Completeness of the Predicate Calculus in the Basic Theory of Predication

THESIS

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Abstract

In the first part of the thesis, we present the Basic Theory of Predication as elaborated by Harvey Friedman in [4] and [3]. Within the Basic Theory of Predication, we develop arithmetic and the basic semantic notions for the predicate calculus. The domains of the structures for the predicate calculus are unrestricted. That is, the quantifiers of predicate calculus are interpreted as ranging over the universe of the metatheory, the Basic Theory of Predication.

In the second part of the thesis, we outline the proof of a completeness theorem for the predicate calculus. According to the theorem, on the assumption that the universe of the metatheory is linearly ordered, every set of sentences consistent with infinity is satisfiable.
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# Contents

Abstract

Acknowledgements

Vita

1 Arithmetic and Semantics in the Basic Theory of Predication
   1.1 The Basic Theory of Predication
   1.2 Arithmetic in BTP
   1.3 Remarks
   1.4 Finite sequences
   1.5 Semantics
   1.6 Schröder-Bernstein Theorem

2 Outline of the Completeness Theorem
   2.1 Stage 1
   2.2 Stage 2
   2.3 Stage 3
   2.4 Stage 4
Chapter 1

Arithmetic and Semantics in the Basic Theory of Predication

1.1 The Basic Theory of Predication

In this section we present the Basic Theory of Predication (BTP henceforth) as elaborated by Harvey Friedman in [4] and [3]. The theory will be introduced fully formally and will be used later as a metatheory for the predicate calculus to prove a completeness theorem. The proof will rely on the assumption that the universe of the metatheory is linearly ordered.\(^1\)

The language of BTP comprises the following symbols:

A. Constant(s): 0.
B. Variables:

\(^1\)For further developments in the absence of this assumption, see [4] and [3].
B1. Individual variables with or without numerical subscripts: \( x, y, z, \ldots, x_1, x_2, \ldots, y_1, \ldots \);

B2. Pure predicative variables with the superscript \( p \) (for "pure"), with or without numerical subscripts: \( P^p, Q^p, \ldots, P_1^p, P_2^p, \ldots, Q_1^p, \ldots \);

B3. General predicative variables with the superscript \( g \) (for "general"), with or without numerical subscripts: \( P^g, Q^g, \ldots, P_1^g, P_2^g, \ldots, Q_1^g, \ldots \);

C. *Logical symbols*: \( \neg, \land, \lor, \rightarrow, \leftrightarrow, \exists, \forall, =. \)

D. *Function symbol(s)*: ( , ).

We will use \( u, v, \ldots \), as metavariables ranging over individual variables of BTP, \( s \) and \( t \) as metavariables ranging over terms (defined below), \( \Pi, \Pi', \ldots \), as metavariables ranging over pure predicative variables, and \( \Gamma, \Gamma', \ldots \), as metavariables ranging over general predicative variables. However, we will not be consistent in this use. When there is no risk of confusion, we will often have particular variables of BTP play the role of metavariables. Moreover, we will drop the superscripts when it is clear from the context which kind of predicative variables is intended. By ‘predicate’ we will mean a predicative variable or what a predicative variable ranges over.

A *term* is defined by the following clauses:

(1) Every individual variable is a term, and so is 0;

(2) If \( s \) and \( t \) are terms, then \((s, t)\) is a term.

Now we define the notion of *atomic formula* of BTP:

(3) if \( s \) and \( t \) are terms, then \( s = t \) is an atomic formula;
(4) if $t$ is a term, $\Pi$ is a pure predicative variable, and $\Gamma$ is a general predicative variable, then $\Pi(t)$ and $\Gamma(t)$ are atomic formulas.

An atomic predication in BTP is always monadic.

Finally, we define the notion of formula of BTP and of free occurrences of individual and predicative variables in a formula:

(5) if $\alpha$ is an atomic formula, then $\alpha$ is a formula and every occurrence of a variable (individual or predicative) in it is free;

(6) if $\alpha$ is a formula, then $\neg\alpha$ is a formula where any occurrence of a variable (individual or predicative) is free if and only if it is free in $\alpha$;

(7) if $\alpha$ and $\beta$ are formulas, and $\ast$ is one of $\land$, $\lor$, $\rightarrow$, $\leftrightarrow$, then $(\alpha \ast \beta)$ is a formula where any occurrence of a variable (individual or predicative) is free if and only if it is free in $\alpha$ or $\beta$;

(8a) if $u$ is an individual variable and $\alpha(u)$ is a formula with at least one free occurrence of $u$, then $(\exists u)\alpha(u)$ and $(\forall u)\alpha(u)$ are formulas where any occurrence of a variable (individual or predicative) is free if and only if it is free in $\alpha(u)$ and it is not an occurrence of $u$;

(8b) if $\Pi$ is a pure predicative variable and $\alpha(\Pi)$ is a formula with at least one free occurrence of $\Pi$, then $(\exists \Pi)\alpha(\Pi)$ and $(\forall \Pi)\alpha(\Pi)$ are formulas where any occurrence of a variable (individual or predicative) is free if and only if it is free in $\alpha(\Pi)$ and it is not an occurrence of $\Pi$;

(8c) if $\Gamma$ is a pure predicative variable and $\alpha(\Gamma)$ is a formula with at least one free occurrence of $\Gamma$, then $(\exists \Gamma)\alpha(\Gamma)$ and $(\forall \Gamma)\alpha(\Gamma)$ are formulas where any occurrence
of a variable (individual or predicative) is free if and only if it is free in $\alpha(\Gamma)$ and it is not an occurrence of $\Gamma$;

BTP is formulated here in a two-sorted first-order language. The notation adopted, with formulas such as $(\exists P^p)P^p(x)$, resembles the usual one for second-order languages. We could avoid this effect by introducing a symbol $\eta$ for predication and by expressing atomic predication as $t \eta \Pi$ and $t \eta \Gamma$. For the sake of notational economy, however, we will not use a distinguished symbol for predication. Other usual conventions to facilitate readability will be in place, such as the elimination of parentheses or the use of English paraphrases. Occasionally, we will read $P^p(x)$ as ‘$P^p$ holds of $x$’ or ‘$x$ is in $P^p$’.

The axioms of BTP are the following:

I. The usual classical axioms concerning the logical symbols;

II. *Extensionality of Pairing*: $(\forall x)(\forall y)(\forall z)(\forall w) ((x, y) = (z, w) \iff (x = z \land y = w));$

III. *Zero*: $(\forall x)(\forall y) 0 \neq (x, y);$ 

IV. *Pure Comprehension*: $(\exists P^p)(\forall x)(P^p(x) \iff \phi(x))$, where $\phi(x)$ is a formula in which no individual variable other than $x$ occurs free and, moreover, neither $P^p$ nor any general predicative variable occurs free;

V. *General Comprehension*: $(\exists P^g)(\forall x)(P^g(x) \iff \phi(x))$ where $\phi(x)$ is a formula in which $P^g$ does not occur free.

The distinction between pure and general comprehension concerns the use of individual parameters in the comprehension formula $\phi(x)$. Notice that Pure Comprehension

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2Parentheses will be eliminated even in cases of predication. For example, we will typically write $P^p(x, y)$ instead of $P^p((x, y))$. 

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4
permits the use only of pure predicative parameters (not general ones) in the comprehension formula. By contrast, General Comprehension permits the use of pure or general predicative parameters in the comprehension formula. Since quantification over predicates is allowed in \( \phi(x) \), both comprehension schemas are to be regarded as impredicative. The type distinction between predicates and objects avoids Russell’s Paradox. There is no identity for predicates so uniqueness with respect to predicates cannot be expressed. For this reason, we may assume that identity and uniqueness with respect to predicates is to be understood in terms of coextensiveness.

We can distinguish three Basic Theories of Predication: a pure one, a general one, and an extended one including both the pure and the general one. The pure BTP (BTPp henceforth) is characterized by axioms I–IV restricting the language to exclude general predicates. The general BTP (BTPg henceforth) is characterized by axioms I–III and IV, restricting the language to exclude pure predicates. The extended one (BTPpg henceforth) is the union of BTPp and BTPg, with no restriction on the language.

Our main goal is to prove a version of the completeness theorem for the predicate calculus in BTPp. However, a detour into BTPg will prove useful at one point. In what follows, unless otherwise noted, we will be working in BTPp.

1.2 Arithmetic in BTP

We will now develop arithmetic within BTPp. The development within BTPg is easier to carry out, since the comprehension axiom of BTPg is less restrictive.

We first deal with successor. For every \( x \), \( S(x) \) will abbreviate \((x, 0)\).

**Theorem 1.** \( \forall x \) \( 0 \neq S(x) \).
Proof. Immediate by the Zero axiom.

Theorem 2. \((\forall x)(\forall y) (S(x) = S(y) \rightarrow x = y)\).

Proof. Since \(S(x) = (x, 0)\) and \(S(y) = (y, 0)\), by the extensionality of pairing \((x, 0) = (y, 0)\) entails \(x = y\).

Let us say that a pure predicate \(\Pi\) is \textit{inductive} if it satisfies the formula

\[
\Pi(0) \land (\forall x)(\Pi(x) \rightarrow \Pi(S(x))).
\]

It is easy to show that there is an inductive predicate.

**Proposition 1.** \((\exists P^p) P^p\) is inductive.

Proof. Pure Comprehension implies the existence of a universal pure predicate:

\[
(\exists P^p)(\forall x)(P^p(x) \iff x = x).
\]

Such a predicate is universal. Thus, it is inductive.

Now we show that there is a minimal inductive predicate.

**Proposition 2.** There is a pure predicate \(P^p\) such that

\[
P^p\ is\ inductive \land (\forall x)(\forall Q^p)((Q^p\ is\ inductive \land P^p(x)) \rightarrow Q^p(x)).
\]

Proof. By Pure Comprehension there is a predicate \(P^p\) such that for all \(y\),

\[
P^p(y) \iff (\forall Q^p)(Q^p\ is\ inductive \rightarrow Q^p(y)).
\]

This is a legitimate instance of Pure Comprehension since no object parameter is used in the comprehension formula. Now it is immediate to verify that \(P^p\) is inductive, hence \(P^p\) satisfies the condition of the Proposition.
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

Since we cannot express identity between predicates, we cannot hope to identify a unique predicate to play the role of $\mathbb{N}$. However, all minimal inductive predicates are coextensive. So we can understand being in $\mathbb{N}$ as being in any minimal inductive predicate, without having to think of $\mathbb{N}$ as a particular predicate. Thus, we introduce $\mathbb{N}$ as an abbreviation:

$$N(x) \iff \text{there is a minimal inductive predicate } P^p \text{ such that } P^p(x).$$

For convenience, we will adopt the traditional set-theoretic notation and write $x \in \mathbb{N}$ for $N(x)$. This should not be understood as suggesting a set-theoretic reading of BTP. It should just be understood as a notational expedient.

Our next goal is to prove an analogue of the principle of induction on $\mathbb{N}$ within BTP.

**Theorem 3 (Induction).** Let $\phi(x)$ be a formula of BTP where no variable other than $x$ occurs free. Suppose that $\phi(0)$ and suppose that for every $x \in \mathbb{N}$, if $\phi(x)$ then $\phi(S(x))$. Then for every $x \in \mathbb{N} \phi(x)$.

**Proof.** Assume that $\phi(0)$ and that for every $x \in \mathbb{N}$ if $\phi(x)$ then $\phi(S(x))$. Since $\phi(x)$ has only $x$ occurring free, and similarly for the formula abbreviated by $x \in \mathbb{N}$, we can apply pure comprehension and obtain:

$$(\star) \quad (\exists P^p)(\forall x)(P^p(x) \iff x \in \mathbb{N} \text{ and } \phi(x)).$$

Now let $P^p$ be such that

$$(\star\star) \quad (\forall x)(P^p(x) \iff x \in \mathbb{N} \text{ and } \phi(x)).$$

First we show that $P^p$ is inductive. Clearly $0 \in \mathbb{N}$ and $\phi(0)$ by assumption. Also, suppose that $x \in \mathbb{N}$ and $\phi(x)$. Then since membership in $\mathbb{N}$ is inductive, $S(x) \in \mathbb{N}$,
and by assumption we also have $\phi(S(x))$. So $P^p(x)$ is inductive. Suppose that $n \in \mathbb{N}$, by minimality of $\mathbb{N}$, $n$ is in every inductive predicate. Therefore, $P^p(n)$. By (**), we conclude that $\phi(n)$. But $n$ is an arbitrary object in $\mathbb{N}$. Thus, for every $y \in \mathbb{N}$, $\phi(y)$. 

Next, in order to be able to define addition and multiplication on $\mathbb{N}$, we prove a recursion theorem on $\mathbb{N}$. The proof will mirror the usual one in set theory done by approximating functions. To carry out the proof we first need to define analogues of functional concepts in BTP.

Let us say that a pure predicate $\Pi$ is *functional* if

$$\forall x \left( \Pi(x) \rightarrow \exists y \exists z \left( x = (y, z) \right) \right) \land \left( \forall x \forall y \forall z \left( \Pi(x, y) \land \Pi(x, z) \Rightarrow y = z \right) \right).$$

If $\Pi$ is a functional pure predicate, we can define its domain and its range as any pure predicates $\text{dom}(\Pi)$ and $\text{ran}(\Pi)$ that satisfy, respectively, the conditions:

$$\forall x \left( \text{dom}(\Pi)(x) \leftrightarrow (\exists y)\Pi(x, y) \right),$$

$$\forall x \left( \text{ran}(\Pi)(x) \leftrightarrow (\exists y)\Pi(y, x) \right).$$

For any functional pure predicate $\Pi$, the existence of a $\text{dom}(\Pi)$ and a $\text{ran}(\Pi)$ is secured by the following two applications of Pure Comprehension:

$$\exists P^p \left( \forall x \left( P^p(x) \leftrightarrow (\exists y)\Pi(x, y) \right) \right),$$

$$\exists P^p \left( \forall x \left( P^p(x) \leftrightarrow (\exists y)\Pi(y, x) \right) \right).$$

As we observed above, the Pure Comprehension schema allows for the use of pure predicative variables as parameters. So these are legitimate instances of the schema. Moreover, as before, uniqueness of domain and range cannot be expressed but we have that domains of a given predicate are coextensive. For any functional pure predicate
Π and any \( x \) in its range, we denote by \( \Pi[x] \) the unique \( y \) such that \( \Pi(x, y) \). Now we are ready to state and prove the recursion theorem.

**Theorem 4 (Pure recursion on \( \mathbb{N} \)).** For every pure predicate \( P^p \) and term \( a \) without variables such that \( P^p(a) \), and for every functional pure predicate \( Q^p \) with \( P^p \) as its domain and range, there is a functional pure predicate \( R^p \) with domain \( \mathbb{N} \) such that

\[
R^p(0, a) \quad \text{and} \quad \forall n \in \mathbb{N}, \, R^p(S(n), Q^p[R^p[n]]).
\]

**Proof.** Let us define a pure predicate \( \Pi \) to be *acceptable* if the following holds:

(i) \( \Pi \) is functional;

(ii) for every \( x \), if \( \text{dom}(\Pi)(x) \) then \( x \in \mathbb{N} \);

(iii) if \( \text{dom}(\Pi)(0) \), then \( \Pi(0, a) \);

(iv) for every \( n \in \mathbb{N} \), if \( \text{dom}(\Pi)(S(n)) \), then \( \text{dom}(\Pi)(n) \) and \( \Pi(S(n), Q^p[\Pi[n]]) \).

Observe that the following instance of Pure Comprehension is legitimate:

\[
(\exists R^p)(\forall x)(R^p(x) \leftrightarrow (\exists \Pi) (\Pi \text{ is acceptable and } \Pi(x)) ),
\]

since the formula on the right-hand side of the biconditional contains no parameter. This is the case since talk of \( \mathbb{N} \) and talk of \( \text{dom}(\Pi) \) reduces to existential formulas, as seen above. Let \( R^p \) be any witness for \((\ast)\). We must verify that \( R^p \) satisfies the conditions of the theorem, i.e., we must verify that (1) \( R^p \) is functional, (2) the domain of \( R^p \) is \( \mathbb{N} \), (3) \( R^p(0, a) \), and (4) for every \( n \in \mathbb{N} \), \( R^p(S(n), Q^p[R^p[n]]) \).

(1) Functionality is immediate. If \( R^p(x) \), then for some acceptable, hence functional \( \Pi, \Pi(x) \). Thus, \( R^p \) is functional.
(2) Next we need to show that $\mathbb{N}$ is a $\text{dom}(R^p)$, i.e., for every $x$, $x \in \mathbb{N}$ if and only if $\text{dom}(R^p)(x)$. To prove the right-to-left direction, suppose that $\text{dom}(R^p)(x)$. Then $\text{dom}(\Pi)(x)$ for some acceptable $\Pi$. By condition (ii) on acceptability, it follows that $x \in \mathbb{N}$.

To prove the left-to-right direction, we proceed by induction applied to the formula

$$\phi(x) : (\exists \Pi)(\Pi \text{ is acceptable } \land (\exists y)\Pi(x, y)).$$

To show that $\phi(0)$, we invoke an instance of Pure Comprehension to obtain a pure predicate $P^p$ such that

$$(**) (\forall x)(P^p(x) \leftrightarrow x = (0, a)).$$

It is easy to verify that $P^p$ meets conditions (i)–(iv). Thus, $P^p$ acceptable. Therefore, $\phi(0)$.

For the inductive step, assume that $\phi(n)$. This means that for some acceptable pure predicate $\Pi$ there is $y$ such that $\Pi(n, y)$. We want to show that $\phi(S(n))$. By Pure Comprehension we have

$$(***) (\exists P^p)(\forall x)(P^p(x) \leftrightarrow \Pi(x) \lor (\exists y)(\exists z)(\Pi(y, z) \land x = (S(y), Q^p[\Pi[y]])).$$

Let $P^p$ now be any witness for $(***)$. This means that, for every $n \in \mathbb{N}$ such that $\text{dom}(\Pi)$ holds of $n$, $P^p$ extends $\Pi$ with $(S(n), Q^p[\Pi[y]])$. The predicates $Q^p$ and $\Pi$ can appear on the right-hand side of the biconditional as pure pure predicative parameters. So $(***)$ is a legitimate application of Pure Comprehension. In order to show that $\phi(S(n))$, we need to verify that $P^p$ is acceptable. Since $P^p$ extends $\Pi$ in the way described, conditions (i) and (iii) on acceptability are clearly satisfied. Let us verify that the other two conditions are satisfied as well.
As for (ii), suppose that $dom(P_p)(x)$. Then either $dom(\Pi)(x)$ or $x = S(y)$ for some $y$ such that $dom(\Pi)(y)$. If $dom(\Pi)(x)$, since $\Pi$ is acceptable, $x \in \mathbb{N}$. If $x = S(y)$ for some $y$ such that $dom(\Pi)(y)$, $y \in \mathbb{N}$ hence $S(y) \in \mathbb{N}$. So $x \in \mathbb{N}$.

To verify condition (iv), suppose that $n \in \mathbb{N}$ and $dom(P_p)(S(n))$. We need to show that $dom(P_p)(n)$ and $P_p(S(n), Q_p[P_p[n]])$. If $dom(\Pi)(S(n))$, then $dom(P_p)(n)$ and $\Pi(S(n), Q_p[\Pi[n]])$ given that $\Pi$ is acceptable. Because $P_p$ extends $\Pi$, it follows that $dom(P_p)(n)$ and $P_p(S(n), Q_p[P_p[n]])$. So suppose that $dom(\Pi)$ does not hold of $S(n)$. Then, since $dom(P_p)(S(n))$, there is $y$ such that $S(n) = S(y)$, $dom(\Pi)(y)$, and $P_p(S(y), Q_p[\Pi[y]])$. Since $S(n) = S(y)$, $y = n$. Again, since now $dom(\Pi)(n)$ and $P_p$ extends $\Pi$, it follows that $dom(P_p)(n)$ and, because $P_p(S(n), Q_p[\Pi[n]])$, it also follows that $P_p(S(n), Q_p[P_p[n]])$. This shows that (iv) is satisfied.

We have verified that $P_p$ satisfies conditions (i)–(iv). So $P_p$, the witness for (**), is acceptable. Therefore, if $\phi(n)$, then $\phi(S(n))$. This completes the induction on $\phi(x)$, establishing the left-to-right direction of (2). Thus, both directions hold. We conclude that (2) holds, that is, for every $x$, $x \in \mathbb{N}$ if and only if $dom(R_p)(x)$.

(3) The next step in our proof of the recursion theorem is to show that $R_p(0, a)$. This is the case if there is an acceptable pure predicate $\Pi$ such that $\Pi(0, a)$. As we have seen, (***) provides such a predicate.

(4) The last step is then to show that for all $n \in \mathbb{N}$, $R_p(S(n), Q_p[R_p[n]])$. So suppose $n \in \mathbb{N}$. We have shown that (3) holds, that is, for every $x$, $x \in \mathbb{N}$ if and only if $dom(R_p)(x)$. In particular, then, $dom(R_p)(S(n))$. By (**) — the characterization of $R_p$ — it follows that there is an acceptable pure $\Pi$ such that $\Pi(S(n), y)$ for some $y$. Since $\Pi$ is acceptable, condition (iv) entails that $dom(\Pi)(n)$ and $\Pi(S(n), Q_p[\Pi[n]])$. Therefore $R_p(S(n), Q_p[\Pi[n]])$. It remains to show that $\Pi[n] = R_p[n]$. Now by definition $\Pi[n]$ is the unique $y$ such that $\Pi(n, y)$. By the acceptability of $\Pi$ and the characteri-
zation of $R^p$, it follows that $R^p(n, y)$. Then by functionality of $R^p$, $R^p[n] = y$. Hence $\Pi[n] = R^p[n]$. We conclude that $R^p(S(n), Q^p[R^p[n]])$ for every $n \in \mathbb{N}$.

Thus, the pure predicate $R^p$ witnessing $(\ast)$ is the sought predicate, satisfying conditions (1)–(4).

We also have a general version of the recursion theorem in which the term $a$ is arbitrary.

**Theorem 5 (General recursion on $\mathbb{N}$).** For every general predicate $P^g$ and every object $x$ such that $P^g(x)$, and for every functional general predicate $Q^g$ with $P^g$ as its domain and range, there is a functional general predicate $R^g$ with domain $\mathbb{N}$ such that

$$R^p(0, x) \quad \text{and}$$

$$\text{for all } n \in \mathbb{N}, \ R^g(S(n), Q^g[R^g[n]]).$$

**Proof.** The proof is completely analogous to that of the pure version. In the pure version, because parameters are not allowed in Pure Comprehension, $a$ was assumed to be a term without variables. Now, we can relax the requirement and use an arbitrary object $x$. Since General Comprehension permits the use of parameters, $x$ will be taken to be the parameter of the instances of Comprehension invoked in the proof. This is the only difference in the construction of the proof.

By Pure Comprehension we have the existence of a functional successor predicate $S^p$ such that:

$$(\forall x)(S^p(x) \leftrightarrow (\exists y)(y \in \mathbb{N} \land x = (y, S(y)))).$$

Now we show that, for any $m \in \mathbb{N}$, there is a functional predicate with domain $\mathbb{N}$ that amounts to adding $m$ to its argument. This is expressed by the following proposition.
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

**Proposition 3.** For any $m \in \mathbb{N}$ there is a functional pure predicate $R_m^p$ with domain $\mathbb{N}$ such that

\[ R_m^p(0, m) \quad \text{and} \quad \text{for all } n \in \mathbb{N}, \ R_m^p(S(n), S^p[R_m^p[n]]). \]

**Proof.** We proceed by induction. If $m = 0$, the existence of the sought $R_0^p$ is given by the Recursion Theorem with 0 taken as $a$ and $S^p$ taken as $Q^p$. Now suppose, as inductive hypothesis, that there is a pure predicate $R_m^p$ with domain $\mathbb{N}$ such that

\[ R_m^p(0, m) \quad \text{and for all } n \in \mathbb{N}, \ R_m^p(S(n), S^p[R_m^p[n]]). \]

We want to show that there is a pure predicate $R_{S(m)}^p$ such that

\[ R_{S(m)}^p(0, S(m)) \quad \text{and for all } n \in \mathbb{N}, \ R_{S(m)}^p(S(n), S^p[R_{S(m)}^p[n]]). \]

By Pure Comprehension we have that

\[ (\exists R_{S(m)}^p)(\forall x)(R_{S(m)}^p(x) \leftrightarrow (\exists y)(\exists z)(R_m^p(y, z) \land x = (y, S(z)))). \]

Now $R_{S(m)}^p$ is the sought pure predicate, so this completes the induction. \qed

For every $n \in \mathbb{N}$ we will call an \textit{m-addition} any pure predicate coextensive with the one whose existence is asserted in Proposition 3. Using $m$-additions we can define an addition operation.

**Proposition 4.** There is a functional pure predicate $A^p$ satisfying the following condition: for every $x$, $A^p(x)$ if and only if there are $y, z, w \in \mathbb{N}$ such that $x = ((y, z), w)$ and $w = R_y^p[z]$ for some $y$-addition $R_y^p$. 

13
Proof. The Proposition follows immediately by Pure Comprehension:

\[(\exists A^p)(\forall x)(A^p(x) \iff (\exists y)(\exists z)(\exists w)(y \in \mathbb{N} \land z \in \mathbb{N} \land \text{there is a } y\text{-multiplication } T^p_y \text{ such that } w = T^p_y[z] \land x = ((y, z), w))).\]

Proposition 3 guarantees that the right-hand side of the biconditional is not empty and that \(A^p\) captures the intended operation. We will call an \(addition\) any pure predicate coextensive with \(A^p\). As is customary, we will write \(x + y = z\) to abbreviate \(A^p(((x, y), z))\). Before we may move to multiplication we need to verify that the the right axioms hold of addition.

**Theorem 6.** Let \(A^p\) be an addition predicate. Then for all \(x, y \in \mathbb{N}\)

\[x + 0 = x \quad \text{and} \quad x + S(y) = S(x + y).\]

Proof. We need to show that for every \(x, y \in \mathbb{N}\), \(A^p((x, 0), x)\) and \(A^p((x, S(y)), S(z))\) where \(z\) is the unique thing such that \(A^p((x, y), z)\). For the first claim, let \(x \in \mathbb{N}\). By definition of \(A^p\), \(A^p((x, 0), x)\) if and only if \(x = R^p_x(0)\) for an \(x\)-addition \(R^p_x\). By Proposition 3, this is true of every \(x\)-addition.

For the second claim, let \(x, y \in \mathbb{N}\) and let \(z\) be the unique thing such that \(A^p((x, y), z)\). Let \(R^p_x\) denote an \(x\)-addition. We want to show that \(A^p((x, S(y)), S(z))\), i.e., \(S(z) = R^p_x[S(y)]\). Since \(A^p((x, y), z)\), \(z = R^p_x[y]\). Let \(S^p\) be a successor predicate. Then, by functionality of successor, \(S^p[z] = S^p[R^p_x[S(y)]\]. Therefore, using the characterization of \(x\)-addition, we can conclude that \(S^p[R^p_x[y]] = R^p_x[S^p[y]]\). Thus, \(S^p[z] = R^p_x[S^p[y]]\), which means that \(S(z) = R^p_x[S(y)]\). Hence, \(A^p((x, S(y)), S(z))\), namely, \(x + S(y) = S(z) = S(x + y)\). \(\square\)
We now move to multiplication, proceeding in a way analogous to that of addition. We first define \( m \)-multiplication and then use it to define multiplication on \( \mathbb{N} \). Finally, we verify that multiplication has the desired properties. Since \( m \)-multiplications and multiplication are pure properties, they can be freely invoked in the instances of Pure Comprehension.

**Proposition 5.** For every \( m \in \mathbb{N} \), there is a functional pure predicate \( T^p \) with domain \( \mathbb{N} \) such that, for any \( m \)-addition \( R^p_m \),

\[
T^p(0,0) \quad \text{and} \quad T^p(n,S(n),R^p_m[T^p(n)]).
\]

Proof. We prove the Proposition by induction. If \( m = 0 \), the existence of the corresponding \( T^p_0 \) is given by the Recursion Theorem with 0 in place of \( a \), \( \mathbb{N} \) in place of \( P \), \( R^p_m \) in place of \( Q \), and \( T^p \) in place of \( R^p \). As an inductive hypothesis, suppose that there is a pure predicate \( T^p_m \) with domain \( \mathbb{N} \) such that

\[
T^p(0,0) \quad \text{and for all } n \in \mathbb{N}, \ T^p_m(S(n),R^p_m[T^p_m(n)]).
\]

We show that there is a pure predicate \( T^p_{S(m)} \) such that

\[
T^p_{S(m)}(0,0) \quad \text{and for all } n \in \mathbb{N}, \ T^p_{S(m)}(S(n),R^p_{S(m)}[T^p_{S(m)}[n]]).
\]

As in the proof of Proposition 3, the inductive step is carried out by appeal to Pure Comprehension:

\[
(\exists T^p_{S(m)})(\forall x)(T^p_{S(m)}(x) \leftrightarrow (\exists y)(\exists z)(T^p_m(y,z) \land x = (y,R^p_y[z])))
\]

Since \( y \) is bounded and \( R^p_y \) is pure, this is a legitimate instance of Pure Comprehension. This completes the induction.
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

The next Proposition defines full-fledged multiplication using \( m \)-multiplication.

**Proposition 6.** There is a functional pure predicate \( M^p \) satisfying the following condition: for every \( x \), \( M^p(x) \) if and only if there are \( y, z, w \in \mathbb{N} \) such that \( x = ((y, z), w) \) and \( w = T^p_y[z] \) for some \( y \)-multiplication \( T^p_y \).

**Proof.** Once the existence of \( y \)-multiplication has been established (Proposition 5), Pure Comprehension gives us the desired predicate as follows:

\[
(\exists M^p)(\forall x)(M^p(x) \iff (\exists y)(\exists z)(\exists w)(y \in \mathbb{N} \land z \in \mathbb{N} \land \text{there is a } y\text{-multiplication } T^p_y \text{ such that } w = T^p_y[z] \land x = ((y, z), w))).
\]

We verify that multiplication, i.e., \( M^p \) or any coextensive pure predicate, has the right properties. We write \( x \cdot y = z \) for \( M^p((x, y), z) \).

**Theorem 7.** For all \( x, y \in \mathbb{N} \)

\[
x \cdot 0 = 0 \quad \text{and} \quad x \cdot S(y) = (x \cdot y) + x
\]

**Proof.** For the first claim, we need to show that for every \( x \in \mathbb{N} \), \( M^p((x, 0), 0) \). By definition of \( M^p \), \( M^p((x, 0), 0) \) if and only if \( 0 = T^p_x(0) \) for some \( x \)-multiplication \( T^p_x \). However, it follows from Proposition 5 that every \( 0 = T^p_x(0) \) for every \( x \)-multiplication.

For the second claim, we need to show that for every \( x, y \in \mathbb{N} \),

\[
M^p[(x, S(y))] = A^p[(M^p[(x, y)], x)].
\]

By definition of multiplication, \( M^p[(x, S(y))] = M^p_x[S(y)] \) for some \( x \)-multiplication \( M^p_x \). Proposition 5 implies that \( M^p_x[S(y)] = R^p_x[M^p_x[y]] \) for some \( x \)-addition \( R^p_x \).
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

The characterization of addition entails that $R^p_x[M^p_y[x]] = A^p[(M^p_y[x], x)]$. Since $A^p[(M^p_y[x], x)] = A^p[(M^p[(x, y)], x)]$, we can conclude that

$$M^p[(x, S(y))] = A^p[(M^p[(x, y)], x)].$$

For exponentiation, we proceed analogously. We first define $m$-exponentiation and then use it to define exponentiation on $\mathbb{N}$. Later, we verify that exponentiation so defined has the desired properties.

**Proposition 7.** For every $m \in \mathbb{N}$, there is a functional pure predicate $U^p_m$ with domain $\mathbb{N}$ such that, for any $m$-multiplication $M^p_m$,

$$U^p_m(0, S(0)) \quad \text{and} \quad \text{for all } n \in \mathbb{N}, \ U^p_m(S(n), M^p_m[U^p_m[n]]).$$

**Proof.** The Proposition is proved, again, by induction. If $m = 0$, then $U^p_0$ exists by the Recursion Theorem with $S(0)$ in place of $a$ and $M^p_0$ in place of $Q^p$. Now suppose that that there is a pure predicate $U^p_m$ with domain $\mathbb{N}$ such that

$$U^p_m(0, S(0)) \quad \text{and for all } n \in \mathbb{N}, \ U^p_m(S(n), M^p_m[U^p_m[n]]).$$

We need to show that there is a pure predicate $U^p_{S(m)}$ such that

$$U^p_{S(m)}(0, 0) \quad \text{and for all } n \in \mathbb{N}, \ U^p_{S(m)}(S(n), M^p_{S(m)}[U^p_{S(m)}[n]]).$$

As in the proof of Proposition 3 and 5, the inductive step is carried out by appeal to Pure Comprehension:

$$(\exists U^p_{S(m)})(\forall x)(U^p_{S(m)}(x) \leftrightarrow (\exists y)(\exists z)(U^p_m(y, z) \land x = (y, M^p_y[z]))).$$

\qed
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

For any \( m \in \mathbb{N} \), we call the predicate defined in the previous Proposition an \( m \)-exponentiation. Since \( m \)-multiplication is pure, \( m \)-exponentiation is pure, thus it can be used in instances of Pure Comprehension. Next, we define full-fledged exponentiation using \( m \)-exponentiation.

**Proposition 8.** There is a functional pure predicate \( E^p \) satisfying the following condition: for every \( x \), \( M^p(x) \) if and only if there are \( y, z, w \in \mathbb{N} \) such that \( x = ((y, z), w) \) and \( w = U^p_y[z] \) for some \( y \)-exponentiation \( U^p_y \).

**Proof.** Proposition 7 gives us \( m \)-exponentiation (where \( m \) is the base) for each \( m \in \mathbb{N} \), so we can use Pure Comprehension as follows to obtain the desired predicate:

\[
(\exists E^p)(\forall x)((E^p(x) \leftrightarrow \\
(\exists y)(\exists z)(\exists w)(y \in \mathbb{N} \land z \in \mathbb{N} \land \text{there is a } y\text{-exponentiation } U^p_y \text{ such that} \\
w = U^p_y[z] \land x = ((y, z), w)))).
\]

As remarked above, the condition defining \( z \)-exponentiation is pure. So this is a legitimate instance of Pure Comprehension.

Finally, we must verify that \( E^p \), exponentiation, has the desired properties. This is done in Theorem 8. We write \( x^y = z \) for \( E^p((x, y), z) \) and, if that is the case, \( x^y \) for \( E^p[(x, y)] \).

**Theorem 8.** For all \( x, y \in \mathbb{N} \)

\[
x^0 = S(0) \text{ and } x^{S(y)} = x^y \cdot x.
\]

**Proof.** For the first claim of the Theorem, we need to show that for every \( x \in \mathbb{N} \), \( E^p((x, 0), S(0)) \). By definition of \( E^p \), \( E^p((x, 0), S(0)) \) if and only if \( S(0) = U^p_x[0] \)
for an $x$-exponentiation $U^p_x$. However, by the characterization of $z$-exponentiation in Proposition in 8, for any $z$, $S(0) = U^p_x[0]$. For the second claim, we need to show that for every $x, y \in \mathbb{N}$,

$$E^p[(x, S(y))] = M^p[(E^p[(x, y)], x)].$$

By definition of exponentiation, $E^p[(x, S(y))] = U^p_x[S(u)]$, for some $x$-exponentiation $U^p_x$. Also, it follows from the characterization of $x$-exponentiation in Proposition 8 that $U^p_x[S(u)] = M^p_x[U^p_x[y]]$, for some $x$-multiplication $M^p_x$. Since, for every $\alpha$, we have that

$$U^p_x[y] = E^p[(x, y)] \text{ and } M^p_x[\alpha] = M^p[(x, \alpha)],$$

we can conclude that $M^p_x[U^p_x[y]] = M^p[(E^p[(x, y)], x)]$. Thus, $E^p[(x, S(y))] = M^p[(E^p[(x, y)], x)]$.

Theorems 1—8 complete our development of arithmetic in BTPp.

1.3 Remarks

Once arithmetic has been developed in BTP, it is easy to define an order on $\mathbb{N}$. Define a predicate $\Pi$ to be an order on $\mathbb{N}$ if it satisfies the following (pure) condition:

$$(\forall x)(\Pi(x) \leftrightarrow (\exists y)(\exists z)(y \in \mathbb{N} \land z \in \mathbb{N} \land x = (y, z) \land (\exists w)(y + w = z))).$$

The existence of an order is given by a straightforward application of Pure Comprehension. Then we use $x \leq y$ to abbreviate

$$(\exists P^p)(\exists w)(P^p \text{ is an order on } \mathbb{N} \land w = (x, y) \land P^p(w)).$$

The Induction Theorem 3 can be strengthened by allowing parameters.
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

Theorem 9 (Induction with parameters). Let $\phi(x, y_1, \ldots, y_n)$ be a formula of BTPp in which no variable other than $x, y_1, \ldots, y_n$ occurs free. Suppose that for every $y_1, \ldots, y_n, \phi(0, y_1, \ldots, y_n)$, and for every $x \in \mathbb{N}$, if $\phi(x, y_1, \ldots, y_n)$, then $\phi(S(x), y_1, \ldots, y_n)$. Then, for every $y_1, \ldots, y_n$ and every $x \in \mathbb{N}$, $\phi(x, y_1, \ldots, y_n)$.

Proof. Suppose that, for every $y_1, \ldots, y_n, \phi(0, y_1, \ldots, y_n)$ and, for every $x \in \mathbb{N}$, if $\phi(x, y_1, \ldots, y_n)$ then $\phi(S(x), y_1, \ldots, y_n)$.

Define by Pure Comprehension the unary predicate $P^p$ such that

$$(\forall x)(P^p(x) \leftrightarrow x \in \mathbb{N} \land \forall y_1, \ldots, \forall y_n \phi(x, y_1, \ldots, y_n)).$$

We observe that $P^p$ is inductive. For the basis, $0 \in \mathbb{N}$ and we are supposing that for every $y_1, \ldots, y_n, \phi(0, y_1, \ldots, y_n)$. So $P^p(0)$. For the inductive step, assume that $P^p(n)$, i.e., $n \in \mathbb{N}$ and for every $y_1, \ldots, y_n, \phi(n, y_1, \ldots, y_n)$. We want to show that $P^p(S(n))$. Since membership in $\mathbb{N}$ is inductive, $S(n) \in \mathbb{N}$. Also, from our assumption that for every $y_1, \ldots, y_n, \phi(n, y_1, \ldots, y_n)$, we get that for every $y_1, \ldots, y_n, \phi(S(n), y_1, \ldots, y_n)$. Therefore, $P^p(S(n))$. Hence, $P^p$ is inductive.

Let $m \in \mathbb{N}$. $P^p$ is inductive. So $P^p(m)$. This means that for every $y_1, \ldots, y_n, \phi(m, y_1, \ldots, y_n)$. Thus, we can conclude that for every $y_1, \ldots, y_n$ and for every $n \in \mathbb{N}$, $\phi(n, y_1, \ldots, y_n)$.

\[ \square \]

1.4 Finite sequences

In this section, we provide a way to code finite sequences of arbitrary objects. First, let us define a finite sequence. For any predicate $\Pi$, we say that $\Pi$ contains the finite sequences if

$$\Pi(0) \land (\forall x)(\forall y)(\Pi(x) \rightarrow \Pi(x, y)).$$
Notice that the universal predicate provides an example of a predicate containing the finite sequences. We then define finite sequences by taking the minimal predicates containing the finite sequences. A predicate \( \Pi \) defines the finite sequences (note that this is not circular) if it satisfies the following condition:

\[
(\forall P^p)(\forall x)(\Pi(x) \land P^p \text{ contains the finite sequences} \rightarrow P^p(x)).
\]

For any \( x \),

\[
x \text{ is a finite sequence} \iff (\exists P^p)(P^p \text{ defines the finite sequences and } P^p(x)).
\]

We now want to characterize a length function and a function that, for any finite sequence, gives the \( i \)-th term of the sequence. Before proceeding, let us remark that for any finite sequence \( x \), if \( x \neq 0 \), there are \( y \) and \( z \) such that \( x = (y,z) \). For convenience, we introduce some notation. If \( x \) is a finite sequence and \( x \neq 0 \), then \( (x)_1 \) is the unique \( y \) such that, for some \( z \), \( x = (y,z) \). Analogously, we denote by \( (x)_2 \) the unique \( z \) such that, for some \( y \), \( x = (y,z) \). A sequence \( x \) is non-empty if \( x \neq 0 \). For example, if \( x \) is the sequence \( ((0,0),0) \) of length two, then \( (x)_1 \) is \((0,0)\) and \( (x)_2 \) is \(0\). We may stipulate that both components of \(0\), the empty sequence, are \(0\).

The existence of a length function can be obtained purely via pure comprehension. We want our length function to meet the following conditions in the appropriate way:

(i) the domain of the function consists of the finite sequences and its range is \( \mathbb{N} \);

(ii) the value of the function at \( 0 \) is \( 0 \);

(iii) for any finite sequence \( x \) and object \( y \), if the value of the function at \( x \) is \( n \), the value of the function at \((x,y)\) is \( n + 1 \).
Luckily, these conditions are pure. However, we need an intermediate step before we are able to use comprehension to obtain a length function. Let us say that a pure predicate $\Pi$ is an \textit{intermediate length function} if it satisfies the following conditions:

$\Pi$ is functional $\land (\forall x)(\Pi(x) \rightarrow (\exists y)(\exists n)(y \text{ is a finite sequence } \land n \in \mathbb{N} \land x = (y, n)) \land (\forall n)(x = (0, n) \rightarrow n = 0))$.

Now, we can characterize as a \textit{length function} any predicate coextensive with the formula $\phi$:

$\phi(x) \equiv (\exists P^p)(P^p \text{ is an intermediate length function } \land P^p(x) \land (\forall y)(\forall z)(\forall n)(P^p(y, n) \rightarrow P^p((y, z), n + 1)))$.

The following Proposition follows immediately.

\textbf{Proposition 9.} A \textit{length function exists}.

\textit{Proof.} Since the conditions on intermediate functions are pure and the extra condition required for a length function is pure, we can simply apply pure comprehension to obtain:

$(\exists L^p)(\forall x)(L^p(x) \leftrightarrow \phi(x))$.

That is, rewriting the comprehension formula explicitly:

$(\exists L^p)(\forall x)(L^p(x) \leftrightarrow (\exists P^p)(P^p \text{ is an intermediate length function } \land P^p(x) \land (\forall y)(\forall z)(\forall n)(P^p(y, n) \rightarrow P^p((y, z), n + 1))))$. 

\qed
We introduce an abbreviation to talk about the length of a finite sequence. For any finite sequence $x$ and for any $n \in \mathbb{N}$:

$$\text{length}(x) = n \iff (\exists L^p)(L^p \text{ is a length function } \land L^p(x, n)).$$

The next step is to characterize an $i$-term function that, for each finite sequence $x$, gives the $i$-term of $x$. For example, with respect to the sequence $(((0, z), y), x)$ of length 3, the terms are counted as follows: the first term is $x$, the second is $y$, and the third is $z$.

We take a brief detour and prove a general result that is relevant to our task and will be helpful later. A pure predicate $P^p$ is a pure binary relation if for any $x$, if $P^p(x)$, then $x = (y, z)$ for some objects $y$ and $z$. Let us define as purely general any pure predicate $Q^p$ such that

$$(\exists P^p)(P^p \text{ is a pure binary relation } \land (\exists y)(\forall x)(Q^p(x) \iff P^p(y, x))).$$

In effect, a purely general predicate is any cross section of a pure binary relation. Now, we want to prove a metatheorem that gives us a pure version of comprehension that resembles the general one. General predicates will be interpreted as purely general predicates.

We define a interpretation $\tau$ (i.e., a translation map) from formulas of BTP$\text{pg}$ to formulas of BTP$\text{p}$ that leaves the pure formulas unchanged.$^3$ For each general predicate $\Gamma$ in the language of BTP$\text{pg}$, we introduce in the language of BTP$\text{p}$ a new correlative pure predicate $P^\Gamma$. We also introduce a new stock of object variables such that, for each general predicate $\Gamma$, there is a new correlative object variable $x^\Gamma$. The translation $\tau$ is recursively characterized as follows:

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$^3$See [3], pp. 79-80.
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

(\(\tau_1\)) for every general atomic formula \(\Gamma(t)\),

\[ \tau[\Gamma(t)] = P_{x_{\Gamma}}(t); \]

(\(\tau_2\)) for any general formula \(\phi(\Gamma)\), with free variable \(\Gamma\),

\[ \tau[(\exists \Gamma)\phi(\Gamma)] = (\exists P_{x_{\Gamma}})\tau[\phi(\Gamma)]; \]

(\(\tau_3\)) for any binary connective \(*\), and formulas \(\phi\) and \(\psi\),

\[ \tau[\neg \phi] = \neg \tau[\phi], \]

\[ \tau[\phi * \psi] = \tau[\phi] * \tau[\psi]; \]

(\(\tau_4\)) pure formulas are left unchanged.

Now, \(\tau\) provides an interpretation of BTP\(pg\) in BTP\(p\), as shown by the next theorem.

**Theorem 10.** For any theorem \(\theta\) of BTP\(pg\), \(\tau[\theta]\) is a theorem of BTP\(p\).

**Proof.** We provide a proof sketch. The crucial step of the proof is to show that instances of General Comprehension are theorems of BTP\(p\) under \(\tau\). We proceed by induction on the complexity of the comprehension formula \(\phi\). We must also consider whether or not the comprehension formula has parameters. By the notation \(\gamma_\phi\), we mean an instance of General Comprehension whose comprehension formula is \(\phi\).

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\(^4\)As an illustration, clauses \((\tau_1)\) and \((\tau_2)\) entail that

\[ \tau[(\exists G^g)G^g(t)] = (\exists P^p_{x_{G^g}})(\exists x_{G^g})P^p(x_{G^g}, t). \]
Let us start with the case in which $\phi$ is a general atomic formula. First, let us assume that $\phi$ does not have parameters. This means that $\phi$ is $P^g(x)$ for some general predicate $P^g$. The instance of General Comprehension is trivial:

$$(\forall P^g)(\exists \Gamma)(\forall x)((\Gamma(x) \iff P^g(x))).$$

We want to show that BTPp proves:

$$(\forall P^g)(\forall x)(P^g(x) \iff P^g(x)).$$

This follows through elementary logical steps from the trivial fact that

$$(\forall P^g)(\forall x)(P^g(x) \iff P^g(x)).$$

So, if $\phi$ is atomic and does not have parameters, BTPp proves $\tau[\gamma_\phi]$. Now suppose that $\phi$ is general atomic and it has one parameter $y$. That is, $\phi$ is $P^g(x, y)$ for some general predicate $P^g$. Then $\gamma_\phi$ is

$$(\forall y)(\forall P^g)(\exists \Gamma)(\forall x)((\Gamma(x) \iff P^g(x, y))).$$

We want to show that BTPp proves:

$$(\forall P^g)(\forall x)(P^g(x) \iff P^g(x, y)).$$

We observe that, by Pure Comprehension, for any pure predicate $P_{p^g}$, there is a pure predicate $P^*_p$ such that,

$$(\forall x)(\forall y)(\forall z)(P^*_p((z, y), x) \iff P_{p^g}(z, (x, y))).$$

Therefore,

$$(\forall y)(\forall x)(P^*_p((x_{p^g}, y), x) \iff P_{p^g}(x_{p^g}, (x, y))).$$
Through elementary logical steps, it follows that:

\[(\forall y)(\forall x_{P_\phi})(\exists P_1)(\exists x_\Gamma)(\forall x)(P_\Gamma(x_\Gamma, x) \leftrightarrow P_{P_\phi}(x_{P_\phi}, (x, y))).\]

Since \(P_{P_\phi}\) is arbitrary, we have:

\[(\forall P_{P_\phi})(\forall y)(\forall x_{P_\phi})(\exists P_1)(\exists x_\Gamma)(\forall x)(P_\Gamma(x_\Gamma, x) \leftrightarrow P_{P_\phi}(x_{P_\phi}, (x, y))),\]

which is equivalent to \((\ast)\). A rigorous induction on the number of parameters would show that adding a parameter to general atomic comprehension formulas yields theorems of BTPp under \(\tau\). The case just proved is the base case of the induction. This completes the proof of the theorem for instances of General Comprehension in the atomic case.

Let us now consider the cases of complex comprehension formulas. Let \(\phi\) be \(\psi_1 \land \psi_2\). We want to show that \(\tau[\gamma_{\psi_1 \land \psi_2}]\) is a theorem of BTPp. By inductive hypothesis, both \(\tau[\gamma_{\psi_1}]\) and \(\tau[\gamma_{\psi_2}]\) are theorems of BTPp. Now, \(\gamma_{\psi_1 \land \psi_2}\) is

\[(\exists \Gamma)(\forall x)(\Gamma(x) \leftrightarrow \psi_1(x) \land \psi_2(x)),\]

and \(\tau[\gamma_{\psi_1 \land \psi_2}]\) is

\[(\exists P_1)(\exists x_\Gamma)(\forall x)(P_\Gamma(x_\Gamma, x) \leftrightarrow \tau[\psi_1(x)] \land \tau[\psi_2(x)]).\]

By inductive hypothesis, we have that both of the following pure formulas are theorems of BTPp:

\[(i) \quad (\exists P_{\psi_1})(\exists x_{\psi_1})(\forall x)(P_{\psi_1}(x_{\psi_1}, x) \leftrightarrow \tau[\psi_1(x)]),\]

\[(ii) \quad (\exists P_{\psi_2})(\exists x_{\psi_2})(\forall x)(P_{\psi_2}(x_{\psi_2}, x) \leftrightarrow \tau[\psi_2(x)]).\]

By Pure Comprehension, which allows the use of predicative parameters in the comprehension formula, we have:

\[(\exists P_{\psi_1 \land \psi_2})(\forall y_1)(\forall y_2)(\forall x)(P_{\psi_1 \land \psi_2}(y_1, y_2, x) \leftrightarrow P_{\psi_1}(y_1, x) \land P_{\psi_2}(y_2, x)).\]
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

It then follows that

\[(\forall x)(P_{\psi_1 \land \psi_2}((x_{\psi_1}, x_{\psi_2}), x) \leftrightarrow P_{\psi_1}(x_{\psi_1}, x) \land P_{\psi_2}(x_{\psi_2}, x)).\]

So, from \((i)\) and \((ii)\):

\[(\forall x)(P_{\psi_1 \land \psi_2}((x_{\psi_1}, x_{\psi_2}), x) \leftrightarrow \tau[\psi_1(x)] \land \tau[\psi_2(x)]).\]

Thus,

\[(\exists x_{\Gamma})(\forall x)(P_{\psi_1 \land \psi_2}(x_{\Gamma}, x) \leftrightarrow \tau[\psi_1(x)] \land \tau[\psi_2(x)]).\]

Therefore,

\[(\exists P_{\Gamma})(\exists x_{\Gamma})(\forall x)(P_{\Gamma}(x_{\Gamma}, x) \leftrightarrow \tau[\psi_1(x)] \land \tau[\psi_2(x)]).\]

This shows that \(\tau[\gamma_{\psi_1 \land \psi_2}]\) is a theorem of BTPp.

The inductive case we have just proved involved no parameter in the comprehension formula. To show that the inductive step goes through even when the comprehension formula contains parameters, one needs to argue by induction on the number of parameters. The proof of the base case mirrors that of the previous case. The instance of General Comprehension with one parameter is, in the case at hand,

\[(\forall y)(\exists \Gamma)(\forall x)(\Gamma(x) \leftrightarrow \psi_1(x, y) \land \psi_2(x)).\]

We want to show that the following is a theorem of BTPp:

\[(\forall y)(\exists P_{\Gamma})(\exists x_{\Gamma})(\forall x)(P_{\Gamma}(x_{\Gamma}, x) \leftrightarrow \tau[\psi_1(x, y)] \land \tau[\psi_2(x)]).\]

One may proceed as above, using the inductive hypotheses

\[(i^*) \quad (\forall y)(\exists P_{\psi_1})(\exists x_{\psi_1})(\forall x)(P_{\psi_1}(x_{\psi_1}, x) \leftrightarrow \tau[\psi_1(x, y)]),\]

\[(ii) \quad (\exists P_{\psi_2})(\exists x_{\psi_2})(\forall x)(P_{\psi_2}(x_{\psi_2}, x) \leftrightarrow \tau[\psi_2(x)]).\]
For the negation case, let $\phi$ be $\neg \psi$ and let us assume as inductive hypothesis that $\tau[\gamma_\psi]$, namely,

$\tag{iii} (\exists P_\psi) (\exists x_\psi)(\forall x)(P_\psi(x_\psi, x) \leftrightarrow \tau[\psi(x)])$,

is a theorem of BTPp. We want to show that $\tau[\gamma_{\neg \psi}]$, namely,

$\tag{**} (\exists P_\Gamma)(\exists x_\Gamma)(\forall x)(P_\Gamma(x_\Gamma, x) \leftrightarrow \tau[\neg \psi(x)])$,

is also a theorem of BTP. It follows from (iii) and Pure Comprehension that

$$(\exists P_\Gamma)(\forall y)(\forall x)(P_\Gamma(y, x) \leftrightarrow \neg P_\psi(y, x)).$$

Together with (iii), this entails (**) through elementary logical steps.

Quantified formulas do not need to be considered in the induction, since every instance of General Comprehension is a closed formula. So the proof of the crucial step of the theorem is complete.

Via Theorem 10, we can prove results in the General Theory of Predication and translate them into the Pure Theory. An application concerns the existence of an $i$-term function. For convenience, the existence of such a function will be proved in the General Theory first.

**Proposition 10.** For any finite sequence $x$, there are functional $R^g_1$ and $R^g_2$ with domain $\mathbb{N}$ such that, for every $n \in \mathbb{N}$, the following inductive condition is satisfied:

1. $R^g_1[0] = x$ and $R^g_2[0] = x$;
2. $R^g_2[n + 1] = (R^g_1[n])_1$ and $R^g_2[n + 1] = (R^g_2[n])_2$.

**Proof.** Let $x$ be any non-empty finite sequence. The proof makes use of the General Recursion Theorem (Theorem 5). By General Comprehension (but also Pure Comprehension) we have:

$$(\exists Q_1^g)(\forall y)(Q^g(y) \leftrightarrow (\exists z)(\exists w)(y = (z, w) \land y \text{ is a finite sequence } \land w = (z)_1)),$$
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

\((\exists Q^9_2)(\forall y)(Q^9(y) \leftrightarrow (\exists z)(\exists w)(y = (z, w) \land y \text{ is a finite sequence } \land w = (z)_2)).\)

We can now apply Theorem 5 to \(Q^9_1\) and \(Q^9_2\) and obtain two functional predicates \(R^9_1\) and \(R^9_2\) such that

(i) \(R^9_1(0, x)\) and \(R^9_2(0, x)\);

(ii) for all \(n \in \mathbb{N}\), \(R^9_1(S(n), Q^9_1[R^9[n]])\) and \(R^9_2(S(n), Q^9_2[R^9[n]])\).

It follows from (ii) and the definition of \(Q^9_1\) and \(Q^9_2\) that, for every \(n \in \mathbb{N}\), \(R^9_1[S(n)] = (R^9[n])_1\) and \(R^9_2[S(n)] = (R^9[n])_2\). So \(R^9_1\) and \(R^9_2\) are the sought functional predicates.

For any finite sequence \(x\), we call the associated predicate \(R^9_2\) given by the Theorem the \(i\)-term function for \(x\). We want to combine individual \(i\)-term functions and obtain a universal \(i\)-term function. The following instance of General Comprehension gives us the desired function:

\((\exists R^9)(\forall x)(R^9(x) \leftrightarrow (\exists y)(\exists z)(\exists w)(x = ((y, z), w) \land y \in \mathbb{N} \land z \text{ is a finite sequence } \land (\forall R^9_2)(R^9_2 \text{ is an } i\text{-term function for } z \rightarrow w = R^9_2[y])))\).

This ends our excursus in BTPg. We want to translate this result in BTPp and obtain a pure analogue of the \(i\)-term function whose existence has been just proved in BTPg. The following Proposition establishes this result.

**Proposition 11.** For any finite sequence \(x\), there are pure functional \(R^9_1\) and \(R^9_2\) and there are objects \(x_{R^9_1}\) and \(x_{R^9_2}\) such that, for every \(n \in \mathbb{N}\), the following inductive condition is satisfied
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

(1) \( R^p_1[(x_{R^p_1}, 0)] = x \) and \( R^p_2[(x_{R^p_2}, 0)] = x \);

(2) \( R^p_2[(x_{R^p_1}, n + 1)] = (R^p_1[(x_{R^p_1}, n)])_1 \) and \( R^p_2[(x_{R^p_2}, n + 1)] = (R^p_2[(x_{R^p_2}, n)])_2 \).

Proof. The Proposition follows from Theorem 10 and Proposition 10, which entail the existence of three-place pure predicates that, when fixing the first argument with some objects \( x_1 \) and \( x_2 \), behave like an \( i \)-term function with respect to the second and third coordinate. However, via Pure Comprehension, the extension of such predicates can be so reorganized as to yield the functional predicates in the statement of Proposition. \( \Box \)

Proposition 11 gives us, for each finite sequence \( x \), a pure \( i \)-term function for \( x \). The Proposition asserts that a pure \( i \)-term function \( R^p \) for a given sequence \( x \) is relative to an object \( x_{R^p} \). So we say that \( R^p \) is a pure \( i \)-term function for \( x \) relative to \( x_{R^p} \). As in the case of general \( i \)-term functions, we want to combine the individual functions. Via Pure Comprehension we have:

\[
(\exists R^p)(\forall x)(R^p(x) \leftrightarrow (\exists y)(\exists z)(\exists w)\\\hspace{1cm} x = (((y, z), w) \land y \in \mathbb{N} \land z \text{ is a finite sequence}) \land (\forall R^p_2)(\forall x_{R^p_2})(R^p_2 \text{ is an } i \text{-term function for } z \text{ relative to } x_{R^p_2} \rightarrow 2 = R^p_2[(x_{R^p_2}, y)])).
\]

For any sequence \( x \) of length \( n \), we then say in BTPp that \( y \) is the \( i \)-th term of \( x \) if and only if, for a pure \( i \)-term function \( R^p \), \( R^p[(i, x)] = y \). This concludes our treatment of finite sequences. We now move to characterize semantical notions in BTPp.

1.5 Semantics

Let \( \mathcal{L}_{PC} \) be the standard language of the predicate calculus, including identity and countably many variables, constants, \( n \)-relations. Most symbols will be in bold so
that they can be distinguished from those of the metalanguage. So $\mathcal{L}_{PC}$ is composed of the following symbols:

A. **Constants**: $c_1, c_2, \ldots, c_n, \ldots$

B. **Variables**: $v_1, v_2, \ldots, v_n, \ldots$

C. **Relations of any finite arity**: $R^1_1, R^1_2, R^1_3, \ldots, R^2_1, R^2_2, \ldots, R^n_1, \ldots$ \(^5\)

D. **Functions of any finite arity**: $f^1_1, f^1_2, f^1_3, \ldots, f^2_1, f^2_2, \ldots, f^n_1, \ldots$

E. **Logical symbols**: $\neg, \land, \exists, =$. \(^6\)

F. **Parentheses**: $(, )$.

The clauses defining well-formed formulas and other syntactic notions of $\mathcal{L}_{PC}$ are the usual ones. Since we have already developed arithmetic within BTPp, we can employ arithmetical notions to code the symbols and the syntax of $\mathcal{L}_{PC}$. \(^7\) We will actually make the standard Gödel numbers of the syntactic expressions of the language to be themselves the items of the language. But, for readability, we will mention the standard symbols rather than their codes.

Since the syntactic notions needed to develop the semantics can all be defined arithmetically, we can employ the ordinary syntactic notions to stand for their arithmetical counterparts. This means that, within BTPp, we can legitimately make use of notions such as **constant**, **variable**, **index of a constant**, **relation**, **index of a relation**, **arity of a relation**, **(well-formed) formula**, etc. Especially in the next chapter, we will

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\(^5\) As usual, the arity of relations and functions is indicated by the superscript.

\(^6\) The other standard connectives and the universal quantifier will be regarded as abbreviations.

\(^7\) See [2]. In [2], however, the language coded is that of arithmetic rather than $\mathcal{L}_{PC}$. Nevertheless, it is straightforward to extend the coding to $\mathcal{L}_{PC}$. 

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31
indulge in set-theoretic talk. For example, we will speak of sets of formulas, terms, variables, etc. However, that is not problematic. Since these notions are explicitly defined in \( \mathbb{N} \), corresponding predicates are available.

We begin by defining purely the notion of a *structure*. A structure is any binary pure predicate \( S^p \) satisfying the following conditions:

(i) for any \( x \), if \( x \) is a constant, there is a unique \( y \) such that \( S^p(x, y) \);

(ii) for any \( x \), if \( x \) is an \( n \)-ary relation symbol, then for every \( y \) such that \( S^p(x, y) \), \( y \) is a sequence of length \( n \);

(iii) for any \( x \), if \( x \) is an \( n \)-ary function symbol, then for every \( y \) such that \( S^p(x, y) \), \( y = (z, w) \) for some sequence \( z \) of length \( n \) and some object \( w \) such that for no other \( w' \), \( S^p(x, (z, w')) \). We call \( w \) the value of \( x \) at \( z \) with respect to \( S^p \).

Next we deal with the notion of satisfaction. For any structure \( S^p \), we characterize a satisfaction predicate with respect to \( S^p \), that is, satisfaction in the structure \( S^p \). The satisfaction predicate with respect to \( S^p \) will be denoted by \( \text{Sat}_{S^p} \). The existence of \( \text{Sat}_{S^p} \) will be proved by induction. We proceed in three stages. First, we introduce a notion of complexity of a formula. Then we define the notion of \( n \)-satisfaction with respect to \( S^p \), namely, satisfaction of formulas of complexity up to \( n \). Finally, we obtain a general notion of satisfaction with respect to \( S^p \).

Define the complexity of a formula \( \phi \) as the number of occurrences of logical operators \( (\neg, \land, \exists) \) in \( \phi \). It follows that

(a) any atomic formula has complexity 0;

(b) if \( \phi \) has complexity \( n \geq 1 \), then exactly one of the following alternatives obtains:

   (b1) there is a unique formula \( \psi \) of complexity \( n - 1 \) such that \( \phi = \neg \psi \);
(b2) there are exactly two formulas \( \psi_1 \) and \( \psi_2 \) of complexities \( m_1 \) and \( m_2 \) such that \( m_1 + m_2 = n - 1 \) and \( \phi \) is \( (\psi_1 \land \psi_2) \);

(b3) there is a unique formula \( \psi \) of complexity \( n - 1 \) and a unique variable \( v \) such that \( \phi \) is \( (\exists v)\psi \).

It should be noted that the existence of a functional predicate giving the complexity of a formula can be derived via Pure Comprehension:

\[ (\exists C^p)(\forall x)(C^p(x) \iff (\exists y)(\exists z)(x = (y,z) \land y \text{ is a formula} \land z \in \mathbb{N} \land \text{z the number of occurrences of a logical operator in } x)) \].

Let us define the complex formula \( \Psi(S^p, n, Sat_{(S^p, n)}) \) with parameters for, respectively, a structure \( S^p \), a natural number \( n \), and a pure satisfaction predicate \( Sat_{(S^p, n)} \) with respect to \( S^p \) for formulas of complexity up to \( n \).

**Definition.** \( \Psi(S^p, n, Sat_{(S^p, n)}) \) if and only if the following conditions hold:

(i) For every \( x \), \( Sat_{(S^p, n)}(x) \) only if there is \( y \) and \( z \), such that \( y \) is a formula of complexity less than or equal to \( n \) and \( z \) is a finite sequence whose length equals the number of free variables of \( y \), and \( x = (y,z) \).

(ii-At) If \( x \) is an atomic formula (i.e., \( n = 0 \)) that does not contain the identity sign, then for any \( y \), \( Sat_{(S^p, n)}(x, y) \) if and only if \( S^p(z, w) \), where \( z \) is the predicate of the formula \( x \) and \( w \) is a sequence of length \( m \) equal to the arity of \( z \) and such that, for each \( i \) (\( 1 \leq i \leq m \)), if \( w_i \) is the \( i \)-term of \( w \), then

(ii-At)(a) if the \( i \)-th term of the formula \( x \) is the constant \( c \), then \( w_i = w^* = S^p[c] \), and
Chapter 1. Arithmetic and Semantics in the Basic Theory of Predication

(ii-At)(b) if the \( i \)-th term of the formula \( x \) is the variable \( v \), and \( v \) is the \( j \)-th variable of \( x \) \((j \leq i)\), then \( w_i = v^* \) with \( v^* \) the \( j \)-term of \( y \);\(^8\) and

(ii-At)(c) if the \( i \)-th term of the formula \( x \) is the \( k \)-ary function \( f \) followed by \( v_{j_1}, \ldots, v_{j_k} \), the \( j_1 \)-th through \( j_k \)-th variables of \( x \), \( w_i \) is the value of the function \( f \) at \((v^*_{j_1}, \ldots, v^*_{j_k})\) with respect to \( S^p \), with each \( v^*_{j_r} \) \((1 \leq r \leq k)\) equal to the \( j_r \)-term of \( y \).

(ii-=) If \( x \) is an atomic identity formula \((n = 0)\), for every \( y \), \( Sat\)\(_{(S^p,n)}(x,y) \) if and only if the length of \( y \) is greater than or equal to the number of free variables in \( x \) and \( w_1 = w_2 \), where

(ii-=)(a) \( w_1 \) is the first term of \( y \) if the first term of \( x \) is a variable, and \( w_1 = S^p[c] \) if the first term of \( x \) is the constant \( c \);

(ii-=)(b) if \( x \) has only one variable, then \( w_2 \) is the first term of \( y \);

(ii-=)(c) if \( x \) has two variables, then \( w_2 \) is the second term of \( y \); and

(ii-=)(d) if \( x \) has no variable and \( d \) is the second term of \( x \), then \( w_2 = S^p[d] \).

(iii) if \( x \) has complexity \( n \geq 1 \), then

(iii)(a) if \( x \) has the form \( \top \phi_1 \land \phi_2 \bot \), then \( Sat\)\(_{(S^p,n)}(x,y) \) if and only if \( Sat\)\(_{(S^p,n)}(\phi_1, y^*) \) and \( Sat\)\(_{(S^p,n)}(\phi_2, y^{**}) \), where \( y^* \) is a finite sequence whose length is the number of free variables in \( \phi_1 \), say \( m_1 \), and whose \( i \)-term \((0 \leq i \leq m_1)\) is the same \( i \)-term of \( y \), and \( y^{**} \) is a sequence whose length is the same as the number of free variables in \( \phi_2 \), say \( m_2 \), and whose \( j \)-term \((0 \leq j \leq m_2)\) is the same as the \((i + m_1)\)-term of \( y \);

---

\(^8\)The order of the variables in a formula is given here by order of their occurrence in the formula.
(iii)(b) if $x$ has the form $\neg \phi$, then $\operatorname{Sat}_{(S^p, n)}(x, y)$ if and only if it is not the case that $\operatorname{Sat}_{(S^p, n)}(\phi, y)$; and

(iii)(c) if $x$ has the form $\forall (\exists v) \phi$, then $\operatorname{Sat}_{(S^p, n)}(x, y)$ if and only if there is $y^*$ such that $\operatorname{Sat}_{(S^p, n)}(\phi, y^*)$, where $y^*$ is a finite sequence that has length 1 plus the length of $y$ and, if $v$ is the $m$-th variable of $\phi$, $y^*$ coincides with $y$ for each term $i < m$ and, for each term $j > m$, the $j$-term of $y^*$ is the $(j - 1)$-term of $y$.

So $\Psi(S^p, n, \operatorname{Sat}_{(S^p, n)})$ stands for the long formula just specified. We want to prove by induction that for every $n$, and every structure $S^p$, there is a unique pure predicate $\operatorname{Sat}_{(S^p, n)}$ such that $\Psi(S^p, n, \operatorname{Sat}_{(S^p, n)})$, where uniqueness is understood in terms of extension.

**Proposition 12.** For every $n$, and every structure $S^p$, there is a (unique) pure predicate $\operatorname{Sat}_{(S^p, n)}$ such that $\Psi(S^p, n, \operatorname{Sat}_{(S^p, n)})$.

**Proof.** Fix a pure predicate $S^p$. We proceed by induction on $n$.

For $n = 0$, we produce the sought $\operatorname{Sat}_{(S^p, n)}$ via Pure comprehension using clauses (i), (ii-At), and (ii-) above. That is,

$$
(\exists \operatorname{Sat}_{(S^p, 0)})(\forall x)(\operatorname{Sat}_{(S^p, 0)}(x) \leftrightarrow
(\exists y)(\exists z)(x = (y, z) \land y \text{ is an atomic formula} \land z \text{ is a finite sequence whose length equals the number of free variables of } y \land
\begin{cases}
\text{if } y \text{ is not identity, then } (\text{ii-At}^*)
& \land
\text{if } y \text{ is an identity, then } (\text{ii-=}^*)
\end{cases}).
$$
where $(\text{ii-At}^*)$ and $(\text{ii-=}^*)$ stand, respectively, for the right-hand sides of the bi-
conditionals in $(\text{ii-At})$ and $(\text{ii-=})$. Since $(\text{ii-At}^*)$ and $(\text{ii-=}^*)$ contain only $S^p$
as a parameter, this is a legitimate instance of Pure Comprehension. By definition of $\text{Sat}_{(S^p,0)}$, it is clear that $\Psi(S^p,0,\text{Sat}_{(S^p,0)})$

Now suppose that there is $\text{Sat}_{(S^p,n)}$ such that $\Psi(S^p,n,\text{Sat}_{(S^p,n)})$. We show that
there is $\text{Sat}_{(S^p,n+1)}$ such that $\Psi(S^p,n+1,\text{Sat}_{(S^p,n+1)})$. Since $\text{Sat}_{(S^p,n)}$ is pure, we
can use it as a parameter in an instance of Pure Comprehension. By using clauses (iii)(a), (iii)(b), and (iii)(c), we have:

$$(\exists \text{Sat}_{(S^p,n+1)})(\forall x)(\text{Sat}_{(S^p,n+1)}(x) \leftrightarrow$$
$$(\exists y)(\exists z)(x = (y,z) \land y \text{ is a non-atomic formula} \land$$

$z \text{ is a finite sequence whose length equals the number of free variables of } y \land$

if $y$ has the form $\neg \phi_1 \land \phi_2$, then $\text{Sat}_{(S^p,n)}(\phi_1, y^*)$ and $\text{Sat}_{(S^p,n)}(\phi_2, y^{**})$,

where $y^*$ is a finite sequence whose length is the number of free variables in $\phi_1$, say $m_1$,
and whose $i$-term ($0 \leq i \leq m_1$) is the same $i$-term of $y$, and $y^{**}$ is a sequence
whose length is the same as the number of free variables in $\phi_2$, say $m_2$, and

whose $j$-term ($0 \leq j \leq m_2$) is the same as the $(i + m_1)$-term of $y \land$

if $x$ has the form $\neg \phi_1 \land \phi_2$, then it is not the case that $\text{Sat}_{(S^p,n)}(\phi, y) \land$

if $x$ has the form $\exists v \phi$, then there is $y^*$ such that $\text{Sat}_{(S^p,n)}(\phi, y^*)$,

where $y^*$ is a finite sequence that has length 1 plus the length of $y$ and,

if $v$ is the $m$-th variable of $\phi$, $y^*$ corresponds to $y$ for each term $i < m$
and, for each term $j > m$, the $j$-term of $y^*$ is the $(j - 1)$-term of $y$).
By definition of $\text{Sat}_{(Sp,n+1)}$, it follows immediately that $\Psi(Sp, n+1, \text{Sat}_{(Sp,n+1)})$. We can conclude by induction that for every $n$ there is a pure predicate $\text{Sat}_{(Sp,n)}$ such that $\Psi(Sp, n, \text{Sat}_{(Sp,n)})$, where $Sp$ is an arbitrary structure.

Uniqueness, in terms of coextensiveness, is easily proved. Suppose that there is a structure $Sp$, an $n$, and two non-coextensive pure predicates $\text{Sat}_{(Sp,n)}$ and $\text{Sat}^*_{(Sp,n)}$ such that $\Psi(Sp, n, \text{Sat}_{(Sp,n)})$ and $\Psi(Sp, n, \text{Sat}^*_{(Sp,n)})$. Let $n$ be the least natural number for which such non-coextensive satisfaction predicates exist. If $n = 0$, a contradiction can be derived using the fact that the atomic formulas satisfied according to the two predicates are determined by $Sp$. If $n > 0$, a contradiction can be derived from the fact that the complex formulas satisfied according to the two predicates are determined by $\text{Sat}_{(Sp,n-1)}$ and $\text{Sat}^*_{(Sp,n-1)}$ which are coextensive, since $n$ is the least number for which non-coextensive satisfaction predicates exist.

In light of Proposition 12, for any given structure $Sp$, we can define a single satisfaction predicate:

$$(\exists \text{Sat}(Sp))(\forall x)(\text{Sat}(Sp)(x) \iff (\exists n)(\exists \text{Sat}(Sp,n))\text{Sat}(Sp,n)(x)).$$

Following the standard model-theoretic notation, for any structure $Sp$, any formula $\phi$, and any sequence $s$ whose length equals the number of free variables of $\phi$, we write $Sp \models \phi [s]$ for $\text{Sat}_{(Sp)}(\phi, s)$.

The notions of structure and satisfaction just defined are based on the idea that the domain of the interpretation is the universe of the metatheory. Such notions have no analogues in standard model theory, since the domains of interpretation are set-sized. In the context of BTPp, we can easily recapture the notion of a model (and a corresponding satisfaction relation) with a restricted domain. In particular, we may have pure predicates serve as domains.
We define $S^p_p$ to be a structure with a *restricted domain* if there is a pure predicate $P^p$ such that

(i) for any $x$, if $x$ is a constant, there is a unique $y$ such that $P^p(x)$ and $S^p_p(x, y)$;

(ii) for any $x$, if $x$ is an $n$-ary relation symbol, then for every $y$ such that $S^p_p(x, y)$, $y$ is a sequence of length $n$ and, for each term $z$ of $y$, $P^p(z)$;

(iii) for any $x$, if $x$ is an $n$-ary function symbol, then for every $y$ such that $S^p_p(x, y)$, $y = (z, w)$ for some sequence $z$ of length $n$ whose terms are in $P^p$ and some object $w$ in $P^p$ such that for no other $w'$, $S^p_p(x, (z, w'))$.

By $S^p_p$ we denote a structure restricted to $P^p$. We call $P^p$ the domain of $S^p_p$. Satisfaction for restricted structures, $\text{Sat}_{(S^p_p, n)}$ can be defined by imposing the following conditions on the unrestricted notion of satisfaction:

(a) finite sequences providing the interpretation of free variables must be built out of objects in the domain, that is, for every sequence $y$ providing the interpretation of free variables, each term $z$ of $y$ must be such that $P^p(z)$; and

(b) the clause for the existential quantifier must refer only to sequences built out of objects in the domain.

The introduction of restricted structures will enable us to recapitulate a number of results in standard model theory, which will be invoked in the proof of the completeness theorem in the next chapter.

We assume a standard classical deductive system for $\mathcal{L}_{PC}$, say that found in [2]. It is a routine exercise to show that the deductive system is sound for the semantics

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9See [2], Chapter 2, section 4.
we have just laid out. Our goal is to prove a version of the completeness theorem. We take up this task in Chapter 2. Before we do, it will be useful to prove an analogue of the Schöder-Bernstein Theorem in the setting of BTPp.

1.6 Schröder-Bernstein Theorem

In this section, we prove an analogue of the Schröder-Bernstein Theorem in BTPp. It will play an important role in the proof of the Completeness Theorem (Stage 6) in the next chapter.

**Theorem 11.** (Schröder-Bernstein) For any pure predicates $A^p$ and $B^p$, if there is an injective functional predicate $F^p$ from $A^p$ to $B^p$ and an injective functional predicate $G^p$ from $B^p$ to $A^p$, then there is bijective functional predicate $H^p$ from $A^p$ to $B^p$.

**Proof.** We imitate, within BTPp, König’s proof of the theorem. For every $a$ such that $A^p(a)$, we say that $a$ terminates if and only if there is a finite sequence $y$ of length $n$ such that

(i) $a$ is the first term of $y$;

(ii) for every $z$ and $w$, if $w$ is the $i$-th term of $y$, $1 < i \leq n$, and $z$ is the $(i - 1)$-th term of $y$, then

$$w = \begin{cases} G^{-1}[z] & \text{if } i \text{ is even,} \\ F^{-1}[z] & \text{if } i \text{ is odd; } \end{cases}$$

(iii) for every $z$, $z$ is the $n$-th term of $y$ if and only if, if $n$ is even, there is no $w$ such that $F^{-1}[z] = w$, and if $n$ is odd, there is no $w$ such that $G^{-1}[z] = w$. 

39
So $a$ terminates if the sequence

$$a, G^{-1}[a], F^{-1}[G^{-1}[a]], G^{-1}[F^{-1}[G^{-1}[a]]], \ldots$$

terminates. It is easy to show that if $a$ terminates, the sequence described above is unique. We call it the terminating $a$-sequence.

If $a$ terminates, we say that $a$ terminates in $A^p$ if the length of the terminating $a$-sequence is odd. If the length of the terminating $a$-sequence is even, we say that $a$ terminates in $B^p$.

By Pure Comprehension we can define the following pure functional predicate $H^p$ with domain $A^p$. For any $x$ such that $A^p$,

$$H^p(x) = \begin{cases} 
  G^{-1}[x] & \text{if } x \text{ terminates in } B^p, \\
  F^p[x] & \text{if } x \text{ terminates in } A^p \text{ or does not terminate.}
\end{cases}$$

We must show that $H^p$ is a bijection between $A^p$ and $B^p$.

First, $H^p$ is surjective. Let $b$ be such that $B^p(b)$. If $b$ is not in the range of $F^p$, then $G^p[b]$ terminates in $B^p$, hence $H^p[G^p[b]] = b$. If $b$ is in the range of $F^p$, then $b = F^p[x]$ for some $x$ such that $A^p(x)$. If $x$ terminates in $B^p$, then $G^p[b]$ also terminates in $B^p$ and, as before, $H^p[G^p[b]] = b$. If $x$ terminates in $A^p$ or does not terminate, $b = F^p[x]$.

In either case, $b$ is in the range of $H^p$. So $H^p$ is surjective.

Moreover, $H^p$ is injective. Suppose that there are $x$ and $y$ such that $A^p(x)$ and $A^p(y)$, and $H^p[x] = H^p[y]$. Let $z$ be $H^p[x]$, that is, $H^p[y]$. We have three cases.

(Case 1) If both $x$ and $y$ terminate in $B^p$, then $G^{-1}[x] = z = G^{-1}[y]$. But $G^p$ is injective, therefore $x = y$.

(Case 2) If both $x$ and $y$ terminate in $A^p$ or do not terminate, then $F^p[x] = z = F^p[y]$. Since $F^p$ is injective, it must be that $x = y$. 

40
(Case 3) If $x$ terminates in $B^p$ but $y$ terminates in $A^p$ or does not terminate, we have that $G^{-1}[x] = z = F[y]$. But then $y = F^{-1}[z] = F^{-1}[G^{-1}[x]]$. It follows that $x$ must have the same status as $y$. Indeed, in this case, these two sequences are identical:

$$x, \ G^{-1}[x], \ F^{-1}[G^{-1}[x]], \ ...$$

$$x, \ z, \ y, \ ...$$

So $x$, like $y$, must either terminate in $A^p$ or not terminate. Contradiction.

This completes the proof that $H^p$ is injective, thereby completing the proof of the Schröder-Bernstein Theorem in BTPp. \qed
Chapter 2

Outline of the Completeness
Theorem

Let INF be the set of formulas

\[ \{ (\exists x_1)...(\exists x_n) \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j : n \in \mathbb{N} \} \]

Our goal is to outline a proof of the following completeness theorem in BTPp.

**Theorem 12. (Completeness Theorem)** If the universe is linearly ordered, then, for any set \( T \) of sentences that is deductively consistent with INF, \( T \) is satisfiable, that is, there is a pure (unrestricted) structure \( S^p \) such that \( S^p \models \sigma \) for any \( \sigma \) in \( T \).

Let \( T_0 \) be any set of sentences (theory) that is deductively consistent with INF. We want to find a structure \( S^p \) that satisfies \( T_0 \). The proof of the theorem proceeds in six main stages in which both the language and the theory are repeatedly expanded. At each stage the given expansion is shown to be satisfiable. The following chart displays this process schematically. The relevant notation will be introduced in due course.
Chapter 2. Outline of the Completeness Theorem

<table>
<thead>
<tr>
<th>Language</th>
<th>Theory</th>
<th>Constants</th>
<th>Structure</th>
<th>Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}_{PC}$</td>
<td>$T_0$</td>
<td>$C(T_0)$</td>
<td>Completeness in $\mathbb{Z}_2$</td>
<td>$M^p$</td>
</tr>
<tr>
<td>$&lt;$</td>
<td>$\mathcal{L}_{PC(&lt;)}$</td>
<td>$T_0(&lt;)$</td>
<td></td>
<td>$M^p$</td>
</tr>
</tbody>
</table>

| Skolem functions | $\mathcal{L}$ | $T$ | $C(T)$ | $M^p$ | $\mathbb{N}$ |

| $c, q \in \mathbb{Q}$ and $x$ in the universe | $\mathcal{L}'$ | $T'$ | $C(T')$ | Compactness and Ramsey Theorem | $M'$ |

| $c, q \in \mathbb{Q}$ and $x$ in the universe | $\mathcal{L}^*$ | $T^*$ | $C(T^*)$ | | $M^*$ |

For later reference, the columns report, respectively, the expansion of the language, the resulting language, the expanded theory, the main theorems used to show that the theory is satisfiable, the structure satisfying the theory, and its domain.
2.1 Stage 1

In the first stage, we argue that $T_0$ is satisfiable. In particular, there is a pure structure $M^p$ with domain $\mathbb{N}$ that satisfies every sentence in $T_0$. Since $T_0$ is consistent with INF, it is consistent. It follows from a version of the completeness theorem for restricted domains, specifically countable domains, that $T_0$ is satisfiable in a restricted structure with domain $\mathbb{N}$. The availability of the completeness theorem for restricted domains is entailed by the fact that this theorem is provable in $Z_2$ (second-order arithmetic) and there is a straightforward interpretation of $Z_2$ in BTPp (see below).\(^1\) Thus, completeness for countable domains is available in BTPp as well.

Let $\pi$ be a translation map from the language of $Z_2$ into the language of BTPp such that

(i) $\pi$ is the identity on logical connectives, $+$, $\cdot$, $<$, and object variables;

(ii) set membership is translated as predication: for every atomic formula $t \in X$ of $Z_2$, $\pi[t \in X]$ is $P^p_X(t)$;

(iii) $Z_2$ quantifiers are restricted to $\mathbb{N}$, that is, for every formula $\phi$ of $Z_2$ and object variable $n$, $\pi[\exists n \phi]$ is $(\exists n)(n \in \mathbb{N} \land \pi[\phi])$ and $\pi[\forall n \phi]$ is $(\forall n)(n \in \mathbb{N} \rightarrow \pi[\phi])$;

(iv) set quantifiers of $Z_2$ are translated as quantifiers over pure predicates, that is, for every formula $\phi$ of $Z_2$ and set variable $X$ free in $\phi$, $\pi[\exists X \phi]$ is $(\exists P^p_X)\pi[\phi]$ and $\pi[\forall X \phi]$ is $(\forall P^p_X)\pi[\phi]$.

\(^1\)For a presentation of $Z_2$ and results concerning logical theorems provable in it, or in its subsystems, see [6], Chapters I, II.8, and IV.3.
Chapter 2. Outline of the Completeness Theorem

For clauses (ii) and (iv) we assume any one-one correspondence between the set variables of the language of $\mathbb{Z}_2$ and the variables for pure predicates in the language of BTPp. So $P^p_X$ is the pure predicative variable associated with $X$.

Theorems 1-8 proved in Chapter 1 show that the translation of every basic axiom of $\mathbb{Z}_2$ is provable in BTPp. In addition to the basic axioms, $\mathbb{Z}_2$ has an induction axiom and a comprehension axiom. Here are these axioms and their translations under $\pi$, where $\varphi$ is any formula in the language of $\mathbb{Z}_2$ in which $X$ does not occur free:

$$\forall X(0 \in X \land \forall n (n \in X \rightarrow n + 1 \in X)) \rightarrow \forall n(n \in X)$$

$$(\forall P^p) (P^p(0) \land (\forall n) (n \in \mathbb{N} \rightarrow (P^p(n) \rightarrow P^p(n + 1))) \rightarrow (\forall n)(n \in \mathbb{N} \rightarrow P^p(n))),$$

$$\forall x_1...\forall x_m \exists X \forall n (n \in X \leftrightarrow \varphi(n, x_1, ..., x_m))$$

$$(\forall x_1)...(\forall x_m)((x_1 \in \mathbb{N} \land ... \land x_m \in \mathbb{N}) \rightarrow (\exists P^p)(\forall n) (n \in \mathbb{N} \rightarrow (P^p(n) \leftrightarrow \varphi(n, x_1, ..., x_m)))).$$

It is straightforward to show that the translation of the induction axiom is provable in BTPp. The axiom of Pure Comprehension of BTP does not allow for object parameters, so we must show that the translation of the comprehension axiom of $\mathbb{Z}_2$, with parameters from natural numbers, is provable in BTPp.

**Proposition 13.** BTPp proves

$$(\forall x_1)...(\forall x_m)((x_1 \in \mathbb{N} \land ... \land x_m \in \mathbb{N}) \rightarrow (\exists P^p)(\forall n)(n \in \mathbb{N} \rightarrow (P^p(n) \leftrightarrow \varphi(n, x_1, ..., x_m)))).$$

---

2For the basic axioms of $\mathbb{Z}_2$, see [6], p. 4.
Chapter 2. Outline of the Completeness Theorem

**Proof.** We proceed by induction on $m$, the number of parameters. For $m = 0$, the claim follows immediately from Pure Comprehension.

As an inductive hypothesis, suppose that comprehension with parameters holds for a formula with an arbitrary number $m$ of parameters. We show that it holds for any formula $\varphi$ with $m + 1$ parameters. For $m + 1$, we have:

\[(\forall x_1) \cdots (\forall x_{m+1})( (x_1 \in \mathbb{N} \land \ldots \land x_m \in \mathbb{N}) \rightarrow \exists P^p \forall n (n \in \mathbb{N} \rightarrow (P^p(n) \leftrightarrow \varphi(n, x_1, \ldots, x_{m+1}))).)\]

Suppose that it is false, namely, there is a formula $\varphi$ such that, for some $a_1, \ldots, a_{m+1} \in \mathbb{N}$,

\[(1) \quad \neg(\exists P^p)(\forall n)(n \in \mathbb{N} \rightarrow (P^p(n) \leftrightarrow \varphi(n, a_1, \ldots, a_{m+1}))).\]

Call $\psi(x)$ the following formula

\[\psi(x) := x \text{ is the least natural number such that } (\exists x_1) \cdots (\exists x_m)
\]

\[(x_1 \in \mathbb{N} \land \ldots \land x_m \in \mathbb{N} \land \neg(\exists P^p)(\forall n)(n \in \mathbb{N} \rightarrow (P^p(n) \leftrightarrow \varphi(n, x_1, \ldots, x_m, x))).)\]

Since $\psi$ has no free variable other than $x$, $\psi$ can be used in Pure Comprehension to obtain

\[(2) \quad \exists Q^p)(\forall x)(Q^p(x) \leftrightarrow \psi(x)).\]

It follows from (1) that there is at least one object satisfying $\psi$. Moreover, by the characterization of $\psi$, there is at most one object satisfying $\psi$. Let $Q^p$ be a witness for (2). The predicate $Q^p$ then holds of exactly one object. Call it $b$. By inductive hypothesis, using $Q^p$ as parameter, we have:

\[(\forall x_1) \cdots (\forall x_m)(x_1 \in \mathbb{N} \land \ldots \land x_m \in \mathbb{N} \rightarrow \exists P^p)(\forall n)(n \in \mathbb{N} \rightarrow (P^p(n) \leftrightarrow (\forall x)(Q^p(x) \rightarrow \varphi(n, x_1, \ldots, x_m, x))).)\]
Chapter 2. Outline of the Completeness Theorem

Since \( b \) is the unique object in \( Q^p \), it follows that

\[
(3) \quad (\forall x_1)\ldots(\forall x_m)(x_1 \in \mathbb{N} \land \ldots \land x_m \in \mathbb{N} \to \\
(\exists P^p)(\forall n)(n \in \mathbb{N} \to (P^p(n) \iff \varphi(n, x_1, \ldots, x_m, b))).
\]

Also, since \( b \) is in \( Q^p \), by definition of \( Q^p \), there are \( b_1, \ldots, b_m \in \mathbb{N} \) such that

\[
(4) \quad \neg(\exists P^p)(\forall n)(n \in \mathbb{N} \to (P^p(n) \iff \varphi(n, b_1, \ldots, b_m, b))).
\]

If we instantiate the universal quantifiers of (3) with \( b_1, \ldots, b_n \), we have:

\[
(5) \quad (\exists P^p)(\forall n)(n \in \mathbb{N} \to (P^p(n) \iff \varphi(n, b_1, \ldots, b_m, b))).
\]

But (4) contradicts (5). We must conclude that

\[
(\forall x_1)\ldots(\forall x_{m+1})(x_1 \in \mathbb{N} \land \ldots \land x_{m+1} \in \mathbb{N} \to \\
(\exists P^p)(\forall n)(n \in \mathbb{N} \to (P^p(n) \iff \varphi(n, x_1, \ldots, x_{m+1}))))
\]

for any formula \( \varphi \).

Via the translation map \( \pi \), \( Z_2 \) can be interpreted in \( \text{BTPp} \). We know that the completeness theorem for countable domains can already be proved in the subsystem \( \text{WKL}_0 \) of \( Z_2 \).\(^3\) It follows that it is provable in \( \text{BTPp} \) as well. Thus, \( T_0 \) is satisfied in a pure structure with domain \( \mathbb{N} \).

### 2.2 Stage 2

Next we expand \( \mathcal{L}_{PC} \) by introducing a new binary relation \( \prec \). Let us call the expanded language \( \mathcal{L}_{PC_{\prec}} \). We add to \( T_0 \) an axiom stating that \( \prec \) is a linear ordering:

\[
(\forall x)(\forall y)(\forall z)(x \not< y \land (x < y \lor x = y \lor y < x) \land ((x < y \land y < z) \to x < z)).
\]

\(^3\)See [6], pp. 137-139.
We call the resulting set of sentences $T_0(\prec)$. By interpreting $\prec$ as the $<$ on $\mathbb{N}$, $M^p$ can be seen to be a model of $T_0(\prec)$.

### 2.3 Stage 3

Now we expand again both the object language and the theory $T_0(\prec)$. As for the object language, we first expand $\mathcal{L}_{PC\prec}$ by adding Skolem functions, that is, new function symbols $F_\psi$ for every formula $\psi$ in $\mathcal{L}_{PC\prec}$ of the form $(\exists x)\phi$. Let us call the resulting language $\mathcal{L}$. The theory $T_0(\prec)$ is expanded as follows.

For every formula $\psi(x_1, ..., x_n)$ of the form $(\exists x)\phi(x, x_1, ..., x_n)$, we add the axiom:

(a) $(\forall x_1) ... (\forall x_n)(\psi(x_1, ..., x_n) \rightarrow \psi(F_\psi(x_1, ..., x_n), x_1, ..., x_n))$.\(^4\)

Since $T_0(\prec)$ has now built-in Skolem functions, it has a universal axiomatization. Let us denote by $T$ the universal axiomatization of $T_0(\prec)$. For most of what follows we will discuss $T$ rather $T_0$ or $T_0(\prec)t$. At the end of the proof (Stage 6), we will return to $T_0$ to prove what we are after, namely, that $T_0$ is satisfiable. Let $C(T)$ denote the constants of $T$.

We further expand $\mathcal{L}$ by introducing new constants. For every $q$ in the field $\mathbb{Q}$ of rationals, we add a constant $c_q$. Let us call the resulting language $\mathcal{L}'$. Moreover, we add new axioms to $T$.

(b) For any $q$ and $r$ in $\mathbb{Q}$ such that $q < r$ in the natural ordering on $\mathbb{Q}$, we add the axiom $c_q < c_r$.

\(^4\)For details and facts about introducing Skolem functions in standard model theory, see [1], chapter 3, section 3.
(c) Finally, we add indiscernibility axioms for the new constants of the form $c_{q_i}$.

That is, for any $q_1 < \ldots < q_n$ and $r_1 < \ldots < r_n$ with each $q_i$ and $r_i$ ($1 \leq i \leq n$) in $\mathbb{Q}$, and for any atomic formula $\phi(x_1,\ldots,x_n)$ of $\mathcal{L}'$ (the expanded language, including both the new constants with rational indices and the Skolem functions), we add the axiom:

$$\phi(c_{q_1},\ldots,c_{q_n}) \leftrightarrow \phi(c_{r_1},\ldots,c_{r_n}).$$

Call $T'$ the resulting set of sentences and call $C(T')$ the set of constants in $T'$.

By compactness and a version of the Finite Ramsey Theorem, both available in $\mathbb{Z}_2$ (hence in BTPp), it can be shown there is a pure structure $M'$ that satisfies $T'$.

Let us sketch this argument.

Since $T_0$ is satisfied in a pure structure with domain $\mathbb{N}$, the same holds of $T$. Let $M^p$ be a pure structure with domain $\mathbb{N}$ satisfying $T$. We want to show that any finite subset $X$ of $T'$ is satisfiable. Any sentence of $X$ in the language of $\mathcal{L}$ is satisfied in $M^p$. Since $T$ has been expanded with axioms of the form $c_q < c_r$ and with indiscernibility axioms, finitely many of these axioms occur in $X$. Let $X'$ be the subset of $X$ containing these axioms. It suffices to show that the finitely many constants in $C(T') - C(T)$ occurring in $X'$ can be interpreted in $M^p$ so as to make the new axioms in $X'$ true.

Let $A = \{\phi_1,\ldots,\phi_m\}$ be the atomic formulas occurring in the indiscernibility axioms of $X'$ and let $K = \{c_{q_1},\ldots,c_{q_k}\}$ be the constants appearing in $A$. Let us assume that $q_1 < \ldots < q_k$.

We can label every subset of $A$ with a number in $\{1,\ldots,2^m\}$. For any $\phi_i \in A$ whose constants are $c_{i_1},\ldots,c_{i_j}$ ($c_{i_1},\ldots,c_{i_j}$ among $c_{q_1},\ldots,c_{q_k}$) and any subset $Z$ of $\mathbb{N}$ of cardinality $k$, we say that $Z$ satisfies $\phi_i$ if and only if $M^p \models \phi_i$ when $c_{i_1},\ldots,c_{i_j}$ are interpreted, respectively, by the first $j$ elements of $Z$ in the order given by $\prec$. 

49
Chapter 2. Outline of the Completeness Theorem

For any $N \subseteq \mathbb{N}$ of cardinality at least $k$, define a function $f_{N,k}$ mapping subsets of $N$ of cardinality $k$ into $\{1, \ldots, 2^m\}$ according to the following rule:

for every $Z \in dom(f_{N,k})$, $f_{N,k}(Z)$ is the label of the subset of $A$ whose formulas are all and only those satisfied by $Z$.

It follows from a version of the Finite Ramsey Theorem that there is $N \in \mathbb{N}$ such that $f_{N,k}$ is constant. This means, in particular, that all the subsets of $N$ of cardinality $k$ satisfy the same formulas of $A$. Let $I$ be the set of the first $k$ elements of $N$ in the order given by $<$. We propose to interpret $c_{q_1}, \ldots, c_{q_k}$ with the elements of $I$ according to their order. We show that this interpretation satisfies every indiscernibility axiom in $X'$. Consider any indiscernibility axiom in $X'$:

$$\phi(c_{p_1}, \ldots, c_{p_n}) \leftrightarrow \phi(c_{r_1}, \ldots, c_{r_n}),$$

where $c_{p_1}, \ldots, c_{p_n}$ and $c_{r_1}, \ldots, c_{r_n}$ are among $c_{q_1}, \ldots, c_{q_k}$. The axiom is true in the proposed interpretation if and only if both $\phi(c_{p_1}, \ldots, c_{p_n})$ and $\phi(c_{r_1}, \ldots, c_{r_n})$ are satisfied in $M^p$ when the constants are interpreted according to the proposed interpretation, or both formulas fail to be so satisfied. For every $q \in \{q_1, \ldots, q_k\}$, let $\iota[c_q]$ be the proposed interpretation of $c_q$. Let $F_1$ and $F_2$ be any subsets of $N$ of cardinality $k$ whose first $n$ elements are, respectively, $\iota[c_{p_1}], \ldots, \iota[c_{p_n}]$ and $\iota[c_{r_1}], \ldots, \iota[c_{r_n}]$. Since $N$ can be chosen to be greater than $2k$, two such subsets of $N$ exist. The Finite Ramsey Theorem entails that $F_1$ and $F_2$ satisfy the same formulas of $A$, hence, they both satisfy $\phi$ or both fail to satisfy $\phi$. It follows that the indiscernibility axiom is true in the proposed interpretation.

Since the proposed interpretation of the constants $c_{q_1}, \ldots, c_{q_k}$ satisfies the indiscernibility axioms in $X'$ and it clearly satisfies every axiom $c_q < c_r$ (for $q < r$), we

---

5See [5], pp. 176-177.
can conclude that any finite subset $X$ of $T$ is satisfiable. Thus, by compactness, there is a pure structure $M'$ satisfying $T'$.

2.4 Stage 4

Let $M'$ be the submodel of $M'$ generated by the set of indiscernibles $C(T') - C(T)$. We now carry out another expansion of the object language and of $T'$. We use $M'$ to obtain a structure satisfying the theory that results from this expansion. We call this a pre-structure, since it does not give the identity sign its intended interpretation.

For every $x$ in the universe and every $q$ in $\mathbb{Q}$, we introduce a constant $c_{x,q}$. Call the expanded language $L^*$. Moreover, we add to $T'$ indiscernibility axioms for the constants of the form $c_{x,q}$. Given the natural ordering on $\mathbb{Q}$ and the linear ordering of the universe, we can define the following lexicographic ordering on the new constants and on their indices:

$$c_{x,q} < c_{y,r} \iff ((x \neq y \rightarrow x < y) \land (x = y \rightarrow q < r)).$$

So, for every $c_{x_1,q_1} < ... < c_{x_n,q_n}$ and $c_{y_1,r_1} < ... < c_{y_n,r_n}$, and for any atomic formula $\phi$ of $L^*$, we have the indiscernibility axiom:

$$\phi(c_{x_1,q_1}, ..., c_{x_n,q_n}) \leftrightarrow \phi(c_{y_1,r_1}, ..., c_{y_n,r_n}).$$

We call $T^*$ the set of sentences obtained by adding these indiscernibility axioms to $T'$. Also, we denote by $C(T^*)$ the constants of $T^*$.

The last two stages in the proof of the completeness theorem concern the construction of a model $M^*$ of $T^*$, starting from the model $M'$ of $T'$. The model $M^*$ is a pre-structure in the above sense. After showing that $M^*$ satisfies $T^*$, we must
find a way to turn $M^*$ into a structure where the identity sign has its intended interpretation. This situation is familiar from the proof of the completeness theorem in standard model theory where, in the pre-structure, the identity sign is interpreted by an equivalence relation. In the standard setting, one obtains the sought model by taking the equivalence classes as the elements of domain. This move is not available here. We must find an explicit way of picking a canonical name for each ‘equivalence class’. We will prove a key lemma that we will allow us to accomplish this.

2.5 Stage 5

The domain of $M^*$ consists of all the closed terms of $\mathcal{L}^*$. We define the following dense linear ordering $<^*$ on the terms.

(i) First come the terms in $C(T)$ ordered according to $M'$.\(^6\)

(ii) Then we have the terms in $C(T') - C(T)$, i.e., the terms of the form $c_q$ with $q$ in $\mathbb{Q}$. They are ordered according to the natural ordering on their indices.

(iii) Finally we have the terms in $C(T^*) - C(T')$, i.e., the terms of the form $c_{x,q}$. They are ordered lexicographically, as specified above.

We define a notion of reduction. A reduction $\alpha$ is any partial mapping from a finite subset of $C(T^*) - C(T)$ into $C(T') - C(T)$ satisfying two conditions:

\(^6\)See Stage 3.
(1) it is identity on $C(T') - C(T)$;

(2) it is order-preserving with respect to $<^*$, that is, for any finite subset $X$ of $C(T^*) - C(T)$ and for any $s$ and $t$ in $X$,

$$s <^* t \rightarrow \alpha(s) <^* \alpha(t).$$

A definition of a reduction of a sentence $\sigma$ of $T^*$ can be derived from that of a reduction of a set of constants: for any reduction $\alpha$, $\alpha(\sigma)$ is the sentence obtained by replacing any constant $c$ in $\sigma$ with $\alpha(c)$. If $\sigma$ contains a constant for which $\alpha$ is not defined, then $\alpha(\sigma)$ is not defined. A reduction for a sequence of terms is any reduction that is defined for all the terms of the sequence.

Using the notion of reduction, let us define an equivalence relation $=^*$ on the closed terms of $L^*$.

**Definition 1.** For any terms $s$ and $t$ of $L^*$, $s =^* t$ if and only if there is a reduction $\alpha$ such that $M' \vDash \lbrack \alpha(s) = \alpha(t) \rbrack$.

There is an alternative but, as we will show, equivalent definition of $=^*$.

**Definition 2.** For any terms $s$ and $t$ of $L^*$, $s =^* t$ if and only if, for any reduction $\alpha$, $M' \vDash \lbrack \alpha(s) = \alpha(t) \rbrack$.

The following lemma shows that these two definitions are equivalent.

**Lemma 1.** The two preceding definitions of $=^*$ are equivalent.

**Proof.** Let $s, t$ be terms of $L^*$. Suppose that there is a reduction $\alpha$ such that

$$(*') M' \vDash \lbrack \alpha(s) = \alpha(t) \rbrack.$$
Let $\beta$ any reduction for $s, t$. We want to show that

$$M' \models \gamma \beta(s) = \beta(t).$$

If $s = t$, then obviously

$$M' \models \gamma \beta(s) = \beta(t).$$

Suppose that $s <^* t$. Then, since reductions are order-preserving, $\alpha(s) < \alpha(t)$. As we saw at stage 3, for any $q$ and $r$ in $Q$ such that $q < r$, $T'$ includes the axiom $c_q < c_r$. This means that the constants in the range of reduction functions are linearly ordered. So, if $\alpha(s) < \alpha(t)$, then

\begin{equation}
(\star\star) \quad M' \models \gamma \alpha(s) < \alpha(t),
\end{equation}

where $M'$ satisfies $T'$. But $T'$ includes $T(<)$. Thus, since $<$ is a strict linear ordering, $(\star)$ and $(\star\star)$ are inconsistent. Therefore, it is not the case that $s <^* t$. For the same reason, it cannot be the case that $t <^* s$. So it must be that $s = t$. Therefore,

$$M' \models \gamma \beta(s) = \beta(t)$$

for any reduction $\beta$.

Next we show that $=^*$ is an equivalence relation.

**Lemma 2.** The relation $=^*$ defined above is an equivalence relations on the terms of $L^*$.

**Proof.** Reflexivity is obvious given that reductions are functions. Suppose that $s =^* t$, then, by definition, for any reduction $\alpha$,

$$M' \models \gamma \alpha(s) = \alpha(t).$$

This means that

$$M' \models \gamma \alpha(t) = \alpha(s).$$

54
So \( t =^* s \). That is, \( =^* \) is symmetric. To show that \( =^* \) is transitive, assume that \( s =^* t \) and \( t =^* u \). So there is a reduction \( \alpha \) such that

\[
M' \models \gamma \alpha(s) = \alpha(t) \quad \text{and} \quad M' \models \gamma \alpha(t) = \alpha(u).
\]

So, by the characterization of satisfaction,

\[
M' \models \gamma \alpha(s) = \alpha(u) \]

Thus, \( s =^* u \).

For every relation \( R_{n}^{i} \) of \( L^* \), we define (often omitting super- and subscripts for convenience) a relation \( R_{n}^{i} \) holding among terms of \( L^* \) and such that, for any terms \( t_1, ..., t_n \) of \( L^* \),

\[
R_{n}^{i}(t_1, ..., t_n) \iff M' \models \gamma R_{n}^{i}(\alpha(t_1), ..., \alpha(t_n)) \]

for some reduction \( \alpha \) on \( t_1, ..., t_n \).

We now show that relations so defined are ‘respectful’ in the following sense.

**Lemma 3** (Respectfulness). *For any terms \( s_1, ..., s_n \) and \( t_1, ..., t_n \) of \( L^* \) and relation \( R_{n}^{i} \), if \( s_1 =^* t_1, ..., s_n =^* t_n \), then

\[
R_{n}^{i}(s_1, ..., s_n) \text{ if and only if } R_{n}^{i}(t_1, ..., t_n).
\]

*Proof.* By the characterization of \( R^* \), for any reduction \( \alpha \),

\[
R_{n}^{i}(s_1, ..., s_n) \iff M' \models \gamma R_{n}^{i}(\alpha(s_1), ..., \alpha(s_n)) \]

Since \( s_1 =^* t_1, ..., s_n =^* t_n \), it follows that, for each \( i \) \( (1 \leq i \leq n) \),

\[
M' \models \gamma \alpha(s_i) = \alpha(t_i) \]
Thus,
\[ M' \models \forall \mathbf{R}^n_i(\alpha(s_1), ..., \alpha(s_n)) \iff M' \models \forall \mathbf{R}^n_i(\alpha(t_1), ..., \alpha(t_n)). \]

Given that
\[ M' \models \forall \mathbf{R}^n_i(\alpha(t_1), ..., \alpha(t_n)) \iff R^*(t_1, ..., t_n), \]
we can conclude that
\[ R^*(s_1, ..., s_n) \iff R^*(t_1, ..., t_n). \]

An analogous result can be proved for functions.

**Lemma 4.** For any terms \( s_1, ..., s_n \) and \( t_1, ..., t_n \) of \( \mathcal{L}^* \) and function term \( f_i \), if
\[ s_1 =^* t_1, ..., s_n =^* t_n, \]
then
\[ f_i(s_1, ..., s_n) =^* f_i(t_1, ..., t_n). \]

**Proof.** By definition of \( =^* \), for any reduction \( \alpha \) on the function terms \( f_i(s_1, ..., s_n) \) and \( f_i(t_1, ..., t_n) \),
\[ f_i(s_1, ..., s_n) =^* f_i(t_1, ..., t_n) \iff M' \models \forall f_i(\alpha(s_1), ..., \alpha(s_n)) = f_i(\alpha(t_1), ..., \alpha(t_n)). \]

As in the proof of the previous lemma, since \( s_1 =^* t_1, ..., s_n =^* t_n \), it follows that, for each \( i \) \( (1 \leq i \leq n) \),
\[ M' \models \forall \alpha(s_i) = \alpha(t_i). \]

Therefore,
\[ M' \models \forall f_i(\alpha(s_1), ..., \alpha(s_n)) = f_i(\alpha(t_1), ..., \alpha(t_n)). \]
Now we characterize a pre-structure, that is, a preliminary interpretation for the language $\mathcal{L}^*$. As mentioned above, the identity sign will not receive its intended interpretation in the pre-structure, as it is interpreted by the equivalence relation $=^*$. In the last stage of the proof of the completeness theorem, this problem will be remedied.

In order to specify the pure structure $M^*$ that interprets the non-logical terminology of $\mathcal{L}^*$, we must specify the interpretation of each non-logical item in the language plus equality.

(A) For any constant $c$ in the language $\mathcal{L}^*$, $M^*[c] = c$.

(B) For any function symbol $f_i$ and terms $t_1, ..., t_n$, $M^*[f_i(t_1, ..., t_n)] = f_i(t_1, ..., t_n)$, the term itself.

(C) For any relation symbol $R^n_i$, $M^*[R^n_i] = R^*_i$, the relation defined above.

(C1) As a particular case of (C), we have that $M^*[<] = <^*$.

(C1) As another particular case, we also have that $M^*[=] = =^*$.

This gives us a preliminary interpretation of $\mathcal{L}^*$ satisfying $T^*$. In order show that $M^*$ satisfies $T^*$, it suffices to show that $M^*$ satisfies the indiscernibility axioms of $\mathcal{L}^*$ and that, for any universal sentence $\sigma$ of $\mathcal{L}'$,

$$M' \models \sigma \Rightarrow M^* \models \sigma.$$ 

Since $M' \models T$ and all sentences in $T$ are universal, if $M^*$ satisfies the universal sentences satisfied by $M'$ and the indiscernibility axioms, it satisfies $T^*$.
Let $c_{x_1,q_1} < ... < c_{x_n,q_n}$ and $c_{y_1,r_1} < ... < c_{y_n,r_n}$, and let $\phi$ be an atomic formula of $\mathcal{L}^*$. Then the indiscernibility axiom for $\phi$ is:

$$\phi(c_{x_1,q_1},...,c_{x_n,q_n}) \iff \phi(c_{y_1,r_1},...,c_{y_n,r_n}).$$

Suppose that $\phi(c_{x_1,q_1},...,c_{x_n,q_n})$ is $R^n_i(c_{x_1,q_1},...,c_{x_n,q_n})$. Then

$$M^* \models R^n_i(c_{x_1,q_1},...,c_{x_n,q_n}) \iff R^n_i(c_{y_1,r_1},...,c_{y_n,r_n})$$

if and only if

$$M' \models \forall \alpha(c_{x_1,q_1}),...\forall \alpha(c_{x_n,q_n}) \cdot R^n_i(\alpha(c_{x_1,q_1}),...,\alpha(c_{x_n,q_n})) \iff R^n_i(\alpha(c_{y_1,r_1}),...,\alpha(c_{y_n,r_n}))^\gamma,$$

for any reduction $\alpha$. However, since reductions are order-preserving,

$$\alpha(c_{x_1,q_1}) < ... < \alpha(c_{x_n,q_n})$$

and

$$\alpha(c_{y_1,r_1}) < ... < \alpha(c_{y_n,r_n}).$$

Since $M'$ satisfies the indiscernibility axioms for $\mathcal{L}'$, we have (2), hence, (1). In the other cases of atomic formulas, one proceeds in a similar fashion. Therefore, $M^*$ satisfies the indiscernibility axioms of $\mathcal{L}^*$.

We now show that $M^*$ satisfies all the universal sentences satisfied by $M'$. We argue for the contrapositive. Let $(\forall v_1)...(\forall v_n)\phi(v_1,...,v_n)$ be a universal sentence of $\mathcal{L}'$. This means that $\phi$ is quantifier-free. Suppose that

$$M^* \not\models (\forall v_1)...(\forall v_n)\phi(v_1,...,v_n).$$

Thus, there are terms $t_1,...,t_n$ of $\mathcal{L}^*$ such that

$$M^* \not\models \phi(t_1,...,t_n).$$
By induction on the quantifier-free formulas of $\mathcal{L}'$, it can be shown that

\[(3) \quad M^* \models \phi(t_1, \ldots, t_n) \iff M' \models [\phi(\alpha(t_1), \ldots, \alpha(t_n))].\]

Suppose that $\phi$ is atomic. If $\phi(t_1, \ldots, t_n)$ is $R^i_n(t_1, \ldots, t_n)$, then (3) is true by construction of $M^*$. The cases of negation and conjunction are also straightforward. Given that

$$M^* \not\models \phi(t_1, \ldots, t_n),$$

(3) entails that

$$M' \not\models [\phi(\alpha(t_1), \ldots, \alpha(t_n))].$$

Thus,

$$M' \not\models (\forall v_1) \ldots (\forall v_n) \phi(v_1, \ldots, v_n).$$

We can conclude that universal sentences of $\mathcal{L}'$ are preserved in $M^*$. So $M^*$ satisfies $T^*$.

### 2.6 Stage 6

In the last stage of the proof, we refine the pre-structure to obtain the intended interpretation of equality. In order to avoid factoring out by the equivalence classes, the goal is to find a canonical name for each equivalence class determined by $=^*$. The following Key Lemma, proved by Friedman (personal communication), suggests a way in which a canonical name can be determined.

Let $c_{x_1}, \ldots, c_{x_n}$ and $c_{y_1}, \ldots, c_{y_n}$ be constants in $C(T^*) - C(T)$. By $t(c_{x_1}, \ldots, c_{x_n})$ we denote a closed term $t$ of $\mathcal{L}^*$ where $c_{x_1}, \ldots, c_{x_n}$ are the constants that appear in $t$ in strictly increasing order with respect to $<^*$. Also, if $c_{y_1}, \ldots, c_{y_n}$ are not the original
constants in $t$, $t(c_{y_1}, ..., c_{y_n})$ denotes the term obtained by replacing each $c_{x_i}$ with $c_{y_i}$, with $c_{y_1}, ..., c_{y_n}$ in strictly increasing order.

**Lemma 5** (Key Lemma). For any $t(c_{x_1}, ..., c_{x_n})$, let $A_t$ be a set defined as follows:

$$A_t = \{ i : 1 \leq i \leq n \text{ and there exists } i^* \in C(T^*) - C(T), \text{ such that } t(c_{x_1}, ..., c_{x_n}) \neq* t(c_{x_1}, ..., c_{x_{i-1}}, c_{x_{i^*}}, c_{x_{i+1}}, ..., c_{x_n}) \}.$$

Then $t(c_{x_1}, ..., c_{x_n}) =* t(c_{y_1}, ..., c_{y_n})$ if and only if, for every $i \in A_t$, $x_i = y_i$.

**Proof.** Let $t(c_{x_1}, ..., c_{x_n})$ be a term in $L^*$. ($\Rightarrow$) Suppose that, for every $i \in A_t$, $x_i = y_i$. We want to show that

$$(*) \quad t(c_{x_1}, ..., c_{x_n}) =* t(c_{y_1}, ..., c_{y_n}).$$

Let $B = \{1, ..., n\} - A$. If $B$ is empty then $t(c_{x_1}, ..., c_{x_n})$ just is $t(c_{y_1}, ..., c_{y_n})$. So (*) follows. Let us assume that $B$ is non-empty and of cardinality $k \leq n$. Without loss of generality, we may assume that $B = \{1, ..., k\}$. Consider the following sequence of terms:

$$t_0 = t,$$

for each $i$ ($i \leq n$), $t_i = t(c_{y_1}, ..., c_{y_i}, c_{x_{i+1}}, ..., c_{x_n}).$

That is, $t_i$ is the term obtained from $t$ by changing each $c_{x_j}$, $j \leq i$, with $c_{y_j}$. It is easy to show that $t_i =* t_{i+1}$ ($0 \leq i \leq n - 1$). For instance, since $1 \in B$, there is no $c_y$ such that

$$t(c_{x_1}, c_{x_2}, ..., c_{x_n}) \neq* t(c_{y_1}, c_{x_2}, ..., c_{x_n}).$$

So

$$t(c_{x_1}, c_{x_2}, ..., c_{x_n}) =* t(c_{y_1}, c_{x_2}, ..., c_{x_n}).$$
Thus, $t_0 =^* t_1$. Similarly for each $t_i =^* t_{i+1}$. Since $=^*$ is an equivalence relation, we conclude that $t_0 =^* t_n$, that is,

$$t(c_{x_1}, ..., c_{x_n}) =^* t(c_{y_1}, ..., c_{y_n}).$$

This completes the proof of the right-to-left direction.

($\Rightarrow$) For the forward direction, suppose that there is $i \in A$, such that $x_i \neq y_i$. Assume ($\ast$). We derive a contradiction. Since $i \in A$, there is $i^*$ such that

$$t(c_{x_1}, ..., c_{x_n}) \neq^* t(c_{x_1}, ..., c_{x_i-1}, c_{x_i^*}, c_{x_{i+1}}, ..., c_{x_n}).$$

Suppose that $c_{x_i} <^* c_{x_i^*}$. (The case in which $c_{x_i} <^* c_{x_i}$ is completely analogous.) Define the set $Y$ as follows:

$$Y = \{ j \leq n : c_{x_i} <^* c_{y_j} <^* c_{x_i^*} \}.$$

We have two cases.

**Case 1**: $Y$ is empty. Then these sequences have the same order type:

$$x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n, y_1, ..., y_n,$$

$$x_1, ..., x_{i-1}, x_i^*, x_{i+1}, ..., x_n, y_1, ..., y_n.$$

By indiscernibility for $\mathcal{L}'$,

$$M' \models \tau t(\alpha(c_{x_1}), ..., \alpha(c_{x_i-1}), \alpha(c_{x_i}), \alpha(c_{x_{i+1}}), ..., \alpha(c_{x_n})) = t(\alpha(c_{y_1}), ..., \alpha(c_{y_n})) \leftrightarrow$$

$$t(\alpha(c_{x_1}), ..., \alpha(c_{x_i-1}), \alpha(c_{x_i^*}), \alpha(c_{x_{i+1}}), ..., \alpha(c_{x_n})) = t(\alpha(c_{y_1}), ..., \alpha(c_{y_n})).$$

It follows from

$$t(c_{x_1}, ..., c_{x_n}) =^* t(c_{y_1}, ..., c_{y_n}).$$
that

\[ M' \models \neg t(\alpha(c_{x_1}), \ldots, \alpha(c_{x_{i-1}}), \alpha(c_{x_i}), \alpha(c_{x_{i+1}}), \ldots, \alpha(c_{x_n})) = t(\alpha(c_{y_1}), \ldots, \alpha(c_{y_n})) \wedge. \]

Thus,

\[ M' \models \neg t(\alpha(c_{x_1}), \ldots, \alpha(c_{x_{i-1}}), \alpha(c_{x_i}), \alpha(c_{x_{i+1}}), \ldots, \alpha(c_{x_n})) = t(\alpha(c_{y_1}), \ldots, \alpha(c_{y_n})) \wedge. \]

Therefore,

\[ t(c_{x_1}, \ldots, c_{x_{i-1}}, c_{x_i}, c_{x_{i+1}}, \ldots, c_{x_n}) = t(c_{y_1}, \ldots, c_{y_n}). \]

This, together with (\*) and the fact that =\* is an equivalence relation, contradicts our assumption that

\[ t(c_{x_1}, \ldots, c_{x_n}) \neq t(c_{x_1}, \ldots, c_{x_{i-1}}, c_{x_i}, c_{x_{i+1}}, \ldots, c_{x_n}). \]

**Case 2:** \( Y \) is not empty, say \( Y = \{j_1, \ldots, j_m\} \). This means that

\[ c_{x_i} <\* c_{y_{j_1}} <\* \cdots <\* c_{y_{j_m}} <\* c_{x_i}, \]

with \( y_{j_1}, \ldots, y_{j_m} \) among \( y_1, \ldots, y_n \), and \( y_{j_m} \) equal to \( y_k \) (\( k \leq n \)). If \( k = n \), let \( f \) be any order-preserving function from \( Y \) into the interval \((x_i^*, x_{i+1})\). If \( k < n \), let \( f \) be any order-preserving function from \( Y \) into the interval \((x_i^*, \min\{y_{k+1}, x_{i+1}\})\). It follows that these sequences have the same order type:

\[ x_1, \ldots, x_n, y_1, \ldots, y_n, \]

\[ x_1, \ldots, x_n, y_1, \ldots, f(y_{j_1}), \ldots, f(y_{j_k}), \ldots, y_n, \quad \text{and} \]

\[ x_1, \ldots, x_{i-1}, x_i^*, x_{i+1}, \ldots, x_n, y_1, \ldots, f(y_{j_1}), \ldots, f(y_{j_k}), \ldots, y_n. \]

An argument analogous to the one employed in **Case 1** shows that

\[ t(c_{x_1}, \ldots, c_{x_n}) = t(c_{y_1}, \ldots, c_{f(y_{j_1})}, \ldots, c_{f(y_{j_k})}, \ldots, c_{y_n}). \]
Chapter 2. Outline of the Completeness Theorem

\[ t(c_{x_1}, \ldots, c_{x_{i-1}}, c_{x_i^*}, c_{x_{i+1}}, \ldots, c_{x_n}) =^* t(c_{y_1}, \ldots, c_{f(y_{j_1})}, \ldots, c_{f(y_{j_k})}, \ldots, c_{y_n}). \]

Thus,

\[ t(c_{x_1}, \ldots, c_{x_n}) =^* (c_{x_1}, \ldots, c_{x_{i-1}}, c_{x_i^*}, c_{x_{i+1}}, \ldots, c_{x_n}). \]

Contradiction. This completes Case 2 and the proof of the forward direction of the Lemma.

We say that two terms \( t \) and \( t' \) have the same splitting if \( A_t = A_{t'} \). Before describing how to choose a canonical name for each equivalence class determined by \( =^* \), we prove the following lemma.

**Lemma 6.** Let \( s \) and \( s' \) be any two terms such that \( s = t(c_{x_1}, \ldots, c_{x_n}) \), \( s' = t(c_{y_1}, \ldots, c_{y_n}) \), and \( t(c_{x_1}, \ldots, c_{x_n}) =^* t(c_{y_1}, \ldots, c_{y_n}) \). Then \( s \) and \( s' \) have the same splitting.

**Proof.** Suppose for *reductio* that the splitting differs with respect to \( i \) (\( 1 \leq i \leq n \)), namely, there is \( i \in A_s \) but \( i \not\in A_{s'} \). Now, \( i \in A_t \), so \( x_i = y_i \) and there is \( c_{x_i^*} \) such that

\[ t(c_{x_1}, \ldots, c_{x_n}) \neq^* t(c_{x_1}, \ldots, c_{x_{i-1}}, c_{x_i^*}, c_{x_{i+1}}, \ldots, c_{x_n}). \]

Suppose that \( x_i <^* x_i^* \). (The case in which \( x_i^* <^* x_i \) is completely analogous.) Since \( x_i = y_i \) and the ordering is dense, there is \( i^{**} \) such that

\[ x_{i-1} <^* x_i <^* x_i^{**} <^* x_{i+1}, \quad \text{and} \]

\[ y_{i-1} <^* x_i = y_i <^* x_i^{**} <^* y_{i+1}. \]

Since the following sequences have the same order type

\[ x_1, \ldots, x_n, x_1, \ldots, x_{i-1}, x_i^{**}, x_{i+1}, \ldots, x_n \]

\[ x_1, \ldots, x_n, x_1, \ldots, x_{i-1}, x_i^{**}, x_{i+1}, \ldots, x_n, \]

63
the fact that
\[ t(c_{x_1}, \ldots, c_{x_n}) \neq^* t(c_{x_1}, \ldots, c_{x_{i-1}}, c_{x_i}, c_{x_{i+1}}, \ldots, c_{x_n}) \]
entails that
\[ t(c_{x_1}, \ldots, c_{x_n}) \neq^* t(c_{x_1}, \ldots, c_{x_{i-1}}, c_{x_i}, c_{x_{i+1}}, \ldots, c_{x_n}). \]

However, \( i \notin A_s' \), thus
\[ t(c_{y_1}, \ldots, c_{y_n}) =^* t(c_{y_1}, \ldots, c_{y_{i-1}}, c_{x_i}, c_{y_{i+1}}, \ldots, c_{y_n}). \]
Since \( t(c_{x_1}, \ldots, c_{x_n}) =^* t(c_{y_1}, \ldots, c_{y_n}) \), it follows that
\[ t(c_{x_1}, \ldots, c_{x_n}) =^* t(c_{y_1}, \ldots, c_{y_{i-1}}, c_{x_{i+1}}, c_{y_{i+1}}, \ldots, c_{y_n}). \]

Finally, the following sequences have the same order type:
\[ x_1, \ldots, x_n, y_1, \ldots, y_{i-1}, x_{i+1}, y_{i+1}, \ldots, y_n, \]
\[ x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y_1, \ldots, y_{i-1}, x_{i+1}, y_{i+1}, \ldots, y_n. \]
Given that
\[ t(c_{x_1}, \ldots, c_{x_n}) =^* t(c_{y_1}, \ldots, c_{y_{i-1}}, c_{x_{i+1}}, c_{y_{i+1}}, \ldots, c_{y_n}), \]
we have that
\[ t(c_{x_1}, \ldots, c_{x_{i-1}}, c_{x_{i+1}}, c_{x_{i+1}}, \ldots, c_{x_n}) =^* t(c_{y_1}, \ldots, c_{y_{i-1}}, c_{x_{i+1}}, c_{y_{i+1}}, \ldots, c_{y_n}). \]
This is inconsistent with what has already been established, namely,
\[ t(c_{x_1}, \ldots, c_{x_n}) =^* t(c_{y_1}, \ldots, c_{y_{i-1}}, c_{x_{i+1}}, c_{y_{i+1}}, \ldots, c_{y_n}), \quad \text{and} \]
\[ t(c_{x_1}, \ldots, c_{x_n}) \neq^* t(c_{x_1}, \ldots, c_{x_{i-1}}, c_{x_{i+1}}, c_{x_{i+1}}, \ldots, c_{x_n}). \]
So we conclude that \( s \) and \( s' \) have the same splitting. □
We are now ready to describe how to choose a canonical name for each equivalence class determined by \( =^* \). For any term \( s(c_{z_1}, ..., c_{z_m}) \),

1. Consider any open term \( t(v_1, ..., v_n) \), with no constant from \( C(T^*) - C(T) \), such that for some terms \( c_{y_1}, ..., c_{y_n} \) of \( C(T^*) - C(T) \),

\[
t(c_{y_1}, ..., c_{y_n}) =^* s(c_{z_1}, ..., c_{z_m}).
\]

Of all the open terms so selected, we choose the one with the least Gödel number. Say it is \( t(v_1, ..., v_n) \).

2. Consider any sequences of terms \( c_{x_1}, ..., c_{x_n} \) and \( c_{y_1}, ..., c_{y_n} \). Let

\[
t'(c_{x_1}, ..., c_{x_n}) = t(c_{x_1}, ..., c_{x_n}), \text{ and}
\]

\[
t''(c_{y_1}, ..., c_{y_n}) = t(c_{y_1}, ..., c_{y_n}).
\]

By Lemma 6, \( t' \) and \( t'' \) have the same splitting, that is, \( A_{t'} = A_{t''} \). Thus, for each \( i \in A_{t'} \), \( c_{x_i} = c_{y_i} \). Let us call the indices in \( A_{t'} \) the \( A \)-positions of \( t \). So the terms in the \( A \)-positions are uniquely individuated. For any term \( c_i \) in an \( A \)-position (i.e., \( i \in A_{t'} \)), the canonical name for the equivalence class of \( s(c_{z_1}, ..., c_{z_m}) \) will be obtained by first replacing each variable \( v_i \) of \( t(v_1, ..., v_n) \) with \( c_i \).

3. Finally we must choose a term for any index \( j \notin A_{t'} \). There are many acceptable ways of doing this explicitly. We illustrate one of them. Fix some constants \( c_{x_1}, ..., c_{x_n} \) and let \( t' = t(c_{x_1}, ..., c_{x_n}) \). Let \( A_s \) be the set \( A_{t'} \). By the above results, such a set is independent of the constants \( c_{x_1}, ..., c_{x_n} \). Say that \( A_s = \{i_1, ..., i_m\} \).

Define

\[
X = \{1, ..., n\} - A_s,
\]
where $X$ is, say, $\{j_1, \ldots, j_k\}$ ($m + k = n$). We recall that the subscript $x_i$ of a term in $C(T^*) - C(T')$ is of the form $\bar{x}_i, q_i$ with $q_i \in \mathbb{Q}$ and $\bar{x}_i$ an object. Let $p \in \mathbb{Q}$ be the smallest distance between the rational parts of any two distinct indices of constants occupying an $A$-position and having the same non-rational part. Define a function $f$ with domain $X$ as follows:

$$f(j) = \begin{cases} 
\text{the greatest } i \in A_s \text{ such that } c_i < ^{*} c_j, & \text{if such } i \text{ exists,} \\
0 & \text{otherwise.}
\end{cases}$$

For every index $j \in X$ such that $f(j) > 0$, we replace each variable $v_j$ of $t(v_1, ..., v_n)$ with

$$c_{\bar{x}_{f(j)}}, q_i + j \cdot \frac{p}{n+1}.$$

For every index $j \in X$ such that $f(j) = 0$, we replace each variable $v_j$ of $t(v_1, ..., v_n)$ with

$$c_{\bar{x}_1}, q_i - \frac{p}{n+1}.$$

It follows from Lemma 5 that the term obtained by substituting each $v_i$ of $t(v_1, ..., v_n)$ ($i \in A_s$) in the way indicated in (2) and by substituting each $v_j$ ($j \in X$) in the way just indicated is within the equivalence class of $s(c_{z_1}, ..., c_{z_m})$.

So we have a canonical name for each equivalence class based on the steps (1)–(3). As in standard model theory, this allows us to turn the pre-structure into a structure $S_{cn}$ that interprets the identity sign as identity. The domain of $S_{cn}$ contains the canonical names of all the equivalence classes determined by $=^*$. In order to show that there is a structure $S^p$ satisfying $T^*$ whose domain is the universe, we need to show that there is a bijection between the domain of $S_{cn}$ and the universe. Since the identity map provides an injection from the domain of $S_{cn}$ into the universe, in light of the
Schröder-Bernstein Theorem proved in Chapter 1, it suffices to show that there is an injection from the universe into the domain of \( S_{cn} \).

For every \( x \) in the universe, let \( F^p \) be a functional predicate such that \( F^p[x] \) is the canonical name of the equivalence class of \( c_{x,0} \). Let us denote such a name by \( t_{[c_{x,0}]} = \ldots \). Suppose that \( x \neq y \) with \( x < y \) (\(<\) is here the order on the universe). Then, \( c_{x,0} <^* c_{y,0} \). It follows that, for any reduction \( \alpha \), \( \alpha(c_{x,0}) <^* \alpha(c_{y,0}) \). Thus, \( c_{x,0} \) and \( c_{y,0} \) are in two different equivalence classes. As a consequence, \( t_{[c_{x,0}]} \neq t_{[c_{y,0}]} \). A similar conclusion follows if \( y < x \). So \( F^p \) is injective.

We conclude that there is a structure \( S^p \) that satisfies \( T^* \) whose domain is the universe. Therefore, \( S^p \) satisfies \( T \). Recall, however, that we began with the set of sentences \( T_0 \) of which \( T \) was a universal axiomatization. So \( T \vdash \sigma \) for every \( \sigma \in T_0 \). By soundness with respect to structures with unrestricted domains, it follows that \( S^p \) satisfies \( T_0 \). This completes the proof of the Completeness Theorem: any set of sentences \( T_0 \) that is deductively consistent with INF is satisfiable in a pure structure \( S^p \) whose domain is unrestricted.
References


