-12 is a graphical presentation of the moduli mentioned above, and shows the relative importance of the various components in determining the tensile modulus of a belt. The predicted modulus is 2.5 percent lower than the actual modulus which is well within the accuracy range of the data.

Using the same method as was used for determining the tensile moduli of the belt components, the compressive modulus of the tension-carrying cords may be obtained from the data given in Figures 2-9 and 2-10. The resulting compressive modulus for the cords is 261 pounds per cord. The crosswise modulus of the whole belt has been given previously as 750 pounds per inch of length.

Moduli based on cross-sectional area. Many of the moduli thus far discussed have been on the basis of pounds per belt/inch per inch. In order to allow these results to be used in determining the moduli of components of belts other than B-section single ply belts, the moduli must be converted to the units of pounds per square inch/inch per inch. To do this, it is necessary to divide each of the moduli by the cross-sectional area of the component to which it applies. For the rubber and the cover components these areas can be measured with sufficient accuracy, and were found to be:

$$A^*_M = 0.1137 \text{ square inch}$$

and

$$A^*_W = 0.080 \text{ square inch}.$$
Figure 2-12. Graphical representation of tensile moduli data showing the relative importance of belt components on tensile modulus of a whole belt.
The resulting moduli are

\[ E_M = 1,147 \text{ pounds per square inch/inch per inch} \]

and

\[ E_W = 4,500 \text{ pounds per square inch/inch per inch} \]

where \( E \) has been used to indicate the modulus in terms of pounds per square inch/inch per inch.

For the cords in a belt, it is impossible to define an area on which a modulus may be based such that the modulus applies to cords of various sizes and constructions. The moduli for the cords will therefore be defined only in terms of pounds per cord/inch per inch.

**Conclusions**

Based on data of the type presented here, it can be concluded that the only component of a V-belt which is effective in supporting static tensile load is the tension member, or the bundle of cords. The static tensile modulus of the cords in a whole belt in terms of pounds per belt/inch per inch is approximately 95 percent of the modulus of the entire belt. This means that about 95 percent of any static tensile load applied to a V-belt will be carried by these cords. It is reasonable to assume that approximately the same situation would exist if the load were applied dynamically.

The static tensile moduli of the belt components tested were found to be 1,147 psi for rubber, 4,500 psi for the cover, and 2,385 pounds per cord for a typical rayon cord. The compressive modulus of the same cord was computed to be approximately 261 pounds per cord.
CHAPTER III

STRESS ANALYSIS

Introduction

At the beginning of this investigation attempts were made to apply the methods of the theory of elasticity to the analysis of the stresses and strains present in a V-belt under load. It soon became apparent, however, that the complex structure of the belt and the poorly defined boundary conditions complicated this analysis to such an extent that its further pursuit was considered impractical. As an alternative, simplifying assumptions were made on both the belt and the boundary conditions such that a routine strength-of-materials approach to the analysis could be used. The stresses thus obtained were then modified by factors which were intended to correct, at least in part, for the failure of the belt and the boundary conditions to comply with the simplifying assumptions.

As has been pointed out in other V-belt analyses, the V-belt on a drive is subjected to three tensile stresses: (1) a direct pull due to the forces exerted on the belt by the pulley, (2) a pull due to the action of centrifugal force, and (3) a tensile stress in the outer

---


fibers of the belt where it is bent over the pulleys. Compressive stresses are present in the lower fibers of the belt due to bending and also due to the wedging action of the belt in the pulley groove. Since the pulley exerts tangential frictional forces on the belt only at its surface, shear stresses must be present to transmit these forces to the core of the belt. Although these and other stresses combine to form the true stress pattern in the belt, the present analysis considers only the tensile stresses due to the three effects mentioned above. Obviously, such an analysis leaves much to be desired from the academic point of view but from the standpoint of practicality it proves to be very satisfactory. The resulting expressions for the three stress components are simple and easily computed, and yet provide a fairly accurate design criterion as is shown in Chapter 4. Much of the success of such a simplified analysis could be due to the fact that many of the stresses which have not been considered are proportional in some way to the computed tensile stress. For example, an increase in the direct pull exerted on the belt by the pulleys would be reflected by a proportionate increase not only in the tensile stress, but also in the compressive stress due to the wedging in the pulley groove. While the increase in compressive stress is not computed, the change in the value of the tensile stress may account for the effect of that increase. Future research should be directed toward the refinement of this analysis by the inclusion of the effects of compressive and shear stresses.

The critical point in the belt has been assumed to be at the outer surface of the bundle of tension-carrying cords which form the core of the V-belt. The tensile stress at this point is therefore con-
sidered to be the criterion which governs the life of the belt. The choice of this critical point is suggested by the tensile data given in Chapter 2. These data show the elastic modulus of the cords to account for about 95 percent of the elastic modulus of the whole belt, which means that the cords carry 95 percent of any tensile pull applied to the belt. Since, by experience, the life of a V-belt has been observed to be highly dependent on the magnitude of the tensile pull, it would appear that the maximum tensile stress in the cords must play an important part in determining the life expectancy of a belt. The maximum tensile stress occurs in the outer fibers of the cords where the stresses due to pulley forces and centrifugal force combine with the maximum tensile stress due to bending.

**Bending Stress**

In the unstressed state, the V-belt is assumed to be a straight composite beam composed of three layers of homogeneous, isotropic, elastic materials. The cross-section of the belt will be assumed to consist of three areas, $A_1$, $A_2$, and $A_3$, as shown in Figure 3-1, each area having its individual modulus of elasticity and tensile strength. Area $A_1$ represents the area occupied by the compression rubber; $A_2$ the area containing the tension-carrying cords; and $A_3$, the area of the tension rubber. The cover, or wrap, of the belt has been treated with the rubber as a composite having a single representative modulus. The moduli of elasticity corresponding to the three areas will be designated as $E_1$, $E_2$, and $E_3$.

Figure 3-2 shows the belt bent over a pulley such that its inner
radius is \( R_1 \). The surfaces bounding the areas \( A_1 \), \( A_2 \), and \( A_3 \) are at distances \( r_1 \), \( r_2 \), \( r_3 \), and \( r_4 \) from the center of the pulley. The radii of the centroids of the three areas are \( \bar{r}_1 \), \( \bar{r}_2 \), and \( \bar{r}_3 \). Since the radius of curvature of the belt is very nearly constant over the pulley, the

![Diagram](image)

Figure 3-1. Side view and cross-sectional view of V-belt before bending.

belt will be assumed to be bent by a pure moment. Due to bending, the outer fibers of the belt will be elongated, while those nearest the center of the pulley will be shortened. At some radius inside the belt, the fibers will suffer no strain as the belt is bent. This radius locates the neutral surface of the belt, and will be called \( r_0 \). Any general radius in the belt will be denoted by \( r \).

In Figure 3-1, consider the rectangle, \( mnop \), in the belt before bending. After the belt is bent, it will be assumed that the lines \( mp \)}
Figure 3-2. Side view and cross-sectional view of V-belt after bending.
and no remain straight, and pass through the center of the pulley. In Figure 3-2, let m′p′ and n′o′ be the new positions of these lines, and let ϑ be the angle between them.

The length of any surface bounded by m′p′ and n′o′ at a radius r in the bent belt is ϑr. Before bending, the length of that surface was mn. Thus, the unit elongation, due to bending, of a surface at radius r is given by

\[ e = (ϑr - mn)/mn \]  

(3-1)

At the radius r_0, no elongation took place, so

\[ mn = ϑ r_0 \]  

(3-2)

or

\[ e = (ϑr - ϑr_0)/ϑr_0 = (r/r_0) - 1 \]  

(3-3)

The stress at r is then

\[ S = Ee = E \left[ (r/r_0) - 1 \right] \]  

(3-4)

and the elemental tangential force on an element of area dA at radius r is

\[ dF = S dA = E \left[ (r/r_0) - 1 \right] dA \]  

(3-5)

If the initial force acting on the surface no is assumed to be zero and only a pure moment is considered, then the total force acting on n′o′ must also be zero, or

\[ \int_{r_1}^{r_4} dF = \int_{r_1}^{r_4} E \left[ (r/r_0) - 1 \right] dA = 0 \]  

(3-6)

Since the modulus of elasticity E has been assumed constant throughout each of the areas A_1, A_2, and A_3, the integral of Equation 3-6 may be broken up into three parts,
\[
\int_{r_1}^{r_4} E \left[ \frac{r}{r_o} - 1 \right] dA = E_1 \int_{r_1}^{r_2} \left[ \frac{r}{r_o} - 1 \right] dA + \\
E_2 \int_{r_2}^{r_3} \left[ \frac{r}{r_o} - 1 \right] dA + E_3 \int_{r_3}^{r_4} \left[ \frac{r}{r_o} - 1 \right] dA 
\]

(3-7)

But

\[
E_1 \int_{r_1}^{r_2} \left[ \frac{r}{r_o} - 1 \right] dA = \frac{E_1}{r_o} \int_{r_1}^{r_2} r dA = E_1 \int_{r_1}^{r_2} dA
\]

\[
= E_1 r_1^2 / r_o = E_1 A_1
\]

(3-8)

where \( r_1 \) is the radius at the centroid of area \( A_1 \).

Using similar expressions for the two remaining terms on the right-hand side of Equation 3-7 this equation may now be written as

\[
0 = (E_1 A_1 r_1 / r_o) - (E_1 A_1) + (E_2 A_2 r_2 / r_o) - (E_2 A_2) + \\
(E_3 A_3 r_3 / r_o) - (E_3 A_3)
\]

(3-9)

or

\[
r_o = \frac{E_1 A_1 r_1 + E_2 A_2 r_2 + E_3 A_3 r_3}{E_1 A_1 + E_2 A_2 + E_3 A_3}
\]

(3-10)

To evaluate \( r_o \) for any given belt and pulley combination, certain properties and dimensions of the belt must be known. The centroids of the various cross-sectional areas may be obtained readily from measurements of the belt. For a trapezoid of height \( s_x \) and parallel sides \( a \) and \( b \), the distance from side \( a \) to the centroid is:

\[
\bar{r}_a = \frac{s(a + 2b)}{3(a + b)}
\]

(3-11)
This allows the location of the centroid of each of the areas with respect to the inner surface of the belt. If an assumption is now made as to the level of the inner surface of the belt with respect to the pitch radius to the pulley, then the radii $\bar{r}_1$, $\bar{r}_2$, and $\bar{r}_3$ may be obtained by simple addition.

The quantities $E_1A_1$ and $E_3A_3$ may be evaluated from area and modulus measurements on the belt. The product $E_2A_2$ represents the force required to cause an elongation of one inch per inch in the neutral axis section of a belt. From cord tensile data, the force required to give a single cord a unit elongation of one inch per inch may be obtained, which when multiplied by the number of cords in the belt gives the required value of $E_2A_2$.

The value of $r_0$ from Equation 3-10 may now be used in Equation 3-3 to define the unit elongation $e$ at any radius $r$.

The critical stress has been assumed to be the maximum tensile stress in the tension-carrying cords of the belt. The maximum tensile stress in these cords caused by bending will occur at the radius $r_3$. Here the elongation is

$$e_{r_3} = \frac{r_3}{r_0} = 1$$  \hspace{1cm} (3-12)

The stress in pounds per cord due to this elongation is

$$G_b = Me_{r_3}$$  \hspace{1cm} (3-13)

where $M$ is the tensile modulus of the cord in question in pounds per cord per inch/inch as obtained from single cord tensile data.
Horsepower Stress

The V-belt under load is subjected to two direct tensile loads, one due to centrifugal force and one due to forces applied to the belt by the pulleys. Since only those forces exerted on the belt by the pulleys are capable of transmitting a load, the stress due to these forces will be called the horsepower stress.

In Chapter 1, methods were developed by which the forces that cause the horsepower stress can be evaluated. These forces are called $T_1$ and $T_2$ for the tight and slack sides of the belt, respectively. The pulley exerts frictional forces on the belt only at its surface, so the tension in the cords caused by these forces would be expected to be highest near the interface between the belt and pulley and to fall off to some lower value at the center of the belt. For the purposes of this analysis the horsepower stress in the cords is computed by assuming that each cord carries an equal share of the load and then applying a correction factor to the result to account for the non-uniform distribution of load. The horsepower stress in pounds per cord is therefore given by

$$G_{hp} = k_{hp} \frac{T}{q} \quad (3-14)$$

where $k_{hp}$ is some constant, $q$ is the number of cords in the belt and $T$ is one of the horsepower-transmitting tensions, $T_1$ or $T_2$.

Centrifugal Stress

In Appendix B, the tension throughout the belt due to centrifugal force $T_c$ is shown to be, from Equation B-9,

$$T_c = \rho V^2/g \quad (3-15)$$
where \( \rho \) is the weight of the belt in pounds per unit length, \( V \) is the belt velocity, and \( g \) is the gravitational constant. Since the weight of the belt is evenly distributed across its width, it appears reasonable that the centrifugal tension should be uniformly distributed among the tension-carrying cords. The stress due to this tension is then

\[
G_c = \frac{\rho V^2}{qg}
\]  

(3-16)

where \( q \) is the number of cords in the belt.

**Combined Stresses**

In metals, tensile stresses due to different types of loading can be added directly to obtain a total effective stress. It would be very optimistic, however, to expect that the stresses due to various loads would be independent of each other in such a complex structure as a twisted cord. The effect of coupling between the stresses must therefore be considered in obtaining a single stress due to combined loads.

Since the horsepower stress and the centrifugal stress result from exactly the same type of loading, it would appear that these two stresses could be added directly to obtain their combined effect. However, from the following discussion, it is reasonable to assume that the bending stress in the cords is not independent of the combined effect of the other two stresses.

Consider the bent cord shown schematically in Figure 3-3. It will be noted that due to the twist of the cord a single fiber may be located at the top of the cord at one point such as \( A \), while at point \( B \), a short distance from \( A \) the same fiber is located at the bottom of the
Figure 3-3. Twisted fibers under bending.

cord. Due to bending the fiber is therefore in tension at A and in compression at B. Now if the fiber were made free to move along its length, it would slip from B toward A in such a manner as to completely relieve the tensile and compressive stresses. This results in bending stresses in each fiber due only to the fiber being bent around its own axis. These stresses are small, so that only a small moment is required to cause the cord to bend. This situation exists when a piece of cotton wrapping cord is bent between the fingers, in which case the cord offers very little resistance to bending.

Now let the cord shown in Figure 3-3 be acted upon by a tension T during bending. Because of the helical shape of the cord fibers, the tension causes the fibers to press against each other. A frictional force can therefore act along the surface of a fiber and allow a change in stress along its length. For high values of tension the action of
the cord under bending might be expected to approach that assumed in the analysis of bending given earlier in this chapter, but for low tensions, stresses obtained from such an analysis would almost certainly be too high. It would appear then that the tensile stress due to bending as given by Equation 3-13 should be modified by some function of the tensile stresses in the belt due to power transmission and centrifugal force. Using Equation 3-13, the modified bending stress $G_b^s$ can be written as

$$G_b^s = f(G_{hp} + G_c)ME_{T3} \quad (3-17)$$

where $f(G_{hp} + G_c)$ indicates some function of the horsepower and centrifugal stresses as given by Equations 3-14 and 3-16, respectively. No attempt has been made to define this function analytically, but the empirical expression discussed in Chapter 4 appears to be quite successful in accounting for the effect of coupling between the two types of tensile stresses. The function would be expected to be dependent on many factors including such things as the surface condition of the fibers and the average angle at which the fibers cross.
CHAPTER IV

FAILURE THEORY

Introduction

An examination of V-belt test data reveals that identical belts run under different load conditions have different average useful lives. This suggests that the belts may fail by the mechanism of fatigue as do metal parts when subjected to alternating stresses. If this assumption is made it would appear then that the allowable load a belt drive can carry satisfactorily must in some way be made a function of the expected life. It is the purpose of this chapter to develop an empirical relationship between belt stresses and belt life and thereby to make possible the definition of a limiting stress for any given desired life.

Definition of Stress Cycle

The critical stress in the belt has already been chosen as the tensile stress in the outer fibers of the tension-carrying cords. The three stresses to be considered, the horsepower stress $G_{hp}$, the centrifugal stress $G_c$ and the modified bending stress $G_b$ are given by Equations 3-11, 3-16, and 3-17 as follows:

\[
G_{hp} = k_{hp} \left( \frac{T}{q} \right) \quad (4-1)
\]

\[
G_c = \frac{\rho v^2}{2ag} \quad (4-2)
\]

and

\[
G_b^3 = f(G_{hp} + G_c)M_{re}^3 \quad (4-3)
\]
The total stress $G$ is given by

$$ G = G_{hp} + G_c + G_b $$

Figure 4-1 shows schematically the manner in which each of these stresses varies in a two-pulley belt drive. The bending stress $G_b$ has been represented as a constant during the passage of the belt over each of the pulleys although the function which modifies the bending term is dependent on the horsepower stress which varies over the pulleys. However, the reason for coupling between the stresses holds only for horsepower and centrifugal stresses present in the cords when bending occurs. Therefore, the variation in tension as the belt passes around the pulley would not be expected to influence the bending stress. To simplify the analysis, the function which modifies the bending stress is assumed to be defined by the tight side tension $T_1$ for both pulleys. This is conservative, since the lower slack side tension $T_2$ would lead to lower values of the modified bending stress on the driven pulley where $T_2$ exists at the entrance.

The total stress is seen to have two peaks, at points B and C where the belt leaves the driven pulley and enters the driver pulley, respectively. The magnitudes of the total stress at these peaks are assumed to be the quantities which define the amount of damage done to the cords in a belt throughout one belt revolution. For a drive of the type depicted in Figure 4-1 in which the pulley diameters are unequal, the stress peaks at B and C are also unequal, so that throughout its life, the belt is subjected to two types of stress cycle, one with a maximum stress equal to the stress at B and one with a maximum stress equal to
Figure 4-1. Variation in belt stress around a two-pulley drive.
that at C. At failure the belt will have experienced an equal number of cycles of each type.

**Cumulative Damage Concept**

To estimate the effect of two different stress cycles on the life of a belt, Miner's theory of cumulative damage\(^1\) will be employed. Miner's theory hypothesizes that the amount of damage done to a part (or the fraction of its life used up) by the application of a stress cycle of amplitude \(S\) is equal to \(1/N\), where \(N\) is the number of these stress cycles which would cause the part to fail. If \(n\) cycles with stress amplitude \(S\) are applied, then the part of the life used up would be \(n/N\) which becomes 1 at failure. Miner's theory further assumes that if \(n_1\) cycles of stress with amplitude \(S_1\), \(n_2\) cycles of stress with amplitude \(S_2\), and, in general, \(n_i\) cycles of stress with amplitude \(S_i\) are applied to a part, then at failure

\[
\sum_i n_i/N_i = 1 \tag{4-5}
\]

where \(N_1\) would be the life in cycles at stress \(S_1\), \(N_2\) the life in cycles at stress \(S_2\), etc.

For the present case of a two-pulley drive Equation 4-5 becomes

\[
(n_1/N_1) + (n_2/N_2) = 1 \tag{4-6}
\]

But the number of cycles applied at each stress is the same, so \(n_1 = n_2 = n\), the number of belt revolutions at failure, and Equation 4-6 can be written

\[
n/N_1 + n/N_2 = 1 \tag{4-7}
\]

---

or
\[ \frac{1}{n} = \frac{1}{N_1} + \frac{1}{N_2} \] (4-8)

**Stress Factor**

The stresses represented by Equations 4-1 through 4-4 and plotted in Figure 4-1 have the units pounds per cord, and therefore do not necessarily indicate the degree to which a cord is loaded. For instance, for a small cord, a stress of 20 pounds per cord might cause an early failure, while the same stress applied to a larger cord could give long life. To place all cords on a similar basis, the stresses given by Equations 4-1 through 4-4 have been divided by a constant times the ultimate strength of the cord under question. The resulting ratio is called a stress factor and is denoted by \( Q \). Equations 4-1 through 4-4 now become

\[ Q_{hp} = \frac{k_{hp} T}{q_{k_e} G_u} \] (4-9)

\[ Q_c = \frac{\rho y^2}{g q_{k_e} G_u} \] (4-10)

\[ Q_b = f(G_{hp} + G_c) \frac{M}{k_e G_u} \] (4-11)

and

\[ Q = Q_{hp} + Q_c + Q_b \] (4-12)

where \( G_u \) denotes the ultimate strength of the cord in pounds per cord and \( k_e \) is a constant. The ultimate strength is modified by the constant \( k_e \) to account for differences in the relationship between ultimate strength and fatigue resistance from one cord to another. This constant would therefore be expected to be the same for all cords of the same material.
Life Test Data

The stress factor $Q$ as defined by Equation 4-12 contains three quantities which must be determined before a number can be computed to represent $Q$ for a given belt on a given drive. These quantities are $k_{hp}$, $k_e$, and $f(G_{hp} + G_o)$, and at the present time can be obtained only by the examination of life-test data. A large amount of this type of data has been investigated and the unknown quantities mentioned above have been defined for many different rayon and dacron cords. For proprietary reasons it is impossible to present the actual values of the empirically determined quantities required for the definition of the stress factor $Q$. However, Figure 4-2 shows a large number of test results on a plot of the stress factor $Q$ versus the logarithm of the cycles to failure $\log N$. As shown by the envelope, all of the points can be closely approximated by a straight line, the equation of which is

$$Q = p \log_{10} N + b \quad (4-13)$$

The data upon which the plot of Figure 4-2 is based were obtained on test drives in which two equal diameter pulleys were used. For this case $N_1 = N_2 = N$ in Equation 4-8, so that

$$\frac{1}{n} = \frac{2}{N} \quad (4-14)$$

or

$$N = 2n \quad (4-15)$$

which states simply that the belt is subjected to two equal stress peaks throughout each belt revolution. After the straight line in Figure 4-2 has been established, Equation 4-8 can be used to estimate the life of a belt which is to be run on two pulleys of different diameters. For example, suppose the stress factors obtained from Equation 4-12 are
Figure 4-2. Stress-life curve for V-belts.
Q₁ for one pulley and Q₂ for the other. The cycles to cause failure corresponding to each of these stress factors can be obtained from Equation 4-13 or directly from Figure 4-2, and are denoted by N₁ and N₂. The numbers can then be used in Equation 4-8 to obtain the approximate expected life of the belt in terms of belt revolutions which can be readily converted to a life in hours. If a certain belt life is desired on a drive and the life as computed above is below this number, then either more than one belt may be used to give a lower load per belt, or the horsepower carried by the drive must be reduced.
APPENDIX A

ANALYSIS OF FORCES AND MOTIONS INVOLVED IN THE
PASSAGE OF A V-BELT OVER A V-PULLEY

General Remarks

Zones 1 and 2. Before attempting an analysis of the forces and motions involved in a V-belt running in a V-shaped groove, a qualitative examination will be made of the overall force and motion picture in a V-belt drive. Figure A-1 shows such a drive divided into eight zones, each zone representing an area throughout which a particular type of loading is imposed on the belt. In zones 1 and 2 no forces are applied to the lateral surfaces of the belt, but the ends of the belt segments located in these zones are loaded in tension and shear accompanied by a pure moment. The response of the belt to these external forces is analyzed in Appendix C.

Zones 3 and 4. Throughout zones 3 and 4 the belt is firmly seated in the pulley groove, and is bent to a nearly constant radius of curvature. The belt segments in these zones are therefore subjected to an essentially constant moment and a nearly zero radial shear stress. The shape of the belt is entirely determined by its position in the pulley groove, and is not influenced by the bending modulus. In these zones, then, the belt behaves as though it were segmented like a chain and could neither support shear nor resist bending.

The level at which the belt rides in the groove is governed by
Figure A-1. Belt drive divided into zones.
the establishment of equilibrium between the normal and frictional forces on the belt surface and the internal forces in the belt, i.e., compression in the axial direction and tension in the circumferential direction. The magnitude of the normal force on the belt surface multiplied by the appropriate coefficient of friction determines the maximum possible magnitude of the frictional force acting on the belt. If the belt is not in motion with respect to the pulley, then the magnitude of the frictional force may have any positive value less than the maximum, but if there is relative motion between the belt and pulley, then the maximum force will be applied to the belt. In the latter case, the direction of the frictional force on the belt will always be in the direction opposite to the direction of the belt velocity relative to the pulley.

The belt may or may not be in motion with respect to the pulley at the entrance to zone 3 or 4, but changes in the velocity of the belt relative to the pulley groove would be expected to occur as the result of a tension change between two points in zone 3 or 4. Due to a tension change in a belt segment, the length of the segment must change, and since the length of pulley groove in contact with the segment does not change, some change in the tangential component of the relative velocity must occur. Also, the tension change causes a change in the radial force tending to push the belt downward in the groove, so that a change in the radial position of the belt in the groove is required to maintain equilibrium between the radial forces. The velocity of the belt with respect to the pulley may thus be expected to be in a direction other than radial or tangential.
Since the radius of curvature of the belt is approximately constant in either zone 3 or 4, the tangential strain in either zone due to bending at any given radial position may be assumed to be constant throughout the zone. This means that changes in tangential strain from one circumferential position to another are due only to tension changes, and might be expected to be the same for all radial locations. The tangential component of the belt velocity relative to the pulley can thus be assumed to be constant along the radial dimension of the belt at any given circumferential position. The radial component of relative velocity may also be assumed constant along the radial dimension, which leads to the assumption that the velocity of the belt relative to the pulley is independent of radial position. The entire problem can then be greatly simplified by considering the normal and frictional forces to act only along a circumferential line at some average radius.

Zones 5, 6, 7, and 8. In zones 5, 6, 7, and 8 the belt is in the process of either entering or leaving the pulley groove. For example, at point a the belt has just touched the surfaces of the pulley groove. At this point no normal force has been developed between the belt and pulley, while at point b the belt is completely seated in the groove. The segment of belt located instantaneously between the points a and b therefore represents a transition zone between zones 2 and 3. The radial shear present in the belt at a is reduced to zero at b as a result of the pulley forces acting on the belt surface. Since the belt is subjected to shear between a and b, the moment in the belt must increase from a to b, with a resultant decrease in the radius of curvature. The shear stress in the belt increases as one approaches a from the left,
but the shear is essentially zero at \( b \), so that a maximum shear must exist between \( a \) and \( b \). At this same point the rate of change of the radius of curvature reaches a maximum and then decreases to zero at \( b \) as the radius of curvature becomes nearly constant.

As was the case for zones 3 and 4, the magnitude of the frictional force available to be exerted on the belt by the pulley at any point in zone 5 is the product of the normal force and the coefficient of friction. Since the belt must move downward in the pulley groove in order to build up a normal force between the belt and pulley, it appears that the belt must be in motion relative to the pulley throughout all of zone 5. If this is the case, then all of the available frictional force acts on the belt at every point in this zone.

The direction of the frictional force in zone 5 is defined, as before, by the direction of the belt velocity relative to the pulley. At any given point in zone 5 this relative velocity is the result of an inlet velocity at \( a \) plus any changes in velocity occurring between \( a \) and the point in question. The inlet velocity can be established only by considering zones 5, 3, and 6 together, and requiring that the total tension change in the three zones be from \( T_2 \) to \( T_1 \). Changes in the radial velocity of the belt may be expected to occur as the belt performs the process of seating itself in the groove. The tangential velocity changes which occur in zone 5 are again the result of changes in tangential strain. In this case, however, unlike in zone 3 the change in tangential strain from one circumferential position to another depends on the radial as well as the circumferential location since the radius of curvature of the belt is not constant. This means that the direction
in which the frictional force acts also varies at each point throughout the two-dimensional contact surface located in zone 5. Also, the normal force on the belt may be expected to vary with radial position, so that the magnitude of the frictional force must be considered as a variable at every point in zone 5. To define the overall effect of the frictional force on the belt segment located in this zone would thus require the description of the magnitude and direction of the frictional force as a function of the radial and circumferential position and then integration of this force over the contact surface. The major difficulty involved in such an analysis is the lack of any knowledge of the manner in which the normal force on the belt varies along its radial depth. All of the forces acting on the belt surface are involved in the variation of the radius of curvature which in turn helps to define the velocity distribution over the contact surface. The extreme complexity of the force and motion situation and the lack of any information on which to base simplifying assumptions discouraged any attempt to analyze the force and motion variations in zones 5, 6, 7 and 8.

At first glance, it would appear that zone 5 and the others like it could be neglected without any serious error in the analysis of the belt drive. These zones may be looked upon as buffer areas between zones such as 2 and 3 in which one general type of loading changes gradually to another type. From observations on belt drives it appears that the belt becomes seated in the groove within a very short circumferential distance after contact has occurred, so that the forces exerted on the belt in the zones in question could be assumed to be a small percentage of the total of all forces on the belt. From a force standpoint, therefore, zone 5 could be neglected.
The difficulty arising from ignoring zone 5 involves the definition of boundary conditions for zones 2 and 3. The unknown radius of curvature at a should be used as the boundary condition for the analysis of zone 2 rather than the known radius at b. Similarly, the direction of the velocity of the belt relative to the pulley is defined at point a by the shape of the belt at that point and the location of the contact point on the pulley, but the direction of the relative velocity at point b is the quantity needed as a boundary condition for the analysis of zone 3. It would appear that a rigorous analysis of the forces and motions would require the complete solution of the equations of motion for all of the zones simultaneously, each providing boundary conditions for the other.

Without the analysis for zone 5, it becomes necessary to find boundary conditions for zones 2 and 3 by other methods. In the investigation of zone 2, in Appendix C, the assumption is made that the radius of curvature at a is equal to that at b, and the close agreement between the resulting analysis and experimental data suggests that the assumption is sufficiently accurate. For the case of zone 3 the assumptions leading to usable boundary conditions will be discussed in the following.

Mathematical Model for Zone 3 or 4

Attention will now be centered on a segment of belt located in a zone such as 3 or 4 in which the tension may change, but the change in radius of curvature is negligible. According to the previous discussion, the belt can be considered as a thin ribbon located at some average radius in the pulley groove. This thin ribbon will be assumed to
have all of the physical properties such as elastic moduli and a coefficient of friction which would be characteristic of the actual belt.

Figure A-2 shows the belt and pulley under examination. The pulley turns clockwise at an angular velocity $\omega_p$. The exit tension $T_e$ in the case shown is higher than the inlet tension $T_i$, resulting in a belt radius $r$ at any angle $\theta$ equal to or smaller than the inlet radius $r_i$. This change in the radial position is due to the belt wedging deeper in the groove under increased tension, and has been greatly exaggerated in Figure A-2. The angle $\theta$ is measured in space from a fixed radial line through the point corresponding to point b in Figure A-1.

In Figure A-3, the area surrounding the angular position $\theta$ in Figure A-2 has been enlarged, and the coordinates $u$ and $v$ are shown. These coordinates measure the relative motion which occurred between the belt and the pulley while the point c on the pulley moved from $\theta = 0$ to the present position. The point e is on the belt, and is that point which was in contact with point c at the position $\theta = 0$. The points c and d are both on the pulley, and are separated by the incremental angle $d\theta$, where $d\theta$ is defined by

$$d\theta = \omega_p \, dt$$  \hspace{1cm} (A-1)

where $\omega_p$ is the angular velocity of the pulley and $dt$ is an increment of time. The corresponding increment of belt length can be defined as that quantity of belt passing the origin $\theta = 0$ during the time interval $dt$. Using $dt$ from Equation A-1, the length of the belt increment at the inlet is given by

$$ds_i = V_{Bi}(d\theta/\omega_p)$$  \hspace{1cm} (A-2)

where $V_{Bi}$ is the absolute velocity of the belt at the inlet. The belt
Figure A-2. Belt and pulley system.
Figure A-3. Belt and pulley system showing all coordinates.
velocity can be obtained from

\[ \vec{V}_{Bi} = \vec{V}_{pi} + \vec{V}_{R_{ti}} + \vec{V}_{R_{ri}} \]  \hspace{1cm} (A-3)

where vector quantities have been used, and \( \vec{V}_{pi} \) represents the velocity of the pulley at the inlet, \( \vec{V}_{R_{ti}} \) is the tangential component of the velocity of the belt relative to the pulley at the inlet, and \( \vec{V}_{R_{ri}} \) is the corresponding radial component of that velocity. Since the relative velocities are expected to be very small compared to the absolute velocity, the vector addition of Equation A-3 would appear as in Figure A-4. From

\[ \vec{V}_{pi} \]
\[ \vec{V}_{R_{ti}} \]
\[ \vec{V}_{R_{ri}} \]

Figure A-4. Graphical solution for belt velocity.

this figure it is apparent that the magnitude of the velocity \( \vec{V}_{Bi} \) can be taken with good approximation as

\[ \vec{V}_{Bi} = \vec{V}_{pi} + \vec{V}_{R_{ti}} \]  \hspace{1cm} (A-4)

The inlet pulley velocity is given by

\[ \vec{V}_{pi} = \omega_p \vec{r}_i \]  \hspace{1cm} (A-5)
and the tangential component of the relative velocity at the inlet is

\[ V_{R_{t_i}} = (dv/d\tau)_i \]  \hspace{1cm} (A-6)

But

\[ \frac{dv}{d\tau}_i = \frac{dv}{d\theta} \frac{d\theta}{d\tau}_i \]  \hspace{1cm} (A-7)

or

\[ V_{R_{t_i}} = \omega_p (dv/d\theta)_i \]  \hspace{1cm} (A-8)

so that, with the aid of Equations A-4, A-5, and A-8, Equation A-2 becomes

\[ ds_i = r_i d\theta + (dv/d\theta)_i d\theta \]  \hspace{1cm} (A-9)

**Effect of Tangential Elasticity**

In the case under observation the tension is assumed to be increasing with increasing \( \theta \) so it is reasonable to assume also that the element of belt increases in length with increasing \( \theta \). In Figure A-3 the length of the belt element \( ds \) at the position \( \theta \) is the length of the line ef. From the figure the length \( ds \) is given by

\[ ds = \left[ (r_i - u)/r_i \right] (cd + v + dv - v) \]  \hspace{1cm} (A-10)

But the length \( cd \) is defined as \( cd = r_i d\theta \), so Equation A-10 can be written as

\[ ds = \left[ (r_i - u)/r_i \right] (r_i d\theta + dv) \]  \hspace{1cm} (A-11)

Expanding Equation A-11 gives

\[ ds = r_i d\theta - u d\theta + dv - (u/r_i)dv \]  \hspace{1cm} (A-12)

Subtracting the initial length of the belt element as given by Equation A-9 from the length at \( \theta \) as given by Equation A-12 and dividing by the
initial length gives the change in strain occurring in the belt between 
\( \Theta = 0 \) and \( \Theta = \Theta \) as

\[
\frac{ds \ - \ ds_i}{ds_i} = \frac{dv \ - \ u \ dv \ - \ u \ d\Theta \ - \ \left[ \frac{dy}{d\Theta} \right]_i \ d\Theta}{\r_i \ d\Theta + \left[ \frac{dy}{d\Theta} \right]_i \ d\Theta}
\]

(A-13)

The corresponding change in tension is

\[
\Delta T = M_t \frac{ds \ - \ ds_i}{ds_i}
\]

(A-14)

where \( M_t \) is the modulus of elasticity of the belt in the tangential di-
rection. The tension at \( \Theta \) is

\[
T = T_i + \Delta T
\]

(A-15)

or, using Equations A-13 and A-14 in Equation A-15,

\[
T = T_i + \frac{M_t}{r_i \ d\Theta + \left[ \frac{dy}{d\Theta} \right]_i \ d\Theta} \left[ \frac{dy}{r_i^2} \ dv \ - \ u \ d\Theta \ - \ \left[ \frac{dy}{d\Theta} \right]_i \ d\Theta \right]
\]

(A-16)

Effect of Axial Elasticity

The change in the internal axial force in the belt due to its radial motion in the groove can be evaluated from

\[
\Delta P_a = \Delta e_a M_a
\]

(A-17)

where \( P_a \) is the distributed axial force in the belt in pounds per tan-
gential inch, \( e_a \) is the axial strain, and \( M \) is the axial modulus of the belt in pounds per tangential inch. The change in belt width \( W \) due to an inward radial motion \( u \) can be obtained by referring to Figure A-5.
Figure A-5. Cross-section of belt in pulley grooves.
which shows a cross-sectional view of the pulley. From the figure

\[ \Delta W = 2u \tan \left( \frac{a}{2} \right) \quad (A-18) \]

and the resulting change in strain is

\[ \Delta \varepsilon_a = \frac{2u \tan \left( \frac{a}{2} \right)}{W} \quad (A-19) \]

Combining Equations A-17 and A-19 gives

\[ \Delta P_a = \frac{2u M_a \tan \left( \frac{a}{2} \right)}{W} \quad (A-20) \]

The axial force at any angular position is given by

\[ P_a = P_{ai} + \Delta P_a \quad (A-21) \]

where \( P_{ai} \) is the internal axial force in the belt at the inlet to the pulley. It should be noted that the value of \( P_{ai} \) is not zero in general because the belt has been assumed to be fully seated in the groove at the inlet to the pulley.

Using Equation A-20 in Equation A-21 results in

\[ P_a = P_{ai} + \frac{2u M_a \tan \left( \frac{a}{2} \right)}{W} \quad (A-22) \]

**Friction Angle**

In order to determine the various components of the frictional force, the direction in which it acts must be defined. The frictional force will act opposite to the direction of motion of the belt relative to the pulley, so that the required direction can be established by finding the slope of the path along which the belt moves with respect to the pulley. The friction angle \( \phi \) will be used to describe the direc-
tion of the frictional force and is defined as shown in Figure A-6. This figure shows a cross section of the belt and a segment of the side of the belt as viewed normal to the conical side surface.

An expression from which $\phi$ may be evaluated can be obtained by reference to Figure A-3. Here the curves representing belt motion relative to the pulley are shown projected into a plane perpendicular to the pulley axis. Since the points c and d on the pulley and the points e and f on the belt are pairs of points which are identical in every respect except circumferential location, it follows that the curve dge is identical to the curve cde except for its length. If the pulley is now turned through an angle $d\theta$, the point e on the pulley will move to the point g in space while point e on the belt moves to point f in space. The slope of the relative motion curve at f can thus be determined by considering the radial and tangential components of the line gf. The tangential component of gf is given by

$$(gf)_t = \frac{r_1 - u}{r_1} \, dv$$  (A-23)

and the radial component by

$$(gf)_r = du$$  (A-24)

The slope of the line gf is thus

$$\text{slope (gf)} = \frac{r_1 \, du}{(r_1 - u)dv}$$  (A-25)

Because the slope of Equation A-25 and the angle $\phi$ in Figure A-6 are measured in planes which are displaced by the angle $\alpha/2$, the two are related by

$$\text{slope (gf)} = \tan \phi \cos (\alpha/2)$$  (A-26)
Figure A-6. Axial and radial forces on a V-belt.
By combining Equations A-25 and A-26 the equation defining the friction angle $\phi$ can be written as

\[ \tan \phi = \frac{r_i}{(r_i - u) \cos (\alpha/2)} \frac{du}{dv} \quad (A-27) \]

Equation A-27 can be rearranged and divided by $d\theta$ to give

\[ \frac{dv}{d\theta} \left( 1 - \frac{u}{r_i} \right) = \frac{du}{d\theta \tan \phi \cos (\alpha/2)} \quad (A-28) \]

At $\theta = 0$, $u = 0$, so Equation A-28 becomes

\[ \left| \frac{dv}{d\theta} \right|_1 = \frac{1}{\tan \phi \cos (\alpha/2)} \left| \frac{du}{d\theta} \right|_1 \quad (A-29) \]

Combining Equations A-28, A-29, and A-16 eliminates the variable $v$ and gives

\[ T = T_1 + \frac{M_t}{r_i + \frac{1}{\tan \phi \cos (\alpha/2)} \left| \frac{du}{d\theta} \right|_1 \left[ \frac{du}{d\theta \tan \phi \cos (\alpha/2)} - \frac{1}{\tan \phi \cos (\alpha/2)} \left| \frac{du}{d\theta} \right|_1 - u \right]} \quad (A-30) \]

**Force Summations**

Three force summations can now be written for the belt element located at $\theta$ — one in the tangential direction, one in the axial direction, and one in the radial direction. The summations will be equated to zero under the assumption that velocity changes due to strains will be so small that the resultant inertia forces may be neglected when compared to the other forces on the element.
Tangential force summation. The forces acting on the segment of belt represented by the line ef in Figure A-3 are shown in Figure A-7. From e to f the radius varies by the amount du, but radial changes in the system will be very small for any given circumferential displacement, so the radius will be assumed constant from e to f. The forces acting on the element are: $F_r \cos \phi$, the tangential component of the frictional forces; $F_r$, the radial force exerted on the element by the pulley; $F_c$, the centrifugal force on the element; $T + (\rho V^2 / g)$, the tangential force in the belt at e; and $T + \frac{\rho V^2}{g} + d(T + \frac{\rho V^2}{g})$, the tangential force in the belt at f.

Summing forces in the tangential direction gives

$$ F_r \cos \phi + (T + \frac{\rho V^2}{g}) \cos \frac{\phi}{2} = \left[ T + \frac{\rho V^2}{g} + d(T + \frac{\rho V^2}{g}) \right] \cos \frac{\phi}{2} \quad (A-31) $$

Since the angle $\phi$ is infinitesimal in size cos ($\phi/2$) may be taken as 1.0, so that Equation A-31 becomes

$$ F_r \cos \phi = d(T + \frac{\rho V^2}{g}) \quad (A-32) $$

But $\rho$, $V$, and $g$ are constants and Equation A-32 can be written

$$ dT = F_r \cos \phi \quad (A-33) $$

The frictional force $F_f$ is the result of the distributed force $P_f$ acting on both sides of the belt over the length ef, or

$$ F_f = 2P_f \text{ef} \quad (A-34) $$

The distributed frictional force is

$$ P_f = C_f P_N \quad (A-35) $$
Figure A-7. Tangential and radial forces on a V-belt.
and the length ef is given by Equation A-12

$$ds = ef = (r_i - u)d\varphi + dv(1 - \frac{u}{r_i})$$  \hspace{1cm} (A-36)

However, the second term on the right side of Equation A-36 represents the change in the length of the belt element due to the change in tension around the pulley, and in practice will not be more than about 0.5 percent of the length of the element itself. This term can thus be neglected with good approximation and Equation A-36 becomes

$$ef = (r_i - u)d\varphi$$  \hspace{1cm} (A-37)

Using Equations A-35 and A-37 in A-34 gives

$$F_f = 2C_f P_N(r_i - u)d\varphi$$  \hspace{1cm} (A-38)

Combining Equations A-33 and A-38 gives

$$\frac{dT}{d\varphi} = 2C_f P_N(r_i - u) \cos \varphi$$  \hspace{1cm} (A-39)

**Radial force summation.** Equating the inward and outward radial forces in Figure A-7 gives

$$F_r + F_c = \left[2(T + \frac{\rho V^2}{g}) + d(T + \frac{\rho V^2}{g})\right] \sin (\delta/2)$$  \hspace{1cm} (A-40)

But $\delta$ is an infinitesimal angle, so

$$\sin (\delta/2) = \delta/2$$  \hspace{1cm} (A-41)

Using this approximation in Equation A-40 yields

$$F_r + F_c = (T + \frac{\rho V^2}{g})\delta + \frac{\delta}{2}d(T + \frac{\rho V^2}{g})$$  \hspace{1cm} (A-42)

Here the last term is a second order differential and may be neglected, so
\[ F_r + F_C = T \delta + \frac{\rho v^2 \delta}{g} \]  \hspace{1cm} (A-43)

The centrifugal force acting on the element ef is given by

\[ F_C = \frac{\rho v^2}{r_1 - u} \]  \hspace{1cm} (A-44)

where \( \rho \) is the mass of the element and can be obtained by multiplying the mass of the belt per unit length \( \rho / g \) by the length of the element \( \delta(r_1 - u) \). This gives

\[ \rho = (\rho / g)(r_1 - u) \delta \]  \hspace{1cm} (A-45)

Combining Equations A-44 and A-45 results in

\[ F_C = (\rho v^2 \delta) / g \]  \hspace{1cm} (A-46)

Substituting Equation A-46 for \( F_C \) in Equation A-43 gives

\[ F_r = T \delta \]  \hspace{1cm} (A-47)

The corresponding distributed force is obtained by dividing the force in Equation A-47 by the length of the element, or

\[ P_r = \frac{T \delta}{\delta (r_1 - u)} = \frac{T}{r_1 - u} \]  \hspace{1cm} (A-48)

A radial force summation on the cross-section of the belt shown in Figure A-6 results in

\[ P_r = 2P_N \sin (a/2) + 2P_f \sin \delta \cos (a/2) \]  \hspace{1cm} (A-49)

or, with the aid of Equation A-35,

\[ P_r = 2P_N (\sin \frac{\alpha}{2} + C_f \sin \delta \cos \frac{\alpha}{2}) \]  \hspace{1cm} (A-50)

Eliminating \( P_r \) from Equations A-48 and A-50 yields

\[ T = 2P_N (r_1 - u) (\sin \frac{\alpha}{2} + C_f \sin \delta \cos \frac{\alpha}{2}) \]  \hspace{1cm} (A-51)
**Axial force summation.** The axial forces acting on the external surface of the belt can be written with the aid of Figure A-6 as

\[ P_a = P_N \cos (\alpha/2) - P_f \sin \phi \sin (\alpha/2) \]  

(A-52)

Equating this external force to the internal axial force as given by Equation A-22 and using Equation A-35 results in

\[ P_N(\cos \frac{\alpha}{2} - C_f \sin \phi \sin \frac{\alpha}{2}) = P_a = P_{ai} + \frac{2u M_a \tan \frac{\alpha}{2}}{W} \]  

(A-53)

The value of \( P_{ai} \) can be obtained by examining Equations A-51 and A-53 at the point \( \Theta = 0 \) where \( u = 0 \), \( \phi = \phi_1 \), and \( T = T_1 \). Using these values, Equation A-51 can be solved for \( P_{Ni} \) to give

\[ P_{Ni} = \frac{T_1}{2r_1(\sin \frac{\alpha}{2} + C_f \sin \phi_1 \cos \frac{\alpha}{2})} \]  

(A-54)

while Equation A-53 gives

\[ P_{Ni} = \frac{P_{ai}}{\cos (\alpha/2) - C_f \sin \phi_1 \sin (\alpha/2)} \]  

(A-55)

Combining Equations A-54 and A-55 yields

\[ P_{ai} = \frac{T_1(\cos \frac{\alpha}{2} - C_f \sin \phi_1 \sin \frac{\alpha}{2})}{2r_1(\sin \frac{\alpha}{2} + C_f \sin \phi_1 \cos \frac{\alpha}{2})} \]  

(A-56)

For the sake of brevity, let

\[ f_1(\phi) = \cos (\alpha/2) - C_f \sin \phi \sin (\alpha/2) \]  

(A-57)

and

\[ f_2(\phi) = \sin (\alpha/2) + C_f \sin \phi \cos (\alpha/2) \]  

(A-58)
Using these definitions and Equation A-56 in Equation A-53 one obtains

\[ P_N f_1(\phi) = \frac{T_1 f_1(\phi_1)}{2r_1 f_2(\phi_1)} + \frac{2u M_a \tan(\alpha/2)}{W} \quad (A-59) \]

**Equation Summary**

The four final equations describing the action of the belt in the pulley groove are collected and listed below. The definitions of Equations A-57 and A-58 have been used throughout.

\[ T = T_i + \frac{M_b}{r_i + \frac{1}{\tan \phi_1 \cos \frac{\alpha}{2}}} \left[ \frac{du}{d\phi} \tan \phi \cos \frac{\alpha}{2} \right] - \frac{1}{\tan \phi_1 \cos \frac{\alpha}{2}} \left[ \frac{du}{d\phi} \right] - u \quad (A-30) \]

\[ \frac{dT}{d\phi} = 2C_f P_N (r_i - u) \cos \phi \quad (A-39) \]

\[ T = 2P_N (r_i - u) f_2(\phi) \quad (A-51) \]

\[ P_N f_1(\phi) = \frac{T_1 f_1(\phi_1)}{2r_1 f_2(\phi_1)} + \frac{2u M_a \tan(\alpha/2)}{W} \quad (A-59) \]

These four equations are presumably independent since they arise from the application of totally independent basic principles, such as force balances and elastic action. Five variables are involved in the equations — \( T, \phi, u, \phi_1, \) and \( P_N \) — with \( \phi \) independent and the others dependent. It would appear then that these equations might provide solutions for any one of the dependent variables in terms of the independent variable \( \phi \).
Boundary Conditions

At the entrance to the pulley, $\theta$ is zero, and $T$ is a known value $T_1$ — the tight or slack side tension depending on whether the driver or driven pulley respectively is under consideration. The value of $u$ at this point is always zero, but for the general case where the belt is in motion with respect to the pulley at $\theta = 0$, the value of $(du/d\theta)_1$ is unknown. If the value of $\theta_1$ were known, then $P_{N1}$ could be obtained from Equation A-54, but for the general case mentioned above, the method by which $\theta_1$ can be determined has not been worked out. A discussion of the analysis which would be required has been given in the General Remarks at the beginning of this appendix.

A special case exists when the tension change occurring over a pulley is less than that which could be attained over the same pulley if the tension varied throughout the entire arc of contact. Presumably this condition would result in a shift of the point $\theta = 0$ to some point on the pulley other than the inlet. The contact angle would thus be divided in two parts — an inactive angle throughout which the tension is constant and consequently no relative motion occurs between the belt and pulley and an active angle in which the tension changes and relative motions are described by the equations under discussion. For this special case the belt enters the pulley in the inactive zone and therefore has no gross tangential velocity relative to the pulley although small local tangential velocity components may exist due to bending as discussed in the General Remarks. In the radial direction, however, a definite relative velocity component is present, since the belt must move inward radially in the groove in order to build up the elastic forces in the belt.
required to establish radial force equilibrium. It would appear reasonable, then, to assume that when the active angle is equal to or less than the contact angle the velocity of the belt relative to the pulley is directed radially inward at the inlet to the pulley. This is equivalent to the assumption that $\phi$ is $90^\circ$ at this point, and, since no changes take place throughout the inactive angle, it may be further assumed that the value $\phi_1$ at $\theta = 0$ is $90^\circ$.

**Solution of Force-Motion Equations for the Special Case $\phi_1 = 90^\circ$**

After a great amount of effort had been expended in an unsuccessful attempt to obtain a closed solution to Equations A=30, A=39, A=51, and A=59, it was decided that the aid of an analog computer should be enlisted in the hope that chart solutions to the equations could be obtained. To explore this possibility, the special case mentioned under Boundary Conditions was investigated on a Pace electronic analog computer. The physical constants used in the investigation were those corresponding to a standard C cross-section V-belt. Their values were:

- $M_t = 45,000$ pounds per inch/inch
- $a = 36$ degrees
- $C_f = 0.50$ pounds/pound
- $M_a = 1000$ pounds/tangential inch per inch/inch
- $W = 0.757$ inch

The boundary conditions used for the special case of $\phi_1 = 90^\circ$ were:

- $\phi_1 = 90$ degrees
- $u_1 = 0$
- $(du/d\theta)_1 = 0$
Various values of $T_i$ and $r_i$ were used in the investigation.

**Driven pulley.** A driven pulley was considered first, which required that $\phi$ be restricted to the interval from $-90^\circ$ to $+90^\circ$. This assures that the tangential component of the frictional force on the belt always acts in the direction opposite to the pulley rotation, or that tension increases in the direction of rotation. This was accomplished in the machine by forcing $\cos \phi$ to be plus at all times.

The results of the analog study on the driven pulley are shown as plotted by the machine itself in Figures A-8 through A-12. A complete set of curves is included for $r_i$ values of 6 and 8 inches and at each radius the inlet tension $T_i$ is varied in steps from 5 to 100 pounds.

The quantities $S_t$ and $F_a$ appearing in Figures A-11 and A-12 are not involved in the equations under consideration but are derived quantities of interest in a V-belt drive. The axial force $F_a$ represents the total force exerted on the belt by the pulley in the axial direction. It is obtained by multiplying $P_a$ as given by the left-hand side of Equation A-53 by the incremental length $r_i d\phi$ and integrating the result from 0 to $\phi$.

The slip $S_t$ is defined as the tangential component of the belt velocity relative to the pulley at a point divided by the pulley velocity at that point. From Figure A-3, it can be established that the tangential component of the relative velocity at any point can be written as

$$V_{Rt} = \frac{dv}{d\tau} \frac{r_i - u}{r_i}$$  \hspace{1cm} (A-60)
TABLE A-1

Curve Identification for Figures A-8 through A-12.

\[ M_t = 45,000 \text{ pounds per belt}; \ \alpha = 36^\circ; \]
\[ G_f = 0.50; \ M_a = 1000 \text{ pounds per inch}; \]
\[ W = 0.757 \text{ inch}. \]

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<th>( T_i ) (lbs)</th>
<th>( S_{ti} )</th>
<th>( r_i ) (in.)</th>
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Figure A-6. $\sin \beta$ versus $\theta$ for C-section driven pulley. For curve identification see Table A-1.
Figure A-9. $\sin \phi$ versus $\theta$ for C-section driven pulley.
For curve identification see Table A-1.
Figure A-10. $T$ versus $\Theta$ for C-section driven pulley. For curve identification see Table A-1.
Figure A-11. \( S_t \) versus \( \theta \) for C-section driven pulley.
For curve identification see Table A-1.
Figure A-12. $F_a$ versus $\Theta$ for C-section driven pulley.
For curve identification see Table A-1.
and the pulley velocity at the same point is

\[ V_p = \frac{d\phi}{dr}(r_1 - u) \quad (A-61) \]

Combining Equations A-60 and A-61 gives the slip \( S_t \) as

\[ S_t = \frac{V_{Rt}}{V_p} = \left(\frac{1}{r_1}\right)\frac{dv}{d\phi} \quad (A-62) \]

By using the relationship between \( du \) and \( dv \) given in Equation A-28, Equation A-62 can be written as

\[ S_t = \frac{1}{(r_1 - u) \tan \phi \cos (\alpha/2)} \frac{du}{d\phi} \quad (A-63) \]

All of the terms in Equation A-63 were available as parts of the set of equations under consideration, so that the slip \( S_t \) could be obtained from the computer with little added effort.

The plots of \( \sin \phi \) versus \( \phi \) shown in Figures A-3 and A-9 exhibit a very interesting variation. It can be seen by observing the beginning of the curves at \( \phi = 0 \) that although various values of \( \phi_1 \) other than \( 90^\circ \) are used as boundary conditions, the curve very rapidly approaches that curve which would have resulted if \( \phi_1 \) had been taken as \( 90^\circ \). This means that \( \phi_1 \) can be taken as some angle slightly less than \( 90^\circ \), and the resultant curves will be nearly correct. This is a fortunate situation, since at \( 90^\circ \) the \( \tan \phi \) in Equation A-30 becomes undefined and causes the computer solution to break down. The values of \( \phi_1 \) used to obtain these plots were the largest possible without causing the computer components to overload.

The curves of \( T \), \( S_t \), and \( F_t \) versus \( \phi \) given in Figures A-10, A-11, and A-12 appear to be qualitatively about what might be expected. Each of the curves exhibits an exponential character such as has been suggested by early analyses of flat belt drives and is thus not surprising
as a result of V-belt analysis. However, the quantitative verification of these results is impossible at the present time. Neither the tension T nor the slip $S_t$ can be measured simply on a pulley, and attempts to perform these measurements have, to date, been unsuccessful. Since no split sheaves are available for the C-section belt, the total axial force on the belt cannot be measured without an expensive experimental set-up. A small amount of quantitative information is available on variable speed V-belt drives, and some of this data is compared to computer solutions in the last section of Chapter I.

In spite of this unfortunate lack of experimental data to check these curves, they do supply two encouraging bits of information. First, the two dotted lines shown in Figure A-10 connect points on the two families of curves at which the tension becomes five times that at the beginning of the curve. The value of $\Theta$ will be observed to be approximately constant for each dotted line, and is numerically slightly below 3 radians. Now this would mean that on the pulleys in question a tension ratio of 5 could be attained without incurring slip at the inlet to the pulley. It would appear to be more than a coincidence that the tension ratio of 5 has been considered for many years to be "about right" for a satisfactory V-belt drive.

The second significant piece of information can be obtained by considering the sections of the curves for which $\Theta$ is approximately constant. To make use of this information Equations A-39 and A-51 can be combined to give, with the aid of the definition of Equation A-58,

$$\frac{dT}{T \, d\Theta} = \frac{C_f \cos \Theta}{\sin (\alpha/2) + C_f \sin \Theta \cos (\alpha/2)}$$

(A-64)
For \( \alpha/2 = 18^\circ \), \( C_f = 0.5 \), and taking \( \phi = 26.7^\circ \) as a constant from Figure A-8, Equation A-64 integrates to give

\[
T/C = e^{0.855 \phi} \tag{A-65}
\]

where \( C \) is a constant of integration. If two angular positions \( \phi_1 \) and \( \phi_2 \) are considered the corresponding tension ratio is given by Equation A-65 as

\[
T_1/T_2 = e^{0.855(\phi_1 - \phi_2)} \tag{A-66}
\]

For \( \phi_1 = \phi_2 = \pi \), the ratio given by Equation A-66 is

\[
T_1/T_2 = 14.7 \tag{A-67}
\]

This number agrees very well with maximum tension ratios obtained by other investigators\(^1\) for V-belts of similar construction and provides another point of agreement between the analysis and practice.

**Driver pulley.** On the driver pulley tension must decrease with increasing \( \phi \), which means that the tangential component of the frictional force on the belt must act in the direction of increasing \( \phi \). For this case then \( \phi \) must be in the interval from \(+90^\circ\) to \(-90^\circ\). This was accomplished by requiring \( \cos \phi \) to be negative at all times.

Only a small number of exploratory computations had been performed on the driver pulley when it became extremely difficult to obtain further operating time on the analog computer. However, it is believed that these computations point the way for future analytical work along this line, and for this purpose they are included here.

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\(^1\)C. A. Norman, "High Speed Belt Drives." Bulletin No. 83, Engineering Experiment Station, The Ohio State University, 1934, p. 28.
The group of curves shown in Figures A-13 through A-16 represent the first attempt to solve the equations for the driver pulley. For this case it was again assumed that the angle of contact between the belt and pulley was greater than the angle required to cause the tension change on the drive. As discussed in the General Remarks at the beginning of this appendix, this means that the first motion of the belt will have a component radially inward. But since tension decreases with increasing $\theta$, it would be expected that the radial motion of the belt in the groove should ultimately be outward. This would require that the friction angle $\varphi$ start in the second quadrant and then swing into the third quadrant at some later $\theta$.

The physical constants used to obtain these curves were the same as those for the driven pulley listed on page 129. For no slip at the inlet, the boundary conditions were

$$ u_i = 0 $$

$$ (\frac{du}{d\theta})_i = 0 $$

and $\varphi_i$, $T_i$, and $r_i$ were given several different values as indicated.

From the curves of $\sin \varphi$ versus $\theta$ in Figure A-13 it can be seen that unlike the case of the driven pulley — any variation in the value of $\varphi_i$ results in an entirely new set of curves. Thus, on the driver pulley, the effect of the friction angle at the inlet is not quickly damped out, but influences all variations throughout the active angle. The magnitude of $\varphi_i$ must therefore be nearly correct in order to give an accurate picture of the variations occurring around the pulley.

Figures A-14, A-15, and A-16 provide further evidence of the critical nature of $\varphi_i$. A good example of this is given by curves 4, 5,
TABLE A-2

Curve Identification for Figures A-13 through A-16.

\[ M_t = 45,000 \text{ pounds per belt}; \quad \alpha = 36^\circ; \]
\[ C_f = 0.50; \quad M_a = 1000 \text{ pounds per inch}; \]
\[ W = 0.757 \text{ inch}. \]

<table>
<thead>
<tr>
<th>Curve</th>
<th>Sin ( \phi_i )</th>
<th>( T_i ) (lbs)</th>
<th>( S_{ti} )</th>
<th>( r_i ) (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0</td>
<td>6.0</td>
</tr>
<tr>
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<td>0.430</td>
<td>400</td>
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<td>0.520</td>
<td>100</td>
<td>0</td>
<td>8.0</td>
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Figure A-13. $\sin \phi$ versus $\phi$ for C-section driver pulley.
For curve identification see Table A-2.
Figure A-14. $T$ versus $\theta$ for C-section driver pulley. For curve identification see Table A-2.
Figure A-15. $S_t$ versus $\Theta$ for C-section driver pulley.
For curve identification see Table A-2.
Figure A-16. \( P_a \) versus \( \Theta \) for C-section driver pulley. For curve identification see Table A-2.
6, and 7 of Figure A-11. Here the tension variation is seen to be very radically affected by a change in the value of \( \phi_i \) with all other conditions identical.

Another very important point can be established by reference to Figure A-11. The tension changes indicated by curves 4, 1, and 8 show that for a value of \( \phi_i \) near 90\(^\circ\) only very small tension difference could be achieved throughout \( \pi \) radians of contact angle. Since for no slip at the inlet it has been reasoned that \( \phi_i \) should be 90\(^\circ\), these results suggest that slip is almost always present at the inlet to the driver pulley. This information makes fruitless further study of Figures A-13 through A-16 because all of these curves are based on an inlet slip \( S_{t1} = 0 \).

Figures A-17 through A-20 show a study of the effect of varying inlet slip at two different values of \( \phi_i \). It should be noted that for the driver pulley all values of slip are negative, since the tangential motion of the belt relative to the pulley is in the direction opposite to that of the pulley motion. However, it is the magnitude of the slip \( |S_t| \) which is important and will be used in this discussion.

Looking first at Figure A-19 a very significant variation can be observed as various values of inlet slip are imposed on the system while \( \phi_i \) is held constant. Consider, for example, curves 1, 2, 3, 4, and 5 for which \( \phi_i \) is constant at 51.2\(^\circ\) and \( |S_{t1}| \) takes on values of 0, 0.001, 0.002, 0.005, and 0.010, respectively. For all values of \( |S_{t1}| \) greater than zero, the curves show \( |S_t| \) decreasing to a minimum, then increasing again with increasing \( \theta \). Now while no analytical proof of the fact can be advanced at this time, it appears highly unlikely that the belt would over-slip at the inlet and thus require a decrease in slip before the
TABLE A-3

Curve Identification for Figures A-17 through A-20.

\[ M_t = 45,000 \text{ pounds per belt}; \ a = 36^\circ; \]
\[ C_F = 0.50; \ M_a = 1000 \text{ pounds per inch}; \]
\[ W = 0.757 \text{ inch.} \]

<table>
<thead>
<tr>
<th>Curve</th>
<th>( \sin \phi_i )</th>
<th>( T_i ) (lbs)</th>
<th>( S_{ti} )</th>
<th>( r_i ) (in.)</th>
</tr>
</thead>
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<td>400</td>
<td>0</td>
<td>6.0</td>
</tr>
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<td>0.780</td>
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<td>0.010</td>
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<td>7</td>
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<td>0.001</td>
<td>6.0</td>
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<tr>
<td>8</td>
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<td>400</td>
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<td>6.0</td>
</tr>
<tr>
<td>10</td>
<td>0.440</td>
<td>400</td>
<td>0.010</td>
<td>6.0</td>
</tr>
</tbody>
</table>
Figure A-17. $\sin \phi$ versus $\theta$ for C-section driver pulley. For curve identification see Table A-3.
Figure A-18. T versus θ for C-section driver pulley.
For curve identification see Table A-3.
Figure A-19. $S_t$ versus $\theta$ for C-section driver pulley. For curve identification see Table A-3.
Figure A-20. $F_a$ versus $\theta$ for C-section driver pulley. For curve identification see Table A-3.
expected increase could occur. If the assumption is made that $|S_t|$ must always increase with increasing $\Theta$, then for $\phi_1 = 51.2^0$, the value of $|S_{ti}|$ must have some value between 0 and 0.001. Now comparing curves 1 and 2 it appears that neither of these is the most likely curve, since 2 contains a minimum and 1 tends to straighten out or even take on a reversed curvature at low values of $\Theta$. By this reasoning then, a "best guess" for the value of $|S_{ti}|$ which should be paired with $\phi_1$ of 51.2$^0$ would be about 0.0005. Considering the proximity of curves 1 and 2 in Figures A-18 and A-20, this combination of $|S_{ti}|$ and $\phi_1$ should give close approximations of the correct curves of $T$ and $F_a$ versus $\Theta$, if it be assumed that the correct value of $|S_{ti}|$ is between 0 and 0.001.

Using this same line of reasoning in conjunction with the curves for $\phi_1 = 25.6$ results in a likely value of $|S_{ti}|$ between 0.002 and 0.005 — say at 0.003. Consideration of curves 8 and 9 in Figures A-18 and A-20 shows that again the indicated choice of $|S_{ti}|$ should give good approximate curves for $F_a$ and $T$ if the true value of $|S_{ti}|$ lies in the chosen range.

It is believed that by the extension of the process just outlined for two values of $\phi_1$ a system of charts could be obtained which would allow complete analysis of the driver pulley in any drive. A group of curves representing several combinations of $\phi_1$ and $|S_{ti}|$ would be necessary for each of many values of $T_i$. A complete set of these curves could then be worked out for various values of inlet radii $r_i$. The only apparent alternative to this perhaps rather unsophisticated approach to the analysis of the driver pulley would be the complete mathematical description of the seating process as discussed in the General Remarks at the beginning of this appendix.
APPENDIX B

CENTRIFUGAL FORCE

The effect of centrifugal force on belt tension can be described by the following derivation performed for a general loop such as that shown in Figure B-1. The loop shown is assumed to be continuous and to be composed of a material which offers no resistance to bending. The loop is considered to be sliding on a horizontal frictionless surface at a velocity \( \mathbf{V} \) along its periphery. The velocity \( \mathbf{V} \) is assumed to be constant in magnitude but is directed along the tangent to the loop at every point. A polar coordinate system will be used to describe the curve formed by the loop. The location of an arbitrary point on the loop is given by the head of the vector \( \mathbf{R} \).

An incremental length of the loop is cut out by the vectors \( \mathbf{R} \) and \( \mathbf{R} + d\mathbf{R} \) at \( \Theta \) and \( \Theta + d\Theta \), respectively. This element is shown in detail in Figure B-2. As shown here, the velocity of a particle on the loop varies by an amount \( d\mathbf{V} \) in moving through the angle \( d\Theta \). If the element of time \( dt \) is defined as the time required for this movement, then the acceleration of the particle at \( \Theta \) is

\[
\mathbf{a} = \frac{d\mathbf{V}}{dt} \tag{B-1}
\]

The forces acting on the element at \( \Theta \) are shown in Figure B-3. The tension \( T \) is the internal resisting force acting inside the loop. This tension has been taken as plus in the plus direction of \( \mathbf{V} \), so the force acting on the right-hand end of the element is negative. As shown, the unbalanced force acting on the element is \( dt \). If the mass
Figure B-1. Continuous loop in motion.

Figure B-2. Velocity diagram for segment of loop.
of the element is $dm$, then from Newton's second law

$$dT = dm \frac{dV}{dt}$$  \hspace{1cm} (B-2)

However, the mass $dm$ can be obtained from

$$dm = \frac{\rho V \, dt}{g}$$  \hspace{1cm} (B-3)

where $\rho$ is the density of the material of the loop in pounds per unit length, $V$ is the magnitude of the velocity, and $g$ is the acceleration of gravity.

Combining Equations B-2 and B-3 gives

$$dT = \frac{\rho V}{g} \, dV$$  \hspace{1cm} (B-4)

If one differentiates the product $VV$ he obtains

$$d(VV) = V \, dV + V \, dV$$  \hspace{1cm} (B-5)

but the magnitude of the velocity has been taken as constant, so $dV = 0$, 

Figure B-3. Force diagram for segment of loop.
and
\[ d(VV) = V \, dV \] (B-6)

Using this result to integrate Equation B-4 gives
\[ T = C + (\rho/g) \, VV \] (B-7)

If one considers only that tension due to motion so that \( T = 0 \) at \( V = 0 \), then \( C = 0 \) and
\[ T = (\rho/g) \, VV \] (B-8)

Thus the magnitude of the tension is given by
\[ T = \rho V^2 / g \] (B-9)

This tension \( T \) is an internal force in the loop, and since \( V \) is constant \( T \) must also be constant regardless of the shape of the curve followed by the loop. The loop is everywhere in equilibrium under the action of an internal resisting force and a body force due to the change in direction of the vector velocity \( V \). There appears to be no reason whatsoever for the shape of the loop to be affected by the magnitude of the velocity \( V \). That is, if the magnitude of the velocity were made very large, there would be no tendency for the loop to become circular in shape.
APPENDIX C

BELT CURVE BETWEEN PULLEYS

In the process of investigating the forces and motions involved in a locked center drive, it became apparent that an equation of the curve of the belt between the pulleys would be most useful. The following derivation was carried out to obtain this curve. To simplify the derivation equal diameter pulleys have been used, but the results should be extendable to other cases.

Consider the belt drive shown in Figure C-1. In this figure the curvature of the belt between the pulleys and the effect of this curvature on the location of the contact point C have been exaggerated. The center distance is denoted by \( L_c \) and the radius of the pulleys by \( r \). The \( x-y \) coordinate system will be oriented as shown in the figure.

Figure C-2 shows an enlargement of the segment of the belt pertinent to the derivation. Since the radius of curvature of the belt is not necessarily infinite at the origin, the moment \( M_1 \) as shown must be present in the belt at this point along with the tension \( T \). At the point \( C \), the belt is assumed to be tangent to the pulley and to have the same curvature. These assumptions may not be exactly true, but should be very good approximations.

The radius of curvature \( R \) of a bent beam is related to the moment \( H \) acting on the beam by

\[
\frac{1}{R} = \frac{H}{K} \quad (C-1)
\]

For homogeneous isotropic beams, the constant \( K \) is given by the product
Figure C-1. Belt drive showing coordinate system.
Figure C-2. Segment of belt used for determination of belt curve.
of the elastic modulus $E$ and the moment of inertia of the beam cross-
section $I$, but for the present case $K$ will be considered simply as a
bending constant peculiar to a particular belt.

The radius of curvature of any curve is given by

$$R = \frac{\left[1 + (y')^2\right]^{3/2}}{y''} \quad (C-2)$$

For small slopes this expression becomes

$$\frac{1}{R} = y'' \quad (C-3)$$

In this analysis one cannot be sure at the outset whether or not
small slopes are involved, but this assumption will be made here and
will then be checked later.

Combining Equations $C-2$ and $C-3$ gives the familiar expression

$$y'' = \frac{H}{K} \quad (C-4)$$

The moment at any arbitrary point such as $p$ is

$$H = H_i + Ty \quad (C-5)$$

Substituting in Equation $C-4$ gives

$$y'' = \left(\frac{1}{K}\right)(H_i + Ty) \quad (C-6)$$

The solution of this differential equation is

$$y = c_1 e^{-(T/K)\frac{1}{2}x} + c_2 e^{(T/K)\frac{1}{2}x} = \frac{H_i}{K} \quad (C-7)$$

where $e$ is the base of the natural logarithms. Using the boundary con-
ditions that

$$y = y'' = 0 \text{ at } x = 0 \quad (C-8)$$

the constants $c_1$ and $c_2$ can be evaluated and Equation $C-7$ can be written as
\[ y = \frac{H_1}{T} \left[ \frac{e^{-(T/K)^{1/2}x} + e^{(T/K)^{1/2}x}}{2} - 1 \right] \]  
\[ (C-9) \]

or, more compactly

\[ y = \left( \frac{H_1}{T} \right) \left[ \cosh \left( \frac{T}{K} \right)^{1/2} x - 1 \right] \]  
\[ (C-10) \]

A further boundary condition is known, since at the point C the radius of curvature of the belt is equal to that of the pulley or \( R = r \). Substituting Equations C-3 and C-10 into Equation C-6 yields

\[ \frac{1}{R} = \left( \frac{H_1}{X} \right) \cosh \left( \frac{T}{K} \right)^{1/2} x \]
\[ (C-11) \]

Then, using the above boundary condition at \( x = x_c \) gives

\[ \frac{1}{r} = \left( \frac{H_1}{K} \right) \cosh \left( \frac{T}{K} \right)^{1/2} x_c \]
\[ (C-12) \]

or

\[ H_1 = \frac{K}{r \cosh \left( \frac{T}{K} \right)^{1/2} x_c} \]
\[ (C-13) \]

Replacing \( H_1 \) in Equation C-10 by \( H_1 \) from Equation C-13 gives

\[ y = \frac{K}{Tr \cosh \left( \frac{T}{K} \right)^{1/2} x_c} \left[ \cosh \left( \frac{T}{K} \right)^{1/2} x - 1 \right] \]
\[ (C-14) \]

Now, using this equation for the curve of the belt it is possible to estimate the magnitude of the error introduced by the assumption made in Equation C-3 by obtaining the maximum value of the slope, \( y^\prime \). From Equation C-14

\[ y^\prime = \frac{(K/T)^{1/2} \sinh \left( \frac{T}{K} \right)^{1/2} x}{r \cosh \left( \frac{T}{K} \right)^{1/2} x_c} \]
\[ (C-15) \]

At the point of maximum slope, \( x = x_c \), so

\[ y^\prime_{\text{max}} = \left( \frac{1}{r} \right) \left( \frac{K}{T} \right)^{1/2} \tanh \left( \frac{T}{K} \right)^{1/2} x_c \]
\[ (C-16) \]
The maximum value taken on by $\tanh A$ is 1.0 regardless of the size of $A$, so for purposes of estimating the maximum slope the term $\tanh (T/K)^{1/2}x_c$ in Equation C-16 may be replaced by 1.0. Equation C-16 then becomes

$$y_{max} = (1/r)(K/T)^{1/2}$$  \hspace{1cm} (C-17)

While no accurate measurements of the bending constant $K$ are available, it appears from the work of Professor Norman\(^1\) and from data taken on the present project that for a C-section belt a value of $K = 20$ is reasonable. For a C-section drive, in normal use the values of $T$ and $r$ would not be expected to go below 20 pounds and 3 inches, respectively. Using these values in Equation C-17 gives

$$y_{max} = 1/3$$

Using this value in Equation C-2 yields

$$R = 1.17/y^\#$$  \hspace{1cm} (C-18)

instead of

$$R = 1/y^\#$$  \hspace{1cm} (C-19)

as was assumed in Equation C-3. This means that Equation C-14 will be fairly accurate for values of $(1/r)(K/T)^{1/2}$ less than 1/3. Either a large $T$ or a large $r$ would insure this condition. Physically this means that the belt is forced to leave the pulley on a curve approaching a straight-line tangent.

Equation C-14 can now be used to define the length of the belt between the pulleys and thereby show the effect of bending on the variations of $T_1$ and $T_2$ as torque is applied to a locked center drive.

From elementary calculus the length of the segment of the curve between

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\(^1\) C. A. Norman, "High Speed Belt Drives." Bulletin No. 83, Engineering Experiment Station, The Ohio State University, 1934, p. 20.
points 1 and 2 is given by
\[ \int_1^2 ds = \int_1^2 \left[ 1 + (y')^2 \right]^{1/2} dx \] (C-20)

To apply Equation C-20 to the present case, one must first establish the limits of the integration. In Figure C-3 the integration is to be between \( x = 0 \) and \( x = x_c \), so the distance \( x_c \) must be defined. From the figure this distance can be written as
\[ x_c = (L_c/2) + t = (L_c/2) + r \sin \Theta \] (C-21)

The angle \( \Theta \) is defined by
\[ \tan \Theta = (y')_{x=x_c} \] (C-22)

where \((y')_{x=x_c}\) denotes the slope of the curve \( dy/dx \) at the point \( C \).

But \( y' \) has been shown to be small for most practical cases, so that \( \sin \Theta \) is approximately equal to \( \tan \Theta \). Replacing \( \sin \Theta \) in Equation C-21 by \( \tan \Theta \) from Equation C-22 gives
\[ x_c = (L_c/2) + r(y')_{x=x_c} \] (C-23)

From Equation C-16
\[ (y')_{x=x_c} = \left( 1/r \right)(K/T)^{1/2} \tanh \left( T/K \right)^{1/2} x_c \] (C-24)

The term \( \tanh \left( T/K \right)^{1/2} x_c \) can never be greater than 1.0, and approaches that value as \( (T/K)^{1/2} x_c \) increases in size. For a C-section belt \( K \) will be of the order of 20, so for minimum practical values of \( T \) and \( x_c \) of 10 and 7, respectively,
\[ \left[ (T/K)^{1/2} x_c \right]_{\text{min}} = \left( 10/20 \right)^{1/2}(7) = 4.95 \]

But
\[ \tanh 4.95 = 0.9999 \]

so the value of \( \tanh (T/K)^{1/2} x_c \) may be taken as 1.0 for all cases with
Figure C-3. Coordinate system for belt length determination.
very good approximation. Equation C–24 now becomes

\[(y^i)_{x = x_c} = \frac{1}{r} (K/T)^{1/2}\]  \hspace{1cm} (C-25)

Combining Equations C–23 and C–25 gives

\[x_c = \frac{L_c}{2} + (K/T)^{1/2}\]  \hspace{1cm} (C–26)

The integration limits are therefore zero to \((L_c/2) + (K/T)^{1/2}\).

The integrand of Equation C–20 will now be examined. From the binomial theorem

\[(1 + z)^{1/2} = 1 + z/2 - z^2/8 + z^3/16 - \ldots\]  \hspace{1cm} (C–27)

for \(|z| < 1\). For the present case \(z = (y^i)^2\) and from Equation C–15

\[(y^i)^2 = \left[ \frac{K}{r^2 T} \right] \left[ \frac{\sinh^2 (T/K)^{1/2} x}{\cosh^2 (T/K)^{1/2} x_c} \right]\]  \hspace{1cm} (C–28)

The maximum value for the second fraction of Equation C–28 is 1.0, and for a C-section belt the extreme conditions which might arise in practice would give values for the parameters in the first fraction of \(K = 20\), \(T = 10\), and \(r = 3\). It should be noted that the tension considered here is lower than that used previously to evaluate \(y^i_{\text{max}}\) from Equation C–17. This means that the assumptions made in the following sections are less likely to cause errors than is the assumption made in the beginning that \(1/\rho = y^i\). The maximum expected value of \((y^i)^2\) will therefore be

\[(y^i)_{\text{max}}^2 = \frac{20}{(10)(9)} = 0.222 < 1\]  \hspace{1cm} (C–29)

so the condition \(|z| < 1\) is met for this case, and the expansion of Equation C–27 is valid. If the length of the curve from \(x = 0\) to \(x = x_c\) is denoted by \(L_{x_c}\) then, with aid of the series expansion of Equation C–27, Equation C–20 can be written as
\[ L_{xc} = \int_0^{x_c} \left[ 1 + \left(\frac{y^*}{2}\right)^2 - \left(\frac{y^*}{8}\right)^4 + \left(\frac{y^*}{16}\right)^6 + \ldots \right] \, dx \quad (C=30) \]

To examine the terms of this series, let
\[ B = \frac{K}{(Tr)^2} \quad (C=31) \]
so that Equation C-28 becomes
\[ (y^*)^2 = B \frac{\sinh^2 \left(\frac{T}{K}\right)^{1/2} x}{\cosh^2 \left(\frac{T}{K}\right)^{1/2} x_c} \quad (C=32) \]

The terms of the series after the first term will be larger as the value of \((y^*)^2\) increases so the value of \((y^*)^2\) \text{max} = 0.222 from Equation C-29 may be used to determine the importance of each term. Using this value, the series becomes
\[ 1 + 0.222/2 - (0.222)^2/8 + (0.222)^3/16 = \ldots \]
or
\[ 1 + 0.111 = 0.0062 + 0.00068 = \ldots \]
so using three terms of the series should give very good accuracy. For the sake of brevity, let
\[ A = \frac{1}{r \cosh \left(\frac{T}{K}\right)^{1/2} x_c} \left(\frac{K}{T}\right)^{1/2} \quad (C=33) \]
so that Equation C-28 can be written as
\[ (y^*)^2 = A^2 \sinh^2 \left(\frac{T}{K}\right)^{1/2} x \quad (C=34) \]

Then Equation C-30 can be closely approximated by
\[ L_{xc} = \int_0^{x_c} \left[ 1 + A^2/2 \sinh^2 \left(\frac{T}{K}\right)^{1/2} x \right. \]
\[ - A^4/8 \sinh^4 \left(\frac{T}{K}\right)^{1/2} x \right] \, dx \quad (C=35) \]
From fundamental identities involving hyperbolic functions

\[ \sinh^2 z = \frac{\cosh 2z - 1}{2} \quad (C-36) \]

and

\[ \cosh^2 z = \frac{\cosh 2z + 1}{2} \quad (C-37) \]

Using these identities Equation C-35 can be reduced to

\[ I_{xc} = \int_0^{x_c} \left\{ 1 + \frac{A^2}{4} \left[ \cosh \left( \frac{T}{K} \right)^{1/2}x - 1 \right] \right\} dx \quad (C-38) \]

Integrating Equation C-38 and putting in the limits gives

\[ I_{xc} = x_c + \frac{A^2}{4} \left[ \frac{\sinh \left( \frac{T}{K} \right)^{1/2}x_c}{2 \left( \frac{T}{K} \right)^{1/2}} - x_c \right] \]

\[ = \frac{A^2}{8} \left[ \frac{\sinh \left( \frac{T}{K} \right)^{1/2}x_c}{\left( \frac{T}{K} \right)^{1/2}} - \frac{l_4 \sinh \left( \frac{T}{K} \right)^{1/2}x_c}{2 \left( \frac{T}{K} \right)^{1/2}} + 3x_c \right] \quad (C-39) \]

For a C-section belt the smallest practical value of

\[ \frac{\sinh \left( \frac{T}{K} \right)^{1/2}x_c}{2 \left( \frac{T}{K} \right)^{1/2}} \]

will occur when \( T = 10, K = 20, \) and \( x_c = 7 \). Using these values gives

\[ \frac{\sinh \left( \frac{T}{K} \right)^{1/2}x_c}{2 \left( \frac{T}{K} \right)^{1/2}} = \frac{\sinh 9.9}{1.414} = 7.060 \]

This means that \( x_c \) can be neglected in the first bracketed quantity of Equation C-39.

In a similar manner it can be shown that for practical cases the second and third terms of the second bracketed quantity in Equation
C-39 can also be neglected. Equation C-39 thus becomes

\[ L_{xc} = x_c + \frac{A^2}{4} \frac{\sinh 2(T/K)^{1/2}x_c}{2(T/K)^{1/2}} = \frac{A^4}{324} \frac{\sinh 4(T/K)^{1/2}x_c}{4(T/K)^{1/2}} \]  
(C-40)

From the definition of Equation C-33

\[ A^2 = \frac{K}{Tr^2} \frac{1}{\cosh^2 (T/K)^{1/2}x_c} \]  
(C-41)

Again using the identity of Equation C-37

\[ \cosh^2 (T/K)^{1/2}x_c = \frac{\cosh 2(T/K)^{1/2}x_c + 1}{2} \]  
(C-42)

But the minimum practical value of \( \cosh 2(T/K)^{1/2}x_c \) is about 10,000 so with very good approximation

\[ \cosh^2 (T/K)^{1/2}x_c = \frac{\cosh 2(T/K)^{1/2}x_c}{2} \]  
(C-43)

so that Equation C-41 becomes

\[ A^2 = \frac{K}{Tr^2} \frac{2}{\cosh 2(T/K)^{1/2}x_c} \]  
(C-44)

By similar approximations it can be shown that

\[ A^4 = \frac{8k^2}{T^2 Tr_4} \frac{1}{\cosh 4(T/K)^{1/2}x_c} \]  
(C-45)

Using Equations C-44 and C-45 in Equation C-40 gives

\[ L_{xc} = x_c + \frac{1}{Lx^2} \left[ \frac{K}{T} \right]^{3/2} \left[ \tanh 2(T/K)^{1/2}x_c \right] \]
\[ - \frac{1}{32x^4} \left[ \frac{K}{T} \right]^{5/2} \left[ \tanh 4(T/K)^{1/2}x_c \right] \]  
(C-46)

However, for all reasonable situations
tanh 2(T/K)^{1/2}x_C = \tanh \frac{4(T/K)^{1/2}x_C}{4} = 1.0 \quad (C-47)

so Equation C-46 can be written as

\[ L_{x_C} = x_C + \frac{1}{4r^2} \left( \frac{K}{T} \right)^{3/2} - \frac{1}{32r^4} \left( \frac{K}{T} \right)^{5/2} \quad (C-48) \]

Equation C-48 represents the length of the belt curve from the origin to the point C in Figure C-3. Let

\[ L_b/2 = L_{x_C} + CB \quad (C-49) \]

where CB is the length of the circle arc from C to B. For equal pulley diameters and equal tensions in the two belt strands, \( L_b \) then represents one-half of the total length of the belt. From Figure C-3 the length CB is given by

\[ CB = (\frac{n}{2} - \theta) r \quad (C-50) \]

Combining Equations C-22 and C-25 and solving for \( \theta \) gives

\[ \theta = \tan^{-1} \left( \frac{1}{r} \right) (K/T)^{1/2} \quad (C-51) \]

Using Equation C-51 in Equation C-50 yields

\[ CB = (\pi r/2) - r \tan^{-1} \left( \frac{1}{r} \right) (K/T)^{1/2} \quad (C-52) \]

Combining Equations C-48, C-49, and C-52 and solving for \( L_b \) gives

\[ L_b = 2x_C + \frac{1}{2r^2} \left( \frac{K}{T} \right)^{3/2} - \frac{1}{16r^4} \left( \frac{K}{T} \right)^{5/2} \]

\[ + \pi r - 2r \tan^{-1} \left( \frac{1}{r} \right) (K/T)^{1/2} \quad (C-53) \]

Finally, using \( x_C \) as given by Equation C-26 in Equation C-53 results in

\[ L_b = L_c + 2(K/T)^{1/2} + \pi r - 2r \tan^{-1} \left( \frac{1}{r} \right) (K/T)^{1/2} \]

\[ + \frac{1}{2r^2} \left( \frac{K}{T} \right)^{3/2} - \frac{1}{16r^4} \left( \frac{K}{T} \right)^{5/2} \quad (C-54) \]
This equation provides an analytical relationship between the tension in a belt on a no-load drive and the center distance between the pulleys over which the belt is mounted. The relationship involves the known quantities of belt length $2L_b$ and pulley radius $r$, and the unknown constant $K$. The method for obtaining $K$ from experimental data is described in Chapter I in the analysis of locked center drives.
BIBLIOGRAPHY


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