FAULT DIAGNOSTICS STUDY FOR LINEAR UNCERTAIN SYSTEMS USING DYNAMIC THRESHOLD WITH APPLICATION TO PROPULSION SYSTEM

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Wenfei Li, B.E., M.S.

Graduate Program in Aeronautical and Astronautical Engineering

The Ohio State University

2010

Dissertation Committee:

Rama K. Yedavalli, Advisor

Jen-Ping Chen

Mo-how Herman Shen
Fault detection and isolation plays a critical role in aircraft engines and the performance of their control systems. A great amount of research on model-based fault detection and isolation of aircraft engines has been studied since the 1970s. Model-based fault detection and isolation methods rely on the accuracy of the model. Model uncertainty, disturbances and noise, etc., all have a great impact on the fault detection and isolation design results. A challenge in the fault detection applications is the design of a scheme which can distinguish between model uncertainties, disturbances and the occurrence of faults. Most of the current approaches use a constant detection threshold. Currently, there are no useful guidelines for constant optimal threshold selection. In the absence of faults, a predetermined constant threshold would lead to more false alarms and missed detections under modeling uncertainties. Hence a technique to accommodate uncertainties and disturbances in the model, help in reducing false alarms and missed detections is essential for the enhancement of aircraft engine operations. In this work, a dynamic threshold algorithm is developed for aircraft engine fault detection and isolation that accommodates parametric uncertainties and disturbances. The algorithm takes the parametric uncertainties into consideration and proposes a dynamic threshold that makes use of the bounds on the parametric uncertainties which can thus distinguish an actual fault from the model uncertainties. First we design Kalman filters or unknown input observers based on the linearized engine model about a given nominal operating point, but the filters or observers use the measurements from the nonlinear engine
model which includes uncertainty description. Using the robustness analysis of parametric uncertain systems, we generate upper-bound and lower-bound time response trajectories of the dynamic threshold. The extent of parametric uncertainties is assumed to be such that the perturbed eigenvalues reside in a set of distinct circular regions. A set of “structured” Kalman filters or unknown input observers are used for engine sensor or actuator fault diagnosis design. The residuals are errors between measured outputs and estimated outputs from a set of Kalman filters or a set of unknown input observers. With the dynamic threshold design approach, the residual crossing the upper bound or lower bound of the dynamic threshold indicates the occurrence of fault. Application to an aircraft turbofan engine model illustrates the performance of the proposed method.
Dedicated to my parents: Zhiyuan and Mingjun
ACKNOWLEDGMENTS

First and foremost, I would like to give my gratitude to my advisor Dr. Rama K. Yedavalli for his guidance and support throughout my graduate study at The Ohio State University. I would like to thank Dr. Jen-Ping Chen and Dr. Mo-how Herman Shen for agreeing to be my dissertation committee members and for their helpful advices. I would also like to thank Dr. Jose B. Cruz, Jr., Dr. Mike Dunn and Dr. Jen-Ping Chen for serving on my candidacy exam committee and for their encouragements.

I would like to acknowledge the financial support and research expertise from Collaborative Center of Control Science, and NASA Glenn. I would also like to thank Dr. John Merrill, Director of the First-Year Engineering Program at OSU, for providing me with Graduate Teaching Assistant funding and the opportunity to teach.

I am grateful to my colleagues – Ms. Hsun-Hsuan Huang, Ms. Nagini Devarakonda, and Mr. Rohit Belapurkar for their discussions, suggestions and encouragements, and for creating an excellent research environment. I am very thankful to my friends and relatives in U.S.A. and in China for their advices and support throughout the entire study.

I deeply appreciate my husband Misha for his love, company and support. Most of all, I would like to dedicate my dissertation to my parents for their endless love, understanding, enormous encouragement and patience. Without their love and care, I would not have had this work done.
VITA

July 16, 1979 ............................ Born - Xian, ShaanXi, China

2001 ................................. B.E. in Automatic Control,
               Beijing University of Aeronautics
               and Astronautics

2005 ................................. M.S. in Aeronautical and Astronautical
               Engineering,
               The Ohio State University

2005-present ......................... Graduate Teaching Associate,
               Graduate Research Associate,
               The Ohio State University

PUBLICATIONS


FIELDS OF STUDY

Major Field: Aeronautical and Astronautical Engineering
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iv</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>v</td>
</tr>
<tr>
<td>Vita</td>
<td>vi</td>
</tr>
<tr>
<td>List of Tables</td>
<td>x</td>
</tr>
<tr>
<td>List of Figures</td>
<td>xi</td>
</tr>
</tbody>
</table>

## Chapters:

1. Introduction .......................... 1
   1.1 Literature Review .................. 1
   1.2 Background and Motivation ........ 5
   1.3 Notation ................................ 9
   1.4 Dissertation Organization ........ 10

2. Preliminaries .......................... 12
   2.1 Mathematical Preliminaries ....... 12
      2.1.1 Vector Norms .................. 13
      2.1.2 Matrix Norms ................. 15
   2.2 Kalman Filter ...................... 17
   2.3 Unknown Input Observer .......... 21
   2.4 Summary ................................ 27
3. Time Response Analysis for Linear Discrete-Time Parametric Uncertain Stochastic Systems .......................................................... 28
   3.1 Problem Formulation .............................................................. 28
      3.1.1 Norm-Bounded Parametric Uncertainties ......................... 36
      3.1.2 Structured Parametric Uncertainties .............................. 37
      3.1.3 Time Response Bounds ............................................... 38
   3.2 An Illustrated Example ...................................................... 40
   3.3 Summary ............................................................................. 43
4. Dynamic Threshold Design ....................................................... 47
   4.1 Dynamic Threshold Design Using Kalman Filter ...................... 47
      4.1.1 Robust Residual Generation Based on Kalman Filter .......... 48
      4.1.2 Dynamic Threshold Design Based on Kalman Filter .......... 48
   4.2 Dynamic Threshold Design Using Unknown Input Observer ........ 53
      4.2.1 Robust Residual Generation Based on UIO ..................... 54
      4.2.2 Dynamic Threshold Design Based on UIO ..................... 56
   4.3 Summary ............................................................................. 58
5. Sensor and Actuator Fault Isolation Schemes ............................... 59
   5.1 Sensor Fault Isolation Schemes Using Kalman Filters .............. 59
      5.1.1 Dedicated Observer Scheme ........................................ 61
      5.1.2 Improved Dedicated Observer Scheme ........................... 64
      5.1.3 Generalized Observer Scheme ..................................... 66
   5.2 Actuator Fault Isolation Scheme Using UIOs .......................... 68
   5.3 Summary ............................................................................. 72
6. Aircraft Engine Sensor and Actuator FDI Application .................... 74
   6.1 Introduction ........................................................................ 74
      6.1.1 Turbofan Engine Overview ....................................... 74
      6.1.2 Aircraft Engine FDI Survey ....................................... 76
   6.2 GE90-115B Turbofan Engine Sensor and Actuator FDI ............. 79
      6.2.1 Engine Model ............................................................ 79
      6.2.2 Engine Sensor FDI Using DOS .................................... 87
      6.2.3 Engine Sensor FDI Using GOS .................................... 111
      6.2.4 Engine Actuator FDI ................................................. 121
   6.3 Summary ............................................................................. 129
7. Conclusions ................................................................. 136
   7.1 Contributions ......................................................... 136
   7.2 Future Work .............................................................. 137

Appendices:

A. List of Nomenclature ...................................................... 139

B. Theorem Proof ............................................................... 142

Bibliography ................................................................. 144
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Nominal Parameters</td>
<td>42</td>
</tr>
<tr>
<td>5.1 Sensor FDI Decision Making Logic for Basic DOS</td>
<td>62</td>
</tr>
<tr>
<td>5.2 Sensor FDI Decision Making Logic for Improved DOS</td>
<td>65</td>
</tr>
<tr>
<td>5.3 Sensor FDI Decision Making Logic for GOS</td>
<td>67</td>
</tr>
<tr>
<td>5.4 Actuator FDI Decision Making Logic</td>
<td>72</td>
</tr>
<tr>
<td>6.1 GE90-115B Engine Model Parameter Notation</td>
<td>78</td>
</tr>
<tr>
<td>6.2 GE90-115B Engine State Variables, Actuators and Sensors</td>
<td>81</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>5</td>
</tr>
<tr>
<td>1.2</td>
<td>7</td>
</tr>
<tr>
<td>1.3</td>
<td>8</td>
</tr>
<tr>
<td>3.1</td>
<td>33</td>
</tr>
<tr>
<td>3.2</td>
<td>43</td>
</tr>
<tr>
<td>3.3</td>
<td>44</td>
</tr>
<tr>
<td>3.4</td>
<td>45</td>
</tr>
<tr>
<td>3.5</td>
<td>45</td>
</tr>
<tr>
<td>3.6</td>
<td>46</td>
</tr>
<tr>
<td>3.7</td>
<td>46</td>
</tr>
<tr>
<td>5.1</td>
<td>60</td>
</tr>
<tr>
<td>5.2</td>
<td>61</td>
</tr>
<tr>
<td>5.3</td>
<td>64</td>
</tr>
<tr>
<td>5.4</td>
<td>71</td>
</tr>
<tr>
<td>6.1</td>
<td>75</td>
</tr>
<tr>
<td>Section</td>
<td>Title</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------------------------------------------------</td>
</tr>
<tr>
<td>6.23</td>
<td>Sensor $z_6$ Fault Diagnostics (Improved DOS)–3</td>
</tr>
<tr>
<td>6.24</td>
<td>Sensor $z_6$ Fault Diagnostics (Improved DOS)–4</td>
</tr>
<tr>
<td>6.25</td>
<td>Sensors $z_4$ and $z_5$ Fault Diagnostics (Improved DOS)–1</td>
</tr>
<tr>
<td>6.26</td>
<td>Sensors $z_4$ and $z_5$ Fault Diagnostics (Improved DOS)–2</td>
</tr>
<tr>
<td>6.27</td>
<td>Sensors $z_4$ and $z_5$ Fault Diagnostics (Improved DOS)–3</td>
</tr>
<tr>
<td>6.28</td>
<td>Sensors $z_4$ and $z_5$ Fault Diagnostics (Improved DOS)–4</td>
</tr>
<tr>
<td>6.29</td>
<td>Aircraft Engine Sensor FDI Scheme Using GOS</td>
</tr>
<tr>
<td>6.30</td>
<td>Sensor $z_1$ Fault Diagnostics (GOS)–1</td>
</tr>
<tr>
<td>6.31</td>
<td>Sensor $z_1$ Fault Diagnostics (GOS)–2</td>
</tr>
<tr>
<td>6.32</td>
<td>Sensor $z_1$ Fault Diagnostics (GOS)–3</td>
</tr>
<tr>
<td>6.33</td>
<td>Sensor $z_1$ Fault Diagnostics (GOS)–4</td>
</tr>
<tr>
<td>6.34</td>
<td>Sensor $z_5$ Fault Diagnostics (GOS)–1</td>
</tr>
<tr>
<td>6.35</td>
<td>Sensor $z_5$ Fault Diagnostics (GOS)–2</td>
</tr>
<tr>
<td>6.36</td>
<td>Sensor $z_5$ Fault Diagnostics (GOS)–3</td>
</tr>
<tr>
<td>6.37</td>
<td>Sensor $z_5$ Fault Diagnostics (GOS)–4</td>
</tr>
<tr>
<td>6.38</td>
<td>Aircraft Engine Actuator FDI Scheme</td>
</tr>
<tr>
<td>6.39</td>
<td>Actuator $u_3$ Fault Diagnostics–1</td>
</tr>
<tr>
<td>6.40</td>
<td>Actuator $u_3$ Fault Diagnostics–2</td>
</tr>
<tr>
<td>6.41</td>
<td>Actuator $u_1$ Fault Diagnostics–1</td>
</tr>
<tr>
<td>6.42</td>
<td>Actuator $u_1$ Fault Diagnostics–2</td>
</tr>
</tbody>
</table>
6.43 Sensor $z_2$ Fault Diagnostics (GOS)–1 .......................... 130
6.44 Sensor $z_2$ Fault Diagnostics (GOS)–2 .......................... 131
6.45 Sensor $z_2$ Fault Diagnostics (GOS)–3 .......................... 132
6.46 Sensor $z_2$ Fault Diagnostics (GOS)–4 .......................... 133
6.47 Actuator $u_3$ Fault Diagnostics1–1 ................................. 134
6.48 Actuator $u_3$ Fault Diagnostics1–2 ................................. 135
CHAPTER 1

INTRODUCTION

1.1 Literature Review

The demand for a safer and more reliable automatic control system has stimulated considerable research on fault detection, isolation and identification (also called accommodation) approaches and technologies over decades. With the development of the complex and large-scale systems, such as aircrafts, automotive vehicles, high-speed railways, power systems and many other applications, the issues of reliability, affordability, safety and system integrity have become significantly important and been addressed in many research fields. A “fault” is defined as a deviation from the normal behavior of the system function [9, 32].

Fault detection is a necessary first step to make a binary decision in any monitored systems; fault isolation is the second step, which provides the information of the location and the cause of the fault; fault identification is to assess the size and type of the fault and how the fault will affect the process. Fault isolation and identification are together called fault diagnosis. Fault detection and fault isolation both have equally important meanings for any practical system. Fault identification, however, may not be necessary if there is no reconfiguration action involved. Therefore, for most practical systems, only fault detection and
isolation stages (referred as FDI system) are considered. FDI system can help avoid system breakdown or failure, and maintain or increase the quality of the monitored systems, if the fault is detected and isolated early enough for actions to take. The approaches to fault detection and isolation may be classified into three categories: model-based methods, knowledge-based methods and signal-processing-based methods. The three categories are not mutually exclusive, but are complementary. The literature review will focus on the model-based methods in this dissertation. Before addressing the model-based techniques, the knowledge-based techniques and signal-processing-based techniques will be briefly introduced below.

The knowledge-based methods are done within the artificial intelligence domain, using expert reasoning, fuzzy reasoning and neural networks, etc. These methods are appealing because they do not require explicit mathematical model of the monitored system. An expert system uses computer to perform a task requiring the reasoning ability of a human expert. Expert systems are highly specialized according to their application areas. Expert systems apply the knowledge using three approaches: database, rule base and inference strategy [1]. Fuzzy-logic FDI method belongs to the rule-based approach where diagnostic rules can be formulated from process structure and system functions [74]. The neural networks are trained such that the relationship between the faults and the causes can be identified and stored in the network [1, 4, 62, 63]. In order to develop the knowledge-based FDI system, the knowledge about the process structure and function of the monitored plants under different faulty conditions is required in advance. The basic knowledge to conduct this approach is training data which contains faults and the corresponding symptoms. The development of a knowledge-based FDI system usually takes considerable time and effort.
to make it effective. A large amount of work has been devoted to develop the knowledge-based method in order to reduce the development time. The signal-processing-based techniques without model application are also used as FDI approaches, which include spectral analysis (time-frequency, time-scale analysis, etc) and statistical methods (signal classification, pattern recognition, etc).

With the development of digital computers and the availability of state variable and transfer function models of many practical systems, model-based FDI methods have received considerable attention. Model-based FDI methods use mathematical models of the monitored systems, the advantage of which is that no additional hardwares are required to realize an FDI system. The idea behind model-based FDI is to use the redundancy information obtained from the physical system measurements combined with a process model. The techniques used for model-based FDI are various, including linear and nonlinear observer, parity equations, parameter estimation, frequency spectral analysis and other algorithms. A large number of books and papers have documented the research of model-based FDI, for example, the books written by Gertler [32], Chen and Patton [9], Patton, Frank and Clark [74], Isermann [39] and Witczak [87]; and survey papers by Willsky [86], Isermann [36, 37, 38], Gertle [31] and Frank [26]. The model-based FDI methods have been studied in many kinds of dynamic systems, for instance: linear systems [42, 47], nonlinear systems [6, 98], stochastic systems [7, 8, 57, 33], time-delay systems [12, 93], hybrid systems [13, 84], periodic systems [97] and multirate sampled-data systems [24, 96, 40], etc. Moreover, the model-based FDI methods have been also applied to various physical system objects either under real operations or in laboratory simulations, such as helicopter rotors [27], aircrafts [65], actuators/sensors [4], automotive vehicles [46], industrial furnace [94], electro-hydraulic cylinders [29], diesel engines [69], induction motors [90, 56],
satellite systems [10], UAVs (Uninhabited Aerial Vehicles) [78], space shuttle main engines [22], rocket engines [99], etc. The three most popular observer-based FDI methods concluded by Isermann in [39] are parameter estimation FDI approach, parity equation FDI approach and observer-based FDI approach. The introduction of the three methods are as follows.

**Parameter estimation FDI approach:** The process parameters of a system are usually partially unknown or not known at all in many practical cases. Isermann explained that the changes of the parameters $\theta$, which are expressed dependent on process coefficients $p_j$ may tell about the process fault [36, 38]. In paper [38], Isermann gave a generalized structure of FDI based on parameter estimation for input-output model and state-space model. The neural network and fuzzy logic algorithms have been recently applied to the parameter estimation FDI applications [3, 62, 43].

**Parity equation FDI approach:** The parity equation method is used to generate the residual for checking input and output signals. In a linear context, the residual generic form is $R(s) = W(s)[Y(s) - M(s)U(s)]$, where $Y(s)$ is the measured output, $M(s)U(s)$ is the estimated output, $W(s)$ is the filter gain. Gertler gave a parity equation in the discrete-time $Z$-domain [32]. Patton and Chen gave a tutorial review on parity space FDI approaches for aerospace systems [73]. Ding, Guo and Jeinsch derived a characterization of parity vectors and a relationship between the order of parity relations and the dimension of the parity space [21]. A new parity space FDI method based on stationary wavelet transform was developed by Ye, Wang and Ding [89].

**Observer-based FDI approach:** The basic idea behind the observer-based (or filter-based in the stochastic case) FDI approach is to obtain the estimates of the measured outputs. This FDI scheme is built on the knowledge of state-space model of the modern control
theory. In most cases, the estimates of the measured outputs are compared with the sensor measurements, that is, the difference between the sensor measurements and the measured output estimates is used to generate a residual signal. Many different observers can be employed based on the characteristics of different systems, for example, Luenberger observers, Kalman filters, nonlinear observers, extended Kalman filters, unknown-input observers, etc [82, 34, 54, 11, 17, 41].

1.2 Background and Motivation

The model-based fault detection based on analytical redundancy comprises two principal steps: residual generation and residual evaluation. Figure 1.1 shows a general FDI process scheme. The purpose of residual generation is to generate a fault indicating signal – residual, using the available input and output information from the monitored plant. The residual signal is supposed to be nonzero in the occurrence of fault and zero when no fault is present. The residual is usually generated by comparing the measured plant output with the mathematical model measured output estimates. There are two main properties in a model-based fault detection algorithm: robustness and sensitivity. Robustness means that the fault detection system does not produce false alarms due to unknown disturbances and modeling errors; while sensitivity means the fault detection scheme should be known

Figure 1.1: Fault Detection and Isolation General Process
as sensitive to faults and not cause missed detections. In general, it is obvious to find out that the two properties are conflicting. The issues of sensitivity and robustness have been addressed by optimizing the residual generator to be sensitive to faults and insensitive to disturbances. Residual generator has been studied in the literature based on observers (or filters) and parity equations, etc.

In an ideal situation, assuming that the monitored plant is modeled correctly and there are no disturbances and the noise is negligible, the residual would be near zero when the plant is fault-free. However, it is not the case in reality. Even if no faults occur, the residual will defer from zero due to unknown disturbances and model uncertainty (unmodeled dynamics, etc). The purpose of the second step – residual evaluation is thus to assess the residual and make a decision on the occurrence of a fault. This is done by comparing the residual signal to a threshold and then to claim the presence of a fault if the residual exceeds the threshold. Hence, it is not sufficient to set the threshold to zero since the existence of model uncertainty and disturbances can cause a false alarm. Except disturbances such as noise, the main source for nonzero residuals is often caused by model uncertainty. Therefore, it is important to design a threshold to accommodate uncertainties in the model that would help in minimizing false alarms and missed detections.

Robust threshold dealing with frequency domain uncertainty is developed by Emami-Naeini, Akhter and Rock [23], Ding and Frank [88], Rank and Niemann [77]. Patton, Frank and Clark [74] showed a threshold selection method using $L_\infty$ norm. The thresholds generated from the above uncertainty description are usually norms of the signals which result in constant thresholds. In current literature, a threshold for aircraft engine FDI is usually predetermined and constant based on empirical data. There are no useful guidelines for constant optimal threshold selections [81, 64, 20]. Simani [81] introduced a simple
threshold detection methodology using a state estimation approach. Lughofer, Efendic, Del Re and Klement [64] used a threshold which is set to 3 or 4 times the accuracy of the corresponding sensor. An empirically trained fault detection threshold method was presented by Depold, Rajamani, Morrison and Pattipati [20]. However, if a fixed threshold in Figure 1.2 is set too high, it has reduced sensitivity to faults; if it is too low, the false alarm rate increases. In case of rapid maneuvers and component degradation, it may happen

![Dynamic Threshold v.s. Constant Threshold Demonstration](image)

Figure 1.2: Dynamic Threshold v.s. Constant Threshold Demonstration

that there is no fixed threshold that allows satisfactory FDI at a tolerable false alarm rate. Kobayashi and Simon also mentioned in [51] that a fixed threshold was not good enough to cover the range of an engine operation, although a constant threshold was still used in their papers [51, 52, 49, 50, 53]. In the absence of faults, a predetermined constant threshold
would lead to more false alarms under modeling uncertainty, which is not tolerant under flight conditions.

In conclusion, this motivates the research of deriving a threshold method in the time-domain, which is a function of time and control activity. The basic idea of a dynamic threshold was elaborated in automobile fault detection applications [75] when we started the research. The notion of dynamic threshold is also discussed by Kobayashi [51], but no details and no algorithms are presented. However, the phrase “adaptive threshold” is used in the above papers [75, 51]. In essence we think that the threshold is dynamic rather than adaptive. So the phrase “dynamic threshold” is used. Preliminary versions of the dynamic threshold approach have been discussed by the author in [92, 58]. A similar idea of dynamic threshold has been expressed by Johansson, Bask and Norlander [44] for a linear, continuous-time interval analysis problem, dealing with multiplicative uncertainties.

In this dissertation the monitored plant is modeled as a linear discrete-time system. In the case of a nonlinear system, a model linearization around an operating point will be applied. The system dynamics shown in Figure 1.3 can be described as:

\[
\begin{align*}
\mathbf{x}_{e,k+1} &= (\Phi_k + \Delta \Phi)\mathbf{x}_{e,k} + (\Psi_k + \Delta \Psi)(u_k + f_{a,k}) + \Gamma_k d_k + w_k + f_{c,k} \\
\mathbf{z}_{e,k+1} &= H_{k+1}\mathbf{x}_{e,k+1} + v_{k+1} + f_{s,k+1}
\end{align*}
\]

Figure 1.3: System Dynamics
where $k = 0, 1, \ldots$, $x_k \in \mathbb{R}^n$ is a state vector, $u_k \in \mathbb{R}^l$ is a control input vector, $d_k \in \mathbb{R}^p$ is a disturbance vector (or unknown input vector), $z_{ek} \in \mathbb{R}^m$ is a measured output vector, $f_{ak} \in \mathbb{R}^l$, $f_{ck} \in \mathbb{R}^n$ and $f_{sk} \in \mathbb{R}^m$ represent actuator fault vector, component fault vector and sensor fault vector, respectively. $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are mutually uncorrelated jointly Gaussian white noise sequences:

$$
\mathcal{E}\{w_i w_j^T\} = Q_i \delta_{ij} \quad (1.2)
$$

$$
\mathcal{E}\{v_i v_j^T\} = R_i \delta_{ij} \quad (1.3)
$$

$$
\mathcal{E}\{w_i v_j^T\} = S = 0, \forall i \text{ and } j \quad (1.4)
$$

Covariance matrix $Q_i$ is positive semidefinite and $R_i$ is positive definite. $\Delta \Phi$ represents system parametric uncertainties and $\Delta \Psi$ is the control input uncertainties. The dimensions of matrices $\Phi$, $\Psi$, $\Gamma$, $H$, $Q$ and $R$ are $n \times n$, $n \times l$, $n \times p$, $m \times n$, $n \times n$, and $m \times m$, respectively.

When applying an observer to the system of the form (1.1) under fault-free condition, i.e. $f_{ak} \equiv 0$, $f_{ck} \equiv 0$ and $f_{sk} \equiv 0$, the state error dynamics (the dynamics of the state estimation error: $x_{ek} - \hat{x}_k$) will also take the form in Equation (1.1). Since the output of the error dynamics is a residual ($z_{ek} - \hat{z}_k = H_k(x_{ek} - \hat{x}_k) + v_k$), it is motivated to search for a tube-shaped time response bound for ($x_{ek} - \hat{x}_k$). This time response bound is a dynamic system with $u_k$, $w_k$ and $v_k$, assuming the noises $w_k$ and $v_k$ are bounded under known values.

### 1.3 Notation

In this section the basic notation will be introduced. A summary of all notation is given in appendix A at the end of the dissertation.
The real numbers will be denoted by $\mathbb{R}$. The space of all $n$-dimensional column vectors with all real elements will be denoted by $\mathbb{R}^n$. The set of all $m \times n$ matrices with all real entries will be denoted by $\mathbb{R}^{m \times n}$. The complex numbers will be denoted by $\mathbb{C}$.

Generally the lower-case Latin or Greek letters will be used for scalars and the bold lower-case Latin or Greek letters will be used for vectors; the bold upper-case Latin or Greek letters will be used for matrices, however, some non-bold upper-case Latin or Greek letters will also be used for scalars. The zero matrix or vector will be written as $0$. The identity matrix will be written as $I$ or $I_n$ when it is necessary to specify the order of the matrix. The transpose of the matrix $X$ will be written as $X^T$. The rank of matrix $X$ is denoted by $\text{rank}(X)$ and its trace is written as $\text{trace}(X)$. The condition number of the matrix $X$ is written as $K(X)$.

We will write vector 2-norm $\|x\|_2$ as $\|x\|$ from Chapter 3, where the subscript 2 will be neglected. The 2-norm is also called Euclidean norm, which is defined as $\|x\|_2 = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$, since in real two or three dimensional space it is the euclidean length of its argument. The induced matrix 2-norm $\|X\|_2$ will be denoted by $\|X\|$ from Chapter 3, where the subscript 2 will also be neglected. The matrix 2-norm is induced by the euclidean vector norm, which is the largest singular value $\sigma_{\max}(X)$ of matrix $X$. The properties of other vector norms or matrix norms are detailed in Chapter 2.

1.4 Dissertation Organization

The dissertation is organized as follows: Chapter 2 reviews the definitions of vector norms and matrix norms in matrix theory and linear algebra, and outlines the concept of estimation and principles for Kalman filter and unknown input observer of the linear discrete-time stochastic system. Chapter 3 presents the idea of time response analysis for
linear parametric uncertain system, and details the derivation of the time response bound for a linear discrete-time parametric uncertain stochastic system. The upper-and-lower bound of the time response for the discrete-time system is described. Chapter 4 applies the time response bound technique to develop the Kalman filter based and unknown input observer based dynamic threshold algorithms. Chapter 5 describes the fault isolation schemes for sensors and actuators, respectively. In Chapter 6, the dynamic threshold method is applied to the sensor and actuator fault diagnostics of GE90-115B high bypass two-spool turbofan aircraft engine. Chapter 7 summarizes the major contributions of the dissertation and provides an outline of future research directions.
CHAPTER 2

PRELIMINARIES

In this chapter, a brief review of mathematical preliminaries (Matrix Theory and Linear Algebra) needed for robust stability analysis will be presented. In addition, concept of estimation theory and algorithms for Kalman filter and unknown input observer (UIO) will be presented to improve the understanding of later part of the dissertation. Vector norm and matrix norm definitions and formulae are addressed in the book by Meyer [70], and the use of Kalman filter and UIO as estimation tools is well discussed in the books by Mendel [68], Chen and Patton [9].

2.1 Mathematical Preliminaries

In this section, the norms of a vector $\mathbf{x}$, where

$$\mathbf{x} = [x_1, x_2, \cdots, x_n]$$  \hspace{1cm} (2.1)

and the norms of a matrix $A$, where

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$  \hspace{1cm} (2.2)

will be reviewed. We are primarily concerned with the vector spaces $\mathbb{R}^n$ or $\mathbb{R}^{m \times n}$. Vector spaces $\mathbb{C}^n$ and $\mathbb{C}^{m \times n}$ in the complex domain will not be covered in the dissertation.
2.1.1 Vector Norms

Definition: A norm of a vector $x$ is a nonnegative number denoted by $\|x\|$, associated with $x$, satisfying the following conditions:

(a) $\|x\| > 0$ for $x \neq 0$, and $\|x\| = 0$ when $x = 0$.
(b) $\|\alpha x\| = |\alpha| \|x\|$ for any scalar $\alpha$.
(c) $\|x + y\| \leq \|x\| + \|y\|$ (the triangle inequality).

Euclidean Vector Norm:
For a vector $x \in \mathbb{R}^{n \times 1}$, the euclidean norm of $x$ is defined to be

$$\|x\| = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} = \sqrt{x^T x}. \quad (2.3)$$

The euclidean vector norm computes the length of a vector in the vector space $\mathbb{R}^n$.

Standard Inner Product:
The scalar term defined by

$$x^T y = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}. \quad (2.4)$$

is called the standard inner product for $\mathbb{R}^n$.

Cauchy-Schwarz Inequality:
For $x, y \in \mathbb{R}^{n \times 1}$,

$$|x^T y| \leq \|x\| \|y\|. \quad (2.5)$$

Equality holds if and only if $y = \alpha x$ for $\alpha = \frac{x^T y}{x^T x}$.

Backward Triangle Inequality:
The triangle inequality $\|x + y\| \leq \|x\| + \|y\|$ shown in the definition of vector norm produces an upper bound for a sum, but it also yields the following lower bound for a difference:

$$\|x - y\| \geq \||x| - |y||. \quad (2.6)$$
\(p\)-Norms:

For \(p \geq 1\), the \(p\)-norm of \(\mathbf{x} \in \mathbb{R}^{n \times 1}\) is defined as

\[
\|\mathbf{x}\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.
\] (2.7)

In practice, only three of the \(p\)-norms are used, and they are

- grid norm/vector 1-norm:
  \[
  \|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|.
  \] (2.8)

- euclidean norm/vector 2-norm:
  \[
  \|\mathbf{x}\|_2 = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}.
  \] (2.9)

- max norm/vector \(\infty\)-norm:
  \[
  \|\mathbf{x}\|_{\infty} = \lim_{p \to \infty} \|\mathbf{x}\|_p = \lim_{p \to \infty} \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = \max_i |x_i|.
  \] (2.10)

For the above three \(p\)-norms, we have the following properties:

\[
\|\mathbf{x}\|_{\infty} \leq \|\mathbf{x}\|_1 \leq n \|\mathbf{x}\|_{\infty} \tag{2.11}
\]

\[
\|\mathbf{x}\|_{\infty} \leq \|\mathbf{x}\|_2 \leq \sqrt{n} \|\mathbf{x}\|_{\infty} \tag{2.12}
\]

Also we inspect that \(\|\mathbf{x}\|_1 \leq \sqrt{n} \|\mathbf{x}\|_2\) and \(\|\mathbf{x}\|_2^2 \leq \|\mathbf{x}\|_1^2\). Hence we can derive

\[
\frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_{\infty} \leq \|\mathbf{x}\|_2 \tag{2.13}
\]

\[
\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n} \|\mathbf{x}\|_2 \tag{2.14}
\]

\[
\frac{1}{n} \|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_{\infty} \leq \|\mathbf{x}\|_1 \tag{2.15}
\]

Vector norms are basic tools for defining and analyzing limiting behavior in vector space. If the vector space is finite-dimensional, all norms are equivalent. Equivalent norms define the same notions of continuity and convergence, and for many purposes do not need to be distinguished.
2.1.2 Matrix Norms

**Definition:** A matrix norm is a function $\| \cdot \|$ from the set of all matrices (of all finite orders) into $\mathbb{R}$ that satisfies the following conditions:

(a) $\|A\| \geq 0$ and $\|A\| = 0 \iff A = 0$.

(b) $\|\alpha A\| = |\alpha| \|A\|$ for any scalar $\alpha$.

(c) $\|A + B\| \leq \|A\| + \|B\|$ for all matrices of the same size.

(d) $\|AB\| \leq \|A\| \|B\|$ for all conformable matrices.

**Frobenius Matrix Norm:**

The Frobenius norm of $A \in \mathbb{R}^{m \times n}$ is defined as

$$\|A\|_F^2 = \sum_{i,j} |a_{ij}|^2 = \sum_i \|A_{i,:}\|_2^2 = \sum_j \|A_{:j}\|_2^2 = \text{trace}(A^T A). \quad (2.16)$$

**Induced Matrix Norm:**

A vector norm that is defined on $\mathbb{R}^p$ for $p = m, n$ induces a matrix norm on $\mathbb{R}^{m \times n}$ by setting

$$\|A\| = \max_{\|x\|=1} \|Ax\| \text{ for } A \in \mathbb{R}^{m \times n}, \ x \in \mathbb{R}^{n \times 1}. \quad (2.17)$$

- An induced matrix norm is compatible with its underlying vector norm in the sense that
  $$\|Ax\| \leq \|A\| \|x\|. \quad (2.18)$$

- When $A$ is nonsingular,
  $$\min_{\|x\|=1} \|Ax\| = \frac{1}{\|A^{-1}\|}. \quad (2.19)$$

**Matrix 2-Norm:**

- The matrix norm induced by the euclidean vector norm is
  $$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sqrt{\lambda_{\text{max}}}. \quad (2.20)$$
where $\lambda_{\text{max}}$ is the largest eigenvalue of matrix $A^T A$ such that $(A^T A - \lambda I)$ is singular.

- When $A$ is nonsingular,

\[
\|A^{-1}\|_2 = \frac{1}{\min \|Ax\|_2} = \frac{1}{\sqrt{\lambda_{\text{min}}}}. \tag{2.21}
\]

where $\lambda_{\text{min}}$ is the smallest eigenvalue of matrix $A^T A$ such that $(A^T A - \lambda I)$ is singular.

**Note:** $(\lambda_{\text{max}})^{\frac{1}{2}} = \sigma_1$ and $(\lambda_{\text{min}})^{\frac{1}{2}} = \sigma_n$ are the largest and smallest singular values of $A$.

**Properties of Matrix 2-Norm:**

In addition to the properties shared by all the induced norms, the matrix 2-norm has the following special properties:

\[
\|A\|_2 = \max_{\|x\|_2=1} \max_{\|y\|_2=1} |y^T Ax|. \tag{2.22}
\]

\[
\|A\|_2 = \|A^T\|_2. \tag{2.23}
\]

\[
\|A^T A\|_2 = \|A\|_2^2. \tag{2.24}
\]

\[
\left\| \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \right\|_2 = \max\{\|A\|_2, \|B\|_2\}. \tag{2.25}
\]

\[
\|U^T AV\|_2 = \|A\|_2 \text{ when } UU^T = I_m \text{ and } V^TV = I_n. \tag{2.26}
\]

**Matrix 1-Norm and Matrix $\infty$-Norm:**

The matrix norms induced by vector 1-norm and $\infty$-norm are defined below:

\[
\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 = \max_j \sum_i |a_{ij}| \tag{2.27}
\]

\[
= \text{ the largest absolute column sum.}
\]

\[
\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}| \tag{2.28}
\]

\[
= \text{ the largest absolute row sum.}
\]

16
Since we can compare vector norms in Equations (2.11)-(2.15), we can easily deduce comparisons for operator norms for matrix $A_{m \times n}$ using Equations (2.11)-(2.15), we find that:

$$\|Ax\|_1 \leq m \|Ax\|_\infty \leq m \|A\|_\infty \|x\|_\infty \leq m \|A\|_\infty \|x\|_1$$  \hspace{1cm} (2.29)

so that $\|A\|_1 \leq m \|A\|_\infty$. By similar argument we have:

$$\frac{1}{\sqrt{m}} \|A\|_2 \leq \|A\|_\infty \leq \sqrt{n} \|A\|_2$$  \hspace{1cm} (2.30)

$$\frac{1}{\sqrt{n}} \|A\|_2 \leq \|A\|_1 \leq \sqrt{m} \|A\|_2$$  \hspace{1cm} (2.31)

$$\frac{1}{n} \|A\|_\infty \leq \|A\|_1 \leq m \|A\|_\infty$$  \hspace{1cm} (2.32)

We observe that in general $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$ and $\|A\|_F$ are not equal. But for all $n \times n$ matrices, the above four matrix norms are in the same range. However, $\|A\|_2$ is more “natural” by virtue of being induced by the euclidean vector norm, which defines the length of a vector.

### 2.2 Kalman Filter

Estimation is a modeling problem. The techniques of estimation include parameter or state estimation or the combination of the two, which are applied to either linear or nonlinear systems. In this dissertation, we will deal with discrete-time systems because in many recent works of engineering analysis and design, a lot of data is collected in a digitized manner, and the mathematics associated with discrete-time estimation theory is simpler than that associated with continuous-time estimation theory. If a plant is modeled in the continuous time, discretization of the model will be applied at the front of the problem.

Kalman filter is a recursive mean-squared state filter developed by Kalman around 1959. It is a time-varying digital filter that uses information from both the state and measurement
equations. Consider a basic linear, (possibly) time-varying, discrete-time, stochastic state-variable model in Equation (2.33):

\[
\begin{align*}
\mathbf{x}_{k+1} &= \Phi_k \mathbf{x}_k + \Psi_k \mathbf{u}_k + \mathbf{w}_k \\
\mathbf{z}_{k+1} &= H_{k+1} \mathbf{x}_{k+1} + \mathbf{v}_{k+1}
\end{align*}
\]  

(2.33)

where \( k = 0, 1, \ldots \cdot \) \( \mathbf{x}_k \in \mathbb{R}^n \) is a state vector, \( \mathbf{u}_k \in \mathbb{R}^l \) is a control input vector, \( \mathbf{z}_k \in \mathbb{R}^m \) is a measured output vector, \( \mathbf{w}_k \in \mathbb{R}^n \) and \( \mathbf{v}_k \in \mathbb{R}^m \) are mutually uncorrelated jointly Gaussian white noise sequences represented in Equations (1.2) – (1.3). The initial state vector \( \mathbf{x}(0) \) is multivariate Gaussian, with mean \( \mathbf{m}_x(0) \) and covariance \( \mathbf{P}_{x}\). The dimensions of matrices \( \Phi, \Psi, H, Q \) and \( R \) are \( n \times n, n \times l, m \times n, n \times n, \) and \( m \times m \), respectively.

A recursive linear Kalman filter is composed of two steps: \textit{predicted state estimates} \( \hat{\mathbf{x}}_{k+1|k} \) and \textit{filtered state estimates} \( \hat{\mathbf{x}}_{k+1|k+1} \). The stages of prediction and filtering are described below.

**Prediction:** A single-stage predicted estimate of \( \mathbf{x}_{k+1} \) is denoted by \( \hat{\mathbf{x}}_{k+1|k} \). It is the mean-squared estimate of \( \mathbf{x}_{k+1} \) which uses all the measurements up to and including the one made at time \( t_k \). Based on the fundamental theorem of estimation, we have

\[
\hat{\mathbf{x}}_{k+1|k} = \mathcal{E}\{ \mathbf{x}_{k+1} | \mathbf{Z}_k \}
\]  

(2.34)

By derivation, we obtain

\[
\hat{\mathbf{x}}_{k+1|k} = \Phi_k \hat{\mathbf{x}}_{k|k} + \Psi_k \mathbf{u}_k
\]  

(2.35)

\[
\hat{\mathbf{z}}_{k+1|k} = H_{k+1} \hat{\mathbf{x}}_{k+1|k}
\]  

(2.36)

where \( k = 0, 1, \ldots \cdot \)

**Filtering:** The recursive filter is written in a predictor-corrector form:

\[
\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + K_{k+1} \hat{\mathbf{z}}_{k+1|k}
\]  

(2.37)
where \( k = 0, 1, \cdots, \hat{x}_{0|0} = \mathcal{E}\{x_0|\text{no measurement}\} = m_{x_0}, \) \( K_{k+1} \) is an \( n \times m \) Kalman gain matrix or weighting matrix, and \( \tilde{z}_{k+1|k} \) is the innovation process

\[
\tilde{z}_{k+1|k} = z_{k+1} - H_{k+1} \hat{x}_{k+1|k}
\] (2.38)

The filtered state estimate of \( x_k \) is obtained by a predictor step, \( \hat{x}_{k+1|k} \), and a corrector step, \( K_{k+1} \tilde{z}_{k+1|k} \). The following result provides the recursive algorithm for evaluating \( \hat{x}_{k+1|k+1} \).

The Kalman gain matrix is specified by

\[
K_{k+1} = P_{k+1|k} H_{k+1}^T (H_{k+1} P_{k+1|k} H_{k+1}^T + R_{k+1})^{-1}
\] (2.39)

\[
P_{k+1|k} = \Phi_k P_k |k \Phi_k^T + Q_k
\] (2.40)

\[
P_{k+1|k+1} = [I_n - K_{k+1} H_{k+1}] P_{k+1|k}
\] (2.41)

where \( P_{k+1|k} \) and \( P_{k+1|k+1} \) are state prediction-error covariance matrix and state filtering-error covariance matrix associated with \( \hat{x}_{k+1|k} \) and \( \hat{x}_{k+1|k+1} \), respectively. \( I_n \) is the \( n \times n \) identity matrix, and \( P_{0|0} = \mathcal{E}\{(x_0 - \hat{x}_{0|0})(x_0 - \hat{x}_{0|0})^T\} = P_{x_0} \). Hence, the linear recursive algorithm for the Kalman filter is summarized by Equations (2.35) – (2.41).

In order to evaluate the estimation results, we usually tend to link the estimated states with the actual states. The difference between the actual states and the estimated states is called “error dynamics”. For a Kalman filter, it is more straight-forward and convenient to relate the filtered estimate of \( x_{k+1}, \hat{x}_{k+1|k+1} \), to the filtered estimate of \( x_k, \hat{x}_{k|k} \), such that the state-filtering error dynamics defined as

\[
e_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}
\] (2.42)

can be easily derived. Equation (2.37) represents a time-varying recursive digital filter once the gain matrix \( K_{k+1} \) is calculated. Substituting Equations (2.35) and (2.38) into (2.37).
The resulting equation, also called recursive filter form, can be written as

\[
\hat{x}_{k+1|k+1} = \Phi_k \hat{x}_{k|k} + \Psi_k u_k + K_{k+1}(H_{k+1} x_{k+1} + v_{k+1} - H_{k+1} \hat{x}_{k+1|k})
\]

\[
= \Phi_k \hat{x}_{k|k} + \Psi_k u_k + K_{k+1} [H_{k+1} x_{k+1} + v_{k+1} - H_{k+1} \hat{x}_{k|k}]
\]

\[
= (I_n - K_{k+1} H_{k+1}) \Phi_k \hat{x}_{k|k} + (I_n - K_{k+1} H_{k+1}) \Psi_k u_k
\]

\[
+ K_{k+1} H_{k+1} x_{k+1} + K_{k+1} v_{k+1}
\]

(2.43)

Then after substituting Equation (2.43) into (2.42), the resulting equation of state-filtering error dynamics is written as

\[
e_{k+1|k+1} = x_{k+1} - (I_n - K_{k+1} H_{k+1}) \Phi_k \hat{x}_{k|k} - (I_n - K_{k+1} H_{k+1}) \Psi_k u_k
\]

\[
- K_{k+1} H_{k+1} x_{k+1} - K_{k+1} v_{k+1}
\]

\[
= (I_n - K_{k+1} H_{k+1}) x_{k+1} - (I_n - K_{k+1} H_{k+1}) \Phi_k \hat{x}_{k|k}
\]

\[
-(I_n - K_{k+1} H_{k+1}) \Psi_k u_k - K_{k+1} v_{k+1}
\]

\[
= (I_n - K_{k+1} H_{k+1}) \Phi_k x_k + (I_n - K_{k+1} H_{k+1}) \Psi_k u_k
\]

\[
+ (I_n - K_{k+1} H_{k+1}) w_k - (I_n - K_{k+1} H_{k+1}) \Phi_k \hat{x}_{k|k}
\]

\[
-(I_n - K_{k+1} H_{k+1}) \Psi_k u_k - K_{k+1} v_{k+1}
\]

(2.44)

The state-filtering error dynamics can be represented in the following form by grouping the noises in Equation (2.44) together:

\[
e_{k+1|k+1} = (I_n - K_{k+1} H_{k+1}) \Phi_k e_{k|k}
\]

\[
+ [I_n - K_{k+1} H_{k+1} - K_{k+1}] \begin{bmatrix} w_k \\ v_{k+1} \end{bmatrix}
\]

(2.45)
So we can see Equation (2.45) is a state equation for state vector $e_{k|k}$, of which plant matrix is $(I_n - K_{k+1}H_{k+1})\Phi_k$.

Using substitutions similar to those used in the derivation of Equations (2.43) and (2.45), we can also obtain the following recursive predictor form and state-prediction error dynamics of the Kalman filter, they are

\[
\hat{x}_{k+1|k} = \Phi_k[\hat{x}_{k|k-1} + K_k(H_kx_k + v_k - H_k\hat{x}_{k|k-1})] + \Psi_k u_k
\]

and

\[
e_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k}
\]

\[
= \Phi_kx_k + \Psi_k u_k + w_k - \Phi_k(I_n - K_kH_k)\hat{x}_{k|k-1} - \Phi_kK_kH_kx_k - \Phi_kK_kv_k + \Psi_k u_k
\]

\[
= \Phi_k(I_n - K_kH_k)e_{k|k-1} + w_k - \Phi_kK_kv_k
\]

\[
= \Phi_k(I_n - K_kH_k)e_{k|k-1} + \begin{bmatrix} I_n & -\Phi_kK_k \end{bmatrix} \begin{bmatrix} w_k \\ v_k \end{bmatrix}
\]

Equation (2.47) is a state equation for state vector $e_{k|k-1}$, of which plant matrix is $\Phi_k(I_n - K_kH_k)$.

### 2.3 Unknown Input Observer

Observers for systems with unknown inputs play an essential role in robust model-based fault detection [7, 8, 9, 87]. When the system under consideration is subject to unknown inputs (disturbances), their effect has to be decoupled from the state estimation error. Thus a residual can also be decoupled from the disturbances to avoid false alarms. The system uncertainty is modeled as an additive unknown disturbance term in the linear discrete-time
stochastic description of the system below:

\[
\begin{align*}
    x_{k+1} &= \Phi_k x_k + \Psi_k u_k + \Gamma_k d_k + w_k \\
    z_{k+1} &= H_{k+1} x_{k+1} + v_{k+1}
\end{align*}
\]  

(2.48)

where \( x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^l, d_k \in \mathbb{R}^p \) and \( z_k \in \mathbb{R}^m \) represent state variable vector, control command input vector (or known inputs), disturbance vector (or unknown inputs), and measured output vector. \( w_k \in \mathbb{R}^n, v_k \in \mathbb{R}^m \) are independent zero-mean white noise sequences with covariance matrices \( Q_k \) and \( R_k \). \( \Phi_k, \Psi_k, \Gamma_k \) and \( H_k \) are known matrices with appropriate dimensions.

Design an optimal observer to estimate the state of the stochastic system (2.48). The structure of the optimal observer is:

\[
\begin{align*}
    q_{k+1} &= F_{k+1} q_k + T_{k+1} \Psi_k u_k + K_{k+1} z_k \\
    \hat{x}_{k+1} &= q_{k+1} + N_{k+1} z_{k+1}
\end{align*}
\]  

(2.49)

where the matrices \( F_{k+1}, T_{k+1}, K_{k+1}, \) and \( N_{k+1} \) are to be designed to achieve disturbance decoupling minimum variance estimation. The state error dynamics is defined as:

\[
e_{k+1} = x_{k+1} - \hat{x}_{k+1}
\]  

(2.50)

Hence, we can derive the estimation error by substituting the Equation (2.48) and (2.49) into Equation (2.50), such as

\[
e_{k+1} = x_{k+1} - \hat{x}_{k+1} \\
  = x_{k+1} - (q_{k+1} + N_{k+1} z_{k+1}) \\
  = x_{k+1} - q_{k+1} - N_{k+1} (H_{k+1} x_{k+1} + v_{k+1}) \\
  = (I - N_{k+1} H_{k+1}) x_{k+1} - q_{k+1} - N_{k+1} v_{k+1} \\
  = (I - N_{k+1} H_{k+1}) x_{k+1} - N_{k+1} v_{k+1} - [F_{k+1} q_k + T_{k+1} \Psi_k u_k + (K^1_{k+1} + K^2_{k+1}) z_k]
\]
\[
(I - N_{k+1} H_{k+1}) x_{k+1} - N_{k+1} v_{k+1} - T_{k+1} \Psi_k u_k \\
-F_{k+1} (x_k - e_k - N_k z_k) - K_{k+1}^1 (H_k x_k + v_k) - K_{k+1}^2 z_k
\]

\[
F_{k+1} e_k - K_{k+1}^1 v_k - N_{k+1} v_{k+1} + (I - N_{k+1} H_{k+1}) w_k \\
-[F_{k+1} - (I - N_{k+1} H_{k+1}) \Phi_k + K_{k+1}^1 H_k] x_k \\
+(I - N_{k+1} H_{k+1}) \Gamma_k d_k - (K_{k+1}^2 - F_{k+1} N_k) z_k \\
-[T_{k+1} - (I - N_{k+1} H_{k+1})] \Psi_k u_k
\]

(2.51)

where \( K_{k+1} = K_{k+1}^1 + K_{k+1}^2 \). If one can make the following relations hold:

\[
\Gamma_k = N_{k+1} H_{k+1} \Gamma_k 
\]

(2.52)

\[
T_{k+1} = I - N_{k+1} H_{k+1} 
\]

(2.53)

\[
F_{k+1} = \Phi_k - N_{k+1} H_{k+1} \Phi_k - K_{k+1}^1 H_k 
\]

(2.54)

\[
K_{k+1}^2 = F_{k+1} N_k 
\]

(2.55)

the error dynamics would be:

\[
e_{k+1} = F_{k+1} e_k - K_{k+1}^1 v_k - N_{k+1} v_{k+1} + T_{k+1} w_k
\]

(2.56)

Loosely speaking, if the matrix \( F_{k+1} \) is stable, \( \mathcal{E}\{e_k\} \rightarrow 0 \), then \( \mathcal{E}\{\hat{x}_k\} \rightarrow \mathcal{E}\{x_k\} \). That means the state estimation will approach the real state asymptotically. Based on Equation (2.56), we can find that the unknown disturbance vector has been decoupled when Equations (2.52) – (2.55) hold. First we need to choose \( N_{k+1} \), then choose \( K_{k+1} \) to stabilize \( F_{k+1} \).

**Note:** The necessary and sufficient condition for the existence of a solution of Equation (2.52) is

\[
\text{rank}(H_{k+1} \Gamma_k) = \text{rank}(\Gamma_k)
\]

(2.57)
This is the only condition achieving disturbance (unknown input) decoupling. That is to say, the maximum number of disturbances which can be decoupled cannot be larger than the number of independent measurements.

The algorithm to determine the Equations (2.52)-(2.55) provided in [9] is summarized below:

- To satisfy the condition in Equation (2.57), the general solution for Equation (2.52) can be constructed as

\[ N_{k+1} = \Gamma_k (H_{k+1} \Gamma_k)^+ \]  \hspace{1cm} (2.58)

where the left inverse is defined as

\[ (H_{k+1} \Gamma_k)^+ = [(H_{k+1} \Gamma_k)^T (H_{k+1} \Gamma_k)]^{-1} (H_{k+1} \Gamma_k)^T \]  \hspace{1cm} (2.59)

- The stability of the observer is dependent on the matrix \( F_{k+1} \),

\[ F_{k+1} = \Phi_{k+1}^1 - K_{k+1}^1 H_k \]  \hspace{1cm} (2.60)

where

\[ \Phi_{k+1}^1 = \Phi_k - N_{k+1} H_{k+1} \Phi_k \]  \hspace{1cm} (2.61)

\( K_{k+1}^1 \) is designed to stabilize the observer.

- The error covariance matrix \( P_k \), which measures how good the estimate \( \hat{x}_k \) is, defined as

\[ P_k = \mathcal{E}\{[x_k - \hat{x}_k][x_k - \hat{x}_k]^T]\} \]  \hspace{1cm} (2.62)

i.e.,

\[ P_{k+1} = \mathcal{E}\{e_{k+1}e_{k+1}^T\} \]  \hspace{1cm} (2.63)
Then

\[
P_{k+1} = (\Phi_{k+1} - K_{k+1} H_k) P_k (\Phi_{k+1} - K_{k+1} H_k)^T + K_{k+1} R_k (K_{k+1})^T + T_{k+1} Q_k T_{k+1}^T + N_{k+1} R_{k+1} N_{k+1}^T
\]

So \( P_{k+1} \) is controlled by \( K_{k+1} \).

- Design \( K_{k+1} \).

To make the state estimation error \( e_{k+1} \) have the minimum variance, the matrix \( K_{k+1} \) should be determined by:

\[
K_{k+1} = \Phi_{k+1} P_k H_k^T [H_k P_k H_k^T + R_k]^{-1}
\]  

(2.65)

Proof:

\[
P_{k+1} = \Phi_{k+1} P_k (\Phi_{k+1})^T + T_{k+1} Q_k T_{k+1}^T + N_{k+1} R_{k+1} N_{k+1}^T
\]

\[
- K_{k+1} H_k P_k (\Phi_{k+1})^T - \Phi_{k+1} P_k H_k^T (K_{k+1})^T
\]

\[
+ K_{k+1} (H_k P_k H_k^T + R_k) (K_{k+1})^T
\]

As \( R_k \) is a positive definite matrix, \( H_k P_k H_k^T + R_k \) is also positive definite, and there exists an invertible matrix \( S \), such that

\[
SS^T = H_k P_k H_k^T + R_k
\]

Let \( D = \Phi_{k+1} P_k H_k^T (S^T)^{-1} \), then

\[
P_{k+1} = \Phi_{k+1} P_k (\Phi_{k+1})^T + T_{k+1} Q_k T_{k+1}^T + N_{k+1} R_{k+1} N_{k+1}^T
\]

\[
+ [K_{k+1} S - D] [K_{k+1} S - D]^T - D D^T
\]

To minimize \( \text{var}(e_{k+1}) = \text{trace}(P_{k+1}) \), one should make \( K_{k+1} S - D = 0 \), then we have

\[
K_{k+1} = \Phi_{k+1} P_k H_k^T [H_k P_k H_k^T + R_k]^{-1}
\]
and

\[ P_{k+1} = \Phi_{k+1}^1 P_k (\Phi_{k+1}^1)^T + T_{k+1} Q_k T_{k+1}^T + N_{k+1} R_{k+1} N_{k+1}^T + K_{k+1}^1 H_k P_k (\Phi_{k+1}^1)^T. \]  

The optimal filtering algorithm presented above [9] is equivalent to a standard Kalman filter for systems (2.33) without unknown disturbances, by setting the matrices \( N_{k+1} = 0 \) and \( T_{k+1} = I_n \) when there is no disturbances for \( \Gamma_k = 0 \).

Thus, the optimal disturbance decoupling observer design procedure is summarized as follows:

1. Set initial values:

\[ P_0 = P(0), \ q_0 = x_0 - \Gamma_0 (H_0 \Gamma_0)^+ z_0, \ N_0 = 0 \text{ and } k = 0. \]

2. Compute \( k + 1 \) using Equation (2.58):

\[ N_{k+1} = \Gamma_k (H_{k+1} \Gamma_k)^+. \]

where \( (H_{k+1} \Gamma_k)^+ = [(H_{k+1} \Gamma_k)^T (H_{k+1} \Gamma_k)]^{-1} (H_{k+1} \Gamma_k)^T. \)

3. Compute \( K_{k+1}^1 \) using Equation (2.65):

\[ K_{k+1}^1 = \Phi_{k+1}^1 P_k H_k^T (H_k P_k H_k^T + R_k)^{-1}. \]

where \( \Phi_{k+1}^1 = \Phi_k - N_{k+1} H_{k+1} \Phi_k. \)

4. Compute \( T_{k+1}, F_{k+1}, K_{k+1}^2, \) and \( K_{k+1} \) using Equations (2.52), (2.53), (2.54) and (2.55):

\[ T_{k+1} = I - N_{k+1} H_{k+1}, \]

\[ F_{k+1} = \Phi_k - N_{k+1} H_{k+1} \Phi_k - K_{k+1}^1 H_k, \]

\[ K_{k+1}^2 = F_{k+1} N_k, \]

\[ K_{k+1} = K_{k+1}^1 + K_{k+1}^2. \]
5. Compute the state estimation $\hat{x}_{k+1}$ and $q_{k+1}$ using Equation (2.49):

$$q_{k+1} = F_{k+1} q_k + T_{k+1} \Psi_k u_k + K_{k+1} z_k,$$

$$\hat{x}_{k+1} = q_{k+1} + N_{k+1} z_{k+1}.$$

6. Compute $P_{k+1}$ using Equation (2.66):

$$P_{k+1} = \Phi_{k+1}^T P_k (\Phi_{k+1})^T + T_{k+1} Q_{k+1} T_{k+1}^T + N_{k+1} R_{k+1} N_{k+1}^T - K_{k+1}^1 H_k P_k (\Phi_{k+1})^T.$$

7. set $k = k + 1$, go to step 2.

### 2.4 Summary

The purpose of this chapter has been the review of definitions and concepts of vector and matrix norms in matrix theory and linear algebra, introduction to algorithms of Kalman filter and unknown input observer. Euclidean vector norms (vector 2-norms) and induced matrix 2-norms will be applied in the robust stability analysis in Chapter 3. The error dynamics of Kalman filter in both state-filtering form and state-prediction form are derived in this chapter. The main advantage of representing the error dynamics in either of the two forms is that the error dynamics system can be viewed as a state equation of the error dynamics state vector, whose plant matrix involves with the original system matrix and the Kalman filter gain. The state-prediction form of the error dynamics will be adopted in Chapter 4 to develop the residual and the dynamic threshold when Kalman filters are used as observers in design. A full-order unknown input observer structure for linear discrete-time stochastic system has been described, and the existence conditions and design procedures have been introduced. The unknown input observer will be used for actuator FDI in Chapter 6.
3.1 Problem Formulation

In the area of uncertain system theory, time response bounds are derived that result in a tube-shaped time response around the nominal trajectory of a scalar component of the state vector for linear continuous-time parametric uncertain systems such as those discussed by Yedavalli and Ashokkumar [91]. Time response analysis of state variables is a standard means to study the transient and steady-state time-domain performance of a system. Parametric uncertainties introduce perturbations in the eigenvalues and eigenvectors (eigenstructure) of the nominal system, which alter the characteristics of the perturbed system. The bounds on eigenvectors and parametric uncertainties need to be established for the perturbed eigenvalues to reside in a set of distinct circular regions. The perturbed eigenvalue and eigenvector bounds greatly affect the upper-bound and lower-bound on the perturbed state variable trajectories. The upper-bound and lower-bound of one state variable trajectory generate a tube-shaped region around the nominal time response of the corresponding state variable. The time response within the tube-shaped region guarantees the performance
of the parametric uncertain system. The methodology and concept of generating a tube-shaped region for a linear continuous-time parametric uncertain system developed in [91] is extended to the case of a linear discrete-time parametric uncertain stochastic system.

**Definition** [91]: Let \( \mathcal{D}(\lambda_i, R_\theta) \) be the disjoint neighborhoods for the nominal eigenvalues \( \lambda_i \) with radius \( R_\theta \), the system of matrices \( (\Phi + \Delta \Phi) \) is said to be \( \mathcal{D}(\lambda_i, R_\theta) \)–stable if an eigenvalue \( \lambda_i \in \text{sp}(\Phi) \) and \( \mu_i \in \text{sp}(\Phi + \Delta \Phi) \) confine to the disjoint circular domains \( \mathcal{D}(\lambda_i, R_\theta) \in \Omega \) such that \( |\mu_i - \lambda_i| < R_\theta \) holds \( \forall i = 1, ..., n \).

**Assumption 1** [91]: Because the perturbed eigenvalues \( \mu_i \) and \( \mu_j \) can have themselves repeated in a non-empty region \( \mathcal{D}(\lambda_i, R_i + \epsilon) \cap \mathcal{D}(\lambda_j, R_j) \) for some \( \epsilon > 0 \). Assume distinct disjoint circular regions \( \mathcal{D}(\lambda_i, R_\theta) \) in the set \( \Omega \) with radius \( R_\theta \leq R_i, \forall i = 1, ..., n \). Thus the geometric multiplicities of the nominal and the perturbed systems are preserved under parametric uncertainties.

**Assumption 2** [91]: All the distinct neighborhoods \( \mathcal{D}(\lambda_i, R_\theta) \) are assumed to lie in the unit circle of the \( \mathcal{Z} \)-plane, since time response bounds are meaningful for a stable uncertain system.

Assume a nominal linear discrete-time stochastic system model is of the form,

\[
x_{k+1} = \Phi x_k + \Psi u_k + w_k
\]  

(3.1)

where \( x(0) = x_0, x_k \in \mathbb{R}^n \) and \( u_k \in \mathbb{R}^l \) represent state variables and control command inputs. \( w_k \in \mathbb{R}^n \) is a zero mean white noise sequence with covariance matrix \( Q_k \). Assume the nominal system (3.1) is stable, i.e., all the eigenvalues of the system matrix \( \Phi \) lie in the unit circle of the \( \mathcal{Z} \)-plane. The solution to the nominal system (3.1) is given by

\[
x_k = \Phi^k x_0 + \sum_{j=0}^{k-1} \Phi^{k-j-1} (\Psi u_j + w_j)
\]  

(3.2)
Assume the perturbed system with real parametric uncertainties is represented as,

$$ x_{e_{k+1}} = (\Phi + \Delta \Phi)x_{e_k} + (\Psi + \Delta \Psi)u_k + w_k \quad (3.3) $$

where $x_e(0) = x_0$, $x_{e_k} \in \mathbb{R}^n$ represents state variables under uncertainties, $\Delta \Phi \in \mathbb{R}^{n \times n}$ represents the system’s parametric uncertainty matrix and $\Delta \Psi \in \mathbb{R}^{n \times l}$ represents the control input parametric uncertainty matrix. Assume all the eigenvalues of the perturbed system (3.3) matrix $(\Phi + \Delta \Phi)$ lie in the unit circle of the $Z$-plane, i.e. the perturbed system is assumed to be stable. The solution to the perturbed system (3.3) is given by

$$ x_{e_k} = (\Phi + \Delta \Phi)^k x_0 + \sum_{j=0}^{k-1} (\Phi + \Delta \Phi)^{k-1-j}[(\Psi + \Delta \Psi)u_j + w_j] \quad (3.4) $$

Assume that the nominal system matrix $\Phi$ has distinct eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and the perturbed system matrix $(\Phi + \Delta \Phi)$ has distinct eigenvalues $(\mu_1, \mu_2, \ldots, \mu_n)$. For the nominal system, if $V$ is a modal matrix diagonalizing $\Phi$, then

$$ \Phi = V \Lambda V^{-1} = V \Lambda W \quad (3.5) $$

where $W = V^{-1}$. Let

$$ W^T = [\omega_1^T \omega_2^T \cdots \omega_n^T], \quad (3.6) $$

$$ V = [\nu_1 \nu_2 \cdots \nu_n], \quad (3.7) $$

$$ \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n). \quad (3.8) $$

Note that $\omega_i$ is a row vector, and $\nu_i$ is a column vector, for $i = 1, 2, \ldots, n$. Then

$$ \Phi \nu_i = \lambda_i \nu_i, \quad (3.9) $$

and

$$ \omega_i \Phi = \lambda_i \omega_i. \quad (3.10) $$
which satisfies the right and left eigenvalue-eigenvector constraints.

For the perturbed system, if $V_e$ is a modal matrix diagonalizing $(\Phi + \Delta \Phi)$, then

$$(\Phi + \Delta \Phi) = V_e \Lambda_c V_e^{-1} = V_e \Lambda_e W_e$$  \hspace{1cm} (3.11)

where $W_e = V_e^{-1}$. Let

$$W_e^T = [\omega^T_{e1} \omega^T_{e2} \cdots \omega^T_{en}], \hspace{1cm} (3.12)$$

$$V_e = [\nu_{e1} \nu_{e2} \cdots \nu_{en}], \hspace{1cm} (3.13)$$

$$\Lambda_e = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_n \end{bmatrix} = \text{diag}(\mu_1, \mu_2, \cdots, \mu_n). \hspace{1cm} (3.14)$$

Then the perturbed vectors $\nu_{ei}$ and $\omega_{ei}$ satisfy

$$\mu_i = \lambda_i + \bar{\rho}_{ii}, \hspace{1cm} (3.15)$$

$$\nu_{ei} = \nu_i + \Delta \nu_i, \hspace{1cm} (3.16)$$

$$\omega_{ei} = \omega_i + \Delta \omega_i. \hspace{1cm} (3.17)$$

so we have

$$(\Phi + \Delta \Phi) \nu_{ei} = \mu_i \nu_{ei}, \hspace{1cm} (3.18)$$

and

$$\omega_{ei} (\Phi + \Delta \Phi) = \mu_i \omega_{ei}. \hspace{1cm} (3.19)$$

where $\bar{\rho}_{ii}$ is the perturbation of the $i$-th eigenvalue, and $\Delta \nu_i$ and $\Delta \omega_i$ are the perturbations of the $i$-th eigenvectors. So the time response of the $q$-th component of the nominal state vector $x_k$ and the perturbed state variable vector $x_{ek}$ can be written as:

$$x_q(k) = \sum_{i=1}^{n} \nu_{qi} \lambda_i^k \omega_i x_0 + \sum_{j=0}^{k-1} \sum_{i=1}^{n} \nu_{qi} \lambda_i^{k-1-j} \omega_i (\Psi u_j + w_j) \hspace{1cm} (3.20)$$
\[ x_{eq}(k) = \sum_{i=1}^{n} (\nu_{qi} + \Delta \nu_{qi}) \mu_{i}^{k} \omega_{ei} x_{0} \]

\[ + \sum_{j=0}^{k-1} \sum_{i=1}^{n} (\nu_{qi} + \Delta \nu_{qi}) \mu_{i}^{k-1-j} \omega_{ei} [(\Psi + \Delta \Psi) u_{j} + w_{j}] \]  

(3.21)

where \((\cdot)_{q}\) refers to the \(q\)-th component of a column vector \((\cdot)\), and \((\cdot)_{qi}\) refers to the \(q\)-th component of a column vector \((\cdot)_{i}\); where \(\nu_{qi}\) is the \(q\)-th component in vector \(\nu_{i}\), and \(\Delta \nu_{qi}\) is a perturbation on the component \(\nu_{qi}\) due to a parametric subset \(\mathcal{D}(\lambda_{i}, R_{\theta})\).

For a perturbed system, the time response trajectories for a perturbed state variables can be checked to determine whether they lie in a tube-shaped region around the nominal time response trajectory. Both initial conditions and input forcing functions are used to develop the time response trajectories for the perturbed state variable of interest which should lie within a tube-shaped region along the time coordinate. We assume that the nominal eigenvalues are distinct, and the perturbed eigenvalues lie in disjoint neighborhoods \(\mathcal{D}(\lambda_{i}, R_{\theta})\) of the nominal eigenvalues. Let us consider the perturbed right eigenvectors \(\nu_{ei} = \nu_{i} + \Delta \nu_{i}\), for example, which are corresponding to the perturbed eigenvalues \(\mu_{i}\) in the regions \(\mathcal{D}(\lambda_{i}, R_{\theta})\). Eigenstructure perturbation theory illustrates that all possible parametric subsets of the eigenvector perturbations \(\Delta \nu_{i}\) are bounded and continuous. Suppose that a bound \(r_{i}\) satisfies \(\| \Delta \nu_{i} \| < r_{i}\), then the continuity constraint requests a parametric subset to keep the perturbed eigenvalues within the regions \(\mathcal{D}(\lambda_{i}, R_{\theta})\) for some \(R_{\theta} < \bar{R}_{\min}\), where \(\bar{R}_{\min} = \min_{1 \leq i \leq n} \bar{R}_{i}\). The regions \(\mathcal{D}(\lambda_{i}, R_{\theta})\) in \(\mathcal{Z}\)-plane for such parametric subsets are shown in Figure 3.1 by dotted lines. Similar discussion holds for the perturbed left eigenvector family \(\omega_{ei}\).

From the input-output time response of a state variable shown in Equations (3.20) and (3.21), we denote the zero-input time response of a state variable by \((\cdot)^{0}\) and the zero-state
Figure 3.1: Disjoint Domains in $\mathcal{Z}$ Plane

time response by ($u^u$). Thus

$$x^0_q(k) = \sum_{i=1}^{n} \nu_{qi} \lambda_i^k \omega_i x_0$$

(3.22)

$$x^u_q(k) = \sum_{j=0}^{k-1} \sum_{i=1}^{n} \nu_{qi} \lambda_i^{k-1-j} \omega_i (\Psi u_j + w_j)$$

(3.23)

$$x^0_e_q(k) = \sum_{i=1}^{n} (\nu_{qi} + \Delta \nu_{qi}) \mu_i^k \omega_i x_0$$

(3.24)

$$x^u_e_q(k) = \sum_{j=0}^{k-1} \sum_{i=1}^{n} (\nu_{qi} + \Delta \nu_{qi}) \mu_i^{k-1-j} \omega_i [(\Psi + \Delta \Psi) u_j + w_j]$$

(3.25)

where $x^0_q(k)$ and $x^u_q(k)$ are the zero-input time response and zero-state time response for the nominal system (3.1), respectively; $x^0_e_q(k)$ and $x^u_e_q(k)$ are the zero-input time response and zero-state time response for the perturbed system (3.3), respectively. Then we have

$$x_q(k) = x^0_q(k) + x^u_q(k)$$

$$x_e_q(k) = x^0_e_q(k) + x^u_e_q(k)$$
Hence, the time response trajectory bounds between the perturbed state variable and the nominal state variable are

\[ |x_{eq}(k) - x_q(k)| = |[x_{eq}^0(k) + x_{eq}^u(k)] - [x_q^0(k) + x_q^u(k)]| \]
\[ \leq |x_{eq}^0(k) - x_q^0(k)| + |x_{eq}^u(k) - x_q^u(k)| \]
\[ \leq \epsilon_q^0(k) + \epsilon_q^u(k) = \epsilon_q(k) \] (3.26)

Based on the analysis and derivation, in order to determine the bound for \(|x_{eq}(k) - x_q(k)|\), we need to find the zero-input response bound of \(\epsilon_q^0(k)\) and the zero-state response bound of \(\epsilon_q^u(k)\). The zero-input time response trajectory of Equation (3.24) can be written as,

\[ x_{eq}^0(k) = \sum_{i=1}^{n} (\nu_q, \mu_i^k \omega_i + \nu_q, \mu_i^k \Delta \omega) + \Delta \nu_q, \mu_i^k \omega_c) x_0 \] (3.27)

Then the difference between the zero-input time response trajectories of the perturbed state variable in Equation (3.27) and the nominal state variable in Equation (3.24) is

\[ x_{eq}^0(k) - x_q^0(k) = \sum_{i=1}^{n} [\nu_q, \mu_i^k - \lambda_i^k \omega_i + \nu_q, \mu_i^k \Delta \omega + \Delta \nu_q, \mu_i^k \omega_c] x_0 \] (3.28)

So the zero-input time response bound is given by,

\[ |x_{eq}^0(k) - x_q^0(k)| \leq \sum_{i=1}^{n} (|\nu_q, | \mu_i^k - \lambda_i^k \|| \omega_i)| + |\nu_q, | \mu_i^k \|| \Delta \omega_i)|
+ |\Delta \nu_q, | \mu_i^k \|| \omega_c)| ||x_0|| \] (3.29)

Now we split and re-order the terms in the zero-state time response trajectory of Equation (3.25), it can be re-written as,

\[ x_{eq}^u(k) = \sum_{j=0}^{k-1} \sum_{i=1}^{n} [\nu_q, \mu_i^{k-j} \omega_i + \nu_q, \mu_i^{k-j} \Delta \omega_i]
+ \Delta \nu_q, \mu_i^{k-j} (\omega_i + \Delta \omega_i)]](\Psi + \Delta \Psi)u_j + w_j \] (3.30)
Then the difference between the zero-state time response trajectories of the perturbed state variable in Equation (3.30) and the nominal state variable in Equation (3.25) is

$$x^u_{eq}(k) - x^u_q(k) = \sum_{j=0}^{k-1} \sum_{i=1}^{n} \{ \nu_i (\mu_i^{k-1-j} - \lambda_i^{k-1-j}) \omega_i (\Psi u_j + w_j) + \nu_i \mu_i^{k-j} \omega_i \Delta \Psi u_j + [\nu_i \mu_i^{k-j} \Delta \omega_i + \Delta \nu_i \mu_i^{k-j} (\omega_i + \Delta \omega_i)] \cdot [(\Psi + \Delta \Psi) u_j + w_j] \}$$

(3.31)

So the zero-state time response bound is given by,

$$\left| x^u_{eq}(k) - x^u_q(k) \right| \leq \sum_{j=0}^{k-1} \sum_{i=1}^{n} \{ |\nu_i| \left| \mu_i^{k-1-j} - \lambda_i^{k-1-j} \right| ||\omega_i|| (\sigma_{\max}(\Psi) ||u_j|| + ||w_j||) + |\nu_i| \left| \mu_i^{k-1-j} \right| ||\omega_i|| \sigma_{\max}(\Delta \Psi) ||u_j|| + |\nu_i| \left| \mu_i^{k-1-j} \right| ||\Delta \omega_i|| + |\Delta \nu_i| \left| \mu_i^{k-1-j} \right| ||\omega_i + \Delta \omega_i|| \cdot [(\sigma_{\max}(\Psi) + \sigma_{\max}(\Delta \Psi)) ||u_j|| + ||w_j||] \}$$

(3.32)

where in Equations (3.29) and (3.32), the bounds for $|\nu_i|$ and $||\omega_i||$ are from the nominal system, which are known values.

**Theorem 1.** Consider $\lambda_i$ ($i = 1, 2, \cdots, n$) are the distinct eigenvalues for the nominal system $\Phi \in \mathbb{R}^{n \times n}$, and $\mu_i$ are the distinct eigenvalues lying in the disjoint neighborhoods $\mathcal{D}(\lambda_i, R_\theta)$ for the perturbed system $(\Phi + \Delta \Phi) \in \mathbb{R}^{n \times n}$, then the bound for $|\mu_i^k - \lambda_i^k|$ is

$$|\mu_i^k - \lambda_i^k| \leq |\lambda_i|^k \left[ \left( 1 + \frac{R_\theta}{|\lambda_i|} \right)^k - 1 \right]$$

(3.33)

**Proof:** See Appendix A.

**Corollary 1.** Consider $\lambda_i$ ($i = 1, 2, \cdots, n$) are the distinct eigenvalues for the nominal system $\Phi \in \mathbb{R}^{n \times n}$, and $\mu_i$ are the distinct eigenvalues lying in the disjoint neighborhoods
\[ \mathcal{D}(\lambda_i, R_{\theta}) \text{ for the perturbed system } (\Phi + \Delta \Phi) \in \mathbb{R}^{n \times n}, \text{ then the bound for } |\mu^k_i| \text{ is} \]

\[ |\mu^k_i| \leq |\lambda_i|^k \left( 1 + \frac{R_{\theta}}{|\lambda_i|} \right)^k \]  \hspace{1cm} (3.34)

**Proof:** See Appendix A.

### 3.1.1 Norm-Bounded Parametric Uncertainties

We assume that the uncertainty locations of the entries in the nominal system matrix \( \Phi \) are unknown. A parametric subset \( r \) satisfying \( \sigma_{\text{max}}(\Delta \Phi) \leq r \) (the perturbed parameter radius for \( \mathcal{D}(\lambda_i, R) \) stability) is discussed in the context of disjoint neighborhood analysis in [2].

**Theorem 2** [91]. Let \( \bar{R}_{\min} = \min_{1 \leq i \leq n} \bar{R}_i \); \( (\lambda_i, \nu_i), (\lambda_i, \omega_i) \) be the \( i \)-th right and left eigenpairs for the nominal matrix \( \Phi \in \mathbb{R}^{n \times n} \) with \( \|\nu_i\| = \delta_i \) and \( \|\omega_i\| = \pi_i \), and \( (\mu_i, \nu_e_i), (\mu_i, \omega_e_i) \) be the \( i \)-th right and left eigenpairs for the parametric uncertain matrix \( (\Phi + \Delta \Phi) \in \mathbb{R}^{n \times n} \) with \( \nu_{e_i} = (\nu_i + \Delta \nu_i), \omega_{e_i} = (\omega_i + \Delta \omega_i) \) and \( \mu_i = (\lambda_i + \bar{\rho}_{ii}) \). Then for all \( R_{\theta} \leq \frac{\mathcal{K}(V)}{\mathcal{K}(V) + 1} \bar{R}_{\min} \) and \( \sigma_{\text{max}}(\Delta \Phi) < \frac{R_{\theta}}{\mathcal{K}(V)} \), the bounds on \( \Delta \nu_i \) and \( \Delta \omega_i \), respectively, are

\[ \|\Delta \nu_i\| < \frac{R_{\theta} \delta_i}{\alpha_i - R_{\theta}} \]  \hspace{1cm} (3.35)

\[ \|\Delta \omega_i\| < \frac{R_{\theta} \pi_i}{\alpha_i - R_{\theta}} \]  \hspace{1cm} (3.36)

where \( \alpha_i = \sigma_{\text{max}}(\Phi - \lambda_i I), \mathcal{K}(V) \) is the condition number for the modal matrix \( V \).

**Proof:** Yedavalli and Ashokkumar have given the proof of this theorem in [91].

**Remark 1:** The following results can be derived from Theorem 2:

\[ \|\Delta \nu_{q_i}\| < \|\Delta \nu_i\| < \frac{R_{\theta} \delta_i}{\alpha_i - R_{\theta}} \]  \hspace{1cm} (3.37)

\[ \|\omega_{e_i}\| = \|\omega_i + \Delta \omega_i\| \leq \frac{\pi_i \alpha_i}{\alpha_i - R_{\theta}} \]  \hspace{1cm} (3.38)
3.1.2 Structured Parametric Uncertainties

Different from the norm bounded uncertainties, the perturbation locations of the entries in the nominal matrix $\Phi$ are assumed to be known. In this way, the error matrix perturbing the nominal matrix is treated as

$$\Delta \Phi = \sum_{l=1}^{r} q_l \Delta \Phi_l$$

(3.39)

where the matrices $\Delta \Phi_l$ are constant with non-zero entries at the locations where $q_l$ appear.

To determine $R_\theta$ in $\mathcal{D}(\lambda_i, R_\theta)$ and the uncertain parameter intervals $|q_l|$, for $l = 1, 2, \cdots, r$, eigenstructure perturbations by disjoint neighborhood analysis states the following theorem.

**Theorem 3** [91]. Consider the circular regions $\mathcal{D}(\lambda_i, R_{\min}) \in \Omega$ and $\bar{R}_{\min} = \min_{1 \leq i \leq n} \bar{R}_i$. Let $(\lambda_i, \nu_i), (\lambda_i, \omega_i)$ be the $i$-th right and left eigenpairs for the nominal matrix $\Phi \in \mathbb{R}^{n \times n}$ with $\|\nu_i\| = \delta_i$ and $\|\omega_i\| = \pi_i$, and $(\mu_i, \nu_{e_i}), (\mu_i, \omega_{e_i})$ be the $i$-th right and left eigenpairs for the parametric uncertain matrix $(\Phi + \Delta \Phi) \in \mathbb{R}^{n \times n}$, where $\Delta \Phi = \sum_{l=1}^{r} q_l \Delta \Phi_l$. Then for the bounds on $q_l$ given by

$$\left( \sum_{l=1}^{r} q_l^2 \right)^{\frac{1}{2}} \sigma_{\text{max}}(P_e) < R_\theta \quad \forall \quad R_\theta \leq \frac{\bar{R}_{\min} \sigma_{\text{max}}(P_e)}{\sigma_{\text{max}}(P_e) + \sigma_{\text{max}}(E_e)},$$

(3.40)

$$\sum_{l=1}^{r} |q_l| \sigma_{\text{max}}(P_l) < R_\theta \quad \forall \quad R_\theta \leq \frac{\bar{R}_{\min} \mathcal{K}(V)}{\mathcal{K}(V) + 1},$$

(3.41)

$$\max_j |q_j| \sigma_{\text{max}} \left( \sum_{l=1}^{r} |P_l| \right) < R_\theta \quad \forall \quad R_\theta \leq \frac{\bar{R}_{\min} \left( \sum_{j=1}^{r} |P_j| \right)}{\sigma_{\text{max}} \left( \sum_{l=1}^{r} |P_l| \right) + \sum_{i=1}^{r} \sigma_{\text{max}}(\Delta \Phi_i)}$$

for $j = 1, 2, \cdots, r$.

(3.42)

the bounds on $\Delta \nu_i$ and $\Delta \omega_i$, respectively, are

$$\|\Delta \nu_i\| < \frac{R_\theta \delta_i}{\alpha_i - R_\theta},$$

(3.43)

37
\[ \| \Delta \omega_i \| < \frac{R_\theta \pi_i}{\alpha_i - R_\theta} . \] (3.44)

where \( \alpha_i = \sigma_{\text{max}}(\Phi - \lambda_i I) \), \( K(V) \) is the condition number for the modal matrix \( V \), and

\[
P_i := V^{-1} \Delta \Phi_i V, \tag{3.45}
\]

\[
|P_i| := [P_{ij}], \ i, j = 1, 2, \cdots, n, \tag{3.46}
\]

\[
P_e := [P_1, P_2, \cdots, P_r], \tag{3.47}
\]

\[
\Delta \Phi_e := [\Delta \Phi_1, \Delta \Phi_2, \cdots, \Delta \Phi_r]. \tag{3.48}
\]

**Proof:** Yedavalli and Ashokkumar have given the proof of this theorem in [91].

### 3.1.3 Time Response Bounds

**Theorem 4:** Let \( \mathcal{D}(\lambda_i, R_\theta) \subseteq \Omega \) be the disjoint neighborhoods for the distinct eigenvalues of the matrix \( \Phi \in \mathbb{R}^{n \times n} \) and let \( \Omega \) be confined to the unit circle of the \( \mathbb{Z} \)-plane, then for any \( \Delta \Phi \in \mathbb{R}^{n \times n} \) which confines the perturbed eigenvalues \( \mu_i \) of \( (\Phi + \Delta \Phi) \) in the disjoint disks \( \mathcal{D}(\lambda_i, R_\theta) \subseteq \Omega \), assume \( ||w_i|| \leq a \) where \( w_i \) is the zero mean white noise sequence with covariance \( Q_i \), we have

\[
|x_{i}^{q}(k) - x_{i}^{q}(k)| \leq \sum_{i=1}^{\lambda_i} |\lambda_i|^k \left\{ \left( 1 + \frac{R_\theta}{|\lambda_i|} \right)^k \left[ |\nu_q| \pi_i \alpha_i \epsilon_i \right. \right.
\]

\[
- |\nu_q| \pi_i \alpha_i \epsilon_i - |\nu_q| \pi_i \alpha_i \epsilon_i \right\} \|x_0\| = \epsilon_{q}^{i}(k) \tag{3.49}
\]

\[
|x_{i}^{u}(k) - x_{i}^{u}(k)| \leq \sum_{i=1}^{\lambda_i} \sum_{j=0}^{k-1} |\lambda_i|^k \left( 1 + \frac{R_\theta}{|\lambda_i|} \right)^{k-1-j} \cdot \left[ |\nu_q| \pi_i \alpha_i \sigma_{\text{max}}(\Psi) \|u_j\| + a \right]
\]

\[
+ |\nu_q| \pi_i \sigma_{\text{max}}(\Delta \Psi) \|u_j\| + \left( |\nu_q| \pi_i \alpha_i \epsilon_i + \frac{R_\theta \pi_i \alpha_i}{\alpha_i - R_\theta} \frac{R_\theta \pi_i \alpha_i}{(\alpha_i - R_\theta)^2} \right)
\]

\[
\cdot \left( (\sigma_{\text{max}}(\Psi) + \sigma_{\text{max}}(\Delta \Psi)) \|u_j\| + a \right) \right\] \left[ \epsilon_{q}^{i}(k) \right.
\]

\[
- |\nu_q| \pi_i \alpha_i \sigma_{\text{max}}(\Psi) \|u_j\| + a \right\} = \epsilon_{q}^{u}(k) \tag{3.50}
\]
where } \alpha_i = \sigma_{\text{max}}(\Phi - \lambda_i I), \nu_{q_i} = k\text{-th component in the eigenvector } \nu_i, \|\nu_i\| = \delta_i \text{ and } \|\omega_i\| = \pi_i.

**Proof:** For the zero-input response, substituting the bounds in Equations (3.33) (3.34) (3.35) (3.36) (3.37) and (3.38) into (3.29), the zero-input time response is obtained:

\[
\left| x_{q_0}(k) - x_{q_0}^0(k) \right| \leq \sum_{i=1}^{n} \{ |\nu_{q_i}| |\lambda_i|^k \left[ \left( 1 + \frac{R_{\theta}}{|\lambda_i|} \right)^k - 1 \right] \pi_i + \frac{R_{\theta} \pi_i}{\alpha_i - R_{\theta}} \} \|x_0\| \\
= \sum_{i=1}^{n} \{ \frac{|\nu_{q_i}| |\lambda_i|^k \left[ \left( 1 + \frac{R_{\theta}}{|\lambda_i|} \right)^k - 1 \right] \pi_i}{\alpha_i - R_{\theta}} \} \|x_0\| \\
= \sum_{i=1}^{n} \{ |\lambda_i|^k \left[ \left( 1 + \frac{R_{\theta}}{|\lambda_i|} \right)^k - 1 \right] \pi_i \} \|x_0\| = \epsilon_q^0(k)
\]

For the zero-state response, using the bounds as earlier we have,

\[
\left| x_{q_0}(k) - x_{q_0}^u(k) \right| \leq \sum_{j=0}^{k-1} \sum_{i=1}^{n} \{ |\nu_{q_i}| |\lambda_i|^{k-1-j} \left[ \left( 1 + \frac{R_{\theta}}{|\lambda_i|} \right)^{k-1-j} - 1 \right] \pi_i \cdot (\sigma_{\text{max}}(\Psi) \|u_j\| + \|w_j\|) \\
+ \frac{R_{\theta} \pi_i}{\alpha_i - R_{\theta}} |\lambda_i|^{k-1-j} \left( 1 + \frac{R_{\theta}}{|\lambda_i|} \right)^{k-1-j} \pi_i \sigma_{\text{max}}(\Delta \Psi) \|u_j\| \\
+ \frac{R_{\theta} \pi_i}{\alpha_i - R_{\theta}} |\lambda_i|^{k-1-j} \left( 1 + \frac{R_{\theta}}{|\lambda_i|} \right)^{k-1-j} \pi_i \alpha_i \} \\
\cdot \left[ (\sigma_{\text{max}}(\Psi) + \sigma_{\text{max}}(\Delta \Psi)) \|u_j\| + \|w_j\| \right] \} \\
= \sum_{j=0}^{k-1} \sum_{i=1}^{n} \{ |\lambda_i|^{k-1-j} \left( 1 + \frac{R_{\theta}}{|\lambda_i|} \right)^{k-1-j} \cdot \left[ |\nu_{q_i}| \pi_i (\sigma_{\text{max}}(\Psi) \|u_j\| + \alpha) \right] \}
\]

39
+ |\nu_q| \pi_i \sigma_{\text{max}}(\Delta \Psi) \|u_j\| + \left( \frac{|\nu_q| R_\theta \pi_i}{\alpha_i - R_\theta} + \frac{R_\theta \delta_i \pi_i \alpha_i}{(\alpha_i - R_\theta)^2} \right) \\
\cdot \left( (\sigma_{\text{max}}(\Psi) + \sigma_{\text{max}}(\Delta \Psi)) \|u_j\| + a \right) \\
- |\nu_q| \pi_i (\sigma_{\text{max}}(\Psi) \|u_j\| + a) \right] \cdot \left( \sigma_{\text{max}}(\Psi) + \sigma_{\text{max}}(\Delta \Psi) \right) \|u_j\| + \|w_j\| \right) \\
- |\nu_q| \pi_i (\sigma_{\text{max}}(\Psi) \|u_j\| + \|w_j\|) \right) = \epsilon^0_q(k) + \epsilon^u_q(k) = \epsilon_q(k) \quad (3.51)

Therefore, the time response bound of the q-th component of perturbed state vector \( x_e(k) \) is expressed as:

\[ x_q(k) - \epsilon_q(k) \leq x_{e_q}(k) \leq x_q(k) + \epsilon_q(k) \quad (3.52) \]

### 3.2 An Illustrated Example

Consider a linear discrete-time model of an advanced high-bypass two-spool turbofan engine. The engine is simulated from stillness (0 RPM) to idle under non-deteriorated condition. The nominal discrete-time aircraft engine model is described as:

\[ x_{k+1} = \Phi x_k + \Psi u_k + w_k \quad (3.53) \]
where

$$\Phi_{6 \times 6} = \begin{bmatrix}
0.6022 & 0.0211 & -0.0972 & 0.0559 & 0.0384 & 0.0812 \\
-0.2610 & 0.6058 & 0.6774 & 0.1738 & 0.1365 & -0.0540 \\
0.0039 & 0 & 0.9358 & 0.0001 & 0.0001 & 0.0002 \\
0.0039 & 0.0087 & 0.0610 & 0.8663 & 0.0022 & 0 \\
-0.0345 & 0.0187 & 0.0674 & 0.0266 & 0.8151 & -0.0025 \\
-0.0361 & 0.0026 & 0.0233 & 0.0158 & 0.9014 & -0.0259 \\
\end{bmatrix},$$

$$\Psi_{6 \times 3} = \begin{bmatrix}
0.0064 & 0 & -2.6131 \\
0.0199 & 0 & 39.2472 \\
0 & 0 & 0.0372 \\
0.0003 & 0 & -0.6095 \\
0.0025 & 0 & -1.5843 \\
0.0024 & 0 & -0.1412 \\
\end{bmatrix}. \quad (3.54)$$

$w_k$ is zero mean white Gaussian noise sequence with covariance matrices $Q$:

$$Q_{3 \times 3} = \begin{bmatrix}
0.001 & 0 & 0 \\
0 & 0.001 & 0 \\
0 & 0 & 0.00001 \\
\end{bmatrix}.$$

The nominal matrix $\Phi$ has eigenvalues:

$$\lambda_1, \lambda_2 = 0.6076 \pm 0.0625i,$$
$$\lambda_3 = 0.8138,$$
$$\lambda_4 = 0.9384,$$
$$\lambda_5, \lambda_6 = 0.8796 \pm 0.0076i.$$

$\bar{R}_{\min} = 0.0076$ and a modal matrix $V$ having a condition number $K(V) = 13.3446$. We assume $\sigma_{\max}(\Delta \Psi) = 0.05$ and $\|w_j\| = 0.05$, where $j = 1, 2, \cdots$. For the uncertain parameters at $\Delta \Phi(2, 4), \Delta \Phi(4, 4), \Delta \Phi(5, 3)$, let an error matrix $\Delta \Phi$ perturbing $\Phi$ be

$$\Delta \Phi = q_1 \Delta \Phi_{24} + q_2 \Delta \Phi_{44} + q_3 \Delta \Phi_{53}.$$ 

where $\Delta \Phi_{ij}$ are zero matrices except with a unity value at the location $ij$.

Assume the $q$-th component of $x_k$ and perturbed step input function for analysis. Then,

$$|x_{eq}(k) - x_q(k)| \leq \epsilon_q(k) = \epsilon^0_q(k) + \epsilon^u_q(k) \quad (3.55)$$

41
| State | $\alpha_i$ | $\delta_i$ | $\pi_i$ | $|\nu_{1i}|$ | $|\nu_{2i}|$ | $|\nu_{3i}|$ | $|\nu_{4i}|$ | $|\nu_{5i}|$ | $|\nu_{6i}|$ |
|-------|-------------|------------|---------|----------------|----------------|----------------|----------------|----------------|----------------|
| $x_1$ | 0.8393      | 1          | 2.6919  | 0.2601         | 0.9596         | 0.0031         | 0.0300         | 0.0985         | 0.0286         |
| $x_2$ | 0.8393      | 1          | 2.6919  | 0.2601         | 0.9596         | 0.0031         | 0.0300         | 0.0985         | 0.0286         |
| $x_3$ | 0.8017      | 1          | 1.6172  | 0.1426         | 0.3465         | 0.0049         | 0.1002         | 0.9173         | 0.0895         |
| $x_4$ | 0.8336      | 1          | 3.1927  | 0.1543         | 0.7083         | 0.2927         | 0.3500         | 0.2913         | 0.4260         |
| $x_5$ | 0.8140      | 1          | 4.3996  | 0.1050         | 0.5426         | 0.0092         | 0.3552         | 0.3359         | 0.6749         |
| $x_6$ | 0.8140      | 1          | 4.3996  | 0.1050         | 0.5426         | 0.0092         | 0.3552         | 0.3359         | 0.6749         |

The nominal parameters contributing to the perturbed time response are summarized in Table 3.1. The $q$-th components in vectors $\nu_i$, i.e. the $q$-th row of modal matrix $V$ are denoted by $\nu_{qi}$.

The eigenvalue perturbation for eigenvector continuity are:

- for norm bounded uncertainties, $R_\theta \leq 0.0071$ (by theorem 2).
- for structured uncertainties, $R_\theta \leq 0.0048$ (by theorem 3) Equation (3.40).

Then the subsets are:

- for norm bounded uncertainties, $\sigma_{\text{max}}(\Delta \Phi) \leq 5.3126 \times 10^{-4}$.
- for structured uncertainties, $(q_1^2 + q_2^2 + q_3^2)^{\frac{1}{2}} < 0.0028, |q_1|, |q_2|, |q_3| < 0.0016$.

Plots for the time response norm bounded uncertainties and structured uncertainties of each state variable $x_q(k) \pm \epsilon_q(k)$ with the corresponding zero-input response bound of $x_q^0(k) \pm \epsilon_q^0(k)$ and zero-state response bound of $x_q^u(k) \pm \epsilon_q^u(k)$ are shown in Figures 3.2-3.7.
where $q = 1, 2, \cdots, 6$. The perturbed response analysis shows that the bounds have less departure from the nominal response for a larger uncertainty size in the structured uncertainty case; moreover, the bounds have more departure for a smaller uncertainty size in the norm-bounded uncertainty case. Based on this observation, we see that the time response bounds for the structured uncertainty are less conservative than those of the norm-bounded uncertainty. One reason for the less conservative bound in the structured uncertainty case is due to a smaller estimate of the radius $R$ for the eigenvalue perturbation regions.

### 3.3 Summary

In this chapter, the time response bound analysis for a linear discrete-time parametric uncertain stochastic system is discussed. Time response bounds in the area of uncertain
Figure 3.3: Time Response of $x_2$ for Norm Bounded and Structured Uncertainties

system theory result in a tube-shaped time response around the nominal trajectory of a scalar component of the state vector. The parametric uncertainty has affected the property of the eigenvalues and eigenvectors (eigenstructure) of the system, which in turn changes the performance of the nominal system. The tube-shaped region generated by the time response bounds inspires the research on designing a dynamic threshold which considers the effect of parametric uncertainty of the system. In the next chapter, we will apply the concept and algorithm of the time response analysis to develop the dynamic threshold.
Figure 3.4: Time Response of $x_3$ for Norm Bounded and Structured Uncertainties

Figure 3.5: Time Response of $x_4$ for Norm Bounded and Structured Uncertainties
Figure 3.6: Time Response of $x_5$ for Norm Bounded and Structured Uncertainties

Figure 3.7: Time Response of $x_6$ for Norm Bounded and Structured Uncertainties
CHAPTER 4

DYNAMIC THRESHOLD DESIGN

4.1 Dynamic Threshold Design Using Kalman Filter

Consider a linear time-invariant discrete-time stochastic system with parametric uncertainty represented in Equation (1.1) with all faults and unknown inputs set to be zero, then the system is expressed as,

\[
\begin{align*}
\dot{x}_{e,k+1} &= (\Phi + \Delta \Phi)x_{e,k} + (\Psi + \Delta \Psi)u_k + w_k \\
z_{e,k+1} &= Hx_{e,k+1} + v_{k+1}
\end{align*}
\]

(4.1)

The nominal system of Equation (4.1) is modeled as Equation (2.33) in a time-invariant case, i.e.

\[
\begin{align*}
\dot{x}_{k+1} &= \Phi x_k + \Psi u_k + w_k \\
z_{k+1} &= Hx_{k+1} + v_{k+1}
\end{align*}
\]

(4.2)

Then the Kalman filter for the system (4.1) is shown as,

\[
\begin{align*}
\hat{x}_{k+1|k} &= \Phi \hat{x}_{k|k} + \Psi u_k \\
\hat{z}_{k+1|k} &= H \hat{x}_{k+1|k} \\
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}(Hx_{e,k+1} + v_{k+1} - H \hat{x}_{k+1|k})
\end{align*}
\]

(4.3) (4.4) (4.5)
4.1.1 Robust Residual Generation Based on Kalman Filter

From the above Kalman filter algorithm, we notice that the perturbed state variable \( x_{ek} \) brings the parametric uncertainty information into the Equation (4.5), which is different from the standard predictor-corrector form in Equation (2.37). Then the recursive predictor form containing the parametric uncertainties is

\[
\hat{x}_{k+1|k} = \Phi[\hat{x}_{k|k-1} + K_k(H x_{ek} + v_k - H \hat{x}_{k|k-1})] + \Psi_k u_k
\]

\[
= \Phi(I_n - K_k H)\hat{x}_{k|k-1} + \Phi K_k H x_{ek} + \Phi K_k v_k + \Psi u_k \quad (4.6)
\]

Based on the above derivation, the state-prediction error dynamics of the Kalman filter with parametric uncertainties is given below

\[
e_{k+1|k} = x_{ek+1} - \hat{x}_{k+1|k}
\]

\[
= (\Phi + \Delta \Phi)x_{ek} + (\Psi + \Delta \Psi)u_k + w_k - \Phi(I_n - K_k H)\hat{x}_{k|k-1}
\]

\[
-\Phi K_k H x_{ek} - \Phi K_k v_k - \Psi u_k
\]

\[
= \Phi(I_n - K_k H)e_{k|k-1} + \Delta \Psi u_k + \Delta \Phi x_{ek} + w_k + (-\Phi K_k)v_k \quad (4.7)
\]

The reason to choose the state-prediction form instead of the state-filtering form is that the output estimate \( \hat{z}_{k+1|k} \) is computed based on the predictor, i.e. \( \hat{z}_{k+1|k} = H\hat{x}_{k+1|k} \), which is used to generate the residual defined in the following form of:

\[
r_{k+1} = z_{ek+1} - \hat{z}_{k+1|k} = H e_{k+1|k} + v_{k+1} \quad (4.8)
\]

4.1.2 Dynamic Threshold Design Based on Kalman Filter

The system (4.1) with possible actuator and sensor faults can be described as:

\[
\begin{align*}
x_{ek+1} &= (\Phi + \Delta \Phi)x_{ek} + (\Psi + \Delta \Psi)(u_k + f_{ak}) + w_k \\
z_{ek+1} &= H x_{ek+1} + v_{k+1} + f_{sk+1}
\end{align*}
\quad (4.9)
\]
where $f_{ak} \in \mathbb{R}^l$ is the actuator fault vector and $f_{sk} \in \mathbb{R}^m$ is the sensor fault vector. For this system, the state-prediction error dynamics and the residual are governed by the following equations:

$$e_{k+1|k} = \Phi(I_n - K_kH)e_{k|k-1} + \Delta \Psi u_k + \Delta \Phi x_{e_k} + w_k + (-\Phi K_k)v_k$$
$$+ (\Psi + \Delta \Psi)f_{ak} + (-\Phi K_k)f_{sk} \quad (4.10)$$

$$r_{k+1} = He_{k+1|k} + v_{k+1} + f_{sk+1} \quad (4.11)$$

A fault can be detected by comparing the residual signal $r_{k+1}$ with a threshold function $\zeta_{k+1}$ according to the test given below:

$$\begin{cases}
\|r_{k+1}\| \leq \zeta_{k+1} & \text{for } f_{k+1} = 0 \\
\|r_{k+1}\| > \zeta_{k+1} & \text{for } f_{k+1} \neq 0
\end{cases}$$

If the test is positive, i.e. the residual is greater than the threshold, we can conclude that a fault exists in the system. There are many ways of defining the residual functions and determining the thresholds. In this dissertation, we will find the time response bound of the threshold for each component of the residual vector such that a structured residual set can be set up for fault detection and isolation (see Chapter 5).

When there is no fault, the Equation (4.11) will be close to zero, but it will never be zero due to the existence of the parametric uncertainties and noises; we want to find an upper-and-lower time response bound for the error dynamics $e_{k+1|k}$ in Equation (4.7) in order to further determine a tube-shaped bound for the residual $r_{k+1}$ in Equation (4.8). Without loss of generality, the steady-state of the Kalman filter gain will be used to simplify the computation of Equation (4.7).

Assume $\bar{K}$ is the steady-state of Kalman filter gain $K_{k+1}$, $Q$ and $R$ are the covariance matrices for the process noise $w_k$ and $v_k$, respectively. The theorem from [67] states:
“If \((\Phi, \sqrt{Q})\) is stabilizable and \((\Phi, H)\) is detectable, then there is a unique positive definite solution \(P_p\) for the matrix Riccati Equation

\[
P_p = \Phi P_p \Phi^T - \Phi P_p H^T (HP_p H^T + R)^{-1} HP_p \Phi^T + Q \tag{4.12}
\]

where \(\lim_{k \to \infty} P_{k+1|k} = P_p\) with \(P_p\) independent of \(P_{0|1} \).”

Given \((\Phi, \Psi, H, Q, R)\), we can first compute \(P_p\), which is the positive definite solution of Equation (4.12); then we can calculate \(\bar{K}\) as

\[
\bar{K} = P_p H^T (HP_p H^T + R)^{-1} \tag{4.13}
\]

Use Equation (4.13) in the error dynamics Equation (4.7), we have

\[
e_{k+1|k} = \bar{\Phi} e_{k|k-1} + \Delta \Psi u_k + \Delta \Phi x_k + w_k + (-\Phi \bar{K}) v_k \tag{4.14}
\]

where

\[
\bar{\Phi} := \Phi (I_n - \bar{K} H) \tag{4.15}
\]

Equation (4.14) is a state equation for state vector \(e_{k+1|k}\), whose plant matrix is \(\bar{\Phi}\).

Next, we need to find the time response bound for each component in the vector \(e_{k+1|k}\) in Equation (4.14). Assume \(\bar{\Phi}\) has distinct eigenvalues \(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n\). Let \(\bar{V}\) be a modal matrix diagonalizing \(\bar{\Phi}\), then

\[
\bar{\Phi} = \bar{V} \bar{\Lambda} \bar{V}^{-1} = \bar{V} \bar{\Lambda} \bar{W} \tag{4.16}
\]

where \(\bar{W} = \bar{V}^{-1}\). Let

\[
\bar{W}^T = [\bar{\omega}_1^T \quad \bar{\omega}_2^T \quad \cdots \quad \bar{\omega}_n^T], \tag{4.17}
\]

\[
\bar{V} = [\bar{\nu}_1 \quad \bar{\nu}_2 \quad \cdots \quad \bar{\nu}_n], \tag{4.18}
\]

\[
\bar{\Lambda} = \begin{bmatrix}
\bar{\lambda}_1 & 0 & \cdots & 0 \\
0 & \bar{\lambda}_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \bar{\lambda}_n
\end{bmatrix} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n). \tag{4.19}
\]
Note that $\vec{\omega}_i$ is a row vector, and $\vec{\nu}_i$ is a column vector, for $i = 1, 2, \ldots, n$. Then

$$\vec{\Phi} \vec{\nu}_i = \lambda_i \vec{\nu}_i,$$  \hspace{1cm} (4.20)

and

$$\vec{\omega}_i \vec{\Phi} = \lambda_i \vec{\omega}_i.$$  \hspace{1cm} (4.21)

which satisfies the right and left eigenvalue-eigenvector constraints.

Thus the time response of the $q$-th component of error dynamics state vector $e_{k|k-1}$ can be written as:

$$e_q(k|k-1) = \sum_{i=1}^{n} \bar{\nu}_i \bar{\lambda}_i^k \vec{\omega}_i e_{0|-1} + \sum_{j=0}^{k-1} \sum_{i=1}^{n} \bar{\nu}_i \bar{\lambda}_i^{k-1-j} \vec{\omega}_i \cdot [\Delta \Psi u_j + \Delta \Phi x_{e_j} + w_j + (-\Phi \bar{K}) v_j]$$  \hspace{1cm} (4.22)

where $e_{0|-1}$ is the initial condition for the error dynamics vector. The bound for $e_q(k|k-1)$ can be expressed as

$$|e_q(k|k-1)| \leq \sum_{i=1}^{n} \left| \bar{\nu}_i \bar{\lambda}_i^k \vec{\omega}_i e_{0|-1} \right| + \sum_{j=0}^{k-1} \sum_{i=1}^{n} \left| \bar{\nu}_i \bar{\lambda}_i^{k-1-j} \vec{\omega}_i \right| \cdot \left[ \| \Delta \Psi u_j + \Delta \Phi x_{e_j} + w_j + (-\Phi \bar{K}) v_j \| \right]$$

$$\leq \sum_{i=1}^{n} \left| \bar{\nu}_i \bar{\lambda}_i^k \vec{\omega}_i e_{0|-1} \right| + \sum_{j=0}^{k-1} \sum_{i=1}^{n} \left| \bar{\nu}_i \bar{\lambda}_i^{k-1-j} \vec{\omega}_i \right| \cdot [\sigma_{\text{max}}(\Delta \Psi) \| u_j \| + \sigma_{\text{max}}(\Delta \Phi) \| x_{e_j} \|]$$

$$+ \| w_j \| + \| (-\Phi \bar{K}) \| \| v_j \|]$$  \hspace{1cm} (4.23)

where the noises $w_k, v_k$ are assumed to be bounded such as $\| w_k \| \leq a$ and $\| v_k \| \leq b$. And $\sigma_{\text{max}}(\Delta \Phi)$ is obtained from the parametric uncertainty analysis in Chapter 3. The
perturbed state vector bound \( \| \mathbf{x}_{e_k} \| \) is expressed as
\[
\| \mathbf{x}_{e_k} \| = \left\| \begin{bmatrix} x_{e1}(k) \\ x_{e2}(k) \\ \vdots \\ x_{en}(k) \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x_1(k) + \epsilon_1(k) \\ x_2(k) + \epsilon_2(k) \\ \vdots \\ x_n(k) + \epsilon_n(k) \end{bmatrix} \right\| = (4.24)
\]
where \( x_q(k) + \epsilon_q(k) \) for \( q = 1, 2, \ldots, n \), is the \( q \)-th component time response upper bound. Hence the dynamic threshold of the error dynamics \( e_q(k|k-1) \) using the Kalman filter is obtained as
\[
|e_q(k|k-1)| \leq \sum_{i=1}^{n} |\tilde{v}_q \bar{\lambda}_i^k \omega, e_{0|-1}| + \sum_{j=0}^{k-1} \sum_{i=1}^{n} |\tilde{v}_q \bar{\lambda}_i^{k-j} \omega_i| \\
\cdot[\sigma_{\text{max}}(\Delta \Psi) \| \mathbf{u}_j \| + \sigma_{\text{max}}(\Delta \Phi)] \\
+ a + \| \Phi \bar{K} \| b \\
= \eta_q(k) = \eta_q(k) \] (4.25)
where \( \eta_q(k) \) can be called the dynamic threshold for \( e_q(k|k-1) \). For the residual of the \( q \)-th sensor, we have
\[
r_q(k) = H_q e_{k|k-1} + v_q(k) = \eta_q(k) \] (4.26)
where
\[
H = \begin{bmatrix} H_1^T, & H_2^T, & \cdots, & H_q^T, & \cdots, & H_m^T \end{bmatrix}^T, \\
r = \begin{bmatrix} r_1, & r_2, & \cdots, & r_q, & \cdots, & r_m \end{bmatrix}^T, \\
v = \begin{bmatrix} v_1, & v_2, & \cdots, & v_q, & \cdots, & v_m \end{bmatrix}^T.
\]
Then the dynamic threshold for the residual of the $q$-th sensor is:

$$ |r_q(k)| = |H_q e_{k|k-1} + v_q(k)| $$

$$ \leq H_q \begin{bmatrix} \eta_1(k) \\ \vdots \\ \eta_q(k) \\ \vdots \\ \eta_n(k) \end{bmatrix} + |v_q(k)| $$

$$ = \zeta_q(k) $$

where $\zeta_q(k)$ is the dynamic threshold for $r_q(k)$, $|v_q(k)|$ is assumed to be $|v_q(k)| = \frac{b}{m}$.

### 4.2 Dynamic Threshold Design Using Unknown Input Observer

Consider a linear time-invariant discrete-time stochastic system with parametric uncertainty and unknown disturbance represented in Equation (1.1) with all faults set to be zero, then the system is given by

$$ \begin{cases} 
    x_{e_{k+1}} &= (\Phi + \Delta \Phi)x_{e_k} + (\Psi + \Delta \Psi)u_k + \Gamma d_k + w_k \\
    z_{e_{k+1}} &= Hx_{e_{k+1}} + v_{k+1} 
\end{cases} $$

(4.28)

The nominal system of (4.28) is modeled as system (2.48) in a time-invariant case, i.e.

$$ \begin{cases} 
    x_{k+1} &= \Phi x_k + \Psi u_k + \Gamma d_k + w_k \\
    z_{k+1} &= Hx_{k+1} + v_{k+1} 
\end{cases} $$

(4.29)

The structure of an optimal observer is expressed as

$$ \begin{cases} 
    q_{k+1} &= F_{k+1} q_k + T_{k+1} \Psi u_k + K_{k+1} z_{e_k} \\
    \hat{x}_{k+1} &= q_{k+1} + N_{k+1} z_{e_{k+1}} 
\end{cases} $$

(4.30)
4.2.1 Robust Residual Generation Based on UIO

The state estimation error of the observer (4.30) is:

\[
e_{k+1} = x_{ek+1} - \hat{x}_{k+1}
\]
\[
= x_{ek+1} - (q_{k+1} + N_{k+1}z_{ek+1})
\]
\[
= x_{ek+1} - q_{k+1} - N_{k+1}(Hx_{ek+1} + v_{k+1})
\]
\[
= (I - N_{k+1}H)x_{ek+1} - q_{k+1} - N_{k+1}v_{k+1}
\]
\[
= (I - N_{k+1}H)x_{ek+1} - N_{k+1}v_{k+1}
\]
\[
- [F_{k+1}q_k + T_{k+1}\Psi u_k + (K^1_{k+1} + K^2_{k+1})z_{ek}]
\]
\[
= (I - N_{k+1}H)x_{ek+1} - N_{k+1}v_{k+1} - T_{k+1}\Psi u_k
\]
\[
-F_{k+1}(x_{ek} - e_k - N_kz_k) - K^1_{k+1}(Hx_{ek} + v_k) - K^2_{k+1}z_{ek}
\]
\[
= F_{k+1}e_k - K^1_{k+1}v_k - N_{k+1}v_{k+1} + (I - N_{k+1}H)w_k
\]
\[
- [F_{k+1} - (I - N_{k+1}H)(\Phi + \Delta \Phi) + K^1_{k+1}H]x_{ek}
\]
\[
+(I - N_{k+1}H)\Gamma d_k - (K^2_{k+1} - F_{k+1}N_k)z_{ek}
\]
\[
- [T_{k+1}\Psi - (I - N_{k+1}H)(\Psi + \Delta \Psi)]u_k
\]
\[
= F_{k+1}e_k - K^1_{k+1}v_k - N_{k+1}v_{k+1} + (I - N_{k+1}H)w_k
\]
\[
+(I - N_{k+1}H)\Delta \Phi x_{ek} + (I - N_{k+1}H)\Delta \Psi u_k
\]
\[
- [F_{k+1} - (I - N_{k+1}H)\Phi + K^1_{k+1}H]x_{ek}
\]
\[
+(I - N_{k+1}H)\Gamma d_k - (K^2_{k+1} - F_{k+1}N_k)z_{ek}
\]
\[
- [T_{k+1} - (I - N_{k+1}H)]\Psi u_k
\]

(4.31)

where \(K_{k+1} = K^1_{k+1} + K^2_{k+1}\).
We notice that the perturbed state variable $x_{e_k}$ brings the parametric uncertain information into the state error dynamics in Equation (4.31). However, the parametric uncertainties will not affect the structure of the unknown input observer design; on the contrary, they will have an impact on the state error dynamics once the relations in Equations (2.52) – (2.55) hold. Based on the algorithm of the UIO design in Section 2.3, we realize that the error covariance matrix $P_k$ has a steady-state solution in Equation (2.66) if the following statement is satisfied:

“\( \Phi^1, \sqrt{TQT^T + N R N^T} \) is stabilizable and \( (\Phi^1, H) \) is detectable, then there is a unique positive definite solution $\bar{P}$ for the matrix Riccati Equation

\[
\bar{P} = \Phi^1 \bar{P} (\Phi^1)^T - \Phi^1 \bar{P} H^T (H \bar{P} H^T + R)^{-1} H \bar{P} (\Phi^1)^T + TQT^T + N R N^T
\]

where $\Phi^1 = \Phi - N H \Phi$, $N = \Gamma (H \Gamma)^+$, and $\lim_{k \to \infty} P_k = \bar{P}$ with $\bar{P}$ independent of $P_0$.”

Hence, we have the steady-state of the observer gain:

\[
\bar{K}^1 = \Phi^1 \bar{P} H^T [H \bar{P} H^T + R]^{-1}
\]

Therefore by using the steady-state of the observer gain $\bar{K}^1$, the relations in Equations (2.52) – (2.55) will have constant results, such as

\[
\Gamma = N H \Gamma
\]

\[
T = I - N H
\]

\[
F = \Phi - N H \Phi - \bar{K}^1 H
\]

\[
\bar{K}^2 = F N
\]

then the state error dynamics would be:

\[
e_{k+1} = F e_{k} - \bar{K}^1 v_k - N v_{k+1} + T w_k + T \Delta \Phi x_{e_k} + T \Delta \Psi u_k
\]
and the optimal observer using the steady state observer gain has the form of

\[
\begin{align*}
q_{k+1} &= Fq_k + T\Psi u_k + \bar{K}z_{e_k} \\
\hat{x}_{k+1} &= q_{k+1} + Nz_{e_{k+1}}
\end{align*}
\] (4.39)

where \( \bar{K} = \bar{K}^1 + \bar{K}^2 \).

### 4.2.2 Dynamic Threshold Design Based on UIO

The system (4.28) with possible actuator and sensor faults can be described as:

\[
\begin{align*}
x_{e_{k+1}} &= (\Phi + \Delta \Phi)x_{e_k} + (\Psi + \Delta \Psi)(u_k + f_{a_k}) + \Gamma d_k + w_k \\
z_{e_{k+1}} &= Hx_{e_{k+1}} + v_{k+1} + f_{s_{k+1}}
\end{align*}
\] (4.40)

where \( f_{a_k} \in \mathbb{R}^l \) is the actuator fault vector and \( f_{s_k} \in \mathbb{R}^m \) is the sensor fault vector. For this system, the state estimation error dynamics and the residual are governed by the following equations:

\[
\begin{align*}
e_{k+1} &= Fe_k - \bar{K}^1 v_k - Nv_{k+1} + Tw_k + T\Delta \Phi x_{e_k} + T\Delta \Psi u_k \\
&\quad + T(\Psi + \Delta \Psi)f_{a_k} - \bar{K}^1 f_{s_k} - Nf_{s_{k+1}} \\
r_{k+1} &= He_{k+1} + v_{k+1} + f_{s_{k+1}}
\end{align*}
\] (4.41, 4.42)

We want to find an upper-and-lower time response bound for the error dynamics \( e_{k+1} \) in Equation (4.38). Assume \( F \) has distinct eigenvalues \( \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n \). Let \( \bar{V} \) be a modal matrix diagonalizing \( F \), then

\[
F = \bar{V}\bar{\Lambda}\bar{V}^{-1} = \bar{V}\bar{\Lambda}\bar{W}
\] (4.43)

where \( \bar{W} = \bar{V}^{-1} \). Let

\[
\bar{W}^T = \begin{bmatrix} \bar{\omega}_1^T & \bar{\omega}_2^T & \cdots & \bar{\omega}_n^T \end{bmatrix},
\] (4.44)

\[
\bar{V} = \begin{bmatrix} \bar{\nu}_1 & \bar{\nu}_2 & \cdots & \bar{\nu}_n \end{bmatrix},
\] (4.45)
\[
\bar{\Lambda} = \begin{bmatrix}
\bar{\lambda}_1 & 0 & \cdots & 0 \\
0 & \bar{\lambda}_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \bar{\lambda}_n
\end{bmatrix} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n).
\] (4.46)

Note that \(\bar{\omega}_i\) is a row vector, and \(\bar{\nu}_i\) is a column vector, for \(i = 1, 2, \ldots, n\). Then

\[
F\bar{\nu}_i = \bar{\lambda}_i\bar{\nu}_i,
\] (4.47)

and

\[
\bar{\omega}_iF = \bar{\lambda}_i\bar{\omega}_i.
\] (4.48)

which satisfies the right and left eigenvalue-eigenvector constraints.

Thus the time response of the \(q\)-th component of error dynamics state vector \(e_k\) in Equation (4.38) can be written as:

\[
e_q(k) = \sum_{i=1}^{n} \bar{\nu}_{qi} \bar{\lambda}_i^k \bar{\omega}_i e_0 + \sum_{j=0}^{k-1} \sum_{i=1}^{n} \bar{\nu}_{qi} \bar{\lambda}_i^{k-j} \bar{\omega}_i \\
\cdot [T \Delta \Psi u_j + T \Delta \Phi x_e_j + Tw_j + (-\bar{K}^{-1})v_j + (-N)v_{j+1}]
\] (4.49)

where \(e_0\) is the initial condition for the error dynamics vector. The bound for \(e_q(k)\) can be expressed as

\[
|e_q(k)| \leq \sum_{i=1}^{n} \left| \bar{\nu}_{qi} \bar{\lambda}_i^k \bar{\omega}_i e_0 \right| + \sum_{j=0}^{k-1} \sum_{i=1}^{n} \left| \bar{\nu}_{qi} \bar{\lambda}_i^{k-j} \bar{\omega}_i \right| \\
\cdot \left| T \Delta \Psi u_j + T \Delta \Phi x_e_j + T w_j + (-\bar{K}^{-1})v_j + (-N)v_{j+1} \right|
\]

\[
\leq \sum_{i=1}^{n} \left| \bar{\nu}_{qi} \bar{\lambda}_i^k \bar{\omega}_i e_0 \right| - 1 + \sum_{j=0}^{k-1} \sum_{i=1}^{n} \left| \bar{\nu}_{qi} \bar{\lambda}_i^{k-j} \bar{\omega}_i \right| \\
\cdot \left[ \|T\| \sigma_{\text{max}}(\Delta \Psi) \|u_j\| + \|T\| \sigma_{\text{max}}(\Delta \Phi) \|x_{e_j}\| \\
+ \|T\| \|w_j\| + \|\bar{K}^{-1}\| \|v_j\| + \|-N\| \|v_{j+1}\| \right]
\] (4.50)

where the noises \(w_k, v_k\) are assumed to be bounded such as \(\|w_k\| \leq a\) and \(\|v_k\| \leq b\). And \(\sigma_{\text{max}}(\Delta \Phi)\) is obtained from the parametric uncertainty analysis in Chapter 3. The
perturbed state vector bound $\|x_{e_k}\|$ is expressed in Equation (4.24). Hence the dynamic threshold of the error dynamics $e_q(k)$ using the UIO is obtained as

$$|e_q(k)| \leq \sum_{i=1}^{n} |\bar{\nu}_q \tilde{\lambda}_i^k \bar{\omega}_i e_0| + \sum_{j=0}^{k-1} \sum_{i=1}^{n} \left( |\bar{\nu}_q \tilde{\lambda}_i^{k-j} \bar{\omega}_i| \cdot \left\{ \left\| T \right\| \cdot \left\| \sigma_{\text{max}}(\Delta \Psi) \right\| \left\| u_j \right\| + \sigma_{\text{max}}(\Delta \Phi) \right\} \cdot \left\| [x_1(j) + \epsilon_1(j)] \right\| \right) + a$$

$$+ b \left( \left\| -K^1 \right\| + \left\| -N \right\| \right)$$

$$= \eta_q(k) \quad (4.51)$$

The formula of the dynamic threshold for the residual of the $q$-th sensor is represented by Equation (4.27).

### 4.3 Summary

We have developed the algorithms of dynamic thresholds based on two kinds of observers: Kalmam filter and unknown input observer. The dynamic threshold is derived from the state error dynamics and contains upper-and-lower bounds by the time response analysis theories. The dynamic threshold of each component of the residual vector is given in the algorithm. Since the existence of the parametric uncertainties, the residual under fault-free condition will never be zero. We have considered the bounded parametric uncertainties in the dynamic threshold such that the unmodeled uncertainty will not cause false alarms.
chapter 5

sensor and actuator fault isolation schemes

5.1 Sensor Fault Isolation Schemes Using Kalman Filters

The purpose of fault isolation is to locate in which sensor or actuator the fault has occurred. Two observer schemes are applied for sensor fault isolation: Dedicated Observer Scheme (DOS) and Generalized Observer Scheme (GOS).

The dedicated observer scheme was proposed by Clark [14] in 1978. Several other papers [16, 15, 25, 28] also discussed the applications and improvements to this scheme. Dedicated observer scheme is designed for sensor fault detection and isolation, which uses observers dedicated to each sensor, i.e., each observer uses a different single sensor measurement to estimate the state variables of the system. Figure 5.1 shows the layout of a basic dedicated observer design. By using the DOS, multiple simultaneous faults in sensors can be principally detected and isolated with the aid of threshold logic. One disadvantage of DOS is that it has no freedom in the design to increase robustness to unknown inputs [26]. In Section 2.3, we stated the necessary and sufficient condition for the existence a UIO: “the maximum number of disturbance which can be decoupled cannot be larger than the number of independent measurements”. Thus, with only one sensor measurement in DOS, an unknown input observer cannot be achieved in general for multiple unknown inputs.
The generalized observer scheme introduced by Frank [25] in 1987 uses a bank of \( m \) observers similar to the DOS with \( m \) equal to the number of sensors. Each observer uses the plant input \( u \), plus outputs from all but the \( i \)-th sensor in order to evaluate the \( i \)-th sensor. This scheme provides single sensor fault detection ability with robustness to unknown input. Figure 5.2 shows the layout of a typical generalized observer design. To design sensor fault isolation scheme, all actuators are assumed to be fault-free and the nominal system with sensor fault are expressed as

\[
\begin{align*}
    x_{k+1} &= \Phi x_k + \Psi u_k + w_k \\
    z_{k+1} &= H x_{k+1} + M u_{k+1} + v_{k+1} + f_{s_{k+1}}
\end{align*}
\]

where \( k = 0, 1, \cdots \) and \( M \in \mathbb{R}^{m \times l} \).
Figure 5.2: Generalized Observer Scheme

5.1.1 Dedicated Observer Scheme

A set of Kalman filters based on each single output measurement are developed to estimate the state variables and the measured output variables for the system in Equation (5.1), assuming all actuators are fault-free. The observability of each individual sensor output is assumed in the scheme, which guarantees the design of the $m$-set Kalman filters. Sensor faults can be isolated through a comparison of $\hat{x}^{(1)}$ with $\hat{x}^{(2)}, \hat{x}^{(3)}, \ldots, \hat{x}^{(i)}, \ldots, \hat{x}^{(m)}$, respectively, where $\hat{x}^{(i)}$ represents the state estimation for the $i$-th Kalman filter using the $i$-th sensor output. Thus, in absence of faults, $\hat{x}^{(i)} = x$ for $i = 1, 2, \ldots, m$. If sensor $z_p$
is faulty, then \( \hat{x}^{(p)} \neq x \) and \( \hat{x}^{(i)} = x \) for \( i = 1, 2, \ldots, m \) and \( i \neq p \), i.e., the state estimation based on the \( p \)-th faulty sensor is different from those based on the non-faulty sensor outputs.

<table>
<thead>
<tr>
<th>fault in sensor 1</th>
<th>fault in sensor 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r^{(1)} = \hat{x}^{(1)} - \hat{x}^{(2)} \neq 0 )</td>
<td>( r^{(1)} = \hat{x}^{(1)} - \hat{x}^{(2)} \neq 0 )</td>
</tr>
<tr>
<td>( r^{(2)} = \hat{x}^{(2)} - \hat{x}^{(3)} = 0 )</td>
<td>( r^{(2)} = \hat{x}^{(2)} - \hat{x}^{(3)} \neq 0 )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( r^{(m)} = \hat{x}^{(m)} - \hat{x}^{(1)} \neq 0 )</td>
<td>( r^{(m)} = \hat{x}^{(m)} - \hat{x}^{(1)} = 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>fault in sensor 3</th>
<th>fault in sensor 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r^{(1)} = \hat{x}^{(1)} - \hat{x}^{(2)} = 0 )</td>
<td>( r^{(1)} = \hat{x}^{(1)} - \hat{x}^{(2)} = 0 )</td>
</tr>
<tr>
<td>( r^{(2)} = \hat{x}^{(2)} - \hat{x}^{(3)} \neq 0 )</td>
<td>( r^{(2)} = \hat{x}^{(2)} - \hat{x}^{(3)} = 0 )</td>
</tr>
<tr>
<td>( r^{(3)} = \hat{x}^{(3)} - \hat{x}^{(4)} \neq 0 )</td>
<td>( r^{(3)} = \hat{x}^{(3)} - \hat{x}^{(4)} \neq 0 )</td>
</tr>
<tr>
<td>( r^{(4)} = \hat{x}^{(4)} - \hat{x}^{(5)} = 0 )</td>
<td>( r^{(4)} = \hat{x}^{(4)} - \hat{x}^{(5)} \neq 0 )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( r^{(m)} = \hat{x}^{(m)} - \hat{x}^{(1)} = 0 )</td>
<td>( r^{(m)} = \hat{x}^{(m)} - \hat{x}^{(1)} = 0 )</td>
</tr>
</tbody>
</table>

\[
\text{Decompose the output space in Equation (5.1) as}
\]

\[
\begin{align*}
z_{i_{k+1}} &= H_i x_{k+1} + M_i u_{k+1} + v_{i_{k+1}} + f_{s_{i_{k+1}}} \\
&= H_i x_{k+1} + M_i u_{k+1} + v_{i_{k+1}} + 0 \quad (5.2)
\end{align*}
\]
where \( z_i \) is the \( i \)-th sensor output, for \( i = 1, 2, \ldots, m \), and

\[
H = \begin{bmatrix}
H_1^T, & H_2^T, & \ldots, & H_i^T, & \ldots, & H_m^T
\end{bmatrix}^T,
\]

\[
M = \begin{bmatrix}
M_1^T, & M_2^T, & \ldots, & M_i^T, & \ldots, & M_m^T
\end{bmatrix}^T,
\]

\[
v = \begin{bmatrix}
v_1, & v_2, & \ldots, & v_i, & \ldots, & v_m
\end{bmatrix}^T,
\]

\[
f = \begin{bmatrix}
f_{s_1}, & f_{s_2}, & \ldots, & f_{s_i}, & \ldots, & f_{s_m}
\end{bmatrix}^T.
\]

Then design Kalman filters using each decomposed output. Note \((\Phi, H_i)\) pairs, for \( i = 1, 2, \ldots, m \), should be observable. The \( i \)-th Kalman filter is:

\[
\hat{x}_{k+1|k+1}^{(i)} = \hat{x}_{k+1|k}^{(i)} + K_{k+1}^{(i)} (z_{ik+1} - \hat{z}_{ik+1|k}^{(i)})
\]

(5.3)

where \( K^{(i)} \in \mathbb{R}^{n \times 1} \) is the filter gain for the \( i \)-th Kalman filter.

The residuals are defined using the state estimations as:

\[
\begin{align*}
   r_k^{(1)} &= \hat{x}_k^{(1)} - \hat{x}_k^{(2)} \\
   r_k^{(2)} &= \hat{x}_k^{(2)} - \hat{x}_k^{(3)} \\
   \vdots \\
   r_k^{(m)} &= \hat{x}_k^{(m)} - \hat{x}_k^{(1)}
\end{align*}
\]

(5.4)

The basic dedicated observer scheme FDI decision making logic is given in Table 5.1. As can be seen from Table 5.1, for example, when sensor 1 is faulty, the first and last residuals \( r_k^{(1)}, r_k^{(m)} \) will be corrupted by the faulty sensor signal. Thus follow the FDI logic, we can conclude that sensor 1 is faulty. The dedicated observer scheme can be used to detect and isolate multiple sensor faults for \( m \geq 4 \).

For each set of state estimates \( \hat{x}_k^{(i)} (i = 1, 2, \ldots, m) \), the error dynamics of the state estimations is:

\[
e_{k+1|k}^{(i)} = \Phi(I_n - K_k^{(i)} H_i)e_{k|k-1}^{(i)} + \Delta \Psi u_k + \Delta \Phi \tilde{x}_{ek} + w_k \\
+ (-\Phi K_k^{(i)}) v_i + (-\Phi K_k^{(i)}) f_{s_i}
\]

(5.5)
where the sensor fault $f_{s_{ik}}$ will affect the state error dynamics through a multiplication with $-\Phi K_k^{(i)}$. However, in some cases, Kalman filter gains converge to zero values in a short time compare to the time when the sensor fault acts on the system. Under this circumstance, the sensor fault signal will not be shown in the state error dynamics. Thus an existing sensor fault is not detectable using the basic dedicated observer scheme design. Therefore, an improved dedicated observer scheme needs to be designed to compensate this problem.

5.1.2 Improved Dedicated Observer Scheme

The improved dedicated observer scheme is introduced by Frank [25] in 1987. Figure 5.3 shows the improved dedicated observer scheme. Each sensor output, $z_i$, is used
Table 5.2: Sensor FDI Decision Making Logic for Improved DOS

<table>
<thead>
<tr>
<th>fault in sensor 1</th>
<th>fault in sensor m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1^{(1)} = z_1 - \hat{z}_1^{(1)} \neq 0$</td>
<td>$r_1^{(1)} = z_1 - \hat{z}_1^{(1)} = 0$</td>
</tr>
<tr>
<td>$r_1^{(2)} = z_1 - \hat{z}_1^{(2)} \neq 0$</td>
<td>$r_1^{(2)} = z_1 - \hat{z}_1^{(2)} = 0$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$r_1^{(m)} = z_1 - \hat{z}_1^{(m)} \neq 0$</td>
<td>$r_1^{(m)} = z_1 - \hat{z}_1^{(m)} = 0$</td>
</tr>
<tr>
<td>$r_2^{(1)} = z_2 - \hat{z}_2^{(1)} = 0$</td>
<td>$r_2^{(1)} = z_2 - \hat{z}_2^{(1)} = 0$</td>
</tr>
<tr>
<td>$r_2^{(2)} = z_2 - \hat{z}_2^{(2)} = 0$</td>
<td>$r_2^{(2)} = z_2 - \hat{z}_2^{(2)} = 0$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$r_2^{(m)} = z_2 - \hat{z}_2^{(m)} = 0$</td>
<td>$r_2^{(m)} = z_2 - \hat{z}_2^{(m)} = 0$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$r_m^{(1)} = z_m - \hat{z}_m^{(1)} = 0$</td>
<td>$r_m^{(1)} = z_m - \hat{z}_m^{(1)} \neq 0$</td>
</tr>
<tr>
<td>$r_m^{(2)} = z_m - \hat{z}_m^{(2)} = 0$</td>
<td>$r_m^{(2)} = z_m - \hat{z}_m^{(2)} \neq 0$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$r_m^{(m)} = z_m - \hat{z}_m^{(m)} = 0$</td>
<td>$r_m^{(m)} = z_m - \hat{z}_m^{(m)} \neq 0$</td>
</tr>
</tbody>
</table>

to drive one Kalman filter, which is the same as the basic dedicated observer scheme. The dedicated Kalman filter is designed of full order or reduced order to observe as many of the remaining measured outputs as possible. Assume the entire process is completely observable, we have $m$-set output estimates. In the fault-free case, $\hat{z}_i^{(p)} = \hat{z}_i^{(q)} = z_i$ holds for $i = 1, 2, \cdots, m$ and $p, q = 1, 2, \cdots, m$. If a fault occurs in the $p$-th sensor, $\hat{z}_i^{(p)} \neq \hat{z}_i^{(q)} = z_i$ for $i = 1, 2, \cdots, m$, $q = 1, 2, \cdots, m$ and $p \neq q$, i.e., the $p$-th Kalman filter that is driven by the faulty sensor provides wrong output estimates whereas the estimates of all the other Kalman filters match the corresponding $z_i$. The output estimates of the $i$-th Kalman filter is:

$$\hat{z}_{k+1|k}^{(i)} = H \hat{x}_{k+1|k}^{(i)} + M u_{k+1} \quad (5.6)$$
where $\hat{x}_{k+1|k}^{(i)}$ is the same as in Equation (5.3).

The residuals are defined as:

$$r_{ik}^{(j)} = z_{ik} - \hat{z}_{ik}^{(j)}$$

(5.7)

where $i, j = 1, 2, \ldots, m$. Thus the sensor fault enters the error dynamics without losing the impact on it. The improved dedicated observer scheme FDI decision making logic is given in Table 5.2. As can be seen from Table 5.2, for example, when sensor 1 is faulty, the residuals $r_{1}^{(i)}$ for $i = 1, 2, \ldots, m$ are corrupted by the faulty sensor signal. Thus based on the FDI logic, we can conclude that sensor 1 is faulty. The dedicated observer scheme can be used to detect and isolate multiple sensor faults.

### 5.1.3 Generalized Observer Scheme

Generalized observer scheme uses an observer dedicated to one specific sensor. The corresponding observer using all sensor measurements except the one from the sensor to be diagnosed. A set of Kalman filters are designed based on this scheme for the system in Equation (5.1), assuming all actuators are fault-free. The observability of each set of sensors is assumed in the scheme, which guarantees the design of the $m$-set Kalman filters. Single sensor fault can be isolated through checking the residual sets. In absence of faults, all the residual sets will be zero, which means they are not crossing the threshold. If the $p$-th sensor is faulty, all the residual sets except the $p$-th one which does not use the $p$-th sensor as one of the measurements will cross the threshold, which indicates the $p$-th sensor is faulty. Decompose the output space in Equation (5.1) as

$$z_{k+1}^{(i)} = H^{(i)} x_{k+1} + M^{(i)} u_{k+1} + v_{k+1}^{(i)} + f_{k+1}^{(i)}$$

(5.8)
<table>
<thead>
<tr>
<th>fault in sensor 1</th>
<th>fault in sensor 2</th>
<th>fault in sensor 3</th>
<th>fault in sensor 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r^{(1)} = z^{(1)} - \hat{z}^{(1)} = 0$</td>
<td>$r^{(1)} = z^{(1)} - \hat{z}^{(1)} \neq 0$</td>
<td>$r^{(1)} = z^{(1)} - \hat{z}^{(1)} \neq 0$</td>
<td>$r^{(1)} = z^{(1)} - \hat{z}^{(1)} \neq 0$</td>
</tr>
<tr>
<td>$r^{(2)} = z^{(2)} - \hat{z}^{(2)} \neq 0$</td>
<td>$r^{(2)} = z^{(2)} - \hat{z}^{(2)} = 0$</td>
<td>$r^{(2)} = z^{(2)} - \hat{z}^{(2)} \neq 0$</td>
<td>$r^{(2)} = z^{(2)} - \hat{z}^{(2)} = 0$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$r^{(m)} = z^{(m)} - \hat{z}^{(m)} \neq 0$</td>
<td>$r^{(m)} = z^{(m)} - \hat{z}^{(m)} \neq 0$</td>
<td>$r^{(m)} = z^{(m)} - \hat{z}^{(m)} \neq 0$</td>
<td>$r^{(m)} = z^{(m)} - \hat{z}^{(m)} = 0$</td>
</tr>
<tr>
<td>fault in sensor $m - 1$</td>
<td>fault in sensor $m$</td>
<td>fault in sensor $m$</td>
<td>fault in sensor $m$</td>
</tr>
<tr>
<td>$r^{(1)} = z^{(1)} - \hat{z}^{(1)} \neq 0$</td>
<td>$r^{(1)} = z^{(1)} - \hat{z}^{(1)} \neq 0$</td>
<td>$r^{(1)} = z^{(1)} - \hat{z}^{(1)} \neq 0$</td>
<td>$r^{(1)} = z^{(1)} - \hat{z}^{(1)} \neq 0$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$r^{(m-2)} = z^{(m-2)} - \hat{z}^{(m-2)} \neq 0$</td>
<td>$r^{(m-2)} = z^{(m-2)} - \hat{z}^{(m-2)} \neq 0$</td>
<td>$r^{(m-2)} = z^{(m-2)} - \hat{z}^{(m-2)} \neq 0$</td>
<td>$r^{(m-2)} = z^{(m-2)} - \hat{z}^{(m-2)} = 0$</td>
</tr>
<tr>
<td>$r^{(m-1)} = z^{(m-1)} - \hat{z}^{(m-1)} = 0$</td>
<td>$r^{(m-1)} = z^{(m-1)} - \hat{z}^{(m-1)} = 0$</td>
<td>$r^{(m-1)} = z^{(m-1)} - \hat{z}^{(m-1)} = 0$</td>
<td>$r^{(m-1)} = z^{(m-1)} - \hat{z}^{(m-1)} \neq 0$</td>
</tr>
<tr>
<td>$r^{(m)} = z^{(m)} - \hat{z}^{(m)} \neq 0$</td>
<td>$r^{(m)} = z^{(m)} - \hat{z}^{(m)} = 0$</td>
<td>$r^{(m)} = z^{(m)} - \hat{z}^{(m)} = 0$</td>
<td>$r^{(m)} = z^{(m)} - \hat{z}^{(m)} \neq 0$</td>
</tr>
</tbody>
</table>

where for $i = 1, 2, \cdots, m$:

\[
\begin{align*}
\mathbf{z}^{(i)} &= \begin{bmatrix} z_1, \cdots, z_{i-1}, \hat{z}_{i+1}, \cdots, \hat{z}_m \end{bmatrix}^T, \\
\mathbf{H}^{(i)} &= \begin{bmatrix} \mathbf{H}_{1}^T, \cdots, \mathbf{H}_{i-1}^T, \mathbf{H}_{i+1}^T, \cdots, \mathbf{H}_{m}^T \end{bmatrix}^T, \\
\mathbf{M}^{(i)} &= \begin{bmatrix} \mathbf{M}_{1}^T, \cdots, \mathbf{M}_{i-1}^T, \mathbf{M}_{i+1}^T, \cdots, \mathbf{M}_{m}^T \end{bmatrix}^T, \\
\mathbf{v}^{(i)} &= \begin{bmatrix} v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_m \end{bmatrix}^T, \\
\mathbf{f}^{(i)} &= \begin{bmatrix} f_1, \cdots, f_{i-1}, f_{i+1}, \cdots, f_m \end{bmatrix}^T.
\end{align*}
\]
Then design Kalman filters using each decomposed output. Note \((\Phi, H^{(i)})\) pairs, for \(i = 1, 2, \cdots, m\), should be observable. The \(i\)-th Kalman filter is:

\[
\hat{x}^{(i)}_{k+1|k+1} = \hat{x}^{(i)}_{k+1|k} + K^{(i)}_{k+1} \left( z^{(i)}_{k+1} - \hat{z}^{(i)}_{k+1|k} \right)
\]  

(5.9)

where \(K^{(i)} \in \mathbb{R}^{n \times (m-1)}\) is the filter gain for the \(i\)-th Kalman filter.

The residuals are defined using the measured output estimations as:

\[
r^{(i)}_k = z^{(i)}_k - \hat{z}^{(i)}_k
\]  

(5.10)

for \(i = 1, 2, \cdots, m\). The generalized observer scheme FDI decision making logic is given in Table 5.3. As can be seen in Table 5.3, for example, when sensor 1 is faulty, all the residuals except the first one will be corrupted by the faulty sensor signal. Following the FDI logic, we conclude that sensor 1 is faulty. The generalized observer scheme can be only used to detect and isolate single sensor fault with the robustness to unknown inputs.

### 5.2 Actuator Fault Isolation Scheme Using UIOs

The robust actuator fault isolation scheme is discussed by Chen and Patton [9]. The idea for actuator fault isolation is very similar to that of the generalized observer scheme. The \(i\)-th observer is designed for using all the actuators except the \(i\)-th one. When a fault occurs in the \(i\)-th actuator, the observer which does not use this actuator would not show a faulty residual, whereas all the other residuals would. That is how an actuator fault is located. To design the robust actuator fault isolation scheme, all sensors are assumed to be fault-free and the nominal system with actuator fault is represented as:

\[
\begin{cases}
\dot{x}_{k+1} = \Phi x_k + \Psi (u_k + f_{a_k}) + w_k \\
\dot{z}_{k+1} = H x_{k+1} + M (u_{k+1} + f_{a_{k+1}}) + v_{k+1}
\end{cases}
\]  

(5.11)
Suppose there is a fault in the $i$-th actuator, we write Equation (5.11) in the form of

\[
\begin{align*}
    x_{k+1} &= \Phi x_k + \Psi^{(i)} u^{(i)}_k + \Psi^{(i)} f^{(i)}_{a_k} + \psi_i (u_{i_k} + f_{i_a_k}) + w_k \\
    &= \Phi x_k + \Psi^{(i)} u^{(i)}_k + \Psi^{(i)} f^{(i)}_{a_k} + \Gamma^{(i)} d^{(i)}_k + w_k \\
    z_{k+1} &= H x_{k+1} + M (u_{k+1} + f_{a_{k+1}}) + v_{k+1} \quad \text{for } i = 1, 2, \ldots, l
\end{align*}
\]  

(5.12)

where $\psi_i$ is the $i$-th column of the matrix $\Psi$, $\Psi^{(i)} \in \mathbb{R}^{n \times (l-1)}$ is from the matrix $\Psi$ by deleting the $i$-th column $\psi_i$, $u_i$ is the $i$-th element of control input vector $u$, $u^{(i)} \in \mathbb{R}^{l-1}$ is from the vector $u$ by deleting the $i$-th element $u_i$, and $\Gamma^{(i)} = [\psi_i], d^{(i)} = [u_{i_k} + f_{i_a_k}]$.

Under this re-constructon, the faulty actuator is separated from all the other actuators and is treated as an unknown input (disturbance) of the original system. However, the output equation is not in the standard form of system (4.29) and the control $u$ term in the output equation would cause a sensor-fault like signal, which would make fault isolation difficult.

In order to remove the $u$ term in the output, we can introduce a transformation in the output equation such that $u$ term can be nulled, i.e.

\[
\tilde{z}_{k+1} = T_z z_{k+1} = T_z H x_{k+1} + T_z M u_{k+1} + T_z v_{k+1} = \tilde{H} x_{k+1} + \tilde{v}_{k+1}
\]  

(5.13)

where $T_z M = 0$, $\tilde{H} = T_z H$ and $\tilde{v} = T_z v$. $T_z$ can be chosen by the following lemma:

**Lemma 1:** Assume matrices $M \in \mathbb{R}^{m \times l}$ ($m > l$) of full rank and $T_z \in \mathbb{R}^{m \times m}$, the matrix $T_z$ is in the form of

\[
T_z = M (M^T M)^{-1} M^T - I_m
\]  

(5.14)

such that $T_z M = 0$, where $I_m \in \mathbb{R}^{m \times m}$.

**Proof:** Let $T_z = T_{z_1} - I_m$, where $T$, $I_m \in \mathbb{R}^{m \times m}$. Then

\[
(T_{z_1} - I_m) M = 0.
\]

i.e. $T_{z_1} M = M$. 

69
so \( M \) has a left inverse (for \( m > l \)): 
\[
M^+ = (M^T M)^{-1} M^T.
\]
Thus we have
\[
T_{z_1} = M M^+ = M (M^T M)^{-1} M^T.
\]
Hence,
\[
T_z = M (M^T M)^{-1} M^T - I_m.
\]

Now we do a simple transformation of \( \bar{z}_{k+1} = T_z z_{k+1} \) for Equation (5.11). Then the \( u \) term in the output will be deleted, thus the \( z \) term in Equation (5.12) will be changed to \( \bar{z} \). Based on the above description, \( l \)-set UIO-based residual generators can be derived as:
\[
\begin{align*}
q^{(i)}_{k+1} &= F^{(i)}_{k+1} q^{(i)}_k + T^{(i)}_{k+1} \Psi^{(i)}_k u_k + K^{(i)}_{k+1} \bar{z}_k \\
\hat{x}^{(i)}_{k+1} &= q^{(i)}_{k+1} + N^{(i)}_{k+1} \bar{z}_{k+1}
\end{align*}
\tag{5.15}
\]

We can see from Equation (5.15), once there exists an actuator fault, this fault would corrupt with all the state variables in the entire system. The estimated outputs based on the control command inputs and sensor outputs for each observer would differ from the actual outputs.

All the parameter matrices must satisfy the conditions as follows:
\[
\begin{align*}
\Gamma^{(i)}_k &= N^{(i)}_{k+1} \bar{H} \Gamma^{(i)}_k \\
T^{(i)}_{k+1} &= I - N^{(i)}_{k+1} \bar{H} \\
F^{(i)}_{k+1} &= \Phi - N^{(i)}_{k+1} \bar{H} \Phi - K^{(i)}_{k+1} \bar{H} \\
K^{(i)}_{k+1} &= F^{(i)}_{k+1} N^{(i)}_k
\end{align*}
\tag{5.16-19}
\]

The residuals are defined using the output estimations as:
\[
r^{(i)}_k = \bar{z}_k - \hat{z}^{(i)}_k
\tag{5.20}
\]
for \( i = 1, 2, \ldots, l \). Each observer can generate the residuals driven by all measured outputs and all control inputs except the \( i \)-th one. When all sensors are fault-free and a fault occurs
in the $i$-th actuator, all the other residuals will exceed the threshold but the $i$-th one. A UIO-based actuator fault isolation scheme is show in Figure 5.4.

![Figure 5.4: Actuator Fault Isolation Scheme](image)

The actuator FDI decision making logic is given in Table 5.4. As can be seen in Table 5.4, for example, when actuator 1 is faulty, all the residuals but the first one will be corrupted by the faulty actuator signal. Following the FDI logic, we conclude that actuator 1 is faulty.

The actuator fault isolation scheme presented in this section can be only used to detect and isolate single actuator fault.
## Table 5.4: Actuator FDI Decision Making Logic

<table>
<thead>
<tr>
<th>fault in actuator 1</th>
<th>fault in actuator 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r^{(1)} = \bar{z} - \hat{z}^{(1)} = 0 )</td>
<td>( r^{(1)} = \bar{z} - \hat{z}^{(1)} \neq 0 )</td>
</tr>
<tr>
<td>( r^{(2)} = \bar{z} - \hat{z}^{(2)} \neq 0 )</td>
<td>( r^{(2)} = \bar{z} - \hat{z}^{(2)} = 0 )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( r^{(l)} = \bar{z} - \hat{z}^{(l)} \neq 0 )</td>
<td>( r^{(l)} = \bar{z} - \hat{z}^{(l)} \neq 0 )</td>
</tr>
<tr>
<td>fault in actuator ( l - 1 )</td>
<td>fault in actuator ( l )</td>
</tr>
<tr>
<td>( r^{(1)} = \bar{z} - \hat{z}^{(1)} \neq 0 )</td>
<td>( r^{(1)} = \bar{z} - \hat{z}^{(1)} \neq 0 )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( r^{(l-1)} = \bar{z} - \hat{z}^{(l-1)} = 0 )</td>
<td>( r^{(l-1)} = \bar{z} - \hat{z}^{(l-1)} \neq 0 )</td>
</tr>
<tr>
<td>( r^{(l)} = \bar{z} - \hat{z}^{(l)} \neq 0 )</td>
<td>( r^{(l)} = \bar{z} - \hat{z}^{(l)} = 0 )</td>
</tr>
</tbody>
</table>

### 5.3 Summary

In this section the commonly accepted sensor and actuator fault isolation schemes are discussed in detail. The fault isolation approach aims to design a structured residual set. The “structure” term means that each residual is designed to be sensitive to a certain fault or a certain group of faults, whereas insensitive to others. The fault isolation scheme introduced here is to make each residual to be sensitive to faults in all but one sensors or actuators. The drawback of the basic dedicated observer scheme has been analyzed and an improved dedicated observer scheme is given. Dedicated observer scheme can be used to detect and isolate multiple simultaneous sensor faults; however, it has no robustness to unknown inputs. Generalized observer scheme for sensor fault isolation is accepted more widely due to its design freedom such as robustness to disturbances (unknown inputs).
Actuator fault isolation is designed under similar scope as generalized observer scheme. Sensor fault isolation is always possible, however, actuator fault isolation is not always possible because the residual dedicated to one actuator might be corrupted by other actuator faults.
6.1 Introduction

The intent of this chapter is to apply the dynamic threshold approach to the aircraft engine sensor fault and actuator fault detection and isolation. The engine model is the component level model (CLM) of GE90-115B, which is an advanced high-bypass two-spool commercial turbofan engine model.

6.1.1 Turbofan Engine Overview

A turbofan engine is composed of several main components: fan, low pressure compressor (LPC) or booster, high pressure compressor (HPC), combustion chamber, high pressure turbine (HPT), low pressure turbine (LPT) and nozzle. They are arranged in the direction along the gas path. Figure 6.1 shows a schematic of a high-bypass turbofan engine and the corresponding gas path. Air entering the engine nacelle is drawn up by the fan and can follow two paths: a vast quantity of air enters the bypass duct, and the remaining travels around the engine core (i.e. HPC, combustion chamber and HPT), where it is exhausted out of the nozzle to provide thrust. The air that enters the core is compressed by the low
pressure compressor or booster, which rotates on the same shaft as the fan. After the air is compressed by the low pressure compressor, it is further fed to the high pressure compressor, which rotates at a much faster speed and brings the airflow up to the design pressure ratio. The highly compressed air is sent to the combustion chamber, where fuel is injected and burned. The resulting hot gas from the combustion chamber drives the high pressure turbine and the low pressure turbine. After the combustion gas leaves the low pressure turbine, the core exhaust leaves the engine through the nozzle.

From a structural point of view, the engine consists of rotating part and non-rotating part. The non-rotating components include the bypass ducts and stator vanes. The shafts in turn support the fan, compressor and turbine disks, which are the rotating components. The fan shaft (or called LPC shaft) passes through the core shaft and is separated from it by bearings. The fan shaft supports the fan, low pressure compressor and low pressure turbine, where the low pressure turbine drives the low pressure compressor and the fan. The
core shaft supports the high pressure compressor and high pressure turbine, where the high pressure turbine drives the high pressure compressor. This type of arrangement is called a two spool engine: one “spool” for the fan, the other “spool” for the core.

### 6.1.2 Aircraft Engine FDI Survey

The demand for a safer and more reliable aircraft gas turbine engine control system has stimulated considerable research on aircraft engine FDI approaches and technologies over decades [79, 19]. Litt et al. presented a survey of general health management technologies for aircraft propulsion systems in [61]. It described the existing state of engine control and on-board health management and included the technology of promising control, diagnostics and prognostics algorithms, etc. A number of FDI methods have been specifically applied to turbine engine systems. One can view the FDI approaches in many different perspectives, in this survey, we review the FDI methods by the modeling type: (i) data-driven or empirical methods; (ii) model-based methods.

Data-driven FDI method, also known as one aspect of the knowledge-based FDI techniques, includes contributions such as in papers [76, 80, 5]. The data-driven FDI approach uses more generic computer learning algorithms, such as artificial neural networks [5]. Artificial neural networks are built based on the neural structure of the human brain, in which the weights among neurons are automatically tuned so that the model fits the data dynamically. If several samples of each fault are available from a system, they can be used to train a neural network without detailed information of that system. The data-driven prognosis using artificial neural networks to model the system can be used to detect the precursors of a failure and predict how much time remains before a likely failure [80].

76
Within the model-based approach, different modeling concepts such as inverse dynamic model [59], qualitative modeling [45] and component level model [48] are applied. Recently, efforts are being made to consider using hybrid model [84], which is a combination of both the data driven information as well as an analytical mathematical model based information. In general, the FDI methods for gas turbine engine include contributions by approaches based on eigenstructure assignment [72], sensor based FDI for a T700 turboshaft engine in [60], gas-path analysis for gas turbine engine FDI in [83, 18], gas-path fault diagnosis in [71, 30]. Methods based on optimization techniques for gas turbine engine FDI are presented in [95]. A popular technique widely used in turbine engine community is the Kalman filter based one, pioneered by [49, 50, 51]. In [85], Volponi compared the advantages and disadvantages of using Kalman filter based and neural network based estimators. Additional approaches include probabilistic methods and information fusion such as in [55].

A difficult challenge in fault detection and isolation applications is the design of a scheme which can distinguish between model uncertainties and the occurrence of faults. Currently in literature, a threshold for aircraft engine FDI is usually predetermined and constant based on empirical data. There are no useful guidelines for constant optimal threshold selections [81, 64, 20]. Simani [81] introduced a simple threshold detection methodology using a state estimation approach. Lughofer, Efendic, Del Re and Klement [64] used a threshold which is set to 3 or 4 times the accuracy of the corresponding sensor. An empirically trained fault detection threshold method was presented by Depold, Rajamani, Morrison and Pattipati [20]. As discussed in Chapter 1, a fixed threshold is not good enough to accommodate the existence of model uncertainty and change of control input, which inspires
the research of finding a dynamic threshold which is robust to the model uncertainty and disturbances in this work.

Table 6.1: GE90-115B Engine Model Parameter Notation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>HPC</td>
<td>high pressure compressor</td>
</tr>
<tr>
<td>HPT</td>
<td>high pressure turbine</td>
</tr>
<tr>
<td>LPC</td>
<td>low pressure compressor</td>
</tr>
<tr>
<td>LPT</td>
<td>low pressure turbine</td>
</tr>
<tr>
<td>WF36</td>
<td>fuel flow</td>
</tr>
<tr>
<td>AE24</td>
<td>variable bleed valve</td>
</tr>
<tr>
<td>STP25</td>
<td>variable stator vane angle</td>
</tr>
<tr>
<td>XN2</td>
<td>low pressure rotor speed sensor, rpm</td>
</tr>
<tr>
<td>XN25</td>
<td>high pressure rotor speed sensor, rpm</td>
</tr>
<tr>
<td>T25</td>
<td>HPC inlet temperature, °C</td>
</tr>
<tr>
<td>P25</td>
<td>HPC inlet pressure</td>
</tr>
<tr>
<td>T3</td>
<td>combustor inlet temperature, °C</td>
</tr>
<tr>
<td>P3</td>
<td>combustor inlet pressure</td>
</tr>
<tr>
<td>T49</td>
<td>LPT inlet temperature, °C</td>
</tr>
<tr>
<td>TMHS23</td>
<td>LPC metal temperature, °C</td>
</tr>
<tr>
<td>TMHS3</td>
<td>HPC metal temperature, °C</td>
</tr>
<tr>
<td>TMHS41</td>
<td>HPT nozzle metal temperature, °C</td>
</tr>
<tr>
<td>TMHS49</td>
<td>LPT metal temperature, °C</td>
</tr>
<tr>
<td>ALT</td>
<td>altitude, ft</td>
</tr>
<tr>
<td>XM</td>
<td>Mach number</td>
</tr>
</tbody>
</table>
6.2 GE90-115B Turbofan Engine Sensor and Actuator FDI

GE90-115B engine (see Figure 6.2) is designed for Boeing 777-300ER aircraft, which has a standard seating capacity of 359 persons, a range of 13,380 km (7,225 n-mile), and a maximum takeoff weight of 340,200 kg (750,000 lb) with a maximum takeoff thrust of 511 kN (115,000 lb) [35].

![Figure 6.2: General Electric GE90 Turbofan Engine](image)

The GE90-115B turbofan engine model being studied in this section is a nonlinear simulation built by GE Global Research. This simulation is constructed as a component level model, which assembles the major components of an aircraft engine (see Figure 6.3) [66]: fan, LPC, HPC, combustor, HPT and LPT. The list of CLM engine model parameter notation is shown in Table 6.1.

6.2.1 Engine Model

The nonlinear dynamic model of the engine is given by

\[
\begin{align*}
\dot{X} &= g(X, U) \\
Z &= h(X, U) + v
\end{align*}
\]

(6.1)

where the state vector \(X \in \mathbb{R}^6\), the control input vector \(U \in \mathbb{R}^3\) and the measured output vector \(Z \in \mathbb{R}^7\). \(v \in \mathbb{R}^7\) is a zero-mean white Gaussian sensor noise sequence with
Figure 6.3: Schematic Diagram of a High-Bypass-Ratio Turbofan

covariance matrix $R$. The engine state variables, actuators and sensor measurements in the model are listed in Table 6.2. The turbofan engine is a very complicated nonlinear dynamic system. The nonlinear functions of $g(X, U)$ and $h(X, U)$ cannot be written out in the analytic equations. The system behavior is simulated by the nonlinear dynamic model CLM. CLM is capable of simulating the different steady-states and transient operating envelope of the engine, and it can also generate linearized models for selected operating points.

In this study, we first simulate the engine from stillness to idle condition. The fan spool idle speed is 2482 rpm. The nonlinear model is linearized around the operating point which is chosen as the steady-state of the control input vector $U$ and the state vector $X$, they are:

$$\mathbf{u}_{ss} = \begin{bmatrix} \text{WF36} \\ \text{AE24} \\ \text{STP25} \end{bmatrix}_{ss} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_{ss} = \begin{bmatrix} 40700 \\ 0 \\ 0.9233 \end{bmatrix}.$$  \hspace{1cm} (6.2)
Table 6.2: GE90-115B Engine State Variables, Actuators and Sensors

<table>
<thead>
<tr>
<th>State Variables</th>
<th>Actuators</th>
<th>Sensors</th>
</tr>
</thead>
<tbody>
<tr>
<td>XN2</td>
<td>WF36</td>
<td>XN2</td>
</tr>
<tr>
<td>XN25</td>
<td>AE24</td>
<td>XN25</td>
</tr>
<tr>
<td>TMHS23</td>
<td>STP25</td>
<td>T25</td>
</tr>
<tr>
<td>TMHS3</td>
<td>STP25</td>
<td>P25</td>
</tr>
<tr>
<td>TMHS41</td>
<td>T3</td>
<td>P3</td>
</tr>
<tr>
<td>TMHS49</td>
<td>T49</td>
<td></td>
</tr>
</tbody>
</table>

and

\[
\begin{bmatrix}
XN2 \\
XN25 \\
TMHS23 \\
TMHS3 \\
TMHS41 \\
TMHS49
\end{bmatrix}_{ss} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}_{ss} = \begin{bmatrix} 2482.8 \\ 10666 \\ 602.1 \\ 1187.6 \\ 2182.8 \\ 2322.1 \end{bmatrix}.
\]

Correspondingly, the steady-state for the measured output \( Z \) is

\[
\begin{bmatrix}
XN2 \\
XN25 \\
T25 \\
P25 \\
T3 \\
P3 \\
T49
\end{bmatrix}_{ss} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{bmatrix}_{ss} = \begin{bmatrix} 2482.8 \\ 10666 \\ 685.6 \\ 33.7 \\ 660.0 \\ 2321.5 \end{bmatrix}.
\]

We define

\[
x = X - x_{ss}
\]

\[
u = U - u_{ss}
\]

\[
z = Z - z_{ss}
\]
Thus the linearized model is a linear perturbation state-variable model. If \( u_i (i = 1, 2, 3) \), is small (e.g. 1\%), \( x_j (j = 1, 2, \ldots, 6) \), will be small, i.e. all variables have a small variation around the operating point and the linear model is derived:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
z(t) &= Cx(t) + Du(t) + v(t), \quad t = t_i \quad i = 1, 2, \ldots
\end{align*}
\]

(6.5)

where \( v(t_i) \) is a discrete-time white noise sequence.

So when the nonlinear simulation model is linearized around the operating point, the linear model matrices are:

\[
A_{6\times6} = \begin{bmatrix}
-2.4769 & 0.1667 & -0.7350 & 0.3521 & 0.2501 & 0.5493 \\
-2.1505 & -2.4918 & 4.2445 & 1.2542 & 1.0216 & 0.2500 \\
0.0258 & -0.0007 & -0.3300 & 0.0000 & 0.0000 & 0.0000 \\
0.0378 & 0.0588 & 0.3182 & -0.7256 & 0.0069 & 0.0000 \\
-0.2187 & 0.1356 & 0.3148 & 0.1507 & -0.1093 & 0.0000 \\
-0.2372 & 0.0189 & 0.0977 & 0.1348 & 0.0960 & -0.5071 \\
\end{bmatrix},
\]

\[
B_{6\times3} = \begin{bmatrix}
0.0374 & 0.0000 & -19.9215 \\
0.1328 & 0.0000 & 247.0555 \\
0.0000 & 0.0000 & 0.2468 \\
0.0010 & 0.0000 & -4.4892 \\
0.0132 & 0.0000 & -12.0208 \\
0.0129 & 0.0000 & -1.3782 \\
\end{bmatrix},
\]

(6.6)

\[
C_{7\times6} = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\
0.1114 & -0.0031 & 0.5727 & 0 & 0 & 0 & 0 \\
0.0201 & -0.0012 & 0.0156 & 0 & 0 & 0 & 0 \\
-0.0383 & 0.1171 & 0.0454 & 0.5908 & 0.0134 & 0 & 0 \\
0.2409 & 0.0278 & -0.0937 & 0.0456 & 0.0242 & 0.0001 & 0 \\
-0.3281 & 0.0264 & 0.1348 & 0.1865 & 0.1329 & 0.2977 & 0 \\
\end{bmatrix},
\]

\[
D_{7\times3} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1.0682 \\
0 & 0 & 0.4122 \\
0.0020 & 0 & -9.9880 \\
0.0036 & 0 & -1.0746 \\
0.0179 & 0 & -1.7047 \\
\end{bmatrix}.
\]
In the next step, we need to discretize the linearized model. The sampling time is set as $T_s = 0.2$ sec. Then the discretized model is expressed as:

$$
\begin{align*}
  x_{k+1} &= \Phi x_k + \Psi u_k \\
  z_{k+1} &= H x_{k+1} + M u_{k+1} + v_{k+1}
\end{align*}
$$

(6.7)

where the discretized model matrices are:

$$
\Phi_{6 \times 6} = 
\begin{bmatrix}
  0.6022 & 0.0211 & -0.0972 & 0.0559 & 0.0384 & 0.0812 \\
  -0.2610 & 0.6058 & 0.6774 & 0.1738 & 0.1365 & -0.0540 \\
  0.0039 & 0.0087 & 0.9358 & 0.0001 & 0.0001 & 0.0002 \\
  0.0345 & 0.0187 & 0.0674 & 0.0266 & 0.8151 & -0.0025 \\
  -0.0361 & 0.0026 & 0.0233 & 0.0231 & 0.0158 & 0.9014 \\
  & & & & & \\
\end{bmatrix}
$$

(6.8)

$$
\Psi_{6 \times 3} = 
\begin{bmatrix}
  0.0064 & 0 & -2.6131 \\
  0.0199 & 0 & 39.2472 \\
  0 & 0 & 0.0372 \\
  0.0003 & 0 & -0.6095 \\
  0.0025 & 0 & -1.5843 \\
  0.0024 & 0 & -0.1412 \\
\end{bmatrix}
$$

$$
H_{7 \times 6} = 
\begin{bmatrix}
  1.0000 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1.0000 & 0 & 0 & 0 & 0 \\
  0.1114 & -0.0031 & 0.5727 & 0 & 0 & 0 \\
  0.0201 & -0.0012 & 0.0156 & 0 & 0 & 0 \\
  -0.0383 & 0.1171 & 0.0454 & 0.5908 & 0.0134 & 0 \\
  0.2409 & 0.0278 & -0.0937 & 0.0456 & 0.0242 & 0.0001 \\
  -0.3281 & 0.0264 & 0.1348 & 0.1865 & 0.1329 & 0.2977 \\
\end{bmatrix}
$$

$$
M_{7 \times 3} = 
\begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1.0682 \\
  0 & 0 & 0.4122 \\
  0.0020 & 0 & -9.9880 \\
  0.0036 & 0 & -1.0746 \\
  0.0179 & 0 & -1.7047 \\
\end{bmatrix}
$$

Kalman filter is used in the observer design for sensor fault diagnosis. We design a full-order Kalman filter to check the output estimation results of the GE90-115B engine system under fault-free condition. The standard Kalman filter design is introduced in Section 2.2.
**Simulation 1**: A step perturbation of 1% of the operating point value (1%$u_{3,ss} = 0.9233 \times 0.01 = 0.009$ deg) for the control STP25 is injected. Figure 6.4 shows the plots of the three control inputs, and Figures 6.5 – 6.6 compare the nonlinear model outputs, the estimated outputs and error dynamics for each individual sensor.

**Simulation 2**: A sinusoidal perturbation with the magnitude of 1% of the operating point value for the control STP25 is injected. Figure 6.7 shows the plots of the three control inputs, and Figures 6.8 – 6.9 compare the nonlinear model outputs, the estimated outputs and error dynamics for each individual sensor.

Based on the plots, we can observe that the steady-states of all the measured output error dynamics vary around zero and very close to it. It shows that Kalman filter gives fair estimation for the measured outputs.
Figure 6.5: XN2, XN25, T25 and P25 Estimations (step input)
Figure 6.6: T3, P3 and T49 Estimations (step input)
6.2.2 Engine Sensor FDI Using DOS

Sensor fault detection is determined by comparing the residual with the dynamic threshold. When the residuals cross the thresholds, it indicates the existence of a fault in the system. Following fault detection, dedicated observer scheme is used for the fault isolation. Figure 6.10 shows the sensor fault diagnostics scheme using the basic dedicated observer structure. The dynamic threshold algorithm is developed in Section 4.1. A structured bound discussed in the example of Section 3 is used to derive the upper-bound of the state vector $x_{e}(k)$, i.e. each element in the state vector has the time response in the form of

$$x_{e_{q}}(k) \leq x_{q}(k) + \epsilon_{q}(k).$$

The dynamic threshold $\eta_{q}(k)$ of the state error dynamics $e_{q}(k|k-1) = x_{e_{q}}(k) - \hat{x}_{q}(k-1|k)$ is computed by Equation (4.25).
Figure 6.8: XN2, XN25, T25 and P25 Estimations (sinusoidal input)
Figure 6.9: T3, P3 and T49 Estimations (sinusoidal input)
Simulation of basic DOS: The simulation is carried out for $t = 40$ sec with sampling time $T_s = 0.2$ sec. Various types of faults are introduced to the system. The list of simulated faults are:

1. A fault occurs in the third sensor $z_3$ (T25: HPC inlet temperature) when $t > 5$ sec, the fault signal is a step and square-wave input, of which the magnitude is $1\%z_{3_{ss}}$ ($1\%z_{3_{ss}} = 685.6 \times 0.01 = 6.86$ °C). See Figure 6.11(a).

2. A fault occurs in the sixth sensor $z_6$ (P3: combustor inlet pressure) when $t > 20$ sec, the fault signal is a step input, of which the magnitude is $1\%z_{6_{ss}}$ ($1\%z_{6_{ss}} = 660 \times 0.01 = 6.6$ psi). See Figure 6.15(a).
Figure 6.11: Sensor $z_3$ Fault Diagnostics (Basic DOS)–1
Figure 6.12: Sensor $z_3$ Fault Diagnostics (Basic DOS)–2
Figure 6.13: Sensor $z_3$ Fault Diagnostics (Basic DOS)–3

(a) Residual $r^{(4)}$

(b) Residual $r^{(5)}$
Figure 6.14: Sensor $z_3$ Fault Diagnostics (Basic DOS)–4
Figure 6.15: Sensor $z_6$ Fault Diagnostics (Basic DOS)–1
Figure 6.16: Sensor $z_0$ Fault Diagnostics (Basic DOS)–2
Figure 6.17: Sensor $z_6$ Fault Diagnostics (Basic DOS)–3
Figure 6.18: Sensor $z_0$ Fault Diagnostics (Basic DOS)--4
Case 1 of basic DOS: The simulation results are shown in Figures 6.11 – 6.14, from which we can see that the residuals $r^{(2)}$ and $r^{(3)}$ cross the dynamic thresholds when the fault signal acts on sensor 3 at $t = 5$ sec. The results of the residuals are:

$$r^{(1)} = \hat{x}^{(1)} - \hat{x}^{(2)} = [0, 0, 0, 0, 0]^T = 0,$$
$$r^{(2)} = \hat{x}^{(2)} - \hat{x}^{(3)} = [1, 1, 1, 1, 1]^T \neq 0,$$
$$r^{(3)} = \hat{x}^{(3)} - \hat{x}^{(4)} = [1, 1, 1, 1, 1]^T \neq 0,$$
$$r^{(4)} = \hat{x}^{(4)} - \hat{x}^{(5)} = [0, 0, 0, 0, 0]^T = 0,$$
$$r^{(5)} = \hat{x}^{(5)} - \hat{x}^{(6)} = [0, 0, 0, 0, 0]^T = 0,$$
$$r^{(6)} = \hat{x}^{(6)} - \hat{x}^{(7)} = [0, 0, 0, 0, 0]^T = 0,$$
$$r^{(7)} = \hat{x}^{(7)} - \hat{x}^{(1)} = [0, 0, 0, 0, 0]^T = 0.$$

Based on Table 5.1, we conclude that sensor 3 is faulty. But we also find the residuals influenced by the faulty sensor signal fall back within the dynamic threshold after 10 sec in the simulation, indicating no-fault in the sensor. We will discuss the cause of this problem in the next simulation case.

Case 2 of basic DOS: The simulation results are shown in Figures 6.15 – 6.18, from which we can see that no residuals cross the dynamic thresholds when the fault signal acts on sensor 6 at $t = 20$ sec, which means a missed detection. The reason for the missed detection is due to the fast convergence of Kalman gains, which make the impact of the sensor fault diminish in the residuals of Equation (5.5). The plots of Kalman filter gains for each observer are shown in Figure 6.19. It is obvious that all the Kalman gains converge to zero at about 10 sec. That means any fault occurs after 10 sec would not be shown by the basic dedicated observer design. It also explains why all the residuals indicating fault in Figure 6.12 fall back within the dynamic threshold about 10 sec. Thus, the basic dedicated observer scheme
Figure 6.19: Kalman Gains for Dedicated Observer Scheme
is not sufficient for effective sensor fault detection and isolation. Due to this reason, we use the improved dedicated observer scheme to avoid the chance of missed detections.

**Sensor FDI Using Improved DOS**

Figure 6.20 shows the sensor fault diagnostics scheme using the improved dedicated observer scheme.

![Diagram of Aircraft Engine Sensor FDI Scheme Using Improved DOS](image)

**Simulation of improved DOS**: The simulation is carried out for \( t = 40 \) sec with sampling time \( T_s = 0.2 \) sec. We simulate two cases for the improved dedicated observer scheme with dynamic threshold for sensor(s) FDI:
1. We apply the same fault condition for the sixth sensor \( z_6 \) as in case 2 of the basic observer scheme simulation: a fault occurs in the sixth sensor \( z_6 \) (P3: combustor inlet pressure) when \( t > 20 \) sec, the fault signal is a step input, of which the magnitude is \( 1\% z_{6ss} = 660 \times 0.01 = 6.6 \) psi. See Figure 6.21(a).

2. We consider the case when multiple faults occur in the system: two faults occur in both the fourth sensor \( z_4 \) (P25: HPC inlet pressure) when \( t > 5 \) sec, and the fifth sensor \( z_5 \) (T3: combustor inlet temperature) when \( t > 15 \) sec; the fault signal of the fourth sensor \( z_4 \) is a step input \( -10\% z_{4ss} = -33.7 \times 0.1 = -3.37 \) psi, the fault signal of the fifth sensor \( z_5 \) is a square-wave input, of which the magnitude is \( 0.5\% z_{5ss} = 1689.6 \times 0.005 = 8.45 \) °C. See Figure 6.25(a).

**Case 1 of improved DOS:** The simulation results of sensor 6 fault using the improved dedicated observer scheme are shown in Figures 6.21 – 6.24. The results of the residuals are:

\[
\begin{align*}
    r_6^{(i)} &= z_6^{(i)} - \hat{z}_6^{(i)} \neq 0, \\
    r_j^{(i)} &= z_j^{(i)} - \hat{z}_j^{(i)} = 0, \quad \text{for } i, j = 1, 2, \cdots, 7, \text{ and } j \neq 6.
\end{align*}
\]

Based on the FDI logic in Table 5.2, we conclude sensor 6 is faulty.

**Case 2 of improved DOS:** The simulation results are shown in Figures 6.25 – 6.28. The results of the residuals are:

\[
\begin{align*}
    r_4^{(i)} &= z_4^{(i)} - \hat{z}_4^{(i)} \neq 0, \\
    r_5^{(i)} &= z_5^{(i)} - \hat{z}_5^{(i)} \neq 0, \\
    r_j^{(i)} &= z_j^{(i)} - \hat{z}_j^{(i)} = 0, \quad \text{for } i, j = 1, 2, \cdots, 7, \text{ and } j \neq 4, 5.
\end{align*}
\]
Figure 6.21: Sensor $z_0$ Fault Diagnostics (Improved DOS)–1
Figure 6.22: Sensor $z_6$ Fault Diagnostics (Improved DOS)–2
Figure 6.23: Sensor \( z_6 \) Fault Diagnostics (Improved DOS)–3
Figure 6.24: Sensor $z_6$ Fault Diagnostics (Improved DOS)–4

(a) Residual $r_6^{(i)}$

(b) Residual $r_7^{(i)}$
Figure 6.25: Sensors $z_4$ and $z_5$ Fault Diagnostics (Improved DOS)–1
Figure 6.26: Sensors $z_4$ and $z_5$ Fault Diagnostics (Improved DOS)--2
Figure 6.27: Sensors $z_4$ and $z_5$ Fault Diagnostics (Improved DOS)–3

(a) Residual $r_4^{(i)}$

(b) Residual $r_5^{(i)}$
Figure 6.28: Sensors $z_4$ and $z_5$ Fault Diagnostics (Improved DOS)--4
For sensor 4 fault, it acts on the sensor at $t = 5$ sec, so we can see the transient response of the residual crosses the dynamic threshold before Kalman gains converge to zero. Based on Table 5.2, we conclude that sensors 4 and 5 are faulty.

### 6.2.3 Engine Sensor FDI Using GOS

Generalized observer scheme can isolate single sensor fault with robustness to unknown inputs. We apply generalized observer scheme for the engine sensor FDI and compare the results with the dedicated observer scheme method. Figure 6.29 shows the sensor fault diagnostics scheme using generalized observer structure. The dynamic threshold $\zeta_q(k)$ for

![Diagram of Aircraft Engine Sensor FDI Scheme Using GOS](image)

Figure 6.29: Aircraft Engine Sensor FDI Scheme Using GOS
the residual $r_q^{(i)}(k) = z_q^{(i)}(k) - \hat{z}_q^{(i)}(k)$ is computed by Equation (4.27).

**Simulation of GOS:** The simulation is carried our for $t = 40$ sec with sampling time $T_s = 0.2$ sec. Two types of sensor faults are simulated in the system, respectively. The list of simulated faults are:

1. A fault occurs in the first sensor $z_1$ (XN2: LP rotor speed) when $t > 4$ sec, the fault signal is a step and square-wave input, of which the magnitude is $0.1\% z_{1ss}$ ($0.1\% z_{1ss} = 2482.8 \times 0.001 = 2.48$ rpm). See Figure 6.30(a).

2. A fault occurs in the fifth sensor $z_5$ (T3: combustor inlet temperature) when $t > 10$ sec, the fault signal is a sinusoidal input $0.3\% \sin(0.8(t - 2))z_{5ss}$ ($0.3\% \sin(0.8(t - 2))z_{5ss} = 0.003 \sin(0.8(t - 2)) \times 1689.6 = 5.07 \sin(0.8(t - 2)) ^\circ C$). See Figure 6.34(a).

**Case 1 of GOS:** The simulation results are shown in Figures 6.30 – 6.33, from which we can see that except residual $r_1^{(1)}$, the residuals $r_2^{(2)}, \ldots, r_7^{(7)}$ all cross the thresholds when the fault signal acts on sensor 1 at $t = 4$ sec. The results of the residuals are:

$$
\begin{align*}
    r_1^{(1)} &= z^{(1)}(1) - \hat{z}^{(1)}(1) = [0, 0, 0, 0, 0, 0]^T = \mathbf{0}, \\
    r_2^{(2)} &= z^{(2)}(1) - \hat{z}^{(2)}(1) = [1, 0, 0, 0, 0, 0]^T \neq \mathbf{0}, \\
    r_3^{(3)} &= z^{(3)}(1) - \hat{z}^{(3)}(1) = [1, 0, 0, 0, 0, 0]^T \neq \mathbf{0}, \\
    r_4^{(4)} &= z^{(4)}(1) - \hat{z}^{(4)}(1) = [1, 0, 0, 0, 0, 0]^T \neq \mathbf{0}, \\
    r_5^{(5)} &= z^{(5)}(1) - \hat{z}^{(5)}(1) = [1, 0, 0, 0, 0, 0]^T \neq \mathbf{0}, \\
    r_6^{(6)} &= z^{(6)}(1) - \hat{z}^{(6)}(1) = [1, 0, 0, 0, 0, 0]^T \neq \mathbf{0}, \\
    r_7^{(7)} &= z^{(7)}(1) - \hat{z}^{(7)}(1) = [1, 0, 0, 0, 0, 0]^T \neq \mathbf{0}.
\end{align*}
$$

Then based on Table 5.3, we conclude that sensor 1 is faulty.
Figure 6.30: Sensor $z_1$ Fault Diagnostics (GOS)–1
Figure 6.31: Sensor $z_1$ Fault Diagnostics (GOS)--2
Figure 6.32: Sensor $z_1$ Fault Diagnostics (GOS)–3
Figure 6.33: Sensor $z_1$ Fault Diagnostics (GOS)–4
Figure 6.34: Sensor $z_5$ Fault Diagnostics (GOS)–1
Figure 6.35: Sensor $z_5$ Fault Diagnostics (GOS)–2
Figure 6.36: Sensor $z_5$ Fault Diagnostics (GOS)–3
Figure 6.37: Sensor $z_5$ Fault Diagnostics (GOS)–4
Case 2 of GOS: The simulation results are shown in Figures 6.34 – 6.37, from which we can see that except residual \( r^{(5)} \), the residuals \( r^{(1)}, \ldots, r^{(4)}, r^{(6)} \) and \( r^{(7)} \) all cross the thresholds when the fault signal acts on sensor 5 at \( t = 10 \) sec. The results of the residuals are:

\[
\begin{align*}
  r^{(1)} &= z^{(1)} - \hat{z}^{(1)} = [0, 0, 0, 1, 0, 0]^T \neq 0, \\
  r^{(2)} &= z^{(2)} - \hat{z}^{(2)} = [0, 0, 0, 1, 0, 0]^T \neq 0, \\
  r^{(3)} &= z^{(3)} - \hat{z}^{(3)} = [0, 0, 0, 1, 0, 0]^T \neq 0, \\
  r^{(4)} &= z^{(4)} - \hat{z}^{(4)} = [0, 0, 0, 1, 0, 0]^T \neq 0, \\
  r^{(5)} &= z^{(5)} - \hat{z}^{(5)} = [0, 0, 0, 0, 0, 0]^T = 0, \\
  r^{(6)} &= z^{(6)} - \hat{z}^{(6)} = [0, 0, 0, 0, 1, 0]^T \neq 0, \\
  r^{(7)} &= z^{(7)} - \hat{z}^{(7)} = [0, 0, 0, 0, 1, 0]^T \neq 0.
\end{align*}
\]

Based on Table 5.3, we conclude that sensor 5 is faulty.

The sensor FDI using different fault isolation schemes shows reasonable simulation results. Incipient fault can be detected by dynamic threshold without causing false alarms since the dynamic threshold is designed to be robust to the parametric uncertainty and noises. The dynamic threshold varies with the change of control input and initial conditions, which follows the dynamics of the plant.

6.2.4 Engine Actuator FDI

Three actuators are actively working in the engine system. We design three sets of UIOs for actuator FDI. Each UIO is driven by all outputs and inputs except the one that is to isolate for fault. Figure 6.38 shows the actuator fault diagnostics scheme.
Simulation of actuator FDI: The simulation is carried out for $t = 40$ sec with sampling time $T_s = 0.2$ sec. Two types of actuator faults are simulated in the system:

1. A fault occurs in the third actuator $u_3$ (STP25: variable stator vane angle) when $t > 15$ sec, the fault signal is a step input, of which the magnitude is $30\%u_{3ss}$ ($30\%u_{3ss} = 0.9233 \times 0.3 = 0.277$ deg). The simulated control inputs are in Figure 6.39(a).

2. A fault occurs in the first actuator $u_1$ (WF36: fuel flow) when $t > 7.5$ sec, the fault signal is a sinusoidal input, of which the magnitude is $0.1\%u_{1ss}$ ($0.1\%u_{1ss} = 40700 \times 0.001 = 40.7$ kg/sec). The simulated control inputs are in Figure 6.41(a).
Figure 6.39: Actuator $u_3$ Fault Diagnostics–1

(a) Inputs

(b) Residual $r^{(1)}$
Figure 6.40: Actuator $u_3$ Fault Diagnostics–2
Figure 6.41: Actuator $u_1$ Fault Diagnostics–1
Figure 6.42: Actuator $u_1$ Fault Diagnostics–2
**Case 1 of actuator FDI:** Actuator 3 FDI simulation results are shown in Figures 6.39 – 6.40, from which we can see that except residual \( r^{(3)} \), the residuals \( r^{(1)} \) and \( r^{(2)} \) both exceed the dynamic thresholds when the fault signal acts on actuator 3 at \( t = 15 \) sec. The results of the residuals are:

\[
\begin{align*}
    r^{(1)} &= z^{(1)} - \hat{z}^{(1)} = [1, 1, 1, 1, 1, 1]^T \neq 0, \\
    r^{(2)} &= z^{(2)} - \hat{z}^{(2)} = [1, 1, 1, 1, 1, 1]^T \neq 0, \\
    r^{(3)} &= z^{(3)} - \hat{z}^{(3)} = [0, 0, 0, 0, 0, 0]^T = 0.
\end{align*}
\]

Then based on Table 5.4, we conclude that a fault occurs in actuator 3.

**Case 2 of actuator FDI:** Actuator 1 FDI simulation results are shown in Figures 6.41 – 6.42, from which we can see that except residual \( r^{(1)} \), the residuals \( r^{(2)} \) and \( r^{(3)} \) both exceed the dynamic thresholds when the fault signal acts on actuator 1 at \( t = 7.5 \) sec. The results of the residuals are:

\[
\begin{align*}
    r^{(1)} &= z^{(1)} - \hat{z}^{(1)} = [0, 0, 0, 0, 0, 0]^T = 0, \\
    r^{(2)} &= z^{(2)} - \hat{z}^{(2)} = [0, 1, 1, 1, 1, 1]^T \neq 0, \\
    r^{(3)} &= z^{(3)} - \hat{z}^{(3)} = [1, 1, 1, 1, 1, 1]^T \neq 0.
\end{align*}
\]

Then based on Table 5.4, we conclude that a fault occurs in actuator 1.

**GE90-115B Engine Model Simulation 2**

The engine is simulated at a flying condition of altitude = 10000 ft and mach = 0.25. The nonlinear model is linearized around the operating point which is chosen as the steady-state of the control input vector \( U \) and the state vector \( X \), they are:

\[
\begin{align*}
    u_{ss} &= \begin{bmatrix} \text{WF36} \\ \text{AE24} \\ \text{STP25} \end{bmatrix}_{ss} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}_{ss} = \begin{bmatrix} 26760 \\ 0 \\ 1.692 \end{bmatrix}.
\end{align*}
\]
and
\[
\begin{pmatrix}
XN2 \\
XN25 \\
TMHS23 \\
TMHS3 \\
TMHS41 \\
TMHS49
\end{pmatrix}
= \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{pmatrix}
= \begin{pmatrix}
2385.2 \\
10254.6 \\
565 \\
1108.8 \\
2035 \\
2144.5
\end{pmatrix}
\]  
(6.10)

Correspondingly, the steady-state for the measured output \( Z \) is
\[
\begin{pmatrix}
XN2 \\
XN25 \\
TMHS23 \\
TMHS3 \\
TMHS41 \\
TMHS49
\end{pmatrix}
= \begin{pmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5 \\
z_6
\end{pmatrix}
= \begin{pmatrix}
2385.2 \\
10254.6 \\
641 \\
23.9 \\
1576.5 \\
461.2
\end{pmatrix}
\]  
(6.11)

The nonlinear simulation model is linearized around the operating point and discretized the linearized model for \( T_s = 0.2 \) sec. Then the discretized model is given as:

\[
\Phi_{6 \times 6} = \begin{bmatrix}
0.7026 & 0.0182 & -0.1519 & 0.0468 & 0.0325 & 0.0717 \\
-0.2616 & 0.7914 & 0.2722 & 0.1581 & 0.1313 & -0.0355 \\
0.0032 & -0.0001 & 0.9508 & 0.0001 & 0.0001 & 0.0001 \\
0.0054 & 0.0050 & 0.0625 & 0.8944 & 0.0015 & 0.0002 \\
-0.0209 & 0.0100 & 0.0871 & 0.0230 & 0.8533 & -0.0011 \\
-0.0255 & 0.0002 & 0.0306 & 0.0199 & 0.0138 & 0.9230
\end{bmatrix}
\]

\[
\Psi_{6 \times 3} = \begin{bmatrix}
0.0068 & 0 & -2.7089 \\
0.0242 & 0 & 32.2925 \\
0 & 0 & 0.0350 \\
0.0003 & 0 & -0.4456 \\
0.0028 & 0 & -1.1287 \\
0.0026 & 0 & -0.1106
\end{bmatrix}
\]  
(6.12)

\[
H_{7 \times 6} = \begin{bmatrix}
1.0000 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 \\
0.1044 & -0.0046 & 0.6410 & 0 & 0 & 0 \\
0.0152 & -0.0013 & 0.0204 & 0 & 0 & 0 \\
-0.0115 & 0.0753 & 0.1790 & 0.6376 & 0.0136 & -0.0001 \\
0.1609 & 0.0216 & -0.1230 & 0.0376 & 0.0185 & -0.0001 \\
-0.2701 & 0.0036 & 0.2442 & 0.1934 & 0.1382 & 0.3197
\end{bmatrix}
\]
\[ M_{7 \times 3} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1.2083 \\
0 & 0 & 0.3456 \\
0.0026 & 0 & -8.2028 \\
0.0035 & 0 & -1.0635 \\
0.0240 & 0 & -1.0671
\end{bmatrix}. \]

**Sensor fault simulation using GOS:** a step input \((0.2\%z_{2ss} = 10254.6 \times 0.002 = 20.5\) rpm) fault occurs in the second sensor \(z_2\) (XN25: high pressure rotor speed) when \(t > 20\) sec. The outputs and all the seven sets of residuals are shown in Figures 6.43 – 6.46. Based on Table 5.3, we conclude that sensor 2 is faulty.

**Actuator fault simulation:** a step input \((25\%u_{3ss} = 1.692 \times 0.25 = 0.42\) deg) fault occurs in the third actuator \(u_3\) (STP25: variable stator vane angle) when \(t > 7.5\) sec. The inputs and all the three sets of residuals are shown in Figures 6.47 – 6.48. Based on Table 5.4, we conclude that actuator 3 is faulty.

### 6.3 Summary

The dynamic threshold with the sensor and actuator fault isolation schemes is applied to the GE90-115B two-spool turbofan engine model in this chapter. The engine model is a component level model. The dynamic threshold technique is capable of detecting sensor and actuator fault without false alarms and missed detections. Using the dynamic threshold and the fault isolation schemes discussed in Chapter 5, the engine sensor/actuator FDI is simulated based on different types of fault signals. The dynamic threshold is a function of time, which also varies with the system initial condition, control inputs and modeling error.
Figure 6.43: Sensor $z_2$ Fault Diagnostics (GOS)–1
(a) Residual $r^{(2)}$

(b) Residual $r^{(3)}$

Figure 6.44: Sensor $z_2$ Fault Diagnostics (GOS)–2
Figure 6.45: Sensor $z_2$ Fault Diagnostics (GOS)–3
Figure 6.46: Sensor $z_2$ Fault Diagnostics (GOS)–4
Figure 6.47: Actuator $u_3$ Fault Diagnostics1–1
Figure 6.48: Actuator $u_3$ Fault Diagnostics1–2
CHAPTER 7

CONCLUSIONS

In this dissertation, a systematic design of the fault detection dynamic threshold based on linear discrete-time parametric uncertain stochastic system has been developed. The theme is to employ the model-based approach to let us use the knowledge of the system to generate threshold which is a dynamic function varying with time, control inputs, modeling errors and disturbances. The proposed technique helps the system accommodate parametric uncertainties in the model and reduce false alarms and missed detections.

7.1 Contributions

We have formulated the problem for the linear discrete-time uncertain stochastic systems, and we have extended the time response bound analysis from linear continuous-time parametric uncertain system [91] to the discrete-time stochastic system. This formulation and its condition serve as the basis for deriving the dynamic threshold for the linear uncertain systems.

We have designed the dynamic threshold based on the time response analysis using two different observers: Kalman filters and unknown input observers. The dynamic threshold method derived in the dissertation gives the tube-shaped upper-and-lower bound for each
component (system variable) of the residual vector, which provides the insight of the residual dynamics. We can use this knowledge to analyze which sensor output is more sensitive to a fault.

We have compared the sensor fault isolation schemes in this dissertation. The fast converging Kalman filter gain makes the sensor fault effect diminishing in the residual such that the sensor fault isolation using dedicated observer scheme does not function. An improved dedicated observer scheme [25] is applied to solve the above issue. The actuator fault isolation method using unknown input observer has been expanded to consider the control input in the system output model.

We have applied the dynamic threshold method, the sensor and actuator isolation schemes to a nonlinear two-spool high bypass turbofan aircraft engine simulation model. The simulation results have shown that the proposed dynamic threshold is capable of detecting incipient fault in the system sensor or actuator and does not cause false alarms or missed detections.

7.2 Future Work

Despite the contributions made in this dissertation, there are several issues that need further investigation.

The main recommendation of this work is to expand the investigation of dynamic threshold using a more thorough time response analysis. The current time response bound study is based on the assumption of the disjoint domains, which has limitations in sensitivity and robustness of the time response bound generation. A well conditioned modal matrix of the system matrix is the conservative part of the application of this method. The component fault detection and isolation is not discussed in this dissertation. However, component
FDI is important and instructive of the monitored system performance. Thus as one future research work, the component FDI should be elaborated.

Although we have given a thorough derivation of the dynamic threshold based on Kalman filters and unknown input observers, an expansion to a more general observer adaptation is desirable for practical application. In addition, the theoretical proofs of the parameter convergence and the stability of the dynamic threshold are needed.

The dynamic threshold with the fault isolation framework has been studied under the open-loop system. A future research in the closed-loop system with controller design and FDI architecture design should be conducted. A robust controller can make the fault effects insensitive and cause the diagnosis very tough [9]. The fault diagnosis in the closed-loop system has more practical applications.
APPENDIX A

LIST OF NOMENCLATURE

FDI fault detection and isolation
UIO unknown input observer
CLM component level model
DOS dedicated observer scheme
GOS generalized observer scheme
$A, B, C, D$ linear continuous-time system matrices
$\Phi, \Psi, H, M$ linear discrete-time system matrices
$\Gamma$ unknown input matrix
$\Phi + \Delta \Phi$ perturbed system matrix
$\Psi + \Delta \Psi$ perturbed control input matrix
$x$ nominal system state vector
$y$ nominal system output vector
$z$ nominal system measured output vector
$u$ control command vector
$u_{\text{real}}$ actuation input vector
$v$ sensor noise vector
$w$ process noise vector
$f_a$ actuator fault vector
$f_c$ component fault vector
$f_s$ sensor fault vector
$Q$ covariance matrix for process noise $w$
$R$ covariance matrix for sensor noise $v$
$\hat{x}_{k+1|k}$ predicted estimator of $x_{k+1}$
$\hat{x}_{k+1|k+1}$ filtered estimator of $x_{k+1}$
$\hat{z}_{k+1|k}$ predicted estimator of $z_{k+1}$
$\tilde{z}_{k+1|k}$ innovation process
$e_{k+1|k}$ state-prediction error dynamics
$e_{k+1|k+1}$ state-filtering error dynamics
$K_{k+1}$ Kalman filter gain matrix
\(P_{k+1|k}\) state prediction-error covariance matrix
\(P_{k+1|k+1}\) state filtering-error covariance matrix
\(q, \hat{x}, F, T, N\) UIO observer vectors and matrices
\(e_k\) UIO state estimation error dynamics
\(K_k^1, K_k^2\) UIO Kalman filter gain matrices, where \(K_k = K_k^1 + K_k^2\)
\(P_k\) UIO state estimation error covariance matrix
\(\text{rank}(X)\) rank of a matrix \(X\)
\(X^+\) left inverse of a matrix \(X\): \(X^+ = [X^T X]^{-1} X^T\)
\(x_e\) perturbed system state vector
\(z_e\) perturbed system measured output vector
\(sp(X)\) eigenvalue spectrum for a matrix \(X\)
\(V\) nominal system modal matrix
\(W\) nominal system modal matrix inverse
\(\Lambda\) nominal system eigenvalue diagonal matrix
\(\nu_i\) nominal system modal matrix’s column vector
\(\omega_i\) nominal system modal matrix inverse’s row vector
\(\lambda_i\) nominal system eigenvalue
\(V_e\) perturbed system modal matrix
\(W_e\) perturbed system modal matrix inverse
\(\Lambda_e\) perturbed system eigenvalue diagonal matrix
\(\nu_e_i\) perturbed system modal matrix’s column vector
\(\omega_e_i\) perturbed system modal matrix inverse’s row vector
\(\mu_i\) perturbed system eigenvalue
\((\cdot)_q\) the \(q\)-th element of a column vector \((\cdot)\)
\((\cdot)_q_i\) the \(q\)-th element of a column vector \((\cdot)_i\)
\(\Delta \nu_i\) perturbation of the right eigenvector, where \(\nu_e_i = \nu_i + \Delta \nu_i\)
\(\Delta \omega_i\) perturbation of the left eigenvector, where \(\omega_e_i = \omega_i + \Delta \omega_i\)
\(\bar{R}_i\) radii for \(D(\lambda_i, R_i)\) such that \(D(\lambda_i, \bar{R}_i) \cap D(\lambda_j, \bar{R}_j + \epsilon)\) is non-empty for \(i \neq j, \epsilon > 0\)
\(\bar{R}\) minimum of \(\bar{R}_1, \ldots, \bar{R}_n\)
\(R_0\) radius \(R_0 \leq \bar{R}\) for \(D(\lambda_i, R) \in \Omega\)
\(|X|\) matrix whose elements are \(|x_{ij}|\)
\(\|X\|\) induced matrix 2-norm, i.e. \(\|X\|_2\)
\(\|y\|\) vector 2-norm or Euclidean vector norm, i.e. \(\|y\|_2\)
\(K(X)\) condition number of a matrix \(X\)
\(\bar{\rho}_{ij}\) \(\mu_i - \lambda_j\)
\(\sigma_{\text{max}}(X)\) maximum singular value of a matrix \(X\), i.e. the matrix 2-norm
\(e^0_q(k)\) zero-input response bound of \(x_q(k)\)
\(e^u_q(k)\) zero-state response bound of \(x_q(k)\)
\(e_q(k)\) time response bound of \(x_q(k)\), where \(e_q(k) = e^0_q(k) + e^u_q(k)\)
\(r^{(i)}\) residual vector
\(\eta\) dynamic threshold vector for state estimation error dynamics
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>dynamic threshold vector</td>
</tr>
<tr>
<td>$\bar{K}$</td>
<td>steady-state of Kalman filter gain</td>
</tr>
<tr>
<td>$\bar{K}^1, \bar{K}^2$</td>
<td>steady-states of UIO filter gains</td>
</tr>
<tr>
<td>HPC</td>
<td>high pressure compressor</td>
</tr>
<tr>
<td>HPT</td>
<td>high pressure turbine</td>
</tr>
<tr>
<td>LPC</td>
<td>low pressure compressor</td>
</tr>
<tr>
<td>LPT</td>
<td>low pressure turbine</td>
</tr>
<tr>
<td>WF36</td>
<td>fuel flow, kg/sec</td>
</tr>
<tr>
<td>AE24</td>
<td>variable bleed valve</td>
</tr>
<tr>
<td>STP25</td>
<td>variable stator vane angle, deg</td>
</tr>
<tr>
<td>XN2</td>
<td>low pressure rotor speed sensor, rpm</td>
</tr>
<tr>
<td>XN25</td>
<td>high pressure rotor speed sensor, rpm</td>
</tr>
<tr>
<td>T25</td>
<td>HPC inlet temperature, °C</td>
</tr>
<tr>
<td>P25</td>
<td>HPC inlet pressure, psi</td>
</tr>
<tr>
<td>T3</td>
<td>combustor inlet temperature, °C</td>
</tr>
<tr>
<td>P3</td>
<td>combustor inlet pressure, psi</td>
</tr>
<tr>
<td>T49</td>
<td>LPT inlet temperature, °C</td>
</tr>
<tr>
<td>TMHS23</td>
<td>LPC metal temperature, °C</td>
</tr>
<tr>
<td>TMHS3</td>
<td>HPC metal temperature, °C</td>
</tr>
<tr>
<td>TMHS41</td>
<td>HPT nozzle metal temperature, °C</td>
</tr>
<tr>
<td>TMHS49</td>
<td>LPT metal temperature, °C</td>
</tr>
<tr>
<td>ALT</td>
<td>altitude, ft</td>
</tr>
<tr>
<td>XM</td>
<td>Mach number</td>
</tr>
</tbody>
</table>
APPENDIX B

THEOREM PROOF

Proof for Theorem 1: Use the Binomial Theorem,

\[ |\mu_i^k - \lambda_i^k| = |(\lambda_i + \tilde{\rho}_{ii})^k - \lambda_i^k| \]
\[ = |\lambda_i^k | \left| \left( 1 + \frac{\tilde{\rho}_{ii}}{\lambda_i} \right)^k - 1 \right| \]
\[ = |\lambda_i|^k \left| \sum_{j=0}^{k} \binom{k}{j} \left( \frac{\tilde{\rho}_{ii}}{\lambda_i} \right)^j - 1 \right| \]
\[ = |\lambda_i|^k \left| \sum_{j=1}^{k} \binom{k}{j} \left( \frac{\tilde{\rho}_{ii}}{\lambda_i} \right)^j \right| \]
\[ \leq |\lambda_i|^k \sum_{j=1}^{k} \binom{k}{j} \left| \frac{\tilde{\rho}_{ii}}{\lambda_i} \right|^j \]
\[ \leq |\lambda_i|^k \sum_{j=1}^{k} \binom{k}{j} \left( \frac{R_\theta}{|\lambda_i|} \right)^j \]
\[ = |\lambda_i|^k \left[ \sum_{j=1}^{k} \binom{k}{j} \left( \frac{R_\theta}{|\lambda_i|} \right)^j + 1 - 1 \right] \]
\[ = |\lambda_i|^k \left[ \sum_{j=0}^{k} \binom{k}{j} \left( \frac{R_\theta}{|\lambda_i|} \right)^j - 1 \right] \]
\[ = |\lambda_i|^k \left[ \left( 1 + \frac{R_\theta}{|\lambda_i|} \right)^k - 1 \right]. \quad \square \]
**Proof for Corollary 1:** Use the Binomial Theorem,

\[
|\mu_i^k| = |(\lambda_i + \bar{\rho}_{ii})^k|
\]

\[
= |\lambda_i^k| \left| \left(1 + \frac{\bar{\rho}_{ii}}{\lambda_i} \right)^k \right|
\]

\[
= |\lambda_i^k| \sum_{j=0}^{k} \binom{k}{j} \left| \frac{\bar{\rho}_{ii}}{\lambda_i} \right|^j
\]

\[
\leq |\lambda_i^k| \sum_{j=0}^{k} \binom{k}{j} \left| \frac{\bar{\rho}_{ii}}{\lambda_i} \right|^j
\]

\[
\leq |\lambda_i^k| \sum_{j=0}^{k} \binom{k}{j} \left( \frac{R_\theta}{|\lambda_i|} \right)^j
\]

\[
= |\lambda_i^k| \left[ \left(1 + \frac{R_\theta}{|\lambda_i|} \right)^k \right]. \quad \qed
\]
BIBLIOGRAPHY


