Stress Intensity Factors and Effective Spring Stiffness for Interfaces with Two and
Three Dimensional Cracks at the Interface between Two Dissimilar Materials

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ABSTRACT

Stress intensity factors and effective spring stiffnesses for two-dimensional and three-dimensional cracks at the interface between dissimilar solids are investigated. First explicit analytical expressions are obtained for the longitudinal and transverse effective spring stiffnesses of a planar periodic array of collinear cracks at the interface between two dissimilar isotropic materials; they are shown to be identical in a general case of elastic dissimilarity (the well-known open interface crack model is employed for the solution). The effects of elastic dissimilarity, crack density and crack interaction on the effective spring stiffness are clearly represented in the solution. It is shown that in general the crack interaction weakly depends on material dissimilarity and, for most practical cases, the crack interaction is nearly the same as that for crack arrays between identical solids. This allows approximate factorization of the effective spring stiffness for an array of cracks between dissimilar materials in terms of an elastic dissimilarity factor and two factors obtained for cracks in a homogeneous material: the effective spring stiffness for non-interacting (independent) cracks and the crack interaction factor.

Second, the longitudinal and transverse effective spring stiffnesses of non-interacting penny-shaped cracks at the interface between two dissimilar, isotropic,
linearly elastic materials is obtained based on classical fracture mechanics. Special care is taken to avoid crack surface interpenetration for transverse loading, and the valid loading range is obtained to assure negligibility of crack surface interpenetration for all possible ranges of isotropic, linearly elastic material combinations. For linear ultrasound applications, it is shown that the expression obtained for transverse springs can be used for most isotropic, linearly elastic material combinations, if the initial maximum crack opening displacement is more than $10^{-6}$ of the crack radius.

Third, based on Kachanov’s approximate method for crack interaction problems, the stress intensity factors for a periodic array of coplanar penny-shaped cracks are obtained as a function of angle around the crack circumference and crack density for square and hexagonal crack configurations. Crack interactions in the hexagonal configuration are shown to be more than those in the square configuration. Numerical errors and errors due to the approximate nature of the method are estimated, and obtained results are shown to be valid within 8% error for crack density up to 95%.

Fourth, following the approximate factorization of the effective spring stiffness for an array of cracks between dissimilar materials, approximate expressions for the longitudinal and transverse effective spring stiffnesses of co-planar penny-shaped cracks with square or hexagonal configurations at the interface between two dissimilar isotropic materials are proposed. They are based on elastic dissimilarity factors and two factors obtained for cracks in a homogeneous material: the effective spring stiffnesses for non-interacting (independent)
cracks and the crack interaction factor. The crack interactions as a function of crack density for square and hexagonal configuration are obtained by comparing the effective spring stiffnesses for coplanar penny shaped cracks and those for non-interacting cracks. By comparing expressions of the effective spring stiffnesses for non-interacting penny shaped cracks at the interface between two dissimilar materials and those in a homogeneous material, the effect of material dissimilarity on the equivalent spring stiffnesses are expressed in terms of material dissimilarity functions. Since the interfacial spring stiffnesses can be experimentally determined from ultrasound reflection and transmission analysis, the obtained expressions can be useful in estimating the percentage of disbond area between two dissimilar materials, which is directly related to the residual strength of the interface.
Dedicated to my family
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CHAPTER 1. Effective Spring Stiffness for a Planar Periodic Array of Collinear Cracks at an Interface between Two Dissimilar Isotropic Materials

1.1 Introduction

Layered materials are extensively used in various products and devices to improve structural performance such as strength and durability. Thermal barrier coatings, for example, are used in turbine blades to protect the core material from thermal fatigue (Miller, 1987). In the field of dentistry, resin-retained ceramic restorations are performed to protect remaining teeth and restore mechanical function without loss of aesthetics (Wang et al., 2007). Wide applications of adhesive bonds in aerospace industry for both aluminum and composite structures are well documented. Failures of these layered structures are often attributed to interfacial damage in the form of micro-cracks or debonded zones, which are either preexisting or developed during service due to mechanical/thermal fatigue and environmental degradations.

Ultrasonic methods have been widely used to detect and characterize interfacial imperfections such as distributed micro-cracks or micro-disbonds (Thompson and Thompson, 1991; Rokhlin et al., 2004). For modeling of ultrasonic wave interactions at
imperfect interfaces, a quasi-static approximation (Baik and Thompson, 1984; Margetan et al., 1988) has been widely used. In this model the reduction in static stiffness of the overall structure due to compromised interfaces (micro-cracks or micro-disbands) is represented by continuous, uniform distributions of interfacial springs. It has been demonstrated by Angel and Achenbach (1985) that the quasi-static approximation is applicable at low frequencies, when the size of the imperfections is much smaller than the wavelength.

Significant experimental and theoretical advances have been made towards inversion of the interfacial stiffness distributions from ultrasonic measurements (Wang and Rokhlin, 1991; Rokhlin et al., 2004; Baltazar et al., 2003; Wang et al., 2006; Leiderman et al., 2007) and relating the spring stiffness constants to the micromechanical and geometric properties of the micro-cracks (Baik and Thompson, 1984; Lavrentyev and Rokhlin, 1994; Pecorari and Kelly, 2000). Explicit expressions of effective spring stiffness in terms of the crack geometry and density are important since they can be used to estimate the percentage of disbond area (Palmer et al., 1988), which is critical in assessing the bond integrity and the remaining life.

Using an available fracture mechanical model, Baik and Thompson (1984) have obtained the expression for effective normal spring stiffness for a planar array of periodically spaced strip cracks in a homogeneous material. For this geometry, Angel and Achenbach (1985) have obtained an exact solution of the elastodynamic reflection/refraction boundary-value problem and have numerically compared the exact
solution with the quasi-static (spring) approximation. They have demonstrated that for this problem the approximation is suitable for \( a/b < 0.8 \) (\( a/b \): crack density, see Figure 1.1 (a)) at \( 2b/\lambda_T < 0.25 \) (\( \lambda_T \) is a transverse wavelength). The applicability range of the \( 2b/\lambda_T \) ratio decreases with increase of crack density \( a/b \) (increase of crack interaction).

Excellent agreement between the spring approximation and experiment has been obtained by Palmer et al. (1988) who have reported measurements of the ultrasonic reflectivity on imperfectly diffusion-bonded samples.

For non-interacting cracks at the interface between two dissimilar materials, Pecorari and Kelly (2000) have obtained an explicit expression of the effective normal spring stiffness and have shown that their expression reduces to that of Baik and Thompson (1984) when elastic moduli difference and crack area fraction approach zero. While crack interactions have been investigated in regard to overall effective material properties (Nemat-Nasser et al., 1993), the analysis of effective spring stiffness for interacting cracks between dissimilar materials is not available.

In this work, for a planar periodic array of collinear cracks between two dissimilar isotropic materials (Figure 1.1 (a)), explicit expressions for effective transverse and normal spring stiffnesses have been obtained; they are shown to be identical in a general case of elastic dissimilarity. We have examined the effect of an elastic dissimilarity and crack interaction on the effective interfacial spring stiffness (Figure 1.1 (b)). Our derivation is based on the elastic analysis of a periodic array of cracks between two dissimilar materials under normal and transverse loading in the framework of the open
crack model (Williams, 1959; England, 1965; Erdogan, 1965; Rice, 1988; Hutchinson and Suo, 1992). By inserting the stress intensity factor (Rice and Sih, 1965; Sih, 1973) into the expression for the strain energy release rate (Hutchinson and Suo, 1992) and using Castigliano’s theorem extended for cracked bodies (Tada et al., 2000), the effective spring stiffness for cracks between dissimilar materials is obtained.

The effect of elastic dissimilarity, crack density and crack interaction on the effective spring stiffness is explicitly expressed, and it is demonstrated that the effect of elastic dissimilarity on the crack interaction is insignificant and the crack interaction is nearly the same as that for crack arrays between identical solids. This allows approximate factorization of the effective spring stiffness for an array of cracks between dissimilar materials in terms of the elastic dissimilarity factor (elastic mismatch), the crack interaction factor, and the effective spring stiffness for non-interacting (independent) cracks in a homogeneous material as shown in Figure 1.1 (c). Both the theoretical limitations related to the open crack assumption and the practical usability of the obtained results in ultrasonic method applications are discussed.
Figure 1.1 Effective spring stiffness for a planar periodic array of collinear cracks between two dissimilar isotropic materials
1.2 Strain energy release rate for an array of interfacial cracks between two dissimilar materials

1.2.1 The strain energy release rate expressed in terms of $\sigma$ and $\tau$

Consider an infinite array of equal cracks of length $2a$ spaced at a constant interval $2b$ along the bond line of two dissimilar isotropic materials with Young’s moduli of $E_1$, $E_2$ and Poisson’s ratios of $\nu_1$ and $\nu_2$. As shown in Figure 1.1 (a), uniform normal ($\sigma$) and shear ($\tau$) stresses are applied at infinity. The complex stress intensity factor, $K$, is obtained by Rice and Sih (1965) as

$$K = K_I + iK_{II} = \sqrt{\pi} \cosh(\pi\epsilon)(k_1 + ik_2)$$  \hspace{1cm} (1.1)$$

where parameters $k_1$, $k_2$ are given as following

$$k_1 = \frac{2\sqrt{\frac{b}{\pi \sin(\pi a / b)}}}{\cosh(\pi \epsilon)} \left[ \sigma \left\{ \sin(\pi a / 2b) \cosh(\pi a\epsilon / b) \cos \left( \epsilon \log \left( \frac{2b}{\pi} \sin(\pi a / b) \right) \right) \right\} + \cos(\pi a / 2b) \sinh(\pi a\epsilon / b) \sin \left( \epsilon \log \left( \frac{2b}{\pi} \sin(\pi a / b) \right) \right) \right]$$

$$+ \tau \left\{ \sin(\pi a / 2b) \cosh(\pi a\epsilon / b) \cos \left( \epsilon \log \left( \frac{2b}{\pi} \sin(\pi a / b) \right) \right) \right\}$$  \hspace{1cm} (1.2)$$

and

$$k_2 = -\frac{2\sqrt{\frac{b}{\pi \sin(\pi a / b)}}}{\cosh(\pi \epsilon)} \left[ \sigma \left\{ -\sin(\pi a / 2b) \cosh(\pi a\epsilon / b) \sin \left( \epsilon \log \left( \frac{2b}{\pi} \sin(\pi a / b) \right) \right) \right\} + \cos(\pi a / 2b) \sinh(\pi a\epsilon / b) \cos \left( \epsilon \log \left( \frac{2b}{\pi} \sin(\pi a / b) \right) \right) \right]$$

$$+ \tau \left\{ \sin(\pi a / 2b) \cosh(\pi a\epsilon / b) \cos \left( \epsilon \log \left( \frac{2b}{\pi} \sin(\pi a / b) \right) \right) \right\}$$  \hspace{1cm} (1.3)$$
(Derivation of $K_i + iK_{ii}$ is given in Appendix A.)

$\varepsilon$ is called the oscillation index that can be expressed in terms of the second Dundurs’ parameter $\beta$ (Dundurs, 1967, Dundurs and Bogy, 1969):

$$\varepsilon = \frac{1}{2\pi} \log \left( \frac{1+\beta}{1-\beta} \right), \tag{1.4}$$

$$\beta = \frac{G_2(\kappa_i - 1) - G_1(\kappa_i - 1)}{G_2(\kappa_i + 1) + G_1(\kappa_i + 1)}, \tag{1.5}$$

where $G_1$ and $G_2$ are the shear moduli of materials 1 and 2, and

$$\kappa_i = 3 - 4\nu_i, \quad i = 1, 2 \quad \text{for plane strain,} \tag{1.6}$$

$$\kappa_i = (3 - \nu_i)/(1 + \nu_i), \quad i = 1, 2 \quad \text{for plane stress.} \tag{1.7}$$

We use the strain energy release rate (SERR) in a general form given by Hutchinson and Suo (1992) (Eq. (2.29))

$$W = \frac{1}{E^* \cosh^2(\pi \varepsilon)} K \overline{K}, \tag{1.8}$$

where $E^*$ is

$$\frac{1}{E^*} = \frac{1}{2} \left( \frac{1}{E_1} + \frac{1}{E_2} \right) = \frac{(1 + \kappa_i)}{8(1 + \alpha)G_1}, \tag{1.9}$$

(See Appendix B) and
\[ \bar{E}_i = \frac{E_i}{1 - v_i^2} \text{ for plane strain}, \quad (1.10) \]

\[ \bar{E}_i = E_i \text{ for plane stress } (i=1,2). \quad (1.11) \]

The parameter \( \alpha \) in Eq. (1.9) is the first Dundurs’ parameter [Dundurs (1967); Dundurs and Bogy (1969)], which is expressed in the form:

\[
\alpha = \frac{G_2(\kappa_1 + 1) - G_1(\kappa_2 + 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)} = \frac{\bar{E}_2 - \bar{E}_1}{\bar{E}_2 + \bar{E}_1}, \quad (1.12)
\]

Thus both Dundurs’ parameters \( \alpha \) and \( \beta \) characterize the elastic mismatch between two semispaces.

By inserting Eqs. (1.1), (1.2) and (1.3) into Eq.(1.8), the SERR can be expressed in terms of \( \sigma \) and \( \tau \) as follows:

\[
W_{\text{array \ dissimilar}}^\text{array} = \frac{1}{4} \frac{(1 + \kappa_1)(1 - \beta^2)}{G_1(1+\alpha)} \pi a \left\{ \cosh^2 \left( \frac{\pi a \varepsilon}{b} \right) - \cos^2 \left( \frac{\pi a}{2b} \right) \right\} \left( \sigma^2 + \tau^2 \right) \quad (1.13)
\]

(Derivation of Eq. (1.13) is given in Appendix C). The subscript ‘dissimilar’ indicates that the two materials are different, and the superscript ‘array’ indicates that the cracks are periodically aligned. By using Taylor series expansions and letting \( a/b \) approach zero, Eq. (1.13) reduces to the SERR for non-interacting (independent) interfacial cracks.

\[
W_{\text{non-interacting \ dissimilar}}^\text{non-interacting} = \frac{1}{8} \frac{(1 + \kappa_1)(1 - \beta^2)}{G_1(1+\alpha)} \pi a \left( 1 + 4\varepsilon^2 \right) \left( \sigma^2 + \tau^2 \right) \quad (1.14)
\]
(See Appendix D for detailed derivation). The same expression can be obtained directly by using the SIFs for non-interacting cracks given in Rice and Sih (1965).

By setting $\alpha = \beta = \varepsilon = 0$, Eq. (1.13) reduces to the SERR for an array of cracks in a homogenous material.

$$W_{\text{array}}^{\text{homogenous}} = \frac{1}{8} \frac{(1 + \kappa_1)}{G_1} \pi a \left\{ \frac{\tan \left( \frac{\pi a}{2b} \right)}{\left( \frac{\pi a}{2b} \right)} \right\} \left( \sigma^2 + \tau^2 \right)$$

(1.15)

Similarly, Eq. (1.14) reduces to the SERR for a non-interacting crack in a homogenous material.

$$W_{\text{non-interacting}}^{\text{homogenous}} = \frac{1}{8} \frac{(1 + \kappa_1)}{G_1} \pi a \left( \sigma^2 + \tau^2 \right)$$

(1.16)

1.2.2 Effect of Material Dissimilarity and Crack Interaction on the SERR

It is advantageous to represent the SERR equation (1.13) with use of Eqs. (1.14) and (1.16) as

$$W_{\text{array}}^{\text{dissimilar}} = I_{W} \times M_{W} \times W_{\text{non-interacting}}^{\text{homogenous}},$$

(1.17)

where

$$I_{W} \left( \frac{a}{b}, \varepsilon \right) = \frac{W_{\text{array}}^{\text{dissimilar}}}{W_{\text{non-interacting}}^{\text{dissimilar}}} = \frac{\cosh^2 \left( \frac{\pi a \varepsilon}{b} \right) - \cos^2 \left( \frac{\pi a}{2b} \right)}{\left( \frac{\pi a}{2b} \right) \sin \left( \frac{\pi a}{b} \right)} \frac{2}{\left( 1 + 4\varepsilon^2 \right)},$$

(1.18)
and

\[ M_W(\alpha, \beta) = \frac{W_{\text{non-interacting}}^{\text{dissimilar}}}{W_{\text{non-interacting}}^{\text{homogeneous}}} = \frac{(1 - \beta^2)(1 + 4\varepsilon^2)}{(1 + \alpha)}. \] (1.19)

It is important to note that \( I_W \), defined by Eq.(1.18), describes the effect of crack interaction on the SERR and depends on both the oscillation index \( \varepsilon \), which is a function of the elastic properties of both semispaces, and the crack density \( a/b \). The factor \( M_W \) (Eq.(1.19)) depends solely on the dissimilarity of the elastic properties of both semispaces, and it represents the effect of material dissimilarity on the SERR. It is also important to note that for an arbitrary bi-material combination \(|\beta| < 0.5\) and \(|\varepsilon| < \frac{1}{2\pi} \ln(3) \approx 0.1748\), where the limits are reached in the extreme cases when the shear modulus ratio \( G_2/G_1 \) and the Poisson’s ratio \( \nu_2 \) (or \( G_1/G_2 \) and \( \nu_1 \)) approach zero under the plane strain condition. In the extreme case where \( \beta = 0.5 \) and \( \alpha = 1 \), material factor \( M_W \) approximately reaches the minimum value of 0.4209, and it becomes equal to 1 for identical semispaces (\( \beta = 0, \alpha = 0 \)). As the value of \( \alpha \) approaches -1 (\( G_2/G_1 \) approaches zero), \( M_W \) becomes unbounded. Therefore, Eq. (1.17) shows that the SERR for a periodic array of cracks at the interface between two dissimilar materials can be exactly factorized as a product of three components: elastic dissimilarity function, \( M_W \), crack interaction function, \( I_W \), and the SERR for non-interacting cracks in a homogenous material.

For most practical applications, \( \varepsilon \) remains small (\(|\varepsilon| < 0.05\)). In this case, using Taylor series expansions, the crack interaction function, \( I_W \), can be approximated as
\[
I_w\left(\frac{a}{b}, \varepsilon\right) \approx I_w^{\text{approximate}}\left(\frac{a}{b}\right) = \tan\left(\frac{\pi a}{2b}\right).
\] (1.20)

(See Appendix E for derivation)

Thus \(I_w^{\text{approximate}}\) depends only on the crack geometry and is independent of the oscillation index, \(\varepsilon\) (elastic mismatch). It is worth noting here that this function is identical to the interaction function for cracks in a homogenous material defined by

\[
I_w^{\text{homogenous}}\left(\frac{a}{b}\right) = \frac{W^{\text{array}}_{\text{homogenous}}}{W_{\text{non-int}}^{\text{homogenous}}} = \tan\left(\frac{\pi a}{2b}\right).
\] (1.21)

To illustrate the accuracy of this approximation we introduce the relative error \(e\) (in percent)

\[
e\left(\frac{a}{b}, \varepsilon\right) = \frac{I_w^{\text{approximate}}\left(\frac{a}{b}\right) - I_w\left(\frac{a}{b}, \varepsilon\right)}{I_w\left(\frac{a}{b}, \varepsilon\right)} \times 100.
\] (1.22)

Figure 1.2 shows the error \(e\) contours in coordinates of the oscillation index \(\varepsilon\) and crack density \(a/b\). Some practical material combinations taken from Hutchinson et al. (1987) are also indicated by horizontal dashed lines. As can be seen, the error reaches maximum as the material mismatch and crack density approach maximum values. However, for most practical applications, \(e\) is less than 2.5% even for crack density \(a/b\) approaching one (when spacing between the cracks is zero). Therefore, it is possible to neglect the
effect of elastic dissimilarity on the crack interaction for most realistic applications and use as an interaction function that obtained for the identical semispaces.

Figure 1.2 Contour plot of percent error of approximate interaction function $I_W(a/b)$ relative to the exact interaction function $I_W(a/b, \varepsilon)$ with some material combination examples (Example combinations are taken from Hutchinson et al. 1987.)
1.3 Effective interfacial spring stiffness

1.3.1 Effective spring stiffness

In the problem considered in Figure 1.1 (a), the far field displacement due to applied tractions can be separated into displacement components of perfectly bonded semispaces (no cracks at the interface) and an additional displacement due to the presence of the crack array.

\[ \Delta = \Delta_{\text{perfect-bonding}} + \Delta_{\text{crack}} \]  

(1.23)

The idea is to replace the array of cracks by continuously distributed interfacial springs, such that they provide the same additional interface compliance (additional displacement \( \Delta_{\text{crack}} \)) as due to the crack array (Figure 1.1 (b)). Following the work by Baik and Thompson (1984), the normal and tangential spring stiffness, \( k_N \) and \( k_T \), are defined by

\[ k_N = \frac{\sigma}{\Delta_{N,\text{crack}}} , \quad k_T = \frac{\tau}{\Delta_{T,\text{crack}}} , \]  

(1.24)

where \( \Delta_{N,\text{crack}} \) and \( \Delta_{T,\text{crack}} \), respectively, represent the additional displacement in the normal and transverse directions. The displacements can be calculated by using Castigliano’s theorem extended to cracked bodies (Tada et al., 2000):

\[ \Delta_{\text{crack}} = \frac{\partial}{\partial Q} \left( 2 \int_0^a W dx \right) , \]  

(1.25)
where $Q$ is the applied load per unit crack length along the coordinate $z$. Eq. (1.25) can also be obtained without invoking Castigliano’s theorem. The strain energy (due to introducing the crack) in the loaded body can be written in terms of the interfacial stiffness and the far-field tractions by considering the work done in applying these tractions. The displacement due to the crack can then be obtained by differentiating the strain energy with respect to the far-field load. Since the strain energy release rate is a known function, the strain energy can be obtained by integrating with respect to the crack length. By inserting Eq. (1.13) into Eq. (1.25) and use of (1.24), we obtain

$$
\left(\begin{array}{c}
G_{i} \\
1 - \beta^2
\end{array}\right) L\left(\frac{a}{b}, \epsilon\right),
$$

(1.26)

where

$$
L\left(\frac{a}{b}, \epsilon\right) = \int_{0}^{\frac{a}{b}} \frac{1}{\sin(\pi x)} \left[ \cosh^2\left(\frac{\pi x}{2}\right) - \cos^2\left(\frac{\pi x}{2}\right) \right] dx
$$

(1.27)

(See Appendix F for detailed derivation)

Equality of normal and transverse spring stiffness equation (1.26) is a generalization of a known result for a 2D crack array in a homogeneous space. The result is not obvious due to coupling of normal and transverse tractions in the stress intensity factors (Eqs. (1.1)-(1.3)).

In Eq.(1.27), crack density, $a/b$, and elastic mismatch represented by the oscillation index, $\epsilon$, are coupled. In Section 1.2.2, however, we have shown that the
The effect of crack interaction and elastic mismatch on the SERR can be approximately decoupled when $\varepsilon$ is small (which is the case for most realistic material pairs). Motivated by the results in Section 1.2.2, the Taylor series expansion is employed in Eq. (1.27) for small $\varepsilon$ ($|\varepsilon|<0.05$) which results in the following approximation for $L\left(\frac{a}{b}, \varepsilon\right)$:

$$L\left(\frac{a}{b}, \varepsilon\right) \approx L_{\text{approximate}}\left(\frac{a}{b}, \varepsilon\right) = \frac{(1+4\varepsilon^2)}{\pi} \ln \left[ \sec \left( \frac{\pi a}{2b} \right) \right].$$

(1.28)

By inserting Eq. (1.28) into Eq.(1.26), we have

$$k_{k, \text{dissimilar}}^{\text{array}, \text{approximate}} = k_{T, \text{dissimilar}}^{\text{array}, \text{approximate}} = \frac{G_1}{b(1+\kappa_1)} \frac{(1+\alpha)}{(1-\beta^2)(1+4\varepsilon^2)} \pi \left( \ln \left[ \sec \left( \frac{\pi a}{2b} \right) \right] \right)^{-1}.$$

(1.29)

The effect of elastic dissimilarity and crack density on the spring stiffness are explicitly expressed in terms of $\alpha$, $\beta$, $\varepsilon$ and $a/b$ in both the exact (Eqs. (1.26) and (1.27)) and the approximate (Eq.(1.29)) forms. In the approximate form (Eq.(1.29)), however, the elastic mismatch and the crack density are no longer coupled as they are in the exact expression for $L(\varepsilon, a/b)$ in Eq.(1.27). It can be shown that both Eqs. (1.26) and (1.29) are symmetric with respect to indices 1 and 2 for material parameters based on the identity $G_1 (1+\alpha)/(1+\kappa_1) = G_2 (1-\alpha)/(1+\kappa_2)$.

The exact spring stiffness for non-interacting interfacial cracks in the dissimilar material case can be similarly obtained by using Eqs.(1.14), (1.24) and (1.25).

$$k_{k, \text{dissimilar}}^{\text{non-interacting}} = k_{T, \text{dissimilar}}^{\text{non-interacting}} = \frac{8}{\pi} \frac{b}{a^2 \left(1+\kappa_1\right)} \frac{G_1 (1+\alpha)}{(1-\beta^2)(1+4\varepsilon^2)}.$$

(1.30)
(Derivation is presented in first part of the Appendix G). It can be shown that as $a/b$ (crack density) approaches zero the approximate spring expression (1.29) for the array of cracks becomes Eq.(1.30), i.e. Eq. (1.30) is a limiting case of Eq.(1.29). Thus, the approximate equation (1.29) for interacting cracks, which neglects the effect of material dissimilarity on the crack interaction, becomes exact for the case of noninteracting cracks between dissimilar semispaces; this is a consistent result since the crack interaction in Eq. (1.30) is absent. One can show that Eq. (1.30) is identical to the result obtained by Pecorari and Kelly (2000), who have examined the spring stiffness for non-interacting cracks (see second part of Appendix G for details).

Eqs. (1.26) and (1.30), respectively, are reduced to the spring stiffness for the homogenous case by setting $\alpha = \beta = \epsilon = 0$, 

$$k_{阵,N, homogenous}^{array} = k_{T, homogenous}^{array} = \frac{G_i}{b(1 + \kappa)} \pi \left[ \ln \left\{ \sec \left( \frac{\pi a}{2b} \right) \right\} \right]^{-1},$$  \hspace{1cm} (1.31)

and

$$k_{N, homogenous}^{non-interacting} = k_{T, homogenous}^{non-interacting} = \frac{8 b}{\pi a^2} \frac{G_i}{(1 + \kappa)},$$  \hspace{1cm} (1.32)

where the results obtained by Baik and Thompson (1984) are recovered with material 1 representing the homogeneous material.

### 1.3.2 Effect of elastic dissimilarity of semi-spaces on the spring stiffness

When comparing the effective spring expressions for non-interacting cracks between homogenous material layers (Eq. (1.32)) and between dissimilar materials
(Eq.(1.30)), we find that they differ by the factor \( (1+\alpha)/[(1-\beta^2)(1+4\varepsilon^2)] \). It is important to note that the same factor appears as the ratio of the approximate form of the effective spring stiffness for an array of cracks between dissimilar materials to that between homogenous material layers. Therefore, we define \( M_k \) by the ratio of Eqs. (1.30) and (1.32) and that of Eqs. (1.29) and (1.31) as follows:

\[
M_k(\alpha, \beta) = \frac{k_{\text{non-interacting}}}{k_{\text{homogenous}}} = \frac{k_{\text{array approximate}}}{k_{\text{array exact}}} = \frac{(1+\alpha)}{(1-\beta^2)(1+4\varepsilon^2)} \tag{1.33}
\]

where the subscript \( N \) and \( T \) are dropped since the normal and tangential spring stiffness are identical. Note that \( M_k = 1/M_w \), where \( M_w \) was introduced by Eq.(1.19); the relation is consistent with Eqs ((1.24), (1.25)). Since the Dundurs’ parameters \( \alpha, \beta \) and the oscillation index, \( \varepsilon \), all characterize the elastic dissimilarity between the semispaces, we can interpret \( M_k \) as the elastic dissimilarity factor (representing the effect of elastic mismatch) for the effective spring stiffness for both non-interacting cracks and an array of cracks, and \( M_k \) reduces to one for the homogeneous case (\( \alpha=\beta=\varepsilon=0 \)).

Figure 1.3 shows the elastic dissimilarity factor, \( M_k \), as contour plots as a function of \( \alpha \) and \( \beta \) for all possible material combinations. Since \( M_k \) is the inverse of \( M_w \) which varies from 0.4209 to infinity (see Section 1.2.2), it is shown that \( M_w \) varies from 0 to 2.376. Each cross symbol in Figure 1.3 represents a material combination provided by Suga et al. (1988) under the plane strain condition. Some commonly used material combinations are labeled with diamond shaped points. The \( M_k \) contours are nearly parallel to the \( \beta \) axis indicating that the \( \alpha \) parameter is the dominant one in determining the value of \( M_k \). At an \( \alpha \) near \( \alpha \geq -1 \), the contours are straight vertical lines. However,
the curvature of contours is increasing with $\alpha$ reaching a maximum at $\alpha = +1$. Note that $M_k$ is not symmetric with respect to indices 1 and 2 for the material parameters since transition to the homogeneous material is made by assuming that both materials are material 1 (Eq.(1.32)).

Figure 1.3 Contour plot of elastic dissimilarity function for effective spring stiffness. Each cross symbol (x) represents a material combination provided by Suga et al. (1988) for plane strain. Some material combinations are indicated by circle and labeled.
Gorbatikh and Popova (2005) have shown that normal elastic compliances of non-interacting rectilinear, penny-shaped, elliptical, and annular non-interacting cracks between two dissimilar materials can be approximated by solutions for a homogeneous case through the common material parameter (Eq. (9) in Gorbatikh and Popova (2005)). It can be shown that their common material parameter, when expressed in terms of the Dundurs’ parameters, is \((1+\alpha)/(1-\beta^2)\). One notes that it differs from \(M_A\) equation (1.33) by the factor \(1/(1+4\varepsilon^2)\), which is close to one since the maximum value of \(\varepsilon\) is \(\frac{1}{2\pi}\ln(3) \approx 0.1748\); and thus \(M_A\) can be approximated by the material dissimilarity parameter introduced by Gorbatikh and Popova (2005). It is a significant result, since the elastic dissimilarity factor equation (1.33) is applicable to the case of crack interaction, and thus one can propose to extend the applicability of Eq. (1.33) from the 2D crack array to the interacting 3D plane cracks on an interface between dissimilar materials.

1.3.3 The effect of crack interaction on the spring stiffness

It is well known (Kachanov (1994), Pecorari and Kelly (2000)) that interference between the crack tips for an array of periodic cracks in a homogenous material can be neglected if the crack density is less than 50%. Exact equations (1.26) and (1.27) allow us to examine the validity of this non-interacting (independent) crack approximation in the context of effective interfacial spring stiffness between dissimilar materials.

In order to make a quantitative error estimate of crack interaction, we consider the ratio of effective spring stiffness for an array of cracks to that for non-interacting cracks.
\[ I_k \left( \frac{a}{b}, \varepsilon \right) = \frac{k_{\text{array \, dissimilar}}}{k_{\text{non-interacting} \, \text{dissimilar}}} = \frac{\pi}{8} \frac{a^2}{b^2} \left( 1 + 4\varepsilon^2 \right), \]

which depends on the integral representation of \( L \left( \frac{a}{b}, \varepsilon \right) \) in Eq.(1.27). Given specific values of crack density \( a/b \) and the oscillation index \( \varepsilon \), \( I_k \) describes the effect of crack interactions on the effective spring stiffness. Inserting Eq. (1.28) into Eq.(1.34), we obtain the approximate form of the interaction factor, which is identical to that for the homogeneous case (\( \varepsilon = 0 \) in Eq.(1.34))

\[ I_k \left( \frac{a}{b}, \varepsilon \right) \approx \frac{k_{\text{array \, approximate} \, \text{dissimilar}}}{k_{\text{non-interacting} \, \text{dissimilar}}} = I_k \left( \frac{a}{b}, \varepsilon \right) \approx \frac{\pi}{8} \frac{a^2}{b^2} \left[ \ln \left( \sec \left( \frac{\pi a}{2b} \right) \right) \right]^{-1}. \]

(See Appendix I for the proof of equivalence of Eq. (1.35) and the one obtained from Baik and Thompson, 1984). Based on Eqs. ((1.34), (1.35)), the effect of elastic mismatch on crack interactions can be examined for all possible values of crack density, \( a/b \), ranging from one (non-interacting cracks or “independent” cracks) to zero (complete debond).

The exact crack interactions factor, equation (1.34), versus \( a/b \) is shown with solid lines in Figure 1.4 for the maximum value of \( \varepsilon = \frac{1}{2\pi} \ln(3) \approx 0.1748 \) and \( \varepsilon = 0 \) (the \( \varepsilon = 0 \) case is identical to Eq.(1.35)). The interaction functions for both \( \varepsilon = 0 \) (homogeneous case) and \( \varepsilon = 0.1748 \) remain close to 0.9 at \( a/b = 0.5 \) in the crack array, indicating that the crack interaction should be taken into account at crack density above 50% for both homogeneous and dissimilar material cases. In order to elucidate the effect of \( \varepsilon \) on the
interaction function, the ratios of the exact interaction function ($\varepsilon = 0.05, 0.1$ and $0.1748$) and the interaction function for the homogeneous case ($\varepsilon = 0$) are also shown by dashed lines in Figure 1.4.

Figure 1.4 Exact interaction function for effective spring stiffness with $\varepsilon \approx 0.1748$ and $\varepsilon = 0$ (solid lines) and ratios of exact interaction function ($\varepsilon \approx 0.1748$, $\varepsilon = 0.1$ and $\varepsilon = 0.05$) with the interaction function to homogenous case $\varepsilon = 0$ (dashed lines)
It is interesting to note that the $I_k$ curve shown by the solid line for the homogeneous case ($\varepsilon=0$) is slightly higher than the $I_k$ curve with $\varepsilon=0.1748$ over the wide $a/b$ range, and consequently the dashed $I_k$ ratio lines with $\varepsilon=0.05$, 0.1 and 0.1748 are all below 1.0. This indicates that the interaction between cracks is slightly stronger for the dissimilar material case. With the maximum value of $\varepsilon=0.1748$, the interaction is highest. However the differences are not large; even for the extreme case of 95% of the crack density ($a/b=0.95$), the $I_k$ ratio remains larger than 85% as shown by the dashed line for $\varepsilon=0.1748$. Considering that for most practical material pairs $\varepsilon$ is smaller than 0.05, the effect of elastic dissimilarity on the crack interaction is negligible as is evident from Figure 1.4, where the dashed $I_k$ ratio line with $\varepsilon=0.05$ remains at almost 1.0 for all values of crack density, $a/b$.

It is important to note that combination of Eqs.(1.26), (1.33) and (1.34) and that of Eqs.(1.29), (1.33) and (1.35), respectively, leads to

\[
k^{\text{array}}_{\text{dissimilar}} = I_k \times M_k \times k^{\text{non-interacting}}_{\text{homogenous}}, \tag{1.36}
\]

\[
k^{\text{array, approximate}}_{\text{dissimilar}} = I_k^{\text{homogenous}} \times M_k \times k^{\text{non-interacting}}_{\text{homogenous}}. \tag{1.37}
\]

As for the SERR, the effective spring stiffness for the array of cracks between dissimilar materials can be factorized in terms of crack interaction function $I_k$, elastic dissimilarity function $M_k$, and effective spring stiffness of non-interacting cracks for a homogenous material $k^{\text{non-interacting}}_{\text{homogenous}}$. For most realistic material pairs, the crack interaction in the dissimilar-material case can be approximated by that in a homogenous material.
It was noted (Kachanov and Laures, 1989) that the crack interaction in the homogeneous space is strongest for 2D strip cracks relative to 3D crack shapes. Thus our estimate provides an upper bound for the error when one replaces the spring interaction function for cracks between dissimilar semispaces by the same plane crack distribution in the homogeneous space.

1.4 Load validity range for ultrasound applications

1.4.1 Evaluation of the valid loading range based on the open model

All the derivations presented in Sections 1.2 and 1.3 are based on the assumption that crack faces are traction free (open) (Erdogan (1963), Williams (1959), Rice and Sih (1965)). The solution obtained under this condition, however, leads to oscillatory crack interpenetration zones at the crack tips, where the traction free condition, and thus solution, are not applicable. Comninou and co-workers (Comninou (1977), Comninou and Schmueser (1979), Schmueser and Comninou (1979)) have addressed the interpenetration zone problem by assuming a frictionless contact between the crack faces near the crack tips (the length of this contact zone is determined as a part of the problem solution). The replacement of the traction free boundary conditions by a unilateral contact zone at the crack tips leads to a nonlinear problem and a complicated iterative solution (See Appendix K for details). However, if the interpenetration zone is much smaller than the crack length, the “small-scale contact conditions” (Rice, 1988) are satisfied, and the crack tip conditions are completely characterized by the open crack solution. Therefore, we would like to identify the range of loading for which small-scale contact conditions are satisfied and Eqs. (1.26) and (1.27) can be used.
Hills and Barber (1993) have provided excellent discussions of both the open formulation and the contact formulation. They have proposed a general expression for estimating the size of the interpenetration zone and have estimated the Comninou contact zone size from that of the interpenetration zone, which is larger than the contact zone. Graciani et al. (2007) have noticed that the Hills and Barber (1993) expression for the interpenetration zone is valid only for positive $\varepsilon$ and have modified it to be suitable for both positive and negative $\varepsilon$.

We employ the work by Graciani et al. (2007) to predict the interpenetration region for an array of cracks. Since materials 1 and 2 can be interchanged by allowing both positive and negative values of $\varepsilon$ without loss of generality, $\tau$ is applied in the positive $x$ direction. By combining Eq. (11) in Graciani et al. (2007) and Eqs. ((1.1)-(1.3)), we obtain the extent of the interpenetration zone $r_i$ from the right crack tip as

$$
\frac{r_i}{2a} = \frac{\sin \left( \frac{\pi a}{b} \right)}{\pi \frac{a}{b}} \exp \left[ \frac{1}{|\varepsilon|} \left( 2n - \frac{1}{2} \right) \pi - \tan^{-1} \left( \frac{\tanh \left( \frac{\pi a e}{b} \right)}{\tan \left( \frac{\pi a}{2b} \right)} \right) \text{sgn} \varepsilon - \cot^{-1} \left( \frac{\sigma}{\tau} \right) \text{sgn} \varepsilon + \tan^{-1} \left( 2|\varepsilon| \right) \right] \quad (1.38)
$$

(See Appendix J for detailed derivation).

Based on Eq. (1.38), it can be shown that the maximum interpenetration occurs at the right crack tip for negative $\varepsilon$. When the sign of $\varepsilon$ is switched to positive, the location of the maximum interpenetration zone is switched to the left crack tip. Therefore, for the purpose of evaluating the maximum interpenetration zone, it is sufficient to consider negative values of $\varepsilon$. When the interpenetration zone is limited to less than 1% of the
full crack length (Rice, 1988) \((r_l/2a<0.01)\), the valid loading range is estimated from Eq. (1.38) as follows:

\[
\left(\frac{\sigma}{\tau}\right) \geq \cot \left( \frac{\pi}{2} - \tan^{-1} \left( \frac{\tanh(\pi a \varepsilon / b)}{\tan(\pi a / 2b)} \right) + \tan^{-1} \left( 2 \varepsilon - \varepsilon \ln \left[ \frac{0.01 (na)}{b} \right] \right) \right) \tag{1.39}
\]

The minimal values of stress ratio \(\sigma/\tau\) required for negligible contact zone, as prescribed by Eq.(1.39), are given in Figure 1.5 as contour plots. For most material combinations, the minimal stress ratio \(\sigma/\tau\) varies from zero to one, indicating that the interpenetration zone is negligible as long as the tensile stress is larger than the shear stress. Only when \(\varepsilon\) is zero (zero contour line, i.e. the homogeneous case), can far field pure shear be applied without encountering problems related to the interpenetration zone. Contour lines are approximately parallel to the \(a/b\) axis (for \(a/b<0.7\)). Thus in this crack density range the minimal stress ratio required for negligible contact zone is mostly controlled by the parameter \(\varepsilon\) and is independent of crack density (this is clear since the crack interaction is small).
1.4.2 Ultrasound applications

Ultrasonic wave interaction with cracks is linear only if the crack surface displacements produced do not induce contact of crack surfaces. It is well known that ultrasonic waves reflect poorly from tightly closed cracks; and for closed interfacial cracks the ultrasonic wave reflection from a dissimilar material interface occurs in the same way as for the case without cracks. Thus experimental detection of interfacial...
cracks by ultrasonic waves is possible only when the cracks are slightly open, which is often the case due to loading history and plastic deformation. If under external or residual compressive stresses the crack is partially closed, the ultrasonic waves are perturbed only by its open part. If the elastodynamic displacements associated with the ultrasonic wave bridges a crack opening displacement, the wave interaction is nonlinear. The displacement amplitude of ultrasonic waves may vary by several orders of magnitude. Absolute amplitude measurements have been performed by optical interferometry methods and have been found to be in the sub-Angstrom to several nanometer range (Palmer et al., 1977 and Fick and Palmer, 2001).

Our model is based on the assumption that the crack surfaces are stress-free while the gap between the crack surfaces is zero in the absence of normal tension. To avoid crack surface interpenetration under ultrasonic shear load, one need to apply a tension load to open the crack, thus mimicking realistic open crack conditions. Below we will estimate the degree of crack opening needed for the validity of the spring model (Eq. (1.26)) under an incident shear wave. The condition obtained will also satisfy the open crack model for the incident longitudinal wave.

By applying far field equivalent tensile stress, we can produce the required crack opening displacement and open the closed crack by the required amount. Therefore, even under pure shear load condition produced by an ultrasonic wave, the loading may be viewed as a combination of shear and equivalent tensile loads where the small-scale contact condition remains valid. This leads to a practical question as to what is the minimum initial crack opening required for the small-scale contact condition to be
satisfied. To estimate this, we shift our attention from an array of cracks to independent (non-interacting) cracks (Hills et al., 1996). Since the crack opening is smallest when the cracks are not interacting (as can be shown in the homogeneous case based on the work by Koiter, 1959), the estimation we are seeking will provide us the most conservative estimation of minimal initial crack opening required. The midpoint of crack opening displacement, \( \delta_{\text{max}} \), of a single crack subjected to infinity equivalent tensile stress, \( \sigma_0 \), (Hills et al., 1996, Eq. (4.8)) can be expressed as

\[
\delta_{\text{max}} = \frac{\sigma_0}{2G_1} \frac{\tau}{\alpha} A(\alpha, \beta, \kappa_1),
\]

(1.40)

where

\[
A(\alpha, \beta, \kappa_1) = (1 + \kappa_1) \frac{\sqrt{1 - \beta^2}}{(1 + \alpha)}.
\]

(1.41)

In Eq. (1.40), we have used multiplication and division by an arbitrary magnitude of shear stress \( \tau \), so the effect of shear and tensile combined loads on the maximum crack opening displacement can be examined. In Figure 1.6, a contour plot of \( A \) as a function of \( \alpha \) and \( \beta \) for \( \nu_1 = 0.33 \) is shown with several examples of material combinations labeled by circular points. Extreme material lines with \( \kappa_2 = 1 \) and 3 (\( \nu_2 = 0.5 \) and 0) are also shown in the figure. The contour lines are almost parallel to the \( \beta \) axis, indicating that the effect of \( \beta \) on the value of \( A \) is small. Utilizing the weak dependence on \( \beta \), we can write approximately
Figure 1.6 Contour plot of parameter $A(\alpha, \beta, \kappa_1)$ for $\nu_1=0.33$. Some material combinations with $\nu_1=0.33$ are indicated by circle and labeled.

$$ A \approx \frac{(\kappa_1 + 1)}{(1 + \alpha)} = \frac{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}{2G_2}. \quad (1.42) $$

As can be seen from Eqs. (1.41) and (1.42), $A$ is unbounded as $\alpha$ approaches -1 ($G_2/G_1=0$). The value of $A$ reaches approximately 100 for all values of $\beta$ when $\alpha$ is -0.98. For all other values of $\nu_1$ varying from 0 to 0.5, contour plots of $A$ are evaluated in a similar manner, and the maximum possible value of $A$ for most practical material
combinations is estimated to be under 100. As we have already noted in the discussion of Figure 1.5, the minimum value of stress ratio $\sigma_0/\tau$ which satisfies the condition of small scale contact zone for all material combination is approximately one. Assuming $\sigma/\tau = 1$ and $\delta_{\text{max}} / a \approx 10^{-5}$ (10nm for a 1mm crack length), Eq. (1.40) provides an estimation of strain, $\tau / 2G_1$, to be of the order $10^{-7}$; this corresponds well to a reasonable level of strain in linear ultrasonics. Indeed, it will correspond to ultrasonic displacement of about an Å in the low MHz frequency range ($\lambda/2 \sim 1$mm) and in the range of directly measured ($10^{-7} - 10^{-8}$) ultrasonic strain (Alers and Fleury, 1964). This leads to the conclusion that the proposed effective spring stiffness is valid for small cracks as long as a preexisting midpoint crack opening is at least of order $10^{-5}$ of the half crack length. This crack opening also satisfies the open crack model requirement for an incident linear longitudinal ultrasonic wave.

1.5 SUMMARY

The longitudinal and transverse effective spring stiffnesses for a periodic array of collinear cracks located at the interface between two dissimilar materials are obtained based on the well-known open interface crack model. It is shown that they are identical to each other as in the case of elastically equal semispaces (homogeneous case); this result is not obvious due to coupling of normal and transverse tractions in the stress intensity factors. The spring stiffness is explicitly expressed in terms of the geometrical and the micromechanical properties of the interfacial cracks. When the crack density approaches zero, the obtained expression reduces to the effective spring stiffness for non-
interacting cracks (Pecorari and Kelly, 2000). Similarly, the effective spring stiffness for a periodic array of collinear cracks in a homogeneous material (Angel and Achenbach (1985), Baik and Thompson (1984)) is recovered from the results obtained when the elastic dissimilarity of the semispaces vanishes.

It is demonstrated that the spring constant depends on the elastic dissimilarity of semispaces, the crack density and the crack interaction in the array. In general, the crack interaction weakly depends on material dissimilarity and, for most practical cases, this dependence can be replaced approximately by the interaction function of the crack array in the homogeneous space. It is also shown that the effective interfacial spring stiffness for non-interacting cracks can be used approximately for crack density below 50% as in the homogeneous case.

It is well known that in the open crack model the crack opening displacement exhibits an oscillatory crack interpenetration zone at the crack tip, thus violating the open crack assumption. To avoid the effect of the interpenetration zones on the effective spring stiffness the range of the tensile to transverse load ratios is obtained (as a function of the oscillation index, \( \varepsilon \), and the crack density) such that the interpenetration zone is limited to 1% of the crack length. Since real cracks are often slightly open (due to prior loading history and plastic deformation), it is demonstrated for ultrasound applications that the results obtained can be readily used for most practical cases of small interfacial cracks as long as the mid-crack opening normalized by the crack length is at least in the order of \( 10^{-5} \). (It is shown, that this is applicable for the level of shear strain below the order of \( 10^{-7} \) which is suitable for linear ultrasonic waves). Since the interfacial spring
stiffness can be determined from ultrasound reflection and transmission analysis, the proposed expressions can be useful in estimating the percentage of disbond area between two dissimilar materials, which is directly related to the residual strength of the interface.
REFERENCES


CHAPTER 2. Effective Spring Stiffness for Non-Interacting Penny-Shaped Cracks at an Interface between Two Dissimilar, Isotropic, Linearly Elastic Materials

2.1 Introduction

A quasi-static approximation has been widely used to model ultrasonic wave interactions at imperfect interfaces between two linearly elastic materials (Baik and Thompson, 1984, Margetan et al., 1988, Angel and Achenbach, 1985, Nakagawa et al., 2004 and Kim et al., 2004). The reduction in static stiffness of the overall structure due to compromised interfaces (micro-cracks or micro-disbonds) is represented by continuous, uniform distributions of linearly elastic interfacial springs. The accuracy of the quasi-static approximation at low frequencies has been well documented (Baik and Thompson, 1984, Margetan et al., 1988, Angel and Achenbach, 1985). Significant experimental and theoretical efforts have been made to relate the spring stiffness constants to the mechanical and geometric properties of the micro-cracks or micro-disbonds (Palmer et al., 1988, Thompson and Thompson, 1991, Rokhlin et al., 1991, Pecorari and Kelly, 2000, Lavrentyev and Rokhlin, 1998, Baltazar et al., 2003, Wang et al., 2006, Bostrom and Golub, 2008). These relationships are useful in estimating the percentage of disbond area, which is critical in assessing the bond integrity and estimating the remaining life.
For non-interacting penny-shaped cracks or contacts between two identical, isotropic, linearly elastic materials, Baik and Thompson (1984) obtained expressions for normal spring stiffness in terms of disbond area ratio and material properties. The expressions representing the penny-shaped contacts were extended by them to those representing contacts between two dissimilar, isotropic, linearly elastic materials based on the Hertzian contact analysis (Johnson, 1987) of a spherical punch. Margetan et al. (1988) suggested an approximate expression for transverse spring stiffness for non-interacting penny-shaped cracks between two identical materials based on the 2D strip crack analysis. Lavrentyev and Rokhlin (1994) extended their work to the case of two dissimilar, linearly elastic materials by using an average Poisson’s ratio and the Hertzian-based effective Young’s modulus (Baik and Thompson, 1984).

In the present paper, we derive normal and transverse spring stiffness expressions for non-interacting, penny-shaped cracks between two dissimilar, linearly elastic materials based on the classical fracture mechanics work by Mossakovskii and Rybka (1964) and Willis (1972). The valid loading range for the small-scale contact zone (1% of crack radius) at the crack tips is evaluated for all homogeneous, isotropic, linearly elastic material combinations. It is shown that the expressions obtained for transverse spring stiffness can be used for linear ultrasound applications as long as the initial maximum crack opening displacement is $10^{-6}$ of the crack radius (the crack opening displacement may be a result of the prior loading history). The results obtained are used to estimate the error in the approximate spring stiffness expressions (Lavrentyev and Rokhlin, 1994, Baik and Thompson, 1984 and Margetan et al., 1988); the error is shown to be small for most practical linearly elastic material combinations.
2.2 Equivalent Interfacial Spring Stiffness

2.2.1 Normal and Transverse Spring Stiffness

We consider an array of penny-shaped cracks of radius $a$ periodically spaced with distance $2b$ at the interface between two different isotropic half spaces subjected to far field tensile stress $\sigma$ and shear stress $\tau$ as shown schematically in Figure 2.1(a). We focus our attention on the special case where $b$ is significantly larger than $a$ ($b \gg a$).

![Effective spring stiffness model for non-interacting penny-shaped cracks at the interface between two isotropic dissimilar materials](image)

(a) Penny-shaped cracks of radius $a$ located at the interface between two dissimilar materials.

(b) Uniformly distributed effective springs in normal and transverse directions

The spring stiffness expressions obtained are valid for most material combinations if the initial crack opening is larger than $10^4a$.

Figure 2.1 Effective spring stiffness model for non-interacting penny-shaped cracks at the interface between two isotropic dissimilar materials
Far field displacement components due to applied traction can be separated into displacement for the perfect bonding case and additional displacement due to the presence of cracks as follows.

\[ \Delta = \Delta_{\text{perfect-bonding}} + \Delta_{\text{crack}} \] (2.1)

As shown in Figure 2.1(b), the interface with cracks can be replaced by a continuous uniform distribution of springs. Following the work by Baik and Thompson (1984), the normal and tangential spring stiffnesses, \( k_N \) and \( k_T \) (in \( N/m^3 \)), are defined respectively by

\[ k_N = \frac{\sigma}{\Delta_{N,\text{crack}}} , \quad k_T = \frac{\tau}{\Delta_{T,\text{crack}}} , \] (2.2)

where \( \Delta_{N,\text{crack}} \) and \( \Delta_{T,\text{crack}} \) represent respectively the additional displacement in the normal and transverse directions.

The additional displacements can be found by using Castigliano’s theorem, extended for cracked bodies (Tada et al., 2000), as in

\[ \Delta_{\text{crack}} = \frac{\partial}{\partial Q} \left( \int_A WdA \right) , \] (2.3)

where \( W \) is the strain energy release rate, \( A \) is the crack surface, and \( Q \) is the applied load.

Under the assumption of no crack interactions, instead of seeking a solution to our problem of a periodic array of cracks, we focus our attention on the concentric cylindrical unit cell with radius \( b \) (see Figure 2.1(a)). Furthermore, the strain energy release rate for
the crack inside this cylindrical unit cell can be approximated by that for a crack between two half-spaces since the effect of far-field loading outside the corresponding cylindrical region on the crack can be ignored under the assumption of \( b \gg a \). These approximations make it possible to use the solution for a single penny-shaped crack at the interface between two dissimilar half-spaces developed by Willis (1972). Based on Eq. (6.16) in Willis (1972), the strain energy release rate integrated over the penny-shaped crack surface can be obtained as follows:

\[
\int WdA = \frac{2}{3} \frac{(1+\kappa_1)}{G_i(1+\alpha)} \beta^{-1} (1-\beta^2) \pi \epsilon (1+\epsilon^2) \left\{ \frac{4 \beta \gamma \tau^2}{(1-\beta^2) \pi \epsilon (1+\epsilon^2)} + \frac{1}{2} \sigma^2 \right\} a^3
\]  

(2.4)

where

\[
\alpha = \frac{G_2(\kappa_1 + 1) - G_1(\kappa_2 + 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}, \quad \beta = \frac{G_2(\kappa_1 - 1) - G_1(\kappa_2 - 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}
\]

(2.5)

and

\[
\epsilon = \frac{1}{2\pi} \log \left\{ \frac{1+\beta}{1-\beta} \right\}, \quad \gamma = \frac{G_1+G_2}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}
\]

(2.6)

(See Appendix L) \( \alpha, \beta \) are the Dundurs’ parameters (Dundurs and Bogy, 1969) expressed in terms of \( \kappa_i=3-4\nu_i \) (Poisson’s ratio \( \nu_i \), \( i=1,2 \)) and the shear moduli \( G_1 \) and \( G_2 \). The ranges of these non-dimensional parameters for all material combinations are \( \alpha[-1,1] \), \( \beta[-0.5, 0.5] \), \( \epsilon[-0.175, 0.175] \) and \( \gamma[0.25, 0.5] \).

By considering the circular cross sectional area \( \pi b^2 \) of the unit cylindrical cell, the applied normal and shear load can be expressed by \( Q_N = \sigma (\pi b^2) \) and \( Q_T = \tau (\pi b^2) \),
respectively. Substituting Eqs. (2.3) and (2.4) into Eq.(2.2) with use of appropriate expressions of $Q$, the explicit expressions for the normalized effective normal and shear spring stiffness are obtained as follows:

$$k_N^* = \frac{k_N}{G_1(1+\alpha)} = \frac{3}{2} \left( \frac{a}{b} \right)^2 \frac{\beta}{\varepsilon \left(1+\varepsilon^2\right) \left(1-\beta^2\right)}$$ (2.7)

$$k_T^* = \frac{k_T}{G_1(1+\alpha)} = \frac{3\pi}{16} \left( \frac{a}{b} \right)^2 \frac{1}{\varepsilon \left(1+\varepsilon^2\right) \left(1-\beta^2\right)} \left[ 1 + \frac{4\beta\gamma}{\pi \varepsilon \left(1+\varepsilon^2\right) \left(1-\beta^2\right)} \right]$$ (2.8)

(See Appendix M and N) Fabrikant (1987) and Kachanov & Laures (1989) have investigated three dimensional elastic interaction of two coplanar penny-shaped cracks in a homogeneous material and found for both mode I and mode II stress intensity factors that the effect of crack interaction is less than 1% as long as the distance between the center of the cracks is more than two times the diameter of the penny-shaped crack. Also, in our separate study (Lekesiz et al.) of periodically spaced 2D strip cracks at the interface between two dissimilar half-spaces, crack interaction was found to be strongly influenced by crack spacing and not by material dissimilarity for most material combinations. Therefore, even through Eqs. (2.7) and (2.8) are derived based on the assumption $b>>a$, they can be approximately utilized for $a/b$ less than or equal to 0.5.

The normalized normal spring stiffness (Eq. (2.7)) for $a/b=0.5$ is shown as a function $\beta$ in Figure 2.2; for all possible material combinations it varies from 18.8 to 22.2. Each circular symbol in Figure 2.2 represents a material combination provided by
Suga et al. (1988). Some commonly used material combinations are labeled with cross-shaped points. The small change in the value of $k_N^*$ for all range of $\beta$ indicates that the effect of $\beta$ on the normal spring stiffness is relatively small. Since $k_N$ is normalized by $G_i(1+\alpha)/(1+\kappa_1)a$ in Eq.(2.7), parameters $G_i$, $\alpha$, $\kappa_1$ and $a$ will all affect the actual value of $k_N$.

Figure 2.2 Normalized normal spring stiffness for penny-shaped interfacial cracks as a function of material parameter $\beta$ ($a/b=0.5$). Each circular symbol (o) represents a material combination provided by Suga et al. (1988). Some material combinations are indicated by cross-shaped (X) points and labeled.
Similarly, Figure 2.3 shows contour plots of normalized transverse spring stiffness $k_T^*$ as a function of $\beta$ and $\gamma$ for $a/b=0.5$; it varies from 14.5 to 19.5. It is interesting to note that after the normalization by $G_1(1+\alpha)/(1+\kappa)\alpha$, $k_T^*$ depends on both $\beta$ and $\gamma$ while $k_N^*$ depends only on $\beta$. While $\beta$ and $\gamma$ vary from – 0.5 to 0.5 and 0.25 to 0.5, respectively, the physically admissible range of $\beta$ and $\gamma$ are restricted to lie within a triangular region as shown in Figure 2.3 (See Appendix O). As in Figure 2.2, some material combinations by Suga et al. (1988) are shown by circular points and cross-shaped points. Contour lines are almost parallel to the $\beta$ axis, and thus the effect of material dissimilarity on $k_T^*$ is mostly controlled by $\gamma$. 
2.2.2 Stress Ratio Required for Small-Scale Contact Conditions

In Willis’s solution (1972), the crack faces are assumed to be traction-free (open crack); however the corresponding crack opening displacement exhibits an oscillatory crack interpenetration zone at the crack tips, thus violating the open crack assumption. Comminou and her co-workers (Comminou, 1977, 1978, Comminou and Schmueser, 1979) have addressed the problem of the interpenetration zone by assuming that near the crack tips the crack faces are in frictionless contact (the length of this contact zone is
determined as a part of the problem solution). This replacement of the traction-free boundary conditions by a unilateral contact zone at the crack tips leads to a nonlinear problem and a complicated iterative solution. However, if the interpenetration zone is much smaller than the crack length, “small-scale contact conditions” are satisfied (Rice, 1988), and the crack tip conditions are completely characterized by the open crack solution. Therefore, we would like to examine the range of loading for which small-scale contact conditions are satisfied and Eqs. (2.7) and (2.8) can be used.

Hills and Barber (1993) have provided excellent discussions of both the open formulation and the contact formulation. By embedding a universal contact field within the surrounding field of the open crack solution, the use of the open crack formulation is extended to the case where the contact zone is sufficiently smaller than the crack lengths but larger than the process zone. They have proposed a general expression for estimating the size of the interpenetration zone and estimated the contact zone size from that of the interpenetration zone, which is larger than the contact zone. Their work has included explicit expression for predicting the extent of the contact region for a penny-shaped interface crack in combined shear and tension. Graciani et al. (2007) have noted that the expression for estimating the interpenetration zone by Hills and Barber (1993) is valid only for positive $\beta$ and modified the expression so that it is valid for both positive and negative $\beta$.

Since our interest is to assure the validity of Eqs. (2.7), (2.8) obtained from the open crack formulation, we need to satisfy the small-scale contact conditions. Thus we will use results of Graciani et al. (2007), and Hills and Barber (1993) to predict the extent
of the interpenetration region for a penny-shaped interface crack in combined shear and tension. By denoting the extent of the interpenetration region by \( r_I \), the expression for estimating the location of the first interpenetration (Eq. (11) in Graciani et al., 2007) is given by

\[
\frac{r_I}{2a} = \exp \left[ \frac{1}{|\varepsilon|} \left\{ \left( 2n - \frac{1}{2} \right) \pi - \text{sgn}(\varepsilon) \arg(K) + \text{tan}^{-1}(2|\varepsilon|) \right\} \right], \tag{2.9}
\]

where \( r_I \) is measured from crack tip, \( n \) is any integer and \( K \) is complex stress intensity factor. Based on Willis’ solutions (1972) for a penny-shaped crack in combined shear and tension, the stress intensity factor was obtained by Hills and Barber (1993) (Eq. (24)) as follows:

\[
K(\theta) = \frac{(2a)^{1/2}}{\pi^2} \frac{\Gamma(2 + i\varepsilon)}{\Gamma(1/2 + i\varepsilon)} \left\{ \frac{2\tau \cos(\theta)}{(1 - \beta^2)\pi\varepsilon(1 + \varepsilon^2)} + i \right\}, \tag{2.10}
\]

where \( \theta \) is the angle measured from the positive \( x \)-axis which is directed along the applied shear load as shown in Figure 2.4 (a). Combining Eqs. (2.9) and (2.10), we obtain (See Appendix P)

\[
\frac{r_I(\theta)}{2a} = \exp \left[ \frac{1}{|\varepsilon|} \left\{ -\pi - \text{tan}^{-1}(2|\varepsilon|) - \text{sgn}(\varepsilon) \arg \left[ \frac{\Gamma(2 + i\varepsilon)}{\Gamma(1/2 + i\varepsilon)} \right] \right\} \right] \\
- \text{sgn}(\varepsilon) \text{tan}^{-1} \left( \frac{8\beta \tau(\tau/\sigma)\cos(\theta)}{(1 - \beta^2)\pi\varepsilon(1 + \varepsilon^2) + 4\beta^2} \right) \right]. \tag{2.11}
\]
Figure 2.4 (a) A penny-shaped crack at the interface between two dissimilar materials with interpenetration zone size $r_I(\theta)$. The largest interpenetration zone occurs at $\theta = \pi$ when $\beta$ is positive and at $\theta = 0$ when $\beta$ is negative. (b) Contour plots of the minimal stress ratio $\sigma/\tau$ required for negligible contact zone (The maximum interpenetration zone size $r_{I,maximum}$ is below 1% of $a$.)

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When the maximum interpenetration zone is limited to less than 1% of the crack radius,

\[ r_{l,\text{maximum}} / a < 0.01, \]  

(2.12)

the effect of the interpenetration zone on the stress intensity factor obtained by the open


crack model can be neglected (Rice, 1988). As in Hill and Barber (1993), without loss of
generality, \( \tau \) is applied in the positive \( x \) direction. Based on Eq.(2.11), the maximum


interpenetration point is shown to occur when \( \theta \) equals \( \pi \) for positive \( \beta \) and zero for


negative \( \beta \). By setting the right hand side of Eq.(2.11) to be 0.01 with \( \theta = 0 \) or \( \pi \), the


minimum tensile/shear stress ratio necessary to maintain the small-scale contact condition


(2.12) is obtained as follows:

\[
\frac{\sigma}{\tau} \geq \frac{8\beta\gamma}{(1 - \beta^2)\pi\epsilon(1 + \epsilon^2) + 4\beta\gamma} \cot \left( \text{sgn}(\epsilon) \left[ -\frac{\pi}{2} + \tan^{-1} \left( \frac{2}{\epsilon} \right) \right] \right) \\
- \text{sgn}(\epsilon) \arg \left[ \frac{\Gamma(2 + i\epsilon)}{\Gamma(1/2 + i\epsilon)} - |\epsilon|\ln(0.005) \right] 
\]  

(2.13)

Contour plots in Figure 2.4 (b) show the minimal values of stress ratio \( \sigma / \tau \) required for


negligible contact zone as prescribed by Eq. (2.13). For most material combinations, the


minimal stress ratio \( \sigma / \tau \) varies from zero to one, indicating that the interpenetration zone


is negligible as long as the tensile stress is larger than the shear stress. Contour lines are


approximately parallel to the \( \gamma \) axis, and thus the effect of material dissimilarity on the


stress ratio required for negligible contact zone is mostly controlled by \( \beta \). Only when \( \beta \) is


zero, can far field pure shear (zero contour line) be applied without encountering


problems related to the interpenetration zone.
2.2.3 Linear Ultrasonic Applications

The discussion in Section 2.2.1 is based on the model assumption that the surfaces of a penny-shaped crack are stress-free, while the gap between crack surfaces is zero in the absence of normal tension. As we have discussed, to avoid crack surface interpenetration under shear load, one needs to apply tension for opening the crack. Experimental detection of interfacial cracks by ultrasonic waves is possible only when cracks are open, which is often the case due to loading history and plastic deformation. If under external or residual compressive stresses the crack is partially closed, the ultrasonic wave is perturbed only by its open part, and if the crack is completely closed the ultrasonic wave becomes insensitive to the crack. In this last case, wave reflection from an interface between dissimilar materials occurs as for the case without a crack. Below we will estimate the degree of crack opening sufficient to consider noninteracting crack surfaces under an incident shear wave; i.e., we will determine the crack opening conditions for the validity of Eq. (2.8).

Based on linear superposition, the initial crack opening can be approximated by far field equivalent tensile stress which causes the closed crack to open by that amount. Therefore, even under the pure shear load condition, the loading may be viewed as a combined load of shear and equivalent tensile load, where the small-scale contact condition remains valid. This leads to a practical question as to what the minimum initial crack opening required for the small-scale contact condition is. The crack opening displacement for a penny-shaped crack at the interface between two dissimilar semi-spaces subjected to far-field tensile stress is given by Eqs. (1.4) and (4.5) in Mossakovsky
and Rybka (1964). By evaluating their general solutions and using parameters $\alpha$, $\beta$, $\kappa_1$, and $G_1$, one can obtain for the maximum opening displacement $\delta_{\text{max}}$:

$$
\delta_{\text{max}} = \frac{2\sigma_0 a}{2G_1} A(\alpha, \beta, \kappa_1),
$$

(2.14)

where

$$
A(\alpha, \beta, \kappa_1) = \left(\frac{\kappa_1 + 1}{1 + \alpha}\right) \frac{\varepsilon}{\beta} \left(1 - \beta^2 + \sqrt{1 - \beta^2}\right).
$$

(2.15)

(See Appendix Q) It can be shown from Eqs. (2.14), (2.15) that $\delta_{\text{max}}$ is symmetrical in indices 1,2 for material parameters of semispaces.

In Figure 2.5, a contour plot of parameter $A$ as a function of $\alpha$ and $\beta$ for $\nu_1=0.33$ is shown with several examples of material combinations labeled by cross-shaped points. Extreme material lines with $\kappa_2=1$ and 3 ($\nu_2=0.5$ and 0) are also shown in the figure. The contour lines are almost parallel to the $\beta$ axis, indicating that the effect of $\beta$ on the value of $A$ is small. Utilizing the weak dependence on $\beta$ and taking into account that as $\beta \to 0$, $\beta / \varepsilon \to \pi$, we can obtain the approximate equation for $A$ as

$$
A \approx \frac{2(\kappa_1 + 1)}{\pi(1 + \alpha)} = \frac{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}{\pi G_2}.
$$

(2.16)

This simplification is useful for estimates using Eq (2.12). As can be seen from Eqs. (2.15) and (2.16), $A$ is unbounded as $\alpha$ approaches -1 ($G_2 \to 0$). The value of $A$ reaches approximately 650 for all values of $\beta$ when $\alpha$ becomes -0.999. For all other values of $\nu_1$
varying from 0 to 0.5, contour plots of parameter $A$ are evaluated in a similar manner, and the maximum possible value of parameter $A$ for most practical material combinations is estimated to be below of 1000.

Figure 2.5 Contour plots of parameter $A$ as a function of $\alpha$ and $\beta$ for $\nu_l=0.33$
Normalizing the maximum crack opening displacement by the crack radius $a$ and multiplying and dividing by shear stress $\tau$ on the right hand of Eq. (2.14), we can rewrite Eq. (2.14) as

$$\frac{\delta_{\text{max}}}{a} = \frac{\sigma_0}{\tau} \frac{\tau}{2G_1} A(\alpha, \beta, \kappa_1),$$

where the normalized crack opening displacement is written as the product of three non-dimensional terms, $\sigma_0/\tau$, $\tau/2G_1$, and $A(\alpha, \beta, \kappa_1)$.

Based on representation in Eq. (2.17), the normalized crack opening displacement necessary to avoid crack surfaces interaction under an incident shear wave producing strain $\tau/2G_1$ can be easily estimated. As shown in Figure 2.4(b), the small-scale contact condition (Eq. (2.12)) is valid for most material combinations as long as the ratio of the far-field tensile stress to shear stress is equal to or greater than unity. The magnitude of shear strain, $\tau/2G_1$, used in linear ultrasonic applications in the MHz range, is typically in the order of $10^{-9}$ (with ultrasonic displacement below 0.1 Å). Using this strain and the $\sigma_0/\tau$ ratio as 1, we conclude from Eq. (2.17) that it is sufficient to have the normalized initial crack opening $\delta_{\text{max}}/a$ larger than $10^{-6}$, to maintain the small-scale contact zone (this puts the crack opening displacement in the nanometer range for crack size on the mm scale). Such a value of the crack opening displacements also assures the condition for linear wave interaction with the crack: the normal component of wave displacement amplitude at the crack surface should be less than the crack opening displacement (the normal crack surface displacement appears for normal incidence of longitudinal wave or
oblique incidence of longitudinal or transverse waves). This leads to the conclusion that
the proposed effective spring stiffness remains valid for most practical applications of
linear ultrasonic measurements.

2.3  Comparison of the Proposed Effective Spring Stiffness with Prior Work

2.3.1  Normal Spring Stiffness

As mentioned in Section 2.2.1, Baik and Thompson (1988) have suggested that
the normal spring stiffness expression based on the unit-cell circumferential edge crack
model in a homogeneous material can be extended to the case of two dissimilar materials
by utilizing the effective Young’s modulus for two dissimilar materials, as is done in the
spherical Hertzian contact problem (Johnson, 1987). By applying this method to the
unit-cell center penny-shaped crack model with diminishing crack density (Eq. (3b) in
Margetan et al., 1988), the following approximate expression for the normal spring
stiffness in terms of crack radius \( a \) and material parameters \( \alpha, G_1, \kappa_1 \) is obtained:

\[
k_N^{\text{approx.}} = \frac{3}{2} \frac{b^2}{a^3} \frac{G_1}{(1 + \kappa_1)} \pi (1 + \alpha)
\]

(2.18)

(See Appendix R). Figure 2.6 shows relative error (in percent) versus \( \beta \) between the
approximate expression (Eq.(2.18)) and the exact normal spring stiffness \( k_N \) (expressed in
the normalized form in Eq.(2.7)) obtained in this work. Several examples of practical
material combinations based on Suga et al. (1988) are indicated in the figure by circular
dots. The error increases with increase of the material dissimilarity parameter \( \beta \). When \( \beta \)
is 0.5, the error reaches approximately 15%. For most practical material combinations, however, the error remains smaller than 4%.

Figure 2.6 Relative error (in percent) as a function of $\beta$ for normal spring stiffness between Eq. (7) and approximate Eq. (16). Each circular symbol (o) represents a material combination provided by Suga et al. [23]. Several material combinations are indicated by cross-shaped points and labeled.
Gorbatikh (2005) considered a representative volume element which contains an interface and obtained normal compliance of a penny-shaped crack between two dissimilar materials. While the focus of her work is to obtain the overall elastic property for a representative volume element rather than interfacial spring stiffness, Eq. (2.7) can be obtained from her expression of compliance by considering the special representative volume element which is infinitely large in the direction perpendicular to the interfacial crack.

2.3.2 Transverse Spring Stiffness

Based on the array of 2D strip crack model, Margetan et al. (1988) have suggested an approximate estimation method of the transverse spring stiffness from the normal spring stiffness for 3D penny-shaped cracks in a homogeneous material. Based on this, Lavrentyev and Rokhlin (1994) have proposed an approximate form for the transverse spring stiffness for a periodic array of circular disbonds between two dissimilar materials, by using an average Poisson’s ratio and a Hertzian-based effective Young’s modulus. By considering the special case with diminishing crack density in Eq. (9) in Lavrentyev and Rokhlin (1994) and using $\alpha$, $\beta$, $\kappa$, $G_1$ for material parameters, an approximate expression for transverse spring stiffness for non-interacting penny-shaped cracks can be written as follows:

\[
k_T^{\text{approximate}} = \frac{3\pi}{16} \frac{b^2}{a^3} \frac{G_1(1+\alpha)}{(1+\kappa)} \left( 5 + \frac{\kappa (\frac{\kappa}{\kappa_1})^2 (2\kappa - \alpha + \beta)}{2\left[ \kappa_1 (1+\alpha) + (1+\beta) \right]} \right)
\]

(2.19)

(See Appendix R). For the purpose of comparison, in Figure 2.7, contour plots of relative percent error between our exact transverse spring stiffness (expressed in the normalized
form in Eq.(2.8)) and the above approximate expression for the case with \( \nu_1=0.33 \) are shown as a function of \( \alpha \) and \( \beta \). Several material combinations with \( \nu_1=0.33 \) are shown by cross-shaped points. Extreme material lines with \( \kappa_2=1 \) and \( 3 \) (\( \nu_2=0.5 \) and \( 0 \)) are also shown in the figure. While the error can reach 15% in extreme material combinations (\( \alpha = -1, \beta = -0.5 \)), for most practical material combinations, the error remains small.

Figure 2.7 Contour plots of relative error (in percent) for the transverse spring stiffness between Eq. (2.8) and the approximate expression (Eq.(2.19)) as a function of \( \alpha \) and \( \beta \) for \( \nu_1=0.33 \). Four material combinations are indicated by cross-shaped points.
2.4 Summary

The expressions for the normal and transverse spring stiffness for non-interacting penny-shaped cracks at the interface between two dissimilar materials are obtained based on the classical fracture mechanics model of Willis (1972). The stress ratio necessary to maintain the small scale contact condition (the maximum interpenetration zone below 1% of crack radius) is evaluated for all material combinations. It is shown that for linear ultrasound applications the obtained spring stiffness expressions can be used for arbitrarily material combinations as long as initial crack opening displacement is above $10^{-6}$ of the crack radius. They are compared with the empirical equations obtained for the dissimilar case based on use of the Hertzian-based effective modulus and average Poisson’s ratio. For most practical materials, the differences between these two expressions are shown to be below 5%.

The present work provides rigorous expressions for spring stiffness for interfacial penny-shaped cracks between two dissimilar materials and confirms theoretically the validity of the approximate spring stiffness expressions obtained in an ad-hoc manner. The expressions obtained will be useful in estimating the disbond area fraction, which is critical in assessment of bond integrity.
REFERENCES


CHAPTER 3. The stress intensity factors for a periodic array of strongly interacting coplanar penny-shaped cracks based on Kachanov’s method

3.1 Introduction

Problems related to interaction of multiple penny-shaped cracks have wide applications in analyzing damage mechanisms of brittle materials such as concrete, rocks and ceramics. They can also be used in assessing the durability of bonded structures. In brittle materials, the discontinuous macroflow can be idealized as multiple penny-shaped cracks in the plane of the eventual fracture and the progressive damage can be modeled by the growth of these cracks. Durability of bonded structures is determined by the integrity of the interfaces which are subjected to various mechanical loads and environmental factors (Lavrentyev and Rokhlin, 1994). Initial flow and its progressive growth of flaw can be modeled by multiple penny-shaped cracks at the interface.

In regard to the importance of the interacting penny-shaped cracks as indicated above, a wide range of investigations can be found in the literature in which a good review with an extensive list of references is provided by Panasyuk et al. (1981). For two interacting penny-shaped cracks, subjected to applied pressure, Collins (1962) reduced the problem to the solutions of Fredholm integral equations of the second kind with the use of potential functions.
Later, Fabrikant (1987, 1989) obtained a new type of integral equations for strongly interacting cracks which are non-singular. The iteration procedure is rapidly convergent even for very close crack interactions. By using simple form of analytical expressions of stress field around a single crack subjected to uniform arbitrary loading by Fabrikant (1990), Kachanov and Laures (1989) established a simple approximate method (Kachanov, 1985) for strongly interacting penny-shaped cracks. Their approximate method is verified against exact solution by Fabrikant (1987, 1989).

Recently, Zhan and Wang (2006) obtained new numerical results based on Legendre polynomial representation of displacements and boundary collocation method to solve governing equations. They compared the results for two coplanar cracks with Kachanov and Laures (1989) and Fabrikant (1987) and a good agreement is observed.

For the interaction of penny-shaped cracks more than two, Collins (1963) provided an approximate solution for infinite row of periodic coplanar penny-shaped cracks under the assumption of widely spaced cracks. Later, Fabrikant (1987) gave another approximate form based on a different approach but proved that two methods matches well for widely spaced cracks.

For the case of periodic array of penny-shaped cracks, the literature mainly focuses on two topics; the local properties of an interacting crack such as stress intensity factor and the estimation of effective elastic properties of solids with multiple cracks. In the field of effective elastic properties of cracked solids, early studies provided various approximate methods such self-consistent model (Budiansky and O’Conell, 1976), the differential scheme (Hashin, 1988) and Mori-Takana method (Mori and Takana, 1973).
However, in these methods, the microcrack interaction are entirely neglected or indirectly taken into account and therefore, they are valid for moderate crack distributions.

For high density of cracks, Nemat-Nasser et al. (1993) obtained a solution based on micromechanics approach along with the homogenization of eigenstrains of periodic structures. Due to the periodic nature of the problem, the effective elastic properties such as normalized Young modulus and shear modulus can be written in terms of Fourier series. The average stress intensity factor can also be obtained from the same series.

In the field of the local properties of the interacting cracks, Sekine and Mura (1979) studied the problem of periodic array of cracks with square configuration (equal periodicity in both coplanar directions) where they utilized the Somigliana dislocation method by adopting the method from single crack problem to the multiple cracks problem. They obtained mode I, II and III SIFs approximately by truncating the series in displacement gradients. The results presented in their work are limited up to \( \frac{a}{b} = 0.92 \) where \( a \) is the crack radius and \( b \) is the half of the distance between two crack centers.

Huang and Karihaloo (1992) examined the reduction in the stress transfer capacity for the quasi-brittle materials weakened by periodic array of cracks with square pattern. Their analysis is based on Fredholm integral equation of the second kind and an approximate method where the crack interactions are formulated based on displacement averages instead of traction averages (Kachanov, 1985). The results presented in their work are limited to normal loading and average stress intensity factor is presented instead of full stress intensity factor as a function of angle through the crack edge.
Figure 3.1 (a) An array of periodic coplanar penny shaped cracks in an infinite media subjected to remote tension and shear ($p_0^0$, $t_0^0$), (b) Square configuration of cracks, (c) Hexagonal configuration of cracks
In this study, the approximate method developed by Kachanov and Laures (1989) is applied to the problem of an array of periodic coplanar periodic cracks in an infinite media subjected to remote tension and shear (Figure 3.1(a)). Two coplanar crack configurations are considered as shown in Figure 3.1(b) and (c): square where the cracks are equally periodic in both $x$ and $y$ directions and hexagonal where each crack is surrounded by six equidistant cracks. The square configuration has a wide applicability in literature and the available results in literature serves as a verification of the results presented in this paper. Hexagonal configuration is chosen because it comprises the possible densest packing and exists in biological application such as dentin tubules, however, there is no solution available in literature known to the authors.

The mode I, II and III SIFs for center crack (crack #1 in Figure 3.1(b) and (c)) are calculated numerically as a function of crack density($a/b$) and the angle trough crack edge($\phi$). The empirical mode I SIF expressions for both square and hexagonal configurations are obtained based on surface fitting of numerical values using least square method. Mode II and III SIF for both configurations are presented in the form of contour lines as a function of the angle around the crack edge ($\phi$) and crack density parameter ($a/b$). The approximation for widely spaced cracks developed by Kachanov (1985) for two cracks is extended for the problem of infinite number of cracks shown in Figure 3.1. In the last section, the results obtained in this work are compared with the available results presented in the literature.
3.2 Procedure

3.2.1 Normal Loading

In this section, we apply the approximate method by Kachanov (1985) to examine crack interactions of an array of periodic coplanar penny-shaped cracks in an infinite media subjected to a remote normal traction, \( p^0 \), at infinity (Figure 3.1 (a)). Square and hexagonal configurations of cracks with radius \( a \) and periodicity \( b \) (\( 2b \): the distance between two adjacent cracks) as shown in Figure 3.1(b) and (c), respectively, will be considered. This problem can be replaced by an equivalent problem where crack faces are subjected to a compression \( p^0 \). This equivalent problem with many cracks can then be separated into multiple problems where each containing single crack loaded by tractions which include crack interactions. Without loss of generality, we choose this single crack to be crack #1 (see Figure 3.1 (b), (c)), and the method for determination of traction averages of surrounding crack #\( j \) (\( j=1, 2, 3, \ldots \)) is briefly summarized below.

First, we consider a single penny-shaped crack #\( j \) of radius \( a \) in an infinite homogenous material subjected to a uniform unit compression (See Figure 3.2(a)). The axisymmetric stress field around this non-interacting penny-shaped crack is given by Kachanov and Laures (1989) (Appendix A1).

\[
\sigma_j(\rho_j) = \frac{2}{\pi} \left[ \left( \frac{\rho_j^2}{a^2} - 1 \right)^{1/2} - \sin^{-1} \left( \frac{\rho_j}{a} \right) \right],
\]

(3.1)

where \( \rho_j = \rho_j/a \) and \( \rho_j \) is the radial distance measured from the crack #\( j \) center.
a) Axisymmetric stress field around a single crack, $\hat{\sigma}_j(\rho_j)$ (crack #j) subjected to a uniform unit compression

b) Transmission factor, $A_i^j$, the average stress distribution of the imaginary crack #1 region when a single real crack #j surface is subjected to a uniform unit compression

Figure 3.2 Procedure to determine transmission factor for coplanar penny-shaped cracks
The transmission factor is defined by the average of the stress distribution on an imaginary crack #1 region (See Figure 3.2 (b)) as in

\[ A_{ji}^{zz} = \frac{1}{\pi a^2} \int_{S_i} \sigma_j(\rho_j) dS_i \quad (j=1, 2, 3, \ldots) \]  

(3.2)

where \( S_i \) represents the imaginary crack #1 region and the corresponding coordinate system, \( \rho_j = \rho / a \) and \( \phi_j \) are used. The first and second superscripts ‘zz’, respectively, indicates the direction of the traction applied on the real crack #j surface and that of the traction averaged over the imaginary crack #1 region. The first and second subscripts ‘j1’, respectively, indicate the real crack #j and the imaginary crack #1.

Second, for an infinite coplanar cracks, the average traction for crack #1, \( \langle p_1 \rangle \), can be written as the summation of the applied compression, \( p^0 \), and the effect of all the surrounding cracks on the crack #1.

\[ \langle p_1 \rangle = p^0 + A_{21}^{zz} \langle p_2 \rangle + A_{31}^{zz} \langle p_3 \rangle + \cdots + A_{i1}^{zz} \langle p_i \rangle + \cdots \]  

(3.3)

where \( \langle p_i \rangle \) \((j=1, 2, 3, \ldots)\) is the average traction for crack #j and remains unknown.

For periodic configurations as in Figure 3.1 (b) and (c), average traction \( \langle p_j \rangle \) are the same for all the cracks.

\[ \langle p_1 \rangle = \langle p_2 \rangle = \langle p_3 \rangle = \langle p_4 \rangle = \cdots \]  

(3.4)

Eqs. (3.3) and (3.4) lead to

\[ \frac{\langle p \rangle}{p^0} = \frac{1}{1 - \sum_{j=2}^{\infty} A_{ji}^{zz}} \]  

(3.5)

The traction distribution on crack #1 surface then can be written as
\[
 p_1(\rho_1, \phi_1) = p^0 + <p> \sigma_2(\rho_1, \phi_1) + <p> \sigma_3(\rho_1, \phi_1) + \ldots \quad (3.6)
\]

The mode I stress intensity factor (SIF) at the given point \( \phi \) of the edge of a crack due to an arbitrary distribution of the normal traction \( p_1(\rho_1, \phi_1) \) is given by Eq. (2.5) in Kachanov and Laures (1989).

\[
 K_i(\phi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sqrt{1 - \rho_1^2} p_1(\rho_1, \phi_1) \rho_1 d\rho_1 d\phi}{1 - 2\rho_1^2 - \rho_1^2 \cos(\phi - \phi_1)} \quad (3.7)
\]

where \( K_i^0 \) is the SIF for a crack loaded by a uniform normal pressure, \( p^0 \), \( \left( K_i^0 = p^0 \sqrt{2a/\pi} \right) \) and \( p_1(\rho_1, \phi_1) = p_1(\rho_1, \phi_1)/p^0 \).

Since the stress field around crack \#j, \( \sigma_j \), plays an important role in determination of \( p_1 \), in Figure 3.3, \( \sigma_j \) evaluated at points \( A(\rho_1 = a, \phi_1 = \psi_{ij}) \), \( B(\rho_1 = a, \phi_1 = \psi_{ij} + \pi) \) and \( C(\rho_1 = 0, \phi_1 = 0) \) along the imaginary crack \#1 region are plotted as a function of the normalized crack distance, \( l_{ij}/a \), in log-log scale. The distance between the two crack centers and the angle measured from the x axis to the center of crack \#j, respectively, are denoted by \( l_{ij} \) and \( \psi_{ij} \). As \( l_{ij}/a \) becomes large, the difference between the maximum and minimum stresses represented by A (dashed line) and B (dotted line) disappears and both values quickly decay. For example, even
for $l_{ij}/a = 6$, the normalized stress values at points A, B and C are all in the order of $10^{-3}$ when the unit traction is applied to the crack $#j$. The stress distribution within the imaginary crack #1 region, therefore, can be assumed to be uniform and represented by the constant stress at point C (center).

Figure 3.3 The normal stress values at various points (A, B and C) along the imaginary crack #1 region due to the unit normal traction applied at the crack $#i$ face as a function of $l/a$ ($l$: center to center distance, $a$: crack radius)
As shown in Figure 3.1(b), all cracks numbered larger than 25 in the square configuration satisfies the condition, \( l_{ij}/a > 6 \). Similarly in the hexagonal configuration (Figure 3.1(c)), the same condition is satisfied for all cracks numbered larger than 31.

\[
\overline{\sigma}_j (\rho_i, \phi_i) \approx \overline{\sigma}_j (0,0) = \text{const.} \quad \left\{ \begin{array}{ll} j \geq n = 26 \text{ for the square configuration} \\ j \geq n = 32 \text{ for the hexagonal configuration} \end{array} \right.
\]

For periodic crack configurations, the number of cracks located at a given value of \( l_{ij}/a \) increases as \( l_{ij}/a \) increases. Therefore, the contribution of the small normalized stress due to surrounding cracks to the SIF of crack #1 can be significant. A question arises as to what is the number of cracks to practically represent an infinite number of cracks in terms of the SIF calculation. In order to investigate the accumulation of small normalized stress by a large number of cracks, \( \sum_{j=n+1}^{m} \overline{\sigma}_j (0,0) \) is plotted as a function of crack number, \( m \), in Figure 3.4 for the crack density \( a/b = 0.95 \) with the square and hexagonal configurations. As can be seen from the figure, \( \sum_{j=n+1}^{m} \overline{\sigma}_j (0,0) \) increases as \( m \) increases for both configurations and become almost flat after 10,000 cracks. The effect of additional 510,000 cracks on the summation of the normalized stress (0.04854 vs. 0.05073 for square configuration) remains small and the corresponding SIF can be shown to differ only around 0.3%. Therefore, for the cracks larger than 10,201, we can write

\[
\overline{\sigma}_j (\rho_i, \phi_i) \approx 0, \quad j > 10,201 \text{ for both configurations}
\]
By using approximate Eqs (3.8) and (3.9) in Eq. (3.6), the SIF in Eq. (3.7) is numerically calculated.

Figure 3.4 The sum of constant stress terms for $a/b=0.95$ as function of total crack number, $m$

3.2.2 Transverse Loading

In this section, we extend our analysis to the case where an infinite number of coplanar cracks are subjected to a constant shear traction in the $x$ direction, $t^\circ$. A constant shear traction applied to an isolated single crack in the $x$ direction produces a
shear stress distribution both in the $x$ and $y$ directions around that crack. Due to this coupling, in the calculation of the transmission factor, we need to employ the stress field around a single crack (crack #j) subjected to a constant shear in both directions, $t^0 + is^0$ ($t^0$ in the $x$ direction and $s^0$ in the $y$ direction). The stress distribution for this case is given by Sankar and Fabrikant (1983) as follows.

\[
\tau_j^x \left( \overline{\rho_j, \phi_j} \right) + i \tau_j^y \left( \overline{\rho_j, \phi_j} \right) = \frac{2(t^0 + is^0)}{\pi} \left[ \frac{1}{\sqrt{\rho_j^2 - 1}} \sin^{-1} \frac{1}{\rho_j} \right] + \frac{2(t^0 - is^0)}{\pi (2 - \nu)} \left( \frac{\cos 2\phi_j + i \sin 2\phi_j}{\rho_j^2 \sqrt{\rho_j^2 - 1}} \right)
\]

where $\tau_j^x$ and $\tau_j^y$, respectively, are the shear stresses in the $x$ and $y$ directions around the crack #j and $\nu$ is the Poisson’s ratio of the material.

Corresponding to Eq. (3.3), the average shear traction of the crack #1 in the $x$ and $y$ directions, $<t_1>$, $<s_1>$, respectively, can be written as follows.

\[
< t_1 > = t^0 + A_{21}' < t_2 > + A_{31}' < t_3 > + \cdots + A_{21}'' < s_2 > + A_{31}'' < s_3 > + \cdots \\
< s_1 > = A_{21}' < t_2 > + A_{31}' < t_3 > + \cdots + A_{21}'' < s_2 > + A_{31}'' < s_3 > + \cdots
\]

(3.11)

where $A$ represents the corresponding transmission factor. The first and second superscripts for $A$, respectively, indicates the direction of shear stress applied on crack #j surface and that of the stress averaged over the crack #1 region. For an infinite number of cracks with a periodic configuration, average tractions, $<t_j>$ and $<s_j>$ are identical for all cracks. Therefore, we can write
Eqs. (3.11) and (3.12) lead to following system of equations.

\[
\begin{bmatrix}
1 - \sum_{j} A_{ji}^{xy} & -\sum_{j} A_{ji}^{yx} \\
-\sum_{j} A_{ji}^{xy} & 1 - \sum_{j} A_{ji}^{yx}
\end{bmatrix}
\begin{bmatrix}
<t> \\
<s>
\end{bmatrix}
= \begin{bmatrix}
t^0 \\
0
\end{bmatrix}
\]  

(3.13)

Based on Eq. (3.10), \( A_{ji}^{xy} \) and \( A_{ji}^{yx} \) can be shown to be identical. For square configuration, the signs of \( A_{ji}^{xy} \) or \( A_{ji}^{yx} \) values for two cracks symmetrically located with respect to the \( y=\pm x \) are opposite, and therefore the summation of \( A_{ji}^{xy} \) values for these two cracks is zero and therefore diagonal terms \( -\sum_{j} A_{ji}^{xy} \) and \( -\sum_{j} A_{ji}^{yx} \) are equal to zero. For hexagonal configuration, it can also be shown that diagonal terms of Eq. (3.13) are zero.

Eq. (3.13), then, leads to

\[
<t> = \frac{t^0}{1 - \sum_{j} A_{ji}^{xx}}, \quad <s> = 0
\]  

(3.14)

The traction distribution on crack #1 surface can be given by

\[
t_1(\bar{\rho}_i, \phi_i) + is_1(\bar{\rho}_i, \phi_i) = t^0 + <t> \left\{ \sum_{j=2}^{\infty} \tau_j^x(\bar{\rho}_i, \phi_i) \bigg|_{\nu=0} + i \tau_j^y(\bar{\rho}_i, \phi_i) \bigg|_{\nu=0} \right\}
\]

(3.15)

where

\[
\tau_j^x(\bar{\rho}_i, \phi_i) \bigg|_{\nu=0} = \frac{\tau_j^x(\bar{\rho}_i, \phi_i) \bigg|_{\nu=0}}{t^0} \quad \text{and} \quad \tau_j^y(\bar{\rho}_i, \phi_i) \bigg|_{\nu=0} = \frac{\tau_j^y(\bar{\rho}_i, \phi_i) \bigg|_{\nu=0}}{t^0}
\]  

(3.16)
As in \( \overline{\sigma}_j(\rho_i, \phi_i) \) shown in Figure 3.3, both \( \overline{\tau}_j^x(\rho_i, \phi_i) \) and \( \overline{\tau}_j^y(\rho_i, \phi_i) \) can be approximated by their values evaluated at the crack #1 center when \( l_{ij}/a \) is larger than six. Consequently, corresponding to Eq. (3.8), we have

\[
\begin{align*}
\overline{\tau}_j^x(\rho_i, \phi_i) \bigg|_{\rho_i=0} & \approx \overline{\tau}_j^x(0,0) \\
\overline{\tau}_j^y(\rho_i, \phi_i) \bigg|_{\rho_i=0} & \approx \overline{\tau}_j^y(0,0)
\end{align*}
\]

\( j \geq n = 26 \) for the square configuration

\( j \geq n = 32 \) for the hexagonal configuration. 

(3.17)

Similarly, corresponding to Eq. (3.9), for cracks numbered larger than 10,201, both \( \overline{\tau}_j^x(\rho_i, \phi_i) \) and \( \overline{\tau}_j^y(\rho_i, \phi_i) \) can be assumed to be zero.

\[
\begin{align*}
\overline{\tau}_j^x(\rho_i, \phi_i) \bigg|_{\rho_i=0} & \approx 0 \\
\overline{\tau}_j^y(\rho_i, \phi_i) \bigg|_{\rho_i=0} & \approx 0
\end{align*}
\]

(3.18)

The mode II and III SIFs for the crack#1 subjected to the traction of \((t_i + is_i)\) is given by Kachanov and Laures (1989) in Eq. (2.6) as follows.

\[
K_{II}(\phi) + iK_{III}(\phi) = \frac{\sqrt{a}}{\pi\sqrt{2}} \int_0^{\frac{\pi}{1}} \int_0^{2\pi} \left\{ -i\rho_i^2 \right\} \frac{e^{-i\phi}(t_i + is_i)}{1 + \rho_i^2 - 2\rho_i \cos(\phi - \phi_i)} \\
+ \frac{\nu}{2-\nu} \frac{e^{i\phi}(3 - \rho_i e^{i(\phi - \phi_i)})(t_i - is_i)}{(1 - \rho_i e^{i(\phi - \phi_i)})^2} \rho_i d\rho_i d\phi_i
\]

(3.19)

As \( a/b \) approaches zero, the traction distribution in the x and y directions, \( t_i(\rho_i, \phi_i) \) and \( s_i(\rho_i, \phi_i) \), respectively, approaches \( t^0 \) and zero. Therefore, Eq. (3.19) reduces to the mode II and III SIF for single crack as follows.
\[
K_{II}^0(\phi) = \frac{4t^0 \sqrt{a}}{\sqrt{2\pi (2{-}\nu)}} \cos(\phi),
K_{III}^0(\phi) = -\frac{4t^0 \sqrt{a (1{-}\nu)}}{\sqrt{2\pi (2{-}\nu)}} \sin(\phi).
\] (3.20)

Utilizing the constant coefficients in Eq. (3.20), the normalized mode II and III SIFs,

\[
K_{II}^\infty(\phi) \left/ \left(\frac{4t^0 \sqrt{a}}{\sqrt{2\pi (2{-}\nu)}}\right)\right. \quad \text{and} \quad K_{III}^\infty(\phi) \left/ \left(-\frac{4t^0 \sqrt{a (1{-}\nu)}}{\sqrt{2\pi (2{-}\nu)}}\right)\right.
\]

are numerically calculated using approximate Eqs. (3.17) and (3.18) in Eq. (3.15).

### 3.3 Results

In Figure 3.5 (a) and (b), the normalized mode I SIF values \(K_i(\phi)/K_{i}^0\) calculated for the square and hexagonal configurations are indicated by solid circles as a function of angle around the crack edge, \(\phi\), and crack density parameter, \(a/b\). Due to the symmetry, the ranges of \(\phi\) for the square and hexagonal configurations, respectively, are limited to \([0, 45^\circ]\) and \([0, 30^\circ]\). For \(a/b < 0.5\), in both configurations, \(K_i(\phi)/K_{i}^0\) remains close to one for all values of \(\phi\), indicating almost no crack interaction. The normalized SIF increases as crack density parameter \((a/b)\) increases and the angle, \(\phi\), approaches zero. It reaches the maximum value of 4.24 and 5.62, respectively, for the square and hexagonal configurations. As expected, crack interactions in the hexagonal configuration are stronger than those in the square configuration.
Figure 3.5 Normalized mode I SIF for periodic array of coplanar penny-shaped cracks as function of angle around the circumference and crack density parameter $a/b$; data points based on accurate numerical calculations and fitted surface, (a) square configuration and (b) hexagonal configuration.
The following empirical equations are obtained by surface fitting the numerical results based on the least square method.

\[
\frac{K_{I_{\text{square}}}}{K_{I_0}} = 1 + 0.2907 \ln \left[ \sec \left( 1.5142 \left( \frac{a}{b} \right)^2 \right) \right] \sqrt{\cos \phi} + 0.0376 \tan \left( 1.5558 \left( \frac{a}{b} \right)^2 \cos^2 \phi \right)
\]

for the square configuration. \( (3.21) \)

\[
\frac{K_{I_{\text{hexagonal}}}}{K_{I_0}} = 1 + 0.3794 \ln \left[ \sec \left( 1.5572 \left( \frac{a}{b} \right)^2 \right) \right] \sqrt{\cos \phi} + 0.06297 \tan \left( 1.5508 \left( \frac{a}{b} \right)^3 \cos^3 \phi \right)
\]

for the hexagonal configuration. \( (3.22) \)

Eqs. (3.21) and (3.22) allow us to evaluate the SIF of an infinite number of interacting coplanar cracks as a function of \( \phi \) and \( a/b \) with the maximum error of 3.5%.

In Figure 3.6 (a) and (b), the normalized mode II SIF, \( K_{II_{\infty}}(\phi) \), for the square and hexagonal configurations, respectively, are plotted in the form of contour lines as a function of angle around the crack edge, \( \phi \), and crack density parameter \( a/b \) for the case of \( \nu=0.5 \). Due to the symmetry with respect to the \( x \) axis and antisymmetry with respect to the \( y \)-axis, the range of \( \phi \) is bounded by \([0, 90^\circ]\) for both configurations. As \( a/b \) approaches one, the normalized mode II SIFs reach their maximum value when \( \phi = 0 \). Since crack interactions for the hexagonal configuration are more than those for the square configuration, given the same \( a/b \) value, the SIF value for the hexagonal configuration (~6.1 for \( a/b =0.9995 \)) is larger than that for square configuration (~4.2 for \( a/b =0.9995 \)).
Figure 3.6 Contour plot of normalized mode II SIF \( (K_{II}^\infty (\phi) \left/ \left( \frac{4\pi^0 \sqrt{a}}{\sqrt{2\pi} (2-\nu)} \right) \right) \) for periodic array of coplanar penny-shaped cracks as function of angle around the circumference and crack density parameter \( a/b \) \( (\nu = 0.5) \); data points based on accurate numerical calculations (a) square configuration (b) hexagonal configuration
As shown in Figure 3.1(b) and (c), the center to center distance between crack #1 and crack #3 for the square and hexagonal configurations are $2\sqrt{2}b$ and $2b$, respectively, while the distance between crack #1 and crack #2 are $2b$ for both configurations. Therefore, crack #3 located at $\phi = 60^\circ$ for the hexagonal configuration contributes to a local peak for high crack densities while there is no local peak at $\phi = 45^\circ$ for square configuration.

In Figure 3.7 (a) and (b), the normalized mode III SIF values, $K_{III} = \frac{4t^0}{\pi} \sqrt{a(1-\nu)} \left( \frac{1}{\sqrt{2\pi(2-\nu)}} \right)$, are shown in the form of contour lines for the square and the hexagonal configurations, respectively as a function of angle around the crack edge, $\phi$, and crack density parameter $a/b$ for the case of $\nu=0.5$. For the square configuration (see Figure 3.1 (b)), crack #4 located at distance $2b$ significantly contributes to the mode III SIF at $\phi = 90^\circ$ of crack #1. Correspondingly, for the hexagonal configuration (See Figure 3.1 (c)), crack #3 located at distance $2b$ plays a significant role in the mode III SIF at $\phi = 60^\circ$ for high crack densities. Therefore, the maximum for square and hexagonal configurations, respectively, occur at $\phi = 90^\circ$ and $\phi = 60^\circ$. It can be shown that the effect of Poisson’s ratio on mode II and mode III SIF for both configurations resembles that for an isolated crack when $a/b<0.75$. Therefore, the mode II and III SIFs for other $\nu$ values can be respectively obtained by multiplying the corresponding numerical values of contour lines by $\frac{4t^0}{\sqrt{2\pi(2-\nu)}}$ and $\frac{-4t^0}{\sqrt{2\pi(1-\nu)}}$. By
comparing this approximation against accurate numerical results based on Eq. (3.19), the maximum error remains smaller than 3% for all $\nu$ when $a/b < 0.75$.

Figure 3.7 Contour plot of normalized mode III SIF ($K_{III}^\infty (\phi) / \left( -\frac{4\pi^6 a (1-\nu)}{\sqrt{2\pi} (2-\nu)} \right)$) for periodic array of coplanar penny-shaped cracks as function of angle around the circumference and crack density parameter $a/b$ ($\nu = 0.5$); data points based on accurate numerical calculations (a) square configuration (b) hexagonal configuration (cont.)
3.4 Widely spaced cracks approximation

For two interacting coplanar cracks located away from each other, Kachanov (1985) employed the asymptotic far-field stress approximation at the center and obtained the approximate form of mode I SIF (Eq. (9) in Kachanov, 1985). As can be seen in Figure 3.5 (a) and (b), the interactions for an infinite number of cracks remain weak for crack density, \( a/b \), smaller than 0.75 and the normalized SIF is almost independent of \( \phi \). Therefore, the asymptotic far-field stress approximation for two cracks can be extended to an infinite number of cracks as long as the condition \( a/b < 0.75 \) is satisfied. By using
the asymptotic stress, \( \bar{\sigma}_j \left( \bar{\rho}_j = \frac{l_{ij}}{a} \right) = \frac{2}{3\pi} \left( \frac{a}{l_{ij}} \right)^3 \), an approximate form of the mode I SIF can be obtained as follows.

\[
\frac{K_i^\infty}{K_i^0} \approx 1 + \frac{1}{12\pi} \left( \frac{a}{b} \right)^3 \left( \sum_{p=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \left( p^2 + r^2 \right)^{-3/2} \right) = 1 + \frac{1}{12\pi} \left( \frac{a}{b} \right)^3 \quad (9.033)
\]

for the square configuration \( (3.23) \)

\[
\frac{K_i^\infty}{K_i^0} \approx 1 + \frac{1}{12\pi} \left( \frac{a}{b} \right)^3 \left( \sum_{p=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \left( \frac{2r + 1}{2} \right)^2 + \left( \frac{\sqrt{5} p}{2} \right)^2 \right)^{3/2} + \sum_{p=-\infty}^{\infty} \left[ r^2 + \left( \frac{\sqrt{5} p}{2} \right)^2 \right]^{3/2} \right) \right) 
= 1 + \frac{1}{12\pi} \left( \frac{a}{b} \right)^3 \quad (11.03)
\]

for the hexagonal configuration \( (3.24) \)

Prime sign, ‘\( ' \), indicates that the term, \( p=r=0 \) is excluded in the summation. The summations in Eqs (3.23) and (3.24) can be shown to converge and the results are accurate within the four significant digits. As expected, the configuration parameter for hexagonal configuration is larger than that for square configuration. The error in the approximate value of SIF given by Eqs (3.23) and (3.24) compared to the maximum value of SIF \( (\phi = 0) \) calculated by Eq. (3.21) and (3.22) remains smaller than 5 % if \( a/b < 0.75 \).

Similarly, approximate forms of mode II and mode III SIF can be obtained using the Taylor expansion of Eq. (3.10) for large \( \rho_j \) values and evaluating Eq. (3.10) at the center \( \rho_j = l_{ij} \). Then, the asymptotic far-field shear stress distribution can be written as
\[
\overline{\tau}_j \left( \frac{l_j}{a}, \phi_j \right) \approx \frac{2}{3\pi} \left( 1 + \frac{\nu}{2-\nu} \cos 2\phi \right) \left( \frac{a}{l_j} \right)^3 = \text{const. for } l_j \gg a. \quad (3.25)
\]

By and inserting Eq. (3.25) into (3.15) along with the assumption \( \langle t \rangle = t^0 \) and defining

\[ l_{ij} = 2b\sqrt{p^2 + r^2}, \quad \phi_i = \tan^{-1} \left( \frac{p}{r} \right) \text{ where } p, r = [\ldots, -2, -1, 0, 1, 2, \ldots], \]

we obtain the following approximate SIF

\[
\frac{K^\infty_{II}(\phi)}{K^0_{II}(\phi)} = \frac{K^\infty_{III}(\phi)}{K^0_{III}(\phi)} \approx 1 + \frac{1}{12\pi} \left( \frac{a}{b} \right)^3 \left[ \sum_{p=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \left( 1 + \frac{\nu}{2-\nu} \cos 2\left( \tan^{-1} \left( \frac{p}{r} \right) \right) \right) (p^2 + r^2)^{-3/2} \right]
\]

\[ = 1 + \frac{1}{12\pi} \left( \frac{a}{b} \right)^3 (9.033) \]

for the square configuration \( (3.26) \)

Similarly, we obtain

\[
\frac{K^\infty_{II}(\phi)}{K^0_{II}(\phi)} = \frac{K^\infty_{III}(\phi)}{K^0_{III}(\phi)} \approx 1 + \frac{1}{12\pi} \left( \frac{a}{b} \right)^3 \left[ \sum_{p=-\infty}^{\infty} \left( \sum_{p: \text{odd}} \left[ 1 + \frac{\nu}{2-\nu} \cos 2\left( \tan^{-1} \left( \sqrt{3} \frac{p}{2} / (2r + 1) \right) \right) \right] \left( \frac{2r + 1}{2} \right)^2 \right)^{3/2} \right]
\]

\[ \times \left[ \left( \frac{2r + 1}{2} \right)^2 + \left( \frac{\sqrt{3} p}{2} \right)^2 \right]^{3/2} + \sum_{p: \text{even}} \left[ 1 + \frac{\nu}{2-\nu} \cos 2\left( \tan^{-1} \left( \sqrt{3} \frac{p}{2r} \right) \right) \right] \left( \frac{\sqrt{3} p}{2} \right)^2 \left( \frac{2r + 1}{2} \right)^2 \right]^{3/2}\]

\[ = 1 + \frac{1}{12\pi} \left( \frac{a}{b} \right)^3 (11.03) \]

for the hexagonal configuration \( (3.27) \)

As may be noted, the term including Poisson’s ratio becomes zero in the summation for both configurations. Therefore, approximate form of normalized mode I,
II and III SIFs are identical for both configurations. Eqs. (3.26) and (3.27) indicates that normalized mode II and III SIF values are not function of $\phi$ and therefore $\phi$ dependence of $K_{II}^\infty$ and $K_{III}^\infty$ is considered through $\cos(\phi)$ and $\sin(\phi)$, respectively, as in single crack case.

The error in the approximate expressions given by Eqs. (3.26) and (3.27) compared to accurate numerical calculations based on Eq. (3.19) remains smaller than 6% for all Poisson’s ratio values when $a/b < 0.75$.

3.5 Comparison with prior work

Sekine and Mura (1979) investigated the three-dimensional stress field for a periodic array of penny-shaped cracks in an infinite isotropic elastic solid under arbitrary uniform loading based on the Somigliana dislocation method. The displacement discontinuity of the somigliana dislocations is assumed to be in the form of $p(x_1',x_2')(1-x_1'^2-x_2'^2)$. The coefficients of the polynomial $p(x_1',x_2')$ are determined using the boundary conditions on the surfaces of penny-shaped cracks where the stresses on the crack surfaces are zero. Numerical results for the stress intensity factors are obtained for the square configuration shown in Figure 3.1 (b).

In Figure 3.8, the maximum values of normalized mode I stress intensity factor as a function of crack density $a/b$ obtained in this work (solid line) is compared against those by Sekine and Mura (1979) (dashed line) for $a/b$ ranging from 0.4 to 0.92. As shown in the figure, the relative percent difference between two methods remains less than 2.8%.
Similarly in Figure 3.9, the normalized mode II and mode III SIFs as a function of angle $\phi$ for two different values of $a/b$ (0.8333, 0.6667) with $\nu=0.3$ obtained by Sekine and Mura (1979) are compared against those obtained in this work. As shown in the figure, the relative percent difference between two methods is less than 1%.
The numerical results by Sekine and Mura (1979) are approximate in the sense that they are obtained by truncating the series for the displacement gradients. The results obtained in this work also involve approximations as outlined in section 3.2. The minor discrepancy between two methods may be attributed to the different nature of approximations and numerical error.

By modeling discontinuous macroflaws by periodic arrays of coplanar periodic shaped cracks, Huang and Karihaloo (1992) investigated the so-called tension softening regime of quasi-brittle materials which exhibits moderate strain hardening behavior.
before ultimate failure. By assuming cracks grow in a self-similar manner, they examined the reduction in the stress transfer capacity with increasing deformation. Their analysis is based on Fredholm integral equation of the second kind and an approximate method where the crack interactions are formulated based on displacement averages instead of traction averages (Kachanov, 1985). The average mode I stress intensity factor defined by

\[ \langle K_I (\phi) \rangle = \frac{1}{2\pi} \int_0^{2\pi} K_I (\phi) d\phi, \]

was numerically obtained as a function of crack density, \( a/b \), in Table 6 in Huang and Karihaloo (1992).

In Figure 3.10, their results in Table 6 (diamond-shaped points) are compared against those obtained in this work (solid line). As can be seen from the figure, the difference between two solutions is indistinguishable even for very high crack densities. While there is no analytical solution available for this problem, excellent agreement between with other prior work as in Figure 3.9 and Figure 3.10 demonstrates the validity of the results presented in this work.
Figure 3.10 Comparison of the averaged value of normalized mode I SIF as a function of $a/b$ calculated in current work and calculated values based on Huang and Karihaloo (1992) for square configuration of periodic array of coplanar penny-shaped cracks.
REFERENCES


CHAPTER 4. Effective Spring Stiffness for a Periodic Array of Coplanar Penny-Shaped Cracks at an Interface between Two Dissimilar Isotropic Materials

4.1 Introduction

The zones of distributed damages may exist in the most materials as a result of manufacturing processes such as welding. Bonding procedure of the two different materials also involves the risk of generating debonded regions at the interface as a result of mechanical and thermal loadings either preexisting or in service. These imperfections reduce the serving life of these structures significantly and therefore needed to be detected.

In a broad range of investigations, these damaged zones and debonded regions are modeled as distribution of cracks and nondestructive detection of these imperfections has a significant attention by researchers (Baik and Thompson, 1984, Margetan et al., 1988, Thompson and Thompson, 1991; Rokhlin et al., 1994).

A quasi-static approximation has been extensively used to model ultrasonic wave interactions with imperfections (Baik and Thompson, 1984, Margetan et al., 1988). In this model, the reduction in static stiffness of the overall structure due to cracked planes is represented by continuous, uniform distributions of springs. It has been demonstrated by Angel and Achenbach (1985) for 2-D periodic array of strip cracks in homogenous materials that the quasi-static approximation is applicable at low frequencies, when the size of the imperfections is much smaller than the wavelength.
Relating the spring stiffness constants to the micromechanical and geometric properties of the micro-cracks such as the crack geometry and density are important since they can be used to estimate the percentage of disbond area (Palmer et al., 1988), which is critical in assessing the bond integrity and the remaining life. Therefore, significant experimental and theoretical advances have been made towards inversion of the interfacial stiffness distributions from ultrasonic measurements (Wang and Rokhlin, 1991; Rokhlin et al., 2004; Baltazar et al., 2003; Wang et al., 2006; Leiderman et al., 2007). A 3-D penny-shaped crack is one of the most common idealization used to model imperfections. The solution for a single penny-shaped crack in a homogenous infinite media is widely used in modeling the wave reflections when the crack interactions can be neglected. In most of these studies, cracks are assumed to be aligned in the square pattern. The hexagonal configuration can be considered as a more realistic configuration especially for the biological systems such as dentin tubules, however, the hexagonal configuration has not been discussed in literature.

Margetan et al. (1988) used a unit-cell model where penny-shaped crack is located at the center of a finite radius cylinder where he fracture mechanics solution for this case is provided by Tada et al. (2000). For the interface cracks, Baik and Thompson (1984) approximately extended the spring stiffness formula for homogenous materials to the dissimilar materials by utilizing the contact model of two spherical punches. (Johnson, 1987)

For the actual problem of an interacting penny-shaped interfacial cracks, there is no fracture mechanics solution available in the literature and therefore an accurate formula for spring stiffness has not been investigated, yet.
In this study, an approximate form of the spring stiffness for square and hexagonal configurations of penny-shaped cracks (Figure 4.2 (a) and (b)) between two dissimilar materials is proposed based on the assumption that cracks interactions are not affected by elastic properties of the bonded materials. The validity of this assumption is examined for an array of 2-D periodic array of strip cracks in a prior work by Lekesiz et al. (2010) and proved that interactions depends on crack density only and independent of the dissimilarity of the bonded materials. As a result of this assumption, the actual problem shown on Figure 4.1(c) can be written as a superposition of three parameters; first material dissimilarity factor, $M$ obtained from a single crack between two dissimilar materials (Figure 4.1(a)), second, the interaction function, $I$ obtained from the periodic array of penny-shaped cracks in homogenous material (Figure 4.1(c)) and third the very well known spring stiffness for a single crack in a homogenous material.

In the section 4.2 and 4.3, the derivation of interaction function (Figure 4.1(b)) is given in detail. The fracture mechanics solution for the corresponding case is obtained by Lekesiz et al., 2010 based on the approximate method by Kachanov and Laures (1989). An empirical spring stiffness formula for periodic array of cracks in a homogenous material is also obtained in Section 4.3. In section 4.4, the elastic dissimilarity function ($M$) is derived based on the prior work by Lekesiz et al., 2010 based on Willis’s work (1972) on single penny-shaped interfacial cracks. In section 4.5, the composition of the all three parameters is discussed in detail and an empirical expression for actual problem is proposed for both square and hexagonal configurations.
Figure 4.1 Schematic of the work
4.2 Strain energy for interacting periodic array of penny-shaped cracks in homogenous materials

4.2.1 Stress intensity factors, Review

Consider an array of periodic coplanar penny-shaped cracks in an infinite media subjected to a remote normal traction, $p^0$, and shear traction in $x$ direction, $t^0$, at infinity (Figure 4.1 (b)). The square and hexagonal configurations of the cracks as shown in Figure 4.2 (a) and (b) are considered. Lekesiz et al. (2010) numerically obtained mode I, II and III SIFs, $K_I$, $K_{II}$ and $K_{III}$ of interacting cracks with these two configurations based on the simple method developed by Kachanov and Laures (1989).

Problem of multiple cracks subjected to remote tractions at infinity are replaced by equivalent problems where crack faces are subjected to a compression $p^0$ and shear in the negative $x$ direction $t^0$. These equivalent problems with an infinite number of cracks can be separated into an infinite number of problems where each containing single crack loaded by tractions which include crack interactions. In Lekesiz et al. (2010), using the fact that these multiple problems are all identical, it is shown that the average normal traction for any crack (say crack #1) is magnified by a constant factor $1 - \sum_{j=2}^{\infty} A_{ji}^{zz}$ due to crack interactions. The factor $A_{ji}^{zz}$ is called the transmission factor (Kachanov, 1985) and describes the normal stress (in z direction) averaged over crack #1 region when a single crack #j in an infinite material is subjected to a unit compression (in z direction) at its crack faces as follows.
Figure 4.2 (a) Square configuration of cracks, (b) Hexagonal configuration of cracks, (c) Crack #1 coordinates

\[ \Lambda_{ij}^{zz} = \frac{1}{\pi a^2} \int_{S_i} \sigma_j(\rho_i, \phi_i) dS_i \quad (j=1, 2, 3, \ldots), \quad (4.1) \]

where

\[ \overline{\sigma_j(\rho_j)} = \frac{2}{\pi} \left[ \frac{1}{\sqrt{\rho_j^2-1}} - \sin^{-1}\left(\frac{1}{\rho_j}\right) \right]. \quad (4.2) \]
In Eq. (3.2), the crack #1 region is represented by $S_1$ and the corresponding coordinate system, $\overline{\rho}_1 (= \rho_1/a)$ and $\phi_1$, are used. In Eq. (3.1), the normalized radial distance measured from the crack #j center is denoted by $\overline{\rho}_j = \rho_j/a$ (Figure 4.2 (c)). Based on Kachanov (1985), the traction of crack #1 is the summation of the applied traction $p^0$ and the superposition of the effect of all surrounding cracks on crack #1, $p^0 \left(1 - \sum_{j=2}^{\infty} A_{ji}^z\right) \sum_{j=2}^{\infty} \overline{\sigma}_j$.

This leads to the normalized mode I SIF as a function of $\phi$ as follows.

$$K^\infty_1(\phi) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \sqrt{1 - \overline{\rho}_1^2} \left[1 + \left(1 - \sum_{j=2}^{\infty} A_{ji}^z\right) \sum_{j=2}^{\infty} \overline{\sigma}_j(\overline{\rho}_1, \phi_1)\right] \overline{\rho}_1 d\overline{\rho}_1 d\phi_1$$

(4.3)

where $K_1^0$ is the SIF for a crack loaded by a uniform normal pressure, $p^0$, $K_1^0 = 2p^0 \sqrt{a}/\sqrt{\pi}$. In this paper, the superscripts, 0 and $\infty$, represents the single crack and infinite cracks solutions, respectively.

Lekesiz et al. (2010) extended this method to the transverse case. When an isolated crack #j in an infinite material is subjected to a unit shear stress in the negative $x$ direction, the shear stress distribution in the plane of cracks has both $x$ and $y$ components (Figure 4.2), $\overline{\tau}_j^x$ and $\overline{\tau}_j^y$, respectively, as follows.

$$\overline{\tau}_j^x(\overline{\rho}_j, \phi_j) + i\overline{\tau}_j^y(\overline{\rho}_j, \phi_j) = \frac{2}{\pi} \left[\frac{1}{\sqrt{\overline{\rho}_j^2 - 1}} - \sin^{-1} \frac{1}{\overline{\rho}_j} + \frac{\nu}{2 - \nu} \left(\frac{\cos 2\phi_j + i \sin 2\phi_j}{\overline{\rho}_j \sqrt{\overline{\rho}_j^2 - 1}}\right)\right].$$

(4.4)
where $\nu$ represents Poisson’s ratio. (Sankar and Fabrikant, 1983). The normalized traction distribution in the $x$ and $y$ directions, $\bar{t}_i, \bar{s}_i$, respectively, are given by

$$
\bar{t}_i(\rho_i, \phi_i) + i\bar{s}_i(\rho_i, \phi_i) = 1 + \left(1 - \sum_{j=2}^{\infty} A_{xi}^{ji}\right)^{-1} \sum_{j=2}^{\infty} \left[ \bar{t}_j(\rho_i, \phi_i) + i\bar{s}_j(\rho_i, \phi_i) \right]. \quad (4.5)
$$

Despite this couple nature of shear stress components, the average shear stress in the $y$ direction for an infinite array of cracks (Figure 4.2 (a) and (b)) is shown to be zero. The average shear stress in the $x$ direction is magnified by a constant factor $\left(1 - \sum_{j=2}^{\infty} A_{xi}^{ji}\right)^{-1}$, where, the transmission factor $A_{xi}^{ji}$ describes the effect of applied shear stress on the crack #j surface by averaging it in the x direction over crack #1 surface. This leads to the expressions for the mode II and mode III SIFs as follows.

$$
K_{II}^{\infty}(\phi) + iK_{III}^{\infty}(\phi) = \frac{t_0}{\pi\sqrt{\pi}} \int_0^{2\pi} \int_0^1 \sqrt{1 - \rho_i^2} \left\{ e^{-i\phi} \left( \bar{t}_i + i\bar{s}_i \right) \frac{e^{-i\phi} \left( \bar{t}_i + i\bar{s}_i \right)}{1 + \rho_i^2 - 2\rho_i \cos(\phi - \phi_i)} \right\} \rho_i d\rho_i d\phi_i + \frac{\nu}{2 - \nu} e^{i\phi} \left( 3 - \rho_i e^{i(\phi - \phi_i)} \right) \left( \bar{t}_i - i\bar{s}_i \right) \rho_i d\rho_i d\phi_i \quad (4.6)
$$

While Eqs. (4.3) and (4.6) involves infinite series, the effect of crack #j on crack #1 diminishes as the distance between two cracks become large as shown in Eq. (4.2) and Eq.(4.4). Based on this, for crack number larger than 26 in the square configuration and 32 in the hexagonal configuration, the effect of crack #j traction on the crack#1 region can be represented by the values at the center of crack #1, $\bar{\sigma}_j(\rho_i = 0)$, $\bar{\tau}_j(\rho_i = 0)$,
\( \tau_j^{\rho_1} = 0 \) and infinite number of cracks can be truncated at \( 10^4 \) cracks for all practical purposes.

4.2.2 The strain energy increase due to crack interaction

The total strain energy for a three-dimensional crack can be expressed as in

\[
U = \int_A \left\{ \frac{(1-\nu^2)}{E} \left( K_I^2 + K_{II}^2 \right) + \frac{(1+\nu)}{E} K_{III}^2 \right\} dA 
\]

(4.7)

where \( A \) represents the crack area. Separating the mode I contribution and the mode II and III contributions in the total strain energy (denoted by \( U_N^{\infty} \) and \( U_T^{\infty} \), respectively) and normalizing them by the corresponding quantities for an isolated (non-interacting) crack \( (U_N^0 \) and \( U_T^0 \)), we obtain following equations.

\[
\frac{U_N^{\infty}}{U_N^0} = \frac{\int_0^{2\pi} \int_0^{\pi} \left( \frac{K_I^\infty}{K_I^0} \right)^2 r d\phi}{\frac{2\pi}{3} \left( \frac{a}{b} \right)^3}
\]

(4.8)

and

\[
\frac{U_T^{\infty}}{U_T^0} = \frac{\int_0^{2\pi} \int_0^{\pi} \left\{ \left( \frac{K_{II}^\infty}{K_{II}^0} \right)^2 \left( \cos \phi \right)^2 + (1-\nu) \left( \frac{K_{III}^\infty}{K_{III}^0} \right)^2 \left( \sin \phi \right)^2 \right\} r d\phi}{\frac{(2-\nu)}{3} \left( \frac{a}{b} \right)^3}
\]

(4.9)

where the mode II and III SIFs for an isolated crack, \( K_{II}^0(\phi) \) and \( K_{III}^0(\phi) \) are given by
\[ K_{II}^0(\phi) + iK_{III}^0(\phi) = \frac{4t^0\sqrt{a}}{\sqrt{\pi(2-\nu)}}\cos(\phi) - i \frac{4t^0\sqrt{a(1-\nu)}}{\sqrt{\pi(2-\nu)}}\sin(\phi) \quad (4.10) \]

and

\[ U_N^0 = \frac{(1-\nu^2)}{E} \frac{4(p^0)^2}{\pi} b^3 \left[ \frac{2\pi}{3} \left( \frac{a}{b} \right)^3 \right], \quad (4.11) \]

\[ U_T^0 = \frac{16(t^0)^2(1-\nu^2)}{\pi(2-\nu)^2 E} b^3 \left[ \frac{(2-\nu)\pi}{3} \left( \frac{a}{b} \right)^3 \right] \quad (4.12) \]

Note that the normal and transverse strain energy ratio, Eqs. (4.8) and (4.9), respectively, represents crack interaction for an infinite array of cracks in terms of strain energy when Eqs. (4.3) and (4.6) are employed. In order to examine the crack interactions, Eqs. (4.8) and (4.9) are numerically evaluated by assuming \(\nu=0.5\) and shown as a function of crack density in Figure 4.3(a) for both square and hexagonal crack configurations. The normal and transverse strain energy ratios \( U_N^\infty / U_N^0 \) and \( U_T^\infty / U_T^0 \), respectively, are shown to be identical for both configurations.

Since cracks in hexagonal configuration are more closely packed than those in the square configuration for the same \(a/b\) value, the strain energy ratio is larger than that in the square configuration. The transverse strain energy ratio depends on Poisson’s ratio as in Eq. (4.9) and \(\nu=0.5\) is used in Figure 4.3(a). When \(\nu=0\), however, the transverse strain energy ratio, \( U_T^\infty / U_T^0 \), can be shown to be identical to the normal strain energy ratio, \( U_N^\infty / U_N^0 \). Therefore, the dependence of transverse strain energy on Poisson’s ratio practically disappears. In Lekesiz et al. (2010), both \( K_{II}^\infty / K_{II}^0 \) and \( K_{III}^\infty / K_{III}^0 \) are
Figure 4.3 Crack energy increase in normal and transverse directions (a) infinite number of cracks with the square and hexagonal configurations, (b) two coplanar penny-shaped cracks and comparison of calculated values in current work and given by Fabrikant (1987, 1989) for the case of $\nu=0.5$
shown to be almost independent of \( \nu \) for most \( a/b \) values due to the periodic and symmetric nature of crack configurations. Furthermore, it can be shown that, the integrated results of the first term, \( \left( \frac{K_{II}}{K_{II}^0} \right)^2 (\cos \phi)^2 \), in the numerator of Eq. (4.9) is almost identical to that of the second term \( \left( \frac{K_{III}}{K_{III}^0} \right)^2 (\sin \phi)^2 \). These facts explain the reason why Eq. (4.9) does not really depend on \( \nu \).

4.2.3 Strain energy increase for two coplanar penny-shaped cracks

Problems of crack interaction have been extensively investigated for two coplanar cracks. Fabrikant (1987, 1989) investigated normal and transverse crack energy ratios for two closely interacting coplanar cracks, \( U_N^{two}/U_N^0 \) and \( U_T^{two}/U_T^0 \). Kachanov and Laures (1989) produced the mode I, II and III SIFs for two closely interacting coplanar cracks using the approximate method outlined in Section 4.2.1 and verified the results against the exact results produced by Fabrikant (1987, 1989).

In order to examine differences in crack interactions between infinite and two coplanar cracks, the normal and transverse strain energy ratios for two coplanar cracks with \( \nu=0.5 \) are evaluated based on Eqs. (4.8) and(4.9). The obtained results are verified against the exact results by Fabrikant (1987, 1989) as shown in Figure 4.3 (b). As opposed to the case for infinite cracks, where the transverse and normal strain energy ratios are identical (Figure 4.3 (a)) for all values of \( \nu \), the transverse energy ratio with \( \nu=0.5 \) is significantly larger than normal strain energy ratio for two coplanar cracks. Only when \( \nu=0 \), the transverse strain energy ratio can be shown to become identical to
the normal strain energy ratio. This difference can be attributed to the fact that two coplanar cracks lack the periodic and symmetric nature of an array of infinite cracks, where interaction of symmetrically located cracks cancels out each other. As in the case for infinite cracks, as $a/b$ increases, the strain energy ratio increases. However, the maximum value (1.115) for transverse strain energy with $\nu=0.5$ remains significantly smaller than those for an infinite number of cracks (2.16 for the hexagonal configuration 1.64 for the square configuration), indicating weaker crack interaction.

4.3 The spring stiffness for an infinite number of penny-shaped cracks in a homogenous media and interaction function

In this section, the effective spring stiffness expressions for the periodic array of coplanar crack with the square and hexagonal configurations (Figure 4.1 (b) and Figure 4.2 (a) and (b)) are derived based on the crack energy expressions provided in the previous section. For and infinite media with coplanar cracks shown in Figure 4.1(b), the far field displacement can be separated into displacement component without crack and an additional displacement due to the presence of cracks

$$\Delta = \Delta_{\text{no-crack}} + \Delta_{\text{crack}} \quad (4.13)$$

The idea is to replace the array of cracks by continuously distributed interfacial springs, such that they provide the same additional interface compliance (additional displacement $\Delta_{\text{crack}}$ ) as due to the crack array.

$$k_N = \frac{P^0}{\Delta_{N,\text{crack}}}, \quad k_T = \frac{\tau^0}{\Delta_{T,\text{crack}}} \quad (4.14)$$
The additional displacements can be determined by using Castigliano’s theorem, extended for cracked bodies (Tada et al., 2000), as in

$$\Delta_{N,\text{crack}} = \frac{\partial U_N}{\partial Q_N}, \Delta_{T,\text{crack}} = \frac{\partial U_T}{\partial Q_T}$$  \hspace{1cm} (4.15)

where $Q_N = \pi b^2 \left( p^0 \right)$ and $Q_T = \pi b^2 \left( t^0 \right)$, respectively, represents the normal and transverse force applied at infinity. By inserting Eqs. (4.8) and (4.9) into Eq. (4.15) and (4.14), the normal and transverse spring stiffness for an array of cracks, $k_{N,\text{homogenous}}^\infty$ and $k_{T,\text{homogenous}}^\infty$ can be obtained as follows.

$$k_{N,\text{homogenous}}^\infty = \frac{\pi^2 E}{8b \left(1 - \nu^2\right)} \left[ \int_0^{2\pi} \int_0^a \left( \frac{K_i}{K_i^0} \right)^2 r^2 \, dr \, d\phi \right]^{-1}$$  \hspace{1cm} (4.16)

$$k_{T,\text{homogenous}}^\infty = \frac{\pi^2 E}{32b \left(1 - \nu^2\right)} \left[ \int_0^{2\pi} \int_0^a \left( \frac{K_H}{K_H^0} \cos \phi \right)^2 + \left(1 - \nu \right) \left( \frac{K_{III}}{K_{III}^0} \sin \phi \right)^2 \right]^{-1}$$  \hspace{1cm} (4.17)

The equivalent spring stiffness for single non-interacting crack can be recovered by letting $a/b$ approaches zero $\left( K_i / K_i^0 = K_H / K_H^0 = K_{III} / K_{III}^0 = 1 \right)$ in Eqs. (4.16) and (4.17) as follows (Margetan et al., 1988).

$$k_{N,\text{homogenous}}^0 = \frac{3\pi E}{16b \left(1 - \nu^2\right)} \left( \frac{a}{b} \right)^3$$  \hspace{1cm} (4.18)

$$k_{T,\text{homogenous}}^0 = \frac{3\pi E}{32b \left(1 - \nu^2\right)} \left(2 - \nu \right) \left( \frac{a}{b} \right)^3$$  \hspace{1cm} (4.19)
As in the crack strain energy, the normal and transverse spring stiffnesses for an array of cracks normalized by those for single crack, defined by $I_N^\infty$ and $I_T^\infty$, respectively, represents crack interactions in terms of spring stiffnesses and are equal to the inverse of crack energy ratios (Eqs. (4.8) and (4.9)) as follows.

$$I_N \left( \frac{a}{b} \right) = \frac{k_{N, \text{homogenous}}}{k_{N, 0}} = \left( \frac{U_N^\infty}{U_N^0} \right)^{-1}$$  \hspace{1cm} (4.20)

and

$$I_T \left( \frac{a}{b} \right) = \frac{k_{T, \text{homogenous}}}{k_{T, 0}} = \left( \frac{U_T^\infty}{U_T^0} \right)^{-1}$$  \hspace{1cm} (4.21)

Since the normal and transverse strain energy ratios $U_N^\infty / U_N^0$ and $U_T^\infty / U_T^0$ are almost identical, we can write

$$I_N \left( \frac{a}{b} \right) \approx I_T \left( \frac{a}{b} \right) = I \left( \frac{a}{b} \right).$$  \hspace{1cm} (4.22)

Margetan et al. (1988) investigated the same problem using a cylindrical unit cell model with a center crack based on Tada (2000) and obtained approximate expressions of normal and transverse spring stiffnesses (Eqs. 3(a) and (b) in Margetan et al., 1988) for the square configuration. Based on their results, the corresponding interaction function for the normal and transverse spring stiffnesses can be shown to be identical and given by
where \( A_d = \frac{\pi a^2}{4b^2} \) represents the areal crack density for the square configuration.

Problems of crack interaction for an array of an infinite number of periodic two-dimensional strip cracks are also investigated by Baik and Thompson (1984) and the corresponding interaction function for the normal and the transverse spring stiffnesses are identical and can be obtained as follows.

\[
I^{2-D} \left( \frac{a}{b} \right) = \frac{\pi^2}{8} \left( \frac{a}{b} \right)^2 \left[ \ln \left\{ \sec \left( \frac{\pi a}{2b} \right) \right\} \right]^{-1}
\]

where \( 2a \) and \( 2b \), respectively, are the full crack length and the periodicity (the distance between the center of two adjacent cracks).

Eqs. (4.22)-(4.24) are a measure of the reduction in the spring stiffness due to crack interaction and may vary from one (no crack interaction) to zero (zero spring stiffness). They are plotted as a function of crack density parameter \( a/b \) in Figure 4.4 (a).
Figure 4.4 (a) Interaction functions for periodic array of penny-shaped cracks with square and hexagonal configuration derived in current work, obtained from Margetan et al. (1988) and obtained for periodic array of 2-D strip cracks (b) Relative percent difference in Margetan’s model and square configuration
As $a/b$ approaches one, the interaction function for 2-D strip cracks approaches zero since the plane of cracks will be completely detached. On the contrary, the minimum value of the interaction function for 3-D penny-shaped cracks is 0.43 since some detached area remain even in the hexagonal configuration. For all values of $a/b$, crack interactions for 2-D strip cracks are significantly larger than those for 3-D penny-shaped cracks.

The relative difference between the interaction function based on Margetan et al. (1988) and that based on this work for the square configuration is plotted in Figure 4.4 (b). As can be seen, the crack interactions based on unit cell model by Margetan et al. (1988) are larger than those obtained in this work. The difference remains small for small crack densities but the difference increases as $a/b$ becomes large, reaching max. 13% difference at $a/b=1$. Since crack interactions are important only for high crack densities, the difference between these two is significant. The approximate results by Margetan et al. (1988) provide conservative estimates.

2-D strip cracks and 3-D penny-shaped cracks may be considered as two extreme limiting cases of elliptical cracks. In fact, crack interactions in terms of the SIFs for two coplanar elliptical cracks subjected to constant normal and shear tractions are shown to have similar trend as aspect ratio varies (Roy and Chatterjee, 1994 and Saha et al., 1999) suggesting that interaction function for spring stiffness may share a common form. Therefore, motivated by the form of interaction function for 2-D periodic array of strip cracks given in Eq.(4.24), the numerically evaluated interaction functions for square and hexagonal configurations are curve fitted based on the least square method and the following approximate analytical expression is obtained.
where $C$ takes constant values, 0.8673 and 0.7140, respectively, for the hexagonal and the square configurations. The maximum curve fitting error remains less than 0.5% for all values of $a/b$.

By inserting Eq. (4.25) into Eqs. (4.20) and (4.21), the normal and transverse spring stiffnesses for the periodic array of coplanar penny-shaped cracks in a homogenous material are obtained as follows.

$$k_{N, \text{homogenous}}^{\infty} = \frac{3 \pi^3 C}{128 b \left(1 - \nu^2\right)} \sqrt{\frac{a}{b}} \left(\ln \left\{ \sec \left[ \frac{\pi}{2} \sqrt{C \left(\frac{a}{b}\right)^{1.75}} \right]\right\}\right)^{-1}$$  \hfill (4.26)

and

$$k_{T, \text{homogenous}}^{\infty} = \frac{3 \pi^3 C}{256 b \left(1 - \nu^2\right)} \sqrt{\frac{a}{b}} \left(\ln \left\{ \sec \left[ \frac{\pi}{2} \sqrt{C \left(\frac{a}{b}\right)^{1.75}} \right]\right\}\right)^{-1}$$  \hfill (4.27)

### 4.4 Elastic dissimilarity function

Based on the work by Willis (1972), the normal and transverse spring stiffnesses for non-interacting penny-shaped cracks between two different isotropic materials, denoted by $k_{N, \text{dissimilar}}^0$ and $k_{T, \text{dissimilar}}^0$, are obtained by Lekesiz et al. (2010) as follows.

$$k_{N, \text{dissimilar}}^0 = \frac{3 \pi}{16 b} \left(\frac{a}{b}\right)^{-3} \frac{E_1}{\left(1 - \nu_i^2\right) \pi e \left(1 + \beta \right) \left(\frac{1 + \alpha}{\beta}\right)}$$  \hfill (4.28)
\[ k_{T,\text{dissimilar}}^0 = \frac{3\pi}{128b} \left( \frac{a}{b} \right)^{-3} \frac{E_1}{(1-\nu_i^2)} (1+\alpha) \left[ 1+\frac{4\beta \gamma}{\pi\varepsilon(1+\varepsilon^2)(1-\beta^2)} \right] \]  

(4.29)

where

\[ \alpha = \frac{G_2(\kappa_1+1)-G_1(\kappa_2+1)}{G_2(\kappa_1+1)+G_1(\kappa_2+1)}, \quad \beta = \frac{G_2(\kappa_1-1)-G_1(\kappa_2-1)}{G_2(\kappa_1+1)+G_1(\kappa_2+1)} \]  

(4.30)

\[ \varepsilon = \frac{1}{2\pi} \log \left( \frac{1+\beta}{1-\beta} \right), \quad \gamma = \frac{G_1+G_2}{G_2(\kappa_1+1)+G_1(\kappa_2+1)} \]  

(4.31)

Dundurs’ parameters, \( \alpha, \beta \) are expressed in terms of \( \kappa_i=3-4\nu_i \) (Poisson’s ratio \( \nu_i \), \( i=1,2 \)) and \( G_1 \) and \( G_2 \) represent shear moduli (Dundurs and Bogy, 1965). The ranges of these non-dimensional parameters for all material combinations are \( \alpha [-1,1] \), \( \beta [-0.5, 0.5], \) \( \varepsilon [-0.175, 0.175] \) and \( \gamma [0.25, 0.5] \). Eqs (4.28) and (4.29) reduce to Eqs. (4.18) and (4.19) when two dissimilar materials become identical.

We introduce factors \( M_N \) and \( M_T \) defined by the ratio of these spring stiffnesses as follows.

\[ M_N(\alpha, \beta) = \frac{k_{N,\text{dissimilar}}^0}{k_{N,\text{homogenous}}^0} = \frac{(1+\alpha)\beta}{\pi\varepsilon(1+\varepsilon^2)(1-\beta^2)} \]  

(4.32)

\[ M_T(\alpha, \beta, \gamma) = \frac{k_{T,\text{dissimilar}}^0}{k_{T,\text{homogenous}}^0} = \frac{(1+\alpha)}{4(2-\nu_1)} \left( 1+\frac{4\beta}{\gamma\pi\varepsilon(1+\varepsilon^2)(1-\beta^2)} \right) \]  

(4.33)
Figure 4.5 Contour plots of the elastic dissimilarity function for the effective normal spring stiffness. Hollow circles (o) represent some material combination provided by Suga et al. (1988). Some material combinations are indicated by cross symbols (x) and labeled.

Factors $M_N$ and $M_T$, respectively, characterize the effect of the material dissimilarity on the normal and transverse spring stiffness and can be called the normal and transverse elastic dissimilarity functions, respectively. In
Figure 4.6 Contour plots of the elastic dissimilarity function for the transverse spring stiffness for $\nu_1=0.33$. Some sample material combinations with $\nu_1=0.33$ are indicated by cross symbols (x) and labeled.

As can be seen, the contour lines are nearly parallel to the $\beta$- axis indicating that $M_N$ is mostly controlled by parameter $\alpha$. As can be seen in Eq.(4.33), $M_T$ depends on $\alpha$, $\beta$ and $\nu_1$, ranging from zero ($\alpha=-1$, $G_2/G_1$ approaches zero) to 2.177 ($\alpha=1$, $G_2/G_1$ approaches infinity and $\nu_1=0.5$). By choosing $\nu_1=0.33$ ($\kappa_2=1.68$), $M_T$ is plotted in the form of contour lines as function of $\alpha$ and $\beta$ in Figure 4.6. Some material combinations with $\nu_1=0.33$ are indicated by cross (x) and labeled. The limits of physically admissible material
combinations are indicated by the lines of $\kappa_2=1$ and $\kappa_2=3$. As in $M_N$, the contour lines are nearly parallel to the $\beta$-axis and $\alpha$ is again a dominant parameter for $M_T$.

Eq. (4.32) can be shown to be almost identical to the elastic dissimilarity function obtained for a periodic array of 2-D strip cracks between two dissimilar isotropic materials by Lekesiz et al. (2010)

\[ M^{2-D}(\alpha, \beta) = \frac{(1 + \alpha)}{(1 + 4\varepsilon^2)(1 - \beta^2)} \]  

(4.34)

The difference between Eqs. (4.32) and (4.34) is less than 0.8% for all the material combinations indicating that material dissimilarity function for the normal spring stiffness is almost independent of the crack shape.

Gorbatikh and Popova (2005) have shown that the common material parameter, $(1+\alpha)/(1-\beta^2)$, can be used in estimating the normal elastic compliances of non-interacting rectilinear, penny-shaped, elliptical, and annular non-interacting cracks between two dissimilar materials from the corresponding solution for the homogeneous case. Their universal material parameter is similar to Eq. (34) and (32). However, Eq. (32) provides an accurate expression for the material dissimilarity function specifically for a penny-shaped crack.

4.5 Spring Stiffness for periodic array of penny-shaped cracks between two dissimilar isotropic materials

An analytical solution of the stress intensity factors for an array of coplanar periodic penny-shaped cracks between two dissimilar isotropic materials (Figure 4.1 (c)) is not
available in the literature known to the authors. Therefore, an exact solution of the spring stiffness for the corresponding case has not been derived. In this section, an approximate expression for spring stiffness is obtained heuristically using the interaction function and elastic dissimilarity functions derived in prior sections. The interaction function for the periodic array of strip cracks between two dissimilar materials can be defined as follows.

\[
I_{\text{dissimilar}}^{2-D} \left( \frac{a}{b}, \varepsilon \right) = \frac{k_{\text{dissimilar}}^{2-D, \infty}}{k_{\text{dissimilar}}^{2-D, 0}}
\]  

(4.35)

\(k_{\text{dissimilar}}^{2-D, \infty}\) is derived by Lekesiz et al. (2010) by following the same procedure as in Eqs. (4.14) and (4.15) using the strain energy for periodic array of strip cracks.

\[
k_{\text{dissimilar}}^{2-D, \infty} = \frac{G_i}{b(1+\kappa_i)} \frac{(1+\alpha)}{(1-\beta^2)} \left\{ \left( \frac{\alpha}{b} \varepsilon \right) L\left( \frac{a}{b}, \varepsilon \right) \right\}
\]  

(4.36)

where

\[
L\left( \frac{a}{b}, \varepsilon \right) = \int_0^{\frac{a}{b}} \frac{1}{\sin\left( \pi \frac{\alpha}{\varepsilon} \right)} \left[ \cosh^2 \left( \pi \frac{\alpha}{\varepsilon} \right) - \cos^2 \left( \frac{\pi \alpha}{2} \right) \right] d\alpha
\]  

(4.37)

\(k_{\text{dissimilar}}^{2-D, 0}\) is given also given by Lekesiz et al. (2010) as follows

\[
k_{\text{dissimilar}}^{2-D, 0} = \frac{8}{\pi} \frac{b}{\alpha^2 (1+\kappa_i)} \frac{(1+\alpha)}{(1-\beta^2)(1+4\varepsilon^2)}
\]  

(4.38)

Substituting Eqs. (4.37) and (4.36) into Eq.(4.35), in Figure 4.7, the interaction function, \(I_{\text{dissimilar}}^{2-D}\) is plotted as a function of crack density. Two extreme cases of material
dissimilarity $\varepsilon = 0$ (homogeneous case) and $\varepsilon = 0.1748$ (See Eq. (4.31)) are indicated in the figure. In order to elucidate the effect of material dissimilarity on the interaction function, the ratios of the exact interaction function ($\varepsilon = 0.05, 0.1$ and $0.1748$) and the interaction function for the homogeneous case ($\varepsilon = 0$) are also shown by dashed lines in Figure 4.7. It is interesting to note that the $I_{\text{dissimilar}}^{2-D}$ curve shown by the solid line for the homogeneous case ($\varepsilon = 0$) is slightly higher than the $I_{\text{dissimilar}}^{2-D}$ curve with $\varepsilon = 0.1748$ over the wide $a/b$ range, and consequently the dashed lines with $\varepsilon = 0.05, 0.1$ and $0.1748$ are all below 1.0. This indicates that the interaction between cracks is slightly stronger for the dissimilar material case. With the maximum value of $\varepsilon = 0.1748$, the interaction is highest. However the differences are not large; even for the extreme case of 95% of the crack density ($a/b = 0.95$), the ratio $I_{\text{dissimilar}}^{2-D}$ remains larger than 85% as shown by the dashed line for $\varepsilon = 0.1748$. As a result, we can approximately write,

$$I_{\text{dissimilar}}^{2-D} \left( \frac{a}{b}, \varepsilon \right) \approx I_{\text{homogenous}}^{2-D} \left( \frac{a}{b} \right)$$

(4.39)

Eq. (4.39) points out the fact that interaction happens in a very similar manner for the 2-D periodic array of interfacial cracks and those in homogenous material in terms of spring stiffnesses.
Authors suggest that this phenomenon is applicable also for 3-D penny-shaped cracks. The validity of this claim cannot be confirmed without the exact solution; however, we can intrinsically prove it by looking back to Figure 4.4(a). Recalling the fact that interaction is much more severe in 2-D strip cracks compared to 3-D penny-shaped cracks for the same a/b value, we can state that influence of material dissimilarity will be less in 3-D penny-shaped cracks and therefore, interaction function for dissimilar case is expected to be very close to the interaction function for homogenous case as in
\[
I_{\text{dissimilar}} \left( \frac{a}{b} \right) = \frac{k_{\text{dissimilar}}^\infty}{k_{\text{dissimilar}}^0} \approx I \left( \frac{a}{b} \right)
\]  

(4.40)

which leads to

\[
k_{\text{dissimilar}}^\infty \approx I \left( \frac{a}{b} \right) \times k_{\text{dissimilar}}^0
\]  

(4.41)

By inserting \(k_{N,\text{dissimilar}}^0\) and \(k_{T,\text{dissimilar}}^0\) from Eqs. (4.28) and (4.29) and \(I\) from Eq. (4.25) into Eq.(4.41), we derive following spring stiffness expressions for array of penny-shaped cracks between two dissimilar elastic materials.

\[
k_{N,\text{array}} = \frac{E_i}{(1-\nu^2_i) b} \frac{3\pi^3 C}{128} \beta(1+\alpha) \left[ \sqrt{\frac{a}{b}} \right] \left( \ln \left\{ \sec \left( \frac{\pi}{2} \sqrt{C \left( \frac{a}{b} \right)^{1.75}} \right) \right\} \right]^{-1}
\]  

(4.42)

\[
k_{T,\text{array}} = \frac{E_i}{(1-\nu^2_i) b} \frac{3\pi^3 C}{1024} \left( 1+\alpha \right) \left[ \frac{4\beta}{\gamma} + \frac{1}{\pi\nu(1+\nu^2)(1-\beta^2)} \right] \left[ \sqrt{\frac{a}{b}} \right] \left( \ln \left\{ \sec \left( \frac{\pi}{2} \sqrt{C \left( \frac{a}{b} \right)^{1.75}} \right) \right\} \right]^{-1}
\]  

(4.43)

Recall \(C=0.8673\) for the hexagonal configuration and \(C=0.7140\) for the square configuration. Eqs. (4.42) and (4.43) are plotted in the Figure 4.8 (a) and (b) as a function of crack density for four material combinations, BC4/Ni, Al2O3/Al Alloy, Steel/Al Alloy and Nylon/Steel, selected from Suga et al. (1988). The range of the plot is limited to \([a/b, 0.5, 1]\) because the spring stiffness for the range of \([a/b, 0, 0.5]\) is very high and happens very similar to non-interacting crack case. As can be seen, the lowest stiffness occurs for Nylon/Steel couple, because the stiffness is determined by the weakest material of the couple. Transverse spring stiffness is smaller compared to normal stiffness; however, the difference is not very large.
Figure 4.8 Effective spring stiffness for four different material combinations with hexagonal array of penny-shaped interfacial cracks as a function of crack density parameter $a/b$ (a) Normal, (b) Transverse
Note that Eq. (4.41) can be rewritten by incorporating $M_N$ and $M_T$ from Eqs. (4.32) and (4.33) as follows.

\[
k_{N, \text{dissimilar}}^\infty \approx I \left( \frac{a}{b} \right) \times M_N \left( \alpha, \beta \right) \times k_{N, \text{homogenous}}^0
\]

(4.44)

\[
k_{T, \text{dissimilar}}^\infty \approx I \left( \frac{a}{b} \right) \times M_T \left( \alpha, \beta, \nu_1 \right) \times k_{N, \text{homogenous}}^0
\]

(4.45)

Eqs. (4.44) and (4.45) allows us to express the most complicated problem of interacting cracks between two dissimilar material in a simple form as a multiplication of interaction function, material dissimilarity and spring stiffness for non-interaction crack in a homogenous material. As also can be seen, Eqs. (4.44) and (4.45) recover all spring stiffnesses derived in this paper, $k_{N, \text{homogenous}}^\infty$ and $k_{\text{dissimilar}}^0$ such that

\[
k_{N, \text{homogenous}}^\infty \approx I \left( \frac{a}{b} \right) \times k_{N, \text{homogenous}}^0 \quad \text{and} \quad k_{T, \text{homogenous}}^\infty \approx I \left( \frac{a}{b} \right) \times k_{N, \text{homogenous}}^0
\]

(4.46)

or

\[
k_{N, \text{dissimilar}}^0 = M_N \left( \alpha, \beta \right) \times k_{N, \text{homogenous}}^0 \quad \text{and} \quad k_{T, \text{dissimilar}}^0 = M_T \left( \alpha, \beta, \nu_1 \right) \times k_{N, \text{homogenous}}^0
\]

(4.47)

The factorization proposed in Eqs. (4.44)-(4.47) are useful because the spring stiffnesses can be related to the crack density and therefore can be used in estimating the percentage of disbond area between two dissimilar materials, which is directly related to the residual strength of the interface.

An important note should be pronounced here. As discussed in great detail by Lekesiz et al. (2010), for interface cracks the open crack model the crack opening displacement exhibits an oscillatory crack interpenetration zone at the crack tip, thus
violating the open crack assumption (Rice, 1988 and Hills and Barber, 1993). In order to avoid this unrealistic interpenetration phenomena to occur, a certain amount of tensile load needed to be maintain however, real cracks are often slightly open (due to prior loading history and plastic deformation), and therefore it is not a problem as long as the mid-crack opening normalized by the crack length is at least in the order of $10^{-6}$. (It is shown, that this is applicable for the level of shear strain below the order of $10^{-7}$ which is suitable for linear ultrasonic waves).
REFERENCES


CONCLUSION

The bonded materials are prone to the failure due to the imperfections occurring at the interfaces between bonded layers. These imperfections may be created initially by the improper bonding procedure or due to service conditions. Detection of these imperfections before a catastrophic failure is crucial in order to improve service life of these structures. Ultrasonic methods have been widely used to detect and characterize interfacial imperfections such as distributed micro-cracks or micro-disbands. A quasi-static approximation has been widely used for modeling of ultrasonic wave interactions at imperfect interfaces. In this model the reduction in static stiffness of the overall structure due to compromised interfaces (micro-cracks or micro-disbands) is represented by continuous, uniform distributions of interfacial springs.

In this dissertation, explicit spring stiffness expressions for the interfaces weakened by periodic cracks are derived by utilizing the stress intensity factors for the corresponding crack geometry. Both two-dimensional strip cracks and three-dimensional penny-shaped cracks are considered in detail.

Spring stiffness for the interfaces weakened by periodic array of 2-D strip cracks depends on the elastic dissimilarity of semispaces, the crack density and the crack interaction in the array. In general, the crack interaction weakly depends on material dissimilarity and, for most practical cases, this dependence can be replaced
approximately by the interaction function (the ratio of spring stiffness for the interacting cracks and non-interacting cracks) of the crack array in the homogeneous space. This allows approximate factorization of the effective spring stiffness for an array of cracks between dissimilar materials in terms of the elastic dissimilarity factor (elastic mismatch), the crack interaction factor, and the effective spring stiffness for non-interacting (independent) cracks in a homogeneous material.

For non-interacting 3-D penny-shaped cracks or contacts between two identical, isotropic, linearly elastic materials, we also derive normal and transverse spring stiffness expressions for non-interacting, penny-shaped cracks between two dissimilar, linearly elastic materials. It is shown that the expressions obtained for transverse spring stiffness can be used for linear ultrasound applications as long as the initial maximum crack opening displacement is $10^{-6}$ of the crack radius (the crack opening displacement may be a result of the prior loading history). The results obtained are used to estimate the error in the approximate spring stiffness expressions in the literature; the error is shown to be small for most practical linearly elastic material combinations.

Multiple interacting penny-shaped cracks in homogenous materials are investigated in two aspects: Stress intensity factors for all modes and interaction function in normal and transverse directions for spring stiffness. For the stress intensity factors, the numerical values of mode I SIF obtained for square and hexagonal configurations are fitted to a surface as a function of crack density parameter, $a/b$ and angle, $\phi$ using the least square method. The mode II and III SIFs are given in the form of contour as a function of $a/b$ and $\phi$. Based on the analysis, the interaction has a magnifying effect on
the stress intensity factor values. The influence of interaction on the mode II and mode III SIFs is slightly more compared to that on the mode I SIF. The SIF value can be considered to be identical to the SIF for single crack if the crack density parameter, $a/b$ is less than 0.5 for both square and hexagonal crack configurations. The mode II and III SIFs are almost identical for Poisson’s ratio values of two extreme values of $\nu=0$ and 0.5 if the crack density parameter, $a/b$ is less than 0.75 for both square and hexagonal crack configurations and therefore the influence of Poisson’s ratio on the mode II and mode III SIFs can be neglected for $a/b<0.75$. The results are compared against the results given in the literature and a perfect agreement is observed between the results.

It is shown that the influence of crack interaction on spring stiffness is much more severe for the 2-D strip cracks compared to 3-D penny-shaped cracks. The interaction function for a periodic array of coplanar penny-shaped cracks in homogenous materials can be used for a periodic array of penny-shaped interface cracks between two different materials. This approximation allows us to write same factorization for 3-D cracks as in the 2-D cracks (Chapter 1) and an empirical expression for the spring stiffness of 3-D periodic array of interaction cracks between two dissimilar solids is obtained.
APPENDIX A  

Stress Intensity factors for a periodic array of interfacial strip cracks

Throughout the following derivation, the notation employed by Rice and Sih (1965) will be used for consistency. The obtained results will be expressed based on notation used in the main text at the end of this appendix. The stress intensity factor for interfacial crack problems is defined by Rice (1988) in page 99 as follows.

\[ K = K_t + iK_{II} = \sqrt{\pi} \cosh(\pi \varepsilon)(k_1 + ik_2) \]  

(A.1)

where

\[ k_1 - ik_2 = 2\sqrt{2}e^{\pi \varepsilon \lim_{z \to a} (z-a)^{1/2+i\varepsilon}} \Phi_1(z) \]  

(A.2)

(Eq. (30) in Rice and Sih (1965)). \( \Phi_1(z) \) for periodic array of strip cracks is defined as

\[
\Phi_1(z) = B \left( \frac{z-d}{b} \right) \left( \frac{z-a}{b} \right)^{-1/2-i\varepsilon} \left( \frac{z+a}{b} \right)^{-1/2+i\varepsilon} \times \prod_{n=1}^{\infty} \left[ 1 - \frac{1}{n^2} \left( \frac{z-d}{b} \right)^2 \right] \left[ 1 - \frac{1}{n^2} \left( \frac{z-a}{b} \right)^2 \right]^{-1/2-i\varepsilon} \left[ 1 - \frac{1}{n^2} \left( \frac{z+a}{b} \right)^2 \right]^{-1/2+i\varepsilon} + A.
\]  

(A.3)

(Page 423 in Rice and Sih (1965)). Noticing here that parameter \( A \) does not depend on \( z \), we can say the term \( (z-a)^{1/2+i\varepsilon} A \) in Eq. (A.2) will go to zero as \( z \) approaches zero. Combining Eqs. (A.2) and (A.3), we have

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As 'z' approaches 'a', Eq. (A.4) becomes

\[
k_1 - ik_2 = 2\sqrt{2}e^{2\pi i} B(b) \lim_{z \to a} \left( \frac{z-a}{1} \right)^{1/2+ie} \left( \frac{z-a}{b} \right)^{1/2-ie} \left( \frac{z-d}{b} \right)^{-1/2+ie} \left( \frac{z+a}{b} \right)^{-1/2+ie}
\]

\[
\times \prod_{n=1}^{\infty} \left[ \left( 1 - \frac{1}{n^2} \left( \frac{z-d}{b} \right)^2 \right) \left( 1 - \frac{1}{n^2} \left( \frac{z-a}{b} \right)^2 \right)^{-1/2-ie} \left( 1 - \frac{1}{n^2} \left( \frac{z+a}{b} \right)^2 \right)^{-1/2+ie} \right].
\]

which leads to

\[
k_1 - ik_2 = 2\sqrt{2}e^{2\pi i} B(b) \left( \frac{a-d}{b} \right)^{1/2+ie} \left( \frac{a+a}{b} \right)^{-1/2+ie}
\]

\[
\times \prod_{n=1}^{\infty} \left[ \left( 1 - \frac{1}{n^2} \left( \frac{a-d}{b} \right)^2 \right) \left( 1 - \frac{1}{n^2} \left( \frac{a-a}{b} \right)^2 \right)^{-1/2-ie} \left( 1 - \frac{1}{n^2} \left( \frac{a+a}{b} \right)^2 \right)^{-1/2+ie} \right],
\]

(A.5)

Let’s reorganize Eq. (A.6) as follows.

\[
k_1 - ik_2 = 2\sqrt{2}e^{2\pi i} B(b) \left( \frac{a-d}{b} \right)^{1/2+ie} \left( \frac{2a}{b} \right)^{-1/2+ie} \prod_{n=1}^{\infty} \left( 1 - \frac{1}{n^2} \left( \frac{a-d}{b} \right)^2 \right) \left( 1 - \frac{1}{n^2} \left( \frac{2a}{b} \right)^2 \right)^{-1/2+ie}
\]

(A.6)

\[
\left( \frac{a-d}{b} \right) \prod_{n=1}^{\infty} \left[ \left( 1 - \frac{1}{n^2} \left( \frac{a-d}{b} \right)^2 \right) \left( 1 - \frac{1}{n^2} \left( \frac{2a}{b} \right)^2 \right)^{-1/2+ie} \right].
\]

(A.7)

We recall the following equations.
\[
\left\{ \frac{2a}{b} \right\} \prod_{n=1}^{\infty} \left[ 1 - \frac{1}{n^2} \left( \frac{2a}{b} \right)^2 \right]^{-1/2 \pi i} = \left\{ \frac{1}{\pi} \sin \left( \frac{2\pi a}{b} \right) \right\}^{-1/2 \pi i},
\]

\[
\left\{ \frac{a-d}{b} \right\} \prod_{n=1}^{\infty} \left[ 1 - \frac{1}{n^2} \left( \frac{a-d}{b} \right)^2 \right] = \left\{ \frac{1}{\pi} \sin \left( \frac{\pi (a-d)}{b} \right) \right\} = \left\{ \frac{1}{\pi} \sin \left( \frac{\pi a (1-2i\varepsilon)}{b} \right) \right\}.
\] (A.8)

(Eq. (37) in Rice and Sih (1965)) where \( d=2i\varepsilon \) (Page 423 in Rice and Sih (1965)).

Inserting Eq. (A.8) into Eq.(A.7), we have

\[
k_1 ik_2 = 2\sqrt{2} e^{\pi} B(b)^{1/2} \left\{ \frac{1}{\pi} \sin \left( \frac{\pi a (1-2i\varepsilon)}{b} \right) \right\} \left\{ \frac{1}{\pi} \sin \left( \frac{2\pi a}{b} \right) \right\}^{-1/2 \pi i}.
\] (A.9)

Separating the power terms in Eq. (A.9), we obtain

\[
k_1 ik_2 = \frac{2\sqrt{2} e^{\pi}}{\sqrt{\pi}} B(b)^{1/2} \left( b \right)^{i\varepsilon} \left\{ \sin \left( \frac{\pi a (1-2i\varepsilon)}{b} \right) \right\} \left\{ \sin \left( \frac{2\pi a}{b} \right) \right\}^{-1/2} \left\{ \frac{1}{\pi} \sin \left( \frac{2\pi a}{b} \right) \right\}.
\] (A.10)

Recalling the following transformations for trigonometric functions, we have

\[
sin \left( \frac{\pi a (1-2i\varepsilon)}{b} \right) = \sin \left( \frac{\pi a}{b} - \frac{2\pi a i\varepsilon}{b} \right) = \sin \left( \frac{\pi a}{b} \right) \cos \left( \frac{2\pi a i\varepsilon}{b} \right) - \cos \left( \frac{\pi a}{b} \right) \sin \left( \frac{2\pi a i\varepsilon}{b} \right),
\]

\[
\cos \left( \frac{2\pi a i\varepsilon}{b} \right) = \cosh \left( \frac{2\pi a \varepsilon}{b} \right), \quad \sin \left( \frac{2\pi a i\varepsilon}{b} \right) = i \sinh \left( \frac{2\pi a \varepsilon}{b} \right),
\] (A.11)

which leads to
\[
\sin \left( \frac{\pi a}{b} - \frac{2 \pi a \varepsilon}{b} \right) = \sin \left( \frac{\pi a}{b} \right) \cosh \left( \frac{2 \pi a \varepsilon}{b} \right) - i \cos \left( \frac{\pi a}{b} \right) \sinh \left( \frac{2 \pi a \varepsilon}{b} \right). \quad (A.12)
\]

Inserting Eq. (A.12) into Eq. (A.10), we have

\[
k_1 - ik_2 = 2 \sqrt{\frac{2b}{\pi \sin \left( \frac{2 \pi a}{b} \right)}} e^{\pi \varepsilon} B \left\{ \sin \left( \frac{\pi a}{b} \right) \cosh \left( \frac{2 \pi a \varepsilon}{b} \right) - i \cos \left( \frac{\pi a}{b} \right) \sinh \left( \frac{2 \pi a \varepsilon}{b} \right) \right\}
\times \left\{ \frac{b \sin \left( \frac{2 \pi a}{b} \right)^{2 \pi \varepsilon}}{\pi} \right\}^{i \varepsilon}
\quad (A.13)
\]

Inserting \( B \) given as \( B = \frac{\sigma - i \tau}{1 + e^{2 \pi \varepsilon}} \) (Eq. (18) in Rice and Sih (1965)) into Eq. (A.13), we obtain

\[
k_1 - ik_2 = 2 \sqrt{\frac{2b}{\pi \sin \left( \frac{2 \pi a}{b} \right)}} e^{\pi \varepsilon} \left\{ \frac{\sigma - i \tau}{1 + e^{2 \pi \varepsilon}} \right\} \left\{ \sin \left( \frac{\pi a}{b} \right) \cosh \left( \frac{2 \pi a \varepsilon}{b} \right) - i \cos \left( \frac{\pi a}{b} \right) \sinh \left( \frac{2 \pi a \varepsilon}{b} \right) \right\}
\times \left\{ \frac{b \sin \left( \frac{2 \pi a}{b} \right)^{2 \pi \varepsilon}}{\pi} \right\}^{i \varepsilon}
\quad (A.14)
\]

and
\[ k_1 - ik_2 = \sqrt{\frac{2b}{\pi \sin \left( \frac{2\pi a}{b} \right)}} \left( \sigma - i\tau \right) \left\{ \sin \left( \frac{\pi a}{b} \right) \cosh \left( \frac{2\pi a \varepsilon}{b} \right) - i \cos \left( \frac{\pi a}{b} \right) \sinh \left( \frac{2\pi a \varepsilon}{b} \right) \right\} \]
\times \left\{ \frac{b}{\pi} \sin \left( \frac{2\pi a}{b} \right) \right\}^{i\varepsilon}.

(A.15)

Let's write the conjugate of Eq. (A.15) as follows.

\[ k_1 + ik_2 = \frac{1}{\cosh (\pi \varepsilon)} \sqrt{\frac{2b}{\pi \sin \left( \frac{2\pi a}{b} \right)}} \left\{ \sin \left( \frac{\pi a}{b} \right) \cosh \left( \frac{2\pi a \varepsilon}{b} \right) + i \cos \left( \frac{\pi a}{b} \right) \sinh \left( \frac{2\pi a \varepsilon}{b} \right) \right\} \]
\times \left\{ \frac{b}{\pi} \sin \left( \frac{2\pi a}{b} \right) \right\}^{-i\varepsilon} \left( \sigma + i\tau \right).

(A.16)

Inserting Eq. (A.16) into Eq. (A.1), we obtain

\[ K = K_1 + iK_2 = \sqrt{\frac{2b}{\sin \left( \frac{2\pi a}{b} \right)}} \left\{ \sin \left( \frac{\pi a}{b} \right) \cosh \left( \frac{2\pi a \varepsilon}{b} \right) + i \cos \left( \frac{\pi a}{b} \right) \sinh \left( \frac{2\pi a \varepsilon}{b} \right) \right\} \]
\times \left\{ \frac{b}{\pi} \sin \left( \frac{2\pi a}{b} \right) \right\}^{-i\varepsilon} \left( \sigma + i\tau \right) \cdot (A.17)

Parameter ‘\(b\)’ in Rice and Sih (1965) corresponds to ‘\(2b\)’ in our paper. By replacing ‘\(b\)’ in Eq. (A.17) by ‘\(2b\)’, we have
\[ K = K_r + i K_i = 2 \sqrt{\frac{b}{\sinh \left( \frac{\pi a}{2b} \right)}} \left\{ \sin \left( \frac{\pi a}{2b} \right) \cosh \left( \frac{\pi \varepsilon}{b} \right) + i \cos \left( \frac{\pi a}{2b} \right) \sinh \left( \frac{\pi \varepsilon}{b} \right) \right\} \times \left\{ \frac{2b}{\pi} \sin \left( \frac{\pi a}{b} \right) \right\}^{-i\varepsilon} (\sigma + i\tau) \] (A.18)
APPENDIX B

E* in terms of Dundurs Parameter for plane strain and plane stress

Plane Strain

E* can be reorganized as follows.

\[
\frac{1}{E^*} = \frac{1}{2} \left( \frac{1}{E_1} + \frac{1}{E_2} \right) = \frac{1}{2} \left( \frac{1}{\frac{1}{1-\nu_1^2} E_1} + \frac{1}{\frac{1}{1-\nu_2^2} E_2} \right) = \frac{1}{2} \left( \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \right) = \frac{E_1(1-\nu_2^2) + E_2(1-\nu_1^2)}{2E_1E_2}
\]

(B.1)

Using the relation \( E = 2(1+\nu)G \), Eq. (B.1) leads to

\[
\frac{1}{E^*} = \frac{2G_1(1+\nu_1)(1-\nu_2^2) + 2G_2(1+\nu_2)(1-\nu_1^2)}{8G_1(1+\nu_1)G_2(1+\nu_2)} = \frac{(1-\nu_2)}{4G_2} + \frac{(1-\nu_1)}{4G_1}.
\]

(B.2)

Recalling \( \kappa_i = 3-4\nu_i \), one can write

\[
(1-\nu_2) = \frac{1+\kappa_2}{4} \quad \text{and} \quad (1-\nu_1) = \frac{1+\kappa_1}{4}.
\]

(B.3)

Combining Eqs. (B.2) and (B.3), we have
Using

\[ \frac{1}{E^*} = \frac{G_1(1+\kappa_1) + G_2(1+\kappa_2)}{16G_1G_2}. \]  \hspace{1cm} (B.4)

Using

\[ \alpha = \frac{G_2(\kappa_1 + 1) - G_1(\kappa_2 + 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)} \quad \text{and} \quad 1 + \alpha = \frac{2G_2(\kappa_1 + 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}, \]  \hspace{1cm} (B.5)

the numerator of Eq. (B.4) can be expressed as follows.

\[ G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1) = \frac{2G_2(1+\kappa_1)}{(1+\alpha)}, \]  \hspace{1cm} (B.6)

Inserting Eq. (B.6) into Eq. (B.4), we have

\[ \frac{1}{E^*} = \frac{(1+\kappa_1)}{8(1+\alpha)G_1}. \]  \hspace{1cm} (B.7)

**Plane Stress:**

Recalling \( E^* \),

\[ \frac{1}{E^*} = \frac{1}{2 \left( \frac{1}{E_1} + \frac{1}{E_2} \right)} = \frac{1}{2E_1} + \frac{1}{2E_2}. \]  \hspace{1cm} (B.8)

Using \( E = 2(1+\nu)G \), Eq. (B.8) leads to

\[ \frac{1}{E^*} = \frac{1}{2 \left( \frac{1}{E_1} + \frac{1}{E_2} \right)} = \frac{1}{4G_1(1+\nu_1)} + \frac{1}{4G_2(1+\nu_2)}. \]  \hspace{1cm} (B.9)
Recalling that \( \kappa = (3 - \nu_i)/(1 + \nu_i) \) for plane stress, we have

\[
\nu_1 = \frac{3 - \kappa_1}{1 + \kappa_1} \quad \text{and} \quad \nu_2 = \frac{3 - \kappa_2}{1 + \kappa_2}.
\]  
(B.10)

Eq. (B.10) leads to

\[
1 + \nu_1 = 1 + \frac{3 - \kappa_1}{1 + \kappa_1} = \frac{4}{1 + \kappa_1} \quad \text{and} \quad 1 + \nu_2 = 1 + \frac{3 - \kappa_2}{1 + \kappa_2} = \frac{4}{1 + \kappa_2}.
\]  
(B.11)

Combining Eqs. (B.9) and (B.11), we have

\[
\frac{1}{E^*} = \frac{(1 + \kappa_1)}{16G_1} + \frac{(1 + \kappa_2)}{16G_2} = \frac{G_1(1 + \kappa_2) + G_2(1 + \kappa_1)}{16G_1G_2}.
\]  
(B.12)

Notice that Eqs. (B.4) and (B.12) are the same. Therefore, steps after Eq. (B.4) will be identical for plane stress.

**Second Dundurs Parameter \( \beta \) in terms of \( E \) and \( \nu \) for plane stress:**

Recalling \( \kappa = (3 - \nu)/(1 + \nu) \) for plane stress and \( E = 2(1 + \nu)G \). Then we have

\[
\beta = \frac{G_2(k_1 - 1) - G_1(k_2 - 1)}{G_2(k_1 + 1) + G_1(k_2 + 1)} = \frac{G_2\left(\frac{3 - \nu_1}{1 + \nu_1} - 1\right) - G_1\left(\frac{3 - \nu_2}{1 + \nu_2} - 1\right)}{G_2\left(\frac{3 - \nu_1}{1 + \nu_1} + 1\right) + G_1\left(\frac{3 - \nu_2}{1 + \nu_2} + 1\right)}
\]

\[
= \frac{\frac{2 - 2\nu_1}{1 + \nu_1} - \frac{2 - 2\nu_2}{1 + \nu_2}}{\frac{4}{1 + \nu_1} + \frac{4}{1 + \nu_2}} = \frac{1}{2} \frac{G_2(1 - \nu_1)(1 + \nu_2) - G_1(1 - \nu_2)(1 + \nu_1)}{G_2(1 + \nu_2) + G_1(1 + \nu_1)}
\]  
(B.13)
Second Dundurs Parameter $\beta$ in terms of $E$ and $\nu$ for plane strain:

Recalling $\kappa_i = 3 - 4\nu_i$ for plane strain and $E = 2(1 + \nu)G$

$$\beta = \frac{G_2(k_1 - 1) - G_1(k_2 - 1)}{G_2(k_1 + 1) + G_1(k_2 + 1)} = \frac{G_2(3 - 4\nu_1 - 1) - G_1(3 - 4\nu_2 - 1)}{G_2(3 - 4\nu_1 + 1) + G_1(3 - 4\nu_2 + 1)}$$

$$\beta = \frac{G_2(2 - 4\nu_1) - G_1(2 - 4\nu_2)}{G_2(4 - 4\nu_1) + G_1(4 - 4\nu_2)} = \frac{1}{2} \frac{G_2(1 - 2\nu_1) - G_1(1 - 2\nu_2)}{G_2(1 - \nu_1) + G_1(1 - \nu_2)} \quad (B.14)$$

Limits of Dundurs’ parameters

It has been shown in Schmauder and Meyer (1992) and Bogy (1979) that the upper and lower limits of $\alpha$ and $\beta$ as follows.

$$-1 \leq \alpha \leq 1, \quad \frac{\alpha}{4} - 0.25 \leq \beta \leq \frac{\alpha}{4} + 0.25 \quad \text{for plane strain} \quad (B.15)$$

and

$$\frac{3\alpha}{8} - 0.125 \leq \beta \leq \frac{3\alpha}{8} + 0.125 \quad \text{for plane stress.} \quad (B.16)$$
APPENDIX C  Strain energy release rate for a periodic array of interfacial strip cracks

Recalling $cosh^{2}(\pi \varepsilon) = \frac{1}{1 - \beta^{2}}$ (Hutchinson and Suo (1992), Page 75), the strain energy release rate for interfacial cracks can be expressed as

$$W = \frac{1}{E^{*}cosh^{2}(\pi \varepsilon)} = \left(1 - \beta^{2}\right) \frac{K^{*}}{E^{*}}.$$  \hspace{1cm} (C.1)

Stress intensity factors (SIF) is defined in the manner of Rice (1988) as follows.

$$K = K_{I} + iK_{II} = \sqrt{\pi} \cosh(\pi \varepsilon)(k_{1} + ik_{2})$$ \hspace{1cm} (C.2)

where

$$k_{1} = 2\sqrt{\frac{b}{\pi \sin(\pi a / b)}} \frac{[ sin(\pi a / 2b) \cosh(\pi a \varepsilon / b) \cos(\varepsilon \log[(2b / \pi) \sin(\pi a / b)]) ]}{ cosh(\pi \varepsilon) }$$

$$[ + \cos(\pi a / 2b) \sinh(\pi a \varepsilon / b) \sin(\varepsilon \log[(2b / \pi) \sin(\pi a / b)]) ]$$

$$[ + \sigma \left( \text{sin}(\pi a / 2b) \cosh(\pi a \varepsilon / b) \text{sin}(\varepsilon \log[(2b / \pi) \sin(\pi a / b)]) \right) ]$$

$$[ - \cos(\pi a / 2b) \sinh(\pi a \varepsilon / b) \cos(\varepsilon \log[(2b / \pi) \sin(\pi a / b)]) ]$$

(C.3)

and
\[
K \bar{K} = K_i^2 + K_{ii}^2 = \frac{4b}{\sin(\pi a/2b)} \left\{ \cosh^2\left(\frac{\pi a}{b}\right) - \cos^2\left(\frac{\pi a}{2b}\right) \right\} (\sigma^2 + \tau^2)
\]  
(C.5)

Inserting Eqs. (B.7) and (C.5) into Eq. (C.1), we obtain

\[
W_{\text{array}} = \frac{(1 + \kappa_i)(1 - \beta^2)}{4G_i(1 + \alpha)} \frac{4b}{\sin(\pi a/2b)} \left\{ \cosh^2\left(\frac{\pi a}{b}\right) - \cos^2\left(\frac{\pi a}{2b}\right) \right\} (\sigma^2 + \tau^2)
\]  
(C.6)

Multiplying and dividing Eq. (C.6) by ‘\pi a’, Eq. (C.6) can be rewritten as follows.

\[
W_{\text{array}} = \frac{1 + \kappa_i}{4G_i(1 + \alpha)} \frac{1 - \beta^2}{\pi a} \left\{ \cosh^2\left(\frac{\pi a}{b}\right) - \cos^2\left(\frac{\pi a}{2b}\right) \right\} (\sigma^2 + \tau^2)
\]  
(C.7)

Maple Work

From Eqs. (C.2)-(C.4), the term \(K \bar{K} = K_i^2 + K_{ii}^2\) have following form.
Using Maple® software, the coefficient of $\sigma^2$-terms, $A$, can be simplified as follows.

\[
A = \left(\sin\left(\frac{\pi a}{2b}\right) \cdot \cosh\left(\frac{\pi \cdot \epsilon a}{b}\right) \cdot \cos\left(\epsilon_{\log}\left(\frac{2b}{\pi} \cdot \sin\left(\frac{\pi a}{b}\right)\right)\right) + \cos\left(\frac{\pi a}{2b}\right) \cdot \sinh\left(\frac{\pi \cdot \epsilon a}{b}\right) \cdot \sin\left(\epsilon_{\log}\left(\frac{2b}{\pi} \cdot \sin\left(\frac{\pi a}{b}\right)\right)\right) \right)^2
\]

\[
+ \cos\left(2b \sin\left(\frac{\pi a}{b}\right)\right) \cdot \cos\left(\epsilon_{\ln}\left(\frac{2b \sin\left(\frac{\pi a}{b}\right)}{\pi}\right)\right) + \left(\sin\left(2b \sin\left(\frac{\pi a}{b}\right)\right) \cdot \sin\left(\epsilon_{\ln}\left(\frac{2b \sin\left(\frac{\pi a}{b}\right)}{\pi}\right)\right)\right)^2
\]

\[
- \cos\left(2b \sin\left(\frac{\pi a}{b}\right)\right)^2 + \cosh^2\left(\frac{\pi \epsilon a}{b}\right)
\]  

\[(C.9)\]

Similarly, $B$ and $C$ can be simplified.
$$B = \left( \sin \left( \frac{\pi \cdot a}{2 \cdot b} \right) \cosh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \cos \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) + \cos \left( \frac{\pi \cdot a}{2 \cdot b} \right) \sinh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \cos \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) \right) + \left( \sin \left( \frac{\pi \cdot a}{2 \cdot b} \right) \cosh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \sin \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) - \cos \left( \frac{\pi \cdot a}{2 \cdot b} \right) \sinh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \cos \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) \right) + \left( \sin \left( \frac{\pi \cdot a}{2 \cdot b} \right) \cosh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \cos \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) + \cos \left( \frac{\pi \cdot a}{2 \cdot b} \right) \sinh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \sin \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) \right)$$

$$C =$$

$$\left( \sin \left( \frac{\pi \cdot a}{2 \cdot b} \right) \cosh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \sin \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) + \cos \left( \frac{\pi \cdot a}{2 \cdot b} \right) \sinh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \cos \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) \right)$$

$$= 0 \quad \text{(C.10)}$$

$$\left( \sin \left( \frac{\pi \cdot a}{2 \cdot b} \right) \cosh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \sin \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) \right)$$

$$- \cos \left( \frac{\pi \cdot a}{2 \cdot b} \right) \sinh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \cos \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right)$$

$$+ \left( \sin \left( \frac{\pi \cdot a}{2 \cdot b} \right) \cosh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \cos \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) + \cos \left( \frac{\pi \cdot a}{2 \cdot b} \right) \sinh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \sin \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) \right)$$

$$+ \left( \sin \left( \frac{\pi \cdot a}{2 \cdot b} \right) \cosh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \cos \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) + \cos \left( \frac{\pi \cdot a}{2 \cdot b} \right) \sinh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right) \sin \left( \text{epsilon} \cdot \log \left( \frac{2 \cdot b}{\pi} \cdot \sin \left( \frac{\pi \cdot a}{b} \right) \right) \right) \right)$$

$$= -\cos \left( \frac{\pi \cdot a}{2 \cdot b} \right)^2 + \cosh \left( \frac{\pi \cdot \text{epsilon} \cdot a}{b} \right)^2 \quad \text{(C.11)}$$

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Eqs. (C.9) - (C.11) can be summarized as follows.

\[ A = C = \cosh^2 \left( \frac{\pi a \varepsilon}{b} \right) - \cos^2 \left( \frac{\pi a}{2b} \right), \quad B=0 \]  \hspace{1cm} (C.12)

By inserting Eq. (C.12) into Eq.(C.8), we have

\[ K\tilde{K} = K_i^2 + K_{\|i}^2 = \frac{4b}{\sin \left( \frac{\pi a}{b} \right)} \left[ \cosh^2 \left( \frac{\pi a \varepsilon}{b} \right) - \cos^2 \left( \frac{\pi a}{2b} \right) \right] (\sigma^2 + \tau^2) \]  \hspace{1cm} (C.13)

which is identical to Eq.(C.5).
APPENDIX D  

**SERR for non-interacting (independent) interfacial strip cracks**

From the double angle definition for hyperbolic cosine function, we can write

\[
\cosh^2\left(\frac{\pi a e}{b}\right) = \frac{\cosh\left(\frac{2\pi a e}{b}\right) + 1}{2} \tag{D.1}
\]

where \( \cosh \) term can be expressed in the form of Taylor expansion as follows.

\[
\cosh\left(\frac{2\pi a e}{b}\right) = 1 + \frac{1}{2} \left(\frac{2\pi a e}{b}\right)^2 + \frac{1}{24} \left(\frac{2\pi a e}{b}\right)^4 + \ldots + \frac{1}{(2m)!} \left(\frac{2\pi a e}{b}\right)^{2m} + \ldots \tag{D.2}
\]

Eqs. (D.1) and (D.2) lead to

\[
\cosh^2\left(\frac{\pi a e}{b}\right) = 1 + \left(\frac{\pi a e}{b}\right)^2 + \frac{1}{3} \left(\frac{\pi a e}{b}\right)^4 + \frac{1}{24} \left(\frac{\pi a e}{b}\right)^6 + \ldots \tag{D.3}
\]

Similarly, using the double angle definition for the cosine term, we have

\[
\cos^2\left(\frac{\pi a}{2b}\right) = \frac{\cos\left(\frac{\pi a}{b}\right) + 1}{2}, \tag{D.4}
\]

where
\[
\cos \left(\frac{\pi a}{b}\right) = 1 - \frac{1}{2} \left(\frac{\pi a}{b}\right)^2 + \frac{1}{24} \left(\frac{\pi a}{b}\right)^4 - \ldots = 1 - 2 \left(\frac{\pi a}{2b}\right)^2 + \frac{2}{3} \left(\frac{\pi a}{2b}\right)^4 - \ldots \quad \text{(D.5)}
\]

Inserting Eq. (D.5) into Eq. (D.4), we have

\[
\cos^2 \left(\frac{\pi a}{2b}\right) = 1 - \left(\frac{\pi a}{2b}\right)^2 + \frac{1}{3} \left(\frac{\pi a}{2b}\right)^4 - \frac{1}{24} \left(\frac{\pi a}{2b}\right)^6 + \frac{1}{576} \left(\frac{\pi a}{2b}\right)^8 - \ldots \quad \text{(D.6)}
\]

Expressing the sine term in the form of Taylor expansion, we have

\[
\sin \left(\frac{\pi a}{b}\right) = \left(\frac{\pi a}{b}\right) - \frac{1}{6} \left(\frac{\pi a}{b}\right)^3 + \frac{1}{120} \left(\frac{\pi a}{b}\right)^5 - \ldots = 2 \left(\frac{\pi a}{2b}\right) - \frac{8}{6} \left(\frac{\pi a}{2b}\right)^3 + \frac{32}{120} \left(\frac{\pi a}{2b}\right)^5 - \ldots \quad \text{(D.7)}
\]

which leads to

\[
\left(\frac{\pi a}{2b}\right) \sin \left(\frac{\pi a}{b}\right) = 2 \left(\frac{\pi a}{2b}\right)^2 - \frac{8}{6} \left(\frac{\pi a}{2b}\right)^4 + \frac{32}{120} \left(\frac{\pi a}{2b}\right)^6 - \ldots \quad \text{(D.8)}
\]

Combining Equations (D.3), (D.6) and (D.8), we have

\[
\frac{\cosh^2 \left(\frac{\pi a \varepsilon}{b}\right) - \cos^2 \left(\frac{\pi a}{2b}\right)}{\left(\frac{\pi a}{2b}\right) \sin \left(\frac{\pi a}{b}\right)} = \left(\frac{\pi a}{2b}\right)^2 \left(1 + 4\varepsilon^2\right) - \frac{1}{3} \left(\frac{\pi a}{2b}\right)^4 \left(1 - 16\varepsilon^4\right) + \ldots
\]

\[
= 2 \left(\frac{\pi a}{2b}\right)^2 - \frac{4}{3} \left(\frac{\pi a}{2b}\right)^4 + \frac{4}{15} \left(\frac{\pi a}{2b}\right)^6 - \ldots \quad \text{(D.9)}
\]

By letting \(a/b\) approach zero for the RHS of Eq. (D.9), we obtain

\[
\lim_{a/b \to 0} \frac{\cosh^2 \left(\frac{\pi a \varepsilon}{b}\right) - \cos^2 \left(\frac{\pi a}{2b}\right)}{\left(\frac{\pi a}{2b}\right) \sin \left(\frac{\pi a}{b}\right)} = \frac{1 + 4\varepsilon^2}{2}
\]

\[
\text{(D.10)}
\]
Combining Eqs. (D.10) and (C.7), we obtain

\[
W_{\text{non-interacting}}^{\text{dissimilar}} = \frac{1}{8} \frac{(1 + \kappa_i)(1 - \beta^2)}{G_i(1 + \alpha)} \pi a \left( 1 + 4a^2 \right) \left( \sigma^2 + \tau^2 \right). \tag{D.11}
\]
APPENDIX E

Approximate form of interaction function for periodic array of strip cracks

Using Eq. (D.3), the following relation can be obtained.

\[
\cosh^2\left(\frac{\pi a \varepsilon}{b}\right) - \cos^2\left(\frac{\pi a}{2b}\right) = \sin^2\left(\frac{\pi a}{2b}\right) + \left(\frac{\varepsilon \pi a}{b}\right)^2 \left[1 + \frac{1}{3} \left(\frac{\varepsilon \pi a}{b}\right)^2 + \frac{1}{24} \left(\frac{\varepsilon \pi a}{b}\right)^4 + \ldots\right].
\] (E.1)

We recall the following limits for \( \varepsilon \) and \( a/b \).

\[
0 \leq \varepsilon \leq 0.175 \quad \text{(E.2)}
\]

\[
0 \leq a/b \leq 1 \quad \text{(E.3)}
\]

Eqs (E.2) and (E.3) lead to

\[
0 \leq \left|\frac{\varepsilon \pi a}{b}\right| \leq 0.5498. \quad \text{(E.4)}
\]

Therefore, higher order terms inside of the bracket in Eq. (E.1) can be neglected compared to one. This yields

\[
cosh^2\left(\frac{\pi a \varepsilon}{b}\right) - \cos^2\left(\frac{\pi a}{2b}\right) \approx \sin^2\left(\frac{\pi a}{2b}\right) + \left(\frac{\varepsilon \pi a}{b}\right)^2 = \sin^2\left(\frac{\pi a}{2b}\right) + 4\varepsilon^2 \left(\frac{\pi a}{2b}\right)^2. \quad \text{(E.5)}
\]

Inserting Eq. (E.5) into interaction function expression for the SERR, we have
\[ I_w \left( \frac{a}{b}, \epsilon \right) = \frac{\cosh^2 \left( \frac{\pi a \epsilon}{b} \right) - \cos^2 \left( \frac{\pi a}{2b} \right)}{\left( \frac{\pi a}{2b} \right) \left( \frac{\pi a}{b} \right)} \approx \frac{\sin^2 \left( \frac{\pi a}{2b} \right) + 4\epsilon^2 \left( \frac{\pi a}{2b} \right)^2}{\left( \frac{\pi a}{2b} \right) \left( \frac{\pi a}{b} \right) (1 + 4\epsilon^2)} \]  

$$(E.6)$$

In this equation, term in the bracket is bounded as follows.

$$1 \leq \left( \frac{\pi a}{2b} \right)^2 / \sin^2 \left( \frac{\pi a}{2b} \right) \leq \left( \frac{\pi}{2} \right)^2$$  

$$(E.7)$$

(Proof: Let’s introduce $x = \frac{\pi a}{2b}$. Therefore $\left( \frac{\pi a}{2b} \right)^2 / \sin^2 \left( \frac{\pi a}{2b} \right) = \frac{x^2}{\sin^2 x}$. Using L’Hospital’s Rule, we obtain

\[ \lim_{x \to 0} \frac{x^2}{\sin^2 x} = \lim_{x \to 0} \frac{x}{\sin x} \cos x = \lim_{x \to 0} \frac{1}{(\cos^2 x - \sin^2 x)} = 1 \text{ and } \lim_{x \to \frac{\pi}{2}} \frac{x^2}{\sin^2 x} = \left( \frac{\pi}{2} \right)^2 \]

For most practical cases, the value of $\epsilon$ remains small as follows.

$$|\epsilon| \leq 0.05$$  

$$(E.8)$$

Combining Eqs. (E.7) and (E.8), $\epsilon^2$ term in Eq. (E.6) can be neglected.
\[ I_w \left( \frac{a}{b}, \varepsilon \right) \approx I_w^{\text{approximate}} \left( \frac{a}{b} \right) = \frac{2 \sin^2 \left( \frac{\pi a}{2b} \right)}{\left( \frac{\pi a}{2b} \right) \sin \left( \frac{\pi a}{b} \right)} = \frac{2 \sin^2 \left( \frac{\pi a}{2b} \right)}{2 \left( \frac{\pi a}{2b} \right) \sin \left( \frac{\pi a}{2b} \right) \cos \left( \frac{\pi a}{2b} \right)} \]  

(E.9)
APPENDIX F Derivation of $k_{N,dissimilar}^{array}$ in section 1.3.1

Displacement is given by Castigliano’s Second theorem for cracked bodies as follows.

$$\Delta = \frac{\partial}{\partial Q} \left( 2 \int_0^a W dx \right)$$  \hspace{1cm} (F.1)

(Tada (2000), Appendix B and Baik and Thompson (1984), Eq. (28))

Using chain rule, we have

$$\frac{\partial}{\partial Q} \sigma = \frac{\partial}{\partial Q} \cdot \frac{\partial}{\partial \sigma}$$ \hspace{1cm} (F.2)

Recalling the definition of $Q_N$ from the main text, $Q_N = \sigma \cdot 2b$, we can rewrite Eq. (F.2) as follows.

$$\frac{\partial}{\partial Q} = \frac{1}{2b} \frac{\partial}{\partial \sigma}$$ \hspace{1cm} (F.3)

By inserting $W_{dissimilar}^{array}$ from Eq. (C.7) into Eq.(F.1), we have
\[
\Delta = \frac{1}{2b} \frac{\partial}{\partial \sigma} \left[ \frac{1}{4} \frac{(1+\kappa_i)(1-\beta^2)}{G_i(1+\alpha)} (\sigma^2 + \tau^2) \int_{-a}^{a} \left\{ \frac{\cosh\left(\frac{x}{b}\right)}{\sin\left(\frac{x}{b}\right)} \right\} dx \right]
\]
\[
= \frac{1}{2b} \frac{\partial}{\partial \sigma} \left[ \frac{(1+\kappa_i)(1-\beta^2)}{G_i(1+\alpha)} (\sigma^2 + \tau^2) b \int_{0}^{a} \left\{ \frac{\cosh\left(\frac{x}{b}\right)}{\sin\left(\frac{x}{b}\right)} \right\} dx \right]
\]
\[(F.4)\]

Introducing \( \bar{x} \) defined by \( \bar{x} = x/b \), \( dx = bd\bar{x} \), we have

\[
\Delta_{\bar{x}} = \frac{1}{2b} \frac{\partial}{\partial \sigma} \left[ \frac{(1+\kappa_i)(1-\beta^2)}{G_i(1+\alpha)} (\sigma^2 + \tau^2) b^2 L \right]
\]
\[(F.5)\]

where

\[
L = \int_{0}^{a/b} \left\{ \frac{\cosh\left(\frac{x}{b}\right)}{\sin\left(\frac{x}{b}\right)} \right\} d\bar{x}.
\]
\[(F.6)\]

This leads to

\[
\Delta_{\bar{x}} = b \frac{(1+\kappa_i)(1-\beta^2)}{G_i(1+\alpha)} \sigma L
\]
\[(F.7)\]

By inserting Eq. (F.7) into spring stiffness expression, we obtain

\[
k^{array}_{N,dissimilar} = \frac{G_i}{b(1+\kappa_i)(1-\beta^2)L}.
\]
\[(F.8)\]
Approximate value of integral $L$

We want to show the validity of following approximation.

$$L = \int_0^\frac{a}{b} \frac{1}{\sin(r \pi x)} \left[ \cosh^2(r \pi x) - \cos^2\left(\frac{r \pi x}{2}\right) \right] dx \approx \frac{1 + 4 \epsilon^2}{\pi} \ln \left[ \sec \left(\frac{r \pi a}{2b}\right) \right] \quad (F.9)$$

Recalling the Eqs (D.3) and (D.6), we can write

$$\cosh^2\left(\frac{r \pi x}{2}\right) - \cos^2\left(\frac{r \pi x}{2}\right) = \left(\frac{r \pi x}{2}\right)^2 \left(1 + 4 \epsilon^2\right) - \frac{1}{3} \left(\frac{r \pi x}{2}\right)^4 \left(1 - 16 \epsilon^4\right) + \frac{1}{24} \left(\frac{r \pi x}{2}\right)^6 \left(1 + 64 \epsilon^6\right) - \ldots \quad (F.10)$$

By factoring $(1+4\epsilon^2)$ out from the RHS of Eq.(F.10), we have

$$\cosh^2\left(\frac{r \pi x}{2}\right) - \cos^2\left(\frac{r \pi x}{2}\right) = \left(1 + 4 \epsilon^2\right) \left\{ \left(\frac{r \pi x}{2}\right)^2 - \frac{1}{3} \left(\frac{r \pi x}{2}\right)^4 \left(1 - 4 \epsilon^2\right) + \frac{1}{24} \left(\frac{r \pi x}{2}\right)^6 \left(1 + 64 \epsilon^6\right) - \ldots \right\} \quad (F.11)$$

Recalling that $\epsilon$ is small ($|\epsilon|<0.175$) and even smaller for most practical applications ($|\epsilon|<0.05$), we can ignore $\epsilon$ terms inside of the braces $\{\}$. Therefore,

$$\cosh^2\left(\frac{r \pi x}{2}\right) - \cos^2\left(\frac{r \pi x}{2}\right) \approx \left(1 + 4 \epsilon^2\right) \left\{ \left(\frac{r \pi x}{2}\right)^2 - \frac{1}{3} \left(\frac{r \pi x}{2}\right)^4 \right\} + \frac{1}{24} \left(\frac{r \pi x}{2}\right)^6 \left(1 + 64 \epsilon^6\right) - \ldots \quad (F.12)$$

The term inside of the braces $\{\}$ now equals to the Taylor expansion of $\sin^2(\pi \frac{r \pi x}{2})$. This leads to the following approximation.

$$\cosh^2\left(\frac{r \pi x}{2}\right) - \cos^2\left(\frac{r \pi x}{2}\right) \approx \left(1 + 4 \epsilon^2\right) \sin^2\left(\frac{r \pi x}{2}\right) \quad (F.13)$$

Inserting Eq. (F.13) into the definition of $L$ in Eq.(F.9), we have
\[ L = \int_{0}^{a} \frac{1}{\sin(\pi x)} \left[ \cosh^{2}(\pi x \varepsilon) - \cos^{2}\left(\frac{\pi x}{2}\right) \right] dx \]

\[ \approx \int_{0}^{a} \frac{1}{\sin(\pi x)} \left[ (1 + 4\varepsilon^2) \sin^{2}\left(\frac{\pi x}{2}\right) \right] dx \]

\[ = (1 + 4\varepsilon^2) \int_{0}^{a} \frac{1}{2 \sin\left(\frac{\pi x}{2}\right) \cos\left(\frac{\pi x}{2}\right)} \left[ \sin^{2}\left(\frac{\pi x}{2}\right) \right] dx \]

\[ L = \frac{1 + 4\varepsilon^2}{2} \int_{0}^{a} tan\left(\frac{\pi x}{2}\right) dx, \quad (F.14) \]

which yields

\[ L \approx \frac{1 + 4\varepsilon^2}{\pi} \ln\left[ \sec\left(\frac{\pi a}{2b}\right) \right]. \quad (F.15) \]
APPENDIX G Derivation of $k_{N,\text{dissimilar}}^{\text{non-interacting}}$ in section 1.3.1

Following the procedure outlined in Appendix F, and inserting $W_{\text{dissimilar}}^{\text{non-interacting}}$ from Eq. (D.11) into Eq. (F.1), we have

$\Delta_n = \frac{1}{2b} \frac{\partial}{\partial \sigma} \left( \frac{1}{8} \frac{(1+\kappa_i)(1-\beta^2)}{G_i(1+\alpha)} \pi \left(1+4\epsilon^2\right) \left(\sigma^2 + \tau^2\right) a_{22} \int_0^a x dx \right)$, \hspace{1cm} (G.1)

which leads to

$\Delta_n = \frac{1}{2b} \frac{\partial}{\partial \sigma} \left( \frac{1}{8} \frac{(1+\kappa_i)(1-\beta^2)}{G_i(1+\alpha)} \pi \left(1+4\epsilon^2\right) \left(\sigma^2 + \tau^2\right) a^2 \right)$. \hspace{1cm} (G.2)

By taking derivative, we have

$\Delta_n = \frac{1}{2b} \left( \frac{1}{4} \frac{(1+\kappa_i)(1-\beta^2)}{G_i(1+\alpha)} \pi \left(1+4\epsilon^2\right) \sigma a^2 \right)$ \hspace{1cm} (G.3)

By inserting Eq. (G.3) into spring stiffness equation, we obtain

$k_{N,\text{dissimilar}}^{\text{non-interacting}} = \frac{8}{\pi} \frac{b \frac{G_i}{(1+\kappa_i)(1-\beta^2)(1+4\epsilon^2)}}{a^2}$. \hspace{1cm} (G.4)

Now, let’s show that the ratio of Eqs. (F.8) and (G.4) equals one when $a/b$ approaches zero.
Using the following Maple® work, it is shown that $I_k$ approaches one when $a/b$ goes to zero.

\[
I_k \left( \frac{a}{b}, \varepsilon \right) = \frac{k_{\text{array}}}{k_{\text{non-interacting}}} = \pi \frac{a^2}{b^2} \frac{1 + 4 \varepsilon^2}{L}
\]

Comparison of Eq. (G.4) with Eq. (11) in Pecorari and Kelly (2000)

The spring stiffness in normal and transverse directions is given by Eq. (11) in Pecorari and Kelly (2000) as follows.

\[
k_N^{\text{non-int}} = k_T^{\text{non-int}} = \Gamma \sqrt{1 - \beta^2} \frac{1}{v a^2 I_c}
\]

where $\nu$ represents the number of cracks in a unit length, and
\[ \Gamma = \frac{2G_1(1+\alpha)}{(1+\kappa_i)(1-\beta^2)}, \]  
\text{(G.8)}

(Eq. (12) in Pecorari and Kelly (2000). Notice the difference in \(\alpha\) definition in Eq. (13).)

\[ I_c = \int_{-1}^{1} \sqrt{1-t^2} \cos \left( \varepsilon \ln \left| \frac{t-1}{t+1} \right| \right) dt. \]  
\text{(G.9)}

(Eq. (14) in Pecorari and Kelly (2000)). In Eq. (G.7), ‘a’ is mentioned as full crack length in Pecorari and Kelly (2000). By inserting Eqs. (G.8) and (G.9) into Eq. (G.7), we have

\[ k_{N, non-int} = \frac{2G_1(1+\alpha)}{(1+\kappa_i)(1-\beta^2)} \left\{ \int_{-1}^{1} \sqrt{1-t^2} \cos \left( \varepsilon \ln \left| \frac{t-1}{t+1} \right| \right) dt \right\}^{-1} \frac{1}{a^2\nu}. \]  
\text{(G.10)}

Eq. (G.4) reduces to spring stiffness for homogenous case when parameters \(\alpha, \beta, \varepsilon\) are equal to zero.

\[ k_{N, homogenous} = \frac{2G_1}{(1+\kappa_i)\pi/2} \frac{1}{a^2\nu} = \frac{4G_1}{\pi(1+\kappa_i)\pi/2} \frac{1}{a^2\nu}. \]  
\text{(G.11)}

(Note that \(I_c=\pi/2\) for the homogenous case, see last paragraph on page 2456 (Pecorari and Kelly (2000)) By inserting \(G=E/2(1+\nu)\) and \(k=3-4\nu\), we have

\[ k_{N, homogenous} = \frac{4E/2(1+\nu)}{\pi(4-4\nu)} \frac{1}{a^2\nu} = \frac{E}{2\pi(1-\nu^2)} \frac{1}{a^2\nu}. \]  
\text{(G.12)}

While ‘a’ in Eq. (G.7) is supposed to be full crack length according to Pecorari and Kelly (2000), it should be half crack length. To prove this point, we will show that Eq. (G.12)
matches with Eq. (2b) in Margetan et. al (1988) only if ‘a’ is half crack length. Eq. (2b) in Margetan et. al (1988) is given as follows.

\[ k_{\text{non-int}}^{\text{N, homogenous}} = \frac{2E}{\pi(1-\nu^2)} s^{-1} A^{-2} \]  

(G.13)

Here, ‘s’ is the unit cell length and ‘A’ is the ratio of ‘cracked length’ to the ‘unit cell length’ which is equal to ‘2a/s’. Therefore, s=1/ν and A=2aν. By inserting these relations into Eq. (A9.12), we obtain,

\[ k_{\text{non-int}}^{\text{N, homogenous}} = \frac{2E}{\pi(1-\nu^2)} \nu \left(\frac{1}{2a\nu}\right)^2 = \frac{E}{2\pi(1-\nu^2)} \frac{1}{a^2\nu}, \]  

(G.14)

which is identical to Eq. (G.11). This shows that ‘a’ in the Eq. (G.10) and hence Eq. (G.7) should be the half crack length. Pecorari and Kelly (2000) list the following equation as the equation corresponding to Eq.(G.12) in Eq. (10).

\[ k_{\text{N, homogenous}} = \frac{2E}{\pi(1-\nu^2)} \frac{1}{a^2\nu} \]  

(G.15)

It does not match with Eq.(G.14). If ‘a’ in Eq. (G.12) represents half crack length and ‘a’ in Eq. (G.15) represents full crack length, then both equations are identical. Based on this, it appears that “a” in Eq. (G.7) represents half crack length instead of full crack length as they claim in their paper.

Now, we want to compare Eq. (G.4) in the text with (G.10). \( \nu = l/2b \) in our study, and if we rewrite Eq.(G.10), we have
$$k_{N\text{ non-int}} = \frac{4G_i(1+\alpha)}{(1+\kappa_1)\sqrt{1-\beta^2}} \left[ \int_{-1}^{1} \sqrt{1-t^2} \cos \left( \varepsilon \ln \frac{|t-1|}{t+1} \right) dt \right]^{-1} \frac{b}{a^2}. \tag{G.16}$$

The solution of the integral term in Eq. (G.16) can be found as follows. Using Euler equation, we can write

$$a^{bi} = \cos(\ln(a^b)) + i \sin(\ln(a^b))$$
$$a^{-bi} = \cos(\ln(a^b)) - i \sin(\ln(a^b)) \tag{G.17}$$

which leads to

$$\cos(\ln(a^b)) = \frac{a^{bi} + a^{-bi}}{2} \tag{G.18}$$

Using relation (G.18), we can write

$$\cos \left( \ln \left| \frac{t-1}{t+1} \right| \right) = \frac{1}{2} \left( \left| \frac{t-1}{t+1} \right| + \left| \frac{t-1}{t+1} \right|^{-i\varepsilon} \right) \tag{G.19}$$

Then the integral term in Eq. (G.16) can be written as

$$\int_{-1}^{1} \sqrt{1-t^2} \cos \left( \ln \left| \frac{t-1}{t+1} \right| \right) dt = \frac{1}{2} \int_{-1}^{1} \sqrt{1-t^2} \left( \left| \frac{t-1}{t+1} \right| + \left| \frac{t-1}{t+1} \right|^{-i\varepsilon} \right) dt \tag{G.20}$$

The integral on the RHS of Eq. (G.20) is given as follows.

$$\int_{0}^{1} \sqrt{1-t^2} \left( \left| \frac{t-1}{t+1} \right| + \left| \frac{t-1}{t+1} \right|^{-i\varepsilon} \right) dt = 4 \frac{\Gamma \left( \frac{3}{2} + i\varepsilon \right) \Gamma \left( \frac{3}{2} - i\varepsilon \right)}{\Gamma(3)} \tag{G.21}$$
(Eq. (4.8) in Mossakovskii and Rybka (1964) with \( a=1, \gamma=i \)). Using the relations about \( \Gamma \)-functions, we can write

\[
\Gamma(3) = 2,
\]

\[
\Gamma\left(\frac{3}{2} + i\epsilon\right) = \left(\frac{1}{2} + i\epsilon\right) \Gamma\left(\frac{1}{2} + i\epsilon\right) \quad \text{and}
\]

\[
\Gamma\left(\frac{3}{2} - i\epsilon\right) = \left(\frac{1}{2} - i\epsilon\right) \Gamma\left(\frac{1}{2} - i\epsilon\right)
\]

(Eq. (6.1.6) and Eq. (6.1.15) in Abbaramovitz and Stegun (1970), notice that \( z = 1/2 \pm i\epsilon \)).

Using relations defined in Eq.(G.22), we can rewrite Eq. (G.21) as follows

\[
4 \Gamma\left(\frac{3}{2} + i\epsilon\right) \Gamma\left(\frac{3}{2} - i\epsilon\right) = 4 \left(\frac{1}{2} + i\epsilon\right) \Gamma\left(\frac{1}{2} + i\epsilon\right) \Gamma\left(\frac{1}{2} - i\epsilon\right)
\]

\[
= 2 \left(\frac{1}{4} + \epsilon^2\right) \Gamma\left(\frac{1}{2} + i\epsilon\right) \Gamma\left(\frac{1}{2} - i\epsilon\right)
\]

(G.23)

From Eq. (6.1.30) in Abbaramovitz and Stegun (1970), we can write

\[
\Gamma\left(\frac{1}{2} + i\epsilon\right) \Gamma\left(\frac{1}{2} - i\epsilon\right) = \frac{\pi}{\cosh(\pi\epsilon)}
\]

(G.24)

By inserting Eq. (G.24) into Eq. (G.23) and then in Eq.(G.16), we obtain
\[ k_{N_{\text{non-int}}} = \frac{4G_i (1+\alpha)}{(1+\kappa_i)\sqrt{1-\beta^2}} \left\{ \frac{(1+4\epsilon^2)}{2} \frac{\pi}{\cosh(\pi\epsilon)} \right\}^{-1} \frac{b}{a^2} \]  

(G.25)

which yields

\[ k_{N_{\text{non-int}}} = \frac{8}{\pi} \frac{b}{a^2} \frac{G_i}{(1+\kappa_i)(1+4\epsilon^2)(1-\beta^2)} \]  

(G.26)

This equation is identical to Eq. (28) in the manuscript.
APPENDIX H  
Comparison of material function $M_k$ for 2-D strip cracks with a similar work done by Gorbatikh and Popova (2005)

Compliance of a non-interacting crack located at the dissimilar material interface is given by Gorbatikh and Popova (2005) as follows.

$$H_{2222} = \frac{B}{\cosh^2(\pi \lambda) 2(1 + \kappa_h)} (H_{2222})_h$$  \hspace{1cm} (H.1)

(Equation (9) in Gorbatikh and Popova (2005)) where $(H_{2222})_h$ represents the compliance of a crack in a homogenous medium along normal direction. $B$ is given as

$$B = \frac{1 + \kappa_1}{\mu_i} + \frac{1 + \kappa_2}{\mu_2},$$  \hspace{1cm} (H.2)

(Eq. (8) in Gorbatikh and Popova (2005)) and

$$\lambda = \epsilon, \; \mu_h = G_i.$$  \hspace{1cm} (H.3)

Since stiffness is inversely proportional to compliance, we can define material function for spring stiffness, $M_k$, by using the ratio of the compliances.

$$M_k = \frac{(H_{2222})_h}{H_{2222}} = \frac{\cosh^2(\pi \epsilon) 2(1 + \kappa_i)}{B \frac{\mu_h}{G_i}}$$  \hspace{1cm} (H.4)

One can derive the following equation for $B$. 

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Recalling Eqs. (B.4) and (B.7), parameter $B$ in Eq. (H.5) can be rewritten as

$$B = \frac{16}{E^*} = \frac{2(1 + \kappa_1)}{(1 + \alpha)G_1} \enspace .$$

(H.6)

Using $\cosh^2(\pi \epsilon) = \frac{1}{(1 - \beta^2)}$ (Hutchinson and Suo (1992), Page 75), we have the following relation for the material parameter $M_k$

$$M_k = \frac{(H_{2222})_b}{H_{2222}} = \frac{1}{(1 - \beta^2)} \frac{1}{(1 - \alpha)} \frac{2(1 + \kappa_1)}{G_1} = \frac{(1 + \alpha)}{(1 - \beta^2)} \enspace .$$

(H.7)

Relationship between compliance and spring stiffness

The compliance of an interfacial crack is given by Gorbatikh (2004) as follows.

$$H_{ijkl} = \frac{\partial}{\partial \sigma_{kl}} \left( \frac{1}{V} \int_L \left( \frac{\partial W}{\partial \sigma_{ij}} \right) dl \right) dL \right)$$

(Eq. (4) in Gorbatikh (2004)). $W$ is given by Eq. (12) in Gorbatikh (2004) as follows.

$$W = \frac{B}{16 \cosh^2(\pi \epsilon)} K \bar{K} \enspace .$$

(H.9)
$K$ is given by Eq. (18) in Gorbatikh (2004) as

$$K = K_i + i K_{II} = (\sigma + i \tau)(1 + 2i\varepsilon)(2a)^{-i\varepsilon}\sqrt{\pi a},$$  \hspace{1cm} (H.10)

which gives

$$K\overline{K} = (\sigma^2 + \tau^2)(1 + 4\varepsilon^2)\pi a.$$  \hspace{1cm} (H.11)

By inserting Eq. (H.11) into Eqs. (H.9) and (H.8), and taking the required integration, we have

$$H_{2222} = H_N = \frac{\partial}{\partial \sigma} \left( \frac{1}{V} \int_L \left( \frac{\partial W}{\partial \sigma} d\ell \right) dL \right)$$

$$= \frac{\partial}{\partial \sigma} \left( \frac{1}{V} \int_0^\alpha \left( \frac{\partial}{\partial \sigma} \left( \frac{B}{16 \cosh^2(\pi\varepsilon)}(\sigma^2 + \tau^2)(1 + 4\varepsilon^2)\pi L \right) \right) dL \right)$$  \hspace{1cm} (H.12)

$$= \frac{B}{V} \frac{(1 + 4\varepsilon^2)\pi}{16 \cosh^2(\pi\varepsilon)} \left[ \frac{\partial}{\partial \sigma} \left( 2 \int_0^\alpha \frac{\partial}{\partial \sigma} (\sigma^2 + \tau^2) \right) dL \right] = \frac{B}{V} \frac{(1 + 4\varepsilon^2)\pi a^2}{8 \cosh^2(\pi\varepsilon)}$$

Inserting Eq. (H.6) into Eq. (H.12), we obtain

$$H_{2222} = H_N = \frac{\partial}{\partial \sigma} \left( \frac{1}{V} \int_L \left( \frac{\partial W}{\partial \sigma} d\ell \right) dL \right) = \frac{1}{V} \frac{(1 + \kappa_1)(1 + 4\varepsilon^2)(1 - \beta^2)}{4G_1 \pi a^2}.$$  \hspace{1cm} (H.13)

Recalling the spring stiffness,
and defining the normalized crack area \( A=2b=V/l \), we have

\[
\kappa_{N,\text{dissimilar}}^{\text{non-interacting}} = \frac{8}{\pi} \frac{b}{a} \frac{G_i}{(1 + \kappa_i)} \frac{(1 + \alpha)}{(1 + 4\varepsilon^2)(1 - \beta^2)},
\]

(H.14)

Therefore, the relationship between Eqs. (H.13) and (H.15) can be written as

\[
\kappa_{N,\text{dissimilar}}^{\text{non-interacting}} = \frac{V}{l} \frac{4G_i}{(1 + \kappa_i)} \frac{(1 + \alpha)}{(1 + 4\varepsilon^2)(1 - \beta^2)} \frac{1}{\pi a^2}.
\]

(H.15)

Therefore, the relationship between Eqs. (H.13) and (H.15) can be written as

\[
k_{N,\text{dissimilar}}^{\text{non-interacting}} = \lim_{V \to \infty} \frac{l}{H_{2222}}.
\]

(H.16)
APPENDIX I Interaction function for 2-D periodic array of strip cracks in homogenous materials obtained from Baik and Thompson (1984)

Spring stiffness for cracks in a homogenous medium is given by

\[ \kappa = \frac{E'}{s} \kappa^* (w/s). \]  
(I.1)

(Eq. (21) in Baik and Thompson (1984)). For a periodic array of cracks in a homogenous medium, \( \kappa^* \) in Eq. (I.1) is given by

\[
\kappa^*_{\text{array homogenous}} (w/s) = \left\{ \frac{4}{\pi} \ln \left[ \sec \left( \frac{\pi (1-w/s)}{2} \right) \right] \right\}^{-1}.
\]  
(I.2)

(Eq. (22) in Baik and Thompson (1984)). For a non-interacting crack in a homogenous medium, \( \kappa^* \) in Eq. (I.1) is given by

\[
\kappa^*_{\text{non-int homogenous}} (w/s) = \frac{2}{\pi [(1-w/s)]^2}.
\]  
(I.3)

(Eq. (26) in Baik and Thompson (1984)). Then, \( I_{k, \text{homogenous}} \) can be introduced as

\[
I_{k, \text{homogenous}} (w/s) = \frac{\kappa^*_{\text{array homogenous}} (w/s)}{\kappa^*_{\text{non-int homogenous}} (w/s)}.
\]  
(I.4)
Combining Eqs. (I.2), (I.3) and (I.4), we have

\[ I_{k, \text{homogenous}} (w/s) = \frac{4}{\pi} \ln \left[ \sec \frac{\pi (1-w/s)}{2} \right]^{-1} \]

\[ = \frac{\pi^2}{8} \left[ (1-w/s) \right]^2 \left\{ \ln \left[ \sec \frac{\pi (1-w/s)}{2} \right] \right\}^{-1} \]

(I.5)

Using our notations, \( w=2b-2a \) and \( s=2b \) (see Figure (6) in Baik and Thompson (1984))

Eq. (I.5) can be rewritten as

\[ I_{k, \text{homogenous}} (w/s) = \frac{\pi^2}{8} \left[ (1-(2b-2a)/2b) \right]^2 \left\{ \ln \left[ \sec \frac{\pi \left(1-(2b-2a)/2b\right)}{2} \right] \right\}^{-1} \]

(I.6)

Eq. (I.6) leads to

\[ I_{k, \text{homogenous}} (w/s) = \frac{\pi^2}{8} \frac{a^2}{b^2} \left\{ \ln \left[ \sec \frac{\pi a}{2b} \right] \right\}^{-1} \]

(I.7)
APPENDIX J  Derivation of interpenetration zone for a periodic array of strip cracks located at dissimilar material interface and required tensile stress for negligible interpenetration zone

The interpenetration zone size, \( r_i \) for interface cracks relative to a reference length, \( l \) is given by Graciani et. al (2007) as follows.

\[
 r_i = l \exp \left[ \frac{(2n - 1/2)\pi - \text{Arg}(K) \text{sgn} \varepsilon + \tan^{-1}\left(2|\varepsilon|\right)}{|\varepsilon|} \right]
\]

(Eq. (11) in Graciani et al. (2007)). Now, let’s focus on \( \text{Arg}(K) \) term in Eq. (J.1). The complex stress intensity factor was derived as follows.

\[
 K' = K_I + iK_{II} = 2 \sqrt{\frac{b}{\sin\left(\frac{\pi a}{b}\right)}} \left\{ \sin\left(\frac{\pi a}{2b}\right) \cosh\left(\frac{\pi a \varepsilon}{b}\right) + i \cos\left(\frac{\pi a}{2b}\right) \sinh\left(\frac{\pi a \varepsilon}{b}\right) \right\} \\
 \times \left\{ \frac{2b}{\pi} \sin\left(\frac{\pi a}{b}\right) \right\}^{-i\varepsilon} (\sigma + i\tau)
\]

(See Appendix A for derivation). However, we want to use stress intensity factor defined with a reference length (Eq. (35) in Rice (1988)) because that definition was used in Graciani’s (2007) and we want to use Graciani’s interpenetration zone size equation given in Eq. (J.1) because it covers all \( \varepsilon \) values. Using Eq. (35) in Rice (1988) and considering reference length of \( \tilde{r} = l \), one can write
\[ K = K' e^{i \varepsilon}. \]  

(J.3)

By inserting Eq. (J.2) into Eq. (J.3), we obtain the stress intensity factor defined in the manner Graciani (2007) used as follows.

\[
K = \frac{2 \sqrt{b}}{\sin \left( \frac{\pi a}{b} \right)} \sqrt{\sin \left( \frac{\pi a}{2b} \right) \cosh \left( \frac{\pi a \varepsilon}{b} \right) + \cos \left( \frac{\pi a}{2b} \right) \sinh \left( \frac{\pi a \varepsilon}{b} \right)}
\times \left\{ \frac{2b}{\pi l \sin \left( \frac{\pi a}{b} \right)} \right\}^{-i \varepsilon} (\sigma + i \tau)
\]

(J.4)

Let’s reorganize ‘\( \text{arg}(K) \)’ term as follows.

\[
\text{Arg}(K) = \text{Arg} \left[ \sin \left( \frac{\pi a}{2b} \right) \cosh \left( \frac{\pi a \varepsilon}{b} \right) + i \cos \left( \frac{\pi a}{2b} \right) \sinh \left( \frac{\pi a \varepsilon}{b} \right) \right] + \text{Arg} \left( \frac{2b}{\pi l \sin \left( \frac{\pi a}{b} \right)} \right)^{-i \varepsilon} + \text{Arg} (\sigma + i \tau) \quad (J.5)
\]

Recalling Euler formula, we have

\[
\left\{ \frac{2b}{\pi l \sin \left( \frac{\pi a}{b} \right)} \right\}^{-i \varepsilon} = \exp \left( -i \varepsilon \ln \left\{ \frac{2b}{\pi l \sin \left( \frac{\pi a}{b} \right)} \right\} \right)
= \cos \left( \varepsilon \ln \left\{ \frac{2b}{\pi l \sin \left( \frac{\pi a}{b} \right)} \right\} \right) - i \sin \left( \varepsilon \ln \left\{ \frac{2b}{\pi l \sin \left( \frac{\pi a}{b} \right)} \right\} \right).
\]  

(J.6)

By inserting Eq. (J.6) into Eq. (J.5), we have
\[
\text{Arg}(K) = \tan^{-1} \left( \frac{\cos \left( \frac{\pi a}{2b} \right) \sinh \left( \frac{\pi a \epsilon}{b} \right)}{\sin \left( \frac{\pi a}{2b} \right) \cosh \left( \frac{\pi a \epsilon}{b} \right)} \right) - \epsilon \ln \left( \frac{2b}{\pi l} \sin \left( \frac{\pi a}{b} \right) \right) + \tan^{-1} \left( \frac{\tau}{\sigma} \right).
\] \quad (J.7)

Inserting Eq. (J.7) into Eq. (J.1), we obtain
\[
\begin{align*}
\begin{bmatrix}
\left(2n - \frac{1}{2}\right) \pi - \tan^{-1} \left( \frac{\tanh(\pi a \epsilon/b)}{\tan(\pi a/2b)} \right) \text{sgn } \epsilon - \cot^{-1} \left( \frac{\sigma}{\tau} \right) \text{sgn } \epsilon \\
+ \epsilon \text{sgn } \epsilon \ln \left( \frac{2b}{\pi l} \sin \left( \frac{\pi a}{b} \right) \right) + \tan^{-1} \left( 2|\epsilon| \right)
\end{bmatrix}
\end{align*}
\] \quad (J.8)

Taking the logarithmic term out of the exponential, one can write
\[
\begin{align*}
\begin{bmatrix}
\left(2n - \frac{1}{2}\right) \pi - \tan^{-1} \left( \frac{\tanh(\pi a \epsilon/b)}{\tan(\pi a/2b)} \right) \text{sgn } \epsilon \\
- \cot^{-1} \left( \frac{\sigma}{\tau} \right) \text{sgn } \epsilon + \tan^{-1} \left( 2|\epsilon| \right)
\end{bmatrix}
\end{align*}
\] \quad (J.9)

Normalizing with crack length and organizing Eq. (J.9), we have
\[
\begin{align*}
\begin{bmatrix}
\sin \left( \frac{\pi a}{b} \right) \\
\pi \frac{a}{b} \sin \left( \frac{\pi a}{b} \right)
\end{bmatrix}
\end{align*}
\] \quad (J.10)

This expression provides the interpenetration zone size at the right crack tip relative to full crack length. In the following figure, interpenetration zone size is plotted as a function of \( \epsilon \) for three different \( \sigma/\tau \) values (0.1, 0.5 and 1) with a/b=0.75.
Figure J.1 Right interpenetration zone size normalized with full crack length as a function of $\varepsilon$ for three different $\sigma/\tau$ values ($\sigma/\tau=0.1$, 0.5 and 1) with $a/b=0.75$

As can be seen from the figure, for positive $\varepsilon$ values, interpenetration zone size is almost zero for any $\sigma/\tau$ value. This is because interpenetration zone occurs at the left crack tip for positive $\varepsilon$ values rather than the right crack tip. The only way to have large contact
zone at the right crack tip for positive $\varepsilon$ values is to change the direction of shear stress.

However, we want to avoid changing shear stress direction. Let’s keep $\sigma/\tau$ value being always positive which means direction of shear is fixed to the direction of positive $x$-axis with no loss of generality. When direction of shear is fixed to be always positive, the interpenetration zone occurs at right crack tip for $\varepsilon<0$ ($\beta<0$) while interpenetration zone occurs at the left crack tip for $\varepsilon>0$ ($\beta>0$). However, the magnitude of the interpenetration zone size will not change. Therefore, analyzing only positive $\varepsilon$ will be enough by keeping in mind that interpenetration zone size will occur at different tips when materials switched. Now, we want to limit $r/2a$ to 0.01, which yields

$$\left[ \sin \left( \frac{\pi a}{b} \right) \right] \exp \left[ -\frac{1}{|\varepsilon|} \left( 2n - \frac{1}{2} \right) \pi - \tan^{-1} \left( \frac{\tanh \left( \frac{\pi a \varepsilon}{b} \right)}{\tan \left( \frac{\pi a}{2b} \right)} \right) \text{sgn} \varepsilon \right] \leq 0.01 \quad (J.11)$$

By organizing Eq. (J.11) further, we have

$$\left\{ n - \frac{1}{2} \pi - \tan^{-1} \left( \frac{\tanh \left( \frac{\pi a \varepsilon}{b} \right)}{\tan \left( \frac{\pi a}{2b} \right)} \right) \text{sgn} \varepsilon \right\} \leq |\varepsilon| \ln \left[ 0.01 \left( \frac{\pi a}{b} \right) \right] \quad \left( \frac{\pi a}{b} \right) \quad (J.12)$$

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For the largest root, \( n=0 \), which yields

\[
-cot^{-1}\left(\frac{\sigma}{\tau}\right) \leq |\varepsilon| \ln \left[ \frac{0.01\left(\frac{\pi a}{b}\right)}{\sin\left(\frac{\pi a}{b}\right)} \right] + \frac{\pi}{2} + tan^{-1}\left(\frac{\tanh(\pi a e/b)}{\tan(\pi a/2b)}\right) sgn \varepsilon - tan^{-1}\left(2|\varepsilon|\right) \quad (J.13)
\]

or

\[
cot^{-1}\left(\frac{\sigma}{\tau}\right) \geq -\frac{\pi}{2} - tan^{-1}\left(\frac{\tanh(\pi a e/b)}{\tan(\pi a/2b)}\right) sgn \varepsilon + tan^{-1}\left(2|\varepsilon|\right) - |\varepsilon| \ln \left[ \frac{0.01\left(\frac{\pi a}{b}\right)}{\sin\left(\frac{\pi a}{b}\right)} \right] \quad (J.14)
\]

Because we consider maximum contact zone size to be happened at the right crack tip under positive shear, we need to consider \( \varepsilon < 0 \). We can write following easily from Eq.(J.14).

\[
-cot^{-1}\left(\frac{\sigma}{\tau}\right) \geq -\frac{\pi}{2} + tan^{-1}\left(\frac{\tanh(\pi a e/b)}{\tan(\pi a/2b)}\right) sgn \varepsilon - tan^{-1}\left(2|\varepsilon|\right) + |\varepsilon| \ln \left[ \frac{0.01\left(\frac{\pi a}{b}\right)}{\sin\left(\frac{\pi a}{b}\right)} \right] \quad (J.15)
\]

By dividing both sides by -1, we obtain

\[
cot^{-1}\left(\frac{\sigma}{\tau}\right) \leq \frac{\pi}{2} - tan^{-1}\left(\frac{\tanh(\pi a e/b)}{\tan(\pi a/2b)}\right) + tan^{-1}\left(2\varepsilon\right) - \varepsilon \ln \left[ \frac{0.01\left(\frac{\pi a}{b}\right)}{\sin\left(\frac{\pi a}{b}\right)} \right] \quad (J.16)
\]

which leads to
\[
\left( \frac{\sigma}{\tau} \right) \geq \cot \left\{ \frac{\pi}{2} - \tan^{-1} \left( \frac{\tanh(\pi \epsilon / b)}{\tan(\pi a/2b)} \right) + \tan^{-1}(2\epsilon) - \epsilon \ln \left( \frac{\sin(\pi a / b)}{\pi (a / b)} \right) \right\} 
\] (J.17)

We can show that Eq. (J.10) reduces to non-interacting case when \( a/b \) approaches zero as shown below. The interpenetration zone size is given in Eq. (J.10) as follows.

\[
\frac{r_i}{2a} \left[ \sin \left( \frac{\pi a}{b} \right) \right] \exp \left[ \frac{1}{|\epsilon|} \right] \left[ \frac{2n - 1}{2} \pi - \tan^{-1} \left( \frac{\tanh(\pi \epsilon / b)}{\tan(\pi a/2b)} \right) \right] \left\{ \cot^{-1} \left( \frac{\sigma}{\epsilon} \right) \text{sgn} \epsilon + \tan^{-1}(2|\epsilon|) \right\} 
\] (J.18)

This leads to the following limit.

\[
\lim_{a/b \to 0} \left( \frac{\pi a}{b} \right)^{-1} \sin \left( \frac{\pi a}{b} \right) = 1 
\] (J.19)

(See Appendix E for details) and

\[
\lim_{a/b \to 0} \left[ \tan^{-1} \left( \frac{\tanh(\pi a \epsilon / b)}{\tan(\pi a/2b)} \right) \right] = \tan^{-1}(2\epsilon). 
\] (J.20)

(See the following Maple® work)

\[
\text{limit} \left( \arctan \left( \frac{\tanh(\pi \cdot \epsilon)}{\tan(\frac{\pi \cdot \epsilon}{2})} \right), x = 0 \right) 
\]

\[
\arctan(2 \epsilon) 
\] (J.21)
By inserting Eqs. (J.19) and (J.20) into Eq. (J.18), we obtain

\[
\left( \frac{r_i}{2a} \right)_{\text{single}} = \exp \left[ \frac{1}{|e|} \left\{ \left( 2n - \frac{1}{2} \right) \pi - \tan^{-1}(2\varepsilon) \text{sgn } \varepsilon \right\} - \tan^{-1}\left( \frac{\sigma}{\tau} \right) \text{sgn } \varepsilon + \tan^{-1}(2|\varepsilon|) \right]\] .

\quad \text{(J.22)}

This expression can be obtained directly using SIF for single crack as shown below.

\[
K_{\text{single}} = \left(1 + 2i\varepsilon \right)^{-ie} \frac{2a}{l} \sqrt{\pi a} \left( \sigma + i\tau \right)
\]

\quad \text{(J.23)}

(Eq. (21) in Rice (1988)) which leads to

\[
\text{Arg}(K_{\text{single}}) = \tan^{-1}(2\varepsilon) - \varepsilon \ln \left( \frac{2a}{l} \right) + \tan^{-1}\left( \frac{\sigma}{\tau} \right)
\]

\quad \text{(J.24)}

By inserting Eq. (J.24) into Eq. (J.1), we obtain

\[
r_i = l \exp \left[ \frac{1}{|e|} \left\{ \left( 2n - \frac{1}{2} \right) \pi - \tan^{-1}(2\varepsilon) - \varepsilon \ln \left( \frac{2a}{l} \right) + \tan^{-1}\left( \frac{\sigma}{\tau} \right) \text{sgn } \varepsilon + \tan^{-1}(2|\varepsilon|) \right\} \right]
\cdot
\]

\[
= l \left( \frac{2a}{l} \right)^{-ie} \sqrt{\pi a} \exp \left[ \frac{1}{|e|} \left\{ \left( 2n - \frac{1}{2} \right) \pi - \tan^{-1}(2\varepsilon) \text{sgn } \varepsilon + \tan^{-1}\left( \frac{\sigma}{\tau} \right) \text{sgn } \varepsilon + \tan^{-1}(2|\varepsilon|) \right\} \right]
\]

\quad \text{(J.25)}

This will yield

\[
\left( \frac{r_i}{2a} \right) = \exp \left[ \frac{1}{|e|} \left\{ \left( 2n - \frac{1}{2} \right) \pi - \left( \tan^{-1}(2\varepsilon) + \psi \right) \text{sgn } \varepsilon + \tan^{-1}(2|\varepsilon|) \right\} \right]
\]

\quad \text{(J.26)}
which is identical to Eq.(J.22).
APPENDIX K Derivation of contact zone from Comninou (1977, 1978) and Comninou and Schmueser (1979)

Pure Tension

The singular integral equation for interfacial single crack under pure tension is given by Eq. (21) in Comninou (1977) as follows.

$$
\left(1 - \beta^2\right) \int_{-1}^{1} \frac{B_s(\omega)}{\omega - \zeta} d\omega - \beta^2 \int_{-1}^{1} k(\omega, \zeta) B_s(\omega) d\omega = \frac{\pi T}{C}
$$

(K.1)

$$B_s$$ can be written as the multiplication of two separate functions as given by Eq. (37) in Comninou (1977) as

$$B_s(\omega) = \frac{(1 - \omega^2)^{1/2} T}{R(\omega) F(\omega)} \phi(\omega)$$

(K.2)

(See also Eq. (3.2) in Erdogan and Gupta (1972)) From Eq. (3.12) in Erdogan and Gupta (1972), the first integral on the LHS can be written in the discrete form as

$$\frac{1}{\pi} \int_{-1}^{1} \frac{B_s(\omega)}{\omega - \zeta} d\omega = \sum_{j=1}^{n} \frac{(1 - \omega_j^2) F(\omega_j)}{(n + 1)(\omega_j - \zeta_k)}$$

(K.3)

where
\[ \omega_j = \cos \left( \frac{\pi j}{n+1} \right) \text{ for } j=1,2,\ldots,n; \quad \zeta_k = \cos \left( \frac{(2k-1)\pi}{2(n+1)} \right) \text{ for } k=1,2,\ldots,n+1. \] (K.4)

(Eq. (3.13) in Erdogan and Gupta (1972)) Based on Eq. (3.14) in Erdogan and Gupta (1972), the second integral on the LHS can be written as

\[ \frac{1}{\pi} \int_{-1}^{1} k(\omega, \zeta) B_j(\omega) d\omega = \frac{1}{\pi} \int_{-1}^{1} k(\omega, \zeta) F(\omega)(1 - \omega^2)^{1/2} d\omega \]
\[ = \sum_{j=1}^{n} k(\omega_j, \zeta_k) \frac{(1 - \omega_j^2) F(\omega_j)}{(n+1)} \] . (K.5)

By inserting Eqs (K.3) and (K.5) into Eq.(K.1),

\[ (1 - \beta^2) \pi \sum_{j=1}^{n} \frac{(1 - \omega_j^2) F(\omega_j)}{(n+1)(\omega_j - \zeta_k)} - \beta^2 \pi \sum_{j=1}^{n} \frac{(1 - \omega_j^2) F(\omega_j)}{(n+1)} k(\omega_j, \zeta_k) = \frac{\pi T}{C}. \] (K.6)

This leads to

\[ \sum_{j=1}^{n} \frac{(1 - \omega_j^2) F(\omega_j)}{(n+1)} \left[ \frac{(1 - \beta^2)}{\omega_j - \zeta_k} - \beta^2 k(\omega_j, \zeta_k) \right] = \frac{T}{C}. \] (K.7)

By inserting Eq. (K.2), we have

\[ \sum_{j=1}^{n} \frac{(1 - \omega_j^2) \phi(\omega_j)}{(n+1)} \left[ \frac{(1 - \beta^2)}{\omega_j - \zeta_k} - \beta^2 k(\omega_j, \zeta_k) \right] = 1. \] (K.8)

(This is identical to Eq. (38) in Comninou (1977).)
Pure Shear

The singular integral equation for interfacial single crack under pure shear is given by Eq. (20) in Comninou (1978) as follows.

\[
(1 - \beta^2) \int_{-1}^{1} \frac{B_y(\omega)}{\omega - \xi} d\omega - \beta^2 \int_{-1}^{1} k(\omega, \zeta) B_y(\omega) d\omega = -\frac{\beta S (\delta\zeta + \sigma) \pi}{C \left[1 - (\delta\zeta + \sigma)^2\right]^{1/2}} \quad (K.9)
\]

\(B_y\) can be written as the multiplication of two separate functions as given by Eq. (32) in Comninou (1978).

\[
B_y(\omega) = (1 - \omega)^{1/2} (1 + \omega)^{-1/2} \frac{\beta S}{R(\omega)} \frac{\phi(\omega)}{F(\omega)} \quad (K.10)
\]

(See also Eq. (B.2) in Hills et. al (1996), \(a=1/2, b=-1/2\)). The first integral term on the LHS can be written in the discrete form by following the same procedure in Eq.(K.3). However, the type of singularity is different for this case. For this specific singularity function, integration points can be obtained from Eq. (B.8) on page 254 in Hills et. al (1996).

\[
\int_{-1}^{1} \frac{B_y(\omega)}{\omega - \xi} d\omega = \int_{-1}^{1} \frac{(1 - \omega)^{1/2} (1 + \omega)^{-1/2}}{\omega - \zeta} \frac{\beta S}{C} \phi(\omega) d\omega = \sum_{j=1}^{n} \frac{2\pi(1 - \omega_j)}{(2n + 1)(\omega_j - \zeta_k)} \frac{\beta S}{C} \phi(\omega_j) \quad (K.11)
\]

where

\[
\omega_j = \cos \left( \frac{2\pi j}{2n + 1} \right) \text{ for } j=1,2,\ldots,n \quad \text{and} \quad \zeta_k = \cos \left( \frac{(2k-1)\pi}{2(n+1)} \right) \text{ for } k=1,2,\ldots,n \quad (K.12)
\]
Similarly for the second integration term on the LHS, we have

$$\int_{-1}^{1} k(\omega, \zeta) B_y(\omega) d\omega = \sum_{j=1}^{n} k(\omega_j, \zeta_k) \frac{2\pi(1-\omega_j)}{(2n+1)} \frac{\beta S}{C} \phi(\omega_j).$$  \hspace{1cm} (K.13)

By inserting Eqs (K.11) and (K.13) into Eq.(K.9), we have

$$\left(1-\beta^2\right) \sum_{j=1}^{n} \frac{2\pi(1-\omega_j)}{(2n+1)\left(\omega_j - \zeta_k\right)} \frac{\beta S}{C} \phi(\omega_j) - \beta^2 \sum_{j=1}^{n} \frac{2\pi(1-\omega_j)}{(2n+1)} \frac{\beta S}{C} \phi(\omega_j) k(\omega_j, \zeta_k)$$

$$= -\frac{\beta S}{C} \pi \left(\delta_{\zeta_k} + \sigma\right) \left[1 - \left(\delta_{\zeta_k} + \sigma\right)^2\right]^{n/2}. \hspace{1cm} (K.14)$$

This leads to

$$\sum_{j=1}^{n} \frac{2(1-\omega_j)}{(2n+1)\left(\omega_j - \zeta_k\right)} \left(1-\beta^2\right) \beta^2 k(\omega_j, \zeta_k) \phi(\omega_j) = -\frac{\left(\delta_{\zeta_k} + \sigma\right)}{\left[1 - \left(\delta_{\zeta_k} + \sigma\right)^2\right]^{n/2}}. \hspace{1cm} (K.15)$$

(This is identical to Eq. (33) in Comninou(1978).)

**Combined Shear and Tension**

The singular integral equation for an interfacial single crack under combined shear and tension is given by Eq. (16) in Comninou and Schmueser (1979) as follows.

$$\left(1-\beta^2\right) \int_{-1}^{1} B_y(\omega) d\omega - \beta^2 \int_{-1}^{1} k(\omega, \zeta) B_y(\omega) d\omega = \frac{\pi S \beta}{C} \left[\frac{\lambda}{\beta} - \frac{\left(\delta_{\sigma} + \sigma\right)}{\left[1 - \left(\delta_{\sigma} + \sigma\right)^2\right]^{n/2}}\right]. \hspace{1cm} (K.16)$$
By can be written as the multiplication of two separate functions as given by Eq. (28) in Comninou and Schmueser (1979).

\[
B_y(\omega) = \left(1 - \omega \right)^{1/2} \left(1 + \omega \right)^{-1/2} \frac{\beta S}{R(\omega)} \phi(\omega) \frac{C}{F(\omega)}
\]  
(17)

(See also Eq. (B.2) in Hills et. al (1996), \(a=1/2, b=-1/2\)). Following (K.11)-(K.13), we have

\[
\left(1 - \beta^2 \right) \sum_{j=1}^{n} \frac{2\pi(1-\omega_j)}{(2n+1)(\omega_j - \zeta_k)} \frac{\beta S}{C} \phi(\omega_j) - \beta^2 \sum_{j=1}^{n} \frac{2\pi(1-\omega_j)}{(2n+1)} \frac{\beta S}{C} \phi(\omega_j) k(\omega_j, \zeta_k)
\]
\[
= \frac{\pi S}{C} \frac{\lambda}{\beta} \left[ \frac{\delta \zeta_k + \sigma}{1 - \left(\delta \zeta_k + \sigma\right)^2} \right]^{1/2}
\]

\[\text{(K.18)}\]

This leads to

\[
\sum_{j=1}^{n} \frac{2(1-\omega_j)}{(2n+1)} \left[ \frac{1-\omega_j}{\omega_j - \zeta_k} \right] - \beta^2 k(\omega_j, \zeta_k) \phi(\omega_j) = \left[ \frac{\lambda}{\beta} \frac{\delta \zeta_k + \sigma}{1 - \left(\delta \zeta_k + \sigma\right)^2} \right]^{1/2}
\]

\[\text{(K.19)}\]

(This is identical to Eq. (29) in Comninou and Schmueser (1979).)

**An Array of Cracks under Combined Shear and Tension**

The integral equation for a periodic array of interfacial cracks with contact zones is given as
\[
\frac{1}{\pi} \int_{\gamma_1}^{\gamma_2} \frac{\phi(u)}{u-s} \frac{\beta^2}{\pi (1-s^2)^{1/2}} \int_{\gamma_1}^{\gamma_2} \phi(u) (1-u^2)^{1/2} \, du = \frac{T}{C (1+d^2s^2)} \left[ 1 - \frac{\beta Ss (1+d^2)^{1/2}}{T (1-s^2)^{1/2}} \right]
\]  
(Eq. (26) in Schmueser and Comninou (1979))

Here \( \gamma_1 \) and \( \gamma_2 \) are defined as

\[
\gamma_1 = \frac{1}{d} \tan \left( \frac{\pi (a-r_{c,\text{left}})}{2b} \right), \quad \gamma_2 = \frac{1}{d} \tan \left( \frac{\pi (a-r_{c,\text{right}})}{2b} \right),
\]  
(K.21)

and

\[
d = \tan \left( \frac{\pi a}{2b} \right).
\]  
(K.22)

(Eq. (18) in Schmueser and Comninou (1979)). By following the change of variables given in Equation (13)-(14) in Schmueser and Comninou (1979), we have

\[
x' = x/a, \quad \xi' = \xi/a \quad \text{and}
\]

\[
ud = \tan \left( \frac{\pi a \xi'}{2b} \right), \quad sd = \tan \left( \frac{\pi ax'}{2b} \right).
\]  
(K.23)

In order to normalize the interval \( (\gamma_1, \gamma_2) \), we introduce \( \zeta \) and \( \omega \) as follows.

\[
s = \delta \zeta + \sigma, \quad u = \delta \omega + \sigma
\]  
(K.24)

(Eq. (14) in Comninou and Schmueser (1979)) where

\[
\delta = \frac{\gamma_2 - \gamma_1}{2}, \quad \sigma = \frac{\gamma_2 + \gamma_1}{2}
\]  
(K.25)
Eq. (K.24) leads to \( du = \delta \, d \omega \) and \( \omega = (u - \sigma) / \delta \). Upper and lower limits of integration in Equation (K.20) can be evaluated from

\[
\omega_{\text{lower}} = \frac{\gamma_1 - \sigma}{\delta} \quad \text{and} \quad \omega_{\text{upper}} = \frac{\gamma_2 - \sigma}{\delta}.
\]  

(K.26)

By using \( \sigma \) and \( \delta \) values from Eq. (K.25), one can obtain \( \omega_{\text{lower}} = -1 \) and \( \omega_{\text{upper}} = 1 \).

By inserting Eq. (A13.24) into Eq. (A13.20) and changing the corresponding limits, we have

\[
\frac{1}{\pi} \int_{-1}^{1} \frac{\phi(\omega)}{(\delta \omega + \phi)(\delta \zeta + \phi)} \delta d\omega - \frac{\beta^2}{\pi \left(1 - (\delta \omega + \sigma)^2\right)^{1/2}} \int_{-1}^{1} \frac{\phi(\omega)}{(\delta \omega + \phi)(\delta \zeta + \phi)} \delta d\omega
\]

\[
= \frac{T}{C \left(1 + d^2 (\delta \zeta + \sigma)^2\right)} \left[ 1 - \frac{\beta S(\delta \zeta + \sigma)(1 + d^2)^{1/2}}{T \left(1 - (\delta \zeta + \sigma)^2\right)^{1/2}} \right].
\]  

(K.27)

Let’s take \( \left(1 - (\delta \zeta + \sigma)^2\right)^{1/2} \) term inside the integral and multiply the entire equation by \( \pi \) as follows.

\[
\int_{-1}^{1} \frac{\phi(\omega)}{(\omega - \zeta)(1 - (\delta \omega + \sigma)^2)^{1/2}} d\omega = \frac{\pi T}{C \left(1 + d^2 (\delta \zeta + \sigma)^2\right)} \left[ 1 - \frac{\beta S(\delta \zeta + \sigma)(1 + d^2)^{1/2}}{T \left(1 - (\delta \zeta + \sigma)^2\right)^{1/2}} \right].
\]  

(K.28)
In order to write Eq. (K.28) in the form of Cauchy type singular integral equation form, we can add and subtract \( \beta^2 \int_{-1}^{1} \left[ \phi(\omega)/(\omega - \zeta) \right] d\omega \) from the left hand side of the equation.

\[
\frac{1}{\omega - \zeta} d\omega - \beta^2 \int_{-1}^{1} \frac{\phi(\omega)}{\omega - \zeta} d\omega = \frac{1}{\omega - \zeta} d\omega - \beta^2 \int_{-1}^{1} \phi(\omega) \left( \frac{1 - (\delta\omega + \sigma)^2}{1 - (\delta\zeta + \sigma)^2} \right)^{1/2} d\omega + \beta^2 \int_{-1}^{1} \frac{\phi(\omega)}{\omega - \zeta} d\omega
\]

which leads to

\[
\frac{\pi T}{C \left( 1 + d^2(\delta\zeta + \sigma)^2 \right)} \left[ 1 - \frac{\beta S (\delta\zeta + \sigma)(1 + d^2)^{1/2}}{T \left( 1 - (\delta\zeta + \sigma)^2 \right)^{1/2}} \right] \tag{K.29}
\]

Let’s reorganize the right-hand side of the integration by taking \( \frac{\pi S \beta}{C} \) term outside the bracket.

\[
(1 - \beta^2) \int_{-1}^{1} \frac{\phi(\omega)}{\omega - \zeta} d\omega - \beta^2 \int_{-1}^{1} \phi(\omega) \left( \frac{1 - (\delta\omega + \sigma)^2}{1 - (\delta\zeta + \sigma)^2} \right)^{1/2} d\omega
\]

\[
= \frac{\pi T}{C \left( 1 + d^2(\delta\zeta + \sigma)^2 \right)} \left[ 1 - \frac{\beta S (\delta\zeta + \sigma)(1 + d^2)^{1/2}}{T \left( 1 - (\delta\zeta + \sigma)^2 \right)^{1/2}} \right] \tag{K.30}
\]

where
\[ k(\omega, \zeta) = \frac{1}{(\omega - \zeta)} \left[ \frac{1 - (\delta\omega + \sigma)^2}{1 - (\delta\zeta + \sigma)^2} - 1 \right]^{1/2} \]  \hspace{1cm} (K.32)

\[ \phi(\omega) \text{ can be written as the multiplication of two separate functions as given by Eqs (B.1) and (B.2) in Hills et. al (1996), } a=1/2, b=-1/2. \]

\[ \phi(\omega) = \frac{(1-\omega)^{1/2}(1+\omega)^{-1/2}}{R(\omega)} \frac{\beta S}{C} \frac{\psi(\omega)}{F(\omega)} \]  \hspace{1cm} (K.33)

(See also Eq. (28) in Comninou and Schmueser (1979)). Following the procedure outlined in Eqs. (K.11)-(K.13), we have

\[ (1 - \beta^2) \sum_{j=1}^{n} \frac{2\pi(1-\omega_j)}{(2n+1)(\omega_j - \zeta_k)} \beta S \frac{\psi(\omega_j)}{C} - \beta^2 \sum_{j=1}^{n} \frac{2\pi(1-\omega_j)}{(2n+1)} \beta S \frac{\psi(\omega_j)}{C} k(\omega_j, \zeta_k) \]

\[ = \frac{\pi S \beta}{C} \left[ \frac{T}{S} - \frac{(\delta \zeta_k + \sigma)(1 + d^2)^{1/2}}{\beta \left[ 1 + d^2 \left( \delta \zeta_k + \sigma \right)^2 \right]} \right] \]  \hspace{1cm} (K.34)

This leads to

\[ \sum_{j=1}^{n} \frac{2(1-\omega_j)}{(2n+1)} \left[ \frac{1 - \beta^2}{\omega_j - \zeta_k} - \beta^2 k(\omega_j, \zeta_k) \right] \psi(\omega_j) \]

\[ = \left\{ \frac{T}{S} - \frac{(\delta \zeta_k + \sigma)(1 + d^2)^{1/2}}{\beta \left[ 1 + d^2 \left( \delta \zeta_k + \sigma \right)^2 \right] \left[ 1 + d^2 \left( \delta \zeta_k + \sigma \right)^2 \right]^{1/2}} \right\} \]  \hspace{1cm} (K.35)

where
\[
\omega_j = \cos \left( \frac{2\pi j}{2n+1} \right) \text{ for } j=1,2,\ldots,n , \quad \zeta_k = \cos \left( \frac{(2k-1)\pi}{2(n+1)} \right) \text{ for } k=1,2,\ldots,n \quad \text{(K.36)}
\]

and

\[
k(\omega_j, \zeta_k) = \frac{1}{(\omega_j - \zeta_k)} \left\{ \frac{1-\left(\delta\omega_j + \sigma\right)^2}{1-\left(\delta\zeta_k + \sigma\right)^2} \right\}^{-1/2} \quad \text{(K.37)}
\]

This equation is solved numerically to determine \(\psi(\omega)\)'s and the large contact zone size \((r_{c,right})\) under the constraint \(\int_{\gamma_1}^{\gamma_2} \phi(u) du = 0\) (Eq. (22) in Schmueser and Comninou (1979)). By changing the basis to \(\omega\), this constraint can be rewritten as

\[
\delta \int_{-1}^{1} \phi(\omega) d\omega = 0 \quad \text{(K.38)}
\]

Inserting Eq. (K.33) into Eq.(K.38), we have

\[
\delta \int_{-1}^{1} (1-\omega)^{1/2} (1+\omega)^{1/2} \frac{B_S}{C} \psi(\omega) d\omega = 0 \quad \text{(K.39)}
\]

Using the integration points for this type of singularity given in Hills et. al (1996) by Eq. (B.8), \(a=1/2\) and \(b=-1/2\), we have

\[
\sum_{j=1}^{n} \frac{2\pi (1-\omega_j)}{2n+1} \psi(\omega_j) = 0 \quad \text{(K.40)}
\]

(Eq. (36) in Comninou(1978)). The solution is not sensitive to small contact zone size. Therefore it is fixed to \((0.8)10^{-6}a\) where \(a\) is half crack length.
APPENDIX L  

Strain energy for a non-interacting penny-shaped crack between two dissimilar materials under the combined loading

Total strain energy for a penny-shaped interfacial crack under combination of normal loading $t_3^0$ and shear loadings in two directions $t_1^0$ and $t_2^0$ is given by Willis (1972) in Eq. (6.16) as follows.

$$E = \frac{8}{3} \pi^2 \kappa \left( 1 + \kappa^2 \right) \left( b^2 - d^2 \right) a^3 \left\{ \frac{\left( t_1^0 \right)^2 + \left( t_2^0 \right)^2}{\left( b - e \right) \pi \kappa \left( 1 + \kappa^2 \right)} + \frac{1}{2} \frac{d^{-1}}{d} \left( t_3^0 \right)^2 \right\} \tag{L.1}$$

Now, in our problem, we will consider a special loading case $t_1^0 = t_2^0 = 0$, $t_3^0$ = $\tau$, $t_3^0$ = $\sigma$.

Then, we can rewrite Eq. (L.1) also multiplying both nominator and denominator by $d^2$

$$E = \frac{8}{3} \pi^2 \kappa \left( 1 + \kappa^2 \right) \left( b^2 - d^2 \right) a^3 \left\{ \frac{\tau^2}{\left( b - e \right) \pi \kappa \left( 1 + \kappa^2 \right)} + \frac{1}{2} \frac{d^{-1}}{d} \frac{d\sigma^2}{d} \right\} \tag{L.2}$$

The material parameters shown in the Eq. (L.2) can be rewritten in terms of Dundurs parameters using the conversions listed below. A brief proof for each conversion will also be provided.

$$d / b = \beta \tag{L.3}$$
Proof:

The parameters \( d \) and \( b \) are defined in Eq. (4.5) in Willis (1972) as follows.

\[
d = \frac{1 - 2\nu_+}{4\pi\mu_+} - \frac{1 - 2\nu_-}{4\pi\mu_-}, \quad b = \frac{1 - \nu_+}{2\pi\mu_+} + \frac{1 - \nu_-}{2\pi\mu_-}
\]  

(L.4)

where ‘+’ represents the material on top side while ‘-’ represents the material on bottom side. Therefore, we can replace subscripts “+” with (1), “−” with (2) and \( \mu \) with \( G \) which yields

\[
d = \frac{1 - 2\nu_1}{4\pi G_1} - \frac{1 - 2\nu_2}{4\pi G_2}, \quad b = \frac{1 - \nu_1}{2\pi G_1} + \frac{1 - \nu_2}{2\pi G_2}
\]

(L.5)

We can now rewrite \( d \) and \( b \) as follows recalling Kosolov’s parameter \( \kappa = 3 - 4\nu \).

\[
d = \frac{1 - 2\nu_1}{4\pi G_1} - \frac{1 - 2\nu_2}{4\pi G_2} = \frac{1 - 2\left(\frac{3 - \kappa_1}{4}\right)}{4\pi G_1} - \frac{1 - 2\left(\frac{3 - \kappa_2}{4}\right)}{4\pi G_2} = \left(\frac{-1 + \kappa_1}{8\pi G_1}\right) - \left(\frac{-1 + \kappa_2}{8\pi G_2}\right),
\]

(L.6)

\[
b = \frac{1 - \nu_1}{2\pi G_1} + \frac{1 - \nu_2}{2\pi G_2} = \frac{1 - \left(\frac{3 - \kappa_1}{4}\right)}{2\pi G_1} + \frac{1 - \left(\frac{3 - \kappa_2}{4}\right)}{2\pi G_2} = \left(\frac{1 + \kappa_1}{8\pi G_1}\right) + \left(\frac{1 + \kappa_2}{8\pi G_2}\right)
\]

(L.6)

Eq. (L.6) will lead to the following relation.

\[
\frac{d}{b} = \frac{G_2(\kappa_1 - 1) - G_1(\kappa_2 - 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}
\]

(L.7)
RHS of the Eq. (L.7) is identical to second Dundurs’ parameter $\beta$ (See Eq. (3) in Hills and Barber (1993)). This completes the proof of the Eq. (L.3).

$$\kappa = \varepsilon$$ \hfill (L.8)

Proof:

$\kappa$ is defined as follows in Eq. (4.7) by Willis (1972).

$$\kappa = \frac{1}{2\pi} \log \left\{ \frac{b + d}{b - d} \right\}$$ \hfill (L.9)

We can rewrite Eq. (L.9) as follows.

$$\kappa = \frac{1}{2\pi} \log \left\{ \frac{1 + d/b}{1 - d/b} \right\}$$ \hfill (L.10)

where $d/b=\beta$. This leads to

$$\kappa = \frac{1}{2\pi} \log \left\{ \frac{1 + \beta}{1 - \beta} \right\}.$$ \hfill (L.11)

This equation is exactly same as Eq. (2) in Hills and Barber (1993) where $\kappa=\varepsilon$ (oscillation index) which completes the proof.

$$\left( \frac{b^2 - d^2}{d (b + c)} \right) = \left( 1 - \beta^2 \right) / (4\beta\gamma)$$ \hfill (L.12)
Proof:

Let’s re-organize the LHS of Eq. (L.12) as follows.

\[
\frac{(b^2-d^2)}{d(b+c)} = \frac{b^2 \left( 1 - \frac{d}{b} \right)^2}{d(b+c)} = \frac{d(b+c)}{d(b+c)} \frac{d(b+c)}{b^2} \frac{d(b+c)}{b} \frac{d \left( 1 + \frac{c}{b} \right)}{b} = \frac{1 - \left( \frac{d}{b} \right)^2}{1 + \frac{c}{b}} \quad \text{(L.13)}
\]

By inserting \( c \) and \( b \) terms from Eq. (4.5) in Willis (1972), we have

\[
\frac{(b^2-d^2)}{d(b+c)} = \frac{(1-\beta^2)}{\beta \left( \frac{G_i + G_j}{G_i(1-\nu_i) + G_j(1-\nu_j)} \right)} = \beta \left( \frac{G_i + G_j}{G_i\left(1-\nu_i\right) + G_j\left(1-\nu_j\right)} \right) \quad \text{(L.14)}
\]

By recalling Kosolov’s parameter \( \kappa = 3 - 4\nu \) and \( \gamma \) definition from Eq.(4) in Hills and Barber (1993), which is

\[
\gamma = \frac{(G_i + G_j)}{G_i\left(\kappa_i + 1\right) + G_j\left(\kappa_j + 1\right)} \quad \text{(L.15)}
\]

we obtain

\[
\frac{(b^2-d^2)}{d(b+c)} = \beta \left( \frac{G_i + G_j}{G_i \left(1 - \frac{3 - \kappa_i}{4}\right) + G_j \left(1 - \frac{3 - \kappa_j}{4}\right)} \right) = \beta \left( \frac{4(G_i + G_j)}{G_i(\kappa_i + 1) + G_j(\kappa_j + 1)} \right) \quad \text{(L.16)}
\]

Eq. (L.16) leads to
\[
\frac{(b^2 - d^2)}{d(b + c)} = \frac{(1 - \beta^2)}{(4\beta\gamma)}
\]  
(L.17)

which completes the proof.

\[
\frac{b-e}{d} = \frac{(b^2 - d^2)}{d(b + c)} = \frac{(1 - \beta^2)}{(4\beta\gamma)}
\]  
(L.18)

Proof:

e is defined in Eq. (4.5) by Willis (1972) as follows.

\[
e = \frac{(bc + d^2)}{(b + c)}
\]  
(L.19)

This leads to

\[
b-e = b - \frac{(bc + d^2)}{(b + c)} = \frac{b^2 + bc - bc - d^2}{(b + c)} = \frac{(b^2 - d^2)}{(b + c)}
\]  
(L.20)

and

\[
\frac{b-e}{d} = \frac{(b^2 - d^2)}{d(b + c)}
\]  
(L.21)

in which the LHS of Eq. (L.21) equals to \(\frac{(1 - \beta^2)}{(4\beta\gamma)}\).

\[
d = \frac{\beta(1 + \kappa_i)}{4\pi G_i(1 + \alpha)}
\]  
(L.22)
Proof:

Recalling ‘d’ from Eq. (L.6), we have

\[ d = \frac{(\kappa_1 - 1)}{8\pi G_1} - \frac{(\kappa_2 - 1)}{8\pi G_2} = \frac{G_2 (\kappa_1 - 1) - G_1 (\kappa_2 - 1)}{8\pi G_1 G_2} \]  

(L.23)

From definition of \( \alpha \), we can obtain

\[ 1 + \alpha = \frac{2G_2 (\kappa_1 + 1)}{G_2 (\kappa_1 + 1) + G_1 (\kappa_2 + 1)} \]  

(L.24)

By dividing \( \beta \) by Eq. (L.24), we have

\[ \frac{\beta}{1 + \alpha} = \frac{G_2 (\kappa_1 - 1) - G_1 (\kappa_2 - 1)}{2G_2 (\kappa_1 + 1)} \]  

(L.25)

By multiplying both the LHS and RHS of Eq. (L.25) by \((1 + \kappa_1)/4\pi G_1\), we get

\[ \frac{\beta (\kappa_1 + 1)}{4\pi G_1 (1 + \alpha)} = \frac{G_2 (\kappa_1 - 1) - G_1 (\kappa_2 - 1)}{8\pi G_1 G_2} \]  

(L.26)

The RHS of the Eq. (L.26) is identical to ‘d’ given in Eq.(L.23). Therefore,

\[ d = \frac{\beta (\kappa_1 + 1)}{4\pi G_1 (1 + \alpha)} \]  

(L.27)

By inserting Eqs. (L.3), (L.8), (L.12), (L.18) and (L.22) into Eq.(L.2), we obtain
\[
E = \frac{8}{3} \pi^2 \varepsilon (1 + \varepsilon^2) \left( \frac{b^2}{d^2} - 1 \right) a^3 \left\{ \frac{\tau^2}{4 \beta \gamma} \left( \frac{1}{\beta^2} \right) \left( \frac{1}{\varepsilon (1 + \varepsilon^2)} \right) + \frac{d}{2} + \frac{d \sigma^2}{2} \right\}
\]

\[
= \frac{8}{3} \pi^2 \varepsilon (1 + \varepsilon^2) \left( \frac{1}{\beta^2} - 1 \right) d \left\{ \frac{4 \beta \gamma \tau^2}{\left( \frac{1}{\beta^2} \right) \varepsilon (1 + \varepsilon^2) + 4 \beta \gamma} + \frac{1}{2} \sigma^2 \right\} a^3
\]

\[
= \frac{8}{3} \pi^2 \varepsilon (1 + \varepsilon^2) \left( \frac{1 - \beta^2}{\beta^2} \right) \left( \frac{1}{\beta} \right) \frac{\beta (1 + \kappa_1)}{4 \pi G_i (1 + \alpha)} \left\{ \frac{4 \beta \gamma \tau^2}{\left( \frac{1}{\beta^2} \right) \varepsilon (1 + \varepsilon^2) + 4 \beta \gamma} + \frac{1}{2} \sigma^2 \right\} a^3
\]

\[
E = \frac{2}{3} \left( \frac{1 + \kappa_1}{G_i (1 + \alpha)} \right) \pi \varepsilon (1 + \varepsilon^2) \left( \frac{1 - \beta^2}{\beta} \right) \left\{ \frac{4 \beta \gamma \tau^2}{\left( \frac{1}{\beta^2} \right) \varepsilon (1 + \varepsilon^2) + 4 \beta \gamma} + \frac{1}{2} \sigma^2 \right\} a^3
\]
APPENDIX M  Normal Spring Stiffness for non-interacting penny-shaped interfacial cracks

The normal displacement of a cracked body in the region far from crack zone can be defined as

\[ \Delta_{N,\text{crack}} = \frac{\partial E}{\partial Q_N} = \frac{\partial \sigma}{\partial Q_N} \frac{\partial E}{\partial \sigma} \tag{M.1} \]

where \( Q_N \) is the normal loading as defined in the main text. By inserting Eq. (L.28) into Eq. (M.1), we obtain

\[ \Delta_{N,\text{cracked}} = \frac{\partial \sigma}{\partial Q_N} \frac{\partial}{\partial \sigma} \left( \frac{2}{3} \frac{(1 + \kappa)}{G_i (1 + \alpha)} \pi \varepsilon (1 + \varepsilon^2) \left( \frac{1 - \beta^2}{\beta} \right) \right) \left\{ \frac{4 \beta \gamma^2}{(1 - \beta^2) \pi \varepsilon (1 + \varepsilon^2) + 4 \beta \gamma + \frac{1}{2} \sigma^2} \right\} a^3 \]

\[ \frac{1}{\pi b^2} \tag{M.2} \]

This leads to

\[ \Delta_{N,\text{cracked}} = \frac{1}{\pi b^2} \frac{2}{3} \frac{(1 + \kappa)}{G_i (1 + \alpha)} \pi \varepsilon (1 + \varepsilon^2) \left( \frac{1 - \beta^2}{\beta} \right) a^3 \sigma = \frac{2 a^3}{3 b^2} \frac{(1 + \kappa)}{G_i (1 + \alpha)} \frac{\varepsilon (1 + \varepsilon^2)(1 - \beta^2)}{\beta} \sigma . \]

\[ \tag{M.3} \]

By inserting Eq. (M.3) into normal spring stiffness definition of \( \sigma / \Delta_{N,\text{crack}} \), we have

\[ k_N = \frac{3 b^2}{2 a^3} \frac{G_i}{(1 + \kappa)} \frac{(1 + \alpha) \beta}{\varepsilon (1 + \varepsilon^2)(1 - \beta^2)} \]

\[ \tag{M.4} \]
We can simply show that Eq. (M.4) can be related to the crack compliance given in Eq. (20) by Gorbatikh. Let’s copy Eq (20) in Gorbatikh (2004) here.

\[
H_{2222} = \frac{B}{V} \frac{\pi a^3}{3 \cosh^2 (\pi \varepsilon)} \frac{\Gamma(2+i\varepsilon)}{\Gamma(2-i\varepsilon)} \frac{\Gamma(\frac{1}{2}+i\varepsilon)}{\Gamma(\frac{1}{2}-i\varepsilon)}
\]  

(M.5)

where subscript ‘2222’ represents the compliance in normal direction due to normal loading (\(\sigma_{22}\)) component.

\[B\] here is the parameter including material properties defined as

\[
B = \frac{1 + \kappa_1}{G_1} + \frac{1 + \kappa_2}{G_2}
\]  

(M.6)

We can quickly show that

\[
B = \frac{2(1 + \kappa_1)}{(1 + \alpha)G_1}
\]  

(M.7)

Proof:

One can derive the following equation for \(B\).

\[
B = \frac{1 + \kappa_1}{G_1} + \frac{1 + \kappa_2}{G_2} = \frac{G_2(1 + \kappa_1) + G_1(1 + \kappa_2)}{G_1 G_2}
\]  

(M.8)

Using the definition of \(\alpha\) and then \((1+\alpha)\)
\[ \alpha = \frac{G_2(\kappa_1 + 1) - G_1(\kappa_2 + 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)} , \quad 1 + \alpha = \frac{2G_2(\kappa_1 + 1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)} , \quad (M.9) \]

we can write

\[ G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1) = \frac{2G_2(\kappa_1 + 1)}{(1 + \alpha)} . \quad (M.10) \]

By inserting Eq. (M.10) into Eq. (M.8), we obtain \( B = 2(1 + \kappa_1)/(1 + \alpha)G_1 \) which completes the proof.

\( V \) is the volume surrounding the crack as shown in Figure M.1 which leads to \( V = \pi b^2 l \).

Figure M.1 Representative volume element in Gorbatikh (2004)
From Eqs. (6.1.30) and (6.1.31) in Abromowitz and Stegun (1970), pg.256, one can show that

\[
\frac{\Gamma(2+i\varepsilon)\Gamma(2-i\varepsilon)}{\Gamma\left(\frac{1}{2}+i\varepsilon\right)\Gamma\left(\frac{1}{2}-i\varepsilon\right)} = \frac{(1+\varepsilon^2)\varepsilon}{\beta}.
\]

(M.11)

Recalling

\[
cosh^2(\pi\varepsilon) = \frac{1}{1-\beta^2}
\]

(M.12)

from Hutchinson and Suo (1992) and inserting Eqs. (M.7), (M.11) and (M.12) into Eq. (M.5), we have

\[
H_{2222} = \frac{2(1+\kappa_i)}{(1+\alpha)G_i \frac{a^3}{\pi b^2 l}} (1-\beta^2) \frac{(1+\varepsilon^2)\varepsilon}{\beta} = \frac{1}{l} \frac{2a^3}{3b^2} \frac{(1+\kappa_i)}{(1+\alpha)G_i} \frac{(1-\beta^2)(1+\varepsilon^2)\varepsilon}{\beta}
\]

(M.13)

We note from (M.4) and (M.13) that

\[
k_N = \lim_{l \to \infty} \frac{l}{H_{2222}}.
\]

(M.14)
APPENDIX N  
Transverse spring stiffness for non-interaction penny-shaped interfacial cracks

Similar to Eq. (M.1), we can write following equation for far away displacement in transverse direction

$$\Delta_{r, \text{cracked}} = \frac{\partial E}{\partial Q_T} = \frac{\partial \tau}{\partial Q_T} \frac{\partial E}{\partial \tau}$$  \hspace{1cm} (N.1)

where $Q_T$ is the transverse loading as defined in the main text. By inserting (L.28) into Eq. (N.1), we have

$$\Delta_{r, \text{crack}} = \frac{\partial \tau}{\partial Q_T} \frac{\partial}{\partial \tau} \left( \frac{2}{3} \frac{(1+\kappa_i)}{G_i (1+\alpha)} \pi \varepsilon (1+\varepsilon^2) \left( \frac{1-\beta^2}{\beta} \right) \left( \frac{4 \beta \gamma \tau^2}{\pi a} \right) \left( \frac{1}{(1-\beta^2)} \pi \varepsilon (1+\varepsilon^2) + 4 \beta \gamma + \frac{1}{2} \sigma^2 \right) \frac{1}{\pi b^2} \right)$$  \hspace{1cm} (N.2)

which leads to

$$\Delta_{r, \text{crack}} = \frac{4}{3} \frac{a^3}{b^2} \frac{(1+\kappa_i)}{G_i (1+\alpha)} \left( \frac{4 \gamma \varepsilon (1+\varepsilon^2) (1-\beta^2)}{(1-\beta^2) \pi \varepsilon (1+\varepsilon^2) + 4 \beta \gamma} \right)$$  \hspace{1cm} (N.3)
By inserting Eq. (N.3) into the equation of transverse spring stiffness, $\tau / \Delta T_{crack}$,

$$k_T = \frac{3}{16} \frac{b^2 G_i(1+\alpha)}{a^3 \gamma (1+\kappa_i)} \left[ \frac{(1-\beta^2) \pi \epsilon (1+\epsilon^2)+4 \beta \gamma}{\epsilon (1+\epsilon^2)(1-\beta^2)} \right]$$

Reorganizing (N.4) further, we get

$$k_T = \frac{3\pi}{16} \frac{b^2 G_i(1+\alpha)}{a^3 \gamma (1+\kappa_i)} \left[ 1+ \frac{4 \beta \gamma}{\pi \epsilon (1+\epsilon^2)(1-\beta^2)} \right].$$

We can normalize $k_T$ value as follows.

$$k_T^* = \frac{k_T}{G_i(1+\alpha)} = \frac{3\pi}{16} \frac{1}{(1+\kappa_i) a} \left[ 1+ \frac{4 \beta \gamma}{\pi \epsilon (1+\epsilon^2)(1-\beta^2)} \right].$$

Let’s prove that Eq. (N.6) reduces to the transverse spring stiffness for homogenous case when, $G_1=G_2$ and $\nu_1=\nu_2$ which yields $\alpha=\beta=0$. Considering this condition, we can write

$$
\begin{align*}
  k_T^\text{homogenous} &= \lim_{\alpha,\beta \to 0} k_T^\text{dissimilar} = \lim_{\alpha,\beta \to 0} \left\{ \frac{3\pi}{16} \frac{b^2 G_i(1+\alpha)}{a^3 (1+\kappa_i)} \frac{1}{\gamma} \left[ 1+ \frac{4 \beta \gamma}{\pi \epsilon (1+\epsilon^2)(1-\beta^2)} \right] \right\} \\
  &= \frac{3\pi}{16} \frac{b^2 G_i}{a^3 (1+\kappa_i)} \left\{ \lim_{\alpha,\beta \to 0} \frac{1}{\gamma} + \lim_{\alpha,\beta \to 0} \frac{4 \beta}{\pi \epsilon (1+\epsilon^2)(1-\beta^2)} \right\} \\
  &= \frac{3\pi}{16} \frac{b^2 G_i}{(1+\kappa_i)} \left\{ \lim_{\alpha,\beta \to 0} \frac{1}{\gamma} + \lim_{\alpha,\beta \to 0} \frac{4 \beta}{\pi \epsilon (1+\epsilon^2)(1-\beta^2)} \right\} \\
  &= \frac{3\pi}{16} \frac{b^2 G_i}{a^3 (1+\kappa_i)} \left\{ \frac{1}{\gamma} + \frac{4 \beta}{\pi \epsilon (1+\epsilon^2)(1-\beta^2)} \right\}.
\end{align*}
\)
\[ \lim_{\alpha, \beta \to 0} \frac{1}{\gamma} = \frac{1}{\gamma_h} = \frac{1}{G_i + G_i} = \frac{1}{2G_i} = (1 + \kappa_i). \quad (N.8) \]

Second limit term will lead to indefiniteness as follows.

\[ \lim_{\alpha, \beta \to 0} \frac{4\beta}{\pi \epsilon (1 + \epsilon^2)(1 - \beta^2)} = 0 \quad (N.9) \]

Using L’Hospital Rule, we have

\[ \lim_{\alpha, \beta \to 0} \frac{d}{d\beta} (4\beta) \]

\[ = \lim_{\alpha, \beta \to 0} \frac{d}{d\beta} \left[ \frac{\pi \epsilon (1 + \epsilon^2)(1 - \beta^2)}{\epsilon} \right] = \frac{4}{\pi} \left\{ \frac{d}{d\beta} \left[ (1 + \epsilon^2)(1 - \beta^2) + \epsilon \frac{d}{d\beta} (1 + \epsilon^2)(1 - \beta^2) + \epsilon (1 + \epsilon^2)(-2\beta) \right] \right\} \quad (N.10) \]

Recalling definition of \( \epsilon \) in terms of \( \beta \), we have

\[ \frac{d}{d\beta} \epsilon = \frac{d}{d\beta} \left[ \frac{1}{2\pi} \log \left( \frac{1 + \beta}{1 - \beta} \right) \right] = \frac{1}{2\pi} \left( \frac{1 - \beta}{1 + \beta} \right) = \frac{1}{\pi} \left( \frac{1}{1 - \beta^2} \right). \quad (N.11) \]

By inserting Eq. (N.11) into Eq. (N.10), we have
\[
\lim_{\alpha, \beta \to 0} \frac{d}{d\beta} \left( \frac{4\beta}{\pi \varepsilon \left( 1 + \varepsilon^2 \right) \left( 1 - \beta^2 \right)} \right) \\
= \lim_{\alpha, \beta \to 0} \frac{4}{\pi} \left\{ \frac{1}{\left( 1 - \beta^2 \right)} \left( 1 + \varepsilon^2 \right) \left( 1 - \beta^2 \right) + \varepsilon \left\{ \frac{1}{\pi \left( 1 - \beta^2 \right)} \left( 1 - \beta^2 \right) + \varepsilon \left( 1 + \varepsilon^2 \right) (-2\beta) \right\} \right\} \\
= 4
\] (N.12)

By inserting resultant limits Eq. (N.12) and (N.8) into Eq. (N.7), we have

\[
k^\text{homogenous}_T = \frac{3\pi b^2}{16 a^3 (1 + \kappa_i)} \left\{ \left( 1 + \kappa_i \right) + 4 \right\}.
\] (N.13)

Recalling \( \kappa_i = 3 - 4\nu_i \) and \( G_i = E_i / (1 + \nu_i) \), we obtain

\[
k^\text{homogenous}_T = \frac{3\pi b^2}{16 a^3 (1 + 3 - 4\nu_i)} \left\{ \left( 1 + 3 - 4\nu_i \right) + 4 \right\} = \frac{3\pi b^2}{16 a^3} \frac{E_i}{8(1 - \nu_i^2)} 4(2 - \nu_i)
\]

\[
k^\text{homogenous}_T = \frac{3\pi b^2}{16 a^3} \frac{E_i}{2 - \nu_i} \] (N.14)

We want to compare this expression with Eq. (3c) in Margetan et. al (1988). For this purpose, first, we assign ratio of crack area to the unit are as \( A_d = (\pi a^2) / (\pi b^2) \) and \( s = 2b \) (See Fig. 2 in Margetan et. al (1988). Note that unit cell is defined as square in Fig.2. However, it is defined as circular area in our work). Now, we can rewrite Eq. (N.14) as follows with the notation in Margetan et al. (1988).
which leads to

\[ k_{T}^{\text{homogenous}} = \frac{3\pi}{16} A_d \frac{1}{a} \left( \frac{E_1}{(1-\nu_1^2)} \right) \frac{2}{2} = \frac{3\pi}{16} \frac{1}{b} \left( \frac{E_1}{(1-\nu_1^2)} \right) \frac{2}{2} \]

\[ k_{T}^{\text{homogenous}} = \frac{3\pi}{16} A_d^{1/2} \frac{2/s}{A_d^{1/2}} \left( \frac{E_1}{(1-\nu_1^2)} \right) \frac{2}{2} \]

This expression is identical to Eq. (3c) in Margetan et al. (1988).
APPENDIX O  
Physical limits of material combinations on $\gamma$-$\beta$ plane

First Dundurs (1967) and then Bogy (1968) showed that the stress under the prescribed surface tractions depends on only two constants for composite structures consisting of two isotropic and elastic materials. These two constants are called Dundurs parameters and expressed in terms of shear modulus ratio, $\Gamma=G_2/G_1$, and Kosolov’s parameters $\kappa_1=3-4\nu_1$, $\kappa_2=3-4\nu_2$ as follows.

\[
\alpha = \frac{\Gamma(\kappa_1 + 1) - (\kappa_2 + 1)}{\Gamma(\kappa_1 + 1) + (\kappa_2 + 1)}, \quad \beta = \frac{\Gamma(\kappa_1 - 1) - (\kappa_2 - 1)}{\Gamma(\kappa_1 + 1) + (\kappa_2 + 1)}
\]  

(O.1)

Dundurs (1969) provided a broad discussion on the valid range of material combinations on $\alpha$-$\beta$ plane considering the following physical limits

\[ \Gamma > 0 \text{ and } 1 < \kappa_1, \kappa_2 < 3. \]  

(O.2)

Analogously, it can be showed that stress fields for three-dimensional problems of composites can be expressed with three parameters. Interestingly, stress intensity factor for a penny-shaped crack under combined loading (See Eqs (24) and (25) in Hills and Barber, 1993) can be written using two combination parameters, $\beta$ and $\gamma$ where we can rewrite

\[ \gamma = \frac{1+\Gamma}{\Gamma(\kappa_1 + 1) + (\kappa_2 + 1)} \]  

(O.3)

and

\[ 0.25 < \gamma < 0.5 \]  

(O.4)
Therefore, it may be beneficial to plot important parameters on $\gamma$-$\beta$ and to discuss the physical limits on $\gamma$-$\beta$. For this purpose, we will consider the special cases in a very similar manner with Dundurs (1969) work.

**Constant $\Gamma$, $\Gamma=a$**

Let’s consider following specific cases

- $\Gamma=a$ and $\kappa_2=1$.

For this condition, $\gamma$ and $\beta$ values from Eqs. (O.1) and (O.3) becomes

\[
\gamma = \frac{a+1}{2a + (\kappa_2 + 1)} , \quad \beta = \frac{-(\kappa_2 - 1)}{2a + (\kappa_2 + 1)} \quad (O.5)
\]

We can relate $\beta$ and $\gamma$ by taking out $\kappa_2$ from $\gamma$ equation and inserting into $\beta$ equation by following the steps shown below. From $\gamma$-expression, we can write

\[
2a\gamma + \kappa_2 \gamma + \gamma = a + 1 , \quad \kappa_2 \gamma = a + 1 - 2a\gamma - \gamma \quad \text{and}
\]

\[
\kappa_2 = \frac{a+1}{\gamma} - 2a - 1 \quad (O.6)
\]

This will lead to

\[
\kappa_2 - 1 = \frac{a+1}{\gamma} - 2a - 2 , \quad \kappa_2 + 1 = \frac{a+1}{\gamma} - 2a \quad (O.7)
\]

By substituting Eq. (O.7) into $\beta$, we have
\[
\beta = \frac{-\frac{a+1}{\gamma} + 2a + 2}{2a + \frac{a+1}{\gamma} - 2a} = \left(\frac{a+1}{\gamma} - \frac{1}{\gamma} + 2\right) = 2\gamma - 1 \quad \text{(O.8)}
\]

As a result of Eq.(O.8), we can say, under the conditions \(\Gamma\) is constant and \(\kappa_1 = 1\), there is linear relation between \(\beta\) and \(\gamma\) which is given by the equation \(\beta = 2\gamma - 1\).

The end points on this line are determined by the limits of \(\kappa_2\):

- If \(\kappa_2 = 1\), from Eq.(O.5), \(\gamma = \frac{a+1}{2a + 2} = 0.5\) and \(\beta = 0\)
- If \(\kappa_2 = 3\), from Eq.(O.5), \(\gamma = \frac{a+1}{2a + 4}\) and \(\beta = \frac{-2}{2a + 4} = \frac{-1}{a + 2}\)

- \(\Gamma = a\) and \(\kappa_1 = 3\).

In this condition, \(\gamma\) and \(\beta\) becomes

\[
\gamma = \frac{a+1}{4a + (\kappa_2 + 1)} \quad \beta = \frac{2a - (\kappa_2 - 1)}{4a + (\kappa_2 + 1)} \quad \text{(O.9)}
\]

Similar to the previous case, we can relate \(\beta\) and \(\gamma\) by taking out \(\kappa_2\) from \(\gamma\) equation and inserting into \(\beta\) equation as shown in the following lines.

\[
4a\gamma + \kappa_2\gamma + \gamma = a + 1, \quad \kappa_2\gamma = a + 1 - 4a\gamma - \gamma
\]

\[
\kappa_2 = \frac{a + 1}{\gamma} - 4a - 1 \quad \text{(O.10)}
\]

This will lead to
\[ \kappa_2 - 1 = \frac{a + 1}{\gamma} - 4a - 2, \quad \kappa_2 + 1 = \frac{a + 1}{\gamma} - 4a \]  \hspace{1cm} (O.11)

By substituting Eq. (O.11) into \( \beta \), we have

\[ \beta = \frac{2a - \frac{a + 1}{\gamma} + 4a + 2}{4a + \frac{a + 1}{\gamma} - 4a} = \frac{6a + 2 - \frac{a + 1}{\gamma}}{(a + 1) - \frac{a + 1}{\gamma}} = 2\left(\frac{3a + 1}{a + 1}\right)\gamma - 1 \]  \hspace{1cm} (O.12)

As a result of Eq.(O.12), under the conditions \( \Gamma \) is constant and \( \kappa_1 = 3 \), there is linear relation between \( \beta \) and \( \gamma \) which is given by the equation \( \beta = 2\left(\frac{3a + 1}{a + 1}\right)\gamma - 1 \).

The end points on this line are determined by the limits of \( \kappa_2 \);

- \( \kappa_2 = 1 \), from Eq.(O.9), \( \gamma = \frac{a + 1}{4a + 2} \) and \( \beta = \frac{2a}{4a + 2} = \frac{a}{2a + 1} \)

- \( \kappa_2 = 3 \), from Eq.(O.9), \( \gamma = \frac{a + 1}{4a + 4} = 0.25 \) and \( \beta = \frac{a - 2}{4a + 4} = \frac{a - 1}{2(a + 1)} \)

- \( \Gamma = a \) and \( \kappa_2 = 1 \).

From Eq. (O.1) and Eq.(O.3), \( \gamma \) and \( \beta \) becomes

\[ \gamma = \frac{a + 1}{a(\kappa_1 + 1) + 2}, \quad \beta = \frac{a(\kappa_1 - 1)}{a(\kappa_1 + 1) + 2} \]  \hspace{1cm} (O.13)

We can relate \( \beta \) and \( \gamma \) by taking out \( \kappa_1 \) from \( \gamma \) equation and inserting into \( \beta \) equation by taking following steps. Using \( \gamma \) expression, we can write

\[ a\gamma \kappa_1 + a\gamma + 2\gamma = a + 1, \quad a\gamma \kappa_1 = a + 1 - a\gamma - 2\gamma \]
\[ \kappa_i = \frac{a+1}{a\gamma} - 1 - \frac{2}{a} \]  

(O.14)

This will lead to

\[ \kappa_i - 1 = \frac{a+1}{a\gamma} - 2 - \frac{2}{a}, \quad \kappa_i + 1 = \frac{a+1}{a\gamma} - \frac{2}{a} \]  

(O.15)

By substituting Eq. (O.15) into \( \beta \), we have

\[
\beta = \frac{a(\kappa_i - 1)}{a(\kappa_i + 1) + 2} = \frac{a \left( \frac{a+1}{a\gamma} - 2 - \frac{2}{a} \right)}{a \left( \frac{a+1}{a\gamma} - 2 \right) + 2} = \frac{a + 1}{2\gamma} - \frac{2(a+1)}{2} = -2\gamma + 1
\]

(O.16)

As a result of Eq.(O.16), we can say, under the conditions \( \Gamma \) is constant and \( \kappa_2=1 \), there is linear relation between \( \beta \) and \( \gamma \) which is given by the equation \( \beta = -2\gamma + 1 \).

The end points on this line are determined by the limits of \( \kappa_i \);

- \( \kappa_i=1, \) from Eq.(O.13), \( \gamma = \frac{a+1}{2a+2} = 0.5 \) and \( \beta = 0 \)
- \( \kappa_i=3, \) from Eq.(O.13), \( \gamma = \frac{a+1}{4a+2} \) and \( \beta = \frac{2a}{4a+2} = \frac{a}{2a+1} \)

\( \Gamma=\alpha \) and \( \kappa_2=3. \)

\( \gamma \) and \( \beta \) definitions for this specific case will lead to

\[ \gamma = \frac{a+1}{a(\kappa_i + 1) + 4}, \quad \beta = \frac{a(\kappa_i - 1) - 2}{a(\kappa_i + 1) + 4} \]  

(O.17)
We can relate \( \beta \) and \( \gamma \) by taking out \( \kappa_1 \) from \( \gamma \) equation and inserting into \( \beta \) equation.

\[
a \gamma \kappa_1 + a \gamma + 4 \gamma = a + 1, \quad a \gamma \kappa_1 = a + 1 - a \gamma - 4 \gamma
\]

\[
\kappa_1 = \frac{a + 1}{a \gamma} - 1 - \frac{4}{a} \quad \text{(O.18)}
\]

This will lead to

\[
\kappa_1 - 1 = \frac{a + 1}{a \gamma} - 2 - \frac{4}{a}, \quad \kappa_1 + 1 = \frac{a + 1}{a \gamma} - \frac{4}{a} \quad \text{(O.19)}
\]

By substituting Eq. (O.19) into \( \beta \), we have

\[
\beta = \frac{a(\kappa_1 - 1) - 2}{a(\kappa_1 + 1) + 4} = \frac{a \left( \frac{a + 1}{a \gamma} - \frac{2 - 4}{a} \right) - 2}{a \left( \frac{a + 1}{a \gamma} - \frac{4}{a} \right) + 4} = \frac{\left( \frac{a + 1}{\gamma} - 2a - 4 \right) - 2}{\left( \frac{a + 1}{\gamma} - \frac{4}{a} \right) + 4}
\]

\[
= \frac{\left( \frac{a + 1}{\gamma} \right) - 2(a + 3)}{\left( \frac{a + 1}{\gamma} \right)} = -2 \left( \frac{a + 3}{a + 1} \right) \gamma + 1
\]

As a result of Eq.(O.20), we can say, under the conditions \( \Gamma \) is constant and \( \kappa_2 = 3 \), there is linear relation between \( \beta \) and \( \gamma \) which is given by the equation \( \beta = -2 \left( \frac{a + 3}{a + 1} \right) \gamma + 1 \).

The end points on this line are determined by the limits of \( \kappa_1 \);

\[\kappa_1 = 1, \text{ from Eq. (O.17), } \gamma = \frac{a + 1}{2a + 4} \text{ and } \beta = -\frac{1}{a + 2}\]
\( \kappa_0 = 3 \), from Eq.(O.17), \( \gamma = \frac{a + 1}{4a + 4} = 0.25 \) and \( \beta = \frac{2a - 2}{4a + 4} = \frac{a - 1}{2(a + 1)} \)

As a result, \( \Gamma = a \) region will be limited by four lines with the following end points:

- \( \beta = 2 \gamma - 1 \) with the end points \((0.5, 0)\) and \((\frac{a + 1}{2a + 4}, -(a + 2)^{-1})\)

- \( \beta = 2 \left( \frac{3a + 1}{a + 1} \right) \gamma - 1 \) with the end points \((\frac{a + 1}{4a + 2}, \frac{a}{2a + 1})\) and \((0.25, \frac{a - 1}{2(a + 1)})\).

- \( \beta = -2 \gamma + 1 \) with the end points \((0.5, 0)\) and \((\frac{a + 1}{4a + 2}, \frac{a}{2a + 1})\).

- \( \beta = -2 \left( \frac{a + 3}{a + 1} \right) \gamma + 1 \) with the end points \((\frac{a + 1}{2a + 4}, -(a + 2)^{-1})\) and \((0.25, \frac{a - 1}{2(a + 1)})\).

These lines on \( \gamma - \beta \) plane produce a quadrangle as shown in the Figure O.1 for various constant \( \Gamma \) values.
As can be seen in Figure O.1, as the $\Gamma$ increases, quadrangle becomes narrower and for $\Gamma=\infty$, the quadrangle becomes a line which passes through $(0.25, 0.5)$ and $(0.5, 0)$ which coincides with $\beta=-2\gamma+1$. Similarly, as $\Gamma$ approaches 0, it becomes a line passing through $(0.25, -0.5)$ and $(0.5, 0)$ it coincides with $\beta=2\gamma-1$. For all values of $\Gamma$, the families of quadrangles scan a triangular region limited by left bound, the line represented by “$\gamma=0.25$”, upper bound, the line represented by “$\beta=-2\gamma+1$” and lower bound, the line represented by “$\beta=2\gamma-1$”.

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**Constant κ values, κ₁=b and κ₂=c**

Consider the case where κ₁ and κ₂ are constant, i.e. κ₁=b and κ₂=c while Γ may vary from 0 to infinity. By inserting κ₁=b and κ₂=c on Equations (O.1) and (O.3), we obtain,

\[
\beta = \frac{\Gamma(b-1)-(c-1)}{\Gamma(b+1)+(c+1)} \quad \text{and} \quad \gamma = \frac{1+\Gamma}{\Gamma(b+1)+(c+1)} \quad (O.21)
\]

We can determine the relation between β and γ by taking out Γ from γ equation and inserting into β equation using following equalities.

\[
\gamma \Gamma(b+1) + \gamma(c+1) = 1 + \Gamma, \quad [\gamma(b+1)−1] \Gamma = 1 − \gamma(c+1)
\]

\[
\Gamma = \frac{1−\gamma(c+1)}{\gamma(b+1)−1} \quad (O.22)
\]

which lead to

\[
\beta = \frac{\left[1−\gamma(c+1)\right](b−1)−(c−1)}{\gamma(b+1)−1 \left[1−\gamma(c+1)\right] (b+1)+(c+1)} = \frac{(b−1)−\gamma(c+1)(b−1)−\gamma(b+1)(c−1)+(c−1)}{(b+1)−\gamma(b+1)(c+1)+\gamma(b+1)(c+1)−(c+1)}
\]

\[
\beta = \frac{(b+c)−2−2\gamma(b+c−1)}{(b−c)}, \quad \beta = -2\frac{(bc-1)}{(b-c)}\gamma + \frac{(b+c)-2}{(b-c)} \quad (O.23)
\]

Consequently, under the conditions κ₁=b and κ₂=c, there is linear relation between β and γ which is given by the equation \( \beta = -2\frac{(bc-1)}{(b-c)}\gamma + \frac{(b+c)-2}{(b-c)} \). The end point of this line can be determined by the limiting values of Γ, as follows.
If $\Gamma=0$, Eq. (O.21) will lead to $\gamma = \frac{1}{(c+1)}$ and $\beta = \frac{-(c-1)}{(c+1)}$.

If $\Gamma=\infty$, Eq. (O.21) will lead to $\gamma = \frac{1}{(b+1)}$ and $\beta = \frac{(b-1)}{(b+1)}$.

As a result, $\kappa_1=b$ and $\kappa_2=c$ while $\Gamma$ vary from 0 to infinity, represents a line connecting $\left(\frac{1}{(c+1)}, -\frac{(c-1)}{(c+1)}\right)$ and $\left(\frac{1}{(b+1)}, \frac{(b-1)}{(b+1)}\right)$.

Some examples of constant $\kappa_1$ and $\kappa_2$ couples are shown on Figure O.2 below.

Figure O.2 Physically admissible values for constant $\kappa_1$ and $\kappa_2$ values
Special Case $\kappa_1 = \kappa_2 = b$

Now, let’s consider the special case of previous section (b) with equal $\kappa$-values. Then Eq. (O.21) becomes

$$\gamma = \frac{1 + \Gamma}{\Gamma (b+1) + (b+1)} = \frac{1}{(b+1)} \quad , \quad \beta = \frac{(b-1)(\Gamma - 1)}{(b+1)(\Gamma + 1)} \tag{O.24}$$

which leads to a vertical line on $\gamma$-$\beta$ plane ($\gamma$ does not depend on $\beta$). $\kappa_1 = \kappa_2 = 1.5$ line is shown in Figure O.2. Thus, the condition of $\kappa_1 = \kappa_2 = b$ represents a vertical line by the equation, $\gamma = \frac{1}{(b+1)}$. The end points of this line can be shown from $\beta$ equation for limiting cases, $\Gamma = 0$ and $\infty$.

- $\Gamma = 0$, Eq. (O.24) will lead to $\beta = \frac{-(b-1)}{(b+1)}$ which is one of the end points on $\gamma$-$\beta$ plane with coordinates \((b+1)^{-1} , \frac{-(b-1)}{(b+1)}\).

- $\Gamma = \infty$, Eq. (O.24) will lead to $\gamma = \frac{1}{(b+1)}$ and $\beta = \frac{(b-1)}{(b+1)}$ which is one of the end points on $\gamma$-$\beta$ plane with coordinates \((b+1)^{-1} , \frac{(b-1)}{(b+1)}\).

One may note that $b$ can vary from 1 to 3. Therefore, families of vertical lines of equal $\kappa$, can be varied from $\gamma = 0.25$ ($\kappa_1 = \kappa_2 = 3$) to $\gamma = 0.5$ ($\kappa_1 = \kappa_2 = 1$) and these produces the left and right bound of the physically admissible material combinations.
Boundaries of physically admissible material combinations on $\gamma$-$\beta$ plane

From the section (a) of this appendix for constant $\Gamma$ values, we emphasized that for $\Gamma=\infty$, the quadrangle becomes a line which passes through (0.25, 0.5) and (0.5, 0) which coincides with $\beta=-2\gamma+1$. Similarly, as $\Gamma$ approaches 0, the quadrangle becomes a line passing through (0.25, -0.5) and (0.5, 0) it coincides with $\beta=2\gamma-1$. Moreover, these lines determine the upper and lower boundaries. Similarly, in the section (c) of this appendix, we highlighted that $\gamma=0.25$ ($\kappa_1=\kappa_2=3$) and $\gamma=0.5$ ($\kappa_1=\kappa_2=1$) determines the left and right boundaries. As a summary, on $\gamma$-$\beta$ plane, physically admissible material combinations generate a triangle as shown in Figure O.3 below. This triangle is bounded by following

- **Left bound**, The line represented by “$\gamma=0.25$”
- **Upper bound**, the line represented by “$\beta=-2\gamma+1$”
- **Lower bound**, the line represented by “$\beta=2\gamma-1$”

This triangle is the analogous of parallelogram on $\alpha$-$\beta$ plane which represents the limits of all physically possible material combinations. The corner points of constant $\Gamma=a$ and $\kappa_1=b$, $\kappa_2=c$ conditions are also indicated in the figure.
Figure O.3 Physically allowable material combination limits and some special cases on $\gamma$-$\beta$ plane
Required tensile load compare to shear load \((\sigma/\tau)\) to maintain small contact zone for non interacting penny-shaped interfacial cracks

Interpenetration zone size for interfacial cracks is provided in Hills and Barber (1993) as follows.

\[
\frac{r_j}{l} = \exp \left[ \frac{1}{\varepsilon} \left\{ \left( 2n - \frac{1}{2} \right) \pi - \arg(K) + \tan^{-1}(2\varepsilon) \right\} \right]
\] (P.1)

As pointed out by Graciani et al. (2007), the formula above is valid only for \(\varepsilon > 0\). They recommended following corrected formula in Eq. (11),

\[
\frac{r_j}{l} = \exp \left[ \frac{1}{|\varepsilon|} \left\{ \left( 2n - \frac{1}{2} \right) \pi - \text{sgn}(\varepsilon) \arg(K) + \tan^{-1}(2|\varepsilon|) \right\} \right]
\] (P.2)

Stress intensity factor for penny-shaped cracks is given by Hills and Barber in Eq. (24).

\[
K(\theta) = \frac{(2a)^{1/2}}{\pi^{1/2}} \frac{\Gamma(2 + i\varepsilon)}{\Gamma(1/2 + i\varepsilon)} \left( \sigma + \frac{2\tau \cos \theta}{\left[ (1 - \beta^2) \pi \varepsilon \right]_{1 + \varepsilon^2}^{4 \beta \gamma} + 1} \right)
\] (P.3)
This leads to

\[
\arg[K(\theta)] = \arg \left[ \frac{\Gamma(2 + i\varepsilon)}{\Gamma(1/2 + i\varepsilon)} \right] + \tan^{-1} \left( \frac{8\beta\gamma / \sigma \cos \theta}{(1 - \beta^2)\pi\varepsilon(1 + \varepsilon^2) + 4\beta\gamma} \right)
\]

(P.4)

By plugging this equation into Eq. (P.2) and considering \(l=2a\), we have

\[
\frac{r(\theta)}{2a} = \exp \left[ \frac{1}{|\varepsilon|} \left( 2n - \frac{1}{2} \right) \pi - \text{sgn}(\varepsilon) \arg \left[ \frac{\Gamma(2 + i\varepsilon)}{\Gamma(1/2 + i\varepsilon)} \right] \right] \left\{ -\text{sgn}(\varepsilon) \tan^{-1} \left( \frac{8\beta\gamma / \sigma \cos \theta}{(1 - \beta^2)\pi\varepsilon(1 + \varepsilon^2) + 4\beta\gamma} \right) + \tan^{-1} \left( 2|\varepsilon| \right) \right\}
\]

(P.5)

\(\theta\) here is the angle measured from positive \(x\)-axis where positive \(x\)-axis is directed along the direction of shear. Now, we want to limit maximum interpenetration zone size along the circumference of crack \(r/a\) to 0.01.

For this purpose, we need to determine the angle where maximum interpenetration zone occurs. This can be achieved by taking derivative of Eq. (P.5) with respect to \(\theta\) and equate to the zero as follows. We will assume all material parameters shown in the Eq. (P.5) are fixed already.
\[
\frac{d}{d\theta} \left( \frac{r_i}{2a} \right) = \frac{d}{d\theta} \left\{ \exp \left\{ -\text{sgn}(\varepsilon) \tan^{-1} \left( \frac{8\beta\gamma r}{\pi\epsilon (1 + \varepsilon^2) + 4\beta\gamma} \frac{\cos \theta}{|\varepsilon|} \right) \right\} \right\} = 0
\]

(P.6)

This will yield to

\[
-A \exp \left( \text{sgn}(\varepsilon) \tan^{-1} \left( \frac{A\cos \theta}{|\varepsilon|} \right) \right) \frac{\sin \theta}{(1 + A^2 \cos^2 \theta |\varepsilon|)^{1/2}} = 0.
\]

(P.7)

The two roots of this equation are \(\theta=0\) and \(\theta=\pi\) and these values can be determined from the plot of the LHS of Eq. (P.7) shown below in Figure P.1.
We cannot determine which one corresponds to maximum. However, by switching the materials we can obtain the maximum point if it is not the maximum one. Therefore it is sufficient to check only one root. Let’s consider $\theta=0$ ($\cos \theta = 1$). By fixing it to be $0.01a$ we obtain
\[
\frac{0.01}{2} = \exp \left[ \frac{1}{|\varepsilon|} \left( 2n - \frac{1}{2} \right) \pi - \text{sgn}(\varepsilon) \arg \left[ \frac{\Gamma(2 + i\varepsilon)}{\Gamma(1/2 + i\varepsilon)} \right] \right] - \text{sgn}(\varepsilon) \tan^{-1} \left( \frac{8\beta\gamma / \sigma}{(1 - \beta^2)\rho(1 + \varepsilon^2) + 4\beta\gamma} \right) + \tan^{-1} \left( 2|\varepsilon| \right) \right].
\]  \quad (P.8)

Considering \(n=0\) for largest root, this equation will lead to

\[
|\varepsilon| \ln \left( \frac{0.01}{2} \right) = \exp \left[ \frac{1}{|\varepsilon|} \left( 2n - \frac{1}{2} \right) \pi - \text{sgn}(\varepsilon) \arg \left[ \frac{\Gamma(2 + i\varepsilon)}{\Gamma(1/2 + i\varepsilon)} \right] \right] - \text{sgn}(\varepsilon) \tan^{-1} \left( \frac{8\beta\gamma / \sigma}{(1 - \beta^2)\rho(1 + \varepsilon^2) + 4\beta\gamma} \right) + \tan^{-1} \left( 2|\varepsilon| \right) \right].
\]  \quad (P.9)

By organizing Eq.(P.9), we have

\[
\text{sgn}(\varepsilon) \tan^{-1} \left( \frac{8\beta\gamma / \sigma}{(1 - \beta^2)\rho(1 + \varepsilon^2) + 4\beta\gamma} \right) = \frac{\pi}{2} + \tan^{-1} \left( 2|\varepsilon| \right) - \text{sgn}(\varepsilon) \arg \left[ \frac{\Gamma(2 + i\varepsilon)}{\Gamma(1/2 + i\varepsilon)} \right] - |\varepsilon| \ln(0.005)
\]  \quad (P.10)

This leads to

\[
\frac{8\beta\gamma}{(1 - \beta^2)\rho(1 + \varepsilon^2) + 4\beta\gamma} = \tan \left( \frac{-\pi}{2} + \tan^{-1} \left( 2|\varepsilon| \right) - \text{sgn}(\varepsilon) \arg \left[ \frac{\Gamma(2 + i\varepsilon)}{\Gamma(1/2 + i\varepsilon)} \right] - |\varepsilon| \ln(0.005) \right)
\]  \quad (P.11)

By leaving \(\tau/\sigma\) on the LHS, we obtain
\[ \frac{\tau}{\sigma} = \frac{(1 - \beta^2) \pi \varepsilon (1 + \varepsilon^2) + 4 \beta \gamma}{8 \beta \gamma} \tan \left( \frac{\pi}{2} + \tan^{-1} (2 \varepsilon) - \text{sgn} (\varepsilon) \arg \left[ \frac{\Gamma (2 + i \varepsilon)}{\Gamma (1/2 + i \varepsilon)} \right] \right) \left[ -|\varepsilon| \ln (0.005) \right] \] 

(P.12)

\( \sigma / \tau \) is more convenient parameter to look at. Therefore, we can take inverse of Eq. (P.12) as follows.

\[ \frac{\sigma}{\tau} = \frac{8 \beta \gamma}{(1 - \beta^2) \pi \varepsilon (1 + \varepsilon^2) + 4 \beta \gamma} \cot \left( \frac{\pi}{2} + \tan^{-1} (2 \varepsilon) - \text{sgn} (\varepsilon) \arg \left[ \frac{\Gamma (2 + i \varepsilon)}{\Gamma (1/2 + i \varepsilon)} \right] \right) \left[ -|\varepsilon| \ln (0.005) \right] \] 

(P.13)
APPENDIX Q Crack opening profile of an interfacial penny-shaped crack under pure tension

Crack opening profile under pure tension is given in Eq. (1.4) by Mossakovskii and Rybka (1964) as the following equation:

$$ b_3(\bar{x}) = w^+(\bar{x},0) - w^-(\bar{x},0) = \phi_3^+(\bar{x},0) - \phi_3^-(\bar{x},0) \quad \text{(Q.1)} $$

$\phi_3^+$ and $\phi_3^-$ is provided in Equation (4.5) by Mossakovskii and Rybka (1964) as follows

$$ \phi_3^+ = \frac{C_2}{A_2 - A_2^{-1}} \left[ A_2 \sqrt{B(A - B)} \pi B \right] \left\{ \int_0^\rho \left[ a_\gamma \left( \frac{a - x}{a + x} \right)^\gamma + \left( \frac{a - x}{a + x} \right)^\gamma \right] \left[ \frac{a - x}{a + x} - \left( \frac{a - x}{a + x} \right)^\gamma \right] \, dx \right\} \sqrt{\rho^2 - x^2} \quad \text{(Q.2)} $$

and

$$ \phi_3^- = \frac{C_2}{A_2 - A_2^{-1}} \left[ \sqrt{B(A - B)} \pi B (EA_2 - H) \right] \left\{ \int_0^\rho \left[ a_\gamma \left( \frac{a - x}{a + x} \right)^\gamma + \left( \frac{a - x}{a + x} \right)^\gamma \right] \left[ \frac{a - x}{a + x} - \left( \frac{a - x}{a + x} \right)^\gamma \right] \, dx \right\} \sqrt{\rho^2 - x^2} \quad \text{(Q.3)} $$

By substituting Eqs. (Q.2) and (Q.3) into Eq.(Q.1), we obtain

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By reorganizing Eq. (Q.4) further, we obtain

$$b_3(\bar{x}) = \frac{C_2}{A_2 - A_2^2} \left( \frac{\sqrt{B(A-B)}}{\pi B} \right) (EA_2 - H - A_2) i \int_0^\bar{x} \left[ a_\gamma \left( \frac{a-x}{a+x} \right)^\gamma + \left( \frac{a-x}{a+x} \right)^{-\gamma} \right] dx
\left[ \left( \frac{a-x}{a+x} \right)^\gamma - \left( \frac{a-x}{a+x} \right)^{-\gamma} \right] \right]$$

$$+ \left( \frac{A_2 E}{B} - \frac{E}{A_2} - \frac{H}{B} + H - A_2 \right) a_\gamma \sqrt{ABi} \right)$$

(Q.4)

The integral term, \(I(x)\) can be rewritten as follows.

$$I(\bar{x}) = \int_0^\bar{x} \left[ a_\gamma \left[ \frac{1-x/a}{1+x/a} \right]^\gamma + \left( \frac{1-x/a}{1+x/a} \right)^{-\gamma} \right] + \left[ \left( \frac{1-x/a}{1+x/a} \right)^\gamma - \left( \frac{1-x/a}{1+x/a} \right)^{-\gamma} \right] \frac{x}{a} \sqrt{\left( \frac{\rho}{a} \right)^2 - \left( \frac{x}{a} \right)^2} \right]$$

(Q.6)

By defining \(u=x/a\) (\(du=dx/a\)), we have

$$I(u) = \int_0^u \left\{ a_\gamma \left[ \frac{1-u}{1+u} \right]^\gamma + \left( \frac{1-u}{1+u} \right)^{-\gamma} \right\} \frac{du}{\sqrt{\left( \frac{\rho}{a} \right)^2 - u^2}}$$

(Q.7)

Recalling \(\gamma = \epsilon_i\) (See next pages for proof) and taking \(ai\) outside of the integral, we obtain
\[ I(u) = ai \int_{0}^{\rho/a} \left\{ e^{i \left( \frac{1-u}{1+u} \right)^{\varepsilon i}} + \left( \frac{1-u}{1+u} \right)^{-\varepsilon i} \right\} \frac{du}{\sqrt{\left( \frac{\rho}{a} \right)^2 - u^2}} \left/ I_i(u) \right. \]  

(Q.8)

This leads to

\[ I(u) = aiI_i(u) \]  

(Q.9)

By substituting Eq. (Q.9) into (Q.5), we get

\[ b_i(\tilde{x}) = \frac{C_2}{A_2 - A_2^{-1}} \left[ \frac{\sqrt{AB - B^2}}{B} \left[ A_2(E - 1) - H \right] i^2 a I_1(u) \pi \right] \left/ \int_{0}^{\rho/a} \left\{ e^{i \left( \frac{1-u}{1+u} \right)^{\varepsilon i}} + \left( \frac{1-u}{1+u} \right)^{-\varepsilon i} \right\} \frac{du}{\sqrt{\left( \frac{\rho}{a} \right)^2 - u^2}} \left/ I_i(u) \right. \right] \]  

(Q.10)

Let’s rewrite Eq. (Q.10) in a more familiar notation using following relations. Proofs are also provided below.

\[ C_2 = \frac{\kappa_2}{G_2(\kappa_2 + 1)} \sigma \]  

(Q.11)

\[ A_2 = \frac{\kappa_2 + 1}{\kappa_2 - 1} \]  

(Q.12)

\[ E = -\frac{G_2 \left( \kappa_1 \kappa_2 + 1 \right)}{G_1 \left( 2\kappa_2 \right)} \]  

(Q.13)

\[ H = -\frac{G_2 \left( \kappa_1 \kappa_2 - 1 \right)}{G_1 \left( 2\kappa_2 \right)} \]  

(Q.14)
\[ A = -\frac{1}{\beta} \left( \frac{\kappa_2 + 1}{\kappa_2 - 1} \right), \quad B = -\beta \left( \frac{\kappa_2 + 1}{\kappa_2 - 1} \right), \quad AB = \left( \frac{\kappa_2 + 1}{\kappa_2 - 1} \right)^2 \] (Q.15)

\[ \gamma = \dot{\varepsilon} \] (Q.16)

Proof:

Material parameters used in Mossakovskii and Rybka (1964) are listed as follows.

\[ A_i = \frac{\lambda_i + 2G_i}{G_i} \quad (i=1,2) \quad \text{where} \quad \lambda_i = \frac{2G_i \nu_i}{1 - 2\nu_i} \] (Q.17)

(See Eq. (1.7) in Mossakovskii and Rybka (1964)) which leads to

\[ A_i = \frac{2G_i \nu_i + 2G_i}{1 - 2\nu_i} = \frac{2\nu_i + 2}{1 - 2\nu_i} = 2 \frac{1 - \nu_i}{1 - 2\nu_i} \] (Q.18)

Similarly for \( B_i \) and \( C_i \) and \( D_i \)

\[ B_i = \frac{G_i (\lambda_i + 2G_i)}{\lambda_i + 3G_i} = \frac{G_i \left( \frac{2G_i \nu_i + 2G_i}{1 - 2\nu_i} \right)}{\lambda_i + 3G_i} = \frac{G_i \left( \frac{2\nu_i + 2}{1 - 2\nu_i} \right)}{\frac{2\nu_i + 3}{1 - 2\nu_i}} = 2G_i \left( \frac{1 - \nu_i}{3 - 4\nu_i} \right) \] (Q.19)

(See Eq. (1.15) in Mossakovskii and Rybka (1964))

\[ C_i = \frac{(\lambda_i + 3G_i)}{2G_i (\lambda_i + 2G_i)} \sigma = \frac{\left( \frac{2G_i \nu_i + 3G_i}{1 - 2\nu_i} \right)}{2G_i \left( \frac{2G_i \nu_i + 2G_i}{1 - 2\nu_i} \right)} \sigma = \frac{(3 - 4\nu_i)}{4G_i (1 - \nu_i)} \sigma \] (Q.20)

(See Eq. (1.9) in Mossakovskii and Rybka (1964))
\[ D_i = \frac{G_i^2}{(\lambda_i + 3G_i)} = \frac{G_i^2}{\frac{2G_i\nu_i}{1-2\nu_i} + 3G_i} = \frac{G_i(1-2\nu_i)}{(3-4\nu_i)} \]  

(Q.21)

(See Eq. (1.15) in Mossakovskii and Rybka (1964))

By recalling \( \nu_i=(3-\kappa_i)/4 \), we can write

\[ 1-\nu_i = 1-\frac{3-\kappa_i}{4} = \frac{\kappa_i+1}{4} \quad \text{and} \quad 1-2\nu_i = 1-\frac{3-\kappa_i}{2} = \frac{\kappa_i-1}{2} \]  

(Q.22)

By inserting Eq.(Q.22) into Eqs. (Q.18), (Q.19), (Q.20) and (Q.21), we obtain

\[ A_i = 2 \frac{\kappa_i+1}{\kappa_i-1}, \quad B_i = \frac{2G_i\kappa_i+1}{\kappa_i}, \quad C_i = \frac{\kappa_i}{4G_i\kappa_i+1} = \frac{\kappa_i}{G_i(\kappa_i+1)}, \quad D_i = \frac{G_i(1-2\nu_i)}{(3-4\nu_i)} = \frac{G_i\kappa_i-1}{2\kappa_i} \]  

(Q.23)

Similarly for \( E, H, A_0 \) and \( A \), we get

\[ E = \frac{B_iB_j + D_iD_j}{D_i^2 - B_i^2} = \frac{G_iG_j(\kappa_i+1)(\kappa_j+1) + G_iG_j(\kappa_i-1)(\kappa_j-1)}{4\kappa_i\kappa_j} \]

\[ = \frac{G_2G_j}{4\kappa_i\kappa_j} \left[ (\kappa_i+1)(\kappa_j+1) + (\kappa_i-1)(\kappa_j-1) \right] \]

\[ = \frac{G_2G_i}{4\kappa_i\kappa_j} \left[ (\kappa_i-1)^2 - (\kappa_i+1)^2 \right] \]

\[ = \frac{G_2}{4\kappa_i\kappa_j} \left[ 2\kappa_i\kappa_j + 2 \right] \]

\[ E = -\frac{G_i(\kappa_i\kappa_j+1)}{2\kappa_j} \]

\[ = \frac{G_2}{G_i} \left[ 2\kappa_i\kappa_j + 2 \right] \]

(Q.24)
\[
H = \frac{B_1D_2 + D_1B_2}{D_1^2 - B_1^2} = \frac{4G_1G_2}{\kappa_1\kappa_2} \left( \frac{\left( \kappa_1 + 1 \right) \left( \kappa_2 - 1 \right)}{4\kappa_1^2} - \frac{\left( \kappa_1 - 1 \right) \left( \kappa_2 + 1 \right)}{4\kappa_2^2} \right) + \frac{4G_1G_2}{\kappa_1\kappa_2} \left( \frac{\left( \kappa_1 - 1 \right) \left( \kappa_2 + 1 \right)}{4\kappa_1^2} - \frac{\left( \kappa_1 + 1 \right) \left( \kappa_2 - 1 \right)}{4\kappa_2^2} \right)
\]

\[
= \frac{G_2G_2}{4\kappa_1\kappa_2} \left[ \frac{(\kappa_1 + 1)(\kappa_2 - 1) + (\kappa_1 - 1)(\kappa_2 + 1)}{4\kappa_1^2} \right] = \frac{G_2}{G_1} \frac{\kappa_1}{\kappa_2} \frac{\left[ 2\kappa_1\kappa_2 - 2 \right]}{\left[ -4\kappa_1 \right]} = \frac{G_2}{G_1} \frac{\left( \kappa_1\kappa_2 - 1 \right)}{2\kappa_2}
\]

(See Eq. (1.22) in Mossakovskii and Rybka (1964))

\[
A_0 = \frac{H}{E - 1} = \frac{-G_2}{G_1} \frac{\left( \kappa_1\kappa_2 - 1 \right)}{2\kappa_2} = \frac{-G_2}{G_1} \frac{\left( \kappa_1\kappa_2 - 1 \right)}{2\kappa_2} = \frac{G_2}{G_1} \frac{\left( \kappa_1\kappa_2 + 1 \right) - 2G_1\kappa_2}{2G_1\kappa_2} \frac{\left( \kappa_1\kappa_2 + 1 \right) + 2G_1\kappa_2}{2G_1\kappa_2}
\]

(See Eq. (1.23) in Mossakovskii and Rybka (1964))
Recalling \( \beta \) definition,

\[
\beta = \frac{G_2 (\kappa_1 - 1) - G_1 (\kappa_2 - 1)}{G_2 (\kappa_1 + 1) + G_1 (\kappa_2 + 1)}
\]  

(Q.28)

we have

\[
A = \frac{1}{-\beta} \times \frac{\kappa_2 + 1}{\kappa_2 - 1}.
\]  

(Q.29)

Similarly for \( B \),
\[
B = \frac{1 - A_2A_0}{A_2 - A_0} = \left(\frac{1 - A_2A_0}{A_2 - A_0}\right)^{-1} \times \frac{\kappa_2 + 1}{\kappa_2 - 1} = \left(\frac{1}{-\beta}\right)^{-1} \times \frac{\kappa_2 + 1}{\kappa_2 - 1}
\]  
(Q.30)

Using Eqs. (Q.29) and (Q.30), we can write following

\[
AB = \frac{1}{-\beta} \times \frac{\kappa_2 + 1}{\kappa_2 - 1} \times \left(\frac{1}{-\beta}\right)^{-1} \times \frac{\kappa_2 + 1}{\kappa_2 - 1} = \left(\frac{\kappa_2 + 1}{\kappa_2 - 1}\right)^2
\]  
(Q.31)

Eventually for \(\gamma\),

\[
\gamma = \frac{1}{2\pi i} \ln\left(\frac{A + \sqrt{AB}}{A - \sqrt{AB}}\right) = \frac{1}{2\pi i} \ln\left(\frac{1 - \beta}{-\beta} + \sqrt{\frac{\kappa_2 + 1}{\kappa_2 - 1}}\right) = \frac{1}{2\pi i} \ln\left(\frac{\kappa_2 + 1}{\kappa_2 - 1}\right) \left(\frac{1}{-\beta} + 1\right)
\]

\[
= \frac{1}{2\pi i} \ln\left(\frac{1 - \beta}{1 + \beta}\right) = -\frac{1}{2\pi i} \ln\left(\frac{1 + \beta}{1 - \beta}\right) = -\frac{1}{i} \epsilon
\]

(Q.32)

Therefore, \(\gamma = \epsilon i\).

By substituting Eqs. (Q.11)-(Q.16) into Eq.(Q.10), we obtain

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By taking $\frac{\varepsilon}{\beta} G_1$-term outside of the parenthesis, we have

$$b_3(\tilde{x}) = \frac{\kappa_2}{G_2(\kappa_2 + 1)} a \frac{1 - \beta^2}{\sqrt{(\kappa_2 - 1)^2 - \beta^2}} \left( - \frac{G_2 (\kappa_1 \kappa_2 - 1) (\kappa_2 + 1)}{G_1 (\kappa_2 - 1) (\kappa_2 - 2) \beta} \left[ \frac{\beta (\kappa_1 \kappa_2 - 1)}{(\kappa_2 - 1) (\kappa_2 - 2)} - \frac{G_1 (\kappa_2 + 1)}{G_2 (\kappa_2 - 1)} \right] \frac{I_4}{\pi} + \frac{\varepsilon}{\beta} \frac{G_2 (\kappa_1 \kappa_2 - 1) (\kappa_2 + 1)}{G_1 (\kappa_2 - 1)} \left[ \frac{\beta (\kappa_1 \kappa_2 - 1)}{(\kappa_2 - 1) (\kappa_2 - 2)} - \frac{G_1 (\kappa_2 + 1)}{G_2 (\kappa_2 - 1)} \right] \right)$$

(Q.33)

Further organization of Eq. (Q.34) will lead to
(Q.35)

Let’s distribute \((\kappa_2-1)\) inside of the parenthesis while taking \((\kappa_1+1)\) outside.

\[
\frac{b_3(\ddot{x})}{\sigma a} = -\left(\kappa_2-1\right)\varepsilon \left(\sqrt{1-\beta^2} \frac{I_1}{\pi \varepsilon} \left[\left(\frac{\kappa_1+1}{\kappa_2-1}\right) + \frac{G_1}{G_2} \left(\kappa_2+1\right)\right] - \left[\frac{\beta}{\kappa_1-1} + \beta \frac{G_1}{G_2} \left(\kappa_2-1\right)\right]\right)
\]

We can write \(G_1/G_2\) in terms of Dundurs parameters.

\[
\frac{G_1}{G_2} = \frac{(1-\alpha)(\kappa_1+1)}{(1+\alpha)(\kappa_2+1)}
\]

(Q.37)
By inserting Eq. (Q.37) into Eq. (Q.36), we obtain

\[
\frac{b_3(\bar{x})}{\sigma a} = -\left(\frac{\kappa_1 + 1}{4G_i}\right) \frac{\varepsilon}{\beta} \left[\sqrt{1 - \beta^2} \frac{I_1}{\pi \varepsilon} \left[1 + \frac{(1 - \alpha)}{(1 + \alpha)} \right] + \left[-\beta \frac{(\kappa_1 - 1)}{(\kappa_1 + 1)} \frac{(1 - \alpha)}{(1 + \alpha)} \frac{(\kappa_2 - 1)}{(1 + \alpha)} \right]\right]
\]  
(Q.38)

which leads to

\[
\frac{b_3(\bar{x})}{\sigma a} = -\left(\frac{\kappa_1 + 1}{4G_i}\right) \frac{\varepsilon}{\beta} \left[\frac{2\sqrt{1 - \beta^2} I_1}{(1 + \alpha) \pi \varepsilon} + \left[-\beta \frac{(\kappa_1 - 1)}{(\kappa_1 + 1)} \frac{(1 - \alpha)}{(1 + \alpha)} \frac{(\kappa_2 - 1)}{(1 + \alpha)} \frac{2}{(1 + \alpha)}\right]\right]
\]  
(Q.39)

Now, let’s rewrite \(\kappa_2\) in terms of \(\alpha, \beta, \text{and} \kappa_1\)

\[
\kappa_2 = \frac{\kappa_1 (1 - \beta) - (\alpha + \beta)}{\kappa_1 (-\alpha + \beta) + (1 + \beta)}
\]  
(Q.40)

Proof:

Following equations can be written based on \(\alpha\) and \(\beta\) definitions.

\[
1 - \beta = \frac{2(G_2 + G_1 \kappa_2)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}
\]  
(Q.41)

\[
-(\alpha + \beta) = \frac{-2(G_2 \kappa_1 - G_1 \kappa_2)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}
\]  
(Q.42)

\[
-\alpha + \beta = \frac{-2(G_2 - G_1)}{G_2(\kappa_1 + 1) + G_1(\kappa_2 + 1)}
\]  
(Q.43)
\[ 1 + \beta = \frac{2 (G_2 \kappa_1 + G_1)}{G_2 (\kappa_1 + 1) + G_1 (\kappa_2 + 1)} \]  

(Q.44)

By inserting these equations into the RHS of Eq.(Q.40), we have

\[
\kappa_1 \frac{2 (G_2 + G_1 \kappa_2)}{G_2 (\kappa_1 + 1) + G_1 (\kappa_2 + 1)} - \frac{2 (G_2 \kappa_1 - G_1 \kappa_2)}{G_2 (\kappa_1 + 1) + G_1 (\kappa_2 + 1)} \\
-2 (G_2 - G_1) + \frac{2 (G_2 \kappa_1 + G_1)}{G_2 (\kappa_1 + 1) + G_1 (\kappa_2 + 1)}
\]

\[
= \frac{G_2 \kappa_1 \kappa_2 k_1 - G_2 \kappa_1 + G_1 \kappa_2}{-G_2 \kappa_1 + G_1 \kappa_1 + G_2 \kappa_1 + G_1}
\]

\[
= \frac{G_1 \kappa_2 \kappa_1 + G_1 \kappa_2}{G_1 \kappa_1 + G_1}
\]

\[
= \frac{\kappa_2 (G_1 \kappa_1 + G_1)}{(G_1 \kappa_1 + G_1)}
\]

\[
= \kappa_2
\]

(Q.45)

Using this definition of \( \kappa_2 \) leads to

\[
\kappa_2 - 1 = \frac{\kappa_1 (1 - \beta) - (\alpha + \beta)}{\kappa_1 (-\alpha + \beta) + (1 + \beta)} - 1 \\
= \frac{\kappa_1 (1 - \beta) - (\alpha + \beta)}{\kappa_1 (-\alpha + \beta) + (1 + \beta)} - \frac{\kappa_1 (-\alpha + \beta) - (1 + \beta)}{\kappa_1 (-\alpha + \beta) + (1 + \beta)}
\]

\[
= \frac{\kappa_1 (1 + \alpha - 2 \beta) - (1 + \alpha + 2 \beta)}{\kappa_1 (-\alpha + \beta) + (1 + \beta)}
\]  

(Q.46)

and
\[
\kappa_2 + 1 = \frac{\kappa_1 (1 - \beta) - (\alpha + \beta)}{\kappa_1 (-\alpha + \beta) + (1 + \beta)} + 1
\]
\[
= \frac{\kappa_1 (1 - \beta) - (\alpha + \beta) + \kappa_1 (-\alpha + \beta) + (1 + \beta)}{\kappa_1 (-\alpha + \beta) + (1 + \beta)}
\]
\[
= \frac{\kappa_1 (1 - \alpha) + (1 - \alpha)}{\kappa_1 (-\alpha + \beta) + (1 + \beta)}
\]

By calculating \((\kappa_2 - 1) / (\kappa_2 + 1)\), we have

\[
\frac{\kappa_2 - 1}{\kappa_2 + 1} = \frac{\kappa_1 (1 + \alpha - 2\beta) - (1 + \alpha + 2\beta)}{\kappa_1 (1 - \alpha) + (1 - \alpha)}
\]
\[
= \frac{(\kappa_1 - 1)(1 + \alpha) - (\kappa_1 + 1)2\beta}{(\kappa_1 + 1)(1 - \alpha)}
\]
\[
= \frac{(\kappa_1 - 1)(1 + \alpha) - 2\beta}{(\kappa_1 + 1)(1 - \alpha)}
\]

By inserting Eq. \((Q.48)\) into Eq.\((Q.39)\), we obtain

\[
b_s(\tilde{x}) = \frac{(\kappa + 1)}{4G_i} \frac{\varepsilon}{\beta} \left( 2\sqrt{1 - \beta^2} I_1 \right) + \left[ -\beta \frac{(\kappa - 1)}{(\kappa + 1)} + \beta \frac{(1 - \alpha)}{(1 + \alpha)} \left( \frac{1 - \alpha}{1 - \alpha} \right) \left( \frac{2\beta}{1 - \alpha} \right) \right]
\]
\[
= -\frac{(\kappa + 1)}{4G_i} \frac{\varepsilon}{\beta} \left( 2\sqrt{1 - \beta^2} I_1 \right) + \left[ -\beta \frac{(\kappa - 1)}{(\kappa + 1)} + \beta \frac{(1 - \alpha)}{(1 + \alpha)} \right]
\]
\[
= -\frac{(\kappa + 1)}{4G_i} \frac{\varepsilon}{\beta} \left( 2\sqrt{1 - \beta^2} I_1 \right) + \frac{2}{(1 + \alpha)} \left[ 1 - \beta^2 \right]
\]

\[(Q.49)\]

which leads to
Now, let’s prove that Eq. (Q.50) will reduce to the crack profile for homogenous case when \( \alpha=\beta=\varepsilon=0 \).

By inserting \( \alpha=\beta=\varepsilon=0 \) into Eq.(Q.50), we obtain

\[
\frac{b_1(\tilde{x})}{\sigma a} = -\frac{(\kappa + 1)\varepsilon}{2G_1} \frac{1}{\beta(1+\alpha)} \left( \frac{I_1}{\pi \varepsilon} + \sqrt{1-\beta^2} \right) \tag{Q.51}
\]

(Recall following relation.

\[
\lim_{\beta \to 0} \frac{\varepsilon}{\beta} = 0
\]

Applying L’Hospital Rule

\[
\lim_{\beta \to 0} \frac{\varepsilon}{\beta} = \lim_{\beta \to 0} \frac{d}{d \beta} \left[ \frac{1}{2\pi} \ln \left( \frac{1+\beta}{1-\beta} \right) \right] = \lim_{\beta \to 0} \frac{1}{2\pi \left( \frac{1+\beta}{1-\beta} \right)} = \lim_{\beta \to 0} \frac{1}{2\pi \left( \frac{1+\beta}{1-\beta} \right)} = \frac{1}{\pi}
\]

where

\[
\frac{I_1 \text{ homogenous}}{\pi \varepsilon} = 2\sqrt{1-\left( \frac{\rho}{a} \right)^2} - 1 \tag{Q.52}
\]

See below for Mathematica® input and output lines for the proof.
By inserting Eq. (Q.52) into Eq. (Q.51) and recalling $\kappa = 3 - 4\nu$, we obtain

$$\left(\frac{b_3(\bar{x})}{\sigma a}\right)_{\text{homogenous}} = -\frac{4(1-\nu)}{2G} \frac{1}{\pi} \left(2\sqrt{1-\left(\frac{\rho}{a}\right)^2} - 1 + 1\right) = -\frac{4(1-\nu)}{\pi G} \left(\frac{\rho}{a}\right)^2 (Q.53)$$

This expression is identical to double of Eq. (23.19) given by Barber (1992).

Now, from Eq. (Q.50), we can derive maximum crack opening displacement throughout the crack region. As we know max. crack opening occurs at the center of crack under pure tension. Then, by plugging $u=0$ in Eq.(Q.50), we obtain

$$\frac{\delta_{\text{max}}}{a} = \frac{b_3(0)}{a} = \sigma \left(\frac{\kappa_i+1}{2G_i}\right) \frac{\varepsilon}{\beta(1+\alpha)} \left(\frac{I_1(0)}{\pi\varepsilon} + \sqrt{1-\beta^2}\right)$$

$(Q.54)$

$I_1(0) / \pi\varepsilon$ value approaches to one as $u$ goes to zero (center of crack).

See below for Mathematica® input and output lines for the proof.
Therefore

\[
\frac{\delta_{\text{max}}}{a} = \frac{b_1(0)}{a} = \frac{\sigma}{2G_i} \frac{\varepsilon}{\beta} \left( 1 + \frac{\sqrt{1 - \beta^2}}{(1 + \alpha)} \right) (1 + \sqrt{1 - \beta^2})
\]  

(Q.55)

We can reorganize Eq. (Q.55) as follows by involving shear stress,

\[
\frac{\delta_{\text{max}}}{a} = \frac{\sigma}{\tau} \frac{\tau}{2G_i} (\kappa_1 + 1) \frac{\varepsilon}{\beta} \left( 1 + \frac{\sqrt{1 - \beta^2}}{(1 + \alpha)} \right) \left( 1 + \sqrt{1 - \beta^2} \right) A(\alpha, \beta, \kappa_1)
\]  

(Q.56)
APPENDIX R  
Approximate expressions of normal and transverse spring stiffness for penny-shaped cracks given in literature

Margetan et al. (1988) suggested the following formula in Eq. (3a) for the normal spring stiffness for homogenous materials.

\[ k_N = \frac{3\pi}{8} E' s^{-1} A_d^{-3/2} \]  
(R.1)

where \( s = 2b \), \( A_d = (a/b)^2 \) and \( E' = E/(1-\nu^2) \)

For dissimilar materials Baik and Thompson (1984) suggests replacing \( E' \) with \( E^* \) in Eq. (40) which is

\[ E^* = \frac{2E_1E_2}{E_1(1-\nu_2^2) + E_2(1-\nu_1^2)} \]  
(R.2)

We can write following conversion for \( E^* \).

\[ E^* = \frac{8(1+\alpha)G_1}{(1+\kappa_1)} \]  
(R.3)

Proof:

Using \( E = 2(1+\nu)G \), we can rewrite Eq. (R.2) as follows
\[
\frac{1}{E^*} = \frac{2G_1(1+\nu)(1-\nu_2^2)+2G_2(1+\nu_2)(1-\nu_1^2)}{8G_1(1+\nu)(G_2(1+\nu_2)} = \frac{1-\nu_2}{4G_2} + \frac{1-\nu_1}{4G_1}, \quad (R.4)
\]

Recalling \(\kappa_i=3-4\nu_i\), one can write

\[
(1-\nu_2) = \frac{1+\kappa_2}{4} \quad \text{and} \quad (1-\nu_1) = \frac{1+\kappa_1}{4}. \quad (R.5)
\]

Combining Eqs. (R.4) and (R.5), we have

\[
\frac{1}{E^*} = \frac{G_1(1+\kappa_2)+G_2(1+\kappa_1)}{16G_1G_2}. \quad (R.6)
\]

Using the definition of

\[
\alpha = \frac{G_2(\kappa_1+1)-G_1(\kappa_2+1)}{G_2(\kappa_1+1)+G_1(\kappa_2+1)} \quad \text{and} \quad 1+\alpha = \frac{2G_2(\kappa_1+1)}{G_2(\kappa_1+1)+G_1(\kappa_2+1)}, \quad (R.7)
\]

the numerator of Eq. (R.7) can be expressed as follows.

\[
G_2(\kappa_1+1)+G_1(\kappa_2+1) = \frac{2G_2(1+\kappa_1)}{(1+\alpha)}. \quad (R.8)
\]

Inserting Eq. (R.8) into Eq. (R.6), we have

\[
\frac{1}{E^*} = \frac{(1+\kappa_1)}{8(1+\alpha)G_1}. \quad (R.9)
\]

By inserting Eq. (R.9) into Eq. (R.1), we obtain

\[
k_N^{\text{approx.}} = \frac{3}{2} b^2 \frac{G_1(1+\alpha)}{\pi a} \left(1+\kappa_1\right) \quad (R.10)
\]
For transverse spring stiffness of interfaces with multiple penny-shaped cracks in homogenous materials, Lavrentyev and Rokhlin (1994) used following formula given in Eq. (9).

\[
k_T = \left( \frac{2 - \nu}{2} \right) \frac{\pi}{8} \frac{E'}{1 - \nu^2} \frac{1}{a} \left( 1.299723 A_d^{1/2} - 0.9952365 A_d + 0.66720233 A_d^{3/2} - 0.42308925 A_d^2 \
+ 0.1406982 A_d^{5/2} - 0.02954016 A_d^3 + \frac{0.149058 A_d^{1/2}}{1 + A_d^{1/2}} - 1.868685 \ln \left( 1 + A_d^{1/2} \right) \right)^{-1}
\]

(R.11)

where \( \nu \) is replaced with \( \nu_{\text{average}} \) for dissimilar material interface problems.

\[
\nu_{\text{average}} = \frac{\nu_1 + \nu_2}{2}
\]

(R.12)

For the diminishing crack density when \( A_d \to 0 \), we can obtain this limiting case for dissimilar material interfaces from Eq. (R.11) as

\[
k_T^{\text{approx}} = \frac{3\pi}{16} E^* A_d^{-3/2} b \left( \frac{2 - \nu_{\text{average}}}{2} \right).
\]

(R.13)

Recalling Eq. (R.3), we can obtain
Recalling the \( \kappa_2 \) definition in terms of \( \alpha, \beta \) and \( \kappa_1 \) given in Eq. (Q.40), we can write \( \kappa_1 + \kappa_2 \) as follows.

\[
\kappa_1 + \kappa_2 = \kappa_1 + \frac{\kappa_1 (1-\beta) - (\alpha + \beta)}{\kappa_1 (-\alpha + \beta) + (1+\beta)}
\]

\[
= \frac{\kappa_1 [(-\alpha + \beta) + (1+\beta)] + \kappa_1 (1-\beta) - (\alpha + \beta)}{\kappa_1 (-\alpha + \beta) + (1+\beta)}
\]

\[
= \frac{\kappa_1 (-\alpha + \beta) + \kappa_1 (1+\beta) + \kappa_1 (1-\beta) - (\alpha + \beta)}{\kappa_1 (-\alpha + \beta) + (1+\beta)}
\]

\[
= \frac{\kappa_1 (-\alpha + \beta) + (2\kappa_1 - \alpha - \beta)}{\kappa_1 (-\alpha + \beta) + (1+\beta)}
\]

By inserting Eq. (R.15) into Eq. (R.14), we have

\[
k_{r, \text{approx}} = \frac{3\pi}{8} \frac{8G_i (1+\alpha)}{(1+\kappa_1)} \frac{1}{2b} \left( \frac{a}{b} \right)^3 \left( \frac{2 - \nu_{\text{average}}}{2} \right)
\]

\[
= \frac{3\pi}{2} \frac{b^2}{a^3} \frac{G_i (1+\alpha)}{(1+\kappa_1)} \left( 2 - \frac{3 - \kappa_1 + \frac{3 - \kappa_2}{4}}{2} \right)
\]

\[
= \frac{3\pi}{2} \frac{b^2}{a^3} \frac{G_i (1+\alpha)}{(1+\kappa_1)} \left( 2 - \frac{6 - (\kappa_1 + \kappa_2)}{8} \right)
\]

\[
= \frac{3\pi}{2} \frac{b^2}{a^3} \frac{G_i (1+\alpha)}{(1+\kappa_1)} \left( \frac{10 + (\kappa_1 + \kappa_2)}{16} \right)
\]

(R.14)
\[ k_T^{\text{approx.}} = \frac{3\pi b^2}{16 a^3} G_i (1 + \alpha) \left( \frac{1}{(1 + \kappa_i)} \left( 5 + \frac{\kappa_i^2 (-\alpha + \beta) + (2\kappa_i - \alpha - \beta)}{2\kappa_i (-\alpha + \beta) + 2(1 + \beta)} \right) \right) \]  
(R.16)

Based on this formula, we can define normalized \( k_T^{\text{approx.}} \) as follows.

\[ k_T^{\text{approx.}*} = \frac{k_T^{\text{approx.}}}{G_i (1 + \alpha) \frac{1}{(1 + \kappa_i)}} = \frac{3\pi \left( \frac{a}{b} \right)^2}{16} \left( 5 + \frac{\kappa_i^2 (-\alpha + \beta) + (2\kappa_i - \alpha - \beta)}{2\kappa_i (-\alpha + \beta) + 2(1 + \beta)} \right) \]  
(R.17)

In order to compare this approximate formula with exact result given in Eq. (A3.5), we define the error as follows.

\[ \text{error} = \frac{k_T^{\text{approx.}*} - k_T^{\text{approx.}*}}{k_T^{\text{approx.}*}} \times 100 = 1 - \frac{\gamma \left( 5 + \frac{\kappa_i^2 (-\alpha + \beta) + (2\kappa_i - \alpha - \beta)}{2\kappa_i (-\alpha + \beta) + 2(1 + \beta)} \right)}{1 + \frac{4\beta \gamma}{\pi \varepsilon (1 + \varepsilon^2)(1 - \beta^2)}} \]  
(R.18)

In order to plot most possible general case, we should reduce all parameters into three i.e., \( \alpha, \beta \) and \( \kappa_i \) (or \( \alpha, \beta \) and \( \gamma \) can be chosen as an alternative set). For this purpose, we need to rewrite \( \gamma \) in terms of \( \alpha, \beta \) and \( \kappa_i \)

\[ \gamma = \frac{(1 + \beta)}{2} + \frac{(1 + \alpha)(1 - \kappa_i)}{2(1 + \kappa_i)} \]  
(R.19)

Proof:

Recalling \( \alpha \) and \( \beta \) definitions, we can write

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\begin{align*}
1 + \alpha &= \frac{2G_2(k_1 + 1)}{G_2(k_1 + 1) + G_1(k_2 + 1)}, \quad 1 + \beta = \frac{2(G_2k_1 + G_1)}{G_2(k_1 + 1) + G_1(k_2 + 1)} \quad (R.20)
\end{align*}

By inserting Eq. (R.20) into the LHS Eq.(R.19), we obtain,

\begin{align*}
\frac{(G_2k_1 + G_1)}{G_2(k_1 + 1) + G_1(k_2 + 1)} + \frac{G_2(k_1 + 1)}{G_2(k_1 + 1) + G_1(k_2 + 1)(1 + k_1)}
&= \frac{(G_2k_1 + G_1) + G_2(1 - k_1)}{G_2(k_1 + 1) + G_1(k_2 + 1)}
&= \frac{G_1 + G_2}{G_2(k_1 + 1) + G_1(k_2 + 1)} = \gamma \quad (R.21)
\end{align*}

As can be seen, the RHS of this expression is identical to the \( \gamma \) definition.
Figure S.1 The effect of $n$ (the number of the surrounding cracks whose effect on stress distribution in the crack #1 region is modeled as non-uniform) and $m$ (the total number of cracks) on the calculation of mode I SIF ($a/b=0.95$) (a) Square configuration (The relative percent difference between $n=25$, $m=10201$ and $n=109$, $m=525625$ is -0.385%), (b) Hexagonal configuration (The relative percent difference between $n=31$, $m=10201$ and $n=121$, $m=525625$ is -0.763%) (Cont.)
Figure S.1 continued

(b)

\[ \frac{K_{i}^{\infty}(\phi)}{K_{y}^{\infty}} \]

Solid
n=31 or 121, m=525625

Dashed
n=13, m=10201

Dotted
n=31 or 121, m=10201
Figure S.2 The effect of \( n \) (the number of the surrounding cracks whose effect on stress distribution in the crack #1 region is modeled as non-uniform) and \( m \) (the total number of cracks) on the calculation of mode II and III SIFs (\( a/b=0.95 \)) (a) Square configuration (The relative percent difference between \( n=25, m=10201 \) and \( n=109, m=525625 \) is -0.397%), (b) Hexagonal configuration (The relative percent difference between \( n=31, m=10201 \) and \( n=121, m=525625 \) is -0.539%) (Cont.)
Figure S.2 continued

Mode II
n=9, 25 or 109
m=10201 or 525625

Mode III
n=9, 25 or 109
m=10201 or 525625

(b)
Figure S.3 Percent error in the surface fitting for the mode I SIF of periodic array of penny-shaped cracks (a) Square configuration (b) Hexagonal configuration
Figure S.4 Normalized mode II SIF for periodic array of coplanar penny-shaped cracks as function of angle around the circumference and crack density parameter $a/b$; data points based on accurate numerical calculations ($\nu=0.5$) (a) Square configuration, (b) Hexagonal configuration
Figure S.5 Normalized mode III SIF for periodic array of coplanar penny-shaped cracks as function of angle around the circumference and crack density parameter $a/b$; data points based on accurate numerical calculations ($\nu=0.5$) (a) Square configuration, (b) Hexagonal configuration
Figure S.6 (a) The comparison of the normalized mode I SIF (identical to mode II SIF) for two collinear strip cracks in a homogenous media between the approximate numerical results produced by Lekesiz et al based on Kachanov (1985) and the exact analytical results by Tada (2000) (b) The relative error in the approximate method.
Figure S.7 (a) The comparison of the normalized mode I SIF (identical to mode II SIF) for infinite periodic array of collinear strip cracks in a homogenous media between the approximate numerical results produced by Lekesiz et al based on Kachanov (1985) and the exact analytical results by Tada (2000) (b) The relative error in the approximate method
Figure S.8 (a) The solid lines represent numerical results produced by Lekesiz et al. based on Kachanov and Laures (1989) for the normalized mode I SIF of two coplanar penny-shaped cracks under a uniform tension with two $a/b$ ratio (0.99975 and 0.95238) as a function of angle around the crack, $\phi$. The exact results by Fabrikant (1987) are shown by circles. Square symbols represent some selected values from Table 1 in Kachanov and Laures (1989)

(b) The relative error of the approximate method (Kachanov-Laures, 1989) compared to exact solution (Fabrikant, 1987) in the maximum mode I SIF at $\phi = 0$ for two coplanar penny-shaped cracks as a function of $a/b$. Circular points represent some values reported by Kachanov and Laures (1989) (Page 295) (cont.)
Figure S.8 continued

![Diagram showing crack propagation and crack modes.](image-url)
Figure S.9 (a) The solid lines represent numerical results produced by Lekesiz et al. based on Kachanov and Laures (1989) for the normalized mode II and III SIFs of two coplanar penny-shaped cracks under a uniform tension with $a/b = 0.995$ as a function of angle around the crack, $\phi$. The exact results by Fabrikant (1987) are shown by squares. Square symbols represent some selected values from Table 3 in Kachanov and Laures (1989).

(b) The relative error of the approximate method (Kachanov-Laures, 1989) compared to exact solution (Fabrikant, 1989) in the maximum mode II SIF at $\phi = 0$ for two coplanar penny-shaped cracks as a function of $a/b$. (Cont.)
Figure S.9 continued

(b)
Figure S.10 Comparison between the normalized mode I SIF averaged over the circumference based on the method by Nemat-Nasser (1993) and those calculated in current work (based on Kachanov and Laures, 1989) for periodic array of coplanar penny-shaped cracks in the square configuration.
Figure S.11 Normalized Mode I SIF along the edges of periodic array of penny-shaped cracks with square configuration and comparison with widely spaced crack approximation ($a/b=0.75$)
Figure S.12 Maximum, averaged and approximate values of normalized mode I stress intensity factor for periodic array of penny-shaped cracks with square configuration as function of $a/b$. 

Normalized maximum Mode I SIF based on accurate numerical evaluation: $\frac{K_{I}^{\phi} (\phi = 0)}{K_{I}^{0}}$

Normalized average Mode I SIF based on accurate numerical evaluation: $\frac{<K_{I}^{\phi}(\phi)>}{K_{I}^{0}}$

Normalized Mode I SIF based on approximation: $\frac{K_{I}^{\phi, \text{approx.}}}{K_{I}^{0}}$
Figure S.13 The relative percent error in the approximate form of mode I SIF for periodic array of penny-shaped cracks with square configuration compared to maximum and averaged values.
APPENDIX T  Additional figures for Chapter 4

Figure T.1 Interaction function for infinite number of coplanar periodic penny-shaped cracks with the square and hexagonal configurations
Figure T.2 (a) The comparison of the normal interaction function (identical to transverse) for two collinear strip cracks in a homogenous media between the approximate numerical results produced by Lekesiz et al based on Kachanov (1985) and the exact analytical results by Tada (2000) (b) The relative error in the approximate method
Figure T.3 (a) The comparison of the normal interaction function (identical to transverse) for infinite periodic array of collinear strip cracks in a homogenous media between the approximate numerical results produced by Lekesiz et al. based on Kachanov (1985) and the exact analytical results by Tada et al. (2000) (b) The relative error in the approximate method.
Figure T.4 The comparison of the normal and transverse interaction function for two-coplanar cracks in a homogenous media between the approximate numerical results produced in the current work based on Kachanov (1985) and the exact analytical results by Fabrikant (1987, 1989)
Figure T.5 The relative percent error in the fitted curves of interaction function for square and hexagonal configurations.


