College Students’ Concept Images of Asymptotes, Limits, and Continuity of Rational Functions

Dissertation

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Abstract

The purpose of this research was to investigate student conceptions of the topic of asymptotes of rational functions and to understand the connections that students developed between the closely related notions of asymptotes, continuity, and limits. The participants of the study were university students taking Calculus 2 and were mostly freshmen.

The study of rational functions and asymptotes follows the study of functions in College Algebra. The function concept is a fundamental topic in the field of mathematics and physical sciences. The concept of asymptotes is closely related to the concepts of limits, continuity, and indeterminate forms in Calculus 1. Therefore, investigating student beliefs of asymptotes and the connections with related topics could possibly shed light onto effective ways of instructing of these concepts.

Qualitative methodology was used to conduct this investigation. The investigation was conducted through two problem-solving interviews and several teaching episodes. The goal of each problem-solving interview was to gain an indepth understanding of students’ thinking processes while solving problems. Nineteen Calculus 2 students participated in the first problem solving interview that investigated student concept images of asymptotes of rational functions and the connections students have developed among these concepts. The interview was about two hours long. Based on the results of this interview, eight students were selected, and seven students completed teaching episodes. The teaching episodes lasted for one hour and thirty minutes and were conducted twice a week for four weeks. The participants of the teaching episodes were divided into two groups; one group consisted of 4 students, and the
other group consisted of 3 students. Thus, adjustments could be made in the teaching episodes of the second group based on the observations of the first group.

The purpose of these teaching episodes was to create a model of student thinking while they re-construct and re-configure the misconceptions revealed during the first interview. Through the teaching experiment the researcher was able to gain first-hand insights into students’ mathematical reasoning (Steffe & Thompson, 2000). This was accomplished by closely observing students’ problem solving procedures, listening to their discussions, and taking notes during the entire process. Data were collected through students’ written work, the researcher comments and field notes, and the transcripts of the videotapes of the interviews as well as teaching episodes.

After the conclusion of the teaching episodes, the second problem-solving interview was conducted with seven students and videotaped. These semi-structured interviews were to determine their concept images after the activities of the teaching episodes and to create models of student conceptions.
Dedication

To my father, with all my love
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Vita

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BOOK REVIEW

PRESENTATIONS
“A Web-based Learning Kit to Help Improve Mathematical Problem Solving Skills”
The International Conference on College Teaching and Learning, April 1999

“Ensuring Possibilities through the Web”
OADE conference 1999

“Problem Solving and Related Issues”
Math, Science, and Technology Educator’s conference, November 2003

“Misconceptions Regarding Functions”
Math, Science, and Technology Educator’s conference proceedings, 2004

Fields of Study

Mathematics Education, Mathematics
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CHAPTER 1

INTRODUCTION

During the past few years, I was confronted with high school and college students’ incomplete conceptions regarding asymptotes of rational functions. An incomplete conception or an alternate conception is a concept or idea that learners develop that is different from expert notions or concepts in the domain (Moschkovich, 1998). Inconsistent conceptions often interfere with students’ learning of mathematical concepts. This research investigates calculus students’ understandings of the concept of rational functions, their asymptotes, and their limits and continuity, beyond the algorithmic knowledge. Interpretations of asymptotes as “lines the curve approaches but never reaches” are very common among students as well as teachers (Yerushalmy, 1997; Louise, 2000). Some instructors with whom I spoke conveyed their notion that a curve must never touch or intersect any of its asymptotes. The National Council for Teachers of Mathematics (NCTM) standards (2000) suggests that students must be adept at describing and analyzing situations in mathematical terms and be able to justify mathematical algorithms in light of related mathematical concepts. NCTM standards encourage the teaching of mathematical topics where mathematical ideas are linked to and build upon one another so that students’ understanding deepens and their ability to apply mathematics expands. According to the standards, an effective mathematics curriculum focuses on important mathematics, mathematics that is well articulated and challenges students to learn increasingly more sophisticated mathematical ideas as they continue their studies. The study of rational functions and asymptotes
follows the study of functions, which is a fundamental topic in the field of mathematics. One reason is that the function concept ties together many apparently unrelated branches of basic mathematics. The other reason is the ever-increasing role of functions in physical sciences. While the topic of functions belongs to one of the many important areas of mathematics, it is one in which students rarely develop an adequate understanding (Dreyfus, 1983). This difficulty centers on its complexity and generality. This multi-faceted concept can be stated at different levels of abstraction pertaining to a variety of seemingly unconnected situations. For example, a student who is familiar with evaluating f(0) when \( f(x) = 2x^2 - 3x + 1 \) often fails to relate this procedure to that of finding the \( y \)-intercepts of the graph of the function. As another example, the notion of vertical asymptotes is seldom coupled with the concept of infinite limit. Also, the notion of horizontal asymptotes is seldom linked with the concept of limit at infinity.

Exploration of a variety of functions such as linear, quadratic, cubic, absolute value, square root, logarithmic, exponential, and rational functions constitutes a major part of advanced algebra, precalculus, and calculus courses. Both high school and college precalculus courses seem to place an emphasis on the exploration of rational functions through graphing. Most textbooks define a rational function, symbolized \( R(x) \), as a function of the form \( R(x) = \frac{g(x)}{h(x)} \), where \( g(x) \) and \( h(x)(\neq 0) \) are polynomial functions with no common factors. Graphing rational functions is a rich mathematical concept that allows students to examine several of their fundamental properties such as its behavior near the zeros of the denominator, identification of intercepts, properties of symmetry, discontinuity, and examination of the domain, range, asymptotes, and limits.

At the precalculus level, identification of asymptotes is mainly for the purpose of analyzing the properties of rational functions and aiding in their graphing. The concept of limits is
not formally introduced in these courses and asymptotes are found using simplified routine algorithms. Such algorithms do not always foster a deeper understanding of the concept under discussion. In calculus courses however, the graphs of rational functions are used to aid the understanding of limits, and in certain situations, asymptotes are further introduced as limits of rational functions. Yet this is done in a mechanical and disconnected way, and students are not usually provided the opportunity to see how the concepts of asymptotes and limits are interconnected (Hornsby & Cole, 1986).

While other functions such as exponential, logarithmic, and trigonometric reciprocal functions may also possess asymptotes, for the first time, the concept of asymptotes is formally introduced through rational functions. Rational functions may possess three types of asymptotes: horizontal, vertical, and oblique (slanted/inclined). Many courses and textbooks mainly discuss vertical and horizontal asymptotes. For example, the UCSMP (University of Chicago School Mathematics Project) precalculus textbook (Foresman, 1992) does not discuss horizontal or oblique asymptotes in depth. During the series of mini studies that I have conducted, all students showed some familiarity with horizontal and vertical asymptotes. At the same time, most of the students showed no familiarity with oblique asymptotes. Oblique asymptotes are assumed to be more difficult to compute than the other two asymptotes, and this topic is not obligatory in most high school courses. At the college level, there is a tendency to postpone the discussion of inclined asymptotes until students are computing derivatives and limits. One of the reasons for this delay was to avoid the complex polynomial (long) division, the process used to find the formula for the oblique asymptote (Yerushalmy, 1997). Many educators argue that long division and polynomial division are nothing more than a waste of time. In the current research, student notions of horizontal and vertical asymptotes are sought out while the study of the concept of oblique asymptote is reserved for another time.
Statement of the Problem

Asymptotes show up for the first time in a precalculus course. The identification of asymptotes plays a central role in the study of rational functions and its limiting properties. The study of asymptotes and the exploration of rational functions are based on algebraic manipulations of expressions and computation of limits. Students don’t always fully understand it. They move onto a calculus course. There, they are introduced to the concepts of limits and continuity. They don’t fully understand them either. During the instruction of limits, asymptotes are touched upon again. The formal definition of continuity is given in terms of limits. Students struggle with this definition as well. May be the concepts of asymptotes, limits, and continuity together could be given more emphasis in hopes that this will foster student understanding of all of these concepts.

The purpose of this research is to investigate student conceptions of the topic of asymptotes of rational functions and to understand the potential connections that students develop between the closely related notions of asymptotes, continuity, and limits. Many researchers extensively studied concept images of functions and limits of functions (Oehrtman, 2009; Juter, 2005; 2006; Clement, 2001; Szydlik, 2000; Williams, 1991, 2001; Carlson & Oehrtman, 2005; Przenioslo, 2004). Studies have shown that students continued to possess misconceptions of limits, even after taking university analysis courses (Juter, 2005). Przenioslo (2004) notes that even those students who held a global perspective of limits ignored or were unable to explain function behavior around its asymptotes. Difficulties with the concepts of indeterminate forms, continuity, asymptotes, etc. were noted as some of the contributing factors. Students’ concept images of limits are tainted by their images of similar concepts that they have experienced. Thus students’ concept images of limits had been affected long before the introduction of limits in calculus courses (Juter, 2005; Louise, 2000).
For the first time the limit concept is introduced to students during the discussion of asymptotes in a precalculus course. The concept of asymptotes is closely related to the concepts of limits, continuity, and indeterminate forms. Therefore, investigating student beliefs of asymptotes and the connections between asymptotes, limits, and continuity could possibly shed light into effective ways of instructing asymptotes, continuity, and limits. With this goal in mind, a series of teaching episodes were carried out to help students attain essential modification in their conceptions of asymptotes, limits, and continuity. During the teaching episodes, a variety of activities that would encourage critical thinking and exploration of conflicting ideas were implemented. Some such activities included definition analysis and/or extended reflection of the problem situations that prompt the need for concept modification.

Research Questions

The questions addressed in this research are:

1. What are student notions of rational functions?

2. What conceptions do students possess regarding asymptotes, in particular the horizontal and vertical asymptotes of rational functions?

3. What connections do students make between the concepts of asymptotes, continuity, and limits of rational functions?

4. What effect does the teaching experiment have on students’ understanding of the concepts of rational functions, their asymptotes, limits, and continuity, and the connections between these closely related concepts?

The investigation was conducted through two problem solving interviews and several teaching episodes. The goal of each Problem Solving Interview was to gain an in-depth understanding of students’ thinking processes while solving problems. Accurate student understanding cannot always be measured based upon the test scores alone, and a correct test
answer doesn’t necessarily imply proper conceptual understanding of the material. Interview questions were semi-structured and were designed to gather information on student thinking. Follow-up questions such as, “What is your way of deciding whether a function possesses horizontal asymptotes or not?” can reveal the thinking processes of an individual in ways that the paper and pencil method may not. The first problem solving interview investigated student concept images of asymptotes of rational functions and the connections students may have developed between the concepts of asymptotes, limits, and continuity. Based on the result of this interview, eight teaching episodes were conducted to help students recognize and reconfigure the misconceptions that they revealed during the first interview. After the conclusion of the teaching episodes, another problem solving interview was conducted to see if students’ concept images have changed after the practices of the teaching episodes.

**Rationale**

The study of the asymptotes of rational functions is a standard part of both precalculus and calculus courses. The concept of asymptotes during the exploration of rational functions is commonly taught by introducing *recipes* for the three types of asymptotes. These recipes, if done correctly, may help identify the asymptotes of rational functions and aid in its graphing. However, the procedural knowledge of finding the equations of asymptotes is not sufficient for facilitating the reasoning and argumentation that is necessary in the analysis of rational functions. It is also not sufficient for understanding behaviors such as continuity and limits, or for understanding the connection between the concepts of asymptotes, limits, and continuity. Inadequate or incomplete knowledge of these concepts can contribute to numerous cognitive problems in the learning of calculus (Cornu, 1991; Louise, 2000).

For students continuing their studies through calculus and beyond, I believe that a thorough understanding of the concept of asymptotes and the behavior of functions around their
asymptotes is essential for the proper understanding of the more sophisticated mathematical idea of limits. Studies on university calculus students and students in other calculus-based mathematics courses have shown that they continue to hold certain types of misconceptions regarding the concept of limits, even after perusing higher level analysis courses (Juter, 2005). It appears as though the misconceptions regarding limits start earlier than the formal introduction of limits at the calculus level (Juter, 2006). Students’ previous conceptions of mathematical ideas play a powerful role in their understanding of later mathematical concepts (Piaget, 1970; Tall & Vinner, 1981).

The introduction to the ideas of infinity, division by zero, the meaning of 0/0, as well as the discussion of asymptotes seem to impact student knowledge of limit concepts (Louise, 2000; Hitt & Lara, 1999). Louise found a strong connection between student notions of asymptotes, continuity, and limits of rational functions. Among the complex mental images that students may posses regarding asymptotes and limits are the notion that a curve can never touch its asymptotes, and the notion that the limit of a function is something the function approaches but never reaches (Oehrtman, 2009; Clement, 2001; Szydlik, 2000; Yerushalmy, 1997; Juter, 2005; Louise, 2000; Williams, 1991, 2001).

Recently, while graphing rational functions in a precalculus and calculus 1 class, I suggested that we find the possible points of intersection and the horizontal asymptote of the curve in question. Students argued that it is not possible to find such a point since a curve simply “approaches” but never intersects any of its asymptotes. I have experienced Calculus students holding similar misconceptions regarding asymptotes and limits. Providing counter examples did not convince students otherwise. Williams (1991, 2001) noted that inconsistent examples do not always convince students to alter their misconceptions because students do not come to our classrooms as blank slates, and there are connections between the knowledge elements that they
already possess. While replacement of such old notions by new ones is not an easy and automatic task to accomplish, people are in fact capable of holding contradicting realities simultaneously (Piaget, 1970; Tall & Vinner, 1981). The transitional process from an old concept to a new one takes place gradually through conscious and unconscious cognitive transactions. While Piaget (1970) explained this development as the process of active compensation, diSessa et al., (1993) acknowledged this as the process of knowledge refinement, and Dubinsky (1991) called it reflective abstraction. To some extent, such knowledge modification could be achieved through active student involvement in the process of knowledge construction. Facilitating cognitive conflict and argumentation through multiple concept-representations could activate increasing student participation and proper conceptual understanding.

Difficulties in mathematics that lead to misunderstanding of mathematical concepts could be analyzed from multiple perspectives. Cornu (1991) attributed these difficulties to students’ spontaneous conceptions, cognitive obstacles, epistemological obstacles, and didactical obstacles. Spontaneous conceptions include either student concept images of a particular concept that was developed as a result of previous instruction, or intuitions that are personal definitions. Besides the discrepancy between such intuitive notions and formal definitions, there are several other factors that are detrimental to students’ proper understanding of mathematical concepts. For example, the limited amount of time spent on concepts during instruction, the constrictions posed by course objectives, the method of instruction and teacher beliefs, and students’ natural (in)ability to formally operate within the fundamental concept and its application (Yerushalmy, 1997) all create severe obstacles in mathematics learning. Cognitive dilemmas that students experience during mathematics learning have been explained from numerous theoretical perspectives. For example, Grey & Tall (1994) used the theory of procept to explain difficulties that students experience with mathematical truth. The innate but often puzzling relevance
between a mathematical concept and mathematical process is called the theory of procept. The process-concept interference, the proceptual divide, obstructs the effective conception of mathematical knowledge. While the details of these theories will be discussed in the theoretical framework section of this dissertation, some issues that are particular to the learning of the concept of asymptotes will be elaborated on in the following section. Other researchers have identified some of these issues as caused by linguistic, technology-based, and instructional complications.

**Linguistic Aspects.** Formal vs. Informal Language: Formality in mathematical language could be associated with the rigorous manner in which words are assembled in the conventional definition of a mathematical concept. Informal language is used when someone tries to explain a concept in his or her own words. Informal language doesn’t necessarily make a concept definition incorrect. However, informal language that is used to articulate a concept may pose the danger of imprecision or incorrectness. The intuition driven interpretation of formal definitions and the use of informal language were found to be the cause of some of the difficulties that students of mathematics encounter (Fischbein, 1987). Regarding intuitions and formality in the terminology used to convey the concept of asymptotes, let us consider a few textbook definitions and/or the language used to explain asymptotes. Thomas (1974) defines an asymptote of a rational function as follows: “As the point P(x, y) on a given curve moves farther and farther away from the neighborhood of the origin, it is possible for the distance between the point P and some fixed line to approach zero” (p. 451). In such an instance, the line is called the asymptote of the curve. James, Redlin, Watson (n.d.) states “informally speaking, an asymptote of a function is a line that the graph of the function gets closer and closer to as one travels along that line (p. 301).” On the same page, the formal definitions of vertical and horizontal asymptotes are further stated with the aid of graphs and symbols. The Thomas (1974) definition is extremely broad and does apply for
all types of asymptotes. This broad definition could make it hard for novice students to fully grasp the concept. The James et al. (n.d.) definition of asymptote could be considered so informal that students may keep the generalization that all asymptotes are lines that the graph of the function gets closer and closer to but never reaches. Although never reaches is not true for all asymptotes, sometimes, the definitions given during instruction seem to add the extra terminology approaches but never reaches for asymptotes as well as limits (Hitt & Lara, 1999). While it is possible for a curve to touch or intersect its horizontal or oblique asymptotes, the UCSMP second edition precalculus textbook (Foresman, 1992) provides no in-depth discussion of what horizontal or oblique asymptotes are. None of the examples or textbook exercises represented a rational function with a horizontal or oblique asymptote crossing its graph.

I believe that by carefully examining textbooks before adopting them, by custom publishing textbooks that clearly and adequately detail the necessary aspects of the concepts discussed, and by the instructor carefully analyzing and re-interpreting the meanings of definitions in a simple and logical manner, some of the complications that are posed by the inadequate and informal presentation of mathematical concepts could be alleviated. Even with a textbook that is perfectly written, careless instruction and sloppy learning habits could pose the threat of pitfalls of inaccuracy in the transcript of mathematical concepts.

**Pitfalls of technology.** In addition to the formal and informal language used to describe concepts, students’ concept images could be affected by the means by which the concepts are introduced. For example, graphing calculators and computer software programs are often used in the graphing of rational functions. Hindering student learning by causing errors and misconceptions was noted as one of the drawbacks of graphing calculators. Some of these misconceptions are attributable to the inherent limitations of technology such as the limited
viewing screen of a graphing calculator, and problems with scaling (Leinhardt, Zaslavsky, & Stein, 1990; Yerushalmy, 1997; Williams, 1991). Zaslavsky noted that some students who use graphing calculators often overlooked the existence of the graph of functions (quadratic, in this case) that were out of view of the calculator screen. Due to the way in which the graph of quadratic functions appeared on the calculator screen, some students thought they may possess vertical asymptotes (Zaslavsky, 2000) causing misconceptions on asymptotes.

Removable discontinuity cannot be readily identified on the graphing calculator, misleading students to feel that the graphs of two functions such as \( j(x) = \frac{x^2 - 1}{x - 1} \) and \( k(x) = x + 1 \) are somehow the same, while they are in fact different. For example,

while \( j(1) = \frac{0}{0} \), the indeterminate form, \( k(1) = 2 \). However, they have similar characteristics. For example, \( \lim_{x \to 1} j(x) = \frac{x^2 - 1}{x - 1} = 2 \) and \( \lim_{x \to 1} k(x) = x + 1 = 2 \). It should be noted that neither of these functions has vertical asymptotes at \( x = 1 \). However, students misunderstand \( j(x) = \frac{x^2 - 1}{x - 1} \) as a function having a vertical asymptote \( x = 1 \). Both concepts \textit{limit} and \textit{asymptote} invoke the image of \textit{getting closer but never reaching}, and it would be beneficial for students to analyze how these two topics are interlocked. Mystification is one of the reasons behind student difficulty with asymptotes and limits when technology is used to teach these concepts (Hitt & Lara, 1999, Oehrtman, 2009, Yerushalmy 1997, Louise 2000).

Since the limitations of technology alone could confuse students’ conceptual understanding, teachers must carefully implement graphing calculators and computer software in their instruction only to enhance and reinforce students’ mathematical understanding.
productive implementation of technology could be accomplished by persuading students to first conjecture and predict the behavior of graphs by using their mathematical knowledge, and then to use the calculator as a secondary means of verification.

**Instructional Pitfalls.** The limited nature in which concepts are introduced in school mathematics contribute to common misconceptions in mathematics learning (Oehrtman, 2009; Clement, 2001; Hitt & Lara, 1999; Szydlik, 2000). The lack of detailed coverage of mathematical concepts during instruction causes students to lack the ability to extend and apply knowledge flexibly and appropriately from one setting to another, such as from precalculus to calculus in this instance. NCTM (2000) reiterates the importance of the demonstration and extension of mathematical knowledge in the activities of both teachers and students. Bransford, Brown, & Cocking, (1999) as well as Zazkis et al (2003) stress the importance of research related to students’ conceptual understanding of mathematical topics and their proficiency to manipulate between factual knowledge and procedural competence. For example, while the graph of a rational function \( R(x) = \frac{2x - 7}{x - 1} \) possesses a vertical asymptote at \( x = 1 \), the graph of \( F(x) = \frac{x^2 - 1}{x - 1} \) is simply a straight line that has a hole at \( x = 1 \). In terms of discontinuity, the function \( R(x) = \frac{2x - 7}{x - 1} \) where \( R(1) = \frac{2x - 7}{x - 1} = \frac{-5}{0} \) has an infinite discontinuity at \( x = 1 \) while the function \( F(x) = \frac{x^2 - 1}{x - 1} \) has a hole at \( x = 1 \).

While researchers have extensively studied misconceptions regarding the general notions of functions and limits, studies focusing on rational functions and their asymptotes and the link between asymptotes, limits, and continuity are not abundant. However, Bridgers (2006), Yerushalmy (1997) and Louise (2000) studied the association between these concepts and
remarked on how concept images of asymptotes and continuity interfere with the conceptual understanding of limits. Misconceptions regarding the concept of asymptotes could be multifaceted in nature. Several researchers discussed how the word “approach” is explained by students as well as instructors as “approaches but never reaches” in the context of limits and asymptotes. This idea leads students to believe that the function and its asymptotes could never be concurrent (Yerushalmy, 1997; Louise, 2000). Students also believed that a rational function always possesses a vertical asymptote at values where the function is undefined, and they ignored the limit behavior of a rational function around its vertical asymptotes.

Based on the categories that others had ascribed student difficulties associated with learning mathematics, I have developed my categories consolidating the ones discussed by others. They are the Innate Nature of Mathematical Knowledge (INK), Student Convictions (SUN), Instructional Features (IF), and Other (OR). In my view, INK attributes to the difficulties encountered while developing the concepts (function notation, for example). In addition, the complications that arise due to the dual nature of mathematics and the process versus concept divide will also be included in the INK category. Because the issue could be made explicit by the curriculum or during instruction, I may include the difficulties caused by the dual nature of mathematics into the IF category as well. The OR category refers to difficulties caused by students not knowing basic mathematics knowledge such as what intercepts are or how to find the intercepts.

Yerushalmy (1997) chronicled her concern that the concurrency between a curve and its asymptotes are not well elaborated by all textbooks. Asymptote was generally accepted as a geometrical object that the function approached without much discussion about where the approaching takes place, or how close graphs should be to be considered asymptotic. A lack of analysis and argumentation of the deep meanings of the concepts of asymptotes was noted by
Yerushalmy as she discussed student struggles with the dissimilarity between function behavior when “approaching vertical and horizontal asymptotes.” She examined students’ cognitive organizations behind their understandings and the transfer of ideas from the vertical asymptote to the horizontal asymptote from a geometric, numeric, and, intuitive perspective.

Researchers have also noted how disengaged presentation of mathematical ideas could seriously impede students’ conceptual flexibility. For example, Louise (2000) studied student notions of rational functions and indeterminate forms in the calculus of limits of functions when x tends to infinity as well as around the points where the function is undefined. He noted a lack of transfer that students exhibited with the ideas of discontinuity and asymptotes. Many times, students did not know about the word discontinuous and used the term space instead to indicate certain parts of a discontinuous graph. Students expressed difficulty with the indeterminate forms, division by zero, and their significance in the calculus of limits. They were unaware of how these two forms could be significant in informing us about the types of discontinuity, the types of limits, and the potential existence of asymptotes. Bridgers (2006), Louise (2000), as well as Hitt & Lara (1999) noted that this disconnect has contributed to the lack of understanding of the limit concept at the calculus level.

Przenioslo (2004), in her research on limit concepts, noted that students sometimes could correctly state a formal concept definition but would not observe a contradiction between this definition and their concept images. Also, they didn’t try to associate the formal and informal aspects of the concept. This demonstrates an inability to interpret and organize different aspects of a concept and the need for instruction to “undoubtedly” take special steps. I believe that among these special steps, reorganization of the order, manner, means, and relevance in which mathematical topics are commenced deserves prime importance.
I discern that asymptotes are fundamentally connected to the notions of domain and range, infinity, division by zero, continuity of functions, and limit of functions. For instance, consider the function \( Z(x) = \frac{2x - 7}{x^2 - 1} \) and analyze its domain. Examining the domain forces students to look at the values at which the function will be undefined. While analyzing the range, students could be stimulated to observe the function behavior around the point \( x = 1 \), where the function is undefined. Think what happens to the function when \( |x| \) gets bigger and bigger. The denominator of the function will always be much bigger than the numerator, and as a result, the function value will approach “0.” The horizontal asymptote will be \( y = 0 \), the limit at infinity will be 0, the vertical asymptotes will be \( x = \pm 1 \), and there is no finite limit as \( x \to \pm 1 \). Thus teaching continuity, asymptotes, and limits in the same course could significantly improve student understanding of many calculus topics. Students in calculus classes have great difficulty understanding the notion of continuity without seeing the function graph. The connection between continuity and differentiability and the formal definitions of both continuity and differentiability are hard to grasp for most calculus students. Yet why introduce so many mathematical concepts prematurely in a shallow and disconnected manner? I believe that proper understanding of the concept of continuity could help foster the understanding of the limit concept, which in turn could foster the understanding of the asymptote concept.

Theoretical contributions of this research would re-iterate the facts that students don’t come to classes as blank slates, their concept images regarding a mathematical topic had already been set by their previous notions. Therefore, related topics need to be re-addressed often to access what students know thus far and how it affects the acquisition of new concept underway. For example, start by exploring student conceptions of infinity, indeterminate forms, and asymptotes when limits are introduced. Another theoretical and practical advantage would be the
realization that misconceptions cannot be eliminated by simply pointing out the flaws or providing counter examples. The process of regaining the conceptual equilibrium will be complex and time consuming and the practices of teaching must actively unravel and challenge student beliefs. It is imperative to re-structure the existing courses, add new courses, and find a practical connection between instruction and educational research.

**Conclusion**

The lack of detailed coverage through activities that connect multiple concepts, in addition to the untimely introduction of new mathematical concepts, can cause students to lack the ability to extend and apply knowledge flexibly from one setting to another, such as, in this instance, from precalculus to calculus. NCTM (2000) reiterates the importance of the demonstration and extension of mathematical knowledge in the activities of both teachers and students. One of the most vital findings of research is that conceptual understanding, along with factual knowledge and procedural facility, is an important component of proficiency, (Bransford et al, 1999) making the investigation of student conceptual understanding of mathematics topics a vital part of research in mathematics education. By means of in-depth interviews and teaching episodes, it would be beneficial to take a deeper look at what students know about asymptotes, continuity and limits, how they came to that understanding, and how they form connections among these concepts.
CHAPTER 2

THEORETICAL FRAMEWORK

For centuries, the notion of functions has been a topic of controversy among mathematicians. As cited by Dreyfus (1983), the word *function* was abstractly introduced by Leibniz in the late 1600s. During the 1700s, Bernoulli and Euler generalized this notion by regarding it as an equation with variables and constants. By the 1800s, the function concept was further developed by Fourier, who introduced trigonometric functions. In 1837, Dirichlet, in conjunction with the ideas of Cauchy, considered a function as a single-valued correspondence between elements of one set and another set. This is the function concept that we have adopted in today’s mathematics curriculum. A variety of alternate representations such as graphical, ordered pairs form, tabular form, real-life word problems form, forms of diagrams, and input-output forms were created by the analogy of a *function machine*. Each of these notions has its own strengths and weaknesses. Identifying functions and non-functions from equations and graphs, evaluating function values, finding the domain and range of a function, and graphing functions are some of the standard tasks performed in mid-level high school and college algebra courses.

The current study explores the complexities that students experienced on the topic of rational functions and their properties such as asymptotes, limits and continuity. In order to recognize how students come to conceptualize these mathematical topics, it is important to consider both how students psychologically organize information about the concepts and how
students learn concepts. I have drawn on three theoretical perspectives to guide this study. They are theories of cognitive development (Piaget, 1970), representation (Goldin & Kaput, 1996; Lesh et al. 1987), and, concept image and concept definition (Tall & Vinner 1981). The first few paragraphs of the next section outline Piaget’s theory of cognitive development as this theory lays the foundation for the theory of representation, and the theory of concept formation, concept definition, and concept images.

Jean Piaget

According to Piaget, cognitive development has four global stages. These stages are sensorimotor or pre-verbal (birth – approximately 2 years), preoperational (ages 2–7), concrete operational (ages 7–11), and formal operational (ages 11–adulthood). Cognition and reasoning become increasingly sophisticated as one progresses through the stages. Since my dissertation is in regards to college students’ cognitive framework regarding the learning of mathematics and the handling of logic, I will focus particularly on the third and fourth stages. The third stage; the concrete operational stage (ages 7 – 11), is characterized by the appropriate use of logic. Piaget (1964) considers this stage as a decisive turning point in cognitive development since this is the stage where intelligence and logic are officially operational. However, at this stage the operations are performed on objects, and not yet verbally articulated hypotheses.

The formal operational period is the fourth and final of the periods of cognitive development in Piaget's (1964) theory. It is characterized by acquisition of the ability to think abstractly, reason logically, and draw conclusions from the information available. While a concrete operation consists of thoughts about the environment, formal operation consists of thoughts of the environment. It takes anywhere form four to five years for the completion of formal operational stage.
I believe that formal operational stage plays a central role in the development of mathematical thinking. Piaget (1970) explained the development from one stage to another using four factors: maturation, experience, social transmission, and equilibration. According to Piaget, the duration of maturation varies across cultures and therefore maturation cannot fully explain everything except for the fact that maturation affects the ability to process information and the potential for intellectual development.

According to Piaget (1970), schema stands for the conceptual associations of a person’s internal conglomeration of objects and/or processes. When learners accommodate new concepts to match their existing schema, they have assimilated knowledge. Conversely, accommodation is apparent when learners modify their internal schemes as a result of the new conceptual input. Equilibration, or conceptual change, occurs when learners seek a balance between their internal schema and new information. Piaget explicited that experience of objects played a central role in cognitive development.

**Experience**

For pedagogical reasons Piaget (1970) identified two categories of experience; physical experience and logico-mathematical experience. Physical experience is the experience in extracting the physical properties from objects through abstraction. Knowledge that a triangle has three sides or that certain objects are hard while certain others are soft are examples of physical experience. Logico-mathematical experience is not drawn from the objects but is drawn from actions effected upon the objects. The acquired knowledge will not always be the knowledge of the objects but it will be the knowledge gained by operating on the object. For example, Piaget (1964) explained how a child lined up 10 pebbles and counted them in two different ways, in addition to rearranging the pebbles into a circle and recounting them again to conclude that there are ten pebbles regardless of how they are arranged. No property of the object of pebbles is
learned though these operations, but the child realized that there were ten pebbles regardless of how they were arranged.

It is not the physical property of the pebbles which the experience uncovered. It is the properties of the actions carried out on the pebbles, and it is quiet another form of experience. It is the point of departure of mathematical deduction. The subsequent deduction requires interiorizing these actions and combining them without needing any pebbles. (Piaget, 1964, p. 180)

Mathematicians are able to combine their operations without any pebbles using symbols and according to Piaget; “the point of departure of this mathematical deduction” is logical-mathematical experience (Piaget, 1964, p.180). Piaget noted that the logic stems from the actions of ordering things and joining things together, which are imperative before there can be any operations. Once the operations have been obtained, there is no need for concrete objects to operate on. According to Piaget, this transition marks the beginning of abstract thinking. After initially acquiring primary perceptions from the physical world, to construct mental entities, these available entities are used and further abstracted to create additional knowledge structures (Grey & Tall, 1999).

Reflective abstraction, according to Piaget (1970), describes an individual’s creation of logico-mathematical constructions during the course of cognitive development, and it reiterates the key distinction between concrete and formal operations. Reflective abstraction also describes how mental objects and actions on mental objects are completed. The objects could be external (such as a physical matter) or a mental object (such as a concept). An individual’s internal collection of objects and/or process was called a schema by Piaget.

Piaget (1970) stated that though experience is fundamental to development and learning, by itself it is not sufficient to warrant proper advancement in learning. He identified social transmission as another process that aids the transition from one stage to another during the process of cognitive development. Social transmission of knowledge could be mediated through
linguistic or educational transmission, or a child’s imitation of a model. As far as a child’s ability to learn from an adult, the child can only take in the new information if he/she has a structure mature enough to support it. Thus, by itself, social transmission is not adequate for cognitive development while it is a primary component of development.

Piaget (1970) stated that the presence of the three factors, maturation, experience and social transmission, calls for the third factor, equilibration. Piaget argued that human beings strive for a constant need for equilibration. When conflicting ideas are presented to a person, disequilibria are created and the person is compelled to reinstate harmony. Piaget describes equilibrium as active compensation or a process of self regulation which leads to reversibility. Equilibration according to Piaget is the fundamental factor of logical-mathematical knowledge. The formation of equilibrium is sequential.

Assimilation and accommodation are two features of maintaining equilibrium that also explain learning. When a person encounters a new experience through the process of assimilation, what is perceived in the outside world is incorporated into the internal world without changing the structure of the internal world. During accommodation, the internal world has to accommodate itself to the evidence with which it is confronted and thus adapt to it, in a way that fit the pre-existing mental constructs. Thus a person adapts to the world through assimilation and accommodation (Piaget, 1964).

The process of restoring equilibrium is slow and it involves gathering experience based on interpretations and re-interpretations. Actions and active construction of knowledge are necessary to make the essential changes (Piaget, 1970). Therefore, with respect to misconceptions, telling a person that their way of concept explanation is flawed and then giving them the clarified version of it may not necessarily facilitate conceptual change. Validating the corrected concept definition through presenting counter examples might not always do the trick.
either. Instead, active construction, examination, and reexamination of knowledge are inevitable to overcome misconceptions and to restore equilibrium. Piagetian views of instruction support the notion of constructivism that involves hands on building of knowledge. A solid perspective on the process of building and representing conceptual knowledge is important to analyze students’ conceptual understanding of asymptotes, limits, and continuity, and the connections they might have developed among these closely related mathematical concepts. Therefore, in the next section, I will elaborate on the theory of representation as proposed by Goldin and Kaput (1996).

**Theory of Representation**

Since student notions of asymptotes, limits, and continuity are the main goal of this research, it would be beneficial to detail the complexities of the process of concept representation. According to Bruner’s (1966) model there are three different types of representations, or ways of capturing experiences: *enactive, iconic,* and *symbolic.* According to Goldin and Kaput (1996), there are internal (mental) and external (physical) representations: “a representation is a configuration of some kind that, as a whole or part by part, corresponds to, is referentially associated with, stands for, symbolizes, interacts in a special manner with, or otherwise represents something else.” (Goldin & Kaput, p. 398). Representation is a complex cognitive process that is signified by personal and idiosyncratic aspects or cultural and conventional aspects (Bruner, 1966, Goldin & Kaput, 1996). Mainly, representation could be internal-mental or external-physical, and the present discussion will mainly focus on representations of mathematical objects.

Iconic representation, (Bruner, 1966) are internal-mental representations (Goldin & Kaput, 1996) that involve the use of mental images. These mental images could be pictures, diagrams, or even written items such as words and definitions. These imageries do not include every detail of the object or actions performed on them but recall their important characteristics. “Internal (mental) representation refers to possible mental configurations of individuals” (Goldin
& Kaput, 1996, p. 399). These internal mental processes, though not directly observable, are often inferred by educators and researchers through external behaviors (words), or actions (problem solving). Thus different people have different mental images of different mathematical contexts since there are different types of representations for human knowledge.

Mathematical learning and development go beyond the acquisition of discrete skills or gaining narrow senses of particular concepts. Development of meaningful and flexible internal representations of varying levels of abstractness appropriate for mathematical understanding occurs and matures through a series of different stages. Goldin (1996) identified these stages as inventive-semiotic, a period of structural development, and an autonomous stage. At the inventive-semiotic stage, new information learned will be used to symbolize the aspect of prior representational systems. Semiotic refers to the “act of assigning meaning,” and prior system acts as a “semantic domain” for new symbols. However, in this stage the new symbols are often considered “not to symbolize, but actually “be” the aspects of the prior system that they represent” and this can lead to cognitive obstacles in the learning of mathematics (Goldin & Kaput, 1996).

At the structural–developmental stage, new information received will be identified with the structural features of the earlier system. At this stage a comparison and contrast between the novel and earlier representations usually will not occur (Goldin & Kaput, 1996). At the autonomous stage, the new system of representation matures and separates from the old one and transfer of meanings between old and new representational systems becomes possible. According to the authors, following these developmental stages and analyzing them appropriately would help reveal some of the cognitive aspects of mathematical learning. These three stages are similar to Piaget’s (1970) notions of assimilation, accommodation and equilibration.
Kaput (1995) claimed that the theory of representation sheds light onto various aspects of mathematical learning and problem solving including student capabilities and limitations. Based on this awareness, teachers can investigate the ways in which valuable mathematical capabilities could be shaped in learners. Thus, understanding different representations and relating them to each other ought to be considered as a basic element of meaningful teaching and learning of mathematics. The theory of representation is closely tied with the notions of concept image and concept definition as portrayed by Tall and Vinner more than a decade earlier. It goes without saying that Piaget’s (1970) theory of cognitive development provided a solid yet pliable foundation for many other theories connected with mathematics learning, such as the theory of concept formation by Tall and Vinner and the theory of representation by Goldin and Kaput.

It is difficult to apply Piaget’s stage theory to college mathematics teaching. According to Piaget, formal operational skills develop during the teen years. However, there are very few college students who are ready to perform at the abstract level of formal operations. In addition, Piaget’s stage theory fails to support an effective teaching strategy since Piaget asserts that advancement from one stage to another cannot be accelerated through teaching. “differences of cognitive demand have often been used in a negative sense to describe students´ difficulties, but rarely to provide positive criteria for designing new approaches to the subject” (Tall, 1991, p. 9). Therefore, Piaget’s stage theory could be extended to accommodate advanced mathematical thinking. “Stage theory may be just a linear trivialization of a far more complex system of change, at least this may be so when the possible routes through a network of ideas become more numerous, as happens in advanced mathematical thinking” (Tall, 1991, p. 9). Piaget focused on the development of mathematical knowledge at the early stages, but rarely went beyond adolescence (Dubinsky, 1991). As far as effective mathematical understanding is concerned, “the act of construction of external representations is critical to the construction of internal
representations” (Goldin & Kaput, 1996, p. 399). “External representations refer to physically embodied, observable configurations such as, words, graphs, pictures, equations, or computer microworlds” (Goldin & Kaput, 1996, p. 400).

Bruner (1966) called the external-physical representation “enactive representation,” a physical process where children represent objects in terms of their immediate action performed upon them. These configurations are observable, though not always objective or perfect since they are affected by the mental processes of the individuals who are doing the interpretations. Internal mental representations and external physical representations are highly structured in nature and one is not a copy of the other. As noted by Goldin & Kaput (1996), in one context, a graph may simply represent an equation while in another context it may represent a function or the relation between position and time of a particular object. Piaget (1970) identified the internal representations as “signified” and the external representations “signifier.”

In mathematics education, formal mathematical concepts and notations are considered as external representations. Formal notational systems are often representations that possess cultural and conventional significances. Bruner (1966) outlined a similar concept of symbolic representation that uses a symbol system to encode knowledge. The symbols can be verbal descriptions, formal definitions, or proceptual which contain process-object duality. Symbols often do not resemble the object but they represent the actual objects. The construction and manipulation of formal mathematical ideas are conducted through a series of internal processes which could be modeled by the discussion of rules and algorithms. These internal representations could be formal or informal with imagistic features. For example, though the formal definition of a horizontal asymptote is introduced, after doing routine problems on finding a horizontal asymptote, the mention of the term “horizontal asymptote” of \( f(t) = \frac{2t + 1}{t - 7} \) may simply evoke
the image of a line $y = 2$, which is obtained by means of application of the algorithm. The student may also perceive the line $y = 2$, for which the graph of the function “gets close to but doesn’t touch,” as an interpretation of the formal definition. Consequently, the student may draw the incorrect conclusion that a graph cannot be concurrent with its horizontal asymptote.

Meaningful mathematical understanding takes place when the relationship between external (formal) representation and internal representation is strengthened by the ability to interpret and discuss why algorithms work, and how to visualize its connection with the formal notational descriptions and definitions.

While building powerful representational systems ought to be the goal of mathematics learning, instruction is often centered on the kinds of problems we as teachers want students to be able to solve, or what particular skills and concepts we want them to have (Goldin, 1996). However, there are limitations to such learning goals: they do not always foster the kind of competence required for the creativity and spontaneity in the extension of an old concept to unfamiliar situations or for the execution of new strategies when necessary (Goldin, 1996).

Goldin and Kaput (1996) place their emphasis on internal representational systems since they are more valuable in providing a means for characterizing the outcomes of learning. “Mathematics learning must focus on the construction of a powerful representational system that will enable a wide and varied domain of applicability, efficient procedural use, and ability to effectively abstract and coagulate essential features from one representation to another” (p. 425). As a result, they suggest that mathematics education should promote the formation of flexible internal systems of representations that facilitate powerful problem solving. To accomplish this goal, imagistic and heuristic systems of mathematical interpretation must also be introduced, in addition to formal symbolic systems of representation. Imagistic and heuristic systems include techniques of visualization, mental rotation, diagram drawing and interpretation, and spatial
projection. This helps create an efficient association between internal and eternal representational systems. As far as asymptotes are concerned, formal definition with exploration of its meaning through multiple perspectives such as algorithmic, arithmetic, geometric, numeric would help build flexible conceptual knowledge with diverse applicability. However, the process of comparison and translation between different representational systems itself is rather complex.

Translations between different representations of mathematical ideas and between common mathematical experiences were noted as one of the problems that mathematics learners are facing. According to Janvier (1987), translation between representations, defined as “the psychological process involved in going from one mode of representation to another” (p. 27), is critical in mathematics learning. Until students reach the level of control to successfully connect the different translations of concepts, such as the multiple conceptual formats of asymptotes, they may experience serious confusions when they are required to interpret this concept though multiple perspective. As Janvier (1987) noted, “mathematics concepts do not start building up the moment they are introduced in class by the teacher” and mathematics is learned through a series of “ever-enriching set of representations” (p. 67).

**Theory of Concept Formation**

Tall and Vinner’s (1981) notion of concept image enables me to focus on how students are dealing with the multiple representations of functions and asymptotes since this theory deals directly with the issues that are more significant in mathematics learning. In addition, it simplifies the intricate details of cognitive descriptions often found in other theories of cognition. In a nutshell, concept images are influenced by internal mental representations of the concept while concept definitions, similar to external representations, are formal and conventional. According to Tall and Vinner (1981), we informally encounter many mathematical concepts before we formally encounter them. Therefore, a complex cognitive system exists that yields a variety of
personal, mental images associated with these concepts. These personal, mental images are called concept images associated with a concept. Many of these concept images are acquired through experiences without formal definitions which may later interfere with the new, formal introduction of that concept. During this intercession process, the concept may be given a symbol or name to enable communication and aid mental manipulation. However, invocation of a single symbol is not the sole cognitive process, which colors the meaning of a concept. Meaning is a complex entity which is more than any mental picture; it could be pictorial, symbolic, or something else entirely.

Many associated notions are consciously or unconsciously summoned while recalling and manipulating “meaning” or concept. For example, the concept of subtraction is first introduced as a process where a smaller number (minuend) is always subtracted from a larger number (subtrahend) resulting in an answer that is smaller than the subtracted. This observation may interfere with the child’s concept image when the idea of signed number subtraction is introduced where subtraction doesn’t always result in an answer that is smaller than the subtrahend. Since new representations are being built on the templates provided by earlier representations, the child might wonder how it is possible to subtract 3 from 2 or why

3 – (-2) is 5, a number bigger than the original number 3. The child’s concept image of subtraction might be that a smaller number is always subtracted from a larger number resulting in a number that is less than the original number. Thus, the concept definition of signed number subtraction is somewhat different from the child’s concept image of the definition of subtraction which is merely whole number subtraction. Concept images could be compared with internal representations while concept definition could be compared with external representations.

We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. As the concept image develops it need not be coherent at all times.
We call the portion of the concept image which is activated at a particular time the evoked concept image. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked simultaneously need there be any actual sense of conflict or confusion (Tall & Vinner, 1981; p. 152).

Concept definition is formal, rigorous, undisputed and widely accepted by experts in the field. The formal definition of a horizontal asymptote as stated in a precalculus textbook was that “the line $y = b$ is a horizontal asymptote of the function $y = f(x)$ if $y$ approaches $b$ as $x$ approaches positive or negative infinity” (Stewart et al., n.d. p. 301). As discussed earlier, the word approach conveys a notion of never reached while in neither of the two graphical illustrations the curve is concurrent with the horizontal asymptotes. Thus students do not even get a chance to ponder a confusion whether the graphs and the horizontal asymptotes are concurrent anywhere.

Concept image is thus affected by the manner in which concept definition is presented as well as acquired, for example, through rote or meaningful learning. Concept image could be a personal definition, or a personal reconstruction of a definition, which can differ from a formal concept definition, accepted by the mathematical community at large. At times, mislaid concept images (personal concept definitions that are inconsistent with the theoretical definition), as the ones described earlier, obstruct the configuration of a coherent, whole concept image.

Tall and Vinner (1981) referred to the portion of the concept image that is triggered at a particular time as evoked concept image. Sometimes seemingly conflicting images are evoked simultaneously, causing a sense of inconsistency or confusion leading to misconceptions. In many situations, during the association with a concept, people do not even consult with definitions. They seem to employ a less technical, more common-sense approach. What a concept name then evokes is an assortment of nonverbal relations with the concept such as visual representations, symbols, impressions, or experiences.
Given a concept name (definition), the learner, who already has a concept image, might accommodate the image to include the definition, keep the same image and forget the definition, or keep both and evoke them independently. Advancement in mathematical thinking happens when concept images are re-configured as concept definitions and are used to extend formal, conventional mathematical knowledge (Tall, 1995). As suggested by Goldin’s (1998) previously discussed stage theory, this re-configuration happens during the autonomous stage of the formation of internal representations. At this stage, concepts are appropriately internalized as they are “inspired by concept images and formalized by concept definitions” (Tall, 1995, p.4).

With respect to asymptotes, students possess the notion of a line which the curve approaches but never reaches, while for functions, students oftentimes imagine a number-producing machine, in which numbers are input to receive further number outputs (Clement, 2001). Additionally, with regards to functions, students possess the notion of a graph passing the vertical line test, or some algebraic form represented by equations (Sajka, 2003). Thus, for students, the notion of function is an amalgam of various symbols and mathematical processes that are related or sometimes unrelated to the concept of functions. The existence of such preexisting symbols may very well complicate the formation of precise concepts and dissuade proper conceptual understanding. The concept image of an asymptote as a line the graph of a function approaches but never reaches could prompt a student from identifying the dotted line in Figure 2.1 as something other than an asymptote since it intersects that graph.
There could be multiple reasons for this misconception that a graph is never concurrent with any of its asymptotes. It is certainly related to the intricacy of mathematical truth and the way in which a child encounters mathematical experience. While symbols play a central role in the understanding of mathematical concepts, they also present additional cognitive stumbling blocks for the proper learning of mathematics. The lack of flexibility associated with mathematical symbol manipulations were studied in detail by many researchers (Tall, 1995; Gray & Tall, 1993; Williams, 1991; Goldin, 1998). Among these studies the theory of procept as formulated by Grey and Tall (1993) is remarkable. The notion of procepts stemmed from Piagetian theory:

mathematical entities move from one level to another; an operation on such “entities” becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by “stronger” structures. (Piaget, 1972, p.70)

Building up on the issues with concept images the notion of uncontrolled imagery (Aspinwall, Shaw, & Presmeg, 1997) is notable. These images are uncontrolled in the sense that “they appear to rise unbidden in an individual’s thought, and also in their persistence even in the face of contrary evidence. An uncontrolled image, then, is one that is beyond the volition of the
cognizing individual” (Aspinwall, Shaw, & Presmeg, 1997, p. 301). These images may “persist and prevent the opening up of more fruitful avenues of thought” (Aspinwall, Shaw, & Presmeg, 1997, p. 301).

One of the reasons for students having conflicting mental images that obstruct learning was attributed to the dual nature of mathematics: the way mathematical entities could be viewed as a process and a product of that process. This clashing nature of mathematical truth is elaborated in the next few paragraphs. This theory is elaborated on in this chapter because of its role in contributing to the epistemological difficulties of mathematical learning.

Theory of Procept and the Formation of Definitions

Theory of procept that explains the confusion caused by the process-product aspect of mathematical truth “highlights the duality of process and concept” (Tall, 1995, p. 2). As described by Grey and Tall (1993, 1994) incompatible concept images and personal concept definitions could also be developed as a part of multiple and at times conflicting symbol manipulations. This theory also holds close resemblance to the theories set forth by Goldin & Kaput (1996) and especially Goldin’s (1998) account of the three stages in which symbols are being manipulated. Thus a symbol in mathematics often represents more than one entity, and success and failure in mathematics result from the difference in the degree of flexibility with which people use mathematical symbols. For example, as mathematics instructors, we encounter the difficulties that students face regarding the topic of fractions. Among the division symbols, \( \div \) and \( / \), the symbol \( / \) plays a duel role in arithmetic. While this symbol in \( \frac{10}{2} \) clearly indicates the division process, in \( \frac{2}{3} \) it represents the concept of fraction along with the process (or operation) of division. Such phenomena materialize in mathematics from elementary to university levels.
Grey and Tall (1981) portrayed the subtle differences in the meaning of symbols in mathematics through their procept theory. The symbol is viewed as a “procept”, a pivot between process and concept. Accordingly, for some people, a symbol is a mathematical object, a thing that can be mentally maneuvered. For others, it denotes a process to be carried out. Those who concentrate on the process may be superior at calculations and may succeed in the short term, but in the long term, they may fall short of the flexibility that will give them definitive success. Grey and Tall (1981, 1993, 2006) defined a procept as a symbol that evokes either a process or the result of that process. Consequently, a procept possesses a vagueness that demands flexible mathematical thinking since it possesses a dual nature that stands for both a process to do and a concept to think about (Tall, 1995). To overcome this vagueness, as a common practice, precise definitions are given for mathematical concepts, which focus mainly on the object of definition rather than the underlying process. However, process itself is a complex entity that involves symbol manipulation, which itself represents a process or the outcome of the process. This further complicates the learning power of some learners who are unable to sense the flexible power of symbols.

The conception of a process giving a product or output, represented by the same symbol for both the process and the product, occurs at all levels in mathematics. Accordingly, a procept is defined to be a collective mental object consisting of a process, a concept (product) produced by that process, and a symbol, which may be used to denote one or both. For a simple example consider the operation of division. While the division symbol, as previously mentioned, represents the process of division in the problem \( \frac{10}{2} \), the same symbol in \( \frac{2}{3} \) represents the
division process as well as the result of that division process in another problem such as \( \frac{10}{15} \) since

\[
\frac{2}{3} = \frac{10}{15} = \frac{10 \div 5}{15 \div 5}.
\]

Another example that illustrates the ambiguity due to the use of symbols is the one that is associated with the concept of limits (Cornu, 1981, 1983). The process of finding the limit of \( \frac{1}{x} \) as \( x \) approaches infinity is symbolically represented as \( \lim_{x \to \infty} \frac{1}{x} \). In this case, the result of this process, the answer to the problem, which is the number zero, is in fact the limit of \( \frac{1}{x} \) as \( x \to \infty \). The mere appearances of the Lt alone in the mathematical equation, \( \lim_{x \to \infty} \frac{1}{x} = 0 \) was found to be confusing to the learners since the answer, number zero, is also called the limit.

Accordingly, due to Grey and Tall (1994), the limit concept could also be considered as another example of a procept. The limit concept and the process of computing limits themselves are considered to be two of the challenging mathematical topics, even without the confusion arising from the convolution of symbols.

While not all mathematical concepts are procepts, they do occur broadly throughout mathematics, mainly in arithmetic, algebra, calculus, and analysis. We may consider functions as procepts, processes that enables us to calculate, and as concepts which can be manipulated in many other forms such as the graphical form, tabular form, ordered pair form, correspondence form, algebraic equation form, etc. For example, the algebraic equation of a function \( f \),

\[ f(x) = 2x - 8, \]

traditionally represents both the process of calculating the function value for definite values of \( x \), the independent variable and the concept or rule that defines a particular linear function. Sfard (1989) and Sajka (2003) point this out as one of the reasons why the
function concept poses difficulty to learners. The function topic, however, could create serious challenges to mathematics learners.

For many students this process itself acts as their concept of vertical asymptote. During my mini and pilot studies, students were asked to find a vertical asymptote of a function such as \( N(x) = \frac{2x}{x^2 + 1} \). The answer was that since the denominator could never be zero there is no vertical asymptote. Students were unable to relate the solving for the denominator process to the concept of asymptotes.

Gray and Tall (1999) proposed that interpretations of mathematical symbols as processes or procepts lead to a “proceptual divide” between the less successful and the more successful mathematical accomplishments. Successful mathematics learners use the flexible notion of procept to their cognitive advantage by deriving great mathematical flexibility from cognitive links associated with process and procept. However, the less successful mathematics learners focus heavily on the process and often follow a procedure oriented approach to mathematical truth.

Procedures help perform mathematical processes in a limited (routine) context by the use of learned procedures, and oftentimes, this does not provide the type of flexibility required to effectively tackle the particular mathematical process. For example, a person who knows how to “add” and “multiply” by performing the process of addition and the process of multiplication as two unrelated procedures may not be able to facilitate the process of addition of 2 + 2 + 2 +2 +2 +2 +2 +2 +2 by performing the process of \( 9 \times 2 \). This could be due to the lack of greater sophistication that allows for the use of alternative procedures for the same process and that allows for students to choose a more competent procedure to carry out the given task quickly and precisely.
As another example of the proceptual divide, consider the finding of horizontal asymptotes. Most precalculus textbooks explain the concept of horizontal asymptotes of a rational function $R(x)$ as something like this: when $x$ gets bigger and bigger beyond limit in the positive or negative direction, the function value approaches a constant. Then several rules or procedures are listed in order to find the horizontal asymptotes. The underlying procedure calls for a comparison of the degrees of the numerator and denominator of the rational function. One of these rules states that if the degree of the numerator is greater than the degree of the denominator the function does not have a horizontal asymptote while if the degree of the numerator is smaller than the degree of the denominator, the Y-axis will be the horizontal asymptote of the function. A student who merely focuses on the procedures involved in finding horizontal asymptote could very easily believe that the rational function $R(x) = \frac{x^2 - 1}{x^3 + 2}$ does not have a horizontal asymptote.

However, a student who has a deeper understanding of the notion of horizontal asymptotes might be able to figure out the horizontal asymptote by reflecting on what would happen to this function when $x$ gets very large. The student will also need the flexibility of utilizing algebraic manipulation to actually realize what would happen to the function values of $R(x)$. In this problem, to avoid the $\frac{\infty}{\infty}$ form that results from directly “plugging in” infinity for $x$, one needs to perform certain algebraic simplification to clearly see the behavior of the function. This process is discussed in the next section. One of the less complicated ways will be to use the table features of the graphing calculator and analyze the function values after inputting large values for $x$. In any case, in order to use graph, algebra, or table features the student must definitely internalize the meaning of the “definition” of horizontal asymptotes. As noted by Skemp (1971), the way of
“learning without reasons” does not allow flexibility in dealing with complex mathematical entities.

Students who take calculus courses will have a formal or semi-formal concept image of the tangent to a graph that is built from their experiences involving the tangent line of a circle that they are familiar with. However, while imagining tangents to the curves depicted in Figure 2.2 in a calculus course, students could have many questions and doubts. Conceptualizing tangents at the indicated points of these curves is ambiguous since a tangent can only intersect a line at exactly one point and this definition does not cover the possibility of a vertical tangent. Most calculus textbooks do not offer a satisfactory explanation to this confusion. This problem poses a deeper cognitive dissonance with the assertion that similar figures do have a tangent at the indicated point.

Figure 2.2. Curves with Ambiguous Tangents.

This is another example of how concept definitions introduced in a narrow sense in the beginning and left unexplored later could contribute to confusing and conflicting concept images in students. Having to cope with these types of disconnected contexts suggests a mismatch between the design of curriculum and students’ cognitive structure (Tall, 1981). As noted by
Skemp (1971), mathematics teaching must focus on the process of mathematical thinking rather than the product of mathematical thought.

**Skemp and the Process of Knowledge Acquisition**

Skemp (1987) explained the process of knowledge acquisition based on his notion of schema. Schema could be considered as a knowledge structure, or mental objects that embody selected aspects of the outside world. To understand something means that the new information is assimilated into the pre-existing schema. Skemp believed that the construction of knowledge structure cannot be communicated directly; it can only be constructed actively. In mathematics learning, direct communication often leads to a fixed plan, a narrow procedure-based problem solving, that Skemp called instrumental mathematics. Relational mathematics involves active schema building facilitates the building of a knowledge structure that is tested and reconstructed against the expectations of the world of experience.

All theorists concur that conceptual understanding necessitates adaptation and linking of all pieces of knowledge, in conjunction with forming connections between old concept images and new information received. But one question remains: How do we, as teachers, know how exactly these connections are being made or what kinds of mental processes are being initiated during the development of the connections and meanings in a learner’s cognitive framework? As a solution to this concern, many researchers and theorists suggested hands-on mathematics instruction by the aid of multiple representations of mathematical notions (Kaput, 1989a 1992; Skemp 1987; NCTM 2000). So how do we, as teachers of mathematics, accomplish teaching that fosters conceptual understanding and student involvement in the active construction of mathematical knowledge? Based on Piaget’s notion of constructivism, Glasersfeld (1996) has made several useful recommendations for mathematical instruction.
The multifaceted nature of mathematics which is an amalgamation of concepts, symbols, process, etc., alone does not necessarily cause ambiguity and misconceptions in the learning of mathematics. Students’ cognitive development and knowledge acquisition are often affected by the manner in which they come to experience these ideas. The instructional strategies and the narrow manner in which mathematical ideas are conveyed via respective courses and textbooks also obstruct cognitive development in students. Goldin noted that “often, the exclusively behavioral characterization of desirable learning outcomes leads educators to rely on the teaching of discrete, disconnected skills in mathematics, rather than on developing meaningful patterns, principles, and insights” (Goldin, 1990, p.36). This view of teaching and learning does not consider learners’ pre-existing mathematical experience. In fact, many educational theories (Piaget, 1964; Tall & Vinner, 1996; Grey & Tall, 1994; Skemp, 1987; Bruner, 1960, 1966) have acknowledged the interaction between pre-existing concept images and newly introduced concept information.

To summarize, Gray & Tall (1994) “brought a new emphasis on the role of symbols, particularly in arithmetic and algebra, that act as a pivot between a do-able process and a thinkable concept that is *manipulable* as a mental object: a *procept*” (Tall, 2004b, p.3). Regarding how people learn, Piaget strongly argued that learners are active builders of knowledge, not passive receivers of information. Regarding knowledge acquisition, Piaget believed that knowledge cannot be *received*; it can only be *constructed* actively through personal experiences. In this constructivist construct, the assumption is that “the new knowledge needs to find some anchor points in the learner’s cognition in order to maintain some cognitive continuity and relevance.” Herscovics, (1988) & Skemp (1987) denounced instrumental mathematics in favor of building up relational understanding. In light of the theories discussed earlier, I will examine the aspects of
active construction of mathematical knowledge and how learning could be accomplished effectively through meaningful learning that is less vulnerable to misconceptions.

**Constructivism - Active Building of Knowledge**

Constructivism has different forms. While they all support the notion of individual construction of knowledge, there are also profound distinction. Paul Ernest (1996) discussed various forms of constructivism: weak constructivism, social constructivism, and radical constructivism. *Weak constructivism*, as Ernest (1996) describes it, assumes that there is a realm of knowledge out there which could be received via senses. Individuals personally construct knowledge to match the body of knowledge out there. *Radical constructivism* is based on the epistemology that all knowledge is being constructed by individuals on the basis of cognitive processes in dialogue with the experienced world. This form of constructivism maintains that individual knowledge is in a state of change through reevaluation via the process of adaptation and accommodation. This form of constructivism assumes no position regarding the world of knowledge outside the experience of the knower. Finally, *social constructivism* acknowledges the existence of a world of knowledge that supports the appearance we have shared access to, but which we have no way of knowing about for sure. Individuals construct knowledge with the force of language and social interaction. Thus knowledge is a social construction, a cultural product consensually negotiated by communities. Learners construct knowledge within the cultural context that gives meaning to that knowledge. From a constructivist viewpoint, knowledge mediation and construction can best be achieved if students are active and interactive participants in the process of learning. According to Piaget (1970), however, social interaction helps build knowledge by simply aiding the restoration of equilibrium.

Glasersfeld (1996) suggested several pointers to help teachers create a classroom that is more constructivist and meaningful in nature. In order for students to build up their own
knowledge, we have to consider that students are not mere blank slates. Rather, we need to acknowledge students’ prior conceptual knowledge and how that can serve as a foundation to construct knowledge that is more meaningful (Piaget, 1970). Accordingly, the teacher must try to find out where they stand as far as the knowledge they hold prior to instruction. For example, before starting to teach signed number operations, the teacher must investigate student knowledge regarding addition and subtraction of positive numbers. This suggestion could be placed in much harmony with different stages of representation and theories of cognitive development discussed earlier in this chapter. Basically, teachers must consider the fact that students come to class with pre-existing concept images of the topics at hand.

Next, a teacher must modify students’ ways of thinking. Students need to recognize that knowledge is not absolute, it is context specific, and that knowledge that is well explored within the context of discussion is more sensible. A constructivist teacher may only claim that in mathematics what is obtained is derived from conventional norms and operations which is context specific. Subsequently, instead of giving out correct answers, a teacher must ask students how they arrived at their answer and then explain why it will not work under another similar circumstance (Glasersfeld, 1996). Thus, teachers must encourage the student to correct mistakes in the current problem on their own. Simply labeling something as “correct” might not be enough for a student to be interested in the concept.

At this point, the focus of this conceptual framework has shifted from the dynamics involved in the attainment of conceptual understanding to the goals of problem solving in our mathematics classrooms. As many researchers and mathematics instructors would agree, high school and most college level undergraduate mathematics courses seem to be designed for students to make good grades on the tests and examinations by simply displaying their procedural knowledge (Kaput, 1992; Tall & Vinner, 1991). Traditional teaching, even in the present era,
often ignores the principles of reflective abstraction where assimilation of new knowledge to pre-existing schema is considered inevitable. For example, the topic of proof by *mathematical induction* is first introduced to learners even before they are exposed to the role of proof in mathematical science. Thus they have no pre-existing schema as to what it actually means by mathematical proof.

Some middle school mathematics textbooks introduces the concept of finding the measurement of an angle \( A \) when \( \tan A = \frac{8}{9} \) by simply expecting students to perform a series of activities accomplished by button pushing. This book offers no explanation to a student who is looking for the logic behind this button pushing. Students are left to wonder why they were not allowed to divide both sides of this equation by \( \tan \), similar to how they would solve for \( A \) in another equation \( RA = 7t \) by dividing both sides by \( R \). This confusion appears to be a very legitimate one for an eighth grader who has no exposure to the notion of inverse relationships or functions. Therefore, the question remains: What is the goal of instructing to solve for \( A \) in \( \tan A = \frac{8}{9} \) as far as the eighth grade geometry curriculum is concerned?

**Summary of Theoretical Framework**

Overall, the theoretical framework presented here explores several crucial aspects of concept, knowledge, concept formation, different categories of knowledge, and glimpses of the theory of procept as well as the importance of representation. The discussion centered on the theories and their interpretations mainly by Piaget, Skemp, Tall and Vinner, Dreyfus, Eisenberg, Carpenter, and Grey. As described earlier, a person’s concept images depend on aspects such as the level of efficiency and flexibility with which the person acquired the conceptual knowledge. According to Piaget’s (1970) and Skemp’s (1971) theory of knowledge acquisition, knowledge or a concept is meaning-driven and is actively constructed by the learner. Tall and Vinner’s (1981) theory of personal concept definition and personal concept images, as discussed in the beginning
of this chapter, are very much in harmony with Piaget and Skemp’s theory of active knowledge construction. Piaget and Skemp argued that a concept is formed through experience. Abstracting is the process of identifying similarities among experiences. Gathering together experiences through similarities is termed classifying.

As noted above, in my research, I focus on student models of the concepts of asymptotes, limits, and continuity, and how they use these concepts to solve related mathematical problems mainly from the perspective of concept images and concept definition. I also focused on how concept images are influenced by previous mathematical experiences. I believe that a student who possesses underdeveloped concept images clearly has not accommodated their new knowledge with the pre-existing knowledge. They have not reflected enough and have not accommodated the connections and differences between the pre-existing and newly introduced conceptions. They are either ignoring the factors that cause disequilibria or they have not reached the cognitive maturity of realizing the disequilibria. Based on the theories discussed so far, students’ concept images are influenced by their previous experiences with the concept. Active encounters with situations that pose disequilibria are necessary to result in eventual concept equilibrium. Even though concepts are learned, not passed on, social interaction is inevitable to invoke cognitive demands that shake down incomplete conceptions. Figure 2.3 portrays my view of how equilibrium could be restored between concept images and concept definition.
In the next chapter, I have outlined some of the previous research studies that focused on student understanding of functions, limits, and continuity. One study particularly investigated student perceptions of the concept of asymptotes. In my research, I am treating the terminology concept images differently from how it was used before. By stating the term concept images, I mean conceptions that are inconsistent with the formal concept definition.
CHAPTER 3

LITERATURE REVIEW

In chapter 2, I discussed how theories of representation (Goldin & Kaput, 1996), concept formation (Tall & Vinner, 1981), procept (Grey & Tall, 1994), and cognition (Piaget, 1964, 1970) explain the significant role that prior knowledge plays in the acquisition of new knowledge. While prior knowledge becomes the foundation on which new knowledge is built, and while it can act as a scaffold for the construction of new knowledge, it can also interfere with the proper attainment of new knowledge (Tall & Vinner, 1981). The mismatch between prior constructs and newly acquired facts generates misconceptions and inadvertent learning outcomes.

Misconceptions and Factors

Mathematics through Passive Learning

Factors that seem to contribute to misconceptions in mathematics and science were extensively sought and studied by numerous researchers (Vinner & Dreyfus, 1989; Tall, 1996; Williams, 2001; Beeth, 1998; Fravelling, Murphy, and Fuson (1999)). According to Fravelling, et al., teachers often fail to promote higher levels of mathematical thinking through activities such as soliciting explanations of student solution methods. Beeth remarked that students participating as active examiners of concepts rather than as passive receivers develop better conceptual understanding. Romberg and Tufte, as cited in the Sabella (1996) article, argued that students’ views of mathematics as a static science that must be mastered through low level cognitive
activities such as rote memorization may potentially cause students to view mathematics learning in a superficial manner. In addition, Artigue (1991) noted that construction and control of meaning could be better achieved through different forms of conceptual representation which emphasize graphical representation to remediate student difficulties in learning functions, limits, and other higher mathematical concepts.

**Mathematics through Routine Procedures**

Researchers have investigated misconceptions regarding functions extensively. Concerning high school and college students, studies had shown that while the concept of functions is central to mathematics learning, it is one of the most difficult and often misunderstood concepts (Vinner, 1991; Tall, 1996; Dreyfus & Eisenberg, 1982; Moschkovich, 1998; Eisenberg, 1991). Students enter classes such as Calculus without being able to provide the definition of a function, and when prompted, the best they can do is to offer routine examples of functions (Clement 2001; Sajka, 2003; Zaslavsky 1997). Zaslavsky outlined some of the conceptual obstacles that interfered with the learning of quadratic functions while Yerushalmy (1997) documented student struggles while developing a definition for asymptotes.

Students’ inability to coherently communicate the mathematical concept of limit of a function was elaborated by many researchers (Cornu, 1991; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Szydlik, 2000; Williams, 1991, 2001; Bridgers 2006; Louise 2000; Juter, 2005, 2006; Przenioslo, 2004; Oehrtman, 2009; Hitt & Lara, 1999). Other studies conducted by Tall & Vinner, 1981; Hitt & Lara, 1999; Moru, 2006; and Doerr, & Tripp, 1999, all accounted for similar difficulties with the concepts of limits and continuity.
**Difficulty of Symbols**

Students especially struggle with mathematical symbol manipulations and interpretations of terminologies that are used to convey mathematical knowledge (Moru, 2006). For example, the symbols used to express limits, \( \lim_{x \to a} f(x) = L \) or as \( x \to a, f(x) \to L \), also convey the notion of unreachability with the use of arrows rather than signs of equality. Thus, the verb *approach* as well as the symbol \( \to \) may imply *a process that is never completed*. The considerations of limits as *unreachable* (Williams, 1991; Tall & Schwarzenberger, 1978; Juter, 2005) and as *bounds that cannot be crossed* (Cornu, 1991; Szydlik 2000) are most prominent among the difficulties identified with the concept of limits. The cognitive aspects behind the concept of limits cannot be adequately conveyed from its definition. This type of obstacle that arises from the very nature of mathematical knowledge is referred to as *epistemological obstacles* (Bachelard, as cited in Cornu, 1991). In-depth discussion of the symbols, their meanings, and the different contexts in which these symbols are interpreted may help learners deal with symbol manipulations more effectively.

**The Process-Product Dilemma**

Grey & Tall (1994) ascribed certain conceptual difficulties surrounding mathematical knowledge to the process-product (procept) dilemma. The ability to move flexibly among different aspects, such as the product and process aspects of the same mathematical concept, is crucial for the in-depth understanding of mathematical concepts (Goldin & Kaput, 1996; Eisenberg, 1991). In the next few paragraphs, I will elaborate some of the research studies that were referred to in earlier paragraphs.
Research on the Function Concept

Clement (2001) examined the development of concept images associated with the topic of functions. Concept images are different from concept definition since the former is influenced by students’ informal intuitions and other everyday experiences (Tall & Vinner, 1981). In her research, Clement investigated students’ conceptual understanding of the function concept using a 28-item instrument that consisted of questions that were atypical of the classic function problems. Thirty-eight public high school pre-calculus students were assessed at the end of the school year, and 6 of the 35 students who represented the high, middle, and low ranges of the paper-and-pencil assessment were further interviewed. Fifty-seven percent of the students revealed the misconception that a function must always be represented by an algebraic equation.

Regarding functions, students also held the image of a machine in which numbers are produced as they are entered (Clement, 2001; Eisenberg, 1991). Additionally, for many students, the notion of functions seemed to evoke merely the image of a graph that passed the vertical line test (Clement, 2001; Even, 1993; Dubinsky, 1991). Other restricted views regarding the function concept were noted as that of an algebraic formula (Dreyfus, 1991; Dubinsky, 1991, Sajka, 2003), as some rule to follow or equation to be manipulated (Dreyfus, 1991), as some graph that must be continuous (Dubinsky, 1991) and as the graph of a one-to-one function (Eisenberg, 1991). These overly narrow perceptions obstruct the process of identifying other function relationships, such as the one described in Figure 3.1 (2001). Clement noted that though some of these textbook-bound images were helpful for beginners to understand certain characteristics of functions, continuing to hold on to just those images may interfere with deciding whether a particular relation is a function.

In Sajka’s (2003) research on the topic of functions, a secondary school female student with an average ability in mathematics was interviewed in Cracow, Poland. The student was
familiar with the definition and multiple representations of functions. Similar to Clement, Sajka used a non-standard task as her research instrument. According to this task, the student was to give an example of a function $f$ such that for any real numbers $x$, and $y$ in the domain of $f$, the equation $f(x) + f(y) = f(x + y)$ holds.

The dialogue between the researcher and the student was recorded and transcribed. Given the nature of the problem, the researcher intervened during the problem-solving endeavor by probing for dialogue through asking questions. The student experienced tremendous difficulty in understanding the task. She brooded over the exact wording of the question and started looking for a certain $x$ and $y$. Then she began substituting numbers as the arguments of the function and was faced with a dead-end. Only after the researcher’s intervention and a lengthy conversation about symbols did the student reveal an accurate understanding of the problem. Sajka noted that the student’s problem solving process was based on certain words of the problem instead of the problem as a whole entity,

After focusing on the word *number* in the problem, the student started substituting numbers and wrote $f(2 + 3) = f(2) + f(3)$. While she was unable to correctly interpret the symbol, she started focusing on the word *equation* and wrote two equations: $f(x) = x^2 - 2x + 3$ and $f(x) = x^2 + 5$. These equations triggered the concept *functional equation* and lead to the discovery that in the current context, the focus of the problem was the equation, not the function notation. Finally, after further brooding over the equations $f(x) = x^2 - 2x + 3$ and $f(x) = x^2 + 5$ through routine processes such as the application of distributive laws and trying to balance the right and left hand sides, the student discovered the relation between the function notation and the argument of the function. In this case, the confusion was caused by the rigid interpretations of the functional symbolism (Sierpinska, 1992; Sajka, 2003; Cornu, 1991).
The vagueness of the function notation can be attributed to the innate nature of mathematical truth (Cornu, 1991; Moru, 2006; Louise, 2000).

Mathematical concepts appearing as processes and the products of processes have convoluting effects on students’ problem solving endeavors (Grey & Tall, 1994). To be specific, a function could be considered as a procept. The function $f(x) = x - 1$ represents the process of evaluating function values for specific values of $x$ as well as the product of that process. Specifically, $f(2)$ tells one to perform the operations, substituting 2 for $x$ and doing the subtraction, $2 - 1$. However, the fact remains that $f(2) = 1$ designates $f(x)$ with the image as the product of the above described process.

**Research on Quadratic Functions**

Zaslavsky’s (1997) research regarding conceptual obstacles that may impede student understanding of quadratic functions disclosed the issues of representation in relation to mathematics learning. Representation in this case referred to the algebraic and graphical illustration of functions. Eight hundred tenth and eleventh graders from 25 mathematics classrooms of affluent Israeli high schools participated in Zaslavsky’s study. The main obstacles that were noted by Zaslavsky (1997) were in the areas of interpretation of graphs, identification of the relation between function and equation representations, and assessment of the analogy between a quadratic function and a linear function. Students assumed that the graph of a quadratic function is limited to the visible part of the function which appeared on the graphing calculator (see Figure 3.1). With this narrow view, some students believed that a quadratic function could possibly have a vertical asymptote.
Research on Limits and Continuity

Several researchers (Cornu, 1991; Bridgers, 2006; Louis, 2000; Juter, 2004; Przenioslo 2004; Tall & Schwarzenberger, 1978; Tall & Vinner, 1981; Sierpinska, 1997; Szydlik, 2000; Williams, S., 1991, 2001) explored student notions of the limits of a function. The concept of limits, known to be the most fundamental idea in standard calculus, is hard to acquire due to its abstract and rigorous nature. The topic of asymptotes is often explained in textbooks in relation to the topic of limits even before the limit concept is formally introduced. The activities concerning asymptotes are usually restricted to applying rules and finding the equations of asymptotes in order to aid the graphing of rational functions. I noted that students often lack the connection between the limit concept and the concept of asymptotes, since there is generally very little attempt in textbooks and classrooms to connect these two important and similar concepts.

Szydlik (2000) examined university calculus students’ beliefs about mathematics and the role those beliefs played in their conceptual understanding of the limits of a function. Student beliefs on topics such as real numbers, infinity, functions, etc. which underline the limit concept are termed content beliefs. Student beliefs on how mathematical truth and validity are recognized are called sources of conviction. Appeals to consistency, logic, intuition, or practical evidence for
the validity in mathematical truth are termed as *internal sources of conviction*, while appeals to authority for the validity of mathematical truth are called *external sources of conviction*. The fact that students may not always rely on consistency, logic, or practical evidence for validity in mathematics was well documented. Instead, they rely on the authority of an instructor or a textbook (Szydlik, 2000).

Szydlik assumed that students’ sources of their convictions played a crucial role in their conceptual understanding. She analyzed students against two dimensions, their content beliefs and sources of conviction, and investigated the role each dimension played in their understanding of limits. She designed a 20-item paper and pencil questionnaire, a Convictions and Belief Instrument (CBI), to bring forth student conceptions of real numbers, infinity, and functions in addition to their source of belief. Thus, the CBI provided a context for dialogue about participant’s mathematical beliefs and sources of conviction. Although all items elicited either multiple choice or Likert-type responses, students were provided free-response opportunities as well. Twenty-seven selected students participated in the second phase of the study, a limit concept interview. The limit concept interview consisted of tasks such as defining the limit concept, solving eight problems aloud, and responding to alternate conceptions of limits as bound, unreachable, and as objects in motion (Szydlik, 2000). While solving the problems aloud, students were asked to explain their problem-solving methods as well as the rationale behind choosing those methods. The responses were scored for mathematical accuracy on a scale from 1 to 5 and were further analyzed. External sources of conviction resulted in increased misconceptions and an inability to support even a correct answer. Students with low scores on sources of convictions produced low scores on content basis as well. These findings seem to support the constructivist assertion that active knowledge construction is imperative for the facilitation of conceptual understanding and logical reasoning (Szydlik, 2000). Investigation and
active knowledge configuration are intrinsic processes that would be more meaningful to the learner.

Juter (2006) also conducted research on the concept of limits. The participants were enrolled in a first-level university calculus course. Data were collected through student solutions to limit tasks and their responses to questions regarding their attitudes towards the limit concept. The tasks were given five times (stages A through E) over a period of ten weeks. The difficulty levels of the tasks got higher and higher as stages progressed. Later in the semester, 38 students participated in two individual interviews each. Each interview lasted 45 minutes. Both formal and informal types of limit tasks were included in the interview. Students were asked to clarify their solutions and to provide written problem solutions.

The second interview was administered during stage E of the study. This interview lasted for about 20 min and fifteen students were interviewed at this point. One of the questions that were asked during this study was, “what will happen to the function \( f(x) = \frac{x^2}{x^2 + 1} \) as \( x \) approaches infinity?” Juter (2006) referred to this problem as a non-routine task. A routine task was to compute \( \lim_{x \to \infty} \frac{x^3 - 2}{x^3 + 1} \) and to explain if the function could attain its limit value in the problem. Another question was to provide the definition of limits of functions using formal and informal language. Juter noted that many students treated limits as being attainable when they solved tasks but unattainable in theoretical discussions. Many students in this study were able to solve easy tasks about limits even though they were unable to explain the meaning of the definition.

Juter’s (2006) study revealed that the process of modifying mental representations of a concept is not a very easy task. If a student’s experiences thus far were consistent with the theory
as he or she perceived it, then when faced with a contradicting new situation, the student tends to consider the new situation as a minor exception. While students were mostly able to perform routine limit finding tasks, they were unable to see connections between the formal definition and the examples they were solving. Nevertheless, these students seemed confident about their ability to grasp the concept, and according to Juter (2005a), this lack of awareness can perhaps be made explicit to them if the students are often placed in challenging and explorative situations.

Juter (2005a) noted that in some cases, student representation of concepts changed eventually over the span of the course, while in some other cases incomplete mental images were left unaltered. According to Juter, in most cases, students needed to see many examples that contradicted with their mental images in order for them to make necessary modifications. One way to accomplish this goal was to provide students with ample opportunity to examine different types of problems and to debate and dispute concepts from multiple perspectives.

There is another notable study, conducted by Przenioslo (2004), pertaining to university calculus students’ images, associations, conceptions and intuitions in connection with the limit concept. The degree of efficiency of conceptual notions and the sources of their formation were analyzed in this study. The research was motivated by the researcher’s own dissatisfaction with his students’ concept images of limits of functions. In describing concept images, words such as “association”, “intuitions”, “efficiency”, “degeneration” and “key element” were used by the researcher. “Association” meant “a connection made in the mind between different things, ideas, etc.” Intuitions or “intuitive conceptions” reflect what somebody thinks is obvious about a given concept although it might be inconsistent with its accepted mathematical meaning. Concept image was considered ‘efficient’ if a student was able to use it to solve a problem correctly. A concept image was considered ‘degenerated’ if it was considerably different from this concept
and was difficult to correct during the conversation with the student. ‘Key element of the image’ is the element, which a student most frequently applied in conflict situations.

Przeniosło’s (2004) extensive research was conducted over a period of years ranging from 1993 – 2001 involving 238 university students from the beginning level to the completion of an analysis course. The researcher was curious to know if and how student perceptions of limit concept changed as the level of sophistication of their mathematical exposure increased. An extended set of “simple but not quite standard” problems was used by Przenioslo. The research process included the analysis of written tests, interviews and group discussions among students. Students’ lecture notes and their informal conversations with teachers and peers were also studied to describe the concept definitions students were familiar with. Written tests composed of specially designed mathematical problems related to limits were followed by an interview asking students to explain their solutions. The problems were non-routine in nature in order to avoid automatic solutions which mainly relied on recall.

Open ended questions such as “What can you say about the limit of the sequence \((a_n)\)? Explain your answer” were asked to gain deeper understanding of students’ associations. Certain problems such as “If we know that the limit of a function, \(f\) at \(x_0 = 1\) equals 2, could the given graph represent this function?” were designed to see if students were aware of any contradictions. In some other cases, students were asked to complete the graph of the functions that had a particular limit behavior. Students’ concept images revealed that those who believed that the limit of the function \(f\) at \(x_0\) was the function value \(f(x_0)\) mainly relied on algorithms for calculating limits. Among the diverse concept images of limits noted, the idea of a graph approaching a point and values approaching a number were prominent. The largest group of students who
participants in the study seemed to have the concept image of a limit as a *graph approaching something*.

Besides the vivid concept images that were exposed, according to Juter (2005), the most significant conclusion drawn from this study was that students continued to hold on to the same misconceptions they had held before, even after taking a series of related courses, including analysis. Additionally, students seemed to have gathered additional misconceptions as they spent more time studying similar mathematical concepts. It should be noted that even at the level of taking analysis courses, many students did not seem to think that the formal definitions of mathematical concepts were important.

Moru (2006) investigated the epistemological obstacles and the effects of language and symbolism encountered by undergraduate students in regards to the concept of limits. The study was conducted at the National University of Lesotho (NUL), in South Africa and data were collected by using interviews and questionnaires.

In investigating the idea of limits within the context of functions, several epistemological obstacles were encountered. They were caused by over-generalizations and the assumptions that the limit value should be the function value. When students encountered situations in which the function value and the limit value were equal, they assumed that the *concept of limit* was the same as the *concept of function value* even though conceptually, they were two different entities.

Moru (2006) suggested that students must be exposed to different kinds of limits of representations using *simple functions* and using various examples of *sequences*. He suggested that students must be exposed to words with multiple meanings and the meanings that they carry in the mathematical contexts. Moru noted that in encapsulating processes into objects, everyday language also acted as an epistemological obstacle. When subjects were asked what they understood the limit to be, they said that the *limit is a boundary, an endpoint, an interval, or a*
Though these interpretations are correct, they are inappropriate if used in technical contexts such as in the mathematical context.

It was also noted that (Moru, 2006), while some subjects referred to the word *limit* as a noun to show that they referred to it as an *object*, other subjects described limit in terms of the *processes* that gave rise to it. According to Moru, this was an indication that full encapsulation of processes into objects was not achieved by the subjects.

Bridgers (2006) investigated the conceptions of continuity of 13 high school calculus teachers and their students by conducting a three-level teaching experiment. At one level, she investigated student conceptions of continuity as revealed through the activities that focused on the concept. At the second level, teachers revealed their own conceptions of continuity while they were also trying to unravel their understanding of students’ conceptions of continuity. At the third level, the researcher tried to develop a model of her own understanding of the students’ and teachers’ conceptions of continuity.

Bridgers noted that teachers who believed continuity to be important viewed it as a complex concept, while teachers who considered continuity to be unimportant viewed it as an easy procedure. Some teachers didn’t relate the concept of continuity with domains and some couldn’t stress the difference between continuity and differentiability. Bridgers noted much similarities in the ways in which students and teachers thought about continuity. Students often got confused between the concepts of continuity and differentiability, and they did not associate the concept of continuity with the limit concept. In addition, Bridgers found that teachers were able to acknowledge student conceptions of continuity accurately.

Hitt & Lara, (1999) studied student conceptions that posed obstacles in the construction of the concept of limit, continuity and discontinuity of functions. They analyzed the lesson plans of 9 high school mathematics teachers those were created without the use of any textbooks. They
also collected data through questionnaires and interviews of students who had just finished high school and were beginning engineering majors. Student misconceptions were found to be very similar to that of their teachers.

In their paper, Hitt, Lara (1999), featured one teacher’s notes in detail. Regarding the limit at infinity, and infinite limits, the teacher specified that it must be done from an intuitive point of view, using a numerical approach with graphical interpretation and by manipulating algebraic knowledge. To help students understand limits better, he used the examples of limit of speed, limit of elasticity of a material, and limit on a beach where you can swim safely. From these examples he expected students to deduce the implications that the limit cannot surpass a mark, the limit value is not reached, and you can be as near as you wish to the limit but not reach it.

In this teacher’s notes, regarding a graph with a hole, the teacher wrote, the function became discontinuous when the function touches the limit value. Hitt and Lara (1999) noted that the teacher was mixing the concepts of limits and continuity without giving the details of either of them while his statements re-iterated the fact that limit cannot be reached.

Williams (2001) and Oehrtman (2009) noted how student usage of metaphors could interfere with the development of limit concepts. Oehrtman engaged 9 students in 60 minute clinical interviews. Among student metaphors, the imagery of limits as approaching and infinity as a number were the most damaging to the study of limits. As for the effect of interference of common sense with scientific notation, Louise (2000) pointed out how students associated the word nothing with ‘a point does not exist’ and correspondingly assumed that the limit does not exist. According to the author, teachers who are familiar with students’ concept-forming mechanisms can choose ways to make conceptual obstacles obvious and try to ‘awaken’ student perceptions of these conditions.
Research on Rational Functions

Louise (2000) investigated college students’ understanding of the limits of rational functions and the ways in which students dealt with division by zero and infinity. The study focused on how ten college students related to rational functions and indeterminate forms as \( x \) approached infinity. The study also focused on how students interpreted undefined forms and division by zero. Two interviews were conducted.

Louise noted several erroneous ideas regarding the behavior of functions by the points where it was undefined; they were categorized as the continuous type, the type presenting a space, the asymptotic type, and the type including a vertical asymptote. Students ignored the discontinuity of a function at a point where a hole existed as well as at a point where infinite discontinuity existed. They drew a continuous type graph at those points. Discontinuity presenting a space was noted in the instances where students left space between parts of the graph at a point where infinite discontinuity occurred.

Asymptotic discontinuity according to Louise was demonstrated by asymptotes drawn at a point where the function had a hole. Louise noted that some of these errors have occurred from students lacking the knowledge regarding the indeterminate and the undefined forms. He also believed that these students possessed incomplete knowledge regarding the notion of asymptote. Students explained division by zero as infinity, and interpreted infinity with nothing in limit situations. Some other students held the belief that indeterminate form meant infinity, only this time infinity meant all real numbers.

In addition, students’ believed that, horizontal asymptotes existed at points where indeterminate form occurred since both hole and finite limit existed at that point (Louise, 2000). Louise documented such acts of borrowing certain aspects of a mathematical form and
associating all or parts of it with another related but different mathematical form as being influenced by *neighboring knowledge*.

**Research on Asymptotes**

Another study related to student conceptions of asymptotes was documented by Yerushalmy (1997) regarding her investigation on the formation of the definitions of asymptotes of rational functions. Using the method of guided inquiry, aided by computer exploration, the researcher tried to facilitate reasoning and argumentation in a high school precalculus class in Israel. Data analysis was based on the observation of group discussions and studies of written work of 30 participants.

The study of Yerushalmy (1997) is included in this paper for two prime reasons. First of all, it elucidates the effect of certain geometric and numeric aspects of mathematics as related to the issue of misconceptions in the field. Vinner (1991) noted that geometric factors could amplify the complexities of certain mathematical concepts. Yerushalmy noted that several geometric factors that contributed to the misconception of asymptotes could be re-phrased as the following questions: (a) are asymptotes and the function concurrent somewhere? (b) What does it mean to say *to approach* infinity? (c) Regarding asymptotes should the focus be on vertical or horizontal distance?

Secondly, Yerushalmy (1997) investigated the most fundamental aspects of asymptotes, the formation of concept definition. The researcher used the method of guided inquiry with the support of computer exploration. This approach is slightly different approach compared to traditional teaching practices. Facilitation of reasoning and argumentation was one of the goals of this research. Analysis was based on observation of group discussions and investigation of written work.
While the participants in this study (Yerushalmy, 1997) were not acquainted with a formal definition of asymptotes, a wide range of definitions evolved from generic examples, using technology as an aid to graphing and exploration. Episodes provided insights into how geometric as well as numerical aspects of asymptotes lead to a broad range of reflections on function and asymptotic properties.

One definition for a vertical asymptote was developed as “a vertical asymptote is a vertical line that exists only at points where the function is not defined and the function’s values approach negative or positive infinity” (Yerushalmy, 1997, p. 8). This definition prompted a question by the teacher whether it is possible for a function not to have a vertical asymptote at points where it is undefined? As a result, students discovered that it is possible for a function to not have a vertical asymptote at a point where the function was undefined if the function had a finite limit there.

Next, with regards to horizontal asymptotes, students were asked to perform a variety of tasks. In addition to constructing a definition, students were asked to write equations of rational functions that possess various horizontal asymptotes. Students made several discoveries, such as “a horizontal asymptote is a horizontal line, is parallel to the $X$-axis, and comes closer to the curve at infinity” (Yerushalmy, 1997, p. 12). Some students also incorrectly stated “if there is a horizontal asymptote, the degree of the numerator has to be larger or equal to the degree of the denominator” (Yerushalmy, 1997, p. 19).

With regard to oblique asymptotes, students were asked to explore graphs of several functions in order to generate a definition (Yerushalmy, 1997, p. 8). One student noticed that a rational function $f(x) = \frac{x^2 - 4x - 5}{x - 3}$ could be re-written as $f(x) = (x - 1) - \frac{8}{x - 3}$ after performing polynomial division. This observation led to the discovery that for large values of $x,$
the function approaches the line $x - 1$, which is in fact the oblique asymptote. In this example, mere graphical representation was not enough to gain understanding of an oblique asymptote. The algebraic procedure of polynomial division and the rewriting of the function equation were necessary to realize the occurrence of oblique asymptotes. The algebraic procedure of polynomial division, and re-writing the function equation accordingly was necessary to realize the occurrence of oblique asymptotes.

Need for Further Research

Louise (2000) noted that students perceived the indeterminate and undefined forms as if they are the same. He cited that some students drew vertical asymptotes at places of holes while some other students drew horizontal asymptotes at places of holes. They drew horizontal asymptotes at holes since finite limit existed at holes. Louise also noted that students drew a straight line with empty space disconnecting it to account for infinite discontinuity. Bridgers (2006) elaborated on students’ inability to connect continuity with the limit concept. Yerushalmy noted students’ confusion regarding the question of concurrence of graphs of functions and their asymptotes. Szydlik (2000) documented the impact of student beliefs on how they learn mathematics. Yerushalmy and Zaslavsky (1997) documented the impact of technology on conceptual understanding.

In Yerushalmy’s (1997) study students were to develop definitions by examining the graphs of various functions on a graphing utility. These students got the opportunity to actually see different types of graphs and compare and differentiate them. Then, based on the pictures they saw they identified specific function behaviors by asymptotes. In this research students depended on a graphing utility that readily allowed them to see the function behavior. According to Dreyfus (1991), concept formation happens through a series of acts, first through a single representation (starts from a concrete case, or a single representation), then using more than one
representation in parallel and making links between parallel representations, integrating representations and flexibly switching between them.

Sajka (2003) and Williams (1991) elaborated on the roles played by the narrow focus of instruction on conceptual understanding. Studies of students' difficulties with limits and other calculus topics (Cottrill et al., 1996; Asiala, Cottrill, Dubinsky, & Schwingerndorf, 1997) while offering pedagogical suggestions stated that students fail to use calculus strategies when dealing with non-routine problems. Some instructors hold the overly simplified attitude that a clear exposition is all it takes for students to gain adequate conceptual understanding (Cornu, 1991).

During instruction, in addition to presenting a clear exposition, students must be made aware of the complexities pertaining to the mathematical concept under discussion. Teach them to reflect on their own ideas. Make them aware of conceptual difficulties. Make students' spontaneous ideas explicit. Juter (2006) noted that students viewed concept definition as something unimportant and disconnected from problems to be solved about those concepts.

In light of what was reported in previous research studies, I believe that research that explicitly investigates student conceptions of asymptotes, limits and continuity of rational functions is required. In addition it is imperative to explore research based instructional proceedings that could facilitate better conceptual understandings of mathematics knowledge.
CHAPTER 4

METHODOLOGY

The purpose of this research was to examine college students’ concept images of rational functions, asymptotes, limits, and continuity and the connections that students may have formed between these concepts. In addition, a teaching experiment, a qualitative method was designed to help students re-configure their incomplete concept images. The teaching experiment also helped the teacher-researcher understand the ways in which students developed their conceptual models of the concepts of asymptotes, limits, and continuity of rational functions.

Generally, the recent history of research processes can be divided into the positivist era and the post-positivist era based on the human view of the world and how to study it (Lincoln & Guba, 1985). Positivism can be identified by scientific (quantitative) research that involves hypothesizing, and testing of hypotheses. On the other hand, post positivism (also called naturalism) uses qualitative methods that are geared towards developing an understanding of the human system. The research goal could be small, such as learning particular student behaviors or large, such as studying a cultural system.

Qualitative Research Methodology

Postpositivists’ methods implement qualitative research methods based upon the assumption that multiple realities exist in various people's perception of the world. In order to
understand reality, multiple research methods must be implemented (Reason & Rowan, 1981; Sabia & Wallulis, 1983). Information sought through a variety of methods needs to be combined and further examined in a meaningful manner. The purpose of qualitative research is to better understand a particular phenomenon. Consequently, this approach emphasizes description, uncovering patterns, giving voice to the participants, and sustaining flexibility as the research project expands.

The choice of research approach must be based on the goal of the research and the position of the researcher within the particular discipline. Qualitative research that is conducted in a natural setting without intentionally influencing the environment usually include research methods such as interviews and observations with the perspective that human beings construct their own reality, and an understanding of why they do what they do is based on why they believe what they believe (Lincoln & Guba, 1985).

**Teaching Experiment Methodology**

I conducted a small scale Teaching Experiment (Steffe & Thompson, 2000) with a small group of participants to gain insights into, and build explanations of students’ mental constructions of mathematical concepts. Through a teaching experiment a researcher will be able to gain first-hand information on students’ mathematical reasoning (Steffe & Thompson, 2000). Teaching experiments are usually situated within a constructivist theory of learning in which learning is believed to be happening because of learners’ active construction of knowledge.

**Components of the Teaching Experiment**

Steffe (1980) identified three major aspects of a teaching experiment as individual interviews, teaching episodes, and modeling. Teaching experiments are concerned with conceptual structures and models of the kinds of change that are considered learning or development. No single observation can be taken as an indication of learning or development.
Since change is the transition from one point to another, at least two observations made at different times must be required. The change itself is subject to interpretation. To this end, usually individual interviews are conducted before and after teaching episodes. Modeling is the formulation of models that would explain the results of the research. In this research, modeling would explain the researcher’s perception of how students construct, interpret, and refine their notions of the mathematical concepts of asymptotes, limits, and continuity. While there is more than one way to perform individual interviews and teaching episodes during a teaching experiment, the order and manner in which this researcher conducted interviews and teaching episodes are now described.

The Role of a Witness.

Steffe and Thompson, (2000) state that communications with students during the teaching episodes can be easily established if the teacher-researcher interacted with students in situations similar to that of the teaching episodes. Steffe suggests that for this purpose the teacher-researcher must engage in exploratory teaching before the conducting of teaching episodes.

Recognizing mathematical language and current students’ actions in an interaction that one has experienced before is a source of confidence for the teacher-researcher that communication is indeed being established. However, the teacher-researcher should expect to encounter students operating in unanticipated and apparently novel ways as well as their making unexpected mistakes and becoming unable to operate. In this case, it is often helpful to be able to appeal to an observer of the teaching episodes for an alternative interpretation of events (Steffe & Thompson, 2000, page 283).

The application of the teaching experiment methodology in the present study is described below in the Teaching Experiment section and Data Collection section.

Ethics

Traditionally, informed consent, confidentiality, and anonymity are key concepts associated with the issues of ethics. In addition, the roles of the researcher also come into play
when judging the issues of ethics. Informed consent requires informing each and every participant about the overall purpose of the research and its key features, as well as of the risks and benefits of participation. Consent is usually given in written format, however, verbal and audio-taped, or videotaped consent is also allowed. Students were notified of their right to withdraw from the study any time they wish.

In ensuring confidentiality, the investigator may not report private data that identifies participants. Any information regarding student performance in the related mathematics course was obtained only through self reports. Student anonymity was maintained by the use of pseudo names and by not videotaping the participant’s face. Pseudo names were later changed to newer pseudo names picked by the researcher, so that no one can recognize the research reports through the old pseudo names. Institutional Review Board (IRB) guidelines were followed and IRB permission was obtained for the conducting of this research.

The research participation was strictly voluntary. The participants were notified of their right to withdraw or discontinue participation at any time including the exclusion of their problem solving interview from the analysis.

Recruitment and Participants

I obtained permission from the department of mathematics of the university where I conducted the research. Then, I contacted the instructors of the calculus 2 classes for their cooperation. As per their request, I agreed to talk to students before class started so that instructional time would not be used for the recruitment. Thus, I sought volunteers to participate in the research; they signed the consent form attached in appendix A to acknowledge their interest in the participation of this research. Students were further asked to complete a Background Feedback Survey (see Appendix B). On the Background Feedback Survey, students were asked to provide their full name, their rank, and their contact information. They were also asked to self identify as
an A-achiever, a B-achiever, a C-achiever, or below C achiever in mathematics. I presumed that
the A, B, or C performers in mathematics are most likely to have some level of knowledge
regarding the concepts of rational functions, asymptotes, limits, and continuity since they
received a passing grade. Gaining insights into their conceptual understanding helped me identify
what *passing* students actually knew about mathematical concepts that they deemed as having
satisfactory knowledge in.

Forty seven students signed up for the research study. I e-mailed 21 students, 7 A-
achievers, 7-B achievers, and 7 C-achievers based on the immediacy of availability to participate
in the first problem solving interview. Nineteen students completed the first interview, eight were
selected for the teaching episodes, and seven completed the episodes. Details of the selection are
given in the procedure section below.

This university was chosen due to easy accessibility and the diversity in its student
backgrounds. Students who attended this major university came from all over the world and
therefore brought a wide variety of concept images contributed by their varied mathematical
experiences. The Calculus 2 course focused on integrals, area, the fundamental theorems of
calculus, logarithmic and exponential functions, trigonometric and inverse trigonometric
functions, methods of integration, applications of integration, and polar coordinates. In these
courses, students were not allowed to use graphing calculators with CAS (Computer Algebra
Systems) capacities.

Each student who participated in the first problem solving interview was awarded 20
dollars for their participation while those who attended the teaching episodes were awarded 15
dollars of each session they attended. Each student who participated in the second interview was
given 15.00 dollars.
The Researcher

According to Spradley (1980) the researcher assumes a variety of roles such as the instrument itself, a prodding outsider, and at times an involved insider. The role of the researcher resembles a rollercoaster ride which could be exciting and fearful at the same time. There was a period of intense data collection and analysis during which time the teacher-researcher may brood over the redundancy of some of the processes. There was also a stage in which the study became more focused and meaningful (Bernard, 1988). A teacher-researcher will have to diligently and systematically work through the ebbs and tides of this period of uncertainty.

The researcher holds two bachelor’s degrees, one from the University of Kerala, India, with mathematics major and statistics and physics minors. The other bachelor’s degree is from Florida Atlantic University, Boca Raton, Florida with mathematics major. Further, she holds a master’s degree in pure mathematics from Florida Atlantic University, and he has have been teaching mathematics for the past 17 years in the United States. The researcher has taught mathematics courses from pre-algebra to calculus, methods courses, linear algebra, elementary statistics and mathematical statistics in a wide range of educational setting such as major universities (7 years), liberal arts colleges (4 years), and community colleges (8 years) and high school (2 years).

While teaching a wide range of mathematics courses from Developmental Mathematics to Calculus III and Linear Algebra solely from a pure mathematics background, I was curious about why certain mathematical concepts are hard for students to acquire. I was questioning why despite of my best explanations, some students still failed to gain proper conceptual understanding. As a part of my graduate work in mathematics education, I was better informed on the nature of mathematical knowledge and the complex aspects of concept formation. It was
fascinating to learn about the diverse concept images students possessed regarding various mathematical topics. The role that prior knowledge played in the acquisition of a new concept was indeed intriguing and worthy of exploration. Thus, for me, the researcher, this endeavor was the fruit of a sincere and passionate attempt to uncover the complexities of important mathematical topics in order to lend a helping hand to the learners of mathematics.

To build models of student conceptions, one should have a picture of the factors that contributed to student difficulties in learning mathematics. Among other things, I believe that the innate nature of mathematics played a serious role in students’ conceptual understanding. For example, the mathematical concepts of functions, limits, asymptotes, and continuity had to be sieved through years and years of refinement before they were presented the way in which they currently exist. Therefore I would attribute student difficulties partly to the innate nature of mathematical knowledge. Students’ difficulties such as whether limits were reached or not could be placed in this category. The instructional features that are commonly implemented in mathematics teaching and learning hold some responsibility for student difficulties with mathematics learning. Not enough attention is given to details that would help students understand the multiple ways in which mathematical concepts could be represented. For example, not enough emphasis was given to the deeper structure of asymptotes; and therefore, according to students, asymptotes are dotted lines that the graph approached but never reached.

**Data Sources**

In the teaching experiment, the sources of data were the participants and the teacher-researcher. Participants provided data through two problem solving interviews and teaching episodes, while the teacher-researcher provided data through her interpretations of student performance and interactions during the teaching episodes. Interviews called for observing and analyzing student actions of problem solving, listening to their arguments supporting their work,
and asking probing, open-ended, and semi-structured questions to foster discussion and extended clarification of concept images. According to Noffke and Somer (2009) observation is one of the most important methods of data collection since it entails the researcher making a record of her impressions of what takes place. During the interview, multiple questioning techniques were implemented to solicit student conceptions. These techniques included repeating the same question after rephrasing it carefully, probing, seeking explanation of their answers and methods, seeking re-explanation, and asking for justifications. In addition, students’ work during the problem solving interview was reviewed and re-evaluated through video tape analysis of the interview.

Teaching episodes were videotaped and the data that came from the transcripts of these videotapes were collected. During the teaching episodes, the teacher-researcher tried to develop a model for student understanding of the concepts under investigation. This was accomplished by closely observing students’ problem solving procedures, listening to their discussions, and taking notes during and after the entire process.

**Development of the Interview Instruments**

The interview instruments were originally modeled after Yerushalmy’s (1997) research in which student understanding of asymptotes were deliberated by having students analyze a series of graphs of rational functions. The instruments were comprehensively modified using a grounded approach in which, data were continuously gathered and analyzed using a series of mini-studies. These peer-edited instruments were used to gain insights into students’ understanding of asymptotes of rational functions and how the concepts of division by zero, continuity, and other basic aspects of limits, were connected with the concept of asymptotes.
Mini Pilot Studies

Mini studies were conducted in 2004, 2006, 2007, and 2009 and four of the participants were my students who were enrolled in a severely rigid, departmentalized precalculus course. In this course, even the homework problems were set by the department. One student, an A-achiever, was in my calculus class. During mini studies, seven problem-solving interviews were conducted and analyzed. The problem-solving instrument, along with the videotaping procedure was developed during these studies. Two of the seven students, who participated in the mini studies performed at the B-level and two others performed at the C-level in mathematics. Two students performed at the A-level in mathematics and one student was a highschool graduate who received a letter grade of A in a rigorous AP calculus course. One of the A-level students was a sophomore majoring in mathematics.

The first mini study focused on how students graphed rational functions and used asymptotes in the graphing process. Manual graphing was given prime importance and students were evaluated on their knowledge of intercepts, and other properties of functions. I realized that examining how students graphed rational functions alone was not adequate for profiling their understanding of asymptotes. Therefore, student intuitions on the concept of asymptotes were sought through a variety of external illustrations: graphical, algebraic, and numeric. Bringing theories of representation into the study shed further light on how students identified mathematical concepts through internal representations and how internal representations could be observed and interpreted through analyzing external outputs which could be words, numbers, and graphs.

As a result of the mini studies, I concluded that students conceived asymptotes as lines the graph of the function simply approached but never reached. Students in general seemed to believe that vertical asymptotes always existed at points where the function was undefined. In
some cases, students were able to use the algorithm to find the equations of asymptotes, they were unable to explain what asymptotes were. For example, most students were unable to use formal terms such as limit at infinity and infinite limit while referring to limits and asymptotic behaviors. Students’ concept images were incomplete at times and they seemed to have stemmed from the images of distinct graphs of functions that they were most familiar with. Most students did not mention oblique asymptotes.

Regarding rational functions, students seemed to have a default belief that all rational functions were undefined at some point. They believed that the presence of a denominator in rational functions implied that the denominator must be zero at some point. Students mainly consulted with their intuitions of undefinedness to decide if a function was a rational function.

Later, I gave them the definition that a rational function has the form \( F(x) = \frac{p(x)}{q(x)}, q(x) \neq 0 \), where \( p(x) \) and \( q(x) \) are polynomial functions that have no common factor. Although originally I was not anticipating the provision of definition during the interview, offering the definition proved fruitful since I was able to obtain first-hand knowledge on how students use the definition in mathematical problem solving contexts. Even after consulting with the definition, some students were conflicted as to whether a function such as \( m(x) = \frac{3x}{x^2 + 1} \) would be a rational function since they believed that a rational function cannot have the domain of all real numbers. Consequently, some students inferred that the above function could not be rational and they considered the formal definition as an exception while their intuitions were considered as the standard.

During the mini-studies, while students’ opinions varied on whether a graph could cross its horizontal or oblique asymptote, no explanation was given supporting their convictions.
Calculus students’ idea of asymptote seemed to be an assortment of notions of continuity, limits, and un-definedness. The student who was majoring in mathematics linked the concepts of continuity and limits with asymptotes and performed at a higher level compared to other participants. In examining graphs, this student was mindful of the limitations of technology and identified the graphing calculator-generated graph of a function correctly with the mention that graphing calculators are unable to display holes in graphs. He stated that at his college, students were not allowed to use graphing calculators in calculus courses. The discoveries made during the mini studies helped the researcher to continuously improve the testing instrument by restructuring questions. In addition, the mini studies helped with the identification of effective and engaging interviewing techniques.

The Interview Instruments: CORI 1 and CORI 2

Student conceptions regarding rational functions, asymptotes, limits, and continuity were assessed using the Concept Organizer Response Instrument (CORI 1) that is included in Appendix C. CORI 1 consisted of several problem solving tasks. Use of CORI 1 invoked qualitative features such as the employment of logic and the explanation of thought processes. The instrument CORI 2 was very similar to that of CORI 1 and was used for the second problem solving interview. Table 4.1 summarizes the relationship between the two instruments. The main objectives of CORI 1 and CORI 2 could be cataloged as follows:

(a) What are students’ concept images of rational function?

(b) Are students able to identify the graph of a rational function from its equation?

(c) How do students differentiate between the graphs and equations of rational and polynomial functions?
(d) What are students’ intuitions of asymptotes and the different types of asymptotes a function could have?

(f) Are students able to make connections between asymptotic nature of graphs and the algebraic description of the function?

(g) What are students’ numeric perspectives of asymptotes?

(h) How do students perceive division by zero?

(i) Can students provide the formal definition of asymptotes?

(j) What are student notions of continuity?

(k) What are student notions of limits?

(l) What connections if any, do students make with respect to the notions of asymptotes, limits, and continuity?

To help answer these questions, CORI 1 and CORI 2 included problems that are both typical and atypical to the types of problems usually appear in traditional textbooks. In most problems, students were allowed to use a TI-84 graphing calculator for the verification of results that they have arrived at. In the next paragraph, the significance of research questions will be discussed along with an analysis of how CORI questions were used to help answer the research questions.

**The Validity of the Instrument**

CORI 1 and CORI 2 instruments were peer evaluated. Other doctoral students taking seminar classes with me, in addition to my colleagues at the OSU Newark campus helped with revision, and recommendations to improve the first problem-solving instrument CORI 1. In addition, it was tested, and further modified through a series of mini studies over a period of four years.
The Research Questions

The purpose of this research was to identity college students’ concept images of asymptotes of rational functions and the connections that they might have formed between these closely related concepts. In addition, the effect of a teaching experiment on students’ existing conceptions was also investigated through this research. The research questions are

1. What are students’ concept images of rational functions?
2. What conceptions do students possess regarding asymptotes: in particular the horizontal and vertical asymptotes of rational functions?
3. What connections do students make between the concepts of asymptotes, limits, and continuity of rational functions?
4. What effect did the teaching experiment have on students’ understanding of the concepts of rational functions, their asymptotes, limits, and continuity and the connections between these closely related concepts?

Relevance and Relationship: Research Questions, CORI 1 and CORI 2 Questions

At the algebra and precalculus levels, students learn polynomial functions and their basic properties such as the domain, range, intercepts, the undefined forms, and the indeterminate forms. Polynomial functions are much less cognitively demanding than rational functions since the former have all real numbers for the domain and range, and they are continuous throughout their domains. However, rational functions that are introduced to students at the precalculus level mostly have restricted domains and their graphs usually come in several pieces. In addition, they are discontinuous at various points, with several different types of discontinuities that signified varied function properties. These functions possessed a variety of asymptotes that predict peculiar function behavior around the asymptotes. Do students believe that all rational functions are discontinuous and have asymptotes and restricted domains? If they do believe so, this itself
could cause confusions and impede the study of rational functions and their properties. The first research question was investigated by the use of questions 1 and 2 of CORI 1 (see appendix C).

Many rational functions have asymptotes. Asymptotes are connected closely with continuity and limits of functions. The concept of asymptotes appears for the first time in a precalculus course. Students don’t always fully understand it, and yet, they move on to taking calculus courses. There they are introduced to limits. During the instruction of limits, asymptotes are touched upon again and introduced as special limits. Later, continuity is explained by the use of limits. At this stage, students don’t fully understand rational functions, asymptotes, limits, or continuity. In many cases, students may not even see the connections among these topics.

If connections between the concepts of asymptotes, limits, and continuity are emphasized, perhaps one of these concepts could help understand the other concepts from a new perspective. Simply put, students who are unable to internalize the meaning of $2 - 7$ might be able to conceptualize $(2) + (-7)$ much easier, if they have the flexibility of representing subtraction as adding the opposite. In a similar manner, infinite limit could be better understood as vertical asymptotes and a hole could be viewed as the absence of vertical asymptote with the existence of finite limit. Accordingly, I was interested to discover student notions regarding each of these concepts and their connections because, such connections are paramount to attain the command and flexibility needed to tackle problem situations more adeptly. Consequently, the second research question evolved: what conceptions or concept images do students of calculus possess regarding asymptotes of rational functions? What mental pictures are evoked by the term asymptote? For example, do students think of any line, dotted line, a vertical line, a horizontal line, a point that is not in the domain of a function, an $x$ value the function is not allowed to have, infinity, a function behaving in a special manner, or some form of discontinuity? All of the CORI
1 (see appendix C) questions were designed to help the researcher understand students’ notions of asymptotes.

Moreover, it was also of interest to study the ways in which students relate to the formal definition of asymptotes. In particular, the definitions and terms students of calculus used to describe the concept of asymptote. Definitions and terms refer to students’ understanding of the formal definition that is used to describe asymptote/s. By analyzing this, I had hoped to be able to recognize whether students were examining if their intuitions were compatible with the formal definition. Research has shown that even those students who knew the formal definition of a mathematical concept did not always associate the meanings of that definition to the way in which they would apply the concept during problem solving (Schoenfeld, 1995; Tall, 1981; Cornu, 1991). CORI 1 (see Appendix C) questions 3, 9, 10, 11, 12 were intended help understand students’ formal conceptions of asymptote/s, limit, and continuity.

Student notions of horizontal and vertical asymptotes were specifically investigated through questions 1 through 4 even though almost all of the CORI 1 questions helped the researcher understand student conceptions of asymptotes from a diverse perspective. Similarly, student notions of the connections between asymptotes, limits, and continuity were sought using the third research question. The fourth research question investigated how students viewed the connections between these concepts during mathematics instruction. Questions 5 through 10 on the CORI 1 helped answer the third research question. The fourth research question investigated how the proceedings of the teaching experiment influenced students’ understanding of the concepts of rational functions, their asymptotes, limits, and continuity. In order to help answer the fourth research question, a second interview was conducted at the conclusion of the teaching episodes. The graphs in CORI 1 might not be clear at some places; however, due to this reason
students initiated a conversation with the researcher right away. Table 4.1 summarizes the correspondences between CORI questions, research concepts, and research questions.

<table>
<thead>
<tr>
<th>CORI 1 Questions</th>
<th>CORI 2 Questions</th>
<th>Concepts Focused</th>
<th>Research Questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 through 4</td>
<td>Rational Functions, Graphs, Matching</td>
<td>Question 1</td>
</tr>
<tr>
<td>2, 3, 4</td>
<td>5 through 16</td>
<td>Rational Function HA, VA, Holes, Graphing, intercepts</td>
<td>Question 1</td>
</tr>
<tr>
<td>2, 3, 4</td>
<td>23, 24, 25</td>
<td>Function Construction, HA, VA, holes. Different possibilities for HA: a horizontal line parallel to X-axis, X-axis itself, no HA</td>
<td>Questions 2, 3</td>
</tr>
<tr>
<td>5, 6, 7, 10</td>
<td>17, 18</td>
<td>Continuity, graphically Continuity by computing limits</td>
<td>Questions 2, 3</td>
</tr>
<tr>
<td>7, 8, 13, 14</td>
<td>19, 20</td>
<td>Finite Limits, limits at infinity, infinite limits</td>
<td>Questions 2, 3</td>
</tr>
<tr>
<td>8, 9</td>
<td>11 through 16</td>
<td>Limits graphically Continuity graphically Finding the equations of HA, VA HA, VA connection to limits Graphing, intercepts, domain</td>
<td>Questions 2, 3</td>
</tr>
</tbody>
</table>

Table 4.1. CORI 1 and CORI 2 Questions and Conceptions
The Teaching Experiment

I will now illuminate my application of the teaching experiment methodology as documented by Steffe et al. (2000) in addition to elaborating on several important aspects from my perspective.

The Philosophy

Once students’ concept images were identified after the exploratory interview, the teacher-researcher carefully outlined the lesson plans to be used during the teaching episodes. Each lesson contained a set of problems that students were expected to solve individually. In some cases, for example, with rational functions, a packet of notes was provided that contained definitions of some concepts with examples and non-examples. Lesson plans were designed bearing in mind that in order for students’ conceptual schemes to change, they should be presented with situations in which they deem their existing schemes inadequate. In such cases, the new situation cannot be solved unless students make a major reorganization in their existing conceptual schemes.

Teaching Episodes

The setting of the teaching episodes was somewhat different from that of a typical classroom. There was no urge to rush; there were no deadlines to meet, and no curriculum that must be covered during these teaching episodes. In addition, direct instruction was not used and students were encouraged to first work on their own and then discuss their ideas in groups while the teacher-researcher took the role of a challenger and a negotiator of concepts. The focus was on how students’ conceptual schemes changed due to specialized mathematical interactions that took place between students, between the student and the researcher, and autonomously within each student. The researcher, however, asked probing questions, provided scaffold at times, and afforded students with additional resources as needed.
The Organization

The teaching episodes were conducted in two groups. The first group met on Mondays and Wednesdays with four students. The second group that met on Tuesdays and Thursdays had three students. Two groups were created mainly to accommodate student schedules and to view the first group as a semi-pilot group so that based on the observation of the first group the researcher could potentially make necessary adjustment in the second group. However, not much adjustment was needed between the two groups. Eight learning episodes or learning sessions were conducted with each of the two groups. Each session lasted one hour and thirty minutes and was held twice a week over a period of 4 weeks. As outlined in table 4.1, each teaching episode was guided by a lesson plan that focused on a core concept.

Before the discussion of any mathematical concepts, in order to help students see the process of problem solving conceptually, I started the teaching episodes with a brief presentation of the nature of mathematical truth. At this time, a list of rules was also discussed and they were nicknamed as the rules to live by. These rules were: do not try to memorize every single rule or fact you may come across, mathematics does not comprise simply of memorization, mathematics is a problem-solving art, not just a collection of facts, do not give up on a problem if you cannot solve it right away, re-read it thoughtfully and try to understand it more clearly, struggle with it until you solve it (Stewart et al, n.d.). These rules were taken from one of the textbooks that was used at this university for calculus courses.

The eight learning episodes were dedicated to mathematics concepts of polynomials, rational numbers, rational functions, and intercepts, usage of correct terminology, definition of asymptotes, vertical asymptotes, horizontal asymptotes, other asymptotes, limits, finite limits, infinite limits, and limits at infinity.
Due to lack of time, continuity was not discussed elaborately during the learning episodes. However, I tried to address students’ incorrect conceptions of continuity while limits and asymptotes were being discussed. The learning sessions could be subdivided roughly into 4 segments. The first segment focused on polynomials, rational numbers, and rational functions. The second segment was focused on intercepts, usage of correct terminology, and definition of asymptotes. The first and second segments lasted one session each. The first segment was the introduction and exploration and identification of rational functions. The second segment focused on polynomial form, more rational functions, and usage of correct terminology. The third segment of three sessions focused on rational functions, asymptotes, the definition, discussion of function properties near its asymptotes, and sketching the function graph tending to asymptotes of rational functions.

The third and fourth segments lasted three sessions each. During the fourth segment when the focus was on limits, concepts such as holes, vertical asymptotes, and horizontal asymptotes were also discussed in addition to the discovery of oblique and non-linear asymptotes. After the introduction of these phases, I will discuss the interactions of students that took place during these sessions and how learning took place as a result of the interactions. The organization of teaching episodes is summarized in Table 4.2.

Next, I will highlight the main points discussed during each of the segments by corresponding sessions. At this time, I will also elaborate on the design of lesson plans and the discussion of the briefings of each segment will include highlights of lesson plans developed for this research.
<table>
<thead>
<tr>
<th>Segments</th>
<th>Number of Sessions</th>
<th>Core Concepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Segment 1</td>
<td>One Session</td>
<td>Introduction, Rational Functions, Rational Numbers</td>
</tr>
<tr>
<td>Segment 2</td>
<td>One Session</td>
<td>Polynomials, Rational functions, Asymptotes</td>
</tr>
<tr>
<td>Segment 3</td>
<td>Three Sessions</td>
<td>Asymptotes</td>
</tr>
<tr>
<td>Segment 4</td>
<td>Three Sessions</td>
<td>Limits, Asymptotes</td>
</tr>
</tbody>
</table>

Table 4.2. Organization of Teaching Episodes

**The Learning Activities**

The lesson plans, more accurately named, learning activities were developed based on the participant’s cognitive difficulties identified during the exploratory interview. As outlined by Steffe (2000), the topic of each lesson as well as the learning activities were to be modified on the basis of new and unexpected discoveries that may uncover during the teaching episodes. During the teaching episodes, at times the teacher-researcher developed instant problems to address certain issues noted at that time.

*Development of the Lesson Plans.* During the exploratory interview, I realized that students were more focused on finishing the problem as quickly as they can. In addition, I felt that students held the belief that as long that they know how to differentiate, and integrate they have accomplished adequate understandings of subject matter of calculus concepts. To help with these situations, I created *the rules to live by* (see Appendix E) whose contents came from Stewart et al. (n.d.). After discussing these rules during the first lesson, based on the results of the CORI 1 interview, I decided to focus on the basic concepts such as rational numbers, identifying rational functions, finding their intercepts and domain. Extra problems of polynomial forms were discussed during the second session since students were unsure of what polynomials were. During the COR 1 interview, all students struggled to use correct terminology in regards to limits.
Therefore during the second session of the learning episodes, students engaged in the discussion of the usage of terminology using a worksheet that I have created. After each session, I examined students’ written work and their explanations. In addition, I took notes during the teaching episodes on student responses and interactions. I further watched the video tapes of the teaching episodes to develop learning materials for the next learning session. After most sessions, I met with Joann to discuss student interpretations and review the worksheets and handouts. She assisted by helping me analyze student conceptions and make decisions for the next session.

Table 4.3 summarizes the materials covered during each learning session.

<table>
<thead>
<tr>
<th>Session</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Discussion: rules to live by Rational function worksheet, Rational numbers</td>
</tr>
<tr>
<td>2</td>
<td>Polynomial worksheet, Rational function redefined, Symbols, intercepts, domain and range of rational functions</td>
</tr>
<tr>
<td>3</td>
<td>Redefinition of asymptotes, Worksheet on vertical asymptote and hole</td>
</tr>
<tr>
<td>4</td>
<td>Graphing rational functions, domain, range, vertical asymptote, horizontal asymptote, hole, how to describe asymptotes in terms of function properties</td>
</tr>
<tr>
<td>5</td>
<td>Graphing rational functions, domain, range, vertical asymptote, horizontal asymptote, hole, how to describe asymptotes in terms of function properties. How to find the equation of horizontal asymptote. Hole versus vertical asymptotes in terms of function properties.</td>
</tr>
<tr>
<td>7</td>
<td>Infinite limit worksheets, different ways to find infinite limits. Vertical asymptotes and undefinedness, holes and undefinedness, hole versus vertical asymptotes, undefined form versus indeterminate form worksheets</td>
</tr>
<tr>
<td>8</td>
<td>Limit at infinity and horizontal asymptotes Limit at infinity and infinite limits Infinite limits and vertical asymptotes Connection between limits and continuity</td>
</tr>
</tbody>
</table>

Table 4.3. Summary of Topics Covered during the Teaching Episodes
Teaching episodes were based on 6 assumptions on student concept formation and conceptual change. The first assumption was that student’s concept images were influenced by their previous exposure and experience with the concept. The second assumption was that students actively construct their own mathematical knowledge and active assimilation, accommodation and equilibration is imperative for the full development of these conceptions.

The third assumption was that, equilibrium can only be maintained through the active encounter with situations that may pose disequilibria in the already existing conceptual framework. Fourth, I assume that social interaction and argumentation are essential features of maintaining conceptual equilibrium. Fifth, teachers could intervene and pose cognitive confusion by simply invoking contradicting problem situations by the usage of appropriate examples or by redirecting students’ attention to formal definitions and probing a constant check between intuitions and formal definitions. The sixth assumption is that in a situation where the second, third, fourth and fifth assumptions are realized, a teacher-researcher might be able to foster concept refinement in learners. I believe that the process of concept refinement becomes complete through the processes of constant re-assimilation, accommodation, and the maintaining of equilibrium invoked by different types of cognitive demands.

During the discussion of rational functions, rational number concept was explored since many students related to rational functions in terms of rational numbers such as \(\sqrt{4}, \sqrt{36}, \sqrt{400}\). First, the definition of a rational number was provided as any rational number \(r\), can be expressed as \(r = \frac{m}{n}\), where \(m\) and \(n\) are integers and \(n \neq 0\) (any typical mathematics textbook). The definition of a rational function that was provided followed that a function of the form
\[ r(x) = \frac{P(x)}{Q(x)} \] where \( P \) and \( Q \) are polynomials and \( Q \) is a non-zero, non-constant polynomial (any typical mathematical textbook). The domain of a rational function was specified as consisting of all real numbers \( x \) except for which the denominator of the function was zero. It was emphasized that when graphing a rational function, one must pay special attention to the behavior of the graph near those \( x \)-values. At this time, polynomial forms and non-polynomial forms were discussed as well. In relation to rational numbers, students worked on re-writing decimals such as \( .235\overline{78} \) as a fraction. Additional worksheets that contained examples and non-examples of polynomials had to be developed since many students did not know the details of the polynomial form.

During the third learning session, the groups focused on the aspects of the textbook definition of asymptotes that they deemed inconsistent with what asymptotic properties were. Consequently, a meaningful definition for asymptotes was formulated and all agreed to follow the new definition as the formal definition of asymptotes. It took two sessions, the first and second sessions, to deliberate on the concepts of rational numbers, rational functions, polynomials, and the correct usage of terminology while describing mathematical entities. The details of the reconstruction of the definition of asymptotes will be elaborated in chapter 5.

The third segment needed roughly three sessions, sessions 3, 4, and 5, to deliberate asymptotes of rational functions. These concepts were explored by the usage of worksheets that asked to describe asymptotes by explaining function behaviors around them, by graphing different rational functions, by finding the equations of asymptotes, by exploring the limit forms of asymptotes, and by re-writing asymptotes by the use the limit notation. During this time, the aspects of holes vs. vertical asymptotes were discussed along with undefined forms and indeterminate forms and the behaviors associated with these forms. In these activities, students’ notions of the related concepts were explored from multiple perspectives. The details of these
problems may be found in the data analysis section, chapter 5 of the dissertation. However, I will discuss few sample problems with the reasons behind their inclusion.

It must be noted that, problems that involved the construction of a rational function from specified properties were not discussed during the learning episodes. During the exploratory interview, all students struggled with the construction of rational functions. I wanted to investigate if students were able to tie together the details of the concepts that were discussed during the learning sessions. I believe that construction of rational functions from the properties indicated in the problem would help the researcher better understand the depths to which a learner has re-configured the concepts discussed during the sessions.

During the fourth segment, the concept of limits was discussed at length. Formal definitions of limits were explored and consensuses were formed. Limits were then explored from the graphical representations of functions. Means of computing infinite limits and limits at infinity without using a graphing calculator were discussed. Tables of values were used sparingly while the focus was on formulating the most proficient ways to compute limits for problems of different types. A variety of activities were explored throughout these sessions some of which included each student correcting the problem solutions of researcher-created imaginary students. In this way, I was able to get students re-focus on certain details that they had overlooked before.

The change in student perspectives could however, be different from what was expected or there could be no change at all. Fundamentally, the researcher is trying to learn about the changes in students’ conceptions and how to explain such changes. It should also be noted that students’ mathematical schemes change only slowly, and there will be extended periods when students operate at the same learning level. For example, during the learning episodes, there were instances where some students simply could not differentiate between the phenomenon of a hole and a vertical asymptote.
Sometimes, students who seemed to have understood a concept suddenly could lose their steadiness when the concept was presented in a newer problem context. For instance, in a question such as: Consider the function \( K(x) = \frac{x^2 - 1}{x + 1} \), which could be simplified as \( K(x) = x - 1 \). Explain the behavior of this function at \( x = -1 \). Some students answered that function \( K(x) \) was defined at \( x = -1 \) since once the function was simplified, \( x + 1 \) was no longer in the denominator. However, in another problem that asked to compare and contrast the behaviors of the functions \( a(x) = \frac{3x - 1}{x + 1} \) and \( K(x) = \frac{x^2 - 1}{x + 1} \) at \( x = -1 \), these students stated that while the function \( a(x) \) will have a vertical asymptote at \( x = -1 \), the function \( K(x) \) will have a hole at the point \((-1, -2)\).

In another situation, a student who was able to distinguish between function behavior around holes and vertical asymptotes, and was able to connect these concepts with finite limits and infinite limits respectively, lost all her logic while analyzing the following situation. The student was asked to comment on the behavior of the function \( U(x) \) at and around \( x = 2 \), and \( x = 7 \) by observing the following table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1 )</th>
<th>( x )</th>
<th>( y_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>-2.118</td>
<td>7.001</td>
<td>210.2</td>
</tr>
<tr>
<td>1.999</td>
<td>-2.199</td>
<td>6.999</td>
<td>209.8</td>
</tr>
<tr>
<td>2.01</td>
<td>-2.208</td>
<td>7.001</td>
<td>210.2</td>
</tr>
<tr>
<td>2.1</td>
<td>-2.285</td>
<td>7.1</td>
<td>212.0</td>
</tr>
</tbody>
</table>

The student stated that the function \( U(x) \) has a vertical asymptote at \( x = 7 \) since the function was undefined at \( x = 7 \) and that the limit was approaching negative infinity as \( x \) approached 7 from the left and positive infinity as \( x \) approached 7 from the right. Then she responded that instead of a vertical asymptote at \( x = 2 \), the function will have a hole at \( x = 2 \) since
the function was approaching finite limit as \( x \) approached 2 from the right and from the left. It took her a while to realize that the function was in fact continuous at \( x = 2 \). I believe that in this case, the student was probably focusing on just two possibilities, a hole, or a vertical asymptote.

During the teaching episodes, at times, the researcher developed instant problems to immediately address certain incomplete student conceptions. For example, during the discussion of indeterminate versus undefined forms the matter of whether an indeterminate form could ever invoke a vertical asymptote arose. Students were wondering if this form always invoked a hole instead of a vertical asymptote. I developed a problem asking to compute \( \lim_{x \to 2} H(x) \), for the function

\[
H(x) = \frac{x^2 - 4}{x^3 - 4x^2 + 4x}.
\]

Students realized that even though, \( H(2) = \frac{0}{0} \) at first sight, once simplified \( H(2) = \frac{4}{0} \), leading to vertical asymptote at \( x = 2 \).

Other instances of unexpected dealings were invoked by student curiosity on what would happen if the degree of the numerator was larger than the degree of the denominator and what if the degree of the numerator was 2 or more degrees higher than the degree of denominator. These curiosities stimulated the discussions of oblique and non-linear asymptotes. To reveal another example, since factoring and canceling common factors were the main activities during the identification of holes and vertical asymptotes, I asked students to solve a problem described by a table of values below. I was simply curious on what students might come up with as a consequence of this particular problem. The problem stated that for the function \( V(x) \), the denominator was
$11x - 4$. By observing the following table, what can you say about the behavior of $V(x)$ at or around $x = \frac{4}{11}$? Note that $\frac{4}{11} \approx .364$.

This problem incited cognitive conflict in students that lead to some prime realizations. While they narrowed down the possibilities to be either a hole with finite limit or a vertical asymptote with infinite limit, the table of values clearly promoted some discussion. Even though the function appeared to approach infinities from either side of $4/11$, the function value at this point appeared to exist. While students curiously perused how this could be possible, consensus were formed except for exact value $x = 4/11$, the function will remain defined. These discussions lead to the realization that it was important to always the use exact value to discriminate between a hole and a vertical asymptote.

All four segments of the teaching episodes were very connected in nature and each segment helped fill the gaps within each of the other segments. In other words, each of the four segments together connected different pieces of the conceptual puzzle by helping students understand and establish the connection between the topics of asymptotes, limits, and continuity of rational functions.

In this dissertation, prior to the teaching episodes, an exploratory interview was conducted with 19 calculus II students. Teaching episodes were followed by an evaluative
interview. The details of both interviews and teaching episodes are detailed next. The exploratory interview was an individual problem solving interview to elicit students’ concept images of asymptotes, limits, and continuity of rational functions. The term *concept images* will be used in the sense of *incomplete conceptions*, or student conceptions that are *inconsistent* with the formal mathematical definition of the concept discussed.

**The Procedure**

Seven students from each of the three groups: A-level performers, B-level performers, and C-level performers, were chosen based on their immediate availability to participate in the Exploratory Interview. Nineteen students completed the interview. To protect their anonymity, students were asked to choose a pseudo name of their choice before the interview. I kept a list of participant’s real name with matching pseudo name in my home-office and the real names were kept separate from interview data. Participants were also informed that the Exploratory Interview may take up to two hours to conclude. A set of follow-up questions were planned as necessary. An *interview box* was created with necessary equipment such as batteries, calculators, videotapes, pens, pencils, graph paper, straight edges, and scrap papers. The problem solving interview was video-taped for further analysis. Only the test booklet (CORI 1) and the student’s hand were videotaped to maintain participant anonymity in addition to replacing participants’ real name using pseudo name of their choice.

**Participants for the Teaching Episodes**

Based on the results of the exploratory interview with 19 participants, I created a short list for the potential participants for the teaching episodes. This list was further narrowed based on student availability for long-term commitment and eight students were chosen to participate in a 4-week long teaching episodes. Each teaching episode, that was conducted twice a week, lasted one hour and thirty minutes. Two groups were formed for the teaching episodes. One group met
on Mondays and Wednesdays and the other group met on Tuesdays and Thursdays. The group that met on Mondays and Wednesdays will be referred as the MW group and the other group will be referred as the TTh group in this dissertation. The MW group had 5 participants while the TTh group had three participants. Students were assigned to the groups based on their class schedules. One participant from the MW session was eliminated due to repeated absence and the other seven students completed the study.

Both groups participated in similarly structured teaching episodes. I had planned based on my experience with the MW group, that I would get the opportunity to make any necessary modifications in the proceedings of the TTh group. However, as the teaching episodes progressed, no significant changes were needed. The week after the completion of the teaching episodes, the evaluative interview using the problem-solving instrument, CORI 2, was administered to seven participants who completed the teaching episodes.

Data Collection

The data from the exploratory interview, the teaching episodes, and the evaluative interview constituted the data for this dissertation. Data from the teaching episodes included the researcher’s informal notes during and immediately after each teaching episodes, students’ written work, and the transcripts of the video-taped teaching episodes.

Exploratory Interview Data Collection: Summary

Though in a qualitative research design it is often a challenge, I tried to provide detailed descriptions of the data by seeking regular patterns of student behavior by coding and sorting data as they were collected. Students’ concept images revealed through this interview were categorized based on the concepts studied. While it is not possible to precisely capture another person’s concept images, it is possible to access traces of concept images from the person’s dealings, records, and words (Juter, 2005). In this dissertation, student concept images of each of
the major concepts, rational functions, asymptotes, horizontal asymptotes, vertical asymptotes, continuity, and limits were cataloged based the commonality in their trends.

Data Collection during the Teaching Episodes

In the teaching experiment, the sources of data were the students and the teacher-researcher. Students and the teacher-researcher provided data in the form of audio and video tapes of the teaching episodes that were transcribed. The researcher provided field notes for all teaching episodes. Video images helped the researcher to go back and review the teaching episodes for clarification and reconfirmation of documented findings during the research. Video segments were also selected to record possible interactions suggesting that students were engaged in personal constructions of strategies to solve a problem or finding explanations for their mathematical realities.

The Witness/Consultant

I recruited a veteran community college mathematics professor Emeritus to be a witness and she originally agreed. Then it became clear that she would be privy to the identities of the participants, and she needed human subjects’ ethics training and IRB approval to be key personnel. Due to the delays, we agreed on a different role as consultant on the lessons. We met regularly every week and reviewed student work. I received feedback on interpretations of student work, and steps to include to further assist students during the teaching episodes. For example, Joann emphasized the importance of students being able to use the correct terminology while communicating mathematical knowledge. She also stressed the importance of students being able to understand the concept of limits in a manner that was more conceptually oriented. In addition, at times Joann reviewed the resources that I had developed and provided feedback.
Models of Student Conceptions

By assembling a model of student conceptions, a researcher hopes to highlight students’ mental transactions while solving problems. Some theorists defined a model as “systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar systems” (Doerr & English, 2003, p.112). In this research, models of concept development were developed for each of the participants of the teaching episodes and the evaluative interview. All of the observations made during the exploratory interview, teaching episodes, and the evaluative interview were used to accomplish the task of modeling. While it is not possible to understand exactly what is going through in someone else’s mind, the teacher-researcher was closely observing student actions and reflectively analyzing the possible reasoning behind the actions that were set forth by the student (Steffe, et al. 2000). In some instances, students were able to elaborate on why they performed the steps they have as per the researcher’s inquiry. For example, a student who ignored sketching parts of a rational function stated that he overlooked the existence of graph in that particular interval since nothing important was happening there. By stating *nothing important*, he meant special function characteristics such as asymptotes, intercepts, etc.

As elaborated by Tripp & Doerr (1999), the investigation focused on how shifts in student thinking occurred and in what ways such shifts in thinking supported the development of viable models. What kinds of events in their interactions with each other, with the problem situation, and with associated representations led to shifts in their thinking? This was accomplished through closely observing changes in students’ interpretations and representations during the teaching episodes as they faced challenging and inconsistent problem situations that conflicted with their existing concept images. This was accomplished mainly through identifying
recurring patterns. As described by Miles and Huberman (1994), patterns are explanatory or inferential codes, which identify an emergent theme, configuration, or explanation.

Another goal of this research was to understand how shifts occurred in the development of student models. Tripp & Doerr (1999) found that technology helped creating a mismatch between students’ certain paper-pencil representations. These researchers termed other elements that mediated a mismatch as conjecturing, impasses to progress, and questioning. Conjecture is a potentially valid statement that a student made about some particular aspect of a problem situation. In my view, faulty conjecturing could sometimes be brought to awareness by the introduction of inconsistent examples. Impasses to progress pose mismatches between internal models and experienced external models and could force students to look for alternate ways of thinking. This mismatch could be within or between words, graphs, and tables, or mismatches between different students’ internal models, or mismatch with in a given student’s way of interpreting the problem situation. Questioning between different students’ representations, and/or between a single student’s older internal models and newer models (self-questioning) could help realize shifts in thinking towards more viable representations.

Summary of Data Collection

Data were gathered through the exploratory interview by the use of the instrument CORI 1. The purpose of the exploratory interview was to identify student concept images (incomplete conceptions) of the concepts of asymptotes, limits, and continuity. The findings of the exploratory interview helped develop the lesson plans used for the teaching episodes. After the teaching episodes, an evaluative interview was conducted to explore any concept modification occurred as a result of the cognitive exercises of the teaching episodes. The purpose of the teaching episodes and the evaluative interview was to facilitate conceptual change by helping
students modify their cognitive structure. The teaching episodes in particular provided students ample opportunity to work collaboratively with other students and with the instructor.

The entire teaching episodes were video-taped and analyzed. CORI 2 was the instrument for the evaluative interview. The items on CORI 2 were very similar to that of CORI 1 in regards to the types of questions and the concepts covered. Table 4.3 summarizes the connection between CORI 1 interview, concepts covered in the teaching episodes, and CORI 2 interview.

**Summary of Data Analysis**

During the analysis, of the first interview data, student concept images were categorized according to the concepts studied. Incomplete concepts were analyzed and the common themes of the various concept images were identified. It must be noted that while analyzing and presenting the data of the exploratory interview, I was particularly looking for inconsistencies in student conceptions Based on student conceptions on various concepts, teaching resources were developed.

Data collected during the teaching episodes were also analyzed to trace concept modifications as well as further conceptual obstacles. During the teaching episodes, the researcher tried to create a model for student thinking and problem solving so that strategies that are more effective could be developed for the benefit of mathematics instruction. The activities that appeared to facilitate necessary concept modification were identified and categorized during the analysis. At the end of the teaching episodes, the second interview or the evaluative interview was conducted.

The data collected during the evaluative interview were used to analyze if and how an incomplete concept was restored as a result of the cognitive exercises of the teaching episodes. While the categories that I have selected are not the same as the categories selected by Tripp & Doerr (1999), similar to these researchers, I too closely observed students’ problem solving
activities to help me understand how student models were developed and modified. To analyze how an incomplete concept was restored, I specifically looked for cognitive dilemmas.

**Trustworthiness**

In qualitative studies the issues of trustworthiness and reliability of the study needs to be addressed (Lincoln & Guba, 1985). The first form of establishing trustworthiness is prolonged engagement with the participants. The technique of triangulation (Lincoln & Guba, 1985) is a way to support observations by using multiple sources and apply different perspectives to interpret the results. I gathered data using video-taped interviews, video-recordings of teaching episodes, and students’ written notes. Videos were transcribed in a timely manner. Any concerns or questions that the researcher may have about the interview was recorded immediately after the interview. The researcher undertook to observe, record, and analyze not just verbal exchanges but also subtle cues by using unassuming measures. For example, if a student appeared as though he or she was not comfortable to discuss a particular problem or concept, I avoided pushing that student to initiate the discussion. At the same time, if I felt that a student was simply agreeing with others on a certain problem situation, I probed that student to openly express his or her concerns. Peer debriefing, and reflective journaling (Lincoln & Guba, 1985) was also employed to reassure trustworthiness. After each teaching episodes, I met with a retired veteran mathematics professor to seek feedback regarding the transactions of the teaching episodes. Together we reviewed the lesson plans, student responses, and their written notes to understand student models from two different perspectives.

The evaluative interview was conducted within one week of the conclusion of the teaching episodes. Therefore, one might wonder if student responses to the evaluative interview could be considered reflective of concept modification that was supported by conceptual
understanding. Or could it be that students were simply responding to questions based on their recollection of what was said during the teaching episodes?

For the past 11 years, during the instruction of the concepts of asymptotes and limits, the teacher-researcher closely observed students as they tried to learn these concepts. During the mini pilot studies, the teacher-researcher explored different ways of asking questions to understand students thinking from multiple perspectives. The problems that were created for the interviews were very diverse in its design and concepts covered so that student notions regarding a concept could be examined from different points of view.

In addition, the questions on CORI 1 concerning the construction of rational functions from their specified characteristics deemed difficult to every student who participated in the interview. I made a deliberate choice not to discuss these types of problems during the learning episodes since I wanted to re-investigate if students were able to tie together and balance all the details of the concepts discussed during the sessions. I believe that creation of rational functions from the properties indicated in the problem would help the researcher better understand the depths to which a learner has re-configured the concepts discussed during the sessions.
This chapter is dedicated to the analysis of data that were collected during the research on College Students’ Concept Images of Asymptotes, Limits, and Continuity of Rational Functions. In this analysis, the term concept image is mainly used to describe student notions of a concept that are different from that of the accepted concept definition. Nineteen students participated in the first exploratory interview that was conducted using the instrument CORI 1. Based on student performance on the first interview and other constraints seven students participated in a one month long teaching experiment that contained eight teaching episodes that met twice a week. Each teaching episode lasted one hour and thirty minutes. After the teaching episodes were concluded, a second evaluative interview was conducted.

After examining the performance of the 19 students during the first interview, common concept images were identified based on the concepts of rational functions, asymptotes, vertical asymptotes, horizontal asymptotes, limits, and continuity. An in-depth analysis of these findings in addition to the student performance on the first and second interview will be presented in this document. Additionally, a comparison of student performance on both interviews will be elaborated in conjunction with researcher’s interpretations of how these changes might have been facilitated as a part of the activities of the teaching episodes.
The Results of the Exploratory Interview

In this section, the observations made regarding students’ concept images of rational functions, asymptotes, vertical asymptotes, horizontal asymptotes, limits, and continuity will be elaborated, in addition to the connections that they might have formed among these concepts.

Rational Functions

Student concept images of rational functions fell mainly into three categories. They were the rational number image, the fraction image, and the discontinuity image. Table 5.1 summarizes the patterns of students’ conceptions of rational functions.

<table>
<thead>
<tr>
<th>Concept Images of Rational Functions</th>
<th>Number of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>[limited] Rational number image</td>
<td>15/19</td>
</tr>
<tr>
<td>“like rational numbers $\sqrt{4}, \sqrt{9}$,” graphs are “nice,” “whole,” “even,” “symmetric” “one-piece,” “continuous,” and “without any complications”</td>
<td></td>
</tr>
<tr>
<td>Fraction image</td>
<td>3/19</td>
</tr>
<tr>
<td>fraction forms with no variable in the denominator, always continuous</td>
<td></td>
</tr>
<tr>
<td>Discontinuity image</td>
<td>3/19</td>
</tr>
<tr>
<td>always discontinuous somewhere, graph comes in several pieces, all rational functions have vertical asymptotes</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1. Common Traits in Students’ Concept Images of Rational Functions

The Rational Number Image. Students who possessed the rational number image described that the graphs of rational functions are “nice,” “whole,” “even,” “symmetric” like that of linear and quadratic functions that are “one-piece,” “continuous,” and “without any
complications.” Some students specified that like rational numbers such as \( \sqrt{4}, \sqrt{9} \) these functions are “whole.” These students’ concept image of rational numbers were definitely restricted to numbers like \( \sqrt{4} \) and \( \sqrt{9} \). The rational number conception was the most prevalent conception of rational functions.

**The Fraction Image.** Some students believed that rational functions assume fraction forms with no variable in the denominator. According to this conception, like fractions, rational functions could only have constants in the denominator.

**The Discontinuity Image.** All rational functions are discontinuous somewhere, their graph comes in several pieces, and all rational functions have vertical asymptotes. This notion could be based on the concept image that a rational form must have a variable in the denominator and therefore, the function would be undefined for some value/s of the variable.

**Asymptotes**

Common traits in students’ conceptions of the general notion of asymptotes consisted of *classic three-piece graph image, and invisible line image*. They are described below.

**Classic three-piece graph and invisible line images.** A majority of students believed that asymptotes were invisible lines the graph approached but never reached. They were called invisible lines since they are graphed using dotted lines. Dotted lines are drawn for asymptotes since the asymptotes just “box graphs in”, they are “not really a part of the function graph”. In regards to asymptotes, many students were guided by the *classic three-piece graph image*. Figure 5.1 represents student conceptions of a classic three-piece graph that was symmetric at the origin.
As portrayed in Figure 5.2, some others held the concept image of a three-piece graph symmetric about the y-axis while discussing asymptotes.

Students acknowledged two types of asymptotes, vertical asymptotes and horizontal asymptotes. In the next few paragraphs, I will detail students’ beliefs regarding vertical and horizontal asymptotes. They were categorized into *Image of undefinedness: type 1*, *Image of undefinedness: type 2*, *Image of undefinedness: type 3*, *holes or vertical asymptote*, *holes and vertical asymptote*. 
At times, vertical asymptote/s will be referred as VA while horizontal asymptote will be referred as HA throughout this dissertation.

**Vertical Asymptotes**

*Image of Undefinedness: Type 1.* Most students stated that *vertical asymptotes occurred at points where the function was undefined.* However, several students did not know when a rational function, such as \( f(x) = \frac{2x - 1}{3x + 5} \) became undefined. Even though every student knew that a fraction became undefined when its denominator was zero, they were unable to extend that knowledge to realize that the function \( f(x) \) became undefined when the denominator \( 3x + 5 \) became zero.

*Image of Undefinedness: Type 2.* Every function that has a variable in the denominator has a vertical asymptote. This belief could be traced back to the difficulty some students may experience with domains of rational functions. *Usually* while covering the concept of domains, simple problems such as find the domain of \( f(x) = \frac{17}{x} \) are introduced first. In this problem, students saw that when \( x = 0, \ f(0) = \frac{17}{0} \) and the fraction became undefined. In the next problem, \( f(x) = \frac{17}{2x - 1} \), some students immediately stated that domain, \( x \neq 0 \) without realizing that they should focus on the entire denominator and weed out those values of \( x \) for which the denominator would become zero. In addition, students commonly believed that a function such as \( f(x) = \frac{17}{x^2 + 1} \) would have to be undefined for some values of \( x \).
**Image of Undefinedness: Type 3.** Some students claimed that a rational function was undefined at the zeros of the numerator. I believe that students who possessed this conviction were probably thinking about finding the x-intercept/s of a rational function.

**Hole or Vertical Asymptote.** Many students were unsure of the conditions under which a hole occurred for a rational function. Even though students had seen graphs with holes and have realized that the function would be undefined there, many of them were unable to differentiate between the function behavior around holes and around vertical asymptotes. The confusion between holes as opposed to vertical asymptotes contributed to the notion that there was no distinction between \( \frac{0}{0} \) and \( \frac{b}{0}, b \neq 0 \) forms. Some students believed that both of these forms implicated the existence of vertical asymptotes, thus, taking the position that a rational function must always have a VA at a point where it was undefined.

**Hole and Vertical Asymptote.** A few students believed that a function could vertical asymptotes at holes and at jump discontinuities if the function was undefined at the point of jump-discontinuity. A vertical asymptote located at a hole was called point asymptote by a couple of students.

The above notions regarding holes versus vertical asymptotes could be traced back to a number of incomplete conceptions regarding several closely related topics. First, the confusion between holes and vertical asymptotes showed a lack of understanding of the function behavior in the region of its vertical asymptotes or holes. Additionally the confusion between holes and vertical asymptotes demonstrated a disconnection between what some students had already established as the properties of functions around its vertical asymptotes and/or holes. Table 5.2 summarizes student concept images of vertical asymptotes as revealed by the CORI 1 interview.
Concept Images of Vertical Asymptotes

<table>
<thead>
<tr>
<th>Concept Images of Vertical Asymptotes</th>
<th>Number of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undefinedness: type 1</td>
<td>VA occurred at points where the rational function was undefined, but did not know when a rational functional function became undefined</td>
</tr>
<tr>
<td>Undefinedness: type 2</td>
<td>Every function with a variable in the denominator must have a vertical asymptote.</td>
</tr>
<tr>
<td>Undefinedness: type 3</td>
<td>Rational functions were undefined at the zeros of the numerator.</td>
</tr>
<tr>
<td>Hole or vertical asymptote</td>
<td>Function is undefined at holes and VA’s. Both 0/0 and b/0, b ≠ 0 meant the same and both implicated vertical asymptotes</td>
</tr>
<tr>
<td>Hole and vertical asymptote</td>
<td>The form 0/0 meant there will be a point-asymptote, a hole on the VA</td>
</tr>
<tr>
<td>Given the VA find f(x) Meaning of</td>
<td>Unable to find f(x) Unable to understand VA from limit form</td>
</tr>
<tr>
<td>$\lim_{x \to a^\pm} f(x) = \pm \infty$</td>
<td>How does the function behave by its VA? What does this mean?</td>
</tr>
<tr>
<td>Other</td>
<td>What is approaching what?</td>
</tr>
<tr>
<td>• Function behavior</td>
<td></td>
</tr>
<tr>
<td>• $\lim_{x \to a^\pm} f(x) = \pm \infty$</td>
<td></td>
</tr>
<tr>
<td>• Terminology</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.2. Student Concept Images of Vertical Asymptotes of Rational Functions

Some students had earlier mentioned that if $x = 3$ was a VA of the function $f(x)$, then

$$
\lim_{x \to 3^-} f(x) = \pm \infty \quad \text{and} \quad \lim_{x \to 3^+} f(x) = \pm \infty.
$$

Later, for a function such as $f(x) = \frac{x-3}{x^2-9}$, they stated that $x = 3$ will be a VA even after computing $\lim_{x \to 3} f(x) = \frac{1}{6}$. This inconsistency in remarks might have been caused by the confounding of concept images evoked simultaneously while dealing with concepts that possess similar characteristics in some ways, and yet are distinct in some other ways. In such situations, being confronted with experiences that cause disequilibrium might
warrant re-examination and re-configuration of confounding ideas. However, the result of re-
configuration might not always be immediate or consistent; it may take more than one instance of
disequilibrium in order for the person to recognize the peculiar problems at hand.

Other reasons that could have complicated the identification of hole versus VA included
students’ inability to distinguish between forms such as \( \frac{0}{0} \), the indeterminate form, and
\( \frac{b}{0}, b \neq 0 \), the undefined form. A student who plugs in 3 and -3 for \( x \) directly in the
function \( f(x) = \frac{x-3}{x^2-9} \), finds that \( f(3) = \frac{0}{0} \), while \( f(-3) = \frac{-6}{0} \) and decides that both forms are
the same, or can’t tell the difference between these two forms but assumes that since the
denominator is zero, there should be a VA there. This also demonstrates a deficiency in noticing
what constitutes an indeterminate form, or what constitutes an undefined form in the behavior of
the function.

Some students, even with the graph of the function available, could not describe function
behavior by the usage of appropriate terminology. Other students were unable to identify vertical
asymptotes from its limit form. Another common trend was that most of the students were unable
to answer problems if they were slightly different from typical textbook problems.

**Horizontal Asymptotes**

In many ways, student notions of horizontal asymptotes were similar to their notions of
vertical asymptotes. The dilemmas were centered around the *inability to find the equation of the
horizontal asymptote*, the belief that a *function’s graph cannot be concurrent with its horizontal
asymptote*, *not knowing the function behavior around its horizontal asymptote*, the *inability to
identify horizontal asymptote from the limit form*, *inability to use appropriate terminology while
describing horizontal asymptotes, and failure to account for a horizontal asymptote while constructing the algebraic form of a function when the equation of the horizontal asymptote was given. Table 5.3 presents the observations made regarding student notions of horizontal asymptotes of rational functions.

<table>
<thead>
<tr>
<th>Concept Images of Horizontal Asymptotes</th>
<th>Number of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>HA and their equations</td>
<td>Horizontal asymptotes occurred at the zeros of the numerator</td>
</tr>
<tr>
<td>Question of concurrence</td>
<td>Curve cannot be concurrent with its horizontal asymptote</td>
</tr>
<tr>
<td>HA and the function behavior</td>
<td>Unable to explain the function behavior around its horizontal asymptote</td>
</tr>
<tr>
<td>HA in the limit form</td>
<td>( \lim_{x \to \pm \infty} f(x) = b ) What does it mean?</td>
</tr>
<tr>
<td>HA and issues of terminology</td>
<td>What is approaching what?</td>
</tr>
<tr>
<td>HA and function construction</td>
<td>From HA to f(x)? If ( \lim_{x \to \pm \infty} f(x) = b ) f(x) =?, What b has to do with it?</td>
</tr>
</tbody>
</table>

Table 5.3. Student Concept Images’ of Horizontal Asymptotes

**HA and their Equations**. The term HA and their equations will be used to refer to students’ inability to find the equation of the horizontal asymptote. Many students believed that horizontal asymptotes occurred at the zeros of the numerator. This view could have stemmed from associating horizontal asymptotes with y-intercepts, and associating vertical asymptotes with x-intercepts. Students may believe that similar to finding the x and y intercepts, the vertical
and horizontal asymptotes are found by performing reciprocal procedures such as solving for the denominator for one while solving for the numerator for the other.

**Question of Concurrence.** The term *question of concurrence* will be used to refer to students’ dilemma on whether a function’s graph could be concurrent with its horizontal asymptote or not. The majority of the students believed that the graph of a rational function cannot be concurrent with its horizontal asymptotes. Several students stated that horizontal asymptotes occurred at points on the function where $y$ is not defined. When asked, they indicated that if $y = 3$ is a horizontal asymptote of a function that meant there was no $x$-value on the graph for which the corresponding $y$-coordinate was 2. Students were unable to go much deeper into their explanation of why it was true.

In my view, yet again, students were eager to seek connections between the *vertical* and the *horizontal* with similar characteristics between them such as when one was specified by $x$, the other was specified by $y$. For another example, the equation for a vertical line is $x = k$, where $k$ is a constant, while the equation for a horizontal line is $y = k$. Not to mention that this way of comparison was in fact helpful and was encouraged in previous mathematics courses. During the learning sessions students’ argumentation on a similar context was explored in detail. The way in which asymptotes were informally defined in one of their textbooks seemed to have encouraged students to believe that a function could not be concurrent with its horizontal asymptotes. In addition, the lack of stress given to the discussion of the differences between horizontal and vertical asymptotes in conjunction with the limited variety of examples that were introduced during instruction could have attributed to the difficulties with HA.

**HA and the Function Behavior.** The phrase *HA and the function behavior* will be used to refer to students not knowing the function behavior around its horizontal asymptote. Similar to the dilemma with the vertical asymptotes, several students did not know the conditions under
which a rational function would have a horizontal asymptote. Even those who stated that as $x$
aproached positive or negative infinity, the function approached a horizontal line, seemed to
contradict themselves when they later solved a problem. An example would be that after
examining the statement, $\lim_{x \to \pm\infty} t(x) = \frac{x + 2}{x^2 + 4} = 0$, and stating that $y = 0$ would be the horizontal
asymptote of the function $\frac{x + 2}{x^2 + 4}$, some students recanted their statement as soon as they viewed
the graph of this function as illustrated in Figure 5.3. Students insisted that unlike what was
indicated by limit at infinity, this function could not have a horizontal asymptote at $y = 0$ since its
graph intersected $y = 0$, the x-axis. According to their concept image, a function could never
intersect its asymptotes. I believe that while many students consider mathematics as heavily rule-
bound, many of them may also believe that “nothing is certain” in math. Therefore, regardless of
what the evidence might indicate, students clung to their personal hunches even though they are
not able to articulate these reasons most of the time.

Figure 5.3. Graph of $y = \frac{x + 2}{x^2 + 4}$
**HA in the Limit Form.** Students’ difficulty to identify a horizontal asymptote from the limit form will be referred as *HA in the limit form*. Following a problem statement, the function $g(x)$ has a horizontal asymptote at $y = -2$, students drew a horizontal dashed line for a horizontal asymptote and a curve that was getting closer and closer to the horizontal line. However, following a limit statement, “$\lim_{x \to \pm \infty} g(x) = -2$”, many students could not tell if the line $y = -2$ was a horizontal asymptote. One reason for this difficulty was due to the belief that a curve could not be concurrent with its HA, and, from the limit statement above, the graph of $g(x)$ could intersect the horizontal line $y = -2$.

In addition to not knowing the properties of asymptotes, students may also have faced confusion due to inadequate concept representation (Goldin & Kaput, 1996) since they may only view asymptotes as dotted lines the graph of the function approached but never reached. They believed that these dotted lines, the asymptotes, are there to mainly *confine* or *box in* the directions in which the graphs could advance. Not many students were able to define their notion of confinement either.

**HA and Issues of Terminology.** *HA and issues of terminology* will be the term used to refer to students’ inability to use appropriate terminology while describing horizontal asymptotes. Even those students who attempted to describe the behavior of a curve around its horizontal asymptote frequently failed to convey it by the use of appropriate terminology. Students had stated that as $y$ approached a certain number, $x$ approached $\pm \infty$, or as $x$ approached $\pm \infty$, “it” approached a number. Students were unable to verbalize and articulate mathematics knowledge or to interpret mathematics from a sense-making point of view. Instead, they possessed a rule bound, procedural point of view which could be attributed to the narrow focus of curriculum and instruction (Tall, 1992; Tall & Vinner, 1981).
**HA and Function Construction.** HA and function construction is the term used to refer to students’ failure to account for a horizontal asymptote while constructing the algebraic form. Out of 19 students I interviewed in the beginning, only two students were able to provide any feedback on finding the term that lead to a horizontal asymptote of the rational function under investigation. Among the very few students who attempted to solve such a problem, some mentioned about comparing the degrees of the numerator and the denominator. While following those rules, several students noted that for a function, 

\[ g(x) = \frac{5x^2 - 9}{7x^2 - 3}, \text{ the HA will be } y = \frac{5}{7}, \]

while, if the HA was \( y = 0 \), for another function, say \( h(x) \), its equation should be \( h(x) = \frac{0x^2}{7x^2} \).

In some other cases, for example, if a function had a VA at \( x = 1 \) and a HA at \( y = 3 \), then equation of the function was written as \( \frac{3}{x-1} \).

While pondering on the reasons behind increased difficulty with writing the equations of horizontal asymptotes, and constructing a function with a given horizontal asymptote, I recognized the following factor. Regarding vertical asymptotes, many students were able to associate with an obvious property of undefinedness of the function. With this association, they are able to think through a condition under which a rational form became undefined. No such easily foreseeable property can be applied in the making of a horizontal asymptote. Therefore, students tend to resort to unfounded strategies such as solving for the numerator.

**Limits**

The following closely related obstacles were noted in students when dealing with limit problems. The problems were categorized as *problems with the usage of correct terminology*, *problems with understanding limits from graph*, *problems with computing limits*, and *problems connecting limits with asymptotic behaviors of functions*. The Term, *limits and use of*
terminology, will identify students’ difficulties related to the usage of correct terminology, while, the term *limits from graphs*, will identify students’ difficulties with understanding limits from graph. Table 5.4 summarizes the observed patterns that emerged in regards to students’ difficulties regarding the concept of limits.

**Limits and use of terminology.** Usage of correct terminology was a problem during the CORI I interview. Instead of stating "as x approached $a$, $f(x)$ approached $L$," the roles of $x$ and $y$ and $x$ and $f(x)$ were often interchanged, or instead of stating $x$, or $f(x)$, or $y$, the word *it* was used.

**Limits from Graphs.** Some students believed that no limit existed at a point if the function was undefined at that point. In some instances, students stated that if $f(x)$ had a hole at the point $(2, -7)$, as $x$ approached 2, the limit of the function would be -7. However, if the function $f(x)$ had a removable discontinuity at $(2, -7)$ such that $f(2) = 3$, then as $x$ approached 2, $f(x)$ would approach 3. Some students, while computing limits stated that in cases such as

$$\lim_{x \to 7} \frac{11}{x - 7}$$

which produced the form, $\frac{11}{0}$, once direct substitution was done, the limit would be 11. In this student’s view, since $\frac{11}{0}$ was *undefined*, and since undefined limit cannot be reached, the limit should be 11 the *defined* part of $\frac{11}{0}$. 

112
Usage of the word “it” while referring to x, and f(x) Not sure if it was as x approached a or as f(x) approached a

<table>
<thead>
<tr>
<th>Limits from Graphs</th>
<th>Is limit the function value?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computation of Limits</td>
<td>Depends on direct substitution</td>
</tr>
<tr>
<td>Limits as Asymptotes</td>
<td>Infinite limits and limit at infinity, what are they?</td>
</tr>
</tbody>
</table>

Table 5.4. Summary of Student Beliefs of Limits

**Computation of Limits.** Computation of limits refers to problems students experienced with computing limits. All except a couple of students were unable to compute limits without a graphing calculator. Most students simply used direct substitution to compute limits if there was no graphing calculator available. In some cases, they plugged-in infinity directly in place of x and wrote that $0/0 = 0$, $\infty/\infty = \infty$, $\infty/0 = 0$, $\infty/\infty = \text{undefined}$. They were unable to compute infinite limits and limit at infinity. While dealing with forms such as $b/0$, $b \neq 0$, $\frac{\infty^2}{\infty}$, $\frac{\infty}{\infty}$, $\frac{\infty}{0}$, students gave up and stated that limit (not necessarily in finite sense) cannot be found, or did not exist.

**Limits as Asymptotes.** The terminology, limits as asymptotes will be used to refer to students’ inability to relate asymptotic behaviors with limits. These difficulties could have been avoided if students were able to relate undefined forms with vertical asymptotes and limit at infinity with horizontal asymptotes. For example, while solving the problem, if $f(x) = \frac{2x}{x-3}$, find $\lim_{x \to 3} \frac{2x}{x-3}$, realizing that $f(3) = 6/0$ might remind students that corresponding to this
undefined form the function has a vertical asymptote, and as such the limit will be positive or negative infinity. Figure 5.4 describes student interpretations of limits from graphs of functions.

Students’ difficulty associated with the concept of limits could have stemmed from a lack of knowledge of function behavior in regards to limits. Students simply focus on the process, the process of finding limits using procedures such as direct substitution, simplification and direct substitution, and the usage of graphing calculator graphs in the event that direct substitution did not work.

**Continuity**

Regarding continuity, some students believed that a function was discontinuous at sharp corners and a function was continuous everywhere in its domain if the domain was all real numbers. They also believed that both jump and removable, discontinuities produced vertical asymptotes at the points of discontinuity. Some others believed that whenever the left hand limit was equal to the right hand limit, the function was continuous at that point.

It must be noted that the majority of students demonstrated correct understanding of the concept of continuity by specifying that in order for the function f(x) to be continuous at a point, say, x = a, \( \lim_{{x \to a}} f(x) \) must exist and must also be equal to \( f(a) \). While answering the question whether a function with all real number domains should always be continuous, several students immediately answered yes, but, as soon as they started sketching a function, they recanted their answer by creating a hole on the curve and placing a point that indicated a removable discontinuity. It should be noted that these students did not display a function with jump discontinuity as a counter example. Understandably, it would be a lot easier to alter a continuous function to one that has a removable discontinuity rather that altering it into a function with a jump discontinuity.
Other

Lack of knowledge of basic concepts such as intercepts, recognizing polynomial forms, factoring polynomials, failure to recognize linear and quadratic forms, failure to understand what linearness meant also posed obstacles during the problem-solving interview. These difficulties were lessened by the researcher engaging students in a variety of actions such as providing the factors of function terms when asked, asking probing questions such as where is an x-intercept located? what is the characteristic of a point on the x-axis? how do you solve a rational form?

In the next section, I have summarized the concept images of each of the seven students who participated in the teaching experiment and in the second interview.

Summary of Concept Images of the Rational Form

The participants were calculus 2 students enrolled in a major public university. Anna was sophomore liberal arts major who identified herself as a B-achiever for mathematics courses in general. Hayden and BlueJay were freshmen, engineering and biology majors respectively, who identified themselves as B-achievers in mathematics. Henry and Tina were both engineering majors, who were self-identified as A-achieving freshmen students. Richie was another freshman, engineering major, who identified himself as a C-achieving mathematics student. Neo was another freshman, engineering student who identified herself as an A achiever in mathematics courses.

Concept Images of Rational Functions -CORI 1 and CORI 2

Since the concept images of the rational function form are relatively smaller in number, I will be presenting the results of both CORI 1 interview and CORI 2 interviews at the same time. Table 5.5 summarizes student concept images of rational functions as identified by the CORI 1
and CORI 2 interviews.

<table>
<thead>
<tr>
<th></th>
<th>CORI 1</th>
<th>CORI 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hayden</td>
<td>did not know</td>
<td>any function with a numerator and a denominator</td>
</tr>
<tr>
<td>Neo, Anna, Tina, BlueJay</td>
<td>Nice, continuous, etc</td>
<td>any function with a numerator and a denominator</td>
</tr>
<tr>
<td>Richie, Henry</td>
<td>Nice, continuous, etc</td>
<td>Knew the rational form</td>
</tr>
</tbody>
</table>

Table 5.5. Rational Functions: CORI 1 and CORI 2 Concept Images

Figure 5.4. Computation of Limits from Graphs – Students’ Perspective

During the CORI 1 interview, Neo, Anna, Tina, BlueJay, Richie, and Henry, stated that rational functions are “nice” “continuous” functions with “no complications” and “no
asymptotes.” They stated that rational functions are like *rational numbers* of the form $\sqrt{4}, \sqrt{9}, \sqrt{25}$. Richie affirmed that rational functions have no wired turns and they are even (symmetrical) shaped. Hayden stated that he did not know what types of function were considered rational functions. Richie elaborated that graphs that are neither even (symmetrical about the Y-axis) nor odd (symmetrical about the origin) could not represent rational functions.

Neo and Anna said $\frac{x^2 - 4}{x + 2}$ and $x - 2$ represented the same functions. BlueJay stated that, $\frac{x^2 - 4}{x + 2}$ and $x - 2$ were not the same functions since $\frac{x^2 - 4}{x + 2}$ will be discontinuous when $x = -2$.

Richie realized that in $d(x) = \frac{x^2 - 4}{x + 2}$, there was a hole at $x = -2$. Henry and Tina stated that $a(x) = x - 2$ and $c(x) = \frac{x^2 - 1}{2}$ are equations of rational functions since rational functions are continuous functions with “nothing going wrong.” Neo and BlueJay seemed to know that to find vertical asymptotes one must solve for when the denominator $= 0$.

During the CORI 2 interview, Henry, and Richie showed understanding of polynomial and rational forms. However, Anna thought “any function with a numerator and a denominator” was a rational function. She then affirmed that the function $\frac{3x - 1}{x + 7}$ was rational except at $x = -7$ (function is discontinuous at $x = -7$), while $\frac{3x^2 - 7x + 9}{x + \sqrt{6}}$ was *irrational* everywhere due to the presence of $\sqrt{6}$ and that radicals were not allowed in a rational form. She described $\frac{-7x^3 + 11x - 8}{6}$ as “always rational,” since it was a continuous function. According to Anna,
\[-\frac{7x^3 + 11x - 8}{1}\] could not be rational since the denominator was not allowed to be 1. The function \[\frac{3}{2x^{\frac{1}{3}} + 11x^2 - 5}\] was also “irrational” due to the term \(2x^{\frac{1}{3}}\). BlueJay stated that rational functions should have “\(x\) in the numerator and in the denominator.” However, he stated that \[\frac{3}{2x^{\frac{1}{3}} + 11x^2 - 5}\] was rational since “numerator doesn’t have to have a variable.” By the time CORI 2 interview was done, Hayden knew that while \[\frac{-7x^3 + 11x - 8}{6}\] could not be considered a rational function due to the fact that the denominator was a constant function. However, he believed that \[\frac{3}{2x^{\frac{1}{3}} + 11x^2 - 5}\] would be a rational function if it weren’t for the exponent \(1/3\) for \(x\). Tina was able to explain that a rational function has “two polynomials, one on the top [numerator] and one at the bottom [denominator], no constants.” Similar to Anna, Tina too stated that \[\frac{3x^2 - 7x + 9}{x + \sqrt{6}}\] was not a rational function due to the presence of radical 6 in the denominator. Again, for Tina, \[\frac{3}{2x^{\frac{1}{3}} + 11x^2 - 5}\] was not rational due to the presence of “fraction exponent” in the denominator although the constant term in the numerator was allowable in rational functions.

During the investigation of students’ concept images of rational functions, I realized that students’ concept images of rational numbers were also limited. They perceived that all rational numbers looked \textit{nice} like \(\sqrt{4}, \sqrt{9}\) and \(\sqrt{16}\). Therefore, during the learning episodes, the formal definition of rational number was discussed along with the formal definition of rational functions.
Rational functions were defined as functions of the form \( \frac{p(x)}{q(x)} \) where \( p(x) \) and \( q(x) \) were polynomial functions and \( q(x) \) a non-zero, non-constant function. Further, the definition, examples, and non-examples of polynomials, rational numbers, and rational functions were discussed during the learning sessions.

**Rational Functions: Comparison between CORI 1, and CORI 2**

*Limited perspective of rational numbers.* In the beginning, Anna associated rational functions with rational numbers. Her concept image of rational number was limited to whole numbers such as \( \sqrt{4}, \sqrt{9}, \text{ and } \sqrt{16} \). The concept image of rational number did not include the mathematical definition of a rational number as a number that could be written in form \( \frac{a}{b}, b \neq 0 \), where \( a \) and \( b \) are integers. After the learning sessions, students accommodated certain parts of the concept definitions of rational number, polynomial, and rational function. *The fraction images* of rational numbers and rational functions have been partially accommodated with parts of the forbidden aspects of the existence of radicals in a polynomial. The traces of previous concept images of a rational function as *a nice function with no discontinuity* were still present. However, some accommodations were made to this concept image by being willing to accept a function as a rational function except at the point of discontinuity.

While Richie and Henry accommodated all details of the polynomial and rational functions, Hayden, BlueJay, Tina, and Anna had accommodated only parts of the rational and polynomial forms and they still had difficulty with certain aspects of these concepts. This difficulty may be resolved by re-examining the definitions of polynomial and rational functions and making sure that the function under scrutiny has met all of the specifics of the definition.
In the next section, I have detailed student concept images of asymptotes, limits, and continuity as observed during the CORI 1 interview. The term concept image was referred as student’ notions of a concept those were different from conventional concept definition. Afterwards, I have described the activities of the teaching episodes that were designed to help modify students’ notions of the concepts investigated. Afterwards, student concept images of asymptotes, limits, and continuity as revealed by the CORI 2 interview are detailed with a comparison of student performances between the two interviews. Conceptual changes as indicated by the differences in performances have been discussed through the lens of the nature of exchanges that took place during the teaching episodes.

**Concept Images of Asymptotes, Holes, Limits, and Continuity: CORI 1**

This section provides student concept images of asymptotes and holes as rendered by the CORI 1 interview.

**Concept Images of Asymptotes and Holes CORI 1.** First, I have included problems 2, 3, 4, 8, and 9 of CORI 1 and a summary of their significances in table format as displayed by tables 5.6 through 5.10. These problems were specifically designed to elicit students’ notions of vertical asymptotes, horizontal asymptotes, and holes in addition to students’ beliefs of function behavior and its limit properties around holes and asymptotes. Through these problems, I also solicited student understanding of how asymptotes and undefinedness contribute to the terms of the equation. Table 5.6 summarizes the relevance of problems 2 and 3. In problem, 3 students’ understanding of limit format of VA and HA were investigated.
Table 5.6 Problems 2 and 3 of CORI 1 with their rationale

In problem 4, even though the function was discontinuous at two points, the nature of discontinuity was explicated only for one point with the specification of a VA at that point. In this problem, the possibility of another VA at the other point of discontinuity was eliminated by the limit statement that was included in the problem. Thus, this problem helped elicit student convictions of VA, limit, and continuity. Table 5.7 summarizes problem 4 and the significances of problem 4

<table>
<thead>
<tr>
<th>Question/s</th>
<th>Concepts Included/significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 2: A rational function g(x) has x = -3 for a vertical asymptote, y = 2 for horizontal asymptote, and is undefined at x = 3 (a) Sketch the graph of the function in the grid given below (b) Explain the behavior of this function at/around the points at which the function is undefined and around the horizontal asymptote (c) Write the equation or parts of the equation of the function g(x)</td>
<td>VA, HA, VA-hole, VA-undefinedness, hole-undefinedness, construction of equation, graph and function properties. Graph the function first and write the equation last</td>
</tr>
<tr>
<td>Problem 3: The rational function h(x) is such that ( \lim_{x \to -3} h(x) = -\infty ) and ( \lim_{x \to -3} h(x) = \infty ) ( \lim_{x \to -3} h(x) = -\infty ) ( \lim_{x \to -3} h(x) = \infty ) (a) Write the equation or parts of the equation of a (b) Explain the behavior of this function around y = 4 and x = -3 (c) Sketch the graph of the function in grid given below</td>
<td>HA – limit, VA – limit, graph and function properties. Write the equation first and then to graph it</td>
</tr>
</tbody>
</table>
Consider the rational function \( l(x) \). Some characteristics of this function are given in the box below.

Function is discontinuous at \( x = \pm 3 \), has a vertical asymptote: \( x = -3 \), \( \lim_{x \to -\infty} l(x) = 4 \), \( \lim_{x \to +\infty} l(x) = 4 \), \( \lim_{x \to -3} l(x) = 2 \)

(a) Sketch the function in the grid given below
(b) Any horizontal asymptote? why or why not
(c) write the equation or parts of the equation of \( l(x) \)

Table 5.7. Problem 4, CORI 1 and its Rationale

In problem 8, students were to write parts of the equation of a rational function that had a horizontal line other than the x-axis as its horizontal asymptote. In problem 9, students were to write parts of the equation of a rational function that had x-axis for a horizontal asymptote. In both problems students were asked to express the HA using limit notation. Table 5.8 displays problems 8 and 9 with a summary of the relevance of these problems.
<table>
<thead>
<tr>
<th>Question/s</th>
<th>Concepts Included/significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 8</td>
<td>Equation construction, HA: line other than the y-axis, limit-HA</td>
</tr>
</tbody>
</table>
| If the function \( p(x) \) has a HA, \( y = 3/5 \)  
(a) Write part of the equation of this function  
(b) Express the HA, \( y = 3/5 \) using the limit notation |  |
| Problem 9  | Equation construction, HA: the y-axis, limit-HA |
| If the function \( q(x) \) have a HA, \( y = 0 \)  
(a) Write part of the equation of this function  
(b) Express the horizontal asymptote, \( y = 0 \) the using limit notation |  |

Table 5.8. Question 8, and 9 of CORI 1

Problem 13, which is displayed in table 5.9, was intended to solicit students’ conceptions of the undefined form and the indeterminate form and how these forms affected the function behavior.

<table>
<thead>
<tr>
<th>Question/s</th>
<th>Concepts Included/significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem 13</td>
<td>Undefined form, indeterminate form, function behavior</td>
</tr>
</tbody>
</table>
| Let \( v(x) \) be a rational function  
(a) if \( \frac{v(5)}{0} = \frac{0}{0} \) what is the behavior of this function at/around \( x = 5 \)?  
(b) if \( \frac{v(5)}{0}, b \neq 0 \) is the behavior of this function at/around \( x = 5 \) |  |

Table 5.9. Question 13 of CORI 1
Discussion of Student Performances

Vertical Asymptotes

Richie was able to recognize vertical asymptotes from their limit format. Richie, Henry, Anna, and BlueJay stated that the behavior of a rational function at the points where it was undefined was unpredictable, since at the *undefined points* the function could have a hole or a vertical asymptote. BlueJay believed that vertical asymptotes occur at jumps and holes as well. Anna, did not know the property that caused undefinedness of a rational function. Richie could not differentiate between the details that created the VA or a hole. He also believed that a VA and a hole could occur simultaneously. However, in problem 4, Richie stated that if this function had a VA at \( x = 3 \), the function will have to be “boxed in” at the vertical asymptote. Thus, in this problem, due to the condition that \( \lim_{x \to 3} f(x) = 2 \), there could not be a VA since the function “deter away” at \( x = 3 \). Anna however, stated that vertical asymptotes can occur on holes.

Problem 4 posed dilemmas for Tina as well. She also commented that it was not possible for a function to be “approaching 2 [as \( x \) approached 3] and be discontinuous at 3 [referring to the problem statement that the function is discontinuous at \( x = \pm 3, \lim_{x \to 3} f(x) = 2 \)].” Nothing more was done by Henry or Tina on this problem. Tina believed that at the points where a rational function was undefined; there would always be vertical asymptotes. In response to problem 2, for the undefinedness at \( x = 3 \), Tina only talked about another VA occurring at \( x = 3 \). Towards the end of CORI 1 interview, Tina realized that holes or vertical asymptotes could occur at the points where a rational function was undefined. All of these students stated that there was no difference between an indeterminate form and an undefined form.

According to Anna, a vertical asymptote could occur on a hole and such an entity will be called a point asymptote. Henry knew that VA was not the only characteristics provoked by
undefinedness, but, he did not know what else occurred at the undefined points. He stated that “everything stops at such [undefined] points.” Henry was unable to recognize vertical asymptotes from their limit format. In problem 4, Henry found the statement, \( \lim_{x \to 3} f(x) = 2 \) confusing and stated that it is not possible for a function to have a limit at \( x = 3 \) while it was discontinuous at \( x = 3 \). Table 5.10 provides a cross-case display of students’ conceptions of vertical asymptotes as portrayed by CORI 1 interview.

<table>
<thead>
<tr>
<th>Vertical Asymptotes/Holes</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Anna, Neo, BlueJay</strong></td>
</tr>
<tr>
<td><strong>Hayden</strong></td>
</tr>
<tr>
<td><strong>Richie</strong></td>
</tr>
<tr>
<td><strong>Henry</strong></td>
</tr>
<tr>
<td><strong>BlueJay</strong></td>
</tr>
<tr>
<td><strong>Henry, Tina</strong></td>
</tr>
<tr>
<td><strong>Richie</strong></td>
</tr>
<tr>
<td><strong>Anna</strong></td>
</tr>
<tr>
<td><strong>Tina</strong></td>
</tr>
<tr>
<td><strong>BlueJay</strong></td>
</tr>
<tr>
<td><strong>Tina, Neo, Richie</strong></td>
</tr>
<tr>
<td><strong>Henry</strong></td>
</tr>
<tr>
<td><strong>BlueJay</strong></td>
</tr>
</tbody>
</table>

| Anna, Neo, BlueJay, Hayden, Richie | Undefinedness Type 3: Holes or vertical asymptotes at \( x = 3 \), a point at which the function was undefined. Anna: holes were called point asymptotes drawn by a dotted line on a hole. Richie: around VA function “boxed in”, around hole function “deter away”. |
| Henry | Unsure: Hole or Vertical Asymptote? Something else other than vertical asymptotes is happening at \( x = 3 \). “everything stops at 3” |
| BlueJay, Henry, Tina | Unsure of function Behavior: Was unable to explain the function behavior from the limit form of asymptotes |
| Richie, Anna | Both: Hole and Vertical asymptote. For \( x = -3 \) there was a vertical asymptotes, for \( x = 3 \) a hole, both terms \( x + 3 \) and \( x - 3 \) will be placed just in the denominator. Anna: holes could appear on the VA, point asymptotes |
| Tina | Undefinedness means VA always. Unsure: Limit Form, Function behavior \( \lim_{x \to 3^+} f(x) = -\infty \), \( \lim_{x \to 3^-} f(x) = \infty \) is not enough to understand the function behavior by its vertical asymptote. “All that you will know would be the end-behavior of the function; you need to know what is happening to the function at other places.” |
| BlueJay | VA occurs at jumps as well |
| BlueJay, Neo, Richie, Henry | No difference between undefined and indeterminate forms |
| BlueJay | Unable to construct function term/s from known VA |

Table 5.10. Cross-case Display of CORI 1 Concept Images of the Vertical Asymptote
**Horizontal Asymptotes**

Regarding horizontal asymptotes, Henry stated that it was possible for a function to cut through its HA. While he was unable to explain the reason, he stated that he remembered seeing such a graph somewhere. Richie, Tina, BlueJay, Anna, and Hayden believed that a curve could never intersect its horizontal asymptote. According to Richie, if a curve did intersect its horizontal asymptote, the point of intersection would be a hole. Tina, in problem 4, sketched a horizontal dashed line at y = 4 stating that at y = 4 “the entire function approached 4, but cannot pass 4 from either side.” Anna recognized horizontal asymptotes from the limit form in problem 4 since the notation “\( \lim_{x \to \pm \infty} f(x) = 4 \)” indicated to Anna that y is only “approaching 4” but “never equal to 4.” The horizontal line that the function approached in this case was a HA, since in Anna’s mind the term *approach* emphasized the absence of concurrency.

BlueJay stated that the limit statements did not insure the existence of asymptotes. For example, he argued that the description of “\( \lim_{x \to -\infty} h(x) = 4, \lim_{x \to \infty} h(x) = 4 \)” could imply that the graph of h(x) intersected the line y = 4. According to BlueJay, since “a curve could not cross its horizontal asymptote”, the line y = 4 could not be a VA to the function h(x). In problem 8, all that BlueJay was able to write was that \( \lim_{x \to \infty} \frac{51}{x^2 + 7} \). As he struggled, I asked him to solve \( \lim_{x \to \infty} \frac{67x^2 - 17x + 3000}{59x^2 + 51x} \), a problem that I instantly created for him. He found the limit to be 67/59 and immediately afterwards he wrote \( p(x) = \frac{3x - 8}{5x + 14} \) as to problem 8, part a. Explicitly seeing the terms, \( \frac{67x^2 - 17x + 3000}{59x^2 + 51x} \), had helped BlueJay realize the behavior of the function when x approached infinity while the degrees of the numerator and the denominator were equal.
He tried to extend this notion for problem 9 by writing a series of steps:

\[ q(x) = \frac{0x}{x+1} \rightarrow \frac{0x+4}{x+1} \rightarrow \frac{4}{x+1} \]

before he finalized the function as \( q(x) = \frac{4x^2 - 7}{12,000x^3} \).

While constructing equations, Neo, Henry, Tina, and Richie could not account for the term of the function that was responsible for the HA. Hayden was able to make some connections between the indicated function term and the HA of the function. Hayden used shifts to write the equation of a function as characterized by its horizontal asymptote. For example, in problem 4, Hayden was able to think through the effect of having the HA, \( y = 2 \) on the graph of \( g(x) \). After writing \( g(x) = \frac{1}{x + 3} \) to account for the VA, \( x = -3 \), since, due to the HA, \( y = 2 \), the graph would “move up by 2” he added 2 to his previous equation leading to \( g(x) = \frac{1}{x+3} + 2 = \frac{2x + 7}{x + 3} \) for his function. I asked Hayden if he could see a connection between the horizontal asymptote, \( y = 2 \), and the equation \( g(x) = \frac{2x + 7}{x + 3} \) that he created. He remarked that he did not see any connections unless he “spilt up” the function [re-write as \( g(x) = \frac{1}{x + 3} + 2 \)] again.

In problems 8 and 9, Hayden used shifts on the reciprocal function \( y = \frac{1}{x} \) to write the equation of the function. Hayden was able to write the function equation by observing the graphical form and visualizing how certain types of horizontal asymptotes will alter its equation. However, Hayden was unable to find the equation of HA from the algebraic form of the function.

Neo was unable to account for horizontal asymptotes while constructing a function from its properties. However, she simply placed a factor in the denominator corresponding to vertical asymptotes and holes. Neo was able to note that if \( y = 3/5 \) was a HA of a function, \( f \), then
\[ \lim_{x \to \pm\infty} f = \frac{3}{5} \] could be held true. In problem 9, Neo started out with \( \lim_{x \to \infty} = 0 \). She then wrote a series of steps: \[ \lim_{x \to \infty} \left( \frac{x}{x-1} \right) \to 1000, \lim_{x \to \infty} \left( \frac{x}{x+1} \right) \to \frac{1000}{1001} \] and stated that “the denominator must grow much faster than the numerator”. While concluding “\( \frac{x}{x^2} \to \frac{x}{x^2+1} \)” Neo realized that the limit was zero; however, she said she did not know why \( y = 0 \) was the HA of this function. Neo realized how the function behaved as \( x \) approached positive or negative infinity. However, she was unable to understand why the limit at infinity was in fact the HA due to the lack of connection between the HA, its properties, and its limit format. Table 5.11 summarizes a cross case analysis of students’ concept images of horizontal asymptotes as revealed by the CORI 1 interview.

<table>
<thead>
<tr>
<th>Hayden</th>
<th>Neo, Henry</th>
<th>Unable to interpret HA from the limit form:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Henry</td>
<td></td>
<td>Found the equation of HA only when the numerator and the denominator had the same degree</td>
</tr>
<tr>
<td>Tina</td>
<td></td>
<td>HA and function construction: Could not form any connection between HA and the corresponding function term. Eventually identified HA from the limit form</td>
</tr>
<tr>
<td>Richie, Henry</td>
<td>Hayden</td>
<td>HA and function construction: Unable to construct function term from given HA, but showed some connection. Hayden used function transformation to find the function equation from HA</td>
</tr>
<tr>
<td>Tina</td>
<td></td>
<td>HA and limit form, HA and function behavior: ( \lim_{x \to \infty} h(x) = 4, \lim_{x \to \infty} h(x) = 4 ) meant some thing is happening at 4. Guessed HA at ( y = 4 ) only because “the entire function approaches 4, but cannot pass 4 from either side”, “this is what was implied by the limit statement”</td>
</tr>
<tr>
<td>Richie, Tina</td>
<td>Hayden, BlueJay</td>
<td>HA cannot cross the graph</td>
</tr>
</tbody>
</table>

Table 5.11. Cross-case Display of CORI 1 Concept Images of the Horizontal Asymptote
Limits and Continuity

Next, I will elaborate on student solutions of problem 5 and 6 of CORI 1. I will state the questions followed by a discussion of students’ performances.

Problem 5, CORI 1: Explain the behavior of the functions $j(x)$, $k(x)$, and $z(x)$ as $x$ approaches 2

Problem 6
(a) Discuss the continuity of the functions $j(x)$, $k(x)$, $z(x)$, and $m(x)$ in question 5
(b) Discuss possible asymptotes if any of the functions $j(x)$, $k(x)$, $z(x)$, and $m(x)$ in question 5

Function $m(x)$
In problem 5, and 6, Henry stated that $\lim_{x \to 2} k(x) = 3$, and $\lim_{x \to 2} k(x) = 2$, while $\lim_{x \to 2} k(x) = 0$.

According to him, if a function is defined at a point, the limit will be the function value at that point. He further explained that in the absence of the condition, $k(2) = 0$, $\lim_{x \to 2} k(x)$ would be undefined. Henry stated that this function will be continuous at $x = 2$ since the function was defined at $x = 2$. He however, stated that the function $j(x)$ will be discontinuous at $x = 2$, since the function was not defined at $x = 2$.

Hayden stated that functions $j(x)$, $k(x)$, and $z(x)$ were all discontinuous at $x = 2$, and that functions $j(x)$ and $k(x)$ could not have vertical asymptotes at $x = 2$ since “there the function was not approaching $\pm \infty$.” Hayden also noted that functions $j(x)$ and $k(x)$ could not have any horizontal asymptotes either while $z(x)$ and $m(x)$ had the x-axis for a horizontal asymptote.

According to Hayden, the discontinuity of $j(x)$ was called a “cusp” while the discontinuity of $k(x)$ was called a “jump discontinuity.” BlueJay stated that the function $j(x)$ could have a VA at $x = 2$, while the function $k(x)$ could only have a VA at $x = 2$ once the point $(2, 0)$ was removed from the graph. However, Anna stated that these functions could have VA at $x = 2$, and HA at $y = 3$.

According to Tina, no limit existed at sharp points. Neo was able to relate to limits and asymptotes correctly in these problems.

Problems 13 and 14 were designed to investigate students’ facility to compute infinite limits without a calculator. While solving problem 13, Anna and Hayden realized that as $x$ approached 3, the function approached $\pm \infty$. However, according to Anna, only a graphing calculator could decide which infinity the function was approaching. Henry realized VA at $\pm 3$, while Hayden stated that the function was undefined at these points. Yet, they could not find the limit of the function. In 14 (b), Anna and Tina substituted 2 for $x$ and wrote “$\frac{6}{0}$, limit
undefined” while Henry could not answer the question. In \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \), Henry found the limit to be 4, though, he was not sure if the function had a VA at \( x = 2 \). In this problem, BlueJay and Anna were also able to simplify and compute the finite limit.

Problem 7 and 14 of CORI 1 had several parts asking for computing limits of different functions. These questions are stated below following a discussion of student performances.

Problem 7
Consider the function \( n(x) = \frac{3x^2 - 13}{x^2 - 9} \). Compute the following limits by explaining the details. Be sure to give the limits and finite/infinite limit

(a) \( \lim_{x \to 3} n(x) \)
(b) \( \lim_{x \to -3} n(x) \)
(c) \( \lim_{x \to \infty} n(x) \)
(d) \( \lim_{x \to -\infty} n(x) \)

(e) Identify the vertical asymptote/s if any
(f) Identify the HA if any
(g) Explain how concepts in parts (a) through (e) are connected

In problem 7, part a of CORI 1, Richie did direct substitution and wrote \( \lim_{x \to 3} m(x) = \frac{14}{0} \), and stated that the function will have a VA at \( x = -3 \). The limit of the function at \( x = -3 \) would be 14 since according to Richie, “14/0 is undefined but in a limit problem the limit will be 14.” It seemed like Richie was invoking the concept image that a function could have a limit at a point where it was undefined. In the expression \( \frac{14}{0} \), he removed the zero in the denominator that made the expression undefined and simply went with 14 as the limit. In 14 (b), Richie once again noted the limit to be 6/0 = 6, while in 14 (c) he noted the limit to be 0/0 = 0.
Problem 14
Create a table of values if needed to answer the following questions. The limit may be a finite number or it can be expressed as in terms of ±∞. In some problems you may compute the left-hand and the right-hand limits separately.

(a) find \( \lim_{x \to 2} \frac{1}{x + 1} \), (b) find \( \lim_{x \to 2} \frac{3x}{x - 2} \), (c) find \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \), (d) find \( \lim_{x \to \pm\infty} \frac{1}{x + 1} \)

(e) find \( \lim_{x \to \pm\infty} \frac{3x}{x - 2} \), (f) find \( \lim_{x \to \pm\infty} \frac{x^2 - 4}{x - 2} \)

While solving, \( \lim_{x \to \infty} \frac{3x^2 - 13}{x^2 - 9} \) and \( \lim_{x \to \infty} \frac{3x}{x - 2} \), Anna applied direct substitution and concluded that the limit was infinity, since according to her \( \frac{\infty}{\infty} = \infty \). In the problem, \( \lim_{x \to \infty} \frac{1}{x + 1} \), Anna found the limit to be \( \frac{1}{\infty + 1} = 0 \). In \( \lim_{x \to \infty} \frac{3x^2 - 13}{x^2 - 9} \), Henry found the limit to be 3, with no work shown or explanation given. BlueJay stated that he remembered doing “something like factoring out \( x^2 \) from the numerator and the denominator to get some of the terms cancelled.”

He performed similar steps in the problem, \( \lim_{x \to \infty} \frac{3x}{x - 2} \) and found the limit to be 3, while he still could not connect limit at infinity with horizontal asymptotes. On this problem, Hayden stated that the limit was 3 since “as a \( x \) approached \( \pm\infty \), the limit was approaching a horizontal asymptote.” He explained that \( \frac{3x}{x - 2} = 3 + \text{some other term} \) and therefore the HA will be \( y = 3 \).

In this problem, Hayden was able to connect the method of applying transformation that he used to find the equation of the function with known HA in the reverse order to find the equation of HA from known function. Table 5.12 summarizes student performance of the limit problems during CORI 1 interview.
<table>
<thead>
<tr>
<th></th>
<th>Infinite Limits CORI 1</th>
<th>Limits at Infinity CORI 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lim_{x \to 3} \frac{3x^2 - 13}{x^2 - 9}$</td>
<td>$\lim_{x \to 2} \frac{3x}{x - 2}$</td>
<td>$\lim_{x \to \infty} \frac{1}{x + 1}$</td>
</tr>
<tr>
<td>Anna</td>
<td>Realized VA at $x = 3$</td>
<td>Plugged in 2 for $x$ Limit undefined</td>
</tr>
<tr>
<td></td>
<td>Said function went to $\pm \infty$, Only calculator can tell which infinity</td>
<td></td>
</tr>
<tr>
<td>BlueJay</td>
<td>function will be undefined</td>
<td>VA at $x = 2$, Did not know any more</td>
</tr>
<tr>
<td>Richie</td>
<td>$\lim_{x \to 3} m(x) = \frac{14}{0}$, “$14/0$ is undefined but in limit problem limit will be 14.” VA at $x = -3$</td>
<td>6/0, Undefined but limit will be 6</td>
</tr>
<tr>
<td></td>
<td>Note: “$14/0$ is undefined but in limit problem limit will be 14.”</td>
<td></td>
</tr>
<tr>
<td>Henry</td>
<td>Noted VA at 3 and -3 No mention of limits</td>
<td>None done</td>
</tr>
<tr>
<td>Tina</td>
<td>“zero in the denominator, feels like there is an asymptote there, -$14/0$”</td>
<td>DNE</td>
</tr>
<tr>
<td>Hayden</td>
<td>functions undefined at 3 and -3</td>
<td>$\pm \infty$, referred to the previous problem</td>
</tr>
</tbody>
</table>

Table 5.12. Summary of Concept Images of Limits

**Summary of Concept Images of Asymptotes, Holes, Limits, Continuity CORI 1**

Richie seemed to believe that the behavior of a rational function at the points where it was undefined, was unpredictable since at the *undefined points* the function could have a hole or a vertical asymptote. He stated that it was also possible for a function to have a hole on the vertical asymptote, though he could not explain when they would occur. While constructing functions with specified hole and a VA, Richie placed the factors for both the hole and the VA merely in the denominator. He did not find any difference between undefined form and
indeterminate form. However, Richie was able to recognize vertical asymptotes from their limit format.

Regarding horizontal asymptotes, Richie believed that a curve could never intersect its HA. He stated that if a curve did intersect its HA, the point of intersection would be a hole. While Richie was able to recognize HA from its limit format, he was unable to find the equation of the HA of by examining the algebraic form of the function. Richie was also unable to write a function term that corresponded to a given HA, though he affirmed that a HA would make graphs go up or down.

Regarding HA, Henry believed that a curve was allowed to intersect its horizontal asymptote. He said that he has seen graphs that intersected its HA. Henry did recognize HA from their limit format; he did find the equation of the HA if the numerator and denominator of the function had same degrees. He stated that he just remembered some rules he had learned though could not tell much about those rules.

Tina believed that at the points where a rational function was undefined, there will always be vertical asymptotes. Tina did not find any differences between the undefined and the indeterminate forms. While constructing functions based on its properties, Tina was unable to write the terms that were responsible for the VA of a function. Tina believed that a curve is not allowed to intersect its horizontal asymptote. She was unable to find the equation of the HA by examining the algebraic form of the function. She was also unable to write the terms that were responsible for the HA of the function from the given horizontal asymptote.

BlueJay believed that at the points where the function was undefined, VA or hole could occur. However, he also believed that vertical asymptotes occurred at jumps and holes as well. He noted no difference between undefined form and intermediate from. While constructing functions he could not write terms responsible for a vertical asymptote. BlueJay argued that the existence of
asymptotes cannot be determined by observing the limit forms. BlueJay believed that HA and the curve cannot intersect; he was unable to write the equation of a HA from observing the algebraic form of the function. He was able to write the equation of a function that was characterized by a given HA only by observing the pattern formed with other functions and their horizontal asymptotes. In some instances (problems 8 and 9), actually seeing the terms of a somewhat similar function helped BlueJay figure out how function terms were affected as x approached ±∞.

Hayden was able to write the equation of vertical asymptotes from the algebraic form of the function. He recognized both VA and HA from the limit form. While constructing functions, Hayden wrote terms responsible for vertical asymptotes. He was unable to find the equation of a HA from the algebraic form of the function. Hayden used shifts to write the equation of a function as characterized by its HA. He was also able to write the function equation by observing the graphical form and visualizing how different types of HA would alter its equation.

Anna stated that at points where the function was undefined vertical asymptotes or holes occurred. Vertical asymptotes can also occur on holes. These were called point asymptotes. While constructing functions, Anna wrote terms responsible for the occurrence of vertical asymptotes. According to Anna, a curve was not allowed to intersect horizontal asymptotes. She solved “numerator = 0” to find the equation of the horizontal asymptotes of functions. She recognized horizontal asymptotes from the limit form only because, “\( \lim_{x \to \pm\infty} f(x) = 4 \)” indicated to Anna that y is only “approaching 4” but “never equal to 4.”

Neo did not find any difference between undefined form and indeterminate form. Neo was unable to account for horizontal asymptotes while constructing the function equation from its
Although Neo seemed to have realized how the function behaves with infinite limits, she did not seem to connect how limits at infinity in these instances were the horizontal asymptote of the functions discussed. Neo believed that holes or VA could occur at the points where a function was undefined. She found no difference between the indeterminate and undefined forms.

Henry stated that at the points where a rational function was undefined in addition to the occurrence of a vertical asymptote, something else too could happen. But, he wasn’t sure what the other possibility was. Henry too did not find any differences between undefined form and indeterminate forms. Henry was unable to recognize vertical asymptotes from their limit format. By examining the function definition, Henry was able to write the terms that were responsible for the VA of the function. According to Henry, if a function was defined at a point the limit would be the function value. In addition, if a function was defined at a point, the function must be continuous at that point. In the case of a hole, the limit of the function was considered did not exist at the hole. BlueJay and Anna believed that a function could have a VA at holes and jump discontinuities. Anna’s convictions revealed that a function could have HA at maximum and minimum points as well. Henry’s and BlueJay’s specifications demanded that in the case of a removable discontinuity, if the isolated point is removed there could be a VA there making their strong-held conviction regarding how asymptotes and graphs could never be concurrent very clear.

Computation of finite limits was accomplished easily by students. However, limit at infinity was mainly attempted by plugging in infinity directly to the function. “Limit undefined,” or “limit did not exist” was the answer provided in these instances. Regarding infinite limits, Anna and Hayden realized the existence of a vertical asymptote, but were unable to narrow it down to which infinity the function was approaching. BlueJay, however, merely responded that
the function was undefined and that there was a vertical asymptote. He was unable to comment about the limit of the function. In one problem, Richie, after direct substitution, stated that even though 14/0 is usually undefined; in a limit problem, the limit will be considered 14. Similarly, according to Richie, 6/0 implied a limit of 6 while 0/0 in another problem implied a limit of 0.

Regarding infinite limits, Henry acknowledged the existence of vertical asymptotes but there was no mentioning of the limit of the function. Tina thought there could be an asymptote with the presence of zero in the denominator but the limit was stated as “DNE.” Hayden realized that since the function was undefined at these points, the limit was ±∞ as he observed in a previous problem. He was unable to find which infinity the function was approaching.

Based on his performance on problem 5, and 6 of CORI 1, Henry seemed to believe that a function will be continuous at a point as long that it was defined at that point. He also stated that the function j(x) will be discontinues at x = 2, since the function was not defined at x = 2. Hayden stated that functions j(x), k(x), and z(x) were all discontinuous at x = 2, and he called the discontinuity of j(x) a “cusp” while the discontinuity of k(x) was called “jump discontinuity” Anna believed that at all points of discontinuity, there was a VA involved.

In light of CORI 1 interview results, I believe that students possess incomplete conceptions regarding the concepts of asymptotes, limits, and continuity and their understanding of these concepts were rather fragmented and disconnected. Students’ inadequate knowledge of the basics such as the rational form, domain, range, and undefinedness was further complicating their understanding of the more sophisticated concepts that were under investigation. In some cases, students failed to attribute any forms of meaning to the mathematical processes that they were performing. For example, the act of computing the limit at infinity was not being associated with the behavior of the function when |x| got infinitely large. Some of these difficulties could be
explained from the epistemological perspective while some other difficulties could be explained from the perspectives of instructional practices and such. An analysis of the possible sources of student difficulties are discussed towards the end of this chapter.

**Teaching Episodes**

The teaching episodes were conducted with two groups. The first group met on Mondays and Wednesdays with four students, Anna, Tina, Hayden, and Neo. The second group that met on Tuesdays and Thursdays had three students, Henry, Richie, and BlueJay. This was done mainly to accommodate student-schedules as well as to view the first group as a semi-pilot group so that based on the observations of the first group the researcher could make necessary adjustment to lessons for the second group. However, very little adjustment was needed between the two groups. This could be due to the fact that in both groups there were students from same sections of the Calculus 2 class they were taking that quarter. In addition, these courses were taught, *home-worked*, and tested in very similar ways due to the departmentalization imposed by the Mathematics Department of this university. In addition, these groups were organized in a manner that both groups had a mixture of students who self-reported as having certain grades for mathematics courses.

During the teaching episodes, a teacher-researcher hopes to develop a model for student understanding of the concepts being investigated. This was accomplished by closely examining, noting, and analyzing student responses and interactions during the teaching episodes and by analyzing their written work, and video and audio-tapes. Based on the observations made, the teacher-researcher tried to modify incomplete perceptions through a variety of appropriate actions such as having students solve problems individually, then discussing solutions as a group, engaging in argumentation supporting their solution, working together in groups, listening to each other, and by participating in other activities initiated by the teacher-researcher. These activities
included grading the work of imaginary students, taking the role of a critic while the teacher-
researcher assumed the role of a student solving problems. In this way, students were faced with
challenging and inconsistent situations that were in conflict with their existing concept images.

There were times when a problem posed obstacles in a way that it could not be solved
based on students’ existing concept images and therefore they were warranted with the need for
seeking alternate conceptions.

Understanding the shifts in student thinking was accomplished by closely observing
changes in student interpretations and representations during their interactions with themselves,
with peers and with the teacher-researcher. The transactions of the teaching episodes were
conducted based on six assumptions on student concept formation and concept modification that
were elaborated in Chapter 4 of this dissertation. To provide a briefing, the first assumption was
that students’ concept images were influenced by their previous, formal, and informal exposure
and experience with the concept. The second assumption was that students actively constructed
their own mathematical knowledge and active assimilation, accommodation and equilibration is
imperative for the full development of these conceptions.

The third assumption was that, equilibrium can only be maintained through the active
encounter with situations that may pose disequilibria in the already existing conceptual
framework. Fourth, I assumed that social interaction and argumentation were essential features of
maintaining conceptual equilibrium. Fifth, teachers could intervene and pose cognitive confusion
by activities such as invoking contradicting problem situations, by the usage of appropriate
examples, by redirecting students’ attention to formal definitions and by probing a constant check
between student concept images and formal concept definitions. Sixth assumption was that in a
situation where the second, third, fourth and fifth assumptions are realized, learners may
apprehend concept refinement. I believe that the process of concept refinement becomes complete
through the processes of constant re-assimilation, accommodation, and the maintaining of equilibrium invoked by different types of cognitive demands.

Teaching episodes were organized into five segments. Table 5.13 summarizes the organization of these segments even though the details could be found in Chapter 4 of this dissertation. There were eight sessions that lasted one hour and thirty minutes twice a week. We met during one entire month.

<table>
<thead>
<tr>
<th>Segments</th>
<th>Number of Sessions</th>
<th>Core Concepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Segment 1</td>
<td>One Session</td>
<td>Introduction, Rational Functions, Rational Numbers</td>
</tr>
<tr>
<td>Segment 2</td>
<td>One Session</td>
<td>Polynomials, Rational Functions, Asymptotes</td>
</tr>
<tr>
<td>Segment 3</td>
<td>Three Sessions</td>
<td>Asymptotes</td>
</tr>
<tr>
<td>Segment 4</td>
<td>Three Sessions</td>
<td>Limits, Asymptotes</td>
</tr>
</tbody>
</table>

Table 5.13. Organization of Teaching Episodes by Concepts Discussed

Figure 5.5 summarizes the highlights of student misconceptions as revealed by the first problem solving interview. First, I will underline the interactions and the intellectual exchanges that took place during the teaching episodes that involved the appropriate use of terminology, and the concepts of asymptotes and holes of rational functions. As noted in Chapter 4, I developed materials (see Appendix D) to help guide the teaching episodes (also called learning sessions). Illustrations of some of the problems discussed will be given in this section. The problems were developed to specifically address the incompleteness identified in students’ concept images. Sixteen out of 19 students interviewed used incorrect terminology while solving mathematics problems. In regards to limit notation students struggled to specify what was approaching or
being approached. Discussions regarding the correct usage of terminology started out with examining logical ways to state *what was approaching* and *what was approached* when discussing the limit behavior of a function.

![Figure 5.5. Organization of the Types of Misconception](image_url)
Dawning of My Activities/Problems

Correct Usage of Terminology. To confer the correct use of terminology, I employed students’ understanding of the function concept where they knew that in the notation, \( y = f(x) \), \( x \) was the input variable and \( y \) or \( f(x) \) was the output variable. With this idea in mind, I developed the question: which of the two scenarios would be more logical to be stated first, is it ‘as \( x \) approaches a number…’ or is it ‘as \( f(x) \) approaches a number’? Henry pointed out that “since \( x \) was the independent variable, we needed to look at the behavior of \( x \) first to see how \( f(x) \) was reacting to it.” Students reached consensus that since \( x \) was the input variable and since the value of \( f(x) \), which was in fact \( y \), depended upon the value of \( x \), it is more sense-making to announce that “as \( x \) approaches \( f(x) \) approaches…..”

Asymptotes and Function Behavior. My next goal was to help students identify and describe the behavior of graphs by its asymptotes using the appropriate terminology. I believed that the best way to accomplish this goal would be to provide the graph of the function depicted below and ask students to complete the partial statements made about its behaviors. I believed that being able to see the graph could help students verbalize the function behavior better. As such, I developed questions based on the graph that is depicted below.
I asked students to complete…the vertical line $x = -1$ was called the VA of the rational function $a(x) = \frac{3x-1}{x+1}$ because… (explain) and the horizontal line $y = 3$ was called the horizontal asymptotes of the rational function $a(x) = \frac{3x-1}{x+1}$ because…. (explain)

Student responses to this question initiated further exploration of the function behavior by its asymptotes, and the examination of the definition of asymptote, and the eventual re-definition of asymptotes. I was happy with the natural progression of discussions that were evolved. Figure 5.12 provides a summary of the progression of interactions as they evolved during the first few learning sessions. The arrows in figure 5.6 indicate the immediate actions that were evolved based on students’ own questions and curiosities.
**Redefinition of Asymptotes.** As elaborated in Chapter 4, the methods section of this document, members unanimously pointed to the informal definition of asymptotes that they were familiar with. According to this definition, ”informally speaking, an asymptote of a function is a line that the graph of the function gets closer and closer to as one travels along that line.” After
examining this definition in regards to both vertical and horizontal asymptotes the dilemma was
resolved and the unanimous decision was reached to amend the informal definition by specifying
that the curve was allowed to cut through the HA as long as it came back and approached the
horizontal line or that the end behavior of the function should be such that it must approach the
asymptote. It was also noted that if a curve was to cut through the vertical asymptote, it could not
come back and approach the vertical line, since this behavior would violate the function
requirement.

Conditions for Vertical Asymptote. I redirected student attention to their previous
statements that “a curve could not intersect its VA”, “the function was undefined at VA” and
“VA occurred at the zeros of the denominator.” I asked students if there were a VA at all values
for which the function was undefined To address this issue, I decided to ask students to compare
the functions \( K(x) = \frac{x^2 - 1}{x + 1} \) and \( a(x) = \frac{3x - 1}{x + 1} \). I also decided to invoke three types of
representations to help students clarify the pitfalls that could arise from only using the algebraic
methods or the graphing calculator without paying attention to the deeper structures of these
functions. As such, students were asked to work first independently, and then as a group to
explore the behavior of functions \( K(x) = \frac{x^2 - 1}{x + 1} \) and \( K(x) = x - 1 \) at \( x = -1 \), first without a
calculator. Then, they were allowed to use a graphing calculator, to graph and to ‘zoom in’ and
then to explain what they saw. Students were further asked to compare and contrast between the
function behaviors of \( K(x) \), and \( a(x) = \frac{3x - 1}{x + 1} \) at \( x = -1 \). This activity lead to a common
consensus that a function \( f(x) = \frac{p(x)}{q(x)} \) will have a VA at \( x = a \), if \( p(a) = 0 \) and \( q(a) \neq 0 \).
Concept reconfiguration through exploration and exchange of ideas was the main goal of the learning sessions. I believed that graphing rational functions would provide students with ample opportunities to explore many important characteristics of these functions. While discussing ways to improve students’ conceptual understanding, Aspinwall et al. stated graphs could be treated as instructional activities that constitute a starting point for students’ mathematical constructions. Such diagrams can make it possible for teachers to guide students into novel experiences by drawing on students’ prior knowledge and experience. Students interpreting mathematical meaning in these activities would form increasingly sophisticated mathematical conceptions (Aspinwall, Shaw, & Presmeg, 1997, p. 301).

In addition, Artigue (1991) noted that construction and control of meaning could be better achieved through different forms of conceptual representation that emphasize on the graphical representation to remediate students’ difficulties associated with the learning of functions, limits, and other higher mathematical concepts.

Therefore, I tried to engage students in graphing different types of rational functions without the aid of a graphing calculator. These students, who were using graphing calculators for graphing, seemed to have forgotten the significance of even basic aspects such as intercepts. I wanted students to be able to examine the deeper structure of functions such as their domain, range, intercepts, asymptotes, and discontinuities and work through them by connecting their various concept images conceptually, procedurally, and graphically. Graphing a variety of rational functions promoted vigorous conceptual exchanges during the learning sessions. These exchanges included questions such as whether a vertical asymptote, or a hole existed at a point, which infinity was the function approaching around a VA, or how did one examine the presence of a HA.
Graphing Rational Functions. The graphing problems were chosen carefully based on my teaching experience, based on my observations of students’ conceptions during the CORI 1 interview, and also based on the feedback of Joann, the veteran math professor. In the first problem, \( m(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2} \), no instructions were given as to how to proceed with the graphing. I had hoped that the attempt of graphing functions of this type would call for situations other than simply plotting points, since graphing functions manually would demand knowledge of function behaviors at different parts of its domain as well as points of discontinuities thereby facilitating student thought and creativity.

BlueJay, Richie, and Henry were simply using the method of plotting points to graph this function. Richie found vertical asymptotes while plotting points, since he had reached undefined answers corresponding to certain x-values that he used. BlueJay found the zeros of the denominator to find the vertical asymptotes and found extra points to help guide the graph. However, due to erroneous computation, and lack of information regarding any other properties of these functions, neither BlueJay, Richie, nor Henry could complete the graph. The first thing Hayden did was trying to find the points where the function was undefined so that he could locate the vertical asymptotes.

Anna, Tina, and Neo were all solving for the denominator \( = 0 \) in order to guide their graphs based on vertical asymptotes. Neo found the horizontal asymptotes as well to help sketch the function. Henry did not include the HA in his graph, but stated that \( y = 2 \) would be the HA from recalling some rules he learned before. Anna and Richie could not relate to horizontal asymptotes in any mode. With the uproar of horizontal asymptotes underway, the discussion focused on how to find the HA of a rational function. BlueJay and Neo decided to examine what would happen to the y-values of the function as \( |x| \) got infinitely large. Neo deduced that it was
only necessary to examine the leading terms of the numerator and denominator to make that determination.

BlueJay initiated the discussion by examining (work shown below) each term by reducing some of them into negligibly small quantities as x approached infinity.

\[
\lim_{x \to \infty} \frac{2x^2 + 7x - 4}{x^2 + x - 2} = \lim_{x \to \infty} \frac{x^2 \left( \frac{2x^2}{x^2} + \frac{7x}{x^2} - \frac{4}{x^2} \right)}{x^2 \left( \frac{x}{x^2} + \frac{x}{x^2} - \frac{2}{x^2} \right)} = \lim_{x \to \infty} \frac{2 + \frac{7}{x} - \frac{4}{x^2}}{1 + \frac{1}{x} - \frac{2}{x^2}} = \frac{2}{1} = 2
\]

Tina and Richie were able to realize the logic behind examining each term as explained by BlueJay, however, they struggled while they solved these problems on their own. Anna was unable to gather the details of horizontal asymptotes, except for knowing that she ought to find the limit at infinity. Hayden was able to conceptualize the earlier rule that determined the horizontal asymptotes with how the function behaved each time according to the different degrees of the numerator and the denominator.

When the focus was back on sketching the graph of the function, some students realized that their graphs were missing intercepts, while some others realized that their functions were approaching the wrong infinity. Students soon realized the importance of recognizing the intercepts to further assist with the behavior of their graphs. Even after finding correct intercepts, Henry, Anna, and Tina included extra intercepts to their graphs while Henry and BlueJay completed their sketches based on the *classic three-piece graph* image that they had held in the past. This is an example of how easily students could fall back into their previous conceptions even in the middle of making new discoveries. This could happen as a matter of habit, or this could be due to not getting enough opportunity to accommodate the new idea into their
conceptual framework. Such uncontrollable mental imageries (Aspinwall, et al., 1997) slow down the process of concept refinement. According to the researchers, the usage of graphs and figures could help reiterate the characteristics of the concepts being re-accommodated.

During the teaching episodes, finding the infinite limit was a challenge for Tina. Tina attributed most of her difficulties to the inadequacies of didactic processes. Tina who had identified herself an A-achiever, made the comment “I cannot believe that I was able to receive A’s in my math classes with this much hole in my understanding.”

The next graphing problem, \( f(x) = \frac{3x + 2}{9x^2 - 1} \) posed challenges for all students since the degree of the denominator was larger than the degree of the numerator, in which case, the HA will be the x-axis. In addition, students had to deal with two vertical asymptotes, one x-intercept, and one y-intercept. Richie had trouble to decide the direction in which the graph would progress around its vertical asymptotes. He could not successfully find HA either. Richie sketched a correct graph by plotting an additional 11 sets of points besides vertical asymptotes. Although, BlueJay and Henry found all aspects of the graph correctly, they still sketched parts of the graph incorrectly. Once again, students evidently ignored the newer mathematical experience in favor of intuitions and instincts.

Tina was unable to sketch any part of the graph and spend most of her time trying to find the horizontal asymptote. Hayden did not think of specifically finding out which infinity the function was approaching. Neo, though confident in the beginning, eventually experienced problems while finding the horizontal asymptote. Anna was unable to do anything except for finding the vertical asymptotes.

Neo’s difficulty stemmed from factoring the denominator, \( 9x^2 - 1 \) into \((3x - 1)(3x + 1)\) and not realizing what would happen as \( x \) approached infinity in this case. I believe that
visualizing what would happen to \((3\times\infty - 1)(3\times\infty + 1)\) appeared more complex compared to imagining the implication of \(9\times\infty^2 - 1\). As soon as Neo re-focused on the leading terms of the numerator and the denominator she was able to realize that as \(x\) approached infinity the function approached zero. Others too found this discussion useful and were able to correct some of the mistakes that they had made in their work.

Another problem, \(t(x) = \frac{2x^3 - 5x^2 - 2x + 5}{2x^2 + 5x + 3}\), that was discussed during the learning sessions asked students to examine the limit at infinity of the function \(t(x)\). Neo started her work by writing “as \(x \rightarrow \pm\infty\), \(t(x) \rightarrow HA\). Then she divided each term first with \(x^3\) and later with \(x^2\) and arrived at “strange” results \(\frac{2}{0}\) and \(\frac{\infty}{2}\) respectively. After realizing that the function was not approaching any horizontal line, Neo concluded that limit at infinity didn’t always produce horizontal asymptotes. Tina concluded that limit at infinity for this function was \(2/0\) and therefore the HA does not exist. Hayden examined the leading terms of the numerator and denominator and concluded that \(\lim_{x \to \infty} t(x) = \infty, \lim_{x \to -\infty} t(x) = -\infty\). BlueJay wrote \(\lim_{x \to \infty} t(x) = \frac{2}{0}\) and left it alone, while Henry wrote \(\lim_{x \to \infty} t(x) = \frac{2}{0} = \infty\). Richie started dividing each term with \(x^3\), but could not think through the problem. At this point, Anna realized that she could simply examine the highest term in the numerator and the highest term in the denominator as \(x\) approached infinity. This time she wrote the eventual behavior of the function as \(\frac{2\times\infty^3}{0}\) and stated that the function too was approaching infinity as \(x\) approached infinity. Everyone agreed that the function is approaching \(\pm\infty\) as \(x\) approached \(\pm\infty\). I graphed this function on a graphing calculator so that students could
explore the function behavior. Students realized that the curve is approaching infinity from an angle. BlueJay commented, “I never understood oblique asymptote, I was wondering if we would ever discuss it.”

We started exploring oblique asymptotes. As I re-wrote the function

$$t(x) = \frac{2x^3 - 5x^2 - 2x + 5}{2x^2 + 5x + 3} = x + \frac{x + 5}{2x^2 + 5x + 3},$$

students explored the process of x approaching infinity. Anna substituted several large values for x in the expression $\frac{x + 5}{2x^2 + 5x + 3}$ and observed the value of the expression kept on decreasing. Richie substituted 3000 for x and examined how the entire expression was becoming negligibly small. Due to the casual and stress-free setting of the teaching episodes, students were eager to verify what they had observed on their own.

Students seemed excited about the discovery of the oblique asymptote, $y = x$, in this case, when the degree of the numerator was exactly one more than the degree of the denominator. Further, in both sessions students wondered if there could be a non-linear asymptote and many realized that it could happen if the degree of the numerator was two or more than the degree of the denominator. At that point, I created the function $v(x) = \frac{2x^3 + 7x^2 - 2x - 7}{x - 3}$ and graphed it on the calculator to briefly discuss non-linear asymptotes. The graph of this function is shown in Figure 5.7.
At this point, I realized that exploring the correct usage of terminology has guided the groups towards investigating a wide range of concepts that were under study. These concepts include function behavior around its vertical asymptotes and holes, function behavior by its HA, how to find the equation of HA for different types of functions, the importance of graphing a variety of functions and the ways in which graphs help concretize the abstract notions of holes, asymptotes and intercepts. In addition, students independently explored the formation of oblique and non-linear asymptotes. My next goal was to get students explore the circumstances under which a hole occurred. The next graphing problem was concerning the function

$$\alpha(x) = \frac{3x^2 - x - 10}{2x^2 - x - 6}$$

which took the factored form $$\alpha(x) = \frac{(3x + 5)(x - 2)}{(2x + 3)(x - 2)}$$. Richie realized that after the cancellation of the factor \((x - 2)\), the function will still be undefined at \(x = 2\). By substituting points close to 2 for \(x\), he found the limit to be approximately 1.6. BlueJay also noticed that as \(x\) approached 2, the function approached 1.6. Richie agreed with Tina, Neo and BlueJay that this function had a hole at \((2, 1.6)\).

Henry first drew a VA at \(x = 2\). However, once he realized that the function did not have infinite limit at \(x = 2\), he recognized the existence of a hole rather than a vertical asymptote. Anna, after listening to others, stated that the “function is not going off to infinity” and therefore,
there was no VA at $x = 2$. I believe that students started gathering connections between how undefinedness affected the function behavior. They have seen that for the function, $\alpha(x)$, there was no VA at $x = 2$, since the function had a finite limit at $x = 2$.

While the graphing problems were concluded, both Neo and Henry noticed that when the degrees of the numerator and denominator were the same, the HA was $y = \text{ratio of the leading coefficient of the numerator to the leading coefficient of the denominator}$. Also, when the degree of the numerator was smaller than the degree of the denominator, $y = 0$ or $x$-axis was always the HA of the rational function. Furthermore, when the degree of the numerator is exactly one more than the degree of the denominator there will be an oblique asymptote for the rational function under discussion. Tina, however, stated that she prefers to examine each of the terms of the numerator and the denominator to find the HA of the function. Finally, students were able to realize how the degree rules that they were introduced in the pre-calculus and calculus courses characterized the asymptotic behaviors of the function. Now that they gained in-depth understanding of the concepts behind these rules, they are more likely to apply them in finding limit at infinity and the equation of HA.

**Solution by Imaginary Learners.** Among other activities that took place during the learning sessions, a series of problems were given to students to reconfigure their conceptions of the topics of vertical asymptote, horizontal asymptote, holes, and limit behavior. Students were presented with a series of problems and their solutions which were written by imaginary learners: Marvin, Mary, Millie, and Melvin. I created some of these problems during the teaching episodes to reflect certain difficulties that students were experiencing while computing limits. In addition, I believe that additional exposures to challenging, disharmonious situations were essential to complete the process of concept modification Table 5.14 summarizes these questions involving the imaginary learners with a specification of their importance.
<table>
<thead>
<tr>
<th>Question/solution presented</th>
<th>Rationale for the question</th>
</tr>
</thead>
</table>
| Marvin’s work | \[
\lim_\limits{x \to -7} \frac{x + 7}{x^2 - 49} = \frac{-7 + 7}{49 - 49} = \frac{0}{0} = DNE
\] | Students often used direct substitution for all limit problems. They interpret 0/0 as DNE |
| Mary’s work | \[
\lim_\limits{x \to -7} \frac{x + 7}{x^2 - 49} = \frac{14}{0}, \text{ limit undefined}
\] | Students often treated indeterminate and undefined form as limit undefined. They don’t realize the implications of the undefined vs. indeterminate forms |
| Millie’s work | \[
a(x) = \frac{x + 7}{x^2 - 49}
\] | VA at all undefined values? Students often treated infinity like a finite number and tend to substitute infinity for x. When infinities are involved in the numerator and in the denominator, students often are unable to understand what the implications were. Correspondingly their answer tend to be DNE, or undefined |
| Millie’s work | \[
\text{VA at } x = 7 \text{ and at } x = -7
\] | |
| | Also, \[
\lim_\limits{x \to -7} \frac{x + 7}{x^2 - 49} = \frac{\infty + 7}{\infty^2 - 49} = DNE
\] | |
| | | |
| Melvin | \[
a(x) = \frac{6x^3 + 7}{11x^2 - 49}
\] | Rote application of rules without paying attention to problem details |
| Melvin | \[
\text{Horizontal asymptote: } \lim_\limits{x \to \infty} \frac{6x^3 + 7}{11x^2 - 49} = \frac{6}{11},
\] | |
| | HA y = 6/11 | |

Table 5.14. Rationale for Questions

In Marvin’s problem, some students did not seem to realize the implication of the 0/0 form that was obtained during the direct substitution. Student comments on Millie’s work were wide-ranging. Anna commented that Millie’s interpretation of vertical asymptotes was correct, while Tina and Neo stated that vertical asymptote was not possible at x = -7 since the limit at x = -7 was a finite number. Richie created a table of values and decided that there was no VA at x = -7.
In Millie’s work, with regard to horizontal asymptotes, Anna stated that there was no HA since “the limit was not a number”, while Tina and Neo stated that there was no horizontal asymptote since the degree of the denominator was greater than the degree of the numerator. After examining Melvin’s work Anna commented that “HA must be y = 6. According to Anna, “since the degree of the numerator was greater that the degree of the denominator, the leading coefficient of the numerator should be taken for the HA.” As revealed by the discussion of students’ comments on Marvin, Millie, and Melvin’s work, the situations under which infinite limit, limit at infinity, and a hole occurred were still not clear to some students. Their concept images seemed to be rather procedural than conceptually based since their focus was on factoring and cancelling common factors and such rather than on the final outcome.

While exploring ways to distinguish between finite and infinite limits using methods other than factoring and canceling common factors, Hayden, BlueJay, and Neo suggested that one could use direct substitution to make the initial discrimination between finite versus infinite limits. Direct substitution was the method most students resorted to while solving limit problems. Therefore, if this method could be extended to help solve limit and vertical asymptote problems in a meaningful manner, students might be able to re-accommodate these concepts easier. Thus the focus was now on the implications of the indeterminate and undefined forms. The following problems were discussed.

1. Guess the behavior of the function $U(x)$ at and around $x = 2$, and $x = 7$. The following table was obtained by using the table features of a graphing calculator where the
function $U(x)$ was entered without any simplifications.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1$</th>
<th>$x$</th>
<th>$y_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-2.05</td>
<td>1.5</td>
<td>-2.11</td>
</tr>
<tr>
<td>0.9</td>
<td>-2.08</td>
<td>1.99</td>
<td>-2.18</td>
</tr>
<tr>
<td>0.99</td>
<td>-2.098</td>
<td>1.999</td>
<td>-2.188</td>
</tr>
<tr>
<td>0.999</td>
<td>-2.0998</td>
<td>2.01</td>
<td>-2.217</td>
</tr>
<tr>
<td>0.9999</td>
<td>-2.09998</td>
<td>2.001</td>
<td>-2.286</td>
</tr>
</tbody>
</table>

Which one/s of the following situations would you consider possible for $U(2)$ and $U(7)$?

(A) $U(2) = -2.2$, (B) $U(2) = 0/0$, (C) $U(2) = b/0$, $b \neq 0$, (D) $U(2)$ DNE (E) $U(7)$ DNE,

(F) $U(7) = 0/0$ (G) $U(7) = b/0$, $b \neq 0$

Richie wrote $\lim_{x \to 2} U(x) = -2.2$, and chose $U(2) = -2.2$ when asked stated the function will be continuous at (2, -2.2). Richie noted that there was a VA for $U(x)$ at $x = 7$. He also noted that $\lim_{x \to 7} U(x) = -\infty$ and $\lim_{x \to 7} U(x) = \infty$. He stated that $U(7)$ = DNE holds in addition to $U(7) = b/0$, $b \neq 0$ for this part. BlueJay, Hayden, Henry, and Tina gave similar explanations to this problem. For $U(2)$, Anna chose $\lim_{x \to 2} U(x) = -2.2$, and sketched a graph that had a hole at (2, -2.2). She said she wouldn’t pick $U(2) = b/0$, $b \neq 0$ since that meant infinite limits and VA. Anna also stated that $U(2) = 0/0$ did not always mean that the function was defined at $x = 2$. As far as $U(7)$ was concerned, Anna stated that there was a VA for $U(x)$ at $x = 7$, she further noted that $\lim_{x \to 7} U(x) = -\infty$ and $\lim_{x \to 7} U(x) = \infty$. In addition, to choosing $U(7)$ DNE and $U(7) = b/0$, $b \neq 0$ for this part, Anna sketched parts of the graph of this function with VA at $x = 7$ and the function approaching infinities as indicated in her limit statement. Neo, stipulated that at $x = 2$, $\lim_{x \to 2} U(x) = -2.2$. She also picked $U(2) = -2.2$ along with $U(2) = 0/0$ as two possibilities.
I noted the details that I wanted to ponder further. For Anna, I wanted to find out why she did not pick \( U(2) = -2.2 \), and why she thought that there was a hole at \((2, -2.2)\)? Then why did not Anna pick \( U(2) = 0/0 \)? Neo’s view seems partially logical since if \( \lim_{x \to 2} U(x) = -2.2 \), the function could be continuous at \( x = 2 \), or that there could be a hole at \( x = 2 \), in which case \( U(2) \) will assume the \( 0/0 \) form. In Neo’s case, I wondered if there was an issue with what she perceived how the calculator would display table of values for functions such as say,

\[
T(x) = x - 1 \quad \text{and} \quad J(x) = \frac{x^2 - 1}{x + 1}.
\]

I introduced a new problem, displayed below with calculator generated table of values.

This time, Anna realized that the function \( Y_1 \) was continuous at \( x = .2 \) while the function \( Y_2 \) had a hole at \( x = 2 \) and therefore, \( Y_2(2) \) assumed \( 0/0 \) form. She also stated that in both cases, the limit at \( x = 2 \) was -2.2. Neo realized that corresponding to \( Y_1 \), there was no indeterminate form at \( x = 2 \), while \( Y_2(2) \) assumed \( 0/0 \) form.

As far as student comments on the previous problem with function \( U(x) \) was concerned, I believe that there was some confusion with calculator displays when dealing with function value at an undefined point. Or it could just be that instances of finite limits somehow reminded students of a hole since, the discussions of these problems originated from the efforts to differentiate between holes and vertical asymptotes on the basis of finite or infinite limits. For the
Tuesday, Thursday session, I discussed the calculator displays beforehand to help clarify the problem details.

In light of the pilot studies that I had conducted, the CORI 1 interview, and my own experience with teaching calculus students, I had realized that anytime students are dealing with infinite limits, the tendency was to call it *undefined* or *DNE*. Even after specifying that they are expected to acknowledge function behavior in the form of infinite limits, most students continued to give answers as limit undefined and/or limit does not exist (DNE). During the CORI 1 interview, students relayed that they are not used to computing limits without a graphing calculator. While the usage of a graphing calculator itself may not cause any problems, not knowing the fundamentals of limits, and not knowing or not being able to envision the function behavior associated with limits at infinity and infinite limits showed a serious lack on conceptual knowledge when it came to student understanding of limits. Consequently, students were not allowed to use graphing calculators during the teaching episodes. They were allowed to use a regular scientific calculator for computation purposes.

**Re-examination of the Concept of Limits.** During the last three sessions of the teaching episodes, the concept of limits was further investigated. At this point, students have already established the connections between limits at infinity and horizontal asymptotes, as well as, infinite limits and vertical asymptotes. Students also established that function value assumed the *undefined* form as an indication of infinite limit. Thus, the major points, yet to be discussed and resolved, narrowed to the exploration of the limit definition, the conditions under which finite limit existed, and how to decide which infinity the function was approaching from either side of a vertical asymptote.

Materials for the teaching episodes were developed based on the results of the CORI 1 interview and the observations made during previous teaching episodes. Two of the problems
discussed will be referred as Plot – 1, and Plot – 2. In addition, once again, imaginary student work was introduced to help bring students’ attention to common misconceptions that were previously identified by the researcher. The imaginary students were Magoo, Rita, Gia, and Eddy. These problems that are stated below will be referred as Magoo’s problem, Rita and Gia’s problem, and Eddy’s problem.

**Plot 1**

“First plot the point (5, 1), then sketch the graph of a function f(x) with x-intercept (-6.5, 0), such that as \( x \to 5^+ \), \( f(x) \to -1 \).

**Plot 2**

“Sketch the graph of a function f(x) with x-intercepts (2, 0), and (6.5, 0), y-intercept (0, -3), such that as \( x \to 5^+ \), \( f(x) \to 1 \), and as \( x \to 5^- \), \( f(x) \to 1 \), and \( f(5) = -7 \).”

During the CORI 1 interview, Tina believed that for a function with a hole in its graph, the limit was the missing y value of the hole, while in a problem with removable discontinuity, the limit was the function value. Henry stated that it is not possible for the function to have a limit at a point while it was discontinuous at that point. Therefore, plot – 1 and plot – 2 were intended to engage students in cognitive dissonance regarding whether a function has to be defined at a point in order for the limit to exist at that point.

In plot 1, BlueJay and Tina asked if the point (5, 1) was a part of the function f(x). Richie, who placed a hole at (5, -1), was later confused due to his conviction that if the point (5, 1) was to be on the graph of the function, the limit of this function as \( x \) approached 5 from the right must be considered 1 instead of -1. In plot 2, in regards to the point (5, -7), once again there
was a dilemma since for example; Henry, BlueJay and Neo just could not imagine the function having a limit at a point different from the function value.

While sorting out student confusion that the limit of a function ought to be the function value, I encouraged students to investigate some limit definitions that they were familiar with. Students found a part of a definition very useful. According to this book, “while computing the limit of a function at a point c, the only thing that matters is if the function is defined near c (Stewart, n.d.)”. In light of this statement, students re-examined the problems in plot – 1, and plot - 2 and decided that in order for limit to exist at a point, the function didn’t have to be defined at that point.

It should be noted that while discussing asymptotes the term approached conveyed the concept image of approached but never reached while during the limit discussion the term approached conveyed the idea of approached and reached. This could be due the usage of the word limit, which might have given students the impression that limits could be reached [but could not be surpassed].

**Computing Infinite limits.** Next, the focus was on the computation of infinite limits without the usage of graphing calculators. Graphing calculators were not allowed because; students seemed perturbed when it came to limit computation without using a graphing calculator. Consequently students lacked the conceptual understanding of how the function would behave at the points where limits were computed.

Currently, after the elaborate discussions of vertical asymptotes, students already knew the situation under which a function would have infinite limit. Henry, Neo, BlueJay, and Tina readily related to \( \frac{b}{0}, b \neq 0 \) as the indication of the occurrence of infinite limit. However, students still had difficulty to decide which infinity the function was approaching as x approached a value.
from the left and from the right. They mostly created table of values to decide this. To enable students’ more than one method at their disposal, so that they would have the flexibility to cross check their work if necessary, I introduced a set of problems. Two of these problems that were created instantly by the researcher during the learning sessions were Magoo’s problem, and Eddy’s problem.

Magoo’s Problem

Question: \( f(x) = \frac{5x-9}{x-2} \), compute \( \lim_{x \to 2} f(x) \)

He found that \( f(2) = 1/0 \), and concluded that the limit will be \( \pm \infty \)

He knew he needed to narrow it down to which infinity exactly? He started by examining two cases: \( \lim_{x \to 2^+} \frac{5x-9}{x-2} \), and \( \lim_{x \to 2^-} \frac{5x-9}{x-2} \). Then he wrote: \( x \to 2^+ \Rightarrow x > 2 \Rightarrow x - 2 > 0 \).

He did not complete this method, instead sought a shortcut. He found \( f(3) = 6 \), a positive number, and therefore concluded that \( \lim_{x \to 2^+} \frac{5x-9}{x-2} = \infty \). Similarly, for \( x \to 2^- \), he found \( f(1) = -4/1 = 4 \), positive. Therefore concluded that \( \lim_{x \to 2^-} \frac{5x-9}{x-2} = \infty \). Comment on Magoo’s work, make corrections if necessary.

Eddy’s Problem

While finding \( \lim_{x \to -7} a(x) \), \( a(x) = \frac{2x}{x+7} \), Eddy spent lots of time plugging in values, making errors and trying to figure out the limit. Finally, he realized something. He wrote

“I will examine the numerator and the denominator separately…”
\[
\lim_{x \to -7} \frac{2x}{x + 7} \quad \quad \lim_{x \to 7} \frac{2x}{x + 7}
\]

\[
x \to -7^+ \Rightarrow x > -7 \Rightarrow x + 7 > 0 \quad \quad x \to 7^- \Rightarrow x < -7 \Rightarrow x + 7 < 0
\]

But, Eddie was unable to finish his method. Can you complete the problem?

While analyzing Magoo’s work, all except Henry and Neo abandoned Magoo’s first method which started out with \(x \to 2^+ \Rightarrow x > 2 \Rightarrow x - 2 > 0\). Students simply commented on the way in which Magoo tested \(x\) values that were not close enough to judge the behavior of the function. To get students re-focus on Magoo’s abandoned work, I had to create Eddy’s problem. In addition to exposing students to different methods of finding infinite limits, this problem helped students reiterate the importance of testing for points that are infinitesimally close to the \(x\)-value in discussion. Every student was able to make that observation.

Next, students were challenged to re-examine the problem \(\lim_{x \to 1} \frac{x + 1}{x^2 - 1}\) that was done previously. This time they were encouraged to solve this problem using Eddie’s method (the method initiated by Eddy in Eddy’s problem). A tie-dye flavored fruit roll-up was the reward offered for accepting the challenge. During the discussion of student independent work, Richie openly displayed his work. After writing as \(x \to -1^+ \Rightarrow x > -1\), Richie drew a number line similar to the one that follows. Realizing that when \(x > -1\), \(x^2 < 1\), and \(x^2 - 1 < 0\),

\[
\begin{array}{cccccc}
2 & -1 & 0 & 1 & 2 & 3 \\
\end{array}
\]

he continued that while \(x + 2 > 0\), \(\frac{x + 2}{x^2 - 1} < 0\). Richie concluded that as \(x \to -1^+\), the function approached negative infinity. Richie won the fruit roll-up for completing this problem (successfully) and explaining it to the group (in a sensible manner). It must be noted that Eddy’s method could get more complicated with more complex functions. Table 5.15 summarizes the problems discussed in previous paragraphs with a highlights of the relevance of these problems.
<table>
<thead>
<tr>
<th>Question</th>
<th>Relevance</th>
<th>Misconceptions identified</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plot 1</td>
<td>Is the point (5, 1) on the function f?</td>
<td>If limit exists at a point, the function will be continuous at that point</td>
</tr>
<tr>
<td></td>
<td>If so, shouldn’t ( \lim_{x \to 5} f(x) = 1 )?</td>
<td>With removable discontinuity the limit was the function value</td>
</tr>
<tr>
<td>Plot 2</td>
<td>Clearly (5, -7) is on the graph. Then, can ( \lim_{x \to 5} f(x) = 1 )?</td>
<td>With a hole in the graph the limit will be the y-coordinate of the hole</td>
</tr>
<tr>
<td>Magoo’s problem</td>
<td>While testing points, how close is sufficiently close?</td>
<td>How close is close</td>
</tr>
<tr>
<td>Rita’s and Gia’s problem</td>
<td>For students who thought Magoo was correct, now they know the graph tells differently. Now they have to re-examine it.</td>
<td>Only one way to tell: substituting values</td>
</tr>
<tr>
<td>Eddy’s problem</td>
<td>Students abandoned Magoo’s original plan of examining the numerator and the denominator now have to re-explore</td>
<td>Multiple ways of finding infinite limits</td>
</tr>
<tr>
<td>Find ( \lim_{x \to -1} \frac{x + 1}{x^2 - 1} )</td>
<td>Did the indeterminate form always imply finite limit?</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.15. Rationale for Magoo’s, Rita and Gia’s, and Eddy’s Problem

**Summary of Teaching Episodes**

During the teaching episodes, the teacher-researcher tried to engage students in a variety of activities to help them reconfigure their incomplete conceptions. This was done by keeping in mind that concept modification was a complex task and there should be ample opportunities to re-assimilate, re-accommodate, and further equilibrate the newly acquired conceptions. The techniques used to accomplish the intricate task of facilitating the re-structuring of student concept images were numerous.
I introduced a variety of problem contexts, and asked thought-provoking questions, seeking explanations of the problem solving processes. I introduced routine as well as non-routine problems that were to be solved graphically, analytically (using algebraic methods), and numerically. After solving problems independently, students were asked to present their problem solutions to other group members. At that time, they defended their solution and received feedback from group members. In addition, students were asked to correct and critique the researchers work while the researcher solved problems. Students also had to correct problem solutions of imaginary students who were born to the researcher’s creativity.

Graphing calculator graphs and table of values were used to create cognitive obstacles at times. Graphing rational functions were used to help students explore a variety of function properties that were tied together contributing to the unique features of the function that were usually discussed in algebra and calculus courses.

Next, I will detail student performance during the second problem solving interview or CORI 2 interview. First I have provided a summary of the findings of this interview in the table format.

*Concept Images of Asymptotes, Holes, Limits, and Continuity: CORI 2*

**Images of Asymptotes and Holes: CORI 2.**

Table 5.16 summarizes student concept images of horizontal asymptotes as explicated by the CORI 2 interview, and the table 5.17 summarizes student concept images of vertical asymptotes and holes as pictured by the CORI 2 interview. In addition, a comparison between CORI 1 and CORI 2 problems are also presented in the table format (see Table 5.18).
As $x$ gets infinitely big what happens to the function?

Horizontal asymptotes could be identified by finding the limit at infinity.

Limit at infinity could be found by substituting large value for $x$.

Limits at infinity could also be found by examining the leading terms of the numerator and the denominator.

A curve is allowed to intersect its HA as long as it approaches the horizontal asymptotes eventually.

Was able to construct the equation of a function with $x$–axis for HA.

Was able to construct the equation of a function with $y = -\frac{2}{7}$ for HA.

Was able to construct the equation of a function with no HA.

Table 5.16. Horizontal Asymptote Conceptions in CORI 2

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anna</td>
<td>As $x$ gets infinitely big what happens to the function?</td>
</tr>
<tr>
<td>BlueJay</td>
<td>Horizontal asymptotes could be identified by finding the limit at infinity</td>
</tr>
<tr>
<td>Richie</td>
<td>Limit at infinity could be found by substituting large value for $x$</td>
</tr>
<tr>
<td>Henry</td>
<td>Limits at infinity could also be found by examining the leading terms of the numerator and the denominator.</td>
</tr>
<tr>
<td>Tina</td>
<td>A curve is allowed to intersect its HA as long as it approaches the horizontal asymptotes eventually.</td>
</tr>
<tr>
<td>Hayden</td>
<td>Was able to construct the equation of a function with $x$–axis for HA</td>
</tr>
<tr>
<td>Neo</td>
<td>Was able to construct the equation of a function with $y = -\frac{2}{7}$ for HA</td>
</tr>
<tr>
<td>Richie</td>
<td>Was able to construct the equation of a function with no HA</td>
</tr>
<tr>
<td>Anna</td>
<td>Was able to identify a non-linear asymptote, while constructing a function with no HA</td>
</tr>
</tbody>
</table>

All students found the intercepts correctly. Anna, after solving for the zeros of the denominator stated that holes occurred at those points, by emphasizing that “vertical asymptotes are the same as holes” and “at the vertical asymptotes, the function goes to $\pm\infty$”. She explicated the function behavior around its horizontal asymptotes correctly and examined the leading terms of the numerator and denominator, and found the HA $y = 3$. Anna further explained that as $x$ gets infinitely big the numerator will be 3 times as big as the denominator.
<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anna</td>
<td>VA occurred at the zeros of the denominator. A function would also have a hole every time it had a VA. If the function was “actually shifting,” VA occurred instead of holes. If factors are cancelled out there will be a “definite hole.” If the factors cannot be cancelled there could be a hole or a vertical asymptote. VA occurred if only the function approached ±∞. “h(2) = \frac{b}{0} meant x – 2 just in the bottom” “h(-4) = \frac{0}{0} meant x + 4 top and bottom”</td>
</tr>
</tbody>
</table>
| Hayden Richie | VA occurred at the zeros of the denominator only if the function value took \( \frac{b}{0}, b \neq 0 \) from. For holes, function took the form 0/0 at the zeros of the denominator.  
\[ h(2) = \frac{b}{0} \text{ meant } x - 2 \text{ in the denominator, finite limit as } x \text{ approached } 2 \]  
\[ h(-4) = \frac{0}{0} \text{ meant } x + 4 \text{ in the numerator and denominator, infinite limit as } x \text{ approached } -4 \] |
| Tina Neo Henry BlueJay | VA occurred at points where the function was undefined if only the function approached ±∞. Holes too occurred at the zeros of the denominator. One may use indeterminate or undefined form to check “things” out.  
\[ h(2) = \frac{b}{0} \text{ meant } x - 2 \text{ in the denominator, VA } \]  
\[ h(-4) = \frac{0}{0} \text{ meant } x + 4 \text{ in the numerator once, and denominator once or twice, could have holes or VA } \] |

Table 5.17. Vertical Asymptotes and Holes. Conceptions of CORI 2
<table>
<thead>
<tr>
<th>CORI 1 Questions</th>
<th>CORI 2 Questions</th>
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<td>Finding the equations of HA, VA</td>
</tr>
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<td></td>
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<td>HA, VA connection to limits</td>
</tr>
<tr>
<td></td>
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<td>Graphing, intercepts, domain</td>
</tr>
</tbody>
</table>

Table 5.18. Concept Organization in CORI 1 and CORI 2 Questions

During the CORI 2 interview, in all problems, students knew to ask for the factored forms of the polynomials involved in the functions. This was done to keep distractions and errors at a minimum, and to manage time efficiently. For CORI 2 Problems 5 through 10 (see table 5.19), the factored form, \( a(x) = \frac{(3x + 4)(2 - x)}{(x - 4)(x + 1)} \) of the rational function was provided by request, and all except Henry asked for the factored form.
Consider the rational function \( a(x) = \frac{-3x^2 + 2x + 8}{x^2 - 3x - 4} \). Find

5. the x-intercept/s
6. the y-intercept
7. the horizontal asymptote
8. the vertical asymptote/s
9. hole/s
10. Using the above features graph the function in the grid given below

Table 5. 19 CORI 2 Problem 5 through 10

While graphing this function, Anna sketched vertical asymptotes at the zeros of the denominator even though she had stated that holes would occur there. When asked, Anna stated that a function will also have a hole every time it had a vertical asymptote. But later she stated that the function was “actually shifting” [changing directions] at these points and therefore have vertical asymptotes, not holes. While sketching the graph, she ignored sketching graph in the interval \((4, \infty)\) since she was not sure “if the function had any graph there”. BlueJay found the intercepts, vertical asymptotes, and HA correctly. For holes, BlueJay commented “since no factors are canceling I know there is no hole.” BlueJay too graphed the function except for in the interval \((4, \infty)\). Later, he completed the graph by stating that he ignored graph in this interval since “nothing important” was happening there.

Neither Anna, nor BlueJay, focused on the function domain to verify if the graph existed in the interval \((4, \infty)\). Henry found the intercepts, and HA correctly. He computed the limit from the right and from the left to decide which infinity the function was approaching. Henry wrote that as \( x \to 4^+ \), \( x > 4 \), \( x - 4 > 0 \), \( x + 1 > 0 \), \( 3x + 4 > 0 \), but since \( 2 - x < 0 \), the product of the
numerator will be < 0. He indicated that the function value is negative and therefore the function approached negative infinity. Similar work was done for the function behavior near its vertical asymptote, $x = -1$. Henry stated that there were no holes since there were “no similarities between the numerator and denominator,” “no terms in the numerator and denominator cancelled out by causing the 0/0 form.”

To find the HA of this function Richie substituted very large and very small numbers such as -1000 and 1000, for $x$ and found the HA to be $y = 3$. Tina while finding the equation of HA used the concept of limit at infinity by directly substituting $\infty$ in $\frac{-3x^2}{x^2}$. She was however, confused by the expression $\frac{-3\infty^2}{\infty^2}$, since she did not know if she could “cancel out infinity.”

Later, after substituting 3 billion for $x$ Tina found the HA to be $y = -3$ since the limit at infinity of this function was 3. After finding $x = 4$ and $x = -1$ as “possible” vertical asymptotes Tina examined the limits of the function as $x$ approached 4 and -1 from both directions. She confirmed that both $x = 4$ and $x = -1$ were vertical asymptotes.

CORI 2 questions 11 – 15 were similar to that of CORI 2 questions 5 – 10 with a different function $b(x) = \frac{2x^2 + 9x - 35}{x^3 + 2x^2 - 31x + 28} = \frac{(2x - 5)(x + 7)}{(x + 7)(x - 4)(x - 1)}$ and with different set of properties to examine. As far as a hole and VA were concerned, Anna realized that if no common factor could be cancelled; there would be vertical asymptotes at the zeros of the denominator. Anna found $b(-7.0001)$ which was -.2159 and $b(-6.9999)$ which was -.2159. Anna said in this case there is no VA at $x = -7$ since the function approached a finite limit.

In comparison to her work in problem 9, Anna stated that in the current problem at $x = -7$ there was a “definite hole” since the factor $(x + 7)$ was canceled from the numerator and
the denominator. She emphasized that without any cancellation there could be a hole or a vertical asymptote; the VA occurring if only the function approached ±∞. Anna found the function behavior by the VA and found the equation of HA correctly to complete the graph of the function.

BlueJay found all requested aspects of the function and successfully completed the graph of the function. Henry placed one hole at (-7, 0) where the x-intercept would have been occurred though he could not explain why the hole was at (-7, 0). Richie realized that when x = -7 the function took the indeterminate form and noted the presence of a hole at x = -7. This time Richie was able to deal with holes much better and knew the characteristics that would lead to the presence of a hole as having the same factor in the numerator and denominator. In addition, viewing the HA as limit at infinity has helped Richie think through the process while finding the equation to the horizontal asymptote.

Tina, while examining the limit of the function $b(x)$, was confused momentarily at the function behavior around x = -7 since she found

$$\lim_{{x \to -7^{-}}} b(x) = -0.2159,$$

and

$$\lim_{{x \to -7^{+}}} b(x) = -0.2161.$$ 

She stated “I am confused,” “the limits are coming together” “so no VA at x = -7, VA at x = 4, and x = 1,” and “a hole at x = -7.” She sketched the graph of function b(x) and stated “cool, I never could draw such a graph [before].” Tina expected the limit at x = -7, to be infinities indicating a VA at x = -7 to be consistent with her previous belief that at the points where the function was undefined a VA always occurred. These types of cognitive confusion are essential for students to re-think the results of their work and to interpret the meaning of their results. At the same time, by observing these behaviors and their resolutions a teacher will be able to understand student models from a clearer perspective. In this problem, Hayden used the 0/0 versus b/0, b ≠ 0 forms to easily differentiate between the existence of VA versus hole and was able to pinpoint the hole by finding the function value.
In problem 23, (see Table 5.20) demanded function construction based on indicated properties.

<table>
<thead>
<tr>
<th>CORI 2. Problem 23</th>
</tr>
</thead>
<tbody>
<tr>
<td>23. Write the equation or parts of the equation of a rational function ( f(x) ) if</td>
</tr>
<tr>
<td>(a) ( x = -7 ) is a vertical asymptote of ( f(x) )</td>
</tr>
<tr>
<td>(b) the function has a hole at ( x = -2 )</td>
</tr>
<tr>
<td>(c) the horizontal asymptote of this function is the X-axis</td>
</tr>
</tbody>
</table>

Table 5.20. CORI 2, Problem 23

Anna first wrote that \( f(x) = \frac{x + 2}{(x + 7)(x + 2)} \), and then multiplied the factors as

\[
f(x) = \frac{x + 2}{x^2 + 9x + 14}.
\]

After examining the leading terms of the numerator and the denominator, \( \frac{x}{x^2} = \frac{1}{x} \), she wrote “\( \lim_{x \to \infty} f(x) = 0 \)” by finalizing \( f(x) = \frac{x + 2}{x^2 + 9x + 14} \) as her function. BlueJay wrote “\( \frac{(x + 2)}{(x + 2)} \to \frac{(x + 2)}{(x + 2)(x + 7)} \)” and commented “HA of this function, will be \( y = 0 \),” since the “degree of the denominator larger than degree of the numerator.” Richie wrote “\( \frac{(x + 2)}{(x + 7)(x + 2)} \to \frac{1}{x + 7} \)” “actually [when] \( x \) goes to \( \pm \infty \), this [function] goes to zero” and finalized \( f(x) = \frac{(x + 2)}{(x + 7)(x + 2)} \) as his function. Henry wrote “\( \frac{x + 2}{(x + 2)(x + 7)} \)” to account for the VA and for the hole. Then he modified the answer to \( \frac{x(x + 2)}{x^2(x + 2)(x + 7)} \), by stating that
when \( x \) approached infinity the limit will be zero. Henry then changed his answer back into \( \frac{x + 2}{(x + 2)(x + 7)} \), while the previous function was still acceptable. Hayden’s work was similar to that of Henry’s except that Hayden sketched the graph before writing its equation.

Tina knew that the function was undefined both at the VA and at the hole and correspondingly the denominator would be zero for those values of \( x \). She placed \((x + 2)\) and \((x + 7)\) in the denominator of her function. She started thinking through “the x-axis as the horizontal asymptote” condition by examining \( \frac{?}{(x + 2)(x + 7)} \). She paused, and wrote \( \frac{(x + 2)}{(x + 2)(x + 7)} \). In terms of limit at infinity, Tina examined \( \lim_{x \to \infty} \frac{x + 2}{x^2 + 9x + 14} \) by dividing both the numerator and the denominator by \( x^2 \), and stated “yes, horizontal asymptote, \( y = 0 \).”

In problem 24 (see Table 5.21), Anna wrote \( \lim_{x \to \infty} \frac{x + 2}{x(x + 2)} = \lim_{x \to \infty} \frac{x + 2}{x^2 + 2x} = \frac{2}{7} \), when?

Then changed into \( \frac{-2}{7} \frac{(x + 1)(x + 2)}{x^2 + 2x} \rightarrow g(x) = \frac{-2x^2 + 3x + 2}{7x^2 + 2x} \) by stating that

<table>
<thead>
<tr>
<th>CORI 2. Problem 24</th>
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</thead>
<tbody>
<tr>
<td>24. Write the equation or parts of the equation of a rational function ( g(x) ) if</td>
</tr>
<tr>
<td>(a) ( x = 0 ) is a vertical asymptote of ( g(x) )</td>
</tr>
<tr>
<td>(b) the function has a hole at ( x = -2 )</td>
</tr>
<tr>
<td>(c) the horizontal asymptote of this function is ( y = -2/7 )</td>
</tr>
</tbody>
</table>

Table 5.21. CORI 2, Problem 24
\[ \lim_{x \to \pm \infty} g(x) = -\frac{2}{7}. \]  I agreed with Anna that her function indeed has HA \( y = -2/7 \). I asked if this function had a VA \( x = 0 \), and a hole at \( x = -2 \). As she paused, I asked if \( -\frac{2x^2 + 3x + 2}{7x^2 + 2x} \) would factor back to her third expression. When Anna realized “it cannot be factored” she re-wrote the function as

\[ g(x) = -\frac{2}{7} \left( \frac{x^2 + 3x + 2}{x^2 + 2x} \right). \]

BlueJay first wrote \( \frac{x + 2}{x(x + 2)} \) and stated that the HA of this function was \( y = -2/7 \). He explained that \( \lim_{x \to \pm \infty} g(x) = -\frac{2}{7} \) and the function should be \( \frac{(x + 2)(-2/7)x^3}{x(x + 2)} \). He multiplied the terms out and found the limit at infinity to be -2/7. While solving this problem, Henry started out like BlueJay but was confused about having \( y = -2/7 \) for HA. After examining problem #7 according to my suggestions, Henry re-wrote \( -\frac{2(x + 2)^2}{7x(x + 2)} \) as his function. Richie started out like Henry, consulted problems 19 and 20 and wrote \( -\frac{2(x + 2)(x + 1)}{7x^2 + 14x} \) which he multiplied out to \( g(x) = -\frac{2x^2 - 6x - 4}{7x^2 + 14x} \). For Tina, the term in the function that constituted the VA \( x = 0 \) was not clear. She said that it is “weird” since she could not think in terms of plugging in “a number for x” in the denominator and watch it becoming 0. Finally, she figured out that \( x \) should go in the denominator for the vertical asymptote. Next, Tina examined if the function

\[ f(x) = \frac{x + 2}{x(x + 2)} = \frac{x + 2}{x^2 + 2x} \]

would have a HA \( y = -2/7 \). After amending the function to
\[ f(x) = \frac{-2x^2 + x + 2}{7x^2 + 2x} \rightarrow f(x) = \frac{-2x(x + 2)}{7x(x + 2)} \]  

Tina wrote \[ f(x) = \frac{-2x(x + 2)}{7x(x + 2)} \] and realized that this function does not have HA \( y = -2/7 \). She changed her equation, back to \( f(x) = \frac{-2x^2}{7x} \rightarrow \frac{-2x}{7} \) as the final answer. Tina was unable to simultaneously account for the various properties of these functions.

Richie stated “VA at \( x = 0 \) means just \( x \) in the denominator,” “hole at \( x = -2 \), so \( (x + 2) \) in the numerator and denominator, HA, \( y = -2/7 \), so \( g(x) = \frac{-2(x + 2)}{7x(x + 2)} \).” “We have \[ \frac{-2x}{7x^2} \rightarrow \frac{-2}{7x} \]; this is tougher” and looked back at problems 19 and 20. Richie continued “denominator \( 7x^2 + 14x \); we need another \( x \), but have to add something to \( x \).

CORI 2 problem 25 was to write the equation of a function \( h(x) \) that had \( b 
eq 0, b \neq 0 \), no horizontal asymptote, and \( h(-4) = 0 \). Anna thought through this problem as “\( h(2) = \frac{b}{0} \) meant \( x - 2 \) just in the bottom, \( h(-4) = 0 \) meant \( x + 4 \) top, and bottom, no HA meant \( \lim_{x \to \pm\infty} \) not a number.” and “the numerator has to be greater than the denominator.”

She wrote \[ \frac{(x + 4)(x + 2)}{x^2 + 2x + 8} \] by stating “in that way when it \([x]\) approaches \( \pm\infty \), it \([\text{function}]\) only approaches \( \pm\infty \).” Anna concluded \[ \frac{x^3 + 4x^2 + 2x + 8}{x^2 + 2x + 8} \] to be her function.

While answering problem 25, BlueJay wrote \[ \frac{(x + 4)}{(x + 4)(x - 2)} \] and proceeded to find limit
at infinity. Stating that the numerator has to have a higher degree than the denominator, he changed his answer to $\frac{(x + 4)x^2}{(x + 4)(x - 2)}$. He computed the limit at infinity again and finalized

$$h(x) = \frac{(x + 4)x^2}{(x + 4)(x - 2)}$$

for this function. At this point I asked BlueJay to explain what the form $h(2) = \frac{b}{0}, b \neq 0$, and $h(-4) = \frac{0}{0}$ meant to him in terms of the limit behavior of the function at $x = 2$ and $x = -4$. He examined the function $h(x) = \frac{(x + 4)x^2}{(x + 4)(x - 2)}$ and stated “as $x$ approaches 2, [due to the presence of] the VA [at $x = 2$], $h(x)$ approaches ±∞,” and “when $x$ approaches -4, okay, $x + 4$ will not be affected [since $x + 4$ will be cancelled out] so $h(x)$ approaches 16/-6.”

Henry wrote

$$h(x) = \frac{x + 4}{(x - 2)(x + 4)} \Rightarrow \frac{(x + 4)^2}{(x - 2)(x + 4)} \Rightarrow \frac{(x + 4)^3}{(x - 2)(x + 4)}$$

for this problem. However, Richie started with $h(x) = \frac{(x + 4)}{(x - 2)(x + 4)}$, and modified it into

$$h(x) = \frac{(x + 4)x^3}{(x - 2)(x + 4)}$$ and into $h(x) = \frac{x^4 + 4x^3}{x^2 - 6x - 8}$.

Later, Richie performed long division to examine the function behavior closely, and stated “if it [the function] doesn’t have a horizontal asymptote, then it should get closer to something else.” He realized that the function, as $x$ approached infinity was getting close to the quadratic function $x^2 + 2x + 4$. I restated what was discussed during the learning sessions about the possibility of oblique and curved asymptotes. We then graphed the above function on a graphing calculator and enjoyed how the graph positioned itself very close the parabola $x^2 + 2x + 4$ as $|x|$ approached large numbers. The graph of this function is pictured in Figure 5.8.
During CORI 2, problem 25, Tina wrote \( \frac{x - 4}{(x - 4)(x - 2)} \) and stated “no HA [horizontal asymptote].” She wrote a series of steps as shown here

\[
\frac{x - 4}{(x - 4)(x - 2)} = \frac{x - 4}{x^2 - 6x + 8}
\]

\[
= \frac{x}{x^2} - \frac{4}{x^2} = 0
\]

and commented “the way I have it now, there would have a HA at 0.”

Later she modified the equation starting from \( \frac{x - 4}{(x - 4)(x - 2)} \) to \( \frac{x^2(x - 4)}{(x - 4)(x - 2)} = \frac{x^3 - 4x^2}{x^2 - 6x - 8} \).

Figure 5.8. Graph of Richie’s Rational Function and its Non-linear Asymptote
Immediately after solving this problem, Tina started reviewing all of her observations regarding the making of horizontal asymptotes. At that point, to help her generalize her own findings, I created three problems and asked Tina to comment on how they behaved as $x$ approached infinity. In the first problem, $p(x) = \frac{3x^2 + 5x - 1}{2x^2 - 8}$, Tina stated “HA = 3/2, degrees are the same” while in the second problem $q(x) = \frac{3x^2 + 5x - 1}{2x - 8}$, Tina stated that “degree higher on top [numerator]” “[degree] 2 on top and [degree] 1 at the bottom and no HA.” In the third problem, which was $R(x) = \frac{2x - 8}{3x^2 + 5x - 1}$, Tina stated “degree higher at the bottom; so $y = 0$ for HA”

Until now, Tina did not seem to realize how the degrees of the numerator and the denominator contributed to the occurrence of horizontal asymptotes even though they were brought up during the learning episodes by her peers. She stated that during the learning sessions she was more focused on “how function terms were affected as $x$ got infinitely large and infinitely small”. Hayden and Neo wrote the function $q(x) = \frac{(x^2 + 1)(x + 4)}{(x + 4)(x + 2)}$ as their final product. Table 5.22 summarizes the cross-case study of the problems involving HA and limit at infinity during CORI 1 and CORI 2.

CORI 2 problems 17 through 22 will be discussed next. First I have displayed these problems and their significances in the table (Table 5.23) format.
Limits at Infinity CORI 1

<table>
<thead>
<tr>
<th>x</th>
<th>( \lim_{x \to \infty} \frac{1}{x+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anna</td>
<td>Direct substitution, limit 0</td>
</tr>
<tr>
<td>BlueJay</td>
<td>0, factored x out. reason: memory</td>
</tr>
<tr>
<td>Richie</td>
<td>None done</td>
</tr>
<tr>
<td>Henry</td>
<td>None done</td>
</tr>
<tr>
<td>Tina</td>
<td>Direct substitution, limit 0</td>
</tr>
<tr>
<td>Hayden</td>
<td>Sketched ( y = \frac{1}{x+1} ), limit = 0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>( \lim_{x \to \infty} \frac{2x^2 + 3x - 2}{2x^3 + 4x^2 + x + 2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( \frac{2x^2}{2x^3} \to \frac{1}{x} \to \frac{1}{\infty} \to 0 \cdot ) limit 0, HA: ( y = 0 )</td>
</tr>
<tr>
<td>Limit 0, eye-balling</td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td>( \frac{x}{x^2} \to \frac{1}{x} \to 0 )</td>
</tr>
<tr>
<td>Limit 0, followed by detailed work</td>
<td></td>
</tr>
<tr>
<td>Examined leading terms and found the limit to be 0, HA, ( y = 0 )</td>
<td></td>
</tr>
</tbody>
</table>

**Table 5.22. Cross-case Display of the HA and the Limits at Infinity, CORI 1 and CORI 2**

<table>
<thead>
<tr>
<th>CORI 2 problems 17 through 22</th>
</tr>
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<tbody>
<tr>
<td>Problem 17</td>
</tr>
<tr>
<td>If ( c(x) = \sqrt{4+5x}, x &gt; 9 ), find ( \lim_{x \to 9} c(x) )</td>
</tr>
<tr>
<td>Ability to compute limit from the right and left hand side using two different definitions</td>
</tr>
<tr>
<td>Problem 18</td>
</tr>
<tr>
<td>Is the function ( f(x) ) continuous at ( x = 9 )? Explain</td>
</tr>
<tr>
<td>Connection between limit and continuity. Is this function continuous because ( \lim_{x \to 9} c(x) ) does exist?</td>
</tr>
<tr>
<td>Problem 19</td>
</tr>
<tr>
<td>Find ( \lim_{x \to 9} \frac{3x^2 - 13}{x^2 - 9} )</td>
</tr>
<tr>
<td>How do students find infinite limits?</td>
</tr>
<tr>
<td>Problem 20</td>
</tr>
<tr>
<td>Find ( \lim_{x \to 9} \frac{3x^2 - 13}{x^2 - 9} )</td>
</tr>
<tr>
<td>How do they find limit at infinity VA, ( y = 3 )</td>
</tr>
<tr>
<td>Problem 21</td>
</tr>
<tr>
<td>Find ( \lim_{x \to 2} c(x) ), if ( c(x) = \frac{2x^2 + 3x - 2}{2x^3 + 4x^2 + x + 2} )</td>
</tr>
<tr>
<td>This is a finite limit problem.</td>
</tr>
<tr>
<td>Problem 22</td>
</tr>
<tr>
<td>Find ( \lim_{x \to 2} \frac{2x^2 + 3x - 2}{2x^3 + 4x^2 + x + 2} )</td>
</tr>
<tr>
<td>Limit at infinity VA ( y = 0 )</td>
</tr>
</tbody>
</table>

**Table 5.23. CORI 2 Problems, 17 through 22**
CORI 2 problems 17 and 18 solicited students’ knowledge of limits and continuity from the algebraic form of functions. In these problems students were to find the limit of a piece-wise defined function as x approached 9 and decide if this function was continuous at x = 9. These questions were compatible with questions 5, 6, 10 of CORI 1. The computation of limits without the aid of graphs was sought out in CORI 1, problem 7 and CORI 2, problems 19 and 20. During both interviews, the problems \( \lim_{x \to 9} \frac{3x^2 - 13}{x^2 - 9} \) and \( \lim_{x \to \infty} \frac{3x^2 - 13}{x^2 - 9} \) were included. CORI 1, problem 14 (b): \( \lim_{x \to 3} \frac{3x^2 - 13}{x^2 - 9} \), was also similar to \( \lim_{x \to 3} \frac{3x^2 - 13}{x^2 - 9} \) of CORI 1 and CORI 2.

Henry’s interpretation of continuity of functions during CORI 1 and CORI 2 were similar. He stated that if the limit existed at a point, the function will be continuous at that point. Tina, Neo, and Hayden answered the continuity questions in CORI 2 correctly. Richie answered the continuity problems correctly during both interviews. While solving \( \lim_{x \to 3} \frac{3x^2 - 13}{x^2 - 9} \) on CORI 2, Anna first substituted -3 for x to distinguish between the undefined form and indeterminate form. After confirming the presence of a VA at x = -3, Anna substituted x = -3.0001 as x approached -3 from the left and x = -2.999 as x approached -3 from the right. Based on the observation Anna concluded that as approached 3 from the left, the function approached \(+\infty\) and as x approached 3 from the right, the function approached \(-\infty\).

This time Anna used direct substitution to recognize the form that would help her discriminate between undefined and indeterminate forms. She seemed to be connecting undefined forms with the occurrence of vertical asymptotes and was using the function behavior by its VA to find limits. The steps Anna took to solve this problem speak to the fact that she has gained
understanding of the conditions under which vertical asymptotes occurred for a rational function. She was also able to state the directions in which the function was approaching infinite limits.

On the same problem, Henry realized that at \( x = 3 \), the numerator was non-zero while the denominator was zero. With that, he established that there is a VA at \( x = -3 \) and that the function will be approaching \( \pm \infty \). He explained that as \( x \to -3^+ \), \( x > -3 \), meaning that
\[
x^2 < 9 \Rightarrow x^2 - 9 < 0 , \text{ while } 3x^2 - 13 > 0 .
\]
Therefore, as \( x \) approached -3 from the right the function approached negative infinity. Similarly, Henry showed that as \( x \) approached -3 from the left, the function approached positive infinity. In addition, Henry too was able to tell that there was a VA at \( x = 3 \) since the numerator, \( p(-3) \neq 0 \), while, the denominator \( q(-3) = 0 \).

While solving this problem, Richie realized that when \( x = -3 \), the function did not assume the indeterminate form and correspondingly he was confident that there was no finite limit as \( x \) approached 3, giving rise to a hole. As he substituted 3.001 and -2.999 for \( x \), due to computation error, the function values appeared to be -1.168 and -1.165 respectively. It seemed as if \( x \) approached -3 from the left and from the right, the function was approaching a finite number which was approximately -1.17. Richie was puzzled while he exclaimed “there is a hole?” After realizing his error in calculation, Richie found the correct values and stated that “as \( x \to -3^- \), \( d(x) \to \infty \)” and “as \( x \to -3^+ \), \( d(x) \to -\infty \).”

This scenario revealed that Richie was indeed re-thinking his steps as he encountered inconsistent situations. Richie realized that without having the \( 0/0 \) form at \( x = -3 \), the function could not approach a finite limit there.

While solving \( \lim_{x \to \infty} \frac{2x^2 + 3x - 2}{2x^3 + 4x^2 + x + 2} \) on CORI 2, Anna related limit at infinity to a possible horizontal asymptote, and stated that as \( x \) gets very big, \( 3x -2 \) in the numerator and
$4x^2 + x + 2$ in the denominator would be negligible. She then wrote $\frac{2x^2}{2x^3} \rightarrow \frac{1}{x} \rightarrow \frac{1}{\infty} \rightarrow 0$ by stating that the limit is 0 in this case. Henry did not respond to the question $\lim_{x \to \infty} \frac{1}{x + 1}$ during CORI 1, while he performed extra work on similar problem on CORI 2. Table 5.24 summarizes the cross-case study of the problems involving VA and infinite limit in light of the results of the CORI 1 and CORI 2 interviews.
<table>
<thead>
<tr>
<th></th>
<th>Infinite Limits CORI 1</th>
<th></th>
<th>Infinite Limits CORI 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\lim_{{x \to -3}} \frac{3x^2 - 13}{x^2 - 9}$</td>
<td>Substituted 2 for $x$</td>
<td>Substituted-3, from $\frac{14}{0}$, convinced of $VA \text{ at } x = -3$</td>
</tr>
<tr>
<td></td>
<td>Limit undefined</td>
<td></td>
<td>Found $d(-3.0001)$ and $d(-2.999)$ To decide which infinity</td>
</tr>
<tr>
<td>Anna</td>
<td>Realized $VA$ at $x = -3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Said function went to $\pm \infty$, Only calculator can tell which infinity</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BlueJay</td>
<td>function will be undefined</td>
<td>$VA$ at $x = 2$ Did not know any more</td>
<td>$VA$ at $x = -3$ Found infinite limits</td>
</tr>
<tr>
<td>Richie</td>
<td>$\lim_{{x \to -3}} m(x) = \frac{14}{0}$</td>
<td>6/0, undefined But limit will be 6</td>
<td>“since the numerator wouldn’t factor we know there are no holes”</td>
</tr>
<tr>
<td></td>
<td>“14/0 is undefined but in a limit problem the limit will be 14.”</td>
<td></td>
<td>Found infinite limits</td>
</tr>
<tr>
<td></td>
<td>$VA$ at $x = -3$</td>
<td></td>
<td>Found $d(-3.0001)$ and $d(-2.999)$ to decide which infinity</td>
</tr>
<tr>
<td>Henry</td>
<td>Noted $VA$ at 3 and -3</td>
<td>None done</td>
<td>Realized $nr \neq 0$ while $dr = 0$ at $x = -3$ Confirmed $VA$ . Analyzed terms: $x \to -3^+, x &gt; -3$, $x^2 &lt; 9 \Rightarrow x^2 - 9 &lt; 0$ .</td>
</tr>
<tr>
<td>Tina</td>
<td>“zero in the denominator, feels like there is an asymptote there, -14/0”</td>
<td>DNE</td>
<td>$VA$ at $x = -3$ Found infinite limits</td>
</tr>
<tr>
<td>Hayden</td>
<td>functions undefined at 3 and -3</td>
<td>$\pm \infty$, referred to the previous problem</td>
<td>$VA$ at $x = -3$ Found infinite limits</td>
</tr>
</tbody>
</table>

Table 5.24. Cross-case Display of the VA and Infinite Limits, CORI 1 and CORI 2
Summary of Concept Images of Asymptotes, Limits, and Continuity CORI 2

During the CORI 2 interview, Tina was able to interpret asymptotes as limits. She identified horizontal asymptotes by computing limit at infinity by examining the function behavior as $|x|$ approached infinity. She did that by exploring the leading terms of the numerator and the denominator and substituting large values for $x$. After closely constructing the equations of functions with different types of HA during the second interview, Tina finally realized how limits at infinity could be determined by comparing the degrees of the numerator and the denominator.

It should be noted that this time Tina specified that it was possible for a curve to cut through its horizontal asymptotes as long as the end behavior of the function must be such that it approached the horizontal line as $x$ approached infinity. As far as relating to the limit format of vertical asymptotes, Tina needed more time to internalize the concept of the “function approaching $\pm \infty$” as $x$ approached “some number” However, once she realized where to find “some number” that $x$ approached, she was able to understand the problem.

While solving problems, Tina was eventually able to relate $b/0, b \neq 0$ form with vertical asymptotes and $0/0$ form with holes. She was able to construct equations of functions from specified properties even though she struggled during the process of construction especially when dealing with horizontal asymptotes of functions. Tina was able to summarize the properties of horizontal and vertical asymptotes elaborately during this interview. In addition, Tina was able to use the correct terminology and eventually her graphs were complete and accurate. Tina’s concept images regarding continuity had been altered in such a way that she now understood why a function was continuous at sharp corners and cusps.
Tina, once realized that her conceptions regarding mathematical topics that she believed to have understood with an A-level of test performance were incomplete with “holes” she was determined to learn mathematics for conceptual understanding.

Hayden found HA by computing the function limit as x approached positive or negative infinity. He also described how to examine the behavior of functions terms for large values of $|x|$ by examining the leading terms of the numerator and denominator. Unlike other students, Hayden did not directly replace x by infinity while examining the function behavior as x approached infinity. He continued that a curve could cut its HA as long as it approaches the asymptote as x goes off to positive or negative infinity. Hayden stated that vertical asymptotes occurred at the zeros of the denominator only if the function value assumed the $\frac{b}{0}, b \neq 0$ form. He realized that the function will have a hole at the zero/s of the denominator if the function takes on the form 0/0 at those particular x values. The terminology was intact during his explanations and he was able to construct functions based on the indicated properties and sketched accurate graphs in all problems.

Henry showed conceptual understanding of the conditions under which vertical and horizontal asymptotes occurred. He specified that it is possible for a HA to intersect a curve with the explanation on why it was possible. He was able to go back and forth between the graphs, limit forms, and the descriptive form (such as, a function has a HA at $x = 6$, etc) of asymptotes of rational functions. He was also able to distinguish between 0/0 and $\frac{b}{0}, b \neq 0$ form and decided which one of these forms invoked holes and which one invoked vertical asymptote. This time, Henry was able to construct rational functions from their stated properties.
Richie showed improvement in his knowledge of horizontal asymptotes, limit at infinity, and function behavior around its horizontal asymptote. He also explicated the circumstances leading to having a horizontal line other than the x-axis for a horizontal asymptote, having x-axis for a horizontal asymptote, and not having any horizontal asymptote at all. Even in the case of not having a HA he performed the long division to observe the function behavior as x got infinitely small or large. Richie made it a point to explain why it was all right for the curve of a rational function to intersect its horizontal asymptotes. Richie knew the difference between the undefined form and the indeterminate form and the function behavior corresponding to these forms. He was able to construct rational functions that held stipulated properties and was able to detail the process accurately.

Anna affirmed that vertical asymptotes are the same as holes and at the vertical asymptotes, the function goes to ±∞. While graphing functions she altered that statement by indicating that at the points where the function is “actually shifting” vertical asymptotes occur instead of holes. Further as Anna was solving more and more problems, she realized that if the factor that contributed to the zero of the denominator was cancelled, there will be vertical asymptotes at those zeros of the denominator. Anna is still not confident about the function behaviors around its vertical asymptote, or holes. For example, she emphasized that without any such cancellation there could be a hole or a VA occurring only if the function approaches ±∞. However, she explicated the function behavior around its horizontal asymptotes correctly and examined the leading terms of the numerator and denominator to find the equations of the horizontal asymptotes. Anna provided reasonable graphs for rational functions. Though struggled at times, Anna was able to construct rational functions that followed specified properties.

BlueJay, in addition to solving the denominator equal to zero, computed the limit of the rational functions at the undefined points to confirm the existence of infinite limits that
characterized vertical asymptotes. BlueJay identified holes by checking if the responsible term was a common factor between the numerator and the denominator. He was able to construct rational functions that possessed specified properties, though he struggled at times during the process. BlueJay first stated that the graph of a rational function must never intersect its horizontal asymptotes. He soon recanted that statement by stating that it is going to take him time to unlearn certain conceptions that he held previously. He explained why it was all right for a curve to intersect its horizontal asymptote. BlueJay was able to recognize the undefined form and the indeterminate form with an explanation of how these two forms affected the function behavior.

Regarding the concept of continuity, Tina had stated that function is discontinuous at sharp corners. During the teaching episodes, continuity was not discussed in depth as a prime concept; however, continuity was touched upon as a part of limits, and asymptotes. While pondering the concept of vertical asymptotes, students realized that only asymptotic discontinuity can cause vertical asymptotes. The limit discussion revealed that a function could have all real number domains and still has a discontinuity at a point. With this, students also realized that at the points where the left hand limit = right hand limit, the function did not have to be continuous. One problem that was asked during the second interview revealed that students understood these aspects of continuity of a function.

**Commonalities across Cases**

Many commonalities were noted across student concept images. Neo, Anna, Tina, and BlueJay conveyed that a rational function was a nice function like a rational number. These students possessed a very limited perspective of rational numbers. They believed that only *nice* numbers like $\sqrt{4}, \sqrt{9}$ are rational numbers and therefore they concluded that rational functions
must also be *nice* with no discontinuity or asymptotes. In the end, Neo, Anna, Tina, and BlueJay associated rational form with fraction form. Therefore, according to them, a rational function was any function that assumed the fraction form. They did not particularly focused on the necessity of the fraction form to be quotient of polynomials. Richie and Henry internalized all details of the rational form.

Concept images of vertical asymptotes possessed certain distinct characteristics. On the issue, where did the vertical asymptote occur, Henry, Anna, BlueJay, Hayden, Neo, and Richie were clear that it (VA) occurred at the point/s where the function was undefined. However, Anna believed that a rational function became undefined at the zeros of the numerator. Richie, Anna, Henry, and BlueJay believed that holes may also happen at the points where the rational function was undefined. Richie and Anna stated that both VA and hole could occur at the same time. None of these students noted any difference between undefined form and indeterminate forms.

While all used incorrect terminology in reference to limit behavior of functions, they also showed difficulty to interpret certain limit behavior as asymptotic characteristics. All students demonstrated lack of understanding regarding the details of horizontal asymptotes except for that HA are dotted horizontal lines that the curve did not intersect but got closer and closer to. Construction of a function from its known asymptotes posed a unanimous challenge while most of the difficulty was regarding the accommodation of HA in the equation.

In general these calculus 2 students seemed to possess elementary views of rational functions and their asymptotes. While these students had learned about asymptotes of rational functions since pre-calculus instruction, they did not even seem to realize that rational functions could have asymptotes. It appears that students take mathematics courses simply because they really have to. They did not seem to make connections between the concepts that they had to learn. It could also be that only certain rules and problems are viewed as important since they are
more emphasized during the examinations. Certain concepts might be ignored in favor of certain other procedures since knowing these procedures was all that mattered to obtain the correct answer to the problems to be solved.

In addition, definitions are seldom discussed during instruction. While textbook language could be sometimes confusing, students make their own interpretations based on their peculiar concept images as to what it meant by certain words such as approach or limit.

Students responded to the extra time and effort they were afforded during the teaching episodes by making conceptual understanding their main focus. In order to accomplish this the nature of mathematics had to be re-defined through the rules to live by as introduced during the first day of the learning sessions. Logical approach to the concept of functions where y, the output value depended on x the input value, therefore it makes more sense to start the limit statement by phrasing as x approached a, f(x) approached L rather than reversing the role of f(x) and x.

Unpacking the definition of asymptotes lead to multi-faceted discoveries for these students. They not only realized that a horizontal asymptote may intersect the graph but they also discovered why a vertical asymptote couldn’t intersect the graph. Moreover, they found that it was possible to draw asymptotes for non-function relations.

Regarding limits, corresponding to infinite limits and limits at infinity, Richie, Anna, BlueJay, Henry, and Tina performed direct substitution and reached the answer limits undefined, or limit did not exist. Without a graphing calculator they were unable to provide any further information. Once again, students lacked conceptual understanding of the limit concept while their focus was mainly process oriented. The issues noted with finite limit were concerning the graphical interpretation of limits. The word approach in the limit definition was taken by students as approached and reached. Correspondingly, students seemed to hold the conviction that limit at a point was the function value at that point. Investigating the deeper meanings of the definition
helped students understand that function behavior *around* a point is what it matters when it came to finding the limit of a function at a point.

**Student Models CORI 1 versus CORI 2**

Overall, my observations of students’ difficulties could be attributed to the *Innate Nature* of Mathematical Knowledge (INK), to *Student Convictions* (SUN), to *Instructional Features* (IF), and to *Other* (OR) reasons. In my view, INK contributes to the difficulties encountered while developing the concepts such as functions (function notation, for example), asymptotes (concurrency of asymptotes and functions), and limits (the limit notation, limit reached or not). In addition, the complications that arise due to the dual nature of mathematics, the *process* versus *concept* divide will also be included in the INK category. However, due to the fact that this issue could be made explicit and tended to during the curriculum setting and instruction, I may include the difficulties caused by the dual nature of mathematics into the IF category.

I attribute student beliefs such as *anything could happen in math or nothing is certain in math*, to the SUN category, while the IF category deals with the issues with curriculum, textbook language, textbook examples, and the methods and goals of teaching. Student comments such as *I was always taught limits by the use of a calculator* or *I never had to explain my methods before*, will also be included in the IF category. The OR category will include issues with the lack of basic knowledge, lack of confidence and such.

Most of the difficulties that Neo had could be attributed to the OR category. This included not knowing the rational form or not knowing the difference between the indeterminate and undefined forms, and so forth. Other than that, Neo was able to relate to asymptotes from the limit form and once a concept was explored and understood she was able to maintain the understanding most of the time without falling back to the old habits of mind. Neo was unable to write the corresponding terms of the equation in relation to HA and while she was able to
correctly perform the procedure and find limits at infinity, she still admitted not knowing why $y = 0$ was the HA of the function she was exploring. This particular difficulty that Neo experienced could be attributed to the lack of connection between the process of computing the limit at infinity and the concept of the behavior of the function as $|x|$ got bigger and bigger. In fact many other difficulties such as not being able to explain the behavior of a function around its asymptotes and not being able to describe the function behavior from the limit format could be traced back to the lack of flexibility to translate between the process – concept configure. This could also be interpreted through the lens of multiple concept representations. I will be attributing such difficulties to INK as well as IF. Once Neo was able to re-accommodate the fact that horizontal asymptotes are horizontal lines the graph approached as $x$ approached positive or negative infinity, Neo was able to understand why the limit at infinity, which was zero for the function in discussion was a HA of that function.

**Horizontal Asymptote and the Issue of ‘Approach’**. All students believed that the term *approach* that was commonly used to define asymptotic properties meant *never intersect*. However, Neo, and Henry stated that it was possible for a graph to intersect HA. At times BlueJay fell back into his old ways of thinking while he recovered from them eventually. The types of examples that all students except for Neo, and Henry were familiar with, did not lead them to believe otherwise. Even though, they believed that approach meant did not intersect, since they had seen graphs with HA intersecting them, they believed that a graph could intersect its HA. In this case, the visual experience of the graph they had seen helped him to overlook the interpretations of the term *approach* in the definition of asymptotes. However, they were unable to provide details. This could be due to the fact that beyond seeing graphs that intersected their asymptotes, these students did not need to ponder further on this situation due to the features of
the instruction or that they might have simply took this observation as one of the strange characteristics of mathematics where nothing was certain.

In addition to the examples illustrated in some textbooks and the problems solved during instruction, formal and/or informal definitions that are used by some textbooks could also create confusions as to what was implied by the term approach. During the teaching episodes, the group explored a textbook definition of asymptotes that they were familiar with. The complications that arose from the textbook ambiguity will also be considered in the IF category.

Hayden was one of the students who initiated new drawings to explore the ways in which the informal definition of asymptotes could be maneuvered. Refining the informal definition through examining why VA could not intersect the graph of its function was helpful for students to modify their belief that no asymptote could ever intersect its function graph. BlueJay was the brain behind the discovery of why graphs of functions cannot intersect its vertical asymptotes. This discussion was initiated by BlueJay’s statement during the CORI 1 interview that limits statements involving infinite limit and limit at infinity did not ensure the existence of asymptotes. His explanation centered on his belief that a curve cannot cross its horizontal asymptote. To clarify his conviction, he wrote \( \lim_{x \to \pm\infty} f(x) = 3 \) could also mean that the graph of \( f(x) \) could intersect the horizontal line \( y = 3 \). According to BlueJay, in that case, the horizontal line \( y = 3 \) would not represent a HA since the curve intersected the line. Next, BlueJay focused on the statement, \( \lim_{x \to -3} f(x) = \infty \), \( \lim_{x \to -3} f(x) = -\infty \), and sketched another graph. Eventually, BlueJay realized that as the curve kept approaching the vertical line after intersecting it once, the curve will no longer represent a function since it failed vertical line test. However, he still insisted that his argument regarding HA and concurrency of the curve was correct according to the definition of asymptotes that he was familiar with.
It took further group discussion and analysis for BlueJay to accept the fact that it was all right for the graph of a rational function to intersect its horizontal asymptote. While, the construction of definition for asymptotes was in progress, the group decided to re-state the informal definition as “an asymptote is a line the curve approached as one moved along the line. It is all right for a curve to cut through its asymptote as long as it comes back and approach the line.” Tina contributed to this new definition by stating that “even though the function values could be less, equal, or greater than b (y = b, the HA), eventually the limit should approach b.” She continued “output values could stray, but eventually they should come closer and closer to b.” Therefore, it was all right for the curve to intersect the horizontal asymptote. Henry made his contribution to this effort by stating that “as \( x \to \pm \infty \), the function must continuously approaches the [horizontal] asymptote.” Piaget’s (1970) description of equilibration seems to have happened in this case, and students were able to re-accommodate the concept of asymptotes in accordance with its formal definition.

Even though BlueJay was behind developing a new definition for HA, while graphing a function during the CORI 2 interview, he once again assumed that the curve cannot intersect a horizontal asymptote. After momentary confusion BlueJay realized that this curve will have to cut through its horizontal asymptote to fit with other properties described in the problem. BlueJay stated that, for him, learning this material is going to be a “slow process” since “he has to first unlearn everything he learned before.” Even after realizing concept modification, some times, people some time fall back to their previous concept images. This is an example of uncontrollable imagery, as elaborated by Aspinwall, et al. (1997), which is so strong that its persistent recurrence poses obstacles in student learning. In such cases, when the concept is in the transitional stage, unless a situation is brought up to their attention that causes disequilibria in their thinking, they may still not realize if they have fallen back to their previous conceptions.
Horizontal Asymptotes and Construction of Functions. All students struggled with accounting for the terms that were responsible for the HA of rational functions. In simple problems, Hayden, was able to account for horizontal asymptotes by invoking the vertical shifts of functions.

During the first interview, even though, both Neo and Hayden were eventually able to write vertical asymptotes using infinite limits, and horizontal asymptotes using limits at infinity, they were unable to use this concept to make sense of the asymptotes since they did not know how to find limits at infinity and infinite limits. This difficulty could be categorized mainly into IF category since the construction of functions is usually not a part of the routine problems solved during the instruction of asymptotes. Even without addressing these questions during the teaching episodes all students managed to put together function terms from the indicated asymptotes. During the teaching episodes, students examined the properties of asymptotes, their limit forms, and how each of the terms of a function contributed to these features of the graph.

By not discussing the problems on the construction of rational functions from the indicated properties of asymptotes, holes, and intercepts given in different formats in different problems, I was trying to look deeper into the nature of concept modifications that student have achieved. To be explicit, I wanted to examine if students are stating the modified conceptions merely from re-collection or not. I believe that rote recollection was not the reason behind students’ newer conceptions as portrayed by CORI 2 interview, since they were able to support their conceptions clearly and they were able to incorporate the new conceptions in diverse and novel problem situations.

Hole, vs. Vertical Asymptote. The distinction between a hole and a VA was another IF area in which all students deemed incomplete understanding. While a few students stated that at the point where a rational function was undefined a hole or VA could occur, they were unable to
elaborate on this. Similarly, students stated that the indeterminate and undefined forms implied only one thing: VA at the corresponding points. During the second interview, all except Tina acknowledged their understanding of the difference between these forms from a multiple perspective. For example, Richie realized that function will have infinite limits by its vertical asymptotes and finite limits by its holes. While solving a problem during CORI 2 interview, based on the undefined form the function assumed at a point, Richie stated that the function had a VA there. However, as he was finding the limit at infinity of this function, due to erroneous calculation, it appeared as though the function was approaching a finite number. Richie realized that the undefined form and finite limit cannot go together, and he re-calculated the limit and re-established that the function was in fact approaching infinity. This scenario reveals that Richie was indeed mindful of his observations while solving problems and he was trying to paint a bigger picture of the properties of functions while exploring them.

In problem 25, students were specifically instructed to write the equation or parts of the equation of a rational function $h(x)$ if (a) $h(2) = \frac{b}{0}, b \neq 0$, (b) the function has no HA(c) $h(-4) = \frac{0}{0}$. While solving this problem, Tina, who had experienced difficulties with indeterminate and undefined forms in a previous problem, was able to re-connect to the implications of these forms.

During the teaching episodes, I presented the function $K(x) = \frac{(x^2-1)}{x+1}$ and its simplified form $K(x) = \frac{(x-1)(x+1)}{x+1} = x - 1$. First I asked if $k(x)$ was defined at $x = -1$, and to explain why or why not? Later, I asked students to sketch the graph of $K(x)$ on a graphing calculator and examine if the function was defined at $x = -1$ using the zoom in feature. I also asked if $x = -1$ was a VA of the graph of this function. Neo, BlueJay, Anna, and Henry first stated that $K(x)$ was defined at $x$
= -1 since “once simplified, the domain was not restricted.” After examining the graph of this function on the graphing calculator, Neo, BlueJay, and Henry recanted that the function was undefined at \( x = -1 \) and that it did not have a VA at \( x = -1 \). They also discovered that the limit of \( K(x) \) as \( x \) approached -1, was -2.

Anna, even after realizing that “\( x = -1 \) is not a VA [of] \( K(x) \),” and that the “function was approaching -2”, stated that the function “still looks continuous [at \( x = -1 \)].” She stated that since it looked continuous at \( x = -1 \), it must be defined at \( x = -1 \). It seemed like according to Anna’s conceptual model, a functions is undefined at a point \( x = -1 \) if and only if the function had a VA at \( x = -1 \) and therefore, the conflict presented by the graph of \( K(x) \), (while zoomed in) was not enough to modify her beliefs. To Anna, \( \frac{x^2 - 1}{x + 1} \) and \( x - 1 \) are just two algebraic entities that are the same. In algebra courses, during the simplification of expressions, students identify these expressions as equal. Therefore, even at the calculus level students hold on to the conception that expressions like the ones discussed above have the same properties. In this case, Anna has trivialized the process of simplification as the concept that shaped the functions represented by \( a(x) \) and \( K(x) \).

However, while comparing the function \( K(x) = \frac{x^2 - 1}{x + 1} \) with another similar function \( a(x) = \frac{3x - 1}{x + 1} \), Anna re-stated that similar to \( a(x) \), \( K(x) \) will also be undefined at \( x = -1 \). This time she was able to generalize her that \( x = -1 \) can only be a VA if the numerator of the function remains non-zero when the denominator becomes zero at \( x = -1 \). For Anna process of equilibration was easier with certain representations; seeing \( K(x) = \frac{x^2 - 1}{x + 1} \) and \( a(x) = \frac{3x - 1}{x + 1} \) side
by side, helped Anna realize that function $K(x)$ even after the cancellation of the factor $x + 1$ will be undefined at $x = -1$. These difficulties could once more fall under IF category.

During the teaching episodes even after realizing the flaws in their conceptions, students at times, fell back into their old habits of thinking. To provide another example, for the function,

$$
\alpha(x) = \frac{3x^2 - x - 10}{2x^2 - x - 6}, 
$$

which he factored as

$$
\alpha(x) = \frac{(3x + 5)(x - 2)}{(2x + 3)(x - 2)},
$$

Henry sketched a VA at $x = 2$, and placed a hole on the asymptote. He later stated that he identified undefined-ness at $x = 2$ with vertical asymptotes and the common factor feature of $(x - 2)$ with hole at the same time, while placing a hole on the vertical asymptote. Later, he found the limit of this function to be $11/7$, as $x$ approached 2, and then realized that $x = 2$ cannot be a vertical asymptote. In this case, I believe that more than one concept image was invoked at once, on two related concepts (VA and hole). Consequently confounding occurred and until the student was confronted with disequilibrium (not having infinite limit at $x = 2$), he did not realize the slip in his thinking.

**Finding HA, computing limits at infinity.** Finding the equation of HA as well as finding the limit at infinity were other two aspects everybody experienced difficulty with. Anna, Tina, Richie, and BlueJay were unable to connect the concepts of HA with limit at infinity. Henry had a procedure bound approach to mathematical problem solving in general. BlueJay and Henry were able to find the equation of HA of a rational function when the degree of the numerator and the denominator was equal. They started remembering parts of some rules but could not explain the rationale behind it. Henry associated HA with a horizontal line the graph approached. BlueJay, Richie, Tina, and Anna were unable to recognize HA from the corresponding limit form. During teaching episodes, several problems were discussed to clarify the concepts of HA and limits at infinity. The rule involving comparing the degrees of the numerator and the denominator was not explicitly brought up, however, students started to recall this rule and they associated this rule
with the detailed process of finding limits at infinity and HA. Problems that involved numerators and denominators of various degrees where explored in detail. Some students pursued HA and limit at infinity by dividing each term of the rational function by the appropriate term, while some other students examined the leading terms of the numerator and the denominator to conclude the function property as $x$ approached $\pm \infty$. The difficulties associated with finding the equation of HA and finding the limit at infinity could partly be due to the innate nature of mathematical knowledge. Notably the process involved while solving problems involving these two concepts are not something that was used in everyday algebraic procedures (Yerushalmy, 1997). The lack of emphasize on the specialty of these procedures and how they are connected with the concepts under investigation provides an IF characteristic to student difficulties associated with the concept of HA and limit at infinity.

At times what a concept or a rule related to that concept conveys will be different from what the student internalizes. While solving problems involving HA and limit at infinity, Anna experienced prolonged difficulty with this concept and the methods of solving related problems. Anna was listening to others’ explanations and at times asked for further clarifications. Eventually she was able to focus on the leading terms of the numerator and denominator to identify the HA and the limit at infinity successfully. However, in the problem, that featured Melvin the imaginary student’s work, on $a(x) = \frac{6x^3 + 7}{11x^2 - 49}$, Anna commented that “HA must be $y = 6$ since the degree of the numerator is greater that the degree of the denominator, the leading coefficient of the numerator should be taken for the HA.” Even though, this reasoning is against the conventional notion, to Anna, this is her reality when it comes to the shortcut of comparing degrees to identify HA. Thus, I believe that errors and difficulties do not occur at random and there seem to be certain regularities to them (Batanero, et al., 1997).
In Anna’s case, the moment she decided to use the shortcut of applying a rule, she simply invoked the rule as she heard it without trying to find the connection between why the rule made sense. This could be considered as one of the obvious and inevitable drawback of rote rule-bound mathematics teaching and learning. In this case, telling Anna that her rule was incorrect might not help her alter her reality of the rule. Therefore, I asked Anna to find the limit at infinity by using the method that she used in previous problems. Then I asked her to explain what the implications of the degree of the numerator being larger than the degree of the denominator might be. In this way, Anna was able to develop the understanding of how the rules of comparing degrees actually worked in finding limits at infinity and equations of HA.

During the teaching episodes, even though the comparing degree rule to identify horizontal asymptote was discussed by some students, Tina insisted on performing the longer method of examining each of function terms to decide what type of HA did the function have. During the CORI 2 interview, after finding the horizontal asymptotes of different types of functions, Tina has discovered the rules (that she did not use during the teaching episodes) on her own. Tina seemed to have reached equilibrium with the logic of the rule of comparing degrees to find horizontal asymptotes and making sense out of this procedure at a level that is acceptable to her. This serves as another excellent example of how autonomy is essential for the learners to consider pre-established mathematical knowledge as credible for them.

To help students understand the concept of infinite limit I encouraged them observe the connections between infinite limit and the asymptotic property of the function. To help them recognize the occurrence of infinite limit, I encouraged them to inspect the function value at the point where limit was computed. Students realized that corresponding to the undefined form infinite limit existed. The next issue was to identify the direction in which the function was approaching infinity. Though struggled in the beginning to establish which infinity the function
was approaching in relation to VA., during the second interview, students were able to narrow
down the type of infinity from diverse perspectives.

During the first interview, Henry acknowledged that if a limit existed at a point, the
function will be continuous at that point. Henry and Tina seem to have the strong conviction that
if the limit was to exist for a function at a point, the function must be defined at that point.
According to them, with removable discontinuity the limit was the function value, while with a
hole in the graph the limit will be the y-coordinate of the missing point. Henry stated that it is not
possible for the function to have a limit at a point while it was discontinuous at that point. Tina
stated that limit cannot exist at sharp corners. BlueJay and Neo just could not imagine the
function having a limit at a point different from the function value. Activities referred as plot 1,
and plot 2, during the teaching episodes were intended to help students understand the concept of
finite limits better. At this time, analyzing the textbook definition, examples, and making note of
the lack of the condition that the function must be defined at a point in order for the limit to exist
at a point helped students re-interpret the concept of finite limit.

During the teaching episodes, students often struggled with the characteristics of
mathematics (INK) that they deem as “anything could happen” and “nothing is certain in math”
(SUN). For example students pointed out a situation where an initial indeterminate form will later
turn out to be undefined form once common factors are cancelled implying that indeterminate
form doesn’t necessarily warrant finite limits and a hole. While graphing functions, each student
tried to focus on the concept with which they struggled most. During the second interview, even
though some students had to struggle to put together all of the function properties, they eventually
demonstrated conceptual refinement and sketched the graphs correctly. While graphing functions,
Richie was mainly using plotting points. However, he was careful to choose enough points to get
a clearer picture of function properties such as its asymptotes and intercepts. Eventually Richie
was able to identify all asymptotes first, and then used plotting points to further guide his graph. Thus, during the teaching episodes, Richie made an effort to focus on one thing at a time, and reached conclusions on his own by using the methods that he was most comfortable with first, and then incorporating what he established already with the unfamiliar methods of solving problems.

Many of Richie’s struggles could be attributed to the didactical aspect of mathematics learning. For example, he needed time to use the method he liked most, the method of plotting points, in each problem before he was willing to explore the connection between his method and other quicker methods. Watching his peers solving problems a lot faster was probably a motivation for Richie to examine the connections between his method of plotting points and others methods such as simply examining the leading terms of the numerator and denominator of the rational functions. In addition, during the teaching episodes, Richie seemed to worry about the correctness of his answer. He was concerned that in his group, he might be the only one getting a wrong answer. However, when he solved problems during both interviews, he showed more confidence and was not hesitant to speak up what he thought of a problem situation. Compared to Tina, or Anna, Richie did not fall back into his old ways of thinking and was diligent while employing the modified concept configuration while solving problems.

Rules without reasons were what Henry was following in most of his mathematics courses. He did not know how to compute limits without a calculator and he told me that nobody ever asked him to explain how things worked while finding the answer of a mathematics problem. I realize some didactical issues and some ontological issues in relation to Henry’s mathematical understanding. Due to the didactical experiences that he had, he believed that the ontology of mathematics was rule without reasons.
Continuity and Limit Connection. During the CORI 1 interview BlueJay and Anna stated that if the domain is all real numbers, then the function have to be continuous everywhere in its domain. They then sketched a continuous curve to prove their point. However, Tina realized that a function with all real number domains doesn’t necessarily have to be a continuous function and she demonstrated it through sketching a diagram. Richie stated that a function with all real number domains doesn’t necessarily have to be continuous since we have no knowledge of its range. Hayden believed that a function with all real number domains must be a continuous function.

Throughout the teaching episodes, the teacher-researcher was mindful to accommodate the six assumptions that were originally set. These assumptions as described earlier in this chapter were that student’ concept images were influenced by their previous exposure and experience with the concept, students actively construct their own mathematical knowledge and active assimilation, accommodation and equilibration are imperative for the full development of these conceptions. Further it was assumed that equilibrium can only be maintained through the active encounter with disequilibrium posing situations, and social interaction and argumentation are essential features of maintaining conceptual equilibrium. It was also assumed that teachers could intervene and pose cognitive confusion by invoking contradicting problem situations and by the user of appropriate examples. Students’ attention could be re-directed towards formal definitions by initiating the unpacking of definitions as a regular classroom practice and by probing a constant check between intuitions and formal definitions. In a situation where the second, third, fourth and fifth assumptions are realized, concept refinement might be possible as a part of instructional practices.

To accommodate for the first assumption, efforts were made by conducting the CORI 1 interview to identify students’ already existing concept images. Lessons were developed bearing
these concept images in mind. For the second assumptions, students were given ample opportunities to delve into the concepts discussed and verify and re-establish conceptions on their own. For the third assumption, carefully designed problems and multiple activities were included to confront and spot incomplete concept images. Students engaged in discussing and defending their thoughts and its products with their peers and with the teacher-researcher. The teacher-researcher often developed instant problems to challenge and engage students in further exploration of the concepts. With all five assumptions met, the teacher researcher was able to observe changes in students’ conceptual models. Some students still experienced difficulties with the concept of concurrency of HA and its function graph, confusion with VA versus holes, and how limits and continuity played in. Even then, as soon as they encountered conflicting situations they re-canted their earlier faulty convictions. This time, students were more focused on concepts rather than the procedures that were done to reach correct answer. The practice that they were exposed to during the teaching episodes in focusing on conceptual understanding helped them realize contradictions as they occurred and re-think what they had earlier stated.
CHAPTER 6

CONCLUSION

The purpose of this study was to examine college students’ concept images of rational functions, asymptotes, limits, and continuity and the connections that students may have formed among these closely related concepts. In addition, the impact of a teaching experiment that was designed to help students re-configure their incomplete concept images was explored. The teaching experiment also helped the teacher-researcher understand the ways in which students developed their conceptual models of the concepts of asymptotes, limits, and continuity of rational functions.

Data were gathered through two interviews and eight teaching episodes. The first interview was called the exploratory interview. This interview was conducted using the problem solving instrument CORI 1. The first interview lasted about two hours and its purpose was to identify student concept images of the concepts of asymptotes, limits, and continuity. Student concept images were then categorized based on the concepts studied. CORI 2, the research instrument for the second interview, was similar to that of CORI 1. The second interview is also called the evaluative interview, or the CORI 2 interview.

The findings of the exploratory interview were used to help develop the lesson plans for the teaching episodes. The purpose of the teaching episodes and the evaluative interview was to facilitate conceptual change by helping students modify their cognitive structure. The teaching
episodes in particular helped provide students ample opportunity to work collaboratively with others. During these episodes, students first engaged in independent problem solving. Afterwards, they came together, discussed, and defended their work. They listened to others’ explanations and made changes in their work only if they understood the rationale behind the change that they needed to make. During this process, the researcher was observing student interactions, and based on the observations more effective strategies were explored to further assist with more meaningful concept formation.

All teaching episodes were video-taped and analyzed. The activities that appeared to facilitate necessary concept modification were identified and categorized during the analysis. The categories were cognitive dilemma caused by technology, cognitive dilemma posed by problem situations that contradicted and challenged student existing conceptual model, and cognitive dilemma posed as a result of questions raised by others.

The items on CORI 2 were similar to the items on CORI 1 in the concepts evaluated and in the type of questions asked. The data collected during the evaluative interview were used to analyze if and how concept modification occurred as a result of the cognitive exercises of the teaching episodes.

The exploratory interview or the first problem-solving interview revealed that students’ concept images of rational functions fell mainly into three categories. They were the rational number image, the fraction image, and the discontinuity image that were elaborated in chapter 5. The common traits among students’ conceptions of asymptotes in general could be categorized into classic three-piece graph image and invisible line image. The classic three-piece graph image included a three-piece graph that was symmetrical about the Y-axis with two vertical asymptotes and one horizontal asymptote.
Regarding the concept of vertical asymptote (VA), students generally believed that there was some form of undefinedness associated with this concept. However, several students were unable to provide explanations as to when a rational function was undefined, or identify the characteristics of a rational function at the points of undefinedness. For example, some students did not realize the difference between a hole and a VA and could not explain the function behavior by its VA. These students also did not know how to express a VA by using the limit notation.

Regarding horizontal asymptotes (HA), students believed that a HA could never be concurrent with the graph of its function. They were also unable to explain the behavior of the function around its HA, and did not know how to express a HA using the limit concept. Infinite limits and limits at infinity were challenging for many students to compute. In addition, they found no difference between the undefined form and the indeterminate forms. The concept of finite limits posed problems to those who believed that in all cases, the function value will be the limit value except when a function was undefined at the point where the limit was computed. In addition, I realized that some students believed that a function with all real number domains must always be continuous.

During the teaching episodes, students’ incomplete concept images were challenged through specially designed problems and worksheets. These resources were intended to create cognitive dissonance in students’ concept images. Often concepts were presented through problem situations that could not be solved using students’ existing notions. Piaget’s (1970) theory of equilibration and re-accommodation of inconsistent concepts laid the groundwork for the activities of the teaching episodes. In addition, as suggested by the theory of multiple representations (Goldin & Kaput, 1996), concepts were placed in different problem contexts so that students could gain multiple perspectives of the same concept.
For example, graphing certain rational functions challenged students’ existing conceptual schema. Group discussions and questions asked during these discussions prompted students to analyze and reflect on their methods and re-configure their conceptions. At times, graphing-calculator-generated tables of values helped create cognitive conflicts that required further investigation of students’ beliefs. Definition analysis, role-playing and critiquing problem solutions presented by imaginary student solutions were some of the other activities that posed cognitive conflicts in students. In addition, at times, horizontal asymptotes were presented as limit at infinity while vertical asymptotes were presented as infinite limits.

After the conclusion of the teaching episodes, another problem-solving interview was conducted (CORI 2, interview) to explore if and how students’ incomplete concept images had altered as the results of the teaching episodes. In most instances, students’ concept images had been altered and were in alignment with the conventional explanation. However, at times, some students held to their previous incomplete concept images. Aspinwall et al. (1997) brought up the theory of \textit{uncontrollable concept} images that impede students from assigning new meanings to their conceptual configurations. Tall and Vinner (1981) relayed that sometimes, more than one concept image is evoked simultaneously in regards to a certain concept. I believe that in that case, confounding of concept images contributes to students falling back to their default concept images even though they might feel a sense of discomfort with these images. Yet, in those instances, unlike previous times, students eventually realized that their conceptions were at odds with established mathematical norms. This happened as they encountered new problem situations that contradicted their convictions in the previous problem. I believe that this time, students’ focus had shifted towards conceptual understanding rather than merely producing a \textit{correct} answer.
During the learning sessions, the nature of mathematics had been discussed and students knew that conceptual understanding was the focus of the learning sessions. In addition, the types of questions asked during the CORI 1 interview helped students realize the deficiency in their understanding regardless of the *good* grades they were receiving in mathematics courses. Overall, it could be stated that students’ incomplete conceptual understanding had altered as a result of the proceedings of the teaching episodes.

**Changes in Student Conceptions**

After the CORI 1 interview, students gained a higher level of cognitive awareness of their own incomplete conceptions. During the eight learning sessions, students were given the opportunity to take their time, work alone, and then in groups to discuss the problems. Students were also made aware of the fact that solid conceptual knowledge was the goal of the learning sessions.

**Rational Functions**

By the time the CORI 2 interview was conducted, students started relating *a rational form* with *a fraction form*. All except 2 students were able to use the definition discussed to identify rational functions. They were able to distinguish between why a function such as 

\[
\frac{x + 11}{7}
\]

was not considered rational, while 

\[
\frac{7}{x + 11}
\]

was considered. In some cases, a function such as 

\[
\frac{x + 11}{x - \sqrt{7}}
\]

was selected as not rational due to the presence of the radical with 7. However, soon after stating that the presence of the radical sign with 7 would make the denominator non-polynomial, students re-stated that as long as there was *no radical sign with the variables*, the expression could be considered a polynomial.
During the discussion of rational functions, rational numbers were discussed. Students previously related to rational numbers as nice numbers such as $\sqrt{4}, \sqrt{9}, \sqrt{36}$. After the learning sessions, they realized that any number that could be written in the fraction form could be considered a rational number. Polynomial forms were also clear to students.

**Usage of Formal Mathematical Symbols**

While referring to limits, students were unable to use correct terminology and they kept switching the role of $x$ and $f(x)$. They settled this problem by realizing that in a function relationship, $y$ or $f(x)$, depends on the values of $x$, $x$ being the input or independent variable. By observing the graphs of rational functions with horizontal and vertical asymptotes, students were able to explain the function behavior around their asymptotes by the use of limit notation. From the graphs, they realized that if $x = a$ is a vertical asymptote of a rational function $f(x)$, as $x \to a^\pm, f(x) \to \pm\infty$. Similarly, students realized that the line if $y = b$ is a horizontal asymptote of a rational function $f(x)$, as $x \to \pm\infty, f(x) \to b$. While re-writing the horizontal asymptote of the graph they were examining, Richie particularly wrote, when $y = 3$ is a horizontal asymptote “$\lim_{x \to \pm\infty} a(x) = 3$” [$a(x) \neq 3$] by making it clear that $a(x)$ cannot touch or cross the HA, $y = 3$.

**Vertical Asymptotes and Undefinedness of Functions**

Students previously believed that a vertical asymptote (VA) always occurred at points where the function was undefined. After the teaching episodes, they realized that at points where the function was undefined either a hole or a VA could occur. It seemed to be clear to students that, VA’s occurred only when the limit of the function was approaching $\pm\infty$. Anytime the limit was a finite number, a hole occurred rather than a VA.
Graphing Using Merely Plotting Points

Students tended to graph a rational function solely by plotting points. However, they realized that plotting points guided by other function properties such as asymptotes, intercepts, and holes would help guide the graph accurately. While finding intercepts, some students who just focused on the procedural parts of intercepts were confused as to whether they should solve for the zeros of the numerator or the denominator. However, the conceptual features of intercepts, the fact that x-intercepts sit on the x-axis, and y-intercepts sit on the y-axis helped students re-connect with how to find the intercepts.

The Indeterminate form and the Undefined Form

Students believed that both forms \( \frac{0}{0} \) and \( \frac{b}{0}, b \neq 0 \) meant the same. However, after examining the function values around these forms students realized that if around \( x = a \), with 
\[
f(a) = \frac{0}{0}
\]
the function had a hole while at \( x = a \), if 
\[
f(a) = \frac{b}{0}, b \neq 0
\]
the function had a VA. Further they learned that the \( \frac{0}{0} \) form was called the indeterminate form, while \( \frac{b}{0}, b \neq 0 \) was called the undefined form.

Finite and Infinite Limits

Computation of limit posed a big challenge for students. They mainly depended on algebraic processes such as factoring, direct substitution, and simplification. Otherwise, they needed a graphing calculator to examine the graph and decide the limit based on the graph. However, students seemed to have gained the ability to view limits from a conceptual perspective. They believed that examining the function value or the form that the function
assumed at a point could be used as a viable model to narrow down whether the limit was finite or infinite at that point.

While dealing with limits at infinity, students were able to apply multiple methods to decide which infinity the function was approaching. These methods were creating a table of values, examining function values at two points from the right hand side and from the left hand side each, and examining the function terms.

**Limit at infinity**

Finding the limit at infinity and finding the equation of a horizontal asymptote appeared sensible to students. They were able to find the connections between different methods that examined the function behavior for large values of x. These methods were examining just the leading terms of the numerator and the denominator, examining each terms in the numerator and denominator by dividing each term algebraically, and by using the rules of comparing the degrees on the numerator and denominator. Students were able to understand the concepts behind each of these methods as well. During the teaching episodes, while finding the equations of HA or finding a limit at infinity, not all students were interested in pursuing the rule related to degrees of the numerator and the denominator. During the CORI 2 interview, after answering such questions using other methods, these students inspected their work and developed the rule comparing degrees on their own.

**Finite Limits**

Students realized that in order for a function to have a limit at a point, the function need not have to be defined at that point. They conferred that with reference to the limit of a function, only the behavior of the function around the vicinity of that point mattered.

**Continuity, Limits and Asymptotes**
Students were able to relate to the concept of discontinuity from multiple perspectives. They related the non-existence of limits with a jump discontinuity and the absence of a VA, while the removable discontinuity was connected with a finite limit existing at a point where the limit was different from the function value. They realized that there was no VA at those points either.

Construction of Function Terms from Stated Properties

During the teaching episodes, construction of a rational function based on pre-set properties was not discussed. Because, in order to construct a function based on multiple predetermined properties, students need to manage and balance concepts from multiple perspectives. All seven students were able to correctly construct functions of all problems asked during the CORI 2 interview. While some students constructed equations relatively easy, some others struggled to manage all indicated properties.

One initiative that was unique to my research was the development of the re-definition of asymptotes. While exploring the dilemma of whether an asymptote could cross the graph of its function, students examined the characteristics of asymptotes in detail and made several amendments to the informal textbook definition that they explored.

Re-Definition of Horizontal Asymptotes

At the end of the teaching episodes, students unanimously agreed that a function is allowed to cross its HA. This realization transpired during the examination of the textbook definition, “informally speaking, an asymptote of a function is a line that the graph of the function gets closer and closer to as one travels along that line.” This definition was first examined in regards to horizontal asymptotes. Diagrams were drawn on the board and were explored in detail. These diagrams are reproduced below. The dotted line represented the possible horizontal asymptote of the function.
Students challenged Figure 6.1 since as $x$ approached negative infinity, the function was not approaching the horizontal line, and while others pointed out that, the definition does not specify the need for the curve approaching the horizontal asymptote from both ends. They agreed that even though the graphs they are familiar with did approach the horizontal line as $x$ approached positive infinity and negative infinity, this condition is not required according to the definition.
Figure 6.2 was considered “flawless” since the graph did approach the horizontal line as $x$ approached positive infinity and negative infinity. However, students vigorously disagreed with figure 6.3, since, in this figure, the graph intersected the horizontal asymptote. Students stated that a curve can only approach the asymptote and cannot intersect.

One student pointed out that even after a curve cut through its horizontal asymptote, the curve could come back to the line and keep approaching it. Henry made the statement “It is the end behavior of the [rational] function that matters. It [the curve of the rational function] could cross the horizontal asymptote as many times as it liked, but in the end, it should approach the horizontal line.” However, before committing to a new definition they explored Figure 6.4 and decided that in this case, the curve stopped approaching the horizontal line as soon as it cut through the line.
The unanimous decision was that the definition needs to specify that *the curve is allowed to cut though the horizontal asymptote as long as it comes back and approaches the horizontal line or that the end behavior of the function should be such that it must approach the asymptote.* Students re-iterated that “even though the function values could be less, equal, or greater than b (y = b, the HA [horizontal asymptote]), eventually the limit should approach b.” She continued “output values could stray, but eventually they should come closer and closer to b.”

Next, students wanted to know why a curve cannot intersect its VA. They analyzed Figures 6.5 and 6.6 and realized that after cutting through the graph if the function would come back to the vertical line the vertical line test would fail.

Figure 6.5. “Asymptotic Behavior Preserved, but, not a Function”
They explored a graph such as the one in Figure 6.7 and accepted it as a possibility for HA to occur.

As the exploration and refinement of the informal definition of asymptotes concluded, consensus were formed that an asymptote is a line the curve approached as one moved along the line. It is all right for a curve to cut through its asymptote as long as it comes back and approach the line and this definition was collectively passed as an acceptable definition.
Student Attitude Change

What I consider as one of the important outcomes of this research was the changes in student attitudes that were demonstrated during the CORI 2 interview. Instead of giving a quick answer and moving on to the next problem, students thought through what they considered as the solution. The progression of equilibration between students’ notions during the CORI 1 interview, during the teaching episodes and during the CORI 2 interview was significant. During the CORI 2 interview, students wanted to double check their answers from multiple perspectives. After answering a question, students went back to a previous related problem to compare and contrast.

To provide examples of the scenarios in the former paragraph, while deciding that a function has a VA at a particular point, some students examined the limit of the function as $x$ approached the value in question. While graphing a rational function, a student stated that a function has a VA at a point. While figuring out which infinity this function was approaching, the student thought the function was approaching a finite limit. So he re-established that the function was just having a hole, not a VA at the point in question. While constructing functions from multiple specified properties, students were very careful to verify several times that the equation that they constructed had all of the properties indicated in the problem.

I believe that a number of things were most influential to the concept modification during this research. Even though these students were passing the course with A, B, or C grades, during the CORI 1 interview, they realized that there were gaps in their conceptual understanding since they couldn’t answer many of the questions on CORI 1. The teaching experiment was conducted to help students make modification in their concept images. At the beginning of the learning sessions, rules were set as to what was the goal of the learning sessions. These rules were: do not try to memorize every single rule or fact you may come across; mathematics does not comprise simply of memorization; mathematics is a problem-solving art, not just a collection of facts; do
not give up on a problem if you cannot solve it right away, re-read it thoughtfully and try to understand it more clearly; struggle with it until you solve it (Stewart et al., n.d.). Students took these rules seriously and tried to follow them as much as they can.

During the discussion of problems, when they didn’t completely understand some things students made comments like, “I am not convinced,” and “what is wrong with this method.” During the eight teaching episodes, students’ concepts were challenged by asking for explanations and posing conflict causing problem-situations in an atmosphere where students were active participants of knowledge construction. Creating an atmosphere where students were actively constructing knowledge was crucial. Theories of conceptual change re-iterate the importance of active knowledge construction (Beeth, 1989; diSessa, et al., 1993; Aspinwall & Miller, 2001). During active knowledge construction students will get the opportunity to be more cognitively aware of the endeavors they encounter.

Another important point was that students need to study concepts in diverse problem contexts. This could be achieved through using non-traditional problems that require thinking, and reflecting beyond rote learning and process focus (Grey & Tall, ; Aspinwall & Miller, 2001; Goldin & Kaput, 1996). During the teaching episodes, I tried to include analytic, graphic, and numeric representations. According to Aspinwall & Miller, in calculus courses, this method helps with the understanding of basic concepts as well.

To analyze how an incomplete concept was enriched, I specifically looked for cognitive dilemmas created by problem situations that could not be solved using students’ existing conceptions. I also asked each student to analyze the mismatches between different student-conceptions. At times, students were asked to critique the problem solution and corresponding problem methods that were used by the researcher, while the researcher role-played as another student. Re-exploring and re-configuring conflict causing textbook definitions and having to
critique imaginary students’ solutions were some of the other activities that were included in the learning episodes.

**Conclusions and Discussion**

Previously researchers have attributed student difficulties with mathematics learning to a number of features. Some of those features included *the innate nature of mathematical knowledge* (Moru, 2006; Cornu, 1991; Sierpinska, 1996; Eisenberg, 1991), limitations of curriculum, instruction, and the type of textbook used (Szydlik, 2000; Sajka, 2003; Moru, 2006; Cornu, 1991; Williams, 1991; Juter, 2005; Schoenfeld, 1992; Przenioslo, 2004, Davis & Vinner, 1986). Some aspects include *student convictions* held regarding mathematics knowledge (Schoenfeld, 1992; Szydlik, 2000; Sajka, 2003) and the process-product dilemma, or the *proceptual divide* as discussed by Grey and Tall (1994). By combining and re-organizing some of these findings, I developed categories, INK, IF, SUN, and OR to identify student difficulties associated with acquiring mathematical concepts.

The acronym INK stands for the Innate Nature of mathematical Knowledge, while IF stands for Instructional Features. The category SUN represents StUdent coNvictions while OR stands for the lack of mathematics knowledge base essential to the concepts studied in this research. In the INK category, I included epistemological difficulties that were unique to the subject of mathematics including difficulties with symbols, mathematical terminologies, and the proceptual characteristics.

Student difficulties associated with the concepts of limits and asymptotes were part of the struggle that the mathematicians faced during the development of these concepts by numerous re-definition and negotiation. While these difficulties could be placed in the INK category, they could also be placed in the IF category. The terminologies such as *approach, limit, and limiting*, used in the definition of these concepts further complicated the internalization of these concepts.
by invoking other familiar concept images such as *approached, approached but never reached, never-ending, and cannot be passed* (Cornu, 1991.)

There was hardly any research that explored student understanding of what contributed to the rational form of rational functions. Even though researchers have studied student understanding of functions (Clement, 2001) and the properties of rational functions such as limits and continuity (Louise, 2004), none that I knew tried to discover student conceptions of rational functions. Of course, all student difficulties could be attributed to the limitations of instruction, but students viewing a rational function and a rational number as *nice* and *whole* entities seemed to have stemmed from their interpretation of the meaning of the word *rational as nice* and *reasonable* and therefore could also be placed in the INK category.

The assumption of no concurrency between the graphs of rational functions and asymptotes was noted by Yerushalmy (1997). The belief that corresponding to holes there too was a VA was all noted by Bridgers (2007) in her dissertation research. She also noted the student notion that a rational function will have a VA corresponding to the indeterminate form. I found that these difficulties relating to the concept of asymptotes stemmed from what students remember from textbook graphical images, and what they remember specifically from the type of problems that were discussed in class during the instruction. During my research, I noted that a VA and a hole could occur simultaneously, and such entity will be called a *point asymptote*.

As noted by Yerushalmy (1997), specifically on the concepts of HA, I noted many students believing that a HA cannot intersect the function graph. They seemed to have made that conclusion from the way asymptotes are informally defined in textbooks and from the way in which asymptotes are informally introduced during instruction. In addition, textbook examples, as well as the problems solved during instruction seemed to have strongly conveyed the notions of non-concurrency between all asymptotes and graphs of corresponding functions. In some
problems, some students believed that functions could have horizontal asymptote (HA) at sharp corners such as the ones of an absolute value function.

In regards to ideas newly noted in my research, I realized that students thought that the point of intersection of a graph and a HA will be a hole. According to another student, the limit form \( \lim_{x \to \pm\infty} f(x) = 4 \) indicated HA, \( y = 4 \) for the function \( f(x) \), since the symbol \( \lim_{x \to \pm\infty} f(x) = 4 \) meant \( f(x) \) approached 4 but was never equal to 4. Another observation of students was the following: the limit statements \( \lim_{x \to -\infty} h(x) = 4 \) and \( \lim_{x \to \infty} h(x) = 4 \) do not ensure the existence of asymptotes since according to the limit notation, function \( h(x) \) could intersect the horizontal line \( y = 4 \).

Regarding limits, most students held the notion that in order for limits to exist at a point, the function must be defined at that point. This assumption was rather complex and detailed. In the event that the left-hand and the right-hand limits were not the same, students assumed that the limit did not exist. However, even when the left-hand and the right-hand limits were the same, in the case of a removable discontinuity, or isolated points, the limit was noted as the function value of the point that constituted the discontinuity. These findings are not consistent with students’ notion of approached but never reached or as bounds that cannot be attained as identified by other researchers such as (Tall & Schwarzenberger, 1978; Juter, 2005, Cornu, 1991; Szydlik 2000). In my study, students indicated that they were under the impression that limits are points the function actually reached or that in order for limit to exist at a point, the function must be defined at that point. This point of view was especially strong when students were asked to find the limit of a function from its graphical representation.

Other researchers (Juter, 2005, 2007; Williams, 2001) have mentioned that some students believed it is possible for continuous function to actually reach the limit even though according to
the definition they believed that reaching the limit was impossible. Thus, students held conflicting notions regarding the attainability of limits. Przenioslo (2004) noted that some students believed the limit of a function, \( f(x) \), at \( x = a \) was \( f(a) \) while some other students seemed confused over the term *approaching* in the limit definition. In addition, Przenioslo commented on students’ beliefs that at the holes the limit of a function did not exist, while at isolated points the limit was considered as the \( y \)-coordinate of the isolated point. During my data collection, in a problem statement that the function \( l(x) \) is undefined at \( x = 3 \), and that \( \lim_{x \to 3} l(x) = 2 \), some students stated that it is not possible for this function to have a limit at 3 while it was discontinuous at \( x = 3 \). This was consistent with some findings that if the limit existed at a point then the function must be continuous at that point.

In my study, students were able to compute finite limits by the use of algebraic procedures and direct substitution. Yerushalmy (1997) noted the rule-bound approach that students relied on while finding the equations of asymptotes. Przenioslo (2004) also noted that the subjects he studied were more inclined to use algorithms and rules to compute limits, while Juter (2006) inferred that most students lacked a strong foundation to understand the concept of limit well enough to be able to form coherent concept images.

I realized that students’ inability to discriminate between indeterminate and undefined forms and their implications on function behaviors at the points where function values assumed these forms, contributed to some extent to their difficulties in computing infinite limit and limits at infinity. Students’ incomplete knowledge of undefined and indeterminate forms and their implications were also documented by Louise (2000). According to Louise, while some students associated \( 0/0 \) form with the non-existence of limit, or infinity some others associated this form to a number. While Louise’s research focused on student knowledge of limits of rational functions
corresponding to the undefined and indeterminate forms, my research focused on student knowledge of other function behaviors such as asymptotes and continuity in addition to the limit behavior of rational functions.

In my study, the concept of continuity was explained well by most students. Everyone seemed to know the definition of continuity in terms of limits. In some problems, after answering that a function with all real number domains must be continuous everywhere in its domain, as soon as they proceeded to sketch the graph of such a function, students realized that such a function could very well be discontinuous. Another notion identified by my research that was consistent with the findings of other researchers was some students’ belief that a function could not be continuous at cusps or sharp corners (Bridgers, 2007).

Other limited conceptual knowledge regarding intercepts, polynomial forms, factoring polynomials, recognizing linear and quadratic forms, and so forth also posed obstacles during the problem-solving interview. A number of students could not tell the condition under which a rational form was undefined. They did not know which one, whether it was the numerator, or was it the denominator, that must be set to zero and solve to identify the VA.

The conceptual model that I developed through this research is depicted in Figure 6.8. As elaborated by Piaget (1970), and Tall and Vinner (1981), during the teaching episodes, I was mindful about students’ previous concept images and how it could interfere with acquiring proper understanding of the concept definition. Drawing from the theory of concept images and concept definition I believed some level of balance between the concept image and the concept definition once the causes of dissonance were identified. I identify the causes of dissonance as belonging to INK, IF, SUN, and OR categories. This dissonance caused could be balanced through instructional activities that highlighted the importance of social interaction and group work. This way student will receive the opportunity to re-examine and re-configure their incomplete
conceptions. In addition, learning and understanding the meaning of definitions must be considered important. Students need to be better informed about the nature of mathematics and the goals of mathematics learning.

Figure 6.8. Nair’s Conceptual Model.
Implications for Practice

I believe that it is important for teachers to be involved in research exploring ways to better conceptual understanding. This could be done alone or in collaboration with other faculty members. At the same time, teachers could benefit from being familiar with existing research findings. I believe that the model of instruction discussed in this research could be used to enhance mathematics learning in classrooms. My model was based on Piaget’s (1964; 1970) theory that accommodation and equilibration are the processes by which students connect their new ideas with pre-existing concept images, then compare, contrast and finalize their understanding of the new knowledge. I believe that the process of equilibration could be expedited by employing a series of actions that pose cognitive conflict, while interacting with the concepts from a multiple point of view.

Definition analysis needs to play a part in student learning. Definition construction must also be an important part of instruction. I believe that the definition of asymptotes as discussed in this research could be used to enhance mathematics learning in classrooms. Current practices mostly involve the teacher merely writing the definition on the board and then, immediately start solving problems that seem unconnected to the definition. In certain cases, such as in the cases of asymptotes and limits, textbook language seemed to have caused confusion in students’ understanding of these concepts.

Regarding the organization of classrooms, I found that students, if left alone to construct knowledge on their own and further explore it through social interaction, were more interested in the learning process. The importance of social interaction in learning was reiterated by Vygotsky (1978). As revealed by my research, the re-definition of asymptotes evolved as a result of team work and negotiation of meanings. Social interaction through this type of group exploration and debate could be very beneficial in acquiring mathematical concepts. To provide another example,
after figuring out the connections between different types of functions and their behaviors, students were curious about what would happen if the degree of the numerator of the function was larger than the degree of its denominator. Through this curiosity, they found the formation of another linear asymptote, the oblique asymptotes, in addition to the possibilities of non-linear asymptotes.

Regarding assessment, I believe that an oral component to the paper-and-pencil testing could be greatly informative as to what students actually know about the topics being tested. During the problem-solving interviews, I found that students knew a lot more than what they wrote on the paper while they solved problems. Many times, what students put on the exam sheet could be interpreted differently by teachers from what students intended to communicate.

Concepts must be taught from multiple perspectives. At least the analytic, the graphic, and the numeric forms must be included while teaching mathematics. The usage of graphs in such a way to promote exploration, interaction, and exchange of ideas among the students and between students and the instructor should help the re-enforcing of conceptual understanding (Aspinwall et al. 1997, Artigue, 1991). In addition, the visual effects of graphs and figures could help weaken conflict causing uncontrollable concept images (Aspinwall et al. 1997).

Some re-organization of mathematics content sequencing is necessary for success in mathematics learning. Others have stated that the order in which mathematics topics are introduced in the curriculum was different from the order in which mathematics concepts evolved. The topic of rational functions must be introduced with the beginning of limits so that students can employ the graphical details of the function to compute limits. Rather than simply stating that infinite limits and limits at infinity do not exist, students will be able picture the function behavior associated with these events. Moreover, students should be cognitively challenged to explore their conceptions with their class mates.
Recommendations for Further Research

As described earlier, the current research attempted to examine how student conceptions could change because of a month-long teaching experiment that took eight sessions. With this type of small-scale teaching experiment, a researcher can only obtain a glimpse of what could happen if the subject matter is taught in a classroom setting that is more constructivist in nature. As far as further research is concerned, a long-term teaching experiment with more students could be conducted to examine how student conceptual understanding of the concepts of asymptotes, limits, and continuity are impacted due to the cognitive exercises of the teaching experiment.

Another recommendation is to conduct similar research with students in other courses such as Calculus 1, Calculus 3 and analysis courses to investigate the conceptions held by more naive and more advanced mathematics students regarding these three basic notions of calculus: asymptotes, limits, and continuity. In particular, it would be beneficial to investigate whether students are making stronger connections between the closely related mathematical concepts of asymptotes, limits, and continuity as their mathematical encounters are broadened.

An additional research recommendation encompasses teaching a special session of a calculus course where asymptotes are introduced side by side with the concept of limits with equal importance, and examine whether student conceptual understanding of asymptotes and limits have improved when these concepts are taught together. This would further help decide the importance of a re-structuring of order and manner in which certain mathematical topics are being taught, and how this curricular change might translate to a more usual class setting. A similar study focusing on functions other than rational functions would be another research avenue to peruse.
REFERENCES


APPENDIX A

Consent to Participate in the Study of Rational Functions

Dear Student,

I am Girija Nair, a PhD candidate at the OSU in the field of mathematics education. I am interested in investigating student understanding of calculus concepts. I invite you to participate in my study.

If you participate in this study, you will be asked to solve several related mathematics problems. It may take up to 2 hours. The problem solving process will be videotaped and you will be asked to talk aloud while solving the problem. While you solve problems, I might ask you questions if some things are unclear. Please note that your face will not be videotaped. In order to protect your anonymity, you will be asked to choose a pseudo name to identify yourself in any report of this research.

After the problem-solving interview, based on student difficulties identified, I will hold several teaching sessions in which a variety of activities will be implemented to help students overcome the difficulty with the material. These teaching sessions will be held twice a week for four weeks. Each teaching session may last form 1 hour to 1 hour and 30 minutes.

After the teaching sessions you will be asked to participate in another problem solving interview similar to the first interview.

If you are willing to participate in this research, you will be offered an incentive. You will be paid 10 dollar for every hour you spend during the problem solving interviews. For the participation in the teaching sessions I will pay $15.00 for each session. I appreciate your participation in this research project. I will be contacting you soon to schedule the interview. Please sign this consent form if you agree to participate in this research. Also, please fill out the background feedback form on the reverse of this page.

Participation in this study is voluntary. Your performance will not in any way affect your grade in the mathematics courses that you are taking. Your performance result will be only viewed by me and my advisor. Your identity will not be revealed to anyone else.

Sincerely,
Girija S. Nair

Your name: ________________________ Please Print
Your signature: _____________________________ Date: _______________________________

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APPENDIX B

Background Feedback Survey

1. Name: ________________________________
   Please Print

2. E-mail: ______________________________

3. Telephone: __________________________

4. Major: ______________________________

5. Your rank: check one box

<table>
<thead>
<tr>
<th>Freshmen</th>
<th>Sophomore</th>
<th>Junior</th>
<th>Senior</th>
<th>Other</th>
</tr>
</thead>
</table>

6. Your typical level of mathematics performance (check one box)

<table>
<thead>
<tr>
<th>A - level</th>
<th>B - level</th>
<th>C - level</th>
<th>Below C-level</th>
</tr>
</thead>
</table>
APPENDIX C

CONCEPT FEEDBACK RESPONSE INSTRUMENT 1 – CORI 1

PLEASE READ AND ANSWER THE FOLLOWING QUESTIONS. WHILE SOLVING PROBLEMS PLEASE TALK ALOUD EXPLAINING YOUR THINKING PROCESS. FEEL FREE TO ASK ME QUESTIONS WHILE I MAY NOT BE ABLE ANSWER ALL OF THEM.

YOUR PSEUDO NAME: ____________________________________________________________

DATE: ________________________________________________________________________

TIME BEGAN: __________________________________________________________________

TIME ENDED: __________________________________________________________________
Identify rational functions. If possible, match the graph of these functions from the list given below.

\[
\begin{array}{|c|c|}
\hline
a(x) = x - 2 & d(x) = \frac{x^2 - 4}{x + 2} \\
\hline
b(x) = \frac{2x - 1}{x + 2} & e(x) = \frac{x^2 - 7x + 6}{2x^2 - 1} \\
\hline
c(x) = \frac{x^2 - 1}{2} & f(x) = \frac{x - 2}{x^2 + 4} \\
\hline
\end{array}
\]

1. A rational function \( g(x) \) has \( x = -3 \) for a vertical asymptote, \( y = 2 \) for horizontal asymptote, and is undefined at \( x = 3 \)
   (a) Sketch the graph of the function in the grid given below
   (b) Explain the behavior of this function at/around the points at which the function is undefined and around the horizontal asymptote
   (c) Write the equation or parts of the equation of the function \( g(x) \)

2. (a) Write the equation or parts of the equation of a rational function \( h(x) \) such that \( \lim_{x \to -\infty} h(x) = 4 \), \( \lim_{x \to \infty} h(x) = 4 \), \( \lim_{x \to -3^+} h(x) = -\infty \), and \( \lim_{x \to -3^-} h(x) = \infty \)
   (b) Explain the behavior of this function around \( y = 4 \) and \( x = -3 \)
   (c) Sketch the graph of the function in grid given below
3. Consider the rational function \( l(x) \). Some characteristics of this function are given in the box below.

| Function is discontinuous at \( x = \pm 3 \) |
| Function has a vertical asymptote: \( x = -3 \) |
| \( \lim_{x \to \infty} l(x) = 4 \), \( \lim_{x \to -\infty} l(x) = 4 \) |
| \( \lim_{x \to 3} l(x) = 2 \) |

(a) Sketch the function in the grid given below.
(b) Does this function have horizontal asymptote? Explain why or why not.
(c) If possible, write the equation or parts of the equation of \( l(x) \).

4. Explain the behavior of the functions \( j(x) \), \( k(x) \), and \( z(x) \) as \( x \) approaches 2.

![Graphs of j(x), k(x), and z(x) as x approaches 2.]

5. (a) Discuss the continuity of the functions \( j(x) \), \( k(x) \), \( Z(x) \), and \( m(x) \) in question 5.
(b) Discuss possible asymptotes if any of the functions \( j(x) \), \( k(x) \), \( Z(x) \), and \( m(x) \) in question 5.
6. Consider the function \( n(x) = \frac{3x^2 - 13}{x^2 - 9} \). Compute the following limits by explaining the details. Be sure to give the limits and finite/infinite limit

(a) \( \lim_{x \to 3} n(x) \)  
(b) \( \lim_{x \to -3} n(x) \)  
(c) \( \lim_{x \to -\infty} n(x) \)  
(d) \( \lim_{x \to \infty} n(x) \)

(e) Identify the vertical asymptote/s if any  
(f) Identify the horizontal asymptote if any  
(g) Explain how concepts in parts (a) through (e) are connected

7. (a) If the function \( p(x) \) has a horizontal asymptote \( y = 3/5 \), write part of the equation of this function  
(b) Express the horizontal asymptote \( y = 3/5 \) using the limit notation

8. (a) If the function \( q(x) \) have a horizontal asymptote \( y = 0 \), write part of the equation of this function  
(b) Express the horizontal asymptote, \( y = 0 \) using limit notation

9. Discuss the following situations, draw diagrams if needed  
(a) If the domain of a function \( r(x) \) is the set of all real numbers, can we assume that the function is continuous everywhere in its domain? Explain.  
(b) A function \( r(x) \) is continuous in \((-\infty, \infty)\). Is it possible for this function to have horizontal and/or vertical asymptote? Explain  
(c) If you know that \( \lim_{x \to 3^+} t(x) = -1 \), \( \lim_{x \to 3^-} t(x) = -1 \), should the function \( t(x) \) be continuous at \( x = 3 \)? Explain.

10. A function \( u(x) \) satisfies the conditions \( \lim_{x \to 2} u(x) = 7 \) and \( u(2) = 3 \)

(a) Sketch the graph of this function in the interval \([7, 7]\)  
(b) Is function \( u \) continuous at \( x = 2 \)?  
(c) How can you alter the graph so that the function will be continuous at \( x = 2 \)?  
(d) How would you change the statements “\( \lim_{x \to 2} u(x) = 7 \) and \( u(2) = 3 \)”, now that you have a new graph of a function that is continuous at \( x = 2 \)?

12. Give a definition for continuity of a function \( D(x) \) at \( x = a \)  
Start: Function \( D(x) \) will be considered continuous at \( x = a \) if….

13. Let \( v(x) \) be a rational function that is defined in the neighborhood (explained during the interview) of \( x = 5 \)

(a) if \( v(5) = \frac{0}{0} \) what can you say about the behavior of this function for value of \( x \) at/around 5?  
(b) if \( v(5) = \frac{b}{0}, b \neq 0 \) what can you say about the behavior of this function for value of \( x \) at/around 5?

14. Create a table of values if needed to answer the following questions. The limit may be a finite number or it can be expressed as in terms of \( \pm \infty \). In some problems you may
compute the left-hand and the right-hand limits separately.

(a) find \( \lim_{x \to 2} \frac{1}{x + 1} \)

(b) find \( \lim_{x \to 2} \frac{3x}{x - 2} \)

(c) find \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \)

(d) find \( \lim_{x \to \pm\infty} \frac{1}{x + 1} \)

(e) find \( \lim_{x \to \pm\infty} \frac{3x}{x - 2} \)

(f) find \( \lim_{x \to \pm\infty} \frac{x^2 - 4}{x - 2} \)

(g) if \( \alpha(x) = \frac{1}{x + 1} \) find \( \alpha(2) \)

(h) if \( \beta(x) = \frac{3x}{x - 2} \) find \( \beta(2) \)

(i) if \( \delta(x) = \frac{x^2 - 4}{x - 2} \) find \( \delta(2) \)

(j) Discuss the observations made in parts (a) through (i)

15. In your own words explain what it means by an asymptote of a rational function. If possible, explain the concept of asymptote using limit notation.
APPENDIX D

CONCEPT FEEDBACK RESPONSE INSTRUMENT 2 - CORI 2

Please read the questions carefully. While solving problems please talk aloud so that I can understand your thinking process. While solving problems, I may ask you for clarifications and further explanations. Thank you for your participation.

Your name: _____________________________

Date: __________________________________
Identify rational functions from the following list of functions. Explain your reasoning.

1. \(\frac{3x-1}{x+7}\)
2. \(\frac{3x^2 - 7x + 9}{x + \sqrt{6}}\)
3. \(\frac{-7x^3 + 11x - 8}{6}\)
4. \(\frac{3}{2x^{\frac{1}{3}} + 11x^2 - 5}\)

For the rational function \(a(x) = \frac{-3x^2 + 2x + 8}{x^2 - 3x - 4}\), find:
5. the x-intercept/s
6. the y-intercept
7. the horizontal asymptote
8. the vertical asymptote/s
9. hole/s

10. Using the above features graph the function \(a(x) = \frac{-3x^2 + 2x + 8}{x^2 - 3x - 4}\) in the grid given below.

For the rational function, \(b(x) = \frac{2x^2 + 9x - 35}{x^3 + 2x^2 - 31x + 28}\), find:
11. the x-intercept/s
12. the y-intercept
13. the horizontal asymptote
14. the vertical asymptote/s
15. hole/s

16. Using the above features graph the function \(b(x) = \frac{2x^2 + 9x - 35}{x^3 + 2x^2 - 31x + 28}\) in the grid given below.

Compute the following limit:

17. \(c(x) = \begin{cases} \sqrt{4+5x}, & x > 9 \\ x - 2, & x < 9 \end{cases}\)

Find \(\lim_{x \to 9} c(x)\)

18. In problem 17, is the function \(c(x)\) continuous at \(x = 9\)? Explain.

Compute the following limit (finite as well as infinite):

19. Find \(\lim_{x \to -3} \frac{3x^2 - 13}{x^2 - 9}\)
20. Find \( \lim_{x \to \infty} \frac{3x^2 - 13}{x^2 - 9} \)

21. Find \( \lim_{x \to 2} e(x) \), if \( e(x) = \frac{2x^2 + 3x - 2}{2x^3 + 4x^2 + x + 2} \)

22. Find \( \lim_{x \to 0} \frac{2x^2 + 3x - 2}{2x^3 + 4x^2 + x + 2} \)

23. Write the equation or parts of the equation of a rational function \( f(x) \) if
   (a) \( x = -7 \) is a vertical asymptote of \( f(x) \)
   (b) the function has a hole at \( x = -2 \)
   (c) the horizontal asymptote of this function is the X-axis

24. Write the equation or parts of the equation of a rational function \( g(x) \) if
   (a) \( x = 0 \) is a vertical asymptote of \( g(x) \)
   (b) the function has a hole at \( x = -2 \)
   (c) the horizontal asymptote of this function is \( y = -2/7 \)

25. Write the equation or parts of the equation of a rational function \( h(x) \) if
   (a) \( h(2) = \frac{b}{0}, b \neq 0 \) (function defined in the neighborhood (will explain) of \( x = 2 \))
   (b) the function has no horizontal asymptote
   (c) \( h(-4) = \frac{0}{0} \) (function defined in the neighborhood (will explain) of \( x = 4 \))
   Can a function intersect its horizontal asymptote? Explain

26. Can a function intersect a vertical asymptote? Explain

27. Describe your understanding of the horizontal asymptote of a rational function

28. Describe your understanding of the vertical asymptote of a rational function
APPENDIX E
LEARNING RESOURCES - CORI 2

1. “The art of teaching is the art of assisting discovery”
   Mark Van Doren (June 13, 1894 – December 10, 1972)

2. Rules to live by
   1. Do not try to memorize every single rule or fact you may come across
   2. Mathematics doesn’t comprise simply of memorization
   3. Mathematics is a problem-solving art, not just a collection of facts
   4. Don’t give up on a problem if you can’t solve it right away
   5. Re-read it thoughtfully and try to understand it more clearly
   6. Struggle with it until you solve it

Once you have done this a few times you will begin to understand what mathematics is really all about

Rational Numbers
Any rational number \( r \), can be expressed as a fraction.
Specifically, any rational number \( r \), can be expressed as \( r = \frac{m}{n} \), where \( m \) and \( n \) are integers and \( n \neq 0 \).

Examine the following statements are explain your reasoning.

1. -18 is a rational number because,
2. 0 is a rational number because,
3. 12.5 is a rational number because,
4. \( \sqrt{16} \) is a rational number because,
5. 0.\overline{3} is a rational number because,
6. \( \sqrt{6} \) is not a rational number because,

4. Rational Functions
A rational function is a function of the form \( r(x) = \frac{P(x)}{Q(x)} \), where \( P \) and \( Q \) are polynomials. The domain of a rational function consists of all real numbers \( x \) except for which the denominator of the function is zero. When graphing a rational function, we must pay special attention to the behavior of the graph near those \( x \)-values.

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1, 2, 3, 4: isbn: 0-495-29487-x  OSU Edition, Authors: James Stewart, Lothar Redlin, Saleem Watson
Domain, Rational Functions

Find the domains of the following functions. Identify rational functions

7. \( h(x) = \sqrt{x+2} \)
8. \( a(x) = \frac{3x-1}{x+1} \)
9. \( j(x) = \frac{5x-1}{9} \)
10. \( k(x) = \frac{x^2-1}{x+1} \)
11. \( c(x) = \frac{3}{x^2+1} \)
12. \( e(x) = \frac{3x-1}{9x^2-1} \)
13. \( g(x) = -9x^3 + 17x^2 - 12x + 2.5 \)
14. \( f(x) = \frac{3x+2}{9x^2-1} \)

Graphing Rational Functions

While it is easier to graph simple rational functions by following their shifts, graphing rational functions that are more complicated would require careful examination of its properties near its vertical and horizontal asymptotes.

Examine the graphs of the rational function: \( a(x) = \frac{3x-1}{x+1} \), and answer the questions stated below

\[ a(x) \text{ is undefined when?} \]

As \( x \to -1^+ \), \( a(x) \to ? \). As \( x \to -1^- \), \( a(x) \to ? \)
While referring to limits, instead of stating ‘as \(a(x) \to a\) number, \(x \to another\ number\)’, why is the terminology ‘as \(x \to a\) number, \(a(x) \to another\ number\)’ usually preferred?

Based on the graph of the function, \(a(x) = \frac{3x-1}{x+1}\) as \(x \to -\infty\), \(a(x) \to ?\)
as \(x \to +\infty\), \(a(x) \to ?\)

The vertical line \(x = -1\) is called the vertical asymptote of the rational function \(a(x) = \frac{3x-1}{x+1}\), because… (examine the graph)

The horizontal line \(y = 3\) is called the horizontal asymptotes of the rational function \(a(x) = \frac{3x-1}{x+1}\), because….

**Informal definition of asymptotes**

"Informally speaking, an asymptote of a function is a line that the graph of the function gets closer and closer to as one travels along that line."

Find the following limits by filling out the table of values to verify what was observed in the graph of the rational function \(a(x) = \frac{3x-1}{x+1}\), on page 247

15. \(\lim_{x \to -1} a(x) = \frac{3x-1}{x+1} = \)

16. \(\lim_{x \to -1} a(x) = \frac{3x-1}{x+1} = \)

17. \(\lim_{x \to -\infty} a(x) = \frac{3x-1}{x+1} = \)

18. \(\lim_{x \to +\infty} a(x) = \frac{3x-1}{x+1} = \)

19. For the function \(a(x) = \frac{3x-1}{x+1}\), find \(\lim_{x \to 2} a(x)\)

20. Is it possible for a function to have a vertical asymptote at a value in its domain? Explain

21. Consider the function \(K(x) = \frac{x^2-1}{x+1}\). Note that this function could be simplified into

\(^5\) (isbn: 0-495-29487-x), OSU Edition, Authors: James Stewart, Lothar Redlin, Saleem Watson
\[ K(x) = \frac{(x-1)(x-1)}{x+1} = x-1 \]. Is \( k(x) \) undefined at \( x = -1 \)? Why or why not?

22. Sketch the graph of \( K(x) \) on a graphing calculator and copy the graph on to this paper. Is \( x = -1 \) a vertical asymptote of the graph of this function?

23. If \( K(x) = \frac{x^2 - 1}{x + 1} \), Find \( \lim_{x \to -1} \frac{x^2 - 1}{x + 1} \).

24. Compare and contrast the behaviors of the functions \( a(x) = \frac{3x-1}{x+1} \) and \( K(x) = \frac{x^2 - 1}{x + 1} \) around the points were they are undefined.

25. Based on your observations so far, state the conditions under which a line \( x = a \) will be a vertical asymptote of a rational function \( r(x) = \frac{P(x)}{Q(x)} \).

26. \( \text{Population Growth:} \) Suppose that the rabbit population on Mr. Jenkins’ farm follows the formula \( p(t) = \frac{3000t}{t+1} \) where \( t \geq 0 \) is the time in months since the beginning of the year.
   (a) What eventually happens to the rabbit population?
   (b) Draw a graph of the rabbit population.

27. \( \text{Drug Concentration:} \) A drug administered to a patient and the concentration of the drug in the bloodstream is monitored. At time \( t \geq 0 \) (in hours since giving the drug), the concentration (in mg/L) is given by \( c(t) = \frac{5t}{t^2 + 1} \).
   (a) What happens to the drug concentration after a long period of time?
   (b) Graph \( c(t) \).
   (c) What is the highest concentration of drug that is reached in the patient’s bloodstream?

28. Find the horizontal asymptote of \( u(x) = \frac{3x^2 - 2x + 11}{5x^2 + 1} \).

29. Graphing Problem

Consider the rational function \( n(x) = \frac{3x - 2}{x^2 + 1} \). Examine the intercepts and asymptotes of this function.

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(67 isbn: 0-495-29487-x) OSU Edition, Authors: James Stewart, Lothar Redlin, Saleem Watson p.314, 315
function and sketch the graph without using a calculator

30. **Graphing Problem**
Consider the function \( v(x) = \frac{2x^3 + 7x^2 - 2x - 7}{x - 3} \). Examine the intercepts and asymptotes of this function and sketch the graph without using a calculator.

31. **Graphing Problem**
Consider the function \( t(x) = \frac{2x^3 - 5x^2 - 2x + 5}{2x^2 + 5x + 3} \). Examine the intercepts and asymptotes of this function and sketch the graph without using a calculator.

**Graphing Problem**
Consider the function \( F(x) = \frac{2x^3 - 7x^2 - 2x + 7}{x^3 - x^2 - 9x + 9} \)

32. Find the x-intercept/s
33. Find the y-intercept
34. Find \( \lim_{x \to 3^+} F(x) \)
35. Find \( \lim_{x \to 3^-} F(x) \)
36. Find \( \lim_{x \to -3^+} F(x) \)
37. Find \( \lim_{x \to -3^-} F(x) \)
38. Find \( \lim_{x \to \infty} F(x) \)
39. Find \( \lim_{x \to -\infty} F(x) \)
40. Find the vertical asymptote/s if any
41. Find the horizontal asymptote if any
42. Sketch the graph incorporating the results found in 1 through 10
Polynomials

Polynomial terms have variables which are raised to whole-number exponents (or else the terms are just plain numbers); there are no square roots of variables, no fractional powers. Here are some examples:

<table>
<thead>
<tr>
<th>Term</th>
<th>Description</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6x^{-2}$</td>
<td>This is NOT a polynomial term...</td>
<td>...because the variable has a negative exponent.</td>
</tr>
<tr>
<td>$\sqrt{x}$</td>
<td>This is NOT a polynomial term...</td>
<td>...because the variable is inside a radical.</td>
</tr>
<tr>
<td>$4x^2$</td>
<td>This IS a polynomial term...</td>
<td>...because it obeys all the rules.</td>
</tr>
<tr>
<td>$6x^{2/3}$</td>
<td>This is NOT a polynomial term...</td>
<td>...because the variable has a “fraction” exponent.</td>
</tr>
</tbody>
</table>

Rational function re-defined

A rational function is of the form $r(x) = \frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials. Particularly, the denominator Q(x) must be a non-zero, non-constant polynomial.

A polynomial function is of the form $ax^n + bx^n + \ldots$ where a, b... are real numbers and m, n ... are whole numbers. Remember that 0 is a whole number.

From the following list, identify rational functions in accordance with the above definition. If a function is not a rational function, explain why it is not considered rational.

43. $f(x) = \frac{x - 7}{x + 11}$
44. $g(x) = \frac{x^3 - 2x - 7}{x^2 + 11}$
45. $h(x) = \frac{x - 7}{11}$
46. $n(x) = \frac{7}{x + 11}$
47. $j(x) = 3x^5 - 7x + 19$

8,9 [http://www.purplemath.com/modules/polydefs.htm](http://www.purplemath.com/modules/polydefs.htm)
48. \( c(x) = \frac{7}{x^{2/3} + 11} \)

49. \( d(x) = \frac{x^3 - 2\sqrt{x} - 7}{x^2 + 11} \)

50. Using the limit notation describe what happens to the function \( f(x) \) as \( x \) approaches 1 from the right and from the left.

51. Milan sketched the following graph and wrote that the function graphed below has a vertical asymptote \( x = 1 \) and a horizontal asymptote \( y = 3 \). Do you agree? Explain your position using the limit notation.
More Rational Function Exploration

52. For a function \( \sigma(x) \), suppose that \( \sigma(7) = \frac{0}{0} \), what can you say about the behavior of the function \( \sigma(x) \), at \( x = 7 \)?

53. What can you say about the behavior of the function \( \sigma(x) \), around \( x = 7 \)?

54. What can you say about \( \lim\limits_{x \to 7} \sigma(x) \)

55. For another function \( \delta(x) \), suppose that \( \delta(7) = \frac{b}{0}, b \neq 0 \)

56. What can you say about the behavior of the function \( \delta(x) \), at \( x = 7 \)?

57. What can you say about the behavior of the function \( \delta(x) \), around \( x = 7 \)?

58. What can you say about \( \lim\limits_{x \to 7} \delta(x) \)

More Limits…

Compute the following limits. Explain the method you used to compute the limit

59. \( \lim\limits_{x \to 3} \sqrt{6-x} \)

60. \( \lim\limits_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} \)

61. If \( t(x) = \begin{cases} x - 5, & x < -2 \\ 1 - 3x, & x \geq -2 \end{cases} \) answer the following questions

62. \( \lim\limits_{x \to -2^+} t(x) \)

63. \( \lim\limits_{x \to -2^-} t(x) \)

64. \( \lim\limits_{x \to -2} t(x) \)
If \( j(x) = \begin{cases} x^2 + 5, & x < -2 \\ 3 - 3x, & x \geq -2 \end{cases} \) find \( \lim_{x \to -2^-} j(x) \), and \( \lim_{x \to -2^+} j(x) \), compute the following limits

65. \( \lim_{x \to -2^-} j(x) \)
66. \( \lim_{x \to -2^+} \left( \frac{4}{x + 2} \right) \)
67. \( \lim_{x \to 3^-} \left( \frac{3}{4-x} \right) \)
68. \( \lim_{x \to 3^+} \left( \frac{2x}{x - 3} \right) \)

How Close is Close?

69. Examine the following tables and discuss the table of values and its appropriateness in computing

the limits \( \lim_{x \to 5^-} \frac{2}{x - 5} \) and \( \lim_{x \to 5^+} \frac{2}{x - 5} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( Y_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0001</td>
<td>20000</td>
</tr>
<tr>
<td>5.001</td>
<td>2000</td>
</tr>
<tr>
<td>5.01</td>
<td>20</td>
</tr>
</tbody>
</table>

\( x = \) 5

<table>
<thead>
<tr>
<th>( x )</th>
<th>( Y_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.8</td>
<td>-10</td>
</tr>
<tr>
<td>4.88</td>
<td>-16.67</td>
</tr>
<tr>
<td>4.888</td>
<td>-17.86</td>
</tr>
<tr>
<td>4.9</td>
<td>-20</td>
</tr>
<tr>
<td>4.99</td>
<td>-200</td>
</tr>
<tr>
<td>4.999</td>
<td>-2000</td>
</tr>
</tbody>
</table>

\( x = \)
More Limit Problems

70. First plot the point (5, 1). Then sketch the graph of \( f(x) \) with x-intercept (-2, 0), y-intercept (0, 3), and as \( x \to 5^+, f(x) \to 1 \)

71. In the problem above, in order for “as \( x \to 5^- , f(x) \to 1^- \)”, must the point (5, 1) be necessarily on function? Explain and illustrate with a graph

72. In the next problem, first plot the point (5, 1). Then sketch the graph of a function \( f(x) \) with x-intercept (-6.5, 0), and as \( x \to 5^+ , f(x) \to -1 \). Should \( f(5) \) be 0? Explain

73. Now, plot the points (5, 1) and (5, -1). Then sketch the graph of a function \( f(x) \) with x-intercepts (2, 0), and (6.5, 0), y-intercept (0, -3), (2, 0), and (6.5, 0), y-intercept (0, -3), and as \( x \to 5^+ , f(x) \to 1 \), as \( x \to 5^- , f(x) \to 1 \), and \( f(5) = -7 \). Find \( \lim \limits_{x \to 5^-} f(x) \)

10 Limits
The limit process is a fundamental concept of calculus. One technique you can use to estimate a limit is to graph the function and then determine the behavior of the graph as the independent variable approaches specific value. Limits of functions could be found numerically, graphically, and analytically.

11 Some students try to learn calculus as if they were simply a collection of new formulas. This is unfortunate. If you reduce calculus to the memorization of differentiation and integration formulas, you will miss a great deal of understanding, self-confidence, and satisfaction.

12 Limits – definition 1
At this stage our approach is completely informal. All we are trying to do here is lay an intuitive foundation for the formal definition of the limit concept. We will start with a number \( c \) and a function \( f \) defined at all numbers \( x \) near \( c \) but not necessarily at \( c \) itself. In any case, whether or not \( f \) is defined at \( c \) and, if so, how is totally irrelevant. Now, let \( L \) be another real number. The notation \( \lim \limits_{x \to c} f(x) = L \) means that by making \( x \) sufficiently close to \( c \), \( f(x) \) can be made to be as close to \( L \) as desired. In that case, we say that "the limit of \( f(x) \), as \( x \) approaches \( c \), is \( L \)".

---

12 Page 54, ISBN: 0-471-69804-0, Salas, Hille, Etgen
13 Limits – definition 2

We write \[ \lim_{x \to c} f(x) = L \] and say "the limit of \( f(x) \) as \( x \) approaches \( c \), equals \( L \)" if we can make the values of \( f(x) \) arbitrarily close to \( L \) (as close to \( L \) as we like) by taking \( x \) to be sufficiently close to \( c \) (on either side of \( c \)) but not equal to \( c \). Notice the phrase "but \( x \neq c \)" in the definition of limit. This means that in finding the limit of \( f(x) \) as \( x \) approaches \( c \), we never consider \( x = c \). In fact, \( f(x) \) need not even be defined when \( x = c \). The only thing that matters is how \( f \) is defined near \( c \).

Example 1
Let \( f(x) = \frac{x-1}{x^2-1} \)

Find \( \lim_{x \to 1} f(x) \) by completing the following table and observing the function values

<table>
<thead>
<tr>
<th>( x &lt; 1 )</th>
<th>( f(x) )</th>
<th>( x &gt; 1 )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.5</td>
<td>1.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.9</td>
<td></td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>.99</td>
<td></td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>.999</td>
<td></td>
<td>1.001</td>
<td></td>
</tr>
<tr>
<td>.9999</td>
<td></td>
<td>1.0001</td>
<td></td>
</tr>
</tbody>
</table>

The limit of \( f \) as \( x \) approaches 1 is _____ because, the only thing that matters is ____________ even though \( f(1) \) is _____

Example 2
Let \( g(x) = \begin{cases} \frac{x-1}{x^2-1}, & x \neq 1 \\ 2, & x = 1 \end{cases} \)

Find \( \lim_{x \to 1} g(x) \) by completing the following table and observing the function values

<table>
<thead>
<tr>
<th>( x &lt; 1 )</th>
<th>( g(x) )</th>
<th>( x &gt; 1 )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.89</td>
<td>1.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>.9</td>
<td></td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>.99</td>
<td></td>
<td>1.01</td>
<td></td>
</tr>
<tr>
<td>.999</td>
<td></td>
<td>1.001</td>
<td></td>
</tr>
<tr>
<td>.9999</td>
<td></td>
<td>1.0001</td>
<td></td>
</tr>
</tbody>
</table>

The limit of \( g \) as \( x \) approaches 1 is _____ because, the only thing that matters is how \( g(x) \) is defined ________________ even though \( g(1) \) is _____

Compute the following limits

---

13 Stewart James, page 99
74. \( \lim_{x \to 2} T(x) \) if \( T(x) = \frac{2x - 3}{x^2 + 5} \)

If \( C(x) = \begin{cases} \sqrt{x - 4}, & x \geq 4 \\ 2x - 5, & x < 4 \end{cases} \), answer questions 2, 3, and 4

75. \( \lim_{x \to 4} C(x) \)
76. \( \lim_{x \to 4} C(x) \)
77. \( \lim_{x \to 4} C(x) \)
78. you may use a regular calculator for this problem
79. you may use a regular calculator for this problem
80. \( \lim_{x \to 5} \frac{2}{x - 5} \)
81. \( \lim_{x \to 5} \frac{1}{x^2} \)
82. \( \lim_{x \to 0} \frac{1}{x^2} \)
83. \( \lim_{x \to 0} \frac{1}{x^2} \)
84. find \( \lim_{x \to 0} \left( \sin \frac{1}{x} \right) \) by examining the graph

but language altered by mentioning the behavior of \( x \) instead.

\(^{1}\)Common types of behavior associated with non-existence of a limit

1. as \( x \) approaches \( c \) from the right side of \( c \), and from the left side of \( c \), \( f(x) \) approaches two different numbers
2. as \( x \) approaches \( c \), \( f(x) \) increases or decreases without bound
3. as \( x \) approaches \( c \), \( f(x) \) oscillates between two fixed values

\(^{15}\)Infinite Limits

Some functions approach \( \infty \) or \( -\infty \) as \( x \) approaches a value \( c \). In this case, the limit \( \lim_{x \to c} f(x) \) does not exist (in the finite sense), but we say that \( f(x) \) has an infinite limit.

\(^{1}\) Page 51, ISBN: 0-618-50298-x
When referring to infinite limits, \( \infty \) and \(-\infty\) must not be treated as numbers. The symbols \( \infty \) and \(-\infty\) are used to indicate that \( f(x) \) increases and decreases respectively without bound as \( x \) approaches \( c \). The symbol \( x \to c^{\pm} \) indicates that the left- and right-hand limits are to be considered separately.

Limit at infinity discusses the end-behavior of a function on an infinite interval. Let \( L \) be a real number. The statement \( \lim_{x \to \infty} f(x) = L \) indicates that as \( x \) increases without bound, \( f(x) \) approaches the line \( y = L \). The statement \( \lim_{x \to -\infty} f(x) = L \) indicates that as \( x \) decreases without bound, \( f(x) \) approaches the line \( y = L \).

85. Graphically illustrate the concept of limit at infinity by drawing figures similar to the following graph:

\[
\lim_{x \to \infty} g(x) = -0.5
\]

Function behavior by examining table values

86. Guess the behavior of the function \( U(x) \) at and around when \( x = 2 \) and \( x = 7 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( U1 )</th>
<th>( x )</th>
<th>( U1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.899</td>
<td>-205.9</td>
<td>1.9</td>
<td>-2.118</td>
</tr>
<tr>
<td>6.9</td>
<td>-208</td>
<td>1.99</td>
<td>-2.192</td>
</tr>
<tr>
<td>6.99</td>
<td>-2098</td>
<td>1.999</td>
<td>-2.199</td>
</tr>
<tr>
<td>7.001</td>
<td>ERROR</td>
<td>2.01</td>
<td>-2.308</td>
</tr>
<tr>
<td>7.01</td>
<td>2102</td>
<td>2.02</td>
<td>-2.217</td>
</tr>
<tr>
<td>7.1</td>
<td>212</td>
<td></td>
<td>-2.286</td>
</tr>
</tbody>
</table>
| \( x=6.899 \) | \( x=2.1 \)

87. For the function \( V(x) \), the denominator was \( 11x - 4 \) while the numerator was unknown. By observing the following table, what can you say about the behavior of \( V(x) \) at or around \( x = \frac{4}{11} \)? Note that \( \frac{4}{11} \approx .364 \)

---

16 Page 198, ISBN: 0-618-50298-x
88. For the function $V(x)$, the denominator was $11x - 4$ while the numerator was unknown. By observing the following table, what can you say about the behavior of $V(x)$ at or around $x = \frac{4}{11}$? Note that $\frac{4}{11} = .363636363636...$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$V(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.35</td>
<td>121</td>
</tr>
<tr>
<td>0.36</td>
<td>166.69</td>
</tr>
<tr>
<td>0.37</td>
<td>266.22</td>
</tr>
<tr>
<td>0.38</td>
<td>180.7</td>
</tr>
<tr>
<td>0.39</td>
<td>-184.9</td>
</tr>
</tbody>
</table>

$x = .367$

**Warm-up 2**
Examine the graphs and decide which one/s of the following statements are correct. Explain your reasoning.

89. $x = -7$

**Choices:**

A) $f(-7) = \frac{0}{0}$

B) $f(-7) = \frac{b}{0}, b \neq 0$

C) $f(-7) = 0$
90. 
Compute the limit: 
\[ \lim_{x \to 0} \frac{9}{x^2} \]

The Mystery of 0/0 form

Does \( f(a) = \frac{0}{0} \) form always indicate a hole at \( x = a \) and not a vertical asymptote \( x = a \)?

\[ f(x) = \frac{x^2 - 7x + 12}{x^2 - 9} \]
\[ f(3) = \frac{9 - 21 + 12}{9 - 9} = \frac{0}{0} \]

91.

Choices:
A) \( h(2) = \frac{0}{0} \)
B) \( h(2) = \frac{b}{0}, b \neq 0 \)
C) \( h(2) = -1 \)

92. Compute the limit: \( r(x) = \lim_{x \to 0} \frac{3x^2}{x^2 + 9x} \) and \( r(x) = \lim_{x \to -9} \frac{3x^2}{x^2 + 9x} \)

93. The Mystery of 0/0 form

Choices:
A) \( \lim_{x \to -\infty} g(x) = -1.5 \)
B) \( y = -1.5 \) is a horizontal asymptote of the graph of \( g(x) \)
C) \( y = -1.5 \) is not a horizontal asymptote of the graph of \( g(x) \) since \( \lim_{x \to -\infty} g(x) \neq \lim_{x \to 0} g(x) \)
\[ f(x) = \frac{x^2 - 7x + 12}{x^2 - 9} = \frac{(x - 3)(x - 4)}{(x - 3)(x + 3)} = \frac{x - 4}{x + 3} \]
\[ f(3) = \frac{-1}{6} \Rightarrow \text{hole at } x = 3 \]

94. \[ f(x) = \frac{x^2 - 4}{(x - 2)^3} \]
\[ f(2) = \frac{0}{0} \]
\[ f(x) = \frac{x^2 - 4}{(x - 2)^3} = \frac{(x - 2)(x + 2)}{(x - 2)(x - 2)(x - 2)} = \frac{x + 2}{(x - 2)^2} \]
\[ = \frac{4}{0} \Rightarrow \text{vertical asymptote at } x = 2 \]

Compute the following limits

95. \[ \lim_{x \to 2^\pm} \frac{6x^2 - 11x + 11}{x - 5} \]
96. \[ \lim_{x \to 2^\pm} \frac{-11x + 11}{7x^2 - 5} \]
97. \[ \lim_{x \to 2^\pm} \frac{2x^3 - 11x + 11}{7x^3 - 5} \]

98. Consider the rational function \( v(x) = \frac{(x^2 + 6x + 7)(x - 1)}{2x^3 + 5x^2 - 7x} \)

Graph this function by finding the following features
(a) X-intercept/s
(b) Y-intercept
(c) Vertical asymptote/s if any
(d) Horizontal asymptote if any
(e) Hole/s if any

\[ \text{Infinite Limits} \]
Some functions approach \( \infty \) or \( -\infty \) as \( x \) approaches a value \( c \). In this case, the limit
\[ \lim_{x \to c} f(x) \] does not exist (in the finite sense), but we say that \( f(x) \) has an infinite limit.

When referring to infinite limits, \( \infty \) and \( -\infty \) must not be treated as numbers. The symbols \( \infty \) and \( -\infty \) are used to indicate that \( f(x) \) increases and decreases respectively without bound as \( x \) approaches \( c \).
The symbol \( x \to c^\pm \) indicates that the left- and right-hand limits are to be considered separately.

\[
\lim_{{x \to 0.5^-}} f(x) = -\infty
\]

\[
\lim_{{x \to -\infty}} g(x) = -0.5
\]

**Limit at infinity** discusses the end-behavior of a function on an infinite interval. Let \( L \) be a real number. The statement \( \lim_{{x \to \infty}} f(x) = L \) indicates that as \( x \) increases without bound, \( f(x) \) approaches the line \( y = L \). The statement \( \lim_{{x \to -\infty}} f(x) = L \) indicates that as \( x \) decreases without bound, \( f(x) \) approaches the line \( y = L \).

**Infinite limit and vertical asymptotes**

A line \( x = a \) is considered the vertical asymptote of a rational function \( F(x) \) if any one of the following conditions are true.

---

\(^{18}\) Page 198, ISBN: 0-618-50298-x

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\begin{align*}
\lim_{x \to a^+} F(x) &= \infty \\
\lim_{x \to a^-} F(x) &= -\infty \\
\lim_{x \to a^+} F(x) &= \infty \\
\lim_{x \to a^-} F(x) &= -\infty
\end{align*}

**Limit at infinity and horizontal asymptotes**

A line \( y = b \) is considered the horizontal asymptote of a rational function \( F(x) \) if any one of the following conditions are true:

\( \lim_{x \to \infty} F(x) = b \)

\( \lim_{x \to -\infty} F(x) = b \)

Which one/s of the following figures represent the existence of a horizontal asymptote at the dotted line?

**Problem 1**

\[ f(x) = \frac{5x - 9}{x - 2} \]

Compute the limit:

\[ \lim_{x \to 2^+} f(x) \]

**Magoo’s Work**

He plugged in 2 for \( x \), and found that \( f(2) = 1/0 \)

He concluded that the limit will be \( \pm \infty \)

He knew he needed to narrow it down to which infinity exactly?

He started by examining two cases:

\[ \lim_{x \to 2^+} \frac{5x - 9}{x - 2} \] and \[ \lim_{x \to 2^-} \frac{5x - 9}{x - 2} \]

\( x \to 2^+ \Rightarrow x > 2 \Rightarrow x - 2 > 0 \)

He paused… and decided to seek a shortcut.

He plugged in \( x = 3 \) for \( x \)

Found \( f(3) = 6 \), positive

Therefore concluded that \( \lim_{x \to 2^+} \frac{5x - 9}{x - 2} = \infty \)

Similarly, for \( x \to 2^- \), he plugged in \( x = 1 \) and found \( f(1) = -4/-1 = 4 \), positive

Therefore concluded that \( \lim_{x \to 2^-} \frac{5x - 9}{x - 2} = \infty \)

Comment on Magoo’s work and make corrections if necessary.

**Problem 2**

Rita thought that Magoo’s work was correct. However, her calculus partner Gia found that according to the graph of the function \( f(x) = \frac{5x - 9}{x - 2} \)
f(x) approached negative infinity as x approaches 2 from the left. However, Gia couldn’t figure out the details.

On a partner quiz, Herald did his share, the first half of the problem as follows.
Examine the following solutions that different students wrote on several limit problems.

a. Are they done correctly? Explain why or why not
b. Between the different methods that were used, which method would you consider most efficient in each problem? Explain

Find all limits, both finite and infinite

Problem 3
\[
a(x) = \frac{2x}{x + 7}
\]
Find \( \lim_{x \to -7^+} a(x) \)

Eva’s work
\[
a(x) = \frac{2x}{x + 7}
\]
\[
\lim_{x \to -7^+} a(x) = \frac{-14}{0} = DNE
\]

Emily’s work
\[
a(x) = \frac{2x}{x + 7}
\]
\[
\lim_{x \to -7^+} a(x) = \frac{-14}{0}
\]
This means that this function must have a vertical asymptote at \( x = -7 \)
The function must approach \( \pm \infty \)
To narrow it down I need a graphing calculator.
Eddy’s work

\[ a(x) = \frac{2x}{x + 7} \]
\[
\lim_{x \to -7} a(x) = \frac{-14}{0}
\]

This means that this function must have either a vertical asymptote at \( x = -7 \) or a hole at \( x = -7 \). Either way the function must approach \( \pm \infty \).

Eddy spent lots of time plugging in values, making errors and trying to figure out the limit. Finally he realized something. He started doing the following work.

“I will examine what the numerator and the denominator was going to be like…”

\[
\lim_{x \to -7} \frac{2x}{x + 7} \quad \quad \lim_{x \to -7} \frac{2x}{x + 7}
\]

\[ x \to -7^+ \Rightarrow x > -7 \Rightarrow x + 7 > 0 \quad \quad x \to -7^- \Rightarrow x < -7 \Rightarrow x + 7 < 0 \]

Before he could complete the problem, the time was up!

If Eddy would have solved this problem correctly, he would have gotten a reward. Could you take Eddy’s place? You too could have a reward. Eddy had to leave the following question unattended. Can you solve it?

Problem 4: compute \( \lim_{x \to 2} \frac{11}{2 - x} \)

Elwin, Eddy’s second cousin attempted the following problem.

Elwin’s work

Problem 5: compute \( \lim_{x \to 5} \frac{2x - 10}{x^2 - 3x - 10} \)

\[ \lim_{x \to 5} \frac{2x - 10}{x^2 - 3x - 10} = \frac{0}{0} \quad \text{This function has a vertical asymptote at } x = 5 \]

Suppose you are in charge of distributing the reward. Would you reward Elwin? Explain

Problem 6: \( \lim_{x \to 4} 2x^3 + x - 11 \)

Eddy’s brother in law’s girlfriend Esther did the following work and found the limit to be -135

<table>
<thead>
<tr>
<th>x</th>
<th>-125.7</th>
<th>4.001</th>
<th>-135</th>
<th>4.1</th>
<th>-136</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.9</td>
<td>-125.7</td>
<td>4.001</td>
<td>-135</td>
<td>4.1</td>
<td>-136</td>
</tr>
<tr>
<td>3.99</td>
<td>-134.1</td>
<td>4.01</td>
<td>-135</td>
<td>4.1</td>
<td>-136</td>
</tr>
<tr>
<td>3.999</td>
<td>-134.9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

She came to you to claim the reward for the most efficient work. What you think? Explain
Based on your observations of the problems described above compute the following limits using a method you consider most appropriate without using a graphing utility.

99. \( f(x) = \begin{cases} 
2x - 17, & x < 6 \\
\sqrt{2x + 13}, & x > 6
\end{cases} \)
find \( \lim_{x \to 6^-} f(x) \), \( \lim_{x \to 6^+} f(x) \), \( \lim_{x \to 6^-} f(x) \)

100. \( g(x) = \frac{x^3 + 2x - 12}{x - 7} \)
find \( \lim_{x \to 2} g(x) \)

101. \( h(x) = \frac{3x}{x^2 - 5x} \)
find \( \lim_{x \to 0} h(x) \), \( \lim_{x \to 5} h(x) \)

102. \( j(x) = \frac{7x^2 - 2x + 1}{x^2 + 5} \)
find \( \lim_{x \to 0} j(x) \), \( \lim_{x \to 0} j(x) \)