On the hypersurfaces of constant curvature in $S^{n+1}$ with boundary

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of The Ohio State University

By

Changhoon Lim, B.S.

Graduate Program in Mathematics

The Ohio State University

2010

Dissertation Committee:

Bo Guan, Adviser

Yuan Lou

Fei-Ran Tian
© Copyright by

Changhoon Lim

2010
ABSTRACT

We prove the existence of a hypersurface of constant Gauss curvature $K$ in $\mathbb{S}^{n+1}$ with $\Gamma$ as a boundary under the condition that $\Gamma$ bounds a certain locally convex hypersurface where $K$ is a given positive constant and $\Gamma$ is a disjoint collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$ of closed smooth embedded $(n-1)$ dimensional submanifolds of $\mathbb{S}^{n+1}$. We prove some important local properties of locally convex hypersurfaces then use this and a Perron method to show the convergence of an area minimizing sequence of hypersurfaces. Regularity of resulting hypersurface is studied.

We are also interested in the extension of the above result to hypersurfaces satisfying more general curvature condition and we need first the existence theorem to the Dirichlet problem of some fully nonlinear elliptic equation. To apply the Evans-Krylov theory and standard existence arguments, we establish \textit{a priori} estimates for principal curvatures of the surface which is the graph of the solution to the above mentioned PDE.
This is dedicated to my family.
ACKNOWLEDGMENTS

First and foremost, I would like to thank my advisor, Prof. Bo Guan, for his patience, guidance and constant support throughout my graduate study. I would like to thank Prof. Yuan Lou and Prof. Fei-Ran Tian for kindly agreeing and taking the time to be members of my dissertation committee. I wish to thank the Ohio State mathematics department for their support during my study. Finally I would like to thank my family, my wife Joo Yun and son Jeayun for their support and love.
VITA

1999 ........................................... B.S. Mathematics, Seoul National University
2003-present ................................. Graduate Teaching Associate, Department of Mathematics, The Ohio State University.

FIELDS OF STUDY

Major Field: Mathematics

Studies in:
    Topic 1  Partial differential equation
    Topic 2  Geometric Analysis
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>ii</td>
</tr>
<tr>
<td>Dedication</td>
<td>iii</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>iv</td>
</tr>
<tr>
<td>Vita</td>
<td>v</td>
</tr>
<tr>
<td><strong>Chapters:</strong></td>
<td></td>
</tr>
<tr>
<td>1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Constant Gauss curvature</td>
<td>2</td>
</tr>
<tr>
<td>1.2 General curvature function</td>
<td>4</td>
</tr>
<tr>
<td>2. Hypersurfaces of constant Gauss curvature</td>
<td>7</td>
</tr>
<tr>
<td>2.1 Coordinates systems and equation</td>
<td>7</td>
</tr>
<tr>
<td>2.2 Local estimates</td>
<td>16</td>
</tr>
<tr>
<td>2.3 Existence</td>
<td>28</td>
</tr>
<tr>
<td>2.4 Regularity</td>
<td>32</td>
</tr>
<tr>
<td>3. Curvature estimates</td>
<td>35</td>
</tr>
<tr>
<td>3.1 Notations and preliminary results</td>
<td>35</td>
</tr>
<tr>
<td>3.2 Interior curvature estimates</td>
<td>39</td>
</tr>
<tr>
<td>3.3 Boundary and global estimates for second derivatives</td>
<td>44</td>
</tr>
<tr>
<td>Bibliography</td>
<td>58</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

Given a disjoint collection $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$ of smooth Jordan curves in $\mathbb{R}^3$, it is a classical question in differential geometry to decide whether there exist surfaces of constant Gauss curvature with boundary $\Gamma$. An elementary necessary condition for $\Gamma \subset \mathbb{R}^3$ to span a surface $S$ of positive Gauss curvature is that each $\Gamma_i$ have no inflection points. However this condition is not sufficient in general as shown by Rosenberg [25]. More generally we can consider the existence of hypersurfaces $M$ in $\mathbb{R}^{n+1}$ with $\partial M = \Gamma$ such that $\Gamma \subset \mathbb{R}^{n+1}$, with each $\Gamma_i$ a closed smooth embedded $(n-1)$ dimensional submanifold of $\mathbb{R}^{n+1}$, and the Gauss curvature of $M$ is constant $K$. We call it a $K$-hypersurface.

Caffarelli, Nirenberg and Spruck [4] proved, using their results in the theory of Monge-Ampère equations, the existence of the $K$-hypersurfaces when the hypersurfaces are global graphs over strictly convex domains. This is a very strong geometric restriction since the surfaces then must be simply connected graphs. This result was extended by Hoffman, Rosenberg and Spruck [18] where the existence of subsolution is used as a condition and subsequently Guan and Spruck [10] obtained an essentially optimal existence result as far as global graphs are concerned. Moreover from
the observation in [10], Guan [11] proved the existence theorem for Monge-Ampère equations in domains of arbitrary geometry.

To solve the problem in parametric generality new techniques such as Perron methods and new geometric conditions were needed. A necessary condition for $\Gamma$ to bound a locally strictly convex hypersurface is that its second fundamental form is nondegenerate everywhere. However this condition is not sufficient. For example, Rosenberg [25] showed that there are topological obstructions. Based on the results in [10], Spruck [27] made the following conjecture: $\Gamma$ must bound an immersed K-hypersurface if it bounds a locally strictly convex immersed hypersurface. This conjecture was proved positively by Guan-Spruck [13] and Trudinger-Wang [28] independently.

Inspired by these results, we study the corresponding question for the hypersurfaces in the space $S^{n+1}$. We also try to extend this problem by changing the Gauss curvature condition to other type of curvature condition.

1.1 Constant Gauss curvature

The goal of the chapter 2 is to prove the existence of the hypersurfaces of constant Gauss curvature in $S^{n+1}$ with prescribed boundary.

A Hypersurface in $S^{n+1}$ can be represented locally by the graph of a function on $S^n$, and if this function is smooth then it satisfies certain Monge-Ampère type equation when the Gauss curvature of the hypersurface is constant. Therefore this problem locally reduces to the Dirichlet problem of Monge-Ampère type equations on a domain in $S^n$. To use the theory of elliptic partial differential equations, we need the resulting equations to be elliptic. This means that the second fundamental
form of the surface should be positive definite when the hypersurface is smooth. We say that a smooth hypersurface is \textit{locally convex (locally strictly convex)} if the second fundamental form is positive definite (semidefinite, respectively). Note that all the principal curvatures of $\mathbb{S}^n$ as a hypersurface in $\mathbb{S}^{n+1}$ are zero.

We need the geometric condition that $\Gamma$ must bound an immersed locally convex hypersurface. Our method is to deform this hypersurface to a $K$-hypersurface that we desire to obtain using the \textit{Perron method}. For this we need to consider non-smooth hypersurfaces. A hypersurface $M$ in $\mathbb{S}^{n+1}$ is said to locally convex if at every point $p \in M$ there exists $\Omega \subset \mathbb{S}^n$ and a neighborhood $\tilde{\Omega} \subset M$ of $p$ such that $\tilde{\Omega}$ is a graph of a nonnegative continuous function $\rho = u(\theta)$, $\theta \in \Omega$, in a coordinate system to be described in section 2.1 below, and locally the region $\rho \geq u(\theta)$ always lies on a fixed side of $M$.

The main theorem of chapter 2 is as follows:

\textbf{Theorem 1.1.1.} Let $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\}$ be a disjoint collection of closed smooth embedded $(n-1)$ dimensional submanifolds of $\mathbb{S}^{n+1}$ which is contained in a hemisphere $\mathbb{S}^n_\pi$. Suppose that there exists a locally convex immersed hypersurface $\Sigma$ in $\mathbb{S}^{n+1}_\pi$ with $\partial \Sigma = \Gamma$ and $K_\Sigma \geq K$ everywhere, where $K_\Sigma$ is the Gauss curvature of $\Sigma$ and $K$ is a positive constant. Assume also that $\Sigma$ is $C^2$ and locally strictly convex along its boundary. Then there exists a smooth locally strictly convex immersed hypersurface $M$ in $\mathbb{S}^{n+1}$ with $\partial M = \Gamma$ such that $K_M \equiv K$.

To prove this theorem, we follow the basic idea of Guan-Spruck [13]. However we need to check the details and make adjustments to make this idea work. In section 2.1, we introduce the local coordinate systems suitable to our argument. The hypersurface we seek to obtain can be locally represented by a graph of a function and this function
should satisfy a certain partial differential equation. We derive this equation using
the coordinate system we introduced. In section 2.2, we prove an important local
property of locally convex hypersurfaces in $\mathbb{S}^{n+1}$ (Theorem 2.2.1). From this we can
obtain the compactness of the sequence of locally convex hypersurfaces. Using the
results of section 2.2 we prove the existence of the K-hypersurface (Theorem 2.3.4)
from the Perron method and the area minimizing property of the deformation in
section 2.3. Finally in section 2.4, we show the hypersurface we obtain is smooth
when $K > 0$ which is Theorem 2.4.1.

1.2 General curvature function

In chapter 3 we will consider the problem of finding a hypersurface $M$ in $\mathbb{S}^{n+1}$
which satisfies

$$f(\kappa[M]) = K \quad (1.2.1)$$

with boundary condition

$$\partial M = \Gamma \quad (1.2.2)$$

where $\kappa[M] = (\kappa_1, \ldots, \kappa_n)$ denotes the principal curvatures of $M$ and $K$ is constant.

The function $f$ is symmetric, smooth and assumed to be defined in the convex
cone

$$\Gamma_n^+ \equiv \{ \lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \}$$

in $\mathbb{R}^n$ and it satisfies

$$f_i(\lambda) \equiv \frac{\partial f(\lambda)}{\partial \lambda_i} > 0 \text{ in } \Gamma_n^+, \ 1 \leq i \leq n, \quad (1.2.3)$$

and

$$f \text{ is a concave function.} \quad (1.2.4)$$
In addition, $f$ will be assumed to satisfy

$$f > 0 \text{ in } \Gamma_n^+, \quad f = 0 \text{ on } \partial \Gamma_n^+, \quad (1.2.5)$$

$$\sum f_i(\lambda) \lambda_i \geq \sigma_0 \quad \text{on } \{ \lambda \in \Gamma_n^+ : \psi_0 \leq f(\lambda) \leq \psi_1 \}, \quad (1.2.6)$$

for any $\psi_1 > \psi_0 > 0$, where $\sigma_0$ is a positive constant depending on $\psi_0$ and $\psi_1$, and for every $C > 0$ and every compact set $E$ in $\Gamma_n^+$ there exists $R = R(E, C) > 0$ such that

$$f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) \geq C \quad \forall \lambda \in E \quad (1.2.7)$$

and

$$f(R\lambda) \geq C \quad \forall \lambda \in E. \quad (1.2.8)$$

To be able to solve the existence problem, we need first to solve the Dirichlet problem of certain type of nonlinear partial differential equation such that the graph of the solution as a hypersurface in $\mathbb{S}^{n+1}$ satisfies the equation (1.2.1), i.e. the equation

$$f(\kappa[u]) = K \quad (1.2.9)$$

where $\kappa[u]$ denotes the principal curvatures of the graph of $u$ as a hypersurface in $\mathbb{S}^{n+1}$. We say that $u$ is a locally convex solution of (1.2.9) if the graph of $u$ is locally convex hypersurface in $\mathbb{S}^{n+1}$.

The critical part of proving local existence is to establish a priori second derivative estimates. The main result of chapter 3 is the following theorem.

**Theorem 1.2.1.** Let $\Omega \subset \mathbb{S}^n$ be a smooth domain which does not contain any hemisphere. Assume that (1.2.3) - (1.2.8) holds. Suppose $u \in C^{0,1}(\overline{\Omega})$ be a locally convex subsolution of (1.2.9) and $u$ is $C^2$ and locally strictly convex up to the boundary in a
neighborhood of $\partial \Omega$. Let $u \in C^\infty(\Omega)$ be a locally convex solution of (1.2.9) satisfying $u \geq \bar{u}$ in $\Omega$ and $u = \bar{u}$ in $\partial \Omega$. Then

$$|u|_{C^2(\Omega)} \leq C$$

(1.2.10)

where $C$ depends on $|u|_{C^2(\partial \Omega)}$, $|u|_{C^0(\Omega)}$ and the geometric quantities of $\partial \Omega$.

We note that from this theorem, we can apply the $C^{2,\alpha}$ estimates of Evans [7], Krylov [21] and the classical Schauder theory to derive a priori estimates for higher-order derivatives. The existence of the solution can then follow from the standard continuity method and degree arguments as in [4].

In section 3.1, we show some properties of symmetric function we consider and review the basic equations from the Riemannian geometry. We then derive formulas which the second fundamental form of the hypersurface should satisfy. In section 3.2 we derive the interior estimates for the curvature of the hypersurfaces which is equivalent to the interior second order derivatives estimates to the solution of the PDE. We use the intrinsic (covariant) calculation to derive the estimates. We prove the boundary estimates for the second derivatives, Theorem 3.3.2, in section 3.3 where we use the basic idea of [14] although the equation itself is not same so the detailed calculation is different. We then finish the proof of the Theorem 1.2.1 by showing $C^0$ and $C^1$ estimates.
CHAPTER 2

HYPERSURFACES OF CONSTANT GAUSS CURVATURE

2.1 Coordinates systems and equation

In this section we introduce a local coordinate system and derive equations which is to be used in the subsequent sections, then we introduce some notations for the proof of the main theorem. Let \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \) be a local coordinate of the unit sphere \( S^n \) and \( \sigma \) be the canonical metric on \( S^n \) induced from \( \mathbb{R}^{n+1} \). That means

\[
\sigma = \sigma_{ij} d\theta_i d\theta_j, \quad \sigma_{ij} = \langle \frac{\partial \mathbf{X}}{\partial \theta_i}, \frac{\partial \mathbf{X}}{\partial \theta_j} \rangle
\]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^{n+1} \) and

\[
\mathbf{X}(\theta) = (x_1(\theta), \ldots, x_{n+1}(\theta)) \in \mathbb{R}^{n+1}
\]

is the position vector of a point \( \theta \) of \( S^n \) in \( \mathbb{R}^{n+1} \). Note that \( x_1^2 + \cdots + x_{n+1}^2 = 1 \) so

\[
x_1 dx_1 + \cdots + x_{n+1} dx_{n+1} = 0 \quad \text{on } S^n. \tag{2.1.1}
\]

Let

\[
Y : (\theta_1, \ldots, \theta_n, \rho) \mapsto (Y_1, \ldots, Y_{n+2}) \in \mathbb{R}^{n+2}
\]
be the position map for \( \rho \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) where

\[
Y_1 = x_1(\theta) \cos \rho, \ldots, Y_{n+1} = x_{n+1}(\theta) \cos \rho, \quad \text{and} \quad Y_{n+2} = \sin \rho.
\]

Then \( Y_1^2 + \cdots + Y_{n+2}^2 = 1 \) and if we denote \( \tilde{g} \) the standard metric on \( S^{n+1} \),

\[
\tilde{g} = dY_1^2 + \cdots + dY_{n+2}^2
\]

\[
= (d(x_1(\theta) \cos \rho))^2 + \cdots + (d(x_{n+1}(\theta) \cos \rho))^2 + (d \sin \rho)^2
\]

\[
= d\rho^2 + \cos^2 \rho \sigma_{ij} d\theta_i d\theta_j
\]

by (2.1.1).

Let \( f \) be a \( C^2 \) function on \( \Omega \), a domain in \( S^n \), with range \( (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( \Sigma \subset S^{n+1} \) be the graph of \( \rho = f(\theta) \), and \( g \) be the Riemannian metric on \( \Sigma \) induced from \( \tilde{g} \). For the frame field \( e_1, \ldots, e_n \) induced from

\[
Z : (\theta_1, \theta_2, \ldots, \theta_n) \mapsto (\theta_1, \ldots, \theta_n, f(\theta)) \in S^{n+1}
\]

that is

\[
e_i \equiv \frac{\partial Z}{\partial \theta_i} = \frac{\partial}{\partial \theta_i} + \frac{\partial f}{\partial \theta_i \partial \rho} \in T\Sigma \subset TS^{n+1},
\]

we have

\[
g_{ij} = \tilde{g}(e_i, e_j) = \cos^2 f \sigma_{ij} + \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j}.
\]  (2.1.2)

Let \( \nabla \) denote the covariant differentiation on \( S^n \). Then \( \nabla f = \sigma^{ij} \frac{\partial f}{\partial \theta_j} \frac{\partial}{\partial \theta_i} \) where \( \sigma^{ij} \)

is the inverse of \( \sigma_{ij} \), and \( \cos^2 f \frac{\partial}{\partial \rho} - \nabla f \in TS^{n+1} \) is perpendicular to \( T\Sigma \) which can be seen by

\[
\tilde{g}(\cos^2 f \frac{\partial}{\partial \rho} - \sigma^{kj} \frac{\partial f}{\partial \theta_j} \frac{\partial}{\partial \theta_k}, \frac{\partial f}{\partial \theta_l} + \frac{\partial f}{\partial \theta_i} \frac{\partial}{\partial \rho}) = 0
\]

for \( i = 1, \ldots, n \). Since

\[
\tilde{g}(\cos^2 f \frac{\partial}{\partial \rho} - \sigma^{kj} \frac{\partial f}{\partial \theta_j} \frac{\partial}{\partial \theta_k}, \cos^2 f \frac{\partial}{\partial \rho} - \sigma^{lm} \frac{\partial f}{\partial \theta_m} \frac{\partial}{\partial \theta_l}) = \sigma^{lm} \frac{\partial f}{\partial \theta_l} \frac{\partial f}{\partial \theta_m} \cos^2 f + \cos^4 f,
\]

1
the (upward) unit normal vector to $\Sigma$ is given by

$$N = \frac{\cos^2 f \frac{\partial}{\partial \rho} - \nabla f}{\sqrt{\cos^2 f |\nabla f|^2 + \cos^4 f}}$$

where $|\nabla f|^2 = \sigma(\nabla f, \nabla f) = \sigma^{ij} \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j}$.

The second fundamental form $B$ of $\Sigma$ is the normal component of the covariant derivative $\tilde{\nabla} W V$ where $V = v^i e_i$ and $W = w^j e_j$ are tangent vector fields on $\Sigma$ and $\tilde{\nabla}$ denote the Riemannian connection on $S^{n+1}$, that is,

$$B(V, W) = \tilde{\nabla}_V W - \nabla_V W.$$

(We implicitly assume that $V$ and $W$ are extended to vector fields on $S^{n+1}$ when we consider $\tilde{\nabla} W V$. We know that it doesn’t depend on the extension.) Let $B_{ij} = B(e_i, e_j) = \tilde{g}(\tilde{\nabla}_{e_i} e_j, N)$.

Let $\Gamma^k_{ij}$ be the Christoffel symbols of $\nabla$ with respect to the frame $\{\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n}\}$ on $S^n$ and $\tilde{\Gamma}^m_{st}$ be the Christoffel symbols of $\tilde{\nabla}$ with respect to the frame $\{\frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_n}, \frac{\partial}{\partial \rho}\}$ on $S^{n+1}$. Note all these frames are coordinate frames so $\Gamma^k_{ij} = \Gamma^k_{ji}$ and $\tilde{\Gamma}^m_{st} = \tilde{\Gamma}^m_{ts}$. Since $\nabla$ is the projection of $\tilde{\nabla}$ onto $TS^n$ and $\tilde{g}(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \theta_l}) = 0$ for $i = 1, \ldots, n$, we have $\Gamma^k_{ij} = \tilde{\Gamma}^k_{ij}$ for $i, j, k \in \{1, \ldots, n\}$ from the calculation

$$\tilde{\nabla} \frac{\partial}{\partial \theta_j} = \tilde{\Gamma}^k_{ij} \frac{\partial}{\partial \theta_k} + \tilde{\Gamma}^\rho_{ij} \frac{\partial}{\partial \rho}$$

$$\nabla \frac{\partial}{\partial \theta_j} = \Gamma^k_{ij} \frac{\partial}{\partial \theta_k}$$

For the remaining types of Christoffel symbols of $\tilde{\nabla}$, we use the formula

$$\Gamma^k_{ij} = \frac{1}{2} \tilde{g}^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$
Then, \((i,j,k)\) are always from 1 to \(n\) below

\[
\tilde{\Gamma}_i^j = \frac{1}{2} \bar{g}^{ij} \left( \frac{\partial}{\partial \theta_i} \tilde{g}_{jp} + \frac{\partial}{\partial \theta_j} \tilde{g}_{ip} - \frac{\partial}{\partial \rho} \tilde{g}_{ij} \right) \\
= \frac{1}{2} (2 \cos \rho \sin \rho \sigma_{ij}) = \sin \rho \cos \rho \sigma_{ij}
\]

\[
\tilde{\Gamma}_i^j = \frac{1}{2} \bar{g}^{ij} \left( \frac{\partial}{\partial \rho} \tilde{g}_{ip} + \frac{\partial}{\partial \rho} \tilde{g}_{jp} - \frac{\partial}{\partial \rho} \tilde{g}_{ij} \right) = 0 = \tilde{\Gamma}_i^j
\]

\[
\tilde{\Gamma}_i^j = \frac{1}{2} \bar{g}^{ij} \left( \frac{\partial}{\partial \theta_j} \tilde{g}_{ip} + \frac{\partial}{\partial \theta_i} \tilde{g}_{ip} - \frac{\partial}{\partial \theta_i} \tilde{g}_{ij} \right) = 0
\]

\[
\tilde{\Gamma}_i^j = \frac{1}{2} \bar{g}^{ij} \left( \frac{\partial}{\partial \theta_j} \tilde{g}_{ip} + \frac{\partial}{\partial \rho} \tilde{g}_{ip} - \frac{\partial}{\partial \rho} \tilde{g}_{ij} \right) = \frac{1}{2} \tilde{\Gamma}_i^j
\]

Now,

\[
\nabla_{e_i e_j} = \nabla_{\bar{e}_i} + \frac{\partial f}{\partial \rho} \left( \frac{\partial}{\partial \rho} \tilde{g}_{ij} + \frac{\partial f}{\partial \rho} \right)
\]

\[
= \nabla_{\bar{e}_i} \frac{\partial}{\partial \rho} \tilde{g}_{ij} + \nabla_{\bar{e}_i} \frac{\partial f}{\partial \rho} \left( \frac{\partial}{\partial \rho} \tilde{g}_{ij} + \frac{\partial f}{\partial \rho} \right)
\]

Therefore

\[
\nabla_{e_i e_j} \left( \Gamma_k^i \frac{\partial}{\partial \theta_k} + \tilde{\Gamma}_j^i \frac{\partial}{\partial \rho} \right) + \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} \frac{\partial}{\partial \rho} + \frac{\partial f}{\partial \theta_i} \left( \tilde{\Gamma}_j^i \frac{\partial}{\partial \theta_k} + \tilde{\Gamma}_j^i \frac{\partial}{\partial \theta_k} \right)
\]

\[
= \Gamma_k^i \frac{\partial}{\partial \theta_k} + \sin \rho \cos \rho \sigma_{ij} \frac{\partial}{\partial \rho} \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} \frac{\partial}{\partial \rho} \tan \rho \delta_k^i \frac{\partial}{\partial \theta_k}
\]

\[
- \frac{\partial f}{\partial \theta_i} \tan \rho \delta_k^i \frac{\partial}{\partial \theta_k}
\]

\[
= \left( \Gamma_k^i \frac{\partial}{\partial \theta_k} + \frac{\partial f}{\partial \theta_k} \tan \rho \delta_k^i \frac{\partial}{\partial \theta_k} \right) \frac{\partial}{\partial \theta_k} \left( \sin \rho \cos \rho \sigma_{ij} + \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} \frac{\partial}{\partial \rho} \right)
\]

\[
= \left( \Gamma_k^i \frac{\partial}{\partial \theta_k} + \frac{\partial f}{\partial \theta_k} \tan \rho \delta_k^i \frac{\partial}{\partial \theta_k} \right) \frac{\partial}{\partial \theta_k} \left( \sin \rho \cos \rho \sigma_{ij} + \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} \right) \frac{\partial}{\partial \rho}.
\]
So on $\Sigma$,

$$
\tilde{g}(\tilde{\nabla}_e e_j, N) = \frac{1}{\sqrt{\cos^2 f|\nabla f|^2 + \cos^4 f}} \cdot
g(\tilde{\nabla}_e e_j, \cos^2 f f \frac{\partial}{\partial \rho} - \sigma^{ls} \frac{\partial f}{\partial \theta_s} \frac{\partial}{\partial \theta_l})
$$

$$
= \frac{1}{\sqrt{\cos^2 f|\nabla f|^2 + \cos^4 f}} \left( \begin{array}{c}
- \cos^2 f \sigma_{kl} \sigma^{ls} \frac{\partial f}{\partial \theta_s} (\Gamma^k_{ij} - \frac{\partial f}{\partial \theta_j} \tan f \delta^k_i) \\
- \frac{\partial f}{\partial \theta_i} \tan f \delta^k_j \end{array} \right) + \cos^2 f \left( \sin f \cos f \sigma_{ij} + \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} \right)
$$

$$
= \frac{\cos^2 f}{\sqrt{\cos^2 f|\nabla f|^2 + \cos^4 f}} \left( \begin{array}{c}
- \delta^k_i \frac{\partial f}{\partial \theta_s} (\Gamma^k_{ij} - \frac{\partial f}{\partial \theta_j} \tan f \delta^k_i) \\
+ \sin f \cos f \sigma_{ij} + \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} \end{array} \right)
$$

$$
= \frac{\cos f}{\sqrt{|\nabla f|^2 + \cos^2 f}} \left( - \Gamma^k_{ij} \frac{\partial f}{\partial \theta_k} + 2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \tan f + \sin f \cos f \sigma_{ij} + \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} \right)
$$

$$
= \frac{\cos f}{\sqrt{|\nabla f|^2 + \cos^2 f}} \left( \nabla_{ij} f + 2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \tan f + \sin f \cos f \sigma_{ij} \right).
$$

where $\nabla_{ij} f$ denote the second covariant derivative of $f$ (or Hessian of $f$), that is

$$
\nabla_{ij} f = \nabla^2_{\alpha_i, \alpha_j} f = \nabla_{\frac{\partial}{\partial \alpha_i}} \nabla_{\frac{\partial}{\partial \alpha_j}} f - \nabla_{\frac{\partial}{\partial \alpha_i}} \frac{\partial}{\partial \alpha_j} f
$$

$$
= \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} - \nabla_{\Gamma^k_{ij} \frac{\partial}{\partial \alpha_k}} f = \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} - \Gamma^k_{ij} \frac{\partial f}{\partial \theta_k}.
$$

Therefore the second fundamental form is

$$
h_{ij} = \frac{\cos f}{\sqrt{|\nabla f|^2 + \cos^2 f}} \left( \nabla_{ij} f + 2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \tan f + \sin f \cos f \sigma_{ij} \right). \quad (2.1.3)
$$

For simplicity let

$$
U_{ij} := \nabla_{ij} f + 2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \tan f + \sin f \cos f \sigma_{ij}.
$$

The Gauss curvature of $\Sigma$ is defined as

$$
K = \frac{\det(h_{ij})}{\det(g_{ij})}
$$

From (2.1.2) and (2.1.3), we obtain

$$
K = (\cos f)^{-n+2} (|\nabla f|^2 + \cos^2 f)^{-\frac{n}{2} - 1} \frac{\det(U_{ij})}{\det(\sigma_{ij})}. \quad (2.1.4)
$$
Let $u(\theta) = \tan f(\theta)$, then

$$\frac{\partial u}{\partial \theta_i} = \sec^2 f \frac{\partial f}{\partial \theta_i},$$

$$\frac{\partial^2 u}{\partial \theta_i \partial \theta_j} = \sec^2 f \left(2 \tan f \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} + \frac{\partial^2 f}{\partial \theta_i \partial \theta_j}\right)$$

so that

$$\cos^2 f \frac{\partial^2 u}{\partial \theta_i \partial \theta_j} = 2 \tan f \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} + \frac{\partial^2 f}{\partial \theta_i \partial \theta_j}$$

Therefore

$$U_{ij} = \nabla_{ij} f + 2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \tan f + \sin f \cos f \sigma_{ij}$$

$$= \frac{\partial^2 f}{\partial \theta_i \partial \theta_j} - \Gamma^k_{ij} \frac{\partial f}{\partial \theta_k} + 2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j} \tan f + \sin f \cos f \sigma_{ij}$$

$$= - \cos^2 f \frac{\partial u}{\partial \theta_i} \frac{\partial u}{\partial \theta_j} + \sin f \cos f \sigma_{ij} + \cos^2 f \frac{\partial^2 u}{\partial \theta_i \partial \theta_j}$$

$$= \cos^2 f \nabla_{ij} u + \sin f \cos f \sigma_{ij}$$

$$= \cos^2 f (\nabla_{ij} u + \tan f \sigma_{ij})$$

$$= \cos^2 f (\nabla_{ij} u + u \sigma_{ij})$$

and

$$\det(U_{ij}) = (\cos f)^{2n} \det(\nabla_{ij} u + u \sigma_{ij})$$

Note also that

$$|\nabla f|^2 = \sigma_{ij} \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_j}$$

$$= \sigma_{ij} \left(\cos^2 f \frac{\partial u}{\partial \theta_i} \right) \left(\cos^2 f \frac{\partial u}{\partial \theta_j} \right)$$

$$= \cos^4 f |\nabla u|^2$$
and \( \cos^2 f = \frac{1}{1+u^2} \), so
\[
(cos f)^{-n+2}(|\nabla f|^2 + \cos^2 f)^{-\frac{n}{2} - 1} = (cos f)^{-n+2}(\cos^2 f)^{-\frac{n}{2} - 1}(1 + \cos^2 f|\nabla u|^2)^{-\frac{n}{2} - 1}
\]
\[
= (\cos f)^{-2n}(1 + \frac{|\nabla u|^2}{1 + u^2})^{-\frac{n}{2} - 1}
\]
Therefore (2.1.4) becomes
\[
K = \left( 1 + \frac{|\nabla u|^2}{1 + u^2} \right)^{-\frac{n}{2} - 1} \frac{\det(\nabla_{ij}u + u\sigma_{ij})}{\det(\sigma_{ij})} \quad (2.1.5)
\]
or
\[
\det(\nabla_{ij}u + u\sigma_{ij}) = K \det(\sigma_{ij}) \left( 1 + \frac{|\nabla u|^2}{1 + u^2} \right)^{\frac{n}{2} + 1} \quad (2.1.6)
\]
We have a following theorem which is from Theorem 5.3 of [11].

**Theorem 2.1.1.** Let \( \Omega \subset S^n \) be a smooth domain that does not contain any hemisphere and \( \phi \in C^\infty(\partial \Omega) \). Then (2.1.6) has a solution \( u \in C^\infty(\overline{\Omega}) \) satisfying \( \{ \nabla_{ij}u + u\sigma_{ij} \} > 0 \) in \( C^\infty(\overline{\Omega}) \) and \( u = \phi \) on \( \overline{\Omega} \), provided that there exists a subsolution \( \bar{u} \in C^2(\overline{\Omega}) \) with \( \{ \nabla_{ij}\bar{u} + u\sigma_{ij} \} > 0 \) in \( C^\infty(\overline{\Omega}) \) and \( \bar{u} = \phi \) on \( \overline{\Omega} \).

To see the corresponding equation in \( \mathbb{R}^n \) using specific coordinate systems on \( S^n \), suppose \( \Omega \), the domain of \( u \), is completely contained in the interior of the upper hemisphere of \( S^n \). Let \( E_1, \ldots, E_{n+1} \) be the standard basis for \( \mathbb{R}^{n+1} \). The indices \( i, j, k \) always runs from 1 to \( n \) below.

Consider the stereographic projection \( \phi \) of upper hemisphere to \( x_{n+1} = 1 \) plane from the center of the sphere,
\[
\phi^{-1}(\theta_1, \ldots, \theta_n) = \left( \frac{\theta_1}{\sqrt{|\theta|^2 + 1}}, \ldots, \frac{\theta_n}{\sqrt{|\theta|^2 + 1}}, \frac{1}{\sqrt{|\theta|^2 + 1}} \right) \in \mathbb{R}^{n+1},
\]
where \( |\theta|^2 = \theta_1^2 + \cdots + \theta_n^2 \). Let \( w = \sqrt{|\theta|^2 + 1} \). Then
\[
\frac{\partial}{\partial \theta_i} = \left( \frac{\delta_{ik}}{w} - \frac{\theta_i \theta_k}{w^3} \right) E_k - \frac{\theta_i}{w^3} E_{n+1}.
\]
So

\[ \sigma_{ij} = \langle \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \rangle = \frac{1}{w^2} (\delta_{ij} - \frac{\theta_i \theta_j}{w^2}) \]

\[ \sigma^{ij} = w^2 (\delta_{ij} + \theta_i \theta_j) \]

and

\[ \Gamma^k_{ij} = \frac{1}{2} \sigma^{kl} (\partial_i \sigma_{jl} + \partial_j \sigma_{il} - \partial_l \sigma_{ij}) \]

\[ = \frac{w^2}{2} (\delta_{kl} + \theta_k \theta_l) \left( \frac{\partial}{\partial \theta_i} w^2 + \frac{\partial}{\partial \theta_j} w^2 - \frac{\partial}{\partial \theta_l} w^2 \right) \]

\[ - (\frac{\partial}{\partial \theta_i} w^4 + \frac{\partial}{\partial \theta_j} w^4 - \frac{\partial}{\partial \theta_l} w^4) \]

\[ = \frac{w^2}{2} (\delta_{kl} + \theta_k \theta_l) \left( \frac{4 \theta_i \theta_j \theta_l}{w^6} - \frac{2 \delta_{jl}}{w^4} - \frac{2 \delta_{il}}{w^4} \right) \]

\[ = -\frac{1}{w^2} (\delta_{jk} \theta_i + \delta_{ik} \theta_j). \]

Therefore for \( u \) above,

\[ \nabla_{ij} u = \frac{\partial^2 u}{\partial \theta_i \partial \theta_j} - \Gamma^k_{ij} \frac{\partial u}{\partial \theta_k} \]

\[ = \frac{\partial^2 u}{\partial \theta_i \partial \theta_j} + \frac{1}{w^2} (\delta_{jk} \theta_i + \delta_{ik} \theta_j) \frac{\partial u}{\partial \theta_k} \]

and

\[ \nabla_{ij} u + u \sigma_{ij} = \frac{\partial^2 u}{\partial \theta_i \partial \theta_j} + \frac{1}{w^2} \left( \frac{\partial u}{\partial \theta_j} \theta_i + \frac{\partial u}{\partial \theta_i} \theta_j \right) + \frac{\delta_{ij} u}{w^2} - \frac{\theta_i \theta_j u}{w^4}. \]

If we let \( v = wu \) then

\[ \frac{\partial v}{\partial \theta_i} = \frac{\theta_i u}{w} + \frac{\partial u}{\partial \theta_i} \]

\[ \frac{1}{w^2} \frac{\partial^2 v}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 u}{\partial \theta_j \partial \theta_i} + \frac{1}{w^2} \left( \frac{\partial u}{\partial \theta_j} \theta_i + \frac{\partial u}{\partial \theta_i} \theta_j \right) + \frac{\delta_{ij} u}{w^2} - \frac{\theta_i \theta_j u}{w^4} \]

so

\[ \frac{1}{w^2} \frac{\partial^2 v}{\partial \theta_i \partial \theta_j} = \nabla_{ij} u + u \sigma_{ij} \]

(2.1.7)
and (2.1.6) becomes
\[
\det\left( \frac{\partial^2 v}{\partial \theta_i \partial \theta_j} \right) = K \left( \frac{1}{|\theta|^2 + 1} + \sum_{i=1}^{n} \left( \frac{\partial v}{\partial \theta_i} \right)^2 - 2v \frac{\partial v}{\partial \theta_i} \theta_i + \frac{|\theta|^2 v}{|\theta|^2 + 1} + \left( \frac{\partial v}{\partial \theta_i} \theta_i \right)^2 \right) \frac{n}{2+1} \tag{2.1.8}
\]

since
\[
\det(\sigma_{ij}) = \frac{1}{w^2}(1 - |\theta|^2) = \frac{1}{w^{2n+2}}
|\nabla u|^2 = w^2(\delta_{ij} + \theta_i \theta_j) \frac{1}{w} \left( \frac{\partial v}{\partial \theta_i} - \theta_i v \right) \frac{1}{w} \left( \frac{\partial v}{\partial \theta_j} - \theta_j v \right)
= \sum_{i=1}^{n} \left( \frac{\partial v}{\partial \theta_i} \right)^2 - 2v \frac{\partial v}{\partial \theta_i} \theta_i + \frac{|\theta|^2 v}{|\theta|^2 + 1} + \left( \frac{\partial v}{\partial \theta_i} \theta_i \right)^2.
\]

We will need the following existence result which follows from Theorem 1.1 of [11] by approximation.

**Theorem 2.1.2.** Let $\Omega$ be a bounded domain in $\mathbb{S}^n$ with $\partial \Omega \in C^{0,1}$. Suppose there exists a locally convex viscosity subsolution $u \in C^{0,1}(\Omega)$ of (2.1.8) where $K \geq 0$ is a constant. Then there exists a unique locally convex viscosity solution $u \in C^{0,1}(\Omega)$ of (2.1.8) satisfying $u = u_0$ on $\partial \Omega$.

We will consider a hypersurface in $\mathbb{S}^{n+1}$, and use the coordinate systems in section 2.1 for $\mathbb{S}^n$. Let $\Phi : \Sigma_0 \to \mathbb{S}^{n+1}$ be an immersion where $\Sigma_0$ is a manifold of dimension $n \geq 2$ with boundary $\partial \Sigma_0$ which may be empty. We will often identify $\Phi$ with its image $\Sigma := \Phi(\Sigma_0)$ and call $\Sigma$ a hypersurface of $\mathbb{S}^{n+1}$. Similarly, the boundary of $\Sigma$, $\partial \Sigma$, means the immersion $\Phi : \partial \Sigma_0 \to \mathbb{S}^{n+1}$. When we consider a point $p \in \Sigma$, it should be understood as one of its preimages in $\Sigma_0$.

In this thesis, any embedded submanifold $\mathbb{S}^n$ of $\mathbb{S}^{n+1}$ will be called a hyper-sphere which will play a similar role as hyperplanes in $\mathbb{R}^{n+1}$.

Let $M$ be a hypersurface of $\mathbb{S}^{n+1}$ with or without boundary. We say that $M$ is a locally convex hypersurface if for every point $p$ of $M$, there is a neighborhood $U(p)$ of
$p$ in $M$, a hyper-sphere $\mathbb{S}^n$ containing $p$ and a nonnegative continuous function $u$ on $\Omega(p) \subset \mathbb{S}^n$ such that $0 \leq u < \frac{\pi}{2}$ on $\Omega(p)$ (note $u = 0$ at $p$), $\Omega(p)$ is strongly convex in $\mathbb{S}^n$ and $U(p)$ is represented as the graph $\rho = u$ on a domain $\Omega(p) \subset \mathbb{S}^n$, and locally the region $\rho \geq u(\theta)$ always lies on a fixed side of $M$. We call such $\mathbb{S}^n$ a \textit{local supporting sphere} at $p$.

\section{2.2 Local estimates}

In this section we prove a $C^{0,1}$ estimates for the hypersurfaces which is Theorem 2.2.1 below. Let $\Sigma$ and $M$ be connected, compact, locally convex hypersurfaces in $\mathbb{S}^{n+1}$ with $\partial \Sigma = \partial M$. We assume as follows:

i) $\Sigma$ and $M$ are oriented in such a way that they induce the same orientation on the boundary.

ii) There is a fixed constant $\delta > 0$ such that the hypersurface $\Sigma_\delta := \{ x \in \Sigma : d_\Sigma(x, \partial \Sigma) < \delta \}$ is $C^2$ up to the boundary and locally strictly convex which means that the second fundamental form is positive definite, where $d_\Sigma$ denotes the intrinsic (Riemannian) distance on $\Sigma$.

iii) $M$ locally lies on the inner side of $\Sigma$ along the boundary. By this, we mean that both $\Sigma$ and $M$ locally lie on the same side of the (unique) local supporting sphere to $\Sigma$ at any point of $\partial \Sigma$.

iv) Any neighborhood of $\partial M$ in $M$ does not intersect $\Sigma_\delta$ in the interior.

Let $\Pi$ denote the second fundamental form of $\partial \Sigma$, which is $C^2$, as a submanifold of $\mathbb{S}^{n+1}$. Then,
Theorem 2.2.1. At every point $p$ on $M$, there is a domain $\Omega_p \subset S^n$ of a fixed lower bound in size and compactly contained in a hemisphere of $S^n$ and a Lipschitz function $u$ defined on $\Omega_p$ such that the graph of $u$ on $\Omega_p$ is a portion of $M$ containing $p$, and

$$\|u\|_{C^{0,1}(\Omega_p)} \leq C_1$$

(2.2.1)

where $C_1$ depends on $\delta, \kappa_{\min}[\Sigma], \kappa_{\max}[\Sigma], \max_{\partial \Sigma} |\Pi|$ and $d_M$.

More precisely, that $\Omega_p$ is of fixed lower bound in size means that there is $r > 0$ which depends only on $\delta, \kappa_{\min}[\Sigma], \kappa_{\max}[\Sigma], \max_{\partial \Sigma} |\Pi|$ and $d_M$ but not $p$, such that $\Omega_p$ contains a geodesic ball of radius $r$ in $S^n$.

**Proof.** Step 1. Let $p$ be an arbitrary point on $\partial \Sigma$ and $X$ a unit tangent vector to $\partial \Sigma$ at $p$. Since $\Sigma$ is locally strictly convex near $\partial \Sigma$, we have

$$\nu_\Sigma(p) \cdot \Pi(X, X) \geq \kappa_{\min}[\Sigma] > 0$$

(2.2.2)

where $\cdot$ is the standard Euclidean inner product in the tangent space which generate the metric in $S^{n+1}_+$. Let $\beta = \frac{1}{2} \sin^{-1}(\kappa_{\min}[\Sigma]/\max_{\partial \Sigma} |\Pi|)$. Then

$$\frac{\nu_\Sigma(p) \cdot \Pi(X, X)}{\|\Pi(X, X)\|} \geq \frac{\kappa_{\min}[\Sigma]}{\max_{\partial \Sigma} |\Pi|} = \sin 2\beta > 0$$

(2.2.3)

Thus the angle between $\nu_\Sigma(p)$ and $\Pi(X, X)$ does not exceed $\frac{\pi}{2} - 2\beta$.

By the local convexity of $M$ and assumptions i), iii), we have $\nu_M(p) \cdot \Pi(X, X) \geq 0$ too. From (2.2.3) we see that the angle between $\nu_M(p)$ and $\nu_\Sigma(p)$ does not exceed $\pi - 2\beta$.

Choose an orthonormal basis $e_1, \ldots, e_{n+1}$ of $T_p S^{n+1}_+$ so that $e_n$ and $e_{n+1}$ are normal to $T_p \partial \Sigma$ and

$$\nu_\Sigma(p) = e_n \cos \beta + e_{n+1} \sin \beta$$

(2.2.4)
Note first that the angle between $\nu_M(p)$ and $e_{n+1}$ does not exceed $\frac{\pi}{2} - \beta$.

Now, there is a hypersphere $P \subset S^+_{n+1}$ such that $e_1, \ldots, e_n$ is a basis for $T_p P$. Note that $T_p \partial \Sigma \subset T_p P$. Choose a coordinate $(\theta_1, \ldots, \theta_n, \rho)$ on $S^+_{n+1}$ near $p$ using $P = S_n \cap S^+_{n+1}$ as in the previous section such that $\frac{\partial}{\partial \rho} = e_{n+1}$ at $p$ and $\theta_i$'s are the normal coordinate of $P$ at $p$ satisfying $\frac{\partial}{\partial \theta_i} = e_i$ at $p$ for $i = 1, \ldots, n$. Recall $\tilde{g} = d\rho^2 + \cos^2 \rho d\theta^2$ where $\tilde{g}$ is the standard metric on $S^+_{n+1}$.

Since the angle between $\nu_\Sigma$ and $\frac{\partial}{\partial \rho}|_p$ is $\frac{\pi}{2} - \beta$ and therefore the angle between $\Pi(X, X)$ and $\frac{\partial}{\partial \rho}|_p$ is not less than $\beta$, $\Sigma_\delta$ can be locally near $p$ expressed as a graph of a $C^2$ function $u$ over a domain $\Omega_p \subset P$ and $\partial \Sigma = \partial M$ is a graph over a portion, which we denote as $\Gamma$, of $\partial \Omega_p$.

By our choice of coordinate, $\frac{\partial}{\partial \theta_1}|_p, \ldots, \frac{\partial}{\partial \theta_{n-1}}|_p$ forms a basis for $T_p \partial \Sigma = T_p \partial M$, and the hyper-sphere $\tilde{S}^n \cap S^+_{n+1}$ at $p$ determined by $\frac{\partial}{\partial \theta_1}|_p, \ldots, \frac{\partial}{\partial \theta_{n-1}}|_p, \frac{\partial}{\partial \rho}|_p$ is a local supporting sphere of $\Sigma$ and $M$ at $p$. Therefore, $\frac{\partial}{\partial \theta_1}|_p, \ldots, \frac{\partial}{\partial \theta_{n-1}}|_p$ are tangent vectors to $\partial \Omega_p$ at $p$ and $\Omega_p$ is contained in the region in $P$ defined by $\theta_n \geq 0$ locally near $p$.

By (2.2.3) and (2.2.4), the angle between $e_n$ and $\Pi(X, X)$ does not exceed $\frac{\pi}{2} - \beta$, that is

$$e_n \cdot \Pi(X, X) \geq |\Pi(X, X)| \sin \beta \geq \kappa_{\min} |\Sigma_\delta| \sin \beta$$

(2.2.5)

for any unit tangent vector $X$ to $\partial \Sigma$ at $p$. Consequently after a rotation of the coordinate $(\theta_1, \ldots, \theta_{n-1})$, $\Gamma$ can be represented as a graph

$$\theta_n = \varphi(\theta') \equiv \sum_{i=1}^{n-1} a_i \theta_i^2 + O(|\theta'|^3), \quad \theta' = (\theta_1, \ldots, \theta_{n-1})$$

(2.2.6)

for some constants $a_i$, $1 \leq i \leq n - 1$, satisfying

$$0 < \kappa_{\min} |\Sigma_\delta| \sin \beta \leq a_i \leq \max_{\partial \Sigma} |\Pi|, \quad 1 \leq i \leq n - 1$$

(2.2.7)
By shrinking $\Omega_p$ as necessary, we may assume $\Omega_p = \{ \varphi < \theta_n < 2r \}$ for some uniform constant $r > 0$ which depends on $\delta, \beta$ and $\kappa_{\text{max}}[\Sigma_\delta]$. Clearly this contains a geodesic ball of radius, say, $\min\{ \frac{r}{2}, \frac{1}{2\kappa_{\text{min}}[\Sigma_\delta]\sin\beta} \}$ which means that $\Omega_p$ has a lower bound in size.

Let us call, for a moment, $L$ an upper function on $\overline{\Omega}_p$ if $L(\theta)$ is a function on $\overline{\Omega}_p$ such that the graph of it is a portion of a submanifold $S^n \cap S^{n+1}_+$ of $S^{n+1}_+$. Define $v$ on $\overline{\Omega}_p$ by

$$v(\theta) = \sup\{ L(\theta) : L \text{ is an upper function on } \overline{\Omega}_p, L \leq u \text{ on } \Gamma \} \tag{2.2.8}$$

We have

$$u \leq v \leq \max_{\Omega} u \text{ in } \overline{\Omega}_p, \quad v = u \text{ on } \Gamma_p$$

and

$$\text{Lip}_{\overline{\Omega}_p}(v) \leq \max_{\Omega_p} \| \nabla u \| + C$$

where $\text{Lip}_{\overline{\Omega}_p}(v)$ denotes the uniform Lipschitz constant of $v$ on $\overline{\Omega}_p$. Note that the graph of $v$ on $\overline{\Omega}_p$ is a locally convex hypersurface $N_p$ in $S^{n+1}_+$ with a portion of a boundary is same with $\Sigma$ and $M$ that is the graph over $\Gamma$.

Now, $M$ can be expressed as a graph of a function $u$ over $\Omega_p$ since the angle between $\nu_M(p)$ and $e_{n+1}$ does not exceed $\frac{\pi}{2} - \beta$ as we noted above, and also $M$ is between $\Sigma$ and $N_p$ by local convexity of $M$ and the assumptions iii) and iv). We observe also that

$$u \leq u \leq v \text{ in } \overline{\Omega}_p \quad \text{and} \quad \text{Lip}_{\overline{\Omega}_p}(u) \leq C \tag{2.2.9}$$

where $C$ depends on $r$ and $\| u \|_{C^1(\overline{\Omega}_p)}$. This shows the theorem holds for any boundary points of $M$. 

19
To go further, let us define some notations. Let \( q \in M, \ Q_q \subset S^{n+1}_+ \) be a hypersphere through \( q \) and \( \mu_q \in T_q S^{n+1}_+ \) be a normal vector of \( Q_q \) at \( q \) such that \( \nu_q : \mu_q > 0 \) where \( \nu_q \) is the normal vector of \( M \) at \( q \). Consider the unit speed geodesic \( \alpha : [0, \pi) \to S^{n+1}_+ \) such that \( \alpha(0) = q, \ \dot{\alpha}(0) = \mu_q \). For \( t \geq 0 \), there is a unique hypersphere \( S^n_{\alpha(t)} \) such that \( \dot{\alpha}(t) \) is a normal vector to \( S^n_{\alpha(t)} \) at \( \alpha(t) \). Note that \( S^n_{\alpha(t)} \) divide \( S^{n+1}_+ \) into two regions. Let \( S^{n+1}_q \) be the one of them which contains \( \alpha([0, t]) \) and let \( U_t := S^{n+1}_q \cap M \). By Lemma 1, \( U_t \) is convex.

**Step 2.** Let \( q \) be an interior point of \( M \) and we first assume that there is a hypersphere \( Q_q \) through \( q \), which either is a local supporting sphere or is transversal to \( M \) at \( q \) such that \( U_t \cap \partial M = \emptyset \) for all \( t > 0 \) sufficiently small.

Let \( t_0 > 0 \) be the smallest value such that \( U_{t_0} \cap \partial M \neq \emptyset \). Since \( -q \), the antipodal point of \( q \), is not contained in \( U_{t_0} \cap \partial M \), choose a point \( p \in U_{t_0} \cap \partial M \) which is closest to \( q \). Note that \( U := U_{t_0} \) is convex. Let \( C_q = C_q(\partial U) \) be the convex cone generated by \( \partial U \) with vertex \( q \). This is understood as follows. Note \( \partial U \) does not contain \( -q \) so for each \( x \in \partial U \) there is a unique minimizing geodesic joining \( q \) and \( x \), say \( \gamma_{q,x}(t) \) with \( \gamma_{q,x}(0) = q, \gamma_{q,x}(\pi) = -q \), then \( \cup_{x \in \partial U} \gamma_{q,x}([0, \pi]) \) is a closed \( n \)-dimensional cone and the closure of its inside region is called \( C_q(\partial U) \).

We will show that \( C_q \) contains a nondegenerate cone of fixed size which is independent of \( q \). At \( p \), as in the step 1, we choose a coordinate \( (\theta_1, \cdots, \theta_n, \rho) \) on \( S^{n+1}_+ \) near \( p \) such that \( \frac{\partial}{\partial \rho} = e_{n+1} \) at \( p \) and \( (\theta_1, \cdots, \theta_n) \) is the normal coordinate of a hypersphere \( P_p \) at \( p \) satisfying \( \frac{\partial}{\partial \theta_i} = e_i \) at \( p \) for \( i = 1, \cdots, n \) and (2.2.4). Recall that the angle beween \( \nu_M \) and \( \nu_{\Sigma} \) at any point on \( \partial \Sigma \) does not exceed \( \pi - 2\beta \). Note \( \partial U \subset S^n_{\alpha(t_0)} \) and since \( U \) is convex, the angle between \( \nu_{\Sigma_{\alpha(t_0)}}(p) \) and \( \nu_{\Sigma}(p) \) does not exceed \( \pi - 2\beta \) too. This implies that, if we let \( \gamma_{p,q} \) be the unique unit speed minimizing geodesic from \( p \)
to $q$ with $\gamma_{p,q}(0) = p$, $\gamma_{p,q}(s_0) = q$, then $\dot{\gamma}_{p,q}(0) \cdot e_n > 0$. We also note that

$$\nu_{S_{\alpha(c_o)}}^\alpha(p) = e_n \cos \beta_0 + e_{n+1} \sin \beta_0$$

for some $\beta_0 \in [\beta, \pi - \beta]$. Rotate $S_{\alpha(t_0)}^n$ fixing $p$ and $-p$ until it contains $\gamma_{p,q}([0, s_0])$ and denote it as $S_{p,q}^n$, then $T_p S_{p,q}^n$ and $T_p S_{p,q}^n$ are subspaces of $T_p S_{\alpha(t_0)}^{n+1}$ and note that $\{ \frac{\partial}{\partial \theta_1}, \cdots, \frac{\partial}{\partial \theta_n}, \frac{\partial}{\partial \rho} \}$ at $p$ forms an orthonormal basis for $T_p S_{\alpha(t_0)}^{n+1}$. We can choose a new coordinate $(\theta'_1, \cdots, \theta'_n, \rho')$ near $p$ such that $\theta'_i$ are a normal coordinates of $S_{p,q}^n$ satisfying $\theta_i = \theta'_i$ for $1 \leq i \leq n - 1$, $\theta'_n(q) > 0$ and $\rho'(q) = 0$. Note that $\theta'_n(q) > \theta_n(q)$.

Let $r > 0$ be as in step 1. If $\theta_n(q) < \frac{r}{2}$ then $q$ is on the graph of $u$ over $\Omega_p$ and we are done. Therefore let us assume that $\theta_n(q) \geq \frac{r}{2}$. Let $\tau_i$ be the unit speed geodesic starting from $p$ such that $\dot{\tau}_i(0) = \frac{\partial}{\partial \theta'_i}(p)$ for $1 \leq i \leq n$ and $\dot{\tau}_{n+1}(0) = \frac{\partial}{\partial \rho}(p)$, then $\dot{\tau}_i(0) = e_i$ for $1 \leq i \leq n - 1$. We have

$$\frac{(\dot{\tau}_k(0) \cdot \gamma_{p,q}(0))^2}{|\gamma_{p,q}(0)|^2} \leq 1 - \left( \frac{r}{2d_M} \right)^2, \quad \text{for all } 1 \leq k \leq n - 1 \quad (2.2.10)$$

From the convexity of $U$, we see that $C_q$ contains the cone generated by $\Gamma(p)$ with vertex $q$ since $\Gamma(p)$ and $q$ are separated by $S_{\alpha(t_0)}^n$. By (2.2.6), (2.2.7) and (2.2.10), the projection of $C_q$ along the direction $-\frac{\partial}{\partial \rho'}$ to $S_{p,q}^n$ contains an $n$-ball $B_{r_0}(p)$ in $S_{p,q}^n$ where $r_0 \geq c_0$ for a uniform constant $c_0 > 0$. To complete the proof, we have to find a point $p_0$ with

$$\dot{\gamma}_{q,p}(0) \cdot \dot{\gamma}_{q,p_0}(0) \leq C'_0 < 1 \quad (2.2.11)$$

for some uniform constant $C'_0 > 0$ such that the cone generated by the convex hull of $\Gamma(p) \cup \{p_0\}$ with vertex $q$ is contained in $C_q$, where $\pi(p_0)$ denote the projection of $p_0$ to $S_{p,q}^n$, that is, $\theta'_i(\pi(p_0)) = \theta'_i(p_0)$ for $1 \leq i \leq n$ and $\rho'(\pi(p_0)) = 0$. Note that (2.2.11) is true if

$$\frac{d(\pi(p_0), q)}{d(\pi(p_0), p_0)} \leq C'_0$$

21
for some uniform constant $\tilde{C}_0$. Since $d(\pi(p_0), q) \leq d_M$ and $d(\pi(p_0), p_0) = \rho'(p_0)$, in terms of coordinate, it suffice to find a point $p_0$ such that

$$\rho'(p_0) > 0 \quad \text{and} \quad \frac{d_M}{\rho'(p_0)} \leq C_0$$

for some uniform constant $C_0 > 0$ such that the cone generated by the convex hull of $\Gamma(p) \cup \{p_0\}$ with vertex $q$ is contained in $C_q$.

For $0 \leq t \leq \theta'_n(q)$, let $S^n_{\tau_n(t)}$ be the hypersphere containing $\tau_n(t)$ such that $\nu_{S^n_{\tau_n(t)}}(p) = \dot{\tau}_n(t)$. $S^n_{\tau_n(t)}$ divide $\mathbb{S}^{n+1}$ into two half spheres and let $\tilde{V}_t$ be the one of them which does not contain $\tau_n([0, t])$. Note $q \in \tilde{V}_t$ and let $W_t := \tilde{V}_t \cap q M$. If $W_{\theta'_n(q)} \cap \partial U \neq \emptyset$ then we are done since we can choose $p_0 \in W_{\theta'_n(q)} \cap \partial U$ and that satisfies (2.2.12).

We thus may assume that $W_{\theta'_n(q)} \cap \partial U = \emptyset$. Note that then $W_{\theta'_n(q)} \subset U$ and is therefore a convex cap. We may find $t_1 \in [0, \theta'_n(q))$ such that $W_t \cap \partial M = \emptyset$ for all $t_1 < t \leq \theta'_n(q)$ and $W_{t_1} \cap \partial M \neq \emptyset$. Note that $W_{t_1}$ is also a convex cap and $W_{t_1} \setminus U \subset C_q$ by convexity since for any $p_1 \in W_{t_1} \setminus U$, the geodesic joining $p_1$ and $q$ should be contained in $C_q$.

If $t_1 = 0$ then

$$\dot{\tau}_n(0) \cdot \Pi(X, X) \leq 0, \quad \text{for all} \quad X \in T_p \partial M \quad (2.2.13)$$

since $\Gamma(p)$ lies in the half space $\theta'_n \leq 0$, that is, $\Gamma(p)$ and $q$ are seperated by the hypersphere $\mathbb{S}^n_{\tau_n(0)}$. This and the condition that $M$ locally lies on the inner side of $\Sigma$ along the boundary implies that $\Gamma(p)$ is contained in the half spaces $\rho' \geq 0$ and $\rho \geq 0$. If follows from (2.2.5) and $\Gamma(p)$ is contained in $\rho' \geq 0$ that $\dot{\tau}_{n+1}(0) \cdot e_n \geq \sin \beta$, that is, the angle between $\dot{\tau}_{n+1}(0)$ and $e_n$ does not exceed $\frac{\pi}{2} - \beta$. 22
Let \( p_0 \) be a point on \( \Gamma(p) \) such that \( \theta_n(p_0) = \frac{\pi}{2} \). Let \( \gamma \) be the unit speed geodesic starting from \( p \) with \( \dot{\gamma}(0) = e_n \) and denote \( \gamma_n(\theta_n(p_0)) \) by \( p_1 \), then \( S_{\gamma_n(\theta_n(p_0))}^n \) contains \( p_0, p_1 \) and \( \tau_{n+1}(t) \) intersects with \( S_{\gamma_n(\theta_n(p_0))}^n \) transversally. Let \( s_0 \) be the smallest value of \( t \) such that \( \tau_{n+1}(s_0) := p_2 \) contained in \( S_{\gamma_n(\theta_n(p_0))}^n \), then \( \rho'(p_2) = s_0 \). Since \( \theta'_n(p_0) \leq 0 \), we have \( \rho'(p_0) \geq s_0 \). Note \( p, p_0 \) and \( p_1 \) form a triangle on \( S_{n+1}^+ \subset S^{n+1} \) with side lengths \( d(p, p_1) = r^2 \), \( d(p, p_1) = s_0 \), and \( d(p_1, p_2) = \rho(p_2) \). By the Law of Cosines on \( S^{n+1} \),
\[
\cos(\rho(p_2)) = \cos\left(\frac{r}{2}\right) \cos(s_0) + \sin\left(\frac{r}{2}\right) \sin(s_0) \cos(b)
\]
where \( b \) is the angle between \( \dot{\tau}_{n+1}(0) \) and \( e_n \). From this, we get
\[
\cot(s_0) = \cot\left(\frac{r}{2}\right) \cos(b) \geq \cot\left(\frac{r}{2}\right) \sin(\beta)
\]
since \( b \) does not exceed \( \frac{\pi}{2} - \beta \), so
\[
\rho'(p_0) \geq s_0 \geq \cot^{-1}(\cot\left(\frac{r}{2}\right) \sin(\beta))
\]
and we see that this \( p_0 \) satisfy (2.2.12).

We now assume \( t_1 > 0 \) and take any point \( p_1 \in W_{t_1} \cap \partial M \). We have \( p_1 \in (W_{t_1} \setminus U) \cup \partial U \subset C_q \) so the geodesic joining \( p_1 \) and \( q \) is contained in \( C_1 \). Since \( W_{t_1} \cap \{ \rho' \leq 0 \} \subset U \), we have
\[
\rho'(p_1) > 0 \text{ and } 0 < \theta_n(p_1) = t_1 < \theta_n(q)
\]
Consider the parallel translate of \( \frac{\partial}{\partial p├\tau_n(t_1)} \) and \( \tilde{\tau}_n(t_1) \), which themselves are parallel translate of \( \frac{\partial}{\partial p├\tau_n(p)} \) and \( \frac{\partial}{\partial p├\tau_n(p)} \) along \( \tau_n \), along the geodesic from \( \tau_n(t_1) \) to \( p_1 \) and denote it by \( \tilde{\tau}_{n=1} \) and \( \tilde{\tau}_n \) respectively. Similar to (2.2.13), we have
\[
\tilde{\tau}_n \cdot X = 0 \text{ and } \tilde{\tau}_n \cdot \Pi(X, X) \leq 0 \text{ for all } X \in T_{p_1} \partial M
\]
Let $\epsilon_0 > 0$ be a small constant which will only depend on $\beta$. We have three cases.

(i) There is a unit vector $X \in T_{p_1}\partial M$ such that $|X \cdot \tilde{\tau}_{n+1}| > \epsilon_0$

(ii) There is a unit vector $X \in T_{p_1}\partial M$ such that $|\Pi(X, X) \cdot \tilde{\tau}_{n+1}| > \epsilon_0$

(iii) $|X \cdot \tilde{\tau}_{n+1}| \leq \epsilon_0$ and $|\Pi(X, X) \cdot \tilde{\tau}_{n+1}| \leq \epsilon_0$ for all unit vector $X \in T_{p_1}\partial M$

If (i) is the case then let $\gamma_X$ be the geodesic in $\partial M$ tangential to $X$ at $p_1$, that is $\dot{\gamma}_X(0) = X$. Recall that the points on $\gamma_X$ do not intersect $S_{\alpha(t_0)}^n$ which contains $\partial U$ unless it is $p$ and $d(p, p_1) \geq t_1$. Therefore we can find a point $p_0$ on $\gamma_X$ near $p_1$ such that $\rho'(p_0) \geq 0$ and $\rho'(p_1) - \rho'(p_0) \geq c_0$ for some universal constant $c_0 > 0$ depends on $\epsilon_0$ hence $\rho'(p_1) \geq c_0$ and (2.2.12) holds. If (ii) is the case then note that, since the angle between $\Pi(X, X)$ and $\nu_\Sigma(p_1)$ does not exceed $\pi/2 - 2\beta$, $\Pi(X, X) \cdot \tilde{\tau}_{n+1} > \epsilon_0$ is not possible because $M$ locally lies on the inner side of $\Sigma$ along the boundary. Therefore we have $\Pi(X, X) \cdot \tilde{\tau}_{n+1} < -\epsilon_0$. We consider $\gamma_X$ as above, then because the extrinsic curvature of $\gamma_X$ in the direction of $-\tilde{\tau}_{n+1}$ is greater than $\epsilon_0$ at $p_1$, we can find $p_0$ as above case and (2.2.12) holds again for $p_1$. Finally suppose (iii). Note this implies

$$\nu_\Sigma(p_1) \cdot \tilde{\tau}_n \leq 0$$  \hspace{1cm} (2.2.16)

when $\epsilon_0$ is sufficiently small depending on $\beta$, since the angle between $\Pi(X, X)$ and $-\tilde{\tau}_n$ is sufficiently small while that between $\Pi(X, X)$ and $\nu_\Sigma(p_1)$ does not exceed $\pi/2 - 2\beta$.

By (2.2.14) and (2.2.16) we obtain

$$\nu_\Sigma(p_1) \cdot \tilde{\tau}_{n+1} \leq 0$$  \hspace{1cm} (2.2.17)

since the geodesic $\gamma_{p_1, q}$ joining $p_1$ and $q$ locally lies on the inner side of $\Sigma$ near $p_1$. Consider the plane section $Sec(V)$ of $S^{n+1}$ determined by $V$ where $V$ is 2-dimensional
subspace of $T_{p_1}S^{n+1}$ with basis $\{\tilde{\tau}_{n+1}, \tilde{\gamma}_{p_1,q}(0)\}$. By (2.2.16), (2.2.17) and the local strict convexity of $\Sigma_\delta$, there exists a point $z \in \Sigma_\delta \cap \text{Sec}(V)$ with

$$\rho'(p_1) - \rho'(z) \geq c_1$$

for some uniform constant $c_1 > 0$ depending on $\delta$ and $\kappa_{\min}[\Sigma_\delta]$. Since $z$ must lie above the geodesic $\gamma_{p_1,q}$ relative to $S^n_{p,q}$ by the convexity of $M$ and the assumption that $\Sigma_\delta$ does not intersect $M$ in interior, we have $\rho'(z) \geq \rho'(q) = 0$. Thus $\rho'(p_1) \geq c_1$ and $p_1$ satisfies (2.2.12) where $C_0 > 0$ depends on $\delta$ and $\kappa_{\min}[\Sigma_\delta]$.

Step 3. We now assume that there is no hypersphere $Q_q$ through $q$ such that $U_t \cap \partial M = \emptyset$ for all $t > 0$ sufficiently small.

Let $P_q$ be a local supporting sphere at $q$ to $M$ and let $E$ denote the set of points on $\partial M$ that belong to $P_q \cap_q M$ then $E \neq \emptyset$ by assumption. Note that $q$ is contained in the convex hull of $E$. Suppose this is not the case, that is, $q$ and $E$ are separated by a hypersphere, then we may assume based on the coordinate system on $P_q$ centered at some point $\tilde{q}$ that $P_q = \{\rho = 0\}$ and $q$ lies in the region $\theta_n > \epsilon$ while $E$ is in $\theta_n < -\epsilon$ for some $\epsilon > 0$. Therefore by rotating $P_q$ about $\tilde{Q}$ sufficiently small we get the contradiction. Now we can conclude that $q$ is contained in an $l$-dimensional simplex $S$ with vertices in $E$ for some $1 \leq l \leq n$. We have then $S \subset P_q \cap_q M$ by the local convexity of $M$.

Let $p$ be a vertex of $S$. Then $P_q$ is a local supporting sphere to $M$ at every point on the geodesic segment $pq$ joining $p$ and $q$. Recall $M$ locally near $p$ is the graph of a convex function over a domain $\Omega_p$ which is a region of the hypersphere $P$ in the above Step 1. Note that based on the coordinate system about $P$,

$$\nu_{P_q} = \epsilon_n \cos \theta + \epsilon_{n+1} \sin \theta$$
for some $\theta \in [\beta, \pi - \beta]$. Therefore $\overline{pq}$ intersects $P$ hence $\Omega_p$ only at $p$. From this, we can see that $\overline{pq}$ transversal to $\partial M$ at $p$ since $\Omega_p$ is strictly convex. Consequently, $P_q$ is the tangent hypersurface to $M$ at $p$ as $P_q$ contains $\overline{pq}$ and is tangential to $\partial M$ at $p$. Next assume furthermore that $\overline{pq} \subset \overline{pp_1} \subseteq S$ for some $p_1 \neq q$. Let $Q$ be a local supporting sphere to $M$ at $p_1$. Then $\overline{pp_1} \subset Q$ and therefore $Q$ is a local supporting hypersphere to $M$ at every point on $\overline{pp_1}$. We have $Q = P_q$ since both are the tangent hypersphere to $M$ at every point on $\overline{pp_1}$. Consequently, $u$ extends along $\overline{pp_1}$.

As we can always find a point $p \in E$ such that the segment $\overline{pq}$ extends in $S$, we see the uniqueness of the local supporting sphere to $M$ at $q$. Using induction on $l$ we will next prove the assertion in the Theorem at point $q$.

Let us first consider the case $l = 1$, that is, $S = \overline{pp_1}$ where $p, p_1 \in \partial M$. Suppose $d(p, q) \leq d(p_1, q)$ and let $\bar{\Omega}$ be the convex hull in $P$ of $\{p'_1\} \cup \Omega_p$ where $p'_1 \in P$ with $p_1 = (p'_1, \rho(p_1))$. (Similar meaning for $q'$ below.) As in Step 2 we may assume $\theta_n(q) \geq \frac{\pi}{2}$. This implies (2.2.10), that is the angle between $\overline{pq}$ and $e_k$ has a uniform positive lower bound for all $1 \leq k \leq n - 1$. Let

$$v(\theta) = \sup\{L(\theta) : L \text{ is an upper function, } L \leq u \text{ at } p'_1 \text{ and on } \Omega_p\}, \theta \in \bar{\Omega}. \quad (2.2.18)$$

Then $v$ is a convex function. We have $u \leq v$ where $u$ is defined in $\bar{\Omega}$. Since $d(p_1, q) \geq d(p, q) \geq \frac{\pi}{2}$, by (2.2.10) there exists a uniform constant $\lambda > 0$ depending on $r$ and $\max_{\partial \Omega} |\Pi|$, such that the geodesic $n$-ball $B_{\lambda}(q')$ is contained in $\bar{\Omega}$. By the local convexity of $M$ we see $u$ is defined on $B_{\lambda}(q') \subset \bar{\Omega}$ with a uniform bound on $||\bar{u}||_{C^0(B_{\lambda}(q'))}$. This completes the proof for $l = 1$.

Assume now $l > 1$ and suppose we have proved the assertion for any point in a simplex of dimension less than $l$ with vertices in $E$. Choose $p \in E$ and $p_1$ on an $(l - 1)$
dimensional face of $S$ such that $q \in \overline{pp_1}$. If $d(p, q) \leq d(p_1, q)$ then the proof follows as exactly in case $l = 1$. Let us therefore assume $d(p, q) \geq d(p_1, q)$. By induction, in a suitable coordinate system $(\theta', \rho')$, where $(\theta', 0)$ is in some hypersphere $Q'$ and coordinate is centered at $p_1$, $M$ locally near $p_1$ is the graph of a convex function $\rho' = u(\theta')$ with a uniform $C^{0,1}$ bound in a geodesic $n$-ball $B_R(0)$ where $R$ is a uniform constant. Since $P_q$ is the tangent hypersphere to $M$ at any point on $\overline{pp_1}$ as observed above, we have $\nu_{P_q} \cdot \frac{\partial}{\partial \rho} \geq c_0$ at every points in $\overline{pp_1}$ for some uniform constant $c_0 > 0$. Thus $u$ extends along $\overline{pp_1}$. Replacing the convex function $v$ in (2.2.18) by

$$v(\theta') = \sup\{L(\theta') : L \text{ is an upper function, } L \leq u \text{ at } p \text{ and on } B_R(0)\},$$

defined in the convex hull of $\{p\} \cup B_R(0)$, the rest of proof follows that of case $l = 1$. This, finally, completes our proof.

As a consequence we have

**Theorem 2.2.2.** There exist uniform constants $R, r > 0$ depending on $\delta$, $\kappa_{\min}[\Sigma_\delta]$, $\kappa_{\max}[\Sigma_\delta]$, $\max_{\Sigma}|\Pi|$ and $d_M$ such that for any $p \in M$, $M \cap_p B_R(p)$ is embedded and the convex body in $B_R(p)$ bounded by $M \cap_p B_R(p)$ contains a geodesic ball of radius $r$.

**Theorem 2.2.3.** Let $\{M_k\}$ be a sequence of locally convex hypersurfaces contained in $\mathbb{S}^{n+1}_+$ with $\partial M_k = \partial \Sigma$ for all $k$. Suppose each $M_k$ lies on the inner side of $\Sigma$ and does not intersect $\partial \Sigma$. Then there exists a subsequence $\{M_k_i\}$ converging in Hausdorff metric to a locally convex hypersurface $M$ with $\partial M = \partial \Sigma$. Moreover, for each $i$ there exists a homeomorphism from $M_{k_i}$ on to $M$ with boundary fixed.
2.3 Existence

Throughout this section, let $\Sigma$ be a locally convex immersed hyper-surface in $S^{n+1}_+$ with embedded boundary $\partial \Sigma$ and Gauss curvature $K_\Sigma \geq K$ everywhere on $\Sigma$ where $K$ is a fixed nonnegative constant.

Let $D \subset \Sigma$ be a disk (a geodesic ball) on $\Sigma$ which, as a hypersurface in $S^{n+1}$, may be represented as the graph of a geodesically convex function $u$ defined in a domain $\Omega$ in some hypersphere with Lipschitz boundary. By Theorem 2.1.2, there is a unique function $u \in C^{0,1}(\Omega)$ whose graph is a convex hypersurface $\tilde{D}$ of constant Gauss curvature $K$ with $\partial \tilde{D} = \partial D$. By the comparison principle, we have $u \geq u$ in $\Omega$. Thus $\tilde{D}$ lies on the inner side of $D$.

This induces a $C^{0,1}$-diffeomorphism $\Psi_D : \Sigma \to \tilde{\Sigma} := \tilde{D} \cup (\Sigma \setminus D)$ which is fixed on $\Sigma \setminus D$. The hypersurface $\tilde{\Sigma}$ is locally convex with $K_{\tilde{\Sigma}} \geq K$ and $\partial \tilde{\Sigma} = \partial \Sigma$. We call $\tilde{\Sigma}$ a basic lifting of $\Sigma$ (by $\tilde{D}$ over $D$). A lifting of $\Sigma$ is a hypersurface which is obtained by a finite number of basic liftings starting from $\Sigma$. We introduce a partial order $\preceq$ between liftings of $\Sigma$, which is, $\Sigma_1 \preceq \Sigma_2$ iff $\Sigma_2$ is a lifting of $\Sigma_1$ or $\Sigma_1 = \Sigma_2$.

**Lemma 2.3.1.** Let $\Sigma_1$ and $\Sigma_2$ be any two liftings of $\Sigma$. Then there is a unique lifting, $\Sigma_1 \lor \Sigma_2$, of $\Sigma$ such that $\Sigma_1 \preceq \Sigma_1 \lor \Sigma_2$, $\Sigma_2 \preceq \Sigma_1 \lor \Sigma_2$ and $\Sigma_1 \lor \Sigma_2 \preceq N$ for any lifting $N$ with $\Sigma_1 \preceq N$ and $\Sigma_2 \preceq N$.

**Proof.** We first assume $\Sigma_1$ is a basic lifting of $\Sigma$ by $\tilde{D}_1$ over a disk $D_1 \subset \Sigma$ and let $A$ be the open region in $S^{n+1}_+$ bounded by $D_1 \cup \tilde{D}_1$. Assume $\Sigma_2$ to be a lifting of $\Sigma$ over a region $D_2$ where $D_2$ is not necessarily a disk. If $\Sigma, \Sigma_1$ and $\Sigma_2$ are all embedded, then the hypersurface

$$\Sigma_1 \lor \Sigma_2 := (\Sigma_2 \setminus (\Sigma_2 \cap \overline{A})) \cup (\tilde{D}_1 \setminus (\tilde{D}_1 \cap B))$$
where $B$ is the open regions in $S_{n+1}$ bounded by $\Sigma_2 \cup \Sigma$, is the lifting of $\Sigma$ with the desired properties. In the general case when some of these hypersurfaces may be immersed, we view $\Sigma$ as an immersion

$$\Psi_0 : \Sigma_0 \to \Sigma \subset S_{n+1}$$

of a differentiable manifold $\Sigma_0$ and let

$$\Psi_i : \Sigma_0 \to \Sigma_i \subset S_{n+1}, \quad i = 1, 2$$

be the immersions induced from the liftings. Note that $\Psi_i = \Psi_0$ on $\Sigma_0 \setminus \Sigma_0^{-1}(D_i)$. The lifting $\Sigma_1 \cupdot \Sigma_2$ is then given by the immersion

$$\Psi : \Sigma_0 \to \Sigma_1 \cupdot \Sigma_2 := \Psi(\Sigma_0) \subset S_{n+1}$$

defined as

$$\Psi(p) := \begin{cases} 
\Psi_1(p) & \text{if } p \in \Psi_0^{-1}(D_1) \setminus \Psi_0^{-1}(D_2) \\
\Psi_1(p) & \text{if } p \in \Psi_0^{-1}(D_1) \cap \Psi_0^{-1}(D_2) \text{ and } \Psi_2(p) \in A \\
\Psi_2(p) & \text{otherwise}
\end{cases}$$

for $p \in \Sigma_0$. The general case follows by induction. \hfill \Box

**Lemma 2.3.2.** Let $\Sigma_1$ and $\Sigma_2$ be liftings of $\Sigma$. If $\Sigma_1 \preceq \Sigma_2$ then $\text{Vol}(\Sigma_1) \geq \text{Vol}(\Sigma_2)$. Moreover, the equality holds iff $\Sigma_1 = \Sigma_2$.

**Proof.** We may assume $\Sigma_2$ is a basic lifting of $\Sigma_1$ over a disk $D_1 \subset \Sigma_1$. Suppose $D_1$ and its lifting $D_2 \subset \Sigma_2$ are the graphs of convex functions $u_1$ and $u_2$ over a domain $\Omega \subset S^n$, respectively. We have $u_1 \leq u_2$ in $\overline{\Omega}$ and $u_1 = u_2$ on $\partial \Omega$. Let

$$N(x) = \frac{\nabla u_2 - \cos^2 u_2 \frac{\partial}{\partial \rho}}{\sqrt{\cos^2 u_2 |\nabla u_2|^2 + \cos^4 u_2}}$$
on $\Omega$

denote the downward unit normal vector field to $D_2$. Thus $\text{div}N$, the distributional mean curvature of $D_2$ with respect to the upward normal vector, is nonnegative almost
every where since $u_2$ is a convex function. Let

$$\omega = \{(x, \rho) \in \mathbb{S}^{n+1}_+; u_1(x) < \rho < u_2(x), x \in \Omega\}$$

and extend $N$ to $\tilde{N}$ on $\omega$ so that $\tilde{N}$ at $(x, \rho) \in \omega$ is equal to $N(x)$, then $div \tilde{N}$ is nonnegative almost everywhere so by the divergence theorem we have

$$0 \leq \int_{\omega} div \tilde{N} dV = \int_{D_1} \tilde{N} \cdot \nu_1 d\sigma - \int_{D_2} d\sigma = Vol(D_1) - Vol(D_2) + \int_{D_1} (\tilde{N} \cdot \nu_1 - 1) d\sigma$$

where $nu_1$ is the downward unit normal vector to $D_1$. Since $0 \leq \tilde{N} \cdot \nu_1 \leq 1$ on $D_1$ ($0 \leq N \cdot \nu_1 \leq 1$ on $\Omega$) we have $Vol(D_1) - Vol(D_2) \geq 0$; Obviously, the equality holds only when $D_1 = D_2$ since $\partial D_1 = \partial D_2$.

We need one more lemma which states that volume is continuous under uniform convergence of uniformly Lipschitz convex functions.

**Lemma 2.3.3.** Let $w_k$ be a sequence of uniformly Lipschitz convex functions on $\Omega$ converging uniformly to $\omega$. Then

$$\int_{\Omega} |\cos^{-1} w| \sqrt{\cos^2 w + |\nabla w|^2} dV = \lim_{k \to \infty} \int_{\Omega} |\cos^{-1} w_k| \sqrt{\cos^2 w_k + |\nabla w_k|^2} dV$$

where $dV$ is the volume form on $\mathbb{S}^n$.

**Proof.** Note that $|\cos^{-1} w_k| \sqrt{\cos^2 w_k + |\nabla w_k|^2} \leq \sqrt{1 + |\nabla w_k|^2}$ so if we can show

$$\int_{\Omega} \sqrt{1 + |\nabla w|^2} dV = \lim_{k \to \infty} \int_{\Omega} \sqrt{1 + |\nabla w_k|^2} dV$$

then by the dominated convergence theorem, we are done. Let $W_k = \sqrt{1 + |\nabla w_k|^2}$. Then $|\nabla W_k| \leq |\nabla^2 w_k| \leq \Delta w_k$ a.e. in $\Omega$ since $w_k$ is convex. Therefore,

$$\int_{\Omega} |\nabla W_k| dV \leq \sup_{\Omega} |\nabla w_k| |\partial \Omega|$$

Hence $W_k$ are uniformly bounded in $W^{1,1}$ and so converge in $L^1$ to $\sqrt{1 + |\nabla w|^2}$.  \qed
Let \( \mathcal{L} \) be the collection of liftings of \( \Sigma \) and set \( \mu = \inf_{L \in \mathcal{L}} \text{Vol}(L) \).

**Theorem 2.3.4.** Suppose \( \Sigma_\delta \) is \( C^2 \) and locally strictly convex up to the boundary for some fixed \( \delta > 0 \). There is a locally convex hypersurface \( M \) in \( S_+^{n+1} \) of class \( C^{0,1} \) up to the boundary with \( \partial M = \partial \Sigma \) and \( K_M \equiv K \). Moreover, \( M \) is homeomorphic to \( \Sigma \) and \( \text{Vol}(M) = \mu \).

**Proof.** For each \( k \geq 1 \), choose \( \Sigma_k \in \mathcal{L} \) such that \( \text{Vol}(\Sigma_i) \leq \mu + \frac{1}{k} \). By Lemma 2.3.1 and 2.3.2, we may assume \( \Sigma_k \leq \Sigma_{k+1} \) for all \( k \geq 1 \). From Theorem 2.2.3, after passing to a subsequence we may assume \( \{\Sigma_k\} \) converges in Hausdorff metric to a locally convex hypersurface \( M \) which, in addition, is homeomorphic to each \( \Sigma_k \). Note \( \partial M = \partial \Sigma \). It remains to show \( \text{Vol}(M) = \mu \) and \( K_M \equiv K \). Consider a point \( p \in M \). There exists a sequence \( p_k \in \Sigma_k, k = 1, 2, \ldots \) converging to \( p \) in \( S_+^{n+1} \) such that \( \Sigma_k \cap p_k B_R(p_k) \) converges to \( M \cap p B_R(p) \) in Hausdorff metric where \( R > 0 \). By Theorem 2.2.1 when \( R \) is chosen sufficiently small each \( \Sigma_k \cap p_k B_R(p_k) \) can be represented as the graph of a convex function \( w_k \) with a uniform \( C^{0,1} \) norm bound (independent of \( k \)). We may choose a coordinate system in \( S_+^{n+1} \) such that, after possibly passing to subsequences, all the functions \( w_k \) are defined in a fixed domain \( \Omega \subset S^n \) satisfying

\[
||w_k||_{C^{0,1}(\Omega)} \leq C_0 \text{ independent of } k \tag{2.3.1}
\]

and \( w_k \) converges uniformly to a function \( w \in C^{0,1}(\overline{\Omega}) \) whose graph represents \( M \) locally. Hence by Lemma 2.3.3 and a covering argument \( \text{Vol}(M) = \mu \). Consider now the Dirichlet problem for the Gauss curvature equation (2.1.8). Using \( w_k \) as a subsolution for each \( k \geq 1 \), by Theorem 2.1.2 we obtain a unique convex solution \( u_k \in C^{0,1}(\overline{\Omega}) \) of (2.1.8) satisfying \( u_k = w_k \) on \( \partial \Omega \). We have \( u_k \geq w_k \) on \( \overline{\Omega} \) and by
(2.3.1),
\[ \|u_k\|_{C^{0,1}(\Omega)} \leq C_0 \] independent of \( k \)

Thus there is a subsequence, which we still denote by \( \{u_k\} \), converging to a convex function \( u \) in \( C^{0,1}(\Omega) \). We see \( u \) satisfies (2.1.8) in the viscosity sense and \( u \geq w \) on \( \Omega \) with \( u = w \) on \( \partial \Omega \). On the other hand, for each \( k \geq 1 \), let \( \tilde{\Sigma}_k \) be the lifting of \( \Sigma_k \) obtained by replacing \( D_k \) with \( \tilde{D}_k \), where \( D_k \) and \( \tilde{D}_k \) are the graphs of \( w_k \) and \( u_k \) over \( \Omega \), respectively. Similarly, let \( \tilde{M} \) be the locally convex hypersurface obtained from \( M \) by replacing the graph of \( w \) over \( \Omega \) by that of \( u \). Clearly \( \tilde{\Sigma}_k \) converges to \( \tilde{M} \) as \( u_k \) converges uniformly to \( u \) on \( \Omega \). Since by Lemma 2.3.2, \( \mu \leq Vol(\tilde{\Sigma}_k) \leq Vol(\Sigma_k) \) for each \( k \) it follows that \( Vol(\tilde{M}) = \mu \) and therefore \( Vol(\tilde{D}) = Vol(D) \). As both \( u \) and \( w \) are convex functions, this implies \( u \equiv w \) on \( \Omega \) by the proof of Lemma 2.3.2. Since \( u \) satisfies (2.1.8) \( M \) has constant Gauss curvature \( K \) in a neighborhood of \( p \).

**2.4 Regularity**

Throughout this section, we assume that \( \Sigma \) is a locally convex immersed hypersurface which is \( C^2 \) and locally strictly convex in a neighborhood of, and up to, \( \partial \Sigma \). In addition, we assume \( \partial \Sigma \) to be embedded and smooth. Let \( K \leq \kappa_{\min}[\Sigma_{\delta}] \) be a nonnegative constant and let \( M \) be the locally convex hypersurface with \( K_M \equiv K \) and \( \partial M = \partial \Sigma \) constructed above. Note \( M \) is \( C^{0,1} \) up to the boundary.

**Theorem 2.4.1.** If \( K > 0 \) then \( M \) is smooth up to the boundary and locally strictly convex.

**Proof.** Consider an interior point \( p \in M \). Since \( M \) is of class \( C^{0,1} \), \( M \) locally near \( p \) can be represented as a graph of a convex function \( \rho = u(\theta) \geq 0 \) over a domain
\( \Omega_1 \subset S^n \) with a \( C^{0,1} \) norm bound

\[
||u||_{C^{0,1}(\Omega)} \leq C_1
\]

Note since \( M \subset S^{n+1}_+ \), \( \Omega_1 \) is contained in the upper hemisphere of \( S^n \). Therefore using stereographic projection as a coordinate on \( S^n \) as in the section 1 and \( v = \tan(u)\sqrt{|\theta|^2 + 1} \), it follows from (2.1.7) and (2.1.8) that \( v \) is convex and satisfies the inequalities in the viscosity sense

\[
K \leq \det(v_{ij}) \leq K(1 + 4C_1^2) \frac{n+2}{2} \text{ in } \Omega \subset \mathbb{R}^n.
\]

By a theorem of Caffarelli [2], the nodal set \( \{v = 0\} \) either is a single point, in which case \( M \) is smooth and strictly convex at \( p \), or does not contain any interior extreme points. We will show that \( \{v = 0\} = \{0\} \). Suppose this is not the case. By the proof (Step 3) of Theorem 2.2.1, we can find two points \( q_1, q_2 \in \partial M \) such that \( \overline{q_1q_2} \subset M \cap \{\rho = 0\} \) and \( \rho = 0 \) is a local supporting sphere of \( M \) at every point on \( \overline{q_1q_2} \) and \( q_1q_2 \) is transversal to \( \partial M \) at the endpoints. We may assume

\[
q_i = (0, \ldots, 0, (-1)^{i}a) \in \mathbb{R}^n, \quad i = 1, 2,
\]

in coordinate where \( a > 0 \). Consequently, there exists a constant \( \delta > 0 \) such that, in a neighborhood of \( \overline{q_1q_2} \), \( M \) is given as a graph \( \rho = \tan^{-1} \frac{v}{\sqrt{|\theta|^2 + 1}} \geq 0 \) over a domain

\[
\Omega_0 := \{ x := (x', x_n) \in \mathbb{R}^n \mid \phi_1(x') < x_n < \phi_2(x') \text{ for } |x'| < \delta \}
\]

where \( \phi_1, \phi_2 \) are smooth functions since \( \partial M \) is smooth and transversal to \( \overline{q_1q_2} \). Let \( \psi \) be a smooth function defined on \( \partial B_r \), where \( \partial B_r \subset \Omega_0 \) is the \( n \)-ball of radius \( r \leq \delta \) centered at the origin, satisfying \( \psi(0, \pm r) = 0 \) and

\[
\psi(x', x_n) \geq \max\{v(x', \phi_1(x')), v(x', \phi_2(x'))\}, \text{ for all } (x', x_n) \in \partial B_r.
\]
This is possible since both $v(x', \phi_1(x'))$ and $v(x', \phi_2(x'))$ are smooth in $x'$ as $\partial M$ is smooth and tangential to $\rho = 0$. By CNSI there exists a unique strictly convex solution $w \in C^\infty(B_r)$ to the Dirichlet problem of the Monge–Ampère equation

$$\det(w_{ij}) = K \text{ in } B_r, \ w = \phi \text{ on } \partial B_r.$$ 

Since $\det(w_{ij}) = K \leq \det(v_{ij})$ in $B_r$ and, by the convexity of $v$,

$$v(x', x_n) \leq \max\{v(x', \phi_1(x')), v(x', \phi_2(x'))\}, \text{ for all } (x', x_n) \in \Omega_0$$

which implies $w \geq v$ on $\partial B_r$, we have $w \geq v \geq 0$ on $\overline{B_r}$ by the comparison principle. Since $w$ is strictly convex and $v(0, a) = v(0, -a) = 0$, however, we have $v(0, 0) < 0$ which is a contradiction. This proves that $M$ is strictly convex and smooth in any interior point. Finally the boundary regularity follows from [4]. The proof is thus complete.

This completes the proof of Theorem 1.1.1.
CHAPTER 3

CURVATURE ESTIMATES

3.1 Notations and preliminary results

Let \( S \) be the set of \( n \times n \) symmetric matrices and \( S^+ \) be the set of positive definite symmetric matrices. We define the function \( F \) by

\[
F(A) = f(\lambda(A)), \quad A \in S^+, \quad (3.1.1)
\]

where \( \lambda(A) = (\lambda_1, \ldots, \lambda_n) \) denotes the eigenvalues of \( A \). We will use the notation

\[
F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ij,kl}(A) = \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A). \quad (3.1.2)
\]

then the matrix \([F^{ij}(A)]\) is symmetric. Let \( \xi_k = (\xi^1_k, \ldots, \xi^n_k) \) be the unit eigenvector of \( A = [a_{ij}] \) with the corresponding eigenvalue \( \lambda_k \) for \( k = 1, \ldots, n \). Then for any \( t \) the matrix \( A + t\xi_k \otimes \xi_k = [a_{ij} + t\xi^i_k \xi^j_k] \) has eigenvectors \( \xi_i \) with corresponding eigenvalues \( \lambda_i \) for \( i \neq k \) and \( \lambda_k + t|\xi_k|^2 \) for \( i = k \). By differentiating \( F(A + t\xi_k \otimes \xi_k) = f(\lambda(A + t\xi_k \otimes \xi_k)) \)

with respect to \( t \), we obtain

\[
\sum_{i,j} F^{ij}(A + t\xi_k \otimes \xi_k) \xi^i_k \xi^j_k = \sum_l f_l(A + t\xi_k \otimes \xi_k) \frac{d}{dt} \lambda_l(A + t\xi_k \otimes \xi_k) = f_k(A + t\xi_k \otimes \xi_k)|\xi_k|^2.
\]

When \( t = 0 \) this becomes

\[
\sum_{i,j} F^{ij}(A) \xi^i_k \xi^j_k = f_k(A)|\xi_k|^2.
\]
This holds for each $k = 1, \ldots, n$, which shows that the matrix $[F_{ij}(A)]$ has eigenvalues $f_i(A)$ with eigenvectors $\xi_i$ that are eigenvectors of $A$ too. By assumption (1.2.3), $[F_{ij}(A)]$ is positive definite for $A \in S^+$. Let $P = [\xi_i]$ then $PAP^{-1} = [\lambda_j \delta_i^j]$, and $P[F_{ij}(A)]P^{-1} = [f_j(A)\delta_i^j]$. Moreover,

$$P[F_{ij}(A)]AP^{-1} = P[F_{ij}(A)]P^{-1}PAP^{-1} = [f_k(A)\delta_i^k][\lambda_j \delta_j^i] = [f_i \lambda_i \delta_i^j],$$

and similarly $PA[F_{ij}(A)]AP^{-1} = [f_i \lambda_i^2 \delta_i^j]$. In particular,

$$\sum_{i,j} F_{ij}(A)a_{ij} = \sum_i f_i \lambda_i, \quad (3.1.3)$$

$$\sum_{i,j,k} F_{ij}(A)a_{ik}a_{kj} = \sum_i f_i \lambda_i^2. \quad (3.1.4)$$

As proved in [5], (1.2.4) implies that $F$ is a concave function of $A \in S^+$, that is,

$$\sum_{i,j,k,l} F_{ij,kl}(A)\xi_{ij}\xi_{kl} \leq 0, \quad \text{for all } [\xi_{ij}] \in S, \ A \in S^+. \quad (3.1.5)$$

Let $M$ be a hypersurface in a $(n + 1)$-dimensional Riemannian manifolds $\tilde{M}$. Let $\tilde{g}, \tilde{\nabla}$ and $\tilde{R}$ be the Riemannian metric, connection and curvature tensor respectively on $\tilde{M}$, and denote the corresponding induced geometric quantities on $M$ by $g, \nabla$ and $R$ respectively. The space of smooth vector fields on any smooth manifold $N$ will be denoted by $\mathcal{X}(N)$. We use the sign convention for $\tilde{R}$ as

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z$$

and

$$\tilde{R}(X,Y,Z,W) = \tilde{g}(\tilde{R}(X,Y)Z,W)$$

for $X,Y,Z,W \in \mathcal{X}(\tilde{M})$. 

36
Let \( \nu \) be a unit normal vector field on \( M \) and \( h \) be the corresponding (scalar) second fundamental form of \( M \). We have the Weingarten equation

\[ \tilde{g}(\tilde{\nabla}_X \nu, Y) = -h(X, Y), \]  

the Gauss equation

\[ \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - h(X, W)h(Y, Z) + h(X, Z)h(Y, W), \]  

and a Ricci identity

\[ (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = \tilde{R}(X, Y, Z, \nu), \quad X, Y, Z \in \mathcal{X}(\tilde{M}). \]  

Suppose \( \tilde{M} \) has constant sectional curvature \( K \). Then

\[ \tilde{R}(X, Y, Z, W) = K[\tilde{g}(X, W)\tilde{g}(Y, Z) - \tilde{g}(X, Z)\tilde{g}(Y, W)]. \]

Hence the Coddazzi (3.1.8) and the Gauss (3.1.6) equations become

\[ (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \]  

and

\[ R(X, Y, Z, W) = h(X, W)h(Y, Z) - h(X, Z)h(Y, W) \]

\[ + K[\tilde{g}(X, W)\tilde{g}(Y, Z) - \tilde{g}(X, Z)\tilde{g}(Y, W)]. \]

when the ambient space \( \tilde{M} \) has constant sectional curvature \( K \).

For the calculations below we will use summation convention. Let \( \{e_1, e_2, \ldots, e_n\} \) be a frame fields on \( M \). The components of \( g \) and \( h \) are \( g_{ij} \) and \( h_{ij} \) respectively, i.e.

\[ g_{ij} = g(e_i, e_j) \quad \text{and} \quad h_{ij} = h(e_i, e_j). \]
Note $g$ and $h$ are symmetric, so $g_{ij} = g_{ji}$ and $h_{ij} = h_{ji}$. Denote the inverse of $g$ by $g^{ij}$, i.e. $g_{ik}g^{kj} = \delta_i^j$. Covariant differentiation will be indicated by indices, only in case of possible ambiguity they will be preceded by semicolon, for example, the covariant derivative of the curvature tensor will be denoted by $R_{ijkl;m}$ while the hessian of a function $u$ on $M$ will be denoted by $u_{ij}$ instead of $u_{;il}$.

The equations (3.1.9), (3.1.7) and (3.1.10) can be written respectively as

\begin{align*}
  h_{ij;k} &= h_{ik;l} \quad (3.1.11) \\
  h_{kl;ij} &= h_{kl;ji} + h_{ml}R_{ijkl}^{m} + h_{km}R_{ijkl}^{m} \quad (3.1.12) \\
  R_{ijkl} &= h_{il}h_{jk} - h_{ik}h_{jl} + K(g_{il}g_{jk} - g_{ik}g_{jl}) \quad (3.1.13)
\end{align*}

where $R_{ijkl}^{m} = g^{ml}R_{ijkl}$. From (3.1.11) and (3.1.12) we can obtain

\[ h_{ij;kl} = h_{kl;ij} + h_{km}R_{jli}^m + h_{im}R_{jik}^m, \]

then using (3.1.13) we have, when $\bar{M}$ has constant sectional curvature $K$,

\[ h_{ij;kl} = h_{kl;ij} + h^n_i(h_{jn}h_{li} - h_{ij}h_{ln}) + h^n_i(h_{jn}h_{lk} - h_{jk}h_{ln}) + K(h_{kj}g_{li} - h_{kl}g_{ij} + h_{ij}g_{kl} - h_{il}g_{jk}). \quad (3.1.14) \]

and

\[ h^i_{j;kl} = g^{im}h_{kl;mj} + h^n_i(h^j_ih_{jn} - h^i_jh_{ln}) + g^{im}h^n_i(h_{jn}h_{lk} - h_{jk}h_{ln}) + K(h_{kj}\delta^i_l - h_{kl}\delta^i_j + h^i_jg_{kl} - h^i_jg_{jk}). \quad (3.1.15) \]

where $h^i_j = g^{ik}h_{kj}$ denote the components of the shape operator.

Now, suppose $M$ satisfies the equation

\[ f(\kappa[M](x)) = \psi(x) \quad \text{for } x \in M, \quad (3.1.16) \]

where $\kappa[M](x) = (\kappa_1(x), \ldots, \kappa_n(x))$ denotes the principal curvatures of $M$ at $x \in M$.

This equation can be written as

\[ F(h) = \psi \quad \text{on } M \quad (3.1.17) \]
where $F(h) = f(\lambda(h))$, $h = (g^{ik}h_{kj})$ is the shape operator and $\lambda(h)$ denotes the eigenvalues of the shape operator $h^i_j$. Let $F^{ij} = F^{ij}(h)$ and $F^{ij,kl} = F^{ij,kl}(h)$ for convenience. By differentiating the equation (3.1.17), we obtain

$$F^{ij}_{i;k} = \psi_k, \quad (3.1.18)$$

$$F^{ij}_{i;kl} = -F^{ij,mn}_{i;k}h^j_{i,m} + \psi_{kl}. \quad (3.1.19)$$

Therefore if $\tilde{M}$ has constant sectional curvature $K$, then we obtain

$$F^{kl}_{i;j,k} = -F^{kl,ns}_{k;i}h^s_{i,m} + \psi_{i,j} + F^{kl}_{i,n}h^n_{i,j}(h_{jn}h_{kl} - h_{jk}h_{ln}) + KF^{kl}_{i,j}(h_{jk}g_{il} - h_{jl}g_{ij} + h_{ij}g_{kl} - h_{il}g_{jk}) \quad (3.1.20)$$

and

$$F^{kl}_{i;j,k} = -g^{im}F^{kl,ns}_{k;m}h^s_{i,m} + g^{im}\psi_{mj} + F^{kl}_{i,n}h^n_{i,j}(h_{jn}h_{kl} - h_{jk}h_{ln}) + KF^{kl}_{i,j}(h_{kj}\delta_{i}^{l} - h_{kl}\delta_{j}^{l} + h_{ij}g_{kl} - h_{il}g_{jk}). \quad (3.1.21)$$

from (3.1.14) and (3.1.15) respectively.

**3.2 Interior curvature estimates**

Suppose $\Sigma$ is a hypersurface in $S^{n+1}$ which is a graph of a smooth function $u$ on a domain $\Omega \subset S^n$. Let $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ be a local coordinates on $\Omega$. Note $\theta$ is a local coordinates on $\Sigma$ too. The induced Riemannian metric, its inverse and the second fundamental form of $\Sigma$ with respect to upward normal $\nu$ are given by

$$g_{ij} = \frac{1}{1 + u^2} \left( \sigma_{ij} + \frac{1}{1 + u^2} u_i u_j \right), \quad (3.2.1)$$

$$g^{ij} = (1 + u^2) \left( \sigma^{ij} - \frac{\sigma^{ik} u_k \sigma^{jl} u_l}{w^2} \right), \quad (3.2.2)$$

and

$$h_{ij} = \frac{1}{w \sqrt{1 + u^2}} \left( \nabla_{ij} u + u \sigma_{ij} \right) \quad (3.2.3)$$
respectively, where \( \sigma_{ij} \) is the standard metric on \( S^n \), \( \sigma^{ij} \) is its inverse, \( \nabla \) is the Riemannian connection on \( S^n \), and \( w = \sqrt{1 + u^2 + |\nabla u|^2} \). The induced connection on \( \Sigma \) will be denoted by \( D \). The function \( u \) is also a function on \( \Sigma \), so from (3.2.2),

\[
g^{ij} u_i = (1 + u^2) \left( \sigma^{ij} u_i - \frac{|\nabla u|^2 \sigma^{ij} u_i}{w^2} \right) = (1 + u^2) \sigma^{ij} u_i \left( 1 - \frac{|\nabla u|^2}{w^2} \right) = \frac{(1 + u^2)^2}{w^2} \sigma^{ij} u_i.
\]

Therefore

\[
|Du|^2 = \frac{(1 + u^2)^2}{w^2} |\nabla u|^2,
\]

and

\[
(1 + u^2)|Du|^2 = ((1 + u^2)^2 - |Du|^2)|\nabla u|^2, \quad (3.2.4)
\]

hence

\[
|\nabla u|^2 = \frac{(1 + u^2)|Du|^2}{(1 + u^2)^2 - |Du|^2}, \quad (3.2.5)
\]

\[
w = \frac{(1 + u^2)\sqrt{1 + u^2}}{\sqrt{(1 + u^2)^2 - |Du|^2}}. \quad (3.2.6)
\]

Note that \( |Du|^2 \leq (1 + u^2)^2 \) by (3.2.4). Let \( \Gamma^{k(g)}_{ij} \) and \( \Gamma^k_{ij} \) be the connection coefficients of \( D \) and \( \nabla \) respectively. Then using the formula

\[
\Gamma^{k(g)}_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial \theta_i} + \frac{\partial g_{il}}{\partial \theta_j} - \frac{\partial g_{ij}}{\partial \theta_l} \right)
\]

we can obtain the relation

\[
\Gamma^{k(g)}_{ij} = \Gamma^k_{ij} - \frac{uu_i}{1 + u^2} \delta^k_j - \frac{uu_j}{1 + u^2} \delta^k_i + \frac{\sigma^{km} u_m}{w^2} (\nabla_{ij} u + u \sigma_{ij}). \quad (3.2.7)
\]

Hence

\[
D_{ij} u = \frac{1 + u^2}{w^2} \nabla_{ij} u + \frac{2uu_i u_j}{1 + u^2} - \frac{|\nabla u|^2 u \sigma_{ij}}{w^2}.
\]

(3.2.8)
Then from (3.2.1) and (3.2.3) we obtain

\[ D_{ij}u = \frac{(1 + u^2)}{w^2} \sqrt{1 + u^2} h_{ij} - u(1 + u^2) g_{ij} + \frac{3u u_i u_j}{1 + u^2}. \]  

(3.2.9)

From now on in this section, covariant differentiation is always done with respect to the connection \( D \) on \( \Sigma \) and we will use the index notation as previous section. i.e. \( u_{ij} \) for \( D_{ij}u \) etc. Let the function \( v \) on \( \Sigma \) be defined by

\[ v := \frac{1}{w} = \frac{\sqrt{(1 + u^2)^2 - |Du|^2}}{(1 + u^2) \sqrt{1 + u^2}} \]  

(3.2.10)

Note first that

\[ 0 < c_0 \leq v \leq 1 \]  

(3.2.11)

on \( \Sigma \) where the constant \( c_0 \) depends on \( |u|_{C^1(\Omega)} \) as a function on \( \Omega \). For later use, let us calculate the hessian of \( v \).

\[
\frac{v_k}{v} = \frac{2uu_k}{1 + u^2} - \frac{2uu_k(1 + u^2) - g^{ps}u_p u_{sk}}{(1 + u^2)^2 - |Du|^2}
\]

\[
= -\frac{3uu_k}{1 + u^2} + \frac{1}{(1 + u^2)^3 v^2} \left[ 2uu_k(1 + u^2) - g^{ps}u_p \{(1 + u^2)_{3/2}v h_{sk}ight. \\
\left. -u(1 + u^2)g_{sk} + 3uu_s u_k \}} \right]
\]

\[
= -\frac{3uu_k}{1 + u^2} + \frac{1}{(1 + u^2)^3 v^2} \left[ 3uu_k \left( 1 - \frac{|Du|^2}{(1 + u^2)^2} \right) - g^{ps}u_p (1 + u^2)^{3/2} v h_{sk} \right]
\]

\[
= -\frac{3uu_k}{1 + u^2} + \frac{1}{(1 + u^2)^2 v^2} \left[ 3uu_k (1 + u^2)v^2 - g^{ps}u_p (1 + u^2)^{3/2} v h_{sk} \right]
\]

\[
= -\frac{g^{ps}u_p h_{sk}}{(1 + u^2)^{3/2}v}.
\]

Here we used (3.2.9). Therefore

\[ v_k = -\frac{g^{ps}u_p h_{ks}}{(1 + u^2)^{3/2}}. \]  

(3.2.12)
From $(1 + u^2)^{\frac{3}{2}}v_k = -g^{ps}u_p h_{ks}$, we have

$$(1 + u^2)^{\frac{3}{2}}v_{kl} = -3uu_l(1 + u^2)^{\frac{1}{2}}v_k - g^{pa}u_p h_{ks} - g^{ps}u_p h_{ks;l}$$

$$= \frac{3uu_l g^{ps}u_p h_{ks}}{1 + u^2} - g^{ps}u_p h_{ks;l}$$

$$- g^{ps}h_{ks}[(1 + u^2)^{\frac{3}{2}}v h_{pl} - u(1 + u^2)g_{pl} + \frac{3uu_l u_p}{1 + u^2}]$$

$$= -g^{ps}u_p h_{ks;l} - (1 + u^2)^{\frac{3}{2}}v g^{ps}h_{ks} h_{pl} + u(1 + u^2)h_{kl}.$$  

Hence

$$v_{kl} = -vg^{ps}h_{sk} h_{pl} + \frac{u}{(1 + u^2)^{\frac{1}{2}}} h_{kl} - \frac{g^{ps}u_p}{(1 + u^2)^{\frac{3}{2}}} h_{ks;l}. \quad (3.2.13)$$

Now, let $\kappa_{\text{max}}(x)$ be the largest principal curvature of $\Sigma$ at $x$.

**Theorem 3.2.1.** Let $\Sigma$ be a locally convex hypersurface in $S^{n+1}$ which is a graph of a smooth function $u$ on a domain $\Omega \subset S^n$ satisfying the equation (3.1.16). Suppose $f$ satisfies conditions (1.2.3) - (1.2.6). Then

$$\max_{\Sigma} \kappa_{\text{max}} \leq C$$

where $C$ depends on $\max_{\partial \Sigma} \kappa_{\text{max}}, \ |u|_{C^1(\Omega)}, \ |\psi|_{C^2(\Sigma)}$ and the constant $\sigma_0$ in the condition (1.2.6).

**Proof.** Let

$$M = \max_{x \in \Sigma} \frac{\kappa_{\text{max}}(x)}{v} \quad (3.2.14)$$

where $v$ is the function as (3.2.10), and assume that $M$ is attained at an interior point $x_0 \in \Sigma$. By choosing a normal coordinate system at $x_0$ followed by rotation, we may assume that $g_{ij}(x_0) = \delta_{ij}$, $h^1_i(x_0) = \kappa_i \delta^j_i$ and $\kappa_1 = \kappa_{\text{max}}(x_0)$ where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of $\Sigma$ at $x_0$. Note also that $h_{ij}(x_0) = \kappa_i \delta_{ij}$. The locally defined function

$$\Phi = \frac{h^1_i}{v} \quad (3.2.15)$$

42
achieves the maximum $M$ at $x_0$. Thus, at $x_0$,

$$
\Phi_k = -\frac{v_k}{v^2} h^1_{1,k} + \frac{1}{v} h^1_{1,k} = 0
$$

(3.2.16)

and the hessian

$$
\Phi_{kl} = \frac{2v_k v_l}{v^3} h^1_{1,k} - \frac{v_{kl}}{v^2} h^1_{1,k} - \frac{v_k}{v^2} h^1_{1,l} + \frac{1}{v} h^1_{1;kl}
$$

$$
= \frac{2v_k v_l}{v^3} h^1_{1,k} - \frac{v_k v_l}{v^2 v} h^1_{1,k} - \frac{v_l v_k}{v^2 v} h^1_{1,l} + \frac{1}{v} h^1_{1;kl}
$$

$$
= \frac{1}{v} h^1_{1;kl} - \frac{v_k v_l}{v^2} h^1_{1,k}
$$

is negative semidefinite, hence

$$
F^{kl} \Phi_{kl} \leq 0 \text{ at } x_0.
$$

(3.2.17)

Since $g_{ij}(x_0) = \delta_{ij}$ and $h_{ij}(x_0) = \kappa_i \delta_{ij}$, we have $F^{ij}(x_0) = f_i \delta^{ij}$ so from (3.1.21) and the fact that $S^{n+1}$ has constant sectional curvature 1,

$$
F^{kl} h^1_{1;kl} = -g^{lm} F^{kl,ns} h^l_{k;m} h^s_{n;1} + g^{lm} \psi_{m1} + F^{kl} h^n_k (h^1_l h^1_n - h^1_l h^1_{ln})
$$

$$
+ F^{kl} g^{lm} h^n_{1,m} (h^1_n h^1_k - h^1_k h^1_{ln}) + F^{kl} (h_{k1} h^1_l - h_{k1} + h^1_{kl} - h^1 g_{kl})
$$

$$
= -F^{kl,ns} h^l_{k;1} h^s_{n;1} + \psi_{11} + f_1(\kappa_1)^3 - \kappa_1 \sum f_i \kappa_i^2
$$

$$
+ \kappa_1^2 \sum f_i \kappa_i - f_1 \kappa_1^3 + f_1 \kappa_1 - \sum f_i \kappa_i + \kappa_1 \sum f_i - f_1 \kappa_1.
$$

Therefore

$$
F^{kl} h^1_{1;kl} = \frac{\kappa_1^2 \sum f_i \kappa_i - \kappa_1 \sum f_i \kappa_i^2 + \kappa_1 \sum f_i - \sum f_i \kappa_i}{F^{kl,ns} h^l_{k;1} h^s_{n;1} + \psi_{11}}.
$$

(3.2.18)

at $x_0$. From (3.2.13) and the Codazzi equation (3.1.11),

$$
F^{kl} v_{kl} = F^{kl} \left( -v g^{ps} h_{skl} h_{pl} + \frac{u}{(1 + u^2)^{\frac{3}{2}}} h_{kl} - \frac{g^{ps} u_p}{(1 + u^2)^{\frac{3}{2}}} h_{ks;l} \right)
$$

$$
= -v \sum f_i \kappa_i^2 + \frac{u}{(1 + u^2)^{\frac{3}{2}}} \sum f_i \kappa_i - \frac{u_s}{(1 + u^2)^{\frac{3}{2}}} F^{kl} h_{kl;1,1}.
$$

43
Hence we get, using (3.1.18),

\[ F_{kl}v_{kl} = -v \sum f_i \kappa_i^2 + \frac{u}{(1 + u^2)^{\frac{1}{2}}} \sum f_i \kappa_i - \frac{u_s \psi_s}{(1 + u^2)^{\frac{3}{2}}} + \frac{u_s \psi_s}{(1 + u^2)^{\frac{3}{2}}} \sum f_i \kappa_i - \frac{u_s \psi_s}{(1 + u^2)^{\frac{3}{2}}} \]  

(3.2.19)

at \( x_0 \). Now (3.2.17) becomes, since \( \kappa_1 \sum f_i - \sum f_i \kappa_i \geq 0 \) and \( F_{kl,ns}h_{kl}^i h_{ns}^i \leq 0 \),

\[ 0 \geq vF_{kl}h_{1;kl} - \kappa_1 F_{kl}v_{kl} \geq v\kappa_1^2 \sum f_i \kappa_i - \kappa_1 \left( \frac{u}{(1 + u^2)^{\frac{1}{2}}} \sum f_i \kappa_i - \frac{u_s \psi_s}{(1 + u^2)^{\frac{3}{2}}} \right) + \psi_{11}. \]

Note that

\[ \left| \frac{u}{(1 + u^2)^{\frac{1}{2}}} \sum f_i \kappa_i - \frac{u_s \psi_s}{(1 + u^2)^{\frac{3}{2}}} \right| \leq \sum f_i \kappa_i + |D\psi| \leq f(\kappa) + |D\psi|, \]

since \( f \) is concave and \( |Du| \leq 1 + u^2 \). Therefore

\[ c_0 \sigma_0 \kappa_1^2 - C(\kappa_1 + 1) \leq 0, \]

where \( \sigma_0 \) and \( c_0 \) are positive constants in (3.3.5) and (3.2.11) respectively and \( C \) depends on \( |D\psi|, |D^2\psi| \) on \( \Sigma \). This implies that \( \kappa_1 \leq C' \) where \( C' \) depends on \( \sigma_0, c_0 \) and \( |\psi|_{C^2(\Sigma)} \). Finally

\[ \max_\Sigma \kappa_{\max} \leq \max \left\{ \frac{C'}{c_0}, \max_\partial \Sigma \kappa_{\max} \right\} \]

and this proves the theorem. \( \square \)

### 3.3 Boundary and global estimates for second derivatives

\( \Sigma \) is a hypersurface in \( S^{n+1} \) which is a graph of a smooth function \( u \) on a domain \( \Omega \subset S^n \) as above. All the covariant differentiation in this section is calculated with respect to the connection \( \nabla \) on \( \Omega \subset S^n \), for example. \( u_{ij} \) means \( \nabla_{ij} u \). For any function \( f \) on \( \Omega \), we have a Ricci identity

\[ f_{ijk} = f_{ikj} + R_{jki}^m u_m \]  

(3.3.1)
where $R_{jkl}^m$ is the component of the (Riemann) curvature endomorphism. Since $(S^n, \sigma)$ has constant sectional curvature 1,

$$R_{i}^{l} \partial_{l} = \sigma_{jk} \partial_{i} - \sigma_{ik} \partial_{j}$$  \hspace{1cm} (3.3.2)

where $\partial_{i}$ denotes the coordinate vector field. Then using symmetry of hessian, we have

$$u_{ijk} = u_{kij} + u_{i} \sigma_{ki} - u_{k} \sigma_{ij}$$  \hspace{1cm} (3.3.3)

Let $[\xi_{ik}]$ be the square root of the metric $\sigma$ with its inverse $[\xi^{ik}]$ depending on the coordinate, that is, $\sigma_{ij} = \xi_{ik} \xi_{kj}$ and $\sigma^{ij} = \xi^{ik} \xi^{kj}$. The square root of induced metric $g$ on $\Sigma$ and its inverse are given, respectively by

$$\gamma_{ik} = \frac{1}{\sqrt{1 + u^2}} \left( \xi_{ik} \frac{\xi^{il} u_l u_k}{\sqrt{1 + u^2} (w + \sqrt{1 + u^2})} \right)$$  \hspace{1cm} (3.3.4)

and

$$\gamma^{ik} = \sqrt{1 + u^2} \left( \xi^{ik} - \frac{\sigma^{in} u_n \xi^{km} u_m}{w (w + \sqrt{1 + u^2})} \right),$$  \hspace{1cm} (3.3.5)

that is, $g_{ij} = \gamma_{ki} \gamma_{kj}$ and $\gamma_{ik} \gamma^{kj} = \delta_{i}^{j}$. Now $\kappa[u] = (\kappa_{1}, \ldots, \kappa_{n})$, the principal curvatures of $\Sigma$ are the eigenvalues of the symmetric matrix $A[u] = [a_{ij}]$:

$$a_{ij} = \gamma^{ki} h_{kl} \gamma^{lj},$$  \hspace{1cm} (3.3.6)

since they are roots of the equation

$$0 = \det(a_{ij} - k \delta_{ij}) = \frac{\det(h_{ij} - k g_{ij})}{\det(g_{ij})}.$$  \hspace{1cm} (3.3.7)

We study the Dirichlet problem

$$f(\kappa[u]) = \psi(x, u) \quad \text{in } \Omega$$

$$u = \varphi \quad \text{on } \partial \Omega$$  \hspace{1cm} (3.3.7)
where \( \psi \in C^\infty(\overline{\Omega} \times \mathbb{R}_+). \) The equation (3.3.7) can be written in the form

\[
F(A[u]) = \psi(x, u). \tag{3.3.8}
\]

Let \( G \) be defined by

\[
G(r, p, z) = F(A(r, p, z)), \quad r \in S^+, \ p \in \mathbb{R}^n, \ z \in \mathbb{R},
\]

then the equation (3.3.8) can be reformulated as

\[
G(\nabla^2 u, \nabla u, u) = \psi(x, u) \tag{3.3.9}
\]

**Lemma 3.3.1.** If \( f \) satisfies (1.2.7), then

\[
\lim_{R \to +\infty} G(r + Rq \otimes q, p, z) = +\infty \quad \forall p, q \in \mathbb{R}^n, q \neq 0, \forall r \in S^+, \forall z \in \mathbb{R} \tag{3.3.10}
\]

**Proof.** Note first that any \( q = (q_1, \ldots, q_n) \in \mathbb{R}^n \) can be written as \( q_i = \xi_{ij} \eta_j \) for some \( \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n. \) Writing \( A(r, p, z) = [a_{ij}] \) and \( A(r + Rq \otimes q, p, z) = [\tilde{a}_{ij}] \), we have

\[
\tilde{a}_{ij} = a_{ij} + \frac{R\xi_{kn} \gamma^k \xi_{nl} \gamma^l}{\sqrt{1 + z^2 + |p|^2} \sqrt{1 + z^2 + |p|^2}},
\]

where \( |p|^2 = \sigma_{ij} p_i p_j. \) Without loss of generality we may assume \( \eta = (0, \ldots, 0, 1). \) We then have

\[
\tilde{a}_{ij} = a_{ij} + \frac{R\xi_{kn} \gamma^k \xi_{nl} \gamma^l}{\sqrt{1 + z^2 + |p|^2} \sqrt{1 + z^2}},
\]

where the expression is not summed over \( n. \) Note that

\[
\xi_{kn} \gamma^k = \sqrt{1 + z^2} \left( \delta^i_n - \frac{\xi_{nl} p_l \xi_{il} p_i}{\sqrt{1 + z^2 + |p|^2} \sqrt{1 + z^2}} \right),
\]

as a matrix has eigenvalues \( \sqrt{1 + z^2} \) of multiplicity \( n - 1 \) and

\[
\frac{1 + z^2}{\sqrt{1 + z^2 + |p|^2}}
\]

46
of multiplicity 1. Therefore, after an orthonormal transformation we may assume

\[ \tilde{a}_{ij} = a_{ij} + \left( \frac{1 + z^2}{1 + z^2 + |p|^2} \right)^{\frac{3}{2}} R \delta^i_k \delta^j_n. \]

By Lemma 1.2 of [5] the eigenvalues of [\tilde{a}_{ij}] behave like

\[ \lambda_\alpha = \lambda'_\alpha + O(1), \quad \alpha \leq n - 1, \quad \lambda_n = R \left( 1 + O \left( \frac{1}{R} \right) \right). \]

as \( R \) tends to infinity, where \( \lambda'_\alpha, 1 \leq \alpha < n \), are the eigenvalues of \([a_{\alpha \beta}]\). Since \((\lambda'_1, \ldots, \lambda'_{n-1}, 1) \in \Gamma_n^+, (\lambda_1, \ldots, \lambda_{n-1}, 1) \) belongs to a compact subset of \( \Gamma_n^+ \) for all \( R \) sufficiently large. Consequently, (3.3.10) follows from (1.2.7).

We will use the notations.

\[ G_{ij}[u] := \frac{\partial G}{\partial r_{ij}} (\nabla^2 u, \nabla u, u), \quad G^k[u] := \frac{\partial G}{\partial p_k} (\nabla^2 u, \nabla u, u), \quad G_u[u] := \frac{\partial G}{\partial z} (\nabla^2 u, \nabla u, u) \]

(3.3.11)

Note that since \( \frac{\partial a_{kl}}{\partial u_{ij}} = \frac{\gamma^i_k \gamma^j_l}{w \sqrt{1 + u^2}} \), we have

\[ G^{ij}[u] = \frac{F^{kl} \gamma^i_k \gamma^j_l}{w \sqrt{1 + u^2}} \]

(3.3.12)

So equation (3.3.9) is elliptic for locally strictly convex solutions. Note also that

\( G^{ij}[u] \sigma_{ij} > 0. \)

**Theorem 3.3.2.** Assume that \( \Omega \) is smooth and (1.2.3) - (1.2.8) holds. Suppose \( u \in C^{0,1}(\Omega) \) be a locally convex subsolution of (3.3.7) and \( u \) is \( C^2 \) and locally strictly convex up to the boundary in a neighborhood of \( \partial \Omega \). Let \( u \in C^\infty(\overline{\Omega}) \) be a locally convex solution of (3.3.7) satisfying \( u \geq u \) in \( \Omega \). Then

\[ |\nabla^2 u| \leq C \quad \text{on} \ \partial \Omega. \]

(3.3.13)

where \( C \) depends on \( |u|_{C^1(\overline{\Omega})}. \)
Proof. We will assume that $\varphi$ has been extended to a harmonic function on the whole $\Omega$. Let $\eta, \zeta$ be some tangential unit vector fields on $\partial \Omega$. Since $u - \varphi = 0$ on $\partial \Omega$ we have

$$\nabla_{\eta \zeta}(u - \varphi) = -\nabla_{\nu}(u - \varphi)\Pi(\eta, \zeta) \quad \text{on } \partial \Omega \quad (3.3.14)$$

where $\nu$ is the interior unit normal vector field to $\partial \Omega$ and $\Pi$ denotes the (scalar) second fundamental form of $\partial \Omega$. We thus have the estimates for the pure tangential second derivatives

$$|\nabla_{\eta \zeta}u| \leq C \quad \text{on } \partial \Omega. \quad (3.3.15)$$

In order to estimate the mixed normal-tangential derivatives and pure second normal derivatives, we need a series of lemmas. Let us consider a linear operator in $\Omega$ defined by

$$L = G^{ij}\nabla_{ij} + G^i \nabla_i \quad (3.3.16)$$

where $G^{ij} = G^{ij}[u]$ and $G^i = G^i[u].$

**Lemma 3.3.3.** If $u$ is locally convex solution of (3.3.9), then

$$\sum |G^i| \leq C \quad (3.3.17)$$

and

$$|G_u[u]| \leq C + G^{ij} \sigma_{ij} \quad (3.3.18)$$

where $C$ depends on $|u|_{C^1(\overline{\Omega})}.$

**Proof.** From (3.3.4) and (3.3.5) we have

$$\frac{\partial \gamma_{ij}}{\partial u_s} = \frac{\sqrt{1 + u^2} \xi^{il} u_l \delta_{js} + u_j \gamma^{si}}{(1 + u^2)\sqrt{1 + u^2}(w + \sqrt{1 + u^2})} \quad (3.3.19)$$

and

$$\frac{\partial \gamma_{ij}}{\partial u} = -\frac{\gamma_{ij} u}{1 + u^2} - \frac{\xi^{il} u_l u_j u}{w(1 + u^2)^2}, \quad (3.3.20)$$

48
\( u_i v^k = (1 + u^2) \frac{u_i \xi^k}{w} \). \hfill (3.3.21)

Note \( a_{ij} \gamma_{js} = \gamma^{ki} h_{ks} \) from (3.3.6). It follows that

\[
h_{ij} \gamma^{ik} \frac{\partial \gamma^{jl}}{\partial u_s} = a_{kp} \gamma^{ip} \frac{\partial \gamma^{jl}}{\partial u_s} = -a_{kp} \gamma^{ij} \frac{\partial \gamma^{pj}}{\partial u_s}
\]

since \( \gamma^{pj} \gamma^{ji} = \delta^l_p \). Then we calculate

\[
G^s[u] = F^{kl} a_{kl} \frac{\partial a_{kl}}{\partial u_s} = F^{kl} a_{kl} \frac{\nabla_{ij} u + u \sigma_{ij}}{\sqrt{1 + u^2}} \frac{\partial (\gamma^{ik} \gamma^{jl})}{\partial u_s}
\]

\[
= -F^{kl} a_{kl} \left( \frac{\sigma_{is} u_i}{u^2} \right) + 2F^{kl} h_{ij} \gamma^{ik} \frac{\partial \gamma^{jl}}{\partial u_s} - 2F^{kl} a_{kp} \gamma^{jl} \frac{\partial \gamma^{pj}}{\partial u_s}
\]

\[
= -F^{kl} a_{kl} \left( \frac{\sigma_{is} u_i}{u^2} \right) - 2F^{kl} a_{kp} \gamma^{jl} \left( \frac{\nabla_{ij} u + u \sigma_{ij}}{\sqrt{1 + u^2}} \right) \frac{\partial (\gamma^{ik} \gamma^{jl})}{\partial u_s}
\]

hence we obtain

\[
G^s[u] = -F^{kl} a_{kl} \left( \frac{\sigma_{is} u_i}{u^2} \right) - 2F^{kl} a_{kp} \frac{\nabla_{ij} u + u \sigma_{ij}}{w(1 + u^2)(w + \sqrt{1 + u^2})} \left( \gamma^{ik} \gamma^{jl} \right)
\]

from (3.3.21). Next,

\[
G_u[u] = F^{kl} a_{kl} \frac{\partial a_{kl}}{\partial u_s} = F^{kl} a_{kl} \gamma^{ik} \gamma^{jl} \frac{\partial \gamma^{ij}}{\partial u} + F^{kl} a_{kl} \frac{\partial (\gamma^{ik} \gamma^{jl})}{\partial u}
\]

We see that

\[
F^{kl} \gamma^{ik} \gamma^{jl} \frac{\partial \gamma^{ij}}{\partial u} = \frac{\sigma_{ij}}{w \sqrt{1 + u^2}} + F^{kl} \gamma^{ik} \gamma^{jl} \left( \nabla_{ij} u + u \sigma_{ij} \right) \frac{\partial}{\partial u} \left( \frac{1}{w \sqrt{1 + u^2}} \right)
\]

\[
= \frac{F^{kl} \gamma^{ik} \gamma^{jl} \sigma_{ij}}{w \sqrt{1 + u^2}} - F^{kl} a_{kl} \left( \frac{u}{1 + u^2} + \frac{u}{w^2} \right)
\]

and

\[
F^{kl} \gamma^{ik} \gamma^{jl} \frac{\partial (\gamma^{ij})}{\partial u} = 2F^{kl} a_{kl} \frac{\partial (\gamma^{ij})}{\partial u} = -2F^{kl} a_{kp} \gamma^{jl} \frac{\partial \gamma^{pj}}{\partial u} = \frac{2F^{kl} a_{kl} u}{1 + u^2} + \frac{2F^{kl} a_{kp} \xi_{ps} \xi_{jt} u}{w^2(1 + u^2)}.
\]
Therefore
\[ G_u[u] = \frac{u}{w^2(1 + u^2)}(\nabla |u|^2 F^{kl} a_{kl} + 2 F^{kl} a_{kp} \xi^pt u \xi^j u_j) + \frac{F^{kl} \gamma^{ik} \gamma^{jl} \sigma_{ij}}{w \sqrt{1 + u^2}}. \] (3.3.23)

Now, from (3.3.22) and (3.3.23), we have
\[ \sum_i |G^i| \leq C_0 \sum f_i |\kappa_i|. \] (3.3.24)
\[ |G_u[u]| \leq C_0 \sum f_i |\kappa_i| + G^{ij} \sigma_{ij} \] (3.3.25)
respectively for some constant \( C_0 > 0 \). Note \( \kappa_i \geq 0 \) by convexity of \( u \). Since \( f \) is concave and \( f(0) = 0 \) by assumption, we have
\[ \sum_i |G^i| \leq C_0 \sum f_i \kappa_i \leq C_0 f(\kappa[u]) \] (3.3.26)
therefore (3.3.17) and similarly (3.3.18) holds. \( \square \)

**Lemma 3.3.4.** Assume \( f \) satisfies (1.2.3)-(1.2.7). For any constant \( C_0 > 0 \), there exist positive constants \( t, \delta, \beta \) sufficiently small and \( N \) sufficiently large such that the function
\[ v = u - u + td - Nd^2 \] (3.3.27)
satisfies
\[ Lv \leq -C_0 - \beta G^{ij} \sigma_{ij} \text{ in } \Omega \cap B_\delta, \quad v \geq 0 \text{ on } \partial(\Omega \cap B_\delta), \] (3.3.28)
where \( B_\delta \) is a ball of radius \( \delta \) centered at a point on \( \partial \Omega \) and \( d \) is the distance function to \( \partial \Omega \) defined in \( \overline{B_\delta} \).

**Proof.** We note first that \( u \) is \( C^2 \) and assumed to be locally strictly convex subsolution in a neighborhood of \( \partial \Omega \). Therefore when \( \delta \) is sufficiently small, we have
\[ \nabla^2 u + u \sigma \geq 5 \beta w \sqrt{1 + u^2} g \geq 5 \beta \sigma \]
in $\Omega \cap B_\delta$ for some fixed $\beta > 0$. Let

$$c_u = \sup_{\Omega \cap B_\delta} |u|.$$  

We can choose $\delta$ small enough so that $c_u \leq \beta$. Note

$$\lambda(\nabla^2 u + u\sigma - 4\beta\sigma) \leq \lambda(\nabla^2 u + c_u\sigma - 4\beta\sigma) \leq \lambda(\nabla^2 u - 3\beta\sigma),$$

hence $\lambda(\nabla^2 u - 3\beta\sigma)$ lies in a compact subset of $\Gamma^+_{\hat{n}}$. Note also that $\nabla^2 d^2 = 2d\nabla^2 d + 2\nabla d \otimes \nabla d$, $|\nabla d| = 1$, and $-C\sigma \leq \nabla^2 d \leq C\sigma$ where $C$ depends only on $\delta$ and the geometric quantities of $\partial \Omega$. Thus we have

$$Ld \leq CG^{ij}\sigma_{ij} + \sum |G^i| \quad \text{in } \Omega \cap B_\delta$$

and

$$\lambda(\nabla^2 (u + Nd^2) - 2\beta\sigma) \geq \lambda(\nabla^2 u - 3\beta\sigma + 2N\nabla d \otimes \nabla d) \quad \text{in } \Omega \cap B_\delta$$

when $\delta$ is small enough so that $2N\delta\nabla^2 d \geq -\beta\sigma$. Now,

$$L(u - u - Nd^2) + 2\beta G^{ij}\sigma_{ij}$$

$$= G^{ij}(u - u - Nd^2)_{ij} + G^i(u - u - Nd^2)_i + 2\beta G^{ij}\sigma_{ij}$$

$$= G^{ij}(u_{ij} - (u_{ij} - 2\beta\sigma_{ij} + N(d^2)_{ij}))$$

$$+ G^i(u - u)_i - 2NdG^i d_i$$

$$\leq G(\nabla^2 u, \nabla u, u) - G(\nabla^2 (u + Nd^2) - 2\beta\sigma, \nabla u, u)$$

$$+ G^i(u - u)_i - 2NdG^i d_i.$$  

where the last inequality holds since the function $G(r, p, z)$ is concave with respect to $r$ which is from the concavity of $F$. By Lemma 3.3.1 and 3.3.3, we may choose $N$ large depending on $|\nabla u|_{C^0(\overline{\Omega})}$ such that

$$G(\nabla^2 (u + Nd^2) - 2\beta\sigma, \nabla u, u) \geq G(\nabla^2 u, \nabla u, u) + G^i(u - u)_i + \sum |G^i| + C_0,$$
in $\Omega \cap B_\delta$ when $\delta$ is sufficiently small depending on $N$. It follow that

$$L(u - u - N d^2) \leq -2\beta G^{ij} \sigma_{ij} - 2NdG^i d_i - \sum |G^i| - C_0 \quad \text{in } \Omega \cap B_\delta.$$ 

Therefore,

$$Lv = L(u - u - N d^2) + tG^{ij} d_{ij} + tG^i d_i$$

$$\leq -C_0 - (2\beta - Ct)G^{ij} \sigma_{ij} - (1 - t + 2Nd) \sum |G^i|$$

$$\leq -C_0 - \beta G^{ij} \sigma_{ij} \quad \text{in } \Omega \cap B_\delta$$

when $t$ and $\delta$ are sufficiently small. Finally, we choose $\delta \leq \frac{t}{N}$ so that $v \geq 0$ on $\partial(\Omega \cap B_\delta)$ for fixed $t$ and $N$. 

Lemma 3.3.5. Assume that $f$ satisfies (1.2.3) - (1.2.8). Let $h \in C^2(\Omega \cap B_\delta)$ where $B_\delta$ is centered at $x_0 \in \partial \Omega$ and let $\rho(x) = d(x_0, x)$. Suppose $h$ satisfies $h(x) \leq C_0 \rho(x)$ on $\partial(\Omega \cap B_\delta)$, $h(x_0) = 0$ and

$$-Lh \leq C_1 (1 + G^{ij} \sigma_{ij}) \quad \text{in } \Omega \cap B_\delta$$

for some constants $C_0, C_1 > 0$. Then

$$\nabla_\nu h(x_0) \leq C \quad \text{(3.3.29)}$$

where $\nu$ is the inward unit normal vector to $\partial \Omega$ at $x_0$ and $C$ depends on $\beta$, $C_0$, $C_1$, $|h|_{C^0(\Sigma \cap B_\delta)}$, and $|u|_{C^1(\Sigma)}$.

Proof. By Lemma 3.3.4 we can choose $A \gg B \gg C_1$ such that $Av + B \rho - h \geq 0$ on $\partial(\Omega \cap B_\delta)$ and $L(Av + B \rho - h) \leq 0$ in $\Omega \cap B_\delta$. It follows from the maximum principle that $Av + B \rho - h \geq 0$ in $\Omega \cap B_\delta$. Consequently,

$$A\nabla_\nu v(x_0) + B\nabla_\nu \rho(x_0) - \nabla_\nu h(x_0) = \nabla_\nu (Av + B \rho - h)(x_0) \geq 0$$

since $Av + B \rho - h = 0$ at $x_0$. This implies (3.3.29).
Let us come back to the remaining proof of the Theorem 3.3.2. By differentiating

\( (3.3.9) \) we see that, for each \( k = 1, \ldots, n \),

\[
G_{ij} u_{ijk} + G^i u_{ik} + G^u [u] u_k = \psi_k + \psi_z u_k.
\]

Then using (3.3.3) we have

\[
L(u_k) = \psi_k + \psi_z u_k - G^u u_k - G^{ij} \sigma_{ki} u_j + G^{ij} \sigma_{ij} u_k.
\]

Therefore from (3.3.18),

\[
|L \nabla_k (u - \varphi)| \leq C(1 + G^{ij} \sigma_{ij}) \quad \text{for all } k = 1, \ldots, n.
\] (3.3.30)

for some \( C > 0 \) depending on \( |u|_{C^1(\Omega)} \).

Let \( x_0 \in \partial \Omega \) and choose a normal coordinate system at \( x_0 \) so that \( \nu(x_0) = \partial_n \).

Now, for each \( \alpha = 1, \ldots, n - 1 \) define \( h_\alpha := \nabla_\alpha (u - \varphi) \). Since \( u - \varphi = 0 \) on \( \partial \Omega \) and

\[
|\nabla (u - \varphi)| \leq C \text{ in } \overline{\Omega},
\]

we have \( h_\alpha(x_0) = 0 \) and,

\[
|h_\alpha(x)| \leq Cd(x_0, x) \quad \text{on } \partial \Omega
\]

near \( x_0 \) from (3.3.15), so from (3.3.30) we can apply Lemma 3.3.5 to \( \pm h_\alpha \) to yield

\[
|u_{\alpha n}(x_0)| \leq C \quad \text{for all } \alpha = 1, \ldots, n - 1
\]

This establishes a bound on \( \partial \Omega \) for the mixed normal-tangential derivatives

\[
|\nabla_{\eta \nu} u| \leq C \text{ on } \partial \Omega.
\] (3.3.31)

We now have to show

\[
|\nabla_{\nu \nu} u| \leq C \text{ on } \partial \Omega.
\] (3.3.32)

Set

\[
M \equiv \min_{\xi \in T_{x_0}(\partial \Omega), |\xi| = 1} \nabla \xi u
\]

53
where $T_x(\partial \Omega)$ denote the tangent space of $\partial \Omega$ at $x \in \partial \Omega$. Assume that $M$ is achieved for $\xi_0 \in T_x(\partial \Omega)$, that is, $M = \nabla_{\xi_0 \xi_0} u(x_0)$. Since $u - \underline{u} = 0$ on $\partial \Omega$, we have

$$\nabla_\eta u = \nabla_\eta \underline{u} - \nabla_\nu (u - \underline{u}) \Pi(\eta, \zeta) \quad \text{on } \partial \Omega \quad (3.3.34)$$

for $\eta, \zeta \in T(\partial \Omega)$ as (3.3.14). Therefore

$$M = \nabla_{\xi_0 \xi_0} u(x_0) = \nabla_{\xi_0 \xi_0} \underline{u}(x_0) - \nabla_\nu (u - \underline{u})(x_0) \Pi(\xi_0, \xi_0) \quad (3.3.35)$$

There are two cases,

$$\nabla_\nu (u - \underline{u})(x_0) \Pi(\xi_0, \xi_0) > \frac{1}{2} (\nabla_{\xi_0 \xi_0} \underline{u}(x_0) + \underline{u}(x_0)) \quad (3.3.36)$$

or

$$\nabla_\nu (u - \underline{u})(x_0) \Pi(\xi_0, \xi_0) \leq \frac{1}{2} (\nabla_{\xi_0 \xi_0} \underline{u}(x_0) + \underline{u}(x_0)) \quad (3.3.37)$$

Suppose (3.3.36) holds. Then $\nabla_\nu (u - \underline{u})(x_0) \Pi(\xi_0, \xi_0) > 0$. Note $0 \leq \nabla_\nu (u - \underline{u}) \leq C$ on $\partial \Omega$. Therefore $\Pi(\xi_0, \xi_0)(x_0) > 0$ and $\nabla_\nu (u - \underline{u})(x_0) > 0$. Choose a unit continuous vector field $\xi$ tangential to $\partial \Omega$ such that $\xi(x_0) = \xi_0$, that is, $\xi(x) \in T_x(\partial \Omega)$, $|\xi| = 1$ and $\xi(x_0) = \xi_0$ (We can consider parallel translations of $\xi_0$ along $\partial \Omega$). Since $\Pi(\xi, \xi)$ and $\nabla_\nu (u - \underline{u})$ are continuous, there exist $c_1 > 0$ and $\delta > 0$ small such that

$$\Pi(\xi, \xi) \geq \frac{1}{2} \Pi(\xi_0, \xi_0) > \frac{\nabla_{\xi_0 \xi_0} \underline{u}(x_0)}{4 \nabla_\nu (u - \underline{u})(x_0)} \geq c_1 \quad \text{in } \Omega \cap B_\delta(x_0). \quad (3.3.38)$$

Define a function

$$\Psi = \frac{\nabla_{\xi \xi} \varphi - M}{\Pi(\xi, \xi)} ,$$

then $\Psi$ is smooth and bounded in $\Omega \cap B_\delta(x_0)$. Let $h = \Psi - \nabla_\nu (u - \varphi)$. Then $h(x_0) = 0$ by (3.3.35), and from (3.3.31) we see that for small enough $\delta > 0$

$$|h(x)| \leq C d(x_0, x) \quad \text{on } \partial(\Omega \cap B_\delta(x_0)).$$
Thus by (3.3.30), we can apply Lemma 3.3.5 to ±h to obtain

\[ |\nabla_{\nu\nu} u(x_0)| \leq C. \quad (3.3.39) \]

Now, if (3.3.37) is the case then

\[ \nabla_{\xi_0\xi_0} u(x_0) + u(x_0) \geq \frac{1}{2} (\nabla_{\xi_0\xi_0} u(x_0) + u(x_0)) \geq \epsilon_0 > 0 \]

for some uniform constant \( \epsilon_0 \) since the graph of \( u \) is strictly convex on the boundary.

Choose a normal coordinate at \( x_0 \) such that \( \nu(x_0) = \partial_n \) as above. From Lemma 1.2 of [5] there exists \( R > 0 \) depending on the bounds in (3.3.15) and (3.3.31) such that if \( \nabla_{nn} u(x_0) + u(x_0) \geq R \), then the eigenvalues \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of the matrix \( [(u_{ij} + u\delta_{ij})(x_0)] \) satisfy

\[ \frac{\epsilon_0}{2} \leq \lambda_i \leq C \text{ for } i < n, \quad \lambda_n \geq \frac{R}{2} \]

From Lemma 3.3.1, we conclude that \( \lambda_n \) is bounded, which gives us a bound for \( \nabla_{nn} u(x_0) + u(x_0) \) hence for \( \nabla_{nn} u(x_0) \). Therefore (3.3.39) holds for either cases (3.3.36) or (3.3.37). Since we choose \( x_0 \in \partial\Omega \) arbitrarily, (3.3.32) holds and this proves the theorem.

Now we are in position to prove the theorem 1.2.1.

\textit{proof of theorem 1.2.1.} From theorem 3.2.1 and theorem 3.3.2, we only need \( C^1 \) estimate to prove the theorem 1.2.1 which can be done using the argument in [10] without modification. Let us reproduce it here for completeness. \( C^0 \) bound is from the following lemma. Note that we used the hypothesis that \( \Omega \) does not contain any hemishpere only in this lemma.
Lemma 3.3.6. Suppose $\Omega \subset \mathbb{S}^n$ does not contain any hemishpere and $u \in C^\infty(\overline{\Omega})$ satisfies

$$(u_{ij} + u\sigma_{ij})\eta_i\eta_j > 0 \quad (3.3.40)$$

for any $0 \neq \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n$ and for any $x \in \overline{\Omega}$. Then there exists $C_0 = C_0(\partial \Omega)$ such that

$$\sup_{\Omega} u \leq C_0 \sup_{\partial\Omega} u \quad (3.3.41)$$

Proof. By the hypothesis, given any $p \in \Omega$, there are a point $q \in \partial\Omega$ and a geodesic in $\Omega$ joining $p$ and $q$, with a total length $l \leq \frac{\pi}{2} - \epsilon$. With respect to the arc-length parametrization we have $u'' + u > 0$ on the geodesic from (3.3.40). Now assume that $u(p) = \sup_{\Omega} u$. Note that $\cos s$ satisfies

$$(\cos s)'' + \cos s = 0, \quad \cos s > 0 \text{ for } 0 \leq s \leq l$$

with $\cos(0) = 1$ and $\cos'(0) = 0$. Set $h = \frac{u}{\cos s}$. Then

$$(h' \cos^2 s)' = u'' \cos s - u(\cos s)'' \geq 0.$$  

Hence $h'(s) \cos^2 s \geq h'(0) \cos^2 0 = 0$. Therefore $h'(s) \geq 0$ and so $h(l) \geq h(0)$, that is, $u(p) \leq u(q)/\cos l \leq C_0 \sup_{\partial\Omega} u$. \qed

From this lemma, we have a uniform bound

$$0 < K_1^{-1} \leq u \leq K_1$$

where $K_1$ depends on $|u|_{\partial\Omega}$. Moreover note that $\sigma^{ij}(u_{ij} + u\sigma_{ij}) = \Delta_{\sigma}u + nu > 0$, where $\Delta_{\sigma}$ is the Laplace - Beltrami operator on $\mathbb{S}^n$. So if $\overline{u}$ is the solution of

$$\Delta_{\sigma}\overline{u} + nK_1 = 0 \quad \text{in } \Omega$$

$$\overline{u} = u \quad \text{on } \partial\Omega,$$

56
we have \( \underline{u} \leq u \leq \bar{u} \) on \( \Omega \). Hence, as the tangential derivatives of \( u \) on \( \partial \Omega \) are known, it follows that

\[
|\nabla u| \leq K_2 \quad \text{on} \quad \partial \Omega \tag{3.3.42}
\]

In order to estimate the gradient on \( \overline{\Omega} \), consider a point \( x_0 \in \overline{\Omega} \) where \( v = \sqrt{u^2 + |\nabla u|^2} \) attains its maximum. In view of inequality (3.3.42), we may assume \( x_0 \in \Omega \). Then there holds

\[
v v_i = (u_{ij} + u\sigma_{ij})\sigma^{jk}u_k = 0, \quad i = 1, \ldots, n.
\]

From the condition (1.2.5), the matrix \([u_{ij} + u\sigma_{ij}]\) is invertible which implies that \( \nabla u(x_0) = 0 \) and so

\[
\max_\Omega |\nabla u| \leq v(x_0) \leq \max_\Omega u.
\]

Note \( u \geq 0 \). We thus have the estimate

\[
|\nabla u| \leq C \quad \text{in} \quad \overline{\Omega}. \tag{3.3.43}
\]

This completes the proof of the theorem.
BIBLIOGRAPHY


