Fifth Graders’ Reasoning on the Enumeration of Cube-Packages in Rectangular Boxes in an Inquiry-Based Classroom

Thesis

Presented in Partial Fulfillment of the Requirements for the Degree Master of Arts in the Graduate School of The Ohio State University

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The Ohio State University

2010

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Abstract

In this study, I am taking what Cobb and Yackel (1996) called an “emergent” perspective, which is a blend of psychological and socio-cultural perspectives from the constructivist paradigm, to investigate how fifth graders construct and modify their mental models and processes for understanding volume measurement in terms of cube-package enumeration problems. In particular, this study is an extension of the Battista’s 1999 study, in which he investigated how fifth grade students predicted and enumerated the number of cubes in a rectangular box in an inquiry-based classroom. I describe the work and reasoning of two pairs of fifth grade boys as they predicted the number of cube-packages that fit into graphically represented boxes in an inquiry-based classroom. This study extends on the essential mental processes that previous research has already defined for cube enumeration problems (Battista, 1999; Battista & Clements, 1996), and describes three additional essential mental processes (locating, positioning, and orienting) that students need to do for cube-package enumeration problems. The results of this study indicate that there are three types of mental models used by the students when dealing with package problems: layer based, non-layer composite-unit based, and non-composite-unit based. Finally, I describe some of the cognitive obstacles and errors that occur when students attempt to solve cube-package problems.
Dedication

Dedicated to my loving and caring wife, Elizabeth.
Acknowledgements

I like to thank my advisor, Michael Battista, for all his help during the writing of this thesis. I would also like to thank my friends and family who supported me throughout my graduate studies at The Ohio State University.
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Chapter 1: Introduction

In our society it is next to impossible to go through our everyday lives without having to deal with measurement in some way. Whether it is measuring ingredients for a recipe or stepping up on scale to weigh ourselves, we are a society that thrives on measurement. In addition, measurement is extremely important and used frequently by professionals in science, engineering, economics, and medical fields. Measurement is so important that the National Council for Teachers of Mathematics (NCTM) has made it one of the five content standards in its Principles and Standards for School Mathematics (National Teachers of Mathematics, 2000). Within the category of measurement, there are many types of measurements such as mass, length, area, velocity, and the focus of this study, volume.

Volume measurement is an important concept in mathematics because it allows us to quantify and measure objects in our three-dimensional world. In 2007, the National Assessment of Educational Progress (NAEP) test results showed that the nation’s eighth grade students’ average was below proficient in the measurement category (National Center for Education Statistics, 2006, 2007). The measurement concepts that are emphasized in the eighth and twelfth grade NAEP test are volume and surface area, as well as converting between different measurements (Kloosterman, 2000). Ben-Haim, Lappan, and Houang (1985) found by looking at large quantitative studies, such as the
NAEP and the Michigan Educational Assessment Program (MEAP), that an area that students struggled with was finding the volume of 3D solids. They found the specific volume questions that students struggled with involved figuring out the number of cubes contained in 3D solid or cube enumeration problems (Ben-Haim et al., 1985).

**Rationale**

Many researchers have investigated why students struggle with enumerating cubes in solids made from cube arrays and where errors occur with these types of problems. Ben-Haim et al. (1985) found that most students who missed cube enumeration problems were using one of four different counting strategies. The four counting strategies were: “(1) counting the actual number of faces showing, (2) counting the actual number of faces showing and doubling that number, (3) counting the actual number of cubes showing, and (4) counting the actual number of cubes showing and doubling that number” (Ben-Haim et al., 1985, p. 396-397). From the results on a pretest and posttest conducted during their study, Ben-Haim et al. (1985) found that counting strategies 1 and 2 were most prevalent on the pretest and counting strategies 3 and 4 were most prevalent on the posttest. These faulty counting strategies explained the common student errors with enumerating cubes in 3D solids. However, they provide little insight as to why these errors occurred. Ben-Haim et al. (1985) concluded at the end of their study that, “The two kinds of frequent errors made by students, namely, dealing with two-dimensions rather than with three-dimensions or missing the hidden portion of the figure, are clearly related to some aspects of spatial visualization ability” (p. 406).
In a 1996 study, Battista and Clements (1996) investigated this idea of spatial visualization ability by looking at the different ways students conceptually construct 3D cube arrays when enumerating cubes, as well as trying to describe the underlying mental processes that occur during these constructions. Unlike the Ben-Haim et al. study, Battista and Clements took a qualitative approach to the topic by looking at student reasoning through a series of student interviews. Instead of categorizing the students’ strategies by their errors like Ben-Haim et al., Battista and Clements categorized the students by the overall strategy they chose to enumerate the 3D cube arrays. They grouped the student’s enumeration strategies into five broad categories: A, B, C, D, and E. (Battista & Clements, 1996). Category A is where the student conceptualizes rectangular cube arrays by organizing them in term of layers or used a layer structuring technique. In Category B, the student mentally visualizes that the cubes fill up a space, but not in terms of layers like in Category A. For Category C, the student’s conception of the cube arrays was focused only on the faces of cubes showing on the outside of the cube arrays. In Category D, the student used the volume formula without a deep understanding. The last category, Category E, represents all other strategies that did not fit into Categories A through D. In terms of getting the correct answer, the results of the study showed that students who used the strategies in Category A were the most successful whereas students who used strategies in Category C were the least successful (Battista & Clements, 1996).
Theoretical Framework

Battista and Clements (1996) introduced a number of constructs to students’ reasoning on cube enumeration problems—spatial structuring, coordination and integration, and mental models. They concluded from their study that in order for the students to meaningfully count cubes, a deep understanding of spatial structuring must occur for them to create valid mental models of 3D arrays of cubes. In addition, they concluded from their study that students who incorrectly enumerated 3D cube arrays did so because of incorrect spatial structuring of the arrays (Battista & Clements, 1996).

In discussing coordination, Battista and Clements (1996) cited work by Morss (1987), who argued that students initially view 3D figures from multiple viewpoints, where each viewpoint is viewed independently of one another. In order for students to overcome this initial perception of disjointed multiple viewpoints, they need to do proper coordination and integration processes. A first part of this process is for students to recognize when cube faces belong to the same cube.

Students must coordinate the different orthogonal views of a 3D figure in order to see how these views spatially interrelate with one another (Battista & Clements, 1996). Students who struggle with this process typically show an error in the enumeration process by double counting cubes and typically do not visualized the 3D array in terms of layers (Battista & Clements, 1996). Coordination is not enough for students to meaningfully enumerate 3D cube arrays. Students also need to integrate the orthogonal views into one complete mental model of the 3D cube array. It is important to understand that integration depends on the student being able to first coordinate all the orthogonal views of the 3D cube array; however, the reverse is not dependent. In the
study, Battista and Clements (1996) found that on an coordination and integration problem, nine out of 47 students could coordinate the different orthogonal views of a 3D figure, but were unable to integrate them correctly when asked to build the figure using cubes.

In addition to the cognitive mechanisms of spatial structuring and creating mental models, Battista (2004) later argued that the processes of abstraction, units-locating, and organizing-by-composites were also critical for meaningful enumeration of 3D cube arrays. Abstraction is the most important cognitive mechanism and is divided into three levels: perceptual, internalized, and interiorized levels (Battista, 1999, 2004; von Glasersfeld, 1995). At the perceptual/recognition level, abstraction isolates an item in the experiential flow and grasps it as an object: “Focused attention picks a chunk of experience, isolates it from what came before and from what came before and from what follows, and treats it as a closed entity” (von Glasersfeld, 1991, p. 47). When items are perceptually abstracted, we become conscious of them, entering them into working memory. Such “empirical” abstraction isolates those sensory properties of an experience that are needed to recognize further instantiations of the experience (von Glasersfeld, 1991). Items that have been abstracted at this level cannot be operated on unless they are physically present. Furthermore, “The abstracted operational pattern necessary to recognize things of a kind does not automatically turn into an image that can be represented” (von Glasersfeld, 1995, p. 93).

Once the student can visualize the item or idea free from perceptual manipulation, the student has reached the internalized level; it is at this level that visualization becomes possible. The interiorized level occurs when the student can visualize the item or idea
and use their imagination to freely decompose or manipulate the item or idea in their minds. The other two cognitive mechanisms of units-locating process and organizing-by-composites are also critical for students to meaningfully enumerating 3D cube arrays. The process of units-locating is crucial to constructing proper mental models and occurs when the student “locates cubes by coordinating their locations along the dimensions that frame an array” (Battista, 2004, p. 192). The organizing-by-composites occur when the student places the units of an array into more complex composite units so that they can efficiently enumerate and generate the entire array (Battista, 2004).

In 1999, Battista did a study on how students’ mental models for 3D arrays of cubes developed during enumeration, and the role that social interaction played in small-group work in an inquiry-based classroom (1999). Battista (1999) found that, “Students who developed an interiorized layer structuring that they could correctly apply in predicting the number of cubes in 3D arrays did so by recursively cycling through sequences of acting (structuring and enumerating), reflecting, and abstracting” (p. 441-442). In addition, he concluded that students who used this layer structuring technique on an earlier task did not necessarily transfer it to a later task (Battista, 1999). He used the three levels of abstraction to provide some insight as to why the layer structuring technique is not always transferred by students (Battista, 1999). He explained that the layer structuring technique is a complex way of thinking and that students might perceive it only at a perceptual level of abstraction. However, it may take a student doing a layer structuring several times before that student can internalize it. Also, it may take multiple reflections on layer structuring in order for it to be interiorized by the student.
In addition, Battista (1999) discussed four different socio-cultural interactions that occur as collaborative pairs of students work together: interpersonal conflict-induced accommodation, interpersonal idea appropriation, communication-induced extension, and discussion-related change in reflective focus. Battista (1999) describes interpersonal conflict-induced accommodation as, “students accommodate their current thinking as they attempt to resolve cognitive conflicts that arise from comparing their views with discrepant views of others” (p. 419). Battista (1999) defines interpersonal idea appropriation when, “elements of the communicated reasoning of others prompt student to personally construct related components of their own reasoning” (p. 419). Communication-induced extension occurs when students try extend their conceptualizations in order (a) to develop and elaborate on their ideas in order effectively communicate them to others and (b) to make sense of ideas that are communicated by others (Battista, 1999). Discussion-related change in relative focus occurs when, as students communicate their ideas to one another, they change the way they conceptualize the situation and the direction they are taking to solve the problem (Battista, 1999).

Battista used his findings from the 1999 study and the different cognitive mechanisms described earlier to create levels of sophistication for students’ understandings of volume measurement (Battista, 2004). Student reasoning on measurement was divided into seven levels of sophistication, with level 1 being the least sophisticated and level 7 being the most sophisticated level. This study came out of larger research study done by Battista called Cognitive Based Assessment (CBA). Battista (2004) described CBA as,
Applying the results, theories, and methods of modern research in mathematics education to create an assessment system that can be used to assess in detail the cognitive underpinnings of progress students make in developing understanding and mastery of elementary school mathematics (Battista, 2001b). (p. 185).

CBA allows elementary teachers to see a particular student’s level of understanding for volume as well as know where the student’s understanding should progress to next. Knowing the level of sophistication is both powerful and helpful for the teacher because it can help them develop instruction tailored to each student’s understandings and needs.

In 2010, Battista (2010) wrote a paper on the current status of students’ reasoning on enumerating 3D cube arrays and volume measurement. He concluded that the reason students struggle with enumeration of 3D cube arrays was not because the diagrams of the cube buildings provided insufficient information for the students to perform correct enumeration, but rather the students struggle to properly construct integrated mental models of the cube buildings (Battista, 2010). In order to construct valid mental models of the cube buildings, students need the opportunity to create hypotheses and to validate or dismiss them through testing (Battista, 2010; Kim & Hahn, 1997). Battista (1999) attempted to implement this predict-and-check approach by having students first predict the number of cubes using two types of diagrams (a pattern picture and an isometric picture) and then test their predictions by building the boxes and filling them with cubes. This encouraged the students to address the coordination process by looking at the relationship between the box sides and the cube faces (Battista, 2010).

**Working memory and cognitive demand.** It is important to note that when students construct mental models of objects, it requires them to hold information and/or
visualizing components in their working memory. Working memory is “the part of your mind in which you combine and manipulate information—it’s pretty much synonymous with consciousness” (Willingham, 2009, p. 26). Working memory has only a limited capacity for information that a person can store at one time. Willingham (2009) suggested that the cognitive load for a person’s working memory does not depend on the number of items, but rather the number of meaningful objects a person is manipulating or retaining in their working memory. A way for people to store more information in their working memory is by chunking or organizing individual pieces of information into larger efficient groups. However, chunking can occur only when a person has “applicable factual knowledge in long-term memory” (Willingham, 2009, p. 26).

As people use their working memory there is a demand, which is placed on their cognitive abilities or what is called a cognitive demand (Edwards & Dall’Alba, 1981). If the cognitive demand exceeds the capacity of a person’s working memory then learning is impaired (Paas, Renkl, & Sweller, 2003). In fact, this may result in the person, “forgetting some of the information needed for processing the primary task and, therefore, some cues are lost or never enter working memory” (Speier, Valacich, & Vessey, 1999, p. 339). Therefore one of the reasons that students make errors in their solutions when solving mathematics problems is when the cognitive load placed on their working memory exceeds the capacity of their memory.

Multiple representations. Research has found that some students struggle to make connections between 2D representation of 3D objects and the actual 3D objects (Lowrie, 2002). In Battista’s 1999 study, the students used multiple representations in order to complete each cube enumeration problem that was given to them. The
representations that were used are: 2D pictures of boxes using a isometric view, 2D pattern pictures of boxes, written descriptions of the dimensions of boxes, 3D mental models of the boxes, and concrete boxes made from grid paper. Using multiple representations of objects, like the ones used in Battista’s 1999 study, can help students get a deeper understanding of the material being taught. Friedlander and Tabach (2001) state that using various representations to learn mathematics helps cater to all students who have individual styles of thinking and learning. Although many researchers point out that each representation has its disadvantages, they state that when multiple representations are used together they cancel out many of the individual disadvantages and prove to be effective instructional tool (Friedlander & Tabach, 2000; Kaput, 1992).

For example, Ben-Haim et al. (1985) found that fifth through eighth graders struggled with cube enumeration problems particularly “relating isometric type drawings to the rectangular solids they represent” (p. 407). However, after they taught the students using an instructional unit where the students represented three-dimensional objects in two-dimensional pictures and vice versa, the students’ spatial visualization test scores increased on items that required them to see links between the two types of representations (2D and 3D) (Ben-Haim et al, 1985). Therefore, this study suggests using multiple representations (2D and 3D) when enumerating cubes in a 3D solid can help students improve their spatial visualization ability. Like the Ben-Haim et al. and Battista 1999 studies, in the current study, students used multiple representations to reason about and solve different cube-package enumeration problems.
Statement of Research Questions

In this study, I will be extending previous research that investigated how students enumerated single cubes in 3D cube buildings to research on how students solve cube-package enumeration problems like the one shown below.

![Diagram of Package B fitting into Box 1]

*Figure 1.1. How many Package B’s will fit into Box 1?*

For this study I will be using the “emergent” perspective, defined by Cobb & Yackel (1996) as, “in which psychological constructivist perspective analyses of individual activity are coordinated with interactionist analyses of classroom interactions and discourse” (p. 175). I will be using an integrated approach that uses both psychological and socio-cultural perspectives of constructivism in order to analyze how students’ individual cognitive constructions of the cube-package problems and volume develop in an inquiry-based mathematics classroom (Battista, 1999). This study takes
this emergent perspective, as I am investigating both the individual students’ and the pairs of students’ reasoning about the different cube-package problems. The intent of this study is to answer the following research questions about students’ reasoning about cube-package problems.

1. How do students reason about cube-package problems? What strategies do they use?

2. What mental models do students construct for cube-package problems and how do they use them in determining the number of packages that fit into a rectangular box?

3. What are the mental processes that students use to construct those mental models?

4. What are common cognitive obstacles and errors that occur as students attempt to solve cube-package problems?

**Essential Mental Processes for Cube Enumeration Problems**

Based on the literature review, there are five essential mental processes that are needed to enumerate cubes in cube buildings. They are coordination, integration, developing proper mental models, iteration of units and composite units, and spatial structuring (Battista, 1999; Battista & Clements, 1996). I will review how research defines these five mental processes. Later I will define the additional mental processes that my research suggests are needed to enumerate cube-packages that fit into a box.
Most of the research investigating the five mental processes was done for cube enumeration problems. So, I will explain how researchers have defined them in this context, and later I will explain how they occur in the cube-packages problems context.

**Coordination.** According to Battista & Clements (1996):

To move beyond the local structuring of the medley-of-viewpoints conception of cube arrays, students must coordinate orthogonal views. That is they must recognize how these views are spatially interrelated. Coordinating views of a 3-D array requires that cube faces depicting the same cube be recognized as such. It also requires that two or more views be considered together in a way that accounts for interrelationship between them. (p. 284)

My interpretation of this definition of coordination is that it is a process of constructing a conceptualization of how the different orthogonal views of the cube array are interrelated.

There are two components of the process of coordination. The first component is a cube-face-to-cube-face matching, which is the recognition of how cube faces from one orthogonal view correspond to the faces of the same cubes from another orthogonal view. The second component of coordination is visualization of how two orthogonal views fit together. The cube-face-to-cube-face matching is a local operation, whereas combining orthogonal views is more of a global operation. The cube-face-to-cube-face matching must precede the orthogonal view combining via a proper coordination of the views.

Although order of these two operations is important, it is more important to understand that these two operations work together to lead to a proper coordination of the views.

**Integration.** For integration, Battista (2008) stated, “To integrate views of a 3D object, students must coordinate the views, then construct a single coherent mental model
of the object that possesses these views” (p. 9). Integration is the process of coordinating all the different orthogonal views of a cube array to create an isometric view of the array. There are two types of integration students can do for cube arrays: complete integration and partial integration. Complete integration is when the student has a complete mental model that correctly locates all the cubes in the array. Partial integration occurs when the student has an incomplete integration of the cubes in the array. For instance, a student is able to visually understand the location of cubes in the outer sides of the cube array, but cannot understand the locations of cubes in the array's interior. As students continue to progress in the levels of abstraction (internalization and interiorization) of the cube array, they will be able to decompose the isometric view in order to be able to visualize the interior cubes of the cube array.

**Mental models.** Battista and Clements (1996) define a mental model as, “an analog mental version of an object whose structure is identical to the structure of the perceived or conceived structure of the object it represents and that is used to simulate interactions with the object (Battista, 1994)” (p.285). Current literature on mental models has found that mental models often do not include all the pictorial detail of the objects they represent (Held, Knauff, & Vosgerau, 2006). For instance, when imagining iterating a cube-package, it is likely that a student's mental model of the package is a spatial token that lacks much of the detail of individual cubes. I interpret a mental model as a mental visualization that a person can mentally manipulate and that lacks some of the pictorial detail of the actual object. Figure 1.1 below shows one possible representation of a mental model of placing copies of Package C in the box.
Spatial structuring and iteration. The last two mental processes needed to enumerate cubes in cube buildings are spatial structuring and iteration. Spatial structuring is the mental process of, “constructing an organization or form for an object or sets of objects” (Battista, 1999, p. 418). Battista and Clements (1996) stated,

The process of spatial structuring includes establishing units, establishing relationships between units (such as how they are placed in relation to each other), and recognizing that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit). (p. 282)

Iteration is the repetition of a cube or composite unit of cubes. In the present context, iteration occurs as students enumerate cube-packages or composite units of cube-packages to fill a box.
Chapter 2: Method

This study analyzes data that was previously collected by Michael Battista as part of research project to develop and study the *Containers and Cubes: 3D Geometry-Volume* textbook in the *Investigations in Number, Data, and Space* elementary mathematics curriculum, and this work was funded by the National Science Foundation (Grant RED 8954664). Battista previously published part of this research about how students’ reason about cube enumeration problems in his 1999 article entitled, *Fifth Graders’ Enumeration of Cubes in 3D Arrays: Conceptual Progress in an Inquiry-Based Classroom*. The present study is an extension of Battista’s 1999 article in that it continues the investigation of students’ reasoning about volume with regards to cube-package enumeration problems. The activities used in both of Battista’s 1999 article and this study were published as part of Investigation 1 and Investigation 2 respectively in the *Containers and Cubes* textbook (Battista & Berle-Carman, 1996).

Instructional Setting and Sample

The research project took place at a suburban elementary school in Northern Ohio during the late winter of 1994. For this study, four pairings of fifth grade students were observed in a classroom taught by a teacher who is highly skilled in teaching mathematics using inquiry and problem solving approaches (Battista, 1999). This teacher
taught a problem-centered inquiry-based volume unit that lasted four weeks to both of her mathematics classes, where each class lasted one hour per day (Battista, 1999).

The teacher explained to her two classes of students that their goal for the activity was to find a correct way to predict the number of cube-packages that would fit into the boxes shown on the activity sheets that were given to the students. There were two different activity sheets of cube-package problems that were given to each student, for which they were to record their answers (see Appendix A). The students collaborated together in pairs to predict the number of cube-packages that would fit in a box using pictorial representations of both objects (see Figure 2.1 below). They then checked their predictions by making the box out of grid paper and placing cube-packages constructed with multilink cubes in the box. The students were required to predict and then check their prediction for one problem before moving on to the next one (Battista, 1999). As the students worked on the activity sheets, “the teacher circulated about the room, interacted with student pairs, encouraging within-pair communication and collaboration, and promoting individual sense making” (Battista, 1999, p. 419).
Students were chosen for the case studies based on the teacher’s recommendation and if they did not use a layering strategy on all three pre-interview problems created by Battista (1999). The teacher chose one pair of boys and one pair of girls from both the high group (Class 1) and the low group (Class 2) (Battista, 1999). However, the case studies discussed in this thesis will be only reporting on the two pairs of boys from the two classes. Class 1 pair of boys completed the package problem activity sheets after one class period (one hour), whereas Class 2 pair of boys took two class periods (two hours) to complete the activity sheets.

**Data Collection and Analysis**

Battista and his research assistants did all the data collection for this study and this author was not involved with the project at that time. The students were asked at the beginning of the activity to verbalize what they were thinking as they attempted to solve the package problems. Battista sat with the two pairs of boys during their respective class
periods and observed them working on the activities and asked them clarifying questions about their reasoning when needed. Battista took field notes as he observed the students working on the package problems to help record what the students said and did during their explanations for each problem. All classroom observations were videotaped, and I transcribed the tapes for Investigation 2, the package problems. Analysis of students’ conceptualizations of package problems was iteratively synthesized from analysis of videotapes, field notes, and interview transcriptions. I discussed all my interpretations with Dr. Battista; often we hypothesized possible interpretations, then I returned to the data—the tapes—to see if these hypotheses were supported.
Chapter 3: Results

In this chapter, I will be defining and presenting the additional mental processes required for cube-package enumeration problems and the cognitive obstacles associated with them through two case studies. The two cases studies are two pairings of fifth grade boys from the two classes observed (J and M from Class 1, P and N from Class 2). I will give detailed descriptions of the students’ work and thinking while they completed the problems in Investigation 2.

Defining the Additional Mental Processes of Package Problems

The difference between cube enumeration problems and package problems. The mental processes that are required in cube-package problems include, but extend, those in cube enumeration problems. It is important to look at the differences between cube enumeration problems and cube-package enumeration problems. The first difference is that with cube enumeration problems the student has to think about two types of items, the box and the array of cubes that fits inside the box. With package problems the student has to be concerned with the same two types of items, but with the added task of thinking about cube-packages—how the cubes are arranged in a package, and how packages are arranged in the box. The second difference is symmetry and orientation. In cube enumeration problems, a cube has 90° (and 180°) rotational symmetry about each of the x-, y-, and z-axes through its center or what we will call 90°
xyz rotational symmetry. Unlike cubes, most of the packages are missing this 90° xyz rotational symmetry for at least one axis, therefore changing the orientation of a package can change how it is placed in the box. The last difference is that a box may not be completely filled with copies of packages.

I hypothesize that there are three additional mental processes that are needed to form proper mental models in package problems: orienting, locating, and positioning.

**Orienting.** It is important to clarify the distinction between “orienting” and “orientation”. We will use “orienting” as a verb when we talk about what students do mentally or physically with the packages and “orientation” as a noun referring to rotational positions of packages.

Orienting is the mental process of choosing an orientation of a package. There are two cases of orienting that occur during spatial structuring of packages in cube-packages problems. Case one is when the student does not change the orientation of the package that is presented in the picture given in the problem. Case two is when the student chooses to reorient the package. For case one, the orienting process might be implicit in that the student might not even consider other orientations of the package. In contrast, with reorienting, the student has to explicitly deal with changing the orientation of the package, which can be difficult to visualize.
**Locating.** Locating is the mental process of determining how the cube faces that appear on one orthogonal view of a package correspond to an array of squares that appears on one side of the box. Locating orthogonal views of a package on the sides of the box requires a determination of how the individual cube faces appearing on the package correspond to the individual squares appearing on the sides of the box. In other words, a student can locate an array of squares that appears on one orthogonal view of a package inside the box by matching that view to a congruent array of squares on a side of the box.

For example, the 2x3 array of squares that appears on the left side of Package C might be located on the left side of the box as shown in Figure 3.2.
Figure 3.2. A locating process for the left orthogonal view of Package C.

Students can have different sets of locating processes depending on how they are visualizing positioning and orienting the packages in the box (see Figure 3.3 below).

Figure 3.3. Examples of some different ways that students might do locating processes for Package B in Box 1.
Once the student has chosen the package’s orientation, they have to match one or more arrays of squares appearing on the package's orthogonal views to corresponding arrays of squares on the sides of the box. Even more, this locating process must be repeated (or iterated). For example, a student might do a set of locating processes for Package B on the bottom of Box 1, locating the single square appearing on the bottom orthogonal view of Package B to various locations in the array of squares appearing on the bottom of Box 1 (see Figure 3.4). This set of locating processes forces all other orthogonal views of Package B to have a one-to-one correspondence with corresponding arrays of squares to other box sides (but this may be difficult for students to see).

![Diagram](image)

*Figure 3.4.* A set of locating processes for Package B that appears on the bottom of Box 1.
As seen in Figure 3.5, for a proper spatial structuring, the left orthogonal view of Package B has to correspond with appropriate arrays of squares on the left side of Box 1 in order to preserve the orientation of the package that was used for the first set of locating processes on the bottom of Box 1. In this case study, we will see examples of students locating a package on one side of the box in way that is inconsistent with the locating processes of the package on another side of the box. Also, clearly how students orient a package affects their locating processes.

![Figure 3.5](image)

*Figure 3.5.* The set of locating processes for Package B that appears on the left side of Box 1.

**Positioning.** Positioning is the process of imagining where a whole package occurs in the box. Positioning is accomplished by integrating the coordinated set of locating processes for the arrays of squares appearing on orthogonal views of a package.
To position a package within a box requires that at least two adjacent sides of the package be located on the array of squares appearing in the corresponding adjacent box sides. This process requires coordinating several locating processes.

*Figure 3.6.* An example of coordinating the locating processes of a Package B.

To illustrate how developing a mental model of a position of a package in a box requires the integration of coordinated locating processes, consider a student attempting to position Package C in the bottom back left corner of Box 1. He or she might first locate the 2x2 array of squares on the bottom orthogonal view of Package C at a 2x2 array of squares on the bottom back left corner of the box (see Figure 3.7). Then the student would locate the arrays of squares on the back and left orthogonal views of Package C at their corresponding arrays of squares on the back and left sides of the box. The student completes a positioning process when he or she integrates these coordinated
locating processes into one coherent three-dimensional visualization of how the package fits inside the box.

Figure 3.7. An example of a positioning process.

**Spatial structuring.** Another mental process required for package problems is spatial structuring. Spatial structuring is the mental process of integrating the entire set of positions of the packages into one coherent mental model for how the set of packages will fit into the box. For instance, Figure 3.8 shows a spatial structuring of Package C in the box as a 2 by 3 array of spatial tokens.
Two additional complexities of dealing with cube-packages. One added complexity to the spatial structuring process for cube-package problems is that the packages do not always completely fill the box, whereas in the cube enumeration problems the cubes always completely filled the box (see Figure 3.9). Therefore, the student needs to account for this possibility of having some empty space left over in the box in their spatial structuring.
Figure 3.9. An example of a student’s mental model of spatial structuring when the packages do not completely fill the box.

A second added complexity in enumerating packages is if the spatial structuring requires more than one package orientation. For example, Package C in Box 2 requires that the student use two different orientations to correctly enumerate packages that fit into the box (see Figure 3.10). Thus, if a student were fixating on a single orientation, they would be unable to come up with correct enumeration of packages.
Throughout the rest of this section and Discussion section, I will document and discuss, how these mental processes occur as students solve cube-package problems. In addition, I will discuss students’ misconceptions for package problems and how are they related to these mental processes.

Case Study 1: J and M, Class 1

J and M, Package A, Box 1, Day 1.
M: Package A is going to be... she [the teacher] said that we could have like top space or space at the top. [Looking at the box by himself, M points to the 2x2 arrays of squares on the bottom of the picture of Box 1.] 1, 2, 3, 4, 5, 6. That is what I came up with. My prediction was five or, no, six.

J: I say six because there is four on the bottom [points to the drawing of Package A] so you could fit three on each thing [pointing at the bottom of the picture of Box 1] and that would take up two [sections of 2x6 arrays of squares the bottom of the box] and there would be three [Package A's] in there [points to the two sections on the bottom of the picture of Box 1].

M: Okay B.
J: No, we are supposed to find it out first. You have to make the box first.

MB: So you both predicted it would be six?

M: Yeah, I think it is because the four [counts the 2x2 arrays of squares on the bottom of the picture of Box 1] 1, 2, 3, 4, 5, 6 and you can’t have any more.

[J begins to make the box, while M writes down the description of what they did.]

[M continually writes, pauses, and erases his description on his paper a couple times as if he is struggling to write a description of their prediction strategy.]

MB: So read your prediction.

M: Okay, so we got the prediction of six. We thought that because we counted the squares on the bottom and divide by 4 because that is how many cubes there are on the bottom of Package A.

MB: So you counted up the number of squares here [points to the bottom of the picture of Box 1] and divided by 4 because that is how many there are on the bottom of this [points to the picture of Package A].

M: Yeah. [Long pause] I think that could be a good theory to work out.

MB: J, did you hear how M figured out how many?

J: No.

MB: Why don’t you tell him?

M: Well it is kind of different than what he probably thought because I screwed up in writing it and then kind of had to redo it and said divided by four. But that is still a good way.

MB: J, how did you figure it out?

J: Well, there are four on the top and bottom of Package A. And you can put three groups of four on one side [points to the right side of the bottom of the picture of Box 2] and then another three groups of four on the other side [points to the left side of the bottom of the picture of Box 2].
Figure 3.13. J’s prediction strategy for Package A’s in Box 1.

M: It is basically the same.

M: So here is our Package A [places a Package A made from cubes in front of J]. I will make Package B.

J: We need enough to fill up the whole box.

[Telling from M’s body language he does not agree with J about the need to make more Package A’s. MB reacts to M’s body language by asking the following question.]

MB: Well, do you think that you can check out whether you are right or wrong without having to completely fill the box?

M: We can, because we are just saying to put it in the corner. [J then moves it around the box and find they have six].

MB: So you were right?

M: Yeah, the answer is really six.

Initially, M locates the 2x2 array of squares appearing on the bottom orthogonal view of Package A on the set of 2x2 array of squares that appear on the bottom of Box 1 (lines 1-4). He iterates this locating process six times (lines 1-4). M predicts that six Package A’s will fit into Box 1. However, when charged with task of writing a
description of their prediction strategy, M seems to struggle with being able to write the
description by the way he is constantly pausing and erasing things as he tries to write the
description (lines 30-31). This also becomes apparent when M states that he got
confused in writing his prediction strategy and decided to rethink his prediction strategy
(lines 51-53). As a result, M’s written description does not seem to match the way he
originally predicted six Package A’s will fit into Box 1.

In M’s written prediction strategy, he states that he counted up the number of
squares that appeared in the array of squares on the bottom of the picture of the Box 1
(lines 35-37). He then took the total number of squares appearing on the bottom of the
box and divided by four because that is the number of squares that appeared on the
bottom of Package A (lines 35-37). Using this new strategy, M again found that six
Package A’s fit into Box 1 (line 35). M’s original prediction involved enumerating the
locating processes of Package A on the bottom of Box 1, whereas his written description
is more of numeric procedure that uses the operation of division to make his prediction.

There are two possibilities as to why this discrepancy occurred between M’s
original prediction and his written description. One possibility is that M is trying to
determine how he got his initial prediction of six packages using numeric operations. M
recognizes that the bottom of Package A has four squares on it and the bottom of Box 1
has 24 squares on it. M recognizes that if he uses the numeric operation of division to
divide 24 by 4, it will give him his original prediction that six Package A’s fit into Box 1.
In this case, his reasoning is only numerically focused and divorced from any spatial
operation or structuring. Therefore, M’s conceptualization would be based on the
recognition that, if he divided the number of squares that appear on the bottom of Box 1
by the number of square appearing on the bottom of Package A, he will get his original prediction of six.

The second possibility is that M’s struggle to articulate his original strategy through written word plays a role in his decision to re-conceptualize his initial strategy, a more spatial strategy, to a more numeric strategy because it is easier to articulate in words. In this case, M’s use of the division operation in the written description comes from his recognition of its use in determining how many groups of four squares are in 24 squares on the bottom of Box 1. Most students probably would not be precise enough in their language to write the spatial description needed to convey M’s original prediction strategy, especially without drawing.

In both his spatial and numeric strategies, M makes no explicit reference to how the height of Package A plays a role in his enumeration of Package A’s in Box 1. It is apparent that M is aware of the height of the box in that he states at the beginning of the episode that they are allowed to have empty space at the top of the box (lines 1-2). Because M makes no explicit mention of the height of the package in his reasoning, this suggests that M does not quite understand the role the height of the package plays in the spatial structuring of packages. We will see the consequences of this lack of explicit attention in M’s strategy for the enumeration of Package B’s in Box 1.

J seems to have a similar prediction strategy to M’s original strategy in that he is conceptualizing how the array of squares on the bottom orthogonal view of Package A are located on array of squares that appears on the bottom of Box 1 (lines 13-17, 57-59). In his explanation of his strategy, J mentions that there are four on the top and four on the bottom of Package A (line 57). Because there is no explicit mention of the units that are
attached to the two sets of four, there is no way of knowing whether J is talking about cube faces or cubes as the units. One possibility is that J is talking about the cube faces of Package A’s in order to figure out how the package will be located on the bottom of the box. In this instance, J would not be addressing the role that the height of the package plays in the spatial structuring of Package A’s in Box 1.

The second possibility is that J is talking about cubes contained in Package A in his statement that there are four on the top and four on the bottom. In this case, although not explicit, J would be making an implicit reference that the height of Package A is two. If this is a reference to the height of Package A, then that suggests that J did a positioning process that would allow him to visualize how the Package A’s are positioned in Box 1.

J mentions that he locates three groups of four on one side of the bottom of the box and then another three groups of four on the other side of the bottom of the box (lines 57-60). Again, J is not explicit whether the four he mentions is referring to squares or cubes. However, as he is talking about the groups of four, he is pointing to the array of squares on the bottom of the box, which suggests that he is talking about the 2x2 arrays of squares on the bottom of the box (lines 58-60). In addition, MB wrote in the field notes that J was talking about seeing two columns of three 2x2 arrays of squares on the bottom of Box 1.

In any case, J’s explanation suggests a little more spatial structuring than M’s original strategy. J’s explanation has more structuring because he conceptualized two composites of three groups of four squares (i.e. 2x2 arrays of squares) (lines 57-60). In a sense, J structures his locating processes into composites of composites. One composite unit is the group of four squares and the other composite unit is a column of three groups
of four squares. Unlike J, M’s initial structuring deals only with finding the locations of the composite of the group of four squares (i.e. 2x2 arrays of squares) on the bottom of the box. It takes more spatial structuring to do column composites of groups of four squares than merely iterating a group of four squares.

MB asks if the boys can figure out how many Packages A’s will fit using just one Package A (lines 79-80). M immediately says yes and that they can put it in the right corner and then move it around to the different positions of that the package fits in Box 1 (line 82). The boys then move the Package A into all the positions it can occur in the bottom of the box to find their prediction of six was correct (lines 82-83). The act of using one package to find the positions of all other packages that will fit into the box is a more difficult action for students to do than using multiple copies of the package. The reason is: when students use one package to check their predictions they have to keep track of the positions that they have already occupied. When using multiple copies of packages to check a prediction, students do not need to pay attention to what positions have been occupied because the box is filled with packages that occupy each position.

J and M, Package B, Box 1, Day 1.
Figure 3.14. How many Package B’s will fit into Box 1?

M: Here is Package B.

J: We have to make a prediction first.

M: Okay. I would say eight.

J: Um, ok.

MB: So how did you figure that out for Package B?

J: We are making a prediction for B.

M: I think it is eight because this [points to the picture of Package B] is three cubes long and there is one and there is two and there is another three, another three [marks to the 3x1 arrays of squares on back bottom side of the picture of Box 1] and that is four and on the other side there is four and that is eight (see Figure 3.15 below).
Figure 3.15. M’s prediction strategy for Package B’s in Box 1.

J: I think it is 24; it has three layers.

M: Oh yeah, because it has three layers. So that would be 8 times 3 equals 24. I have a feeling that our prediction will be exactly the same as the actual.

J: [Checks it by moving around a single Package B with the given orientation in Box 1].

24. [Then J takes Package A and adds a layer of four cubes to it to make Package C].

J: Here is Package C.

M: Where is Packages A and B.

J: I just added one layer to Package A to get C.

M: You added on to Package A!

MB: So you were right, how did you find that you were right?

J: 1, 2, …8 [as he moves the Package B with the given orientation in the different positions it fits in the bottom of Box 1] and there is three layers so 24.
Figure 3.16. J’s prediction strategy for Package B’s in Box 1.

[J writes down the description of the boys’ prediction strategy. In checking Package B, M went back to B after he had made a prediction for Package C.]

M: [Changes now back to Package B]. 1, 2, 3, 4 [as he moves a single Package B with the given orientation in the back four positions it fills on the bottom of Box 1] times 3 is 12 (see Figure 3.17 below).

Figure 3.17. M’s checking strategy for Package B’s in Box 1, part 1.
M: And 1, 2, 3, 4 [as he move a single Package B with the given orientation in the front four positions it fills on the bottom of Box 1] times 3 [motioning as if he were stacking Package B on top of one another three times] is 12 and 12 plus 12 is 24.

Figure 3.18. M’s checking strategy for Package B’s in Box 1, part 2.

Figure 3.19. M’s checking strategy for Package B’s in Box 1, part 3.
M initially predicts that there are eight Package B’s that will fit into Box 1 (lines 5, 13-16). M’s initial conceptualization does not explicitly consider the height of Package B or the height of Box 1. M’s lack of explicit consideration to the height of the box and height of package was also suggested in previous episode (i.e. Package A’s in Box 1). For Package A, M’s prediction strategy was locate and enumerate the 2x2 array of squares that appears on the bottom of Package A on the array of squares appearing on the bottom of Box 1. This worked for enumerating the number of Package A’s that fit into Box 1 because there was only one layer of Package A’s that could fit into the box. Now with Package B, three layers of eight Package B’s with the given orientation can fit in the box, therefore M’s strategy of doing only locating processes on the bottom of Box 1 is not going to be valid by itself. M needs to extend this strategy by considering the locating processes of the packages that occur on one of the lateral sides of the box in order to see that there are three layers of eight Package B’s that can fit into Box 1. Since the prediction strategies for Package A’s and B’s are so similar, this episode also provides stronger evidence that M’s prediction strategy for Package A’s also did not consider the locating process for the height of Package A on the height of Box 1.

J first agrees with M’s prediction of eight Package B’s, but seems reluctant (line 7). M then describes that he got eight packages by locating four packages on one side of the bottom of the box, then recognizing that those four packages took exactly half the space on the bottom of the Box 1 (lines 13-15). M then realized that since half of the array of squares on the bottom of the box was filled by four packages he could also locate four packages on the other half of the box to give him a total of eight Package B’s (lines
M’s prediction strategy for Package B is very similar to J’s prediction strategy for Package A in that both boys structure the locating processes on the bottom of Box 1 into composites of packages. In J’s strategy for Package A, he conceptualizes the locating processes on the bottom of the box in terms of composites of squares appearing on the package bottom. Since M saw J’s strategy for Package A, it seems likely that he applied it to his strategy for Package B.

After thinking about it a little more, J changes his prediction to 24, stating that he visualizes that there are three layers of eight Package B’s with the given orientation fit into Box 1 (line 25). M quickly sees that there are three layers and agrees with J’s prediction of 24 (lines 27-28). M is adapting his mental model of how the Package B’s fit into Box 1 by considering the heights of Box 1 and Package B. J’s communication of his strategy changed how M conceptualized the solution in that he extends his strategy to J’s strategy of thinking in terms of layers of Package B’s (lines 27-28).

When checking his prediction, M structures the Package B’s in a way that is different from both his original strategy and the one he discussed with J (lines 56-58, 67-69). M takes one Package B and uses it to iterate the four positions on the back bottom side of Box 1 (see Figure 3.17). He then multiplies four times three because he has three composites of four Package B’s to get 12 Package B’s that fit into the back half of Box 1 (see Figure 3.17). M then positions four Package B’s on the bottom of the front half of the box by moving a single Package B into the four positions while iterating (see Figure 3.18). He again multiplies 4 times 3 because he has three composites of four Package B’s, which gives him a total of 12 packages that fills the second half of Box 1. M then
structures the Package B’s into two parts, each containing three composites of four packages (see Figure 3.19). M then adds them together to get 24 Package B’s.

There are two interesting aspects about M’s checking strategy. One aspect is that M’s checking strategy is different than the layering structure strategy that J used to get his prediction of 24. J’s strategy is more global than M’s checking strategy in that he structures the eight packages into a composite of a layer that covers the whole bottom and then uniformly iterates that composite to find the total number packages that fit into Box 1 (line 25). Even though M saw J’s layering strategy, he did not use it when he was checking their prediction of 24, which suggests that he might not have fully appreciated the power of J’s conception. M did, however, use composites in his structuring, but his composites were not layer composites that matched the bottom of the box.

The second aspect is that M’s checking strategy has very complex and sophisticated spatial structuring, which can make it difficult to generalize for other package problems. First, M is able to iterate four Packages B’s that fit in the back half of the array of squares appearing on the bottom of Box 1 (see Figures 3.17, 3.18, 3.19 for a visual of M’s checking strategy). Then he is able to conceptualize a composite for the four Package B’s, which fits three times into a back half composite of 12 Package B’s in Box 1. M conceptualizes that the front half composite of Packages B’s will contain 12 Package B’s by doing the same process he did for the back half composite of Package B’s. M now has two composites of 12 packages each and decides to add them together to give him a total of 24 Package B’s. It is apparent from the description above that M’s checking strategy is much more complicated than J’s layering strategy which finds how many packages are in the bottom layer and multiplies it by the number of layers that fit
into the box. Therefore, as the cube-package problems themselves get more complex, it will become more difficult to use M’s checking strategy to find the number of packages that fit into a box.

**J and M, Package C, Box 1, Day 1.**

*Figure 3.20. How many Package C’s will fit into Box 1?*

1. M: This is Package C. [Places Package C with given orientation in the back left corner of Box 1. (See Figure 3.21 below.)] I would say six again because it is just one layer higher than Package A. The actual is....

*Figure 3.21. M places a Package C in the back left corner of Box 1*
J: Wait a second; I have not made a prediction yet.

[While waiting for J, M checks Package B’s in Box 1, which was shown in the previous episode].

MB: And J, how did you predict?

J: I also got six because it is three high and four on the bottom. So like Package A you can fit it three times on one side and three times on the other, but there wouldn’t be any left over because it is three high on Box 1.

M: 1, 2, 3, 4, 5, 6 [iterates as he moves Package C with given orientation in each of its positions inside Box 1].

One thing to note is that in the previous episode (Package B) when building Package C, J added one layer of four cubes to Package A to make Package C (J and M, Package B, Box 1, line 37). When M asked where Package A and Package B was, J told him that he built Package C by adding on a layer of four cubes to Package A (J and M, Package B, Box 1, lines 35-37). This dialogue is interesting because for M’s prediction strategy for Package C, he recognizes that Package C is just one more layer of four cubes added on to Package A (lines 2-3).

In addition to seeing J create Package C by adding on cubes to Package A, M also places a Package C with the given orientation in back left corner of Box 1 and then seems to visualize how many Package C’s could fit into Box 1 (lines 1-2). This suggests that M is still at a perceptual level of abstraction in that he must physically see a Package C with the given orientation positioned in the box in order to be able to visualize the remaining locations of Package C’s that will fit into Box 1. As a result of this recognition, M is able to conceptualize that Package A’s and Package C’s fill the same locations on the bottom
of Box 1. M recognizes that there are no new packages being added or subtracted from the enumeration M did for Package A’s. Therefore, both Package A’s and Package C’s have the same enumeration of six packages for Box 1 (line 2-3).

J also is able to recognize the spatial relationship that Package C is one layer of four cubes higher than Package A. This is apparent when J constructed Package C by adding a layer of four cubes to Package A in the previous episode (J and M, Package B, Box 1, Day 1, line 37). As a result, when J physically constructs the Package C by adding on a layer of four cubes to Package A, it validates the spatial relationship between the two packages in J’s mental model, which is something he will use in the conceptualization of Package C’s in Box 1. In this episode, J states that he also got a prediction of six Package C’s by explaining that a Package C is three cubes high and has four cubes on the bottom (lines 19-21). For his prediction strategy for Package C, J implicitly uses the spatial relationship between the two packages to realize that he can use the same positioning process he used for Package A with the additional structuring that the top layer of cubes in Box 1 will be filled (lines 19-21).

Overall, the important thing is that both M and J were able to recognize the spatial relationship between Package A and C and were able to use that relationship to do the proper spatial structuring of Package C’s that fit into Box 1. The only difference between the two boys’ strategies was J attended specifically to the height of Package C, while M did not. However, since both boys had the proper spatial structuring of the packages they were able to correctly enumerate the number of Package C’s that fit into Box 1.
J and M, Package D, Box 1, Day 1.

Figure 3.22. How many Package D’s will fit into Box 1?

J: Okay, Box 1, D.
M: Let’s just split apart A.
J: No you would die without your A. [So, instead J takes apart Package C to create Package D].
MB: So have you made a prediction for Package D?
J: I have.
MB: So predictions for Package D?
M: I would say this… how many is this [holds up Package C with its given orientation]? Six. I would say 18 of these [Package D’s] because it is the same as this [Package C] except three.
J: Yeah cause it is the same like this [points to Package C] cause it would have three in each row on the bottom.
MB: J, you said six on the bottom?

J: Six on the bottom, six on the middle, and six on the top so that would be 18. (See Figure 3.23 below).

Figure 3.23. J’s prediction strategy for Package D’s in Box 1.

MB: And what did you say M?

M: I said 18 because it was exactly Package C, it is divided by … this [hold up Package C] is three times larger.

MB: It is like C, but C is three times larger.

M: So it would be 3 times 6, which is 18. So my prediction is 18.

M: [In the process of checking their predictions for Package D] So D that would be 1, 2, 3, 4, 5, [J grabs M’s hand and helps him move one Package D with its given orientation in all the positions it fits in Box 1] 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18. Okay, we were right.

M reasons about Package D’s in terms of a spatial and numeric relationship that occurs between Package D and Package C. He conceptualizes the spatial relationship that Package C can be viewed as a composite unit of three Package D’s (lines 14-16, 35-36, see Figure 3.24 below). This allows M to use the same spatial structuring he used in
the previous episode where six Package C’s fit into Box 1. M then realizes that since six Package C’s completely fills Box 1 and each Package C contains three Package D’s, it must take three times as many Package D’s as Package C’s to fill Box 1 (i.e. the numeric relationship between Package D and Package C) (lines 35-36). M decides to multiply six, the number of Package C’s, by three, the number of Package D’s in a Package C, to get 18 Package D’s fitting into Box 1 (line 40). Thus, M is conceptualizing his prediction strategy in terms of Package D’s spatial and numeric relationships to Package C, a package whose enumeration for Box 1 has already been discovered.

Figure 3.24. The spatial relationship that three Package D’s makes one Package C.

J’s conceptualization uses a layer structuring technique in which he visualizes six Packages D’s with the given orientation in each of horizontal layers in Box 1 (line 23). Unlike for Package C, where J conceptualized Package C in terms of its spatial relationship to Package A, J does not seem to conceptualize Package D in terms of one of
the prior packages (as you can see he could have conceptualized it in terms of Packages A or C). Even though J does not use Package C as a composite unit of Package D’s for his prediction strategy, he does recognize that Package C’s and Package D’s have the same locating processes on the bottom of Box 1 and that their first layers of packages will both be six (lines 18-19).

J and M are both using strategies that use the idea of composite units of Package D’s. J is using the composite unit of a layer of six Package D’s that are positioned in the bottom of the box, and there are three layers that fit into the box. On the other hand, M is visualizing the space in the box being divided up into six Package C’s and each of those C’s is three Package D’s. So J seems to be constructing the instructional target reasoning and spatial structuring—to find the number in one maximal layer (matches a box side), multiply by the number of layers, whereas M is not focusing on this target reasoning.

**J and M, Package E, Box 1, Day 1.**

*Figure 3.25. How many Package E’s will fit into Box 1?*
M: Now Package E, now this is the tricky one.
J: Wait, we have to make a prediction.
M: I don’t know because look at it, this is five and then you can’t show…
J: Well you still have to make a prediction.
M: You might need two boxes [Package E’s] for this, which we have got plenty of.

[While looking at the box picture, J grabs a Package E and reorients it with a 90° rotation about the z-axis and does 12 iteration motions with the package as shown in Figure 3.26.]

Figure 3.26. J’s iteration motions with Package E.

J: I have a prediction, but I am going to check it over because there may be another way to do it.
[M is manipulating the blocks not paying attention to what J is saying to him].
MB: So what is your prediction J?
J: 12
M: I don’t think we can do a prediction, well, without putting these [packages] in. [M is filling the bottom of Box 1 with four Package E’s reoriented by a 90° rotation about the z-axis].

J: But now we will know.

M: But I can’t [pauses]; but we can’t fill this in. [Points to the empty row of four squares left after he had put in reoriented four Package E’s in Box 1].

J: He [MB] said that there could be empty space. We have got to make a prediction.

M: So we just kind of black out that right there [points to the four empty squares on the bottom of Box 1] and then you just go 5, 10, 15, 20 times 3 is 60.

Figure 3.27. How M placed Package E’s on the bottom of Box 1.

J: How did you get five here?

M: 5, 10… [Points to the Package E’s].

J: But we aren’t counting how many blocks there are, we are counting how many Package E’s fit in it. So we have to count these [holds up a Package E] as one not five.

M: So 4 times 3 is 12. Okay 12.
J leans over to help M with checking their prediction by placing Package E’s in Box 1.

M: So it is...

J: 12.

M: 12.

MB: Now how did you make your prediction M, I mean J?

J: Well there is six [squares] going up and down that way [going front to back according to the columns of squares on the bottom of the picture of Box 1] that [points to a Package E] is five which is five [cubes] going up and down. So you could fit four from across but since [Package E] is five, you couldn’t fit one going that way [makes a motion from left to right on the picture of the box]. And you couldn’t go up there [sets the Package E in a vertical position] because the box is only three high. So you can fit four [Package E’s] on the bottom and four on each layer and that would be 12. There would still be that one row that you couldn’t fill unless you broke it up.

**Figure 3.28.** J’s conceptualization for Package E’s in Box 1.

MB: Now I sort of saw you motioning like this [holds up the package and moves left to right], were you counting?

J: Yeah I was in the box.
At first, M does not think that they can make a prediction without filling in the box with Package E’s (lines 30-32). J quickly tells M that they must make a prediction first before filling in the box with packages (lines 7, 34). Ignoring J’s comments about making a prediction first, M places four Package E’s reoriented with 90° rotation about the z-axis on the bottom of Box 1 (lines 30-32). This suggests that M is struggling to visualize how the Package E’s will fit into Box 1 without physically placing the packages in the box (lines 30-32). One way visualization of an object occurs, such as structuring packages that fit into a box, is from abstracting the action and perception of physically placing packages in the box. When the student requires this concurrent physical presence of perceptual material (i.e. physical cube-packages and boxes) to visualize an object or objects, the student is at the most basic or perceptual level of abstraction (Battista, 1999). The student then reaches the internalized level of abstraction when they can create a mental image of the object without having to see or perform actions on the physical object. When a student is able to take that mental image of the object and mentally manipulate and operate on it in their imagination, then they have reached the interiorized level of abstraction (Battista, 1999). For cube-package problems, the interiorized level of abstraction produces a spatial structuring that enables one to properly utilize the locating and positioning processes for the packages inside the box.
M’s struggle to visualize the positions of Package E’s suggests that M has not sufficiently abstracted the locating and positioning of Package E to be able to visualize the proper structuring of Package E’s in the box. Therefore, M is not able to reach the interiorized level of abstraction and has to fall back on some perceptual abstraction by physically placing four reoriented Package E’s into Box 1 in order to help him see the positions of Package E’s in Box 1 (lines 30-32). This is only the fifth time that M and J have done a cube-package problem. Therefore, M might not have enough experience with the actions of placing packages in the box to be able to visualize how many Package E’s will completely fit in the box without physically doing so first.

After placing four reoriented packages on the bottom of the box, M tells J that that they will not be able to completely fill the bottom of Box 1 with Package E’s (lines 36-37). J explains to M that there can be empty space left over in the structuring of packages in Box 1 (line 39). However, from the communication with J, M re-conceptualizes his mental model. But instead of re-conceptualizing it as a the number of Package E’s in Box 1, M re-conceptualizes it in terms number of cubes contained in Package E’s that will fit in the box (lines 41-42). J immediately recognizes M’s misconception and makes M aware of it by explaining they are not counting cubes, but rather Package E’s that fit into the box (lines 55-56). J adds a visual aid to his explanation about M’s misconception by holding a Package E in his hand stating that each Package E is one unit (line 56). From J’s explanation, M is able to re-conceptualize his mental model to a layering structuring technique in which he sees three layers of four Package E’s. He then performs the numeric expression of 4 times 3 to give him a total of 12 Package E’s (line 58).
This error might have occurred because M’s re-conceptualization of his strategy or scheme might have placed a high cognitive demand on his working memory (i.e. reaching his cognitive load for his working memory), which caused him to lose track of the units he was supposed to be enumerating in Box 1. This shows that even students who seem to have a good understanding of what is going on in a package problem can lose track of the units they are supposed to be enumerating with. It is also important to point out that the students had previously done cube enumerations of cube buildings in Investigation 1. Therefore, it might be easy for M to revert back to using cubes as the units because he just used them in the previous investigation.

J explains some positioning processes that show the different ways that he cannot position Package E’s in the box (lines 70-77). First, J is able to locate six rows of four squares on the bottom of Box 1 (lines 70-71). While J does a positioning process with the given orientation of package from the picture on the student sheet, he notices that he would have one remaining cube from Package E that will not fit into the box (lines 72-73, see Figure 3.29 below). As a result, J discards the given orientation and decides that he will have to reorient Package E to get it to fit in Box 1.
J then reorients Package E by rotating it 90° about the x-axis so only one cube sits on the bottom of Box 1 (line 74). He states that this orientation of Package E would not work because the box has a height of three and Package E would have two cubes outside the top of Box 1 (lines 74-75, see Figure 3.30 below). By systematically ruling out all other orientations of Package E first, J realizes that there is only one way that Package E can be oriented in order to position any packages inside the box. Thus the only way that Package E can be oriented so that J can position Package E’s in Box 1 is by rotating the given orientation 90° about the z-axis. This is example of a student explicitly reorienting a package in order to find a proper positioning process.
J is able to do positioning processes that allow him to visualize that he can position four reoriented Package E’s on the bottom of the box with an empty row of four squares where he cannot fit any Package E’s. As a result, J is able to conceptualize the four reoriented Package E’s on the bottom of the box as a layer composite. J then recognizes that there are three horizontal layer composites of four Package E’s that can fit into Box 1, which gives him a total of 12 Package E’s that fit into the box (lines 75-76). Because of J’s previous use of layering, there is thus some evidence that he is beginning to interiorize a layering scheme for packages.

MB notices that J did an iteration motion while holding a Package E in his hand during his initial prediction and asks J if he was counting (lines 86-87). J states that he did in fact count by ones (lines 89, 99). So did J just iterate the first layer by ones and then multiply by three because he knew that there were three layers or did he iterate the
entire box by ones to get 12 packages? Looking further back in the interview, notice that J does a motion of 12 iterations while he is coming up with his prediction (lines 11-12). This evidence suggests that he initially spatially iterated individual packages by ones rather than conceptualized a layer of packages. In addition, after J spatially iterates 12 packages he states that he wants to check his prediction to see if there is another way of doing it (lines 21-22). I hypothesize that this is when J re-conceptualizes his mental model to a layer structuring and recognizes that he had three layers of four Package E’s in Box 1.

**J and M, Package A, Box 2, Day 1.**

*Figure 3.31. How many Package A’s will fit into Box 2?*

1. M: Okay, Package A.
2.
J: [M begins to place Package A in Box 2 when J stops him] We need to make a prediction.

M: I would say 12. This is [Package A] kind of double the other one [from Box 1].

J: I would say, wait a second. [He takes his fingers and moving them around the picture of Box 2. He moves them six times as if he was visualizing placing Package A in Box 2]. Yeah I would say 12.

[The boys check their predictions by moving Package A around Box 2 and finds the total is 12.]

J: You can’t fit exactly one [row of cubes].

M appears to be conceptualizing how to enumerate the number of Package A’s for Box 2 in terms of how he enumerated Package A’s in Box 1. In making Box 2, M makes the following statement early on, “Okay, they [Boxes 1 and 2] are the same width... you just have to add two layers I think. You just add two more layers and you have five.” M recognizes that by adding two more layers of cubes to Box 1, he can stack one more layer of Packages A’s in Box 2 than he can in Box 1. M recalls that six Package A’s fit into Box 1, so he doubles them to get 12 Package A’s to fit into Box 2 (line 6). M is adapting his previously used mental model of Package A’s filling Box 1 to create a new mental model for Package A’s filling Box 2.

J moves his finger six times as if it iterating six Package A’s on the bottom of the picture of Box 2 (lines 9-10). He then predicts that 12 Package A’s will fit into Box 2, where it is likely he recognized that a second layer of six Package A’s could be placed in Box 2 (line 10). J also explicitly recognizes that there will be one layer of cubes left over at the top of Box 2 (line 15). This suggests that J was able to do a proper spatial structuring process that allowed him to visualize all the positions of the Package A’s in one coherent mental model. The boys validate their predictions by placing a single
Package A in Box 2 while iterating as they move it around to the different positions it will fit into the box (line 15).

J and M, Package C, Box 2, Day 1.

Figure 3.32. How many Package C’s will fit into Box 2?

M: Okay C, prediction [pauses] six. Wait, my prediction is six.

[J takes his fingers and moves them six times around the bottom of the picture of Box 2 as if he was visualizing placing Package C’s in Box 2.]

MB: We are on Package C. What is your prediction M?

M: I said for Package C, I would say six.

J: I got 12. [He takes a Package C with its original orientation and places it into one of the box’s corners]. One. Wait, let’s make a lot of them [Package C’s] to show my way.

[M takes the Package C that is placed in the box and starts to move it to the other positions it will fit on the bottom of the box].

J: No M, I need a lot more of them [Package C’s] to show my way.
M: Can I just show my way while we are at it. 1,…,6 [moving the Package C as it is oriented on the student sheet around the bottom of Box 2] and it can’t go any higher [holds a Package C with the given orientation in a position that would be on top of the bottom layer of six packages to show J that one layer of cubes of Package C will stick out of the box.] (See Figure 3.33 below).

Figure 3.33. M’s reasoning for his prediction of six Package C’s fitting into Box 2.

J: Yeah, but make more of these [Package C’s].

M: Oh you can set… [rotates the Package C in his hand 90˚ about the y-axis]. But wait, six plus, but then only four can fit because this [Package C] is only three long, see [he places a Package C rotated 90˚ about the y-axis, so that it is three units long in Box 2]. (See Figure 3.33 below).
Figure 3.34. Package C rotated 90° about the y-axis from its given orientation in Box 2.

J: I will show you.

M: There is 1, 2, 3, 4 [as he is moves a reoriented Package C in the different positions it fits on the bottom of Box 2].

J: J takes the Package C out of M hands and reorients it back into its given orientation. He then begins to move it around the bottom of Box 2.] 1, 2, well we know that there are six this way on the bottom (see Figure 3.35 below). [He then grabs another Package C and rotates both packages back 90° about the y-axis]. Then 7, 8, [holds the two Package C’s in the front and back positions in the top layer on left side of Box 2 (see Figure 3.36 below)] 9, 10 [moves the two Package C’s to the front and back positions in the top layer on the right side of Box 2 (see Figure 3.37 below)]. [Then he counts another two Packages C’s in middle of the top layer of Box 2] 11, 12 (see Figure 3.38 below).
Figure 3.35. J positions six Package C’s on the bottom of Box 2.

Figure 3.36. J positions two reoriented Package C’s on top left side of Box 2.

Figure 3.37. J positions two reoriented Package C’s on the top right side of Box 2.
Figure 3.38. J positions two reoriented Package C’s in the top middle of Box 2.

M: Where are you getting these middle ones [points to the two packages labeled 11, 12 shown in Figure 3.38 that J is holding]? But these are two things. Do you see what I mean?

J: Oh yeah.

M: So it’s only ten.

MB: Wait tell me how you… I got lost.

J: [J moves the two Package C’s in the box with the given orientation to the different positions that Package C can occur on the bottom of the box and iterates six Package C’s.]

M: Then you just rotate it.

J: [J then rotates the two Package C’s 90˚ about the y-axis] 7, 8, [hold the two packages as shown in Figure 3.36 above] 9, 10 [moves the two packages to the positions shown in Figure 3.37 above] and that takes up four originally on the bottom.

MB: Oh so you said there are six like this [places a Package C with the given orientation on the bottom of Box 2]. Right. 1, 2, 3, 4, 5, 6 [points to the six locations that Package C’s with the given orientation will fit on the bottom of Box 2]. So what was your answer again?

J: 10.
M is initially positioning Package C’s only in terms of the given orientation of Package C as it appears on the student sheet. At this point, his mental model does not include rotating Package C 90° about the y-axis and placing four more packages on top of his first layer of packages (lines 18-22). His orientation process falls into the case of orienting the package by the way it is presented in the picture on the student sheet.

For this particular task there is added complexity for the students. In order to find the correct number of Package C’s that fit into Box 2, they have to position the packages using mixed orientations. One possible explanation for M’s initial inability to recognize this added complexity might be that for all the previous tasks in Investigation 2, M has not ever explicitly considered nor has done positioning processes for packages using mixed orientations. In addition, until this task all the other enumerations of packages that fit into a specific box could be solved correctly by having a uniform orientation for all the packages during the positioning process. Therefore it makes sense that M might not have considered using mixed orientations of the package because it was not required in the previous tasks to get a correct enumeration of packages.

After M explains his reasoning, J agrees with how M positioned the six packages on the bottom of Box 2, but tells M to build more Package C’s so he can explain his prediction (line 31). It is at this point that M re-conceptualizes his strategy and realizes that he can reorient the package by rotating it 90° about the y-axis in order to be able to stack more packages on top of his bottom layer of Package C’s (lines 33-36). It should be mentioned that in a previous episode, J explicitly discussed reorienting Package E’s to different orientations with MB, a discussion that M was present for and likely heard (J and M, Package E, Box 1, Day 1, lines 70-77). Thus it is possible that M remembered J
reorienting in that earlier episode and used it to re-conceptualize his mental model. M is able to keep his initial spatial structuring the same for the packages on the bottom of Box 2, but changes his mental model to include the four reoriented Package C’s that can be stacked on top of the bottom layer (lines 34-35). He explains to J that he thinks that maybe only four packages fit in the second layer and that will give them a total of 10 Package C’s, instead of 12 (lines 47-48).

J’s conceptualization is an interesting one because he is on the verge of having valid reasoning for his prediction, but makes an error in his spatial structuring that leads to an incorrect prediction. J is able to conceptualize his prediction strategy in terms of two orientations of Package C, an original orientation and horizontal orientation that runs along the length of Box 2 (see Figure 3.39 below). J states that he can position six Package C’s on the bottom of Box 2 using the original orientation and position another set of six Package C’s using a horizontal orientation stacked on top of the bottom layer (lines 50-57). For the second layer of Package C’s, J positions a package in each of the four corners of the box, giving him a subtotal of ten packages in Box 2 (lines 52-56, see Figures 3.36 and 3.37 on page 62). J then makes an error in his spatial structuring when he says there are two more Package C’s positioned in the middle of second layer of packages, which brings his total to 12 (lines 56-57). J shows M the two middle packages’ positions by placing a pair Package C’s in the inner two positions of the top layer (see Figure 3.38 on page 62)...
Figure 3.39. The two orientations used in J’s prediction strategy.

After J’s explanation, M is quick to point out that J has positioned two extra Package C’s, which overlap with the positions of the previously counted four Package C’s in the top layer of packages in Box 2 (lines 81-82, see Figure 3.40). We do not know if J was actually conceptualizing the positioning of the middle packages like the way M explains it, but we do know that J was able to recognize the mistake and agrees with M that only ten Package C’s will fit in Box 2 (line 85).
Figure 3.40. J’s error in his spatial structuring of Package C’s in Box 2.

J and M, Package E, Box 2, Day 1.

Figure 3.41. How many Package E’s will fit into Box 2?
M: Okay, now Package E is easy.

J: Maybe, I want to do a prediction first.

MB: Okay, why don’t you both write a prediction.

J: I got mine.

M: Okay, let’s see, I would say, [looking at the picture of Box 2] that is 5, 6, 12, 18, 24 and then each of these fit there [points to the columns of six squares on the bottom of the picture Box 2]. So I would say 24.

J: That is what I got.

M: So now let’s find the actual.

MB: Now before you do that, can you tell me how you predicted?

M: Well see there is … on the bottom there are 24 squares … [M grabs a Package E as he is talking and rotates it 90° about the x-axis so it stands vertically on one cube].

J: And one of them [Package E] fits all the way up [in Box 2]. [Grabs the vertical Package E out of M’s hands and places it into the box]. (See Figure 3.42 below).

Figure 3.42. The boys’ explanation of their prediction strategy for Package E’s in Box 2.
J: [J places four vertical Package E’s in the back row of the box] Four here, then 8, 12, 16, 20, 24 (see Figure 3.43 below).

M: That makes perfect sense.

MB: You didn’t completely fill it. How did you…

[J points the four vertical Packages E’s that are in the back row of the bottom of Box 2].

M: Because it just goes along the length.

J: There are four on this [points to the four Package E’s on the back side of Box 2] and it just goes along there is 4, 8, 12, 16, 20, 24.

M: You could just imagine that this flying along here [moves one vertical Package E from the back side along the length to the front side of Box 2]. (See Figure 3.44).
Figure 3.44. M’s visualization for checking the prediction of 24 Package E’s in Box 2.

MB: Okay, J you were saying 4, 8, 12, 16, but M did you do the same thing?
M: Yeah.
J: Well, there is four going this way [in terms of the columns] and six going that way [in terms of rows] and 6 times 4 is 24.

In this episode, both boys immediately got a prediction that 24 Package E’s would fit into Box 2 (lines 11-13). The boys first reoriented Package E by rotating it 90° about the x-axis so it would stand vertically on one cube as shown in Figure 3.42. Then the boys did locating processes where they locate the single square that appears on the bottom orthogonal view of Package E on the array of squares that appears on the bottom of Box 2 and found that 24 Package E’s were located on the bottom of the box (lines 19-20). However, the boys enumerated the locations of 24 Package E’s on the bottom of the box using slightly different composite units. M counts the first column of six squares on the bottom of the box and then skip counts by sixes for the three remaining columns of
six squares to get a total of 24 squares on the bottom of the box (lines 9-11). Later, when J is checking their prediction of 24 Package E’s, he skip counts by the six rows of four squares appearing on the bottom of Box 2 to enumerate 24 Package E’s fit into Box 2 (lines 32-33, 50-51). Therefore the difference between the two boys’ strategies is that M used column composites of squares, whereas J used row composites of squares to enumerate the number of locations of Package E’s on the bottom of Box 2.

As a result, J and M agree that their prediction of 24 Package E’s is correct (lines 32-33, 42). MB quickly asks how they found the actual number of Package E’s when they did not completely fill the box with packages (lines 44). J points to the row of four Packages E’s that he had placed in the box and explains that the remaining five rows of four squares can each be filled with four Package E’s (line 50-51). Since J places only the first row of four packages in the box, this suggests that he visualizes that four Package E’s will fit in the each of the five remaining rows of four squares that appear on the bottom of Box 2. J is then able to generalize his reasoning into a more numeric expression of 6 times 4, which give a total 24 Package E’s in Box 2 (lines 67-68).

M states, “You could just imagine that this flying along here [moves one vertical Package E from the back side along the length to the front side of Box 2]” (lines 53-54). This statement can be interpreted two different ways depending on what M was referring to when he says “this.” One possibility is that M is referring to the individual Package E that he is holding in his hand, which seems to suggest that M was visualizing a column composite of six Package E’s. This explanation is supported by M’s previous statement that he located four column composites of six locations of Package E’s in the array of squares appearing on the bottom of Box 2 (lines 9-11).
The second possibility is that the “this” that M is referring to is the row of four Package E’s that are placed in Box 2. When he grabs one Package E from the row of four packages in Box 2 and moves it along the length of Box 2, he could be using the individual Package E to represent the row of four Package E’s. By moving the one package from the back to the front side of Box 2, he is representing the positions that the row of four Package E’s will fit into Box 2. This possibility is supported by the fact that M’s statement is in response to MB’s question about how J checked the actual number of Package E’s without having to completely fill the box with packages (line 44).

Case Study 2: N and P, Class 2.

N and P, Package A, Box 1, Day 1.

Figure 3.44. How many Package A’s will fit into Box 1?

1 P: Oh, she [the teacher] is just wondering how many will fit not how many does it take to fill it all up.
N: So six.

[Both boys proceed to write six on their papers].

MB: So that is your prediction?

N: Yeah.

P: Do we have to build it or something?

MB: Yeah, you have to build the box.

P: Do we just have to build it once and... yeah we just have to build it once.

MB: There is only one box to build but there are several packages that you will have to build with cubes, then you can check.

P: Oh yeah.

MB: So you predicted six. How did you make the prediction, N?

N: You look at this [points to the picture of Package A] there is four on the back [of Package A], four here [points to a 2x2 array of squares found in the back left corner of the back side of box picture], four on the bottom, there is one [points to the 2x2 array of squares on the back left on the bottom of the picture of the Box 1. See Figure 3.45 below].

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*Figure 3.45.* N’s locating processes for the Package A in the back left corner of Box 1.
N: Then the one here [points to the 2x2 array of squares on the back right corner of the bottom of the picture of Box 1]. So then there are two here in this part and three in this part (see Figure 3.46). Wait…

“Then the one here.“

“There is one.“

“And three in this part.“

Figure 3.46. N’s locating processes for Package A on the bottom of Box 1.

[N has drawn six circles around the 2x2 arrays of squares that he found on the bottom of the picture of Box 1]. (See Figure 3.47 below).
Figure 3.47. N’s demarcation of the locations of Package A’s on the bottom of Box 1.

MB: Oh, I see you drew in where the packages go.

N: This is four [points to a 2x2 array of squares on the bottom of the picture of the box] so that is 1, 2, 3, 4, 5, 6 [points to the six 2x2 arrays of squares that he earlier circled on the bottom of the picture of Box 1]. So we can’t get the top row because you can’t break it [Package A] up to fill the top row [points to the top row of squares on the back and left sides of the picture of Box 1]. That is why we have an empty top row. (See Figure 3.48).

Figure 3.48. N’s conceptualization of Package A’s in Box 1.

[P draws the box pattern on grid paper].

MB: Maybe N, you should check to see if P’s pattern is right.

[P gives N his pattern and N looks it over].

N: Yeah it is right, get the tape.

MB: So P, you are going make Package A.

P: [Concentrating deeply while making Package A]. Two in the front and two in the back is four. Do we have to make a whole bunch of them or just make one?
MB: Well I think you just need to make as many as you need to check your answer. If you can do it with one, that is fine.
P: The problem is it is one higher.

[N tapes the sides of the box, except for one. Then P and N continue to make Package A’s to fill the box with.]

[The boys’ place six Package A’s in Box 1].
P: Six. Yeah does it have to fill everything or how many it fits?

MB: It asks how many packages fit.
P: So it doesn’t matter if there is stuff like this [points to the empty layer of cubes at the top of Box 1].

MB: No, cause you can’t fit it.

Even though we do not hear P explicitly state his prediction, we do see him write a prediction of six Package A’s on his paper after N explicitly states his prediction of six (line 6). This seems to suggest that P at least thinks that the prediction of six Packages A’s is correct. Since P was never asked what his reasoning was for his prediction of six, we do not know whether P came to the prediction by himself or if he was just agreeing with N’s prediction. On the other hand, we do get some insight into N’s reasoning for his prediction of six. First, N locates the 2x2 array of squares appearing on the back orthogonal view of Package A on a 2x2 array of squares appearing on the bottom left of the back side of Box 1 (lines 25-27). He then locates the 2x2 array of squares appearing on the bottom orthogonal view of Package A on a 2x2 array of squares appearing on the back left of the bottom of Box 1 (line 28). Next, he does a positioning process where he is able to integrate the two locating processes into one coherent mental model of Package A positioned in the back left corner on the bottom of Box 1 (lines 27-29). After N
positions this Package A, he does not explicitly mention doing similar positioning processes for the remaining positions of Package A in Box 1. Instead, he does a set of locating processes for the remaining 2x2 arrays of squares that appear on the bottom of Box 1 (lines 38-40).

Although N does not explicitly mention positioning processes for the rest of the packages after positioning the first one, there is evidence that suggests that he was able to recognize where the remaining packages were positioned and was able do a proper spatial structuring process to see how the set of Package A’s fit into Box 1. The evidence is that N correctly determines that six Package A’s fit into Box 1 as well as he is able to infer that the top layer of cubes will be empty (lines 63-65).

**N and P, Package B, Box 1, Day 1.**

![Figure 3.49. How many Package B’s will fit into Box 1?](image)

P: The next one would be easier to read because it is three [cubes]. There would be 1, 2, 3, 4, 5, 6, 7, 8 [P motions with his finger to eight 3x1 arrays of squares on the bottom of the picture of Box 1]. (See Figure 3.50 below).
Figure 3.50. P’s prediction strategy for Package B’s in Box 1.

N: [N iterates individual squares on the bottom of the picture of Box 1] 1, 2, 3, 4, 5, …, 24. There are 24.

P: No, there are eight. No you don’t count every single [square].

N: No, look there is 1, 2, 3 boxes [i.e. cubes], so there is three going up [points to the height of Box 1 on the back side of the box picture]. So there is 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, [counting the individual squares on the bottom side of the picture of Box 1] cause it equals the same thing.

P: I got eight down here [points to the bottom of the box picture] and oh yeah 8 times 3 is 24 because there are three up.
Figure 3.51. P recognizes that he is not considering the height of the box in his structuring of Package B’s in Box 1.

[N nods in agreement]

MB: Now, N I saw you counting the individual ones.

N: Yeah.

MB: Then what did you do? Did you count 1, 2, …20 [points to the individual squares on the bottom of the picture of Box 1]? How were you counting?

N: Like one of these [Package B] is one row up, so if you kind of use every square down at the bottom. It will be like standing up like this [holds Package B so it is standing vertically on one cube]. It will be standing like that [places the vertical Package B in the box].

P: The way I did was kind of the same, but what I did was count how many would fit here [placing Package B with its original orientation in the front right corner of the bottom of Box 1]. And eight fits here [in the first layer] and there are three layers going up so there are 24.

MB: Oh, so there are eight that fit lying down and there are three layers.

[P and N are making Package B’s with cubes and placing them in Box 1 to check their predictions using P’s structuring.]

P: That is eight, so we have five more.

MB: So what did you say P?
P: I thought that was it, but we still need to do two more rows [i.e. horizontal layers].

P: All you have to do really is get one [layer].

MB: What P?

P: Just get one [layer], and then get three of them. But I guess you have to do all of it.

[P connects the bottom layer of cubes to get one connected layer.]

[The boys completely fill the box with Package B’s according to P’s structuring of three layers of eight Package B’s.]

P: Okay we are done. Okay there is 1, 2, …, 24 [counting the packages as he is removing them from the box]. [With the last layer P points the Package B’s he has connected and gets a final count of 24.]

MB: So was the prediction right?

P: Yeah.

N states a prediction that 24 Package B’s will fit in Box 1, whereas P at first believes that they can fit only eight Package B’s in Box 1 (lines 1-3, 12). P uses the given orientation of Package B’s to locate and enumerate 3x1 arrays of squares that appear on the bottom orthogonal view of Package B on the array of squares appearing on the bottom of Box 1 (lines 1-3, see Figure 3.50 on page 78). P concludes that eight Package B’s will fit into Box 1 (lines 2-3). However, P makes no explicit mention to the height of Package B or the height of Box 1. This suggests that P does not explicitly consider how Package B’s with the given orientation are located on the height of Box 1. As a result, P errors in not recognizing that he can locate two additional layers of eight Package B’s on top of the bottom layer of packages in Box 1.

Unlike P, N conceptualizes the orientation of Package B in terms of it standing vertically on one cube (lines 17-18, 42). N locates the 3x1 array of squares on the back
orthogonal view of Package B to a 3x1 array of squares on the height of the box appearing on the back side of Box 1 (lines 17-18). N implicitly locates the single square appearing on the bottom orthogonal view of the Package B on each square appearing on the bottom of Box 1. N does a positioning process that integrates the two locating processes into one single three-dimensional visualization of a position of Package B in Box 1. He then recognizes that for each individual square on the bottom of the box he can position one Package B in Box 1 (lines 17-19, 42-44). N then concludes that there are 24 Package B’s that fit into Box 1.

Since N explicitly reorients Package B, we can see that N’s conceptualization represents the second case of the mental process of orientation, in which he explicitly considered orientations of Package B different from the given orientation. On the other hand, P’s conceptualization is an example of the first case of orientation because he does not change the given orientation of package on the student sheet.

At first, P disagrees with N’s prediction of 24 Package B’s, arguing that N should not be counting every individual square on the bottom of Box 1 (line 15). N points out that there are three cubes in Package B and if they reorient the package to stand vertically on one cube, then they can position one package for each square on the bottom of Box 1 (lines 17-18). N’s explicit statement that he sees Package B as three cubes going up a lateral side of Box 1 causes P to adapt his mental model to include doing locating processes of Package B’s with the given orientation on the height of Box 1. This idea taken with N’s prediction of 24 caused P to extend his conceptualization to a layering strategy that allows him to see that he can fit three layers of eight Package B’s into Box 1 (lines 22-23). P then connects this spatial reasoning to numeric reasoning, when he states
that 8 times 3 is 24, the same answer that N predicted. It is interesting that, rather than simply adopting N’s strategy, P uses the idea of considering the height of Box 1 to re-conceptualizes his structuring from a single layer to a three layers.

It is also interesting that even though both boys eventually got the same prediction of 24 Package B’s, they do not seem disturbed that their prediction strategies were vastly different. Additionally, N does not seem disturbed by the fact that when the boys check their prediction with concrete materials, they check only P’s structuring of Package B’s in Box 1 (lines 54-55). This might be because they are accustomed to an inquiry-approach classroom where it is acceptable to have different prediction strategies that give the same prediction. It is also possible that the boys’ understood each other’s prediction strategies after they explain their reasoning to each other. After P extends his initial conceptualization to enumerating three layers of Package B’s, we see that N nods in agreement (line 33). This nodding seems to suggest that N understands and agrees with P’s prediction strategy. However, there is no explicit evidence that suggests P understood N’s prediction strategy.

The interviewer then asks each one of the boys separately to explain the reasoning behind his prediction (line 39). It is interesting that P states his strategy is “kind of the same” as N’s strategy, when in reality they are vastly different (line 47). I think that there are two possibilities as to why P might think that his and N’s strategies are similar. One possibility is that P thinks that the strategies are similar because both strategies produce a prediction of 24 Package B’s in Box 1. The second possibility is that P thinks that since both boys’ do locating processes for Package B’s on the bottom of the box as well as locating Package B on the height of Box 1, that their strategies are similar.
However, they differ in that N structures only one layer of Package B’s with a vertically orientation, whereas P does a layering strategy for three horizontal layers of eight Package B’s with the given orientation. Even though some general steps in the strategies are similar, the different orientations and number of layers makes the strategies very different.

**N and P, Package C, Box 1, Day 1.**

![Diagram showing Package C and Box 1]

*Figure 3.52. How many Package C’s will fit into Box 1?*

1. N: Six. [Draws six circles around 2x2 arrays of squares on the bottom of the picture of Box 1 and counts the six circles.]
2. P: Six, yeah six.
3. MB: P, how did you get yours?
4. P: What I did was there is two right there, so that is one [points to one 2x2 array of squares at the front bottom of the box picture]. There are two going up so it is 2x2, this is a 2x2, that’s a 2x2, that is a 2x2, that is a 2x2, and that is a 2x2. [P circles the six 2x2 arrays of squares in the bottom of the picture of Box 1]. (See Figure 3.53 below).
Figure 3.53. P’s locating processes of Package C’s on the bottom of Box 1.

MB: And you did the same thing N?

P: And there are three going up, so it would be just one up.

N: Yeah, I did the same thing.

MB: [Talking to P]. There are three going up in Package C [points to the height of the picture of Box 1], so it would be just one row.

[P nods his head in agreement].

[The teacher interrupts the students working to clarify some directions that students are misinterpreting none of which apply to this pair’s work. After teacher is done clarifying directions, P and N fill Box 1 with Package C’s using the given orientation].

MB: So was your prediction correct?

P: Yes, six.

Both P and N predict that six Package C’s fit into Box 1 (lines 1, 4). P conceptualizes how Package C’s fit into Box 1 using the same strategy he used to get the correct enumeration of Package A’s in Box 1. The only slight difference between this
episode and the Package A episode is P explicitly locates the height of Package C on the height of Box 1 (line 22), whereas for Package A’s, P makes no explicit mention of this locating process for the height of Package A.

Like P, N also uses the same prediction strategy that he has used for the previous two packages to predict the number of Package C’s in Box 1. However, the major difference between this episode and the previous two episodes’ strategies is N does not explicitly mention anything about the height of Box 1, the height of Package C, or any other explicit evidence that he positioned Package C’s in Box 1. On the other hand, N did explicitly position a Package A and Package B in Box 1 in the previous two episodes. Thus there is explicit evidence that N is able to properly position a package in a box. I hypothesize that combination of using the same locating processes for the bottom of Box 1 for both Packages A and C and the fact that N was able to correctly position Packages A and B suggests that N was probably able to properly position Package C’s using the same strategy he used to structure Package A’s and B’s in Box 1.

N and P, Package D, Box 1, Day 1.
MB: Okay, how about Package D? How are you going to predict Package D?

P: You have to predict first [directed at N].

N: Oh yeah.

[N does counting motions where he repeatedly points to the six 2x2 arrays of squares with his pencil on the bottom of the picture of Box 1. He does his counting motions in cycles of six, meaning that every six 2x2 arrays of squares he counts he is back at his original starting 2x2 array of squares. He does this counting cycle two and half times for the 2x2 arrays of squares appearing on the bottom of the picture of Box 1. On Figure 3.55 shown below, the numbers 1, 7, and 13 are all representative of the same 2x2 array of squares on left corner of the bottom of the box picture. He counts under his breath by ones and gets a total of 15. So N missed the last three locations of Package D in Box 1.]
N counts again the six 2x2 arrays of squares on the bottom of the picture of Box 1. He repeatedly counts the same six 2x2 arrays of squares two more times. Each time he counts six 2x2 arrays of squares on the bottom of the picture of Box 1, he makes a tally mark on his student sheet. When he is done counting groups of six 2x2 arrays of squares, he notices that he has a total of three tally marks on his student sheet.

P: I think it is 18.

N: 18.

MB: Now tell me how you got your prediction?

P: What I did was I saw there are three here [points to three 2x2 arrays of squares on the left side of the bottom of the picture of Box 1] and I just figured that since that is the other half, then there are three right here [points to other three 2x2 arrays of squares on the right side of the bottom of the picture of Box 1]. So that was six because 3 times 2 is 6 and there are three high. So 6 times 3 is 18. (See Figure 3.56 below).
Figure 3.56. P’s prediction strategy for Package D’s in Box 1.

MB: And N, how did you do it?

N: I pretty much did the same, I just counted the six from down here [points to the bottom of the picture of Box 1] and I multiplied it by three [points to the height of the box picture] and I got 18.

MB: So you actually took 6 times 3 or did you count it 1, 2, 3, 4, 5, 6 [and so on]?

N: I took it as 6 times 3.

[The boys’ build Package D’s out of cubes to place into Box 1 to check their predictions. Once Box 1 is full of packages, P counts the number of packages as he removes the packages one by one from the box.]

P: 18.

MB: So you were right?

P: Yeah.

For Package D’s, P uses a similar layer structuring strategy that he has used to predict the previous packages in Investigation 2. P first locates three Package D’s with the given orientation on the left half of arrays of squares appearing on the bottom of Box 1 (lines 35-36, see Figure 3.56). P states that these three locations of Package D’s take up
exactly half of the area that appears on the bottom of Box 1 and infers that three Package D’s must also be located on the other half of the bottom of Box 1 (lines 35-38). P then recognizes that he has two column composites of three locations of Package D on the bottom of Box 1 and is able connect that to the numeric expression 3 times 2 is 6 (lines 38-39). Next, P positions the layer of six Package D’s on the bottom of Box 1 and conceptualizes it as a layer composite of Package D’s. P then locates the height of bottom layer of Package D’s on the height of Box 1, where P recognizes that he can fit three layers of six Package D’s in Box 1 (line 39). P is able to connect this spatial structuring to its numeric expression of 6 times 3 in order to give him a prediction of 18 Package D’s fit into Box 1 (line 39). The only difference between P’s conceptualization for Package D’s and the ones he did for the previous packages is that he conceptualizes two column composites of three locations of Package D’s on the bottom of Box 1, whereas with previous packages he located every individual location of the packages on the bottom of Box 1. This suggests that P has progressed to a more sophisticated level of reasoning for his prediction strategy for cube-package problems.

N begins his strategy by imagining he is spatially iterating Package D’s with the given orientation by using successive locating processes of 2x2 arrays of squares as indicators of Package D’s on the bottom of the picture of Box 1 (lines 7-10). He continues counting his spatial iterations of Package D by ones an additional one and half more times on the bottom of the picture of Box 1 (lines 10-11, see Figure 3.55). N initially counts only 15 Package D’s, but does not say this number aloud (lines 13-14). However, an examination of the videotape indicates that N made exactly 15 counting motions. N seems unsatisfied with his prediction of 15 Package D’s and decides to do
the counting motions again on the bottom of the picture of Box 1. However, this time N makes a tally mark on his paper for every time he spatially iterates six Package D’s on the bottom of the picture of Box 1 (lines 23-26). He does this process three times and looks back at his paper and notices he has three tally marks (lines 26-27). He then remembers that each tally represents a group of six Package D’s, and then he implicitly multiplies 6 times 3 to get a prediction of 18 Package D’s (lines 26-27, 31).

There are two possibilities for how N conceptualized his prediction strategy. The first possibility is N tries to visualize spatially iterating a Package D in each of the positions it fits into Box 1. The second possibility is by looking at the height of Box 1, N realizes that he would have to iterate Package D six times for each of the three layers of Package D’s that fit into Box 1. It is also possible that N does both possibilities during his conceptualization of Package D’s in Box 1. When N initially spatially iterates Package D’s in Box 1 and got a initial prediction of 15, N might have been trying to visualize where each Package D was positioned in Box 1 (lines 7-14). Possibly, N reached the limit of his working memory while attempting to visualize how each Package D fits in the box and this caused him to lose track of what positions of Package D he is already enumerated in Box 1. As a result, N is able to recognize that he lost track of the positions he had already enumerated in Box 1, which causes him to double check his initial prediction of 15 using a slightly different prediction strategy (lines 23-27). N then re-conceptualizes his strategy to the second possibility, where he recognizes that for each horizontal layer of Package D’s in Box 1 he will have to spatially iterate a Package D six times (lines 23-27). In order to lighten N’s cognitive load on his working memory, N
uses tally marks to demarcate each time he has done an iteration of six Package D’s for a layer in Box 1.

Later in the episode, MB asks N to explain his prediction strategy for Package D’s in Box 1 (line 48). N states that his strategy is similar to P’s layering strategy in that he counted six Package D’s in the bottom layer of the box and multiplies 6 by 3 to get 18 Package D’s in Box 1 (lines 50-52). However, N’s explanation of his strategy to MB is not consistent with how N iterated individual Package D’s for each successive layer in Box 1 at the beginning of the episode. Therefore, this suggests that sometime after N initially predicted 18 Package D’s fit in Box 1, he re-conceptualized his strategy by visualizing three layer composites of six Package D’s fit into Box 1. I hypothesize that there are two possibilities that could have caused N to re-conceptualize his mental model.

The first possibility is that after N implicitly recognizes that each tally mark he drew on his paper represented a layer of six Package D’s, N re-conceptualizes his mental model to visualize layers as composite units. N then recognizes that he counted three layers in Box 1. N connects his new spatial structuring to a numeric expression of the multiplication problem of 6 times 3 to get 18 Package D’s in Box 1. While working out his initial strategy, it seems like N has an “ah-ha moment” in viewing layers of Package D’s as composite units, which causes him to manipulate his mental model in a new way.

The second possibility is that N might have re-conceptualized his mental model to include the idea of layers representing composite units after P explained his prediction strategy. It is possible that in listening to P’s layering strategy, N recognizes P’s use of layers as composite units of Package D’s, which causes N to re-conceptualize his mental model to accommodate this idea.
After N had finished explaining his reasoning, MB asks him if he did 6 times 3 or if he counted the individual Package D’s he thought were going to fit into Box 1 (line 54). N responds that he did the numeric expression 6 times 3 is 18 (line 56). N is not wrong for saying that his prediction strategy was that he found three layers of six Package D’s fit in Box 1 and that he multiplied the two numbers together to get a prediction of 18. Since I argued that N re-conceptualized his mental model to this more sophisticated layering strategy, it makes sense that he explains only his final conceptualization and does not explain the initial strategies. It is highly likely that N views his final conceptualization as his overall strategy and may not think it is relevant to explain the initial strategies that he used to get to that final conceptualization.

N and P, Package E, Box 1, Day 1.

![Figure 3.57](image.png)

*Figure 3.57. How many Package E’s will fit into Box 1?*

1. MB: Okay, how about Package E?
2
P: Six. 1, 2, 3, 4, 5, [points to the bottom of the picture of the Box 1]. 15. I think there are 15.

MB: Okay P, explain to me how you got your answer.

P: Okay, so there is five in here [points to the picture of Package E]. So there is 3, 4, 5, okay that is five [draws a line through a 5x1 array of squares found in a vertical column of squares on the bottom of the picture of Box 1]. Yeah so there is one right there, two right there, three right there, four right there [points to the vertical columns of six squares on the bottom of the picture of Box 1]. Then there is five right there [points to the front row of four squares on the bottom of the picture of Box 1]. Wait, that is only four [squares], so I can’t use that so… [During P’s explanation N leaning over and watching what P is pointing at on the picture of Box 1].

Figure 3.58. P’s locating processes for Package E’s on the bottom of Box 1.

N: I think it is 60.

MB: How many?

N: 60.

MB: 60, okay.

P: 4 times [3] is 12… There are only 12 [Package E’s] that will fit. What do you think [directing the question at N]?

N: I think 60.
MB: Okay N, why don’t you explain your thinking to P?

N: Okay, this what I did. There are five going down each row here [points to the four 5x1 arrays of squares found in the vertical columns of six squares on the bottom of the box picture] and that equals 20 and it is three high. So 20 times 3 is 60.

P: There are six [squares] in a row [i.e. columns] though.

N: I know, but for this row we can’t break it up, so that is like empty [points to the front row of four squares on the bottom of the picture of Box 1].

Figure 3.59. N’s locating processes for Package E’s in Box 1.

P: You got 60 though?

N: What is 20 times 3?

P: 60. Okay, tell me how you did this again.

N: Okay, I counted 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 [begins to count each individual square found on the 5x1 arrays of squares in each of the vertical columns of six squares at the bottom of the picture of Box 1]. (See Figure 3.60 below).
Figure 3.60. N’s error of counting squares instead of counting locations of Package E on the bottom of Box 1.

P: No, it is how many packages, not how many blocks.

N: Oh [smiling]! [Then N counts the individual columns by motioning over the columns of five squares on the bottom of Box 1]. Yeah 12.

MB: Okay, you say 12.

[The boys place Package E’s in the box and they count the number of Package E’s as they remove them from the box.]

P: 12 and then we can’t fill [the empty space]. We were right.

MB: You said that there was 12, but there is?

P: There are 12. She [the teacher] said you can’t break them up.

MB: Oh that wouldn’t fit over there [in the empty space] because you can’t break them up.

P initially thinks that five Package E’s are located on the bottom of Box 1 with two different orientations: one Package E with the given orientation and four Package E’s rotated 90’ about the z-axis (lines 3-4, 8-13). First, P locates the four reoriented Package E’s on the 5x1 array of squares found in each of the four vertical columns of six squares
on the bottom of Box 1 (lines 8-12, see Figure 3.58). As a result, he has one remaining horizontal row of four squares on the bottom of Box 1, where he locates the Package E with the given orientation (lines 12-13). Although P does not explicitly state it, it seems that he is able to locate the height of Package E on the height of Box 1 in order to recognized that he could have three layers of five Package E’s fit in Box 1. P implicitly connects his spatial structuring of three layers of five Package E’s to a numeric expression of 5 times 3 to get a prediction of 15 Package E’s (lines 3-4).

For this initial conceptualization, P is making a locating error in that he locates a Package E with the given orientation, which contains five cubes, on a horizontal row that fits only four cubes. What is interesting is even though P’s initial conceptualization is incorrect, he is able to conceptualize that he can mix different orientations of packages together in his structuring of Package E’s in Box 1. Up until this point we have seen neither P nor N use mixed orientations of packages within their mental models.

After P begins to explain his strategy to MB, he recognizes his locating error by noticing that the one horizontal row of four squares, where he located a Package E, can hold only four cubes, whereas Package E contains five cubes (lines 13-14). This perturbation causes P to adapt his mental model to allow for empty space in or above the one horizontal row of four squares on the bottom of Box 1. Then P re-conceptualizes his structuring to four Package E’s fit in a layer with three layers fitting into Box 1. He then does 4 times 3 to get a prediction of 12 Package E’s in Box 1 (line 32).

N’s initially predicts 60 Package E’s will fit into Box 1 (line 24). First, N properly locates the 5x1 array of squares appearing on the bottom orthogonal view of Package E on the four vertical columns of six squares appearing on the bottom of Box 1
As result of these locating processes, N recognizes that one horizontal row of four squares will be empty on the bottom of Box 1. N states that they cannot position a Package E in the space above the empty horizontal row of four squares without breaking apart a Package E (lines 45-46). Then N states that he has 20 Package E’s in the bottom layer and notices that the height of Box 1 is three, therefore he can fit three layers of 20 Packages E’s each into Box 1 (line 41). He is able to connect his structuring to the numeric expression of 20 times 3 gives him a prediction that 60 Package E’s fit into Box 1 (line 41).

N errors in his conceptualization in that he loses track of the unit he is supposed to be enumerating in his prediction strategy. What is interesting is N constructs a proper spatial structuring for Package E’s in Box 1; however, he loses track of the unit he is suppose to be enumerating and counts cubes instead of Package E’s (lines 39-41). So N’s spatial structuring is correct, but his numeric reasoning is inconsistent with that spatial structuring. Since cubes are represented in the pictures of each particular package in these problems it easy for students to mistakenly use cubes as their units instead of packages.

After N initially explains his conceptualization to P, P questions how N got a prediction of 60 Package E’s to fit into Box 1 (lines 55, 59). N then counts the individual squares on the bottom of the picture of Box 1 that corresponds to the locations of four Package E’s on the bottom of the box (lines 61-63). P then recognizes that N is making the mistake of counting the cubes contained in Package E’s, instead of counting Package E’s that fit into Box 1 (line 73). Once P points out this error to N, N immediately sees his mistake and adjusts his mental model to enumerate packages instead
of cubes, which gets him the accurate prediction of 12 (lines 75-76). P and N then check their prediction by placing Package E’s in Box 1 and enumerating the number of Package E's that fit. They find that their prediction of 12 Package E’s was correct.

**N and P, Package A, Box 2, Day 2.**

![Box and Package A](image)

*Figure 3.61. How many Package A’s will fit into Box 2?*

1. [N draws six “X’s” on six 2x2 arrays of squares on the bottom of the picture of Box 2. (See Figure 3.62 below). He then marks two more “X’s” on two 2x2 arrays of squares on the back side of the picture of Box 2. Then he place two more “X’s” on two 2x2 arrays of squares on the left side of the picture of Box 2.]
Figure 3.62. N’s locating processes for Package A’s in Box 2.

P: First, we have to make this box [points to the picture of Box 2], and then we do our prediction.

N: No. Wait, why do we need to make the box first?

MB: You’ve got to predict first and then make the box.

P: Yeah, but I am talking about this box right here [points to the picture of Box 2].

MB: So what are you thinking N?

N: I think the prediction is 10.

MB: For Package A?

N: Yeah, because like I counted all the four squares [points the 2x2 arrays of squares on the bottom of the picture of Box 2] (see Figure 3.63 below) except it is higher and when you count the bottom here, you got those there [points to the bottom two rows of squares that appear on the back and left sides of the picture of Box 2]. Then there is these two up here [points to the two sets of 2x2 arrays of squares on the back side of the picture of Box 2] and there is these two here [points to the two sets of 2x2 arrays of squares on the left side of the picture of Box 2] and that is 10.
Figure 3.63. N’s conceptualization of Package E’s in Box 2.

[MB gets up and walks away from the table]

P: What did you get?

N: 10.

P: There are six right here [points to the bottom of the picture of Box 2], 12. It would be 12.

N: How did you get 12?

P: Look there is 1, 2, 3, 4, 5, 6 in here [points to 2x2 arrays of squares on the bottom of the picture of Box 2]. Then there are six right here [points to third and fourth rows of squares up on the back and left sides of the picture of Box 2]. So there are six down here [points to the bottom of the picture of Box 2] and then 6 plus 6 equals 12. (See Figure 3.64 below).
Figure 3.64. P’s conceptualization of Package A’s in Box 2.

N: [Looks back at his work and counts again]. There are 10. P look there are 1, 2, 3, 4, 5, 6 [points to the bottom of the picture of Box 2]. And then you take this out there [points to the bottom two rows of squares on the back and left sides of the picture of Box 2]. Then there are 7, 8, 9, 10 [points to where he placed X’s on the 2x2 arrays of squares on back and left sides of picture of Box 2].

P: What? Do that again.

N: There is six right here, right [points to the bottom of the picture of Box 2]?

P: Yeah.

N: Then you already counted this [points the bottom two rows of squares on the back and left sides of the picture of Box 2] from counting these [points to the bottom of the picture of Box 2]. Then there are 7, 8, 9, 10 [pointing to where he placed X’s on the back and left sides of picture of Box 2].

[P then draws an X on the 2x2 array of squares for rows 3 and 4 on the left side of the box]. (See red X on Figure 3.65 below)
Figure 3.65. The red X is where P thinks N missed counting a Package A in Box 2.

N: You won’t count that.
P: Why?
N: See you won’t count this here [points to P’s newly marked X] because this one is a part of this one [points to the first X furthest on the left of the back side of the picture of Box 2]. (See Figure 3.66 below).

Figure 3.66. N’s explanation for why they do not count the red “X” as part of their enumeration of Package A’s in Box 2.
P: No see that one, you are counting the sides you are not counting the bottom.

N: You have to count the whole thing though.

P: I don’t know. I told you why don’t we make our box first.

N: I think we should do our prediction first. I think it is 10.

P: This box right here.

N: I think it is 10.

P: I will put down 12. No, because we have to agree.

N: Okay, tell me again how you got 12.

P: Okay, 1, 2, 3, 4, 5, 6 [points to the bottom of the picture of Box 2].

N: So we both agree that this [the bottom of Box 2] is six. So you messed up, up here [pointing the upper rows of squares of the left and back sides of the picture of Box 2].

P: So there are six here and then six times … Since this is part of the bottom right?

N: Yeah.

P: So this is all six. This is all six right here, this has to be all six on top and 6 plus 6 is 12.

[N goes back to picture and begins recounting].

P: You don’t count the top.

N: Why not? Then where do you get your other six?

P: See look, okay all this and all this and all this and all this is six. (See Figure 3.67 below)
N: Yeah.

P: Okay, then this is with that. Then pretend this is the beginning again where this is all filled up [points to back and left sides of the picture of Box 2] and there is nothing on the bottom okay. Pretend that this is the bottom right here [points to the third row of squares up on the backs side]. Then it would be exactly like the bottom. (See Figure 3.68 below).

Figure 3.67. P’s explanation of his conceptualization of Package A’s on the bottom of Box 2.

Figure 3.68. P’s explanation for why there is a second layer of six Package A’s in Box 2.
N: I don’t think there are 12.

P: There are six on the bottom and six on the top because it is shaped exactly the same way. Isn’t it? (See Figure 3.69 below)

Figure 3.69. P’s conceptualization of the two layers of the six Package A’s in Box 2.

[N goes back to his picture and recounts where he sees Package A’s fitting in Box 2]

N: But there is not six on the top

P: There is, look. [P is drawing something on his picture Box 2].

[MB returns to the table.]

MB: How are you guys doing?

P: Well, he thinks it is 10 and I think it is 12 and we have to agree on it.

MB: Oh no you don’t have to agree. This is your prediction. And now you should build the box and should check it out. Now you said it was 12? How did you come up with 12, P?
There are six right here [points to the bottom of the picture of Box 2] and then this is all a row because two going high is like that [points to the picture of Package A on the student sheet] and then this is exactly the same on the top it is just another layer. So it would be 12.

P: [Drawing on the grid paper]. This won’t fit five.

MB: It won’t fit on a single paper. What some students do is they take their box from the last one and they sort of build up the sides on here [points to Box 1].

P: So take this apart.

MB: Yeah, you could take it apart or just add stuff to the sides.

[The boys create the box out of grid paper and tape].

MB: Are you ready to check it out?

[The boys begin to build many Package A’s to fill the box in with. The boys’ place six Package A’s in the bottom layer of Box 2].

MB: If you look right now what would you say?

P: Six fit into the bottom and then I think six more will fit on the top.

MB: What do you think N?

N: I think both of us are wrong. [Long pause]. I think P is right, actually.

P: We are running out of cubes.

MB: Yeah this has so many it might be hard to completely fill it.

[Once Box 2 was completely full with Package A’s, P counts them by taking them out one by one.]

P: 12.

N predicts 10 Package A’s fit into Box 2 (line 24). First, N locates the array of squares appearing on the bottom orthogonal view of Package A on the array of squares appearing on the bottom of Box 2, and recognizes that six Package A’s are located on the bottom of Box 2 (lines 1-2, 28-29). Then N is able to position these Package A’s by
recognizing that the bottom two rows of squares on the back and left sides of Box 2 correspond to the six Package A’s that he located on the bottom of Box 2 (lines 29-31). After N visualizes the bottom layer of Package A’s, he does two locating processes of Package A’s to the remaining area left on the back side of Box 2 (lines 32-33). He then does more two locating processes of Package A’s to the remaining array of squares on the left side of Box 2 (lines 33-34).

N later states the reason that there is only two Packages A’s on left side of Box 2 instead of three is because the 2x2 array of squares on the back corner of the left side of Box 2 coordinates to the same package that he already located and counted on the left corner of the back side of Box 2 (lines 98-100, see Figure 3.66). N had to integrate these two locating processes into one mental model of the position of that Package A to allow him to realize that he had already accounted for that package when he counted its location on the back side of Box 2. Since N was able to position this Package A also suggests that N was also able to position the other packages that he located on the back and left sides of Box 2.

After he has positioned the Package A’s on the left and back sides of Box 2 in space above the first layer, N thinks he has accounted for every package that fits into Box 2 and states that his prediction is 10 (lines 34). However, N did some improper positioning processes for Package A’s found in the second layer of packages in Box 2, which leads N to an incorrect prediction of 10. What N does not realize is that he missing two Package A’s that are positioned on the front right side of Box 2. N needed to do the same positioning processes for the front right side of Box 2 as he did for the front left side of Box 2. However, N’s conceptualization of the space above the first
layer of Package A’s did not just have missing positioning processes, but also had an incomplete mental model. His mental model the positions of Package A’s in the space above the first layer was incomplete because it did not enable him to see the empty space where he could position the two missing packages on front right side of Box 2.

N’s structuring for the space above the first layer seems to be focused on positioning individual packages in the box instead of repeating some composite of Package A’s. N positions only individual Package A’s where there is an explicit 2x2 array of squares to locate it on the sides of Box 2. Thus, for the second layer, N positions only Package A’s to the back and left sides of Box 2 because they are the only two lateral sides that have arrays of squares appearing on them. As a result of focusing on individual packages, N does not realize that he missed positioning two Package A’s that correspond to the front and right lateral sides of Box 2.

N’s focus on positioning individual Package A’s in the second layer caused him to face a very difficult visualization problem. In order for him to visualize that two Package A’s fit into that empty space, he would have to do a very complex set of coordinating and integrating processes. For simplicity, I am just going to explain the process N would need to do in order to visualize how a Package A is positioned in the front right corner of the second layer. N would have to coordinate a reverse locating process on the right back side with the reverse locating process on the front left side of Box 2 (see Figure 3.70a below). He then would have to integrate these processes to form a mental model of Package A positioned as shown in Figure 3.70b.
Figure 3.70. How N would have to visualize the Package A in the front right corner of Box 2 using his current conceptualization.

P conceptualizes two layers of six Package A’s each will fit into Box 2, which gives him a prediction of 12 (lines 49-50). He first does six locating processes of Package A on the bottom of Box 2 (lines 54-55). Next, P had to properly position the six Package A’s in the bottom layer in order to recognize that the bottom two rows of squares on the back and left sides of Box 2 correspond to the same six Package A’s he located on the bottom of Box 2 (line 144). Then, from his mental model of positioning this layer, he sees that he can position another layer of Package A’s on top of the bottom layer in Box 2 (lines 55-57). During one of P’s explanations to N, he explains that after he placed six Package A’s on the bottom layer to pretend the top orthogonal view of the bottom layer of packages is now the bottom of the box. So P realizes that the tops of the six Package A’s that he already placed in the box in his mental model is the same array of squares that appears on the bottom of Box 2 (lines 157-158). This shows that P’s mental model of rectangular arrays of cubes includes the geometric property of opposite sides are congruent.
The communication between the boys and the conflict. N explains that he thinks 10 Package A’s will fit into the Box 2 before P has a chance to finish thinking about his prediction (lines 24, 47). Thus, P is aware that N has a prediction of 10 Package A’s, but this does not persuade him into accepting N’s prediction for his own. Instead P disagrees with N’s prediction, and explains that he used a layer structuring strategy where he found that two layers of six Package A’s fit into the box giving him a total of 12 Package A’s in Box 2 (lines 54-58). In this episode, each boy firmly believes that their prediction is the correct one because neither boy is able to explicitly identify N’s error of missing two Package A’s in the front right side of Box 2.

N then explains his strategy again, but this time P marks an “X” on the second layer on the back left side of Box 2 to try to show N that he missed enumerating a Package A at that location (lines 84-85, see Figure 3.65). This shows that P is attempting to identify an error in N’s prediction strategy, however, the package location that P thinks N missed in his enumeration is actually not an error in N’s reasoning. N is quick to point out that this 2x2 array of squares has already been accounted for in his prediction because it corresponds to the same Package A that was previously located and enumerated in the left corner of the back side of Box 2 (lines 94, 98-100, see Figure 3.66). N is correct in his reasoning in that he has done a proper positioning process that integrates these two coordinated locating processes in one coherent mental model of the positioning of that Package A.

P immediately disagrees with N and states to N that “you are counting the sides, you are not counting the bottom” (line 110). It seems like P abandons trying to identify and understand where N’s error is in his prediction strategy. Instead, P states that N’s
strategy itself is incorrect because N counted the locating processes of the back and left sides of Box 2, rather than again using the same locating processes N did on the bottom of Box 2. The difficulty that the boys are struggling with is their prediction strategies are different for the second layer of Package A’s in Box 2. P’s positions the second layer of Package A’s according to the bottom layer of Box 2, whereas N is positioning individual Package A’s to the back and left sides of the box. Neither boy seems to understand what the other boy’s strategy is for his prediction.

The boys think that they are supposed to agree on a prediction, therefore they continue to debate which of their predictions are correct (line 122). They agree that there are six Package A’s positioned on the bottom of Box 2 and that the difference in their strategies is their positioning of packages in the second layer (lines 128-129). P then explains to N that after they position six packages in the bottom layer, if they pretend that the top of that layer is the bottom of Box 2, then they can position another layer of six Package A’s in Box 2 (lines 157-160, 171-172). This shows that P is still trying to find new ways to communicate his reasoning to N. On the other hand, N immediately states that he does not think the answer is 12 (line 169). Because of the immediacy of his response, it seems that N has become inflexible and less willing to modify his mental model to try understand P’s reasoning.

It seems as though both boys are too concerned with explaining their own strategy to convince the other that their strategy is right that they become unable to understand the other’s reasoning. Neither seems flexible enough to really try to understand and accommodate the other’s mental model. I hypothesize that N is unable to visualize P’s mental model because he is fixated on positioning each Package A in the second layer of
packages. On the other hand, P clearly sees that N’s prediction is incorrect, but is unable to explicitly identify N’s error of missing two Package A’s in front right corner of the second layer of packages in Box 2. It is interesting that when P and N have disagreed on predictions in previous episodes (e.g. Package B Box 1, Package E Box 1), one of the boys has always explicitly identified where the misconception or error occurred in other’s prediction strategy. Therefore, since N does not see what is wrong with his mental model and P is unable explicitly identify N’s error, there is nothing convincing N that his prediction is wrong and P’s prediction is right.

MB then returns to the table and asks the boys how they were doing with Package A’s for Box 2 (line 189). The boys state that since they have to agree on one prediction before they can check their answers using concrete materials that they are still trying to figure out whose predictions is correct (line 191). MB explains to them that they do not have to agree on the same prediction in order to check their predictions with cubes (lines 193-194). The boys then begin to check their predictions by placing Package A’s in Box 2 (lines 215-216). After the boys place six Package A’s in the bottom of Box 2, N realizes that P’s prediction of 12 Package A’s was correct (line 224). Thus, it was not until N explicitly used concrete materials where he saw Packages A’s being placed into Box 2 that he recognized that P’s structuring of Package A’s was correct.

N and P, Package C, Box 2, Day 2.
Figure 3.71. How many Package C’s will fit into Box 2?

MB: Okay, Package C, what do you think? How many Package C’s do you think will fit in there?
P: I think six.
MB: So you made your prediction P by looking at this picture?
P: Yeah, I didn’t have to count I knew that there was six because there isn’t six rows going up [points to the height of the back side of the picture of Box 2] so there are only three and there are three rows up here [points to the picture of Package C].

N: I think that there are nine.
MB: Let me ask you this. Let’s look at this pattern, and here is what a Package C looks like. [P hands MB a Package C. Where MB places the Package C with its given orientation next to Box 2]. Without touching these would you keep your prediction the same if you looked at this [Box 2] and you looked at this [Package C]?
P: [Long pause] I would.
MB: Okay, why don’t you check it out?
P: How did you get nine?
N: What I did was I counted the bottom again, since there are four on the bottom [of Package C] and I got six. And then that eliminated the three up and then there is seven [points to the top row of the front left side of the picture of Box 2] and three more is eight [points to the back left side of the picture of Box 2] and three more is nine [points the left back side of the picture of Box 2]. (See Figure 3.72 below).
Figure 3.72. N’s prediction strategy for Package C’s in Box 2.

P: You can’t break them up.

N: No, you don’t break them up cause there is two rows like this and if you turn this [grabs Package C and reorients it as show in Figure 3.73] sideways there are two rows and it is three across. It goes three across on two rows. (See Figure 3.73 below).

Figure 3.73. N’s explanation as to why he can position Package C’s in the second layer of Box 1.
MB: So you put some down on the bottom and then some at the top.

N: Yeah you put them sideways like this [shows the interviewer].

[The boys test their predictions by filling the box with copies of Package C’s. Note: When the boys previously constructed Box 2 with paper they did not tape up the right side of the box.]

N: Six [points to the six Package A’s on the bottom of Box 2]. See seven, eight [places two more packages in the box sideways as shown in Figure 3.74 below]

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**Figure 3.74.** The boys initial structuring of Package E’s in Box 2 when using concrete materials.

P: [Staring at the eight Package C’s sitting in Box 2]. It would be 10.

[N then pulls out the packages and counts to eight].

P: No, see look there are six on the bottom and then there are eight right here [placing all the packages back into the box]. Then 9, 10 [moving the seven and eight packages to another spot showing that two more could fit into the box as shown in Figure 3.75].
Figure 3.75. P’s explanation as to why the actual answer is ten packages instead of eight packages.

N: Oh yeah, 10.

N predicts that nine Package C’s fit into the box, whereas P predicts six Package C’s fit into Box 2 (lines 4, 12). P says that he does not even need to count for his prediction because he recognizes that the height of Box 2 is not six, so he can locate only the height of Package C with its given orientation once on the height of the Box 2 (lines 8-10). Although P makes no explicit mention of how he got six packages in the bottom layer, I hypothesize that he knows that the bottom of Package C is 2x2 array of squares and that he locates six 2x2 arrays of squares on the bottom of Box 2. We saw P explicitly do these same locating processes for Package A’s on the bottom of Box 2, where like Package C, Package A also has a 2x2 array of squares appearing on its bottom orthogonal view (N and P, Package A, Box 2, Day 2, line 54).
However, P failed to reorient Package C to see if he could fit anymore Package C’s into Box 2. In fact, if P would have reorient Package C by rotating it 90° about the y-axis he might have noticed that he could fit four more Package C’s on top of the first layer of packages, resulting in him getting an accurate prediction of 10 Package C’s in Box 2. Thus far in Investigation 2, none of the tasks required the boys to use mixed orientations of packages to get an accurate enumeration of packages that fit into a box. We have seen P used mixed orientations incorrectly in his initial strategy for Package E’s in Box 1. However, P soon recognized that he was positioning a Package E with its original orientation in a place where it did not actually fit, and therefore did not need to use mixed orientations of packages to get the correct enumeration. Since using mixed orientations in his structuring of Package E’s did not give an accurate prediction for Package E’s it is likely that P discarded or forgot the idea of using mixed orientations in his mental model. Therefore, P may not even consider using mixed orientations of Package C’s in his mental model because he has never needed to use it to get an accurate prediction for the previous tasks.

MB then places a Package C next to the constructed Box 2 and asks the boys, “Without touching these would you keep your prediction the same if you look at this [Box 2] and you look at this [Package C]?” (lines 16-17). Remember the boys are predicting using two-dimensional pictures of the packages and boxes to represent three-dimensional objects. When the interviewer asked this question he was trying to see if having the three-dimensional objects physically placed in front of the boys had any effect on the boys’ conceptualizations of positioning Package C’s in Box 2. For P specifically, I think the interviewer wanted to see if having a concrete Package C in front of P helped
him think about what actions (e.g. reorienting) he could do in order to manipulate more Package C’s to fit in Box 2. P responds that he would keep his prediction that six Package C’s fit in Box 2 (line 19). This shows that in this specific instance that seeing the physical objects of Package C and Box 2 in front of P had no effect on his conceptualization of his prediction. However, if P is not considering mixed orientations of Package C’s in his conceptualization to begin with, it seems likely that he might not see any difference between viewing the two-dimensional pictures of the objects on his student sheet and seeing the physical objects themselves.

Unlike P, N’s conceptualization does explicitly use mixed orientations of Package C’s in his mental model. N locates six Package C’s on the bottom of Box 2 and recognizes these Package C’s will account for three horizontal rows of squares that appear on the back and left sides of Box 2 (line 26). Then N reorients Package C to a horizontal orientation where a 2x3 array of squares is on the bottom orthogonal view of Package C (i.e. a rotates the package 90° about the y-axis). Then N locates the 2x3 array of squares appearing on the left orthogonal view of the Package C two times on top left side of Box 2 (lines 26-28). Then he reorients Package C again by rotating it 90° about the z-axis so that the 2x3 array of squares that was appearing on the left orthogonal view of a Package C is now located on the top back side of Box 2, giving him a total of nine Package C’s in Box 2 (lines 28-29, see Figure 3.72 on page 113).

Although N is correct that he needs to use mixed orientations of Package C’s in order to find the correct enumeration of packages, he makes a misconception in mental model by doing an uncoordinated set of locating processes for this second layer of Package C’s. The misconception occurs when P locates the 2x3 array of squares
appearing on a orthogonal view of Package C to the top left side of Box 2 (see Figure 3.76a below) and then he locates that same orthogonal view to the top back side of the Box 2 (see Figure 3.76b below). What N does not realize is if he were to position Package C’s at each of those locations (see Figure 3.77a and b below) he would overlap two Package C’s in back left corner of Box 2. The overlap is the 2x2x2 cube array shown in Figure 3.77c below.

![Figure 3.76](image)

Figure 3.76. Two of N’s locating processes on the left and back sides of Box 2.
N does not realize that once he orients a package and locates one of its orthogonal views on one side of the box, then all other locating processes for this package on the sides of the box must coordinate to its corresponding orthogonal view of this package with that same orientation. For this episode, when N did the locating process of a Package C for the top two rows of squares on back left side of Box 2, he would also need to located the back orthogonal view of that same Package C on the left back side of Box 2 as shown in Figure 3.78 below. This suggests that N did not visualize proper positioning processes for the second layer because he did not coordinate and integrate his locating processes of these Package C’s to see that he was overlapping two positions of Package C in the back left corner of Box 2.

Figure 3.77. Overlap occurs where the Package is colored black.
Essentially, N is still using the same scheme that he used for the previous task, where he locates explicit $2 \times 3$ arrays of squares that occur on the back and left sides of Box 2 to indicate where he can position individual Package C’s. However, the difference between the two episodes is N did properly position Package A’s using the coordinated locating processes that occurred on the back and left sides of Box 2, whereas N did not do similar positioning processes for the Package C’s he located on the back and left sides of Box 2.

After N explains his reasoning for the second layer of packages, P interjects and says that N cannot break the Package C’s apart (line 38). Recall the earlier hypothesis that P was not considering mixed orientations of Package C in his conceptualization. This suggests that P is conceptualizing that N was positioning Package C’s using the given orientation in the empty space above the bottom layer of packages in Box 2. As a
result, this causes P to think that the only way N is getting three Package C’s with the given orientation to fit in the empty space above the bottom layer of packages is by breaking the Package C’s apart. N then explains to P that he is not breaking apart the Package C’s (line 40). He grabs a Package C and states that if they reoriented the package to horizontal orientation so it will have a height of two and it is three cube long it will fit in the remaining space on top of the bottom layer of packages (lines 40-42). P does not give a response to N’s second explanation, so there is no way of knowing whether P understands N’s reasoning or not. It is important to note that the conflict between the boys’ predictions remains unresolved before they check their predictions using concrete materials.

The boys decide that it is time to check their predictions by filling Box 2 with Packages C’s (lines 56-58). The boys have only enough cubes to create eight Package C’s to place into Box 2. After the boys have filled the first layer with six Package C’s, N then places two more packages with a horizontal orientations (as shown in Figure 3.79 below) on top of the first layer and points out to P that he was right about fitting more packages on top of the bottom layer (lines 60-61).
By looking at the way they placed the eight Package C’s in Box 2 as shown in Figure 3.79, P is able to visualize two more reoriented packages fit into the box giving him a total of 10 Package C’s (line 71). On the other hand, N has to iterate each Package C that was placed in the box and states that are only eight Package C’s fit into Box 2 (line 73). It is interesting that even though N can physically see eight Package C’s sitting in Box 2 that he still is not able to visualize the two additional Package C’s that are missing on the right side of the second layer. It is important to note that when the boys constructed Box 2 using grid paper and tape, they taped only three of the four sides of Box 2 up, leaving the right side of the box open so it would be easier for them to place packages inside the box (see Figure 3.79 above). This suggests that N is still fixating on positioning Package C’s only where he can locate explicit 2x3 arrays of squares that appear on a box side. Since the right side of the box was left open and it is the side where the two missing packages are positioned, then N has no explicit visual representation of
the arrays of squares inside the box for him locate the two missing Package C’s. P then has to physically move the other two Package C’s in the second layer to the two missing positions of Package C on the right side of the box before N realizes that there are 10 Packages C’s that can fit into Box 2 (lines 75-77, 87, see Figure 3.75 on page 115). This movement shows that P is able to translate the packages to different positions in the box without losing the positions of the two original packages. At the end of the episode, both boys agree that 10 Package C’s fit into Box 2 (line 87).

**N and P, Package E, Box 2, Day 2.**

![Figure 3.80. How many Package E’s will fit into Box 2?](image)

1 MB: Now you are doing E?
2 P: Yes, and I think 20.
6 P: How did you get 24?
N: 1, 2, 3… 20, 21, 22, 23, 24 [as he is pointing to each individual square on the bottom of the picture of Box 2]. (See Figure 3.81 below).

Figure 3.81. N’s prediction strategy for Package E in Box 2.

P: Yeah, but there are five in each row, not four.

N seems confused by P’s statement. P then notices that N seems confused and points to the picture of Package E on the student sheet.

N: Oh, okay.

MB: How did you get 20, P?

P: Okay, there are five right there [points to picture of Package E]. There are five [squares] going four times this way, so that would be four [points to the four vertical columns of six squares on the bottom of the picture of Box 2]. There are five going up so it would be umm….

MB: Show me what you did again? I didn’t quite see it.

P: Okay, well, what I did was there are five this way. Maybe I messed up.

MB: Five this way? [Using his finger to make a motion over the columns on the array of squares on the bottom of the picture of Box 2].
P: Yeah, but that is not five that way, there are six that way [meaning in the columns on the bottom of Box 2]. But I did not count the top one [row of four squares] and I just said that there was four in here [points to the bottom of the picture of Box 2] and there are five going up. But now I kind of think that wouldn’t work because this [holding package of two cubes that is stacked vertically] takes up that row. This and this are the same one [points to the bottom of the Box 2 picture on his paper].

There are five Package E’s going up.  

Four Package E’s are located here.  

Do not count the top row of squares because it only has four squares whereas Package E has five cubes.

Figure 3.82. P’s prediction strategy for Package E’s in Box 2.

[The boys then begin to build Package E’s and place them in the box according to P’s strategy].

N: I think I was right when I said 24.

P: I think that both of us are wrong.

MB: You think that you are both wrong?

N: I changed it to 20.

MB: Oh, you changed it to 20.
N: How many is that so far?
P: Um 1, 2 it is 12, 13, 14. [Pauses]. Yeah, I guess I was right.

[L comes over to the table]

L: Oh you are on E, great. Do you have a strategy in mind that you are going to be able
talk about?
P: Yeah kind of.

L: Kind of? Are you kind of going to be able to write something down?
P: Yeah.

L: What did you find? Anything important?
N: I found something.

L: What did you find?
N: Well, all I did was there are four on every side, so when you count every four on the
bottom [points to the bottom of the picture of Box 2] it will equal every four here [points
to the picture of Package A]. And then you will count every four here [points to the
bottom of the picture of Package A].

L: That is for which one?
N: Package A.

MB: What did you get?
P: 20.

P: I think it was 20.
N: It was?
P: Yeah.

MB: So will any more fit into the box? (See Figure 3.83).
Figure 3.83. MB’s question on whether the boys can fit any addition Package E’s to their structuring in Box 2.

N: Yeah, the four over here [points the empty space on the box towards the front] so it will fit 24.

[P then places a Package E in Box 2 so that it lies across the row of four squares].

P: Well it [the package] is five, so it won’t fit.
Figure 3.84. P’s reasoning about why N’s conception that four more Package E’s fitting into the empty space left in Box 2 is wrong.

N: No, if you stand it up. [Reorients Package E so it is standing vertically].

Figure 3.85. N’s checking strategy for Package E’s in Box 2.

“Well it [the package] is five, so it won’t fit”
P: Oh okay, so it would have been 24.

MB: Oh so it is 24.

P: We only got the first one right.

N: Except for B. I put ten down. Remember?

MB: Let me ask you something, N when you originally came up with 24. How did you come up with that number?

N: Well, what I did was I was counting all the four rows [points to the rows of fours squares on the bottom of Box 2] instead of the five [points to the columns of six squares on the bottom of Box 2]. I was counting the four rows.

MB: So you were counting 1, 2, 3, 4, 5, 6, 7, 8 [continues to count all the individual squares in the horizontal rows of four squares on the bottom of Box 2].

N: Yeah.

MB: Now do you think that is right looking back at or do you think it is wrong?

N: 24 is right.

P: Oh yeah, that is right because then after that you add these [points to a lateral side of Box 2] to make five.

MB: Okay, what do you mean P?

P: These right here, on the side, the four we added [points to the front side of Box 2].

MB: Oh, so it is 24. Is that what you were thinking when you originally did 24 of them going in like this? [Orients the Package E vertically and places it into Box 2].

N: Sort of.

MB: But it was hard to see in the picture?

N: Yeah.

P: The picture is kind of hard because you can look at it both ways. Making it looks like it is sticking out this way [makes a hand motion in the horizontal direction] and then in [makes a hand motion in the vertical direction].
Follow-up to Package E for Box 2.

[At the beginning of class, the teacher instructed the students to write their predictions and strategies down so they could discuss them later. After the boys finish with Package E, they wrote down their prediction strategies for Packages A, C, and E for Box 2 on a separate piece of paper].

MB: So N, why don’t you tell me what you wrote?

N: For A, I put down ten as my prediction. And how I got that was I count six on the bottom and four on the top and then I add those together and got ten. And for C, I got nine and there are six on the bottom here and then there was one right here in that space [points to the 2x3 array on the back side of the picture of Box 2]. Then another one here so that was eight and another one here and that is how I got nine [pointing to the two 2x3 arrays that located on the left side of the picture of Box 2]. And for E I started with 24 but changed it to 20 because I was counting the four rows across...

P: Are we supposed to have one for each one? I just wrote one big long one.

MB: Well that is okay; you have got one big long one that applies to all of them, that is okay. So tell me, you wrote four rows across? [Directing the question to N].

N: Well, I was like counting the four rows [i.e. the rows of four squares] across.

MB: Down at the bottom four rows across like this way [does a horizontal motion] or this way [does a vertical motion]?

N: I was like getting confused, and I was really counting each row going up [points to the back side of the picture of Box 2] and that is how I was counting. I thought I was counting the four rows [i.e. rows of four squares] cause I was counting side to side like that [motions horizontally over the rows on the bottom of the picture of Box 2]. And I got confused.

MB: So you were trying to count how many were going up [points to the columns on the back side of the picture of Box 2] and then when you were counting you got confused because you thought you were going back and forth down here [points to the bottom of the picture of Box 2] on the bottom.

N: And then P said that is only four rows [i.e. squares], you need five rows [i.e. squares]. So then I changed to 20 because P said 20.
P: What I did was I counted how many were in the package and looked at the box and asked myself how many could fit into the box and studied it. Finally, it came into my mind. And I wrote it down and checked it. It was wrong, and I said, “What did I do wrong?” And then I hoped the next one would be right. If it wasn’t, I would do it over and over until I got it right.

[P hands over his writing so MB can read it.]

MB: Now when you said that you looked at the box and you thought for a while and you asked yourself how many fit, how did you try to figure that out? How did you try to figure out how many fit?

P: Like if I do it on here [looking at a 4x2x2 grid box that he created] for this one?

MB: So A in that box.

P: I would say there are four on the bottom right here. Two rows going up on the sides [points to the back side of the picture of Package A] and there are two rows going up on the sides right there [points to the 4x2x2 box in his hand]. So that would be one and that would be one and then there would be two in it. (See Figure 3.86 below)

*Figure 3.86. P’s strategy for finding the number of Package A’s in a 4x2x2 box.*
MB: So you sort of visualized this [the package] fitting in there, but you sort of look at the lines on the grid paper to tell you exactly how they fit. How would you figure out how many A’s fit this box [points to Box 2]?

P: Here are four on the bottom [of Package A] and like four right here [points to 2x2 array of squares on bottom of the picture of Box 2] and that is one. There are four right there, so that is one. If there are four right there and so there are three [points to the far left three 2x2 arrays of squares on the bottom of the picture of Box 2]. And since there are three on this side [points to the left three 2x2 arrays of squares on the bottom of the picture of Box 2] there has to be three on this side [points to the right side] so there are six. There are two going up [points to the back side of the picture of Box 2] and there are two up again (see Figure 3.87 below). So this row, [points the bottom two horizontal rows of squares on the left and back sides the picture of Box 2] so that would be six all right here. Then since there is another two going up that would be 12. (See Figure 3.88 below).

Figure 3.87. P’s locating processes for Package A’s in Box 2.
MB: Let me ask you this question, do you think it would be easier to predict like if we, instead of looking at this [points to the box picture of Box 2], if you could look at the box and one copy of Package E, would that make it easier to predict? You couldn’t actually put it in [the box], but if you could just look at those two.

P: Oh yeah, because this [holding the paper box] is more realistic than when you look at it. Say you know since there are five going that way [vertically] and you know that there are five rows going... So there would be one right there, two right there… [moving a vertical Package E around the bottom of the box and iterating for each square on the bottom of Box 2] 16 right there, [pauses] it would be 24. So all you have to do is put this [Package E] so that one of these [Package E’s] can fit there [Box 2] and then you just count blocks inside here [points to the bottom of Box 2].

MB: So one fits this way [place a vertical Package E in the box] and then you have to count the bottom squares because it fits on each square. Okay.

P: It is easier than this [placing Package E horizontally in the box].

MB: Oh, it is easier than doing, okay.

P: But I didn’t see that on this paper cause I couldn’t really tell.

MB: You couldn’t really tell from the picture.

P: Yeah, but then when I saw those boxes I knew my answer was wrong cause I thought there was something else.
MB: Now N, you said a similar thing, really. I think, if I am right you sort of said that at first you thought they fit like this [places a vertical Package E in Box 2] and started counting them, but then when you were looking at the picture and counting you started thinking this way [makes a horizontal motion on the bottom of Box 2] and you got confused.

N: Yeah.

MB: But it turned out that your initial idea was right, but the picture was too hard to see. I just asked P if you could predict by looking at the box and one of the packages like this, would that be easier to do? Okay, if I said how many of these [points to Package E] will fit in here [Box 2]?

[L rings the bell and gives final directions and dismisses class].

P predicts 20 Packages E’s, whereas N predicts 24 Packages E’s fit in Box 2 (lines 3, 5). P immediately asks N how he got his prediction of 24 Package E’s (line 7). N responds by iterating the individual squares on the bottom of the picture of Box 2 (lines 9-10). P quickly points out to N that Package E contains five cubes, not four cubes, which causes N to become confused (lines 19-21). P notices that N is confused and points to the picture of Package E on the student sheet, which causes N to agree with P’s prediction of 20 (lines 21-22). Thus, P thinks that N is improperly locating Package E with its given orientation on each row of four squares on the bottom of Box 2 (see Figure 3.89 below). Since N does not provide a valid reason to P for why he was counting individual squares on the bottom of the picture of Box 2 and did not argue with P’s description of his strategy, it seems that N was unable to complete a proper mental model of his strategy.
Later on in the episode, the interviewer asks N how he originally predicted 24 Package E’s (lines 151-152). N repeats the same reasoning that P conceptualized as N’s error that was discussed in the previous paragraph (lines 19, 154-156). However, this reasoning is not consistent with N counting individual squares on the bottom of Box 2 in his initial explanation to P (lines 9-10). If N did do these improper locating processes on the rows of four squares on bottom of Box 2 and enumerated correctly, then he would have a prediction of 30, not 24 Package E’s fit into Box 2. It was P who initially stated that N was locating Package E’s on rows of four squares on bottom of Box 2, not N. Since N did not fully understand the reasoning behind his prediction of 24, he accepts P’s conceptualization of his strategy, even though it does not match the initial actions performed in his strategy.

After N’s responds to MB’s question, MB points out he saw N counting the individual squares on the bottom of the box picture and asks him if he thinks this strategy
is correct (lines 158-159, 163). N only says that 24 is right and provides no further insight (line 165). P interjects by stating that it is right because if they stand Package E up vertically on one cube then they can fill up the remaining empty space on the bottom of Box 2 (lines 167-168, 172). In this case, P and MB are talking about two different things. P’s response was referring to adding the four vertical Package E’s to the empty space in his initial structuring of Package E’s in Box 2, whereas MB was asking about using a uniformed structuring of vertically oriented Package E’s in Box 2. MB then clarifies his question to N by asking if he was initially thinking about positioning 24 vertically oriented Package E’s in Box 2 when he initially predicted (lines 174-175). N states that it was sort of what he was thinking, but it was hard to see in the picture on the student sheet (lines 177-181). This suggests that N is still not completely sure what his initial prediction strategy was at the beginning of the episode.

If P’s conceptualization of N’s initial strategy was not an accurate one, what is N’s conceptualization of his initial strategy? In the Follow Up episode, N explains that he actually reoriented Package E to stand vertically on one cube and then iterated the locations where it occurred on the bottom of Box 2 (lines 217-218). N explicitly states he got confused because he lost track of the unit during the horizontal motions he did while counting the individual squares along the rows of squares on the bottom of Box 2 (lines 218-221). The horizontal motions he did during his iteration caused him to get confused on whether his units of enumeration were individual squares or rows of four squares on the bottom of the box (lines 218-221).

Thus, according to N, he was already confused about his unit of enumeration by the time P stated that he thought N was improperly locating Package E’s on the rows of
four squares on the bottom of Box 2. By having P state that N uses rows of four squares as the unit of iteration, one of the two choices N is confused by, probably suggested to N that this was his unit of iteration. Sometimes when a student is already confused about his or her strategy and a peer states his or her understanding of that student’s strategy, it becomes easier for that student to accommodate his or her peer’s understanding of their strategy. In addition, P’s description of N’s error caused N to change his prediction to 20 to match P’s prediction (lines 228-229). N’s statement that, “So then I changed to 20 because P said 20,” suggests that N only changed his prediction to 20 because it was what P predicted and not because he actually understood P’s strategy or thought it was correct (lines 228-229).

P uses the same prediction strategy that he has used on many of the previous tasks in Investigation 2, where he finds the number of packages in the bottom layer and then finds how many layers fit in the box. P explains that he can fit one Package E on each of the vertical columns of six squares on the bottom of Box 2 with one square left over for each column (lines 28-31, 40-43). This gives him a total of four Package E’s that fit into the bottom of Box 2 (lines 28-29). P explains that after locating the four Package E’s on the bottom of Box 2, each empty square left over from the columns of six squares will form an empty row of four squares on the front bottom of Box 2 (lines 41-42). P later mentions that he is unable to fit a Package E at that location because it can fit only four cubes across, whereas Package E is five cubes across (lines 114).

P then locates the height of Package E, which is one cube, on the height of Box 2, which is five (lines 42-43). As a result, P conceptualizes that he will have five horizontal layers of four Package E’s that will fit into the box (see Figure 3.82 on page 125). P
implicitly does the numeric expression 4 times 5 equals 20 in order to get his prediction of 20 Package E’s (lines 41-43). At the end of his explanation of his reasoning, P is not so sure his strategy will work, but it is not clear as to the reason why he thinks this (lines 43-45).

In this episode, P does not seem to explicitly consider using a vertical orientation of Package E in his mental model. In fact, for this particular cube-package problem, P would have to structure Package E’s either using a uniform vertical orientation or mixed orientations of both vertical and horizontal orientations in order to get a correct enumeration of 24 Package E’s in Box 2. However, there is evidence that he does consider reorienting Package E in his mental model of Box 2. On the worksheet, the picture of Package E is oriented horizontally on its side going from right to left, whereas in P’s mental model the Package E’s in Box 2 are oriented horizontally on their sides going from front to back (see Figure 3.82 on page 125). This shows that P is reorienting Package E because he has to mentally rotate it 90˚ about the z-axis in order to locate the packages on the vertical columns of squares appearing on the bottom of Box 2.

After the boys seem to validate their prediction of 20 by placing packages in the box according to P’s strategy, MB asks if they can fit any more packages in the remaining empty space in Box 2 (line 109, see Figure 3.83 on page 127). N immediately says he can fit four more packages in the empty space above the empty row of four squares on the bottom of Box 2 (lines 119-120). P disagrees and states that Package E is five cubes across and the empty row of four squares can hold only four cubes (lines 122-124, see Figure 3.84 on page 127). N then shows P that if they reorient Package E to stand vertically on one cube, they could fit four more packages in the remaining empty
space (line 134). P recognizes his misconception and agrees with N that there are 24 Package E’s that would fit into Box 2 (line 143).

Why does P not initially consider a 90° rotation about the x-axis in his orienting process of Package E’s? This is not the first time P has seen an example of someone reorient a package to its vertical orientation. Back for Package B’s in Box 1, N showed P that they could rotate Package B’s 90° about the y-axis when explaining his prediction strategy (N and P, Package B, Box 1, Day 1, lines 17-20). In addition, it is important to note that P and N completed the cube-package problem for Package B’s in Box 1 a day prior to when they predict the number of Package E’s in Box 2. Thus, it seems that P forgot or never completely recognized and/or internalized that N used a vertical orientation of Package B in his structuring.

Additionally, Packages C’s in Box 2 was the first task that the boys were required to use mixed orientations of the packages to get a correct enumeration. For that episode, it was N who suggested that, in order to maximize the number of Package C’s in Box 2, the boys needed to have packages with mixed orientations (i.e. a horizontal and the given orientation) in their structuring in Box 2 (N and P, Package C, Box 2, Day 2, lines 25-29). In fact, P did not initially think that they could fit any more packages after he had structured the bottom layer of six Package C’s with the given orientation in Box 2. Thus, even though P has seen N’s scheme of structuring using vertical and mixed orientations of packages from previous episodes, P has not sufficiently abstracted these ideas so that it would occur to him to use that scheme in his structuring of Package E’s in Box 2. On the other hand, there is evidence that P has abstracted the ideas of vertical and mixed
orientations at low perceptual level because he sees the use of mixed orientation immediately after N explicitly states it (line 143).

*Follow-up episode.* As part of the directions, the boys are required to write their prediction strategies for all the packages they did for Box 2 to turn into the teacher (lines 192-195). Instead of repeating what was discussed earlier, I am going to briefly summarize what happened and attend to the new and interesting ideas that are found this episode. The boys write what their prediction strategies for Packages A, C, and E for Box 2. N writes down a prediction strategy for each of the three packages in Box 2 (lines 199-205), whereas P writes one general prediction strategy that does not give specifics about how he predicted for each package (lines 207, 231-235).

After listening to P’s written strategy, MB asks P if it would be easier to make a prediction if the boys had the box and one package in front of them instead of using the pictures on the worksheet (lines 286-289). P explains that he thinks that it would be easier because he would have been able to visualize reorienting Package E vertically to see how each Package E fits into the bottom of Box 2 (lines 291-297). Interestingly, we saw MB ask a similar question of P and N for Package C’s in Box 2 (N and P, Package C, Box 2, Day 2, lines 14-17). In that episode, both boys had improper mental models of the structuring of Package C’s in Box 2. As result of MB’s question in that episode, we saw that having the physical box and a Package C in front of the boys did not help them recognize the misconceptions in their mental models.
Figure 3.90. Package C and Box 2.

It appears P’s answer to MB’s question in the Package C Box 2 episode is not consistent with the answer that he states in this episode. However, when looking at the progress that P makes from Package C’s in Box 2 episode to this episode, it becomes clear as to why P answers the question differently in the two situations. I earlier hypothesized that a possible explanation as to why seeing the concrete objects did not change P’s mental model for Package C’s in Box 2 is because P initially did not explicitly consider using mixed orientations of Package C’s in his structuring. However, as P progresses through the tasks of Package C’s and Package E’s for Box 2 tasks, he sees that using vertical orientations and/or mixed orientations of packages are required for correct enumerations. Therefore, by the time he gets to the Follow-Up for Package E, P seems to be considering the orientating process in his mental model and reasoning about concrete objects.
Chapter 4: Discussion

Mental Processes

The newly defined mental processes that students used for cube-package enumeration problems, as previously described in the Results, are orienting, locating, and positioning. The Results section clearly illustrated the usefulness of these newly defined mental processes, in addition to the ones previously used in cube enumeration problems, in explaining student reasoning for cube-package problems.

Types and Constructions of Mental Models

In Investigation 1, Battista (1999) defined mental models as, “… nonverbal recall-of-experience mental visions of situations; they have structures isomorphic to the perceived structures of the situations they represent (Battista, 1994)” (p. 418). In following sections, I will discuss various aspects of the mental models that students constructed for cube-package problems.

Mental models versus pictures: How much detail is in mental models.

Current research on mental models has found that they often lack much of the pictorial detail of the objects they represent (Held et al., 2006). In general, the difference between mental models and pictures, is that pictures are images that are fixed in time with a lot of detail, whereas mental models of objects are more general mental visualizations of the objects that include only the detail needed for current tasks. That is, the amount of
pictorial detail that is included in a student’s mental model depends on the purpose that the student is using the mental model for. For example, if a student is spatially iterating cube-packages that fit into a box, the packages are often represented as spatial tokens without all the drawing detail of individual cubes (Held et al., 2006). In that case, imagining all the detail from a picture in one’s mental model would place a high cognitive demand on a person’s working memory, making it very difficult to do any type manipulation freely in one’s mental model.

In contrast, when constructing mental models for cube-package enumeration problems students use individual squares on the box to indicate locations of individual cubes, and arrays of squares on the box are used to indicate locations of one-layer thick arrays of cubes. So when students are locating packages in a box they pay more attention to the pictorial detail of what the faces of packages look like in order to locate a corresponding array of squares on one of the sides of the box.

The results of this study indicate that there are three types of mental models used by the students when dealing with package problems: layer based, non-layer composite-unit based, and non-composite-unit based.

**Layer based mental models.** In this investigation, a layer is defined as a composite unit that matches one of the six sides (faces) of the box. In layer based mental models, students structured the packages in the box in terms of layer composites. However, there are three different approaches that students took to construct a layer based mental model. One approach was how P constructed his mental model for Package B in Box 1. In that episode, P first located eight Package B’s on the bottom of Box 1 and stated it as his initial prediction (Figure 4.1a). However, after listening to N describe his
strategy, P realized that he was not considering the height of Box 1 and that he could locate more packages on top of the bottom layer of eight packages. As P extended his mental model to consider the height of Box 1, he conceptualized the eight Package B’s as a horizontal layer composite (Figure 4.1c). As a result, P recognized that he could fit two additional layers of eight Package B’s on top of his bottom layer (Figure 4.1d). In this case, P was not approaching the construction of his mental model with a layer structuring in mind, but rather it was a result of situational thinking based on how he adapted his mental model to consider the height of Box 1.

Figure 4.1. P’s layer based mental model for Package B’s in Box 1.
However, from that point on P approached constructing mental models for the other packages in a slightly different way. In those cases, P approached constructing mental models with a layer structuring strategy in mind from the beginning, rather than noticing that he could fit additional horizontal layer composites of packages after he had found the number of packages that fit on the bottom of the box. Therefore, P’s mental model for these cube-package problems was constructed by finding how many packages are in a layer and then finding how many layers fit into the box. Also, instead of iterating each individual package in a layer, students sometimes used other composites of packages (e.g. row or column etc.) to form a layer composite, as shown in Figure 4.2.

*Figure 4.2. P’s layer based mental model for Package D’s in Box 1.*
Non-layer composite-unit based. Another type of mental model constructed by students for cube-package problems was the non-layer composite-unit based mental model. In a non-layer composite-unit based mental model a student explicitly uses composites of packages other than layer composites. An example of this type of mental model was found in J’s prediction for the number of Package A’s in Box 1. In this episode, J visualized two column composites of three Package A’s fitting in Box 1. He then connected his structuring to the numeric expression of $3 \times 2$ to get his prediction that six Package A’s fit into Box 1 (see Figure 4.3).

Figure 4.3. J’s conceptualization of Package A’s in Box 1.

Students can also form larger composites of packages out of many smaller composites of packages when they construct a non-layer composite-unit based mental
model. M used this approach when he constructed a non-layer composite-unit mental model to check his prediction for Package B’s in Box 1. M first positioned four Package B’s with the given orientation on the back half of the bottom of Box 1 (Figure 4.4a). He then conceptualized the four Package B’s as a composite of packages (Figure 4.4b) and spatially iterated that composite two additional times along the height of Box 1 (Figure 4.4c). He connected this structuring of three rows of four Package B’s to the numeric expression 4 times 3 to get 12 Package B’s to form a another larger composite of packages (Figure 4.4d). M then did the same processes for the second row of four Package B’s that fit into the front half of bottom of Box 1 and ended up with two vertical halves of 12 Package B’s fitting into Box 1 (Figure 4.4e). He added the two vertical halves of 12 Package B’s together and got an accurate solution of 24 Package B’s fit inside Box 1. Note that once M constructed the composite shown in Figure 4.4d, he did not iterate this composite; instead, he reconstructed it.
Non-composite-unit based. The last type of mental model that students constructed in this investigation was a non-composite-unit based mental model (we are talking about composites of packages—each package is actually a composite unit). For this mental model a student iterates individual packages throughout the whole box instead of structuring packages using composite units. However, students can use different representations of the packages when constructing this mental model. In this investigation, we saw three ways students represented and visualized packages when constructing non-composite-unit based mental models. These three ways are three-dimensional tokens, package locations, and three-dimensional flattened tokens.

One way that packages are visualized is using a three-dimensional glob or spatial token that fills the position of the package it represents. J used three-dimensional spatial tokens when he constructed his initial mental model for Package E’s in Box 1. First, J
grabbed a concrete Package E and reoriented it by rotating its given orientation by 90° about the z-axis. While looking at the picture of Box 1, J took the reoriented Package E and did 12 iteration motions as shown in Figure 4.5 below.

![Figure 4.5](image)

*Figure 4.5. J’s iteration motions with a concrete Package E while looking at the picture of Box 1.*

Thus, J positioned one Package E on the bottom of Box 1 and then conceptualized it as a three-dimensional token where he visualized taking the three-dimensional token of Package E and repeatedly spatially iterating it for all positions that Package E’s fit into Box 1 (see Figure 4.6 below). In order to remember which positions of Package E he already has iterated in his mental model, N needed to be visualizing some kind three-dimensional token for each position. Additionally, as J spatially iterated the three-
dimensional tokens of Package E, he was also enumerating its number of positions and got a total of 12 Package E’s to fit in Box 1.

When enumerating individual packages using a non-composite-unit based mental models, students often represented packages with two-dimensional regions that indicated package locations. Many students who used this representation for packages iterated only the locations of packages where explicit arrays of squares appeared on the box sides. N used this representation of packages when he constructed his mental model for Package C’s in Box 2. N first located the 2x2 array of squares that appears on the bottom orthogonal view of Package C on the array of squares appearing on the bottom of Box 1.
(Figure 4.7a). As a result, N found that six Package C’s are located on the bottom of Box 2. N then recognized that these six Package C’s located on the bottom of the box will also account for the three bottom horizontal rows of squares that appear on the back and left sides of Box 2 (Figure 4.7b). N reoriented Package C by rotating it 90° about the y-axis, and located two Package C’s on the top left side of Box 2 (Figure 4.7c). N reoriented Package C again by a rotation of 90° about the z-axis and located one Package C on the top back side of Box 2 (Figure 4.7d). N counted the number of Package C’s he has located on the sides of Box 2 and got a prediction that nine Package C’s fit into Box 2.

*Figure 4.7. N’s construction of his non-composite-unit based mental model for Package C’s in Box 2.*
The last representation of packages that students’ used to construct non-composite-unit based mental models is three-dimensional flattened tokens. These tokens are visualized as three-dimensional flattened globs that are explicitly connected to the locations of packages on the inside sides of the box. N used this representation of packages when he constructed his mental model for Package A’s in Box 2. N first located six Package A’s on the bottom of Box 2 (Figure 4.8a), and then recognized that the two horizontal rows of squares on the back and left sides of Box 2 are accounted for by the six Package A’s that he located on the bottom of Box 2 (Figure 4.8b). N was then able to visualize the positions of Package A’s as three-dimensional flattened tokens on the bottom of Box 2 (Figure 4.8c). Then he located two Package A’s on the top back side and two Package A’s on top left side of Box 2 as shown in Figure 4.8d. N visualized the three-dimensional flattened tokens of the positions of four Package A’s that he positioned on top of the six Package A’s on the bottom of Box 2 (Figure 4.8e). He enumerated the three-dimensional flattened tokens of Package A’s, which resulted in a prediction that 10 Package A’s fit into Box 2.
Thus what evidence was there that N used a 3D flattened token instead of the other two representations of packages when constructing his mental model? At first glance it might even seem that he could actually have used one of the other two representations of packages to construct his non-composite-unit based mental model. However, there are some subtle details in the construction of his mental model that provide evidence that he was not using the other two representations of packages. First, if N would have used three-dimensional tokens it seems likely that he would have noticed after he constructed his mental model that he was missing two Package A’s on the right side of the top layer of packages. As shown in Figure 4.9 below, it is much easier to
visualize the missing two packages on the right side of the top layer if N is using three-dimensional tokens rather than using three-dimensional flattened tokens. He was not using 2D locations because in the episode N explicitly explained to P that the 2x2 array of squares on the back left corner of Box 2 coordinated with the same package that he already located on the top left corner of the back side of Box 2 (see Figure 4.10 below). This showed that N positioned tokens of Package A that had some 3D depth, therefore eliminating the possibility of using 2D locations to construct his mental model.

*Figure 4.9. N’s conceptualization of Package A’s in Box 2 using three-dimensional tokens and three-dimensional flattened tokens.*
Common Strategies Students use to Solve Cube-Package Problems

There are three common strategies that we saw students use in this investigation:
using arrays of squares as indicators of packages, iterating composite units of packages, and utilizing spatial relationships between packages.

Using arrays of squares as indicators of packages. Students repeatedly used arrays of squares as indicators for packages or composites of packages to enumerate and visualize package locations and positions. When students did locating processes of packages they used the arrays of squares that appear on the sides of the box and located the orthogonal views of the packages on the sides of the box. However, when it came to positioning packages inside the box, students used arrays of squares as indicators in two
different ways. One involved visualizing the 3D spatial tokens of the positions of packages or composites of packages and the other is where students used the arrays of squares as symbols of where the package is positioned inside the box. The best way to describe these two possibilities is by using a specific example.

When enumerating Package B’s in Box 1, N reoriented Package B to a vertical orientation and positioned it in Box 1. After he positioned that one Package B he pointed to and counted 24 individual squares on the bottom of the box and stated that 24 Package B’s fit into Box 1. One hypothesis of what N was doing is that he conceptualized Package B as a 3D spatial token, then visualized spatially iterating the token on the individual squares on the bottom of Box 1 (see Figure 4.11). In order to do this type of visualization, N would have to be at the interiorized level of abstraction because he mentally manipulated or operated on his mental model of Package B by spatially iterating it in its proper positions in Box 1.
The second possibility is that N did not iterate the 3D spatial tokens of Package B, but rather took each square on the bottom of the box as an indicator or symbol for where a vertical Package B is positioned. In discussing levels of abstraction, Battista (2007), following von Glasersfeld (1995) described how words, inscriptions, and sentences, can reach “symbol” status in which they "can act as substitutes for, or pointers to" these objects or operations, with individuals reasoning meaningfully about these objects or operations "without having to re-present the actual operations for which they stand" (p. 860). So, the hypothesis is that when N pointed to squares on the bottom of the box, the
squares acted as symbols for the operation of imagining positioning Package B’s on the squares.

**Iterating composites units of packages.** Another reasoning strategy that students used frequently is forming and iterating composite units of packages. Conceiving these composites of packages makes students’ mental models less complex in terms of what they need to store in their working memory. For example, a student who constructs a mental model of where they spatially iterate all the individual Package B’s that fit into Box 1 has to remember the positions and enumerations of Package B’s in his working memory (see Figure 4.12a). As the boxes get larger and can hold more packages, remembering all these positions places a higher cognitive demand on students’ working memory. On the other hand, if the student uses layer composites to construct their mental model of Package B’s in Box 1, they only have to find the number of Package B’s in one layer then find how many layers fit into Box 1 (see Figure 4.12b). There is significantly fewer spatial and numeric components to remember when using composites units of packages when compared to iterating each individual package.
Utilizing spatial relationships between packages. The last common reasoning strategy was that students saw a spatial relationship between a previously enumerated package and a new package and used that relationship to enumerate the number of new packages. In this strategy, students modified an existing mental model of structuring a previously enumerated package to make a new structuring of the other package. Here are two different examples of how students modified an existing mental model for one package to a new structuring for a new package using spatial relationships between the two packages.

The first example was J’s use of a spatial relationship between Package A and Package C in order to find the number of Package C’s in Box 1. In an earlier episode, J correctly predicted and properly structured two column composites of three Package A’s in Box 1. When J was predicting the number of Package C’s that fit into Box 1, he
recognized that Package C had one more horizontal layer of four cubes added to Package A (see Figure 4.13 below). Therefore, J modified his mental model of Package A’s for Package C’s by conceptualizing that Package C had the same enumeration as Package A (see Figure 4.14 below).

Figure 4.13. A spatial relationship between Package A and Package C.

Figure 4.14. J used a spatial relationship between Package A and Package C to construct his mental model for Package C’s in Box 1.
The second example was M's use of a spatial relationship between Package C and 
Package D. For the Package C’s in the Box 1 episode, M conceptualized that Package C 
could be viewed as a composite unit of three Package D’s. As a result, M recognized that 
he could use the structuring of Package C’s in Box 1 for his structuring of Package D’s in 
Box 1. M saw that each Package C’s contained three Package D’s, so he used the 
numeric expression of 6 times 3 and got a total of 18 Package D’s fit into Box 1. M and J 
are both modifying previously constructed mental models to form new ones for other 
packages.

*Figure 4.15. A spatial relationship between Package D and Package C.*
Common Errors and Cognitive Obstacles

When trying to understand student reasoning it is important to determine what common errors students make and what cognitive obstacles students face. This is specifically important for teachers because if they can recognize and understand students’ errors they are more likely to be able to find ways to address those errors through instruction. In this investigation, I found four common errors and/or cognitive obstacles that students dealt with in cube-package problems. They are: fixation on the given orientation of a package, losing track of the units, inability to deal with lack of specific pictorial indicators of packages, and indicators of packages losing their indicating function.

**Fixation on the given orientation of a package.** One major cognitive obstacle that students faced with cube-package problems was when the problem required students to use a different orientation of the package other than the orientation originally given in the problem statement. In addition, some cube-package problems required students to fill

\[ \text{Figure 4.16. M used a spatial relationship between Packages C and D to modify his mental model for Package C’s to obtain a mental model for Package D’s in Box 1.} \]
the box with packages using a mixture of different orientations in order to get correct enumerations of packages. Thus for package problems that require some sort of reorienting process, if a student was fixating on one orientation of a package, they would be unable to predict the correct enumeration of packages in the box. In this investigation, we saw P and M make this error when they predicted Package C’s in Box 2 and P made the error again when he predicted the number of Package E’s in Box 2.

For example, enumerating Package C’s in Box 2 required students to use both vertical and horizontal orientations of Package C’s in their structuring. At first, both P and M positioned Package C’s using only the given orientation of Package C in Box 2 (see Figure 4.17a). Neither boy explicitly considered reorienting Package C in order to position four more Package C’s on top of the first layer (see Figure 4.17b). It is important to point out that all previous package problems could be enumerated correctly if the student structured the packages using a uniform orientation. Therefore, P and M both seem to have been fixated on using a uniform orientation during their positioning processes because that was what got them correct enumerations for previous package problems.
Figure 4.17. Comparing P and N’s structuring of Package C’s in Box 2 to a proper structuring of Package C’s to fill Box 2.

**Losing track of the units.** Another common error that we saw students make in this investigation was losing track of the unit of enumeration (i.e. packages) and reverting back to using cubes as units. We saw both M and N make this error when they predicted the number of Package E’s that fit into Box 1. In both cases, P and M were able to construct a proper spatial structuring of Package E’s in Box 1; however, when they enumerated Package E’s they enumerated the number of cubes contained in the Package E’s instead (see Figure 4.18). I hypothesize that there are three possible reasons why both of these boys lost track of the units of enumeration during their predictions.
One reason is when the boys’ tried to structure Package E’s in Box 1, it put such a high cognitive demand on each boy’s working memory that it caused them to lose track of the units of enumeration, Package E’s. The structuring of Package E’s was a little more challenging for students than the structuring of previous packages because it was the first time that the packages did not completely fill the bottom layer in Box 1. Therefore the students had to adapt their mental models to allow for empty space on the bottom of the box, which can result in a higher cognitive demand on their working memory.

Another possible reason is that both boys reverted back to their well-established schemes for single-cube enumeration problems from Investigation 1. They had strong understandings and familiarity with using these schemes, which makes it easy to apply them to problems that seem similar. Because of the similarity of the problems, the more familiar and stronger cube enumeration schemes get accidentally activated.
It is also possible that the reason N and M lost track of the unit of enumeration is actually a combination of the two previously stated reasons. There is no way of determining for sure which possibility is the reason why M and N lost track of the unit of enumeration in their conceptualization of Package E’s in Box 1. However, seeing this error occur in two separate cases with two different boys suggests that this error might be one of the more common errors students have in cube-package problems. However, further empirical research needs to be done with a much larger sample to test the frequency of this error among students in order to know whether this error is actually a common one among students.

**Inability to deal with lack of specific pictorial indicators of packages.** As stated earlier students used arrays of squares as pictorial indicators of locations and positions of packages that fit into a box. However, some students who counted the individual packages by ones positioned only the packages where they could locate an explicit array of squares on the box sides. This often results in students missing positions of packages that are located on sides where explicit visual representations of squares are absent. An example of this error was in N’s prediction for Package A’s in Box 2.

For Package A’s in Box 2, N was able to properly position six Package A’s on the bottom of Box 2. Then N properly located and positioned two Package A’s each on the back and left sides of Box 2 as shown in Figure 4.19 to get his prediction of 10. However, N made an error in that he missed positioning two Package A’s on the front and right sides of Box 2. We saw from N’s positioning processes that he was only positioning packages where there was an explicit 2x2 array of squares appearing on a side of the box. As a result, since the front and right sides of the picture of Box 2 do not have
any explicit arrays of squares appearing on them, therefore there are no pictorial indicators for N to use to position the two packages that he missed on those sides. This suggests that students who position individual packages using squares as pictorial indicators often do not see packages that do not have explicit pictorial indicators that they can attribute the packages to.

![Diagram](image)

Figure 4.19. N’s positioning for Package A’s in Box 2.

**Indicators of packages losing their indicating function.** A cognitive obstacle that some students face with using squares as indicators of packages is when the student loses track of the objects (i.e. packages) that the squares represent. In this investigation, we not only saw this cognitive obstacle as difficult for some students to overcome, but also resulted in students making errors in their predictions. There are two examples that exhibit this obstacle.
The first example was N’s prediction for Package C’s in Box 2. In this episode, N initially predicted that nine Package C’s fit into Box 2. Recall, N located six Package C’s with its given orientation on the bottom of the box. It is apparent that N clearly used 2x2 arrays of squares as symbols for the positioning of Package C’s. He then located three more Package C’s at 2x3 arrays of squares on the top two rows on the back and left sides of Box 2 as shown in Figure 4.20. Unlike with the six Package C’s that he located on the bottom, N lost track that the locations of Package C’s that he located in second layer represented positions of Package C’s. Thus, N counted the locations of Package C’s in the second layer without coordinating them to see if positions of Package C could actually fit there. As a result, N made an error in his reasoning because he did not recognize that if he positioned Package C’s at the locations he determined, he would have two packages overlapping in the back left corner of the box (see Figure 4.21).

Figure 4.20. N’s prediction strategy for Package C’s in Box 2.
The second example of this cognitive obstacle occurred in N’s initial prediction strategy for Package E’s in Box 2. In that episode, N initially predicted 24 Package E’s fit into Box 2 by counting the individual squares that appeared on the bottom of Box 2. N used the squares on the bottom of box as symbols for the positions of Package E with a vertical orientation. However, when P questioned N’s prediction strategy, N either lost track of what the squares he was counting represented or never fully conceptualized the squares as symbols for vertically oriented Package E’s. This caused N to discard his correct prediction of 24 Package E’s to adopt P’s incorrect prediction of 20 Package E’s fit into Box 2.
Chapter 5: Conclusions and Instructional Implications

The purpose of this study was to develop a better understanding of student reasoning about cube-package enumeration problems, with the hope that appropriate descriptions of this research will help teachers and curriculum developers construct and implement more effective instruction on volume ideas and spatial reasoning. This study illustrates in detail the mental processes that students have to do in order to solve package problems and the cognitive obstacles they face while doing these problems. For teachers, if they have a deep understanding on how students think and struggle with cube-package problems, they become better in guiding student learning about these problems. More specifically in an inquiry-based classroom, having this type of knowledge of student thinking enables teachers to ask the right type of questions and to give the right kind of representations that will help students progress in their reasoning about package problems.

Battista’s conclusions on problem-centered inquiry-based learning from his 1999 article about cube enumeration problems are supported by this study’s findings. Like that study, this study shows that students often initially make mistakes and struggle to make sense out of ideas when using this instructional approach (Battista, 1999). Teachers need to be aware that a natural component of inquiry-based learning is that students often are grappling to make sense or even sometimes get a little confused as they work through the
material (Battista, 1999). This is part of the students’ learning process that often leads to meaningful learning for the students. Therefore, it can be difficult for teachers who are using inquiry-based learning to know when it is appropriate to provide assistance to students and when to let them work things out for themselves (Battista, 1999). In order for teachers to know when to intervene with a student in an inquiry-based environment, the teachers need to have a deep understanding of how to implement this type of instruction and to understand how students reason and struggle with the mathematical concepts being taught (Battista, 1999). This study provides teachers with information about the latter part of that statement specifically for cube-package enumeration problems.

Additionally, this study shows that students who are in an inquiry-based environment who get the chance to attempt, reflect, and verify their ideas often develop increasingly sophisticated reasoning. However, learning a complex idea like spatial structuring takes time to develop; allowing students time to explore their ideas is critical for them to develop a deep understanding of the material. For example, we saw that M had to go through a long process of using some less sophisticated ways of structuring packages before he started using a more sophisticated structuring. A general trend that occurs in both Battista’s 1999 article and this study is that students are able develop valid strategies that give accurate predictions; however, these strategies are not always the most efficient strategies to use in that situation. In fact, some of these less efficient prediction strategies are sometimes very complex, which makes them difficult to transfer to more difficult problems. So even though using these complex strategies might be getting the students correct answers, teachers must keep in mind the most efficient strategies and how to encourage students to move in that direction. If teachers do not
encourage students to understand and use more efficient strategies, when it comes time to connect student thinking to mathematical formulas like the volume formula, these students will be unable to do so.

Another major instructional implication that is highlighted in this study is that it is important to the development of students’ spatial reasoning that teachers give students problems in which they use visualization to make predictions, and then check their visualizations and predictions using concrete materials. Instead of using concrete materials to initially solve package problems, I think it is more productive for students to make a prediction first and then check their prediction using concrete materials. In this study, the students were first asked to look at a picture of a box and a package to predict the number of packages that fit into the box. Having students first predict the number of packages that fit into a box forces the students to create and use visualizations (i.e. mental models) of the problem. A major conceptual component of spatial reasoning of package problems is being able to visualize how the packages fit into the box, absent of concrete materials. In addition, we want students to be able to do spatial structuring and volume measurement of objects using representations other than using concrete materials. In the real world, students will not always have physical representations to use when they are the solving volume measurement and/or spatial structuring problems, therefore being able spatially visualize what is occurring in these problems becomes critical.

Having students use concrete materials to check their predictions not only helps students validate their mental models when they are correct, but also can reveal to them when errors and misconceptions occur in their reasoning. For example, N struggled with visualizing positions of packages when there were no explicit pictorial indicators on some
of sides of the box. However, when N checked his predictions with concrete materials he was able to see what packages he was missing. Therefore, one way to help students who are struggling to visualize spatial concepts is to have them work with concrete materials to physically see and reflect on the spatial items they struggle with.

Another way that using visual materials can help instruction is by making communication easier for both the teacher and the students because they have something to visually reference during explanations. Spatial structuring and volume are difficult mathematical topics for students, teachers, and even mathematicians to communicate about because of the complexity that goes along with trying to describe three-dimensional objects. So one viable approach that eases the complexity of being able to describe working with 3D objects for teachers and students is giving them visual materials that they can reference and point to during their explanations. In fact, in this investigation there have been many instances when a student pointed to the visual representations of the packages and boxes (whether it be pictorial or concrete) to help him explain his reasoning to another person. A piece of technology that can help teachers during class discussions is a document camera. These cameras allow the whole class to see students demonstrate their prediction strategies using visual materials.

This study concludes that problem-centered inquiry-based instruction, like the instructional activities used in this study, offers teachers an effective approach to teaching students about spatial concepts in mathematics. It also recommends that having students predict then check with visual materials in instruction is critically important when developing students’ spatial reasoning. Overall, the knowledge of students’ mental processes and the cognitive obstacles students face in package problems presented in this
study will help teachers and curriculum developers design instruction and curricula that will further benefit and maximize student learning about solving cube-package enumeration problems.
References


Appendix A: Student Sheets
Student Sheet 3a: How Many Packages?

1. How many packages of each size do you predict will fit in Box 1? You may not break packages apart. Predict, then make the box and fill it with cube-packages to check your predictions. Record your predictions and your actual findings in the table below.

<table>
<thead>
<tr>
<th>Packages</th>
<th>Prediction</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Student Sheet 3b: How Many Packages?**

2. How many packages of each size will fit in Box 2? Predict, then make the box and check. Record your predictions and your actual findings in the table.

**Packages**

<table>
<thead>
<tr>
<th>Packages</th>
<th>Prediction</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Box 2**

4 by 6 cubes on the bottom
5 cubes high

<table>
<thead>
<tr>
<th>A</th>
<th>Prediction</th>
<th>Actual</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>