QUASI-LINEAR DYNAMIC MODELS OF HYDRAULIC ENGINE MOUNT
WITH FOCUS ON INTERFACIAL FORCE ESTIMATION

DISSERTATION

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Jong-Yun Yoon, M.S.
Graduate Program in Mechanical Engineering

The Ohio State University
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Dissertation Committee:
Professor Rajendra Singh, Adviser
Professor Marcelo Dapino
Professor Ahmet Kahraman
Professor Junmin Wang
ABSTRACT

This dissertation proposes indirect methods to estimate dynamic forces that are transmitted by a fixed or free decoupler type hydraulic engine mount to a rigid or compliant base. The estimation processes detailed models of the hydraulic mount, and motion and/or internal pressured measurements. Linear system transfer functions that relate the force transmitted to the top chamber pressure and excitation displacement are first derived in the Laplace and frequency domains; these clearly identify the roles of rubber and hydraulic force paths up to 50 Hz. Since hydraulic mounts are inherently nonlinear, a new quasi-linear model is developed that incorporates spectrally-varying and amplitude-sensitive parameters such as top chamber compliance and rubber path properties (stiffness and damping). Alternate schemes based on a quasi-linear fluid system model of the mount are formulated. These work well in frequency domain as dynamic force spectra over a range of harmonic displacement excitations are successfully predicted given motion and/or pressure measurements. In particular, the force to pressure transfer function model is quite promising. Conversely, the analogous mechanical system model fails as it yields highly inaccurate forces.

Next, experimental data from the non-resonant dynamic stiffness test are investigated in both time and frequency domains. In particular, the super-harmonic contents in fluid chamber pressure and force time histories are examined using both
measurements and mathematical models. Linear time-invariant, nonlinear and quasi-linear fluid and mechanical system models are proposed to predict the transmitted force time history under sinusoidal excitation conditions given measured (or calculated) motion and/or internal pressure time histories. Several alternate indirect schemes for estimating dynamic forces are formulated. In particular, the quasi-linear model with effective system parameters, say in terms of force to pressure or force to motion transfer functions, is found to correlate well with measured dynamic forces though linear time-invariant and nonlinear models could be employed as well.

Finally, a new indirect measurement concept is developed to estimate interfacial dynamic forces by employing the hydraulic mount as a dynamic force sensor. The proposed method utilizes a combination of linear mathematical models and operating motion and/or pressure measurements. A laboratory experiment, consisting of a powertrain, three powertrain mounts including a dynamic load sensing hydraulic mount, a sub-frame, and 4 bushings, is then constructed to verify the proof of the concept. Quasi-linear fluid and mechanical system models of the experiment are proposed and evaluated in terms of eigenvalues, transfer functions and forced sinusoidal responses. The lower chamber pressure in the hydraulic mount is also estimated as it was not measured. This leads to a better estimation of effective rubber and hydraulic path parameters with spectrally-varying and amplitude-sensitive properties up to 50 Hz. The reverse path spectral method is finally employed to predict interfacial forces at both mount ends by
using measured motions and upper chamber pressure signals. Overall, the proposed fluid system model yields better prediction of forces when compared with the direct measurements of the dynamic forces though simpler mechanical models provide some insights. This work also advances prior component and transfer path type studies by providing an improved system perspective.
DEDICATION

Dedicated to my wife Suyoung Lee and Parents
I would like to express my sincere gratitude to my advisor Prof. Rajendra Singh. Without his great guidance throughout my doctoral studies at The Ohio State University, I would never finish my study and work. Particularly, I could feel the interest on my topics with his tremendous insight, which has helped me improve myself and encouraged me to study. I also thank my committee members, Prof. Marcelo Dapino, Prof. Ahmet Kahraman and Prof. Ahmet Selamet for their time and guidance to review my works and dissertation. I would like to express sincere appreciation to Smart Vehicle Concepts Center for the sponsor which made this research possible. I thank all the members of the Acoustics and Dynamics Laboratory for their help and comments during this study. Finally, I would like to thank my wife Suyoung Lee for her support and faith to me, which always gives me big encouragement to finish the doctoral degree.
VITA

February 6, 1971 .............................................. Born – Dong-Doo-Cheon City, Korea
1994 ......................................................... B.S. Mechanical Engineering
........................................ Korea Military Academy
2000 – 2003 ................................................. M.S. Mechanical Engineering,
....................................................... The Ohio State University
........................................ Columbus, OH
2003 – 2004 ................................................... Research and Development Engineer
....................................................... NVH Team (Hyundai Motor Company)
....................................................... HwaSung, Korea
2004 – Present ............................................... Graduate Research Associate,
....................................................... The Ohio State University
....................................................... Columbus, OH

PUBLICATIONS


FIELDS OF STUDY

Major Field: Mechanical Engineering
 System Dynamics and Vibrations
 Nonlinear numerical analysis
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LIST OF SYMBOLS

\( A \)  
Effective piston area

\( a, b, g \)  
Coefficients of polynomial function

\( B \)  
State matrix

\( C \)  
Compliance of upper (lower) chamber

\( C \) (Damping Matrix)

\( c \)  
Damping

\( D \)  
Eigenvector

\( d \)  
Denominator coefficients of dimensionless effective compliance

\( E \)  
Engine (or source)

\( E \)  
Root–mean–square value of error

\( e \)  
Exponential function

\( e_1 \)  
Length from z axis

\( e_2 \)  
Length from y axis

\( F \)  
Amplitude of force

\( \mathcal{F} \)  
Fast Fourier transforms

\( f \)  
Input (force) vector

\( f \)  
Force

\( G \)  
Transfer function between pressure and displacement

\( H \)  
Force transmissibility

\( I \)  
Inertance

\( I \)  
Identity matrix

\( i \)  
\( \sqrt{-1} \)

\( K \)  
Dynamic stiffness

\( K \)  
Stiffness Matrix

\( k \)  
Stiffness

\( l \)  
length or depth on engine and sub-frame

\( M \)  
Mass (or inertia) matrix

\( m \)  
Mass

\( N \)  
Number of points in time domain

\( n \)  
Number of harmonic

\( P \)  
Amplitude of pressure

\( p \)  
Pressure

\( Q \)  
Amplitude of fluid flow

\( q \)  
Fluid flow

\( R \)  
Resistance

\( r \)  
Dimensionless frequency
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>S</td>
<td>Sub-frame (or receiver)</td>
</tr>
<tr>
<td>S</td>
<td>State vector</td>
</tr>
<tr>
<td>S</td>
<td>Effective single degree of freedom system model</td>
</tr>
<tr>
<td>s</td>
<td>Laplace domain variable</td>
</tr>
<tr>
<td>T</td>
<td>Period of time</td>
</tr>
<tr>
<td>t</td>
<td>Time (history)</td>
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<tr>
<td>U</td>
<td>Amplitude of raw time signal</td>
</tr>
<tr>
<td>u</td>
<td>Raw time signal</td>
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<tr>
<td>X</td>
<td>Amplitude of displacement</td>
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<td>x</td>
<td>Displacement coordinate vector</td>
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<tr>
<td>x</td>
<td>Displacement (with ( x ) direction)</td>
</tr>
<tr>
<td>y</td>
<td>Displacement of ( y ) direction</td>
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<tr>
<td>Z</td>
<td>Maximum harmonic term</td>
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<tr>
<td>z</td>
<td>Displacement of ( z ) direction</td>
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<td>( \alpha )</td>
<td>Real value of dimensionless effective compliance</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Imaginary value of dimensionless effective compliance</td>
</tr>
<tr>
<td>( \Gamma )</td>
<td>Smoothening function</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Static stiffness (compliance)</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Net decoupler gap</td>
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<td>( \zeta )</td>
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<td>( \eta )</td>
<td>Normalizing constant</td>
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<tr>
<td>( \theta )</td>
<td>Moment of ( y ) (or ( z )) direction</td>
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<tr>
<td>( \vartheta )</td>
<td>Real part of complex eigenvalue</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>Polynomial function of magnitude</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>Dimensionless effective compliance</td>
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<tr>
<td>( \mu )</td>
<td>Imaginary part of complex eigenvalue</td>
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<tr>
<td>( \Xi )</td>
<td>Amplitude of relative displacement</td>
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<td>( \xi )</td>
<td>Relative displacement</td>
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<td>( \tau )</td>
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<td>Eigenvalue</td>
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<td>( \Phi )</td>
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<td>Phase</td>
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<tr>
<td>( \varphi )</td>
<td>Polynomial function of phase</td>
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<td>( \chi )</td>
<td>Calibration factor</td>
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<td>Symbol</td>
<td>Description</td>
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<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Fundamental frequency with respect to the curve-fits</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Acceleration vector</td>
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<tr>
<td>$\psi$</td>
<td>Amplitude of acceleration</td>
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<td>$\omega$</td>
<td>Acceleration</td>
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<tr>
<td>$\omega$</td>
<td>Frequency</td>
</tr>
<tr>
<td>$#d$</td>
<td>Decoupler element</td>
</tr>
<tr>
<td>$#i$</td>
<td>Inertia track</td>
</tr>
<tr>
<td>$#l$</td>
<td>Lower chamber</td>
</tr>
<tr>
<td>$#u$</td>
<td>Upper chamber</td>
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### Subscript

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<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>$A$</td>
<td>Analogous mechanical system</td>
</tr>
<tr>
<td>$AVE$</td>
<td>Averaged value</td>
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<tr>
<td>$a$</td>
<td>Ambient variable</td>
</tr>
<tr>
<td>$B$</td>
<td>Property of bushing</td>
</tr>
<tr>
<td>$c$</td>
<td>Damping</td>
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<tr>
<td>$D$</td>
<td>Damped natural frequency</td>
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<tr>
<td>$d$</td>
<td>Property of decoupler element</td>
</tr>
<tr>
<td>$E$</td>
<td>Property of engine (or source)</td>
</tr>
<tr>
<td>$e$</td>
<td>Effective value</td>
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<tr>
<td>$F$</td>
<td>Property of force</td>
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<td>Ground</td>
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<td>$H$</td>
<td>Property of force transmissibility</td>
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<tr>
<td>$HF$</td>
<td>Property of hydraulic mount at front side</td>
</tr>
<tr>
<td>$HR$</td>
<td>Property of dynamic load sensing hydraulic mount (at rear side)</td>
</tr>
<tr>
<td>$h$</td>
<td>Property of hydraulic path</td>
</tr>
<tr>
<td>$IB$</td>
<td>Interfacial force on the receiver (or bottom side)</td>
</tr>
<tr>
<td>$IM$</td>
<td>Imaginary number</td>
</tr>
<tr>
<td>$IT$</td>
<td>Interfacial force on the source (or top side)</td>
</tr>
<tr>
<td>$i$</td>
<td>Property of inertia track</td>
</tr>
<tr>
<td>$K$</td>
<td>Property of dynamic stiffness</td>
</tr>
<tr>
<td>$k$</td>
<td>Property of Stiffness</td>
</tr>
<tr>
<td>$L$</td>
<td>Linear model</td>
</tr>
<tr>
<td>$l$</td>
<td>Property of lower chamber</td>
</tr>
<tr>
<td>$M$</td>
<td>Measured value</td>
</tr>
<tr>
<td>$m$</td>
<td>Mean value</td>
</tr>
<tr>
<td>$\text{max}$</td>
<td>Maximum value</td>
</tr>
<tr>
<td>$N$</td>
<td>Natural frequency</td>
</tr>
</tbody>
</table>
\( n \) Number of harmonic
\( n \) Nominal value
\( o \) Fundamental harmonic term
\( P_p \) Property of pressure
\( Q \) Quasi-linear model
\( R \) Property of rubber mount (or rubber path)
\( R \) Property of resistance
\( RE \) Real number
\( r \) Model index
\( r \) Property of rubber path (or rubber material)
\( ref \) Reference value
\( S \) Property of receiver (or sub-frame)
\( S \) Property of effective single degree of freedom system
\( Scheme \ II \) Force estimation with Scheme II-C
\( Scheme \ III \) Force estimation with Scheme III-C
\( T \) Transmitted value
\( u \) Property of the upper chamber
\( V \) Voight model parameter
\( v \) Counting index (for variables or number of DOF)
\( w \) Counting index or Mount (or interfacial force) index
\( X \) Property of displacement
\( y \) Property of \( y \) direction
\( Z \) Maximum harmonic term
\( z \) Property of \( z \) direction
\( \varepsilon \) Error function value
\( \kappa \) Coefficient of polynomial function
\( \varphi \) Property of phase
\( \Omega \) Property of fundamental frequency regimes
\( 1,2,3\ldots \) Counting indices (of Harmonic order, polynomial, parameter or DOF)
\( (fixed) \) Fixed decoupler mount
\( (free) \) Free decoupler mount

**Superscript**

\( T \) Transpose
\( . \) 1\(^{st}\) derivative with respect to time
\( .. \) 2\(^{nd}\) derivative with respect to time
\( \sim \) Complex value indicator
\( – \) Dimensionless (or normalized) property
Operators

Im[ ] Imaginary value of complex number
Re[ ] Real value of complex number
\mathcal{F} Fast Fourier transform
\langle \rangle Averaged value

Abbreviations

C.G. Center of geometry
DOF Degree of freedom
LTI Linear time-invariant
NL Nonlinear
QL Quasi-linear
RMS Root-mean-square
SDOF Single degree of freedom
CHAPTER 1

INTRODUCTION

1.1 Motivation

Precise knowledge of the dynamic forces that are transmitted by machinery mounts and isolators to rigid or compliant bases in vehicles, buildings, and equipment is critical to the dynamic design and vibration control considerations. Direct measurement of forces (for example, using conventional force transducers) is not practical in many real-life applications since the interfacial conditions may change [1.1, 1.2]. Therefore, indirect methods must be utilized to estimate dynamic forces and moments using the internal states or motions of the isolation paths, vibration sources and receiving structures [1.1-1.6]. For instance, one could employ transfer path approaches, though they are applicable primarily in the frequency domain for a linear time-invariant system [1.3, 1.4]. Also, dynamic forces could be estimated by using other measured signals such as operating motions, but then dynamic stiffness must be known a priori [1.7]. Such indirect force estimation methods pose special difficulty for nonlinear mounts or isolators. For instance, hydraulic engine mounts exhibit spectrally-varying and amplitude-sensitive parameters [1.8]. Further, most vibration isolators and machinery mounts are inherently nonlinear [1.9-1.11] and consequently improved system identification models are needed
to assess forces from motion and/or internal state measurement.

Figure 1.1 displays the internal configuration of the hydraulic engine mount and its fluid system model; it will be discussed further in Chapters 2, 3 and 4. Here, \( f_m \) is the preload, \( x(t) = x_m + Re[\tilde{X}e^{j\omega t}] \) is the excitation displacement, \( x_m \) is the mean displacement, \( \tilde{X} = Xe^{j\phi_X} \) is the complex valued excitation amplitude, \( X \) is the amplitude of displacement, \( \phi_X \) is the phase of \( x(t) \), \( \omega_o \) is the excitation (fundamental) frequency (rad/s), and \( Re[\ ] \) is the real value operator; tilde over a symbol implies that is complex valued. Figure 1.2 shows the measured data of the stiffnesses and dampings of hydraulic and rubber mounts given by the context of a non-resonant dynamic stiffness testing procedure under the ISO standard 10846 [1.12].

### 1.2 Literature Review

The dynamic characteristics of hydraulic engine mounts have been examined for almost two decades [1.13-1.20]. For instance, Singh et al. [1.14] analyzed the linear characteristics of both fixed and free decoupler mounts. That article [1.14] defined the mathematical model of system responses in the hydraulic mount using lumped parameter concepts and compared the fixed and free decoupler mounts in frequency domain. Colgate et al. [1.15] examined the dynamic characteristics of hydraulic mount by employing equivalent linear models. Adiguna et al. [1.16] studied the transient responses under step and saw tooth excitations using nonlinear profiles given by the measurement under the static load condition. He and Singh [1.13, 1.17] compared the fluid and analogous mechanical system models and predicted the system responses in both time
Figure 1.1 Force transmitted $f_T(t)$ by a hydraulic mount with fluid model and its parameters for rubber and hydraulic paths.
Figure 1.2 Typical dynamic properties of hydraulic engine mount given

\[ x(t) = \text{Re} \left[ \tilde{X} e^{i\omega t} \right] \text{ at } |\tilde{X}| = 0.1 \text{ mm in frequency domain with stiffness and damping.} \]
and frequency domains. They employed the spectrally-varying and amplitude-sensitive parameters in the dynamic stiffness. Shangguan and Lu [1.18] calculated the dynamic fluid pressure in the upper chamber and investigated the influence of temperature on fluid viscosity. Fan and Lu [1.19] introduced a plate type mount and experimentally studied its function. Truong and Ahn [1.20] determined the performance of an inertia track with variable area using an analogous mechanical system model.

Various investigations with respect to the spectrally-varying and amplitude-sensitive concepts have been conducted to identify the effective mount parameters. He and Singh [1.13] examined the analogous mechanical system models by employing the effective mechanical properties with the amplitude and frequency dependent parameters in both time and frequency domains. Lee and Moon [1.21] developed the mathematical dynamic model of a displacement-sensitive shock absorber using fluid-flow modeling. Kim and Singh [1.22, 1.23] found the nonlinear properties of chamber compliances and fluid resistances. Tiwari et al. [1.24] refined the measurements to find out the fluid system parameters and defined the nonlinear characteristics of inertia track resistances and upper and lower chamber compliances over a range of the static preloads. Lee and Singh [1.25, 1.26] have found super-harmonics in the mount responses when excited harmonically and have suggested that vehicle system behavior would be nonlinear. Geisberger et al. [1.27] identified the nonlinear properties from the effective area and volumetric compliance by isolating the mount component. Mrad and Levitt [1.28] have investigated nonlinear behaviors of the active suspension system in a ¼ vehicle model.
Christopherson and Jazar [1.29] introduced a direct decoupler type mount and investigated its nonlinear model by comparing it with a floating decoupler type mount.

Indirect measurements or force reconstruction methods have been adopted to estimate dynamic forces [1.1-1.6]. Carne et al. [1.30] utilized the frequency response function data to indirectly estimate the input force. Jacquelin et al. [1.31] employed a deconvolution technique to reconstruct the dynamic force. Tao et al. [1.32] identified excitation force in the center of engine using the velocity amplitude and phase at the mounting points. Liu and Shepard [1.33] compared the truncated singular value decomposition and the Tikhonov filter approaches used to enhance the inverse process. Lin and Chen [1.34] identified the contact stiffness and damping properties of mechanical interfaces. However, most of the available indirect or inverse force estimation methods are valid only for linear time-invariant systems since they employ transfer functions or similar concepts.

1.3 Problem Formulation

1.3.1 Simple formulation and initial results

As shown in Figure 1.1, the steady state forces transmitted $f_T(t)$ and $F_T(\omega)$, in both time $(t)$ and frequency $(\omega)$ domains, are related to the excitation displacement $x(t)$ and $X(\omega)$, and upper chamber response $p_u(t)$ and $P_u(\omega)$. Here, the uppercase symbols represent complex value Fourier amplitudes:

\[ f_T(t) = f_{Tr}(t) + f_{Th}(t), \quad (1.1) \]

\[ f_{Tr}(t) = c_r \dot{x}(t) + k_r x(t), \quad (1.2) \]
Here, $f_{Tr}(t)$ is the rubber path force (subscript $r$), $f_{Th}(t)$ is the hydraulic path force (subscript $h$), $k_r$ and $c_r$ are the rubber stiffness and damping coefficient respectively, and $A_r$ is the effective piston area. The experiments show that $p_u(t)$ deviates from the sinusoidal shape depending on $X$ and $\omega_o$. Likewise, $k_r$ and $c_r$ vary as well with $X$ and $\omega_o$.

Figure 1.3 presents some initial results of force estimation in both time and frequency domains. These results illustrate the nature of the problem. The forces are based upon a free decoupler mount under steady state sinusoidal excitation. Both $x(t)$ and $p_u(t)$ signals are assumed to be given under certain dynamic test conditions, and thus the measured $x(t)$ and $p_u(t)$ data set could be employed to estimate $f_{Tr}(t)$ and $F_{Tr}(\omega)$ by using Eqns. (1.1)-(1.6). Figure 1.3(a) compares the time domain results at 8.5 Hz for this particular mount given the nominal parameters as follows: $k_r = 2 \times 10^5$ N/m; $c_r = 496.1$ N-s/m; $A_r = 4 \times 10^{-3}$ m$^2$. Discrepancies between the prediction and the measurement are observed. Figure 1.3(b) compares measured and estimated $F_{Tr}(\omega)$ for a low excitation level up to 50 Hz. The results show significant differences between the measured and estimated $F_{Tr}(\omega)$ as well. Therefore, the force estimation method as given by Eqns. (1.1)-(1.6) must include the effective properties of the mounts and, in particular, their frequency $\omega_o$ and excitation amplitude $X$ dependent properties for both rubber and hydraulic paths.
Figure 1.3 Comparison between measured and predicted transmitted force for the free decoupler mount: (a) transmitted force time histories, given sinusoidal displacement $x(t) = X \sin \omega_o t$ at $\omega_o/2\pi = 8.5$ Hz and $X = 1.5$ mm; (b) magnitude spectra for $f_t(t) = |F_T| \sin (\omega_o t + \phi)$ for the free decoupler mount given $X = 0.15$ mm. Key for (a): , experiment; ..., theory (Eqns. (1.1)-(1.3)); Key for (b): --, experiment; ---, theory (Eqns. (1.4)-(1.6)).
1.3.2 Scope, assumptions and objectives

This dissertation examines the dynamic response of hydraulic engine mounts that exhibit spectrally-varying and amplitude-sensitive properties [1.8, 1.11, 1.13]. Linear time-invariant (LTI), nonlinear (NL), and quasi-linear (QL) models are utilized to predict the force time history under sinusoidal excitation conditions given measured (or calculated) motion and/or internal pressure time histories. In particular, the super-harmonic contents in $p_u(t)$ and $f_T(t)$ time histories are investigated using both measurements and mathematical models. Finally, this study of the dynamic force estimation with the hydraulic mount [1.2, 1.35] is extended into the estimation of the interfacial path forces by using the measured motion and upper chamber pressure with the multi-degree of freedom isolation system. Thus, the hydraulic engine mount will be embedded in a practical system as a load sensing device and interfacial forces are estimated in both frequency and time domains under sinusoidal excitation using measured or calculated motions and/or internal pressure signals.

This research will focus only on the steady state vertical displacements up to 50 Hz over a range of displacement $X$ from 0.15 to 1.5 mm (zero-to-peak) with hydraulic mount alone. The range of $X$ corresponds to typical engine excitations. In all cases, the excitation displacement, upper chamber pressure, and force transmitted to the rigid base are measured on the elastomer test machine for both fixed and free decoupler mounts. With respect to the multi-degree of freedom system, only the translational motions are considered. To estimate the interfacial forces on both ends of hydraulic mount, 2DOF and 3DOF mechanical, and fluid system models are investigated up to 50 Hz. Also, quasi-
linear (QL) models embedded by spectrally-varying and amplitude-sensitive parameters are proposed to improve the force estimation based on the prior studies [1.2, 1.35].

Main assumptions for this research are summarized as follows:

1. Dynamic characteristics of hydraulic mount can be derived first using the linear time-invariant theory with nominal parameters. Thus, the relationship between pressure to displacement (or force to pressure) are expressed by the (2nd order)/(2nd order) transfer function below 50 Hz [1.2, 1.13, 1.14, 1.35].

2. Both displacement \( x(t) \) and upper chamber pressure \( p_u(t) \) signals are available or known under certain dynamic test conditions, and thus the measured \( x(t) \) and \( p_u(t) \) data set could be employed to estimate \( f_T(t) \) and \( F_T(\omega) \).

3. The upper chamber pressure is most affected by the nonlinear phenomena, and thus all nonlinearities are lumped into the effective upper chamber compliance \( \tilde{C}_{ue}(n\omega_o,X_n) \) with \( n \) harmonic term. Thus, the \( n\omega_o \) contents of \( f_T(t) \) and \( F_T(n\omega_o,X_n) \) are directly affected by the corresponding terms of \( p_u(t) \) and \( P_u(n\omega_o,X_n) \).

4. Both rubber and hydraulic force paths are assumed to possess spectrally-varying and amplitude-sensitive properties.

5. The rubber forces are assumed to be identical on top and bottom sides of the hydraulic mount. On the other hand, the hydraulic path force is considered to be asymmetric due to the nonlinearities from fluid parameters.

The specific objectives of this dissertation are as follows.

**Objective 1**: Develop quasi-linear models of the hydraulic mount with focus on the fundamental harmonic term (Chapter 2).
(1a) Determine the linear system transfer functions that relate $F_T$ to $X$ and $P_u$ in the Laplace and frequency domains for both rubber and hydraulic force paths, and predict $f_T(t)$ using $x(t)$ and/or $p_u(t)$.

(1b) Estimate the effective (frequency and amplitude dependent) mount parameters, and embed them in quasi-linear models. All mount nonlinearities will be lumped into upper chamber compliance $C_{ue}(\omega,X)$, rubber stiffness $k_r(\omega,X)$, and damping $c_r(\omega,X)$ properties.

(1c) Estimate force $f_T(t)$ in the time domain by using the quasi-linear model and the Fourier expansion method, and compare sinusoidal force predictions with measurements.

**Objective 2:** Examine super-harmonic terms and compare linear time-invariant, quasi-linear and nonlinear models (Chapter 3).

(2a) Analyze the measured $p_{uM}(t)$ and $f_{TM}(t)$ in both time and frequency domains and examine their spectral contents.

(2b) Develop linear time-invariant and nonlinear models of both fixed and free decoupler mounts and compare their $f_T(t)$ predictions with measurements.

(2c) Propose a quasi-linear model with spectrally-varying and amplitude-sensitive parameters at both $\omega_o$ and $n\omega_o$ ($n = 2,3,4, \cdots$) terms; both fluid and analogous mechanical system models are used to predict $f_T(t)$ and compare with $f_{TM}(t)$.

(2d) Estimate $f_T(t)$ in the time domain by using the Fourier series expansion.
Objective 3: Embed the nonlinear hydraulic mount in a multi-degree of freedom system and then utilize it as a load sensing device to estimate interfacial forces (Chapter 4).

(3a) Construct and instrument a laboratory experiment corresponding to Figure 1.4; one load sensing hydraulic mount will be embedded. The system is excited in the vertical direction by a steady state sinusoidal force $f_E(t)$ of frequency $\omega_0$ and motions in other directions are ignored.

(3b) Conduct experiments under sinusoidal excitation, measure dynamic accelerations (at different points in the system), fluid pressure (in the top chamber of the load sending mount) and forces (at selected interfaces).

(3c) Develop 2 and 3 degree of freedom (DOF) linear and quasi-linear models (with spectrally-varying and amplitude-sensitive parameters as suggested in prior component studies [1.2, 1.35]); this would include determination of the effective parameters including upper and lower chamber compliances.

(3d) Estimate the interfacial forces at both ends of the load sensing hydraulic mount by employing mechanical and fluid models of the load sensing mount, and compare them with direct force measurements in frequency and time domains.
Figure 1.4 Schematic of a multi-degree of freedom isolation system with hydraulic mounts in Paths I and II.
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CHAPTER 2

QUASI-LINEAR MODELS OF HYDRAULIC MOUNT WITH FOCUS ON FUNDAMENTAL HARMONIC

2.1 Introduction

Minimization of dynamic forces that are transmitted to rigid, compliant foundations in machines, vehicles and buildings is an important design paradigm in vibration and structure-borne noise control. Yet, the interfacial forces are difficult to measure using direct methods under realistic conditions [2.1-2.3]. Therefore, indirect methods must be utilized to estimate dynamic forces and moments using the internal states or motions of the isolation paths, vibration sources and receiving structures [2.1-2.5]. Further, most vibration isolators and machinery mounts are inherently nonlinear [2.6-2.8] and consequently improved system identification models are needed to assess forces from motion and/or internal state measurements. This chapter discusses this particular issue with application to hydraulic engine mounts that exhibit spectrally-varying and amplitude-sensitive properties [2.8-2.10].

The dynamic characteristics of hydraulic engine mounts have been examined using fluid and mechanical system models [2.9, 2.11-2.23]. For instances, Singh et al.
[2.11] analyzed the linear characteristics of both fixed and free decoupler mounts. Colgate et al. [2.12] examined the dynamic characteristics of hydraulic mount by employing equivalent linear models. Kim and Singh [2.13, 2.14] found the nonlinear properties of chamber compliances and fluid resistances. Tiwari et al. [2.15] refined these measurements and defined the nonlinear characteristics of inertia track resistances and upper and lower chamber compliances over a range of the static preloads. Adiguna et al. [2.16] studied the transient responses under step and saw tooth excitations. He and Singh [2.9, 2.17] compared the fluid and analogous mechanical system models and predicted the system responses in both time and frequency domains. Shangguan and Lu [2.18] calculated the dynamic fluid pressure in the upper chamber and investigated the influence of temperature on fluid viscosity. Fan and Lu [2.19] introduced a plate type mount and experimentally studied its function. Truong and Ahn [2.20] determined the performance of an inertia track with variable area using an analogous mechanical system model. Christopherson and Jazar [2.21] introduced a direct decoupler type mount and investigated its nonlinear model by comparing it with a floating decoupler type mount. Lee and Singh [2.22, 2.23] have found super-harmonics in the mount responses when excited harmonically and have suggested that vehicle system behavior would be nonlinear. Overall, none of the prior studies has examined the force measurement and related estimation issues which are the focus of this chapter.
2.2 Problem Formulation

Figures 2.1 and 2.2 describe the experimental setup and hydraulic mount along with its fluid system in the context of non-resonant dynamic stiffness testing procedure under the ISO standard 10846 [2.24]. Here, $f_m$ is the preload; $x(t) = x_m + X \sin \omega t$ is the excitation displacement; $x_m$ is the mean displacement; $X$ is the excitation amplitude, but it is usually given in zero-to-peak [2.25]. In this test, the force transmitted to the rigid base $f_I(t)$ is measured under sinusoidal excitation, though a Fourier filter is utilized to retain only the fundamental frequency signal [2.25]. Additionally, we install a pressure transducer in the upper chamber (in our laboratory experiments) and measure the dynamic pressure $p_u(t)$. The force transmitted $f_I(t)$ is related to the excitation displacement $x(t)$ and fluid system response $p_u(t)$ as shown below in both time ($t$) and frequency ($\omega$) domains; here the uppercase symbols represent complex value Fourier amplitudes.

\[
\begin{align*}
    f_I(t) &= f_{Ir}(t) + f_{Ih}(t), \\
    f_{Ir}(t) &= c_r \ddot{x}(t) + k_r x(t), \\
    f_{Ih}(t) &= A_r p_u(t), \\
    F_r(\omega) &= F_{Ir}(\omega) + F_{Ih}(\omega), \\
    F_{Ir}(\omega) &= (i\omega c_r + k_r) X(\omega), \\
    F_{Ih}(\omega) &= A_r P_u(\omega).
\end{align*}
\]

Here, $f_{Ir}(t)$ is the force from the rubber path via rubber (subscript $r$); $f_{Ih}(t)$ is the force from the hydraulic path (subscript $h$); $k_r$ and $c_r$ are the rubber stiffness and damping.
Figure 2.1 Force transmitted $f(t)$ by a hydraulic mount in the context of non-resonant elastomeric test [2.24]: (a) experimental setup with mount; (b) schematic of the mount and measurements; (c) sinusoidal displacement excitation $x(t)$ and dynamic forces transmitted by two paths ($f_{Tr}$ and $f_{Th}$).
Figure 2.2 Fluid model of the hydraulic mount and its parameters for rubber and hydraulic paths.
(using the Voight model), respectively; and \( A_r \) is the effective rubber (piston) area. Initially we assume a linear time-invariant system with nominal parameters, and thus only the sinusoidal responses at \( \omega_o \) are considered in Eqns. (2.2a)-(2.2c).

First, we present some initial results to illustrate the nature of the problem. Consider a free decoupler mount (refer to [2.11, 2.14-2.16] for a detailed description) under steady state sinusoidal excitation. Let us assume that we have both \( x(t) \) and \( p_u(t) \) signals under certain dynamic test conditions, and thus the measured \( x(t) \) and \( p_u(t) \) data set could be employed to estimate \( f_T(t) \) and \( F_T(\omega) \) by using Eqns. (2.1a)-(2.1c) and (2.2a)-(2.2c). Figure 2.3 compares the time domain results at 8.5 Hz for this particular mount given the nominal parameters of Table 2.1. Discrepancies between the prediction and the measurement are seen. Figure 2.4 compares measured and estimated \( F_T(\omega) \) for a low excitation level up to 50 Hz. Again we observe significant differences between the measured and estimated \( F_T(\omega) \). Therefore, the force estimation method as given by Eqns. (2.1a)-(2.1c) and (2.2a)-(2.2c) must include the effective properties of the mounts and, in particular, their frequency \( \omega_o \) and excitation amplitude \( X \) dependent properties for both rubber and hydraulic paths.

The current study will examine the hydraulic mount alone, with fixed or free decoupler design. Predictions of \( f_T(t) \) and \( F_T(\omega,X) \) will be based on fluid and analogous mechanical system models, and the results will be correlated with measured force signals from the elastomer test machine. Only the steady state vertical displacements up to 50 Hz over a range of \( X \) from 0.15 to 1.5 mm (zero-to-peak) will be considered. This range of \( X \) corresponds to typical engine excitations.
<table>
<thead>
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<th>Parameters</th>
<th>Symbols (units)</th>
<th>Nominal Value</th>
</tr>
</thead>
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<td>$k_r$ (N/m)</td>
<td>$c_r$ (N-s/m)</td>
<td>$2 \times 10^5$</td>
</tr>
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<td><strong>Fluid System Model</strong> (Figures 2.1 and 2.2)</td>
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<td>$I_i$ (kg/m$^4$)</td>
<td>$I_d$ (kg/m$^4$)</td>
</tr>
<tr>
<td><strong>Analogous Mechanical System Model</strong> (Figure 2.11)</td>
<td>$m_{ie}$ (kg)</td>
<td>$c_{ie}$ (N-s/m)</td>
<td>$k_{u}$ (N/m)</td>
</tr>
</tbody>
</table>

Table 2.1 Nominal parameters for fluid and mechanical system models.
Figure 2.3 Comparison between measured and predicted transmitted force time histories for the free decoupler mount, given sinusoidal displacement $x(t) = X \sin \omega_0 t$ at $\omega_0/2\pi = 8.5$ Hz and $X = 1.5$ mm: (a) measured $x(t)$ with $X = 1.5$ mm and $p_u(t)$ time histories; Key for part (a): ..., $x(t)$; ---, $p_u(t)$; (b) measured and predicted force, $f_T(t)$; Key for part (b); ——, experiment; ..., theory (Eqns. (2.1a)-(2.1c) given nominal parameters of Table 2.1).
Figure 2.4 Comparison between measured and predicted transmitted force, $f_t(t) = |F_T| \sin(\omega_d t + \phi)$ for the free decoupler mount given $X = 0.15$ mm: (a) magnitude spectra; (b) phase spectra. Key: - , experiment; - , theory (Eqns. (2.2a)-(2.2c) given nominal parameters of Table 2.1).
The specific objectives of this chapter are as follows: (1) determine the linear system transfer functions that relate $F_T$ to $X$ and $P_u$ in the Laplace and frequency domains for both rubber and hydraulic force paths, and predict $f_T(t)$ using $x(t)$ and/or $p_u(t)$; (2) estimate the effective (frequency and amplitude dependent) mount parameters, and embed them in quasi-linear models. All mount nonlinearities will be lumped into upper chamber compliance $C_{ue}(\omega, X)$, rubber stiffness $k_{r}(\omega, X)$, and damping $c_{r}(\omega, X)$ properties; (3) estimate force $f_T(t)$ in the time domain by using the quasi-linear model and the Fourier expansion method, and compare sinusoidal force predictions with measurements. This study will consider only the fundamental harmonic ($\omega_0$), like the dynamic stiffness measurements employed by the non-resonant elastomer test machines [2.25]. The quasi-linear model with effective $C_{ue}(\omega, X)$, $k_{re}(\omega, X)$ and $c_{re}(\omega, X)$ will be employed for such calculations.

2.3 Fluid System Model: Linear System Analysis

2.3.1 Transfer functions relating pressure and displacement

A lumped model of the fluid system could be developed as illustrated in Figures 2.1 and 2.2 based on the following assumptions: (1) the underlying system is a linear time-invariant and assigned with nominal fluid system parameters; (2) the mount is connected to a rigid base, and the top end is excited by the steady state sinusoidal displacement $x(t)$ under a mean load; (3) the force transmitted to the rigid base consists of two path forces. The momentum and continuity equations lead to the following governing
equations and fluid parameters; they are well described in the literature [2.11, 2.13-2.17, 2.22, 2.23].

\[ p_u(t) - p_l(t) = I_i \dot{q}_i(t) + R_i q_i(t) , \]  
(2.3a)

\[ p_u(t) - p_l(t) = I_d \dot{q}_d(t) + R_d q_d(t) , \]  
(2.3b)

\[ C_u \dot{p}_u(t) = A_r \ddot{x}(t) - q_i(t) - q_d(t) , \]  
(2.3c)

\[ C_i \dot{p}_l(t) = q_i(t) + q_d(t) . \]  
(2.3d)

Here, \( C_u \) and \( C_l \) are the upper (#u) and lower (#l) chamber compliances, respectively; \( I_i \) and \( I_d \) are the inertances of the decoupler (#d) and inertia track (#i), respectively; \( R_i \) and \( R_d \) are the resistances of decoupler and inertia track, respectively; and \( q_d \) and \( q_i \) are the fluid flow through decoupler and inertia track, respectively. Transform Eqns. (2.1a)-(2.1c) and (2.3a)-(2.3d) into the Laplace domain (s) with the assumption that the initial conditions are zeros.

\[ F_r(s) = F_{\tau_r}(s) + F_{\tau_h}(s) , \]  
(2.4a)

\[ F_{\tau_r}(s) = (c_r s + k_r) X(s) , \]  
(2.4b)

\[ F_{\tau_h}(s) = A_r P_u(s) , \]  
(2.4c)

\[ P_u(s) - P_l(s) = (I_i s + R_i) Q_i(s) , \]  
(2.4d)

\[ P_u(s) - P_l(s) = (I_d s + R_d) Q_d(s) , \]  
(2.4e)

\[ C_u s P_u(s) = A_r X(s) - Q_i(s) - Q_d(s) , \]  
(2.4f)

\[ C_i s P_l(s) = Q_i(s) + Q_d(s) . \]  
(2.4g)
Note that $F_{Th}(s)$ and $F_{Th}(s)$ in Eqns. (2.4b) and (2.4c) are the dynamic forces transmitted by the rubber and hydraulic paths, respectively. Prior literature [2.9, 2.11, 2.14-2.16, 2.22] has primarily focused on the dynamic stiffness formulation. We extend this concept further and examine other transfer functions. First consider the fixed decoupler mount.

From Eqns. (2.4a)-(2.4g), the relationship between $P_d(s)$ and $X(s)$ is derived by assuming that $I_d = 0$ and $R_d \to \infty$:

$$\frac{P_d(s)}{X(s)} = \frac{A_I \left( C_I I_s s^2 + C_I R_s s + I \right)}{C_u C_I I_s s^2 + C_u C_I R_s s + \left( C_u + C_I \right)} . \quad (2.5a)$$

Next consider the free decoupler mount and assume that $I_d \approx 0$ below 50 Hz. Resulting transfer function is as follows:

$$\frac{P_d(s)}{X(s)} = \frac{A_I \left[ C_I I_s R_d s^2 + (C_I R_d R_d + I) s + (R_d + R_d) \right]}{C_u C_I I_s R_d s^2 + \left[ C_u C_I R_d R_d + \left( C_u + C_I \right) I \right] s + \left( C_u + C_I \right) (R_d + R_d)} . \quad (2.5b)$$

Both transfer functions (now designated by $G(s)$), Eqns. (2.5a) and (2.5b), could be converted into a standard form as shown below:

$$\frac{P_d(s)}{X(s)} = G(s) = \frac{A_I}{C_u + C_I} \frac{s^2 + \frac{2 \zeta_1}{\omega_{N1}} s + \frac{1}{\omega_{N1}^2}}{s^2 + \frac{2 \zeta_2}{\omega_{N2}} s + \frac{1}{\omega_{N2}^2}} . \quad (2.6)$$

Here, $\zeta_1$ and $\zeta_2$ are the damping ratios, and $\omega_{N1}$ and $\omega_{N2}$ are the natural frequencies of the numerator (zero of $G(s)$) and denominator (pole of $G(s)$). They are related to the fluid parameters for fixed and free decoupler mounts in Table 2.2. Note that the standard parameters are slightly different from the one reported previously since the previous study included the assumption that $C_I \geq 100C_u$ [2.11]. However, the current study does
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Fixed Decoupler</th>
<th>Free Decoupler</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta_1$</td>
<td>$\frac{1}{2} \sqrt{\frac{C_i R_i^2}{I_i}}$</td>
<td>$\frac{1}{2} \left( \sqrt{\frac{C_i R_j R_i^2}{I_j (R_j + R_d)}} + \sqrt{\frac{I_i}{C_j R_j (R_i + R_d)}} \right)$</td>
</tr>
<tr>
<td>$\omega_{N_1}$</td>
<td>$\frac{1}{\sqrt{C_i I_i}}$</td>
<td>$\frac{R_j + R_d}{\sqrt{C_i I_i R_d}}$</td>
</tr>
<tr>
<td>(rad/s)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\zeta_2$</td>
<td>$\frac{1}{2} \sqrt{\frac{C_u C_i R_i^2}{I_i (C_u + C_i)}}$</td>
<td>$\frac{1}{2} \left[ \sqrt{\frac{C_u C_i R_j R_i^2}{I_j (C_u + C_i)(R_j + R_d)}} + \sqrt{\frac{(C_u + C_i) I_i}{C_u C_j R_d (R_i + R_d)}} \right]$</td>
</tr>
<tr>
<td>$\omega_{N_2}$</td>
<td>$\sqrt{\frac{C_u + C_i}{C_u C_i I_i}}$</td>
<td>$\sqrt{\frac{(C_u + C_i) (R_j + R_d)}{C_u C_i I_i R_d}}$</td>
</tr>
<tr>
<td>(rad/s)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2 Fluid system parameters for fixed and free decoupler mounts assuming linear time-invariant system. Refer to Eqn. (2.6) and Figure 2.5 for details.
not include this assumption since the system might be affected by a variety of ranges of $C_u$ while the quasi-linear model is applied with the effective value of $C_u$.

2.3.2 Dimensionless formulations

We develop dimensionless formulations by using the following reference values: $X_{ref} = \text{reference excitation displacement amplitude}; k_{ref} = \text{reference stiffness}; P_{uref} = \text{reference pressure} = (k_{ref} X_{ref})/A_r$; and $F_{Tref} = \text{reference force} = k_{ref} X_{ref}$. With these reference values, define the dimensionless variables as follows.

$$\overline{X} = \frac{X}{X_{ref}}, \quad (2.7a)$$

$$\overline{P}_u(s) = \frac{A_r}{k_{ref} X_{ref}} P_u(s), \quad (2.7b)$$

$$\overline{P}_u(s) = \overline{G}(s) = \frac{A_r P_u(s)}{k_{ref} X(s)}, \quad (2.7c)$$

$$\overline{F}_T(s) = \overline{K}(s) = \frac{l}{k_{ref} X(s)}, \quad (2.7d)$$

where $\overline{K}(s)$ is the dimensionless cross point dynamic stiffness. In our study, we select the reference values, $k_{ref} = 2.0\times10^5 \text{ N/m}, A_r = 4.5\times10^{-3} \text{ m}^2$, but $X_{ref} (\times10^{-3} \text{ m})$ is chosen according to the excitation amplitude used in experiments. By using Eqns. (2.6) and (2.7a)-(2.7d), we obtain the dimensionless $\overline{G}(s)$ as follows, where $\gamma_h$ is the dimensionless static compliance resulting from the elastic containers of upper and lower chambers.
\[
\overline{G}(s) = \frac{\overline{P}_u}{\overline{X}}(s) = \gamma_h \left( \frac{s^2}{\omega_{N1}^2} + \frac{2\zeta_{l1}}{\omega_{N1}} s + 1 \right) / \left( \frac{s^2}{\omega_{N2}^2} + \frac{2\zeta_{l2}}{\omega_{N2}} s + 1 \right), \quad (2.8a)
\]

\[
\gamma_h = \frac{A^2}{k_{\text{ref}} (C_u + C_I)}. \quad (2.8b)
\]

Further the dynamic stiffness \( \overline{K}(s) \) and its rubber (\( \overline{K}_r \)) and hydraulic (\( \overline{K}_h \)) components are described using Eqns. (2.7d), (2.8a) and (2.8b).

\[
\overline{K}(s) = \frac{\overline{F}_T}{\overline{X}}(s) = \frac{\overline{F}_{Tr}}{\overline{X}}(s) + \frac{\overline{F}_{Kh}}{\overline{X}}(s), \quad (2.9a)
\]

\[
\overline{K}_r(s) = \frac{\overline{F}_{Tr}}{\overline{X}}(s) = \gamma_r \left( 1 + \tau_r s \right), \quad (2.9b)
\]

\[
\overline{K}_h(s) = \frac{\overline{F}_{Kh}}{\overline{X}}(s) = \gamma_h \left( \frac{s^2}{\omega_{N1}^2} + \frac{2\zeta_{l1}}{\omega_{N1}} s + 1 \right) / \left( \frac{s^2}{\omega_{N2}^2} + \frac{2\zeta_{l2}}{\omega_{N2}} s + 1 \right), \quad (2.9c)
\]

\[
\tau_r = \frac{c_r}{k_r}, \quad (2.9d)
\]

\[
\gamma_r = \frac{k_r}{k_{\text{ref}}}. \quad (2.9e)
\]

Here, \( \tau_r \) is the time constant of the rubber path (modeled using the Voight model), and \( \gamma_r \) is the dimensionless static stiffness. From the relationship between Eqns. (2.8a) and (2.9a), relate dynamic force and upper chamber pressure as follows.

\[
\overline{H}(s) = \frac{\overline{F}_T}{\overline{P}_u}(s) = \frac{\overline{F}_{Tr}}{\overline{P}_u}(s) + \frac{\overline{F}_{Kh}}{\overline{P}_u}(s), \quad (2.10a)
\]

\[
\overline{H}_r(s) = \frac{\overline{F}_{Tr}}{\overline{P}_u}(s) = \gamma_r \left( 1 + \tau_r s \right) \left( \frac{s^2}{\omega_{N1}^2} + \frac{2\zeta_{l1}}{\omega_{N1}} s + 1 \right) / \left( \frac{s^2}{\omega_{N2}^2} + \frac{2\zeta_{l2}}{\omega_{N2}} s + 1 \right), \quad (2.10b)
\]
\[
\overline{H}_h(s) = \frac{F_{Th}(s)}{P_u(s)} = 1.
\] (2.10c)

Note that \(\overline{H}_r(s)\) and \(\overline{H}_h(s)\) represent the rubber and hydraulic path transmissibilities as they relate the force transmitted (to the rigid base) to the interfacial force (inside the fluid system).

2.3.3 Typical frequency responses based on nominal parameters

When the hydraulic mount is harmonically excited under steady state condition, the Laplace transfer functions can be converted to the frequency response functions by changing the variable \(s\) with \(i\omega\) where \(i\) is the imaginary unit. Figures 2.5 to 2.8 show typical frequency responses, in terms of \(\overline{K}_r(\omega)\) \(\overline{H}_r(\omega)\) and \(\overline{K}_h(\omega)\) \(\overline{H}_h(\omega)\) and compare rubber and hydraulic path forces for both fixed and free decoupler mounts. Based on nominal parameters, natural frequencies \(\omega_{N1}\) and \(\omega_{N2}\) are found to be 1.62 (1.9) Hz and 16 (19) Hz for the fixed (free) decoupler mount. In particular, Figures 2.5 (a) and (b) illustrate the dynamic characteristics in terms of \(\overline{K}(\omega)\) and \(\overline{H}(\omega)\) respectively. As illustrated in Figures 2.5 (a) and (b), the system shows that the peak loss angle is located between \(\omega_{N1}\) and \(\omega_{N2}\) as suggested by prior research [2.11].

Figures 2.6 and 2.7 compare \(\overline{K}_r(\omega)\) \(\overline{H}_r(\omega)\) from the rubber path with those from the hydraulic path \(\overline{K}_h(\omega)\) \(\overline{H}_h(\omega)\). Figure 2.6 shows that hydraulic path magnitudes and loss angles are much higher than those from the rubber path forces (magnitudes are nearly constant, and the loss angles change linearly with the frequency). On the other hand, Figure 2.7 shows that the rubber path forces, in terms of \(\overline{H}_r(\omega)\),
exhibit significant changes below 30 Hz. Overall, these curves suggest that the dynamic force transmitted is controlled by the hydraulic path. This could be confirmed by examining Eqns. (2.9a)-(2.9e) and (2.10a)-(2.10c). For example, Eqns. (2.9c) and (2.10b) include fluid parameters in terms of $\zeta_1$, $\zeta_2$, $\omega_{N1}$, and $\omega_{N2}$ (as described in Table 2.2). However, Eqns. (2.9b) and (2.10c) contain only the rubber path parameters, and these are nearly frequency-invariant. Figure 2.8 illustrates the contributions of $F_r(\omega)$ and $F_h(\omega)$ in terms of $K_r(\omega)$ ($H_r(\omega)$) and $K_h(\omega)$ ($H_h(\omega)$). The comparisons show that the hydraulic path, below 30 Hz, affects the dynamic force $F_r(\omega)$ more than the rubber path. However, the loss angle from the rubber path becomes more dominant beyond 30 Hz. These features also could be assessed by comparing $K_r(\omega)$ and $H_r(\omega)$ with $K_h(\omega)$ and $H_h(\omega)$ in Figures 2.6 and 2.7.
Figure 2.5 Typical frequency responses for fixed and free decoupler mounts based on linear system based formulation with nominal parameters as listed in Tables 1 and 2. Natural frequencies are identified here: (a) $|\mathbf{K}(\omega)| = \frac{|\mathbf{F}_{\tau}(\omega)/\mathbf{X}|}{|\mathbf{X}|}$, $\phi_{\tau} = \angle \frac{|\mathbf{F}_{\tau}(\omega)/\mathbf{X}|}{|\mathbf{X}|}$; (b) $|\mathbf{H}(\omega)| = |\mathbf{F}_{\tau}(\omega)/\mathbf{P}_{\nu}|$, $\phi_{\tau} = \angle \frac{|\mathbf{F}_{\tau}(\omega)/\mathbf{P}_{\nu}|}{|\mathbf{P}_{\nu}|}$. Key: — , fixed decoupler mount; --- , free decoupler mount.
Figure 2.6 Dynamic stiffness responses for rubber and hydraulic paths: (a) fixed decoupler mount; (b) free decouple mount. Key: —– , $\overline{K}(\omega) = \overline{F}_r(\omega)/\overline{X}$; ——— , $\overline{K}_s(\omega) = \overline{F}_p(\omega)/\overline{X}$; ——— , $\overline{K}_r(\omega) = \overline{F}_e(\omega)/\overline{X}$. 
Figure 2.7 Force transmissibilities for rubber and hydraulic paths: (a) fixed decoupler mount; (b) free decoupler mount. Key: \( \overline{H}(\omega) = \frac{\overline{F}_T(\omega)}{\overline{P}_u}; \) \( \overline{H}_h(\omega) = \frac{\overline{F}_T(\omega)}{\overline{P}_u}; \) \( \overline{H}_r(\omega) = \frac{\overline{F}_T(\omega)}{\overline{P}_u}. \)
Figure 2.8 Dynamic stiffness and force transmissibility responses for rubber and hydraulic paths: (a) $\overline{K}(\omega) = \overline{F}_r(\omega)/\overline{X}$; (b) $\overline{H}(\omega) = \overline{F}_r(\omega)/\overline{P}_u$. Key: — , $\overline{K}_r(\omega) = \overline{F}_r(\omega)/\overline{X}$ and $\overline{H}_r(\omega) = \overline{F}_r(\omega)/\overline{P}_u$ with fixed decoupler mount; -- , $\overline{K}_s(\omega) = \overline{F}_r(\omega)/\overline{X}$ and $\overline{H}_s(\omega) = \overline{F}_r(\omega)/\overline{P}_u$ with free decoupler mount; --- , $\overline{K}_s(\omega) = \overline{F}_r(\omega)/\overline{X}$ and $\overline{H}_s(\omega) = \overline{F}_r(\omega)/\overline{P}_u$ with fixed decoupler mount; ..., $\overline{K}_s(\omega) = \overline{F}_r(\omega)/\overline{X}$ and $\overline{H}_s(\omega) = \overline{F}_r(\omega)/\overline{P}_u$ with free decoupler mount.
2.4 Spectrally-Varying and Amplitude-Sensitive Properties

2.4.1 Effective upper chamber compliance

The transfer functions of section 2.3 are given for a linear system, but the hydraulic mount is a nonlinear device [2.13-2.17, 2.22, 2.23]. Thus, we must develop a quasi-linear model that would include effective properties. We assume that the upper chamber pressure is most affected by the nonlinear phenomena, and thus, all nonlinearities are lumped into the effective upper chamber compliance $C_{ue}(\omega,X)$. The definition of $C_{un}$ (compliance under nominal conditions for a linear device) is now changed, and the term $C_{ue}(\omega,X)$ is now a complex value parameter that includes both amplitude-sensitive stiffness and damping properties at any frequency. Define it using the empirical $\lambda_u (= \alpha+i\beta)$ as follows, where the coefficients $\alpha$ and $\beta$ would be determined from measurements:

$$C_{ue} = \lambda_u C_{un} = (\alpha + i\beta) C_{un}. \quad (2.11)$$

Then, $C_u$ term of Eqn. (2.5a) for the fixed decoupler is replaced by $C_{ue}$ to determine $\alpha$ and $\beta$ as a function of $\omega$ and $X$. First, the measured $P_{uM}(\omega,X)$ dataset is defined (under sinusoidal excitation) in terms of magnitude $P_{uM}$ and its phase $\phi_{PM}$ as:

$$P_{uM} = P_{uRE} + iP_{uIM}, \quad (2.12a)$$
$$P_{uRE} = Re[P_{uM}] = |P_{uM}| \cos(\phi_{PM}), \quad (2.12b)$$
$$P_{uIM} = Im[P_{uM}] = |P_{uM}| \sin(\phi_{PM}). \quad (2.12c)$$

Then, Eqn. (2.5a) is expressed in the frequency domain as follows:
\[ P_u(\omega, X) = \frac{A_C \left[ (1 - \omega^2 C_i I_i) + i\omega C_i R_i \right] X}{(C_u + C_i - \omega^2 C_u C_i I_i) + i\omega C_u C_i R_i}. \]  

Thus, the coefficients \( \alpha \) and \( \beta \) are estimated by substituting Eqns. (2.11) and (2.12a)-(2.12c) into Eqn. (2.13). These formulations will be investigated further in Chapter 3; the next chapter would determine super-harmonics in the upper chamber pressure and force signals under sinusoidal excitation [2.26].

### 2.4.2 Procedures used to find \( \lambda_u(\omega, X) \) curve-fits

By using the equations of section 2.4.1 and measured \( P_{uM}(\omega, X) \), we can calculate \( \lambda_u(\omega, X) \) values. This discrete data set is curve-fit using piecewise polynomial functions and smoothened using hyper-tangent functions [2.27], as described next, in order to obtain continuous functions of the frequency for modeling purposes. Sample results are first illustrated in Figure 2.9, where three frequency regimes are shown along with their smoothening functions. Although only one example of the magnitude and phase curves is shown in Figure 2.9, the entire range of \( \lambda_u(\omega, X) \) dependent upon \( X \) should be considered.

Steps are described below.

First, the magnitude and phase of \( \lambda_u(\omega, X) \) are grouped in three frequency regimes. A linear function is assumed to curve-fit below \( \Omega_1 (= 2.5 \text{ Hz}) \); a 6th order polynomial function is chosen to define the curves between \( \Omega_1 \) and \( \Omega_2 (= 22.5 \text{ Hz}) \); and, a 4th order polynomial function is utilized above \( \Omega_2 \) as illustrated in Figure 2.9. Second, each polynomial function is multiplied by the relevant smoothening functions \( \Gamma_{\Omega_1}(\omega), \Gamma_{\Omega_2}(\omega) \) and \( \Gamma_{\Omega_3}(\omega) \)). The product of the smoothening function and the polynomial function yields
Figure 2.9 Magnitude (on $\log_e$ scale) and phase curves of $\lambda_d(\omega, X)$ for fixed decoupler mount at $X = 0.15$ mm, smoothening functions, $\Gamma_\Omega$, and frequency regimes defined by $\Omega_1 = 2.5$ Hz and $\Omega_2 = 22.5$ Hz. Key for $\Gamma_\Omega$ curves: $\Gamma_{\Omega_1}(\omega)$; $\Gamma_{\Omega_2}(\omega)$; $\Gamma_{\Omega_3}(\omega)$. 
the continuous $\lambda_u(\omega, X)$. Third, when $|\lambda_u(\omega, X)|$ is estimated in the frequency domain, it is on the $\log_e$ scale since the frequency responses of $|F_T(\omega, X)|$ are sensitive to a variation in $|\lambda_u(\omega, X)|$. The following expressions describe the polynomial functions, $\kappa_i(\omega)$ ($i = 1 - 3$), of $|\lambda_u(\omega, X)|$ for three frequency regimes and their smoothening functions $\Gamma_{\Omega_i}(\omega)$. These are functions of $\Omega (= \omega/2\pi, \text{Hz})$.

$$
\kappa_1(\omega) = a_{\kappa_1}(\omega/2\pi) + a_{\kappa_0}, \quad \Omega < \Omega_1, \tag{2.14a}
$$

$$
\kappa_2(\omega) = b_{\kappa_6}(\omega/2\pi)^6 + b_{\kappa_5}(\omega/2\pi)^5 + b_{\kappa_4}(\omega/2\pi)^4 + b_{\kappa_3}(\omega/2\pi)^3 
+ b_{\kappa_2}(\omega/2\pi)^2 + b_{\kappa_1}(\omega/2\pi) + b_{\kappa_0}, \quad \Omega_1 < \Omega < \Omega_2, \tag{2.14b}
$$

$$
\kappa_3(\omega) = g_{\kappa_4}(\omega/2\pi)^4 + g_{\kappa_3}(\omega/2\pi)^3 + g_{\kappa_2}(\omega/2\pi)^2 + g_{\kappa_1}(\omega/2\pi) + g_{\kappa_0}, \quad \Omega > \Omega_2, \tag{2.14c}
$$

$$
\Gamma_{\Omega_1}(\omega) = 0.5\left\{-\tanh\left[\sigma_1(\omega/2\pi - \Omega_1)\right] + 1\right\}, \tag{2.14d}
$$

$$
\Gamma_{\Omega_2}(\omega) = 0.5\left\{\tanh\left[\sigma_2(\omega/2\pi - \Omega_1)\right] + 1\right\}, \tag{2.14e}
$$

$$
\Gamma_{\Omega_3}(\omega) = 0.5\left\{\tanh\left[\sigma_3(\omega/2\pi - \Omega_2)\right] + 1\right\}. \tag{2.14f}
$$

The following smoothening factors $\sigma_v$ ($v = 1 - 3$) are selected: $\sigma_1 = 1 \times 10^6$; $\sigma_2 = 1 \times 10^6$; and $\sigma_3 = 1$. By applying this method, the entire set of coefficients (at each $X$) is estimated. For example, the coefficients for the fixed decoupler are as follows: $a_{\kappa_1} = 0.12, 0.10, -0.06, -0.14, -0.29, -0.43$ and $-0.55$ with $X = 0.15, 0.25, 0.5, 0.75, 1.0, 1.25$ and $1.5\text{ mm}$ respectively. Also, $a_{\kappa_0} = 4.76, 4.71, 4.93, 5.01, 5.31, 5.52$ and $5.62$ with $X = 0.15, 0.25, 0.5, 0.75, 1.0, 1.25$ and $1.5\text{ mm}$ respectively. Further, curve-fit the coefficients in terms of $X$ as suggested below.
\[ a_{k}(X) = \sum_{w=1}^{7} a_{kw} X^{w-1} (v = 0, 1), \]  
(2.15a)

\[ b_{k}(X) = \sum_{w=1}^{7} b_{kw} X^{w-1} (v = 0 - 5), \]  
(2.15b)

\[ g_{k}(X) = \sum_{w=1}^{7} g_{kw} X^{w-1} (v = 0 - 4), \]  
(2.15c)

\[ a_{k1}(X) = a_{k16} X^6 + a_{k15} X^5 + a_{k14} X^4 + a_{k13} X^3 + a_{k12} X^2 + a_{k11} X + a_{k10}. \]  
(2.15d)

Here, \( a_{k}(X) \), \( b_{k}(X) \) and \( c_{k}(X) \) are the coefficient functions. Index \( w \) in Eqns. (2.15a)-(2.15c) indicates different \( X \) values since the experimental values of \( X \) are given for 7 cases only (\( X = 0.15, 0.25, 0.5, 0.75, 1.0, 1.25 \) and 1.5 mm). Therefore, the coefficient polynomial functions are limited by the 6th order term as illustrated by Eqn. (2.15d). The estimated coefficients for Eqn. (2.15d) are: \( a_{k16} = -6.63; a_{k15} = 32.67; a_{k14} = -62.29; a_{k13} = 57.76; a_{k12} = -26.86; a_{k11} = 5.30; \) and \( a_{k10} = -0.24 \). Using this method, all coefficients as a function of \( X \) are found. Therefore, \( |\lambda_{u}(\omega, X)| \) is described in a continuous manner in terms of \( \omega \) and \( X \). The phase of \( \lambda_{u}(\omega, X) \) is also described in the same way. In order to determine the values of \( |\lambda_{u}(\omega, X)| \), the estimates are calibrated in terms of an exponential function since the empirical values of \( |\lambda_{u}(\omega, X)| \) are curve-fit on a \( \log_e \) scale. Therefore, the procedure to determine a continuous \( |\lambda_{u}(\omega, X)| \) is summarized as follows:

\[ |\lambda_{u}(\omega, X)| = e^{\kappa_{u}(\omega, X)}, \]  
(2.16a)

\[ \kappa_{ue}(\omega, X) = \kappa_{\Omega \Omega}(\omega, X) + \kappa_{\Omega2}(\omega, X) + \kappa_{\Omega3}(\omega, X), \]  
(2.16b)

\[ \kappa_{\Omega1}(\omega, X) = \kappa_{x1}(\omega, X) \Gamma_{\Omega1}(\omega), \]  
(2.16c)

\[ \kappa_{\Omega2}(\omega, X) = \kappa_{x2}(\omega, X) \left[ \Gamma_{\Omega2}(\omega) - \Gamma_{\Omega3}(\omega) \right], \]  
(2.16d)
\[ \kappa_{x3}(\omega, X) = \kappa_{x3}(\omega, X) \Gamma_{\Omega3}(\omega), \]  
(2.16e)

\[ \kappa_{x3}(\omega, X) = a_{\kappa}(X)(\omega/2\pi) + a_{\kappa0}(X), \]  
(2.16f)

\[ \kappa_{x2}(\omega, X) = b_{\kappa}(X)(\omega/2\pi)^6 + b_{x5}(X)(\omega/2\pi)^5 + b_{x4}(X)(\omega/2\pi)^4 + b_{x3}(X)(\omega/2\pi)^3 + b_{x2}(X)(\omega/2\pi)^2 + b_{x1}(X)(\omega/2\pi) + b_{x0}(X), \]  
(2.16g)

\[ \kappa_{x3}(\omega, X) = g_{x4}(X)(\omega/2\pi)^4 + g_{x3}(X)(\omega/2\pi)^3 + g_{x2}(X)(\omega/2\pi)^2 + g_{x1}(X)(\omega/2\pi) + g_{x0}(X). \]  
(2.16h)

Here, \( \kappa_{x3}(\omega, X) \) is \( \log_e|\lambda_{u}(\omega, X)| \), \( \kappa_{x3}(\omega, X) \) \((v = 1 - 3) \) is the smoothened function of \( \kappa_{x3}(\omega, X) \) \((v = 1 - 3) \), \( \kappa_{x3}(\omega, X) \) \((v = 1 - 3) \) is the polynomial function over relevant frequency range with coefficient functions \( a_{k}(X) \) \((v = 1 - 3) \), \( b_{k}(X) \) \((v = 1 - 6) \), and \( g_{k}(X) \) \((v = 1 - 4) \) as described by Eqns. (2.15a)-(2.15d). The smoothening functions \( \Gamma_{\Omega3}(\omega) \) \((v = 1 - 3) \) are given by Eqns. (2.14d)-(2.14f).

The phase of \( \lambda_{u}(\omega, X) \) is also determined as follows:

\[ \varphi_{x1}(\omega, X) = \varphi_{x1}(\omega, X) \Gamma_{\Omega3}(\omega), \]  
(2.17a)

\[ \varphi_{x2}(\omega, X) = \varphi_{x2}(\omega, X) \Gamma_{\Omega3}(\omega), \]  
(2.17b)

\[ \varphi_{x3}(\omega, X) = \varphi_{x3}(\omega, X) \Gamma_{\Omega3}(\omega), \]  
(2.17c)

\[ \varphi_{x4}(\omega, X) = a_{\varphi4}(X)(\omega/2\pi) + a_{\varphi0}(X), \]  
(2.17d)

\[ \varphi_{x5}(\omega, X) = b_{\varphi4}(X)(\omega/2\pi)^6 + b_{\varphi5}(X)(\omega/2\pi)^5 + b_{\varphi4}(X)(\omega/2\pi)^4 + b_{\varphi3}(X)(\omega/2\pi)^3 + b_{\varphi2}(X)(\omega/2\pi)^2 + b_{\varphi1}(X)(\omega/2\pi) + b_{\varphi0}(X), \]  
(2.17e)

\[ \varphi_{x6}(\omega, X) = g_{\varphi4}(X)(\omega/2\pi)^4 + g_{\varphi3}(X)(\omega/2\pi)^3 + g_{\varphi2}(X)(\omega/2\pi)^2 + g_{\varphi1}(X)(\omega/2\pi) + g_{\varphi0}(X). \]  
(2.17f)
Here, $\varphi_{1e}(\omega, X)$ is the phase of $\lambda_{u}(\omega, X)$, $I_{1e}^{\omega}(\omega, X) (v = 1 - 3)$ is the smoothened function of $\varphi_{X1}(\omega, X) (v = 1 - 3)$, $\varphi_{X1}(\omega, X)$ (v = 1 - 3) is the polynomial function for relevant frequency range with coefficient functions $a_{v1}(X) (v = 1 - 3)$, $b_{v1}(X) (v = 1 - 6)$, and $g_{v1}(X) (v = 1 - 4)$. The coefficients are also calculated using the method described by Eqns. (2.15a)-(2.15d).

2.4.3 Effective rubber path properties

The linear system study of section 2.3 has suggested the importance of both force paths, and therefore, we define effective rubber path properties as $C_{ue}(\omega, X)$. The effective stiffness $k_{ve}(\omega, X)$ is defined as $k_{rn} \lambda_{kr}(\omega, X)$, and $c_{ve}(\omega, X)$ as $c_{rn} \lambda_{cr}(\omega, X)$ where $k_{rn}$ and $c_{rn}$ are the nominal (linear system) values of $k_{r}$ and $c_{r}$, respectively. $\lambda_{kr}(\omega, X)$ and $\lambda_{cr}(\omega, X)$ are spectrally-varying and amplitude-sensitive parameters for $k_{r}$ and $c_{r}$, respectively [2.10].

Figure 2.10 shows the measured dynamic stiffness data from the rubber mount test with $X = 0.15$ mm; this test is done when fluid is drained from the hydraulic mount. Thus, this represents the combined effect of the elastomeric parts within the mount. The methods of section 2.4.2 are applied to obtain the continuous curve-fit schemes of $\lambda_{kr}(\omega, X)$ and $\lambda_{cr}(\omega, X)$. This issue will be further investigated in Chapter 3 [2.26].

2.5 Dynamic Force Estimation Using Quasi-Linear System Models

The quasi-linear models with effective parameters in terms of $\lambda_{u}(\omega, X)$, $\lambda_{kr}(\omega, X)$, and $\lambda_{cr}(\omega, X)$ will be employed next to estimate $F_{T}(\omega, X)$ in the frequency domain by using
Eqns. (2.2a)-(2.2c), (2.9a)-(2.9e), and (2.10a)-(2.10c). Also, alternate force estimation methods will be compared.

2.5.1 Analogous mechanical system model

Figure 2.11 illustrates the analogous mechanical system model which has the following equivalent parameters that are converted from the fluid system model: Effective mass of inertia track fluid column \( m_{ie} = A_r^2 I_i \); effective viscous damping of inertia track fluid \( c_{ie} = A_r^2 R_i \); equivalent stiffness of upper chamber compliance \( k_u = A_r^2/C_u \); and equivalent stiffness of lower chamber compliance \( k_l = A_r^2/C_l \). The nominal values are listed in Table 2.1. As illustrated in Figure 2.11, relationship between \( x(t), x_{ie}(t) \) and \( f_T(t) \) is derived as follows:

\[
\begin{align*}
    m_{ie} \ddot{x}_{ie}(t) + c_{ie} \dot{x}_{ie}(t) + (k_u + k_l) x_{ie}(t) &= k_u x(t), \\
    f_T(t) &= c_r \dot{x}(t) + k_r x(t) + k_{ie} x_{ie}(t).
\end{align*}
\]

where effective velocity of inertia track fluid is given by \( \dot{x}_{ie}(t) = q_i(t)/A_r \). By taking the Laplace transform for both Eqns. (2.18) and (2.19), the transfer function of the hydraulic path, \( F_{th}(s)/X(s) = N(s) \) is derived as follows:

\[
\begin{align*}
    X_{ie}(s) &= \frac{k_u}{m_{ie}s^2 + c_{ie}s + k_u + k_l} X(s), \\
    \frac{F_T(s)}{X} &= K_A(s) = \frac{F_{dr}(s)}{X} + \frac{F_{th}(s)}{X}, \\
    \frac{F_{dr}(s)}{X} &= (c_r s + k_r),
\end{align*}
\]
Figure 2.10 Spectrally-varying and amplitude-sensitive properties of the rubber path with $X = 0.15$ mm: (a) effective stiffness quantified by $\lambda_{kr} (= k_{re}/k_{rn})$ and displayed on a linear scale; (b) viscous damping quantified by $\lambda_{cr} (= c_{re}/c_{rn})$ and displayed on a $\log_e$ scale.
Figure 2.11 Analogous mechanical system model and its parameters for rubber and hydraulic paths.
\[
\frac{F_{Ah}(s)}{X} = K_{Ah}(s) = \frac{k_i k_j}{m_i s^2 + c_i s + k_u + k_l}.
\] (2.21c)

Also, the dimensionless dynamic stiffness \( K_A(s) \) is acquired by employing Eqns. (2.7a)-(2.7d) as follows:

\[
\frac{\bar{F}_T(s)}{X} = \bar{K}_A(s) = \frac{\bar{F}_{Ar}(s)}{X}(s) + \frac{\bar{F}_{Ah}(s)}{X}(s),
\] (2.22a)

\[
\bar{F}_{Ar}(s) = \gamma_r (1 + \tau_r s),
\] (2.22b)

\[
\bar{F}_{Ah}(s) = \bar{K}_{Ah}(s) = \gamma_k \left( \frac{s^2}{\omega_{N2}^2} + \frac{2 \zeta}{\omega_{N2}} s + 1 \right).\] (2.22c)

where \( \bar{F}_{Ar}(s)/\bar{X}(s) \) formulation is the same as Eqns. (2.9a) and (2.9b). However, it differs from Eqn. (2.8a) as shown in Eqn. (2.22c).

### 2.5.2 Alternate transfer function schemes

Alternate transfer function schemes, from Eqns. (2.2a)-(2.2c), (2.9a)-(2.9c), (2.10a)-(2.10c), and (2.22a)-(2.22c), could be considered along with quasi-linear model of section 2.4. Table 2.3 lists alternate schemes (and their designations) as well as the type of measured data such as \( X(\omega) \) and/or \( P_u(\omega, X) \). Note that Schemes I, II, III, and IV are based on Eqns. (2.2a)-(2.2c), (2.9a)-(2.9c), (2.10a)-(2.10c) and (2.22a)-(2.22c) respectively whereas Scheme I is the simplest estimation of \( \bar{F}_T(\omega, X) \) with Eqns. (2.2a)-(2.2c). In particular, Scheme III illustrates the direct relationship between \( \bar{F}_T(\omega, X) \) and \( \bar{P}_u(\omega, X) \) and Scheme IV employs the mechanical system parameters. The proposed
schemes can incorporate spectrally-varying and amplitude-sensitive parameters \( \lambda_v(s;X) \) \((v = u, kr, cr)\) under harmonic excitation with amplitude \( X \). In the limiting cases when \( \lambda_v(s;X) = 1 \) \((v = u, kr, cr)\), the \( \mathcal{F}_T(\omega, X) \) is estimated by the linear model with only nominal values. The proposed schemes (I, II, III and IV) are summarized in Figure 2.12, via the block diagrams in Laplace domain including the sub-blocks that dictate \( \mathcal{K}(s) \), \( \mathcal{H}(s) \), and \( \mathcal{K}_A(s) \). Therefore, the steady state responses are examined by changing the \( s \) term to \( j\omega \).

Scheme I of Figure 2.12 (a) and Table 2.3 is the simplest case though it needs both \( X \) and \( P_u \) measurements. By comparing the alternate schemes, the merits of each method could be considered in terms of the number of sensors needed and main transfer function such as \( \mathcal{K}(s) \) or \( \mathcal{H}(s) \).

2.5.3 Estimation of \( \mathcal{F}_T(\omega, X) \) by using quasi-linear model

Figures 2.13 and 2.14 compare the results of both linear and quasi-linear models with experiment. Observe that the estimated dimensionless force is nearly equal to 2 when \( \omega \rightarrow 0 \) since the contribution of the hydraulic path is negligible at very low frequencies. This is analytically confirmed the term \( \mathcal{F}_T(\omega \approx 0) = \gamma \left( I + \tau_s s \right) X \) results in \( \gamma \left( \frac{k_{re}(0;X)}{k_{ref}} \right) = 2 \) with \( \lambda_v(0;X) \) and nominal parameters.

Schemes I and II successfully predict the dynamic force transmitted to a rigid base. Schemes III also shows good correlation except at \( X = 0.15 \) mm. Note that the
dimensionless $\overline{F}_T(\omega, X)$ estimate in fact the stiffness $\overline{K}(\omega, X)$ estimation. Overall, the linear model predicts the tendency of force well, but not the precise magnitude and phase.

### 2.5.4 Error committed by mechanical model

Specifically the linear analysis of Figure 2.14 (a) shows a significant deviation from the experiment. Scheme IV (based on mechanical model) fails to predict the forces as all magnitude and phase values are almost constant in Figures 2.13 and 2.14. This is investigated in Figure 2.15 that compares hydraulic and rubber path forces. The error committed by scheme IV is investigated as follows where the subscript $A$ implies the analogous mechanical model:

$$
\bar{\varepsilon}_1(\omega, X) = \frac{\overline{F}_h(\omega, X) - \overline{F}_A(\omega, X)}{\overline{F}_h(\omega, X)}.
$$

(2.23)

When Eqns. (2.8a) and (2.8b) are incorporated in Eqn. (2.23), the error function (in the Laplace domain) is:

$$
\bar{\varepsilon}_2(s) = 1 - \frac{1}{s^2 + \frac{2\zeta}{\omega_{ni}} s + 1}.
$$

(2.24)

Next, replace $s$ with $j\omega$ to yield the following:

$$
\bar{\varepsilon}_2(r_i) = 1 - \frac{1}{(1-r_i^2) + i2\zeta r_i}, \ r_i = \frac{\omega}{\omega_{ni}}.
$$

(2.25)

Now we consider three cases with respect to $\omega_{ni}$; values are 1.62 and 1.9 Hz for the fixed and free decoupler mounts, respectively. Consider the limiting case close to the zero frequency where $r_i \to 0$:
When $r_j = 1$, we get:

\[
\bar{\varepsilon}_2 (r_j = 1) = 1 + \frac{i}{2\zeta},
\]

\[
\left| \bar{\varepsilon}_2 (r_j = 1) \right| = \sqrt{1 + \frac{1}{(2\zeta)^2}} \approx 1,
\]

\[
\phi_{\varepsilon_2} = \tan^{-1} \left( 2\zeta + i \right) - \tan^{-1} \left( 2\zeta + i \right) = \tan^{-1} \left( 2\zeta + i \right).
\]

where $\phi_{\varepsilon_2} \approx 11.53^\circ \left( 13.40^\circ \right)$ for the fixed (free) decoupler mount by using the nominal values of $\zeta$ and $\omega_{N1}$ as described in Section 2.3.

Finally consider the case when $r_j$ greater than 1:

\[
\bar{\varepsilon}_2 (r_j) = \frac{-r_j^2 + i2\zeta r_j}{(1-r_j^2) + i2\zeta r_j},
\]

\[
\left| \bar{\varepsilon}_2 (r_j) \right| = \sqrt{\left( \frac{-r_j^2}{1-r_j^2} \right)^2 + \left( \frac{2\zeta r_j}{1-r_j^2} \right)^2} \approx 1,
\]

\[
\phi_{\varepsilon_2} = \tan^{-1} \left( \frac{2\zeta r_j}{-r_j^2} \right) - \tan^{-1} \left( \frac{2\zeta r_j}{1-r_j^2} \right) \approx 0.
\]

For each case as described by Eqns. (2.26a)-(2.26d) to (2.28a)-(2.28c), the error $\left| \bar{\varepsilon}_2 (\omega) \right|$
Figure 2.12 Comparison of alternate schemes via block diagrams: (a) Scheme I; (b) Scheme II; (c) Scheme III; (d) Scheme IV.
Table 2.3 Comparison of alternate force estimation schemes based on quasi-linear system model

<table>
<thead>
<tr>
<th>Scheme Designation</th>
<th>Sensor(s) required</th>
<th>Spectrally-varying and amplitude-sensitive parameters</th>
</tr>
</thead>
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<tr>
<td></td>
<td>( X )</td>
<td>( P_u )</td>
</tr>
<tr>
<td></td>
<td>( \lambda_{kr} = 1, \lambda_{cr} = 1 )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( \lambda_{kr} \neq 1, \lambda_{cr} \neq 1 )</td>
<td></td>
</tr>
<tr>
<td>Fluid System Model</td>
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<tr>
<td>II-A</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>( \lambda_{kr} = 1, \lambda_{cr} = 1, \lambda_u \neq 1 )</td>
<td></td>
</tr>
<tr>
<td>II-B</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>( \lambda_{kr} \neq 1, \lambda_{cr} \neq 1, \lambda_u = 1 )</td>
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</tr>
<tr>
<td>II-C</td>
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</tr>
<tr>
<td></td>
<td>( \lambda_{kr} \neq 1, \lambda_{cr} \neq 1, \lambda_u \neq 1 )</td>
<td></td>
</tr>
<tr>
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<td>Yes</td>
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<tr>
<td></td>
<td>( \lambda_{kr} = 1, \lambda_{cr} = 1, \lambda_u \neq 1 )</td>
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<tr>
<td>III-B</td>
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<td>Yes</td>
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<tr>
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<td>Yes</td>
</tr>
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<tr>
<td>IV-B</td>
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<td></td>
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<tr>
<td></td>
<td>( \lambda_{kr} \neq 1, \lambda_{cr} \neq 1, \lambda_u \neq 1 )</td>
<td></td>
</tr>
</tbody>
</table>
Figure 2.13 Force transmitted to the rigid base by a fixed decoupler mount: (a) $X = 0.15$ mm; (b) $X = 1.5$ mm. Key: $\square$, experiment; $--$, linear model; $\cdots\cdots\cdots$, Scheme I; $---$, Scheme II; $\ldots\ldots\ldots$, Scheme III; $\rightarrow$, Scheme IV.
Figure 2.14 Force transmitted to the rigid base by a free decoupler mount: (a) \( X = 0.15 \) mm; (b) \( X = 1.5 \) mm. Key: \( \square \), experiment; \( \rightarrow \), linear model; \( \cdot \cdot \cdot \), Scheme I; \( \cdot \cdot \cdot \cdot \), Scheme II; \( \cdot \cdot \cdot \), Scheme III; \( \cdot \cdot \cdot \cdot \), Scheme IV.
Figure 2.15 Comparison of $\overline{F}_\tau(\omega)$ with path contributions $\overline{F}_{\omega}(\omega)$ and $\overline{F}_{\phi}(\omega)$ using scheme IV at $X = 1.5$ mm: (a) fixed decoupler mount; (b) free decoupler mount. Key:

- , estimated $\overline{F}_\tau(\omega)$; - · , estimated $\overline{F}_{\omega}(\omega)$; - - , estimated $\overline{F}_{\phi}(\omega)$. 
is almost 100% over the entire range of frequencies. This confirms the chief weakness of analogous mechanical models though they have utilized [2.28].

2.5.5 Contribution of rubber and hydraulic paths to the total force \( F_T(\omega, X) \)

Figure 2.16 compares \( F_T(\omega, X) \) estimates based on schemes III-A, B, and C for fixed and free decoupler mounts. The results show that estimates depend upon the contribution of rubber and hydraulic paths. Scheme III-A fails to capture the precise magnitude and phase. Scheme III-B shows moderate results even though the rubber path force is estimated with a quasi-linear model. Scheme III-C yields the most precise predictions when the quasi-linear models for both paths are concurrently applied. Figure 2.17 compares the rubber and hydraulic path forces with Scheme II with \( X = 1.5 \text{ mm} \). As shown in Figures 2.17 (a) and (b), \( F_T(\omega, X) \) is quite sensitive to the rubber path though rubber path forces are almost constant when compared with the hydraulic path forces.

2.6 Dynamic Force Estimation in Time Domain

Based upon the frequency domain results, \( f_T(t) \) could be estimated in time domain by employing the following assumptions: (1) Only the steady state response is of interest under the sinusoidal displacement \( x(t) \) excitation; and (2) only the mean (preload \( f_m \)) and fundamental frequency terms are considered in the Fourier expansion. As illustrated previously, the dynamic stiffness \( K(\omega) \) or force transmissibility \( H(\omega) \) could be estimated by employing \( \lambda_{kr}(\omega, X) \), \( \lambda_{cr}(\omega, X) \), and \( \lambda_u(\omega, X) \) in the quasi-linear model.
Figure 2.16 Force estimations with Scheme III at $X = 1.5$ mm: (a) fixed decoupler; (b) free decoupler. Key: $\square$, experiment; $\cdot$, Scheme III-A; $\longrightarrow$, Scheme III-B; $\cdots$, Scheme III-C.
Figure 2.17 Comparison of $\overline{F}_T(\omega)$ with path contribution $\overline{F}_{Tn}(\omega)$ and $\overline{F}_{Tn}(\omega)$ with scheme II at $X = 1.5$ mm: (a) fixed decoupler mount; (b) free decoupler mount. Key: $\bullet$, measured $\overline{F}_T(\omega)$; $\cdots$, estimated $\overline{F}_T(\omega)$; $\star$, estimated $\overline{F}_{Tn}(\omega)$; $-$, estimated $\overline{F}_{Tn}(\omega)$. 
Therefore, the Fourier expansion is applied by embedding $K(\omega)$ or $H(\omega)$ as quasi-linear parameters with input $X$ or $P_u$ respectively as shown below where $\omega_o$ is the fundamental frequency:

$$f_{T-SchemeII}(t) = f_w + \chi_1 K(\omega_o) \sin(\omega_o t + \phi), \quad (2.29a)$$

$$\chi_1 = k_{ref} X_{ref} X. \quad (2.29b)$$

$$f_{T-SchemeII}(t) = f_w + \chi_2 H(\omega_o) \sin(\omega_o t + \phi), \quad (2.30a)$$

$$\chi_2 = k_{ref} X_{ref} P_u. \quad (2.30b)$$

The $\chi_1$ and $\chi_2$ terms are used for finding the $f_T(t)$ history in force units (N) as the estimation yields the normalized values. Further, $f_{T-SchemeII}(t)$ and $f_{T-SchemeIII}(t)$ are the force estimations from Scheme II-C and III-C respectively. Figure 2.18 compares the time histories as predicted by Eqns. (2.29a), (2.29b), (2.30a) and (2.30b) with measured forces for the free decoupler mount at $\omega_o/2\pi = 8.5$ Hz and $X = 1.5$ mm. This result is a significant improvement over the previous formulation as illustrated in Figure 2.3. When the quasi-linear models for both rubber and hydraulic force paths are employed, the estimation of $f_T(t)$ matches well with measured force specifically in terms of its amplitude. The discrepancies between the experiment and the proposed models are primarily due to the super-harmonic terms, which are not included in the current study. Chapter 3 will address this issue further with focus on the super-harmonic responses.
Figure 2.18 Comparison between measured and predicted forces in time domain for the free decoupler mount, given sinusoidal displacement $x(t) = X \sin \omega_0 t$ at $\omega_0/2\pi = 8.5$ Hz and $X = 1.5$ mm. Key: \textcolor{blue}{-}, experiment; \textcolor{red}{--}, estimation by Eqn. (2.1) given nominal parameters; \textcolor{green}{- -}, estimation by Scheme II-C with quasi-linear parameters; \textcolor{olive}{- - - -}, estimation by Scheme III-C with quasi-linear parameters.
2.7 Conclusion

This chapter has proposed indirect methods to estimate dynamic forces that are transmitted to a rigid base by a fixed or free hydraulic mount. Specific contributions of this chapter include the following. First, alternate transfer function formulations are derived for a linear system that relate sinusoidal motion $x(t)$ and/or chamber pressure $p_u(t)$ to harmonic forces through rubber and hydraulic paths. In particular, the derivation of force to pressure transfer function is very promising as it permits an estimation of forces only by pressure measurements. Second, the effect of mount nonlinearities on forces is quantified only in terms of top chamber compliance and rubber path properties, unlike prior methods that need extensive models [2.9, 2.13-2.16]. This leads to a quasi-linear model with amplitude-sensitive and spectrally-varying parameters such as $C_{ue}(\omega,X)$, $k_r(\omega,X)$ and $c_r(\omega,X)$. Since the hydraulic force path has been successfully characterized only by $C_{ue}(\omega,X)$, other hydraulic path nonlinearities such as inertia track resistance, bottom chamber compliance and decoupler flow resistance could be ignored and thus simpler force estimation schemes could be developed. Nevertheless, the rubber path of the nonlinearities must be included as well in order to accurately estimate $F_T(\omega,X)$. Third, the forces transmitted to a rigid base under harmonic displacement excitation are successfully predicted and compared with measured forces; all schemes (with a quasi-linear formulation) work well except the mechanical system model (Scheme IV) that provides almost 100% error in forces.

This chapter has focused on frequency domain analyses though limited results in time domain are also presented. The force time history $f_T(t)$ is predicted by applying the
Fourier expansion with an embedded quasi-linear model with only the fundamental (excitation) frequency. However, force estimation methods (especially in time domain) need to include multi-harmonic response terms which are observed in measurements (Figures 2.3 and 2.18). In the next chapter, the super-harmonic responses are being investigated and they will be incorporated in order to better estimate the force time histories [2.26].
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CHAPTER 3

EXAMINATION OF SUPER-HARMONICS UNDER SINUSOIDAL
EXCITATION USING NONLINEAR AND QUASI-LINEAR MODELS

3.1 Introduction

Precise knowledge of the dynamic forces that are transmitted by machinery mounts and isolators to rigid or compliant bases in vehicles, buildings, and equipment is critical to the dynamic design and vibration control considerations. Direct measurement of forces (for example, using conventional force transducers) is not practical in many real-life applications since the interfacial conditions may change [3.1, 3.2]. Thus indirect force estimation methods must be developed [3.3, 3.4]. For instance, one could employ transfer path approaches, though they are applicable primarily in the frequency domain for a linear time-invariant system [3.5, 3.6]. Also, dynamic forces could be estimated by using other measured signals such as operating motions, but then dynamic stiffness must be known a priori [3.7]. Such indirect force estimation methods pose special difficulty for nonlinear mounts or isolators. For instance, hydraulic engine mounts exhibit spectrally-varying and amplitude-sensitive parameters [3.8].

To illustrate the concepts of this chapter, consider Figure 3.1 (a) that displays the
internal configuration of the hydraulic engine mount and its fluid system model; it will be discussed further in section 3.3. Figure 3.1 (b) shows the non-resonant dynamic stiffness test concept, as defined by the ISO standard 10846 [3.9]. Here, $f_m$ is the preload, $x(t) = x_m + \text{Re}[\tilde{X}e^{i\omega_t}]$ is the excitation displacement, $x_m$ is the mean displacement, $\tilde{X} = Xe^{i\phi_X}$ is the complex valued excitation amplitude, $X$ is the amplitude of displacement, $\phi_X$ is the phase of $x(t)$, $\omega_o$ is the excitation (fundamental) frequency (rad/s), and $\text{Re}[\ ]$ is the real value operator; tilde over a symbol implies that is complex valued. In addition, a pressure transducer, $p_u(t)$, is also installed in the upper chamber in our laboratory experiments. As shown in Figure 3.1 (b), the steady state force transmitted $f_T(t)$, in time ($t$) domain, is related to the excitation displacement $x(t)$ and upper chamber response $p_u(t)$:

$$f_T(t) = f_{Tr}(t) + f_{Th}(t),$$

$$f_{Tr}(t) = c_r\dot{x}(t) + k_r x(t),$$

$$f_{Th}(t) = A_r p_u(t).$$

Here, $f_{Tr}(t)$ is the rubber path force (subscript $r$), $f_{Th}(t)$ is the hydraulic path force (subscript $h$), $k_r$ and $c_r$ are the rubber stiffness and damping coefficient respectively, and $A_r$ is the effective piston area. Our experiments show that $p_u(t)$ deviates from the sinusoidal shape depending on $X$ and $\omega_o$ as discussed later in this chapter. Likewise, $k_r$ and $c_r$ vary as well with $X$ and $\omega_o$. The chief goal of this chapter is, therefore, to propose linear time-invariant (LTI), nonlinear (NL), and quasi-linear (QL) models that could be utilized to predict the force time history under sinusoidal excitation conditions given
Figure 3.1 Force transmitted $f_I(t)$ by a hydraulic mount in the context of non-resonant elastomeric test [3.9]: (a) Fluid model of the hydraulic mount and its parameters for rubber and hydraulic paths; (b) sinusoidal displacement excitation $x(t)$ and dynamic forces transmitted by two paths ($f_{Tr}$ and $f_{Th}$).
measured (or calculated) motion and/or internal pressure time histories. In particular, the super-harmonic contents in $p_u(t)$ and $f_f(t)$ time histories are investigated using both measurements and mathematical models. Even though the focus of this chapter is on hydraulic engine mounts, its concepts could be extended to a multi-degree of freedom isolation system problems [3.1,3.5-3.6]. For instance, Chapter 4 and the article [3.10] examine the indirect methods of force measurements in a nonlinear isolation system using a dynamic load sensing device.

3.2 Initial Results and Objectives

Figure 3.2 presents the initial results based upon Eqns. (3.1)-(3.3) when only the fundamental measured (designated with subscript $M$) $x_M(t)$ and $p_{uM}(t)$ signal terms (as shown in Figure 3.2 (a)) are considered. At this juncture, the following nominal (and constant) parameters are incorporated in Eqns. (3.1)-(3.3): $k_r = 2 \times 10^5$ N/m; $c_r = 496.1$ N-s/m; $A_r = 4 \times 10^{-3}$ m$^2$. This formulation is designated as a ‘simple prediction model’, and its results are compared in Figure 3.2 (b) with the direct measurements of dynamic force; both single term and multi-term time histories are displayed. The results show the same order of magnitudes, but the precise time history deviates from the sinusoidal shape. This suggests that super-harmonic terms must be included in $p_u(t)$. Also, a better knowledge of the amplitude ($X$) and frequency sensitive parameters is needed.

Fluid and mechanical system models of hydraulic engine mounts, based on the linear time-invariant system theory, have been extensively investigated for both fixed and free decoupler mounts [3.2,3.11,3.12]. Nevertheless, their dynamic characteristics depend
Figure 3.2 Measurement of upper chamber pressure and transmitted force time histories for the free decoupler mount, given sinusoidal displacement 

\[ x(t) = Re\left[ \tilde{X} e^{i\omega t} \right] \] at \( \omega_o/2\pi = 8.5 \text{ Hz} \) and \( X = 1.5 \text{ mm} \): (a) \( x_M(t) \) with \( X = 1.5 \text{ mm} \) and \( p_{uM}(t) \) time histories with single harmonic term; Key for part (a): \( \cdots \), \( x_M(t) \); \( -\cdots- \), \( p_{uM}(t) \) with a single harmonic term; (b) measured and predicted forces, \( f_T(t) \); Key for part (b); \( -\cdots- \), measured force with many harmonics; \( \cdots\cdots\cdots \), measured force with a single harmonic term; \( \cdots\cdots\cdots \), predicted force with nominal parameters and a single harmonic term.
upon the amplitude and frequency of excitation [3.13]. Thus, nonlinear models have been suggested under both steady state and transient conditions [3.14-3.21]. For instance, He and Singh [3.19] developed the discontinuous nonlinear model when the hydraulic mount is excited by a step-up or step-down input. Lee and Singh [3.20,3.21] have found that the nonlinear responses (of a quarter vehicle model with hydraulic mount) are affected by the super-harmonic terms even though the system is excited by a pure sinusoidal force. Unlike prior researches [3.14-3.19] that describe various mount models, this chapter focuses on the prediction of dynamic forces by using alternate methods.

The scope of this chapter is limited to the mount only, and steady state experiments under sinusoidal excitation from 1 to 50 Hz with $X$ from 0.15 to 1.5 mm (zero-to-peak) are considered. In all cases, $x_M(t)$, $p_{\nu M}(t)$, and $f_{TM}(t)$ are measured on the elastomer test machine for both fixed and free decoupler mounts. Specific objectives are as follows: (1) Analyze the measured $p_{\nu M}(t)$ and $f_{TM}(t)$ in both time and frequency domains and examine their spectral contents; (2) Develop linear time-invariant and nonlinear models of both fixed and free decoupler mounts and compare their $f_T(t)$ predictions with measurements; (3) Propose a quasi-linear model with spectrally-varying and amplitude-sensitive parameters at both $\omega_o$ and $n\omega_o$ ($n = 2,3,4, \cdots$) terms; both fluid and analogous mechanical system models are used to predict $f_T(t)$ and compare with $f_{TM}(t)$; and (4) Estimate $f_T(t)$ in the time domain by using the Fourier series expansion.
3.3 Linear Time-Invariant (LTI) Model for Fluid System

First, we develop the linear time-invariant (LTI) model for the fluid system of Figure 3.1 (a) with the following assumptions: (1) the hydraulic mount is excited by a pure sinusoidal displacement \( x(t) \) under a mean load \( f_m \), and it reaches steady state; and (2) the hydraulic mount is attached to a rigid base. The momentum and continuity equations for the hydraulic path are as follows [3.11-3.17]:

\[
f_T(t) = c_r \ddot{x}(t) + k_r x(t) + A_r p_u(t), \tag{3.4}
\]
\[
p_u(t) - p_i(t) = I_i \dot{q}_i(t) + R_i q_i(t), \tag{3.5}
\]
\[
p_u(t) - p_i(t) = I_d \dot{q}_d(t) + R_d q_d(t), \tag{3.6}
\]
\[
C_u \dot{p}_u(t) = A_r \dot{x}(t) - q_i(t) - q_d(t), \tag{3.7}
\]
\[
C_i \dot{p}_i(t) = q_i(t) + q_d(t). \tag{3.8}
\]

Here, \( C_u \) and \( C_i \) are the upper (\#u) and lower (\#l) chamber compliances, respectively; \( I_i \) and \( I_d \) are the inertances of the inertia track (\#i) and decoupler (\#d), respectively; \( R_i \) and \( R_d \) are the resistances of the inertia track and decoupler, respectively; and \( q_i \) and \( q_d \) are the fluid flow through inertia track and decoupler, respectively. Transform Eqns. (3.4)-(3.8) into the Laplace domain (\( s \)) with the assumption that the initial conditions are zeros.

\[
F_T(s) = (c_r s + k_r) X(s) + A_r P_u(s), \tag{3.9}
\]
\[
P_u(s) - P_i(s) = (I_i s + R_i) Q_i(s), \tag{3.10}
\]
\[
P_u(s) - P_i(s) = (I_d s + R_d) Q_d(s), \tag{3.11}
\]
\[
C_u s P_u(s) = A_r s X(s) - Q_i(s) - Q_d(s), \tag{3.12}
\]
To facilitate models and experimental estimations, we define dimensionless variables and parameters as: \( \bar{X} = X / X_{\text{ref}} \) = the dimensionless excitation displacement amplitude; \( X_{\text{ref}} \) = reference displacement amplitude; \( \bar{P}_u = P_u / P_{u\text{ref}} \) = dimensionless pressure; \( P_{u\text{ref}} = (k_{\text{ref}} X_{\text{ref}})/A_r \) = reference pressure; \( k_{\text{ref}} \) = reference stiffness; \( \bar{F}_T = F_T / F_{T\text{ref}} \) = dimensionless force; and \( F_{T\text{ref}} = k_{\text{ref}} X_{\text{ref}} \) = reference force. We now define three dimensionless transfer functions that relate \( \bar{F}_T \) to \( \bar{P}_u \) and \( \bar{X} \).

\[
\bar{G}(s) = \frac{\bar{P}_u}{\bar{X}}(s) = \frac{A_r}{k_{\text{ref}}} \frac{P_u(s)}{X(s)} = \gamma_h \left( \frac{s^2}{\omega_{N1}} + \frac{2\zeta_1}{\omega_{N1}} s + 1 \right) \left( \frac{s^2}{\omega_{N2}^2} + \frac{2\zeta_2}{\omega_{N2}} s + 1 \right)
\]

\[
\bar{K}(s) = \frac{\bar{F}_T}{\bar{X}}(s) = \frac{1}{k_{\text{ref}}} \frac{F_T(s)}{X(s)},
\]

\[
\bar{H}(s) = \frac{\bar{F}_T}{\bar{P}_u}(s).
\]

Here, \( \bar{G}(s) \) is the dimensionless pressure to displacement transfer function, \( \bar{K}(s) \) is the dimensionless cross point dynamic stiffness, and \( \bar{H}(s) \) is the dimensionless force transmissibility. Using Eqns. (3.9)-(3.16), the above transfer functions are expressed in terms of natural frequencies (\( \omega_{N1} \) and \( \omega_{N2} \)), damping ratios (\( \zeta_1 \) and \( \zeta_2 \)), and hydraulic path static stiffness \( \gamma_h \) as expressed below for both fixed and free decoupler mounts; these system parameters are derived in Chapter 2 and paper [3.2].

\[
\omega_{N1(\text{fixed})} = \sqrt{\frac{1}{C_i I_i}},
\]

\( C_i \) and \( I_i \) are the suspension stiffness and moment of inertia, respectively.
\[ \omega_{N1(\text{free})} = \sqrt{\frac{R_i + R_d}{C_i I R_d}}, \]  
(3.18)

\[ \omega_{N2(\text{fixed})} = \sqrt{\frac{C_u + C_i}{C_u C_i I_i}}, \]  
(3.19)

\[ \omega_{N2(\text{free})} = \sqrt{\frac{(C_u + C_i)(R_i + R_d)}{C_u C_i I_i R_d}} \]  
(3.20)

\[ \zeta_{l(\text{fixed})} = \frac{1}{2} \sqrt{\frac{C_i R_i^2}{I_i}}, \]  
(3.21)

\[ \zeta_{l(\text{free})} = \frac{1}{2} \left( \frac{C_i R_i R_d^2}{I_i (R_i + R_d)} + \frac{I_i}{\sqrt{C_i R_d (R_i + R_d)}} \right), \]  
(3.22)

\[ \zeta_{2(\text{fixed})} = \frac{1}{2} \sqrt{\frac{C_u C_i R_i^2}{I_i (C_u + C_i)}}, \]  
(3.23)

\[ \zeta_{2(\text{free})} = \frac{1}{2} \left[ \frac{C_u C_i R_d^2}{I_i (C_u + C_i)(R_i + R_d)} + \frac{(C_u + C_i) I_i}{\sqrt{C_u C_i R_d (R_i + R_d)}} \right], \]  
(3.24)

\[ \gamma_h = \frac{A_r^2}{k_{\text{ref}} (C_u + C_i)}. \]  
(3.25)

Here, subscripts \((\text{fixed})\) and \((\text{free})\) refer to the fixed and free decoupler mount designs, respectively. Next, decompose the dynamic stiffness into rubber (subscript \(r\)) and hydraulic (subscript \(h\)) paths as:

\[ \overline{K}(s) = \overline{K}_r(s) + \overline{K}_h(s), \]  
(3.26)

\[ \overline{K}_r(s) = \frac{\overline{F}_{tr}(s)}{\overline{X}}(s) = \gamma_r \left( I + \tau_r s \right), \]  
(3.27)
Finally, decompose the force transmissibility as well:

$$\overline{K}_h(s) = \overline{F}_{\text{th}}(s) = \gamma_h \left( \frac{s^2}{\omega_{n1}^2} + \frac{2\zeta_1}{\omega_{n1}} s + 1 \right) \left( \frac{s^2}{\omega_{n2}^2} + \frac{2\zeta_2}{\omega_{n2}} s + 1 \right),$$  \hspace{1cm} (3.28)

Here, $\tau_r$ is the time constant and $\gamma_r$ is the rubber path static stiffness. Observe that the fixed decoupler case is derived from the free decoupler formulations by assuming that $I_d = 0$ and $R_d \to \infty$. In our study, the nominal parameters are as follows: $I_i = 4 \times 10^6$ kg/m$^4$; $I_d = 509.3$ kg/m$^4$; $C_u = 2.5 \times 10^{-11}$ m$^5$/N; $C_l = 2.4 \times 10^{-9}$ m$^5$/N; $R_i = 2 \times 10^8$ N-s/m$^5$; $R_d = 5 \times 10^8$ N-s/m$^5$. Reference values are selected as: $k_{rref} = 2.0 \times 10^5$ N/m, $A_r = 4.5 \times 10^{-3}$ m$^2$; and $X_{rref}$ ($\times 10^{-3}$ m) though different values of $X_{rref}$ according to the experimental excitation amplitudes are utilized.
3.4 Linear Time-Invariant (LTI) Model for Analogous Mechanical System

Figure 3.3 illustrates the analogous mechanical system LTI model with effective parameters that could be related to the fluid system properties. The governing equations are:

\[
m_{ie} \ddot{x}_{ie} (t) + c_{ie} \dot{x}_{ie} (t) + (k_u + k_l) x_{ie} (t) = k_a x(t)
\]

(3.34)

\[
f_r (t) = c_r \dot{x}(t) + k_r x(t) + k_i x_{ie} (t)
\]

(3.35)

Here, the mechanical parameters are defined as follows: effective mass of inertia track fluid column \( m_{ie} = A_r^2 l_i \); effective viscous damping of inertia track fluid \( c_{ie} = A_r^2 R_i \); equivalent stiffness of upper chamber compliance \( k_u = A_r^2 / C_u \); and equivalent stiffness of lower chamber compliance \( k_l = A_r^2 / C_l \); and effective velocity of inertia track fluid \( \dot{x}_{ie} (t) = q_i (t) / A_r \). By transforming Eqns. (3.34) and (3.35) into the Laplace domain \((s)\) and ignoring initial conditions, the dimensionless transfer functions, like the fluid system model, are derived.

\[
\frac{F_r}{X}(s) = \bar{K}_A(s) = \frac{F_{dr}}{X}(s) + \frac{F_{dh}}{X}(s) = \bar{K}_{dr}(s) + \bar{K}_{dh}(s),
\]

(3.36)

\[
\bar{K}_{dr}(s) = \gamma_r (1 + \tau_r s),
\]

(3.37)

\[
\bar{K}_{dh}(s) = \gamma_h \left( \frac{s^2}{\omega_n^2} + \frac{2 \zeta_s}{\omega_n} s + 1 \right).
\]

(3.38)

Here, \( \bar{K}_A(s) \) indicates the dimensionless dynamic stiffness (with subscript \( A \)) of the analogous mechanical system. The values of nominal parameters are: \( m_{ie} = 81 \text{ kg} \); \( c_{ie} = 4.1 \times 10^3 \text{ N-s/m} \); \( k_u = 8.1 \times 10^5 \text{ N/m} \); \( k_l = 8.4 \times 10^3 \text{ N/m} \). Our study will focus on \( \bar{K}(s) \) and
Figure 3.3 Analogous mechanical system model and its parameters for rubber and hydraulic paths.
\( \overline{H}(s) \) from the fluid system model as described by Eqns. (3.15)-(3.33), and \( \overline{K}_s(s) \) from the analogous mechanical system model as given by Eqns. (3.36) to (3.38).

### 3.5 Nonlinear (NL) Model for Fluid System

The differential equations for the fluid system as given in Eqns. (3.4)-(3.8) are now modified to include four nonlinearities [3.16,3.17]. The sign convention is as follows: \( p_u(t) \) is positive (in compression) corresponding to the upward (positive) motion of \( x(t) \), \( q_i(t) \); and \( q_d(t) \); and \( p_u(t) \) is negative (in expansion) for the downward motion of \( x(t) \), \( q_i(t) \) and \( q_d(t) \); \( f_T(t) \) follows the \( x(t) \) sign. First we express the governing equations in the state space form where the state variable is defined as \( \mathbf{S}(t)=\begin{bmatrix} p_i(t) & p_s(t) & q_i(t) & \dot{x}_d(t) & f_T(t) \end{bmatrix}^T \); here \( \dot{x}_d(t)=q_d(t)/A_d \), and \( A_d \) is the effective piston area of decoupler element.

\[
\dot{\mathbf{S}}(t) = \mathbf{B} \mathbf{S}(t) + \mathbf{f}(t),
\]

\[
\mathbf{B} = \begin{bmatrix}
0 & 0 & 1/C_u & A_d/C_u & 0 \\
0 & 0 & -1/C_i & -A_d/C_i & 0 \\
-1/I_i & 1/I_i & -R_i/I_i & 0 & 0 \\
-A_d/m_d & A_d/m_d & 0 & -c_d/m_d & 0 \\
0 & 0 & -A_r/C_u & -A_rA_d/C_u & 0
\end{bmatrix},
\]

\[
\mathbf{f}(t) = \begin{bmatrix}
-A_r\dot{x}(t)/C_u \\
0 \\
0 \\
0 \\
\left(A_r^2/C_u+k_r\right)\dddot{x}(t)+c_r\ddot{x}(t)
\end{bmatrix}.
\]
where $\dot{x}(t)$ and $\ddot{x}(t)$ are the excitation velocity and acceleration, respectively; the effective mass of decoupler element is $m_d = A_d^2 I_d$, and effective viscous damping of decoupler fluid is $c_d = A_d^3 R_d$. The nominal parameters are: $m_d = 6.0 \times 10^{-3}$ kg; $c_d = 100$ Ns/m; $A_d = 1.96 \times 10^{-3}$ m$^2$.

The discontinuous motion of the decoupler element is given in terms of the switching mechanism as follows:

$$
\ddot{x}_d(t) = \begin{cases} 
\frac{1}{m_d} \left[ A_d \left\{ -p_u(t) + p_l(t) \right\} - c_d x_d(t) \right]; & -\frac{\delta_d}{2} < x_d(t) < \frac{\delta_d}{2}, \\
0, & \dot{x}(t) = 0; \quad x_d(t) = -\frac{\delta_d}{2} \text{ or } x_d(t) = \frac{\delta_d}{2} 
\end{cases} 
$$

(3.42)

where $x_d(t)$ and $\delta_d$ is the displacement of decoupler element and the net decoupler gap, respectively. The nonlinear functions for $C_u(p_u(t)), C_l(p_l(t))$, and $R_i(q_i(t))$ are described below based on prior studies [3.16,3.17].

$$
C_u(p_u(t)) = a_0 \text{ when } p_u(t) \geq p_a, \quad (3.43)
$$

$$
C_u(p_u(t)) = a_{17} \left[ p_u(t) \right]^7 + a_{10} \text{ when } p_u(t) < p_a, \quad (3.44)
$$

$$
C_l(p_l(t)) = a_{23} \left[ p_l(t) \right]^3 + a_{22} \left[ p_l(t) \right]^2 + a_{21} p_l(t) + a_{20}, \quad (3.45)
$$

$$
R_i(q_i(t)) = a_R \left| q_i(t) \right|. \quad (3.46)
$$

Typical coefficients of $C_u(p_u(t)), C_l(p_l(t))$, and $R_i(q_i(t))$ are as follows under $f_m = 1200$ N:

$a_0 = 1.09 \times 10^{-11}$; $a_{17} = -7 \times 10^{-45}$; $a_{10} = 2.5 \times 10^{-11}$; $a_{23} = 1.51 \times 10^{-18}$; $a_{22} = -6.82 \times 10^{-14}$; $a_{21} = 3.13 \times 10^{-9}$; $a_{20} = 5.19 \times 10^{-6}$; $a_R = 3.45 \times 10^{11}$. Further, the nonlinear model of the
fixed decoupler mount is also examined with the assumptions that \( I_d = 0 \) and \( R_d \to \infty \) by essentially ignoring Eqn. (3.42).

The nonlinear model is solved numerically using the Runge Kutta integration technique. The initial values of \( p_u(t) \) and \( f_T(t) \) are selected from the measurements as follows:

\[
x(0) = \text{Re}\left[\hat{X}\right]; \quad \dot{x}(0) = \text{Re}\left[i\omega_p \hat{X}\right]; \quad \text{and} \quad f_T(0) = c_r \text{Re}\left[i\omega_p \hat{X}\right] + k_r \text{Re}\left[\hat{X}\right] + A_p n(0).
\]

The initial conditions of other states \( p_i(t) \), \( q_i(t) \), and \( \dot{x}_d(t) \) are set to zeros. Here, the mean values \( x_m \) and \( f_m \) are assumed to be known. Thus, only the dynamic terms are considered. Figures 3.4 and 3.5 show typical time histories and their Fourier magnitude spectra under sinusoidal excitation. The switching mechanism of the free decoupler is evident in pressure, decoupler motion and flow rate predictions. Also, the super-harmonic terms are clearly observed in \( x_d(t) \), \( p_u(t) \), \( q_i(t) \) and \( q_d(t) \) as shown in Figure 3.5. The dynamic forces estimated by the nonlinear model including a comparison with measurements will be addressed later.

### 3.6 Quasi-Linear (QL) Model with Effective Parameters and Super-Harmonic Terms

#### 3.6.1 Spectrally-varying and amplitude sensitive parameters

Next, we develop a quasi-linear (QL) model, say in terms of the transfer functions of sections 3.3 and 3.4 that would include effective or empirical properties. Several issues must be considered since the hydraulic mount is a nonlinear device [3.14-3.18]. First, the causality problem must be considered as the measured signals are transferred into the frequency domain by using the Fast Fourier Transform (FFT) routine. This inherent
Figure 3.4 Typical predictions of the nonlinear model for the free decoupler mount given sinusoidal excitation with $X = 1.5$ mm and $\omega_o/2\pi = 8.5$ Hz (observe the effect of the switching phenomena in pressure and flow rate time histories). Key: — $x_d(t)$; — $p_u(t)$ and $q_i(t)$; --- $p_f(t)$ and $q_d(t)$. 
Figure 3.5 Magnitude of $\tilde{X}_d$, $\tilde{P}_u$, $\tilde{Q}$, and $\tilde{Q}_d$ from the FFT analysis with a free decoupler mount by using the nonlinear model given sinusoidal excitation with $X = 1.5$ mm and $\omega_o/2\pi = 8.5$ Hz. Key: $\tilde{X}_d$, FFT of $x_d(t)$; $\tilde{P}_u$, FFT of $p_u(t)$; $\tilde{Q}$, FFT of $q(t)$; $\tilde{Q}_d$, FFT of $q_d(t)$. 
problem could be understood by employing a Hilbert transform pair to represent the
causal system in terms of the real and imaginary parts of a QL model [3.22-3.24]. Second,
we assume that the upper chamber pressure is most affected by the nonlinear phenomena,
and thus, all nonlinearities are lumped into the effective (subscript e) upper chamber
compliance. The definition of $C_{un}$ (static compliance under nominal conditions, with
subscript n) has to be changed, and thus we define complex valued parameter $\tilde{C}_{ue}(\omega, X)$
that includes both amplitude-sensitive stiffness and damping properties at any frequency.
Relate static and dynamic compliances as: $\tilde{C}_{ue} = \tilde{\lambda}_u C_{un} = (\alpha + i\beta) C_{un}$
where $\tilde{\lambda}_u (= \alpha + i\beta)$ is an empirical parameter whose coefficients $\alpha$ and $\beta$ at any frequency
would be determined from measurements. Similarly, the rubber path is formulated as
discussed next. The overall quasi-linear model concept is shown in Figure 3.6; it will be
addressed further in section 3.7.

3.6.2 Estimation of effective rubber force path parameters

To investigate the rubber force path, the anti-freeze mixture (water) is drained
from the mount and then the mount is excited using the same method [3.7,3.9]. The
spectrally-varying and amplitude-sensitive parameters of the rubber path such as $k_{re}(\omega_o, X)$ and $c_{re}(\omega_o, X_i)$ are determined only at the fundamental frequency ($\omega_o$) where
the subscript e designates the effective value. Here $X_i$ is the 'virtual' excitation
displacement amplitude at $\omega_o$ as shown in Table 3.1, which will be further explained in
the next section. Figure 3.7 shows sample $k_{re}(\omega_o, X_i)$ and $c_{re}(\omega_o, X_i)$ data with $X = 0.15$
mm. In our study, the data set consists of 7 cases corresponding to $X = 0.15, 0.25, 0.5,$
Figure 3.6 Force estimation using quasi-linear models of Table 2. Key: $x_M(t)$, measured displacement; $p_M(t)$, measured pressure; $\tilde{K}(n\omega_o, X_n)$, effective value of $K(n\omega_o, X_n)$; $\tilde{H}(n\omega_o, X_n)$, effective value of $H(n\omega_o, X_n)$; $f_T(t)$, force estimated by simple prediction model and quasi-linear models.
<table>
<thead>
<tr>
<th>Experimental excitation</th>
<th>‘Virtual’ displacement (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$X_1$</td>
</tr>
<tr>
<td>Fixed Decoupler</td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>Free Decoupler</td>
<td></td>
</tr>
<tr>
<td>0.15</td>
<td>0.15</td>
</tr>
<tr>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Table 3.1 ‘Virtual’ excitation displacements $X_n$ used for the quasi-linear ($QL$) model.
0.75, 1.0, 1.25, 1.5 mm. Therefore, the quasi-linear model with curve-fit functions can be developed given experimental data [3.2]. The goal is to develop continuous profiles of the quasi-linear parameters $\lambda_{kr}(\omega_o,X_1)$ as a function of frequency $\Omega = \omega_o/2\pi$, Hz. However, the magnitude of $\lambda_{kr}(\omega_o,X_1) = k_{re}(\omega_o,X_1)/k_{rn}$ is considered in two frequency regimes as illustrated in Figure 3.7. The first curve-fit below $\Omega_1 = 2.5$ Hz is a linear function and the second one beyond $\Omega_1$ is represented by a 5th order polynomial function. Two smoothening functions are employed in terms of $\Gamma_{\Omega_1}(\omega_o)$ and $\Gamma_{\Omega_2}(\omega_o)$ as shown in Figure 3.7 [3.25] to yield the continuous profiles of $\lambda_{kr}(\omega_o,X_1)$:

$$ k_{r1}(\omega_o) = a_{k1}(\omega_o/2\pi) + a_{k0}, \; \Omega < \Omega_1, \quad (3.47) $$

$$ k_{r2}(\omega_o) = b_{k5}(\omega_o/2\pi)^5 + b_{k4}(\omega_o/2\pi)^4 + b_{k3}(\omega_o/2\pi)^3 + b_{k2}(\omega_o/2\pi)^2 + b_{k1}(\omega_o/2\pi) + b_{k0}, \; \Omega \geq \Omega_1, \quad (3.48) $$

$$ \Gamma_{\Omega_1}(\omega_o) = 0.5\left\{ -\tanh\left[ \sigma_1(\omega_o/2\pi - \Omega_1) \right] + 1 \right\}, \quad (3.49) $$

$$ \Gamma_{\Omega_2}(\omega_o) = 0.5\left\{ \tanh\left[ \sigma_2(\omega_o/2\pi - \Omega_2) \right] + 1 \right\}. \quad (3.50) $$

Here, $k_{r1}(\omega_o)$ and $k_{r2}(\omega_o)$ are the curve-fits of $\lambda_{kr}(\omega_o,X_1)$ in the range of $\Omega < \Omega_1$ and $\Omega \geq \Omega_1$, respectively, and, $\Gamma_{\Omega_1}(\omega_o)$ and $\Gamma_{\Omega_2}(\omega_o)$ are the smoothening functions over $\Omega < \Omega_1$ and $\Omega \geq \Omega_1$ regimes, respectively. The smoothening factors $\sigma_v$ ($v = 1 - 2$) are selected as $\sigma_1 = \sigma_2 = 1 \times 10^6$. In our study, the coefficients $a_{kv}$ ($v = 0, 1$) and $b_{kv}$ ($v = 0 - 5$) are determined corresponding to each excitation $X$. For example, $a_{k1} = 0.045, 0.038, 0.036, 0.034, 0.0324, 0.033$, and 0.030 with $X = 0.15, 0.25, 0.5, 0.75, 1.0, 1.25$, and 1.5 mm respectively; $a_{k0} = 2.03, 2.02, 1.96, 1.92, 1.89, 1.86$, and 1.85 with $X = 0.15, 0.25, 0.5,$
Figure 3.7 Empirical parameters of the rubber path at $X = 0.15$ mm. Here magnitudes of $\lambda_{kr}(\omega, X_1)$ and $\lambda_{cr}(\omega, X_1)$ are displayed along with the smoothening functions, $\Gamma_{\Omega_v}(\nu)$ ($\nu = 1, 2$). The frequency regimes are separated at $\Omega_1 = 2.5$ Hz. Key for $\Gamma_{\Omega_v}$ curves: $\Gamma_{\Omega_1}(\omega)$; $\Gamma_{\Omega_2}(\omega)$. 

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Likewise, 6 sets of coefficients for $b_{kv} (v = 0 - 5)$ are estimated corresponding to $X = 0.15, 0.25, 0.5, 0.75, 1.0, 1.25, \text{ and } 1.5 \text{ mm}$. Thus, another set of curve-fits with respect to $X_I$ is used as follows:

\[
a_{kv}(X_I) = \sum_{w=1}^{7} a_{k_w} X_I^{w-1} (v = 0, 1), \tag{3.51}
\]

\[
b_{kv}(X_I) = \sum_{w=1}^{7} b_{k_w} X_I^{w-1} (v = 0 - 5), \tag{3.52}
\]

\[
a_{kl}(X_I) = a_{k_{16}} X_I^6 + a_{k_{15}} X_I^5 + a_{k_{14}} X_I^4 + a_{k_{13}} X_I^3 + a_{k_{12}} X_I^2 + a_{k_{11}} X_I + a_{k_{10}}. \tag{3.53}
\]

Here, $a_{kv}(X_I)$ and $b_{kv}(X_I)$ are the coefficient curve-fits for $a_{kv} (v = 1, 0)$ and $b_{kv} (v = 0 - 5)$ with the function of $X_I$ respectively. Eqn. (3.53) describes a typical example of coefficient curve-fit as a function of $X$. Sample coefficients $a_{klw-1}$ are as follows: $a_{16} = 0.098$; $a_{15} = -0.596$; $a_{14} = 1.411$; $a_{13} = -1.652$; $a_{12} = 1.003$; $a_{11} = -0.304$; $a_{10} = 0.073$.

Therefore, the overall procedure to determine the spectrally-varying and amplitude-sensitive parameters for $\lambda_{kr}(\omega_o, X_I)$ is described as follows:

\[
k_{rc}(\omega_o, X_I) = k_{rc} \lambda_{kr}(\omega_o, X_I), \tag{3.54}
\]

\[
\lambda_{kr}(\omega_o, X_I) = k_{\Omega_1}(\omega_o, X_I) + k_{\Omega_2}(\omega_o, X_I), \tag{3.55}
\]

\[
k_{\Omega_1}(\omega_o, X_I) = k_{r_1}(\omega_o, X_I) \Gamma_{\Omega_1}(\omega_o), \tag{3.56}
\]

\[
k_{\Omega_2}(\omega_o, X_I) = k_{r_2}(\omega_o, X_I) \Gamma_{\Omega_2}(\omega_o), \tag{3.57}
\]

\[
k_{r_1}(\omega_o, X_I) = a_{k_1}(X_I)(\omega_o / 2\pi) + a_{k_0}(X_I), \tag{3.58}
\]

\[
k_{r_2}(\omega_o, X_I) = b_{k_1}(X_I)(\omega_o / 2\pi)^2 + b_{k_4}(X_I)(\omega_o / 2\pi)^4 + b_{k_3}(X_I)(\omega_o / 2\pi)^3 + b_{k_2}(X_I)(\omega_o / 2\pi) + b_{k_0}(X_I). \tag{3.59}
\]
Here, $k_{\Omega}(\omega_o, X_1) \ (v = 1, 2)$ is the smoothened function of $k_r(\omega_o, X_1) \ (v = 1, 2)$ and $k_v(\omega_o, X_1) \ (v = 1, 2)$ is the polynomial curve-fit in the relevant frequency range. The coefficients $a_{kv} \ (v = 0, 1)$ and $b_{kv} \ (v = 0 - 5)$ are described in Eqns. (3.51)-(3.53). Similarly, $\lambda_{cr}(\omega_o, X_1) = c_{re}(\omega_o, X_1)/c_{rn}$ is estimated as follows. Note that $\lambda_{cr}(\omega_o, X_1)$ is now considered on a $\log_e$ scale, as indicated below, since the damping varies over a large range:

$$c_{re}(\omega_o, X_1) = c_{rn} e^{\lambda_{fr}(\omega_o, X_1)}, \quad (3.60)$$

$$\lambda_{cr}(\omega_o, X_1) = c_{\Omega 1}(\omega_o, X_1) + c_{\Omega 2}(\omega_o, X_1), \quad (3.61)$$

$$c_{\Omega 1}(\omega_o, X_1) = c_{r 1}(\omega_o, X_1) \Gamma_{\Omega 1}(\omega_o), \quad (3.62)$$

$$c_{\Omega 2}(\omega_o, X_1) = c_{r 2}(\omega_o, X_1) \Gamma_{\Omega 2}(\omega_o), \quad (3.63)$$

$$c_{r 1}(\omega_o, X_1) = a_{c 1}(X_i)(\omega_o / 2\pi) + a_{c 0}(X_i), \quad (3.64)$$

$$c_{r 2}(\omega_o, X_1) = b_{c 5}(X_i)(\omega_o / 2\pi)^5 + b_{c 4}(X_i)(\omega_o / 2\pi)^4 + b_{c 3}(X_i)(\omega_o / 2\pi)^3$$
$$+ b_{c 2}(X_i)(\omega_o / 2\pi)^2 + b_{c 1}(X_i)(\omega_o / 2\pi) + b_{c 0}(X_i). \quad (3.65)$$

Likewise, $c_{\Omega 1}(\omega_o, X_1) \ (v = 1, 2)$ is the smoothened function of $c_{r 1}(\omega_o, X_1) \ (v = 1, 2)$ and $c_{r 2}(\omega_o, X_1) \ (v = 1, 2)$ is the polynomial curve-fit in the relevant frequency range. Again, $a_{cv} \ (v = 0, 1)$ and $b_{cv} \ (v = 0 - 5)$ are also calculated using the method described by Eqns. (3.51)-(3.53). Finally, the estimated $\lambda_{kr}(\omega_o, X_1)$ and $\lambda_{cr}(\omega_o, X_1)$ are embedded in the quasi-linear model at the fundamental harmonic term ($\omega_o$).
3.6.3 Effective hydraulic force path parameter $\tilde{\lambda}_u(n\omega_o, X_n)$ at super-harmonics

The effective dynamic compliance $\tilde{C}_{ue}(n\omega_o, X_n)$ is next evaluated at the super-harmonics ($n\omega_o$). Note that the term $X_n$ should be viewed as the ‘virtual’ excitation displacement amplitude at $n\omega_o$. Even though $x(t)$ is close to a pure sinusoid, super-harmonic amplitudes $X_n$ are observed via the FFT analysis of $x_m(t)$ signals even though their amplitudes are several orders of magnitudes lower, as listed in Table 3.1. These can be employed to estimate $\tilde{C}_{ue}(n\omega_o, X_n)$ in terms of $\tilde{\lambda}_u(n\omega_o, X_n)$ ($\alpha_n + i\beta_n$) as follows

where $C_{un}$ is the nominal (static) value:

$$\tilde{C}_{ue}(n\omega_o, X_n) = C_{un}\tilde{\lambda}_u(n\omega_o, X_n) = C_{un}(\alpha_n + i\beta_n), \quad (3.66)$$

$$\alpha_n = \alpha(n\omega_o, X_n), \quad \beta_n = \beta(n\omega_o, X_n), \quad (n = 1, 2, 3, \cdots). \quad (3.67)$$

From Eqns. (3.9)-(3.13), the relationship between $X(s)$ and $P_u(s)$ is derived as follows for the fixed decoupler mount (with $I_d \approx 0$ and $R_d \rightarrow \infty$) under the steady state condition. The complex valued terms $\tilde{P}_u(n\omega_o, X_n)$ at $n = 1, 2, 3, \cdots$ are expressed in the frequency domain (with multi-harmonic terms) by replacing $s$ with $i(n\omega_o)$:

$$\tilde{P}_u(n\omega_o, X_n) = \frac{A_r\left[\left(1-(n\omega_o)^2\right)C_iI_i\right] + i(n\omega_o)C_iR_i}{\left(C_u+C_i-(n\omega_o)^2C_uC_iI_i\right)+i(n\omega_o)C_uC_iR_i}X_n, \quad (n = 1, 2, 3, \cdots). \quad (3.68)$$

From Eqns. (3.66) and (3.67), $\tilde{\lambda}_u(n\omega_o, X_n)$ can be related in terms of $\alpha_n$ and $\beta_n$ in the frequency domain at the $n\omega_o$ terms, and this step leads to:
\[
\tilde{P}_u(n\omega_o, X_n) = \frac{A_r \left[ \left(1-(n\omega_o)^2 C_i I_i \right) + i(n\omega_o) C_i R_i \right] X_n}{\lambda_u C_{un} + C_i (n\omega_o)^2 C_i I_i (\lambda_u C_{un}) + i(n\omega_o) C_i R_i (\lambda_u C_{un})} \\
= \frac{A_r \left[ \left(1-(n\omega_o)^2 C_i I_i \right) + i(n\omega_o) C_i R_i \right] X_n}{(\alpha_n + i\beta_n) C_{un} + C_i (n\omega_o)^2 C_i I_i (\alpha_n + i\beta_n) C_{un} + i(n\omega_o) C_i R_i (\alpha_n + i\beta_n) C_{un}} \tag{3.69}
\]

The measured \( \tilde{P}_{uM}(n\omega_o, X_n) \) signal is considered as a complex quantity and given by magnitude \( |\tilde{P}_{uMn}| \) and phase \( \phi_{PMn} \) as:

\[
\tilde{P}_{uMn} = P_{uREn} + iP_{uIMn}, \tag{3.70}
\]

\[
P_{uREn} = Re \left[ \tilde{P}_{uMn} \right] = |\tilde{P}_{uMn}| \cos(\phi_{PMn}), \tag{3.71}
\]

\[
P_{uIMn} = Im \left[ \tilde{P}_{uMn} \right] = |\tilde{P}_{uMn}| \sin(\phi_{PMn}). \tag{3.72}
\]

From Eqns. (3.69) and (3.70), the empirical coefficients \( \alpha_n \) and \( \beta_n \) are determined at \( n\omega_o \) and \( X_n \) as follows:

\[
P_{uREn} + iP_{uIMn} = \frac{A_r \left[ \left(1-(n\omega_o)^2 C_i I_i \right) + i(n\omega_o) C_i R_i \right] X_n}{(\alpha_n + i\beta_n) C_{un} + C_i (n\omega_o)^2 C_i I_i (\alpha_n + i\beta_n) C_{un} + i(n\omega_o) C_i R_i (\alpha_n + i\beta_n) C_{un}} \tag{3.73}
\]

\[
\alpha_n = \frac{\alpha_{n4} (n\omega_o)^4 + \alpha_{n2} (n\omega_o)^2 + \alpha_{n0}}{d_{n4} (n\omega_o)^4 + d_{n2} (n\omega_o)^2 + d_{n0}}, \tag{3.74}
\]

\[
\beta_n = \frac{\beta_{n4} (n\omega_o)^4 + \beta_{n2} (n\omega_o)^2 + \beta_{n0}}{d_{n4} (n\omega_o)^4 + d_{n2} (n\omega_o)^2 + d_{n0}}, \tag{3.75}
\]

\[
\alpha_{n4} = X_n A_r C_i^2 I_i^2 P_{uREn}, \tag{3.76}
\]

\[
\alpha_{n2} = X_n A_r P_{uREn} C_i \left( C_i R_i^2 - 2I_i \right) + C_i^2 \left( P_{uREn}^2 + P_{uIMn}^2 \right) I_i, \tag{3.77}
\]
\[ \alpha_{n0} = X_n A_i P_{uREn} - C_1 \left( P_{uREn}^2 + P_{uIMn}^2 \right), \quad (3.78) \]

\[ \beta_{n4} = -X_n A_i C_i^2 I_i^2 P_{uIMn}, \quad (3.79) \]

\[ \beta_{n2} = X_n A_i P_{uIMn} C_1 \left( 2I_i - C_i R_i^2 \right), \quad (3.80) \]

\[ \beta_{n1} = C_i^2 \left( P_{uREn}^2 + P_{uIMn}^2 \right) R_i, \quad (3.81) \]

\[ \beta_{n0} = -X_n A_i P_{uIMn}, \quad (3.82) \]

\[ d_{n4} = C_{un} C_i^2 I_i^2 \left( P_{uREn}^2 + P_{uIMn}^2 \right), \quad (3.83) \]

\[ d_{n2} = C_{un} C_i \left( C_i R_i^2 - 2I_i \right) \left( P_{uREn}^2 + P_{uIMn}^2 \right), \quad (3.84) \]

\[ d_{n0} = C_{un} \left( P_{uREn}^2 + P_{uIMn}^2 \right). \quad (3.85) \]

Here, the subscripts \( RE \) and \( IM \) designate the real and imaginary numbers, respectively.

Similarly, expressions for the free decoupler mount (with assumptions such as \( I_d \approx 0 \) below 50 Hz) are derived leading to:

\[ \overline{P}_u(n\omega_0, X_n) = \frac{X_n A_i \left[ \left( -\omega_0^2 \right) C_i I_i R_d + \omega_0 \left( \omega_0 C_i R_i R_d + I_i \right) \right]}{\omega_0^2 C_u C_i I_i R_d + \omega_0 \left[ \omega_0 C_u C_i R_i R_d + (C_u + C_i) I_i \right] + (C_u + C_i) (R_i + R_d)}, \quad (3.86) \]

\[ \alpha_n = \frac{\alpha_{n4} (n\omega_0)^4 + \alpha_{n2} (n\omega_0)^2 + \alpha_{n0}}{d_{n4} (n\omega_0)^4 + d_{n2} (n\omega_0)^2 + d_{n0}}, \quad (3.87) \]

\[ \beta_n = \frac{\beta_{n4} (n\omega_0)^4 + \beta_{n3} (n\omega_0)^3 + \beta_{n2} (n\omega_0)^2 + \beta_{n1} (n\omega_0) + \beta_{n0}}{d_{n4} (n\omega_0)^4 + d_{n2} (n\omega_0)^2 + d_{n0}}, \quad (3.88) \]

\[ \alpha_{n4} = X_n A_i P_{uREn} C_{un} C_i^2 I_i^2 R_d^2, \quad (3.89) \]

\[ \alpha_{n2} = X_n A_i P_{uREn} C_{un} C_i^2 I_i^2 R_d^2 + \left[ C_i \left( P_{uREn}^2 + P_{uIMn}^2 \right) - 2X_n A_i P_{uREn} \right] C_{un} C_i R_d^2 \]

\[ + \left[ X_n A_i P_{uREn} - C_i \left( P_{uREn}^2 + P_{uIMn}^2 \right) \right] C_{un} I_i^2, \quad (3.90) \]
\[ \alpha_{n\ell} = C_{un} \left[ X_n A, P_{uREn} - C_{i} \left( P_{uREn}^2 + P_{uIMn}^2 \right) \right] \left( R_i + R_d \right)^2, \tag{3.91} \]

\[ \beta_{n\ell} = -X_n A, P_{uIMn} C_{un} C_{i} I_d^2 R_d^2, \tag{3.92} \]

\[ \beta_{n3} = C_{un} C_{i} \left( P_{uREn}^2 + P_{uIMn}^2 \right) I_d^2 R_d^2, \tag{3.93} \]

\[ \beta_{n2} = -X_n A, P_{uIMn} C_{un} \left[ C_{i} R_i^2 R_d^2 - 2C_{i} I_d^2 R_d^2 + I_d^2 \right], \tag{3.94} \]

\[ \beta_{n1} = C_{un} C_{i}^2 R_i R_d \left( P_{uREn}^2 + P_{uIMn}^2 \right) \left( R_i + R_d \right), \tag{3.95} \]

\[ \beta_{n0} = -X_n A, P_{uIMn} C_{un} \left( R_i + R_d \right)^2, \tag{3.96} \]

\[ d_{n4} = \left( P_{uREn}^2 + P_{uIMn}^2 \right) C_{un} C_{i} I_d^2 R_d^2, \tag{3.97} \]

\[ d_{n2} = C_{un} \left( P_{uREn}^2 + P_{uIMn}^2 \right) \left[ C_{i} R_i^2 R_d^2 - 2C_{i} I_d^2 R_d^2 + I_d^2 \right], \tag{3.98} \]

\[ d_{n0} = C_{un} \left( P_{uREn}^2 + P_{uIMn}^2 \right) \left( R_i + R_d \right)^2. \tag{3.99} \]

Figure 3.8 compares the magnitude spectra of \( \tilde{\lambda}_u(n\omega_o, X_n) \) based upon Eqns. (3.66), (3.67), (3.74)-(3.85), and (3.87)-(3.99) up to \( n = 3 \) term with given excitation \( X = 1.5 \text{ mm} \). When the results of both fixed and free decouplers are compared with those with the fundamental \( \omega_o \) only, dynamic characteristics seem to show similar responses. But, the magnitudes of 2\(^{nd}\) and 3\(^{rd}\) harmonics show mount specific behavior. This illustrates the importance of nonlinear dynamics. For instance, the \( \tilde{\lambda}_u(n\omega_o, X_n) \) spectra for the free decoupler mount show higher magnitudes around 20 Hz for the 3\(^{rd}\) harmonic term, when compared with results at \( \omega_o \). This is directly related to the switching mechanism as well as the nonlinear upper chamber compliance [3.20,3.21].
Figure 3.8 Effective parameter for the upper chamber compliance, displayed in terms of the magnitude spectra of $20\log_e |\tilde{\lambda}_u(n\omega_r, X_n)| \ p_{\lambda_u} = 1.0\ dB$ with 3 harmonic terms given $X = 1.5\ mm$: (a) fixed decoupler mount; (b) free decoupler mount. Key: $\square$, $n = 1$; $\circ$, $n = 2$; $\blacksquare$, $n = 3$. 
3.7 Examination of Super-Harmonics and Estimation of Forces in Time Domain

3.7.1 Examination of super-harmonics in measured data

Typical $p_{uM}(t)$ and $f_{TM}(t)$ measurements for the free decoupler mount at 8.5 Hz (with $X = 1.5$ mm) are shown in Figures 3.9 (a), (b) and 3.10 (a). Spectral contents are determined by the FFT algorithm as shown in Figure 3.10 (b). Even though the fundamental $\omega_o$ term (8.5 Hz) is quite dominant, responses reveal several $n\omega_o$ terms (especially the second harmonic at 17 Hz and third harmonic at 25.5 Hz). Also, spectral contents of $f_{TM}(t)$ are similar to those of $p_{uM}(t)$. Therefore, the super-harmonic terms in $f_{TM}(t)$ can be assumed to be controlled by the nonlinear characteristics of the hydraulic path.

Figures 3.11 and 3.12 map the magnitude spectra of $\tilde{P}_{uM}(n\omega_o,X)$ and $\tilde{F}_{TM}(n\omega_o,X)$ respectively up to 3rd harmonic term for the fixed decoupler mount up to 50 Hz with $X = 1.5$ mm. When the contents at $\omega_o$ are compared with those at $n\omega_o$ in Figures 3.11 (a) and (b), the magnitude of the $\omega_o$ term is dominant as observed in Figure 3.10 (b) as well. Also, the super-harmonic contents of $\tilde{F}_{TM}(n\omega_o,X)$ seem to match the $\tilde{P}_{uM}(n\omega_o,X)$ spectra in Figures 3.11 (a), (b) and 3.12. This is further investigated in the time domain by marking key events in Figure 3.10 (a) as “A” and “B”. Note that the fundamental $\tilde{F}_{TM}(\omega_o,X)$ term is also dependent upon the initial conditions.
Figure 3.9 $p_{uM}(t)$ and $f_{TM}(t)$ for the free decoupler mount given $X = 1.5$ mm, $\phi_x = 68.7^\circ$ and $\omega_o/2\pi = 8.5$ Hz: (a) time history of $p_{uM}(t)$; (b) time history of $f_{TM}(t)$. Key: ..., $x_M(t)$; --, $p_{uM}(t)$ and $f_{TM}(t)$; $T_o$, period ($= 2\pi/\omega_o$).
Figure 3.10 Measured pressure (and force) time histories and spectra for the free decoupler mount given \(X = 1.5\) mm, \(\phi = 68.7^\circ\) and \(\omega_o/2\pi = 8.5\) Hz: (a) time history of \(p_{uM}(t)\) and \(f_{TM}(t)\); (b) magnitude spectra \((\tilde{P}_{uM} \text{ and } \tilde{F}_{TM})\). Key for part (a): ---, \(p_{uM}(t)\); ---, \(f_{TM}(t)\); \(T_o\), period \((=2\pi/\omega_o)\). Key for part (b): \(\times\), \(\tilde{F}_{TM}\); \(\otimes\), \(\tilde{P}_{uM}\).
Figure 3.11 Magnitude spectra of $\tilde{P}_{uM}$ and $\tilde{F}_{TM}$ for the fixed decoupler mount given $X = 1.5$ mm: (a) magnitude of $\tilde{P}_{uM}$ with 4 harmonic terms; (b) magnitude of $\tilde{F}_{TM}$ with 4 harmonic terms. Key: ■, 1st harmonic; ■, 2nd harmonic; ■, 3rd harmonic; ■, 4th harmonic.
Figure 3.12 Magnitude spectra of $\tilde{P}_{uM}$ and $\tilde{F}_{TM}$ for the fixed decoupler mount given $X = 1.5$ mm. Key: – , $2^{nd}$ harmonic of $\left|\tilde{F}_{TM}(\omega, X)\right|$; – , $2^{nd}$ harmonic of $\left|\tilde{P}_{uM}(\omega, X)\right|$; · , $3^{rd}$ harmonic of $\left|\tilde{F}_{TM}(\omega, X)\right|$; · , $3^{rd}$ harmonic of $\left|\tilde{P}_{uM}(\omega, X)\right|$. 
3.7.2 Indirect force estimation given measured $X_{Mn}(n\omega_o)$ and $\tilde{P}_{uMn}(n\omega_o,X_n)$

Based on the experimental data, we assume that the $n\omega_o$ contents of $f_T(t)$ and $\tilde{F}_T(n\omega_o,X)$ are directly affected by the corresponding terms of $p_u(t)$ and $\tilde{P}_{uM}(n\omega_o,X)$. Thus, all $n\omega_o$ terms are lumped into the hydraulic path. This premise is employed for estimating $\tilde{C}_{ue}(n\omega_o,X)$. Further, the Fourier series expansion is utilized by employing the reverse path spectral method that has been well described by Richards and Singh [3.26], and Bendat [3.27].

Figure 3.6 illustrates the procedure used to estimate the dynamic forces given measured $x_M(t)$ and $p_{uM}(t)$ signals under harmonic excitation at $\omega_o$. The $x_M(t)$ and $p_{uM}(t)$ time histories are then transformed into $X_{M\omega_o}(n\omega_o)$ and $\tilde{P}_{uM\omega_o}(n\omega_o,X_{Mn})$ respectively, where $X_{M\omega_o}(n\omega_o)$ is viewed as the ‘virtual’ sinusoidal input and $\tilde{P}_{uM\omega_o}(n\omega_o,X_{Mn})$ includes relevant $n\omega_o$ terms. Second, effective parameters from both rubber and hydraulic paths are identified, as illustrated in Figure 3.13, by employing the reverse path spectral method [3.26,3.27]. For example, $\tilde{C}_{ue}(n\omega_o,X_{Mn})$ is calculated by Eqns. (3.66)-(3.99) for both fixed and free decoupler mounts. When the measured amplitudes $\tilde{P}_{uM\omega_o}(n\omega_o,X_{Mn})$ are employed in Eqns. (3.66)-(3.99), $\tilde{C}_{ue}(n\omega_o,X_{Mn})$ should be given at $n\omega_o$ such as $\tilde{C}_{ue}(\omega_o,X_{M1})$ from $\tilde{P}_{uM1}(\omega_o,X_{M1})$ and $\tilde{C}_{ue}(2\omega_o,X_{M2})$ from $\tilde{P}_{uM2}(2\omega_o,X_{M2})$.

Next, the quasi-linear transfer functions $\tilde{G}_e(n\omega_o,X_{Mn})=(k_{rref}/A_e)\tilde{G}(n\omega_o,X_{Mn})$ are used, as described in Figure 3.13 (a), at $n\omega_o$ terms corresponding to $X_{M\omega_o}(n\omega_o)$ and
\( \tilde{P}_{uMn}(n\omega_o, X_{Mn}) \). Also, as illustrated in Figure 3.13 (b), \( k_{re}(\omega_o , X_I) \) and \( c_{re}(\omega_o , X_I) \) are identified by using \( X_{Mf}(\omega_o) \) at \( \omega_o \) as described by Eqns. (3.54)-(3.65). Further, other transfer functions such as \( \tilde{K}(n\omega_o , X_n) = k_{ref} \tilde{K}(n\omega_o , X_n) \) and \( \tilde{H}(n\omega_o , X_n) = A_{ref} \tilde{H}(n\omega_o , X_n) \) from Eqns. (3.14)-(3.33) are now used to estimate the dynamic forces at \( n\omega_o \) terms. The overall procedure with effective parameters \( k_{re}(\omega_o , X_I) \), \( c_{re}(\omega_o , X_I) \) and \( \tilde{C}_{ue}(n\omega_o , X_n) \) is illustrated in Figures 3.6 and 3.14. In particular, Figure 3.14 describes the reverse path spectral method which employs effective parameters of Figure 3.13. Thus, \( \tilde{K}_e(n\omega_o , X_n) \) and \( \tilde{H}_e(n\omega_o , X_n) \) assume effective values at each \( n\omega_o \) term since both are affected by \( \tilde{C}_{ue}(n\omega_o , X_n) \).

### 3.8 Force Estimations with Quasi-Linear Models

Alternate force estimation schemes are summarized in Table 3.2, especially for the quasi-linear (QL) model that depends upon the empirical parameters \( \lambda_{kr}(\omega_o , X_I) \), \( \lambda_{cr}(\omega_o , X_I) \), and \( \tilde{\lambda}_u(n\omega_o , X_n) \). When \( \lambda_{kr}(\omega_o , X_I) = 1 \), \( \lambda_{cr}(\omega_o , X_I) = 1 \), and \( \tilde{\lambda}_u(n\omega_o , X_n) = 1 \), then the model is obviously linear time-invariant (LTI) with nominal values of \( k_r \), \( c_r \), and \( C_u \) as described earlier in sections 3.2 and 3.3. The LTI model (with calculated \( x(t) \) and \( p_d(t) \)) is designated as I-A in Table 3.2; note that it is equivalent to the ‘simple prediction model’ that employs measured \( x_M(t) \) and \( p_{uM}(t) \). As illustrated in Table 3.2, many different calculation schemes can be designed based on the combination of \( \lambda_{kr}(\omega_o , X_I) \), \( \lambda_{cr}(\omega_o , X_I) \) and \( \tilde{\lambda}_u(n\omega_o , X_n) \) along with applicable transfer functions such as
Figure 3.13 Identification of effective parameters for the reverse path spectral method:

(a) identification of effective upper chamber compliance ($\tilde{C}_{ue}$) with multi-harmonic terms; (b) identification of effective stiffness ($k_{re}$) and damping ($c_{re}$) with fundamental harmonic term. Key: $\tilde{K}_R(\omega_o, X_{M1})$, dynamic stiffness from the rubber path; $\tilde{F}_R(\omega_o, X_{M1})$, rubber path force.
Figure 3.14 Force estimation in frequency domain using the reverse path spectral method: (a) dynamic stiffness ($\tilde{K}_e$) concept; (b) force transmissibility ($\tilde{H}_e$) concept.
<table>
<thead>
<tr>
<th>Model and Scheme Designation</th>
<th>Sensor(s) or Variables required</th>
<th>Spectrally-varying and amplitude-sensitive parameters</th>
<th>Harmonic content</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Simple prediction model given measured data</strong></td>
<td>Yes</td>
<td>$\tilde{x}<em>k = 1$, $\tilde{x}</em>{cr} = 1$</td>
<td>$\omega_o$ only</td>
</tr>
<tr>
<td><strong>Nonlinear Model (NL)</strong></td>
<td>Yes</td>
<td>$\tilde{x}<em>k = 1$, $\tilde{x}</em>{cr} = 1$</td>
<td>$n\omega_o$ ($n = 1,2,3 \cdots$)</td>
</tr>
<tr>
<td><strong>Linear time-invariant Model (LTI)</strong></td>
<td><strong>I-A</strong></td>
<td>Yes</td>
<td>$\tilde{x}<em>k = 1$, $\tilde{x}</em>{cr} = 1$</td>
</tr>
<tr>
<td></td>
<td><strong>I-B</strong></td>
<td>Yes</td>
<td>$\tilde{x}<em>k \neq 1$, $\tilde{x}</em>{cr} \neq 1$</td>
</tr>
<tr>
<td></td>
<td><strong>II-A</strong></td>
<td>Yes</td>
<td>$\tilde{x}<em>k = 1$, $\tilde{x}</em>{cr} = 1$, $\tilde{x}_u \neq 1$</td>
</tr>
<tr>
<td></td>
<td><strong>II-B</strong></td>
<td>Yes</td>
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</tr>
<tr>
<td></td>
<td><strong>II-C</strong></td>
<td>Yes</td>
<td>$\tilde{x}<em>k \neq 1$, $\tilde{x}</em>{cr} \neq 1$, $\tilde{x}_u \neq 1$</td>
</tr>
<tr>
<td></td>
<td><strong>III-A</strong></td>
<td>No</td>
<td>$\tilde{x}<em>k = 1$, $\tilde{x}</em>{cr} = 1$, $\tilde{x}_u \neq 1$</td>
</tr>
<tr>
<td></td>
<td><strong>III-B</strong></td>
<td>No</td>
<td>$\tilde{x}<em>k \neq 1$, $\tilde{x}</em>{cr} \neq 1$, $\tilde{x}_u = 1$</td>
</tr>
<tr>
<td></td>
<td><strong>III-C</strong></td>
<td>No</td>
<td>$\tilde{x}<em>k \neq 1$, $\tilde{x}</em>{cr} \neq 1$, $\tilde{x}_u \neq 1$</td>
</tr>
<tr>
<td><strong>Quasi-Linear Model (QL)</strong></td>
<td><strong>IV-A</strong></td>
<td>Yes</td>
<td>$\tilde{x}<em>k = 1$, $\tilde{x}</em>{cr} = 1$, $\tilde{x}_u = 1$</td>
</tr>
<tr>
<td></td>
<td><strong>IV-B</strong></td>
<td>Yes</td>
<td>$\tilde{x}<em>k \neq 1$, $\tilde{x}</em>{cr} \neq 1$, $\tilde{x}_u \neq 1$</td>
</tr>
</tbody>
</table>

Table 3.2 Overview of linear, quasi-linear and nonlinear models used to estimate dynamic forces.
\( \tilde{K}_e(n\omega_o, X_n) \) for schemes II-A, II-B, and II-C, \( \tilde{H}_e(n\omega_o, X_n) \) for schemes III-A, III-B, and III-C, or \( \tilde{K}_e(n\omega_o, X_n) \) for schemes IV-A and IV-B. Finally, observe that the nonlinear model \( (NL) \) employs nominal nonlinear parameters as described in section 3.5, and thus effective (frequency domain) parameters are not considered in this formulation.

In order to estimate \( f_T(t) \) from a transfer function \( \tilde{K}_e \) or \( \tilde{H}_e \) with effective \( k_{re}, c_{re} \) and \( \tilde{C}_{ue} \), the Fourier series expansion is employed for particular QL schemes as illustrated below. Assuming that the mount is excited under steady state condition, define the input displacement \( x(t) \) as \( \text{Re}\left[ X_n e^{j\omega_o t}\right] = \text{Re}\left[ X_n e^{j(\omega_o t + \phi_{X_n})}\right] \). Note that the ‘virtual’ sinusoidal inputs at \( n\omega_o \) are included where \( X_n \) and \( \phi_{X_n} \) are the amplitude and phase at the \( n^{th} \) harmonic. First, the dynamic force is estimated at \( n = 1 \) or \( \omega_o \) term as described below where the subscripts \( L, K, \) and \( H \) indicate the LTI model, dynamic stiffness \( \tilde{K}_e(\omega_o, X_I) \), and force transmissibility \( \tilde{H}_e(\omega_o, X_I) \) formulations respectively:

\[
\chi_{KL} = k_{\text{ref}} X_{\text{ref}}, \quad \chi_{HL} = A_P(\omega_o, X_I).
\]

(3.100)

\[
p_u(t) = \text{Re}\left[ P_u e^{j\omega_o t}\right] = \text{Re}\left[ P_u e^{j(\omega_o t + \phi_{P_u})}\right].
\]

(3.101)

\[
f_{TKL}(t) = f_m + \chi_{KL} \left| \tilde{K}_e(\omega_o, X_I) \right| e^{j(\omega_o t + \phi_{KL})}, \quad \phi_{KL}(\omega_o) = \angle \tilde{K}_e(\omega_o, X_I).
\]

(3.102)

\[
f_{THL}(t) = f_m + \chi_{HL} \left| \tilde{H}_e(\omega_o, X_I) \right| e^{j(\omega_o t + \phi_{HL})}, \quad \phi_{HL}(\omega_o) = \angle \tilde{H}_e(\omega_o, X_I).
\]

(3.103)

Here, \( \phi_{P_m} \) is the phase of \( p_u(t) \) at \( n\omega_o \) term, \( \phi_{KL} \) is the phase of \( \tilde{K}_e(\omega_o, X_I) \), and \( \phi_{HL} \) is the phase of \( \tilde{H}_e(\omega_o, X_I) \). The above models include measured \( X_M \) and \( \tilde{P}_{uM} \) contents.
at \( \omega_o \) with \( X_l \). Next, the time domain force is constructed by using \( QL \) models at relevant \( n \) terms as follows:

\[
\begin{align*}
    f_{TKQ}(t) &= f_m + f_{TKQ1}(t) + f_{TKQ2}(t) + \cdots + f_{TKQZ}(t), \\
    f_{TKQ1}(t) &= \chi_{KL} \mathcal{K}e(\omega_o, X_1) \left[ \text{Re} \left[ e^{i(\omega t + \phi_{1})} + \phi_{KL}(\omega_o, X_1) \right] \right], \\
    f_{TKQ2}(t) &= \chi_{KL} \mathcal{K}e(2\omega_o, X_2) \left[ \text{Re} \left[ e^{i(2\omega t + \phi_{2})} + \phi_{KL}(2\omega_o, X_2) \right] \right], \\
    f_{TKQZ}(t) &= \chi_{KL} \mathcal{K}e(Z\omega_o, X_Z) \left[ \text{Re} \left[ e^{i(Z\omega t + \phi_{Z})} + \phi_{KL}(Z\omega_o, X_Z) \right] \right], \\
    \phi_{KL}(n\omega_o, X_n) &= \angle \mathcal{K}e(n\omega_o, X_n), \ (n = 1, 2, \cdots, Z).
\end{align*}
\]

Summing up Eqns. (3.105)-(3.108), the total force is defined as:

\[
\begin{align*}
    f_{TKQ}(t) &= f_m + \chi_{KL} \sum_{n=1}^{Z} \mathcal{K}e(n\omega_o, X_n) \left[ \text{Re} \left[ e^{i(n\omega t + \phi_{KL}(n\omega_o, X_n))} \right] \right] \\
    f_{TKQ}(t) &= f_m + \sum_{n=1}^{Z} \chi_{KL} \mathcal{K}e(n\omega_o, X_n) \left[ \text{Re} \left[ e^{i(n\omega t + \phi_{KL}(n\omega_o, X_n))} \right] \right], \\
    \chi_{KL}(n\omega_o, X_n) &= A_r \left| P_n(n\omega_o, X_n) \right|, \ (n = 1, 2, \cdots, Z).
\end{align*}
\]

Likewise, the force is alternately estimated by using \( \mathcal{H}e(n\omega_o, X_n) \) as:

\[
\begin{align*}
    f_{THQ}(t) &= f_m + \sum_{n=1}^{Z} \chi_{HL}(n\omega_o, X_n) \left| \mathcal{H}e(n\omega_o, X_n) \right| \left[ \text{Re} \left[ e^{i(n\omega t + \phi_{HL}(n\omega_o, X_n))} \right] \right], \\
    \chi_{HL}(n\omega_o, X_n) &= A_r \left| P_n(n\omega_o, X_n) \right|, \ (n = 1, 2, \cdots, Z).
\end{align*}
\]

Using the same method, the analogous mechanical system model yields the following force:

\[
\begin{align*}
    f_{TKA}(t) &= f_m + \chi_{KL} \sum_{n=1}^{Z} \mathcal{K}a(n\omega_o, X_n) \left[ \text{Re} \left[ e^{i(n\omega t + \phi_{KL}(n\omega_o, X_n))} \right] \right], \\
    \phi_{KL}(n\omega_o, X_n) &= \angle \mathcal{K}a(n\omega_o, X_n), \ (n = 1, 2, \cdots, Z).
\end{align*}
\]
3.9 Results and Discussion

3.9.1 Comparison of models

Our study includes 12 harmonic terms in the quasi-linear (QL) models. Figure 3.15 compares three QL schemes. Here, the LTI (or the simple prediction) model employs measured $x_M(t)$ and $p_{um}(t)$ with $\omega_o$ term only. The QL schemes are based upon Eqns. (1)-(3) and (104)-(111), and are designated by I-B, II-C and III-C. The forces estimated by the QL schemes match well with measured force time histories. Conversely, the mechanical model, with QL scheme IV-C, fails to predict $f_T(t)$ as seen in Figure 3.16. This is due to the $\overline{K}_{ae}$ formulation based upon the system of Figure 3.3. In particular, the numerator of $\overline{K}_{ae}$ does not include any system properties unlike the fluid model. Also, the mechanical system does not properly incorporate the dynamic compliance $\overline{C}_{ue}(n\omega_o, X_n)$ in the $f_T(t)$ expression as observed in Figure 3.3.

Figure 3.17 compares the nonlinear (NL) model with two quasi-linear models (schemes II-C and III-C). Observe the NL model shows some discrepancies in the time domain. To examine the underlying cause, spectral contents are compared in Figure 3.18 for NL and QL models on a logarithmic scale. Specifically, the NL models predict lower magnitudes at $4\omega_o$ and $5\omega_o$ for both fixed and free decouplers; conversely, two QL schemes match experimental data very well.
Figure 3.15 Comparison of LTI and quasi-linear (QL) models with experiment in time domain, given \( x(t) = Re[\bar{X}e^{i\omega_0 t}] \) at \( \omega_0/2\pi = 8.5 \) Hz and \( X = 1.5 \) mm: (a) fixed decoupler; (b) free decoupler. Key: –––, experiment ; –––, LTI (scheme I-A with \( \omega_0 \) term only); ⋄, QL scheme I-B; ⋄, QL scheme II-C; ⋄, QL scheme III-C.
Figure 3.16 Comparison of fluid and analogous mechanical system models with experiment in time domain, given $x(t) = Re\left[\tilde{X}e^{j\omega t}\right]$ at $\omega_o/2\pi = 8.5$ Hz and $X = 1.5$ mm: (a) fixed decoupler; (b) free decoupler. Key: ●, experiment; - , $QL$ scheme II-C; , $QL$ scheme III-C; ..., $QL$ scheme IV-C.
Figure 3.17 Comparison of nonlinear (NL) and quasi-linear (QL) models with experiment in time domain, given \( x(t) = \text{Re}\left[\tilde{X}e^{i\omega t}\right] \) at \( \omega_0/2\pi = 8.5 \text{ Hz} \) and \( X = 1.5 \text{ mm} \): (a) fixed decoupler; (b) free decoupler. Key: , experiment ; , QL scheme II-C; , QL scheme III-C; , nonlinear (NL) model.
Figure 3.18 Comparison of super-harmonics between nonlinear (NL) and quasi-linear (QL) models with experiment given $x(t) = \text{Re} \left[ \tilde{X} e^{i\omega t} \right]$ at $\omega_o/2\pi = 8.5$ Hz and $X = 1.5$ mm: (a) fixed decoupler; (b) free decoupler. Key: $\bullet$, experiment; $\blacksquare$, QL scheme II-C; $\bigcirc$, QL scheme III-C; $\blacklozenge$, nonlinear (NL) model.
3.9.2 Comparison of model errors

The normalized error $\varepsilon(t)$ between measured force $f_{TM}(t)$ and predicted force $f_T(t)$ at any time $t$ is calculated as $\varepsilon(t) = \left( f_T(t) - f_{TM}(t) \right) / f_{TM}(t)$. The overall root-mean-square (RMS in %) error $E$ is then given by $E = 100 \sqrt{\frac{1}{N_{\text{max}}} \sum_{v=1}^{N_{\text{max}}} \left[ \varepsilon(t_v) \right]^2}$ where $N_{\text{max}}$ is the maximum number of points in time domain. Table 3.3 lists $E$ values for all models in Table 3.2. Errors from the $QL$ schemes such as I-B, II-C and III-C are much lower than other models. In particular, the nonlinear model shows more than 10 % error even though it includes four nonlinear expressions. The chief reason is that the nonlinear profiles were measured under the static conditions as thus they do not fully capture the dynamic forces under the sinusoidal excitation conditions.

3.9.3 Comparison of rubber and hydraulic paths

Figures 3.19 and 3.20 compare the quasi-linear models with a combination of $\lambda_b(\omega_0, X_1)$, $\lambda_{cr}(\omega_0, X_1)$, and $\tilde{\lambda}_u(n\omega_0, X_n)$. The results reveal significant discrepancies from the measured force time history. This suggests that $\tilde{\lambda}_u(n\omega_0, X_n)$ is the most important parameter though all must be retained.

Figures 3.21 and 3.22 compare the rubber path force $f_{THb}(t)$ and the hydraulic path force $f_{THh}(t)$ based upon the $QL$ scheme III-C at three excitation frequencies with $X = 1.5$ mm. The peak-to-peak value of $f_{THb}(t)$ is less than $f_{THh}(t)$ below 10 Hz. The $f_{THh}(t)$ is larger than $f_{THb}(t)$ for the fixed decoupler mount between 10 and 20 Hz. Beyond 20 Hz, $f_{THb}(t)$ and
### Model and Scheme Designation

<table>
<thead>
<tr>
<th>Model and Scheme Designation</th>
<th>Fixed decoupler</th>
<th>Free decoupler</th>
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<tr>
<td>RMS error, $E$ in (%)</td>
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<td><strong>Simple prediction model given measured data</strong></td>
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<tr>
<td>$\omega_o$ only</td>
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<td>25.2</td>
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<tr>
<td>with $n \omega_o$ terms</td>
<td>6.9</td>
<td>11.4</td>
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<td>($n = 1, 2, 3 \ldots$)</td>
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<td></td>
</tr>
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<tr>
<td>I-B</td>
<td>1.82</td>
<td>2.56</td>
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<tr>
<td>II-A</td>
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</tr>
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<td>II-B</td>
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<td>II-C</td>
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<td>2.56</td>
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<td>23.0</td>
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<td>III-B</td>
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<td>III-C</td>
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<td></td>
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<tr>
<td>IV-B</td>
<td>23.0</td>
<td>27.3</td>
</tr>
</tbody>
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Table 3.3 Comparison of the model estimation errors at $\omega_o/2\pi = 8.5$ Hz and $X = 1.5$ mm for alternate force estimation schemes.
Figure 3.19 Comparison of alternate quasi-linear (QL) models in time domain for the fixed decoupler mount, given $x(t) = \text{Re}\left[\tilde{X}e^{j\omega t}\right]$ at $\omega_o / 2\pi = 8.5$ Hz and $X = 1.5$ mm:

(a) dynamic stiffness ($\tilde{K}_e$) concept; (b) force transmissibility ($\tilde{H}_e$) concept. Key: —, experiment; --, QL schemes II-C and III-C; ---, QL schemes II-A and III-A; ----, QL schemes II-B and III-B.
Figure 3.20 Comparison of alternate quasi-linear (QL) models in time domain for the free decoupler mount, given sinusoidal displacement \( x(t) = \Re \left[ \tilde{X} e^{i\omega t} \right] \) at \( \omega_o/2\pi = 8.5 \text{ Hz} \) and \( \bar{x} = 1.5 \text{ mm} \): (a) dynamic stiffness (\( \tilde{K}_c \)) concept; (b) force transmissibility (\( \tilde{H}_c \)) concept. Key: , experiment ; , QL schemes II-C and III-C; , QL schemes II-A and III-A; , QL schemes II-B and III-B.
Figure 3.21 Comparison between rubber and hydraulic path forces in time domain for the fixed decoupler mount given sinusoidal displacement $x(t) = Re[\tilde{X}e^{i\omega t}]$ at $X = 1.5$ mm: (a) $\omega_o/2\pi = 8.5$ Hz; (b) $\omega_o/2\pi = 16.5$ Hz; (c) $\omega_o/2\pi = 20.5$ Hz. Key: \textcolor{blue}{-\textcolor{black}{-\textcolor{black}{-}}}, total force $f_{TH}(t)$; \textcolor{red}{-\textcolor{black}{-\textcolor{black}{-}}}, rubber path force $f_{THr}(t)$; \textcolor{black}{\ldots\ldots}, hydraulic path force $f_{THh}(t)$. 
Figure 3.22 Comparison between rubber and hydraulic path forces in time domain for the free decoupler mount given sinusoidal displacement $x(t) = \text{Re}[\tilde{X}e^{i\omega t}]$ at $X = 1.5\text{mm}$: (a) $\omega_o/2\pi = 8.5$ Hz; (b) $\omega_o/2\pi = 16.5$ Hz; (c) $\omega_o/2\pi = 20.5$ Hz. Key: , total force $f_{TH}(t)$; , rubber path force $f_{THr}(t)$; , hydraulic path force $f_{THh}(t)$.
\( f_{THh}(t) \) are almost the same in terms of peak-to-peak values. However, \( f_{THh}(t) \) remains almost sinusoidal at all frequencies, but \( f_{THh}(t) \) consists of multi-harmonic terms as observed in Figures 3.21 and 3.22. Further, the rubber path forces \( f_{THh}(t) \) are similar in fixed and free decoupler mounts, but, the contribution of hydraulic path force \( f_{THh}(t) \) is smaller in the free decoupler mount when compared with the fixed decoupler mount. This is primarily due to the dynamic fluid flow through the decoupler element.

### 3.10 Conclusion

This chapter has proposed new methods to estimate dynamic forces (in both time and frequency domains) that are transmitted by a hydraulic mount under sinusoidal excitation. The super-harmonic contents of measured upper chamber pressure \( p_{uM}(t) \) and force \( f_{TM}(t) \) are examined and correlated. Effective parameters \( k_{re}(\omega_o,X_1) \), \( c_{re}(\omega_o,X_1) \), and \( \tilde{C}_{ue}(n\omega_o,X_n) \) at the fundamental frequency and super-harmonics \( (n\omega_o) \) are quantified for rubber and hydraulic force paths. This leads to the development of an improved quasi-linear model with spectrally-varying and amplitude-sensitive parameters. The rubber force path is considered only at \( \omega_o \), but the hydraulic path compliance \( \tilde{C}_{ue}(n\omega_o,X_n) \) is quantified at \( n\omega_o \) terms. Alternate relevant transfer function formulations with \( n\omega_o \) terms are also examined by employing the Fourier series expansion as well as the reverse path spectral method. The causality problem should be carefully considered before employing such formulations in time domain based on frequency domain measurements [3.22-3.24]. Finally, the hydraulic mount also exhibits a sub-harmonic term response [3.16]. This period-doubling effect and a more refined nonlinear model will be the subject of a future
work. Yet another study [3.10] will incorporate the load sensing device in a real system consisting of a vehicle powertrain and its sub-frame. Overall, the methods of this chapter can be extended to a multi-degree of freedom isolation systems where in-situ dynamic forces must be assessed.
References for chapter 3


3.6 A. Inoue, R. Singh, G. A. Fernandes, Absolute and relative path measures in a discrete system by using two analytical methods, *Journal of Sound and Vibration*


CHAPTER 4

ESTIMATION OF INTERFACIAL FORCES IN A MULTI-DEGREE OF
FREEDOM ISOLATION SYSTEM UNDER SINUSOIDAL EXCITATION

4.1 Introduction

Precise knowledge of the dynamic forces at sub-system junctions or interfaces is of vital interest in the dynamic and vibro-acoustic design of mechanical systems, vehicles, buildings and power plants. In general, it is difficult to install force transducers at the sub-system junctions without altering interfacial or boundary conditions unless there is a mobility mismatch. Thus, indirect measurement or force reconstruction methods must be adopted to estimate dynamic forces [4.1-4.9]. This has been the subject of several recent articles. For instance, Inoue et al. [4.1] and Gunduz et al. [4.2] have suggested new or improved transfer path methods to estimate forces in parallel structural paths of a discrete vibratory system. Yap and Gibbs [4.3] examined the forces at the machine-receiver interface by using the mobility method. Leclere et al. [4.4] assessed the internal loads on bearings with an inverse transfer function method. Carne et al. [4.5] utilized the frequency response function data to indirectly estimate the input force. Jacquelin et al. [4.6] employed a deconvolution technique to reconstruct the dynamic force. Tao et al.
identified excitation force in the center of engine using the velocity amplitude and phase at the mounting points. Liu and Shepard [4.8] compared the truncated singular value decomposition and the Tikhonov filter approaches used to enhance the inverse process. Lin and Chen [4.9] identified the contact stiffness and damping properties of mechanical interfaces. However, most of the available indirect or inverse force estimation methods are valid only for linear time-invariant systems since they employ transfer functions or similar concepts.

In this chapter, we will embed an inherently nonlinear hydraulic mount in a multi-degree of freedom system and then utilize this mount as a load sensing device to estimate interfacial forces (at both ends of the mount). In recent work from the component perspective [4.10, 4.11], we have successfully estimated forces that are transmitted by a hydraulic mount to a rigid base by using linear, quasi-linear and nonlinear models. In this chapter, we adopt a system perspective and construct a laboratory experiment to examine the proof of the concept. The scope is limited to the harmonic interfacial forces in both frequency and time domains, using measured or calculated motions and/or internal pressure signals. The recent articles by Gunduz et al. [4.2] and Yoon and Singh [4.10, 4.11] provide a comprehensive review of the relevant literature on the identification of interfacial forces and hydraulic engine mount models, respectively. Additional review will be reported as we develop material further.
4.2 Problem Formulation

The underlying issues can be illustrated by a generic source-path-receiver system of Figure 4.1. This is a modified version of the systems analyzed by Inoue et al. [4.1] and Gunduz et al. [4.2]. Here, multiple linear and nonlinear isolators (paths) are shown. Path I has two separate force paths; one path is given by linear spring ($k_{ES1}$) and viscous damper ($c_{ES1}$), and the other path incorporates mass $m_1$ that is connected to the source and receiver by two linear springs ($k_{ES11}$ and $k_{ES12}$) and dampers ($c_{ES11}$ and $c_{ES12}$). Path II includes nonlinear spring ($k_{ES2}$) and damper ($c_{ES2}$) along with a nonlinear fluid path that is designated by $p_u(t)$. Also, linear spring and damper (designated as $k_{EG}$ and $c_{EG}$) support the source as well. The receiver is connected to the ground by four linear springs ($k_{SG1}$, $k_{SG2}$, $k_{SG3}$ and $k_{SG4}$) and viscous dampers ($c_{SG1}$, $c_{SG2}$, $c_{SG3}$ and $c_{SG4}$). For the sake of illustration, first assume a linear time-invariant (LTI) system, and define interfacial forces in time domain for path II as follows.

$$f_{IT}(t) = c_{ES2} \dot{\xi}(t) + k_{ES2} \xi(t) + A_r p_u(t).$$  (4.1)

$$f_{IB}(t) = -c_{ES2} \dot{\xi}(t) - k_{ES2} \xi(t) - A_r p_u(t).$$  (4.2)

Here, $f_{IT}(t)$ and $f_{IB}(t)$ are the interfacial forces on the source (subscript IT) and receiver (subscript IB) sides respectively; $A_r$ is the effective piston area in the fluid path; $x_E(t)$ and $x_S(t)$ are the displacements of the source and receiver, respectively; and $\xi(t) = x_E(t) - x_S(t)$ is the relative displacement. By transforming Eqns. (4.1) and (4.2) into Laplace domain ($s$) and assuming that the initial conditions are equal to zero, path forces are:

$$F_{IT}(s) = (c_{ES2} + k_{ES2}) \Xi(s) + A_r P_u(s).$$  (4.3)

$$F_{IB}(s) = -(c_{ES2} + k_{ES2}) \Xi(s) - A_r P_u(s).$$  (4.4)
Figure 4.1 Schematic of nonlinear isolation system in the context of source, path and receiver network. Both paths are assumed to include hydraulic mounts, and thus have parallel force paths. In particular, path II is assumed to possess spectrally-varying and amplitude-sensitive properties.
When Eqns. (4.1)-(4.4) are compared, the top and bottom forces (in path II) are identical except for the sign as long as there are no inertial elements in the fluid path. This obviously is not the case in many practical devices [4.12]. In order to estimate interfacial forces on both source and receiver sides, one must recognize and resolve some difficulties. First, the above equations require a precise knowledge of in-situ parameters such as nonlinear stiffness $k_{ES2}$ and damping $c_{ES2}$ (in the rubber force path), and effective piston area $A_r$ (in the hydraulic force path) [4.13, 4.14]. Second, forces could be estimated by using measured motions and/or pressure signals but then effective (dynamic) stiffness and damping parameters must be known a priori [4.15]. The latter poses special difficulty for both elastomeric and hydraulic isolators [4.13]. For instance, hydraulic engine mounts exhibit spectrally-varying and amplitude-sensitive parameters [4.16].

The specific objectives are as follows. (1) Construct and instrument a laboratory experiment corresponding to Figure 4.1; one load sensing hydraulic mount will be embedded. The system is excited in the vertical direction by a steady state sinusoidal force $f_E(t)$ of frequency $\omega_o$ and motions in other directions are ignored. (2) Conduct experiments under sinusoidal excitation, measure dynamic accelerations (at different points in the system), fluid pressure (in the top chamber of the load sending mount) and forces (at selected interfaces). (3) Develop 2 and 3 degree of freedom (DOF) linear and quasi-linear models (with spectrally-varying and amplitude-sensitive parameters as suggested in prior component studies [4.10, 4.11, 4.16]); this would include determination of the effective parameters including upper and lower chamber compliances. (4) Estimate the interfacial forces at both ends of the load sensing hydraulic
mount by employing mechanical and fluid models of the load sensing mount, and compare them with direct force measurements in frequency and time domains.

4.3 Experiment with a Dynamic Load Sensing Hydraulic Mount

Figure 4.2 illustrates the laboratory experimental setup with powertrain (assembly of engine and transmission) and sub-frame; it is based on the generic multi-degree of freedom isolation system of Figure 4.1. The powertrain is connected to the sub-frame by two hydraulic mounts though it is supported by the third rubber mount that is grounded. The sub-frame is supported by 4 identical elastomeric bushings. The experiment is excited by an electrodynamic shaker that is located on the powertrain. A signal generator is used to generate multiple excitations. One piezoelectric force transducer is between the shaker and powertrain to measure the excitation force $f_E(t)$; it is also used as a reference signal. Here, the excitation force is expressed as $f_E(t) = Re[F_E e^{i\omega_o t}]$, where $F_E$ is the excitation amplitude, $F_E$ is the amplitude of force, $\phi_F$ is the phase of excitation force $f_E(t)$, $\omega_o$ is the excitation (fundamental) frequency (rad/s), and $Re[ ]$ is the real value operator; tilde over a symbol implies that is complex valued. The scope of experiment is limited to 50 Hz with 5 different excitation amplitudes (1, 10, 50, 100, and 150 N).

The following sensors are employed as illustrated in Figure 4.2: (1) two piezoelectric accelerometers are located on top of two hydraulic mounts and two accelerometers are placed on the sub-frame near the hydraulic mounts; (2) two piezoelectric force transducer is are located between hydraulic mounts and sub-frame for
directly measuring the interfacial forces; (3) another piezoelectric force transducer is
located on top of the load sensing hydraulic mount to directly measure the interfacial
forces on the source side; and (4) one piezoelectric pressure transducer is installed within
the upper chamber of the load sensing hydraulic mount.

In order to satisfy the vertical excitation condition as stated earlier, we determine
the location of \( f_E(t) \). Figure 4.3 illustrates the free body diagrams of powertrain and sub-
frame systems by assuming each inertial body has rectangular shape. Here, the following
symbols are utilized: \( k_{HF} \), stiffness of the hydraulic mount at front side; \( k_{HR} \), stiffness of
the dynamic load sensing hydraulic mount; \( k_R \), stiffness of rubber mount; \( l_{E1} \), width of
powertrain; \( l_{E2} \), depth of powertrain; C.G., center of geometry; \( e_1 \) and \( e_2 \), length from \( z \)
and \( y \) axes, respectively; \( l_{S1} \), width of sub-frame; \( l_{S2} \), depth of sub-frame; and \( \theta_z \) and \( \theta_y \) are
the moments about the \( z \) and \( y \) axes, respectively. To determine the location of \( f_E(t) \), we
employ the following assumptions: (a) Relative (for two hydraulic mounts) and absolute
(for rubber mount) displacements are identical; (b) the centers of mass for both
powertrain and sub-frame are located at their geometrical centers; (c) all bushings have
identical spring rates; (d) if the system has no moment under static load condition, no
moment would be generated under the dynamic condition; and (e) the excitation point is
located on top of the powertrain. Define the relative motion as \( \zeta (\equiv x_E - x_S) \) and relate it to
the force based on the static equilibrium as shown in Figure 4.3:

\[
f_E = (k_{HF} + k_{HR} + k_R) \zeta. \tag{4.5}
\]

Next, the summation of moments for \( \theta_{zv} \) and \( \theta_{yv} (v = 1, 2, 3, \cdots) \) should be zero:

\[
\sum v \theta_{zv} = f_E e_1 + k_{HF} \zeta \frac{l_{E1}}{2} - k_{HR} \zeta \frac{l_{E1}}{2} = 0. \tag{4.6}
\]
Figure 4.2 Laboratory experimental setup with load sensing hydraulic mount; (a) powertrain and sub-frame assembly with instrumentation; (b) front view laboratory experiment with focus on vertical motions.
Figure 4.3 Free body diagrams of sub-systems of Figure 4.2 used to determine the excitation force location; (a) free body diagram of mass $m_E$; (b) free body diagram of mass $m_S$. 
\[ \sum_y \theta_{yv} = -f_E e_2 + (k_{HF} + k_{HR}) \frac{\dot{\psi}_{E2}}{2} - k_R \frac{\ddot{\psi}_{E2}}{2} = 0. \]  \hspace{1cm} (4.7)

From Eqns. (4.5)-(4.7), \( e_1 \) and \( e_2 \) are calculated as follows:

\[ e_1 = \frac{-k_{HF} + k_{HR} \ell_{E1}}{k_{HF} + k_{HR} + k_R \ell_{E1}}. \]  \hspace{1cm} (4.8)

\[ e_2 = \frac{k_{HF} + k_{HR} - k_R \ell_{E2}}{k_{HF} + k_{HR} + k_R \ell_{E2}}. \]  \hspace{1cm} (4.9)

Given nominal parameters of mounts \( k_{HF} = 7.5 \times 10^5 \) N/m; \( k_{HR} = 6.0 \times 10^5 \) N/m; \( k_R = 1.2 \times 10^5 \) N/m), the excitation location is determined from Eqns. (4.8) and (4.9) as \( e_1 = -0.05 \ell_{E1} \) and \( e_2 = 0.42 \ell_{E2} \).

### 4.4 Dynamic Measurements and Signal Analysis

In our experimental study, sinusoidal responses are measured up to 50 Hz with amplitudes \( |\bar{F}_E| = 1, 10, 50, 100, 150 \) N. Here, each ‘raw’ time domain signal is designated as \( u_v(t) \) (\( v = 1, 2, \cdots, L \)) where \( L = 20 \) is the ensemble size; the Fast Fourier Transform (FFT) is given by \( U_v(\omega) \) (\( v = 1, 2, \cdots, L \)). Time signals \( u_v(t) \) are acquired for 20 seconds and thus 20 raw data sets of \( u_v(t) \) with time window of \( T = 1 \) second are constructed. The ensemble sets of \( u_v(t) \) are averaged in two ways. First, the synchronous time domain averaging technique is used to yield the averaged time history, designated as \( \langle u_v(t) \rangle_v \); it is then transformed into the frequency domain, as defined below.

\[ \langle u_v(t) \rangle_v = \frac{1}{L} \sum_{v=1}^{L} u_v(t), \]  \hspace{1cm} (4.10)

\[ U_{vAVE}(\omega) = \mathfrak{F}[\langle u_v(t) \rangle_v]. \]  \hspace{1cm} (4.11)

Here, \( \mathfrak{F}[\ ] \) is the FFT operator. Second, the frequency domain averaging technique is utilized as follows:
\( U_v(\omega) = \Im[i u_v(t)], \quad (4.12) \)

\( \langle U_v(\omega) \rangle_v = \frac{1}{L} \sum_{v=1}^{L} U_v(\omega). \quad (4.13) \)

Figures 4.4 and 4.5 compare ‘raw’ and averaged data. For example, Figure 4.4 compares time histories as described by Eqn. (4.10). Here, \( f_{IT}(t) \) and \( f_{IB}(t) \) are the interfacial forces on top and bottom sides of the dynamic load sensing hydraulic mount respectively and \( p_u(t) \) is the upper chamber pressure. Specifically, significant noise is observed in \( p_u(t) \). Thus, the synchronous time domain averaging method yields one clear time history.

Figure 4.5 (a) compares \( U_v(\omega) = \Im[i u_v(t)] \) and \( \langle U_v(\omega) \rangle_v \) as described by Eqn. (4.13). All 20 spectra from \( p_u(t) \) signals also include much noise as observed in the \( \mid \vec{P}_u \mid \) spectra. Again clear FFT data are estimated from the averaged time history per Eqn. (4.11).

Figure 4.5 (b) shows the comparisons between \( U_v(AVE)(\omega) \) and \( \langle U_v(\omega) \rangle_v \) as described in Eqns. (4.11) and (4.13). Two spectra match well. Figure 4.6 shows the time history and its FFT with the acceleration \( \psi_{IT}(t) \) on top of the dynamic load sensing hydraulic mount.

As observed from the time history in Figure 4.6 (a), the acceleration includes super-harmonics when compared with the sinusoidal force input \( f_E(t) \). The nonlinear characteristics of \( \psi_{IT}(t) \) is also observed well in the FFT as shown in Figure 4.6 (b). This shows the second harmonic \( (2\omega_o) \) as well as the fundamental \( \omega_o \) term.
Figure 4.4 Comparison between measured ‘raw’ \( u_v(t) \) and time averaged \( \langle u_v(t) \rangle \) signals for \( f_{IT}(t) \), \( f_{IB}(t) \) and \( p_u(t) \) given sinusoidal force input \( f_E(t) = \text{Re}[\bar{F}_E e^{i\omega_o t}] \) at \( \omega_o/2\pi = 23 \text{ Hz} \) with \( |\bar{F}_E| = 100 \text{ N} \). Key: \( - \), \( f_E(t) \); \( - \), \( u_v(t) \) for \( f_{IT}(t) \), \( f_{IB}(t) \) and \( p_u(t) \); \( - - \), \( \langle u_v(t) \rangle \) for \( f_{IT}(t) \), \( f_{IB}(t) \) and \( p_u(t) \).
Figure 4.5 Spectral contents of $F_{IT}(\omega)$, $F_{IB}(\omega)$ and $P_u(\omega)$ given $f_E(t) = Re[F_E e^{i\omega t}]$ at $\omega_o/2\pi = 23$ Hz with $|F| = 100$ N: (a) comparison between $\langle U_v(\omega) \rangle_v$ and $U_v(\omega)$ for $F_{IT}(\omega)$, $F_{IB}(\omega)$ and $P_u(\omega)$; (b) comparison between $\langle U_v(\omega) \rangle_v$ and $U_{vAVE}(\omega)$ for $F_{IT}(\omega)$, $F_{IB}(\omega)$ and $P_u(\omega)$. Key: $\rightarrow$, $U_v(\omega)$ and $U_{vAVE}(\omega)$ for $F_{IT}(\omega)$, $F_{IB}(\omega)$ and $P_u(\omega)$; $\cdots$, $\langle U_v(\omega) \rangle_v$ for $F_{IT}(\omega)$, $F_{IB}(\omega)$ and $P_u(\omega)$. 
Figure 4.6 Measured acceleration on the bottom side of load sensing hydraulic mount including super-harmonics given sinusoidal force input $f_E(t) = Re\left[F_E e^{j\omega_0 t}\right]$ at $\omega_0/2\pi = 23$ Hz with $|F_E| = 100$ N: (a) time history of measured acceleration $\psi_{IT}(t)$; (b) Spectral content of acceleration, $\Im[\psi_{IT}(t)]$. Key: $f_E(t)$, $\psi_{IT}(t)$ and $\Im[\psi_{IT}(t)]$. 

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4.5 Analytical Modal Analysis of Experiment

4.5.1 2DOF and 3DOF system models

Analogous 2DOF and 3DOF mechanical system models of the experiment are developed in Figure 4.7. Voight models are used to describe two hydraulic and one rubber mount, as well as bushings. Here, following symbols are designated: \( m_E \), mass of powertrain; \( m_S \), mass of sub-frame; \( c_{HF} \), damping of the hydraulic mount at front side; \( c_{HR} \), damping of the dynamic load sensing hydraulic mount; \( m_{ie} \), effective mass of inertia track column; \( \dot{x}_{ie}(t) \), effective velocity of inertia track fluid; \( k_u \) and \( k_l \), effective stiffness of upper and lower chambers respectively; \( k_{HRr} \) and \( c_{HRr} \), rubber stiffness and damping of the dynamic load sensing hydraulic mount respectively; \( c_{R} \), damping of rubber mount; \( k_B \) and \( c_B \), stiffness and damping of bushing respectively. The dynamic load sensing hydraulic mount is described by only stiffness and damping elements in the 2DOF model. On the other hand, the 3DOF model includes effective mechanical properties (including fluid inertia) as derived from the fluid parameters [4.12, 4.17, 4.18].

Assume that both systems of Figure 4.7 are linear time-invariant (LTI). First, the 2DOF model (with subscript 2) is described below. Here, \( \mathbf{x}_2(t) \) and \( f_2(t) \) are the displacement and excitation force vectors respectively, and \( \mathbf{M}_2, \mathbf{K}_2, \) and \( \mathbf{C}_2 \) are the mass, stiffness, and viscous damping matrices.

\[
\mathbf{M}_2 \ddot{\mathbf{x}}_2(t) + \mathbf{C}_2 \dot{\mathbf{x}}_2(t) + \mathbf{K}_2 \mathbf{x}_2(t) = f_2(t),
\]

\[
\mathbf{M}_2 = \begin{bmatrix} m_E & 0 \\ 0 & m_S \end{bmatrix}, \quad \mathbf{K}_2 = \begin{bmatrix} k_R + k_{ES2} & -k_{ES2} \\ -k_{ES2} & k_{ES2} + k_B \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} c_R + c_{ES2} & -c_{ES2} \\ -c_{ES2} & c_{ES2} + c_B \end{bmatrix},
\]
\[
x_2(t) = \begin{bmatrix} x_E(t) \\ x_S(t) \end{bmatrix}, \quad f_2(t) = \begin{bmatrix} \text{Re}[\bar{F}_E e^{i\omega t}] \\ 0 \end{bmatrix}.
\] (4.16)

Here, \( k_{ES2} = k_{HF} + k_{HR}; \ c_{ES2} = c_{HF} + c_{HR} \). Similarly, the 3DOF model (with subscript 3) is derived.

\[
M_3 \ddot{x}_3(t) + C_3 \dot{x}_3(t) + K_3 x_3(t) = f_3(t),
\] (4.17)

\[
M_3 = \begin{bmatrix} m_E & 0 & 0 \\ 0 & m_{ie} & 0 \\ 0 & 0 & m_S \end{bmatrix}, \quad K_3 = \begin{bmatrix} k_R + k_u + k_{ES3} & -k_u & -k_{ES3} \\ -k_u & k_u + k_l & -k_l \\ -k_{ES3} & -k_l & k_{ES3} + k_l + k_B \end{bmatrix},
\]

\[
C_3 = \begin{bmatrix} c_R + c_{ES3} & 0 & -c_{ES3} \\ 0 & c_{ie} & 0 \\ -c_{ES3} & 0 & c_{ES3} + c_B \end{bmatrix}.
\] (4.18)

\[
x_3(t) = \begin{bmatrix} x_E(t) \\ x_{ie}(t) \\ x_S(t) \end{bmatrix}, \quad f_3(t) = \begin{bmatrix} \text{Re}[\bar{F}_E e^{i\omega t}] \\ 0 \\ 0 \end{bmatrix}.
\] (4.19)

where \( k_{ES3} = k_{HF} + k_{Rr}; \ c_{ES3} = c_{HF} + c_{Rr} \). Next, the real eigenvalue problem is considered for with \( v = 2 \) or 3DOF model [4.19].

\[
K_v D_v = \nu_v M_v D_v, \quad r = 1, 2, \ldots, (v = 2 \text{ or } 3\text{DOF model}).
\] (4.20)

Here, \( M_v \) and \( K_v \) are mass and stiffness matrices respectively; \( \nu_v \) and \( D_v \) are eigenvalues and eigenvectors respectively; and \( r \) is the modal index. Apply a normalizing constant \( \eta_{vr} \) with relationships \( \Phi_{vr} = \eta_{vr} D_v \) \((r = 1, 2, \ldots)\) to find the normalized modal vector \( \Phi_{vr} \) such that \( \Phi_{v}^T M_v \Phi_{v} = I_v \). Also, obtain the modal mass \( m_{vr} \) from \( D_v^T M_v D_v = m_{vr} \); this implies that \( \eta_{vr}^2 = 1/m_{vr} \). By applying the above, define \( \bar{x}_v(t) \) as the modal (normal) coordinate as:

\[
x(t) = \Phi_v \bar{x}_v(t).
\] (4.21)
Figure 4.7 Alternate versions of the analogous mechanical system models of experiment; (a) schematic of the 2DOF model; (b) schematic of the 3DOF model with fluid inertia in the mount.
Consider the viscous damping matrix $C_v (v = 2 \text{ or } 3\text{DOF})$ and define $C_v^* = \Phi_v^T C_v \Phi_v$ from Eqn. (4.21). The modal damping ratios $\zeta_v$ are found by ignoring the off-diagonal terms to ensure proportional damping.

$$C_v^* \cong \text{diag}[\cdots, 2\zeta_v \omega_v, \cdots]$$ \hspace{1cm} (4.22)

Next, consider the complex eigenvalue problem for the 2DOF system [4.20] where $S_2$ and $B_2$ are the state vector and matrix, respectively.

$$\dot{S}_2 = B_2 S_2, \quad S_2 = [x_E \quad x_S \quad \dot{x}_E \quad \dot{x}_S]^T, \quad B_2 = \begin{bmatrix} 0 & I \\ -M_2^{-1}K_2 & -M_2^{-1}C_2 \end{bmatrix}. \hspace{1cm} (4.23)$$

Similarly the state space form of the 3DOF model is as follows.

$$\dot{S}_3 = B_3 S_3, \quad S_3 = [x_E \quad x_{ie} \quad x_S \quad \dot{x}_E \quad \dot{x}_{ie} \quad \dot{x}_S]^T, \quad B_3 = \begin{bmatrix} 0 & I \\ -M_3^{-1}K_3 & -M_3^{-1}C_3 \end{bmatrix}. \hspace{1cm} (4.24)$$

Here, $S_3$ and $B_3$ are the state vector and matrix, respectively. The complex eigenvalues as $\vartheta_v + i\mu_v$, $(r = 1, 2, \cdots)$, where, $\vartheta_v = \zeta_v \omega_{N_v}$ and $\mu_v = \omega_{D_v}$. The damped natural frequency is $\omega_{D_v} = \omega_{N_v} \sqrt{1 - \zeta_v^2}$; the natural frequency is $\omega_{N_v} = \sqrt{\vartheta_v^2 + \mu_v^2}$ and the modal damping ratio is $\zeta_v = \vartheta_v / \omega_{N_v}$.

For the linear analysis with both 2DOF and 3DOF models, the following nominal parameters are used: $m_E = 299.5 \text{ kg}$; $m_S = 52 \text{ kg}$; $m_{ie} = 81 \text{ kg}$; $c_{HF} = 1547.2 \text{ N-s/m}$; $c_{HR} = 796.2 \text{ N-s/m}$; $c_R = 18.1 \text{ N-s/m}$; $k_u = 6.75 \times 10^5 \text{ N/m}$; $k_l = 8.44 \times 10^3 \text{ N/m}$; $k_{HRr} = 2 \times 10^5 \text{ N/m}$; $c_{HRr} = 496.1 \text{ N-s/m}$; $k_B = 3.6 \times 10^6 \text{ N/m}$; $c_B = 100.0 \text{ N-s/m}$. Tables 4.1 and 4.2 compare real and complex eigensolutions; note that the complex eigensolutions include the phase of modal vectors. The natural frequencies, damping ratios, and mode shapes from both real and complex eigensolutions match well for both 2DOF and 3DOF models. For instance,
the powertrain mode for the 2DOF model is dominant at 9.6 Hz. On the other hand, the sub-frame shows the dominant mode at 49.5 Hz for both real and complex eigensolutions. Also, the damping ratio at 49.5 Hz is higher than the value at 9.6 Hz. When modes of the 3DOF model are examined, we observe coupled motions between $x_E$ and $x_{ie}$ and both $x_E$ and $x_{ie}$ are higher than $x_S$ around 7.5 Hz. The second mode indicates that $x_{ie}$ is slightly coupled with $x_E$. However, $x_S$ is dominant at the third mode at 47.3 Hz, as also observed by the 2DOF model.

4.5.2 Effective SDOF models

From the modal data of previous section, effective single degree of freedom (SDOF) linear system models could be developed. In order to include the internal mount dynamics (from $m_{ie}$), three effective SDOF models designated as $S_{e1}$, $S_{e2}$, and $S_{e3}$ are considered. Figure 4.8 illustrates these; each is valid for a dominant mode of the experiment. For example, Mode I at 7.5 Hz in Table 4.2 shows a coupled mode in terms of $x_E$ and $x_{ie}$. Thus, the motion of $m_E$ is assumed to be strongly affected by the inertia track fluid column stiffness $k_u$ and damping $c_{ie} + c_{HR}$. The dominant motion of $m_{ie}$ is observed at Mode II (of the 3DOF model). The dominant motion of $m_S$ is seen only at Mode III (of the 3DOF model). Natural frequencies from these SDOF models, 7.6 Hz with $S_{e1}$, 14.6 with $S_{e2}$, and 47.1 with $S_{e3}$ as reposted in Table 4.3, match well with Table 4.2 results of the 3DOF model.
<table>
<thead>
<tr>
<th></th>
<th>Real Eigensolution</th>
<th>Complex Eigensolution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Frequency (Hz)</td>
<td>Modal Vector ([x_E , x_S]^T)</td>
</tr>
<tr>
<td><strong>Mode I</strong></td>
<td>9.6</td>
<td>([1.00 , 0.28]^T)</td>
</tr>
<tr>
<td></td>
<td>(\zeta_{31}=0.02)</td>
<td></td>
</tr>
<tr>
<td><strong>Mode II</strong></td>
<td>49.5</td>
<td>([-0.05 , 1.00]^T)</td>
</tr>
<tr>
<td></td>
<td>(\zeta_{32}=0.05)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1 Real and complex eigensolutions of the 2DOF model of experiment as shown in Figure 4.7

<table>
<thead>
<tr>
<th></th>
<th>Real Eigensolution</th>
<th>Complex Eigensolution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Frequency (Hz)</td>
<td>Modal Vector ([x_E , x_{ie} , x_S]^T)</td>
</tr>
<tr>
<td><strong>Mode I</strong></td>
<td>7.4</td>
<td>([0.72 , 1.00 , 0.16]^T)</td>
</tr>
<tr>
<td></td>
<td>(\zeta_{31}=0.20)</td>
<td></td>
</tr>
<tr>
<td><strong>Mode II</strong></td>
<td>17.0</td>
<td>([-0.36 , 1.00 , -0.08]^T)</td>
</tr>
<tr>
<td></td>
<td>(\zeta_{32}=0.17)</td>
<td></td>
</tr>
<tr>
<td><strong>Mode III</strong></td>
<td>47.3</td>
<td>([-0.04 , 0.00 , 1.00]^T)</td>
</tr>
<tr>
<td></td>
<td>(\zeta_{33}=0.07)</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2 Real and complex eigensolutions of the 3DOF model of experiment as shown in Figure 4.7
<table>
<thead>
<tr>
<th>SDOF Model</th>
<th>Natural Frequency (Hz)</th>
<th>Effective Interfacial Force (N) Used for Accelerance Calculations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Top side (at $m_{e}$)</td>
</tr>
<tr>
<td>$S_{e1}$</td>
<td>7.6</td>
<td>0.75</td>
</tr>
<tr>
<td>$S_{e2}$</td>
<td>14.6</td>
<td>0.20</td>
</tr>
<tr>
<td>$S_{e3}$</td>
<td>47.1</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 4.3 Natural frequencies and effective interfacial forces of the single degree of freedom (SDOF) models of Figure 4.8
Figure 4.8 Alternate SDOF models of the experiment; (a) SDOF model $S_{e1}$ dominated by $x_E$; (b) SDOF model $S_{e2}$ dominated by $x_{ie}$; (c) SDOF model $S_{e3}$ dominated by $x_S$. 
4.6 Linear System Model Predictions and Comparison with Measurements

4.6.1 Accelerance spectra predicted by alternate models

From Eqns. (4.14)-(4.19), accelerances for both 2DOF and 3DOF system models are calculated by the matrix inversion method assuming that both systems are linear time-invariant (LTI). Here, the accelerance is determined applying the harmonic force of unity amplitude on \( m_E \).

\[
\Psi_v(\omega) = -\omega^2[-\omega^2 M_v + i\omega C_v + K_v]^{-1} f_v(\omega) \quad (v = 2 \text{ or } 3 \text{DOF}),
\]

(4.25)

\[
f_2(\omega) = [1 \quad 0]^T, \quad f_3(\omega) = [1 \quad 0 \quad 0]^T.
\]

(4.26)

The similarities and differences between the 2DOF and 3DOF models could be seen from the frequency responses in Figure 4.9. The accelerance spectra \( \Psi_{fT}(\omega) \) for the 2DOF model show one dominant resonance but the 3DOF model displays two resonances corresponding to the dominant motions of \( x_E \) and \( x_{ie} \). However, the 2DOF model shows a higher peak at 9.6 Hz since it does not include \( x_{ie} \) which is related to the fluid motion within the mount. Both models predict the same sub-frame resonance \( x_S \) as seen in the \( \Psi_{fB}(\omega) \) spectra. Further, the governing equations for effective SDOF models of Figure 4.8 are as follows.

\[
m_E \ddot{x}_E(t) + (c_{ie} + c_{HR}) \dot{x}_E(t) + k_i x_E(t) = f_{IT}(t),
\]

(4.27)

\[
m_{ie} \ddot{x}_{ie}(t) + c_{ie} \dot{x}_{ie}(t) + (k_u + k_l) x_{ie}(t) = f_{ie}(t),
\]

(4.28)

\[
m_S \ddot{x}_S(t) + (c_{HF} + c_{HR} + c_B) \dot{x}_S(t) + (k_{HF} + k_{HR} + k_B) x_S(t) = f_{fS}(t).
\]

(4.29)

To estimate the good correlations of accelerances by the \( \mathbf{S}_{ev} \) (\( v = 1,2,3 \)), models, different forces \( (f_{IT}(t), f_{ie}(t), \text{ and } f_{fB}(t)) \) are employed on both \( m_E \) and \( m_S \). Here, \( f_{ie}(t) \) is the
effective interfacial force on $m_{ie}$, $f_{IT}(t)$ is the interfacial force on $m_E$, and $f_{IB}(t)$ is the interfacial force on $m_S$. Table 4.3 shows typical values; observe that effective forces are different on top and bottom sides which highlights the importance of asymmetrical characteristics of the interfacial forces.

From Eqns. (4.27)-(4.29), the accelerances, $\overline{\Psi}_{S1}(\omega)$, $\overline{\Psi}_{S2}(\omega)$ and $\overline{\Psi}_{S3}(\omega)$ for the SDOF models $S_{e1}$, $S_{e2}$ and $S_{e3}$ respectively, are analytically found as follows; note that effective forces of Table 4.3 are used in these expressions.

$$\overline{\Psi}_{S1}(\omega) = -\omega^2 \frac{F_{IT}(\omega)}{\left[k_u - \omega^2 m_E + i \omega (c_{ie} + c_{HR})\right]}, \quad (4.30)$$

$$\overline{\Psi}_{S2}(\omega) = -\omega^2 \frac{F_{ie}(\omega)}{\left[(k_u + k_l) - \omega^2 m_E + i \omega c_{ie}\right]}, \quad (4.31)$$

$$\overline{\Psi}_{S3}(\omega) = -\omega^2 \frac{F_{IB}(\omega)}{\left[(k_{HF} + k_{HR} + k_B) - \omega^2 m_S + i \omega (c_{HF} + c_{HR} + c_B)\right]}, \quad (4.32)$$

Figure 4.10 compares accelerance spectra of the 3DOF and SDOF $S_{ev}$ ($v = 1,2,3$) models as well as a combination thereof. Figure 4.10 (a) shows the comparison at $m_E$ where only $S_{e1}$ and $S_{e2}$ models are included. Overall, the cumulative accelerance spectra of the $S_{ev}$ ($v = 1,2,3$) models show three resonances and follow the accelerances of the 3DOF model well.
Figure 4.9 Predicted accelerances for the experiment using 2 and 3DOF linear system models; (a) comparison of accelerances at $m_E$; (b) comparison of accelerances at $m_S$.

Key: ---, 2DOF model; ––, 3DOF model.
Figure 4.10 Comparison of accelerances from 3DOF and alternate SDOF linear models: (a) accelerance at $x_E$; (b) accelerance at $x_S$. Key: ——, 3DOF model; ——, $S_{e1}$ model; ——, $S_{e2}$ model; ——, $S_{e3}$ model; ——, summation of accelerances from a combination of $S_{e1}$, $S_{e2}$ and $S_{e3}$ models.
4.6.2 Frequency domain comparison between measurements and predictions based on linear system models

Figures 4.11, 4.12, and 4.13 show measured accelerations, interfacial forces, and upper chamber pressures in the frequency domain, essentially forced response only at the fundamental frequency $\omega_0$. Figure 4.11 shows accelerations $\bar{\Psi}_{IT}(\omega)$ and $\bar{\Psi}_{IB}(\omega)$ measured on top and bottom sides of the load sensing hydraulic mount, respectively. When comparing measurements and LTI model predictions of section 4.5, the transfer function magnitudes of the LTI model (such as accelerances) must be multiplied by the force amplitude (such as 100 N in Figure 4.11). Figure 4.11 shows that the LTI model follows the tendency of measured responses and predicts three resonances around 7 Hz, 18 Hz, and 47 Hz as observed in the measured $\bar{\Psi}_{IT}(\omega)$ and $\bar{\Psi}_{IB}(\omega)$ signals. However, significant discrepancies in the magnitudes are seen.

The interfacial forces in frequency domain are estimated by using Eqns. (4.3) and (4.4), replacing $s$ by $i\omega$.

$$F_{IT}(\omega) = (i\omega c_{ES2} + k_{ES2}) \Xi(\omega) + A_r P_u(\omega). \quad (4.33)$$

$$F_{IB}(\omega) = -(i\omega c_{ES2} + k_{ES2}) \Xi(\omega) - A_r P_u(\omega). \quad (4.34)$$

Here, the nominal values of parameters are as follows: $A_r = 4.5\times10^{-3} \text{ m}^2$; $k_{ES2} = 6.0\times10^5 \text{ N/m}$; $c_{ES2} = 796.2 \text{ N-s/m}$. Note that the measured values of $\Xi(\omega)$ and $P_u(\omega)$ are utilized in Eqns. (4.33) and (4.34). The estimation of relative displacement from the measured accelerations will be explained in section 4.7. Figure 4.12 compares measured and estimated interfacial forces $\bar{F}_{IT}(\omega)$ and $\bar{F}_{IB}(\omega)$ on top and bottom sides of the load sensing mount. Observe significant discrepancies in the magnitudes. Thus, the LTI system
Figure 4.11 Comparison of measured accelerations with estimation by LTI system model on both top and bottom sides of the dynamic load sensing hydraulic mount given $f_E(t) = \text{Re}[F_E e^{i\omega t}]$ at $|F_E| = 100$ N in frequency domain: (a) acceleration on top of the mount $\Phi_{IT}(\omega)$; (b) acceleration on bottom of the mount $\Phi_{IB}(\omega)$. Key: - , measurement; -- , estimation by LTI system model.
Figure 4.12 Comparison of measured interfacial forces with estimation by LTI system model on both top and bottom sides of the dynamic load sensing hydraulic mount given 

\[ f_E(t) = Re[\bar{F}_E e^{i\omega t}] \] at \( |\bar{F}_E| = 100 \text{ N} \) in frequency domain: (a) interfacial force on top of the mount \( \bar{F}_{IT}(\omega) \); (b) interfacial force on bottom of the mount \( \bar{F}_{IB}(\omega) \). Key: –, measurement; ––, estimation by LTI system model.
Figure 4.13 Measured upper chamber pressure of the dynamic load sensing hydraulic mount given $f_E(t) = \text{Re} \left[ \vec{P}_E e^{i \omega t} \right]$ at $|\vec{F}_E| = 100$ N and 150 N in frequency domain: (a) $\vec{P}_u(\omega)$ at $|\vec{F}_E| = 100$ N; (b) $\vec{P}_u(\omega)$ at $|\vec{F}_E| = 150$ N.
model cannot predict forces including asymmetrical characteristics introduced by the hydraulic mount. It is clear as the measured forces at either ends $\tilde{F}_{II}(\omega)$ and $\tilde{F}_{IB}(\omega)$ differ quite a bit in magnitudes especially below 20 Hz. Figure 4.13 describes measured $\tilde{P}_u(\omega)$ for two excitation amplitudes ($|F_E| = 100$ N and 150 N). Similar to the spectral contents in $F_{IT}(\omega)$ and $F_{IB}(\omega)$, the magnitude of $\tilde{P}_u(\omega)$ also shows larger values below 20 Hz when compared with the magnitudes above 20 Hz. Also, $\tilde{P}_u(\omega)$ shows one clear resonance around 47 Hz though its magnitude is rather small.

### 4.7 Development of a Quasi-Linear (QL) Mechanical System Model

An interrelationship between the interfacial forces $\tilde{F}_{IT}(\omega)$ and $\tilde{F}_{IB}(\omega)$, and internal pressure $\tilde{P}_u(\omega)$ can be observed from spectral contents in Figure 4.5 with respect to the fundamental and super-harmonic terms. Thus, a quasi-linear (QL) model with effective parameters must be developed to successfully predict the interfacial forces. Two analogous mechanical system models, as illustrated in Figure 4.7 and previously discussed in sections 4.4 and 4.5, will be employed again to develop quasi-linear models. Recall that the 2DOF model includes the Voight formulation in terms of stiffness $k_{HR}$ and damping $c_{HR}$ to represent the load sensing hydraulic mount. However, the 3DOF model includes effective mechanical parameters, $k_u$, $k_l$, and $c_{ie}$, that represent internal fluid system.

First, use the 2DOF model (with subscript 2) to define the interfacial forces on both top and bottom ends of the load sensing hydraulic mount as follows.

$$f_{IT2}(t) = c_{HR}\ddot{x}(t) + k_{HR}\dot{x}^2(t).$$  \hspace{1cm} (4.35)
\[ f_{IB2}(t) = -c_{HR} \ddot{\zeta}(t) - k_{HR} \dot{\zeta}(t). \] (4.36)

Here, \( \zeta(t) = x_E(t) - x_S(t) \) is the relative displacement between \( x_E \) and \( x_S \). By transforming Eqns. (4.35) and (4.36) into the Laplace domain (\( s \)) with the assumption that the initial conditions are equal to zero, interfacial forces are:

\[ F_{IT2}(s) = (c_{HR} s + k_{HR}) \Xi(s). \] (4.37)

\[ F_{IB2}(s) = -(c_{HR} s + k_{HR}) \Xi(s). \] (4.38)

Likewise, the interfacial forces with the 3DOF model (with subscript 3) are calculated as follows.

\[ f_{IT3}(t) = c_{HRr} \ddot{\zeta}(t) + k_{HRr} \dot{\zeta}(t) + k_u \ddot{x}_u(t), \] (4.39)

\[ f_{IB3}(t) = -c_{HRr} \ddot{\zeta}(t) - k_{HRr} \dot{\zeta}(t) - k_l \dot{x}_l(t), \] (4.40)

\[ F_{IT3}(s) = (c_{HRr} s + k_{HRr}) \Xi(s) + k_u \Xi_u(s), \] (4.41)

\[ F_{IB3}(s) = -(c_{HRr} s + k_{HRr}) \Xi(s) - k_l \Xi_l(s). \] (4.42)

Here, the relative displacements are defined as: \( \ddot{x}_u(t) = x_E(t) - x_{ie}(t) \) and \( \ddot{\zeta}(t) = x_{ie}(t) - x_S(t) \).

From Eqns. (4.39)-(4.42), the force estimation with the 3DOF model shows asymmetrical characteristics between \( f_{IT}(t) \) and \( f_{IB}(t) \) by the factors \( k_u \ddot{x}_u(t) \) and \( k_l \dot{x}_l(t) \) which have different relative displacements and effective stiffness values. Thus, the displacement \( x_{ie}(t) \) must be predicted to estimate \( f_{IT}(t) \) and \( f_{IB}(t) \). The relationship between \( x_{ie}(t) \) and other known variables such as measured (with subscript \( M \)) displacements \( x_{EM}(t) \) and \( x_{SM}(t) \), and \( f_{EM}(t) \) can be derived from Eqns. (4.17)-(4.19). Transforming Eqns. (4.17)-(4.19) into the Laplace domain and ignoring initial conditions:

\[
\left[ m_E s^2 + (c_R + c_{HF} + c_{HRr}) s + (k_R + k_{HF} + k_{HRr} + k_u) \right] X_E(s) \\
- \left[ (c_{HF} + c_{HRr}) s + (k_{HF} + k_{HRr}) \right] X_S(s) - k_u x_{ie}(s) = F_E(s). \] (4.43)
\[ [m_Ss^2 + (c_{HF} + c_{HR} + c_B)s + (k_{HF} + k_{HR} + k_i + k_B)]X_S(s) \]
\[ - [(c_{HF} + c_{HR})s + (k_{HF} + k_{HR})]X_E(s) - k_iX_e(s) = 0. \] (4.44)

From Eqns. (4.41)-(4.44), the interfacial forces \( F_{IT}(s) \) and \( F_{IB}(s) \) are derived as follows:

\[ F_{IT3}(s) = F_E(s) + (c_{HF}s + k_{HF})X_S(s) - [m_Es^2 + (c_R + c_{HF})s + (k_R + k_{HF})]X_E(s). \] (4.45)

\[ F_{IB3}(s) = (c_{HF}s + k_{HF})X_E(s) - [m_Ss^2 + (c_{HF} + c_B)s + (k_{HF} + k_B)]X_S(s). \] (4.46)

Next, the interfacial forces in the frequency domain can be calculated by replacing \( s \) by \( i\omega \) in the above equations. Thus, interfacial forces with the 2DOF model in the frequency domain are as follows.

\[ \bar{F}_{IT2}(\omega) = (i\omega c_{HR} + k_{HR})\Xi(\omega). \] (4.47)

\[ \bar{F}_{IB2}(\omega) = -(i\omega c_{HR} + k_{HR})\Xi(\omega). \] (4.48)

Likewise, the interfacial forces with the 3DOF model are described below.

\[ \bar{F}_{IT3}(\omega) = \bar{F}_E(\omega) + (i\omega c_{HF} + k_{HF})\bar{X}_S(\omega) \]
\[ - [\omega^2 m_E + i\omega(c_R + c_{HF}) + (k_R + k_{HF})]\bar{X}_E(\omega). \] (4.49)

\[ \bar{F}_{IB3}(\omega) = (i\omega c_{HF} + k_{HF})\bar{X}_E(\omega) - [\omega^2 m_S + i\omega(c_{HF} + c_B) + (k_{HF} + k_B)]\bar{X}_S(\omega). \] (4.50)

Here, the displacements \( \bar{X}_E(\omega) \) and \( \bar{X}_S(\omega) \) are calculated directly from the measured accelerations \( \bar{Y}_{ITM}(\omega) \) and \( \bar{Y}_{IBM}(\omega) \) by the relationships \( \bar{X}_{EM}(\omega) = -\bar{Y}_{ITM}(\omega)/\omega^2 \) and \( \bar{X}_{SM}(\omega) = -\bar{Y}_{IBM}(\omega)/\omega^2 \) by avoiding double integration in the time domain. Thus, the measured relative displacement \( \bar{X}_M(\omega) = \bar{X}_{EM}(\omega) - \bar{X}_{SM}(\omega) \) is calculated by \( \bar{X}_M(\omega) = -\bar{Y}_{ITM}(\omega)/\omega^2 + \bar{Y}_{IBM}(\omega)/\omega^2 \). Figure 4.14 shows the measured \( \bar{X}_M(\omega) \) in the frequency domain under the sinusoidal excitation force \( f_E(t) = Re[\bar{F}_Ee^{i\omega t}] \) with \( |\bar{F}_E| = 100 \) N.
Recall that Eqns. (4.43) and (4.44) are based upon the linear time-invariant (LTI) system concept and thus, their stiffnesses \((k_{HF}, k_{HR}, k_{HRr}, k_{R} and k_{B})\), and dampings \((c_{HF}, c_{HR}, c_{HRr}, c_{R} and c_{B})\) are assigned constant values. To develop quasi-linear models for both 2DOF and 3DOF mechanical systems, effective spectrally-varying parameters must be included. Figure 4.15 shows measured stiffness and damping data of hydraulic and rubber mounts, from the non-resonant dynamic stiffness testing procedure under the ISO standard 10846 [4.21]. Though such empirical data (typically supplied by the mount vendors) is limited in scope, some useful information can still be gained. Therefore, we develop the quasi-linear model by using the following steps: (1) Start with mechanical system models where the interfacial forces are derived from the LTI system formulation; (2) Embed spectrally-varying stiffness and damping elements in terms of \(k_w(\omega)\) and \(c_w(\omega)\) (where \(w\) is the mount index for \(HF, HR, HRR, R,\) and \(B\)) as shown in Figure 4.15 into the frequency domain formulation; (3) Employ measured motions \(\tilde{X}_{EM}(\omega)\) and \(\tilde{X}_{SM}(\omega)\) in Eqns. (4.47) and (4.48) for the 2DOF model and Eqns. (4.49) and (4.50) for 3DOF model; (4) Calculate the time histories of interfacial forces by using the complex exponential form as shown below. Note that this includes only the fundamental (excitation) frequency as follows.

\[
f_{ITv}(t) = |\tilde{F}_{ITv}(\omega)|\text{Re}\left[e^{i(\omega t + \phi_{FITv})}\right],
\]

\[
\phi_{FITv} = \angle \tilde{F}_{ITv}(\omega), \ (v = 2\text{DOF or 3DOF}).
\]

\[
f_{IBv}(t) = |\tilde{F}_{IBv}(\omega)|\text{Re}\left[e^{i(\omega t + \phi_{FIBv})}\right],
\]

\[
\phi_{FIBv} = \angle \tilde{F}_{IBv}(\omega), \ (v = 2\text{DOF or 3DOF}).
\]
Figure 4.14 Relative displacement between $x_E$ and $x_S$ at the dynamic load sensing hydraulic mount from the measured accelerations $\Psi_{ITM}(\omega)$ and $\Psi_{IBM}(\omega)$ given $f_E(t) = Re[F_E e^{i\omega t}]$ at $|F_E| = 100$ N in frequency domain.
Figure 4.15 Measured dynamic properties of the hydraulic and rubber mounts given
\[ x(t) = Re[\bar{X}e^{i\omega t}] \] with \( |\bar{X}| = 0.1 \) mm: (a) stiffness and damping spectra for the
dynamic load sensing hydraulic mount; (b) stiffness and damping spectra for the rubber
mount.
4.8 Development of a Quasi-Linear (QL) Fluid System Model

Much of the mount research work based on the linear system models was developed by Singh et al. [4.22]. Kim and Singh [4.23, 4.24] and Tiwari et al. [4.17] measured nonlinear inertia track resistances and upper and lower chamber compliances using laboratory experiments. These properties were utilized by Adiguna et al. [4.18] and He and Singh [4.12] to predict steady state and transient responses. Other hydraulic mount studies include work by Shangguan and Lu [4.25], Fan and Lu [4.26], Truong and Ahn [4.27], Geisberger et al. [4.28], Mrad and Levitt [4.29] and Lee and Moon [4.30]. In particular, this chapter will extend prior studies by Yoon and Singh [4.10, 4.11], and estimate in-situ path forces in the context of Figure 4.1. The interfacial forces will be divided into rubber and hydraulic force paths, and the quasi-linear model is developed by embedding the spectrally-varying and amplitude-sensitive parameters and by using measured displacement and upper chamber pressure signals.

Figure 4.16 illustrates the fluid system model of the load sensing hydraulic mount (fixed decoupler type). Here, \( f_{ITr}(t) \) and \( f_{IBr}(t) \) are the top and bottom sides of rubber path forces, respectively, and \( f_{ITh}(t) \) and \( f_{IBh}(t) \) are the top and bottom sides of hydraulic path forces, respectively. The rubber forces are modeled using \( k_{HRr} \) and \( c_{HRr} \) elements, which are assumed to be identical on top and bottom sides of the hydraulic mount. However, the hydraulic path force is assumed to be asymmetric due to the nonlinearities from fluid parameters. The designation of the fluid parameters in Figure 4.16 (b) are as follows: \( p_u(t) \) and \( p_l(t) \) are the upper and lower chamber pressures, respectively; \( C_u \) and \( C_l \) are the upper (#u) and lower (#l) chamber compliances, respectively; \( I_i \) is the inerance of the inertia

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Figure 4.16 Experiment with a fluid system model of the dynamic load sensing hydraulic mount: (a) schematic used for modeling; (b) expanded view of the fluid system model with parallel rubber and hydraulic force paths.
track \((\#i)\); \(R_i\) is the resistance of the inertia track, respectively; and \(q_i\) is the fluid flow through inertia track. The interfacial forces for both top and bottom sides of the mount are derived with the following sign conventions: \(p_u(t)\) and \(p_l(t)\) are positive (in compression) corresponding to the upward (positive) motion of \(x_E(t)\), \(x_S(t)\), \(q_i(t)\), \(f_{IT}(t)\), and \(f_{IB}(t)\); and \(q_d(t)\); and \(p_u(t)\) and \(p_l(t)\) are negative (in expansion) for the downward motion of \(x_E(t)\), \(x_S(t)\), \(q_i(t)\), \(f_{IT}(t)\), and \(f_{IB}(t)\).

\[
f_{IT}(t) = f_{Ir}(t) + f_{ITh}(t), \tag{4.55}
\]

\[
f_{IB}(t) = f_{Ir}(t) + f_{IBh}(t), \tag{4.56}
\]

\[
f_{Ir}(t) = c_{HRr} \dot{\xi}(t) + k_{HRr} \xi(t), \tag{4.57}
\]

\[
f_{ITh}(t) = -2A_r p_u(t) + A_r p_l(t), \tag{4.58}
\]

\[
f_{IBh}(t) = -A_r p_u(t) + 2A_r p_l(t). \tag{4.59}
\]

Here, the relative displacement \(\xi(t) = x_E(t) - x_S(t)\). Also, the momentum and continuity equations for the hydraulic path with respect to \(\xi(t)\) are as follows [4.17, 4.18, 4.22]:

\[
p_u(t) - p_l(t) = I_i \dot{q}_i(t) + R_i q_i(t), \tag{4.60}
\]

\[
C_u \dot{p}_u(t) = A_r \dot{\xi}(t) - q_i(t), \tag{4.61}
\]

\[
C_l \dot{p}_l(t) = q_i(t). \tag{4.62}
\]

Transform Eqns. (4.55)-(4.62) into the Laplace domain \((s)\) and ignore the initial conditions:

\[
F_{IT}(s) = F_{Ir}(s) + F_{ITh}(s), \tag{4.63}
\]

\[
F_{IB}(s) = F_{Ir}(s) + F_{IBh}(s), \tag{4.64}
\]

\[
F_{Ir}(s) = (c_{HRr}s + k_{HRr}) \overline{\xi}(s), \tag{4.65}
\]
\[ F_{Th}(s) = -2A_uP_u(s) + A_rP_i(s), \quad (4.66) \]
\[ F_{Bh}(s) = -A_rP_u(s) + 2A_rP_i(s), \quad (4.67) \]
\[ P_u(s) - P_i(s) = (I_i s + R_i)Q_i(s), \quad (4.68) \]
\[ C_{us}P_u(s) = A_r \xi(s) - Q_i(s), \quad (4.69) \]
\[ C_{ps}P_i(s) = Q_i(s). \quad (4.70) \]

From Eqns. (4.68)-(4.70), several transfer functions \((G_1, G_2 \text{ and } G_3)\) are found that relate \(P_u(s)\) (or \(P_i(s)\) ) and \(\xi(s)\), and then \(P_i(s)\) and \(P_u(s)\):

\[ G_1(s) = \frac{P_u(s)}{\xi(s)} = \frac{A_r}{C_u+C_i} \frac{s^2 + \frac{2\xi_1}{\omega_{N1}^2}s + \frac{2\xi_2}{\omega_{N2}^2}}{s^2 + \frac{2\xi_1}{\omega_{N1}^2}s + \frac{2\xi_2}{\omega_{N2}^2} + 1}, \quad (4.71) \]
\[ G_2(s) = \frac{P_i(s)}{P_u(s)} = \frac{1}{s^2 + \frac{2\xi_1}{\omega_{N1}^2}s + \frac{2\xi_2}{\omega_{N2}^2} + 1}, \quad (4.72) \]
\[ G_3(s) = \frac{P_i(s)}{\xi(s)} = \frac{A_r}{C_u+C_i} \frac{1}{s^2 + \frac{2\xi_1}{\omega_{N1}^2}s + \frac{2\xi_2}{\omega_{N2}^2} + 1}, \quad (4.73) \]

\[ \zeta_1 = \frac{1}{2} \sqrt{\frac{C_i R_i}{I_i}}, \quad (4.74) \]
\[ \omega_{N1} = \frac{1}{\sqrt{C_i I_i}}, \quad (4.75) \]
\[ \zeta_2 = \frac{1}{2} \sqrt{\frac{C_u C_i R_i^2}{I_i (C_u+C_i)^2}}, \quad (4.76) \]
\[ \omega_{N2} = \sqrt{\frac{C_u+C_i}{C_u C_i I_i}}. \quad (4.77) \]

Eqns. (4.58), (4.59), (4.66), and (4.67) reveal the asymmetrical dynamic characteristics since the hydraulic path forces \(f_{Th}(t)\) and \(f_{Bh}(t)\) have different values. The following nominal (and constant) parameters are incorporated: \(k_{Hr} = 2 \times 10^5 \text{ N/m}; \ c_{Hr} = 496.1 \text{ N-} \)
s/m; \( I_i = 4 \times 10^6 \text{ kg/m}^4; \ C_i = 2.5 \times 10^{11} \text{ m}^5/\text{N}; \ C_l = 2.4 \times 10^9 \text{ m}^5/\text{N}; \ R_i = 2 \times 10^8 \text{ N-s/m}^5. \) To predict the interfacial forces \( f_{IT}(t) \) and \( f_{IB}(t) \), measured motion \( \zeta_M(t) \) and upper chamber pressure \( P_uM(t) \) are employed. However, the lower chamber pressure \( P_l(\omega, \Xi) \) in the frequency domain is still unknown. Accordingly, it must be estimated by using Eqns. (4.63)-(4.67). First, by replacing \( s \) with \( i \omega \) in Eqns. (4.63)-(4.67), the equations in frequency domain are expressed in terms of the spectrally-varying and amplitude-sensitive conditions as follows.

\[
F_{IT}(\omega, \Xi) = (i\omega c_{Hr} + k_{Hr})\Xi(\omega) - 2A_r\bar{P}_u(\omega, \Xi) + A_r\bar{P}_l(\omega, \Xi). \tag{4.78}
\]

\[
F_{IB}(\omega, \Xi) = (i\omega c_{Hr} + k_{Hr})\Xi(\omega) - A_r\bar{P}_u(\omega, \Xi) + 2A_r\bar{P}_l(\omega, \Xi). \tag{4.79}
\]

By subtracting Eqn. (4.79) from (4.78), the lower chamber pressure \( \bar{P}_l(\omega, \Xi) \) is derived as follows.

\[
A_r\bar{P}_l(\omega, \Xi) = -A_r\bar{P}_u(\omega, \Xi) - \bar{F}_{IT}(\omega, \Xi) + \bar{F}_{IB}(\omega, \Xi). \tag{4.80}
\]

Second, to estimate \( \bar{P}_l(\omega, \Xi) \), the measured data set \( \bar{F}_{ITM}(\omega, \Xi) \) and \( \bar{F}_{IBM}(\omega, \Xi) \) from the laboratory experiment can be used in the complex valued form as follows.

\[
\bar{F}_{ITM} = F_{ITRE} + iF_{ITIM}, \quad \phi_{ITM} = \angle \bar{F}_{ITM}(\omega, \Xi), \quad \tag{4.81}
\]

\[
F_{ITRE} = \text{Re}[\bar{F}_{ITM}] = |\bar{F}_{ITM}| \cos(\phi_{ITM}), \quad \tag{4.82}
\]

\[
F_{ITIM} = \text{Im}[\bar{F}_{ITM}] = |\bar{F}_{ITM}| \sin(\phi_{ITM}). \quad \tag{4.83}
\]

\[
\bar{F}_{IBM} = F_{IBRE} + iF_{IBIM}, \quad \phi_{IBM} = \angle \bar{F}_{IBM}(\omega, \Xi), \quad \tag{4.84}
\]

\[
F_{IBRE} = \text{Re}[\bar{F}_{IBM}] = |\bar{F}_{IBM}| \cos(\phi_{IBM}), \quad \tag{4.85}
\]

\[
F_{IBIM} = \text{Im}[\bar{F}_{IBM}] = |\bar{F}_{IBM}| \sin(\phi_{IBM}). \quad \tag{4.86}
\]

\[
\bar{P}_{uM} = P_{uRE} + iP_{uIM}, \quad \phi_{P_{uM}} = \angle \bar{P}_{uM}(\omega, \Xi), \quad \tag{4.87}
\]
\[ \dot{P}_{uRE} = Re[\tilde{P}_{uM}] = |\tilde{P}_{uM}| \cos(\phi_{PuM}), \quad (4.88) \]

\[ \dot{P}_{uIM} = Im[\tilde{P}_{uM}] = |\tilde{P}_{uM}| \sin(\phi_{PuM}). \quad (4.89) \]

By substituing Eqns. (4.81)-(4.89) into Eqns. (4.80), real and imaginary parts of \( \tilde{P}_l(\omega, \Xi) \) are calculated as below.

\[ \dot{P}_{RE} = -\dot{P}_{uRE} + \frac{1}{A_r}( -F_{ITRE} + F_{IBRE}), \quad (4.90) \]

\[ \dot{P}_{IM} = -\dot{P}_{uIM} + \frac{1}{A_r}( -F_{ITIM} + F_{IBIM}). \quad (4.91) \]

Figure 4.17 shows estimated \( \tilde{P}_{LM}(\omega_o, \Xi_M) \) along with measured \( \tilde{P}_{uM}(\omega_o, \Xi_M) \). When \( \tilde{P}_{LM}(\omega_o, \Xi_M) \) is compared with \( \tilde{P}_{uM}(\omega_o, \Xi_M) \), \( \tilde{P}_{LM}(\omega_o, \Xi_M) \) seems to be similar in terms of the dynamic behavior and its range of magnitudes. This means that the load sensing hydraulic mount is also affected by the dynamics arising from the lower chamber pressure \( \tilde{P}_l(\omega, \Xi) \). Third, from the calculated \( \tilde{P}_{LM}(\omega_o, \Xi_M) \), the effective stiffness and damping parameters in terms of \( k_{HRre}(\omega, \Xi) \) and \( c_{HRre}(\omega, \Xi) \) respectively are estimated as follows.

\[ \tilde{F}_{IR}(\omega, \Xi) = (i\omega c_{HRre} + k_{HRre})\Xi(\omega) \]

\[ = F_{ITRE} + iF_{ITIM} + 2A_r(\dot{P}_{uRE} + i\dot{P}_{uIM}) - A_r(\dot{P}_{RE} + i\dot{P}_{IM}), \quad (4.92) \]

\[ k_{HRre}(\omega, \Xi) = F_{ITRE} + A_r(2\dot{P}_{uRE} - \dot{P}_{RE}), \quad (4.93) \]

\[ c_{HRre}(\omega, \Xi) = \frac{1}{\omega_o}[F_{ITIM} + A_r(2\dot{P}_{uIM} - \dot{P}_{IM})]. \quad (4.94) \]

Figure 4.18 shows the results from Eqns. (4.93) and (4.94) by using measured \( \Xi_M(\omega_o) \) and \( \tilde{P}_{uM}(\omega_o, \Xi_M) \). Results show that \( k_{HRre}(\omega_o, \Xi_M) \) and \( c_{HRre}(\omega_o, \Xi_M) \) not only include the dynamics of the component itself but some coupling effects from the sub-systems responses since three resonances are observed in \( k_{HRre}(\omega_o, \Xi_M) \). Note that \( c_{HRre}(\omega_o, \Xi_M) \) has
Figure 4.17 Predicted lower chamber pressure spectra $|\bar{P}_l(\omega_o,\Xi_M)|$ given $f_E(t) = Re[\bar{F}_E e^{i\omega_o t}]$ with $|\bar{F}_E| = 100$ N by using measured $|\bar{\Xi}_M(\omega_o)|$ and $|\bar{P}_uM(\omega_o,\Xi_M)|$. Key:

- $\bar{P}_uM(\omega_o,\Xi_M)$
- $\bar{P}_lM(\omega_o,\Xi_M)$.
Figure 4.18 Predicted effective stiffness $k_{HRe}(\omega_o, \Xi_M)$ and damping $c_{HRe}(\omega_o, \Xi_M)$ spectra given $f_E(t) = Re[\bar{F}_E e^{i\omega_o t}]$ with $|\bar{F}_E| = 100$ N by using measured $\Xi_M(\omega_o)$ and $|\bar{P}_{uM}(\omega_o, \Xi_M)|$. 
the highest value around 11 Hz. Finally, both effective lower and upper chamber compliances \( \tilde{C}_{le}(\omega_o, \Xi_M) \) and \( \tilde{C}_{ue}(\omega_o, \Xi_M) \), respectively, are estimated. The \( \tilde{C}_{le}(\omega_o, \Xi_M) \) term is expressed in the complex valued form as follows.

\[
\tilde{C}_{le}(\omega_o, \Xi_M) = C_{ln} \tilde{a}_l(\omega_o, \Xi_M) = C_{ln} \left( \alpha_{pl} + i\beta_{pl} \right),
\]

(4.95)

\[
\alpha_{pl} = \alpha_l(\omega_o, \Xi_M),
\]

(4.96)

\[
\beta_{pl} = \beta_l(\omega_o, \Xi_M).
\]

(4.97)

Here, \( C_{ln} \) is the nominal value of \( C_l \), \( \tilde{a}_l(\omega_o, \Xi_M) \) is the spectrally-varying and amplitude-sensitive parameter for \( C_l \) with the coefficients \( \alpha_{pl} \) and \( \beta_{pl} \). From Eqn. (4.72) by replacing \( s \) by \( i\omega \) and employing Eqns. (4.95)-(4.97), the coefficients \( \alpha_{pl} \) and \( \beta_{pl} \) are estimated as follows.

\[
\frac{P_{uRE}^2 + iP_{uIM}^2}{P_{RE}^2 + iP_{IM}^2} = \frac{i}{1 - \omega^2 I_i C_{ln}(\alpha_{pl} + i\beta_{pl}) + \omega R_i C_{ln}(\alpha_{pl} + i\beta_{pl})^2},
\]

(4.98)

\[
\alpha_{pl} = \frac{\alpha_{pl2}\omega_o^2 + \alpha_{pl1}\omega_o}{d_{pl}(\omega_o^2 + R_l^2 \omega_o^2)},
\]

(4.99)

\[
\beta_{pl} = \frac{\beta_{pl2}\omega_o^2 + \beta_{pl1}\omega_o}{d_{pl}(\omega_o^2 + R_l^2 \omega_o^2)},
\]

(4.100)

\[
d_{pl} = C_{ln}(P_{IRE}^2 + P_{IM}^2),
\]

(4.101)

\[
\alpha_{pl2} = I_l(P_{IRE}^2 + P_{IM}^2 - P_{uRE}P_{IRE} - P_{uIM}P_{IM}),
\]

(4.102)

\[
\alpha_{pl1} = R_l(-P_{uRE}P_{IM} + P_{uIM}P_{IRE}),
\]

(4.103)

\[
\beta_{pl2} = I_l(P_{uRE}P_{IM} - P_{uIM}P_{IRE}),
\]

(4.104)

\[
\beta_{pl1} = R_l(P_{IRE}^2 + P_{IM}^2 - P_{uRE}P_{IRE} - P_{uIM}P_{IM}).
\]

(4.105)
The $\tilde{C}_{ue}(\omega_o, \Xi_M)$ term is also estimated by substituting $i\omega$ for $s$ with Eqn. (4.73) and employing the results $\tilde{C}_{le}(\omega_o, \Xi_M)$ from Eqns. (4.95)-(4.105). Define $\tilde{C}_{ue}(\omega_o, \Xi_M)$ in terms of an empirical spectrally-varying and amplitude-sensitive parameter $\tilde{\lambda}_{ue}(\omega_o, \Xi_M)$ along with coefficients $\alpha_{pu}$ and $\beta_{pu}$.

$$\tilde{C}_{ue}(\omega_o, \Xi_M) = C_{un}\tilde{\lambda}_{ue}(\omega_o, \Xi_M) = C_{un}\left(\alpha_{pu} + i\beta_{pu}\right),$$  \hspace{1cm} (4.106)

$$\alpha_{pu} = \alpha_{u}(\omega_o, \Xi_M),$$ \hspace{1cm} (4.107)

$$\beta_{pu} = \beta_{u}(\omega_o, \Xi_M).$$ \hspace{1cm} (4.108)

$$\rho = |\Xi_{M}(\omega_o)|,$$ \hspace{1cm} (4.109)

$$\phi_{\Xi} = \angle\Xi_{M}(\omega_o),$$ \hspace{1cm} (4.110)

$$P_{ulRE1} = \left|\bar{P}_{ulM}\right|\cos(\phi_{PuM} - \phi_{\Xi}),$$ \hspace{1cm} (4.111)

$$P_{ulIM1} = \left|\bar{P}_{ulM}\right|\sin(\phi_{PuM} - \phi_{\Xi}).$$ \hspace{1cm} (4.112)

Here, $\bar{P}_{ue}(\omega_o, \Xi_M)$ is calibrated using $\Xi_{M}(\omega)$ as a reference value for the sake of deriving the effective values.

$$\frac{\bar{P}_{ulRE1} + iP_{ulIM1}}{\rho} = \frac{A\left[1 - \omega^2\tilde{C}_{le}I_i + i\omega\tilde{C}_{le}R_i\right]}{C_{un}\left(\alpha_{pu} + i\beta_{pu}\right)\tilde{C}_{le}I_i + i\omega C_{un}\left(\alpha_{pu} + i\beta_{pu}\right)\tilde{C}_{le}R_i},$$ \hspace{1cm} (4.113)

$$\alpha_{pu} = \frac{\alpha_{pu4}\omega_o^4 + \alpha_{pu2}\omega_o^2 + \alpha_{pu1}\omega_o + \alpha_{pu0}}{d_{pu}(d_{u4}\omega_o^4 + d_{u2}\omega_o^2 + d_{u1}\omega_o + d_{u0})},$$ \hspace{1cm} (4.114)

$$\beta_{pu} = \frac{\beta_{pu4}\omega_o^4 + \beta_{pu2}\omega_o^2 + \beta_{pu1}\omega_o + \beta_{pu0}}{d_{pu}(d_{u4}\omega_o^4 + d_{u2}\omega_o^2 + d_{u1}\omega_o + d_{u0})},$$ \hspace{1cm} (4.115)

$$d_{pu} = C_{un}^2(P_{ulRE1}^2 + P_{ulIM1}^2),$$ \hspace{1cm} (4.116)

$$d_{u4} = C_{in}^2\left(\alpha_{pl}^2 + \beta_{pl}^2\right),$$ \hspace{1cm} (4.117)
\[ d_{u2} = C_{\text{in}}^2 R_i^2 \left( \alpha_{pl}^2 + \beta_{pl}^2 \right) - 2C_{\text{in}} I_i \alpha_{pl}, \quad (4.118) \]
\[ d_{u1} = -2C_{\text{in}} R_i (P_{uRE1}^2 + P_{uIM1}^2) \beta_{pl}, \quad (4.119) \]
\[ d_{u0} = C_{\text{un}}^2 (P_{uRE1}^2 + P_{uIM1}^2), \quad (4.120) \]
\[ \alpha_{pu4} = A_1 \rho C_{\text{un}} C_{\text{in}}^2 R_i P_{uRE1} \left( \alpha_{pl}^2 + \beta_{pl}^2 \right), \quad (4.121) \]
\[ \alpha_{pu2} = C_{\text{un}} C_{\text{in}}^2 \left[ A_1 \rho R_i^2 P_{uRE1} + I_i (P_{uRE1}^2 + P_{uIM1}^2) \right] \left( \alpha_{pl}^2 + \beta_{pl}^2 \right) - 2A_1 \rho C_{\text{un}} C_{\text{in}} I_i P_{uRE1} \alpha_{pl}, \quad (4.122) \]
\[ \alpha_{pu1} = -2A_1 \rho C_{\text{un}} C_{\text{in}} R_i P_{uRE1} \beta_{pl}, \quad (4.123) \]
\[ \alpha_{pu0} = -C_{\text{un}} C_{\text{in}} \left( P_{uRE1}^2 + P_{uIM1}^2 \right) \alpha_{pl} + A_1 \rho C_{\text{un}} P_{uRE1}, \quad (4.124) \]
\[ \beta_{pu4} = -A_1 \rho C_{\text{un}} C_{\text{in}}^2 R_i^2 P_{uIM1} \left( \alpha_{pl}^2 + \beta_{pl}^2 \right), \quad (4.125) \]
\[ \beta_{pu2} = -A_1 \rho C_{\text{un}} C_{\text{in}}^2 R_i P_{uIM1} \left( \alpha_{pl}^2 + \beta_{pl}^2 \right) + 2A_1 \rho C_{\text{un}} C_{\text{in}} I_i P_{uIM1} \alpha_{pl}, \quad (4.126) \]
\[ \beta_{pu1} = C_{\text{un}} C_{\text{in}}^2 R_i \left( P_{uRE1}^2 + P_{uIM1}^2 \right) \left( \alpha_{pl}^2 + \beta_{pl}^2 \right) + 2A_1 \rho C_{\text{un}} C_{\text{in}} R_i P_{uIM1} \beta_{pl}, \quad (4.127) \]
\[ \beta_{pu0} = -C_{\text{un}} C_{\text{in}} \left( P_{uRE1}^2 + P_{uIM1}^2 \right) \beta_{pl} - A_1 \rho C_{\text{un}} P_{uIM1}. \quad (4.128) \]

Figure 4.19 compares the spectral contents of \( \tilde{C}_{ue}(\omega, \Xi) \) and \( \tilde{C}_{ie}(\omega, \Xi) \). When the range of \( |\tilde{C}_{ue}| \) and \( |\tilde{C}_{ie}| \) is compared, their magnitudes are similar but the phase of \( \tilde{C}_{ue}(\omega, \Xi) \) shows significant dynamic phenomenon above 20 Hz. Thus, the lower chamber must be modeled under in-situ condition.

Overall, the QL model with effective fluid parameters \( k_{HRle}(\omega, \Xi), c_{HRle}(\omega, \Xi), \)
\( \tilde{C}_{ue}(\omega, \Xi) \) and \( \tilde{C}_{ie}(\omega, \Xi) \) is developed based on the reverse path spectral method [4.31, 4.32] as illustrated in Figures. 4.20 and 4.21. Figure 4.20 describes the procedure used to determine effective parameters from relevant transfer functions \( \tilde{G}_1(\omega, \Xi) \) and \( \tilde{G}_2(\omega, \Xi) \)
and dynamic stiffness $\bar{K}_R(\omega, \Xi)$. The $\bar{F}_{IM}(\omega_o, \Xi_M)$ is predicted first using $\bar{F}_{IT}(\omega_o, \Xi_M)$ and $\bar{F}_{IB}(\omega_o, \Xi_M)$. Then $k_{HRre}(\omega_o, \Xi_M)$ and $c_{HRre}(\omega_o, \Xi_M)$ are identified as illustrated in Figure 4.20 (a). Figure 4.20 (b) shows the procedure to determine $\tilde{C}_{ue}(\omega_o, \Xi_M)$ and $\tilde{C}_{le}(\omega_o, \Xi_M)$ that employ $\tilde{G}_1(\omega, \Xi)$ and $\tilde{G}_2(\omega, \Xi)$. Therefore, the interfacial forces are estimated based on Eqns. (4.78) and (4.79).

$$F_{IT}(\omega, \Xi) = F_{IR}(\omega, \Xi) + F_{ITh}(\omega, \Xi), \quad (4.129)$$

$$F_{IB}(\omega, \Xi) = F_{IR}(\omega, \Xi) + F_{IBh}(\omega, \Xi), \quad (4.130)$$

$$F_{IR}(\omega, \Xi) = \bar{K}_R(\omega, \Xi) \Xi(\omega), \quad (4.131)$$

$$\bar{K}_R(\omega, \Xi) = i\omega c_{HRre}(\omega, \Xi) + k_{HRre}(\omega, \Xi). \quad (4.132)$$

Here, $F_{IR}(\omega, \Xi)$ is the rubber path force with $\bar{K}_R(\omega, \Xi)$ as its dynamic stiffness. And $F_{ITh}(\omega, \Xi)$ and $F_{IBh}(\omega, \Xi)$ are the hydraulic path forces on top and bottom sides of the hydraulic mount respectively. The hydraulic path forces $F_{ITh}(\omega, \Xi)$ and $F_{IBh}(\omega, \Xi)$ may be estimated by using two different transfer functions as derived below. First, consider the force transmissibility concept:

$$F_{ITh}(\omega, \Xi) = A_r [\tilde{G}_{2e}(\omega, \Xi) - 2\tilde{P}_{ue}(\omega, \Xi)], \quad (4.133)$$

$$F_{IBh}(\omega, \Xi) = A_r [2\tilde{G}_{2e}(\omega, \Xi) - 1\tilde{P}_{ue}(\omega, \Xi)]. \quad (4.134)$$

Second, consider the dynamic stiffness concept:

$$F_{ITh}(\omega, \Xi) = A_r [-2\tilde{G}_1(\omega, \Xi) + \tilde{G}_3(\omega, \Xi)] \Xi(\omega), \quad (4.135)$$

$$F_{IBh}(\omega, \Xi) = A_r [-\tilde{G}_1(\omega, \Xi) + 2\tilde{G}_3(\omega, \Xi)] \Xi(\omega). \quad (4.136)$$

Figure 4.21 (a) and (b) compare both force estimation methods. The dynamic stiffness concept as illustrated in Figure 4.21 (b) is simpler than the force transmissibility concept.
Figure 4.19 Predicted upper $|\tilde{C}_{ue}(\omega_o,\Xi_M)|$ and lower $|\tilde{C}_{le}(\omega_o,\Xi_M)|$ chamber compliance spectra given $f_E(t) = Re[P_E e^{i\omega_o t}]$ with $|P_E| = 100$ N by using measured $\Xi_M(\omega_o)$ and $P_{uM}(\omega_o,\Xi_M)$. Key: – , $\tilde{C}_{ue}(\omega_o,\Xi_M)$; · , $\tilde{C}_{le}(\omega_o,\Xi_M)$; , nominal value $C_{un}$. 
Figure 4.20 Identification of effective parameters for the reverse path spectral method:
(a) prediction of lower chamber pressure $\tilde{p}_{lm}(\omega_o, \Xi_M)$ and identification of effective stiffness ($k_{HRre}$) and damping ($c_{HRre}$) for the fundamental harmonic term; (b) identification of effective lower chamber compliance $\tilde{c}_{le}(\omega_o, \Xi_M)$ and effective upper chamber compliance $\tilde{c}_{ue}(\omega_o, \Xi_M)$.
Figure 4.21 Force estimation in frequency domain using the reverse path spectral method: (a) force transmissibility ($\tilde{G}_{2e}$) and dynamic stiffness ($\tilde{K}_{Re}$) concepts; (b) dynamic stiffness ($\tilde{G}_{1e}$, $\tilde{G}_{3e}$ and $\tilde{K}_{Re}$) concept.
since it needs only measured displacement $\xi_M(\omega_o)$. Further, the method with both force transmissibility and dynamic stiffness concepts, as illustrated in Figure 4.21 (a), needs two sets of measured data $\xi_M(\omega_o)$ and $P_{uM}(\omega_o, \xi_M)$. However, the number of effective parameters is reduced compared with the case of Figure 4.21 (b). From the frequency domain results, the time histories of interfacial forces with the fluid system model are estimated as follows.

$$f_{IT}(t) = \left| \tilde{F}_{IT}(\omega_o, \xi_M) \right| Re \left[ e^{i(\omega_o t + \phi_{FIT})} \right], \quad (4.137)$$

$$\phi_{FIT} = \angle \tilde{F}_{IT}(\omega_o, \xi_M). \quad (4.138)$$

$$f_{IB}(t) = \left| \tilde{F}_{IB}(\omega_o, \xi_M) \right| Re \left[ e^{i(\omega_o t + \phi_{FIB})} \right], \quad (4.139)$$

$$\phi_{FIB} = \angle \tilde{F}_{IB}(\omega_o, \xi_M). \quad (4.140)$$

4.9 Experimental Validation of Quasi-Linear (QL) Models in Frequency and Time Domains

Figures 4.22 and 4.23 compare measured and predicted interfacial forces by using both mechanical and fluid system QL models which are described in sections 4.7 and 4.8. As observed in Figures. 4.22 and 4.23, the fluid model predicts the interfacial forces better than the 2DOF or 3DOF mechanical model. Specifically, the fluid model reflects the asymmetrical characteristics well. Observe that $\tilde{F}_{IT}(\omega_o, \xi_M)$ of the 2DOF mechanical model follows the measurement quite well. However, the 2DOF model is unable to predict $\tilde{F}_{IB}(\omega_o, \xi_M)$ since it assumes symmetric forces as evident from Eqns. (4.47) and (4.48). The force estimation with the 3DOF mechanical model shows the asymmetrical
behavior for both magnitude and phase spectra as observed in Figures. 4.22 and 4.23. However, its force magnitudes do not match well with direct force measurements since the effective mechanical parameters are not well assessed.

Figures 4.24 and 4.25 show the contributions of rubber and hydraulic paths. The magnitudes of both rubber and hydraulic path forces are similar but much higher than the total force. This suggests that the interfacial forces are affected by the relative phase between paths. For instance, Figures 4.24 (b) and 4.25 (b) show that the $\bar{F}_{IT}(\omega_o,\Xi_M)$ and $\bar{F}_{IB}(\omega_o,\Xi_M)$ spectra are more affected by the rubber path force since the phases of $\bar{F}_{IT}(\omega_o,\Xi_M)$ and $\bar{F}_{IB}(\omega_o,\Xi_M)$ are the almost same as $\bar{F}_{IT}(\omega_o,\Xi_M)$ and $\bar{F}_{IB}(\omega_o,\Xi_M)$.

Figure 4.26 compares measured $f_{IT}(t)$ and $f_{IB}(t)$ with estimated forces in the time domain by using the 2DOF mechanical and fluid system QL models. Note that only the fundamental $\omega_o$ term is included here. To assess the validity of each model, define an error $\varepsilon(t)$ between the measured force $f_{TM}(t)$ and predicted force $f_{IT}(t)$ and $f_{IB}(t)$ at any time $t$ as: $\varepsilon(t) = (f_w(t) - f_{TM}(t))/f_{TM}(t)$, ($w = IT$ or $IB$). The overall root-mean-square error $E$ (RMS) is then given by $E = \sqrt{\frac{1}{N_{max}} \sum_{v=1}^{N_{max}} [\varepsilon_v(t)]^2}$ where $N_{max}$ is the maximum number of points in the time domain. For the top forces $f_{IT}(t)$, errors from the 2DOF and fluid system models are found as 71.0% and 9.0% respectively. Errors for the bottom forces $f_{IB}(t)$ from the 2DOF and fluid system models are 111.0% and 3.7% respectively. Thus, the fluid system model is more accurate than the 2DOF mechanical model. Nevertheless, the fluid system model still shows discrepancies since it does not include any super- or sub-harmonics.
Figure 4.22 Comparison between measured and predicted interfacial force spectra $F_{IT}(\omega_o, \Xi_M)$ using mechanical or fluid system model, given $f_E(t) = Re[F_E e^{i\omega_o t}]$ with $|F_E| = 100$ N, and by employing measured $\Xi_M(\omega_o)$ and $P_{uM}(\omega_o, \Xi_M)$. Key: measured; 2DOF mechanical model; 3DOF mechanical model; fluid system model.
Figure 4.23 Comparison between measured and predicted interfacial force spectra $\overline{F}_{IB}(\omega_o,\Xi_M)$ using mechanical or fluid system model, given $f_E(t) = \text{Re}[\overline{F}_E e^{i\omega_o t}]$ with $|\overline{F}_E| = 100$ N, and by employing measured $\Xi_M(\omega_o)$ and $\overline{P}_{uM}(\omega_o,\Xi_M)$. Key: , measured; -- , 2DOF mechanical model; --- , 3DOF mechanical model; --- , fluid system model.
Figure 4.24 Comparison between rubber $\bar{F}_{ITr}(\omega_o,\Xi_M)$ and hydraulic $\bar{F}_{ITh}(\omega_o,\Xi_M)$ paths with the fluid system model, given $f_E(t) = \text{Re}[\bar{F}_E e^{i\omega_o t}]$ with $|\bar{F}_E| = 100$ N, and by employing measured $\Xi_M(\omega_o)$ and $\bar{P}_{uM}(\omega_o,\Xi_M)$. Key: $\cdot\cdot\cdot$, $\bar{F}_{IT}(\omega_o,\Xi_M)$; $\cdot\cdot\cdot\cdot\cdot$, $\bar{F}_{ITr}(\omega_o,\Xi_M)$; $\cdot\cdot\cdot$, $\bar{F}_{ITh}(\omega_o,\Xi_M)$. 

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Figure 4.25 Comparison between rubber $\vec{F}_{IBr}(\omega_o, \Xi_M)$ and hydraulic $\vec{F}_{IBh}(\omega_o, \Xi_M)$ path with the fluid system model, given $f_E(t) = \text{Re}[\vec{F}_E e^{i\omega_o t}]$ with $|\vec{F}_E| = 100$ N, and by employing measured $\Xi_M(\omega_o)$ and $\vec{P}_{\omega M}(\omega_o, \Xi_M)$. Key: $\square$, $\vec{F}_{IB}(\omega_o, \Xi_M)$; $\blackline$, $\vec{F}_{IBr}(\omega_o, \Xi_M)$; $\blackline$, $\vec{F}_{IBh}(\omega_o, \Xi_M)$. 
Figure 4.26 Comparison between predicted and measured interfacial forces in time domain given $f_E(t) = Re[\tilde{F}_E e^{i\omega_0 t}]$ with $|\tilde{F}_E| = 100$ N and $\omega_0/2\pi = 25$ Hz. Key: ——, measurement; -- , fluid system model; --------, 2DOF mechanical system model.
4.10 Conclusion

This chapter has made several contributions to the state of the art, as summarized below. First, a new concept is developed to indirectly measure interfacial forces by employing the hydraulic mount as a dynamic force sensor. The proposed method utilizes a combination of models and operating motion and/or pressure measurements. Second, a laboratory experiment, consisting of a powertrain, three powertrain mounts including a dynamic load sensing hydraulic mount, a sub-frame, and 4 bushings, is then constructed to verify the proof of the concept. Third, the lower chamber pressure \( p_l(t) \) is estimated as it was not measured in either our study or any prior articles [4.10, 4.11, 4.12, 4.17, 4.18]. This has led to a better estimation of effective lower chamber compliance \( \bar{C}_{le}(\omega, \Xi) \) along with \( k_{Rre}(\omega, \Xi), c_{Rre}(\omega, \Xi) \) and \( \bar{C}_{ue}(\omega, \Xi) \). Overall, the proposed fluid system model yields a better prediction of the interfacial forces though the mechanical models provide some useful insights. This research also advances prior component studies [4.10, 4.11] by providing an improved system perspective.

This chapter has focused on the frequency domain analyses though limited results in time domain are also presented. However, interfacial force measurements exhibit multi-harmonic terms (Figure 4.6) under sinusoidal excitation. As part of future work, the super-harmonic responses in time domain should be incorporated for a better estimation of the interfacial forces. Other excitations such as transients [4.2, 4.18] should be considered as well.
References for chapter 4


CHAPTER 5

CONCLUSION

5.1 Summary

This dissertation has improved the indirect force estimation method with the hydraulic mount alone and in the multi-degree of freedom isolation system. In order to better understand the dynamic characteristics of the hydraulic engine mount, the force transmitted is divided into rubber and hydraulic force paths. And a new quasi-linear model is developed by embedding the spectrally-varying and amplitude-sensitive parameters using measured displacement and upper chamber pressure [5.1, 5.2].

In Chapter 2, indirect force estimation with hydraulic mount has been investigated by using quasi-linear models with focus on the fundamental harmonic term. First, the linear system transfer functions are determined to relate $F_T$ to $X$ and $P_u$ in the Laplace and frequency domains for both rubber and hydraulic force paths, and predict $f(t)$ using $x(t)$ and/or $p_u(t)$. Then, effective (spectrally-varying and amplitude-sensitive) mount parameters are estimated and embed into quasi-linear models. Thus, all mount nonlinearities are lumped into upper chamber compliance $C_{ue}(\omega, X)$, rubber stiffness $k_r(\omega, X)$, and damping $c_r(\omega, X)$ properties. Based on the developed quasi-linear model, the
force transmitted to the rigid base are estimated in the time domain by using the quasi-linear model and compared with measurements.

In Chapter 3, quasi-linear models with spectrally-varying and amplitude-sensitive parameters have been developed including super-harmonic terms and compared with linear time-invariant and nonlinear models. To investigate the relevance between pressure and force, the measured $p_{uM}(t)$ and $f_{TM}(t)$ in both time and frequency domains are compared and their spectral contents examined. Then, a quasi-linear model with spectrally-varying and amplitude-sensitive parameters at both $\omega_o$ and $n\omega_o$ ($n = 2,3,4, \cdots$) terms are proposed with both fluid and analogous mechanical system models to predict $f_f(t)$ and compare with $f_{TM}(t)$. These results in the frequency domains are assessed in the time domain by using the Fourier series expansion as well. Also, the predictions of force $f_f(t)$ are conducted by linear time-invariant and nonlinear models of both fixed and free decoupler mounts and compared with the proposed quasi-linear models and measurements.

In Chapter 4, a nonlinear hydraulic mount has been embedded in a multi-degree of system and then utilized as a load sensing device to estimate interfacial forces at both ends of the mount. First, a laboratory experimental setup consisting of a powertrain, three powertrain mounts including a dynamic load sensing hydraulic mount, a sub-frame, and 4 bushings, is constructed. Then, the system is excited in the vertical direction by a steady state sinusoidal force $f_E(t)$ of frequency $\omega_o$ and motions in other directions are ignored. The experiments are conducted under sinusoidal excitation and dynamic accelerations (at different points in the system), fluid pressure (in the top chamber of the load sending
mount) and forces (at selected interfaces) are measured. Based on the laboratory experimental setup, 2 and 3 degree of freedom (DOF) linear and quasi-linear models are developed with spectrally-varying and amplitude-sensitive parameters as suggested in prior component studies [5.1, 5.2]. This would determine the effective parameters including upper and lower chamber compliances. Finally, the interfacial forces at both ends of the load sensing hydraulic mount are estimated by employing mechanical and fluid models of the load sensing mount, and their results are compared with direct force measurements in frequency and time domains.

5.2 Contributions

Main contribution of this research has been the development of new quasi-linear models of hydraulic engine mount. New methods to estimate interfacial forces are proposed by employing the spectrally-varying and amplitude-sensitive parameters, and super-harmonic terms are successfully predicted. The hydraulic mount is also utilized as a dynamic force sensor in a multi-degree of freedom isolation system. Specific contributions for each chapter are as follows.

Chapter 2:

1. Alternate transfer function formulations are derived for a linear system that relate sinusoidal motion \( x(t) \) and/or chamber pressure \( p_u(t) \) to harmonic forces through rubber and hydraulic paths. The derivation of force to pressure transfer function permits an estimation of forces only by pressure measurements.
2. The effect of mount nonlinearities on forces is quantified only in terms of top chamber compliance and rubber path properties, unlike prior studies [5.3-5.7]. This leads to a quasi-linear model with spectrally-varying and amplitude-sensitive parameters. Since the hydraulic force path has been successfully characterized only by $C_{ue}(\omega, X)$ and thus, simpler force estimation schemes could be developed.

3. The forces transmitted to a rigid base under harmonic displacement excitation are successfully predicted and compared with measured forces with respect to the fundamental harmonic term.

Chapter 3:

1. The super-harmonic contents of measured upper chamber pressure $p_{uM}(t)$ and force $f_{TM}(t)$ are examined and correlated.

2. Effective parameters $k_{re}(\omega_0, X_1)$, $c_{re}(\omega_0, X_1)$ and $\bar{C}_{ue}(n\omega_0, X_n)$ at the fundamental frequency and super-harmonics ($n\omega_0$) are quantified for rubber and hydraulic force paths.

3. Alternate relevant transfer function formulations with $n\omega_0$ terms are examined by employing the Fourier series expansion as well as the reverse path spectral method.

Chapter 4:

1. The hydraulic mount is developed as a dynamic force sensor to indirectly measure interfacial forces. The proposed method utilizes a combination of models and operating motion and/or pressure measurements.

2. A laboratory experimental setup, consisting of a powertrain, three powertrain mounts including a dynamic load sensing hydraulic mount, a sub-frame, and 4 bushings, is constructed to verify the proof of the concept.
3. The lower chamber pressure $p_l(t)$ is estimated as it was not measured in either our study or any prior articles [5.1-5.3, 5.6, 5.7]. This has led to a better estimation of effective lower chamber compliance $\bar{C}_{le}(\omega,\Xi)$ along with $k_{HRre}(\omega,\Xi)$, $c_{HRre}(\omega,\Xi)$, and $\bar{C}_{ue}(\omega,\Xi)$.

5.3 Future Work

Suggestions for future work include the following:

1. Include a sub-harmonic term in the dynamic model of a nonlinear hydraulic mount since experimental data exhibit sub-harmonic responses under sinusoidal excitation.

2. Examine the effective properties of lower chamber compliance within the hydraulic mount since transient responses are also affected by the lower chamber dynamics [5.8].


4. Predict interfacial forces under transient excitations in a multi-degree of freedom nonlinear system.

5. Develop interfacial force estimation methods further for other physical system [5.9, 5.10].
References for chapter 5


MTS Elastomer Test System 831.50, 1000 Hz model, [http://www.mts.com](http://www.mts.com).


