THEORY OF QUANTITY-SETTING FIRM AND RISK AVERSION:  
A CERTAINTY-EQUIVALENT APPROACH

DISSERTATION
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*****

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CHAPTER I
INTRODUCTION

1.1. The Purposes of This Study

The past ten years have witnessed important advances in the theory of the firm. A number of writers have extended the traditional firm theory to the world of uncertainty. Specifically, they have devoted a great deal of attention: (1) to the question of how a firm's production decisions are affected by its attitude towards risk-taking when facing uncertain output price, as well as (2) to conducting comparative-static experiments in this context. Despite the large pile of individual works, each focusing on some subjects within this context,¹ there is the glaring lack of a systematic and comprehensive study of firm theory under uncertainty. This is especially true for the study of tax effects. For instance, the existing literature has nothing to say about the effect of a sales tax, not to mention the differential effect of a specific sales tax from an ad valorem sales tax. So is the effect of a progressive income tax largely ignored; the only two serious attempts yield

¹For example, Tisdell [87] compares certainty output with uncertainty output only. Penner [67] deals exclusively with tax effect. McCall [56] limits his investigation to the case where the firm has constant absolute risk aversion (see Pratt [68]).
doubtful conclusions. As for a proportional income tax, the discussions have been ample, yet most of the results have been incomplete, if not incorrect. All these justify the effort to present a careful and thorough examination of firm behavior under uncertainty. As can be seen later, our study goes far beyond recasting existing results in terms of our model. In most comparative-static cases, our results are either better (in the sense of more powerful in predicting the change in the optimal decisions) or original.

 Practically all previous authors assume that the firm maximizes the expected utility of profits, as compared to mere profits in the traditional context. They presume that randomness in price can be reasonably dealt with by the expectation operator and decision makers' attitudes toward risk incorporated in the utility function. It should be stressed here that in their analysis, the effect of uncertainty depends critically on the attitude of the firm towards risk. This draws attention to the issue of whether a firm should be risk neutral, risk averse, or risk loving in the first place. Many finance authors believe that firms, unlike individual investors who are risk-averse, should be risk-indifferent. This belief seems to be grounded in the so-called "homemade diversification" theorem. Essentially this theorem says that individual investors, by their own diversification, can costlessly eliminate any diversifiable risk present in a firm's investment portfolio so that the firm need not diversify for individuals. While it is true that diversification (i.e., reduction of the variance of the firm's total cash flow) can
be ignored in its capital-budgeting decisions, it is not clear that this implies risk-neutrality on the part of the firm. Although the contribution of a project to the firm's variance of return does not affect the accept-or-reject decision, the correlation of return from this project to that of the market portfolio does enter the criterion. (Rubinstein [73], pp. 171-74.) Therefore, this theorem seems to suggest a risk neutrality toward diversifiable risk rather than a complete risk neutrality. Furthermore, the derivation of this theorem depends on several assumptions, one of which is perfect capital markets.\textsuperscript{2} Lessard and Bower [49] and Van Horne [91] point out that with market imperfections, an approach which recognizes risk-aversion on the part of the management might be desirable. On the other end, the risk-aversion assumption is widely adopted by economics theorists, especially in work dealing with firm's production decision. Unfortunately, in most cases no justification for this assumption is given. Leland [46] is the only exception, arguing that "given that firms are managed according to the wishes of their owners who are typical asset holders, we might surmise that the firm will exhibit risk-averse behavior" (p. 282). In another place he further explains: "Both the notion that investors (who are risk averse) control the firm and the notion that 'security' is a management goal would seem to suggest a risk-averse utility function for the firm" (fn. 11). This type of argument seems to apply best to small and closely-owned.

\textsuperscript{2}The model from which the "homemade diversification" theorem is derived is called "capital asset pricing model" and is further discussed in footnote 1 of Chapter.II.
quantity-setting firms (e.g., a farmer) whose random earnings enter
the utility function of the consumer-producer more directly than is
ture for other types of firm. Apart from such theoretical
considerations, the expected-utility approach may be preferred to
expected-profit approach simply due to the degree of generality, since
the former degenerates to the latter when risk neutrality is assumed.

While the expected-utility approach provides a natural framework
for extending orthodox firm theory to cover the case of uncertainty,
directly working with this objective function incurs some costs.
Common to all the aforementioned articles is the absence of a
diagrammatic exposition of the derivations of their results. The
two-dimensional graphs, which are featured in most articles and texts
in classical firm theory, are not capable of handling the addition
of a utility dimension. It is desirable to maintain a two-dimensional
graphical exposition of the firm's production decisions, and we
consider this too a meaningful goal for this paper to achieve.

1.2. Analytical Tool

Our analytical tool is a certainty-equivalent model based on
Pratt's risk-aversion measures in [68]. The general purpose of
certainty-equivalent models is to circumvent (rather than overcome)
the complications caused by the joint introduction of a utility
dimension and expectation operator. Our tasks call for this kind of

3 A recent article by Hawawini [37] represents an attempt in this
direction. His success is limited by the mean-variance framework
he chooses to work with. For more detailed discussion of this, see
Chapter II.
alternative formulation. To search out a particular certainty-equivalent model, we use three criteria: its applicability, economic plausibility, and reliance on approximation. Some certainty-equivalent models are too general to have practical content. Stone's "two-parameter functional representation (TPFR)" in [86] is such an example. It encompasses all other certainty-equivalent models as special cases, but it itself is void of empirical content. Marschak [54] and Farrar [30], using Taylor's series approximation, work out explicit objective functions, but they contain economically undesirable properties. The derivation of our model can be divided into two stages. In the first stage we show that under assumptions of nonsatiation and monotonicity (in wealth) for the utility function, the basic objective function in our model is essentially the original expected-utility function. Except that our function is defined in dollar space and the other in utility space, they are equivalent. The transformation is carried out through definitions, and it serves to remove the arbitrariness of the utility dimension. In the second stage the objective function is further specified by way of a quadratic Taylor's series approximation and then is expressed as a function of output level for apparent reason. A question naturally arises: how would the approximation affect the results? To answer this, we contrast our results, on one hand, with their counterparts obtained from direct expected-utility maximization whenever possible. On the other hand, we trace the approximation error from the objective function to the partial differential equations resulting from the total differentiations of the first-order condition. Both indicate
that as long as the decision maker is completely and consistently characterized by the Arrow-Pratt risk-aversion indices and their first derivatives, the approximation error has no qualitative effect on the optimal decisions. Strictly speaking, the validity of our model is limited by this qualification. If the utility function of the decision maker is such that the indices and/or their first derivatives change sign frequently, the use of approximation is not warranted. However, we want to point out that the assumption of monotonicity in risk aversion is less restrictive than some of the orthodox justifications, such as assuming a quadratic utility function. As a matter of fact, most of the utility functions employed in the literature, e.g., logarithmic and negative exponential utility functions, do satisfy this requirement. In addition, some authors implicitly assume this property even though no specific utility function is mentioned in their studies.

With the lost generality, we buy ourselves a model that has some advantageous and important features. First, our objective function is decomposed into two separate parts; one readily reflects the substitution effect and the other the wealth (or income) effect. This gives us clear insight into underlying economic forces that determine the final results. Second, our approach not only extends Pratt's definitional concepts of risk premium and cash equivalent into a model, but also brings a less-known risk-aversion measure, the size-of-risk-

---

4 See Arrow [8] and Pratt [68].

5 See, for example, Sandmo [79] and Merton [59].
aversion index, into the center of attention. It is shown that this latter is more than supplemental to Arrow-Pratt absolute and relative risk-aversion indices; it is indispensable in the theory of decision making under uncertainty.

In comparison we believe that our certainty-equivalent model has much to recommend it. Its contribution to the theory of the firm under uncertainty demonstrates one potential use.

1.3. Outline of Chapters

The body of the dissertation is divided into two parts. Chapter II deals with the concept of certainty equivalence and its various in the literature. Our model is developed step by step. Much of what we treated in the preceding section is demonstrated with more rigor and in greater detail. The second part consists of three chapters, all devoted to constructing a theory of the firm under uncertainty, from our certainty-equivalent model. In Chapter III terms in our model are first expressed as functions of output. Optimal output for a competitive firm facing uncertain market price is then derived both graphically and mathematically. Optimal output under uncertainty is contrasted with output under certainty. In the latter portion of this chapter, modifications and extensions of the basic model are attempted. We replace profits by sales revenue in the objective function, first for an imperfectly competitive quantity-setting firm, then for a purely competitive firm. We analyze the resulting change in production decisions in both cases. Moreover, the analysis of decision-making using an additive two-criteria
objective function with profit and sales revenue as its attributes is shown to be within the model's capability. Chapter IV deals mainly with comparative statics. Included are changes in fixed costs, expected price and the variance or the random price. Also included are simple but interesting analyses of: (1) the relationship between the firm's information-collecting activity and its production decision, and (2) the interaction between the firm's investment decision and its production decision. Tax effects on optimal output are discussed in detail in Chapter V. The advantages of our model prove most striking in this chapter. Some original results are presented, along with a thorough evaluation of previous research on tax effects. The latter includes not only works directly addressing tax effects on the firm's production decisions, but also research in portfolio theory focusing in the tax effects on individual investors' portfolio selections. This closing chapter provides a brief summary and conclusion, and some possible future extensions.
CHAPTER II
CERTAINTY-EQUIVALENT APPROACH

2.1. The Essence of Certainty-Equivalent Models

An alternative formulation of the expected-utility approach for attacking the problem of decision making under uncertainty is to employ certainty-equivalent models. This formulation maintains the flavor of the former approach but possesses fewer operational disadvantages. The common procedure of the certainty-equivalent approach is to replace the expected-utility function by a function which contains only two parameters: (1) a measure of the central tendency of the alternative possible outcomes of the random variable, and (2) a measure of relevant risks. In more specified models, in which the two parameters are linked by an explicit functional form, a coefficient of risk aversion usually accompanies the risk measure. This second coefficient carries the risk element forward into an effect on the decision maker’s evaluation of the risky venture. Because this procedure imposes an equivalence (in utility) between a risky venture and two parameters defined on its space of outcomes, it then maps each pair of parameters into a single artificial variate, the value of which is called a certainty equivalent, the equivalent certain amount of the risky venture. For each alternative prospect described by its own pair of parameters, a unique value is assumed by the certainty equivalent by
which to weigh the desirability of each prospect against other opportunities. Thus, the certainty equivalent replaces expected utility as the index of reference or choice criterion.

2.2. Class of General Certainty-Equivalent Models

In this class the certainty equivalent is an implicit function of the return and risk parameters. Certain areas of agreement as to the general nature of the function exist. Most economics theorists agree, for example, that additional units of risk must be compensated by increments in return measure, i.e., there is a trade-off between the two measures. The rate of the trade-off is measured by the marginal rate of substitution (MRS). The MRS is usually assumed to be decreasing, but in this application, it is not explicitly derived.

The best-known model in this class is the mean-variance (EV) or mean-standard deviation (Eσ) approach to one-period portfolio selection. In this framework, the mean is selected as the first measure and the second central moment is the risk measure. Mathematically we have:

\[ EU(W) = U(\bar{W}, \sigma^2_w), \]

or, \[ EU(W) = U(\bar{W}, \sigma_w), \]

where \( E \) denotes the expectation operator, and \( \bar{W} \) and \( \sigma_w \) denote the mean and standard deviation of terminal wealth, \( W \), respectively.

Based on the above assumption, we can present a set of risk-return indifference curves on the \((\bar{W}, \sigma^2_w)\) or \((\bar{W}, \sigma_w)\) plane. The curvature of an indifference curve is governed by the decision maker's MRS between \( \bar{W} \) and \( \sigma^2_w \) (or \( \sigma_w \)). On the other hand, we can plot
numerous points in the same plane, each corresponding to a feasible prospect. A boundary line, called the $\bar{\omega}_w^2$-efficient locus, can be derived by selecting points having highest $\bar{\omega}$ for given $\omega_w^2$ (or $\omega_w$) and/or lowest $\sigma_w^2$ (or $\sigma_w$) for given $\bar{\omega}$. The optimum is the tangency point between the efficient locus and an indifference curve.

Since $\bar{\omega}$ and $\sigma_w^2$ can easily be interpreted in the context of firm theory, employing an EV-model seems promising. A very recent such attempt is by Hawawini [37]. His paper claims to show that "firms' behavior under uncertainty can be easily derived using a geometric approach based on mean-standard deviation framework introduced by Harry Markowitz and extended by James Tobin" (p. 194). The performance of his paper, however, can at best be described as partial success. He manages to analyze only a few of the simplest comparative-static cases, neglecting the more exciting cases of tax effects. The main problem is that only basic characteristics of an individual's preference function can be pictured in the two-dimensional indifference map. Generally speaking, the shape (or slope) of an indifference curve indicates whether one is risk averse; the concavity of an upward-sloping indifference curve distinguishes a diversifier from a plunger (as defined in Tobin [88]); and the change in the degree of concavity from one indifference curve to another at fixed levels of variance indicates whether one becomes more risk averse as one's wealth position changes. (This is exactly what Arrow-Pratt absolute risk-aversion index measures.) Unfortunately, some comparative-static results are conditional on other more-sophisticated characteristics of the individual's utility function. For example, whether one becomes more
risk averse when both variance and wealth change at the same time.

Two-dimensional graphical versions of the general EV-approach are of limited applicability.¹

¹Markowitz' model has been developed into theories of capital-market equilibrium. These models investigate the implications of the two-parameter framework for the equilibrium structure of asset prices. The initial work on this "capital asset pricing model" neglects supply functions for securities and thus the production aspect of the firms. But more recently a whole body of literature has proceeded toward the construction of a general-equilibrium model, in which the capital markets and production are connected under uncertainty. Individuals in this framework act as investors in the frictionless capital market and as shareholders of the firms, making production decisions either to maximize the market value of outstanding stocks (e.g., Mollin [62]) or to serve best their own interests (e.g., Leland [47]). If a firm produces to maximize its share price, pretending that its decision has no effect on the value of other firms (i.e., the assumption of competitive market in the context of uncertainty), then it also operates in the shareholders' interests; the two criteria coincide. Now, the market-value rule in the EV-framework implies a certainty-equivalent objective function of the following form:

\[ V = \frac{E(W) - \lambda \text{cov}(W, M)}{(1+r)}, \]

where \( V \) is the market value of stocks, \( W \) the end-of-period cash flow of the firm, \( M \) the end-of-period return on the market portfolio, \( \lambda \) the market price of risk, and \( r \) the risk-free rate.

This certainty equivalent of random income distinguishes itself from all other certainty-equivalent models in our study in two ways. First, the risk of a well-diversified portfolio is measured by \( \text{cov}(W, M) \), since in the market context, firms need to concern only the systematic portion of the total risks. Secondly, the internal coefficients of risk aversion are replaced by the external market price of risk \( \lambda \). Thus, two firms with identical production functions, the same distribution of demand, and equivalent factor-market conditions must make identical production decisions, regardless of differences in wealth positions or preferences. This independence of optimal decision from individual's tastes is a "separation theorem."

Undoubtedly, this model takes a much broader view of the theory of the firm and has proven to be very fruitful. However, some crucial assumptions draw criticism. For instance, the assumption that investors choose among alternative portfolios solely on the basis of mean and variance of return is valid only under some strict circumstances. Moreover, this separation property no longer holds if investors have heterogeneous subjective probability distributions of return (Lintner [51] and Rubinstein [73]). Thirdly, the objective function (i.e., value of the firm) is evaluated at the point of equilibrium. In other words, production decisions are made only after
An even more general formulation can be found in Stone [86]. This so-called "two parameter functional representation (TPFR)" is an expression for expected utility of the form:

$$ EU(w) = f(w, \psi), $$

where $\psi$ is any measure of risk or reaction to risk. If we substitute $\sigma_w^2$ or $\xi_w$ for $\psi$ in the TPFR, the EV or ES model results. If we substitute Nelson's "composite measure of variance and skewness ($s_Y$)" for $\psi$, the model in Chapter 4 of his book [65] on the term structure of interest rate results. It should be clear that to employ this expression in particular applications, the risk measure, if not also the form of the function $f$, must be specified.

____________________

(con'd)

the firm's shareholders have already traded to their equilibrium positions. Rubinstein [74] considers a situation in which production is simultaneous with share exchange and concludes that this possibility in general invalidates the competitive nature of the market, an assumption crucial to the market-value model. King [43] discusses a similar situation where "the stock market reopen after the announcement of the change in policy" (p. 325) and argues that except in trivial cases shareholders would not consider their position optimal and the production decision may change again. These indicate the possible sources of indeterminacy of the optimal production decisions in the market-value model.

Our model contains subjective elements in the evaluation of the risks. On the one hand, by ignoring the existence of a security market, it is admittedly deficient. On the other hand, though, it does recognize "the special position of management who in reality have both the power and legal responsibility to take the initiative in making decision" (King, p. 320); this is a factor completely ignored by the market-value model. We may add that Stigum's criticism on the market-value rule in [85] is in the same spirit.

As a final justification for our certainty-equivalent maximization, we note that there are still many firms in which decisions are made by one person or by a group of people with sufficiently similar preferences to ensure the existence of a consistent preference ordering of the firm.
2.3. Class of Specified Certainty-Equivalent Models

This class consists of models with operational objective function. Usually they derive their particular functional forms via a Taylor's series expansion of an expected-utility function. For example, Marschak's decision function, when applied to one-risky-security case, is nothing but a quadratic approximation of the expected utility:

\[ EU(W) = U(\bar{W}) + \frac{1}{2} U''(\bar{W}) \sigma_w^2. \]

Farrar's formulation in [30] arbitrarily translates the origin of the above function from \( U(\bar{W}) \) to \( \bar{W} \):

\[ EU(W) = \bar{W} + \frac{1}{2} U''(\bar{W}) \sigma_w^2. \]

While the Marschak and Farrar models are derived from a truncated Taylor's series, Stone's "generalized Markowitz criterion (GMC)" is an exception. The GMC arises from defining a "generalized risk measure (GRM \( \phi \))", which is "the difference in the utility of expected future wealth and the expected utility of future wealth" ([86], p. 12), i.e.:

\[ \phi = U(\bar{W}) - EU(W). \]

By rearranging the terms, the above equation yields the GMC representation:

\[ EU(W) = U(\bar{W}) - \phi. \]

Note that under quadratic expansion of \( EU(W) \), this formulation immediately translates into Marschak's function.

A drawback common to both Marschak's and Farrar's models, (and to GMC when \( \phi \) is approximated by \(-\frac{1}{2} U''(\bar{W}) \sigma_w^2\)), is treating \(-\frac{1}{2} U''(\bar{W})\) as the

\[ \text{To avoid confusion, it should be pointed out that } \phi \text{ is really a measure of the subjective costs of risk-bearing, which is quite different from an objective risk measure, such as variance or range.} \]
coefficient of risk aversion. Since individual preference orderings must be invariant under positive linear transformations of the utility function, to be meaningful a risk-aversion measure must also be invariant under such transformations. The second derivative of the utility function, and therefore \(-\frac{1}{2}U''(\bar{W})\), does not satisfy this requirement.\(^3\)

Another undesirable feature of Marschak's measure and the GMC is the utility index appears in both terms. It will be shown that, in comparative-static analysis, the fewer terms make use of the utility measure, the more concise are the criteria that govern the results. For example, the implication of Farrar's formulation is that assumption on how \(U''(\bar{W})\) changes is mainly what is needed to determine the comparative-static results since the remaining \(\bar{W}\) and \(\bar{S}_w^2\) must change in an objective technical way in response to an outside disturbance.\(^4\) While for Marschak's model, the same disturbance would require us to make additional assumption about the way \(U(\bar{W})\) changes in order to determine its impact on the optimum.

\(^3\)An instructive example which demonstrates the invalidity of \(U''\) as a risk-aversion measure is given by Pratt in [68], p. 127. It should be noted that since Farrar uses his model mainly for empirical fitting an efficient locus in the EV-plane for industrial equities, the above criticism is not relevant to his study.

\(^4\)However the arbitrary transformation of the function's origin seems to discredit Farrar's effort of confining the utility measure to only one term. Farrar himself recognizes this in saying: "Such a transformation, unfortunately, represents an approximation that becomes increasingly tenuous as \(W\) departs from \(\bar{W}\)" ([30], p. 21).
2.4. The Certainty-Equivalent Model Based on Pratt's Measure of Risk Aversion

Pratt [68] certainly did not intend to construct another certainty-equivalent model. Rather, he developed a group of three related concepts: the risk premium, and the absolute and relative risk-aversion functions.\(^5\) For a risk-averse individual,\(^6\) uncertainty means anxiety. The risk premium is the amount he is willing to give up to avoid bearing the anxiety; in other words, the amount he is ready to pay for insurance against the risk. Or we can turn this around to say that a risk averter would be indifferent between taking a risk that has an expected value \(\bar{Z}\) and receiving for sure the excess of \(\bar{Z}\) over the risk premium, or simply the certainty equivalent in the narrow sense (henceforth CE).\(^7\) From the above definition, the following equation results:

\[
EU(w) = U(\omega_0 + \text{CE}), \\
= U(\omega_0 + \bar{Z} - \Omega),
\]

\(^5\)The above concepts were developed independently by Arrow. The two risk-aversion functions are often called Arrow-Pratt risk-aversion indices in the literature. The absolute risk-aversion index is defined as \(r_A = U''(w)/U'(w)\); whereas the relative risk-aversion index is defined as \(r_R = U''(w)w/U'(w) = \omega r_A\).

\(^6\)From now on we focus our attention on the analysis of a risk-averse individual. In most cases, behavior of a risk-indifferent or risk-loving individual can be similarly analyzed by changing the signs of the indices.

\(^7\)Pratt calls this "cash equivalent" or "cash value" of a risk. Unlike the previous broadly defined certainty-equivalent variate, this CE represents an evaluation of the risky prospect in monetary terms. Since its absolute magnitude is meaningful in utility theory, it is the only cardinal measure and the best specified of them all.
where W as before refers to total end-of-period wealth; its expected value W differs from Z by amount of the initial wealth W₀. The symbol Ω denotes the risk premium whose value is dependent on the distribution of the return and the form of the utility function.

Our certainty-equivalent model starts from the same equation by which Pratt defines CE and Ω. We note that since certainty equivalent in the narrow sense is the decision maker's evaluation of the risky prospect, maximizing one is equivalent to maximizing the other; the solution to max EU(W) must also solves max U(W₀+CE). But U(W₀+CE) is maximized only when CE, or equivalent Z-Ω, is maximized due to the nonsatiation and monotonicity of the utility function, the "veil" of the utility scale is thus lifted away from our objective function. Instead of maximizing the expected value of a utility function, we end up merely maximizing the simple difference CE=Z-Ω.

Pratt has shown that "under suitable regularity conditions," (essentially when the variance is very small), the risk premium can be approximated by half the absolute risk-aversion index, evaluated at W, times the variance of the risk:

\[ \Omega = \frac{1}{2} \tau_A(\bar{W}) \sigma_w^2. \]

---

8 These mild assumptions have intuitive appeal and are generally accepted. See, for example, Horowitz [41] and Sandmo [79].

9 The derivation is briefly reproduced here. First we expand both sides of the equation EU(W)=U(\bar{W}-\Omega) around \bar{W} to get:

LHS = U(\bar{W})+\frac{1}{2} U''(\bar{W}) \sigma_w^2,

and RHS = U(\bar{W})-\Omega U'(\bar{W}).

Since U(\bar{W}) cancels itself from both sides, we have:

\[ \Omega = -\frac{1}{2} \sigma_w^2 (U''(\bar{W})/U'(\bar{W})), \]

\[ = \frac{1}{2} \tau_A(\bar{W}) \sigma_w^2. \]
Substituting this approximation for \( \Omega \) into the objective function, we get:

\[ CE = \bar{Z} - \frac{2}{3} r_A (\bar{w}) \sigma^2_w. \]

That the present model is a perfectly legitimate certainty-equivalent model is apparent from the manner in which it is written. Furthermore, the absolute risk-aversion index is also a valid measure of risk aversion since it remains unchanged under positive linear transformations of the utility function.\(^{10}\) Moreover, the utility element appears only in the second term. Consequently, the comparative-static analysis can be less constrained by the arbitrariness of the utility dimension. Finally, we want to mention that Stone, in arguing for his GRM \( \emptyset \), has contended that "\( \emptyset \) may be more accessible in studying risk-return relationship (than \( \Omega \))" ([86], p.19) because \( \emptyset \) is "outside" the utility function whereas \( \Omega \) is "inside" it. On the contrary, we hold that precisely because \( \Omega \) is inside the utility function along with \( \bar{w} \), and is thus independent of the utility scale, we are able to remove the barrier of utility and greatly simplify our study on the impact of uncertainty.

2.5. Quadratic Approximation and Its General Justifications

To translate our model into an operational decision criterion requires us to specify the risk premium. As indicated in footnote 9 above, in general this requires a quadratic approximation of the

\(^{10}\) Let \( V(w) = aU(w) + b \), where \( a \) and \( b \) are positive. Then

\[ r_A = -V''/V' = -aU''/aU' = -\bar{U}'/\bar{U}' \].
expected-utility function via Taylor's expansion. To make the approximation valid requires that the probability distribution or the utility function meet certain conditions. A search into the literature yields the following summarized results. One approach is to assume a quadratic utility function. But this implies implausible behavior. Samuelson [76] proposes a justification called "compact distribution" property. He states that if moments of order three and higher are small in magnitude relative to the first two moments, quadratic approximation is asymptotically valid. Recently Ohlson [66] further extends Samuelson's idea and shows that the approximation is justified if the third central moment vanishes at a faster rate than the first two moments. Tsiang in a controversial paper and a later reply [89,90] argues that quadratic approximation is considered good as long as the ratio of standard deviation to the mean value of total wealth remains sufficiently small, provided the distribution is not extremely skewed. Agnew [1] proves that a so-called "bilateral exponential distribution" is also capable of yielding quadratic expected-utility function even though it does not belong to the stable Paretoian distributions. Other justifications focus on the probability distribution of return. A particularly relevant justification is provided by Sengupta [82] in his book on stochastic models of firms. He demonstrates that with many firms and many buyers interacting in a market, price will tend to be distributed in accordance with a normal

\[11\] As we proceed further, we find in a few cases, the risk premium can be related to the decision variable without approximation.
process.\textsuperscript{12} Since a normal distribution is fully characterized by its first two central moments, the expected utility can be represented by the first two terms in the expansion and the approximation becomes exact under these circumstances.

We have been trying to defend our approximation by arguing that if either the distribution or utility function meets appropriate conditions, there would be no or negligible approximation error. In what follows, we try to justify the use of the truncated Taylor's series in "activating" Pratt's risk premium in a different way. We want to show that for a certain class of utility function, comparative-static results of this paper remain unchanged, even if there exists an approximation error.

2.6. Quadratic Approximation and the Qualitative Comparative-Static Results

Our model aims to compare the production decisions under certainty uncertainty, as well as to investigate the effect of changes in parameters on production in a world of uncertainty. Both tasks belong to the realm of comparative statics. A comparison between optimal output of certainty and uncertainty can also be considered as a comparative-static study since the parameter change (increases from from zero to a positive value) is in the variance or whatever risk measure used. Moreover, it is the qualitative comparative-static results that we try to obtain. All we need to know is the direction

\textsuperscript{12}Empirical evidence, however, seems to run against his assertion. See, for example, Clark [24].
in which a parameter change alters the production levels; in other words, our theory is based on signs rather than magnitudes of the change in optimum.$^{13}$ For our purpose then, approximating $\Delta$ (and henceforth approximating the objective function $CE$) is acceptable so long as the same signs result from the partial derivatives in the comparative statics. The next question then is what signs the partials? To answer this, we notice that in most recent studies concerning decision making under risk, Arrow-Pratt absolute and relative risk-aversion indices have become almost indispensable. Comparative-static results in those studies have typically been governed by the assumed properties of $r_A$ and $r_R$.\textsuperscript{14} It is very important to note that while the two indices and their respective first derivative play a dominant role in signing the partial derivatives in the comparative-static system, no hypothesis on their second derivatives has ever been postulated or needed.\textsuperscript{15} Implicitly, this observation implies two related conditions: (1) that both indices and their first derivatives are monotonic, i.e., their signs can not be reversed by the successively higher derivatives; and (2) that decision makers can be usefully categorized into one of $r_A > 0$, or one of $r_A < 0$.

---

\textsuperscript{13} In the literature this kind of economic theory is called qualitative economics. See Allingham and Morishima [6]. In our model output level is the only decision variable, therefore our comparative-static system is obviously "qualitatively solvable" (p. 9).

\textsuperscript{14} Another relevant and powerful tool is the "stochastic dominance" principle. See, for example, Russell and Smith [75].

\textsuperscript{15} See, for example, Hildreth [40] and Diamond and Stiglitz [27]. Several other sources could easily be cited.
or/and \( r'_R \geq 0 \), but they cannot and need not be further classified as one of \( r''_A \geq 0 \) or \( r''_R \leq 0 \). It is easy to interpret a positive, zero or negative \( r'_A \) or \( r'_R \) and \( r''_A \); but to find an equally significant economic interpretation for the sign of \( r''_A \) or \( r''_R \) is a different story. We can therefore assume away the impact of these second derivatives.

Next we note that:

\[
\frac{r'_A}{U''} = -U''/U',
\]

and

\[
\frac{r''_A}{U''''} = -\left( U'' - \frac{1}{2} U'' U'' \right)/(U')^2,
\]

hence

\[
\frac{r''_A}{U''''} = \frac{-U''U''}{(U')^2} + 2r'_A r''_A.
\]

From the above equations it can be seen that hypotheses on \( r'_A \) and \( r''_A \) impose restrictions on \( U'' \) and \( U'''' \) respectively. Analogously, any particular hypothesis on \( r''_A \) would establish a range of value \( U'''' \) can take. Or conversely, the lack of postulated hypothesis on \( r''_A \) implies that \( U'''' \) is free from restrictions, except that it should not upset the assumed properties of \( r'_A \) and \( r''_A \). Note that this assertion can also be demonstrated with the relative risk-aversion index and its derivatives.

Now we turn to prove that in any partial derivative of output with respect to a parameter, it is exactly \( U'''' \) that would contain the error generated from the quadratic approximation. Then, combined with the assertion we have just derived that the value of \( U'''' \) is not material so long as the Arrow-Pratt indices may properly serve as decision criteria, we can dissolve the dissatisfaction about the approximation for this class of utility function.
That $U'''$ is responsible for "absorbing" the error is because normally the derivation of a partial differential equation requires us to differentiate the quadratic objective function twice: once to get the first-order condition, then totally differentiate with respect to the parameter in interest to derive the partial. During these steps, the quadratic term of the initial objective function is inevitable escalated into $U''''$, carrying within it the difference between truncated expansion and exact expansion.\footnote{17}

In summary, our "quasi-justification" depends on the use of Arrow-Pratt indices as criteria. Since they do not commit themselves to the value of the fourth derivative of the utility function under the assumption of monotonicity in risk aversion, the approximation error is neutralized. It is only incidental that when Pratt develops the two risk-aversion indices, he also provides us with this additional justification for applying his $\bar{Z}-\frac{1}{2}\bar{r}_A \sigma_w^2$ as our objective function.

\footnote{16}By "normally," we preclude corner solutions.

\footnote{17}The whole argument obviously applies to cubic or even higher-order approximation. Quadratic approximation is the "marginal" case.
CHAPTER III

APPLICATION OF THE MODEL TO FIRM THEORY:

BASIC RESULTS AND SOME VARIATIONS

3.1. Introduction and Outline of the Chapter

We have seen that expected-utility maximization is in fact CE-maximization. When applied to the firm theory, this equivalence implies that the task of an expected-utility-maximizing firm is actually one of seeking for an output that generates the highest CE. It is therefore both advantageous and necessary to express CE explicitly as a function of output. For a purely competitive firm, each individual component of CE is ready to be related to output level. To picture the optimization, we graph the risk premium against the quantity of output. We state our results in Section 3.3 but shunt most of the derivations to Appendix A. Then we proceed to compare the production under uncertainty to that under certainty. A brief review on articles explicitly dealing with this comparison concludes the first half of this chapter. Next, we examine some variations of our model. In Section 3.7 we attempt to apply our model to other quantity-setting types of firm. For both theoretical and practical considerations, we replace profit with sales revenue in the objective function. Section 3.8 investigates the possibility of the existence of a finite solution if sales revenue replaces profit
as the goal of a purely competitive firm under uncertainty. Then we ask the question: given the same price distribution, cost function and utility function, is optimal output under expected-utility-of-sales maximization greater than that under expected-utility-of-profit maximization? As expected, the model proves such is the case. In Section 3.10 we consider the production decision when a two-criteria objective function is assumed. Due to the limitation of our CE-approach, decisive results can be derived only in a few restrictive cases.

3.2. Purely Competitive Firm and Its Expected-Profit Function

We assume the firm sells its output in a purely competitive market. It produces product by manufacturing processes that require enough time that the sale price of the product may change materially between the time production begins and the time the output is ready for sale. Thus the firm makes its production decision in the beginning of the period on the basis of a subjectively determined probability density function, \( f(P) \), defined on a finite set of positive prices.\(^1\) The first two central moments of \( f(P) \) are:

\[
E(P) = \int P f(P) dP = \bar{P},
\]

and

\[
\text{var}(P) = E(P - \bar{P})^2 = \sigma_P^2.
\]

The total production costs \( C(Q) + F \) are known with certainty, where \( F \) represents fixed costs. Combining the sales revenue and the costs,

\[^1\] This type of decision making is described by Nelson in his observation about managerial forecasting in [64], pp. 1-2.
we have a profit function:

$$\pi(Q; P) = PQ - C(Q) - F,$$

which is also random with $PQ - C(Q) - F$ as its mean and $\sigma_{F}^{2}$ as its variance.\(^2\)

Each price corresponds to a different profit function, with a different quantity of output that maximizes it. We are particularly interested in the profit function that corresponds to the mean price; that is, $\pi(Q; \bar{P}) = \bar{\pi}$. It is clear that if a firm maximizes expected profit, the optimal output corresponds to the peak of this profit function $\bar{\pi}$. In this sense we can say that expected-profit maximization is achieved if the firm regards the mean price as the market price to prevail.

The expected-profit function is graphed in Figure 1, where $Q^*$ denotes the quantity that maximizes the expected profit, $Q^B$ the short-run break-even (or zero-expected-profit) points, and $Q^T$ the short-run shut-down points. Given an ordinary cost function, the steadily-increasing revenue (at the constant rate of $\bar{P}$) is sooner or later crossed by fast-rising costs (at the rate $C'(Q)$). Thus $\bar{\pi}$ function displays a reversed U-shape. If however, we have a linear cost function which lies below the expected-revenue line, the expected-profit function would be monotonically increasing and fail to have a finite optimal solution. This is depicted in Figure 2.

\(^2\)In this one-period model, the cost of production is sunk when the product is offered to the market, thus $PQ$ is the only random part of $\pi$. 
General Derivation of Expected-Profit Function:

The Profit Function Corresponding to the Mean Price

Derivation of Expected-Profit Function

with Linear Cost Function
3.3. Risk Premium as A Function of Output

As have been shown, the firm's risk premium can be approximated by:

\[ \Omega = \frac{1}{2} r_A(\overline{w}) \sigma_w^2. \]

Substituting \( \overline{\pi} \) for \( \overline{w} \) and \( \sigma_P \sigma_w^2 \) for \( \sigma_w^2 \), the firm's risk-premium function is written as:

\[ \Omega(q) = \frac{1}{2} r_A(\overline{\pi}) \sigma_P^2 q^2. \]

Differentiating \( \Omega(q) \) with respect to \( q \):

\[ \frac{d\Omega}{dq} = \sigma_P^2 r_A(\overline{\pi}) q + \frac{1}{2} \sigma_P^2 (r_A(\overline{\pi})/\overline{\pi}) (d\overline{\pi}/dq) q^2, \]

\[ = \sigma_P^2 r_A Q + \frac{1}{2} \sigma_P^2 r_A \overline{\pi'} q^2. \]

While the first term on the RHS is always positive (for a risk averter), the second term can be either positive, zero or negative, depending on both the type of absolute risk aversion and the output range. For example, when quantity rises above the optimal \( Q^* \), the marginal profit \( \overline{\pi'} \) is negative; but under the assumption of increasing absolute risk aversion, \( r_A' \) is positive. Together they yield a negative second term, and the sign of \( d\Omega/dQ \) is not clear. In Appendix A we explicitly develop the risk premium as a function of \( Q \) in some representative hypothetical cases, using approximation formula for \( \Omega \) only if an exact \( \Omega \) can not be derived. The results are summarized below.

(A) Representing the class of utility functions having constant absolute risk aversion, a negative exponential function is our

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3 We ignore initial wealth temporarily. In Chapter V it is reintroduced and its role discussed.
natural choice.\textsuperscript{4} By assuming the random price takes on normal, gamma and uniform distributions respectively, we derive an exact functional form of the risk premium for each distribution.\textsuperscript{5} In all three cases, the risk-premium function appears to be increasingly increasing with the output. In view of the above derivative, this should be expected since the second term vanishes over the entire range of output when $r_A$ is constant, or equivalently $r'_A = 0$.

(B) The class of decreasing absolute risk aversion is represented by a very basic form: $U'(r) = (\pi^a + b)^c$, where $a$ and $c$ are positive.\textsuperscript{6} Through the approximation formula, general form of the risk-premium function and its first derivative are derived under the assumption of a full quadratic cost function.\textsuperscript{7} Upon further assigning special values for some key parameters in the utility function and cost function, $d\Omega/dQ$ becomes manageable. We exclude the portion of $\Omega$ that corresponds to negative $Q$ for obvious reason. We also ignore the portions outside the

\begin{itemize}
\item \textsuperscript{4} By direct integration of $r_A = k$, a negative exponential function results. See Pratt [68].
\item \textsuperscript{5} The choice of the distributions solely depends on the convenience in deriving $\Omega$.
\item \textsuperscript{6} Pratt proves that certain operations yield decreasing $r_A$ utility function when applied to this form.
\item \textsuperscript{7} A quadratic cost function, on the one hand, captures the crucial rising marginal-cost portion of a cubic cost function; on the other hand, simplifies the derivation to a great extent. If relaxed, the derivation of the risk-premium function and its derivative is still possible for some special cases, e.g., with a logarithmic utility function.
\end{itemize}
short-run closing points since the optimum can not occur at an output which incurs an expected loss greater than the fixed costs. The remaining risk-premium function turns out to be increasing over the relevant range of output in every case. One special case shows that if the cost function is linear without fixed costs, it is possible to derive a linear risk-premium function in addition to a linear expected-profit function. In such a case, a finite solution may not exist for CE-maximization since CE, being the difference between two linear functions, is also linear and is ever-rising. This contradicts the belief that if the firm maximizes expected utility instead of expected profit, a finite optimal output can be found even when a linear cost function is assumed.\(^8\)

(C) We again use a very basic form to represent the family of increasing-\(r_{A}\) utility functions. That is \(U(\pi) = (a-\pi)^b\), where \(\pi \leq a\) and \(b > 1\). Using the approximation formula and assuming a full quadratic cost function, we are able to express the risk premium and its first derivative as functions of output. The risk-premium function is shown to be monotonically rising over the relevant range of \(q\), under an inequality restriction on the values of parameters involved. If the inequality goes the

\(^8\)Bowles and Kendrick [21] uses an EV-diagram to show that "in risky situation the firm's decision variable might stabilize, even though marginal costs of production are constant" (p. 151). Leland also states that in the case when profit is linear in output and the firm is risk averse, "a \(q\) will yield a unique global maximum of expected utility" ([46], p. 281). He mentions this point again in another paper ([45]).
other direction, $\Omega$ could include a declining portion, approaching a positive asymptotic value, for some output levels greater than $Q^*$. This indicates that $r_A$, when sufficiently lowered due to the deviation of $\pi$ from its maximum, may outweigh the $Q^2$ term and pull the risk premium down. In the next two sections, dealing with the construction of CE-function and its maximization, we see that even though $\Omega$ may bend down, as long as this happens for $Q > Q^*$ and $\Omega$ stabilizes at a positive value, our basic results concerning the comparison between certainty output and uncertainty output, as well as the comparative-static results, are not affected.

The above summary suggests that, in general, risk premium may be treated as an increasing function of output. Furthermore, for constant and decreasing absolute risk aversions, risk-premium function is not only increasing but at a non-decreasing rate. The case of increasing-$r_A$ is less conclusive, but one can find comfort in realizing that this is also the least plausible hypothesis on $r_A$ after all.

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9 Baron (11) has a similar statement. He shows that, at the optimal output, the risk premium is increasing. It is obvious that our observation on the shape of $\Omega$ is a much more general one.

10 Postulating increasing-$r_A$ amounts to the belief that risk aversion is more pronounced as one's wealth position is improved. Thus far the consensus is rather against this. See, for example, Arrow (7,8). One especially relevant argument is in Hathaway's book (36) on agriculture industry, a typical competitive industry. He states: "The willingness of farm operators to enter highly uncertain enterprises appears related to their asset position. A farmer whose asset position is weak, in that his equity is low, often avoids uncertain enterprises because if he entered one and the income results were adverse, he would be out of business... Thus, as the asset position of individual farmers improves, it is to be expected that farmers will be less likely to make internal arrangements to mitigate the effects of adverse outcomes,..." (p. 154).
3.4. CE as A Function of Output

Now that both components of CE have been expressed as Functions of output and their shapes explored, we proceed to graph CE against Q. By subtracting an increasing \( \Pi - \)function from the bell-shaped \( \Pi - \)function at each Q, the resulting CE resembles the expected-profit function but is everywhere inside the \( \Pi - \)curve. Ordinarily it has only one hump (Figure 3); if, however, the risk premium includes a declining portion, CE will be double-peaked (Figure 4). But one can be sure that the second, \( Q^X \), is always dominated by a point to the left of \( Q^* \). To prove, let \( Q^Y \) denote an output which generates the same expected profit as \( Q^X \) but is smaller than \( Q^* \). Then \( \Pi(Q^X) > \Pi(Q^Y) \), since \( r_A(\Pi(Q^X)) \) and \( r_A(\Pi(Q^Y)) \) are equal and \( Q^X > Q^* > Q^Y \). Thus:

\[
\text{CE}(Q^X) = \Pi(Q^X) - \Pi(Q^X) < \Pi(Q^Y) - \Pi(Q^Y) = \text{CE}(Q^Y).
\]

Since the second local maximum is immaterial to the optimization, we dismiss this portion of CE in our later studies and graphs.

The CE-maximization is shown in the righthand panel of Figure 5. The optimum falls on \( Q^{**} \), normally referred to as uncertainty output.

3.5. Comparison of Production Decision: Certainty versus Uncertainty

A number of economists have shown interest in this comparison. In any event, before the comparison can be meaningful conducted, one must first specify the meaning of the "certainty output". A popular specification can be found in Sandmo (79) and Leland (46). Both of them define the perfectly competitive certainty output as the optimal quantity when the price to occur is known to be equal to the mean value of its distribution. In our notation, the \( Q^* \). Following this
Figure 3
Construction of CE-Function: Single Peak

Figure 4
Construction of CE-Function: Double Peaks
Figure 5

Comparisons of production Decisions and Utility Levels:

\( \pi \)-Maximization versus CE-Maximization
definition, our model shows that certainty output exceeds uncertainty output, i.e., \( Q^{**} < Q^{*} \). A simple proof is to differentiate CE with respect to \( Q \) and evaluate the derivative at \( Q^{*} \). That is:

\[
\frac{d(CE)}{dQ}\bigg|_{Q^{*}} = \frac{d(\pi - \Omega)}{dQ}\bigg|_{Q^{*}},
\]

\[
= \frac{d\pi}{dQ}\bigg|_{Q^{*}} - \frac{d\Omega}{dQ}\bigg|_{Q^{*}},
\]

\[
= 0 - \frac{d\Omega}{dQ}\bigg|_{Q^{*}};
\]

since the risk premium is rising at \( Q^{*} \), \( \frac{d\Omega}{dQ}\bigg|_{Q^{*}} \) is positive and \( \frac{d(CE)}{dQ}\bigg|_{Q^{*}} \) negative. This implies that \( Q^{*} \) is at the downhill portion of CE, therefore it lies to the right of \( Q^{**} \). Both Sandmo and Leland reach this conclusion.

For a more general model, in which pure competition is only a special case, Dhrymes [26] derives the same result. So too for Horowitz [41] in his chapter 12 and Rothschild and Stiglitz [72] in Section 2F. It should be noted that the above conclusion depends on the risk-aversion assumption. For a risk-preferring firm, the opposite result obtains.

Tisdell [87] adopts a different concept and makes a crucial mistake. The certainty output is assumed to be the "average Bayesian output," the average of profit-maximizing outputs when price takes on different known values; and the uncertainty output is surprisingly our \( Q^{*} \). (It is less surprising when one notices that in his book the firm under certainty maximizes profit and under uncertainty maximizes expected profit). In our notation, it is a comparison between:

\[
\bar{Q} = \int C^{-1}(P)f(P)dP = E[C^{-1}(P)] = E[\xi(P)],
\]

and \( Q^{*} = C^{-1}(\int Pf(P)dP) = C^{-1}[E(P)] = g[E(P)], \)
where \( C' \) denotes the marginal cost function and \( g = C'^{-1} \) is its inverse. By the inverse differential rule, \( g' = 1/C'' \), which is positive if marginal cost is an increasing function.

Since, by Jensen's inequality, \( \mathbb{E}[g(P)] = \frac{1}{2} \mathbb{E}[g(P)] \) according as \( g'' \leq 0 \), Tisdell proceeds to derive \( g'' \). Apparently because he mistakenly identifies \( 1/C'' \) as the inverse of \( C'' \), he uses again, this time improperly, the inverse differential rule on \( g' \) to get \( g'' \) and claims that \( g'' \leq 0 \) as \( C'' \geq 0 \) (p. 81). This leads him to the false conclusion that the certainty \( \bar{Q} \) is greater than the uncertainty \( Q^* \) if the marginal cost is increasing at an increasing rate.

If we differentiate \( g' \) correctly, we get:

\[ g'' = (-C'''/C'') \cdot g', \]

which has opposite, rather than the same, sign as \( C'' \). Hence under Tisdell's specification, the uncertainty output is greater, equal to or smaller than the certainty output according as the marginal cost is increasing at an increasing, constant or decreasing rate.

3.6. **Comparison of Satisfaction Levels: Certainty versus Uncertainty**

A look at the lefthand panel of Figure 5 reveals the undesirability of price uncertainty. A risk averter's utility level of maximized expected profit is always greater than that of maximized CE, i.e.:

\[ U(\text{max } \bar{\pi}) > U(\text{max } \text{CE}) = \max \text{EU}(\pi). \]

The difference between these two measures firm's loss in utility after fully adjusting its production to the now-random price. As mentioned in Chapter II, this difference is also Stone's "GMC \( \emptyset \)," evaluated at the optimal output.
3.7. Other Type of Quantity-Setting Firm and Certainty Equivalent of Sales as A Function of Output

Until now, we assume that the firm is in a purely competitive environment, and is maximizing the expected utility of profit. Before we proceed to apply our CE-model to general quantity-setting firms, we change the argument in the expected-utility function and certainty-equivalent function from profit to sales revenue. It is often argued that the assumption of seeking highest profits is appropriate only for a purely competitive firm. As the firm under consideration deviates from pure competition, a modification of the objective function is warranted.\(^{11}\) One goal that has received considerable attention from both managerial and orthodox economists is sales revenue. Beside its theoretical and empirical plausibility,\(^{12}\) it is the easiest one to analyze. As an attribute that relates to output in a simple way, total revenue fits into our CE-model rather nicely. We start from the now-familiar decomposition of CE into expected revenue and risk premium of sales:

\[
CE_R(Q) = \overline{R}(Q) - \Omega_R(Q),
\]

\[
= \overline{F}(Q) - \frac{1}{2} \sigma_R^2(Q) r_A(\overline{R}),
\]

where subscript \(R\) refers to revenue. The same principle of notation is applied throughout the rest of the paper to all certainty-

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\(^{11}\) Machlup in his review article [52], lists and evaluates various reasons why a modification is needed. See especially pp. 17-18.

\(^{12}\) See Galbraith [32] and Baumol [13] for its theoretical argument, and Evans [29] for an econometric study strongly supporting this goal. A list of other articles concerning sales-maximization can be found in Heidensohn and Robinson [38], chapter 6.
equivalent equations.

For the sake of simplicity, we assume that the variance of price itself is independent of the output.\textsuperscript{13} Then:

$$CE_R = \overline{P}(Q)\sigma_P^2 r_A(\overline{R}).$$

The shape of the first term obviously depends on the assumed demand relationship. If we assume that $\overline{P}(Q)$ is approximately linear in $Q$, then $\overline{R}(Q)$ is bell-shaped just as the previous expected-profit curve. More than this, if the cost function is quadratic, the general shape of the risk premium for sales, $\Omega_R(Q)$, is the same as the risk premium for profit when the utility function are linear transformations of each other. To see this, we compare:

$$\frac{d\Omega\pi}{dQ} = \frac{d\Omega_R}{d\overline{R}} \frac{d\overline{R}}{dQ},$$

with $\frac{d\Omega_R}{dQ} = \frac{d\Omega_A}{d\overline{R}} \frac{d\overline{R}}{dQ}$.

Let $P(Q) = m-nQ$ and $C(Q) = aQ^2 + bQ + F$, where all constants are positive. Then:

$$\frac{d\overline{R}}{dQ} = \frac{d\overline{P}(Q)}{dQ} + \frac{dP(Q)}{dQ} Q,$$

$$= (\overline{P}-b)-2aQ;$$

and $\frac{d\overline{R}}{dQ} = \overline{P}(Q) + \frac{d\overline{P}(Q)}{dQ} Q$,

$$= (m-nQ)-nQ, \text{ since } \overline{P}(Q) = m-nQ,$$

\textsuperscript{13}This means $\sigma^2_R = \sigma^2_P Q^2$, where $\sigma^2_P$ is a constant with respect to $Q$. Note that the variance of revenue is still a function of output. Thus Ieland's "principle of increasing uncertainty (PIU)," which assumes that increasing expected total revenues be accompanied by increasing variance of total revenue, is preserved. Furthermore, the assumption can be relaxed to allow the variance of price to vary directly with output without affecting any of the results below.
Both turn out to be linear in output with differences only in the coefficients. Furthermore, $\partial \Omega \pi / \partial \pi$ and $\partial \Omega R / \partial \bar{R}$ are also alike since the absolute risk-aversion function and its first derivative are invariant to linear transformations of the utility functions. Thus $d \Omega \pi / d Q$ and $d \Omega R / d Q$ indeed have the same functional form.

Having shown that both components of the CE-functions are alike, we can conclude that under quite general conditions an imperfect competitor's certainty equivalent of sales has the same shape of a pure competitor's certainty equivalent of profit.

It should be noted that the linearity in demand is not a necessary condition to have a well-behaved certainty-equivalent function. Conceptually, as long as the expected revenue is intersected from below by the risk-premium function we may have a CE-function that has a global maximum.¹⁵

3.8. Sales Revenue and A Risk-averse Competitive Firm

In the traditional theory of the competitive firm, it is meaningless to consider sales as a maximand. But when uncertainty prevails, and the firm is assumed to be risk-averse, a finite optimal

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¹⁴ The proof of the invariance of $r'_A$ is similar to that for $r_A$. See footnote 11 in Chapter II.

¹⁵ An interesting example is when the demand function is a rectangular hyperbola, i.e. $P = kQ^{-1}$. The certainty equivalent becomes:

$$CE_R = k \frac{\sigma^2}{P} Q^2 r_A(k),$$

and is ever-declining. The optimal output is a corner solution.
output is at least conceptually possible. In Appendix B we derive some exact certainty-equivalent functions for interesting combinations of utility functions and price distributions. For example, a logarithmic utility function (characterized by decreasing absolute risk aversion) is matched with a lognormal distribution, and a negative exponential utility (characterized by constant absolute risk aversion) is matched with normal, gamma and uniform distributions. The results are somewhat unexpected. Although all four yield increasing risk-premium functions, all rising at a non-decreasing rate, only one (resulting from the combination of negative exponential utility and normal price distribution) may rise fast enough to cross the expected-revenue line and ensures a finite optimum. The reason is that in our examples only utility functions with constant or decreasing \( r_A \) are utilized. When a higher output increases the expected revenue, their zero or negative wealth effects tend to restrict the growth of the risk premium. But in any case, we have shown that once firm's attitude toward risk is incorporated, even sales could be a meaningful target.

3.9. Comparison of Optimal Outputs: \( CE_p \)-maximization versus \( CE_P \)-maximization

It is natural to ask then: if a competitive firm switches from profit to sales in its utility function, how would the production decision change?

To construct an answer, let \( Q^{**} \) denote the optimal output to \( CE_P \)-maximization, then:

\[
\frac{dCE_P}{dQ} \mid Q^{**} = \overline{P} - C'(Q^{**}) - \frac{d\Pi}{dQ} \mid Q^{**} = 0,
\]
and \( \bar{P} = C'(Q^{**}) + \left. \frac{d\Omega_{\pi}}{dQ} \right|_{Q^{**}} \).

By differentiating \( CE_{R} \) and evaluating the derivative at \( Q^{**} \):

\[
\frac{dCE_{R}}{dQ}\Bigg|_{Q^{**}} = \frac{d\Omega_{\pi}}{dQ}\Bigg|_{Q^{**}} - \frac{d\Omega_{R}}{dQ}\Bigg|_{Q^{**}},
\]

\[
= \bar{P} - \frac{d\Omega_{R}}{dQ}\bigg|_{Q^{**}},
\]

\[
= C'(Q^{**}) + (\frac{d\Omega_{\pi}}{dQ} - \frac{d\Omega_{R}}{dQ})\bigg|_{Q^{**}}.
\]

But \( \Omega_{\pi} = \frac{3}{2} \sigma_{p}^{2} q^{2} r_{A}(\pi) \) and \( \Omega_{R} = \frac{3}{2} \sigma_{p}^{2} q^{2} r_{A}(R) \). Differentiate both with respect to output, then subtract the second one from the first:

\[
\frac{d\Omega_{\pi}}{dQ} - \frac{d\Omega_{R}}{dQ} = \sigma_{p}^{2} q^{2} [r_{A}(\pi) - r_{A}(R)] + \frac{3}{2} \sigma_{p}^{2} q^{2} [r_{A}'(\pi - C'(Q)) - r_{A}'(\pi)],
\]

\[
= A[r_{A}(\pi) - r_{A}(R)] + B r_{A}'(-C'(Q)).
\]

Assuming constant absolute risk aversion, then \( r_{A}(\pi) = r_{A}(R) \) and \( r_{A}' = 0 \). Therefore:

\[
\frac{d\Omega_{\pi}}{dQ} = \frac{d\Omega_{R}}{dQ},
\]

and \( \frac{dCE_{R}}{dQ}\Bigg|_{Q^{**}} = C'(Q^{**}) > 0. \)

Assuming decreasing absolute risk aversion, then \( r_{A}(\pi) > r_{A}(R) \)

since \( \pi < R \); also \( r_{A}' < 0 \). Therefore:

\[
\frac{d\Omega_{\pi}}{dQ} - \frac{d\Omega_{R}}{dQ} = (+)(+) + (+)(-)(-) > 0,
\]

and \( \frac{dCE_{R}}{dQ}\Bigg|_{Q^{**}} = C'(Q^{**}) + (+) > 0. \)

Assuming increasing absolute risk aversion, then we can prove in a similar way that:

\[
\frac{d\Omega_{\pi}}{dQ} - \frac{d\Omega_{R}}{dQ} < 0,
\]
\[
\frac{dCE_R}{dQ} \bigg|_{Q^{**}} \text{ is positive, zero or negative according as } C'(Q^{**}) \text{ is greater, equal to or smaller than } \left( \frac{d\Omega_\pi}{dQ} - \frac{d\Omega_R}{dQ} \right) \bigg|_{Q^{**}}.
\]

The results can be summarized as follows. If the firm is characterized by constant or decreasing absolute risk aversion, maximizing expected utility of revenue (and thus \( CE_R \)) unambiguously leads to a greater production than maximizing expected utility of profit (and thus \( CE_\pi \)). However constant or decreasing \( r_A \) is sufficient but not necessary. For an increasing-\( r_A \) firm, the above result may also obtain, providing the marginal costs at \( Q^{**} \) are sufficiently high.

Intuitively, these conclusions seem plausible. At \( Q^{**} \), the \( CE_\pi \)-maximizer has already attained the balance between the "benefit" of producing \( Q^{**} \), \( \bar{R}(Q^{**}) \), and the "cost" of producing it, \( C(Q^{**}) + \Omega_\pi(Q^{**}) \); but for the \( CE_R \)-maximizer, part of the cost, \( C(Q^{**}) \), does not concern him. The balance of the latter can be restored by a higher output level, except in the case of increasing absolute risk aversion where the higher risk-aversion cost outweighs the saving in \( C(Q^{**}) \) and thus needs a lower output.

### 3.10. Bi-Criteria Utility Function

O. Williamson [92] develops a revenue-profit model, using both \( \pi \) and \( R \) as arguments. The optimal output appears to be higher than that of using profit in the utility function alone. More recently, Brown and Revankar [22] and Landsberger and Subotnik [44] also select sales revenue to enrich the neoclassical objective function. In this section we apply our CE-approach to the same two-criteria situation under uncertainty.
Due to the limitations of our model, we are able to derive definite results only in a few special cases. A more meaningful case occurs when the firm has an additive utility function, i.e., when:

$$E[U(\pi, R)] = E[U(\pi)] + E[U(R)].$$

It is clear that maximizing the above objective is equivalent to maximizing:

$$CE(\pi, R) = CE_{\pi} + CE_{R}.$$ 

Since both CE's contribute to the total CE, the efficient region does not include the section where both are increasing or decreasing. Thus the optimum to max $E[U(\pi, R)]$, say $Q^{**}_{X}$, must fall between $Q^{**}_{\pi}$ and $Q^{**}_{R}$. The inequalities below summarizes the results:

$$Q^{**}_{\pi} < Q^{**}_{X} < Q^{**}_{R}, \text{ if } r_{A} \leq 0.$$ 

For increasing absolute risk aversion, the inequalities may be reversed.
CHAPTER IV

SOME COMPARATIVE STATICS OF THE FIRM

4.1. Introduction and Formulation of Analysis

By substituting $\bar{P}Q-C(q)-F$ for $\bar{\pi}$, the objective function $CE$ can be written as:

$$CE = \bar{P}Q-C(q)-F-\frac{3}{2}\sigma_P^2 r_A(\pi).$$

Immediately we see three parameters are contained in the certainty-equivalent function, namely, the fixed costs, $F$; the expected price, $\bar{P}$; and the variance of the subjective distribution of the random market price, $\sigma_P^2$. The task of this chapter is to examine how changes in these parameters affect the optimal solution. Mainly what we do in each comparative-static experiment is to construct a new certainty equivalent to reflect the change in the parameter. Then we compare the after-change $CE$ with the initial $CE$ to find the sufficient and/or necessary condition for the new $CE$ to peak at a greater, smaller or the same optimal output level. For instance, if under certain conditions the after-change $CE$ can be expressed as a linear function of the initial $CE$, i.e.,

$$CE_A = aCE + b,$$

where $a$ and $b$ are not functions of output, then we can conclude that the optimum is preserved under such conditions. It should be noted that this can be done without using differential calculus at all, because our objective function is extremely easy
to manipulate. This should be clear in context.

This chapter is organized as follows. Sections 4.2 and 4.3 deal with a change in fixed costs and in the expected price respectively. The output effect of a mean-preserving spread on the subjective distribution of the random price is considered in Section 4.4. A simple analysis of the relationship between a firm's information activity and its output decision is developed in Section 4.5. Indeed, incorporating information activity can be considered either a comparative-static experiment or an extension to our basic model. Yet, we find that by adopting the formulation by Baltensperger and Milde [10], information collection is equivalent to a costly negative mean-preserving spread. Hence it is treated in this chapter. Section 4.6 deals with another extension of the basic model. Here the no-investment assumption is relaxed to allow the firm to seek a balance between financing its production and purchasing bonds with its limited initial wealth. This represents an effort to treat investment decisions and production decisions together.

In each comparative-static experiment, whenever meaningful counterparts can be found, we contrast the results derived to those obtained in traditional and expected-profit-maximization frameworks. These comparisons are summarized in Table 1.

4.2 Effect of A Change in Fixed Costs

In the absence of uncertainty, fixed cost is sunk cost that does not affect the production decision. Even under price uncertainty,
as long as risk-indifferent behavior is assumed, fixed cost exerts no impact on optimality. This is not so when both uncertainty and risk aversion are introduced into the analysis. For convenience, we treat a positive change in fixed costs. We can derive the after-change certainty equivalent by subtracting the change in fixed cost, $\Delta F$, from each $\pi$ that appears in the initial certainty equivalent.

$$CE_p = \bar{\pi} - \Delta F - \frac{1}{2} \sigma_p^2 \sigma^2 r_A(\pi - \Delta F),$$

where the subscript $F$ refers to fixed costs. Note that the variance of $\pi - \Delta F$ is still $\sigma^2 \pi$, since $\Delta F$ is a given amount.

The first two terms on the RHS, $\bar{\pi} - \Delta F$, indicate that the expected-profit function is lowered uniformly by the amount $\Delta F$. This lowers the certainty-equivalent function by exactly $\Delta F$ everywhere. All uncertain prospects are now worth $\Delta F$ less. Furthermore, the wealth argument in the absolute risk aversion function is also lowered by $\Delta F$. This implies a deterioration in the firm's wealth position prior to decision-making, hence the firm may change its demand for risk premium. If so, the change would be in the last term.

Comparing $CE_p$ with the initial $CE$ reveals that the crucial condition that governs the comparative-static result lies in the change in risk premium (from $\frac{1}{2} \sigma_p^2 \sigma^2 r_A(\pi)$ to $\frac{1}{2} \sigma_p^2 \sigma^2 r_A(\pi - \Delta F)$); the term $\Delta F$, being a fixed amount, is of no importance in determining the optimum. Further simplifying the change in risk premium by cancelling out common coefficients indicates that it is the change in absolute risk aversion that dictates the direction in which the optimum changes. If we assume constant absolute risk aversion, i.e., $r_A(\pi - \Delta F) = r_A(\pi)$, then:
\[ CE_F = \bar{\pi} - \Delta F - \frac{1}{2} \sigma^2_F \sigma^2 \bar{r}_A (\bar{\pi} - \Delta F), \]
\[ = \bar{\pi} - \frac{1}{2} \sigma^2_F \sigma^2 \bar{r}_A (\pi) - \Delta F, \]
\[ = CE - \Delta F. \]

The initial CE function is shifted down uniformly over all output levels. The quantity which yields maximal CE would also yield maximal CE*F. The optimum would remain unchanged.

If we assume otherwise, then \( r_A(\bar{\pi} - \Delta F) \) is either greater or smaller than \( r_A(\bar{\pi}) \) since \( \bar{\pi} - \Delta F < \bar{\pi} \). Note that \( r_A \) is accompanied by \( -\sigma^2 \) term in the risk premium, hence a change in \( r_A \) necessarily implies that \( CE_F \) can not be expressed as a linear function of CE. If \( r_A \) is decreased, i.e., \( r_A(\bar{\pi} - \Delta F) < r_A(\bar{\pi}) \) due to increasing risk aversion, the change in risk premium, \( \Delta \Omega_F \), becomes an increasing function of \( Q \). \( CE_F \), being the sum of the initial CE and \( \Delta \Omega_F \), clearly cannot attain its maximum when CE is rising. Thus the new optimal output, \( Q^*_F \), must lie to the right of \( Q^{**} \) where CE starts declining. Similarly if \( r_A \) is increased due to decreasing absolute risk aversion, \( CE_F \) becomes the sum of CE and a decreasing \( \Delta \Omega_F \). The optimal output must be lower than \( Q^{**} \) since \( CE_F \) cannot attain its maximum when both terms are falling.

This case is depicted in Figure 6.

Thus the necessary and sufficient condition for the optimal output to be higher, the same or lower boils down to the assumption of increasing, constant or decreasing absolute risk aversion. This is identical to the results derived by Sandmo, Leland, McCall [57] and Hawawini [37] and others.
Figure 6

Effect of An Increase in Fixed Costs with Decreasing $r_A$

Figure 7

Effect of An Increase in Expected Price with Decreasing $r_A$
4.3. Effect of A Change in Expected Price

In either the profit-maximization or the expected-profit-maximization framework, a rise in price generally causes a competitive firm to produce more if the cost curves are well-behaved, and vice versa. In expected-utility maximization, the same result could be obtained under suitable hypotheses regarding a firm's risk-aversion behavior.

Analysis of this case is analogous to the previous one. First, we write down the after-change certainty equivalent:

\[ CE_{P} = \bar{\pi} + (\Delta \bar{P})Q - \frac{1}{2} \sigma_{Q}^{2} r_{A} [\bar{\pi} + (\Delta \bar{P})Q], \]

where the subscript P refers to price change and \( \Delta \bar{P} \) denotes the increase in expected price.

Unlike \( \Delta F \) in the previous case, \( (\Delta \bar{P})Q \) is a function of \( Q \) and is as crucial as the change in the risk premium in determining the optimum for \( CE_{P} \). Assuming decreasing absolute risk aversion, both the change in expected-profit function and the change in the risk premium exert positive influence on \( CE_{P} \). They both help to raise the certainty-equivalent function and move the optimum further to the right.

Figure 7 on the bottom half of page 48 illustrates this case. For constant-\( r_{A} \) case, \( r_{A} [\bar{\pi} + (\Delta \bar{P})Q] = r_{A} (\bar{\pi}) \), thus \( CE_{P} = CE + (\Delta \bar{P})Q \). A higher optimum results from the change in expected profit alone. For increasing-\( r_{A} \), the change in the two components of the certainty-equivalent function work in opposite directions and create an ambiguous outcome. To state the results in conventional terms, we have shown that decreasing or constant absolute risk aversion is a sufficient condition for a higher optimal output.
Both Sandmo [79] and Leland [46] derive the same result by differentiating the first-order condition, \( dE[U(\pi)]/d\bar{Q}=0 \), with respect to a change in expected price, then rearranging terms to get \( d\bar{Q}/d\bar{P} \). Furthermore, they find that the firm's response to an increase in \( \bar{P} \) can be decomposed into two separate but complicated parts, one is always positive and the sign of the other depends on the hypothesized pattern of \( r_A \). Noticing their resemblance to the components of a Slutsky equation, Leland terms them "revenue-substitution effect" and "risk-income effect" respectively. It is worth noting that our approach tells that the change in certainty equivalent due to an increase in \( \bar{P} \) can also be decomposed into two parts: one that is always positive is the change in total revenue, the other whose sign depends on \( r_A \) is the change in risk premium. Yet when compared to theirs, not only the forms of our terms are greatly simplified but also the meaning of the terms becomes more transparent. Leland's terminology fits in perfectly and obviously.

It is also worth mentioning that Aivazian [4] in examining the effect on the demand for an asset when its price rise distinguishes two sets of income and substitution effects. One corresponds to the traditional income and substitution effects of consumer theory. The other set is called "productive effect" which arises via the "productive characteristics" of the asset, namely, its mean and variance. While an increase in the price of one asset may cause reallocation among assets, it also changes the productive characteristics of the asset and may generate reallocation among the characteristics of the portfolio which in turn has feedback effect on the demand for that
asset. The former set of effects is irrelevant to our single-product firm since there is no alternative production the firm can switch to or from when the relative price changes. Of the latter set we can easily identify our \((\bar{P})Q\) as his "productive-substitution effect" and our \(\frac{1}{2}\sigma_P^2Q^2\) as his "productive-wealth effect." These recognitions help to bring out the fact that the output is affected solely because the higher \(\bar{P}\) changes the productivity of \(Q\) in contributing to the utility-yielding CE. When decreasing absolute risk aversion is assumed, a higher optimal output results because \(Q\) becomes more productive in "producing" expected profit while at the same time less productive in "by-producing" risk aversion. When increasing absolute risk aversion is assumed, a higher \(\bar{P}\) simultaneously causes \(Q\) to be more productive in yielding expected profit and counterproductive in creating risk aversion. We have to weigh one effect against the other to determine the sign of the change in the optimum.

A final point to make before we finish this section concerns the firm's supply curve under uncertainty. In the traditional framework, a competitive firm's responses to changes in price generate its supply curve. It does not, however, make sense to speak about the output effect of "change in price" in our uncertainty world since the price is seen by the firm as a random variable whose realized value depends partly on the market demand for the product. This is the same story as with profit under uncertainty. Thus, strictly speaking there is no supply curve for our competitive firm. It seems natural to associate this with the well-known phenomenon that a monopolist has no supply curve. In both cases, the supply curve cannot to derived
because there is no one-to-one correspondence between a certain price and the quantity to produce. And the lack of such unique relationship is due to the variability in market demand.¹

Sandmo [79] suggests that under uncertainty a firm's supply curve is best considered as a relationship between quantity produced and the mathematical expectation of the price with higher central moments constant. An interesting implication of his definition is that we may observe a downward-sloping supply curve for a firm whose absolute risk aversion is increasing and the risk-income (or productive-wealth) effect outweighs the revenue-substitution (or productive-substitution) effect. A phenomenon analogous to the backward-bending supply for labor or upward-sloping demand for Giffen goods.

4.4. Effect of A Positive Mean-Preserving Spread (MPS) on the Price Distribution

The idea of MPS can be found in Rothschild and Stiglitz [71]. Essentially they use MPS as a criterion to determine whether one distribution is riskier than the other. Applied to our theory of the firm, a positive MPS on the price distribution means an increase in the variation of price with no change in the mean value.² Since

¹For a monopolist, we may construct two demand curves each has its MR curve intersect the MC curve at the same Q. Then it follows that the monopolist produces the same output but charges different prices under these different demand situations. A two-to-one relationship results.

²A positive MPS implies an increase in the variance, but not vice versa. Also, a simple MPS means the same thing as a "stochastic dominance in the second degree." Compare Diamond and Stiglitz [27], equation 5, with Russell and Smith [75], equation 5, to see this.
the price distribution is subjective in the sense that it is based on the entrepreneur's personal experiences and judgment, a positive MPS takes place when he expects the same market price but with less "compact" belief. Heavier weights are placed on prices farther away from the mean, consequently the variance of the distribution is increased by a positive MPS. To see the impact on optimal quantity, we look at the after-change certainty equivalent:

$$CE_V = \pi - \frac{1}{2}(\sigma_p^2 + A)Q^2A(\pi),$$

$$= CE - \frac{1}{2}Ar_A(\pi)Q^2,$$

the subscript V refers to variance, and A denotes the increase in $\sigma_p^2$ and is positive. As can be seen, the expected profit part is not affected but the risk premium is "blown up" proportionately so that the CE function contracts toward lower-left corner and results in a smaller optimal output level. Intuitively, we can think of the mean-preserving stretching of the distribution as a pure increase in uncertainty and a risk-averse firm naturally responds by reducing its production.

4.5. Information Activities and the Production Decisions

We have just considered the effect of a positive MPS on the production decision. It is shown that as long as $\sigma_p^2$ is part of the risk measure, an exogenous increase in the variance of price not only discourages the firm's production but also lowers the CE achievable and hence the expected utility for each output level. It is tempting to ask: can the firm reduce the variation in price and therefore obtain higher satisfaction?
Recent development in the theory of information provides an affirmative answer to the above question. By incorporating information-generating activity into the study of uncertainty, a costly channel is created through which the agent can improve his subjective knowledge about the distribution of the stochastic variable. Or more specifically, he can achieve a reduction in the variance of the stochastic variable via an increase in the amount of information collected. In a study of the portfolio behavior of commercial banks, Baltensperger and Milde [10] link the variance of net reserve outflows to the amount of resources spent on customer-information collecting by a so-called "informative function." By introducing the variance into the bank's objective function through a standardization procedure, they derive conditions for determining optimal portfolio and optimal level of information activity.

The informative function fits into our formulation nicely since the variance of the stochastic variable, $\sigma_p^2$, is already in our objective function and need not be "standardized" in.

Let $q$ measure the level of information, the informative function can be stated as:

$$\sigma_p^2 = i(q),$$

where $i' < 0$, indicating the negative relationship between the variance and the amount of information collected. In addition, $i(0) > 0$ denotes the variance of the price when there is no information gathered. We further assume that the marginal cost of information is constant at $\$c$. The certainty-equivalent function with the possibility of information gathering becomes:
\[ CE_I = \tilde{\pi} - cq - \frac{1}{2} i(q) q^2 r_A(\tilde{\pi} - cq), \]

where the subscript I refers to information gathering.

Note that the second term \( cq \), just as additional fixed costs, has no direct effect on the optimum since it does not contain \( Q \).

Therefore the comparison of the risk-premium part, or equivalently, \( i(q) r_A(\tilde{\pi} - cq) \) versus \( i(0) r_A(\tilde{\pi}) \), is of crucial importance. We know, however, that \( i(q) < i(0) \) for all \( q > 0 \), and \( r_A(\tilde{\pi} - cq) \leq r_A(\tilde{\pi}) \) for increasing, constant or decreasing \( r_A \). Thus, under the hypothesis of either increasing or constant absolute risk aversion,

\[ i(q) r_A(\tilde{\pi} - cq) < i(0) r_A(\tilde{\pi}), \]

and therefore \( CE_I \) can be expressed as the sum of \( CE-cq \) and an increasing function of \( Q \). We reach the conclusion that a sufficient condition for the firm to increase its production when costly information reduces risk is increasing or constant absolute risk aversion. For decreasing \( r_A \), the effect of the negative MPS lowers the degree of uncertainty and stimulates production, but the presence of information costs generates an adverse wealth effect and discourages production. The combined result is unclear. Note that only in decreasing absolute risk aversion does the effect of a costly negative MPS differ from that of a costless negative MPS.

Two complications may be added. The first is to consider the effect of a change in information cost on optimal output. Since the

\[ ^{3} \text{An implicit assumption here is that the optimal level of information must be greater than zero. If the costs of procuring and processing information are too high to be profitable (e.g., if sophisticated forecasting techniques are involved), then the firm may choose not to engage in information activity at all.} \]
change again has no impact on the expected profit and has only a quantitative effect on the risk premium, we can easily conclude that it tends further to encourage or discourage firms having increasing or decreasing $r_A$, respectively. For firms with constant $r_A$, this change is neutral.

The second complication concerns an argument made by Baron in [11]: "As information is obtained, .... it is likely that the best estimate of $\bar{F}$ will change, so output can either increase or decrease" (p. 471). If, in addition to the above informative function, we assume that the mean value of price is also a function of information level, say $\bar{F}=j(q)$ where $j' \neq 0$, and $j(0)=0$ is the expected price when there is no information gathered, then the new certainty-equivalent function becomes:

$$CE_{ij} = j(q)Q-C(Q)-F-cq-q^2i(q)q^2r_A[\bar{f}(q)Q-C(Q)-F-cq],$$

where double-subscript $IJ$ refers to the dual effects of information activity.

Compared with the initial CE, we see that the area where we can derive unambiguous result is further restricted. It is less difficult to see this when one realizes that this case is nothing but a grand mixture of all three previous cases. The term $cq$ has exactly the same effect as $\Delta F$. The functions $j(q)$ and $i(q)$ correspond to $\Delta \bar{F}$ and MPS on price distribution respectively. According to the results we derived, shown in Table 1, the only situation where all three changes point to the same direction of change in the optimal $Q$ is when constant absolute risk aversion and upward revision on expected price are both assumed. That is, without considering the $r_A$ function, the firm with
increased optimism about future market price and strengthened faith produces a even larger amount. For all other combinations, for example increasing $r_A$ and $j' > 0$ or decreasing $r_A$ and $j' < 0$, optimal output gains in one component but loses in the other, so that the net effect is indeterminate.

4.6. Investment Activity and Production Decisions

This last extension aims to relate firm's financial investment decisions to its production decisions. We assume that the firm has limited initial wealth, $W_0$, which is used either to finance the production costs or to buy bonds with a risk-free rate or return $r$; there is no idle funds nor borrowed funds. Thus the addition to its total wealth comes from two sources: profit and interest return:

$$W = W_0 + \pi(Q) + r(W_0 - C(Q) - F),$$

$$= W_0 + \pi(Q) + rB(Q),$$

where the investment $B(Q) > 0$, and $B'(Q) < 0$, indicating that a higher production level leaves less funds for purchasing bonds.

The certainty equivalent with investment option can be stated as:

$$CE_B = \pi + rB(Q) - \frac{1}{2} \sigma Q^2 r_A [W_0 + \pi + rB(Q)],$$

where subscript $B$ refers to bond-purchasing opportunity.

A closer look at the above equation reveals that this case is similar to a decrease in the anticipated price. Specifically, the second term $rB(Q)$ has the same effect on the expected profit as $(4\bar{P}Q)$ when $4\bar{P} < 0$. Intuitively, the availability of bonds increases the opportunity cost of production, or equivalently, decreases the expected "price" of output. However, there is a major difference between the
two in that, in terms of expected wealth, lending earns whereas anticipating a lower price costs. With a given hypothesis on \( r_A \) (except constant \( r_A \)), the wealth effects on output for these two cases are just the opposite.\(^4\)

To ascertain the conditions, we note that since \( r_B(q) \) is a decreasing function of output, the only chance of a higher optimal output is when the risk premium with investment is sufficiently reduced to offset the substitution effect. This can happen only if the firm has decreasing absolute risk aversion.

Finally, if the same investment opportunity is open to a profit-maximizer or an expected-profit-maximizer, the optimal output would decrease only when the risk-free rate is greater than profit or expected profit, per dollar of cost.

\(^4\) As a matter of fact, if the firm is to be a debtor, both substitution and income effects would be identical.
### Table 1

Summary of Comparative-Static Analyses
under Certainty, Uncertainty, and Risk Aversion

<table>
<thead>
<tr>
<th>Parameter Changes</th>
<th>Effect on Optimal Output Level</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\pi$-Max.</td>
</tr>
<tr>
<td>Increase (Decrease) in Fixed Costs</td>
<td>0</td>
</tr>
<tr>
<td>Decreasing (Expected) Price</td>
<td>(+)</td>
</tr>
<tr>
<td>Positive (Negative) MPS</td>
<td>N/A</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Information Collection: $\sigma^2_q = i(q)$.</th>
<th>N/A</th>
<th>N/A</th>
<th>+ increasing or constant</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Information Collection: $\sigma^2_P = i(q)$, $F = j(q)$.</th>
<th>N/A</th>
<th>+ if $j' &gt; 0$</th>
<th>-(-) decreasing</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Risk-free Lending at Market</th>
<th>- iff $r &gt; \frac{\pi}{C(Q) + F}$</th>
<th>- iff $r &gt; \frac{\pi}{C(Q) + F}$</th>
<th>- if constant $r_A$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Risk-free Borrowing at Market</th>
<th>+ iff $r &lt; \frac{\pi}{C(Q) + F}$</th>
<th>+ iff $r &lt; \frac{\pi}{C(Q) + F}$</th>
<th>- if increasing $r_A$</th>
</tr>
</thead>
</table>

(a) The word "if" means sufficient condition.
(b) The word "iff" means necessary and sufficient condition.
(c) A question mark denotes ambiguous results.
(d) "N/A" means not applicable in this case.
CHAPTER V

TAX EFFECTS ON OPTIMAL OUTPUT

5.1. Introduction

The main purpose of this chapter is to examine how under uncertainty imposing different types of taxes on a risk-averse competitive firm affects its output decision. As in the previous chapter, in each case, we compare the results derived with what would be obtained if profit maximization or expected-profit maximization were assumed instead. In this way we can appreciate how incorporating a firm's attitude toward risk alters the theory of the firm. While tax effects certainly belong to the comparative statics of this paper, they seem more interesting and important than the effects of changes in other parameters. In the literature, they have drawn considerably more attention. The attention, however, has been distributed among different taxes in an extremely asymmetric way. While lump-sum taxes and proportional income taxes with full loss-offset provision are widely analyzed, sales taxes are largely ignored. Moreover, conjectured effects of different taxes have occasioned incomplete, questionable, or even conflicting assertions by different authors. For example, in the discussion of a proportional income tax, Sandmo [79] and Rothschild and Stiglitz [72] on one hand, and Stiglitz [83] and Mossin [61] on the other hand, have arrived at conclusions which
conflict with each other when the agent is assumed to have decreasing absolute and constant relative risk aversion. In discussing a degressive income tax, (essentially a flat tax with an exemption), Ahsan's [2] conclusions can be restated in a more complete way. It seems wise to examine tax effects in greater detail and in a separate chapter from other comparative-static results.

Before proceeding to analyze the effects of different tax structures, one modification in the certainty-equivalent formula has to be made. That is, the initial wealth\(^1\) of the firm, which has been deliberately ignored in the previous chapters except for section 4.6, must be brought back into the picture; or more specifically, into the risk-aversion function:

\[
CE = \pi - \frac{1}{2} \sigma^2 \pi A (W_0 + \pi).
\]

An obvious reason for this change is that we want to distinguish a wealth tax from an income tax. Were \(W_0\) being left out, these two would have identical tax base and become indistinguishable. A more fundamental reason, regarding an inevitable misinterpretation of the relative risk aversion if \(W_0\) is ignored, will be brought forward in a later section.

In this chapter, each section deals with one particular tax, ranging from a lump-sum tax to a degressive income tax. In addition, Section 5.5 compares a lump-sum tax with sales taxes—specific and

---

\(^1\)Wealth may be defined as assets, incomes from other ventures, cumulated retained earnings, market value of the firm or any other measures that the firm chooses to use. Leland in [47] formulates an objective function using the sum of the market value and current profits as the argument.
ad valorem—to see whether the traditional argument that a lump-sum tax is always preferable to a sales tax holds under uncertainty and risk aversion. Section 5.8 presents a thorough overview of previous research on proportional income taxation with provision for full loss-offset, since most controversies are generated here. The chapter ends with a table which summarizes our conclusions about the effects of different tax structures. Tax effects under profit or expected-profit maximization are also summarized so that a comparison can easily be made.

5.2. Lump-Sum Tax

One of the central theorems of the traditional theory of the profit-maximizing firm asserts that taxes on corporate profits, as long as they are not progressive, are irrelevant in determining optimal quantity. A lump-sum tax is one such tax. It is obvious that this assertion is still true if uncertainty, but no risk aversion, is admitted into the analysis. As a constant, a lump-sum tax is washed away from the first-order condition no matter whether the maximand is profit or expected profit. But when expected-utility-maximization is assumed, the conclusion is totally different. We can write the after-tax certainty equivalent as:

\[ CE_{LS} = \pi - T - \frac{1}{2} \sigma_{\pi}^2 r_A (w_o + \pi - T), \]

where the subscript LS refers to lump-sum tax, T denotes the lump-sum tax payment. The variance of \( \pi - T \) is still \( \sigma_{\pi}^2 \) since T is a fixed amount.
Note that changes in CE due to the lump-sum tax are formally the same as changes due to an increased fixed cost, considered in the previous chapter. This should be expected since both of them have to be paid regardless of the physical quantity, or value of the sales, or the amount of the profits. Thus we can conclude that the necessary and sufficient condition for an increased, an unchanged or a decreased optimal output is the assumption of increasing, constant or decreasing absolute risk aversion respectively.

5.3. Specific Sales Tax

When a specific sales tax of $t per unit of output is imposed, a profit-maximizing competitive firm responds by decreasing its output. This is because the first-order condition requires the firm to lower marginal costs to match the now-lower unit price \( P-t \). Given increasing marginal costs, the firm does this by reducing its output. For a firm maximizing expected profits, marginal cost is equated to \( \bar{P} \) after the imposition of tax. Here, too, a lower optimal output ensues.

In our model the after-tax certainty equivalent is:

\[
CE_{SS} = \pi - tQ - \frac{1}{2} \sigma^2 r_A \left( \sigma_0 + \pi - tQ \right),
\]

\[
= (\bar{P} - t)Q - C(Q) - P - \frac{1}{2} \sigma^2 r_A \left[ \sigma_0 + (\bar{P} - t)Q - C(Q) - F \right],
\]

where subscript SS refers to specific sales tax. The variance of \( \pi - tQ \) remains to be \( \sigma^2_\pi \) since \( tQ \) is not random.

From the above equation, we see that such a tax is formally equivalent to an additive downward shift in the expected price, such as we considered in the previous chapter. Thus we conclude that
decreasing or constant absolute risk aversion is a sufficient but not necessary condition for a decrease in the optimal output, (see Table 1). If the firm happens to be increasingly risk averse, whether the optimum would be higher or lower depends on whether the revenue-substitution effect outweighs the risk-income effect.

5.4. Ad Valorem Sales Tax

If the sales tax is proportional to the value of sales, results are similar to the previous case under either profit maximization or expected-profit maximization.

Let $s$ denote the tax rate. The first-order conditions for a profit-maximizer and an expected-profit-maximizer are $(1-s)p = c'(q)$ and $(1-s)\bar{p} = c'(q)$, respectively. In each case, optimal output is decreased.

In the context of uncertainty, imposing an ad valorem sales tax not only decreases the expected profit by $s\bar{R}$ but also increases the variance of the net profit. That is:

$$\text{var}(\pi - s\bar{R}) = \text{var}(\pi) + \text{var}(s\bar{R}),$$

$$= \sigma_\pi^2 + s^2 \sigma_R^2,$$

$$= (1+s^2)\sigma_\pi^2,$$

since $\sigma_\pi^2 = \sigma_R^2$. The after-tax CE becomes:

$$CE_{AVS} = \bar{\pi} - s\bar{R} - \frac{1}{2}(1+s^2)\sigma_\pi^2 r_A (\bar{w}_o + \bar{\pi} - s\bar{R}),$$

where subscript AVS refers to ad valorem sales tax.

To see the difference in impact on the production decision between the two sales taxes, we let $s\bar{p} = t$ to equalize tax payment.
Then:

$$CE_{AVS} = (\bar{F} - s\bar{F}) Q - C(Q) - F - \frac{s^2}{s} \sigma^2 r_A \left( w_0 + (\bar{F} - s\bar{F}) Q - C(Q) - F \right) - \frac{s^2}{s} \sigma^2 r_A (w_0 + \bar{w} - sR),$$

$$= CE_{SS} - \frac{s}{s} \sigma^2 F Q^2 r_A (w_0 + \bar{w} - sR).$$

Since $r_A$ is always positive, the additional term (with the negative sign) in $CE_{AVS}$ is always decreasing in $Q$. We can conclude that under uncertainty an ad valorem sales tax generally has a greater negative effect on a risk-averse firm's production decision than an equal specific sales tax.

5.5. Lump-Sum Tax versus Sales Tax under Uncertainty

In the traditional theory, it is frequently argued that a lump-sum tax is preferable to a sales tax. Henderson and Quandt [39] use a numerical example to show that given the amount of tax revenue government desires to collect, both the quantity of output consumers can buy and the after-tax profits the firm can earn will be higher in the case of a lump-sum tax than a specific sales tax (p. 220). It is natural to inquire whether the results hold when there is uncertainty and the firm maximizes expected utility of profits.

In the following we proceed to compare a lump-sum tax with a specific sales tax. We utilize the fact that the after-tax certainty equivalent can be expressed as:

$$CE_{LS} = CE - T - \Delta \Omega_{LS},$$

$$CE_{SS} = CE - T - \Delta \Omega_{SS},$$

where, as usual, $CE$ is the before-tax certainty equivalent, $\Delta \Omega$ is the change in risk premium, and subscripts LS and SS refer to lump-sum tax and specific sales tax respectively. They are positive, zero, or
negative as the absolute risk-aversion function is decreasing, constant or increasing. In addition, they are functions of \( Q \) since they contain the term \( \sigma^2_F = \sigma^2_P \).

We need to prove two things. First, the solution to \( \text{max } CE_{LS} \), \( Q^*_{LS} \), is greater than the solution to \( \text{max } CE_{SS} \), \( Q^*_{SS} \). Second, maximal \( CE_{LS} \) is greater than maximal \( CE_{SS} \). If so, we can argue that by substituting a lump-sum tax for a sales tax raising an equal amount of tax revenue: (1) consumers would be better-off since a larger quantity is offered for sale at a given price; (2) the firm would be better-off since it can obtain a higher certainty equivalent of profit, hence, a higher utility level; and (3) the government would be indifferent since the tax revenue collected is maintained. Therefore from the viewpoint of the entire economy, a lump-sum tax is preferable to a sales tax even when there is uncertainty.\(^2\)

We first examine the case of constant absolute risk aversion in Figure 8. From our previous discussion, this assumption necessarily leads to a lower optimum when a sales tax is imposed. Thus \( Q^*_SS < Q^*SS \), where \( Q^*SS \) denotes the before-tax optimum. The size of the tax revenue at this point is \( tQ^*SS \) if the tax rate is \( t \). Now suppose government imposes a lump-sum tax of equal size, i.e., \( T = tQ^*SS \), then \( CE-T \) and \( CE-tQ \) intersect at \( Q^*SS \). Moreover, \( CE-T \) lies above \( CE-tQ \) for all

\(^2\)Under this set of criteria, a specific sales tax is clearly superior to an ad valorem sales tax. This can be seen from the derived relationship between them in the previous section. Thus, the comparison between a lump-sum tax and an ad valorem sales tax would be redundant if a lump-sum tax proves to be superior to a specific sales tax.
Figure 8
Lump-Sum Tax versus Specific Sales Tax with Constant $r_A$

Figure 9
Lump-Sum Tax versus Specific Sales Tax with Decreasing $r_A$
Q > Q^*_SS since within that range tQ > T. There is no change in the risk premium under the assumption of constant absolute risk aversion, i.e., \( \Delta \Omega_{LS} = \Delta \Omega_{SS} = 0 \). Combining the facts that Q^*_LS = Q^*_SS and \( CE_{LS} = CE - T > CE - tQ = CE_{SS} \) for Q > Q^*_SS, the proof is complete.

The case of decreasing absolute risk aversion, depicted in Figure 9, is slightly more complicated in the sense that changes in the risk premium component have to be compared with each other in addition to running a comparison between CE-tQ. At Q^*_SS, while CE-T intersects CE-tQ from below, \( \Delta \Omega_{LS} \) intersects \( \Delta \Omega_{SS} \) from above. The latter event occurs because for \( Q > Q^*_SS \), \( tQ > T \), which implies \( \pi_T > \pi_tQ \). Since the risk premium is inversely related to wealth under the assumption of decreasing absolute risk aversion, \( \Delta \Omega_{SS} > \Delta \Omega_{LS} > 0 \).

Now we differentiate CE_{LS} and evaluate at Q^*_SS:

\[
\left. \frac{dCE_{LS}}{dQ} \right|_{Q^*_SS} = \left. \frac{d(CE - T)}{dQ} \right|_{Q^*_SS} - \left. \frac{d\Delta \Omega_{LS}}{dQ} \right|_{Q^*_SS}.
\]

But from the above discussion we know that:

\[
\left. \frac{d(CE - T)}{dQ} \right|_{Q^*_SS} > \left. \frac{d(CE - tQ)}{dQ} \right|_{Q^*_SS},
\]

and

\[
\left. \frac{d\Delta \Omega_{LS}}{dQ} \right|_{Q^*_SS} < \left. \frac{d\Delta \Omega_{SS}}{dQ} \right|_{Q^*_SS},
\]

thus

\[
\left. \frac{dCE_{LS}}{dQ} \right|_{Q^*_SS} > \left. \frac{d(CE - tQ)}{dQ} \right|_{Q^*_SS} - \left. \frac{d\Delta \Omega_{SS}}{dQ} \right|_{Q^*_SS} = 0.
\]

The above inequality implies that at \( Q^*_SS \), when CE_{SS} attains its maximum, CE_{LS} not only intersects CE_{SS} but continues to climb. This in turn implies that Q^*_LS > Q^*_SS and \( \max CE_{LS} > \max CE_{SS} \). Again a lump-

\[^3\text{Under decreasing absolute risk aversion, } \Omega(W_0 + \pi_T) < \Omega(W_0 + \pi - tQ), \text{ for } tQ > T. \text{ Thus } \Delta \Omega_{SS} > \Delta \Omega_{LS} \text{ and both are positive.}\]
sum tax is preferable to a sales tax of equal size.

The case of increasing absolute risk aversion confronts us with two difficulties. One is that the location of $Q^*_S$ relative to $Q^{**}_S$, and henceforth to $Q^*_L$, is uncertain. If $Q^*_S$ lies to the left of $Q^{**}_S$ or coincides with $Q^{**}_S$ due to a strong "revenue-substitution effect," then $Q^*_L > Q^*_S$, since $Q^*_L$ is greater than $Q^{**}_S$ when increasing $r_A$ is assumed. Figure 10 depicts this situation. If on the other hand, $Q^*_S$ lies to the right of $Q^{**}_S$ due to a strong "risk-income effect," $Q^*_S$ is mathematically and graphically possible to be greater than $Q^*_L$. This is shown in Figure 11. A more serious problem when assuming increasing $r_A$ is that for $Q^*_S$, $T^*_S t Q$ and $T^{**}_S t Q$, so that not only $C^*_E T^*_S C^*_E T$, but also $\Delta Q^*_L \geq \Delta Q^*_S$. In other words, both $C^*_E T$ intersects $C^*_E T^*_S$ and $\Delta Q^*_L$ intersects $\Delta Q^*_S$ from below. When the two inequalities are combined to derive the after-tax certainty equivalent, they tend to offset rather than reinforce each other. This makes it impossible to determine the magnitude of $C^*_E L_S$ relative to $C^*_E S_S$. The indeterminacies are illustrated in Figures 10 and 11, where we use dashed-lines for $C^*_E L_S$ and $C^*_E S_S$ to indicate the fact that they are drawn for demonstrating purpose only; chances are they may be located differently.

In conclusion, the answer to the question we raised at the beginning of this section is affirmative, except in the case of increasing absolute risk aversion, where a specific sales tax may, but need not, be indifferent or even preferable to a lump-sum tax. As for an ad valorem tax, the likelihood for it to be superior to a lump-sum tax is even slimmer.
Figure 10

Lump-Sum Tax versus Specific Sales Tax with Increasing $r_A$: Case 1
Figure 11

Lump-Sum Tax versus Specific Sales Tax with Increasing $r_A$: Case 2
5.6. Proportional Wealth Tax

Wealth taxation has no role in either profit or expected-profit maximization, but it becomes a relevant factor in our model. In principle, variation in wealth influences a firm's attitude toward risk.

If the tax rate is a constant $t$, the after-tax wealth is $(1-t)(\bar{w}_o + \pi)$ and the variance of $\pi$ becomes $(1-t)^2\sigma_\pi^2$. The after-tax certainty equivalent can be written as:

$$CE_W = (1-t)^2\sigma_\pi^2(1-t)^2r_A[(1-t)(\bar{w}_o + \pi)],$$

where subscript $w$ refers to wealth tax.

Factor out $(1-t)$, we get:

$$CE_W = (1-t)^2\sigma_\pi^2(1-t)r_A[(1-t)(\bar{w}_o + \pi)].$$

By the same fashion of analysis we used in the previous chapter, we know that the solution to max $CE_W$ may be greater, equal or smaller than the solution to max $(1-t)CE$, (and henceforth the solution to max $CE$), according as:

$$\frac{1}{2}\sigma_\pi^2(1-t)r_A[(1-t)(\bar{w}_o + \pi)] \leq \frac{1}{2}\sigma_\pi^2(\bar{w}_o + \pi).$$

If we cancel the common factors from each side and multiply each by $\bar{w}_o + \pi$, the condition becomes:

$$(1-t)(\bar{w}_o + \pi)r_A[(1-t)(\bar{w}_o + \pi)] = (\bar{w}_o + \pi)r_A(\bar{w}_o + \pi),$$

or equivalently:

$$r_R[(1-t)(\bar{w}_o + \pi)] = r_R(\bar{w}_o + \pi),$$

where $r_R(X)$ is Arrow-Pratt relative risk-aversion function and is defined before as $Xr_A(X)$. Thus we have found that a necessary and sufficient condition for the after-tax optimum to be higher, equal or lower is simply increasing, constant or decreasing relative risk
aversion.\footnote{The question of which hypothesis of $r^*$ is more realistic is still open to debate. In Gordon, Paradis and Rorke [33], an interesting portfolio-selection game is designed and a number of graduate students of business administration are asked to participate. It reports that the students exhibited increasing relative risk aversion. Arrow [7] has argued in favor of increasing relative risk aversion on both theoretical and empirical grounds. However his theoretical arguments have been questioned by Tsian [89] (p. 355, fn. 2) and Stiglitz [84], and the validity of his empirical evidence from studies on the demand for money questioned by Stiglitz. Projector and Weiss [69], Bossons [20] and especially Cohn, Lewellen, Lease and Schlarbaum [25], have derived empirical results supporting decreasing $r^*$. Lintner [50] argues that constant-$r^*$ hypothesis is good for firms facing large and more portentous decisions.}

That the effect of a wealth tax is fully governed by relative risk aversion should not be surprising since, by definition, relative risk aversion describes how one responds to a change which affects both initial wealth and the risky prospect in the same proportion, and a wealth tax does so affect wealth and the risky prospect.

5.7. Proportional Income Tax with Full Loss-Offset

A proportional corporate income tax (also known as profit tax) is levied on current profits earned by the firm. As pointed out before, such a tax has no short-run effect in either profit or expected-profit maximization since the fixed tax element always washes out of the first-order condition.

This is not true in our model. In this section, we assume full loss-offset so that:

$$CE^F_{PPI} = (1-t)(1-\pi)^2(1-t)^2r_A [W_o +(1-t)\pi],$$
$$= (1-t)(1-\pi)^2(1-t)r_A [W_o +(1-t)\pi],$$

where superscript $F$ denotes full loss-offset and subscript PPI refers
to proportional income tax.

This time the crucial condition for the optimum to be greater, equal, or smaller is seen to be:

\[ \frac{\sigma^2}{\mu} (1-t) r_A (\bar{W} + (1-t) \bar{W}) \leq \frac{\sigma^2}{\mu} r_A (\bar{W} + \bar{W}). \]

Contrary to the previous cases, the necessary and sufficient conditions in this case can not be expressed in terms of either absolute or relative risk aversion. Absolute risk aversion is relevant when considering a change in wealth while the risk remain unchanged. Relative risk aversion is relevant when considering a proportional change in both wealth and the risk. To analyze a corporate income tax, we need a measure of firm's aversion toward risk when risk varies but wealth remains fixed.

Such a measure has been formulated, first by Hanson and Menezes [34, 58] and also independently developed later by Zeckhauser and Keeler [96]. Although to my knowledge, it has received only meager attention. Following Z-K, we call the index the "size-of-risk aversion," \( r_S \). Technically:

\[
\begin{align*}
    r_S (\mu; \bar{W}) &= -\pi U''(\bar{W} + \pi)/U'(\bar{W} + \pi), \\
    &= r_R (\bar{W} + \pi) - \bar{W} r_A (\bar{W} + \pi), \\
    &= \pi r_A (\bar{W} + \pi). 
\end{align*}
\]

---

5 It is called "partial relative risk aversion" function in M-H papers and "size-of-risk aversion" function in Z-K paper. In the literature, this measure has been largely ignored. We find only Diamond and Stiglitz [27] utilize it in the context of tax effects on portfolio selection.

6 Unlike \( r_A \) and \( r_R \), \( r_S \) is stated in a one-variable/one-parameter form; that is, \( r_S (\pi; \bar{W}) \) rather than \( r_S (\bar{W} + \pi) \). This is because in \( r_S \), \( \pi \) appears separately from \( \bar{W} \) whereas in \( r_A \) and \( r_R \), they enter together.
Furthermore since the relationships between \( r_S' \) on one hand, and \( r_A' \) and \( r_R' \) on the other hand, have not received a thorough examination in either of these papers, it is interesting to know whether a hypothesized pattern on one index is consistent with that on another. Or in other words, what are the restrictions they place on one another? We investigate and present the results in Table 2. The proof is given in Appendix C.

Now back to our tax example. To express our condition in terms of \( r_S \), we cancel \( \frac{1}{2} \sigma^2 \) from the before- and after-tax risk premia and multiply each by \( \bar{\pi} \), so they become:

\[
(1-t)\bar{\pi} r_A (W_0 + (1-t)\bar{\pi}) \leq r_A (\bar{W}_0 + \bar{\pi}),
\]

or equivalently,

\[
r_S [(1-t)\bar{\pi}; W_0] \leq r_S [\bar{\pi}; \bar{W}_0].
\]

Thus a proportional income tax with full loss-offset will increase, leave constant or decrease the optimal output according as size-of-risk aversion is increasing, constant or decreasing in the risk \( \pi \).

---

7Intuitively, the hypothesis of an increasing \( r_S \) seems plausible. This amounts to saying that a given proportional increase in the risk, wealth held constant, will result in a more than proportional increase in the risk premium. Or to state in another way, given fixed wealth, a risk-averse individual is less willing to accept a risk which promises probable greater gain and greater loss at the same time. In the literature, Mason [55] seems to postulate an increasing \( r_S \) (although the concept of \( r_S \) is not mentioned), in saying that: "The person who loses everything if he errs is likely to be more risk-averse than the individual who goes on the welfare rolls if he errs" (p. 84).
Table 2
Relationships between $r_{G}$, and $r_{A}$ and $r_{R}$

<table>
<thead>
<tr>
<th>Relative Risk Aversion</th>
<th>Absolute Risk Aversion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Decreasing</td>
</tr>
<tr>
<td>Decreasing</td>
<td>$\frac{dr_{S}}{d\pi} \not\geq 0$</td>
</tr>
<tr>
<td>Constant</td>
<td>$\frac{dr_{S}}{d\pi} &gt; 0$</td>
</tr>
<tr>
<td>Increasing</td>
<td>$\frac{dr_{S}}{d\pi} &gt; 0$</td>
</tr>
</tbody>
</table>

Note: Asterisks indicate that it is impossible to have nondecreasing absolute risk aversion with nonincreasing relative risk aversion.
5.8. Overview of Previous Works on the Effect of A Proportional Income Tax with Full Loss-Offset under Uncertainty

Research on the risk-taking effect of a proportional income tax with full loss-offset is extensive. Work focusing on individual investor's portfolio selection includes Richter [70], Stiglitz [83], and Mossin [61]. Work addressing the firm's output decision under uncertainty includes Penner [67], Horowitz [41], Baron [11], Sandmo [79], Rothschild and Stiglitz [72]. Two slightly different models are presented in Feldstein [31] and Diamond and Stiglitz [27]. Feldstein assumes that the argument of individual's utility function is not restricted to the outcome of portfolio investment but includes incomes from all relevant decisions, such as one's choice of occupation, etc. Diamond and Stiglitz derive their results through a more general and complicated concept, namely, the "mean utility preserving spread". It is worth noting here that the research on portfolio selection cited above uniformly presumes the existence of only two assets, typically one is risk-free and the other risky. This simplifying assumption establishes an analogy between portfolio choice and production decision-making, and makes their conclusions comparable to ours.\(^8\) The analogy is depicted in Table 3.

\(^8\) In a recent article [81], Sandmo generates the traditional one-safe-one-risky model to one-safe-many-risky case. He finds, among other things, that the effect of imposing general proportional tax on gross income (no exemption for return from the safe asset) consists of the usually income and substitution effects. However no general qualitative conclusion, regarding (1) the demand for a particular risky asset, or (2) total demand for risky assets, can be drawn.

Furthermore, the existence of a riskless asset plays a crucial part in the analysis of taxation and risk-taking.
Table 3

Analogy between Output Decision and Portfolio Selection

<table>
<thead>
<tr>
<th>Contexts</th>
<th>Less Risk-Taking</th>
<th>Neutral</th>
<th>More Risk-Taking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm's Output Decision</td>
<td>decreased supply of Output</td>
<td>initial optimal production</td>
<td>increased supply of output</td>
</tr>
<tr>
<td>Individual's Portfolio Decision</td>
<td>decreased demand for risky asset</td>
<td>initial optimal allocation</td>
<td>increased demand for risky asset</td>
</tr>
<tr>
<td></td>
<td></td>
<td>unchanged</td>
<td></td>
</tr>
</tbody>
</table>

Thus, conditions which give rise to an increase in the demand for the risky asset should be compared to our conditions for a higher optimal output, and vice versa. Conditions which cause the investor to maintain his initial allocation between the two assets should be compared to our conditions that cause the firm to maintain its production level.

In the following subsections we inspect conclusions derived in the aforementioned articles to see whether they are consistent with our conclusions in the previous section. Our review are summarized at the end of this section in Table 4.

5.8.1. Richter

In Richter's paper, the investor's utility function is restricted by the assumption that it is quadratic over the relevant range. He shown that a proportional tax always increases risk-taking. It should
be immediately suspected that his determinate result depends upon very crucial assumptions. To see this, we write down three causal relationships derived before: (a) A quadratic utility function implies increasing absolute risk aversion, (b) increasing absolute risk aversion implies increasing size-of-risk aversion, and (c) increasing size-of-risk aversion leads to higher risk-taking when a proportional income tax is imposed. Thus Richter is correct for a very restrictive case.

5.8.2. Stiglitz and Mossin

Both Stiglitz and Mossin derive the same results that a proportional income tax increases risk-taking if (a) absolute risk aversion is constant or increasing, or (b) absolute risk aversion is decreasing and relative risk aversion increasing or constant. As have shown, either of these two conditions implies increasing size-of-risk aversion. Hence their conclusions are consistent with ours. Nevertheless, for those firms that happen to have decreasing absolute and relative risk aversion, their criteria cease to predict the tax effect. Utilizing our criteria, on the other hand, we can further differentiate that type of firms into increasing, constant or decreasing size-of-risk averters and predict accordingly. In short, where they derive some sufficient conditions, we obtain necessary and sufficient condition.

5.8.3. Penner

Penner's conclusions on the effect of a proportional income tax on a risk-averse firm's output decision can be summarized in two points. First, when the firm has a quadratic utility function,
optimal output would be increased. Second, when the marginal utility is diminishing at an increasing rate, optimal output might be reduced. The first assertion is identical to Richter's, although in a different context, and is seen to be correct.

To see what the second point amounts to, we write \( r_s \) as:

\[
    r_s = -\frac{\pi U''}{U'}
\]

By differentiating:

\[
    \frac{dr_s}{d\pi} = \frac{U'(U''+\pi U''') - \pi(U'')^2}{(U')^2}
\]

where \( U' > 0, \ U'' < 0 \) for a risk-averse firm.

First, if \( U''' < 0 \), or equivalently, marginal utility is diminishing at a slower and slower rate, then:

\[
    \frac{dr_s}{d\pi} = -\frac{(+)(-)(+)(-)(+)-(+) (+)}{(+)} > 0;
\]

that is, the size-of-risk aversion is increasing. Hence the optimal output cannot be decreased if \( U''' < 0 \).

Since less risk-taking can be observed only when \( r_s \) is decreasing, it is clear that the only possibility of a lower optimal output is when \( U''' \) is sufficiently greater than zero so that, when \( U''' \) is multiplied by \( \pi \), the product term outweighs the negative \( U'' \) and furthermore makes the numerator of the derivative positive. Combined with the minus sign in front of the ratio, decreasing \( r_s \) results. Thus a large positive value for \( U''' \) is not only possible, as Penner suggests, but necessary for the optimal output to decrease.
5.8.4. Horowitz and Dhrymes

Horowitz, using Dhrymes' framework in [26], also considers a proportional profit tax with full loss offset in a section of his chapter 13. Expressed in our notation, the firm has a risk-preference function of the following form:

$$E(U) = \pi^{1+\alpha} \sigma^2 p q^2.$$

He derives the first-order condition and concludes that when $\alpha > 0$ the firm will increase its output. But note that the positive constant $\alpha$ in this utility function implies constant absolute risk aversion for the firm, one of the many hypotheses that give rise to increasing size-of-risk aversion and higher risk-taking. This work emerges, again, as only a special case of ours.

5.8.5. Baron

Baron's discussion of the effect of a profit tax is different from others in that he examines the tax effect when the industry is in short-run equilibrium. He argues that a risk averter must receive a positive expected profit at equilibrium to compensate for his positive risk premium. If an income tax is imposed, it reduces the payoffs for any risk and therefore reduces the risk-premium (by the factor $(1-t)^2$); and a lower risk premium implies a lower opportunity cost due to aversion to risk and therefore induces a risk-averse firm to produce more. What seems to be lacking in this deduction is the discussion on the effect of the reduced payoff on the risk-aversion function. If $r_A$ is assumed to be constant when the risk reduces from $\pi$ to $(1-t)\pi$, then there is no question that firms will respond to the tax by producing more since they are not just risk-aversers, but
constant-$r_A$ risk averters. The puzzling inconsistency between Baron's conclusion and ours is resolved by noting that constant absolute risk aversion is implied by increasing size-of-risk aversion; the latter always leads to higher risk-taking.

5.8.6. Sandmo and Rothschild and Stiglitz

Work in this subsection suffers from a common mistake, namely, the omission of initial wealth of the firm. This is a safe practice so long as only absolute risk aversion is involved in the analysis. For in $r_A$ the initial wealth $W_o$ and the risky prospect $\pi$ are always added together. Thus only $\pi$ needs to be present to carry through the wealth effect, be it originated from a change in the wealth position or a change in the risky prospect. This feature is not shared by the other two indices. Consider the following two mutually exclusive cases. In one the risk varies with changed wealth; in the other the risk varies but wealth does not. Failure to include initial wealth not only obscures the distinction between the two but also makes relative risk aversion, the appropriate criterion for the first case, and size-of-risk aversion, the appropriate criterion for the second case, indistinguishable.\(^9\)

---

\(^9\) Pratt's parenthesized "warning" in the introduction of his paper [68] says: "Throughout this paper, utility is regarded as a function of total assets rather than of change which may result from a certain decision,... (This is essential only in connection with proportional risk aversion)." (p. 123)

\(^10\) To see this, we plug zero for $W_o$ in the two measures, then:

$$r_R = (W_o + \pi)r_A(W_o + \pi) = \pi r_A(\pi)$$

and

$$r_S = \pi r_A(W_o + \pi) = \pi r_A(\pi).$$
This is exactly what causes the above writers to employ the relative risk-aversion index in discussing the effect of a profit tax, when actually the size-of-risk aversion governs the result. It is also the reason why their results coincide with what we derive in the case of a wealth tax (Section 5.6).

5.8.7. Feldstein

Feldstein's paper aims to offer a counter-example to the then-popular proposition that taxation increases risk-taking. He employs a marginal utility function with constant elasticity and shows that upon imposing a proportional income tax the expected-utility ordering is preserved. Unfortunately the initial wealth is also assumed away from the utility function. Were he to add back $\bar{W}$, he would find such utility function supports rather than disproves the proposition he attempts to counter. This is simply because constant elasticity in marginal utility implies nothing but constant relative risk aversion, which in turn implies increasing size-of-risk aversion and higher risk-taking.

5.8.8. Diamond and Stiglitz

Diamond and Stiglitz is the only work that utilizes the size-of-risk aversion. They introduce the concept of "mean utility preserving increase in risk" and define a equivalent measure, called "index of risk aversion." This latter translates into size-of-risk aversion

\[ r_R = - \frac{XU''(X)}{U'(X)} = - \frac{\text{dln}U'(X)}{\text{dln}x} = \varepsilon(U(X), x). \]

If \( \varepsilon \) is a constant, \( r_R \) is also a constant.
upon the imposition of a proportional income tax. The fact that through a different approach they arrive at identical conclusions as ours helps to confirm our assertion concerning the use of approximation in our model.

5.9. Degressive Income Tax

In this section, we move beyond a proportional tax to consider a simple tax structure of progressive nature. Essentially, this tax has an exemption level $K$ and a constant marginal tax rate $t$ which applies both above and below $K$. For profits higher than $K$, the average tax rate $T/\pi$, where $T=t(\pi-K)$, is positive; for profits less than $K$, $T/\pi$ becomes negative. Following Blum and Kalven, we call this a "degressive" income tax.\(^{12}\) The after tax profit is:

$$\pi_T = \pi - T,$$
$$= \pi - t(\pi - K),$$
$$= (1-t)\pi + tK.\(^{13}\)$$

\(^{12}\)Blum and Kalven ([17]) uses the term "degressive" to mean a "progressive tax curve with decelerating rates." That this tax contains decelerating progressiveness can be seen by twice differentiating $T/\pi$ with respect to $\pi$.

$$\frac{d(T/\pi)}{d\pi} = tK\pi^{-2} > 0 \text{ for all } \pi,$$
$$\frac{d^2(T/\pi)}{d\pi^2} = -2tK\pi^{-3} < 0 \text{ for } \pi > 0.$$

This type of tax is sometimes called "linear progressive tax" or "negative income tax."

\(^{13}\)When $\pi$ happens to be exactly equal to $K$, the optimum would not be affected by such tax since the before-tax and after-tax profits are the same.
<table>
<thead>
<tr>
<th>Authors</th>
<th>Type of Utility Function</th>
<th>Positted Hypotheses on ( r_A ) and ( r_R )</th>
<th>Implied Hypothesis on ( r_S )</th>
<th>Derived Risk-Taking Effect and Consistency with Our Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Richter</td>
<td>Quadratic</td>
<td>Increasing ( r_A )</td>
<td>Increasing</td>
<td>Higher; Consistent</td>
</tr>
<tr>
<td>Mossin; Stiglitz</td>
<td></td>
<td>(a) Increasing or constant ( r_A )</td>
<td>Increasing</td>
<td>Higher; Consistent</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(b) Decreasing ( r_A ) and constant ( r_A ) or increasing ( r_R )</td>
<td>Increasing</td>
<td>Higher; Consistent</td>
</tr>
<tr>
<td>Feldstein</td>
<td>Constant-elasticity MU</td>
<td>Constant ( r_R )</td>
<td>Increasing</td>
<td>Unchanged; Incorrect</td>
</tr>
<tr>
<td>Penner</td>
<td>(a) Quadratic</td>
<td>Increasing ( r_A )</td>
<td>Increasing</td>
<td>Higher; Consistent Lower; Consistent</td>
</tr>
<tr>
<td></td>
<td>(b) ( U'' ) substantially greater than zero</td>
<td></td>
<td>Decreasing</td>
<td></td>
</tr>
<tr>
<td>Horowitz (Dhrymes)</td>
<td>( EU = \bar{\pi} + \frac{1}{2} \alpha \sigma_\pi^2 )</td>
<td>Constant ( r_A )</td>
<td>Increasing</td>
<td>Higher; Consistent</td>
</tr>
<tr>
<td>Sandmo; Rothschild &amp; Stiglitz</td>
<td></td>
<td>(a) Increasing ( r_R )</td>
<td>Increasing</td>
<td>Higher; Consistent Unchanged; Incorrect Lower; Incomplete</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(b) Constant ( r_R )</td>
<td>Increasing</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c) Decreasing ( r_R )</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>Baron</td>
<td>Strictly concave convex</td>
<td></td>
<td></td>
<td>Higher; Incomplete Lower; Incomplete</td>
</tr>
<tr>
<td>Diamond &amp; Stiglitz</td>
<td></td>
<td></td>
<td></td>
<td>Higher; Unchanged; Identical Lower;</td>
</tr>
</tbody>
</table>
It should be immediately clear that such a tax has no impact on firm's output decision if the firm maximizes either profits or expected profits since the factor \((1-t)\) vanishes from the first-order condition as usual.

In our model the after-tax certainty equivalent becomes:

\[
CE_{DI} = (1-t)(\pi + tk - \frac{1}{2}d^2\pi)(1-t)\nu_A^2(\mu_0 + \mu_T),
\]

\[
= (1-t)(\pi - \frac{1}{2}d^2\pi)(1-t)r_A^0[\mu_0 + (1-t)(\pi + tk)] + tk,
\]

where subscript DI refers to degressive income tax. Note that the variance of after-tax profit is still \((1-t)^2d^2\pi\) since the extra term \(tk\) is a constant with respect to \(Q\).

Disregarding the multiplicative factor \((1-t)\) and the last term \(t(\mu_0 + k)\), we can see that the optimal output increases, remains constant of decreases as \((1-t)r_A^0[\mu_0 + (1-t)(\pi + tk)]\) is smaller than, equal to or greater than \(r_A^0(\mu_0 + \pi)\).

A closer examination reveals that neither of those previously developed risk-aversion functions alone is capable of characterizing the necessary and sufficient condition in full. More specifically, we have seen that, given initial wealth, the effect of a multiplicative change in \(\pi\) is governed by hypotheses about size-of-risk aversion, and the effect of an additive change in \(\pi\) by hypotheses about absolute risk aversion. Generalizing it should be obvious that changes which result in a full linear function in \(\pi\), consisting of both multiplicative and additive elements, naturally demand hypotheses on both \(r_A\) and \(r_S\). A degressive income tax provides an example of such changes.
In Table 5, we examine tax effects in all possible and legitimate combinations of hypotheses on $r_A$ and $r_S$. For the term $tK$, the effect on optimal output is positive, neutral or negative as $r_A$ is decreasing, constant or increasing. For the term $(1-t)$, the effect on quantity is positive, neutral or negative as $r_S$ is increasing, constant or decreasing. We can get unambiguously positive effects for some cases. However, when both $r_A$ and $r_S$ are increasing or decreasing, their respective effects offset each other, yielding indeterminate results. Nevertheless, it is worth noting that under the most plausible combination of hypotheses on $r_A$ and $r_S$, namely, decreasing $r_A$ and increasing $r_S$, we obtain definitely positive effect.

In the literature, we find Ahsan [2] uses this particular tax structure to represent a progressive tax and examines its effect on

---

14 For the meaning of "legitimate," see Note (d) of Table 5.

15 Unlike the case of an increase in lump-sum tax, where the tax change is subtracted from $\pi$, $tK$ enters as a positive factor here. Thus the conclusion is just the opposite of what we derived for an increase in lump-sum tax.

16 One can check this out very easily mathematically. First we assume $\pi > K$, or equivalently, $\pi > \pi_T$, then increasing $r_S$ implies that:

$$\overline{\pi} r_A (w_0 + \overline{\pi}) > [(1-t)\overline{\pi}+tK] r_A (w_0 + \overline{\pi}_T).$$

But $\text{RHS} > (1-t)\overline{\pi} r_A (w_0 + \overline{\pi}_T),$

thus, $r_A (w_0 + \overline{\pi}) > (1-t) r_A (w_0 + \overline{\pi}_T).$

This implies that $CE_{DI}$ is the sum of $CE$ and an increasing function of $Q$. This proves that the optimum is increased.

If $\pi < K$, or equivalently, $\pi < \overline{\pi}_T$, decreasing $r_A$ implies that:

$$r_A (w_0 + \overline{\pi}) > r_A (w_0 + \overline{\pi}_T).$$

But $\text{RHS} > (1-t) r_A (w_0 + \overline{\pi}_T),$

thus, $r_A (w_0 + \overline{\pi}) > (1-t) r_A (w_0 + \overline{\pi}_T).$

We arrive at the same conclusion.
Table 5
Decisive and Indecisive Effects of A Degressive Tax
with Alternative Hypotheses on $r_A$ and $r_S$

<table>
<thead>
<tr>
<th>Size-of-Risk Aversion</th>
<th>Absolute Risk Aversion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Decreasing</td>
</tr>
<tr>
<td>Decreasing</td>
<td>$r_A: Q \uparrow \uparrow$</td>
</tr>
<tr>
<td></td>
<td>$r_S: Q \downarrow \downarrow$</td>
</tr>
<tr>
<td>Constant</td>
<td>$r_A: Q \uparrow \uparrow$</td>
</tr>
<tr>
<td></td>
<td>$r_S: Q \uparrow \uparrow$</td>
</tr>
<tr>
<td>Increasing</td>
<td>$r_A: Q \uparrow \uparrow$</td>
</tr>
<tr>
<td></td>
<td>$r_S: Q \uparrow \uparrow$</td>
</tr>
</tbody>
</table>

Note: (a) The directions of the arrowheads indicate the directions of changes in optimal output.
(b) $\varnothing$ denotes neutral effect.
(c) A question mark denotes ambiguous combined effect.
(d) Asterisks indicate the fact that increasing size-of-risk aversion is the only assumption on $r_S$ that is compatible with nondecreasing absolute risk aversion.
the demand for risky asset by an individual investor. Some of his conclusions which are related to ours state that: (a) if absolute risk aversion is decreasing and relative risk aversion increasing or constant, such tax leads to higher risk-taking, and (b) if absolute risk aversion is increasing, the effect is ambiguous. While his (b) is identical to our result, his (a) is only part of our decreasing-$r_A$-increasing-$r_S$ case since increasing and constant $r_R$ are subsets of increasing $r_S$.

5.10. **Discussions on General Progressive Taxation**

We have yet to examine the impact of a general progressive tax. As we have pointed out in the introduction, our certainty-equivalent approach seems incapable of handling taxes that change the expected profit in a nonlinear way. Without explicitly specifying the form of the progressivity of the tax rate, the probability distribution of the random price, and the nature of firm's risk aversion, there is no way to derive the relationship between the before-tax and after-tax certainty equivalents. As a matter of fact, this inability extends to most models based on general expected-utility maximization.\(^{17}\) This may explain why writers like Baron, Mossin,

\(^{17}\)Russel and Smith [75] examine the effect of a progressive tax by a stochastic-dominance approach. They show that such a tax with full loss offset preserves the initial ordering of the expected utility, i.e., stochastic dominance, of two random arguments. Lepper [48] also examines the effect of a progressive tax in the mean-variance framework through numerical solution to representative problems. She shows that such tax establishes disincentives against risk-taking.
Sandmo and Stiglitz do not even mention a progressive tax. On the other hand, some authors express interest in progressive taxation but offer no result. For example, Naslund in [63] says: "It would be interesting to study the effect of a graduated tax" (p. 304). Ziemba and Vickson [97] have a similar statement (p. 214), yet at the same time they admit that the solution is unknown or in doubt to them. Feldstein in [31] also considers a progressive tax of the form \( \pi_T = \alpha \pi \), \( 0 < \alpha, \beta < 1 \). Even under the assumption of an isoelastic utility function he concludes that "it is not possible to say whether this tax will increase or decrease risk-taking" (p. 763).

Two serious attempts to analyze the effect of a graduated tax exist in the literature. One is Penner [67], who asserts that the progressivity of the tax rate would seem to have negative effect on risk-taking. The other is Batra [12], who asserts that "the nature of the tax (proportional or progressive) makes little difference" to his analysis of the incidence of corporation income tax (p. 347). Unfortunately, both authors commit an identical mistake in their argument, leaving their assertions doubtful. They start with a proportional tax. Upon differentiating the expected utility of after-tax profits with respect to an input factor \( L \) and setting the derivative equal to zero, they get:

\[
\frac{d\mathbb{E}U_W(1-t)\pi}{dL} = \mathbb{E}((1-t)U_{\frac{d\pi}{dL}}) = 0.
\]

By cancelling \((1-t)\), the first-order condition becomes:

\[
\mathbb{E}(U_{\frac{d\pi}{dL}}) = 0.
\]

Having derived some implications from the above equation for the case of proportional income tax, they turn to consider a progressive
tax. They both note that \((1-t)\) is now a random element and therefore can not be taken outside the brackets of the expected-value operator. Thus the first-order condition when \(t\) is random goes back to:

\[
E[(1-t)U' \frac{d\pi}{dL}] = 0.
\]

Then Penner derives his assertion about progressivity from this equation and Batra sees no major difference between the two first-order conditions.

The procedure described, however, is clearly questionable. It is totally unwarranted to derive a maximizing condition for a progressive tax by treating the tax rate as a constant when differentiating, and then making some adjustments on the resulting condition to cope with a random tax rate. Mathematically, the true first-order condition should consist of two terms, i.e.:

\[
E[(1-t)U' \frac{d\pi}{dL}] - E[U'\pi \frac{\partial t}{\partial \pi} \frac{d\pi}{dL}] = 0;
\]

where the first term represents the direct marginal effect of \(\pi\) on expected utility level due to a change in \(L\), whereas the second term represents the indirect marginal effect of \(\pi\) through the tax rate on expected utility level due to the same change. Apparently, the explicit form of the tax-rate structure is necessary for evaluating the term \(\frac{\partial t}{\partial \pi}\) and for making any proposition. Meanwhile it should be clear that works which neglect this tax-rate factor should not be considered reliable.
5.11. Conclusions and Summary of the Chapter

In this chapter the effects of six different taxes are studied. Table 6 summarizes our results, as well as the results which would have been obtained if those taxes were levied on a neoclassical or an expected-profit-maximizing firm. A remarkable observation is that, in the presence of uncertainty, optimal output may change in any direction when an expected-utility-maximizing firm faces a tax of any kind. It is shown that properties of the three risk-aversion functions, \( r_A \), \( r_R \) and \( r_S \), preeminent dictate the conclusions. On one hand, a firm may react differently to various taxes. On the other hand, firms with different risk-aversion indices may react oppositely to a particular tax. In our opinion, these indeterminacies due to "unlikeness" or asymmetry among firms make the study of the theory of the firm under uncertainty exciting and worthwhile.

While the two Arrow-Pratt risk-aversion measures have been widely utilized, the size-of-risk aversion function has not received the attention it deserves. In studying the effects of two relatively more complicated taxes -- proportional income tax and regressive income tax -- \( r_S \) enables us to draw more precise and complete conclusions than those derived by others using only \( r_A \) and \( r_R \).
Table 6

Summary of Tax Effects under Certainty, Uncertainty and Risk Aversion

<table>
<thead>
<tr>
<th>Tax Structure</th>
<th>Response of Optimal Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Certainty: $\pi$-Max.</td>
</tr>
<tr>
<td></td>
<td>or</td>
</tr>
<tr>
<td></td>
<td>Uncertainty: $E(\pi)$-Max.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Lump-Sum Tax</td>
<td>0</td>
</tr>
<tr>
<td>Specific or Ad Valorem Sales Tax</td>
<td>-</td>
</tr>
<tr>
<td>Proportional Wealth Tax</td>
<td>N/A</td>
</tr>
<tr>
<td>Proportional Income Tax: Full Loss-Offset</td>
<td>0</td>
</tr>
<tr>
<td>Degressive Income Tax</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>?</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: (a) The word "if" implies sufficient condition only.
(b) The word "iff" implies necessary and sufficient condition.
(c) A question mark denotes ambiguous result.
CHAPTER VI
SUMMARY AND FUTURE EXTENSIONS

6.1. Summary and Conclusions

This dissertation has two principal parts. In the first part we discuss both the favorable and unfavorable features of using an expected-utility function to cope with the existence of uncertainty and to incorporate the decision maker's attitude toward risk. We then review some of the certainty-equivalent models, aiming to simplify the expected-utility approach. We find, however, they suffer either from a misspecification of the coefficient of risk aversion or from the inherent inconvenience in conducting comparative-static analysis. Some are simply too general to have practical content. Our CE-model is then developed based on the ideas of Pratt's risk premium and narrowly-defined certainty equivalent (CE). It seems that the only factor that could have hindered others from developing Pratt's concepts into a model is the approximation involved. In addition to some standard justifications employed in the literature to defend a quadratic approximation by Taylor's series, we set out to investigate the impact of the approximation error on the qualitative comparative-static results. We find that in the literature of decision-making under uncertainty, the utility functions employed are usually assumed to be characterized by Arrow-Pratt risk-aversion
indices and their respective first derivatives. Furthermore, the monotonicity of the indices and their first derivatives are usually taken for granted. Under these circumstances, the approximation error can be ignored. Since expected-utility-maximization is equated to the CE-maximization and the CE is decomposed into expected return and risk premium, we find it readily applicable to the theory of firm under uncertainty. Although the first component of CE relates naturally to output, we use the approximation to make the risk premium a function of output too, so that the firm's certainty-equivalent maximization becomes mathematically and graphically tractable. We then extend our model to cover the analysis of any quantity-setting firms' production decision under uncertainty, but at the same time change the attribute in the utility function from profit to sales revenue. We find that such firms' certainty equivalent of sales has exactly the same shape as a purely competitive firm's certainty equivalent of profit, provided some general assumptions are made. This exempts us from deriving all over again the risk premium and CE in this new argument. Other interesting findings include the possibility of a finite optimal output to the expected-utility-of-sales maximization for a purely competitive firm and the tendency for an increasing-\(r_A\) producer to act "unorthodoxically." Comparative-static experiments comprise the major contents of the second part. Here we demonstrate how our CE-approach handles exogenous changes. In each case, we derive an after-change CE by making some simple corrections in the expected-profit part and the risk-premium part to reflect the exogenous change or changes. Then we establish a
relationship between the new and old CE's. The results, decisive or
indecisive, as well as the conditions -- sufficient and/or necessary --
become intuitively clear upon examining the resulting relationship.
Beside the ordinary changes in parameters, we also consider the effect
of information activity on production, using a formulation by
Baltensperger and MIlde. Impact of a simple investment option on
production is also explored. We then study the tax effects
extensively and thoroughly. Compared to numerous other studies on
this subject, our model undeniably yields more complete results. Our
theorems often include the results derived by other authors as special
cases. It should be noted that the utilization of a rather less-
known risk-avoidance index, the "size-of-risk aversion" function, is
of great help.

Overall, our model performs well in this application. Not only
are better results derived, but they are derived in a way that stresses
the underlying determinants and thus yields intuitive economic
interpretations. This is perhaps our approach's greatest advantage.
The two-part decomposition of CE, which turns out to parallel a
Slutsky-type separation of income effects from substitution effects,
is more than a convenient analytical device. It is the key factor in
achieving interpretable findings.

6.2. Future Extensions

In this study we consider the impact of uncertainty in sale
price. Other possible sources of uncertainty exist: uncertain
production technology, uncertain input prices, uncertain sales volume
or even uncertain tax-rate changes, to name just a few. As if to complicate matters further, one may consider a situation where various combinations of the above uncertainties arises. For example, the dichotomy of a firm into sales department and production department may necessitate the incorporation of at least two types of uncertainties. Conceptually, some of these additional sources of uncertainties can be incorporated into the model through the variance in the risk-premium part of the maximand. Exactly how the risk-averse firm changes its demand for risk premium must be specified before we can investigate how such uncertainties would affect the optimal decision and the decision criterion.

Other possible extensions include the examination of a multi-product or a multi-location firm facing uncertain prices in different markets, independent or correlated. This is attractive because, in comparative statics, assuming more than one product market would enable us to observe the set of ordinary substitution and income effects in addition to the productive substitution and wealth effects that served as the focus of this dissertation.

Testing our results empirically represents another direction for future work. Great difficulties link in the unobservability of subjective risk-aversion coefficients as well as the first two central moments of the price distribution. An alternative to observing real-world behavior is data that may be obtained from experimental games designed to provide information on entrepreneurs' risk-aversion patterns and their expectation formation. Gordon, Paradis and Rorke [33] report on one such experimental game. Information on
behavior and expectation obtained may be used as proxies in the empirical study of, say, tax effects. Of course, until experimental results are replicated, results are always suspect.
Appendix A

Derivations of Risk-Premium Functions for A Purely Competitive Firm

Having Increasing, Constant or Decreasing Absolute Risk Aversion

A.1. Constant Absolute Risk Aversion

For this class of utility functions, the negative exponential function, \( U(\pi) = e^{-c\pi} \), is adopted due to its generality. We find that when the price distribution is assumed to be normal, gamma or uniform, we can derive an exact risk-premium function, using the above utility function. It should be noted that Case (a) has been derived by Baron [11].

(a) Normal distribution: \( P \sim N(\bar{P}, \sigma_P^2) \).

\[
EU(\pi) = \int U(\pi) f(P) dP, \\
= \int (-e^{-c\pi}) k \cdot \exp(-\frac{1}{2} \frac{\sigma_P^2}{(P-\bar{P})^2}) dP,
\]

where \( k \) is the normalizing constant and "exp" refers to the exponent. Completing the square in the exponent and integrating gives:

\[
EU(\pi) = -\exp[-c(\bar{P} - \frac{1}{2} \sigma_P^2 \bar{Q}^2)].
\]

On the other hand:

\[
U(\bar{P} - \Omega) = -\exp[-c(\bar{P} - \Omega)].
\]

By equating the two, we find:

\[
\Omega = \frac{1}{2} \sigma_P^2 \bar{Q}^2.
\]
Its first and second derivatives are:

\[ \frac{d\Omega}{dq} = \alpha \beta (1 + c\beta q)^{-1} > 0, \]
\[ \frac{d^2\Omega}{dq^2} = \alpha^2 (1 + c\beta q)^{-2} > 0. \]

Thus the risk premium is increasing at an increasing rate.

(b) Gamma distribution: \( P \sim \Gamma(\alpha, \beta); \ \alpha, \beta > 0, \ 0 < P < \infty. \)

\[ EU(\pi) = (e^{-\alpha p})(\beta^{-\alpha}(\alpha-1))^{-1}P^{\alpha-1}e^{-P/(\beta/1+c\beta q)}dP, \]
\[ = (1+\alpha\beta q)^{-\alpha}e^{c(Q)+F} \]
\[ \cdot \int(\beta/(1+c\beta q))^{-\alpha}(\alpha-1)^{-1}P^{\alpha-1}e^{-P/(\beta/1+c\beta q)}dP, \]
\[ = (1+c\beta q)^{-\alpha}e^{c(Q)+F}, \]

since the last integral represents the area under \( \Gamma(\alpha, \beta/1+c\beta q). \)

Now \( U(\pi - \Omega) = \exp[-c(\pi - \Omega)]. \) By equating the above two equations, we have:

\[ \exp[-c(\pi - \Omega)] = (1+c\beta q)^{-\alpha}e^{c(Q)+F}, \]

or, \( -c(\pi - \Omega) = -\alpha \ln(1+c\beta q) + c(Q)+F, \)

or, \( \Omega = \frac{\pi - \alpha^{-1}}{\ln(1+c\beta q)}. \)

Taking derivatives, we have:

\[ \frac{d\Omega}{dq} = \alpha \beta (1 + c\beta q)^{-1}, \]
\[ = \frac{\pi}{1-c^{-1}(1+c\beta q)^{-1}}, \text{ since } \pi = \alpha \beta. \]
\[ > 0; \]

and \[ \frac{d^2\Omega}{dq^2} = \alpha^2 (1 + c\beta q)^{-2}, \]
\[ = \sigma^2 P(1+c\beta q)^{-2}, \text{ since } \sigma^2 = \alpha \beta^2, \]
\[ > 0. \]

Again, the risk premium is increasing at an increasing rate.
(c) Uniform distribution: \( P \sim \mathcal{U}(\alpha, \beta); \alpha, \beta > 0 \).

\[
EU(\pi) = \int_{\pi}^{\beta} (e^{-c\pi})(\beta-\pi)^{-1}d\pi,
\]

\[
= [c(\beta-\pi)^{-1} - c(C(Q) + F)(e^{-\alpha Q} - e^{-\beta Q})].
\]

Setting the above equation equal to \( U(\pi - \Omega) \):

\[
e^{-c(\pi - \Omega)} = [c(\beta-\pi)^{-1} - c(C(Q) + F)(e^{-\alpha Q} - e^{-\beta Q})].
\]

\[
\Omega = \frac{c}{\pi} - \ln(\beta - \alpha) - \ln(\pi - \pi^Q) - \ln(e^{-\alpha Q} - e^{-\beta Q}) - \ln(e^{-\alpha Q} - e^{-\beta Q}).
\]

By differentiating \( \Omega \), we get:

\[
\frac{d\Omega}{dQ} = \frac{c}{\pi} - \ln(\beta - \alpha) - \ln(\pi - \pi^Q) - \ln(e^{-\alpha Q} - e^{-\beta Q}).
\]

But \( \frac{d\gamma}{dQ} = \frac{1}{2}(\alpha + \beta) \), thus:

\[
\frac{d\Omega}{dQ} = -\frac{1}{2}c^{-1}(\beta - \alpha)(1 - e^{-\beta Q})^{-1}(1 + e^{-\beta Q}).
\]

Taking limit:

\[
\lim_{Q \to \infty} \frac{d\Omega}{dQ} = \frac{1}{2}c^{-1}(\beta - \alpha) > 0.
\]

Thus the risk-premium function is increasing and is approaching a finite asymptote.

A.2. Decreasing Absolute Risk Aversion

Unlike the previous class, there is no known combinations of utility function and price distribution that are capable of producing an exact risk-premium function for the family of decreasing-\( r_A \). We therefore utilize the approximation formula to derive the risk-premium function for a very basic form of decreasing-\( r_A \) utility function, namely, \( U'(\pi) = (\pi^a + b)^{-c} \), \( a, c > 0 \).
Further differentiating $U'(\pi)$ yields:

$$U''(\pi) = -c(\pi^{a+b})^{-(c+1)}a_{n}^{a-1}. $$

Dividing $-U''$ by $U'$, we get:

$$r_{A} = ac\pi^{a-1}(\pi^{a+b})^{-1}. $$

Since $\Omega = \frac{1}{2}\sigma_{A}^{2}r_{A}(\pi)$, upon substituting the above equation for $r_{A}$:

$$\Omega = \frac{1}{2}\sigma_{P}^{2}Q'a_{c}\pi^{a-1}(\pi^{a+b})^{-1}, $$

$$= kQ'a_{c}\pi^{a-1}(\pi^{a+b})^{-1}, $$

where $k = \frac{1}{2}\sigma_{P}^{2}ac$. This function is not defined at the point $\pi = b$.

Taking derivative of $\Omega$ gives:

$$\frac{d\Omega}{dQ} = k(\pi^{a+b})^{-2}\pi^{a-2}Q\{2(\pi^{a+b})\pi^{a-(F-C'(Q))}Q(\pi^{a-1}+b(a-1)-a_{n}^{a-1})\}. $$

Assuming a full quadratic cost function, $mQ^{2}+nQ+F$, then $d\Omega/dQ$ becomes:

$$\frac{d\Omega}{dQ} = K\{\pi^{a-1}[(F-n)Q-2F]+b[-2amQ^{2}+(F-n)Q(a+1)-2F]\}. $$

where $K = k(\pi^{a+b})^{-2}\pi^{a-2}Q$.

To investigate the shape of the risk-premium function, we assign values to some key parameters to simplify the derivation.

(a) $a=1$.

$$\frac{d\Omega}{dQ} = \frac{1}{2}\sigma_{P}^{2}(\pi+b)^{-2}\{[(F-n)Q-2(F-b)]Q\}. $$

Two turning points are $Q_1 = 0$ and $Q_2 = \frac{2(F-b)}{F-n}$. The second-order derivative is:

$$\frac{d^{2}\Omega}{dQ^{2}} = \sigma_{P}^{2}(\pi+b)^{-3}\{m(F-n)Q-3m(F-b)Q^{2}+(F-b)^{2}\}. $$

Evaluating at $Q_1$ and $Q_2$:

$$\frac{d^{2}\Omega}{dQ^{2}} \bigg|_{Q_1} = -\sigma_{P}^{2}(F-b)^{-1}, $$
and \( \frac{d^2 \Omega}{dq^2} \bigg|_{q_2} = \sigma_p^2 \left( (F-n)^2 - 4m(F-b) \right)^{-2} (F-n) (F-b)^{-1} \).

To sign the above two second-derivatives we note two constraints on the parameters. The first comes from the definition of a risk averter requiring as someone who requires a positive risk premium; the second comes from the short-run shut-down consideration. From the risk-premium function, we see that with \( a = 1 \), a positive risk premium requires that \( \bar{\pi} > b \). The closing-down option requires that \( \bar{\pi} > F \). In combination, they require that \( F - b < 0 \). If \( F - b = 0 \), then:

\[
\frac{d \Omega}{dq} = \frac{1}{2} \sigma_p^2 (\bar{\pi} + b)^{-2} (F-n) q_1^2 > 0,
\]

the risk premium is monotonically rising. If \( F - b < 0 \):

\[
\frac{d^2 \Omega}{dq^2} \bigg|_{q_1} > 0,
\]

\[
\frac{d^2 \Omega}{dq^2} \bigg|_{q_2} < 0,
\]

and \( q_2 < 0 \). These results indicate that the risk-premium function has a local minimum at the origin and a local maximum at a negative output level. Figure 12 depicts this. The location of the solution to \( \max \bar{\pi}, Q^* = \frac{(F-n)}{2m} \), the short-run shut-down point, \( Q^* = \frac{(F-n)}{m} \), and the discontinuities of the risk-premium function, \( Q^L \), are all determined without ambiguity. The portion of the risk premium that is within the shut-down point is seen to be rising at an increasing rate.
Figure 12

Approximate Risk-Premium Function with Decreasing $r_A$
(b) \( a=1 \) and \( c=1 \).

Then \( U'(\pi) = \pi + b \), we have a logarithmic utility function. The shape of the risk premium is essentially the same as the previous case since the additional assumption \( c=1 \) has no qualitative effect on \( \Omega \).

(c) \( b=0 \), and \( m=0 \) and \( F=0 \).

A simple logarithmic utility with a linear cost function without fixed costs yields interesting results.

\[
\frac{d\Omega}{dQ} = \frac{1}{2} \sigma_F^2 \pi \rho (\bar{r} - \pi)^{-1},
\]

constant

\( > 0 \).

This is a very special case where both expected-profit and risk-premium functions turn out to be linear. The implication is that the certainty equivalent is also rising steadily and yields no finite optimal output.

A.3. Increasing Absolute Risk Aversion

We assume the utility function is \( U(\pi) = (a - \pi)^b \), where \( \pi \leq a \), \( b > 1 \).

This function includes the quadratic utility function as a special case \( b=2 \). We can derive the following:

\[
U'(\pi) = b(a - \pi)^{b-1},
\]

and \( U''(\pi) = -b(b-1)(a - \pi)^{b-2} \),

\[
x_A = -U''/U' = (b-1)(a - \pi)^{-1},
\]

and \( \Omega = \frac{1}{2} \sigma_F^2 (b-1)Q^2(a - \pi)^{-1} \).

Assuming a full quadratic cost function, then:

\[
\frac{d\Omega}{dQ} = \frac{1}{2} (b-1) \sigma_F^2 (a - \pi)^{-2} [2Q(a - \pi) + Q^2(\bar{r} - 2mQ - n)],
\]
where \( A = \frac{1}{2}(b-1)\sigma_p^2 (a-\pi)^{-2} > 0 \). Setting the derivative equal to zero, two turning points are \( Q_1 = 0 \) and \( Q_2 = 2(a+F)/(\bar{P}-\pi) > 0 \). Further differentiating and evaluating at \( Q_1 \) and \( Q_2 \), we get:

\[
\frac{d^2n}{dQ^2} \bigg|_{Q_1} = \sigma_p^2 (b-1)(a+F)^{-1} > 0,
\]

and,

\[
\frac{d^2n}{dQ^2} \bigg|_{Q_2} = -\sigma_p^2 (b-1)^3 (\bar{P}-\pi)^4 (a+F)^{-1} [(b-1)(\bar{P}-\pi)^2 - 2m(a+F)]^{-2} < 0.
\]

Thus the risk premium has a local minimum at the origin and a local maximum at \( Q_2 \). To locate \( Q^* \) in relative to \( Q_2 \), we subtract \( Q^* \) from \( Q_2 \):

\[
Q_2 - Q^* = \left[ 2(a+F)/(\bar{P}-\pi) \right] - \left[ (\bar{P}-\pi)/2m \right],
\]

\[
= \left[ 2m(\bar{P}-\pi) \right]^{-1} \left[ 4m(a+F) - (\bar{P}-\pi)^2 \right].
\]

But note that \( \pi \) is restricted to be smaller than \( a \) to exclude the implausible declining portion of the utility function. This requirement implies that:

\[-mQ^2 + (\bar{P}-\pi)Q < a.\]

Substituting \( Q^* \) into the inequality, we get:

\[4m(a+F) > (\bar{P}-\pi)^2.\]

Thus \( Q_2 \) lies to the right of \( Q^* \).

In order to determine the shape of the risk premium over the relevant range of output we must locate the short-run closing point in relative to \( Q_2 \) too. If \( Q_2 \) is greater than \( \bar{Q}^{\text{T}} \), then only the rising portion of the risk premium is relevant to the firm's production decisions. If however \( Q_2 \) is smaller than \( \bar{Q}^{\text{T}} \), relevant risk-premium function contains a declining part. Now:

\[Q_2 - \bar{Q}^{\text{T}} = \left[ 2(a+F)/(\bar{P}-\pi) \right] - \left[ (\bar{P}-\pi)/m \right],\]
\[ (2m(a+F)-(\bar{F}-n)^2)/m(\bar{F}-n), \]

\[ \gamma > 0, \]

according as \( 2m(a+F) \frac{n}{2} (\bar{F}-n) \). Since there is no a priori requirement why one should be greater than the other, our risk-premium function may consist of both a rising and a declining portions. Furthermore, the falling portion of the risk premium is approaching a positive value since:

\[ \lim_{Q \to \infty} \Omega = \frac{1}{2} \sigma^2 (b-1) m^{-1} > 0. \]

Figure 13 illustrates the above results. The implication of the results is discussed in the text, especially in Section 3.4.
Figure 13
Approximate Risk-Premium Function with Increasing $r_A$
Appendix B

Some Exact Risk-Premium and Certainty-Equivalent Functions
for A Purely Competitive Firm

Maximizing Expected-Utility-of-Revenue

B.1. Constant Absolute Risk Aversion

We have seen in Appendix A that given the negative exponential utility function $U(X) = e^{-cX}$, the exact risk premium can be derived by assuming the price distribution is normal, gamma or uniform. The results obtain when profit is replaced by sales revenue. Yet the certainty-equivalent function of sales here are not directly analogous to that of profit since the first component is now a straight line rather than a curve. The existence of a finite optimum needs to be ascertained. In what follows we merely state the resulting risk-premium functions without formal proof since their derivations and forms are exactly the same as that of $\Omega_\pi$.

(a) Normal distribution: $F \sim N(\bar{F}, \sigma_F^2)$.

The risk premium and its first-order and second-order derivatives are:

$$\Omega_R = \frac{1}{2} c \sigma_F^2 Q^2,$$

$$\frac{d\Omega_R}{dQ} = c \sigma_F^2 Q > 0,$$

$$\frac{d^2 \Omega_R}{dQ^2} = c \sigma_F^2 > 0.$$
The risk premium is seen to be increasing at an increasing rate. Now we turn to the certainty-equivalent function.

Note that:
\[
CE_R = P - 2c_p^2 Q^2.
\]

By differentiating and setting the derivative to zero, an optimal output can be found.

\[
\frac{dCE_R}{dQ} = P - c_p^2 Q = 0,
\]

\[
Q^* = \frac{P}{c_p^2}.
\]

(b) Gamma distribution: \( P \sim \Gamma(\alpha, \beta); \ \bar{P} = \alpha \beta, \ \sigma_P^2 = \alpha \beta^2. \)

The risk premium and its two derivatives are:

\[
\Omega_R = \bar{P}Q - \alpha c^{-1} \ln(1 + c \beta Q),
\]

\[
\frac{d\Omega_R}{dQ} = \bar{P}[1 - c^{-1}(1 + c \beta Q)^{-1}] > 0,
\]

and \( \frac{d^2 \Omega_R}{dQ^2} = \sigma_P^2(1 + c \beta Q)^{-2} > 0. \)

Again, the risk premium is rising at an increasing rate.

But the certainty equivalent is:

\[
CE_R = R - \Omega_R,
\]

\[
= \alpha c^{-1} \ln(1 + c \beta Q).
\]

For positive output, \( CE_R \) is ever-increasing. Figure 14 shows this nonexistence of a finite optimum.

(c) Uniform distribution: \( P \sim M(\alpha, \beta); \ \alpha, \beta > 0. \)

The risk premium and the first-order derivative are:

\[
\Omega_R = \bar{P} - c^{-1} \ln(\beta - \alpha)c\theta + c^{-1} \ln(e^{-\alpha Q - e^{-\beta Q}}),
\]

and \( \frac{d\Omega_R}{dQ} = c^{-1} Q^{-1} + 2c^{-1}(\beta - \alpha)(1 - e^{-(\beta - \alpha)Q})^{-2} - (\beta - \alpha)Q. \)
Figure 14
Nonexistence of A Finite Optimum:
Negative Exponential Utility of Sales with Gamma Distribution of Price

Figure 15
Nonexistence of A Finite Optimum:
Logarithmic Utility of Sales with Lognormal Distribution of Price
If $Q$ is sufficiently large, we have shown that:

$$\lim_{Q \to \infty} \frac{d \Omega_R}{dQ} = \frac{1}{2}(\beta - \alpha) e^{-1}.$$ 

But $\frac{1}{2}c^{-1}(\beta - \alpha) < \frac{1}{2}(\beta + \alpha) = \bar{P}$, thus the expected-revenue function is steeper than the risk-premium function. Again, $CR_\bar{R}$ is ever-rising and no finite optimum exists.

### B.2. Decreasing Absolute Risk Aversion

This is a new case in which an exact risk premium of sales can be derived but not risk premium of profit. The derivation is shown in detail below.

A logarithmic utility function is employed to represent this class; price is assumed to be distributed according to a lognormal process, i.e., $\ln P \sim N(m, s^2)$.

It can be shown (Aitchison and Brown [3]) that:

$$\bar{P} = e^{m + \frac{1}{2}s^2},$$

and

$$\sigma_P^2 = \left(e^{m + \frac{1}{2}s^2}ight)^2(e^s - 1).$$

Taking logarithms of the first one yields $m = \ln \bar{P} - \frac{1}{2}s^2$. Dividing the second one by the square of the first shows:

$$\sigma_P^2 = e^{s^2 - 1},$$

or,

$$s^2 = \ln(\sigma_P^2 + 1).$$

Combining the results we obtain:

$$m = \ln \bar{P} - \frac{1}{2} \ln(\sigma_P^2 + 1).$$

Now:

$$EU(R) = \int (\ln R)f(P)dP,$$
= \int \ln Pf(P) dP + \int \ln Qf(P) dP,
= m + \ln Q,
= \ln \bar{R} - \frac{1}{2} \ln (\sigma_P^2 + 1) + \ln Q,
= \ln [\bar{R}(\sigma_P^2 + 1)^{-\frac{1}{2}}].

But \( U(\bar{R} - \Omega_R) = \ln (\bar{R} - \Omega_R) \). Therefore:

\[ \bar{R} - \Omega_R = \bar{R}(\sigma_P^2 + 1)^{-\frac{1}{2}}, \]

\[ \Omega_R = \bar{R}Q[1-(\sigma_P^2 + 1)^{-\frac{1}{2}}]. \]

The first and second derivatives are:

\[ \frac{d\Omega_R}{dQ} = \bar{R}[1-(\sigma_P^2 + 1)^{-\frac{1}{2}}], \]

\[ < \bar{R}. \]

and \( \frac{d^2\Omega_R}{dQ^2} = 0. \)

Thus the risk-premium function is monotonically increasing with a slope smaller than \( \bar{R} \), which is the slope of \( \bar{R} \). The certainty-equivalent function is therefore ever-increasing and does not have a finite optimum. Figure 15 shows this graphically.
Appendix C

Relationship between Size-of-Risk Aversion and Absolute and Relative Risk Aversions

In the following we prove that non-increasing size-of-risk aversion may occur only when both absolute and relative risk aversion are decreasing. We prove this by showing that all other combinations of \( r_A \) and \( r_R \) lead to increasing \( r_S \).

Since \( r_S \) is defined as:
\[
 r_S = -\pi U'(w_o + \pi)/(w_o + \pi),
 = \pi r_A;
\]
by differentiating \( r_S \) we get:
\[
 r_S' = \frac{dr_S}{d\pi} = r_A + \pi \frac{\partial r_A}{\partial (w_o + \pi)} \frac{d(w_o + \pi)}{d\pi},
 = r_A + \pi r_A'.
\]

Two sufficient conditions for \( r_S' \) to be positive are:

(a) Increasing or constant absolute risk aversion, i.e., \( r_A' \geq 0 \).

For then \( r_S' \) must be positive since \( r_A \) and \( \pi \) are positive.

(b) Decreasing absolute risk aversion and non-decreasing relative risk aversion, i.e., \( r_A' < 0 \) and \( r_R \geq 0 \).

We first differentiate \( r_R = (w_o + \pi) r_A \) and then set the derivative greater than or equal to zero:

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\[ r'_R = r'_A + (\omega + \pi) r'_A, \]
\[ = r'_A + \pi r'_A + \omega r'_A, \]
\[ = r'_S + \omega r'_A, \]
\[ \geq 0; \]

thus:

\[ r'_S \geq -\omega r'_A. \]

But \( r'_A \) is assumed to be negative, therefore size-of-risk aversion is strictly increasing.

This completes our proof of Table 2.
BIBLIOGRAPHY

Journal Abbreviations

AER American Economics Review
BJEMS Bell Journal of Economics and Management Science
EA Economica
EFA Econometrica
IER International Economic Review
JEL Journal of Economic Literature
JET Journal of Economic Theory
JE Journal of Finance
JFQA Journal of Financial and Quantitative Analysis
JPE Journal of Political Economics
JPUE Journal of Public Economics
OEP Oxford Economic Paper
QJE Quarterly Journal of Economics
RES Review of Economic Studies
RESt Review of Economics and Statistics
SEJ Southern Economic Journal


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