INFERENC E ON STUDENTS' PROBLEM SOLVING PERFORMANCES
THROUGH THREE CASE STUDIES

Thesis

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By

Pingping Zhang, B.A.

Graduate Program in Education, School of Teaching and Learning

The Ohio State University

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Thesis Committee

Dr. Azita Manouchehri, Advisor

Dr. Douglas T. Owens
ABSTRACT

This study investigated the problem solving behaviors as each solved four common non-routine problems. The major goals of the researcher were to determine whether the individual's performances were consistent across different subject areas and problem types that could be solved using different heuristics and to identify possible factors that influenced children's choices and strategy use in different contexts. Data analysis included search for isolating common and unique patterns of behaviors that children exhibited as well as possible factors that led to institution of those patterns. Background interviews and problem solving interviews were conducted with each participant individually. Based on the interview episodes, a case study report for each participant was prepared. The three cases served as data sources for the overall analysis.

The results showed that self-monitoring was positively correlated with success in performance on certain mathematical activities. The results suggested that intra-task strategy flexibility does not imply success at reaching correct answers to tasks, yet further proposed that the level of intra-task strategy flexibility might depend largely on the individual's confidence and preference for the use of certain strategies. Inconsistency in the same individual's mathematics problem solving behaviors across different subject areas and/or heuristics usage was revealed. The results also supported the findings that the nature of the individual's knowledge and its organization serves as a major influence
on their success or failure as problem solvers. Contrary to the previous studies, the study found no relationship between stated confidence in mathematics and success in non-routine problem solving performance.
DEDICATION

I dedicate this work to my parents
ACKNOWLEDGEMENTS

First I want to thank my academic advisor Dr. Azita Manouchehri. She has offered me not only tremendous and inspirational advice on academic study, but also generous care since I arrived in US. She has been a great mentor (although I am not sure whether she likes this word or not) and a great friend to me. I extremely appreciate the opportunity to study and work with her in the past 2 years and in future. Because of her encouragement and help, I was able to complete this thesis. I also want to thank Dr. Douglas Owens for his time and effort on reviewing my thesis and offering suggestions.

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Finally, I want to thank my peers who had offered me generous help and advice on my thesis: Aaron, Manju, Scott, and Yating. All of you have been my closest friends since I came here.
VITA

March 5, 1985 .......................................................... Born in Beijing, P.R. China

September, 2003 ......................... B.A., English; B.S. Mathematics, Beihang University

FIELDS OF STUDY

Major field: .................................................. Education

Specialization: ............................................. Mathematics Education
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CHAPTER 1
INTRODUCTION

The development of problem-solving ability has been among the primary goals of mathematics teaching and learning (Wilson, Fernandez, & Hadaway, 1993). Among the five general goals for students in the area of mathematics identified in *Curriculum and Evaluation Standards for School Mathematics*, "becoming a mathematical problem solver: essential to becoming a productive citizen, which requires experience in solving a variety of extended and non-routine problems (cited in Romberg, 1994, p.288)" is the one most prominently cited. While this goal has been a persistent part of mathematics education community for over a century (Manouchehri, Zhang, & Gilchrist, in press), issues regarding how to develop problem solving skills among learners through instruction continues to be a major dilemma (Manouchehri, Zhang, Gilchrist, & Somayajulu, 2010).

Previous research on problem solving has focused on effective implementation of teachers' problem solving approaches and improvement of students' problem solving performance (Anderson & White, 2004). This body of work has considered ways in which instruction on particular strategies, heuristics and cognitive engagement might impact learners' problem solving performance. Muir, Beswick, and Williamon (2008) suggested that researchers must focus on understanding what successful problem solvers do and help individuals develop their problem solving behaviors to complement this.
They further argued that instead of focusing on whether particular strategies should be taught or not and how, greater attention must be devoted to understanding processes that individuals use when engaged in problem solving.

I have been engaged in a longitudinal study of potential influences of an after school mathematics enrichment program designed for approximately 80 children from historically underrepresented communities. The program focuses on helping children develop their mathematical thinking skills, including mathematical problem solving, by experiencing an inquiry based curriculum. The program caters to students from 7th to 9th grades for 3 years. The children are enrolled in the Young Scholars Program supported by the Office of Minority Affairs at The Ohio State University. Over the course of the implementation of the program I observed ways in which children from different grade levels interacted with problem solving activities, behaviors they exhibited when working on open explorations and working with technology. I noticed that some students exhibited highly sophisticated problem solving skills when tackling certain problems, failed to perform similarly in other contexts. This inconsistency in performance was particularly puzzling to me. I began to wonder what factors might contribute to the phenomenon of successful problem solving on some tasks and not others.

On the other hand, based on my own school experience in China, I observed that the problem solving behaviors exhibited by students enrolled in the enrichment program were significantly different from Chinese students. Under the experience-based mathematics education system, Chinese students complete more standardized and procedural tasks (Cai & Nie, 2007). The most sophisticated approach which a problem is designed to foster is always the expected solution. Approaches besides standard methods
are not accepted as correct solutions, even when the reported final answer is correct. Due to this fact, Chinese students' problem solving behaviors are less varied and they tend to be more successful performing on school tasks.

Compared to Chinese students, the students I observed, while not always proficient in algorithms and school-taught procedures, managed to show facility with the use of various approaches when solving problems. They exhibited much more non-standardized mathematical behaviors and yet were successful in solving problems. I found the inconsistent patterns of performance also intriguing. This phenomenon further motivated me to examine carefully mathematical problem solving performance of children who were involved in the enrichment program as a way of understanding their motives, reasoning and thinking when confronted with non-routine tasks.

**Purpose of the Study**

The purpose of this study was to examine performance of three students when solving non-routine problems. I was particularly interested in learning whether the individuals' performances were consistent across different subject areas and problem types that could be solved using different heuristics. Moreover, I was interested in identifying those factors that influenced children's choices and strategy use in different contexts to account for potential inconsistencies in their performance. Lastly, I was curious to isolate common and unique patterns of behaviors that children exhibited as well as those factors that led to the institution of those patterns. Interviews were conducted with three participants who exhibited outstanding problem solving performances in a research group.
Rationale

Previous studies have provided evidence highlighting factors for the success or failure of implementation of problem solving approaches in mathematics teaching. However, detailed studies on individuals' problem solving behaviors are limited. De Hoyos, Gray, and Simpson (2002) examined two undergraduate students' problem solving processes, describing their approaches to solving one problem and providing insights into their understanding and their conjecture about mathematics. Moving further from this study, Muir, Beswick, and Willamson (2008) investigated the consistency of individual's problem solving approaches across six different problems concerning patterns and algebra that also utilized different heuristics. Instead of characterizing students as novice and expert problem solvers, they suggested a more detailed model for differentiating among different levels of problem solving behaviors: naive, routine, and sophisticated. Based on their model, they concluded that individual's problem solving behaviors were consistent across the problems, exhibiting characteristics within a same level on all tasks. Whether individual's problem solving behaviors would be consistent across different subject areas with different heuristics was not concerned in this study. Whether individuals perform consistently in all subject areas and heuristics remains an open line of inquiry. In agreement with Muir, Beswick, and Willamson (2008) thesis regarding lack of productivity of labeling individuals as "poor" or "good" problem solvers, it is important to consider specific factors that contribute to their success or failure on tasks accordingly. This knowledge would better position a teacher in helping students develop their problem solving skills.
Statement of the Problem

This study investigated participants' problem solving behaviors when solving non-routine problems. Problems were selected to study the consistency of individual's problem solving behaviors across different subject areas and heuristics. The researcher also studied possible patterns of problem solving processes for each individual. Strategies and representations applied by participants were considered. Similarities and differences in problem solving behaviors/performances among three individuals were examined. Possible factors that impacted the consistency and performances were investigated. The following four questions focused data collection and analysis:

1. What mathematical problem solving processes do children exhibit when working on non-routine tasks?

2. What mathematical problem solving strategies do children naturally use as they work on non-routine tasks?

3. What types of metacognitive behaviors do children naturally exhibit as they engage in mathematical problem solving?

4. Are children's strategy uses and problem solving processes consistent across different problem types depending on the kind of heuristic needed or the mathematical subject area used in context?
CHAPTER 2

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

The issue of improving mathematical problem solving of school children has been the subject of passionate debates for over a century (Bottge, 2001). Research on mathematical problem solving began in the early 1970s and continued on until mid 90s. An overview of problem solving research emphases and methodologies used in research from 1970 to 1994 is shown in Table 1.

<table>
<thead>
<tr>
<th>Dates</th>
<th>Problem-solving research emphases</th>
<th>Research methodologies used</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970 – 1982</td>
<td>Isolation of key determinants of problem difficulty; Identification of characteristics of successful problem solvers; Heuristics training</td>
<td>Statistical regression analysis; early &quot;teaching experiments&quot;</td>
</tr>
<tr>
<td>1978 – 1985</td>
<td>Comparison of successful and unsuccessful problem solvers (experts vs. novices); Strategy training</td>
<td>Case studies; &quot;think aloud&quot; protocol analysis</td>
</tr>
<tr>
<td>1982 – 1990</td>
<td>Metacognition; relation of affects/beliefs to problem solving; Metacognition training</td>
<td>Case studies; &quot;think aloud&quot; protocol analysis</td>
</tr>
<tr>
<td>1990 – 1994</td>
<td>Social influences; problem solving in context (situated problem solving)</td>
<td>Ethnographic methods</td>
</tr>
</tbody>
</table>

Table 1. An overview of problem solving research emphases and methodologies 1970-1994 (Lester, 1994, p. 664)
Research projects that have focused on the study of mathematical problem solving among children generally fall under three broad categories, including: (1) characterization of mathematical problem solving, (2) novice–expert studies aimed at defining attributes of successful mathematical problem solvers (affective and cognitive); and (3) metacognitive behaviors exhibited during problem solving process. The following section will offer the most prominent results in each of these domains. A description of how the current study connects to past research is also offered.

**Problem Solving: What is it and what it is not?**

A problem is defined as a "situation in which a goal is to be attained and a direct route to the goal is blocked" (Kilpatrick, 1985, p.2). Thus, for different individuals, the same task could be viewed as either an exercise or a problem, where a direct route to the goal is clear for an exercise. Mayer (1985) stated problem solving as "a series of mental operations that are directed toward some goal" (p. 124). Therefore, the problem solving process could be a representation of an individual's own internal exploration towards an unknown path, instead of one's ability to directly retrieve known techniques. Research on mathematics problem solving dated back to 1930s, later largely inspired by Polya's (1945) *How to solve it*, introduced the concepts of heuristics and strategies when problem solving. Since then, various dimensions of problem solving has been studied using different theoretical stances. Among many include: computer-simulated problem solving, expert problem solving, problem solving strategies and heuristics, metacognitive process, problem posing, mathematical modeling, and interdisciplinary problem solving (English & Sriraman, 2010).
A survey of the collective body of work indicates consensus within the community that good problem solvers are distinguished from poor problem solvers in several respects, summarized by Lester (1994) as: 1. Good problem solvers know more and their knowledge is well connected and composed of rich schemata. 2. Good problem solvers focus more on structural features instead of literal features of problems. 3. Good problem solvers are more aware of their own strengths and weakness in terms of problem solving. 4. Good problem solvers monitor and regular their problem-solving efforts in a better way. 5. Good problem solvers are more concerned about obtaining best solutions to problems.

However, in synthesizing the content, results and impact of a large body of research on problem solving, English and Sriraman (2010) indicated that previous research had focused mainly on solving word problems that are emphasized in school textbooks, primarily routine and procedural (which is distinguished as solving problem from problem solving (Manouchehri, Zhang, & Gilchrist, in press). Problem solving concerns struggling with non-routine problems. What's more, teaching students about Polya-style heuristics and strategies has not been found successful in improving students' problem solving ability (Schoenfeld, 1992; Lester & Kehle, 2003). The authors posited that an explanation for this result is due to community's little understanding of how individuals come to make decisions about when, where, why, and how to use heuristics, strategies, and metacognitive actions. Focusing on applying these strategies, without understanding how and why individuals make decisions about pathways for solving problems is non-productive (English, Lesh, & Fennelwal, 2008). In studying problem
solving pathways that individuals might take, the use of a comprehensive mathematical problem solving theory is essential as described below.

**Mathematical Problem Solving**

Mason (1985) divided the process of problem solving into three phases: entry, attack, and review. The entry phase includes thinking about "what do I know," "what do I want," and "what can I introduce." The review phase further contains "checking," "reflecting," and "extending." However, a typical problem solving activity is seldom linear; an individual always goes back and forth when proceeding to the desired outcome. Also it is possible that the attack phase is difficult to observe or the review phase is missing. Generally speaking, this perspective could provide an overview of the entire problem solving process so that a clearer relationship between steps could be identified and studied.

Mason proposed a visual model to display how the processes and the emotional states were linked together dynamically (see Figure 1).

![Figure 1. Mason's dynamic model for problem solving process](image)
In his model, the back segment of the band represents the process between the achieved generality, the original state and the experience in Attack, while the forward segment represents the process between the achieved generality and further questions which it provoked. Each loop represents an opportunity to extend understanding to a higher level.

This model, however, does not indicate the internal or external forces which impact the movement from one phase of process to next. Between each process, distinct motives/stimuli might exist, initiated by the individual or outside information that can either facilitate or prevent making progress towards a more general understanding and ultimately, more efficient problem solving performance. These forces could be crucial to study the movement of each step, where positive influences are needed to be identified and generalized. Mason's model applied in this research was modified in order to demonstrate certain internal (self-initiated) and external (interviewer-initiated) forces influencing processes involved.

**Skills Central to Mathematical Problem Solving**

Problem solving strategies are one of the fundamental components of mathematical thinking (Schoenfeld, 1992). There is evidence indicating that students' use of heuristic strategies is positively related to success in problem solving, although the effect may not always be significant (Kantowski, 1977). Yet a number of studies have shown the deficiencies that students exhibit when applying heuristics and metacognitive strategies to their problem solving processes (Schoenfeld, 1985; 1992).
A study of four 6th grade students' problem solving behaviors was conducted by Muir, Beswick, and Williamson (2008). The strategies students used in solving 6 problems were analyzed and three categories of performance were proposed to associate with the levels of problem solving behaviors students exhibited including, naive, routine and sophisticated. The authors stated that the unsuccessful problem solvers tended to manipulate the numbers in the problem and stick with one or two strategies. The consistency of approaches across problems for each individual was also studied, and the conclusion was that most individuals consistently exhibited behaviors characteristic in one category. The limit of this study is that the 6 problems used were about number and number sense, thus the consistency in performance across different content areas was not revealed. What's more, the interviews in this study were divided into 2 parts: individual paper work and a verbal interview following the completion of each problem that explaining what the subjects had done. This could have led to a loss of important information at each thinking stage as the participants may forget something that lead to each switch of strategy, or only be willing to explain the steps that they think they are "right."

Flexibility refers to the quantity of variations that can be introduced by an individual in the concepts and mental operations one already possess (Demetriou, 2004). Elia, Heuvel-Panhuizen, and Kolovou (2009) discussed two methods for studying strategy flexibility usage: inter-task flexibility (changing strategies across problems) and intra-task flexibility (changing strategies within problems). They used three non-routine problems to study the strategy use and strategy flexibility by 4th grade high achievers. An implicative statistical method using the computer software CHIC (Classification
Hierarchique, Implicative et Cohesitive) was performed to determine whether the strategies used by students to solve the three problems were successful or not. Guess-and-check strategy was found to be the most crucial strategy that led to the success of the three pattern/algebra problems. An important finding was that higher inter-task strategy flexibility was displayed by more successful problem solvers, while intra-task strategy flexibility did not support the problem solvers in reaching a correct answer. An intra-task strategy flexibility study showed that the understanding to the problem influenced the correctness of the answer, instead of the flexibility of the strategies.

**Multi-representations**

There is some evidence that the use of various representations could help students gain a better understanding of subject, and build a connectionist model, which is beneficial for encoding and retrieving knowledge in long-term memory (Bruning, 2004). What's more, multi-representations that highlight various aspects of different concepts could help an individual to draw a larger picture of the structure (Tripathi, 2008). Also, different representations could provide different perspectives which may help develop better insight into mathematical ideas. Therefore, the use of various representations (either within problems or across problems) could be considered as one of the important components of problem solving ability.
Metacognition

Metacognition refers to one's knowledge concerning one's own cognitive processes and products or anything related to them (Flavell, 1976). Metacognition plays an important role in problem solving development, especially in the process of reflection. Research on metacognition has focused on three categories of intellectual behavior: 1. knowledge about one's own thought processes. 2. Control, or self-regulation. 3. Beliefs and intuitions (Schoenfeld, 1987).

Garofalo and Lester (1985) believed knowledge in mathematics includes the assessment of one's own capabilities and limitations both in general and on specific topics or tasks. The importance of monitoring and controlling of mathematical activities was studied by Cohors-Fressenborg, Kramer, Pundsack, Sjuts, and Sommer (2010). The authors provided evidence, drawing on data from three studies, that self-monitoring was significantly correlated with success in certain mathematical activities. However, the test-items used in this study were all procedural and routine. At this point, evidence is limited in terms of metacognitive behaviors that children exhibit or their impact on performance when conceptual and non-routine problem are used.

Lester (1982) believed that an individual's metacognition before, during, and after a problem solving phase as well as the ability to control his/her sense of monitoring and self-regulation should significantly affect successful problem solving in mathematics. On the other hand, Verschaffel (1999) claimed that metacognition is important during the initial stage of problem solving when an individual is trying to construct an appropriate representation of the problem, as well as during the final stage of problem solving when an individual is trying to justify his/her own answer. In this research, evidence of
metacognitive behaviors across the four problems was analyzed to study their effect on students' performance.

**Gestures**

Gestures are the most observable behaviors during the problem solving process. Studies concerned about gestures have largely focused on the importance of the role of children's gestures in learning (Pine, Lufkin, & Messer, 2004; Broaders et al., 2007). There are limited studies concerned about students' gestures during problem solving activities, and insights have been mostly generated instead of general principle and a solid body of theory (Garber, 1998). Goldin-Meadow (2003) stated that the spontaneous gestures a problem solver produces during problem solving often reveal the way he/she represents the task. Evidence that gestures could reveal implicit or immediate knowledge which may be expressed in speech later was reviewed by Roth (2001). Bieda and Nathan (2009) stated that one's gestures could be used to infer about the characteristics of complex cognitive behaviors during explanations of problem solving. They conducted a study about gestures along with strategies in problem solving. They investigated the gestures through three types of grounding: physical grounding, spatial grounding, and interpretative grounding, where grounding refers to the mapping that an individual makes between an unfamiliar or abstract representation and a more concrete or familiar referent (Glenberg, De Vaga, & Graesser, 2008; Nathan, 2008). Their conclusion was that grounding provided a way to map the unfamiliar to the familiar; however, grounding to concrete instances and interpretations could have a negative effect on problem solving.
Summary

Aside from using the modified version of Mason's (1985) model, I used a linear model in this project to illustrate detailed description of problem solving processes of the participants as they worked on four problems. These two models were used to capture the entry phase, strategy used by the participant (identified as initial, modified, and alternative strategies), phase of justification, phase of reflection, questions, answers, and insights. The phases of justification, reflection and questions were further identified as self-initiated and interviewer-initiated. The length of the line segment for each phase was proportionally scaled along the whole time line to indicate the relative amount of time the participant spent on each phase. This model provided a direct view of the problem solving phases, which was more beneficial when describing the problem solving behaviors.

Additionally, drawing from past research, in this work I relied on analysis of gestures to offer perspectives on not only their emotional state as they worked on tasks but to shed light on ways their cognitive monitoring was enacted. Lastly, in responding to the calls for more careful analysis of children's mathematical performance of children as a way of theorizing about why certain pathways they use when solving problems I focused attention also on studying their choices along the particular orientations they may have exhibited toward tasks, choice of representations, and their preferred strategies.
CHAPTER 3
METHODOLOGY

In answering the research questions driving this study, a task-based interview methodology was used to closely observe and study three students as they worked on non-routine mathematical tasks. A case study report was developed for each student, describing in detail their actions during interview sessions. These case study reports were used, first, to identify and analyze the processes students used and patterns of problem solving behaviors they exhibited while solving problems; second, to describe and analyze the problem solving strategies children utilized, their choice of representations and metacognitive behaviors they accessed and used. This chapter describes the nature of the study, interview protocol, subjects, and methods of data analysis.

Nature of the Study

The purpose of the study was to investigate the patterns of students' problem solving behaviors and thinking as they solved non-routine mathematical problems. Goetz and Lecompte (1984) suggested that the case study approach was one that allowed the researcher to a suitable means for a careful study of a complex phenomenon in an attempt to add theoretical understanding.

Detailed description of the participants' mathematical behaviors and practices
were studied to enable me to keep track of emerging patterns. One of my primary motives was to build a model of problem solving process of children as they engaged, naturally, in solving non-routine problems. This is in line with the goals of case based study research.

**Participants**

Although the major research project from which the data for the study was selected involved interviews with nearly 60 children, only three participants, Jazzy in 8th grade, Liza and Yoni in 9th grade (all pseudo names), were selected to serve as case subjects for the current study. This selection was due to several important considerations as described below.

First, all three participants had signed consent forms to participate in the longitudinal research project. Second, all three had worked on the same four tasks used as data collection sources. This would allow me to draw inferences regarding comparisons among their thinking and orientations, processes they used during problem solving episodes and metacognitive behaviors they exhibited. Additionally, the three participants offered a wide range of backgrounds and habits that would strengthen the potential for generalizability of the results. Despite their differences, the participants shared similar attributes; they were characterized as "successful" students of mathematics as measured by grades they had secured in their mathematics courses and results of the State mandated standardized exams. Lastly, their engagement, efficiency, and success in exploring non-routine problems given to all students in the enrichment program were noticeable among students in the same grade level. Yet each of them displayed distinct behaviors when
interacting with problems: Liza exhibited flexible explorations when tackling problems, Jazzy showed strong reliance on calculator and numbers, and Yoni possessed the most sophisticated mathematics technique learned in school curriculum. Diverse performances were expected among the three participants.

**Data Sources**

The data sources consisted of two interviews with each of the participants. Each interview consisted of approximately 35-40 minutes. Each interview was tape–recorded and used in analysis. The first interview consisted of two parts: the first part was assessing participants' mathematics background information, their beliefs about mathematics, and their views on value of mathematics for their lives. The second part of the first interview the children contained problem solving episodes. During the second interview it was reassured that the participants solved the remaining problem selected for data analysis. The length of time difference between the first and second interview in each case was varied and ranged from one to 6 months.

The participants were interviewed individually. The background interview for each participant followed the same protocol (see Appendix A). The protocol for problem solving interviews suggested the least interruption from interviewers during students' problem solving work. The protocol also suggested that interventions be made only when a clear understanding what children were doing was not evident, when questions existed about the choice of strategies they used when solving problems and if reasoning and justification was not shared by the participants. The children were not restricted by the amount of time they could work on a problem or their choice of representations.
Data Collection Instrument

Four non-routine problems were selected to access participants' problem solving performances in patterns, functions, and geometry (Table 2). The selected subject areas were identified as among the most important ones in school mathematics (NCTM, 2000). Different heuristics were needed and could be used to solve these problems, including working backwards, drawing graph, and looking for patterns. The diversity of subject areas and heuristics served the aim of studying the consistency of individual problem solving behaviors/performances across problem types and subject matter contexts.

1. Joe gives Nick and Tom as much money as each already has. Then Nick gives Joe and Tom as much money as each of them then has. If at the end each has 8 dollars, how much money did each have at the beginning?
2. Water Lilies are growing on a lake. The water lilies grow rapidly, so that the amount of water surface covered by lilies doubles every 24 hours. On the first day of the summer, there was just one water lily. On the 90th day of summer, the lake was entirely covered. On what day was the lake half covered?
3. Consider the two pay options: $300 a week or $7.50 an hour. What factors will affect your choice of option to take. Draw a graph that compares the two pay options, allowing the reader to determine which option might be best for them to take.
4. Consider the graph below: What can we say about the areas of triangles BEC and BFC?

Table 2. Description of problems

These problems were distinct from the problems with which the participants were familiar in school. Implicit problem solving behaviors were expected from the participants.
Data Analysis

Each interview episode was videotaped. The researcher first went through an episode repetitively to track the distinct problem solving phases during each problem solving episode. A problem by problem performance model was developed for each child and then a cross problem performance analysis was completed.

Using two different theoretical lenses, problem solving processes of each child was analyzed and modeled. A detailed description of the processes each child used, strategies they employed and metacognitive behaviors they exhibited was completed. Every strategy and representation used by the participants was identified and recorded. The researcher then took notes on noticeable gestures exhibited during the problem solving phase. The process was the same for each episode.

Following the completion of each case study analysis, the three cases served as data sources for the overall analysis. Constant comparison of these cases along common dimensions allows me to formulate answers to the four research questions. Conclusions were generated from the study results.
CHAPTER 4
THREE CASE STUDIES: LIZA, JAZZY, YONI

This chapter provides a detailed description of the three participants presented in the form of individual cases. Each case consists of four sections. The first section of each case provides background information on the subject elicited during the first part of the first interview. The second part offers an elaborate discussion of the participant's mathematical practices on each of the four problems used for data collection individually (intra–problem analysis). Analysis of the performance is offered along three dimensions: Problem solving phases according to Mason's (1985) model, problem solving strategies used, gestures, and their choice of representations. The third portion of each case study is devoted to an inter–problem solving performance analysis; comparison of performance of each participant on all four problems, along each of the selected themes. The last section presents an overall analysis of each case organized as responses to the four research questions guiding the study.

The Case of Liza

A. Liza's Background

During the first part of the first interview, Liza was asked to articulate her feelings and attitudes towards mathematics, explain what was easy or difficult for her to do and
whether she felt confident in her mathematical ability. This portion of the interview allowed us to build a profile of Liza's thinking about and attitude towards mathematics that could potentially explain her specific mathematical behaviors during the problem solving sessions.

During the second part of the interview, Liza was asked to solve four mathematical problems. She was reassured that our primary interest was in her thinking and the processes she used in tackling problems. We noted that we wanted her to think out loud and to explain her thinking as she interacted with problems. Liza was not restricted to solve the problems in a specific amount of time and she was also reassured she could take as much time as she needed to resolve problems using any method she deemed appropriate. Data on Liza's problem solving performance was compiled during two interview sessions. Problem 1 was administered approximately a month later than the session during which she attempted problems 2 through 4.

Who is Liza?

At the time of data collection Liza was in 9th grade and taking a course in Geometry. She had completed a year of Algebra I in the previous year. Liza identified English as her favorite subject in school. She argued this was so since she aspired to pursue a career in writing. She explained that what she most appreciated about learning mathematics in school was using manipulatives and "extracurricular" activities as she considered those "fun" to do. She said she was confident in her ability to control school mathematics, referring OGT exercises as "easy stuff."

When she was asked whether she liked algebra or geometry more, Liza clearly distinguished between liking the subject and liking a teacher by explaining that while she
liked the algebra teacher from previous year, she enjoyed geometry more. Liza did not seem to be much interested in standard content covered in the school since she could not recall specific topics they were studying at the time of data collection. She also explained that she did not like working on "big long problems" that demanded manipulation of too many numbers. According to her, such tedious manipulation effort made it difficult for her to organize the problem solving process.

Liza believed mathematics to be all about "proving," because standard mathematics questions always requested that the person must either prove statements or explain why things were true. Liza explained that while she considered herself "good at mathematics," she did not particularly like the subject. According to Liza, being "good at mathematics" implied that the person could "solve most problems eventually." She also considered herself different from her classmates who immediately asked the teacher for help when they encountered difficulties with solving assigned tasks. She explained that she believed she could solve all problems only if she "kept trying." Upon further reflection on her current experiences in her geometry course she explained that she did not consider her classroom environment particularly positive since most of her classmates were either taking this course for the second time or were behind their credits. For these reasons, she suggested, her classmates did not particularly care about mathematics.

Liza stated that her feelings about mathematics changed after 2nd grade, once the focus of instruction was shifted from doing "fun" manipulative activities to solely solving word problems. In response to the question of whether she considered mathematics useful in real life, she leaned back, laughed and said "oh, now you sound like my math teacher." This particular episode seemed illustrative of her attitude towards her teacher. The
example she shared resembled typical exercises done in school, cited some "real life exercises" from homework assignments she had done as examples.

When Liza was encouraged to consider and share instances of how mathematics might be used in her own life, she referenced using mathematics when shopping and explained that in real life she would only use mathematics instead of thinking about it, which showed that she distinguished between daily and academic mathematics. She considered solving equations as the easiest activity for her in mathematics, while the problems involving complex processes (i.e. double back) were the most difficult. She explained that although she only encountered such problems on quarter tests, she "always got stuck" on them.

Liza was outspoken and confident about sharing her ideas. She had an outgoing personality and willing to engage in discussions without much prompting.

**B. Liza's Intra-problem Analysis**

Table 3 illustrates an overview of Liza's four problem solving episodes. Column 1 offers the list of problems Liza was asked to solve. Column 2 presents the order in which problems were administered. Column 3 of the table lists the amount of time Liza spent on solving each of the problems. Problem 1 was administered a month after the first interview.

<table>
<thead>
<tr>
<th>Interview episodes</th>
<th>Order</th>
<th>Time of the episode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Money Transaction</td>
<td>Session II</td>
<td>14'02&quot;</td>
</tr>
<tr>
<td>2. Water Lily</td>
<td>Session I</td>
<td>04'01&quot;</td>
</tr>
<tr>
<td>3. Pay Option</td>
<td>Session I</td>
<td>10'42&quot;</td>
</tr>
<tr>
<td>4. Compare Area</td>
<td>Session I</td>
<td>14'49&quot;</td>
</tr>
</tbody>
</table>

Table 3. General overview of Liza's episodes.
a. Money Transaction problem

Joe gives Nick and Tom as much money as each already has. Then Nick gives Joe and Tom as much money as each of them then has. If at the end each has 8 dollars, how much money did each have at the beginning?

Liza spent 14 minutes and 2 seconds on this problem. The recurrent process of enter, attack, and review in this episode revealed her well-developed problem solving ability. Her justifications were effective, which led to either realization of misunderstanding or self-initiated reflection. In this episode, her intra-task strategy flexibility was noticeable. Her confidence changed when she was switching strategies, which was related to whether she thought the numbers/answers made sense or not.

i. Problem solving phases: Global description

Liza's process for solving this problem was: entry – using initial strategy – getting an answer – justifying the answer – using the same strategy with modified information – justifying the new answer – reflecting on previous work – revisiting the problem – using alternative strategy – getting a new answer – justifying and confirming the answer (shown in Figure 2). The self-initiated justification and reflection is represented by shaded figures; the interviewer-initiated justification and reflection is represented by non-shaded figures.

Figure 2. Linear model of Liza's process for solving the Money Transaction problem
This process is also illustrated using Mason's model as shown in Figure 3. The "self-initiated force" column and the "interviewer initiated force" column refer to the self-initiated and the interviewer initiated activities that impacted the dynamic flow. Framing her work using the spiral model of Mason, the recurrent process of revisiting and tackling the problem can be viewed as indicative of mature problem-solving ability.

Figure 3. Mason's model of Liza's process for solving the Money Transaction problem

Liza started her initial strategy soon after she read the problem. After she quickly figured out the answer, she justified her answer upon the request of the interviewer, and she realized that she had misunderstood the information in the problem. Hence, she did not change the strategy, but modified the numerical values she had used in the initial entry phase. As she began to justify her new answer under the modified strategy, she hesitated, seemingly reflecting on her strategy and stating that the answer "did not make sense."
This reflection apparently motivated her to re-enter the problem by carefully revisiting the problem. In the second level of entry, Liza applied an alternative strategy and successfully solved the problem. She showed confidence in her result as she stated that the answer "makes more sense." She successfully justified her final answer at the interviewer's request.

ii. Strategies

The initial strategy Liza used is illustrated in Figure 4. She first wrote an 8 under each person's initial letter, then subtracted 4 from Joe's and Tom's final amount of money because according to her, "they each received as much of money as they got from Nick." Then she subtracted 4 from Nick's amount of money and 2 from Tom's, explaining again that "they each had received as much money as they got from Joe." Finally, she concluded Joe had 4 dollars, Nick had 4 dollars, and Tom had 2 dollars at the beginning.

Figure 4. Liza's initial strategy for the Money Transaction problem

When trying to justify her answer, Liza realized Nick and Joe also had given money instead of only receiving money at each stage of transaction. She modified her strategy (as seen in Figure 5), wrote three 8s the same as before, then subtracted 4 from
both 8s above N and T. Next she subtracted 8 from the 8 above J, and wrote the results 0, 4, 4, respectively. She explained that Nick and Tom would have 4 because "that's double 8," and Joe would have no amount left because he gave all his away. At this step, she took the "giving money" information into consideration, but she maintained the manner she had dealt with "receiving money" in her previous strategy to deal with the "giving money" activity - subtracting a number from 8. She then reasoned that Nick could not have given any money to Joe because Joe had nothing, and wrote a 0 above the previous row of 0. She then argued that Tom would be given 4 dollars, and she added 4 to the previous 4 above T. Finally, she crossed the 4 above N and wrote 0 above it. This time she claimed that Joe had 0 dollars, Nick had 0 dollars, and Tom had 8 dollars prior to any transaction having taken place. At this step, Liza modified the way she dealt with "receiving money" activity - adding money to the original number instead of subtracting, while maintaining the "giving money" activity as subtracting. It is noticeable that in this strategy her direction of writing was in bottom to top order, which coincided with the "working backwards" heuristic. However, the motivation for this change may not have been conscious, since she was using the "working forwards" approach to justify her computation (subtracting when giving money and adding when receiving money). I argue that this may have been done because the space under those three initial letters was occupied by the previous work from the initial strategy so she had to write from bottom to top.
The alternative strategy Lisa applied after her self-initiated reflection is shown in Figure 6. She started her work using a fresh piece of paper, wrote three 8s at the bottom of the paper, she then wrote the three initial letters at the very top of the paper and created a formal table, which indicated a desire to work backwards. After staring at the paper for 5 seconds, she declared, "Now I want to work backwards." She clearly stated that the first step she wanted to work on was the second transaction, and wrote two 4s above the two 8s in the columns of J and T. After thinking for a few seconds, wrote 16 above the 8 in the column of N. When she was asked to explain the 16, she reasoned that in order to end up with 8 and to give the other two money, one would have to add up the three numbers together (4, 8, and 4), yielding 16. She then wrote 8 above 16 and 2 above the 4 in the third column, and finally produced 14 by adding 4, 8, and 2. She claimed that Joe had 14 dollars, Nick had 8 dollars, and Tom had 2 dollars at the beginning. When she was asked why she seemed to be more confident with this new answer, she answered "it makes more sense."
Whether a result "made sense" or not seemed to be an important criterion for Liza to judge her answer. This judgment could happen right after she reached an answer, and influenced her confidence towards the answer. "Making sense" for her did not necessarily mean the answer was correct, but whether the numbers looked reasonable based on her understanding of the problem and what she was expected to find.

iii. Gestures

Liza exhibited some special behaviors at certain steps during problem solving phase. After the first episode of justification, Liza grabbed her pen but then decided to change to a new color. This revealed her awareness of the need to organize her work. After the reflection phase, she turned back to the problem and made different marks under certain words, indicating her more careful study of the problem. The connection between the end of "revisiting problem" and "the start of the alternative strategy" was continuous, with no pause in between. This could be interpreted as an indication that her confidence about her understanding towards the problem became so high that she was
ready to re-attack the problem without hesitation. The behaviors during the last justification episode also revealed her confidence by the agile way she tapped the calculator and her assured tone, which was clearly different from the behaviors she had exhibited in the previous two justifying processes.

iv. Representations

Liza used an organized table to solve the Money Transaction problem. The first table combined two steps simultaneously. In the second table she separately listed each step. The last table was more standard than the previous two, having borders to separate each column, and the writing was clean and clear. The representation slightly changed as her understanding towards the problem became deeper and in return, offered her a more effective vehicle to keep track of her thoughts.

b. Water Lily problem:

Water Lilies are growing on a lake. The water lilies grow rapidly, so that the amount of water surface covered by lilies doubles every 24 hours. On the first day of the summer, there was just one water lily. On the 90th day of summer, the lake was entirely covered. On what day was the lake half covered?

Liza spent about 4 minutes on this problem. As she read the problem, she immediately stated that she had seen a similar problem before, but she could not remember "the equation" to solve it. The answer that she later produced was not a direct product of her initial comments on the problem or her initial strategy, but the result of an insight, that was clearly articulated by her at the end of the episode.

i. Problem solving phases

Liza's process for solving this problem was: entry – using initial strategy – developing an insight – getting an answer – revisiting the problem – justifying the
answer – reflecting on the answer (shown in Figure 7). Using Mason's model as the framework, this process is illustrated by Figure 8.

![Figure 7. Linear model of Liza's process for solving the Water Lily problem](image)

In this problem, Liza's entry was quick, "I remember this... with rice and I watched it on a TV show." She immediately started drawing a table after reading the
problem, but she paused after computing 13 rows of the table and said "wait a minute." Then she pointed to the problem (around "the 90th day") and stated that the answer to the problem was 89. When she was asked whether she was sure about her answer, she studied the problem by reading it again and she drew a picture to justify why she believed her answer was right. The justification is considered to be self-initiated because there was no clear request for it from the interviewer (i.e. "can you justify/check your answer" or "can you convince me that this is the answer"). The reason that she switched strategies or decided on the accuracy of her answer was not clear until she was asked to reflect on the key moment at the end of the problem solving episode – she explained that initially she was considering to use an equation instead of repetitive computation and then was inspired by the "wording" of the problem. In the problem solving process, Mason (1985) referred this as "insight," which is usually an unexpected resolution after a few calculations or years of mulling (Mason, 1985).

ii. Strategies

The initial strategy that Liza used is illustrated in Table 4. Instead of the standard "initial strategy – answer – justification – modify/adjust the strategy" circulation, this strategy provoked an insight which led to her arriving at a correct answer, instead of getting a direct answer from the strategy itself. The switching of the strategies was not really observable, but the existence of the switch was clear according to the reflection under the request of the interviewer.
Table 4. Liza's initial strategy for the Water Lily problem

<table>
<thead>
<tr>
<th>day</th>
<th>lilies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
<tr>
<td>6</td>
<td>32</td>
</tr>
<tr>
<td>7</td>
<td>64</td>
</tr>
<tr>
<td>8</td>
<td>128</td>
</tr>
<tr>
<td>9</td>
<td>256</td>
</tr>
<tr>
<td>10</td>
<td>512</td>
</tr>
<tr>
<td>11</td>
<td>1024</td>
</tr>
<tr>
<td>12</td>
<td>2048</td>
</tr>
<tr>
<td>13</td>
<td>4096</td>
</tr>
</tbody>
</table>

iii. Gestures

Liza spent only 4 minutes on the Water Lily problem, from reading the problem to the end of her final reflection. When she started drawing the table, she seemed quite confident through her facial expression (smiling), her writing (quick), and her tone (agile). Her writing gradually became slower and finally stopped, indicating a doubt in utility of her strategy to her strategy. At the revelation of insight, her deliberate mode of pointing to the problem revealed her confidence in her answer.

iv. Representations

In addition to the table she created at the point of entry, Liza drew a picture to support her reasoning when she was trying to justify her answer. She first drew the pond on the last day, which was full of water lilies (the left picture in Figure 9). Then she stated that the day prior to the full coverage, the lake would be half full, placing her pen
in the middle of the pond (the right picture in Figure 9). In this manner, she was working backwards, according to her use of representation.

![Liza's visual representation for the Water Lily problem](image)

**Figure 9** Liza's visual representation for the Water Lily problem

If the table was the stimulator of her insight, this picture could be considered as an intuitive representation after she launched the answer. There was evidence that she did not have this picture in her mind when she declared the answer: it was only after she was asked whether she was sure about the answer that she re-read the problem, pointing at each word by the pen instead of immediately drawing the picture.

c. **Pay Option problem**

Consider the two pay options: $300 a week or 7.5 an hour. What factors will affect your choice of option to take. Draw a graph that compares the two pay options, allowing the reader to determine which option might be best for them to take.

Liza spent 10 minutes and 42 seconds on this problem. She stayed with her initial strategy during the second part of the problem for over 3 minutes without modifying or
switching. Two extended questions were asked by the interviewer to access her understanding towards the problem after she finished her work.

i. Problem solving phases

Liza's process for solving the Pay Option problem was: entry – using initial strategy to answer the first part of the problem – using initial strategy to answer the second part of the problem – answering a question asked by the interviewer – answering a follow up question asked by the interviewer (shown in figure 10). The non-shaded Q represents a question asked by the interviewer to access the interviewee's understanding and to provoke her/his thoughts. Using Mason's model as the framework, this process is illustrated by Figure 11.

![Figure 10. Linear model of Liza's process for solving the Pay Option problem](image-url)
For the first part of the problem (what factors would affect the choice of pay option), Liza directly claimed that the answer depended on "how many hours you are going to work for the day." Elaborating further on this statement, she assigned a number as the working hours and compared the payments. When she hesitated about the computation, the interviewer asked if she could use estimation (as opposed to computation) to answer the question. She paused for a second upon hearing the question. However, she continued following her previous approach, assigning different numbers as working hours and comparing the results. Different from other calculator-dependent students, she was able to divide the multiplication into 2 parts which made it easier for her to compute without a calculator, saying "50 cents time 12 is 6 dollars, and 7 times 12 would be..." This evidenced number sense and immediate recall of multiplication facts on her part. After obtaining the amounts of payment, she further stated that if working fewer hours, the weekly payment would be better. When she was asked whether there would be a situation that the two payments were the same, she answered "it might be equal" when
one had the number of working hours in a day and the number of working days in a week to make the amount of hourly payment end up at 300.

For the second part of the problem (drawing a graph to compare the two pay options), she drew 2 graphs to indicate the two factors she claimed before (how many hours a day and how many days a week), and added the weekly payment to one of the graphs under the request from the interviewer. When she finished the graph, the interviewer asked, "for 40 hours a week, is it possible to quickly determine how much the pay would be according to your graph?" Lisa hesitated and went back to the method she had used in the first part of the problem. After she found that the two payments would be the same, the interviewer further asked if she could show the amounts in her picture. Liza first stated that a much taller graph would be needed. When she was about to draw it, the interviewer encouraged her to provide a general description about how the two lines would look like without worrying about the scale. Liza's answer was that "the weekly payment line would still be steep and the hourly payment line would still be shallow, but they will meet at a point" and "that point would be 40 hours."

ii. Strategies

The strategy that Lisa used to examine the factors is shown in Figure 12. She first wrote 300 and 7.50 in two columns as the two pay options. Then she assumed that the person worked 12 hours a day and got the amount of 90 dollars per day if paid hourly. Then she timed the 90 dollars by 7, saying that "how many days you work per week is another factor." The final result was found to be 630 dollars per week if paid by hour.
For the second part of the problem, Lisa immediately drew x and y axes and claimed that she needed to figure out "what would go along x-axis and y-axis" (as shown in Figure 13). After thinking for 20 seconds, she marked y-axis as "hours a day" and x-axis as "pay." Then she set up the scales on x and y axes, marked 0 to 12 on the "hours a day" axis (which was her choice of number in the first part of the problem) and the first two amounts of payment on the x-axis (7.5 and 15), and drew a line by the coordinates. After this, she moved on to the other graph, marked 1 and 2 on the "days" axis and 630 and 1260 on the x-axis, then drew the line. She explained that these two lines "basically have the same rate" because "it is the same way." This statement probably meant that the slope of the line for the same pay option would remain the same with different types of units (hours versus days).
When she was asked to present the other pay option on the same graph, she chose the days-graph and claimed that one would have to start from 7 days. She extended the y-axis coordinate to 7 and marked 300 at x-axis (as shown in Figure 14). Then she divided 300 by 7 and got 42.86. She tried to mark this number on the x-axis but the paper space available was insufficient, so she only explained that this number would be the corresponding amount of payment on day 1. Then she connected the origin and the intersection of day 7 and the payment, saying "you will have a much steeper graph."
When Liza was using her initial strategy during the first part of the problem, she seemed to be more confident and natural. When she later encountered questions related to her graph, she hesitated and went back to the initial strategy. In this episode, Liza looked to be more comfortable with the numerical strategy than the graphic one.

iii. Gestures

As Liza decided to figure out the payment amounts for different hours, she held her head and murmured "it is terrible." This indicated her dislike of repetitive computation. But when she was actually doing the computation, she did not show any visible signs of having trouble with it. At the end of the interview, when she was indicating that the two lines would meet at a place by moving her finger along the lines, the movement was slow and her tone was hesitant.

iv. Representations

Liza used 2 kinds of representations in this episode. Compared to the numerical representation she used in the first part of the problem, she seemed to be less familiar with graphing conventions and could not use a graph to generate a more coherent representation of the problem and its solution. Her graph also failed to support her final statement, that the two lines would meet at some point.

d. Compare Areas problem

Consider the graph below: What can we say about the areas of triangles BEC and BFC?
Liza spent 14 minutes and 49 seconds on the Compare Area problem. In this episode, the role of the interviewer was different from the previous three. There were more scaffolding questions provided by the interviewer, and Liza's behaviors were heavily influenced by these questions.

Liza's process for solving this problem was: entry – using initial strategy – using an alternative strategy based a question asked by the interviewer – getting an answer – justifying the answer – using another alternative strategy based on a question asked by the interviewer – getting an answer – justifying the answer – answering a question asked by the interviewer – answering a following question asked by the interviewer – using another alternative strategy based on a question asked by the interviewer (shown in Figure 15). Using Mason's model as the framework, this process is illustrated by Figure 16.

![Figure 15. Linear model of Liza's process for solving the Compare Area problem](image)
The initial strategy Liza used was finding equalities in the picture. When Liza encountered problems after finding one pair of equal angles and one pair of equal sides, the interviewer asked her a series of scaffolding questions to help her further study the problem. A table containing the series of scaffolding questions asked by the interviewer and Liza's corresponding behaviors is used to demonstrate the interactional process between the interviewer and Liza (Table 5). Each question is corresponded to a non-shaded Q (representing an interviewer-initiated question) in Figure 15, respectively.

Figure 16. Mason's model of Liza's process for solving the Compare Area problem
<table>
<thead>
<tr>
<th>Question asked by the interviewer</th>
<th>Student's corresponding behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Which has a larger area, triangles BEC or BFC?</td>
<td>Stated that outside areas were the same (Figure 1.d.4)</td>
</tr>
<tr>
<td>2. If the outside areas are not the same, is there a different way of looking at this problem?</td>
<td>Copied the outer areas side by side and compare (Figure 1.d.5).</td>
</tr>
<tr>
<td>3. How would you determine the sizes would even out completely?</td>
<td>Stated that she could measure the outer areas to see if they are the same.</td>
</tr>
<tr>
<td>4. Is there a way of telling how much of the area of the rectangle that the triangle is?</td>
<td>Stated that she could shift the outer area to see whether the sum was the same as the inner area but didn't know how to do it.</td>
</tr>
<tr>
<td>5. What information would you want me to give you in order to determine how much of the area of the rectangle that the triangle is?</td>
<td>Stated that she needed to know &quot;how big the unit box is,&quot; so that she could add up the unit boxes together and see how many boxes the triangle has.</td>
</tr>
</tbody>
</table>

Table 5. Scaffolding questions asked by the interviewer and Liza's corresponding behaviors in the Compare Area problem

Liza did not formally reach the correct answer at the end of the episode. However, her understanding towards the problem was improved and her strategy became more considerate and systematic each time she responded to a scaffolding question.

ii. Strategies

The first strategy Liza used is shown in Figure 17. She stated that the two angles were the same because they were opposite angles, and claimed that the two marked sides were the same because it was an isosceles triangle. Liza's first strategy was not relevant to the question. This may be due to an association she made between the problem and the school content she was learning (since she was in geometry class that time).
The alternative strategy Lisa used was inspired by the question of "larger area" involved comparing the outside areas of the two triangles (shaded areas shown in Figure 18). After staring at the picture for 15 seconds, she claimed that these two areas were the same. She was asked to explain it, and in response she indicated that the two pairs of corresponding areas (1a and 1b, 2a and 2b in Figure 18) were the same, respectively, because those two pairs of triangles were congruent. This (the assertion of congruent) could be also considered as a connection she made to school content.

The next alternative strategy Lisa used is shown in Figure 19. She first traced the two smaller parts of shaded areas by copying them onto a new paper, and directly compared the areas. She determined that the smaller shaded area of the left triangle in
figure 18 (the darker area on the left in figure 19) was bigger than the smaller shaded area of the right triangle in figure 18 (the lighter area on the left in figure 19). But when she applied the same strategy to the larger shaded areas, she realized that the darker area was smaller than the lighter area, which contradicted her assumption that the darker area would be always bigger than the lighter one. At the end of this phase, she made a guess that the areas might be the same "because the sizes could even it out."

Figure 19. Liza's another strategy for the Compare Area problem

The last strategy Liza used is shown in Figure 20. After being given the information that one side of the rectangle was 5 units long, and the other side was 20 units long, she drew unit boxes and stated that she could add up the partial boxes to see how many whole boxes there were, and then she would be able to determine what was the relationship between the area of the rectangle and the area of the triangle.
Although Liza switched strategies 3 times, she remained with visual model instead of seeking a more abstract method, that is, the triangle area formula. The absence of triangle area formula in this problem solving episode could be interpreted as a failed application of related knowledge.

iii. Gestures

According to Liza herself, she liked geometry more than algebra. She showed confidence throughout the episode, noticeable excitement each time she reached a better understanding, and outstanding patience when she drew graphs again and again. In spite of the different properties of this particular interview, she spent the longest time on the task and tried the greatest number of strategies, although it was the only one she didn't reach a "correct answer." This indicated that her preference for the subject might have influenced her confidence in tackling an unfamiliar problem from the subject, the time she spent on the problem, and the number of strategies she tried, but not necessarily the success of solving the problem.

iv. Representations

Liza manipulated different visual graphs in this problem solving episode. She was able to provoke and check her ideas with the graphs. Visual representation seemed to be a
comfortable tool for her to explore problems. But it also appeared to be an obstacle for her to reach a more abstract/systematic understanding.

C. Liza's Inter-problem Analysis

a. Problem solving phases

The overall comparison of Lisa's 4 problem solving processes is illustrated in Figure 21. The length of each line is adjusted to indicate the relative amount of time Liza spent on each problem.

Figure 21. Overview of Liza's four problem solving processes
In the Water Lily episode, which is the shortest episode among the four, an insight transpired which yielded immediately to launching a correct answer. Possible reasons for the occurrence of the insight include the impact from previous experience (the TV show), her confidence, and her preference for efficient strategies (which caused her attempt to switch to a non-computational strategy).

There is a noticeable difference between the first two problem solving episodes. In the first episode, the reflection occurred (spontaneously) during the problem-solving process and led to the modification of her strategy, while in the second episode, the reflection occurred (initiated by the interviewer) at the end of the problem solving episode, after she successfully justified her answer and concluded that it was the right answer. The nature of these two reflections is also different: the first one concerned about the misunderstanding / "wrong" idea, which seems to be a natural human behavior, while the latter one concerned about the inspiration / "right" idea, which is less natural for students.

Another noticeable difference is that although the problem solving process was very systematic in the Money Transaction problem (even in the Water Lily problem), as there are justifications, reflections and an under-controlled exploration of strategies, the problem solving processes in the Pay Option problem and the Compare Area problem are less systematic. Her work not only lacked reflections or justifications, but she seemed unaware of her own mistakes. What's more, the "revisit problem" step is missing in the latter two processes, and she did not fully reach a "right answer" in either situation.

b. Strategies
Among the four problem solving episodes, Liza tended to switch strategies when she felt her strategy was inefficient (Water Lily episode), or had developed a more systematic understanding of the problem (Money Transaction episode). But if the previous misunderstanding was found to be content-related (i.e. the neglect of "giving money" in Money Transaction episode), she would try to modify the numbers instead of switching strategy. In the Compare Area episode, she switched strategies partially because she was asked specific questions, which were intended to evoke deeper thoughts on her part. The only episode she stayed with one strategy was the Pay Option episode, which was probably due to her unfamiliarity and lack of confidence with problem type or choice of heuristic. Liza appeared to be confident most of the time, which was possibly because she considered herself as "good at mathematics," which she defined as "can solve most problems eventually." This perception can also account for the persistence she showed towards solving problems during all episodes.

c. Representations

Liza used various kinds of representations in the four problem contexts. Although she was able to use numerical representations when she felt she had to, she showed a clear preference for visual representations. She seemed more patient and enthusiastic when she worked with visual representations (Compare Area episode), but she entered a problem more naturally with a numerical approach, which seems to help her delve into the problem more quickly. She seldom self-initiated the use of multiple representations within one problem, unless she was requested to (Water Lily problem and Pay Option problem).
D. Summary

Based on Liza’s four problem solving episodes, her general problem solving orientation is illustrated in Figure 22.

![Figure 22. Liza's general problem solving orientation](image)

After reading the question, Liza would recall a particular instance or a familiar strategy based on her understanding the problem. When she was working with the initial strategy, she would constantly try to make sense of what she obtained or determined the efficiency of the strategy. If she thought the answer or method did not make sense (to her), or she felt the strategy was not sufficiently efficient, she attempted to adjust her approach. After she reached an answer, she would first examine the value based on whether it made sense or not. If the answer did not make sense to her, she would try to
justify it, switch to other strategies, or revisit the problem to check her understanding of what was asked. If the answer did make sense to her, she did not try to justify it unless she was asked to do so.

The strategies that Liza naturally chose depended largely on how well she understood the problem. If she was not very familiar with the strategy she chose, she tended to stay with it even when she came across difficulties. She had a preference for the efficiency of strategies, but did not seem to be particularly committed to a specific heuristic.

Among the four problem solving episodes, Liza exhibited two natural metacognitive behaviors. In the Money Transaction problem, she self-initiated a reflection after she failed to justify her answer. As a result, she modified her strategy. Here, a need for change of strategy might be her motive for reflection. In the Water Lily problem, she self-initiatedly justified her answer that she had reached by insight. A possible motive for her self-initiated justification could be her feeling of the need to explain her answer, since she did not have any paper work for that answer (she only had a table which could not support her answer). An evidence for this assertion was that she never justified her answer when she had the working process written on the paper, unless she was asked to.

Liza's gestures during problem solving processes generally revealed her confidence towards what she was working on (i.e. the strategy she was using), as well as her enthusiasm for the problem. Liza was comfortable to talk about what she was doing when she was confident, yet talked less when she was less confident or getting more
engaged in the problem solving activity (i.e. revisiting the problem after getting a wrong answer).

Liza's performance was not consistent throughout either different subject areas used in the context or heuristic needed to solve the problem. Her performance was better in the content area of number sense than patterns and geometry, yet she was confident in both subject areas. She was able to develop her strategy to reach the heuristic of working backwards. She was not able to further her progress with linear graphs.

The Case of Jazzy

A. Jazzy's Background

During the first part of the first interview, Jazzy was asked to articulate her feelings and attitudes towards mathematics, explain what was easy or difficult for her to do and whether she felt confident in her mathematical ability. This portion of the interview allowed us to build a profile of Jazzy's thinking about and attitude towards mathematics that could potentially explain her specific mathematical behaviors during the problem solving sessions.

During the second part of the interview, Jazzy was asked to solve four mathematical problems. She was reassured, as all other participants were that our primary interest was in her thinking and the processes she used in tackling problems. We noted that we wished for her to think out loud and to explain her thinking as she interacted with problems. Jazzy was not restricted to solve the problems in a specific amount of time and she was also reassured she could take as much time she needed to resolve problems using any method she deemed appropriate.
Who is Jazzy?

At the time of data collection, Jazzy was in 8th grade, and she had completed a course in geometry in 7th grade. Jazzy identified mathematics as her favorite subject in school and explained she had consistently felt positive about mathematics, confident in her ability to do mathematics. She explained she liked mathematics because "it was easy." Being easy to her meant that, "there was only one answer to a problem, although there might not be only one way to get that answer." The perception of existence of "only one answer" revealed her view that mathematics problems are well-defined, that using different methods one could reach the same answer. When she was asked to provide an example of the type of problem that she considered to have more than one answer, thinking for a few seconds, she shrugged and said she couldn't.

When Jazzy was asked whether her feelings towards mathematics had changed since the previous year, her response was negative explaining that the course (geometry) was "pretty much the same thing" and "nothing really new" about angles discussed. In terms of how the concepts she was learning in geometry could be useful, she offered a classical textbook exercise on similar triangles, where given a triangle and measure of one side and an angle, measures of the lengths of other sides could also be found. She used this example to also argue why she viewed "geometry as being useful," which was not related to daily life.

Jazzy was a quiet and reflective individual. She seemed confident in her mathematical work. She trusted she could solve any problem she was given. Her general tendency was to provide short and brief responses when she was asked. Indeed, these
characteristics were also evident during all four problem solving episodes, as described below.

**B. Jazzy’s Intra-problem Analysis**

Table 6 illustrates an overview of Jazzy's four problem solving episodes.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Order</th>
<th>Time of the episode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Money Transaction</td>
<td>Session I</td>
<td>11'26&quot;</td>
</tr>
<tr>
<td>2. Water Lily</td>
<td>Session II</td>
<td>05'11&quot;</td>
</tr>
<tr>
<td>3. Pay Option</td>
<td>Session II</td>
<td>12'21&quot;</td>
</tr>
<tr>
<td>4. Compare Area</td>
<td>Session II</td>
<td>10'50&quot;</td>
</tr>
</tbody>
</table>

Table 6. General overview of Jazzy's episodes

A major difference between Jazzy's first and second problem solving episodes was that in the first interview, she stayed with one strategy until she reached the correct answer. But in the second interview, she switched strategies once or twice in each episode. Another difference is that in the first interview Jazzy did not talk during the episode unless she was asked by the interviewer, while in the second interview she was more willing to explain what she was doing when she was solving the problems.

*a. Money Transaction problem*

Joe gives Nick and Tom as much money as each already has. Then Nick gives Joe and Tom as much money as each of them then has. If at the end each has 16 dollars, how much money did each have at the beginning?

This problem was given to Jazzy first. The version of problem used in this interview was slightly different from the one used in the interviews with other participants.
Jazzy spent 11 minutes and 26 seconds on this problem. She appeared to be confident when she was using the calculator. But even after she adjusted the numbers many times and finally got the same answer for the second time, she was not certain about the answer.

i. Problem solving phases

Jazzy's process for solving this problem was: entry – using initial strategy – getting an answer – justifying the answer under the interviewer's request – self-initiated reflection – using the same strategy with modified information – getting a new answer – justifying the answer under the interviewer's request (shown in Figure 23). In this episode, the entry and initial strategy phases were relatively long, as indicated by the ratio of the lengths of segment on the line. Using Mason's model as the framework, this process is illustrated by Figure 24.

Figure 23. Linear model of Jazzy's process for solving the Money Transaction problem
It took Jazzy 2 minutes to start writing (the entry phase), which revealed that she would not write anything before she was sure. She did not speak at all when she was working on the problem using the initial strategy. When she was justifying/explaining her answer, she realized that she had taken a wrong step and said "hold on." She crossed the wrong answers and corrected them on the original work. When she was trying to justify the corrected answer (which was right), she was not certain about her answer, saying "I don't know," "that's not right." Tapping on the calculator for a while, she tried to adjust more numbers. Since the original work became messy, she decided to start a new table. But after finishing the first two columns, she shouted "no, hold on," "I'm getting confused." She returned to the calculator and then announced "I was right." She corrected the number again, and confirmed the previous answer under the first correction. She still did not seem certain about this answer, saying "maybe it is the right answer," which revealed her focus on "the only right answer" when solving problems, although she had
no doubt about the accuracy of each step of computation she had taken. After she justified the answer with the interviewer, she finally looked more certain.

Jazzy's reflection was prompted right after the first justification. She successfully realized her mistake and corrected it. However, the reflection did not seem to stop after the first correction, it generated more mathematical behaviors (i.e. more adjustments on numbers). It is possible that her confusion was caused by the way she traced the amount of money after each transaction, which did not include the final stage (that each had 16 dollars).

ii. Strategies

The initial strategy Jazzy used was working backwards (illustrated in Figure 25). She first wrote "Joe - 8" and "Tom - 8." Then after tapping on the calculator for 16 seconds, she wrote "Nick - 32" on top. Re-reading the problem for a while, she wrote " - 4" on the Tom row, and " - 64" on the Nick row. Tapping on the calculator for a few more seconds, she wrote " - 76" next to the Joe row and finally drew a vertical line to indicate the answer. She might have checked the answer on the calculator before she wrote it. Being asked to explain her answer, she claimed that she was working backwards. She stated that Nick ended with 16 and he gave Joe and Tom 16, so he must have 32. She reasoned that if Tom was given the amount of money he had, he should have had 4 prior to the transaction. When she was trying to justify Nick's money (64), she realized she had not gotten it right.
She adjusted her numbers by crossing the number "64" and wrote "16" beside it. After tapping on the calculator, she crossed the number "76" and wrote "28" instead. But when she was trying to justify this new answer (Figure 26(a)), she hesitated, returned to the calculator, and crossed the number "32" with "48" (Figure 26(b)). It indicated that she was trying to add 16, 28, and 4 to adjust Nick's amount of money after the first transaction, revealing her attempts at self-regulation through numerical computation.

Jazzy started a new table after these corrections (Figure 27(a)), probably because she thought it was too messy (which could make her more confused) to allow for computation of a new answer.
As soon as she started testing with these numbers, she stopped, claimed she had been right, and replaced the "48" with "32." She then wrote the numbers "16," "28," "4," respectively. Noticeably, she tapped the calculator for a while before writing each number, but she got the number "28" before "4," which should be the sum of 16, 8, and 4. I interpreted this as that she was referring to the previous result (Figure 26(a)) when accomplishing the new result (Figure 27(b)); the tapping calculator behavior was more like "checking the computation" instead of restarting the whole reasoning process.

Despite the mistake on the second step of Nick's transaction (I interpret this as an mistake instead of misunderstanding because she clearly understood the process very well by her reasoning for the second step of Tom's transaction), Jazzy's strategy was clear and helpful. Self-regulation was conducted through numerical computation throughout the application of the strategy.

iii. Gestures

Jazzy never talked during her problem solving process unless she was asked by the interviewer; sometimes she did not talk even after she was asked to do so (she was once asked to explain what she was tapping on the calculator in this episode, but she was so concentrated on her work that she did not talk until she finished the computation). She
would not write anything unless she was certain of the accuracy of what she had done. Under these circumstances, her gestures were a crucial signal to understanding what she might have been thinking. During the long entry phase, she tapped three times on the paper with her pen in a left-to-right direction, which could be a sign of tracking the transaction process. Instead of writing any computation on paper and keeping track of values by writing, she mostly relied on the calculator. She always looked confident and proficient when she was working on the calculator.

iv. Representations

The elimination of the final status of the money (three 16s) could have served as an obstacle for Jazzy to gain a clearer view of the whole transaction process. She directly started with the amount of money each person had before the second money transaction, which would have been a second step if she had started with the amount of money after the second money transaction (which was 16 dollars each). The reason for this choice could be that she considered starting with 16 dollars would be too trivial and/or she might believe she was able to process that part in her mind instead of writing it.

b. Water Lily problem

Water Lilies are growing on a lake. The water lilies grow rapidly, so that the amount of water surface covered by lilies doubles every 24 hours. On the first day of the summer, there was just one water lily. On the 90th day of summer, the lake was entirely covered. On what day was the lake half covered?

Jazzy spent only 5 minutes and 11 seconds on this problem. During the entire episode, she did not write anything except drawing a picture to justify her answer. She did not seem confident (i.e. kept saying "I don't know") and was easy to hesitate during
the whole process. Even with a convincing picture, she was not sure about her answer and was eager to know what the right answer was.

i. Problem solving phases

Jazzy's process for solving this problem was: entry – using initial strategy (half of 90) – getting an answer – self-initiated reflection – asking a key question – using an alternative strategy (working with specific area of the pond) – getting an answer – justifying the answer (shown in Figure 28). Using Mason's model as the framework, this process is illustrated by Figure 29.

Figure 28. Linear model of Jazzy's process for solving the Water Lily problem
Jazzy spent 30 seconds on the entry phase, and directly provided the answer "45 days," reasoning that "it's half of 90." When she was asked to justify this answer by drawing a picture, she thought for 8 seconds and said "it's probably not 45." The immediate hesitation towards her first answer may be due to the absence of persuasive numerical values with which she could reason, since she seemed to be more comfortable with numerical facts. Reflecting on the problem, she stated "I don't wanna go though everyday; it takes forever," "try to think." These statements revealed an attempt to switch to a more efficient strategy. During the reflection phase, she tried to use the calculator but did not test any numbers. After 30 seconds, she asked the interviewer for "how many is full?" since she believed it would make it easier to solve the problem. This question could be seen as a representation of her preference for numeral sense – making (but not tedious computation). When she was told that 1000 square feet was full, she quickly claimed that
"when there be like 80 something, it was half full." This sentence revealed that she had a sense that the answer should be some number close to 90. Then she immediately spoke out the answer, 89th day. When she was asked to explain her answer, she first said she was not sure about the answer, then she stated that "500 times 2 is 1000," "so it would be the day before." Then the interviewer changed the condition from 1000 square feet to 500,000 square feet, and Jazzy immediately answered it would be still the day before. But when she was questioned whether she was sure, she hesitated again, saying "wait, I don't know." She then tried to draw a picture to illustrate the answer. After she successfully justified her answer with the picture, she claimed she was sure her answer was correct. However, she said she was still a little doubtful and wanted to know the "exact right answer" even when she admitted that the picture was convincing to her. Her uncertainty towards the answer which was supported by convincing visual evidence could be interpreted as her strong reliance on numeral evidence; before she could find a numeral way to solve the problem, she may not be truly sure. This may also be the influence of her beliefs about mathematical problems that concerned the there is always an exact way of solving problems.

ii. Strategies

The initial strategy Jazzy applied was very direct: half of 90 days would be the answer. But she immediately hesitated when she was asked to justify this answer and then reflected on the context again.

The alternative strategy she used was numerical: provided that 1000sq ft was full, since 500 times 2 is 1000, so it would be the day before. Using this strategy, Jazzy was able to extend her answer to other condition (i.e. if 500,000 sq ft was full). This strategy
revealed her numerical orientation; she tended to manipulate numbers to make sense of the problem. The need for the target value could be indicating of her goal of working backwards.

iii. Gestures

Besides drawing a picture to justify her final answer, Jazzy did not write anything. Her two major gestures during this problem solving episode were: 1. holding the pen in a way like she was about to write something, and 2. grabbing the calculator in a way like she was about to tap some numbers. The first gesture implied that she was not sure about what she was thinking, since she would not write anything before she was sure. The second gesture implied her attempt to gain idea from numbers/computation. Other gestures included shrugging (when she was saying "I don't know") and leaning forward (when she was reflecting).

iv. Representations

The one representation Jazzy used in this problem solving episode is shown in Figure 30.

Figure 30. Jazzy's visual representation for the Water Lily problem
She first drew the big circle, which was to represent the pond. Then, she divided it in half, and filled the left side, stating that this was the day before the last day. She then filled the other side to demonstrate the "doubled to be full."

This "half and then doubled" way of representation corresponded to her previous numerical reasoning, which was "500 times 2 is 1000." Hence, this representation should be considered more as an alternative expression of the same idea instead of a different way of thinking. To Jazzy, visual representation was probably more like a different form of numerical representation, not a different way of thinking about the problem.

c. Pay Option problem

Consider the two pay options: $300 a week or 7.5 an hour. What factors will affect your choice of option to take. Draw a graph that compares the two pay options, allowing the reader to determine which option might be best for them to take.

Jazzy spent 12 minutes and 21 seconds on the Pay Option problem. She was first confused by how many hours a person would work during a week, then by how to label both pay options within one graph. She switched strategies three times, as well as the representations. She did not have confidence in her strategies or her understanding towards the problem. This could be because she was uncomfortable with questions that are hard to be directly solved numerically.

i. Problem solving phases

Jazzy's process for solving this problem was: enter the first part of the problem – give an answer – use initial strategy for the second part of the problem (linear graph) – use alternative strategy – answer a question asked by the interviewer – answer another question asked by the interviewer – use an alternative strategy (shown in Figure 31).
Figure 31. Linear model of Jazzy's process for solving the Pay Option problem

Using Mason's model as the framework, this process is illustrated by Figure 32.

Figure 32. Mason's model of Jazzy's process for solving the Pay Option problem
After hearing the first part of the question, Jazzy immediately answered that it depended on how many hours a person worked. An additional question was asked following this answer: "How many hours would you say a typical person work in a week?" At first she said 8 hours, probably thinking 8 hours a day. When she heard the interviewer repeating her answer, she realized her mistake and contemplated the answer for a while. She declared "40 hours," not writing anything or tapping on the calculator (she did not have a calculator at that time). For the second part of the question, she first drew two axes, but soon found that she had some problems setting up the scale. At this point, she was given a calculator. After computing for a while with the calculator, she concluded that there was "no point to make a graph" since if a person worked 40 hours a week then the two pay options would have the same amount, which would be two lines overlapping each other. This conclusion was directly affected by the previous answer that a typical person worked 40 hours a week. When she was asked what would happen if the person did not work 40 hours a week, she answered it was "still no point to make a graph" although she could, since it would take too much time to compute by hours. Jazzy said "no point to make a graph" twice but each had a different meaning in my judgment. The first one meant it was useless to draw one line to indicate two options, which were the same. This could be interpreted as she believed that it was more necessary to show "differences" than "sameness," in that sameness was more obvious or easier to understand. The second one meant it was too tedious/trivial to use that method under that circumstance. This revealed her preference to more advanced strategies, which did not include repeating similar steps. When she was encouraged to graph them, she hesitated for two seconds and asked what kind of graph she was expected to produce. She chose to
draw a table. When she was trying to fill out the table, she asked whether the week included weekends, to which the interviewer didn't answer. She continued to compute 2 lines and then stopped. She stated that this was not making sense to her so she did not think her attempt seemed worked either. Upon request, she further explained what "making sense" meant: she wanted to break the payment into hours, but she did not know how many hours there were in a week in terms of the weekly pay option. Then she was asked whether there was any additional information that might be helpful to her. She thought for 15 seconds, answered: "how many hours would this person be working?" The concentration on the number of working hours in a week indicated her awareness of the need to work under the same measurement (turning weekly payment into an hourly form). Instead of answering her question, the interviewer asked her a series of questions. These are shown in Table 7.

<table>
<thead>
<tr>
<th>Questions asked by the interviewer</th>
<th>Answers from the interviewee</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. If the person was expected to work 20 hours a week, which would be a better option?</td>
<td>(After computation) The weekly one.</td>
</tr>
<tr>
<td>2. What if he worked 30 hours a week?</td>
<td>(Thought for 3 seconds) Still the weekly one.</td>
</tr>
<tr>
<td>3. What if 40 hours a week?</td>
<td>(Thought for 1 seconds) They will be the same.</td>
</tr>
<tr>
<td>4. What if more than 40 hours a week?</td>
<td>(Immediately) It will be the hourly one.</td>
</tr>
</tbody>
</table>

Table 7. Questions asked by the interviewer and Jazzy's corresponding behaviors in Pay Option problem

At this point the interviewer asked a synthesizing question: "putting that information together, could we draw a graph that captures all that information?" Jazzy thought for 20 seconds, stating that she could draw a bar graph but she didn't know how to label it.
When trying to draw it, she explained she couldn't resolve the difference between "week" and "hour," and couldn't label the graph. She spent about 1 minute drawing the bar graph and labeling it. At last, being asked how she could tell which pay option would be better if someone worked an amount of time that was not included in the bar graph, she simply answered "you can tell it by instinct."

ii. Strategies

The initial strategy Jazzy applied is shown in Figure 33.

![Graph](image)

Figure 33. Jazzy's initial strategy for the Pay Option problem

She first drew two axes, and carefully labeled them as "weeks" and "money." Then she stopped, stating she didn't know how to scale it since she didn't know how many hours the person worked in a week. After she was given a calculator, she tapped some numbers and concluded there was no point to make a graph. When she was indicating that if a person worked 40 hours a week the two pay options would have the same amount, she sketched two overlapping lines on the graph.

The alternative strategy she used is illustrated in Table 8.
Table 8. Jazzy's alternative strategy in Pay Option problem

<table>
<thead>
<tr>
<th></th>
<th>300</th>
<th>7.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1h</td>
<td>1.78</td>
<td>7.50</td>
</tr>
<tr>
<td>2h</td>
<td>15.0</td>
<td></td>
</tr>
</tbody>
</table>

She filled 300 in the left column and 7.50 in the right column to indicate the weekly and hourly pay options. After thinking for a while, she marked "1h" on the left side of the second row, indicating the condition that the number of working hour was one. Then she filled the 7.50 at the intersection at the second column and the second row. In terms of the intersection at the first column and the second row (how much was the amount of weekly payment if the amount of working time was 1 hour), she hesitated for 20 seconds and wrote 1.78 after tapping on the calculator, explaining that "24 times 7 is 168," and "300 divided by 168 is 1.78." In this step, Jazzy was computing the amount of payment for 1 hour under the weekly pay option by taking all the hours (24 hours a day and 7 days a week) within a week into consideration. She did not continue finishing the rest of the table, but concluded that she did not know how many hours there were in a week. Jazzy was focusing on the idea of "breaking payment into hours," but failed to realize that the payment under weekly pay option would always be 300 dollars no matter how many hours a person worked in a week.

The last representation she used is shown in Figure 34.
Jazzy did not label the axes in this bar graph as she did in her initial strategy. What's more, the way she scaled the x-axis was unusual: it was a combination of hourly and weekly pay options mixed in a distinct way. She first scaled the x-axis by 10-hour, starting at 30 hours, and drew bars to indicate the amount of payment for each amount of time the person worked in a week. When she finished the 40-hour bar, she drew a bar marked by "weekly" next to it, which had the same height and number, representing the weekly payment would be always the same. This could be considered as her benchmark for the two pay options. Then she continued drawing the hourly payment bar to 50-hour. This graph revealed that she was aware of the equal condition for the two pay options and using it as a way to compare values. Additionally, it indicated she was aware of a general relationship between the two options when the number of hours was more or less than 40 hours.

Each change of the strategy revealed a better understanding of the problem on her part and contained more information.

iii. Gestures
At the time when Jazzy was trying to figure out how many hours a person typically worked in a week (she had already stated 8 hours so what she needed to figure out was what was 8 times 5), she looked up and leaned forward. After 5 seconds she said 40 hours in a questioning tone. This revealed that Jazzy was able to do basic computation in her mind but she was not confident in value she obtained. Considering her strong reliance on numbers and calculator, it might be helpful to extend her choice on problem solving strategies by reducing her access to calculator, since she might be able to focus more on general idea instead of being confused by numbers.

iv. Representations

Jazzy used 3 kinds of representations in this problem: linear graph, table, and bar graph. The first two representations seemed more appropriate for this problem, yet she did not have a clear idea about how to represent/manipulate the two pay options when she used those two representations. She did not try to stay with one representation and modify it, but changed representations each time she was asked to draw a graph. I interpret this as that she might not want to stay with the same representation if she thought it was not working, although it might be that the understanding was not appropriate but not the representation itself.

\textit{d. Compare Area problem}

Consider the graph below: What can we say about the areas of triangles BEC and BFC?
Jazzy spent 10 minutes and 50 seconds on this problem. She did not use calculator in this episode. She used only visual representations for this problem. The interviewer asked two series of scaffolding questions when Jazzy insisted that she needed numbers to solve the problem after two attempts at finding an answer. Each series of questions successfully inspired her understanding towards the problem. The role of the teacher during student's problem solving process was significant in this episode.

i. Problem solving phases

Jazzy's problem solving process on this task was: entering by asking a question – using initial strategy – answering a question asked by the interviewer – using an alternative strategy – answering 7 questions asked by the interviewer respectively – using another alternative strategy – answering 8 questions asked by the interviewer respectively (shown in Figure 35). Using Mason's model as the framework, this process is illustrated by Figure 36.

Figure 35. Linear model of Jazzy's process for solving the Compare Area problem
During the entry phase of the problem, Jazzy asked if there had to be some numbers in this picture. This question could be seen as revealing two facts. The first fact is that the figures in "similar" geometry problems that Jazzy had previously experienced were mostly labeled with numbers. The second fact is that Jazzy tended to rely on numeral information when she came across a geometry problem that resembled "scale" problem types. After she was asked if numbers were given, how she would use those numbers, she answered that she could compute each area by the given numbers and then compare the results. Since the interviewer did not give her any numbers, Jazzy continued to think about the problem without considering numbers. After thinking for a while, she stated that she did not know because the left triangle was "more up-and-down" while the right one was "more slide." She also argued that the lengths of the sides were different. Then
the interviewer asked, if the base was 10 units long and the height was 5 units, whether she could answer the question. Jazzy thought for 15 seconds, then described that she would determine the lengths by the ratio between line segments. This could be considered as a modified strategy of the initial one, which was using proportion to compute the sides, then figuring out the areas. Since Jazzy had been stating that the problem could not be solved without numbers, the interviewer asked a series of questions. The 7 questions in this series, as well as the answers from the interviewee, were listed as following in Table 9.

<table>
<thead>
<tr>
<th>Questions asked by the interviewer</th>
<th>Answers provided by the interviewee</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. How do we find the area of a triangle?</td>
<td>1/2 * base * height</td>
</tr>
<tr>
<td>2. Could you draw a picture to show what that formula means to you?</td>
<td>(drew a right triangle, marked the height and base) (see Figure 37(a))</td>
</tr>
<tr>
<td>3. Why do we time them together and divided to half?</td>
<td>Because the area of the triangle is half of the rectangle.</td>
</tr>
<tr>
<td>4. Is it only true for right triangles?</td>
<td>No?</td>
</tr>
<tr>
<td>5. Draw a non-right triangle, is it still one half base times height?</td>
<td>Yes? (drew a picture) (see Figure 37(b))</td>
</tr>
<tr>
<td>6. Convince me if it is still one half base times height.</td>
<td>The left half of the triangle could be moved to the right to make a rectangle (see Figure 37(c))</td>
</tr>
<tr>
<td>7. (drew a different triangle) Would this area still be one half base times height? (Figure 37(d))</td>
<td>No. But the formula should work all the time and I don't know why.</td>
</tr>
</tbody>
</table>

Table 9. First series of scaffolding questions asked by the interviewer and Jazzy's corresponding behaviors in the Compare Area problem
The last question in this series revealed that Jazzy had her own criterion for applying the triangle area formula, which was flipping half of the triangle to make a rectangle (like Figure 37(c)). When her own criterion for the formula did not work, her thinking was influenced by two competing forces: one was her personal belief that "the area would not be one half base times height"; the other one was her school knowledge, which was "the formula should work all the time." This contradiction confused her.

The interviewer asked Jazzy whether the formula could be used in the problem. Jazzy looked at the problem for 5 seconds and answered yes in a hesitative tone. After that, she said "oh I see" in a more certain tone, and used an alternative strategy to reach the correct answer – they would have the same area. After she solved the problem, the interviewer asked another series of questions. The 8 questions were listed as following in Table 10.
<table>
<thead>
<tr>
<th>Questions asked by the interviewer</th>
<th>Answers provided by the interviewee</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Would it be true that they still have the areas if those numbers are different? Like we didn't have 5 and 10, but 30 and 40.</td>
<td>Yes?</td>
</tr>
<tr>
<td>2. Convince me that it doesn't matter what those numbers might be.</td>
<td>(Marked with a different color) Because if the height is 30 and the base is 40, the height of the triangle would still be 30. (see Figure 38(a))</td>
</tr>
<tr>
<td>3. What if we don't have the measurements?</td>
<td>Now after I thought about it, it wouldn't matter still because the heights and bases would be still the same.</td>
</tr>
<tr>
<td>4. What is the relation between either of the triangles and the big rectangle?</td>
<td>I think it will be half of it.</td>
</tr>
<tr>
<td>5. Are you sure? How can we be sure?</td>
<td>No. (Thought for 20 seconds) The rectangle's area would be 50. The area of each triangle will be half, which is 25. (She returned to the numerical evidence to support her statement)</td>
</tr>
<tr>
<td>6. Always?</td>
<td>I don't know about always, But I know it works for these two triangles (the two triangles in Figure 38(a)).</td>
</tr>
<tr>
<td>7. Do you think the area would still be one half if I move the top point of the triangle along the upper side of the rectangle?</td>
<td>Yes, it will still be one half, because the height doesn't change.</td>
</tr>
<tr>
<td>8. What if I gave you this image, is there any relationship between the area of the triangle ABC and triangle BCD? (see Figure 38(b))</td>
<td>They would be the same, because the heights and bases are still the same.</td>
</tr>
</tbody>
</table>

Table 10. Second series of scaffolding questions asked by the interviewer and Jazzy's corresponding behaviors in the Compare Area problem

Figure 38. Pictures in the second series of scaffolding questions for Jazzy's Compare Area problem
This series of questions revealed Jazzy's ability to extend a specific case to more general case, which is an important skill in problem solving.

ii. Strategies

The initial strategy Jazzy described was computing each area with the given numbers and then comparing the results.

After being provided the lengths of the height and base of the rectangle, Jazzy modified her initial strategy and indicated that she could find out the length of each line segment along the top side of the rectangle using proportion. But it was not clear how she would use those lengths to determine the areas of the triangles.

The alternative strategy she used is shown in Figure 39.

![Figure 39. Jazzy's alternative strategy for the Compare Area problem](image)

Following the question of whether the triangle area formula could be used in this problem, she soon drew the heights of the triangles and marked them with 5, and further concluded that the areas would be the same because the heights and bases were the same. This strategy revealed her ability of applying appropriate knowledge to solve certain problems. Provided basic knowledge, she was able to adjust it and transfer the knowledge into the current situation.
iii. Gestures

Jazzy did not use calculator during this problem solving episode, since it was a geometry problem and was not designed to involve computation (no measurement). Most of her gestures were drawing pictures, and she showed most of her work either on the paper or by talking, which was unusual compared to her other episodes. This difference might be due to the different content — geometry — which was more visual instead of numerical.

iv. Representations

Although Jazzy tried to use numbers at the point of entry into the problem, she was able to further understand the problem through visual graphs. This could be interpreted as that although she tends to prefer numerical representations, she relies on visual representations easily and when accessible to her.

C. Jazzy's Inter-problem Analysis

a. Problem solving phases

The overall comparison of the 4 problem solving processes is illustrated in Figure 40. The length of each line is adjusted to indicate the relative amount of time Jazzy spent on each problem.
The four entry phases in these episodes had one common ground: Jazzy tended to seek out a numerical information/strategy to initiate her work, including the geometry problem, in which measurement was neither necessary nor provided. This common ground could be due to her preference for numeral facts and her confidence with manipulating numbers.
Another noticeable feature is that Jazzy did not justify her answer unless she was asked to do so. But she always had effective self-initiated reflection following each justification, and was able to move forward towards the right answer after reflection.

b. Strategies

Among the four episodes, Jazzy's choice of switching strategies or staying with one strategy was not consistent. For the Money Transaction problem, she stayed with her initial strategy throughout the whole problem solving process, only modified the numbers she used in that strategy. When she came cross errors, she didn't doubt her strategy, but focused on checking the numbers. This probably could be because she was more certain/confident about the approach. For the Water Lily problem, she immediately tried to switch her strategy after she was asked to explain her first answer. It was possible that the absence of numerical reasoning in her initial strategy made her seek out other strategies. For the Pay Option problem, she used a completely different strategy each time she started a new attempt. Although the first two strategies were more suitable to the problem, she did not try to modify them after having gained a better understanding of the problem. These frequent switches could be due to her lack of confidence in the strategies she used. She may probably doubt about her strategies, although it was not the strategy, but her understanding that was not adequate. For the Compare Area problem, she first stayed with her initial strategy and modified it, then switched to a different strategy after she had a better understanding to the problem. In the context of the 3 problems she completed in one session, her intra-task strategy flexibility was higher and consistent. Her confidence on a strategy seemed to depend on how much she believed she understood the problem.
c. Representations

Jazzy used various representations in the four problem solving episodes. It is noticeable that she used 3 different representations in the Pay Option episode. It was not natural for her to use more than one type of representations for any given problem. The reason she tried to use different types of representations within this problem could be because she thought the previous representations were not appropriate, or that she was challenged when she attempted to solve the problem with this. However, she did not deliberately seek out multiple representations in as a means to gain a better understanding.

Jazzy seemed to prefer numerical representations. In the Water Lily problem, she reasoned her answer in a numerical way. Only when she was asked to draw a picture, she used a visual representation, which was interpreted as a visual form of her numerical reasoning instead of an original way of thinking. In the Pay Option problem, Jazzy acted negatively by saying "no point to make a graph" when she was asked to draw a graph, and "you can tell it by instinct" when she was asked whether the graph was appropriate.

D. Summary

Based on Jazzy's four problem solving episodes, her general problem solving process is illustrated in Figure 41.
During the entry phase, Jazzy would first manipulate given numbers in order to understand the problem. She exhibited different types of behaviors depending on the type of strategy she used: if the strategy was to use numerical data and manipulating numerical information, she tended to stay with the strategy, only modified it by changing numbers. If the strategy was non-numerical, she tended to more flexibly switch to another strategy if she found her approach ineffective or insufficient. She also comfortably switched strategies once she gained a better understanding of the problem at hand.

Jazzy's natural choice of strategies was mostly numerical. Her numerical orientation was revealed throughout all four problem solving episodes. She relied on numerical facts to gain a better understanding of the problem, although she was able to generalize the answer and abstract ideas.
Jazzy never self-initiated a justification for her answer, but she had self-initiated reflections and questions. Both self-initiated reflections occurred after she realized she might be wrong, while the two self-initiated questions were asked to request more numerical information.

Jazzy did not tend to talk during the problem solving episodes, especially when she was not ready to write what she was thinking. Thus gestures became a major way to assess her possible thoughts and problem solving processes. Some usual gestures for Jazzy included tapping on the calculator, leaning back and forth, and playing with the pen.

Jazzy's performance was consistent across different content areas. She always tried to seek or depend her understanding of the problem using numerical data. Using numbers as a way of specializing was a particular approach when she was not familiar with the problem. But her performance for the use of heuristics varied: she was confident and successful in her strategy usage of working backwards, but was easily confused when a graph was required.

The Case of Yoni

A. Yoni's Background

During the first part of the first interview, Yoni was asked to articulate her feelings and attitudes towards mathematics, explain what was easy or difficult for him to do and whether he felt confident in his mathematical ability. This portion of the interview allowed us to build a profile of Yoni's thinking about and attitude towards mathematics.
that could potentially explain his specific mathematical behaviors during the problem solving sessions.

During the second part of the interview, Yoni was asked to solve the four mathematical problems as the others. He was reassured, as all other participants were, that our primary interest was in his thinking and the processes he used in tackling problems. We noted that we wished for him to think out loud and to explain his thinking as he interacted with problems. Yoni was not restricted to solve the problems in a specific amount of time and he was also reassured he could take as much time he needed to resolve problems using any method he deemed appropriate.

Who is Yoni?

At the time of data collection, Yoni was in 9th grade and taking Algebra 2, having completed a yearlong geometry course the previous year. Yoni stated that for him Algebra was easier than Geometry because Algebra was about equations instead of shapes, thus he liked Algebra more than Geometry. Upon request, he provided an example of the type of equations he was working on \([(5x^3 + 4x^2 + 2x + 17)/(x + 2)]\).

He further explained that he had just learned how to solve the equation without knowing when he could use this kind of equations. Yoni identified solving equations as the part he liked most about Algebra 2, while the long division of polynomials as the least part he liked. He preferred to deal with numbers. He also stated that shapes, angles, and formula made Geometry difficult for him. Yoni considered himself as "good at mathematics" because he could understand something if he had done anything similar. He agreed that mathematics was useful in his life, because he could use for "baking," "check," and
"taxes." When he was asked how he could use mathematics in his real life, he mentioned "grocery shopping" and "budget." Yoni identified mathematics as his most favorite subject in school. He argued that his feeling towards mathematics had improved significantly since the previous year when he had considered "gym" to his favorite subject. His professional goal was to be involved in medical field; he had just joined a biomedical program in his school.

Yoni had an outgoing personality. He responded to questions in detail and seemed excited and comfortable with sharing his ideas.

**B. Yoni's Intra-problem Analysis**

Table 11 illustrates an overview of Yoni's four problem solving episodes.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Order</th>
<th>Time of the episode</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Money Transaction</td>
<td>Session II</td>
<td>08'07&quot;</td>
</tr>
<tr>
<td>2. Water Lily</td>
<td>Session I</td>
<td>10'57&quot;</td>
</tr>
<tr>
<td>3. Pay Option</td>
<td>Session I</td>
<td>08'04&quot;</td>
</tr>
<tr>
<td>4. Compare Area</td>
<td>Session I</td>
<td>04'38&quot;</td>
</tr>
</tbody>
</table>

Table 11. General overview of Yoni's episodes

*a. Money Transaction problem*

Joe gives Nick and Tom as much money as each already has. Then Nick gives Joe and Tom as much money as each of them then has. If at the end each has 8 dollars, how much money did each have at the beginning?

Yoni spent 8 minutes and 7 seconds on the Money Transaction problem. He began the episode, stayed with the guess-and-check strategy until he solved the problem. He appeared confident when he was using this strategy. When he was reminded that Joe was not giving money during all transactions, he appeared less sure and worked more
slowly. After he had tried several numbers, his computations were completed more rapidly and finally solved the problem.

i. Problem solving phases

Yoni’s process for solving this problem was: entry – using initial strategy (guess and check) – getting an answer – using the same strategy with different numbers – getting a new answer as shown in Figure 42. Using Mason’s model as the framework, this process is illustrated by Figure 43.

Figure 42. Linear model of Yoni’s process for solving the Money Transaction problem

Figure 43. Mason’s model of Yoni’s process for solving the Money Transaction problem
After reading the problem, Yoni immediately set up a table of values as his initial strategy. After he reached an answer based on false assumptions, the interviewer reminded him about an important piece of information which he hadn't considered. He considered the question for a while, redrew the table and filled in numbers more carefully. He tried 5 sets of numbers and finally found the correct answer. When he was asked whether he could solve the problem in other ways, he tried to multiply the initial numbers (14, 8, 2) by 2. Manipulating the numbers for a while, he concluded that he could not use other ways to solve it.

In this problem, Yoni was never asked to justify his answer and he never attempted to do so. After quickly figuring out his first answer, the interviewer directly pointed out his misunderstanding of the information instead of asking him to justify the answer as a way of helping him realize the misunderstanding. After he correctly solved the problem, he did not test his answer, and the interviewer did not ask him either to justify his answer or whether he was sure about the answer.

ii. Strategies

The initial strategy Yoni applied is guess-and-check (working forwards), illustrated in Figure 44.

<table>
<thead>
<tr>
<th>J</th>
<th>N</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

Figure 44. Yoni’s initial strategy for the Money Transaction problem
He started with the columns of N and T, filling two 2s, two 4s, and two 8s from top to bottom. Then he turned to the column of J, wrote "16 - 4 = 12 - 8 = 20," and "20 - 4 = 16 - 8 = 8." It is possible that for the "12 - 8 = 20" he was thinking about "12 + 8." The 16 was probably the sum of the two 8s. Then he concluded that Joe had 20 dollars at the beginning, and filled the first column with 20, 16, and 8 without providing further explanation, yet it was clear that he was subtracting the sum of two 2s from 20, and the sum of two 4s from 16 to representing that Joe gave the other two people the amount of money they already had.

After being reminded by the interviewer that Joe didn't always give money during all transactions, Yoni thought for 20 seconds and redrew the table. He first filled the second row with 8, 2, 2, respectively, then filled two 4s under the two 2s (Figure 44(a)). Then he replaced the 2 under column N with 4, and replaced the 4 under column N with 8. Thinking for another 5 seconds, he replaced the 8 under column J with 12. This number was probably the sum of 8 and 4. The adjusted work is shown in Figure 45(b).

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![Figure 45. Yoni's modified strategy for the Money Transaction problem](image-url)
Yoni stared at the table for 10 seconds, then replaced the 12 with 10. He further filled 4 under 12, and paused for 15 seconds. When the interviewer asked him to explain what he was doing, he answered that he knew Joe had 10 dollars, but Nick might not have 4 at the beginning. He tried to adjust the two numbers in the column of N to 6 and 12, but soon found that the table became too messy (Figure 46(a)). So he started again with a new table, filling 10, 6, 2, respectively, as the amounts of money they started with, following with 12 and 4 under 6 and 2. He then adjusted the 10 to 12 again, filling 4 under 12. Then he doubled the two 4s and got two 8s (Figure 46(b)). When he had 1 cell left to fill in, he paused for 30 seconds.

![Figure 46. Yoni's modified strategy for the Money Transaction problem](image)

Drawing a new table, he started with 14, 8, 2, respectively. Reasoning that Joe gave 2 dollars and 8 dollars to the other two people, he quickly concluded that Joe had 4 dollars left after the first transaction (14 - 2 - 8 = 4). The remaining two cells on the same row were immediately filled with 16 and 4. Then he doubled the two 4s and determined the value of the last cell by a quick computation (Figure 47). He claimed that 14, 8, 2 was the answer.

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Yoni used "guess-and-check" as his strategy to solve this problem. From the numbers he adjusted each time, it could be considered as "educated guess-and-check."

iii. Gestures

Yoni tended to talk more when he was confident. When he was drawing the first table, he drew and filled it fast, with explanations to what he was doing. All of these could be interpreted as that he felt sure with the strategy he was using. When he moved on to the using modified strategy, he drew and wrote more slowly, as well as keeping silent unless he was asked to explain, which indicated a deeper engagement into the strategy. The speed of writing became faster each time he adjusted the initial numbers, revealing that he became more comfortable with the strategy.

iv. Representations

Yoni immediately chose to draw a table to solve the problem. He stayed with the table, as well as the top-down (working forwards) order of computation. This representation served his strategy, but failed to inspire the idea of working backwards. He seemed to focus more on the strategy instead of the type of representation he used.

b. Water Lily problem
Water Lilies are growing on a lake. The water lilies grow rapidly, so that the amount of water surface covered by lilies doubles every 24 hours. On the first day of the summer, there was just one water lily. On the 90th day of summer, the lake was entirely covered. On what day was the lake half covered?

Yoni spent 10 minutes and 57 seconds on the Water Lily problem. He switched strategies twice and self-initiated two reflections in the middle of his work. However, the second reflection led to discovery of an incorrect pattern, which Yoni failed to recognize during justification.

i. Problem solving phases

Yoni's process for solving this problem was: entry – using initial strategy (half of 90) – getting an answer – using alternative strategy under the interviewer's request – self-initiated reflection on previous work – using alternative strategy – self-initiated reflection – justifying his previous work (shown in Figure 48). Using Mason's model as the framework, this process is illustrated by Figure 49.

Figure 48. Linear model of Yoni's process for solving the Water Lily problem
After Yoni finished reading the problem, he said "I think I need to know how big the lake actually is." But he did not expand on this statement. Instead, he immediately applied his initial strategy "half of 90" and reached his answer, which was "the 45th day." When he was asked to sketch the problem, he drew a rectangle and filled it with circles that represented lilies. Proceeding to the fourth day, he paused in the middle of drawing circles, thought for a while, and claimed that "a chart would be better." When he started drawing a table, he kept track of the amount of water lilies on day 1, day 5, day 10 etc. instead of on day 1, day 2, day 3 etc.. He stated that he needed a pattern for the table when he was trying to figure out how many water lilies there would be on the 4th day, and continued computing in the way as he did for the previous days. He claimed that he
found the pattern when he moved on to day 15. Being asked to justify the numbers in his table, he checked the computation using the pattern he had found previously and restated that he believed 45th day was the answer because it was half of 90. Yoni seemed to focus on approaching his answer (45th day) in a different way instead of reframing the problem, which led to his choice of the way keeping track of the lilies (day 1, day 5, and day 10) and further caused the conclusion of an incorrect pattern.

In this problem, self-initiated reflection occurred twice. The first reflection led to a more efficient strategy, while the second reflection led to a discovery of incorrect pattern. Both self-initiated reflections revealed positive motivations, although the latter one seemed to be not successful. When he was asked to justify his answer, he applied the incorrect pattern and thus, failed to justify correctly.

ii. Strategies

The initial strategy Yoni applied was directly to divide 90 days by 2. He seemed to be sure about his answer strategy because he rephrased the statement "it would be half of 90" three times until the interviewer asked him to sketch the problem.

The strategy Yoni used to sketch the problem is illustrated in Figure 50. He first drew a rectangle as a representation of the pond. Then he started with 1 circle, explaining that on the first day there was 1 water lily, drawing a second circle under the first one. He stated that on the second day there would be 1 more. For the third day, he reasoned that you would have 2 more, and "2 times 2 is 4," so he drew 4 more circles under the previous 2. Then he moved on to the fourth day, claimed it was 12 at that time and drew 8 circles in a new column. Having found the drawing and counting inefficient, he paused, stating that "a chart would be better."
Yoni was inconsistent with his drawing in this strategy. During the first two steps, he seemed trying to add the number of water lilies increased on each day, that is, the number of circles in the rectangle would represent the total number of water lilies there were on a certain day. During the latter two steps, he was adding the total amount of water lilies that would be added on a certain day. This inconsistency might be due to his unfamiliarity with the strategy or the problem. The reason he decided to change the strategy may have been due to realizing the number of circles would be too many to draw as evidenced by his pause at the 8th circle instead of finishing all the 12th levels of iteration.

The alternative strategy Yoni applied is shown in Table 12. He set up a table of values. Instead of writing each day in a sequence, he claimed that he would "skip days" although he would "still count them." He first filled in the day-column with 1, 5, 10, 15, 20, ..., 90. Then he wrote "1" next to day 1 and explained there was 1 lily on the first day. He tapped numbers on the calculator to compute the numbers of water lilies for each connective day, multiplying 2 by each number he obtained for the previous day (without writing anything). When he was trying to figure out how many water lilies there would be on the fourth day, he stated that he needed a pattern. He restarted the computation by
mind, timing 1 by 2 for five times, and filled 32 next to the Day 5. Then he timed another five 2s to 32 on the calculator, and wrote 1024 next to the Day 10. Tapping on the calculator for 1 minute and 4 seconds, he claimed that he had "a pattern that multiplies by 32 every time." When he was asked how he got the pattern, he explained that the number on Day 5 was 1 times 32, and the number on Day 10 was 32 times 32. When he was asked to justify his original answer (45th day) by this table, he multiplied each number by 9 all the way to the 45th day, stated that he could go through the rest of the table in this way, and then jumped to the last row to multiply the number on the 45th day by 32 for 9 times.

<table>
<thead>
<tr>
<th>Day 1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day 5</td>
<td>32</td>
</tr>
<tr>
<td>Day 10</td>
<td>1024</td>
</tr>
<tr>
<td>Day 15</td>
<td></td>
</tr>
<tr>
<td>Day 20</td>
<td></td>
</tr>
<tr>
<td>Day 25</td>
<td></td>
</tr>
<tr>
<td>Day 30</td>
<td></td>
</tr>
<tr>
<td>Day 35</td>
<td></td>
</tr>
<tr>
<td>Day 40</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td></td>
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<td>50</td>
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<td>80</td>
<td></td>
</tr>
<tr>
<td>85</td>
<td></td>
</tr>
<tr>
<td>90</td>
<td></td>
</tr>
</tbody>
</table>

Table 12. Yoni's alternative strategy to sketch the Water Lily problem
Yoni's method of setting up the table made him confused when computing the number of water lilies on the 5\textsuperscript{th} day (there were only 4 days from Day 1 to Day 5, thus he should have multiplied 2 for 4 times instead of 5). The confusion could have been also due to the absence of written work during the computation. The reason Yoni chose to set up the table in this way could also be that he wanted to compute values all the way to the 90\textsuperscript{th} day yet trying to avoid producing a table containing 90 rows.

iii. Gestures

Yoni looked and sounded self-assured when he was presenting his initial strategy. He repeated some gestures when he was explaining his initial strategy: showing the day would be half by shortening the distance between two palms. This specific gesture could be interpreted as a sense of visual representation.

When Yoni was drawing the table, he explained what he was doing for each step and proceeded the whole computation without pause. This could be interpreted as that he was confident in the computation.

iv. Representations

The first graph Yoni drew was more visual. The way he kept track of the days and the number of lilies was not clear in that graph, thus the graph became an obstacle for him to gain a more abstract/systematic understanding of the problem. But it was a more natural way for him to start.

The alternative representation (table) was more commonly used for this problem. However, the way Yoni marked the days (Day 1, Day 5, Day 10...) was unique. The reason he chose this way of keeping records of the number of water lilies might be that he wanted to track all the way to the 90\textsuperscript{th} day, yet 90 rows in a table would be too many for
him to write. This special way of tracking was confusing and became an obstacle to generating a correct pattern.

Although Yoni used two different representations in this problem, neither of them worked as well as they could have because the focus of his work was on approaching his answer (45th day) in a different way instead of reframing the problem.

c. Pay Option problem

Consider the two pay options: $300 a week or 7.5 an hour. What factors will affect your choice of option to take. Draw a graph that compares the two pay options, allowing the reader to determine which option might be best for them to take.

Yoni spent 8 minutes and 4 seconds on this problem. He chose a bar graph as his strategy and stayed with it without modifying or switching. He chose to look at the payments with a different number of days as a compromise between hourly and weekly pay options, yet he tended to fix the number of hours in each day when he was considering the number of days in a week. His graph only represented an instance among all the situations.

i. Problem solving phases

Yoni's process for solving this problem was: entering the first part of the problem – giving an answer – asking a question at the beginning of the second part of the problem – using initial strategy – answering a question asked by the interviewer – answering another question asked by the interviewer (shown in Figure 51). Using Mason's model as the framework, this process is illustrated by Figure 52.
When he was asked what was the factor that could affect the choice of option, Yoni immediately answered "how many hours I'm working." He further gave an example: 7 days a week or 5 days. The example he provided was in the unit of "days" instead of "hours." This inconsistency between his answer and his example could be because Yoni was focused on different pieces of information: when he gave the answer, he was focused on the hourly payment; when he provided the example, he was focused on the weekly payment. Lack of awareness of a need for consistency served as an obstacle for him to systematically review the problem globally.
For the second part of the problem, Yoni first asked whether the person was working 8 hours a day but did not receive a specific answer from the interviewer. This question revealed that Yoni was thinking about a specific number of hours instead of considering various cases. Then he tapped on the calculator for a while, stating that he was dividing 300 by 5 since there were 5 working days in a week and then dividing by 8 since each day he would work 8 hours. Upon request, he explained that he chose 5 days and 8 hours because it was the most common condition under which people worked. He spent the next 4 minutes and 30 seconds finishing the graph with his initial strategy, during which he concluded that the two options would be the same, yet the weekly payment would be better on the long term. In response to asking how the number of the hours of work affected his choice, he stated that if working 8 hours, then the two options would be the same; but if working less hours, "like 4 hours," the weekly payment would be way more than the hourly one. When he was asked how to tell which pay option would be better from his graph, he answered it didn't matter if the person worked 8 hours a day, but if he worked more than 5 days a week, then the weekly payment would be a better option. Yoni was able to judge various-hour conditions when he was asked to, yet he seemed not to be able to consider both variables of hour and day at the same time — he tended to fix the number of hours when he was considering the number of days in a week. Thus his graph only represented one case of all possible situations.

ii. Strategies

Yoni stayed with one strategy in this problem. After drawing x and y axes, he first marked the x-axis by "day" and y-axis by "pay." Then he scaled the y-axis by 10, marking from 10 to 300. Tapping on the calculator for 5 seconds, he drew a bar, shaded it
by wave. Then he did another computation, drew a bar with the same height, shaded it by line segments. Upon request, he explained that he first divided 300 by 5 and got 60, since there were 5 days in a week; then he multiplied 7.5 by 8 and obtained 60, because each day the person would work 8 hours. Then he moved on to the second pair of bars, tapping on the calculator before drawing each bar. He paused after finishing the second day (Figure 53), and claimed that he believed the two options were the same, although the weekly pay option would be better in the long term because there were seven days in a week.

![Figure 53. Yoni’s initial strategy for the Pay Option problem](image)

At this point the interviewer suggested that he may extend the bar graph to seven days. The motivation for this suggestion might have been to see what he meant by "the weekly one would be better if there were seven days in a week." Yoni extended the graph and tried to draw the remaining 5 days. For the rest of bars, he only computed the wave bar (the payment with weekly pay option) and then drew the other bar with the same
height without computation (Figure 54). He did not follow the scale on the y-axis like he did for the first two days anymore, but focused on the illustrative goal that the two payments would be the same.

![Graph showing Yoni's final answer for the Pay Option problem](image)

Figure 54. Yoni’s final answer for the Pay Option problem

Yoni claimed the weekly pay option would be better without showing it on his graph. Although he extended the bar graph to day 7, he kept using 60 dollars as the daily payment for the weekly pay option. I interpreted this to mean that he wanted to show the situation of working 7 days a week but failed to consider the change of daily payment between working 5 days a week and working 7 days a week, since he was claiming about the situation of working 7 days in a week.

iii. Gestures

When writing from 10 to 300 by scale of 10, Yoni was patient but quick. Besides the short pause when he finished drawing the first two pairs of bars, he drew the graph continuously without any hesitation. He seemed to be confident with the strategy he chose and his conclusion.
iv. Representations

The representation Yoni chose was bar graph, and he did not switch to other representations during the whole problem solving episode. The reason for choosing this particular representation might be because of previous experience; he might be more familiar with a bar graph when dealing with graphs about variable comparison. He was confident with setting scales and drawing the graph, which could be interpreted as that he was confident with using this kind of representation.

d. Compare area:

Consider the graph below: compare the areas of triangles BEC and BFC, which one is larger and why? What can we say about the relationship between the area of ABCD and the area of triangle BEC? Is it possible for either of these two triangles to be larger than one half of the area of the rectangle?

The Compare Area problem given to Yoni was a little different from the problem given to the other two interviewees. It had two extended parts.

Yoni spent 4 minutes and 38 seconds on the Compare Area problem, which was relatively short. His answers to the three parts of the problem were quick and he did not seem to doubt them. He referred to the relationship between a rectangle and a right triangle with the same base and height when he was answering the last two parts of the problem, but did not extend it to more general situations.

i. Problem solving phases
Yoni's process for solving this problem was: entry – asking a question – using initial strategy – getting an answer for the first part of the problem – answering the second part of the problem – answering the third part of the problem (shown in Figure 55). Using Mason's model as the framework, this process is illustrated by Figure 56.

Figure 55. Linear model of Yoni's process for solving the Compare Area problem

Figure 56. Mason's model of Yoni's process for solving the Compare Area problem

After reading the problem Yoni asked whether there should be some numbers in the picture. This question revealed his previous experience with area problems involving pictures, which was probably about measurements. Instead of waiting for an answer to
his question he quickly reasoned that the area of triangle BEC would be bigger with his initial strategy. For the second part of the problem, Yoni paused for 10 seconds and stated that if he could move the point E to the position of point A, then the area of triangle BEC would be half of the area of rectangle ABCD. This statement revealed that he was familiar with the relationship between the area of a rectangle and the area of a right triangle with the same base and height. He did not stop at this statement, but paused for another 5 seconds and claimed that without moving the point E, the area of rectangle BEC was a little less than half of the area of rectangle ABCD because the area of triangle ABE made the triangle BEC smaller than half of the rectangle ABCD. For the last part of the problem, Yoni immediately answered that although he didn't know about bigger, he knew it could be exactly the same as half of the area of the rectangle, providing the condition that if you move point E to the position of point A, or move point F to the position of point D. Yoni did not seem to realize moving those two points would not change the areas of the triangles. Thinking for another 10 seconds, he claimed that the area of triangles could not be larger than one half of the area of rectangle because “two triangles make up a rectangle.” Yoni seemed to stay with the relationship between a rectangle and a right triangle with the same base and height. He kept trying to transform the unknown object (the picture in the problem) into something he knew (rectangle and right triangle), but did not draw from the essential part of the knowledge (how two triangles made up a rectangle). Thus he failed to reach a conclusion for the general case.

ii. Strategies

The initial strategy Yoni used to answer the first part of the problem is illustrated in Figure 57. He identified the two areas ABE and CDF (shaded areas in the picture), and
stated that the area ABE was smaller than the area CDF. Then he reasoned that since the area ABE was smaller than the area CDF, the triangle BEC would be bigger than the triangle BFC.

![Diagram of a geometric figure with labeled points A, B, C, D, E, and F.](image)

Figure 57. Yoni’s initial strategy for the first part of the Compare Area problem

This strategy was incomplete. Yoni did not consider the other part of the outer area, nor extend this strategy or try any other strategies. He did not seem to doubt his answers, which he reached in very short time.

iii. Gestures

All of Yoni’s demonstration about how he reasoned with the problem was carried out by gestures. He did not mark or write anything. Besides oral explanation, gestures served as the only way he presented his work, helping him communicate with the interviewer. As he stated before, angles, shapes, and formulae made geometry difficult for him. It might be his attitude towards geometry that stopped him from writing things and further carefully studying the problem.

iv. Representations
Yoni did not write anything during this episode, yet he demonstrated his strategy and kept track of everything with his finger. This revealed that he was clear about his reasoning and was comfortable with the way he worked with the problem.

C. Yoni's Inter-problem Analysis

a. Problem solving phases

The overall comparison of the 4 problem solving processes is illustrated in Figure 58. The length of each line is adjusted to indicate the relative amount of time Yoni spent on each problem.

Figure 58. Overview of Yoni's four problem solving processes
Among the four episodes, Yoni had 2 self-initiated key questions prior to his initial strategies. Both questions were raised but not further discussed. In the Pay Option episode, the initial strategy used by Yoni following with the self-initiated question was related to the self-initiated question – he asked whether the person worked 8 hours a day, and later he chose 8 as the number of hours when he computed the payment for each number of days. In the Compare Area episode, the initial strategy following with the self-initiated question was not related to the self-initiated question. The difference revealed that a key self-initiated question may not impact further behaviors if no response was provided by the interviewer.

Both justification and reflection were rare among the four episodes. Compared to the other self-initiated behaviors (questions and reflections), self-initiated justification was absent. The only justification that was initiated by the interviewer did not lead to a realization of the misunderstanding, since it was only a check of computation instead of a check of pattern. The two self-initiated reflections in the Water Lily episode played positive roles in the problem solving process: promoting a switch to a more effective strategy and finding a pattern.

b. Strategies

Except the Water Lily episode, Yoni stayed with his initial strategy during the entire problem solving episodes. He was patient with repetitive computations, as well as guess-and-check strategy. The only time he switched strategies was from a visual strategy to a more numerical strategy. Based on what he said about the preference to algebra rather than geometry and his problem solving strategy flexibility, it seems that Yoni was more likely to stay with his strategy if he was working on numbers. For the Compare
Area problem, Yoni's problem solving episode was short and there was no switch on strategies. It was probably due to his negative attitude towards geometry, which made him try to avoid more engagement with the problem.

c. Representations

Yoni used more numerical representations in the four problem solving episodes. He used limited visual representations: the visual graph he drew in the Water Lily problem was more intuitive instead of an effective way to visualize the problem. The unbalance of his preference on representation might be due to his preference to numbers instead of geometrical objects. This became one of his obstacles to be fully benefit from his problem solving behaviors.

D. Summary

Based on Yoni's four problem solving episodes, his general problem solving process is illustrated in Figure 59.

![Figure 59. Yoni's general problem solving orientation](image)
Yoni's problem solving process followed a linear progression. He would ask questions if he was not clear about the problem, but his further exploration of the problem may or may not have been influenced by the questions or the answer he received. For instance, in the Pay Option problem, his initial strategy was influenced by the number of hours a person worked a day "working 8 hours," as evidenced in his question, but in the Compare Area problem, he continued with solving the problem without worrying about whether numbers were given in the picture. Once he chose his initial strategy, he would usually stay with it until he solved the problem (unless he was asked to consider other strategies).

The problem solving strategies Yoni naturally relied on and used were generally his most familiar strategies (i.e. guess-and-check) or the most reasonable strategies he could think of (i.e. bar graph and partial outer areas of the triangles). He was confident in appropriateness of his strategies and persisted on using them as long as he could.

Yoni had self-initiated reflections and questions during all problem solving episodes, but never justified his answers unless he was asked to do so. The two self-initiated reflections he exhibited led to a modification of an inefficient strategy and a discovery of pattern. Although the pattern was incorrect due to the choice of representation, the two reflections could be considered as having positive impact on his problem solving process.

Yoni's performance was consistent across different content areas and different heuristics. He was confident, working forwards, and stayed with his initial strategy disregarding of the content and heuristic needed. Even when he was asked to use other strategies (upon the request from the interviewer), he retained his initial answer (which
was incorrect) and justified it using the pattern he had found. His confidence in each process was probably due to his belief that he was good at mathematics.
CHAPTER 5
CONCLUSIONS AND DISCUSSION

Summary and Conclusions

The overarching goal of the current research project was to study the mathematical problem solving practices of three 8th and 9th grade children as they worked on novel problem solving tasks, using a clinical interview methodology. In carrying out the data collection and analysis, four research questions guided the researcher's efforts:

1. What mathematical problem solving processes do children exhibit when working on non-routine tasks?
2. What mathematical problem solving strategies do children naturally use as they work on non-routine tasks?
3. What types of metacognitive behaviors do children naturally exhibit as they engage in mathematical problem solving?
4. Are children's strategy uses and problem solving processes consistent across different problem types depending on the kind of heuristic needed or the mathematical subject area used in context?

Drawing from the case studies presented in the previous chapter, this chapter will offer a cross case analysis of the mathematical behaviors of the three children to outline responses to each of the guiding research questions.
Children's Problem Solving Behaviors: A Brief Overview

All three participants showed a high degree of involvement in solving the assigned problems during the interviews. They took ownership of the tasks and attempted to attack, review and solve them. The intensity of their involvement in tasks was reduced or diminished if they failed to see patterns, fully understand the information given, make connections between what they knew and the contexts under study. An additional inhibitive factor included their ability to define mechanisms for gauging their own success in solving problems when using strategies most familiar to them.

In all three cases, the interviewers' questions and their probing comments motivated further exploration of the tasks (e.g., Compare Area). This was paramount in the context of problem types least familiar to children or those most ambiguously stated (Pay Option). The children's ability to identify relevant from irrelevant data, either embedded in the problem or deduced as the result of their own work, was a pivotal influence on their successful problem solving as evidenced by their willingness to reflect on options or reconsider approaches. In case of Yoni, preoccupation with using familiar techniques learned in school curriculum served as a primary motive for his reluctance to focus on extending his understanding of the problems, reflecting systemically on what was given, or to even test and justify his answers.

The children's particular orientation and their personalities also played a crucial role in not only how they engaged in problems but also the degree of persistence they showed in solving them and whether they tried to access additional strategies or to engage in metacognitive actions. Liza, the one subject with least amount of interest in school mathematics and its content, seemed most flexible in changing strategies. Her
need for understanding and sense-making, as articulated during both interview sessions, may have been the primary force behind her natural desire to remove from context at hand and switch her approaches.

On the other hand, Yoni, the most successful student among the three, and the one with most sophisticated mathematical tools, appeared least flexible in his thinking and choices. Indeed, his attempts to use procedures he had learned in school prohibited him from taking the initiative to justify or verify his own answers.

All three participants showed the tendency to enter problems using the technique of testing numerical values. They made, or refused to make modifications to their initial choices of numerical values to understand the problem better. Once and if a deeper understanding of the problem was achieved they were more willing to switch strategies. They are also more successful in the use of newly adopted approaches. Despite this, those subjects with greater control over numerical manipulation tended to remain loyal to the use of this approach. A shift from one strategy to another was the result of either a significant change in level of understanding of the problem, or provoked by interviewers' questions. In all cases when a shift in the choice of strategy occurred, problem was not solved unless a good understanding of the problem was achieved.

All three subjects showed the tendency to use concepts most recently addressed in school at the time of data collection, regardless of whether these concepts were relevant to the problem under study. This was most notable when they worked on the geometry task. Indeed, that problem revealed the major gap between understanding the concept of area and the ability to apply algorithm. With the exception of Jazzy, the two other
participants experienced difficulty when abstractions of specific knowledge became a focus of work.

The children's ability to access different representational modes was also driven by the contexts they had previously experienced in school experiences. The use of drawing a picture for illustrating the problem became only natural for two of the participants (Jazzy and Liza) when their initial attempt at using numerical data for answering questions seemed too cumbersome to be practical. Even when they were successful in use of the strategy they remained skeptical of the accuracy of their own responses. Formalizing and authenticating the final answer derived using this approach was endorsed to an outside authority (the interviewer), as opposed to self conviction.

Patterns of refinement of thinking and monitoring progress was evident in the work of all three participants, however, the frequency of its occurrence seemed closely linked to scaffolding techniques used by the interviewer.

Table 13 provides an overview of particular behaviors and performances of children as they relate to average amount of time they spent on tasks, average number of instances of self-initiated questions, average number of times they switched strategies, average number of self-initiated testing and justifying episodes.

<table>
<thead>
<tr>
<th></th>
<th>Average length of PS episode</th>
<th>Average number of self initiated questions</th>
<th>Average number of times shifts in strategy usage</th>
<th>Average number of justifying answers</th>
<th>Average number of interviewers' scaffolding questions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liza</td>
<td>10'54&quot;</td>
<td>0</td>
<td>1</td>
<td>1.5</td>
<td>1.75</td>
</tr>
<tr>
<td>Jazzy</td>
<td>9'57&quot;</td>
<td>0.5</td>
<td>1</td>
<td>0.75</td>
<td>5.25</td>
</tr>
<tr>
<td>Yoni</td>
<td>7'57&quot;</td>
<td>0.5</td>
<td>0.5</td>
<td>0.25</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 13. Overview of participants' particular behaviors and performances
As the data illustrate, although the three participants varied greatly in the average amount of time they spent on tasks, their performance along the self-initiated constitutive elements of the problem solving process was consistent, suggesting potential patterns of thinking and actions. The average numbers of self-initiated questions and times shifts in strategy usage had the least variety of items, indicating a less diverse use of these two behaviors. The average numbers of justifying answers, which included self-initiated justification and interviewer-initiated justification, varied more than the previous two items. The average number of scaffolding questions that the interviewers asked was the most item with the most variety, which was also the least natural item for participants. A detailed cross case analysis of subjects' mathematical practices is offered in the following section.

What mathematical problem solving processes do children exhibit when working on non-routine tasks? What mathematical problem solving strategies do children naturally use as they work on non-routine tasks?

Mathematical Problem Solving Process: An Intra Problem-Analysis

A. Money Transaction problem

The problem solving processes used by all three participants for this problem are illustrated in Figure 60.
All the three students successfully solved this problem. They all chose a table (or table-like format) as the only type of representation for this problem.

The initial strategy each student used was different. Liza started with the final amount of money but actually "worked forward," according to her way of manipulating numbers (i.e. subtracting when giving money and adding when receiving money). Jazzy started the process by working backward, while Yoni initiated the problem solving process by using guess-and-check strategy, working from amount of money that remained at the end of the three transactions and then moving forward by adding value.

These initial strategies represented their first choice based on their understanding of the problem during the entry phase, which may reveal their most familiar/confident
strategy when coming across such problem situations. The choice of initial strategy could be influenced by their previous experiences with mathematical problems, preference/confidence with certain strategies, and their level of understanding of the problem. For this problem, Jazzy and Yoni showed their familiarity and confidence with "working backward" and "guess and check" strategies, respectively, since they stayed with their initial strategies, modifying numbers instead of attempting to switch strategies.

Following the failure of their initial strategy to help obtain an answer, Liza changed her strategy to working backward heuristic, while Jazzy and Yoni stayed with the strategies until they solved the problem. The motivation for Liza to change her strategy was her reflection on her initial strategy, as well as development of an improved understanding of the problem following the reflection, as evidenced by her comments.

Liza's initial strategy was not successful. But after she gained a better understanding of the problem, she switched to a successful alternative strategy. Her alternative strategy was clearer and more systematic compared to Jazzy's and Yoni's methods. In my assessment, Liza's performance on this problem is more efficient than the other two, since a conscious seeking for improvement of understanding (revisiting the problem), an efficient adjustment to the strategy, and a clear and systematic solution to the problem were presented.

Liza and Jazzy had self-initiated reflection phase before they proceeded to their final step of solving the problem. Each self-initiated reflection followed by an interviewer-initiated justification to the answer. Both types of justification (self-initiated and interviewer-initiated) were absent in Yoni's problem solving episode.

_B. Water Lily problem_
The problem solving processes used by all three participants on this problem are illustrated in Figure 61.

Yoni spent twice as much time as the other two students on this problem; yet he did not successfully solve this problem the other two, who solved it correctly.

All the three students used visual representations in this episode, yet each representation played a different role. Liza's graph was used to justify her answer, which was in a "working backward" order (drawing the pond on 90th day, then showing half of them would be the day before). Jazzy's graph was in a "working forward" order,
corresponding to her previous numerical reasoning (500 times 2 is 1000). Her visual representation here, which was requested by the interviewer, is more like the visual form of her numerical representation, instead of an efficient way for her to think about the problem. Yoni's graph was a detailed iconic representation of the problem (circles which represented lilies), which was soon switched to a table. The different role of visual representations could reveal their ability to use this type of representation in problem solving activities, where Liza's is considered to be the most developed among the three.

Both Jazzy's and Yoni's initial strategy was to find "half of 90." The immediate answer revealed their tendency to directly manipulate given information when they came across unfamiliar and info–limited problems ("double" and "90th day"). But they showed different behaviors after this initial approach: Jazzy immediately doubted her initial strategy and self-initiatedly reflected on the problem, while Yoni showed no doubt and even confirmed this answer. Jazzy's immediate hesitation towards her initial strategy, as well as her self-initiated question followed the reflection phase "how many is full?" could be interpreted as a preference for numerical sense-making. Liza's initial strategy was to set a table, which led to the development of an insight to the problem. Her previous experience to a similar problem (the rice problem) and her preference for efficient strategies may probably have impacted her performance.

Both Jazzy and Yoni had self-initiated reflection during the problem solving process. Jazzy's self-initiated reflection followed an interviewer-initiated justification, and led to a key self-initiated question. On the contrary, Yoni's two self-initiated reflections occurred in the middle of his strategy use, without any interviewer-initiated activities. His reflections led to a switch to a more efficient strategy, but were interrupted.
by recognition of an incorrect pattern. Both students' reflections played positive roles in their problem solving process, yet data suggest that a self-initiated reflection (even occurred without interviewer-initiated activities) may not guarantee launching correct answers.

C. Pay Option problem

The problem solving processes used by all participants for this problem are illustrated in Figure 62.

Figure 62. Overview of three participants' problem solving processes for the Pay Option problem

For the first part of the problem, all the students immediately provided the answer, it depends on "how many hours a person worked."

For the second part of the problem, both Liza and Jazzy used a linear graph, yet Liza stayed with her strategy throughout the problem solving process (except a reference
to her previous strategy using in the first part of the problem) and Jazzy switched strategy as soon as she considered the graph as meaningless and inefficient. One difference between their linear graphs was labeling the axes: Liza set the x-axis as payment and y-axes as hours and days (she had two linear graphs at first), while Jazzy set the x-axis as weeks and y-axis as payment amount. Jazzy's way of setting axes is more conventional, where x-axis represented the independent variable. Yet her choice of the independent variable (weeks) made it problematic for her to stay with the graph: she was reluctant about how to deal with the consideration of different hours within this graph.

Yoni chose a bar graph as his initial strategy and stayed with it until he was finished. The bar graph was carefully set up, including detailed amounts of payment on y-axis and pairs of bars with different identifications representing each pay option. However, the information contained in his bar graph was limited, as compared to the bar graph Jazzy produced in her episode. Yoni only illustrated one case of all possible situations, although he was able to reason about viability of other situations when he was asked to do so. Jazzy's bar graph was less conventional (the x-axis was labeled as "30h – 40h – weekly – 50h"), but she tried to include all important information in the image (the equalization of the two pay options at 40 hours, and the tendency of the change of payment when the amount of hours increased or decreased).

In terms of representation, Liza did not appear to be familiar with linear graphs, although she continued working with it throughout the episode. She had a sense of the answer based on her numerical reasoning in the first part of the problem, and she tried to confirm the answer by the linear graph in the second part. Her final graph contradicted her conclusion, but she still tried to use it to support her conclusion. This could be due to
her strong confidence in the previous numerical reasoning, which was hard to be doubted even in the present of contradicting data (graph). Jazzy used three kinds of representations in all. Her motivation to use different representations was generally caused by her lack of security in the representation she was using, which could be due to her insufficient understanding of the problem. Yoni stayed with bar graph throughout the episode, indicating his familiarity with this type of representation.

At the end of this problem solving process, both Jazzy and Yoni were able to reason the tendency of the change for the amounts of payment as the result of change in the number of working hours. They also identified the equal status of the two pay options, although Yoni did not show it in his work. Liza reached the conclusion that the two pay options would be equal at a point (which she failed to show in her graph), but did not explicitly state the relationship between the amounts of payment and the number of working hours. Jazzy's final level of understanding towards the problem could be considered to be the highest among the three.

D. Compare Area problem

The problem solving processes used by all participants for this problem are illustrated in Figure 63.
For this problem, Liza's and Jazzy's episodes contained scaffolding questions asked by the interviewer, which impacted their problem solving performance. The interviewer did not intervene in the episode with Yoni.

The initial strategy each student applied was different. Liza looked at similar/congruent triangles in the graph, which was probably the topic she had covered in her geometry class. Jazzy's initial strategy was computing each area and comparing the results, and she insisted that the information was not sufficient to solve the problem unless numbers were provided. Her initial strategy and conclusion indicated her previous experience with such problems as well. Yoni considered part of the outer areas for each triangle, and directly compared the two triangles by the two partial outer triangles. His initial strategy indicated the most conceptually efficient start compared to the other two,
since he concentrated more on the relationship between areas, instead of less general relationships (similar/congruent) or numerical relationships (computing the area).

The scaffolding questions were generally used to provoke deeper thoughts based on their existing reasoning in order to gain a better understanding towards the problem. Liza started with a visual perspective, and she remained with visual approaches although she switched strategy three times. She did not try to represent the problem in a non–visual form (i.e. numbers and formulae); she even manipulated the areas by copying them onto a new sheet. This action corresponded to her preference for manipulatives.

Different from Liza, Jazzy started the problem using a numerical perspective, and she explored the triangle formula inspired by a series of scaffolding questions. Once she was provided with values for base and height, Jazzy successfully reached the correct answer. With another series of scaffolding questions, she was able to extend the conclusion to the general case. Jazzy's need for manipulating numbers to better understand a problem was a consistent pattern. Her ability to apply meaningful knowledge to solve the problem was also noticeable.

Both Jazzy and Yoni asked a self-initiated question at the beginning of their episodes and both questions were related to the existence of numbers in the picture. These questions were indicative of previous experiences with such images. However, Jazzy stayed with her question and did not try to solve the problem without numbers, while Yoni did not wait for an answer from the interviewer but started solving the problem using a visual strategy. Jazzy's confidence in relevance of her question made her reluctant to try a new method, while Yoni chose to ignore his doubt and believe the
problem was solvable. If Yoni had been provided a response to his self-initiated question, he might have used other strategies to solve the problem.

**Problem Solving Processes: An Inter-Analysis**

Based on each student's problem solving episodes, Liza, Jazzy, and Yoni's general problem solving processes (despite content and heuristics) are illustrated in Figure 64, 65, and 66, respectively.

![Diagram of Problem Solving Processes](image)

*Figure 64. Liza's general problem solving orientation*

Compared to the other two students, Liza's general problem solving process had a unique feature: sense-making. Sense-making was her way of self-monitoring, which was presented throughout all problem solving episodes. After she reached an answer, sense-making was her premier way to justify her response. It was one of the factors that could influence her switch of strategy. Liza had a belief that she could solve most problems
eventually, thus she was more likely to deliberately re-enter the problem in order to gain a better understanding of the problem.

Figure 65. Jazzy's general problem solving orientation

Jazzy's entry in the general problem solving process was different from other students. She always started with manipulating numbers to get a sense of the problem. She visibly preferred numerical dependent strategies among all others. Her intra-task strategy flexibility largely depended on the type of the strategy she used (numerical or non-numerical), where she tended to switch strategy instead of modifying information when she was not using a numerical strategy.
Yoni's general problem solving process was more linear: unless he was asked to use other strategies, he tended to stay with his initial strategy until he solved the problem. Yoni seldom justified his answer, he considered the problem was finished whenever he reached an answer (either right or wrong). His low intra-task strategy flexibility may be probably due to his belief that he was good at mathematics.

Among the three students Liza was the one most flexible in her strategy use, although Jazzy showed greater success when using alternate representations for solving problems.

The reliance on numerical clues frequently interfered with children's ability to focus on understanding the problem.

**Children's Choice of Representations: A Cross-Case Analysis**

The most prominent type of representation used by the three children was numerical, with preferred technique being setting up a table of values for either finding a pattern or generalizing answers. This choice was naturally used and reliance on other modes of representation occurred either as the result of the interviewers' questions or elicitation (external demand, nonetheless).
What types of metacognitive behaviors do children naturally exhibit as they engage in mathematical problem solving?

**Children's Metacognitive Activities: A Cross-Case Analysis**

Among the three participants, evidence of metacognitive presence in the way of either self-monitoring or self-regulating solution process was least visible in Yoni's work. This was interesting since it contrasts sharply with the body of work highlighting connections between confidence and problem solving performance. Yoni was the most confident of the three participants and perhaps the one most articulate about his taste for mathematics. However, his reflections on problems, compared to others, were least self motivated. This could be explained, at least in part, to be the result of his success with school mathematics and his success in controlling school exercises.

In contrast, Liza, least skilled in the use of school mathematics was most motivated by whether answers made sense to her or not. Indeed, it was this particular desire that allowed her to move flexibly among different representations and strategies when solving problems. The primary motive for her monitoring and regulating answers and her own actions was whether the process was internally meaningful. On the other hand, Jazzy's regulating and monitoring processes were guided by testing and justifying answers. In places where she was unable to justify or explain accuracy of her answers she naturally folded back and considered other options.

In all three cases, scaffolding questions posed by the interviewer served as fundamental impetus for the "looking back" phase of problem solving process of children. While in varying degrees, all three participants' problem solving process was positively impacted by scaffolding questions.
Are children's strategy uses and problem solving processes consistent across different problem types depending on the kind of heuristic needed or the mathematical subject area used in context?

Consistency in Performance: A Cross-Case Analysis

The participants' performances across different problems according to the types of heuristic needed in the solution process and subject areas were not always consistent. Liza's performance was not consistent in either area, Jazzy's performance was consistent across different subject areas but not with the use of different heuristic, while Yoni was consistent in both areas. The various performances/styles could be the result of the student's orientation, belief, and preference for certain subject, which were already discussed in this chapter.

Discussion

The results of this work provide additional evidence, supporting findings of previous research on problem solving, to suggest that self-monitoring is positively correlated with success in performance on certain mathematical activities (Cohors-Fressenborg, Sjuts, & Sommer, 2004; Cohors–Fressenborg et al., 2010). However, unlike other research studies in which students' problem solving performance was examined either on routine tasks or word problems (see for instance, Cohors–Fressenborg's study; Malloy & Jones, 1998), in this research I used non-routine problems. Liza, who used sense-making as a way of self-monitoring her progress on tasks and towards gauging her problem solving process accordingly, and Jazzy, who used numerical computation as a
way to self-regulating her actions and increasing her control over tasks, performed better than Yoni, who did not exhibit self-monitoring/regulating consistently during his problem solving processes. Hence, we highlight that self-monitoring/regulating could be a significant influence on successful problem solving on both routine and non-routine tasks.

Consistent with findings of previous research, the results of this work suggest that intra-task strategy flexibility does not imply success at reaching correct answers to tasks (Elia, Heuvel-Panhuizen, & Kolovou, 2010). However, I do posit further that the level of intra-task strategy flexibility might depend largely on the individual's confidence and preference for the use of certain strategies. These constructs may not ensure that correct answers across different subject areas and heuristics might be reached. Instead, they may prevent the individuals from moving forward in securing an enhanced level of understanding of the problem.

According to the data, high intra-task strategy flexibility is associated with personal preference for efficient strategies, lack of confidence on a currently utilized strategy, and significant change in level of understanding of the problem. In contrast, low intra-task strategy flexibility could be associated with confidence in the strategy currently used and an insufficient change of understanding.

My study reveals inconsistency in the same individual's mathematics problem solving behaviors across different subject areas and/or heuristics usage. This result is distinct from the conclusion of previous research that indicates most individuals exhibit consistent problem solving behaviors (Muir, Beswick, & Williamson, 2008). According to our data, the factors that may impact the consistency in behaviors include the preference for the use of specific approaches and orientations (visual, graphical, pictorial,
etc.), experience with specific subject area (number theory, algebra, geometry), familiarity with the heuristic needed to solve the problem, and personal belief about one's own mathematical ability and confidence in problem solving.

One's preference for a specific subject area could impact one's endeavor devoted to the problem, disregarding negative effects (i.e. unfamiliarity and frustration) he/she encounters during the problem solving process. If an individual is working on a problem that involves familiar heuristic, he/she is more likely to successfully develop an appropriate strategy to reach the correct answer even when he/she is not familiar with the problem. The preference for specific types of strategies could result in perseverance or persistence on the use of preferred strategy. Personal orientation could largely impact one's problem solving behaviors throughout the entire process (i.e. Jazzy's numerical orientation). Personal belief about one's own knowledge and ability could impact one's confidence: Liza believed she could eventually solve all problems and Yoni believed he was good at mathematics; both of them exhibited noticeable confidence during their problem solving episodes. On the other hand, Jazzy's belief about problem solving ("the only right answer") influenced her attitude during the problem solving process: she asked for the right answer even when she was convinced by visual evidence.

The findings of this study further support the results of previous research indicating that the nature of the individual's knowledge and its organization serves as a major influence on their success or failure as problem solvers (Lawson & Chinnappan, 2000). In the current study, the data provided evidence that in places where children had a conceptual understanding of relationships and concepts, they moved more flexibility, within both the problem and solution spaces. Indeed, they managed to change strategies,
assess and monitor their own work and progress. In contrast, when knowledge was disconnected and algorithm-based, children tended to imitate mathematical behaviors perceived to be expected, using their school experiences as a framework for judgment.

The result of this study proposes that the understanding of a non-routine problem is a key factor that impacts individuals' performance of solving that problem. What's more, the understanding is formed at the entry phase of the problem solving process, and developed dynamically throughout the entire process. The level of understanding could impact the choice, modification, or switch of strategies, and certain metacognitive behaviors, as well as the efficiency of these activities.

Lastly, perhaps a puzzling finding of my study, is the relationship between children's claimed level of confidence with mathematics and their problem solving performance. In virtually all past literature focused on the connections between affect and problem solving performance of children, the conclusion had been drawn that confidence and success in problem solving are directly related: the more confident an individual was in his/her mathematical ability, the better performance on problem solving was observed. My findings, at least in two cases provide conflicting results. As described earlier, the most mathematically confident individual in this study, Yoni, showed the lowest degree of flexibility in thinking or control over tasks. It is not quite clear at this point what measures were used in previous work for determining the problem solvers' levels of confidence and whether claims were carefully studied. This issue merits further study.
Implications for Practice

The findings of the current study while supporting some of the results of past literature on this topic suggest new insights on children's problem solving processes and their strategy use on non-routine tasks. Most prominently, I established evidence that children's problem solving performance may not be consistent across all problem types or subject areas. Secondly, data indicated that rigidity in the use of heuristics was closely linked to the individual's school experiences. Lastly, in places where children focused on understanding tasks, they were most successful in solving problems. These findings have immediate implications for practice, as discussed below.

Previous research on problem solving performance of children had either assumed or implied that performance might be uniform across problem types. My findings contrast sharply with this result. The data indicated that a student's problem solving behaviors/performances should not be expected to be consistent at all times. A successful problem solving performance may not guarantee a successful performance in a different context; an unsuccessful problem solving performance on one task may also not suggest failed performance in all situations. Teachers need to pay particular attention to how different mathematical practices might be exhibited according to the subject area. They may need to analyze sources that either impede or enhance mathematical practices of children, some of which might be due to the nature of the concept or the sequencing of the curricula. These issues can be addressed in instruction. Characterizing children using general descriptions of being either "good" or "bad" at problem solving is not sufficient in helping them advance in their thinking. With this knowledge, the teacher would be able to offer specific help to individuals in corresponding situations.
Consistent with previous literature, my findings also suggest that metacognitive behaviors are crucial to success when problem solving. I further emphasize the need to devote instructional time to helping children develop the skills to plan, monitor and regulate their mathematical actions. The teacher needs to encourage self-monitoring and self-regulating during students' problem solving activities instead of emphasizing specific strategies or techniques to be used for certain problems or situations. Knowledge of a specific strategy may not be so useful as it could be utilized without necessary adjustment or modification, which is usually conducted by self-monitoring/regulating. As it has already been established, metacognitive skills are not particularly well developed by children in the absence of an instructional guide (Schoenfeld, 1992; Flavell, Miller, & Miller, 2002). Recent research provides some evidence that instruction can have a significant impact on children's growth of planning, monitoring and regulating both problem and solution spaces. Indeed, this body of work suggests that the earlier (grade level) such instruction takes place, the greater the likelihood of development (Schmidt & Ford, 2003).

Findings of this study also suggest that repeated exposure to procedural tasks and routine problems can have negative effect on students' ability to explore non-routine problems. Students might either limit their exploration within a "routine" bound, or try to make connections between a non-routine problem and a more familiar routine problem (Compare Area problem in this study). Findings of previous research on the impact of instruction on heuristics usage has been inconclusive (Schoenfeld, 2007). Based on the episodes of scaffolding portions of the interview, I argue that introducing a strategy without relating it to a certain type of problem might be a productive venue to pursue in
the classroom. A potentially beneficial way to fulfill such a goal could be using various types of examples/problems for one strategy, or applying a variety of strategies for one problem.

**Implication for Future Research**

The non-routine problems used in the study contained only one geometry problem. What's more, two of the participants were provided scaffolding questions in their geometry problem episode. Students' natural behaviors and performances in geometry could not be explicitly studied as in the three algebra problems, which involved little intervention from interviewer. In future research, the consistency of behaviors and performances across different subject areas needs to be studied under a more balanced distribution of subject areas, as well as more consistent interviewer interventions (i.e. only ask participants to explain their thoughts or justify their answers instead of scaffolding questions).

The mathematics courses that the participants were taking and had taken in the previous year were not the same. The grade level of one participant was also different from the other two participants at the time of data collection. These may have impacted their behaviors and performances in different subject areas. Future research may keep more consistency in participants’ background and grade level.

Although the issue of gender difference was not considered as a framing perspective for data analysis, it is plausible to suggest that the study of children's problem solving performance might benefit from adopting such a theoretical lens. The two female participants in this study exhibited somewhat different patterns of mathematical practices compared to Yoni, the male participant. It is not clear whether gender plays a part in
describing certain orientations towards strategy use, consistency in performance across subject areas and heuristics adoption and use. A larger and more non-homogeneous sample of participants would be needed to shed light on these issues.

Perhaps an unintended data source in this study was the role that interviewers' interventions played on mathematical practices of children. The role of scaffolding questions from the teacher/interviewer was not explicitly examined in this study, yet data and analysis revealed positive impact of certain scaffolding questions in provoking students' mathematics thinking during their problem solving process. Future research could more closely inspect the short and long term impact of teacher/researcher interventions on students' problem behaviors/performance during their problem solving processes as well as the transfer of knowledge from one context to another. Such studies could capitalize on extending the community's understanding of how metacognitive skills among children might be nurtured as the result of exposure to these conditions.
REFERENCES


APPENDIX A

QUESTIONS FOR BACKGROUND INTERVIEW
a. What course are you taking at school?

b. What do you want to be when you grow up? Do you think mathematics is useful in that profession?

c. What is your favorite subject? Why is it your favorite subject?

d. Do you consider yourself good at math? If yes, why? If no, what does it mean to be good in math?

e. Give me an example of the kind of a problem you are doing at school?

f. Give me an example of the kind of mathematics problem you like to do?

g. Give me an example of the kind of mathematics problem you don't like to do?

h. Generally, how do you feel about math?

i. What do you typically do in class? Do you like how you do things?