TEST VECTORS OF RANKIN-SELBERG
CONVOLUTIONS FOR GENERAL LINEAR GROUPS

DISSERTATION

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In the integral representation of the local Rankin-Selberg $L$-function of a pair $(\pi_1, \pi_2)$ for $GL_n(F) \times GL_m(F)$ with $n \geq m$ over a non-archimedian field $F$, we know that $L(s, \pi_1 \times \pi_2)$ is a finite sum $\sum I(s, W_{\pi_1,i}, W_{\pi_2,i})$ (or $\sum I(s, W_{\pi_1,i}, W_{\pi_2,i}, \Phi_i)$ for $n = m$) of local integrals, where $(W_{\pi_1,i}, W_{\pi_2,i})$ or $(W_{\pi_1,i}, W_{\pi_2,i}, \Phi_i)$ are test vectors of the pair $(\pi_1, \pi_2)$. Our goal is to find explicit formulas for test vectors for any pair $(\pi_1, \pi_2)$ with $\pi_1 \in Irr(GL(n))$ and $\pi_2 \in Irr(GL(m))$. In this dissertation, we prove that one local integral computes $L(s, \pi_1 \times \pi_2)$ for any unramified $\pi_2$. We construct a test vector of any pair $(\pi_1, \pi_2)$ for $GL(n) \times GL(1)$. We also show that most pairs $(\pi_1 \times \pi_2)$ for $GL(n) \times GL(2)$ need at most two test vectors. As a generalization, we prove some pairs for $GL(n) \times GL(m)$ are optimal 1-regular.
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CHAPTER 1
INTRODUCTION

1.1 Introduction

Let $F$ be a non-Archimedean local field and denote by $Irr(GL(n))$ the set of all irreducible admissible and generic representations of $GL(n)$, where $GL(n) = GL_n(F)$. Let $\pi_1 \in Irr(GL(n))$ and $\pi_2 \in Irr(GL(m))$ with $n \geq m \geq 1$. My project concerns non-archimedean local Rankin-Selberg convolutions for a pair of representations on general linear groups:

Find explicit Whittaker functions $W_{1,i}$ and $W_{2,i}$ of $\pi_1$ and $\pi_2$ respectively (along with Schwartz functions $\Phi_i$ on $F^n$ for the case $n = m$) such that the $L$-function $L(s, \pi_1 \times \pi_2)$ for the pair $(\pi_1, \pi_2)$ of representations on $GL(n)$ and $GL(m)$ respectively, $n \geq m$ is equal to $\sum I(s, W_{1,i}, W_{2,i})$, or $\sum I(s, W_{1,i}, W_{2,i}, \Phi_i)$. When such functions exist, we call them test vectors.

We call $(\pi_1, \pi_2)$ an $r$-regular pair if there are $r$ test vectors $(W_{1,i}, W_{2,i})$ or $(W_{1,i}, W_{2,i}, \Phi_i)$ such that

$$L(s, \pi_1 \times \pi_2) = \begin{cases} \sum_{i=1}^{r} I(s, W_{1,i}, W_{2,i}) & \text{if } n > m, \\ \sum_{i=1}^{r} I(s, W_{1,i}, W_{2,i}, \Phi_i) & \text{if } n = m. \end{cases}$$

Abstractly we know that any pair $(\pi_1, \pi_2)$ for $GL(n) \times GL(m)$ is $r$-regular for some $r$. 

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possibly depending on the pair. But for arithmetic applications such as special values of $L$-functions or $p$-adic interpolation, we need to know, as explicitly as possible, a precise expression of the $L$-function as a sum of local integrals.

We will study which pairs $(\pi_1, \pi_2)$ are $r$-regular, in particular 1-regular. We prefer to choose $\hat{W}_2 = W_2^0$ the new vector of $\pi_2$ whenever possible because the new vector is unique (up to constant multiple) and it has a canonical invariance property. In particular, for any pair $(\pi_1, \pi_2)$ and the new vector $W_2^0$ of $\pi_2$, if $(W_1, W_2^0)$ (or $(W_1, W_2^0, \Phi)$ for $n = m$) is a test vector, then we call $(\pi_1, \pi_2)$ an optimal 1-regular pair.

In this dissertation, our goal is to find explicit formulas for test vectors for any pair $(\pi_1, \pi_2)$ with $\pi_1 \in \text{Irr}(\text{GL}(n))$ and $\pi_2 \in \text{Irr}(\text{GL}(m))$ $(n \geq m)$. The main idea is to apply the theory of derivatives of representations developed by Bernstein and Zelevinski in [2, 27] which relates any irreducible admissible representation on $\text{GL}(n)$ with its “derivatives”, which are representations on $\text{GL}(d)$ with $d \leq n$. This in general allows us to study representations inductively on the size of a general linear group. Moreover, Cogdell and Piatetski-Shapiro in [8] calculated $L$-functions of a representation in terms of the $L$-functions of its derivatives, hence we may study $L$-functions inductively as well.

There are some classical results: Due to Shintani [25], for the unramified representation $\pi$ of $\text{GL}(n)$, there is the unique normalized Whittaker functions $W_\pi^0$, which can be explicitly written down and are called the spherical vectors fixed by the respective maximal compact subgroups. So when both representations are unramified, the spherical vectors $W_1^0$ and $W_2^0$ satisfy $L(s, \pi_1 \times \pi_2) = I(s, W_1^0, W_2^0)$ which was proved
by Jacquet- Piatetski-Shapiro-Shalika [16]. Thus, the pair \((\pi_1, \pi_2)\) is optimal 1-regular if both \(\pi_1\) and \(\pi_2\) are unramified. In [18], Jacquet- Piatetski-Shapiro-Shalika proved that for unramified \(\pi_2 \in Irr(GL(m))\) and its spherical vector \(W_2^0\), there is a suitable Whittaker function, called the new vector \(W_1^0\) of \(\pi_1 \in Irr(GL(m+1))\) so that \(I(s, W_1^0, W_2^0) = L(s, \pi_1 \times \pi_2)\). Thus, for any \(m \geq 1\), and \(\pi_1 \in Irr(GL(m+1))\), the pair \((\pi_1, \pi_2)\) is optimal 1-regular if \(\pi_2 \in Irr(GL(m))\) is unramified, and \((W_1^0, W_2^0)\) is its test vector where \(W_1^0\) and \(W_2^0\) are new vectors of \(\pi_1\) and \(\pi_2\) respectively.

The corresponding theory for the Archimedean fields is more complicated and it is unknown if the local \(L\)-function has an expression of a finite sum of local integrals except a certain case. More on the theory of the Archimedean fields and Global theory can be found in [9].

### 1.2 Outlines

Let us briefly describe the contents of this dissertation. Chapter 1 begins with the general notations and theories to be used throughout this dissertation. In Chapter 2, we will study the case when \(\pi_2\) is an unramified representation of \(GL(m)\). In particular, we will show that \((\pi_1, \pi_2)\) is optimal 1-regular for any \(\pi_1 \in Irr(GL(n))\) with \(n \geq m\). In this case we have a test vector \((W_1^0, W_2^0)\) for \(n > m\) as in Theorem 2.2.1 and \((W_1^0, W_2^0, \Phi)\) for some \(\Phi \in S(F^m)\) for \(n = m\) for the pair \((\pi_1, \pi_2)\) as in Theorem 2.1.1. This is a generalization of the classical result by Jacquet, Piatetski-Shapiro and Shalika in [18]. Moreover, we will prove that \(I(s, W_1^0, W_2^0) = 0\) if \(\pi_2\) ramified for \(n > m\) in Proposition 2.2.2. After this
Chapter, we assume that $\pi_2$ is a ramified representation of $GL(m)$ unless otherwise stated. For a possibly ramified representation $\pi_2$ on $GL(m)$, we take $W_2 = W_2^0$ to be its new vector in the sense of Jacquet, Piatetski-Shapiro and Shalika in [18], and attempt to find an explicit Whittaker function $W_1$ of $\pi_1$ on $GL(n)$ when $n > m$, so that $I(s, W_1, W_2^0) = L(s, \pi_1 \times \pi_2)$. In the case when $n = m$ we need also an explicit Schwartz function on $F^n$. This is carried out for the cases $GL(n) \times GL(1)$ and $GL(n) \times GL(2)$ for $n \geq 2$. Moreover, the case of $GL(n) \times GL(m)$ with $n > m$ can be reduced inductively to the case of $GL(m) \times GL(m)$. For this we constructed test vectors along with Schwartz functions for $m = 1$ and $m = 2$, and then we will generalize the construction to $m \geq 3$. In Chapter 3, we will discuss the case when $\pi_2$ is a ramified representation of $GL(1)$ and show that $(\pi_1, \pi_2)$ is optimal 1-regular for any $\pi_1 \in Irr(GL(n))$ with $n \geq 1$. We will prove that most pair $(\pi_1, \pi_2)$ of $GL(2) \times GL(2)$ are $r$-regular with $r \leq 2$ in Chapter 4 and 5. And then we apply these result to study the case $\pi_1 \in Irr(GL(n))$ with $n \geq 3$ and a ramified representation $\pi_2 \in Irr(GL(2))$ in Chapter 6. Specially, the case $GL(m) \times GL(m)$ is connected to E. Lapid’s question: whether we can use the new vectors of $\pi$ and $\tilde{\pi}$ as a test vector of a pair $(\pi, \tilde{\pi})$ along with an appropriate Schwartz function. We will answer this question for $m = 1, 2$ in Chapter 3 and 5.
1.3 Preliminaries

1.3.1 General notations

Let $F$ be a non-Archimedean local field, $\varpi$ the uniformizer of $F$, and $p = (\varpi)$ the maximal ideal of $F$. We normalize the absolute value on $F$ by $|\varpi| = q^{-1}$ with $q$ the cardinality of the residue field. Let $1$ be the trivial character of $F^\times$ and $\nu = |det(\cdot)|$.

Let $G = GL_n(F) = GL(n)$ be the general linear group over $F$, $B_n$ the standard upper triangular Borel subgroup of $G_n$, $N_n$ the unipotent radical of $B_n$, $K_n = GL_n(O)$ the maximal compact group of $G_n$, and $I_n$ the identity matrix in $G_n$. Let $P_n$ denote the mirabolic subgroup of $G_n$, that is the subgroup consisting of matrices in $G_n$ with the last row $(0, \cdots, 0, 1)$ and $Z_n$ is the center of $G_n$. Whenever necessary, we use the subscript to indicate the size of the matrices in discussion except the unipotent radical $N_n$.

Fix a non-trivial $\mathbb{C}$-valued continuous additive character $\psi$ of $F$ whose conductor is $O$, the ring of integers of $F$. We can define a character of $N_n$ as follows:

$$\psi(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}) \quad \text{for } u = (u_{i,j}) \in N_n. \quad (1.1)$$

By abuse of notation, we still denote it by $\psi$.

Let $\pi$ be an irreducible admissible generic representation of $G$ with representation space $V = V_\pi$. Then $V_\pi$ can be realized in a space of functions, the Whittaker Model denoted $W(\pi, \psi)$. The Whittaker function $W$ in the Whittaker model satisfy the following property:

$$W(ug) = \psi(u)W(g) \quad \text{for any } u \in N_n, g \in G.$$
And for $W \in \mathcal{W}(\pi, \psi)$, we have $W(zg) = \omega_{\pi}(z)W(g)$ for any $z \in Z$ and $g \in G$, where we call $\omega_{\pi}$ the central character of $\pi$. We may use the notation of a Whittaker function $W_1$ and the central character $\omega_1$ instead of $W_{\pi_1}$ and $\omega_{\pi_1}$, specially when we consider pairs $(\pi_1, \pi_2)$. For $W \in \mathcal{W}(\pi, \psi)$, if we let $\tilde{W}(g) = W(w^tg^{-1})$ where $w = \begin{pmatrix} 1 & \cdots & 1 \\ \\ 1 & \cdots & 1 \end{pmatrix}$, then $\tilde{W}(g)$ is in $\mathcal{W}(\tilde{\pi}, \psi^{-1})$, where $\tilde{\pi}$ is the contragradient of $\pi$.

Let $\mathcal{S}(F^n)$ be the space of Schwartz functions on $F^n$, which are locally constant and compactly supported. Denote by $\mathcal{S}_0(F^n)$ the subspace of $\Phi \in \mathcal{S}(F^n)$ with $\Phi(0) = 0$. Given an additive character $\psi$ of $F$ and a Haar measure $dx$ of $F$, we can consider the Fourier transformation, for $\varphi \in \mathcal{S}(F)$, $\hat{\varphi}(y) = \int_F \varphi(x)\psi(xy)dx$. By the Fourier inversion formula, we have $\hat{\varphi}(x) = c\varphi(-x)$ for some $c$. Usually, by adjusting $dx$, one may assume $c = 1$, in which case $dx$ is called self-dual with respect to $\psi$. Let $dk$ be the normalized Haar measure on $K$ such that $Vol(K) = 1$ and for a fixed integer $N$, denote by $V_N$ the volume of $K_0(N)$, where

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_n(O) \left| c \equiv 0 \pmod{\varpi^N} \right. \right\}.$$

1.3.2 Rankin-Selberg Convolution

We review some basic facts about $L$-functions attached to a pair of irreducible generic representations of $GL(n)$ and $GL(m)$, the details can be found in [16].
Let $W_1 \in \mathcal{W}(\pi_1, \psi)$ and $W_2 \in \mathcal{W}(\pi_2, \psi^{-1})$, the local integrals of a pair $(\pi_1, \pi_2)$ are given by

$$I(s, W_1, W_2, \Phi) = \int_{N_m \backslash G_m} W_1(g)W_2(g)\Phi(e_m g)|\text{det}(g)|^s dg \quad \text{if} \quad n = m, \Phi \in \mathcal{S}(F^m),$$

where $e_m = (0, 0, \cdots, 1) \in F^m$, and

$$I(s, W_1, W_2) = \int_{N_m \backslash G_m} W_1\left(\begin{array}{c} g \\ I_{n-m} \end{array}\right)W_2(g)|\text{det}(g)|^{s - \frac{n-m}{2}} dg \quad \text{if} \quad n > m. \quad (1.3)$$

Jacquet, Piatetski-Shapiro and Shalika [16, Theorem 2.7] showed that each of the integrals is absolutely convergent of $\text{Re}(s)$ large, and they are rational functions of $q^{-s}$. In it, they also proved that the local integrals span a fractional ideal $\mathbb{C}[q^s, q^{-s}]L(s, \pi_1 \times \pi_2)$ of the ring $\mathbb{C}[q^s, q^{-s}]$ such that the factor $L(s, \pi_1 \times \pi_2)$ has the form $P(q^{-s})^{-1}$, where $P \in \mathbb{C}[X]$, $X = q^{-s}$ and $P(0) = 1$. Moreover, they proved the existence of the local functional equation for a pair $(\pi_1, \pi_2)$. We may assume that $L(s, \pi_1 \times \pi_2) = L(s, \pi_2 \times \pi_1)$ for the case $n = m$ based on the argument from [16].

For $n = m$, we divide the poles of $L(s, \pi_1 \times \pi_2)$ into two types. We call a pole of $L(s, \pi_1 \times \pi_2)$ regular if it is a pole of some $I(s, W_1, W_2, \Phi)$ with $\Phi \in \mathcal{S}_0(F^m)$. A pole of $L(s, \pi_1 \times \pi_2)$ is called exceptional if it is not a pole of any $I(s, W_1, W_2, \Phi)$ when $\Phi$ is restricted to lie in $\mathcal{S}_0(F^n)$. In particular, from [8, Proposition 2.1], we know that the exceptional poles $s_0$ of the family $I(s, W_1, W_2, \Phi)$ can only occur among those $s$ for which $\tilde{\pi}_1 \cong \pi_2 \nu^s$. 7
1.3.3 Derivatives of representation and $L$-function of a pair

Bernstein and Zelevinski discussed the theory of derivatives of generic representations in [2] [27]. Cogdell and Piatetski-Shapiro explicitly computed the $GL(n) \times GL(m)$ local $L$-factor for generic representation in terms of $L$-functions for supercuspidal representations in [8]. From [8], we can see the computation of the local $L$-functions $L(s, \pi \times \sigma)$ for representations $\pi$ of $GL(n)$ and $\sigma$ of $GL(m)$ in terms of the (exceptional) $L$-functions of their derivatives.

First we review derivatives of representation, more details can be found in [2]. Let $\tau$ be a smooth representation of $P_n$ and denote by $\text{Rep}(P_n)$ the set of all smooth representations of $P_n$ up to equivalence. Let $U_n = \left\{ \begin{pmatrix} I_{n-1} & \ast \\ 1 & \end{pmatrix} \right\} \subset P_n$. For given $\tau \in \text{Rep}(P_n)$, we consider functors $\Phi^-$ and $\Psi^-$ as follows: $\Phi^-(\tau)$ is the normalized representation of $P_{n-1}$ on the space $V_\tau/V_\tau(U_n, \psi)$ with $V_\tau(U_n, \psi) = \langle \tau(u)v - \psi(u)v | u \in U_n, v \in V_\tau \rangle$, and $\Psi^-(\tau)$ is the normalized representation of $GL(n-1)$ on the quotient $V_\tau/V_\tau(U_n, 1)$. Now, for each $k = 1, 2, \cdots, n$, there are representations $\tau^{(k)} \in \text{Rep}(P_{n-k})$ and $\pi^{(k)} \in \text{Rep}(GL(n-k))$ associated to $\tau$ defined by

$$\tau^{(k)} = (\Phi^-)^k(\tau) \quad \text{and} \quad \pi^{(k)} = \Psi^-(\Phi^-)^{k-1}(\tau).$$

Let $\pi \in \text{Rep}(GL(n))$, then the derivative of $\pi$ can be defined by using the restriction of $\pi$ to $P_n$, that is, if we set $\tau = \pi(0) = \pi|_{P_n}$, then $\pi^{(k)} = \tau^{(k)}$ and $\pi^{(k)} = \tau^{(k)}$ for $k = 1, 2, \cdots, n$. From the following lemmas, we know that $\pi^{(k)}$ is unramified in $\text{Rep}(GL(n-k))$ when $\pi$ is unramified in $\text{Rep}(P_n)$, which means that $\pi_{(0)}$ has a nonzero $K_n \cap P_n$-fixed vector. Note that $\pi^{(k)}$ need not be irreducible.
Lemma 1.3.1. Let $\pi_{(0)}$ be unramified in $\text{Rep}(P_n)$.

1. $\pi_{(1)} \in \text{Rep}(GL(n - 1))$ is unramified.

2. $\pi_{(1)} \in \text{Rep}(P_{n-1})$ is unramified, that is, $\pi_{(1)}$ has a nonzero $P_{n-1} \cap K_{n-1}$-fixed vector.

Here we will use the characterization of $V_{\tau}(U_n, \psi)$ and $V_{\tau}(U_n, 1)$ in terms of the Whittaker model of $\tau$ as in [8]. There are equivalent characterizations of the space $V_{\tau}(U_n, \psi)$ and $V_{\tau}(U_n, 1)$ in [1, 2]. Once we prove this, by induction, we know that $\pi^{(k)}$ is unramified in $\text{Rep}(GL(n - k))$ for all $k$.

Proof. Let $\tau = \pi_{(0)} \in \text{Irr}(P_n)$ be unramified and $v$ be a nonzero $K_n \cap P_n$-fixed vector of $\tau$. Then, we have natural projection

$$v \mapsto \bar{v} = v + V_{\tau}(U_n, 1) \quad \text{in the space of } \Psi^{-}(\tau).$$

Let $k' \in K_{n-1} \subseteq P_n \cap K_n$. Then

$$\Psi^{-}((\pi_{(0)})(k')(\bar{v})) = |\det(k')|^{-1/2}(\tau(k')v + V_{\tau}(U_n, 1)) = v + V_{\tau}(U_n, 1) = \bar{v}.$$

To complete the proof, we have to show that $v$ is not in $V_{\tau}(U_n, 1)$. In fact, for any nonzero $K_n$-fixed vector, we can show that $v \not\in V_{\tau}(U_n, 1)$. Let $W_v$ be the function corresponding to $v$, note that $v \mapsto W_v$ is injective, so $W_v \not= 0$ if $v \not= 0$. In particular, for $J = (f_1, f_2, \cdots, f_{n-1}) \in \mathbb{Z}^{n-1}$, set $\varpi^J = \text{diag}(\varpi^{f_1}, \varpi^{f_2}, \cdots, \varpi^{f_{n-1}})$, then $W_v \left( \begin{array}{c} \varpi^J \\ 1 \end{array} \right) \not= 0$ only if $f_1 \geq f_2 \geq \cdots \geq f_{n-1} \geq 0$ since $\psi$ is an unramified additive character. Note that we have the characterization of $V_{\tau}(U_n, 1)$ as follows:
\[
\begin{cases}
W_v \left( \begin{array}{c} g \\ 1 \end{array} \right) \\
\text{there exists } N > 0 \text{ such that }
\end{cases}
\]

max \{ |g_{n-1,i}| \} < q^{-N} \text{ implies } W_v \left( \begin{array}{c} g \\ 1 \end{array} \right) \equiv 0.

Suppose \( W_v \left( \begin{array}{c} g \\ 1 \end{array} \right) \in V_\tau(U_n, 1) \). Then there exists \( N > 0 \) (depending on \( v \)) such that if \( g = diag(\varpi^{N+1}, \ldots, \varpi^{N+1}) \), then \( W_v \left( \begin{array}{c} g \\ 1 \end{array} \right) = 0 \) since \( \max \{ |g_{n-1,i}| \} = |\varpi^{N+1}| = q^{-(N+1)} < q^{-N} \). But for this \( g \), \( W_v \left( \begin{array}{c} g \\ 1 \end{array} \right) \) is not zero by Shintani’s formular in [25] for \( W_v \). Thus, nonzero \( K_n \)-fixed vector can not be in \( V_\tau(U_n, 1) \), which proves that \( \Psi^{-}(\pi(0)) = \pi^{(1)} \) has a nonzero \( K_{n-1} \)-fixed vector.

Now similarly, \( \tilde{v} = v + V_\tau(U_n, \psi) \) is a \( P_{n-1} \cap K_{n-1} \)-fixed vector. Note that

\[
V_\tau(U_n, \psi) = \left\{ W_v \left( \begin{array}{c} g \\ 1 \end{array} \right) \left| W_v \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right) = 0 \text{ for } p \in P_{n-1} \right. \right\}.
\]

If \( W_v \) is normalized \( K \)-invariant, that is, \( W_v(I_n) = 1 \), and \( I_{n-1} \in P_{n-1} \), so if \( v \in V_\tau(U_n, \psi) \), this is impossible unless \( W_v = 0 \) by the injectivity of \( v \mapsto W_v \). Thus, nonzero \( K \)-fixed vector can not be in \( V_\tau(U_n, \psi) \), that is \( \pi^{(1)} = \Phi^{-}(\pi(0)) \) has a nonzero \( K_{n-1} \cap P_{n-1} \)-fixed vector, as desired. \( \square \)

Let \( \pi \) be the irreducible generic representation of \( GL(n) \). From [27], we have the
classification of such $\pi$ and in this dissertation, we follow the notation from [8] as follows:

$$\pi \cong Ind(\Delta_1 \otimes \cdots \otimes \Delta_t), \quad \Delta_i = [\rho_i, \rho_i \nu, \cdots, \rho_i \nu^{i-1}] \in Irr(GL(r_i)),$$

where $\rho_i$ is a supercuspidal representation of $GL(m_i)$, such that $r_i = m_i l_i$, and $n = r_1 + \cdots + r_t$. (1.4)

Note that $\Delta_i$ above is the unique irreducible quotient of $Ind(\rho_i \otimes \rho_i \nu \otimes \cdots \otimes \rho_i \nu^{l_i-1})$ and $\Delta_i$ are non-linked in the sense of [2, 27]. The derivative of these representations are following [8, Section 2.4]:

- Let $\rho$ be a supercuspidal representation of $GL(r)$, then $\rho^{(0)} = \rho$, $\rho^{(k)} = 0$ for $1 \leq k \leq r-1$, and $\rho^{(r)} = 1$.

- Let $\pi = \Delta$ with $\Delta = [\rho, \rho \nu, \cdots, \rho \nu^{l-1}]$ and $\rho$ a supercuspidal representation of $GL(r)$. Then $\pi^{(k)} = 0$ if $k$ is not a multiple of $r$, $\pi^{(0)} = \Delta$, $\pi^{(kr)} = [\rho \nu^k, \rho \nu^{k+1}, \cdots, \rho \nu^{l-1}]$ for $1 \leq k \leq l-1$, and $\pi^{(lr)} = 1$.

- Let $\pi$ be generic and $\pi = Ind(\Delta_1 \otimes \cdots \otimes \Delta_t)$, then $\pi^{(k)}$ is glued from those representation of the form $Ind(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)})$.

Throughout this dissertation, we assume that $\pi$ is in general position, which means that the derivatives $\pi^{(k)}$ are completely reducible and the subquotients $Ind(\Delta_1^{(k_1)} \otimes \cdots \Delta_t^{(k_t)})$ are generic and irreducible. We denote by $\{\pi_i^{(k)}\}$ the finite set of irreducible generic constituents of $\pi^{(k)}$.

Let $\pi \in Irr(GL(n))$ and $L(s, \pi)$ be the local $L$-function attached by $\pi$ defined in [13]. In it, the $L$-function has the form $L(s, \pi) = P_\pi(q^{-s})^{-1}$, where $P_\pi(X) \in \mathbb{C}[X]$ satisfies
\( P(0) = 1 \), and we will refer to the degree of \( P_\pi(X) \) as the degree of \( L(s, \pi) \). We know that if \( u \) is the degree of \( L(s, \pi) \), then \( u \leq n \) for \( \pi \in \text{Irr}(GL(n)) \) by \([18\text{, Proposition 2.5}]\).

**Theorem 1.3.2.** Let \( \pi \in \text{Irr}(GL(n)) \) be in general position. Then \( \pi^{(k)} \) has an unramified constituent, for \( n \geq k \geq n - u \), in particular, \( \pi^{(n-u)} \) has a unique unramified constituent \( \pi_0^{(n-u)} \). Moreover, \( L(s, \pi) = L(s, \pi_0^{(n-u)}) \).

**Proof.** Let \( \pi \in \text{Irr}(GL(n)) \) as in \(1.4\), then by \([19]\) we know that

\[
L(s, \pi) = \prod_{i=1}^{t} L(s, \Delta_i), \quad \text{where}
\]

\[
L(s, \Delta_i) = \begin{cases} 
L(s, \rho_i \nu_i^{\lambda_i-1}) & \text{if } \rho_i \text{ is unramified representation of } GL(1), \\
1 & \text{otherwise.}
\end{cases}
\]

Since the degree of \( P_\pi \) is \( u \), there are \( u \) \( \rho_i \)'s which are unramified characters. Without loss of generality, we may assume that \( \rho_1, \cdots, \rho_u \) are unramified characters. Then

\[
L(s, \pi) = \prod_{i=1}^{u} L(s, \rho_i \nu_i^{\lambda_i-1}).
\]

By \([27]\) and the general position assumption,

\[
\pi^{(k)} = \sum_{k_1 + \cdots + k_t = k} \text{Ind}(\Delta_1^{(k_1)} \otimes \cdots \otimes \Delta_t^{(k_t)}).
\]

If \( \pi^{(k)} \) has an unramified constituent, then this constituent is fully induced from
unramified character. So $\pi^{(k)}$ has a unramified constituent if and only if $k = k_1 + \cdots + k_t$ and $k_j$ is as below:

$$k_j = \begin{cases} r_{j-1} \text{ or } r_j & \text{if } 1 \leq j \leq u, \\ r_j & \text{if } u+1 \leq j \leq t. \end{cases}$$

Thus, for $k = n - j$, $0 \leq j \leq u$, $\pi^{(k)}$ has $\binom{u}{j}$ unramified constituents. Consequently, $\pi^{(n-u)}$ has only one unramified constituent, and denote $\pi_0^{(n-u)}$ the unique unramified constituent of $\pi^{(n-u)}$. By (1.6), $\pi_0^{(n-u)} = Ind(\rho_1 \nu_1^{u-1} \otimes \cdots \otimes \rho_u \nu_u^{u-1})$. And from (1.5), we conclude that $L(s, \pi) = L(s, \pi_0^{(n-u)})$. □

**Corollary 1.3.3.** For any $\pi_1 \in \text{Irr}(GL(n))$ with $u = \text{degree of } L(s, \pi_1)$:

$$L(s, \pi_1 \times \pi_2) = L(s, \pi_1^{(n-u)} \times \pi_2),$$

if $\pi_2 \in \text{Irr}(GL(m))$ is an unramified generic representation with $m \leq n$.

**Proof.** Let $\pi_2 = Ind(\mu_1 \otimes \cdots \otimes \mu_m)$ with all $\mu_i$s are unramified characters, then by [16, Section 8], $L(s, \pi_1 \times \pi_2) = \prod_{i=1}^{m} L(s, \pi_1 \times \mu_i)$, and for each $L(s, \pi_1 \times \mu_i)$, by [19, 14], we have

$$L(s, \pi_1 \times \mu_i) = L(s, (\pi_1 \otimes \mu_i) \times 1) = L(s, \pi_1 \otimes \mu_i).$$

Thus,

$$L(s, \pi_1 \times \pi_2) = \prod_{i=1}^{m} L(s, \pi_1 \times \mu_i) = \prod_{i=1}^{m} L(s, \pi_1 \otimes \mu_i),$$

and $L(s, \pi_1 \otimes \mu_i)$ has the same degree as $L(s, \pi_1)$, say $u$, and

$$L(s, \pi_1 \otimes \mu_i) = L(s + s_i, \pi_1) = L(s + s_i, \pi_1^{(n-u)}),$$

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where $s_i \in \mathbb{C}$ such that $\mu_i = \nu^{s_i}$. Thus, for each $i = 1, 2, \cdots, m$,

$$L(s, \pi_1 \otimes \mu_i) = L(s, \pi_{1,0}^{(n-u)} \otimes \mu_i).$$

So

$$\prod_{i=1}^{m} L(s, \pi_1 \otimes \mu_i) = \prod_{i=1}^{m} L(s, \pi_{1,0}^{(n-u)} \otimes \mu_i) = L(s, \pi_{1,0}^{(n-u)} \times \pi_2).$$

\[\square\]

1.3.4 New vector of the irreducible admissible generic representation of $GL(n)$

In [18, Section 4], Jacquet, Piatetski-Shapiro, and Shalika showed that for any $\pi \in \text{Irr}(GL(n))$ there is $N \geq 0$ depending on $\pi$ such that $\text{dim} V_{\pi}^{K_1(N)} = 1$. We then call $N$ the conductor of $\pi$ and denote it by $N = c(\pi)$. In other words, there is unique $W_0^\pi \in \mathcal{W}(\pi, \psi)$ such that

- $W_0^\pi \left( g \begin{pmatrix} k \\ 1 \end{pmatrix} \right) = W_0^\pi(g)$, for all $k \in GL_{n-1}(\mathcal{O})$,

- $L(s, \pi \times \sigma) = I(s, W_0^\pi, W_0^\sigma)$, for any unramified $\sigma$ of $GL_{n-1}$ and $GL_{n-1}(\mathcal{O})$-fixed vector $W_0^\sigma$.

We call this $W_0^\pi$ the new vector of $\pi$, which is invariant under $K_1(N)$, where

$$K_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_n(\mathcal{O}) \bigg| c \equiv 0 \pmod{\varpi^N}, \quad d \equiv 1 \pmod{\varpi^N} \right\}.$$
In fact, by the properties of the new vector $W_\pi^0$, we have

$$W_\pi^0\left(g \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \omega_\pi(d)W_\pi^0(g), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N).$$

So if the central character $\omega_\pi$ of $\pi$ is unramified, then the new vector $W_\pi^0$ of $\pi$ is also invariant under $K_0(N)$.

Now, we realize the new vector $W_\pi^0 \in \mathcal{W}(\pi, \psi)$ of $\pi$ in the Kirillov model. By [18, Lemma 5.2], we know that for any new vector $W_\pi^0 \in \mathcal{W}(\pi, \psi)$,

$$W_\pi^0\left(g \begin{pmatrix} 1 \\ \end{pmatrix}\right) \neq 0, \quad g \in G_{n-1},$$

implies that the last row of $g$ has integer coefficients. From [18, Section 5], we rewrite the new vector formula in terms of derivatives. More precisely, We express the new vector $W_\pi^0$ of $\pi$ in terms of the new vector $W_{\pi_0}^0(n-u)$ of the unramified constituent of $\pi^{(n-u)}$. Note that from Theorem 1.3.2, we know that $\pi^{(n-u)}$ has the unique unramified constituent $\pi_0^{(n-u)} \in Irr(GL(u))$ where $u$ is the degree of $L(s, \pi)$.

First, consider the case when $u = n$, which is equivalent to $\pi$ being unramified. For this, Shintani [25] showed the explicit formula for the spherical vector as follows:

Let $\pi = \text{Ind}(\chi_1 \otimes \chi_2 \otimes \cdots \otimes \chi_n)$, $\chi_i$ unramified character for all $i$, and set $\mu_i = \chi_i(\varpi)$ and let $f = (f_1, f_2, \cdots, f_n) \in \mathbb{Z}^n$. Then

$$W_\pi^0(\varpi^f) = q\sum_{i=1}^n \frac{(i-u)f_i}{2} \chi_f(\mu), \quad (1.7)$$
where
\[
\chi_f(\mu) = \begin{vmatrix}
\mu_1^{f_1+n-1} & \mu_2^{f_1+n-1} & \cdots & \mu_n^{f_1+n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_1^{f_n} & \mu_2^{f_n} & \cdots & \mu_n^{f_n}
\end{vmatrix}
\prod_{i<j}(\mu_i - \mu_j),
\]
if \( f_1 \geq f_2 \geq \cdots \geq f_n \) and is equal to zero otherwise.

Next, if \( u = n - 1 \), then the result from [18, Section 5] and Theorem 1.3.2 give
\[
W_\pi^0 \left( \begin{array}{c} g \\ 1 \end{array} \right) = W_{\pi_0^{(1)}}^0(\Phi_0(e_{n-1}g)|\det(g)|^{1/2},
\]
where \( W_{\pi_0^{(1)}}^0 \) is the \( K \)-fixed function of \( \pi_0^{(1)} \) and \( \Phi_0 \) is the characteristic function of \( O^{n-1} \).

Finally, if \( u < n - 1 \), then as in [18, Section 5], define \( \varphi_\pi(g) \) on \( GL_{n-1} \) as follows:
\[
\varphi_\pi(nak) = \psi(n)\varphi_\pi(a),
\]
where \( g = nak \) by the Iwasawa decomposition and \( \psi(n) \) as in (1.1). And for \( a = \text{diag}(a_1, \cdots, a_{n-1}) \),
\[
\varphi_\pi(a) = \begin{cases}
W_{\pi_0^{(n-u)}}^0 \left( \begin{array}{c}
a_1 \\
\vdots \\
a_u \end{array} \right) & \text{if } |a_{u+1}| = \cdots = |a_{n-1}| = 1, \\
0 & \text{otherwise}.
\end{cases}
\]
Then
\[
W_\pi^0 \left( \begin{array}{c} g \\ 1 \end{array} \right) = \varphi_\pi(g)|\det(g)|^{\frac{n-u}{2}}.
\]
Remark 1.3.4. 1. We normalize the new vector $W^0_\pi$ such that $W^0_\pi(I_n) = 1$.

2. In later chapter, we mostly consider the pair $(\pi_1, \pi_2)$ with a ramified representation $\pi_2$ and its new vector $W^0_{\pi_2}$. Since the new vector $W^0_\pi$ above is defined in $\mathcal{W}(\pi, \psi)$, we need to modify $W^0_{\pi_2}$ for $\pi_2$ so that $W^0_{\pi_2} \in \mathcal{W}(\pi_2, \psi^{-1})$ to consider the Rankin-Selberg convolution. For this, we can apply the following result from [15]: for given $W \in \mathcal{W}(\pi, \psi)$, $W'(g) = W(\eta g) \in \mathcal{W}(\pi, \psi^{-1})$ where $\eta = \text{diag}(-1, 1, -1, \cdots, (-1)^n)$.

3. $L(s, \pi)$ can be thought of as the $L$-function of $\pi$ twisted by the trivial character of $GL(1)$, more generally by [16], $I(s, W^0_\pi, W^0_\sigma) = L(s, \pi \times \sigma)$ if both $\pi$ and $\sigma$ are unramified. From Theorem 1.3.2, we can conclude $I(s, W^0_\pi) = L(s, \pi)$. And by the above definition of the new vector $W^0_\pi$, we have $I(s, W^0_\pi) = I(s, W_{\pi(n-u)})$, for any $\pi \in \text{Irr}(GL(n))$ with $n \geq 2$ and $u$ be the degree of $L(s, \pi)$. 
CHAPTER 2

L-FUNCTIONS OF A PAIR \((\pi_1, \pi_2)\) FOR \(GL(N) \times GL(M)\)

WITH ANY UNRAMIFIED REPRESENTATION \(\pi_2\)

From [18], it is known that for \(\pi_2\) an unramified representation of \(GL(m)\) and its spherical vector \(W_2^0\), there is unique \(W_1^0 \in W(\pi_1, \psi)\), where \(\pi_1 \in Irr(GL(m + 1))\) so that \(I(s, W_1^0, W_2^0) = L(s, \pi_1 \times \pi_2)\). Thus, for any \(m\) and \(\pi_1 \in Irr(GL(m + 1))\), \((\pi_1, \pi_2)\) is 1-regular and \((W_1^0, W_2^0)\) is its test vector if \(\pi_2 \in Irr(GL(m))\) is unramified. In this chapter, we generalize this result to the case for any \(\pi_1 \in Irr(GL(n))\), \(n \geq m\).

Some computations in this chapter are based on [11, Lecture7].

2.1 A pair \((\pi_1, \pi_2)\) for \(GL(m) \times GL(m)\)

Let \(\pi_2\) be an unramified representation of \(GL(m)\). We discuss the pair \((\pi_1, \pi_2)\) for \(GL(m) \times GL(m)\) here.

Theorem 2.1.1. For any \(m\), \(\pi_i \in Irr(GL(m))\), \(i = 1, 2\), \((\pi_1, \pi_2)\) is optimal 1-regular if \(\pi_2\) is unramified. More precisely, for fixed new vectors \(W_i^0\), \(i = 1, 2\), there is an appropriate \(\Phi \in S(F^m)\) so that \(L(s, \pi_1 \times \pi_2) = I(s, W_1^0, W_2^0, \Phi)\).

Before proving this, we study the general ideas of the local integral \(I = I(s, W_1, W_2, \Phi)\).
for a pair $(\pi_1, \pi_2)$ in $GL(m) \times GL(m)$ as follows: for $W_1 \in \mathcal{W}(\pi_1, \psi)$, $W_2 \in \mathcal{W}(\pi_2, \psi^{-1})$ and $\Phi \in \mathcal{S}(F^m)$, from the equation  \ref{eq:1.2},

\[
I = \int_{N_m \backslash G_m} W_1(g)W_2(g)\Phi(e_m g) |\text{det}g|^s \text{d}g.
\]

By Iwasawa’s decomposition, for $g \in G_m$

\[
g = nz \begin{pmatrix} h \\ 1 \end{pmatrix} k, \quad n \in N_m, z = z \cdot I_m, z \in F^\times, h \in G_{m-1}, k \in K_m.
\]

We can decompose our integral as

\[
I = \int \int W_1 \begin{pmatrix} h \\ 1 \end{pmatrix} W_2 \begin{pmatrix} h \\ 1 \end{pmatrix} Z(ms, \omega, \Phi, k) |\text{det}(h)|^{s-1} \text{d}h \text{d}k,
\]

where the integration is over $h \in N_{m-1} \setminus G_{m-1}$ and $k \in K_m$, $\omega = \omega_1 \omega_2$ and

\[
Z(ms, \omega, \Phi, k) = \int_{F^\times} \omega(z) \Phi(e_m z k)|z|^{ms} \text{d}^\times z, \quad k \in K_m. \tag{2.1}
\]

Now, for a fixed character $\chi$ and an integer $N \geq 0$, define a Schwartz function $\Phi^0_{\chi,N}$ on $F^m$ by

\[
\Phi^0_{\chi,N}(x_1, \cdots, x_m) = \begin{cases} 
\chi^{-1}(x_m) & \text{if } x_1, \cdots, x_{m-1} \in \mathcal{O}^N, x_m \in \mathcal{O}^\times, \\
0 & \text{otherwise}.
\end{cases} \tag{2.3}
\]

For any $A \subseteq F^m$, let $\Xi_A$ be the characteristic function of $A$, that is,

\[
\Xi_A(z) = \begin{cases} 
1 & \text{if } z \in A, \\
0 & \text{otherwise}.
\end{cases}
\]
Lemma 2.1.2. Let $W_i^0$ be the new vectors of $\pi_i$, $i = 1, 2$. Set $N = \max\{c(\pi_1), c(\pi_2)\}$ and $\omega = \omega_1\omega_2$, and $\Phi_{\omega,N}^0$ as in (2.3). Then $I(s, W_1^0, W_2^0, \Phi_{\omega,N}^0) = V_N B(s, W_1^0, W_2^0)$ where

$$B(s, W_1^0, W_2^0) = \int_{N_{m-1}\setminus G_{m-1}} W_1^0 \begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix} W_2^0 \begin{pmatrix} h & 1 \\ 1 & 1 \end{pmatrix} |\det(h)|^{s-1} dh. \quad (2.4)$$

Proof. By the definition of $\Phi_{\omega,N}^0$ in (2.3), $\Phi_{\omega,N}^0(e_m z k) \neq 0$ only if $k \in K_0(N)$ and $z \in \mathcal{O}^\times$. In other words, $\Phi_{\omega,N}^0(e_m z k) = \begin{cases} \omega^{-1}(k_{mm} z) & \text{if } k \in K_0(N) \text{ and } z \in \mathcal{O}^\times, \\ 0 & \text{otherwise}. \end{cases}$

So

$$Z(ms, \omega, \Phi_{\omega,N}^0, k) = \int_{F^\times} \Xi_{\mathcal{O}^\times}(z) \omega(z) \omega^{-1}(k_{mm} z)|z|^{ms} d^\times z = \omega^{-1}(k_{mm}),$$

if $k \in K_0(N)$ and is equal to zero otherwise. Combining with the equation (2.1) gives

$$I(s, W_1^0, W_2^0, \Phi_{\omega,N}^0) = \int \int W_1^0 \begin{pmatrix} h & k \\ 1 & 1 \end{pmatrix} W_2^0 \begin{pmatrix} h & k \\ 1 & 1 \end{pmatrix} \omega^{-1}(k_{mm}) |\det(h)|^{s-1} dhdk$$

$$= V_N \int W_1^0 \begin{pmatrix} h \\ 1 \end{pmatrix} W_2^0 \begin{pmatrix} h \\ 1 \end{pmatrix} |\det(h)|^{s-1} dh,$$

where the integration is over $N_{m-1}\setminus G_{m-1}$ for $h$ and over $K_0(N)$ for $k$. Thus

$$I(s, W_1^0, W_2^0, \Phi_{\omega,N}^0) = V_N B(s, W_1^0, W_2^0).$$

□

Using this Lemma, we prove Theorem 2.1.1 as follows:
Proof. Let $u$ be the degree of $L(s, \pi_1)$, and $N$ as before. Note that since $\pi_2$ is unramified, $N = c(\pi_1)$. We may assume that $u \leq m - 1$, since if $u = m$, then both $\pi_1$ and $\pi_2$ are unramified and which is well known 1-regular such that

$$I(s, W_1^0, W_2^0, \Phi) = L(s, \pi_1 \times \pi_2),$$

where $\Phi$ is the characteristic function of $\mathcal{O}^m$. If we take $\Phi = V_{N^{-1}}^\circ \Phi_{0,N}$, then by the Lemma 2.1.2 we have $I(s, W_1^0, W_2^0, \Phi) = B(s, W_1^0, W_2^0)$. Now we study $B(s, W_1^0, W_2^0)$.

Notice that for $h \in N_{m-1} \setminus G_{m-1}$, $W_2^0 \begin{pmatrix} h \\ 1 \end{pmatrix} = 0$ unless last row of $h$ is in $\mathcal{O}^{m-1}$ by [18] Lemma 5.2.

1. If $u = m - 1$:

By Theorem 1.3.2 $\pi_1$ has a unique unramified constituent $\pi_{1,0}^{(1)}$ of $\pi_1^{(1)}$ and the new vector $W_1^0$ of $\pi_1$ is from the new vector $W_{\pi_{1,0}^{(1)}}^0$ of $\pi_{1,0}^{(1)}$ as shown (1.8). Then,

$$B(s, W_1^0, W_2^0) = \int_{N_{m-1} \setminus G_{m-1}} W_{\pi_{1,0}^{(1)}}^0(h) \Phi_0(\epsilon_{m-1} h) W_2^0 \begin{pmatrix} h \\ 1 \end{pmatrix} |\text{det}(h)|^{s-1/2} dh,$$

and by the property of $W_2^0 \begin{pmatrix} h \\ 1 \end{pmatrix}$ as mentioned above, we have

$$B(s, W_1^0, W_2^0) = \int_{N_{m-1} \setminus G_{m-1}} W_{\pi_{1,0}^{(1)}}^0(h) W_2^0 \begin{pmatrix} h \\ 1 \end{pmatrix} |\text{det}(h)|^{s-1/2} dh$$

$$= I(s, W_2^0, W_{\pi_{1,0}^{(1)}}^0) = L(s, \pi_2 \times \pi_{1,0}^{(1)}).$$

On the other hand, by the Corollary 1.3.3 $L(s, \pi_1 \times \pi_2) = L(s, \pi_{1,0}^{(1)} \times \pi_2)$. Thus, $I(s, W_1^0, W_2^0, \Phi) = L(s, \pi_1 \times \pi_2)$ as desired.
2. If \( u < m - 1 \):

Note that \( \pi_1^{(m-u)} \) has a unique unramified constituent \( \pi_1^{(m-u)} \) and \( W_1^0 \) its new vector, which gives the new vector \( W_1^0 \) of \( \pi_1 \) as shown in (1.10).

\[
B(s, W_1^0, W_2^0) = \int_{\N m-1 \setminus \G m-1} \varphi_{\pi_1}(h) W_2^0 \begin{pmatrix} h \\ 1 \end{pmatrix} |\det(h)|^{s-1 + \frac{m-u}{2}} dh. \quad (2.5)
\]

Let \( h = nak \in GL_{m-1} \), that is, \( h = n\varpi^j k \), \( J = (j_1, j_2, \ldots, j_{m-1}) \in \mathbb{Z}^{m-1} \), \( dh = dn \delta_b^{-1}(\varpi^j) dk \). Since \( \varphi_{\pi_1} \) and \( W_2^0 \) both are \( K_{m-1} \)-invariant, the equation (2.5) becomes

\[
\sum_{J \in \mathbb{Z}^{m-1}} W_2^0 \begin{pmatrix} \varpi^j \\ 1 \end{pmatrix} \varphi_{\pi_1}(\varpi^j) |\det(\varpi^j)|^{s-1 + \frac{m-u}{2}} \delta_b^{-1}(\varpi^j).
\]

From (1.9), \( \varphi_{\pi_1}(\varpi^j) = 0 \) unless \( j_{u+1} = \cdots = j_{m-1} = 0 \). If we let \( J = (J', 0) \) with \( J' = (j_1, \ldots, j_u) \), then

\[
\varphi_{\pi_1}(\varpi^{(J', 0)}) = \varphi_{\pi_1} \begin{pmatrix} \varpi^{J'} \\ I_{m-u-1} \end{pmatrix} = W_0^{(m-u)}(\varpi^{J'}).\]

Note that \( W_0^{(m-u)}(\varpi^{J'}) = 0 \) unless \( j_1 \geq j_2 \geq \cdots \geq j_u \geq 0 \) since \( W_0^{(m-u)} \) is \( K_u \)-fixed vector and we know that \( |J| = |J'|, \delta_{b_{m-1}}^{-1}(\varpi^J) = \delta_{b_u}^{-1}(\varpi^{J'}) |\det(\varpi^{J'})|^{m-u-1} \).

From the above observation, we can express (2.5) of \( B(s, W_1^0, W_2^0) \) as

\[
\sum_{j_1 \geq \cdots \geq j_u \geq 0} W_2^0 \begin{pmatrix} \varpi^{J'} \\ I_{m-u} \end{pmatrix} W_0^{(m-u)}(\varpi^{J'}) q^{-|J'|(s-\frac{m-u}{2})} \delta_{b_u}^{-1}(\varpi^{J'}).\]

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On the other hand, since \((\pi_{1,0}^{(m-u)}, \pi_2)\) is an unramified pair of \(GL(u) \times GL(m)\) with \(u < m\), the pair \((\pi_2, \pi_{1,0}^{(m-u)})\) is 1-regular and \((W_2^0, W_1^0_{\pi_{1,0}^{(m-u)}})\) is its test vector such that \(I(s, W_2^0, W_1^0_{\pi_{1,0}^{(m-u)}}) = L(s, \pi_2 \times \pi_{1,0}^{(m-u)})\). By the Corollary 1.3.3, \(I(s, W_2^0, W_1^0_{\pi_{1,0}^{(m-u)}}) = L(s, \pi_1 \times \pi_2)\). Now, the local integral \(I(s, W_2^0, W_1^0_{\pi_{1,0}^{(m-u)}})\) is

\[
\int_{N_u \backslash G_u} W_2^0 \begin{pmatrix} g & \rho \end{pmatrix} W_1^0_{\pi_{1,0}^{(m-u)}}(g) |\det(g)| s^{-m-u} dg.
\]

And similarly, we have

\[
I(s, W_2^0, W_1^0_{\pi_{1,0}^{(m-u)}}) = \sum_{j_1 \geq \cdots \geq j_u \geq 0} W_2^0 \begin{pmatrix} \rho J & \rho \end{pmatrix} W_1^0_{\pi_{1,0}^{(m-u)}}(\rho J) q^{-|J|(s-m-u)/2} \delta_{B_u}^{-1}(\rho J),
\]

which is equal to the result above. Thus, \(I(s, W_2^0, W_1^0_{\pi_{1,0}^{(m-u)}}) = L(s, \pi_1 \times \pi_2)\).

Therefore, the pair \((\pi_1, \pi_2)\) is 1-regular and \((W_1^0, W_2^0, \Phi)\) is its test vector with \(\Phi = V_{N^{-1}}^{\psi, N}. \)

2.2 A pair \((\pi_1, \pi_2)\) for \(GL(n) \times GL(m)\) with \(n \geq m\)

Theorem 2.2.1. For fixed \(m\) and any \(n > m\) and \(\pi_1 \in \text{Irr}(GL(n))\), a pair \((\pi_1, \pi_2)\) is optimal 1-regular for any unramified \(\pi_2 \in \text{Irr}(GL(m))\) with test vector \((W_1^0, W_2^0)\).

The case \(n = m + 1\) is proven in [18], so we may assume that \(n \geq m + 2\).

Recall the definition of the local integral \(I = I(s, W_1^0, W_2^0)\) for the pair \((\pi_1, \pi_2)\) in \(GL(n) \times GL(m)\) for \(n > m\). For \(W_1 \in \mathcal{W}(\pi_1, \psi), W_2 \in \mathcal{W}(\pi_2, \psi^{-1})\), from the equation (1.3),
\[
I = I(s, W_1, W_2) = \int_{N_m \backslash G_m} W_1 \left( \begin{array}{c} g \\ I_{n-m} \end{array} \right) W_2(g) |\det g|^{s - \frac{n-m}{2}} dg.
\]

**Proof.** Let \( u \) be the degree of \( L(s, \pi_1) \) as before. We know that there is a unique unramified constituent \( \pi_{1,0}^{(n-u)} \) of \( \pi_1^{(n-u)} \) by Theorem 1.3.2.

The main point is the fact that an unramified pair \((\pi, \sigma)\) is optimal 1-regular for any \( n \geq m \) with a test vector \((W_0^\pi, W_0^\sigma)\) (or \((W_0^\pi, W_0^\sigma, \Phi_0)\) if \( n = m \)) and the new vector \( W_1^0 \) of \( \pi_1 \) is from the new vector \( W_0^{\pi_{1,0}^{(n-u)}} \) of an unramified constituent \( \pi_{1,0}^{(n-u)} \in \text{Irr}(GL(u)) \).

1. If \( u = n \), then \( \pi_1 \) is unramified, that is, \((\pi_1, \pi_2)\) is an unramified pair, so we are done.

2. If \( u = n - 1 \), then \( W_1^0 \left( \begin{array}{c} g \\ 1 \end{array} \right) = W_0^{\pi_1^{(1)}}(g) \Phi(e_{n-1}g) |\det(g)|^{1/2} \) with \( \Phi \) the characteristic function of \( O^{n-1} \) as given in (1.8).

Since both \( \pi_1^{(1)} \) and \( \pi_2 \) are unramified, the pair \((\pi_1^{(1)}, \pi_2)\) is 1-regular with test vector \((W_0^{\pi_1^{(1)}}, W_2^0)\), and by the Corollary 1.3.3 we have \( L(s, \pi_1^{(1)} \times \pi_2) = L(s, \pi_1 \times \pi_2) \). So we are done.

3. If \( u < n - 1 \), then let \( W_1^0 \) as in (1.10), and \( \varphi_1 := \varphi_{\pi_1} \) as in (1.9)

\[
I = \int_{N_m \backslash G_m} \varphi_1 \left( \begin{array}{c} g \\ I_{n-m-1} \end{array} \right) W_2^0(g) |\det(g)|^{s - \frac{u-m}{2}} dg.
\]

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Let $g = n\omega^k \in GL_m$, that is, $g = n\omega^Jk$, $J = (j_1, j_2, \ldots, j_m) \in \mathbb{Z}^m$, $dg = dn\delta_{B_m}^{-1}(\omega^J)dk$. Since $\varphi_1$ and $W_2^0$ both are $K_m$-invariant,

$$I = \sum_{J \in \mathbb{Z}^m} \varphi \begin{pmatrix} \omega^J & \cdot \\ I_{n-m-1} & \cdot \end{pmatrix} W_2^0(\omega^J)|\det(\omega^J)|^{s-\frac{u-m}{2}} \delta_{B_m}^{-1}(\omega^J).$$ (2.6)

From now on, we study the equation (2.6) depending on the relation between $u$ and $m$:

- If $u > m \geq 1$, then we have
  
  (a) $W_2^0(\omega^J) = 0$ unless $j_1 \geq j_2 \geq \cdots \geq j_m$,

  (b) Let $J = (j_1, \cdots, j_m)$, then from the definition of $\varphi_1$, we have
  
  $$\varphi_1 \begin{pmatrix} \omega^J & \cdot \\ I_{n-m-1} & \cdot \end{pmatrix} = W_2^0(\pi_{1,0}^{(n-u)}) \begin{pmatrix} \omega^J & \cdot \\ I_{u-m} & \cdot \end{pmatrix},$$

  which is zero unless $j_1 \geq j_2 \geq \cdots \geq j_m \geq 0$ since $W_{\pi_{1,0}^{(n-u)}}$ is $K_u$ fixed vector.

  Thus, the equation (2.6) becomes

  $$\sum_{j_1 \geq \cdots \geq j_m \geq 0} W_{\pi_{1,0}^{(n-u)}}^0 \begin{pmatrix} \omega^J & \cdot \\ I_{u-m} & \cdot \end{pmatrix} W_2^0(\omega^J)q^{-|J|(s-\frac{u-m}{2})} \delta_{B_m}^{-1}(\omega^J),$$

  where $|J| = \sum_{i=1}^m j_i$. On the other hand, since $\pi_{1,0}^{(n-u)}$ and $\pi_2$ are unramified representations of $GL(u)$ and $GL(m)$ respectively, $m < u < n$, $(\pi_{1,0}^{(n-u)}, \pi_2)$ is a 1-regular pair and $I(s, W_{\pi_{1,0}^{(n-u)}}^0, W_2^0) = L(s, \pi_{1,0}^{(n-u)} \times \pi_2)$. And

  $$I(s, W_{\pi_{1,0}^{(n-u)}}^0, W_2^0) = \int_{N_m \setminus G_m} W_{\pi_{1,0}^{(n-u)}}^0 \begin{pmatrix} g & \cdot \\ 1 & \cdot \end{pmatrix} W_2^0(g)|\det(g)|^{s-\frac{u-m}{2}} dg.$$
Similarly, we have

\[
I(s, W^0_{\pi_{1,0}^{(n-u)}}, W^0_{2}) = \sum_{j_1 \geq \cdots \geq j_m \geq 0} W^0_{\pi_{1,0}^{(n-u)}} \begin{pmatrix} \varpi^J \\ I_{u-m} \end{pmatrix} W^0_2 (\varpi^J) q^{-|J|(s-n+u)} \delta_{B_m}^{-1}(\varpi^J). \]

Thus, \( I(s, W^0_1, W^0_2) = L(s, \pi_1 \times \pi_2) \) as desired.

- If \( 1 \leq u < m < n - 1 \). Then by the definition of \( \varphi_1 \) as in (1.9),

\[
\varphi_1 \begin{pmatrix} \varpi^J \\ I_{n-m-1} \end{pmatrix} = 0 \text{ unless } j_{u+1} = \cdots = j_m = 0. \]

So from (2.6), let \( J = (J', 0) \) with \( J' = (j_1, \cdots, j_u) \), then

(a) \[
\varphi_1 \begin{pmatrix} \varpi^J \\ I_{n-m-1} \end{pmatrix} = \varphi_1 \begin{pmatrix} \varpi^{J'} \\ I_{n-u-1} \end{pmatrix} = W^0_{\pi_{1,0}^{(n-u)}} (\varpi^{J'}),
\]

(b) \[
W^0_2 (\varpi^J) = W^0_2 \begin{pmatrix} \varpi^{J'} \\ I_{m-u} \end{pmatrix},
\]

and \( W^0_2 \begin{pmatrix} \varpi^{J'} \\ I_{m-u} \end{pmatrix} = 0 \) unless \( j_1 \geq \cdots \geq j_u \geq 0 \), and

(c) \( |J| = |J'| \) and \( \delta_{B_m}^{-1}(\varpi^J) = \delta_{B_u}^{-1}(\varpi^{J'}) \det(\varpi^{J'})^{-(m-u)}. \)

So, applying all these in the equation (2.6) gives

\[
\sum_{j_1 \geq \cdots \geq j_u \geq 0} W^0_2 \begin{pmatrix} \varpi^{J'} \\ I_{m-u} \end{pmatrix} W^0_{\pi_{1,0}^{(n-u)}} (\varpi^{J'}) q^{-|J'|(s-n+u)} \delta_{B_u}^{-1}(\varpi^{J'}). \]
On the other hand, since \((\pi_{1,0}^{(n-u)}, \pi_2)\) is an unramified pair,

\[
L(s, \pi_{1,0}^{(n-u)} \times \pi_2) = L(s, \pi_2 \times \pi_1^{(n-u)}) = I(s, W_2^0, W_2^{(n-u)}) = \int_{N_u \setminus G_u} W_2^0 \left( \begin{array}{cc} g & W_2^0(\pi_{1,0}^{(n-u)}(g)) \det(g) |^{s - \frac{m-u}{2}} \\ I_{m-u} & \end{array} \right) dg.
\]

Similarly, we have

\[
L(s, \pi_{1,0}^{(n-u)} \times \pi_2) = \sum_{j_1 \geq \cdots \geq j_m \geq 0} W_2^0(\varpi^{(n-u)}_{1,0}(\varpi^{j'} J') q^{-|J'| s - \frac{m-u}{2}}) \delta_{B_m}^{-1}(\varpi^{j'} J').
\]

Thus \(I(s, W_1^0, W_2^0) = L(s, \pi_{1,0}^{(n-u)} \times \pi_2) = L(s, \pi_1 \times \pi_2)\) by the Corollary 1.3.3.

- If \(1 \leq m = u\), then as above, we can rewrite the equation (2.6) as:

\[
\sum_{j_1 \geq \cdots \geq j_m \geq 0} W_2^0(\varpi^{(n-u)}_{1,0}(\varpi^{j'} J') q^{-|J'| s - \frac{m-u}{2}}) \delta_{B_m}^{-1}(\varpi^{j'} J').
\]

Now, \(\pi_1^{(n-u)}\) and \(\pi_2\) are both unramified representations of \(GL(m)\). Let \(\Phi \in \mathcal{S}(F^m)\) be the characteristic function of \(\mathcal{O}^m\), then

\[
L(s, \pi_{1,0}^{(n-u)} \times \pi_2) = I(s, W_1^0, W_2^0, \Phi) = \int_{N_m \setminus G_m} W_2^0(\pi_{1,0}^{(n-u)}(g)) W_2^0(\pi_{1,0}^{(n-u)}(g)) \det(g) |^{s} dg = \sum_{j_1 \geq \cdots \geq j_m \geq 0} W_2^0(\varpi^{j'} J') q^{-|J'| s - \frac{m-u}{2}} \delta_{B_m}^{-1}(\varpi^{j'} J').
\]

Thus, \(I(s, W_1^0, W_2^0) = L(s, \pi_{1,0}^{(n-u)} \times \pi_2) = L(s, \pi_1 \times \pi_2)\) as desired.
• If \( u = 0 \), then \( \varphi_1 \) vanishes unless \( g = nk \), that is,

\[
\varphi_1 \left( \begin{array}{c} \ g \\ \ I_{n-m-1} \end{array} \right) = \begin{cases} \psi(n) & \text{if } g = nk, \\ 0 & \text{otherwise.} \end{cases}
\]

So, the equation (2.6) becomes

\[
\varphi_1(I_{n-m-1})W_0^1(J_m)|\det(J_m)|^{s+m/2}\delta_{B_m}^{-1}(I_m) = 1 = L(s, \pi_1 \times \pi_2).
\]

Thus, \( I(s, W_1^0, W_2^0) = L(s, \pi_1^{(n-u)} \times \pi_2) = L(s, \pi_1 \times \pi_2) \) by the Corollary 1.3.3.

Therefore, for any \( \pi_1 \in Irr(GL(n)) \), a pair \( (\pi_1, \pi_2) \) is 1-regular if \( \pi_2 \) is an unramified representation of \( GL(m) \) with \( n \geq m \) as desired. \( \square \)

Some natural question arises here. Is it always true that \( I(s, W_1^0, W_2^0) = L(s, \pi_1 \times \pi_2) \)? Unfortunately, if we remove the unramified condition for \( \pi_2 \) in the Theorem 2.2.1, we can not conclude that \( (W_1^0, W_2^0) \) is a test vector for the pair \( (\pi_1, \pi_2) \). If we let \( \pi_2 \in Irr(GL(m)) \) be ramified, and \( W_2^0 \) its new vector, then we have the following result:

**Proposition 2.2.2.** Let \( \pi_2 \) be a ramified representation of \( GL(m) \) with \( W_2^0 \in W(\pi_2, \psi^{-1}) \) its new vector and \( \pi_1 \) be any irreducible admissible generic representation of \( GL(n) \) with its new vector \( W_1^0 \). Then \( I(s, W_1^0, W_2^0) = 0. \)
Proof. Note that the maximal compact subgroup $K_m = GL_m(\mathcal{O})$ ($n > m$) is embedded in any $K_1(N) \subset K_n = GL_n(\mathcal{O})$. So $W_1^0$ is fixed by $K_m$. Now we compute

\[ I(s,W_1^0,W_2^0) = \int_{N_m \backslash G_m} W_1^0 \left( \begin{array}{cc} g & \; \\ I_{n-m} \end{array} \right) W_2^0(g) |\text{det}(g)|^{s-(n-m)/2} dg \]

\[ = \int_{A_m} \int_{K_m} W_1^0 \left( \begin{array}{c} a k \\ I_{n-m} \end{array} \right) W_2^0(ak) |\text{det}(a)|^{s-(n-m)/2} dk da \]

\[ = \int_{A_m} W_1^0 \left( \begin{array}{c} a \\ I_{n-m} \end{array} \right) \left[ \int_{K_m} W_2^0(ak) dk \right] |\text{det}(a)|^{s-(n-2)/2} da = 0. \]

The last equality is true since $W_2^0 \mapsto \int_{K_m} W_2^0(ak) dk$ is $K_m$-invariant of $\pi_2$. But $\pi_2^{K_m} = \{0\}$, since $\pi_2$ is ramified, that is, $\pi_2$ does not have a nonzero $K_m$-fixed vector. □
CHAPTER 3

L-FUNCTIONS OF A PAIR \((\pi_1, \pi_2)\) FOR \(GL(N) \times GL(1)\)

Here we consider the case when \(\pi_2\) is ramified and \(m = 1\), that is, \(\pi_2\) is a ramified character, and \(\pi_1\) is any representation of \(GL(n)\), \(n \geq 1\). Note that the new vector of a character is itself. Before we prove the main results, Theorem 3.2.1 and Theorem 3.3.1 we will review the \(L\)-function of a character, which is a representation of \(GL(1)\). Main reference for this chapter is [26].

3.1 Preliminaries

Let \(\chi\) be the character of \(F^\times\) then

\[
L(s, \chi) = \begin{cases} 
(1 - \chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified}, \\
1 & \text{otherwise.} 
\end{cases} \tag{3.1}
\]

For \(\phi \in S(F)\), consider the following local integral \(I = I(s, \chi, \phi)\),

\[
I = I(s, \chi, \phi) = \int_{F^\times} \phi(a)\chi(a)|a|^s d^\times a,
\]

which is convergent in some half planes, but has meromorphic continuation to all of \(\mathbb{C}\). And these integrals represent rational functions of \(q^{-s}\). Since \(F^\times = \bigsqcup_{m \in \mathbb{Z}} \varpi^m \mathcal{O}^\times\), above local integral \(I\) can be rewritten as

\[
\int_{F^\times} \phi(a)\chi(a)|a|^s d^\times a = \sum_m q^{-ms} \left( \int_{\mathcal{O}^\times} \phi(\varpi^m u)\chi(\varpi^m u)d^\times u \right).
\]
If \( \chi \) is unramified, that is, \( \chi \) is trivial on the the group \( \mathcal{O}^\times \), and \( \phi = Char(\mathcal{O}) \), then we have

\[
I = \sum_{m \geq 0} q^{-ms} \chi(\varpi)^m \int_{\mathcal{O}^\times} \chi(u) d^\times u = \frac{1}{1 - \chi(\varpi)q^{-s}},
\]

where the last equality follows from summing the geometric series and from \( Vol(\mathcal{O}^\times) = 1 \). So this \( \phi \) is the desired Schwartz function to compute \( L(s, \chi) \) when \( \chi \) is unramified, that is, we have

\[
I(s, \chi, \phi) = \frac{1}{1 - \chi(\varpi)q^{-s}} = L(s, \chi),
\]
as in (3.1).

On the other hand, if \( \chi \) is ramified, then there exists an integer \( c = c(\chi) \geq 1 \) such that \( \chi \) is trivial on the group \( U_c = 1 + \varpi^c \mathcal{O} \) and nontrivial on \( 1 + \varpi^a \mathcal{O} \), when \( 0 < a < c \). We call \( c \) the conductor of \( \chi \).

So if we take

\[
\phi(a) = \begin{cases} 
Vol(U_c)^{-1} & \text{if } a \in U_c, \\
0 & \text{otherwise,}
\end{cases}
\]

then \( \phi \in \mathcal{S}(F) \), and we can compute

\[
I = Vol(U_c)^{-1} \int_{U_c} \chi(u) d^\times u = 1 = L(s, \chi),
\]
as in (3.1).

**Remark 3.1.1.** Note that the test vector \( \phi \) for ramified \( \chi \) is not unique: if we choose \( \phi \) as follows:

\[
\phi(x) = \begin{cases} 
\chi(x)^{-1} & \text{if } x \in \mathcal{O}^\times, \\
0 & \text{otherwise,}
\end{cases}
\]

then we also have \( I(s, \chi, \phi) = 1 = L(s, \chi) \) as in (3.1).
3.2  $L$-function of pair $(\pi_1, \pi_2)$ in $GL(1) \times GL(1)$

**Theorem 3.2.1.** Any pair $(\pi_1, \pi_2)$ for $GL(1) \times GL(1)$ is optimal 1-regular. In particular, there is an appropriate $\phi$ such that $(W_1^0, W_2^0, \phi)$ is a test vector for this pair.

**Proof.** Let $\pi_1 = \chi$ and $\pi_2 = \mu$, then $L(s, \pi_1 \times \pi_2) = L(s, \chi \times \mu) = L(s, \chi \otimes \mu)$ by [11]. Consider the local integral as in (1.2), note that the new vector $W_i^0$ of $\pi_i$ is itself,

$$I(s, \chi, \mu, \phi) = \int_{F^\times} \chi(a)\mu(a)|a|^sd^xa. \quad (3.4)$$

Note that Our goal is to find an appropriate Schwartz function $\phi$ so that

$$I(s, \chi, \mu, \phi) = L(s, \pi_1 \times \pi_2).$$

1. If both $\chi, \mu$ are unramified, then choose $\phi = \text{Char}(\mathcal{O})$, then

$$I(s, \chi, \mu, \phi) = I(s, \chi\mu, \phi) = L(s, \chi \otimes \mu) = L(s, \pi_1 \times \pi_2),$$

as desired.

2. If one of $\chi, \mu$ is ramified and $\chi\mu$ is also ramified. Let $N = c(\chi\mu)$ conductor of $\chi\mu$, which is equal to $\max\{c(\chi), c(\mu)\}$. Then we take $\phi$ as one of the following forms (3.2) and (3.3) :

$$\phi(x) = \begin{cases} 
Vol(U_N)^{-1} & \text{if } x \in U_N, \\
0 & \text{otherwise},
\end{cases}$$

or

$$\phi(x) = \begin{cases} 
(\chi\mu)^{-1}(x) & \text{if } x \in \mathcal{O}^\times, \\
0 & \text{otherwise},
\end{cases}$$

which gives that $I(s, \chi, \mu, \phi) = 1 = L(s, \pi_1 \times \pi_2)$ as desired.
3. If $\chi, \mu$ are ramified and $\chi\mu$ is unramified, that is, $\chi\mu = \nu^{s_0}$ for some $s_0$, then

$$L(s, \chi \times \mu) = \frac{1}{1 - q^{-s_0} - s}.$$ If we choose $\phi = \text{Char}(O)$, then

$$I(s, \chi, \mu, \phi) = \int_{F^\times} \phi(a) |a|^{s+s_0} d^\times a = \sum_{n \geq 0} q^{-n(s+s_0)} = \frac{1}{1 - q^{-s-s_0} - s} = L(s, \pi_1 \times \pi_2).$$

Therefore, any pair $(\pi_1, \pi_2)$ is optimal 1-regular for $\pi_i \in \text{Irr}(GL(1))$ as desired. $\square$

3.3 **$L$-function of a pair $(\pi_1, \pi_2)$ for $GL(n) \times GL(1)$ ($n \geq 2$)**

The notion of the new vector can be generalized: Given any character $\mu$ on $F^\times$, the representation $\mu \otimes \pi$ is still irreducible admissible generic and so has a new vector $W^0_{\mu \otimes \pi}$. Denote by $W^0_{\pi,\mu}$ the $\mu$-twisted new vector of $\pi$ defined as

$$W^0_{\pi,\mu} : g \mapsto \mu^{-1}(\det g)W^0_{\mu \otimes \pi}(g), \quad (3.5)$$

then $W^0_{\pi,\mu}$ is an element in the Whittaker model of $\pi$. Notice that if $\mu$ is unramified, then $W^0_{\pi,\mu} = W^0_{\pi}$. And for any pair $(\pi_1, \pi_2)$ of representations and any given character $\mu$, if we let $\pi'_1 \cong \mu \otimes \pi_1$ and $\pi'_2 \cong \mu^{-1} \otimes \pi_2$, then the pair $(\pi'_1, \pi'_2)$ has the same $L$-function with the original pair $(\pi_1, \pi_2)$. Twisted new vectors will be also used in Chapter 4 and Chapter 6.

**Theorem 3.3.1.** For $n \geq 2$, any pair $(\pi_1, \pi_2)$ for $GL(n) \times GL(1)$ is optimal 1-regular. More precisely,

$$L(s, \pi_1 \times \pi_2) = \begin{cases} I(s, W^0_{1,\pi}, W^0_{2}) & \text{if } \pi_2 = \mu \text{ unramified,} \\ I(s, W^0_{1,\pi}, W^0_{2}) & \text{if } \pi_2 = \mu \text{ ramified.} \end{cases}$$
Proof. If $\mu$ is unramified, then we are done by the Theorem 2.2.1. Let $\pi_2 = \mu$ be a ramified representation of $GL(1)$. By Proposition 2.2.2, we know that the local integral $I(s, W_0^0, \mu) = 0$ where $W_1^0$ is the new vector for $\pi_1 \in \text{Irr}(GL(n))$. Thus, $(W_1^0, W_2^0)$ cannot be a test vector for the pair $(\pi_1, \pi_2)$. But, if we consider the $\mu$-twisted new vector $W_{1,\mu}^0$ of $\pi_1$ as defined in (3.5), then $(W_{1,\mu}^0, W_2^0)$ is a test vector for this pair as follows:

$$I(s, W_{1,\mu}^0, W_2^0) = \int_{F^\times} W_{1,\mu}^0 \begin{pmatrix} a \\ I_{n-1} \end{pmatrix} \mu(a)|a|^{s-n-1/2} d^\times a$$

$$= \int_{F^\times} W_0^0 \otimes \mu \begin{pmatrix} a \\ I_{n-1} \end{pmatrix} \mu^{-1}(a)\mu(a)|a|^{s-n-1/2} d^\times a$$

$$= \int_{F^\times} W_0^0 \otimes \mu \begin{pmatrix} a \\ I_{n-1} \end{pmatrix} |a|^{s-n-1/2} d^\times a$$

$$= I(s, W_0^0 \otimes \mu, 1) = L(s, \pi_1 \otimes \mu) = L(s, \pi_1 \times \mu).$$

Thus, the pair $(\pi_1, \pi_2)$ is optimal 1-regular for any $\pi_1 \in \text{Irr}(GL(n))$ and $\pi_2 \in \text{Irr}(GL(1))$. □
CHAPTER 4

L-FUNCTIONS OF A PAIR \((\pi_1, \pi_2)\) FOR \(GL(2) \times GL(2)\),

PART (I) - NON-LINKED PAIRS

4.1 Preliminaries

We call \((\pi_1, \pi_2)\) a linked pair for \(GL(n) \times GL(n)\) for \(n \geq 1\) if \(\tilde{\pi}_2 \cong \pi_1 \nu^{s_0}\) for some \(s_0 \in \mathbb{C}\). For \(n = 1\), we count only ramified character pairs. Even if \((\pi_1, \pi_2)\) is not a linked pair for \(GL(n) \times GL(n)\), we may have a linked pair \((\pi_1^{(n-k)}, \pi_2^{(n-k)})\) for \(GL(k) \times GL(k)\). If this happens, we call \((\pi_1, \pi_2)\) partially non-linked. And we call \((\pi_1, \pi_2)\) a totally non-linked pair if every pair \((\pi_1^{(n-k)}, \pi_2^{(n-k)})\) for \(GL(k) \times GL(k)\) of constituents of \(\pi_1^{(n-k)}\) and \(\pi_2^{(n-k)}\) is non-linked. Our goal is to show that A pair \((\pi_1, \pi_2)\) is \(r\)-regular (\(r \leq 2\)), for any \(\pi_1, \pi_2 \in \text{Irr}(GL(2))\) except one case(which we will describe later). To show this, we will separate cases as follows:

1. a pair \((\pi_1, \pi_2)\) is totally non-linked.

2. a pair \((\pi_1, \pi_2)\) is partially non-linked.

3. a pair \((\pi_1, \pi_2)\) is linked.

We will cover the first two cases in this chapter. In the next chapter, we will discuss third case.
By Theorem 2.1.1, we already know that \((\pi_1, \pi_2)\) is 1-regular, if either one of \(\pi_i\) is full-induced unramified representation. So from this point of on, we assume that \(\pi_i\) is a ramified representation of \(GL(2)\).

### 4.2 Totally Non-Linked Pairs

**Theorem 4.2.1.** Let \((\pi_1, \pi_2)\) be a totally non-linked pair for \(GL(2) \times GL(2)\). Then the pair \((\pi_1, \pi_2)\) is optimal 1-regular, and \((W_0^0, W_0^0, \Phi)\) is its test vector, where \(\Phi = V_N^{-1}\Phi_{\omega,N}^0\) with \(N = \max\{c(\pi_1), c(\pi_2)\} \geq 1\).

**Proof.** Let \(\Phi = V_N^{-1}\Phi_{\omega,N}^0\), where \(\Phi_{\omega,N}^0\) is defined in (2.3).

1. If \(\pi_2\) such that \(L(s, \pi_2)\) has degree 1, that is, \(\pi_2 = Ind(\mu_1 \otimes \mu_2)\) with one of \(\mu_i\) is unramified, say \(\pi_1\) or \(\pi_2 = St_2(\mu) = [\mu\nu^{-1/2}, \mu\nu^{1/2}]\) with unramified \(\mu\). Then by [17], Chapter IV: Local theory for \(GL(2) \times GL(2)\), \(L(s, \pi_1 \times \pi_2) = L(s, \pi_1 \times \mu_1)\) for the first case of \(\pi_2\) or \(L(s, \pi_1 \times \pi_2) = L(s, \pi_1 \times \mu\nu^{1/2})\) for the second case of \(\pi_2\). Either case, \(L(s, \pi_1 \times \pi_2) = L(s, \pi_1 \times \pi_2^{(1)})\), where \(\pi_2^{(1)}\) is the unramified constituent of \(\pi_2^{(1)}\) and from (1.8), the new vector of \(\pi_2\) is as follows:

\[
W_2^0 \begin{pmatrix}
  a \\
  1
\end{pmatrix} = \Xi_{\mathcal{O}}(a)\pi_{2,0}^{(1)}(a)|a|^{1/2}.
\]

Now, for any \(\pi_1 \in Irr(GL(2))\) by Lemma 2.1.2, \(I(s, W_1^0, W_2^0, \Phi) = B(s, W_1^0, W_2^0)\). Using the fact \((\pi_1, \pi_2^{(1)})\) is 1-regular such that \((W_1^0, \pi_2^{(1)})\) is a test vector of this pair by Theorem 2.2.1 and \(W_1^0 \begin{pmatrix}
  a \\
  1
\end{pmatrix}\) is nonzero only if \(a \in \mathcal{O}\) provides
\[ B(s, W_1^0, W_2^0) = \int_{F^\times} W_1^0 \begin{pmatrix} a \\ 1 \end{pmatrix} \pi_{2,0}^{(1)}(a) \Xi_{\mathcal{O}}(a) |a|^{s-1/2} d^\times a \]
\[ = I(s, W_1^0, \pi_{2,0}^{(1)}) = L(s, \pi_1 \times \pi_{2,0}^{(1)}) \]
\[ = L(s, \pi_1 \times \pi_2). \]

2. If \( \pi_2 \) such that \( L(s, \pi_2)^{-1} \) has degree 0, then \( W_2^0 \begin{pmatrix} a \\ 1 \end{pmatrix} \) has a small compact support in \( F^\times \) by (1.10), and \( L(s, \pi_1 \times \pi_2) = 1 \). With similar argument,
\[ I(s, W_1^0, W_2^0, \Phi) = B(s, W_1^0, W_2^0) \]
\[ = \int_{\mathcal{O}^\times} W_1^0 \begin{pmatrix} a \\ 1 \end{pmatrix} \Xi_{\mathcal{O}^\times} (a) |a|^s d^\times a \]
\[ = 1 = L(s, \pi_1 \times \pi_2). \]

Thus, we are done. \( \square \)

4.3 Partially Non-Linked Pairs

Assume that a pair \((\pi_1, \pi_2)\) is partially non-linked for \( GL(2) \times GL(2) \), that is, \( \tilde{\pi}_2 \not\cong \pi_1 \nu^t \) for any \( t \in \mathbb{C} \), but \( \tilde{\pi}_2^{(1)} = \pi_1^{(1)} \nu^{s_i} \) for some \( s_i \). By [\cite{8}, Proposition 2.1], we know that for a partially non-linked pair \((\pi_1, \pi_2)\), there is no exceptional pole from the \( GL(2) \)-pairing but a ramified pair \((\pi_1^{(1)}, \pi_2^{(1)})\) has an exceptional pole from the \( GL(1) \)-paring. Note that if a pole at \( s = s_0 \) of the family \( I(\pi_1, \pi_2) \) of local integrals \( I(s, W_1, W_2, \Phi) \) is exceptional if \( I(s_0, W_1, W_2, \Phi) \) vanishes identically on \( \mathcal{S}_0(F^n) \).
And from [8, Proposition 2.2], the poles of the family \( I(\pi_1, \pi_2) \) which are not exceptional are precisely the poles of the family \( I_0(\pi_1, \pi_2) \) spanned by \( B(s, W_1, W_2) \). But if we use \( W_1^0 \) and \( W_2^0 \) with \( \Phi^0 \) as in (2.3), then \( I(s, W_1^0, W_2^0, \Phi^0) \) does not give a pole of \( L \) function for a ramified pair \( (\pi_{1,j}^{(1)}, \pi_{2,i}^{(1)}) \). So instead of the new vectors of the pair \( (\pi_1, \pi_2) \) along with \( \Phi \in S_0(F^2) \), we consider the twisted new vectors also.

**Theorem 4.3.1.** Let \( (\pi_1, \pi_2) \) be a partially non-linked pair for \( GL(2) \times GL(2) \). Then the pair \( (\pi_1, \pi_2) \) is \( r \)-regular with \( r \leq 2 \).

**Proof.** Let \( \pi_2 = Ind(\mu_1 \otimes \mu_2) \) and \( \mu_2 \) ramified. Note that \( \pi_2^{(1)} = \mu_1 \oplus \mu_2 \). We separate cases as follows:

1. \( \pi_1 = [\chi_2 \nu^{-1/2}, \chi_2 \nu^{1/2}] \) with \( \pi_1^{(1)} = \chi_2 \nu^{1/2} \) or \( \pi_1 = Ind(\chi_1 \otimes \chi_2) \) with \( \pi_1^{(1)} = \chi_1 \oplus \chi_2 \) such that only \( (\chi_2, \mu_2) \) is a ramified linked pair. Denote by this case \( R_0 \).

2. \( \pi_1 = Ind(\chi_1 \otimes \chi_2) \) with \( (\chi_2, \mu_2) \) linked, and \( (\chi_1, \mu_1) \) is an unramified pair. Denote by this case \( R_1 \).

3. \( \pi_1 = Ind(\chi_1 \otimes \chi_2) \) with \( (\chi_2, \mu_2) \) linked and the pair \( (\chi_1, \mu_1) \) is linked. Denote by this case \( R_2 \).

Note that for the case \( R_1 \) and \( R_2 \), we have \( \chi_1 \mu_1 = \nu^{s_1} \) and \( \chi_2 \nu_2 = \nu^{s_2} \) for some \( s_1 \neq s_2 \) (If \( s_1 = s_2 \), then the pair \( (\pi_1, \pi_2) \) is linked which we will talk in next Chapter). And we will show that the pair \( (\pi_1, \pi_2) \) is 1-regular if a pair is of the case \( R_0 \) and 2-regular otherwise.
For $\mathcal{R}_0$ case, consider $\pi'_1 = \mu_2 \otimes \pi_1$ and $\pi'_2 = \mu_2^{-1} \otimes \pi_2$, then $(\pi'_1, \pi'_2)$ is totally non-linked. So if we let $N' = \max\{c(\pi'_1), c(\pi'_2)\}$, and use the same argument in Theorem 4.2.1 then we have

$$I(s, W_{\pi'_1}^0, W_{\pi'_2}^0, \Phi') = L(s, \pi'_1 \times \pi'_2),$$

where $\Phi' = V^{-1}_{N'} \Phi_0^{N'}$.

If we choose $\mu_2$-twisted new vector $W_{1, \mu_2}^0$ for $\pi_1$ and $\mu_2^{-1}$-twisted new vector $W_{2, \mu_2^{-1}}^0$ of $\pi_2$ as follows:

$$W_{1, \mu_2}^0(g) = W_{\mu_2 \otimes \pi_1}^0(g) \mu_2^{-1}(\det(g)) = W_{\pi'_1}^0(g) \mu_2^{-1}(\det(g)),$$

and

$$W_{2, \mu_2^{-1}}^0(g) = W_{\mu_2^{-1} \otimes \pi_2}^0(g) \mu_2(\det(g)) = W_{\pi'_2}^0(g) \mu_2(\det(g)).$$

Then,

$$I(s, W_{\mu_2 \otimes \pi_1}^0, W_{\mu_2^{-1} \otimes \pi_2}^0, \Phi') = L(s, \pi'_1 \times \pi'_2).$$

Moreover, $L(s, \pi'_1 \times \pi'_2) = L(s, \pi_1 \times \pi_2)$ as we discussed in Section 3.3. This proves that pairs of the case $\mathcal{R}_0$ are 1-regular.

For $\mathcal{R}_1$ case, Jacquet showed that $L(s, \pi_1 \times \pi_2) = L(s, \chi_1 \times \mu_1) L(s, \chi_2 \times \mu_2)$ in [17] Chapter IV: local theory for $GL(2) \times GL(2)]$. First, we can consider the new vectors $W_i^0$ of $\pi_i$ respectively, and $\Phi = V^{-1}_N \Phi_0^{\omega, N}$ with $N = \max\{c(\pi_1), c(\pi_2)\}$ and $\omega = \omega_1 \omega_2$ as before. Then by Lemma 2.1.2,

$$I(s, W_1^0, W_2^0, \Phi) = B(s, W_1^0, W_2^0).$$
By definition of the new vectors $W^0_i$ as in (1.8),

$$W^0_i \begin{pmatrix} a \\ 1 \end{pmatrix} = \Xi_\mathcal{O}(a)\pi_{1,0}^{(1)}(a)|a|^{1/2},$$

where $\pi_{1,0}^{(1)} = \chi_1$ and $\pi_{2,0}^{(1)} = \mu_1$, which are unramified constituents of $\pi_i^{(1)}$.

$$B(s, W^0_1, W^0_2) = \int_{F^\times} \Xi_\mathcal{O}(a)\chi_1(a)\mu_1(a)|a|^s d^\times a$$

$$= I(s, \chi_1, \mu_1, \phi) = L(s, \chi_1 \times \mu_1).$$

This local integral does not compute the exact $L$-function of the pair $(\pi_1, \pi_2)$.

On the other hand, if we use the twisted argument like the case $\mathcal{R}_0$, then an argument similar to the one use in above shows that

$$I(s, W^0_1, W^0_2, \Phi') = B(s, W^0_{\pi'_1}, W^0_{\pi'_2}) = L(s, \chi_2 \times \mu_2),$$

where $\Phi' = V_{-1}^{N'} \Phi_0^{N'}$ with $N' = \max\{c(\pi'_1), c(\pi'_2)\}$. Since $(\pi'_1, \pi'_2)$ is again partially non-linked in the case $\mathcal{R}_1$, this twisted local integral does not compute the exact $L(s, \pi_1 \times \pi_2)$ either. However, consider the sum of the above two local integrals, $I(s, W^0_1, W^0_2, \Phi)$ and $I(s, W^0_{\pi_1, \mu_2}, W^0_{\pi_2, \mu_2}^{-1}, \Phi')$, as follows: Note that $\chi_1\mu_1 \neq \chi_2\mu_2$ since the pair $(\pi_1, \pi_2)$ is non-linked. If we let

$$\alpha_1 = \frac{\chi_1\mu_1(\varpi)}{\chi_1\mu_1(\varpi) - \chi_2\mu_2(\varpi)} \quad \text{and} \quad \alpha_1 + \alpha_2 = 1, \quad (4.1)$$

and set $\Phi_1 = \alpha_1\Phi$ and $\Phi_2 = \alpha_2\Phi'$. Then it is straightforward to show that

$$I(s, W^0_1, W^0_2, \Phi_1) + I(s, W^0_{\pi_1, \mu_2}, W^0_{\pi_2, \mu_2}^{-1}, \Phi_2) = L(s, \pi_1 \times \pi_2).$$

Thus, pairs of case $\mathcal{R}_1$ are 2-regular.
For the case $R_2$, we use similar argument with the case $R_1$, note that $L(s, \chi_1 \times \mu_1) L(s, \chi_2 \times \mu_2)$ by [17, Chapter IV: Local theory for $GL(2) \times GL(2)$].

Let $\chi_1 \mu_1 = \nu^{s_1}$ and $\chi_2 \mu_2 = \nu^{s_2}$ with $s_1 \neq s_2$. First, consider $(\pi'_1, \pi'_2)$ as $\pi'_1 = \mu_1 \otimes \pi_1$ and $\pi'_2 = \mu_1^{-1} \otimes \pi_2$, then

$$I(s, W^0_{1, \mu_1}, W^0_{2, \mu_1}, \Phi''') = B(s, W^0_{1, \mu_1}, W^0_{2, \mu_1}) = L(s, \chi_1 \times \mu_1),$$

where $\Phi''' = V^{-1}_{N''} \Phi_{\omega''}^0 \Phi''_0$ with $N'' = \max\{c(\pi''_1), c(\pi''_2)\}$. And we already had

$$I(s, W^0_{1, \mu_2}, W^0_{2, \mu_2}, \Phi') = B(s, W^0_{1, \mu_2}, W^0_{2, \mu_2}) = L(s, \chi_2 \times \mu_2),$$

where $\Phi' = V^{-1}_{N'} \Phi_{\omega'}^0 \Phi'_0$ with $N' = \max\{c(\pi'_1), c(\pi'_2)\}$.

If we choose $\Phi = \alpha_1 \Phi'''$ and $\Phi_2 = \alpha_2 \Phi'$ with $\alpha_1, \alpha_2$ from (4.1), then

$$I(s, W^0_{1, \mu_1}, W^0_{2, \mu_1}, \Phi_1) + I(s, W^0_{1, \mu_2}, W^0_{2, \mu_2}, \Phi_2) = L(s, \pi_1 \times \pi_2).$$

So pairs of the case $R_2$ are 2-regular.

Therefore we conclude that partially non-linked pairs are 1-regular if they are of the case $R_0$, or 2-regular otherwise.

\[\square\]

**Remark 4.3.2.** 1. The result of Theorem 4.2.1 can be generalized for any totally non-linked pair $(\pi_1, \pi_2)$ for $GL(n) \times GL(n)$ with $n \geq 3$.

2. In the result of Theorem 4.3.1, we cannot conclude that the 2-regular pairs are not 1-regular. In this Chapter, I want to find a test vector which is easy to describe so more useful to apply in other areas. For further study, we may consider the the test vector $(W^0_1, W^0_2, \Phi)$ with $\Phi$ which is quite complicated but non vanishing at origin, to try to find a 1-regular test vector.
CHAPTER 5

L-FUNCTIONS OF A PAIR \((\pi_1, \pi_2)\) FOR \(GL(2) \times GL(2)\),

PART (II) - LINKED PAIRS

This chapter is for a linked pair \((\pi_1, \pi_2)\) for \(GL(2) \times GL(2)\) such that \(\tilde{\pi}_2 \cong \pi_1 \nu^{s_0}\) for some \(s_0\). Our goal here is to prove the following: Let \((\pi_1, \pi_2)\) be a linked pair for \(GL(2) \times GL(2)\). Then \((\pi_1, \pi_2)\) is \(r\)-regular with \(r \leq 2\). It suffices to show that the case \((\pi_1, \tilde{\pi}_1)\) by the twisted argument discussed in the Section 3.3. More details are following: for any Whittaker function \(W \in \mathcal{W}(\pi_2, \psi^{-1})\) with \(\pi_2 \cong \tilde{\pi}_1 \nu^{s_0}\), we define a \(\nu^{-s_0}\)-twisted Whittaker function of \(\pi_2\) as

\[
W_{2, \nu^{-s_0}} : g \mapsto \nu^{s_0}(\text{det}(g))W_{\pi_2 \otimes \nu^{-s_0}}(g).
\]

As discussed in the Section 3.3, \(W_{2, \nu^{-s_0}} \in \mathcal{W}(\pi_2, \psi^{-1})\). And since \(\pi_2 \otimes \nu^{-s_0} \cong \tilde{\pi}_1\), \(W_{\pi_2 \otimes \nu^{-s_0}} \in \mathcal{W}(\tilde{\pi}_1, \psi^{-1})\). Thus, for any \(W_1 \in \mathcal{W}(\pi_1, \psi)\) and \(\Phi \in \mathcal{S}(F^2)\), we have

\[
I(s, W_1, W_{2, \nu^{-s_0}}, \Phi) = I(s + s_0, W_1, W_{\pi_2 \otimes \nu^{-s_0}}, \Phi).
\]

If \(\pi_1\) is an unramified representation of \(GL(2)\), then by Theorem 2.1.1 the pair \((\pi_1, \tilde{\pi}_1)\) is optimal 1-regular. From now on, throughout this Chapter, we assume that \(\pi_1\) is a ramified representation of \(GL(2)\) and \(\pi_2 = \tilde{\pi}_1\).
Each section of this Chapter includes many technical computations. Let us briefly explain the main result for each section. In Section 5.2, we will show that for \( \pi_1 = \rho(E/F, \xi) \), the pair \((\pi_1, \tilde{\pi}_1)\) is optimal 1-regular if either \(E/F\) unramified or \(E/F\) ramified with \(n(\pi_1 \times \tilde{\pi}_1)\) odd. In Section 5.3, for a generic constituent \(\pi_1\) of an induced representation \(\text{Ind}(\chi_1 \otimes \chi_2)\), we will show that the pair \((\pi_1, \tilde{\pi}_1)\) is \(r\)-regular with \(r \leq 2\). In particular, the pair is optimal 1-regular if \(c(\pi_1) = 1\). And the pair \((\pi_1, \tilde{\pi}_1)\) with \(c(\pi_1) \geq 2\) is 1-regular if \(\chi_1\chi_2^{-1}\) is unramified. Otherwise, the pair \((\pi_1, \tilde{\pi}_1)\) is 2-regular.

To show that the pair \((\pi_1, \tilde{\pi}_1)\) is 1-regular, we will use the follow steps: First, we consider the various local integral with some special Schwartz functions as shown in Section 5.1.2. We know that the \(L\)-function of the linked pair has an exceptional pole by [8]. In other words, if we fix new vectors \(W_i^0\) of \(\pi_i\), \(i = 1, 2\), and use a Schwartz function \(\Phi\) which vanishes at origin such as \(\Phi_{0,\omega,N}^0\) in (2.3), then \(I(s, W_1^0, W_2^0, \Phi)\) does not give all poles of \(L(s, \pi_1 \times \pi_2)\). So we need to compute the local integral with a Schwartz function which is non-vanishing at origin along with fixed new vectors also.

We need to know \(W_{\pi}(g)\) for all \(g \in GL(2)\), not only \(W_{\pi}(a)\) with \(a = \text{diag}(a, 1)\) in the Kirillov model. Schmidt [22] shows the explicit new vector \(f_i \in V_{\pi_i}\), where \(\pi_i \in \text{Irr}(GL(2))\). From a modification of the result of [22], we compute the corresponding new vector \(W_i^0 \in \mathcal{W}(\pi_i, \psi)\), which is compatible with our new vector formula in the Kirillov model as discussed in the Chapter 1. And then for the new vectors \(W_1^0\) and \(W_2^0\), we compute local integrals \(I(s, W_1^0, W_2^0, \ast)\) with \(\ast\) the Schwartz functions discussed in Section 5.1.2 either directly or applying the Functional equation (5.1). Finally, for some \(\Phi\) which is a linear combination of various explicit Schwartz
functions, we have \[
\frac{I(s, W_1^0, W_2^0, \Phi)}{L(s, \pi_1 \times \pi_2)}
\] is a nonzero constant. Consequently, we show that the linked pair \((\pi_1, \pi_2)\) for \(\text{GL}(2) \times \text{GL}(2)\) is optimal 1-regular for some cases. Similarly, we can show the other cases.

5.1 Preliminaries

5.1.1 Functional Equations and \(\epsilon\)-factor

Direct computation of the local integral \(I(s, W_1^0, W_2^0, \Phi)\) with a certain Schwartz function \(\Phi\) is quite difficult at some point even if we have the explicit formula, so we use the Functional equation for the Rankin-Selberg Convolution. First, we review the functional equation from [16]: For any \(W_1 \in \mathcal{W}(\pi_1, \psi), W_2 \in \mathcal{W}(\pi_2, \psi^{-1}), \Phi \in \mathcal{S}(F^2)\), there is \(\epsilon(s, \pi_1 \times \pi_2, \psi) \in \mathbb{C}[q^s]\) of the form \(\epsilon(1/2, \pi_1 \times \pi_2, \psi)q^{-n(s-1/2)}\) such that

\[
\frac{I(1-s, \tilde{W}_1, \tilde{W}_2, \hat{\Phi})}{L(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2)} = \omega_{\pi_2}(-1)\epsilon(s, \pi_1 \times \pi_2, \psi)\frac{I(s, W_1, W_2, \Phi)}{L(s, \pi_1 \times \pi_2)}, \tag{5.1}
\]

where \(\tilde{W}_\pi(g) = W_\pi(w^g) \in \mathcal{W}(\tilde{\pi}, \psi^{-1})\) and \(\hat{\Phi}(u, v) = \int_F \int_F \Phi(x, y)\psi(ux + vy)dx\,dy\), where the measure \(dx, dy\) are the self-dual with respect to \(\psi_F\).

Remark 5.1.1. 1. We call \(n = c(\pi_1 \times \pi_2)\) and \(\epsilon(1/2, \pi_1 \times \pi_2, \psi)\) the conductor and the root number of the pair \((\pi_1, \pi_2)\) respectively.

2. Our concern is \(\pi_2 = \tilde{\pi}_1\), so \(L(1-s, \tilde{\pi}_1 \times \tilde{\pi}_2) = L(1-s, \pi_2 \times \pi_1) = L(1-s, \pi_1 \times \pi_2)\), and \(\omega_{\pi_2} = \omega_{\pi_1}^{-1}\) in the equation \((5.1)\).
3. Instead of using the equation (5.1) directly we use the following form more often,
\[ I(s, W_1, W_2, \Phi) = \omega_{\pi_1}(-1)Aq^{ns}I(1 - s, \tilde{W}_1^0, \tilde{W}_2^0, \tilde{\Phi}) \frac{L(s, \pi_1 \times \pi_2)}{L(1 - s, \pi_1 \times \pi_2)}, \quad \text{(5.2)} \]
where \( A = \epsilon^{-1}(1/2, \pi_1 \times \pi_2, \psi)q^{-n/2} \).

Once we know the conductor \( n = c(\pi_1 \times \pi_2) \) of the pair \((\pi_1, \pi_2)\), then using the Functional equation (5.2) to compute the local integral \( I(s, W_1^0, W_2^0, \Phi) \) is more efficient. In fact, if \( \pi_1 \) is a generic constituent of an induced representation \( Ind(\chi_1 \otimes \chi_2) \), then we can compute \( n \) depending on the inducing data \( \chi_1 \) and \( \chi_2 \). So we will compute \( \epsilon \)-factor \( \epsilon(s, \pi, \psi) \) for such \( \pi_1 \) in Lemma 5.3.1. On the other hands, for the pair \((\pi_1, \pi_2)\) of supercuspidal representations of \( GL(2) \), it is more complicated to compute \( n \) and many people are still working on the conductor \( n \) of supercuspidal pairs using the local Langlands correspondence \cite{5, 3, 6, 4}. In the next section, we will show that an interesting result for the conductor \( n \) of the certain supercuspidal pairs in Proposition 5.2.2.

For now, we review the theories for \( \epsilon \)-factor and the conductor of \( \pi \in Irr(GL(n)) \), \( n \geq 1 \), which are found in \cite{15, 18, 7, 10}. Let \( \psi \) be a fixed nontrivial additive character of \( F \).

1. For any \( \pi \in Irr(GL_n(F)) \),
\[ \epsilon(s, \pi, \psi)\epsilon(1 - s, \tilde{\pi}, \psi) = \omega_{\pi}(-1). \quad \text{(5.3)} \]

2. For any \( \pi \in Irr(GL_n(F)) \),
\[ \epsilon(s, \pi, \psi) = cq^{\epsilon(c(\psi) - c(\pi))s}. \quad \text{(5.4)} \]

If \( \pi \) and \( \psi \) are both unramified, then \( \epsilon(s, \pi, \psi) = 1 \).
3. For any unramified character $\chi$ and $\pi \in \text{Irr}(GL_n(F))$,

$$\epsilon(s, \pi \otimes \chi, \psi) = \chi(\varpi)^{c(\pi)}\epsilon(s, \pi, \psi).$$

(5.5)

4. Let $\psi^a(x) = \psi(ax)$ with $a \in F^\times$, then

- for $\chi \in \text{Irr}(GL(1))$,

$$\epsilon(s, \chi, \psi^a) = |a|^{s-1/2}\chi(a)\epsilon(s, \chi, \psi).$$

(5.6)

- for $\pi_1\pi_2 \in \text{Irr}(GL(n))$, $\pi_2 \in GL(m)$ with $m < n$,

$$\epsilon(s, \pi_1 \times \pi_2, \psi^a) = \omega_{\pi_1}(a)^m\omega_{\pi_2}(a)^n|a|^{nm-s-d}\epsilon(s, \pi_1 \times \pi_2, \psi),$$

(5.7)

where $d = nm - \frac{m(m+1)}{2}$.

5.1.2 Local integrals for special Schwartz functions

In the proof of Lemma 2.1.2, we used the Schwartz function $\Phi^0_{\omega,N}$, which depends on the given representations $\pi_1$ and $\pi_2$ with $\omega = \omega_1\omega_2$ and $N = \max\{c(\pi_1), c(\pi_2)\} \geq 1$. Similarly, we define Schwartz functions in $S(F^2)$, which are depending on the given representation $\pi_1$, and then we study the corresponding local integrals with the new vectors $W_1^0$ and $W_2^0$. Let $\pi_1 \in \text{Irr}(GL(2))$ be such that $c(\pi_1) = N$. For fixed $N$, and new vectors $W_1^0$ and $W_2^0$ of $\pi_1$ and $\pi_2 = \bar{\pi}_1$ respectively:

1. For any integer $l$, let

$$\Phi^0_l = \text{Char}(\varpi^{N+l} O \times \varpi^l O^\times),$$

(5.8)
which vanishes at the origin. If we take \( l = 0 \), then \( \Phi^0_\omega \) is the same as the Schwartz function \( \Phi^0_{\omega, N} \) in (2.3) since \( \omega = 1 \). Here for simplicity of notation, we omit the subscript \( N \).

Note that \( \Phi^0_l((0, z)k) \neq 0 \) only if \( z \in \omega^l \mathcal{O}^\times \) and \( k \in K_0(N) \). So in the equation (2.1), we have

\[
Z(2s, 1, \Phi^0_l, k) = \begin{cases} 
q^{-2ls} & \text{if } k \in K_0(N), \\
0 & \text{otherwise}.
\end{cases}
\]

Moreover, \( W^0_1 \cdot W^0_2 \) is invariant under \( K_0(N) \) since central characters of \( \pi_1 \) and \( \pi_2 \) are inverse to each other. Thus, an argument similar to the one used in the proof of the Lemma 2.1.2 shows that

\[
I(s, W^0_1, W^0_2, \Phi^0_l) = V_N q^{-2ls} B(s, W^0_1, W^0_2),
\]

where \( B(s, W^0_1, W^0_2) \) is from (2.4).

2. For any integer \( m \), let

\[
\Phi_m = Char(\omega^m \mathcal{O} \times \omega^m \mathcal{O}),
\]

whose support contains the origin. Then we have

\[
I(s, W^0_1, W^0_2, \Phi_m) = V_N \left( \frac{q^{-2ms}}{1 - q^{-2s}} \right) \sum_{k_i \in K/K_0(N)} I_{k_i},
\]

where

\[
I_{k_i} = \int_{\mathcal{F}^\times} W^0_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k_i \right) W^0_2 \left( \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} k_i \right) |a|^{s-1} d^\times a.
\]
Since $\Phi_m$ is $K$-invariant and $\Phi_m(e z k) \neq 0$ only if $z \in \varpi^m \mathcal{O}$ and $I(s, W_1^0, W_2^0, \Phi_m) = I_1 \cdot I_2$, where

$$I_1 = \int_K \int_{F^\times} W_1^0 \left( \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix} k \right) W_2^0 \left( \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix} k \right) |\text{det}(a)|^{-1} d^\times adk,$$

and

$$I_2 = \int_{F^\times} \Xi_{\mathcal{O}^m}(z) |z|^{2s} d^\times z = \sum_{n \geq m} q^{-2ns} = \frac{q^{-2ms}}{1 - q^{-2s}}.$$

Note that $I_1$ is independent of $m$ and

$$I_1 = V_N \sum_{k_i \in K/K_0(N)} \int_{F^\times} W_1^0 \left( \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix} k_i \right) W_2^0 \left( \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix} k_i \right) |a|^{-1} d^\times a$$

$$= V_N \sum_{k_i \in K/K_0(N)} I_{k_i}.$$

3. Let $\Psi_m \in S(F^2)$ be such that $\hat{\Psi}_m = \Phi_m$ as in (5.10). Then we have

$$I(1 - s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi_m) = V_N \left( \frac{q^{-2m(1-s)}}{1 - q^{-2(1-s)}} \right) \sum_{k_i \in K/K_0(N)} \tilde{I}_{k_i}, \quad (5.13)$$

where

$$\tilde{I}_{k_i} = \int_{F^\times} W_1^0 \left( w \begin{pmatrix} a^{-1} & 0 \\ 1 & 0 \end{pmatrix} k_i \right) W_2^0 \left( w \begin{pmatrix} -a^{-1} & 0 \\ 1 & 0 \end{pmatrix} k_i \right) |a|^{-1} d^\times a. \quad (5.14)$$

Using similar argument with before, first consider $\tilde{I} = I(1 - s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi_m) = \tilde{I}_1 \cdot \tilde{I}_2$ where

$$\tilde{I}_1 = \int_K \int_{F^\times} \tilde{W}_1^0 \left( \begin{pmatrix} a & 0 \\ 1 & 0 \end{pmatrix} k \right) \tilde{W}_2^0 \left( \begin{pmatrix} -a & 0 \\ 1 & 0 \end{pmatrix} k \right) |a|^{-1} d^\times adk,$$
by using the coset representation of \( K/K_0(N) \) as before, we have

\[
\tilde{I}_1 = V_N \sum_{k_i \in K/K_0(N)} \tilde{I}_{k_i},
\]

and

\[
\tilde{I}_2 = \int_{F^\times} \Xi_{\omega^m \mathcal{O}}(z)|z|^{2(1-s)} d^\times z = \frac{q^{-2m(1-s)}}{1 - q^{-2(1-s)}}.
\]

Combining them gives the result.

4. For any integer \( l \), let

\[
\Psi^0_l \in \mathcal{S}(F^2) \text{ such that } \hat{\Psi}^0_l = \Phi^*_l,
\]

where \( \Phi^*_l = \text{Char}(\omega^l \mathcal{O}^\times \times \omega^{N+l} \mathcal{O}) \).

Then we have

\[
I(1-s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi^*_l) = V_N q^{-2l(1-s)} \tilde{I}_{w_1}, \tag{5.16}
\]

where \( w_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and

\[
\tilde{I}_{w_1} = \int_{F^\times} W_1^0 \left( w \begin{pmatrix} a^{-1} \\ 1 \end{pmatrix} w_1 \right) W_2^0 \left( w \begin{pmatrix} -a^{-1} \\ 1 \end{pmatrix} w_1 \right) |a|^{-s} d^\times a. \tag{5.17}
\]

First, observe the following:

\textbf{Observation 1: } \( \Phi^*_l((0, z)k) \neq 0 \) only if \( k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R \) and \( z \in \omega^l \mathcal{O}^\times \),

where \( R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O})|d \equiv 0 \pmod{\omega^N} \right\}. \)
Since If \((cz, dz) \in \mathcal{O}^\times \times \mathcal{O}^{N+l}\), then \(ord(cz) = l\) and \(ord(dz) \geq N + l\). If \(d \in \mathcal{O}^\times\) then, \(ord(z) \geq N + l\). But \(l = ord(cz) = ord(c) + ord(z) \geq N + l\), which is impossible since \(N \geq 1\). So \(c \in \mathcal{O}^\times\), then \(z \in \mathcal{O}^{N+l}\) and \(ord(d) \geq N\).

**Observation 2:** Let \(R\) as above, then \(R = \omega^t \mathcal{O}_0(N)\).

Since if \(k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K\) with \(ord(d) \geq N\), then

\[
\begin{pmatrix} -1 \\ 1 \end{pmatrix} k = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \in \omega^t \mathcal{O}_0(N).
\]

So, \(k \in \omega^t \mathcal{O}_0(N)\), which gives \(R \subseteq \omega^t \mathcal{O}_0(N)\). And clearly, \(
\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\n\)
with \(b \equiv 0 \pmod{\mathcal{O}^N}\) is in \(R\), which shows that \(\omega^t \mathcal{O}_0(N) \subseteq R\).

Using these two observations, we compute \(\bar{I} = I(1 - s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi^*_l)\) as follows:

Note that \(\tilde{W}_1^0 \cdot \tilde{W}_2^0\) is invariant under \(\omega^t \mathcal{O}_0(N)\):

\[
\bar{I} = \int_K \int_{F^\times} \tilde{W}_1^0 \left( \begin{pmatrix} a \\ 1 \end{pmatrix} k \right) \tilde{W}_2^0 \left( \begin{pmatrix} -a \\ 1 \end{pmatrix} k \right) |a|^{-s} d^\times a
\]

\[
\int_{F^\times} \Phi^*_l((0, z)k)|z|^{2(1-s)}d^\times z dk
\]

\[
= q^{2(1-s)} \int_{\omega^t \mathcal{O}_0(N)} \int_{F^\times} \tilde{W}_1^0 \left( \begin{pmatrix} a \\ 1 \end{pmatrix} w_1 k \right) \tilde{W}_2^0 \left( \begin{pmatrix} -a \\ 1 \end{pmatrix} w_1 k \right) |a|^{-s} d^\times a dk
\]

\[
= V_N q^{2(1-s)} \bar{I}_{w_1}
\]
as desired.
Remark 5.1.2. For the equation (5.11) and (5.13), we need to know the coset representatives of $K/K_0(N)$. In fact,

$$K/K_0(N) \cong B_1 \cup B_2,$$

where

\[
B_1 = \left\{ \begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix} \mid \beta \in \mathcal{O}/\mathcal{O}^N \right\} \quad \text{and} \quad B_2 = \left\{ \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \mid \alpha \in \mathcal{O}/\mathcal{O}^N \right\}.
\]

First we show that $B_1 \cup B_2 \subseteq K/K_0(N)$, that is, if $g_1, g_2 \in B_1 \cup B_2$ with $g_1 \neq g_2$, then $g_2 \notin g_1K_0(N)$.

- If both $g_1, g_2 \in B_1$: set $g_1 = \begin{pmatrix} \beta_1 & 1 \\ -1 & 0 \end{pmatrix}$ and $g_2 = \begin{pmatrix} \beta_2 & 1 \\ -1 & 0 \end{pmatrix}$. Note that
  \[
g_1K_0(N) = \left\{ \begin{pmatrix} \beta_1 a + c & \beta_1 b + d \\ -a & -b \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \right\}.
\]

So if $g_2 \in g_1K_0(N)$, then

\[
\begin{pmatrix} \beta_2 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \beta_1 a + c & \beta_1 b + d \\ -a & -b \end{pmatrix},
\]

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N)$. But then $a = 1$, and $\beta_2 = \beta_1 + c$, that is, $\beta_2 - \beta_1 = c \in \mathcal{O}^N$, which is impossible since $g_1 \neq g_2$. 


• If both $g_1, g_2 \in B_2$: set $g_1 = \begin{pmatrix} 1 & 0 \\ \alpha_1 & 1 \end{pmatrix},$ and $g_2 = \begin{pmatrix} 1 & 0 \\ \alpha_2 & 1 \end{pmatrix}$. Note that

$$g_1 K_0(N) = \left\{ \begin{pmatrix} a & b \\ \alpha a + c & \alpha b + d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \right\}. $$

So if $g_2 \in g_1 K_0(N)$, then

$$\begin{pmatrix} 1 & 0 \\ \alpha_2 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ \alpha a + c & \alpha b + d \end{pmatrix},$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N)$. But then $a = 1$, and so $\beta_2 - \alpha = c \in \varpi^N \mathcal{O}$, which is impossible since $g_1 \neq g_2$.

• If $g_1 \in B_1$ and $g_2 \in B_2$: set $g_1 = \begin{pmatrix} \beta & 1 \\ -1 & 0 \end{pmatrix},$ and $g_2 = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$. So if $g_2 \in g_1 K_0(N)$, then

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ \alpha a + c & \alpha b + d \end{pmatrix},$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N)$. But then $\alpha = -a$ and so $1 = -\alpha \beta + c$, which is impossible since $\alpha \in \mathcal{O}, \beta \in \varpi \mathcal{O}, c \in \varpi^N \mathcal{O}$. 

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It remains to show that for any \( k \in K = \text{GL}_2(\mathcal{O}) \), there is \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \) such that
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B_1 \cup B_2.
\]

Let \( k = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K \) and \( \Delta = \det(k) \in \mathcal{O}^\times \).

- If \( A \in \mathcal{O}^\times \), find \( x \in \varpi^N \mathcal{O} \) such that \( \frac{C}{A} + x \in \mathcal{O}/\varpi^N \mathcal{O} \). Compute
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -\frac{B}{A} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & \frac{A}{x} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{C}{A} + x & 1 \end{pmatrix} \in B_2.
\]

- If \( A \notin \mathcal{O}^\times \). So \( A \in \varpi \mathcal{O} \), which implies that \( B, C \in \mathcal{O}^\times \), then find \( y \in \varpi^N \mathcal{O} \) such that \( -\frac{A}{C} + y \in \mathcal{O}/\varpi^N \mathcal{O} \). Compute
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -\frac{D}{C} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -C^{-1} & 0 \\ 0 & -\frac{C}{A} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} -\frac{A}{C} + y & 1 \\ -1 & 0 \end{pmatrix} \in B_1.
\]

So, we are done.

Next we summarize some useful \( p \)-adic integrals which we will use later.

1. Let \( \psi \) be a nontrivial additive character of \( F \), then
\[
\int_{\varpi^m \mathcal{O}} \psi(xy)dy = \begin{cases} 
q^{-m} \text{Vol}(\mathcal{O}) & \text{if } x \in \varpi^{-m+c(\psi)}, \\
0 & \text{otherwise}.
\end{cases}
\] (5.18)
2. For fixed $m$, if $c(\psi) = 0$ and $V = Vol(\mathcal{O})$, then

$$
\int_{\mathbb{Z}m \times \mathbb{O}} \psi(xy)dy = \begin{cases} 
0 & \text{if } \text{ord}(x) \leq -m - 2, \\
-Vq^{-m-1} & \text{if } \text{ord}(x) = -m - 1, \\
Vq^{-m}(1 - q^{-1}) & \text{if } \text{ord}(x) \geq -m. 
\end{cases} \quad (5.19)
$$

Since

$$
\int_{\mathbb{Z}m \times \mathbb{O}} \psi(xy)dy = \int_{\mathbb{Z}m \mathcal{O}} \psi(xy)dy - \int_{\mathbb{Z}m+1 \mathcal{O}} \psi(xy)dy.
$$

By the equation (5.18) with the assumption $c(\psi) = 0$, we have

$$
\int_{\mathbb{Z}m \mathcal{O}} \psi(xy)dy = \begin{cases} 
Vq^{-m} & \text{if } \text{ord}(x) \geq -m, \\
0 & \text{otherwise.} 
\end{cases}
$$

3. Let $\xi$ be a ramified character of $F^\times$, $\psi$ be a nontrivial additive character of $F$ and $dx$ be the self-dual Haar measure on $F$, that is, $Vol(\mathcal{O}) = q^{c(\psi)/2}$, then by the change variable and the Lemma 1.1.1 in [22] we have

$$
\int_{\mathcal{O}^\times} \xi^{-1}(y)\psi(xy)dy = \begin{cases} 
\frac{\xi(x)}{|x|}\epsilon(0, \xi, \psi) & \text{if } \text{ord}(x) = c(\psi) - c(\xi), \\
0 & \text{otherwise.} 
\end{cases} \quad (5.20)
$$

**Remark 5.1.3.** Compute $\Psi_m$ and $\Psi_0^l$ explicitly for the case $c(\psi) = 0$. Note that $\hat{\Phi}(u, v)$ is defined in (5.1).

1. For $\Psi_m$: we use the definition $\hat{\Psi}_m = \Phi_m$ and $\hat{\Psi}_m = \Psi_m$. 

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\[ \hat{\Phi}_m(u, v) = \int \int \Phi_m(x, y) \psi(ux + vy) dx dy \]
\[ = \int_{\mathcal{O}} \int_{\mathcal{O}} \psi(ux + vy) dx dy \]
\[ = \left( \int_{\mathcal{O}} \psi(ux) dx \right) \left( \int_{\mathcal{O}} \psi(vy) dy \right) \]
\[ = \begin{cases} Vol(\mathcal{O})^2 & \text{if } (u, v) \in \mathcal{O} \times \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases} \]  

Since \( Vol(\mathcal{O}) = q^{-m} \), \( \Psi_m = q^{-2m} \Phi_m \).

2. For \( \Psi_0^0 \): we use the definition of \( \hat{\Psi}_0^0 = \hat{\Phi}_0^* \) and \( \hat{\Psi}_0 = \Psi_0^0 \).

\[ \hat{\Phi}_0^*(u, v) = \int \int \hat{\Phi}_0^*(x, y) \psi(ux + vy) dx dy \]
\[ = \int_{\mathcal{O}} \int_{\mathcal{O}} \psi(ux + vy) dx dy \]
\[ = \left( \int_{\mathcal{O}} \psi(ux) dx \right) \left( \int_{\mathcal{O}} \psi(vy) dy \right). \]

Then,
\[ \int_{\mathcal{O}} \psi(ux) dx = \begin{cases} q^{-l}(1 - \frac{1}{q}) & \text{if } \text{ord}(u) \geq -l, \\ -q^{-l-1} & \text{if } \text{ord}(u) = -l - 1, \\ 0 & \text{otherwise,} \end{cases} \]

and
\[ \int_{\mathcal{O}} \psi(vy) dy = \begin{cases} q^{-N-l} & \text{if } \text{ord}(v) \geq -N - l, \\ 0 & \text{otherwise.} \end{cases} \]
Thus

\[
\Psi^0_l(u,v) = \Phi_l^* = \begin{cases} 
q^{-N-2l}(1-q^{-1}) & \text{if } (u,v) \in \mathcal{O} \times \mathcal{O}, \\
-q^{-N-2l-1} & \text{if } (u,v) \in \mathcal{O} \times \mathcal{O}, \\
0 & \text{otherwise.}
\end{cases}
\]

Moreover, we have \(\text{Supp}(\Psi^0_l) = \mathcal{O} \times \mathcal{O},\) that is, \(\Psi^0_l(0) \neq 0.\)

5.1.3 Generating Constant Lemma

By [16], we know that \(\frac{I(s, W_1, W_2, \Phi)}{L(s, \pi_1 \times \pi_2)}\) is a polynomial in \(\mathbb{C}[q^s, q^{-s}]\) for any \(W_1 \in \mathcal{W}(\pi_1, \psi),\) \(W_2 \in \mathcal{W}(\pi_2, \psi^{-1})\), and \(\Phi \in \mathcal{S}(F^2).\) In particular, for fixed new vectors \(W_1^0, W_2^0\) for \(\pi_1, \pi_2\) respectively, and \(\Phi\) defined in the previous section, if we compute the ratio of local integrals \(I(s, W_1^0, W_2^0, \Phi)\) to \(L(s, \pi_1 \times \pi_2)\), then we will have three different kinds of polynomial as follows: let \(Y = q^{-s},\)

\[
\begin{align*}
R & := R(Y) = \sum_{i=0}^{d} r_i Y^i, \\
A_m & := A_m(Y) = \alpha Y^{2m} - Y^{2m+1} \quad (m \in \mathbb{Z}), \\
B_l & := B_l(Y) = \beta Y^{2l-1} - Y^{2l} \quad (l \in \mathbb{Z}),
\end{align*}
\]

(5.21)

Note that \(d, r_i (1 \leq i \leq d), \alpha \) and \(\beta\) are depending on the given representation \(\pi_1.\)

**Lemma 5.1.4.** For \(R, A_i, \) and \(B_i\) as above. There is \(\hat{R} \in \mathbb{C}\) such that

\[
\hat{R} = R + \sum_{i=0}^{m(d)} a_i A_i + \sum_{i=1}^{l(d)} b_i B_i,
\]

for some \(m(d)\) and \(l(d).\) In particular, \(\hat{R}\) is written in terms of \(\alpha, \beta, r_i\) as follows:
\[ \hat{R} = \begin{cases} 
  r_0 + \alpha r_1 + \alpha \beta r_2 + \alpha^2 \beta r_3 + \cdots + \alpha^{d-1} \beta^{\frac{d-1}{2}} r_{d-1} + \alpha^{\frac{d+1}{2}} \beta^{\frac{d+1}{2}} r_d & \text{for } d \text{ odd}, \\
  r_0 + \alpha r_1 + \alpha \beta r_2 + \alpha^2 \beta r_3 + \cdots + \alpha^\frac{d}{2} \beta^\frac{d}{2} r_{d-1} + \alpha^\frac{d}{2} \beta^\frac{d}{2} r_d & \text{for } d \text{ even.} 
\end{cases} \] 

(5.22)

**Proof.** The key of the argument here is using \( A_i \) and \( B_i \) alternatively, to decrease the degree of the fixed polynomial \( R = r_0 + r_1 Y + \cdots + r_{d-1} Y^{d-1} + r_d Y^d \) to the constant term. In other words, we will choose \( a_i \) and \( b_i \) depending on \( r_i, \alpha \) and \( \beta \) so that the summing up gives the desired constant \( \hat{R} \).

First, consider the following sum \( S \):

\[ S = \sum_{i=0}^{m(d)} a_i A_i + \sum_{i=1}^{l(d)} b_i B_i. \]

If \( m(d) = l(d) \), then this sum \( S \) becomes

\[ S_o = \alpha a_0 + \sum_{i=1}^{l(d)} (\beta b_i - a_{i-1}) Y^{2i-1} + \sum_{i=1}^{m(d)} (\alpha a_i - b_i) Y^{2i} - a_{m(d)} Y^{2m(d)+1}, \]

and if \( m(d) = l(d) - 1 \), then this sum \( S \) becomes

\[ S_e = \alpha a_0 + \sum_{i=1}^{l(d)} (\beta b_i - a_{i-1}) Y^{2i-1} + \sum_{i=1}^{m(d)} (\alpha a_i - b_i) Y^{2i} - b_{l(d)} Y^{2l(d)}. \]

Now, for given \( d \) and \( r_i \) \((0 \leq i \leq d)\) of the polynomial \( R(Y) \):

- if \( d \) is odd, then from letting \( m(d) = l(d) = \frac{d-1}{2} \), we have

\[ R + S_o = (r_0 + \alpha a_0) + \sum_{i=1}^{\frac{d-1}{2}} (r_{2i-1} + \beta b_i - a_{i-1}) Y^{2i-1} + \sum_{i=1}^{\frac{d-1}{2}} (r_{2i} + \alpha a_i - b_i) Y^{2i} + (r_d - a_{\frac{d-1}{2}}) Y^d. \]

To get the desired result, we take \( a_{m(d)} = a_{\frac{d-1}{2}} = r_d \) and then find \( a_i \) \((0 \leq i \leq \frac{d-3}{2})\) and \( b_i \) \((1 \leq i \leq \frac{d-1}{2})\) satisfying the following:
\[
\begin{cases}
 r_{2i-1} + b_i \beta - a_{i-1} = 0 & \text{for } 1 \leq i \leq \frac{d-1}{2}, \\
 r_{2i} + \alpha a_i - b_i = 0 & \text{for } 1 \leq i \leq \frac{d-1}{2}.
\end{cases}
\]

(5.23)

Accordingly we have a recursive formula for \(a_i\) and \(b_i\) in terms of \(\alpha, \beta\) and \(r_i\) as follows:

\[
\begin{cases}
 a_{\frac{d-1}{2}} = r_d, \\
 b_i = r_{2i} + \alpha a_i, & \text{for } 1 \leq i \leq \frac{d-1}{2}, \\
 a_{i-1} = r_{2i-1} + \beta b_i & \text{for } 1 \leq i \leq \frac{d-1}{2}.
\end{cases}
\]

(5.24)

Then we have a constant \(\hat{R} = r_0 + \alpha a_0\) and by the above recursive formula, this becomes

\[
\hat{R} = r_0 + \alpha r_1 + \alpha \beta r_2 + \cdots + \alpha^{\frac{d-1}{2}} \beta^{\frac{d-1}{2}} r_{d-1} + \alpha^{\frac{d+1}{2}} \beta^{\frac{d+1}{2}} r_d.
\]

- if \(d\) is even, then after setting \(m(d) = \frac{d}{2} - 1\) and \(l(d) = \frac{d}{2}\), we have

\[
R + S_e = (r_0 + \alpha a_0) + \sum_{i=1}^{\frac{d}{2}} (r_{2i-1} + b_i \beta - a_{i-1}) Y^{2i-1} + \sum_{i=1}^{\frac{d-1}{2}} (r_{2i} + \alpha a_i - b_i) Y^{2i} + (r_d - b_{\frac{d}{2}}) Y^d.
\]

To get the desired result, similarly, we take \(b_{\frac{d}{2}} = r_d\) and then find \(a_i\) (0 \(\leq i \leq \frac{d}{2} - 1\)) and \(b_i\) (1 \(\leq i \leq \frac{d}{2} - 1\)) satisfying (5.23). Consequently, we have a recursive formula for \(a_i\) and \(b_i\) in terms of \(\alpha, \beta\) and \(r_i\) as follows:

\[
\begin{cases}
 b_{\frac{d}{2}} = r_d, \\
 a_{i-1} = r_{2i-1} + \beta b_i & \text{for } 1 \leq i \leq \frac{d}{2}, \\
 b_i = r_{2i} + \alpha a_i & \text{for } 1 \leq i \leq \frac{d}{2} - 1.
\end{cases}
\]

(5.25)
Finally we have a constant \( \widehat{R} = r_0 + \alpha a_0 \), which becomes, by the recursive formula (5.24) or (5.25),

\[
\widehat{R} = r_0 + \alpha r_1 + \alpha^2 \beta r_2 + \alpha^3 \beta^2 r_3 + \cdots + \alpha^d \beta^{d-1} r_d - 1 + \alpha^d \beta^d r_d.
\]

\[\square\]

5.2 Pair of supercuspidal representations

Let \( \pi_1 \) be the supercuspidal representation of \( GL(2) \) with the conductor \( N \) and \( \pi_2 = \tilde{\pi}_1 \). From (1.10), we know the new vectors \( W^0_1 \) and \( W^0_2 \) of \( \pi_1 \) and \( \pi_2 \) respectively in their Kirillov models:

\[
W^0_1 \begin{pmatrix} a \\ 1 \end{pmatrix} = \Xi_{\mathcal{O}^\times}(a), \quad \text{and} \quad W^0_2 \begin{pmatrix} a \\ 1 \end{pmatrix} = \Xi_{\mathcal{O}^\times}(a).
\]

And so

\[
B(s, W^0_1, W^0_2) \overset{2.4}{=} \int_{F^\times} \Xi_{\mathcal{O}^\times}(a)|a|^{s-1} d^\times a = 1. \quad (5.26)
\]

Recall that for the new vectors \( W^0_1 \) and \( W^0_2 \), \( W^0_1 \cdot W^0_2 \) is invariant under \( K_0(N) \) since central characters of \( \pi_1 \) and \( \pi_2 \) are inverse to each other. With (5.26) and the equation (5.9), we have

\[
I(s, W^0_1, W^0_2, \Phi^0) = V_N q^{-2ls}. \quad (5.27)
\]

Let \( \eta \) be the unramified quadratic character of \( F^\times \) which is given by \( \eta(x) = (-1)^{ord_F(x)} \) for all \( x \in F^\times \). Gelbart and Jacquet [12, Corollary (1.3)] showed that for the supercuspidal representation \( \pi_1 \) and the unramified quadratic character \( \eta \) of \( F^\times \), if \( \pi_1 \) is
not equivalent to \( \pi_1 \otimes \eta \) then \( L(s, \pi_1 \times \tilde{\pi}_1) = (1 - q^{-s})^{-1} \), otherwise \( L(s, \pi_1 \times \tilde{\pi}_1) = (1 - q^{-2s})^{-1} \).

### 5.2.1 Preliminaries

First, we review some preliminaries on the supercuspidal representation attached to \((\xi, E/F)\). The construction and properties of this representation are based on [15]. Let \( \rho = \rho(E/F, \xi) \) be the supercuspidal representation associated with a quadratic field extension \(E/F\) and a character \( \xi \) of \( E^\times \) that is not trivial on \( U^1 \), the kernel of norm map \( N_{E/F}(E^\times) \). Fix a nontrivial additive character \( \psi \) of \( F \) with \( c(\psi) = 0 \), and let \( \psi_E = \psi \circ \text{tr} \). Let \( \chi \) be the quadratic character of \( F^\times \) which is trivial on \( N(E^\times) \). Throughout this section, for the simplicity of notation, we will omit the subscript for \( F \). To start with construction of the supercuspidal representation \( \rho(E/F, \xi) \), we review the Weil representation \( w_E \) of \( SL_2(F) \) acting on \( C_\infty^c(E) \) based on [15]. For \( \varphi \in C_\infty^c(E) \) and \( v \in E \),

\[
\begin{align*}
\left( w_E \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) (v) &= \psi(xN_{E/F}(v))\varphi(v) \\
\left( w_E \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) (v) &= |a|\chi(a)\varphi(av) \\
\left( w_E \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right) (v) &= \gamma \int_E \varphi(u)\psi_E(u\overline{v})du,
\end{align*}
\]  

(5.28), (5.29), (5.30)
where $\gamma^2 = \chi(-1)$ and $du$ is the Haar measure on $E$ such that $Vol(\mathcal{O}_E) = q_E^{c(\psi_E)/2}$, that is, $du$ is the self-dual Haar measure on $E$.

Now let $G^+ = GL_2(F)^+ = \{g \in GL_2(F)|\det(g) \in N_{E/F}(E^\times)\}$ and

$$U_\xi = \{\varphi \in C^\infty_c(E)|\varphi_\xi(xy) = \xi^{-1}(x)\varphi_\xi(y) \text{ for all } y \in E \text{ and } x \in U^1\}.$$  

We then define the representation $w_{\xi,\psi}$ of $G^+$ on $U_\xi$ by extending from $w_E$ as

$$w_{\xi,\psi} \left( \begin{pmatrix} a & s \\ 1 & 1 \end{pmatrix} \varphi_\xi \right)(v) = |a|^{1/2}\xi(b)(w_E(s)\varphi_\xi)(bv) \tag{5.31}$$

if $b \in E^\times$ with $N_{E/F}(b) = a \in F^\times$ and $s \in SL_2(F)$.

Note that the definition of $w_{\xi,\psi}$ above is independent of the choice of $b$, since if there is another $x \in E$ so that $N_{E/F}(x) = a$, the right hand side of (5.31) is $|a|^{1/2}\xi(x)((w_E(s)\varphi_\xi)(xv)$, let $\varphi'_\xi = w_E(s)\varphi_\xi \in C^\infty_c(E)$ such that $\varphi'_\xi(xv) = \varphi'_\xi(xb^{-1}bv)$, but $N_{E/F}(xb^{-1}) = aa^{-1} = 1$, so $\varphi'_\xi(xv) = \xi^{-1}(xb^{-1})\varphi'_\xi(bv)$. Finally, the right hand side of (5.31) for any choice of $x$ with $N_{E/F}(x) = a$ is same with the choice of $b$.

Finally, we define $\rho = Ind_{G^+}^{GL_2(F)}(w_{\xi,\psi})$. From [15], we know that

- the central character of $\rho$ is $\chi \cdot [F^\times]$.
- for any character $\mu$ of $F^\times$, $\rho \otimes \mu \cong \rho'$, where $\rho'$ is the supercuspidal representation associated with $\xi \cdot (\mu \circ N_{E/F})$ and $E/F$.

For $f \in V_\rho$ where $\rho = \rho(E/F,\xi)$, we have the corresponding function $W_f \in W(\rho,\psi)$ with

$$W_f(g) = \lambda(\rho(g)f) = (\rho(g)f)(I_2)(1_E) = f(g)(1_E). \tag{5.32}$$
In particular, if \( f \) is the new vector for \( \rho \), then we have the new vector \( W_f \in \mathcal{W}(\rho, \psi) \).

Similarly, we obtain \( \tilde{\rho} = \rho(E/F, \xi^{-1}) \) and \( W_{\tilde{f}} \in \mathcal{W}(\tilde{\rho}, \psi) \) for \( \tilde{f} \in V(\tilde{\rho}, \xi^{-1}) \). Then

\[
W_{\tilde{f}} \left( \begin{pmatrix} -1 & \ 1 \\ \ 0 & 1 \end{pmatrix} g \right) \in \mathcal{W}(\tilde{\rho}, \psi^{-1}) \text{ as stated in Remark (1.3.4).}
\]

In [22, Section 2.3], Schmidt computed the explicit formula for the new vector \( f \in V_\rho \) where \( \rho = \rho(E/F, \xi) \) as follows: Let

\[
\varphi_{\xi}^0(v) = \begin{cases} 
\xi^{-1}(v) & \text{if } v \in \mathcal{O}_{E}^{\times}, \\
0 & \text{otherwise.}
\end{cases}
\tag{5.33}
\]

1. For \( E/F \) unramified,

\[
f(g) = \begin{cases} 
w_{\xi, \psi}(g)\varphi_{\xi}^0 & \text{if } g \in G^+, \\
0 & \text{otherwise.}
\end{cases}
\tag{5.34}
\]

2. For \( E/F \) ramified,

\[
f(g) = \begin{cases} 
w_{\xi, \psi}(g)\varphi_{\xi}^0 & \text{if } g \in G^+, \\
w_{\xi, \psi}(gt^{-1})\varphi_{\xi}^0 & \text{if } g \notin G^+.
\end{cases}
\tag{5.35}
\]

where \( t = \begin{pmatrix} x \\ 1 \end{pmatrix} \) such that \( x \in \mathcal{O}_{F}^{\times} \) but \( x \notin N(E^{\times}) \).

And we have the formular for the conductor of \( \psi_E \) in [22, Lemma 2.3.1] and \( \rho = \rho(E/F, \xi) \) in [22, Theorem 2.3.2]:

\[
\begin{align*}
c(\psi_E) &= e(E/F)c(\psi) - f(E/F)^{-1}d(E/F), \\
c(\rho) &= f(E/F)c(\xi) + d(E/F),
\end{align*}
\]
where the ramification index $e(E/F)$, the degree of the residue field extension $f(E/F)$ and the valuation of the discriminant of the field extension $e(E/F)$. With our assumption $c(\psi) = 0$ and $q$ is odd if $E/F$ is ramified, we have the following results:

$$c(\psi_E) = \begin{cases} 
0 & \text{if } E/F \text{ is unramified}, \\
-1 & \text{if } E/F \text{ is ramified and } q \text{ is odd}, 
\end{cases} \quad (5.36)$$

and

$$c(\rho) = \begin{cases} 
2c(\xi) & \text{if } E/F \text{ is unramified}, \\
c(\xi) + 1 & \text{if } E/F \text{ is ramified and } q \text{ is odd}. 
\end{cases} \quad (5.37)$$

To use the Schmidt’s formula above, we need to review some facts on the number theory of quadratic extension from [23, 24]. In particular, we often use norm map $N_{E/F} : E^\times \mapsto F^\times$ with the following properties:

- $N_{E/F}(b) = b^2$ for any $b \in F^\times$.
- $N_{E/F}(b) \in \mathcal{O}^\times$ implies $b \in \mathcal{O}_E^\times$, so $\text{Ker}(N_{E/F}) \subseteq \mathcal{O}_E^\times$.
- $N_{E/F}(\mathcal{O}_E^\times) = \mathcal{O}_F^\times$ if $E/F$ unramified, so $-1 \in N_{E/F}(E^\times)$ if $E/F$ unramified.
- if $E/F$ ramified, $[\mathcal{O}_F^\times : N_{E/F}(\mathcal{O}_E^\times)] = 2$, i.e., there is $x \in \mathcal{O}_F^\times - N_{E/F}(\mathcal{O}_E^\times)$ such that for each $y \in \mathcal{O}_F^\times$, either $y \in N_{E/F}(\mathcal{O}_E^\times)$ or $xy \in N_{E/F}(\mathcal{O}_E^\times)$.

If $\varpi_E$ is the uniformizer of $E$, then we have $\varpi_F \mathcal{O}_E = \begin{cases} 
\varpi_E \mathcal{O}_E & \text{if } E/F \text{ unramified}, \\
\varpi_E^2 \mathcal{O}_E & \text{if } E/F \text{ ramified},
\end{cases}$ which gives

$$q_E = |\mathcal{O}_E / \varpi_E \mathcal{O}_E|_E = \begin{cases} 
q^2 & \text{if } E/F \text{ unramified}, \\
q & \text{if } E/F \text{ ramified}. 
\end{cases}$$
From this, if $N_{E/F}(b) = a$ with $b \in F^\times$, then $ord_F(a) = ord_E(a) = 2ord_E(b)$ if $E/F$ unramified and $2ord_F(a) = ord_E(a) = 2ord_E(b)$ if $E/F$ ramified. This implies that for any $a \in F^\times$, $|a|_E^{1/2} = |a|_F$ if $E/F$ unramified, and $|a|_E^{1/2} = |a|_F$ if $E/F$ is ramified.

For (additive) Haar measure, $d_E(xy) = |x|_Ed_E(y)$. Assume that $c(\psi) = 0$ and $dx$ as normalized Haar measure of $F$ with $Vol(\mathcal{O}) = 1$, then we have $Vol(\mathcal{O}_F^\times) = (q - 1)/q$, and the straightforward computation gives the following:

$$
\begin{cases}
Vol(\mathcal{O}_E) = 1, & \text{if } E/F \text{ unramified} \\
Vol(\mathcal{O}_E) = q^{-1}, & \text{if } E/F \text{ ramified}
\end{cases}
$$

$$
\begin{cases}
Vol(\varpi_E^m \mathcal{O}_E) = q^{-2m}, & \text{if } E/F \text{ unramified} \\
Vol(\varpi_E^m \mathcal{O}_E) = q^{-m-1/2}, & \text{if } E/F \text{ ramified}
\end{cases}
$$

$$
\begin{cases}
Vol(\mathcal{O}_E^\times) = (q^2 - 1)/q^2, & \text{if } E/F \text{ unramified} \\
Vol(\mathcal{O}_E^\times) = (q - 1)/q^2, & \text{if } E/F \text{ ramified}
\end{cases}
$$

5.2.2 Supercuspidal representation $\pi_1$ attached to $(\xi, E/F)$; $E/F$ unramified

Assume $E/F$ unramified, let $\pi_1 = \rho(E/F, \xi)$, then

$$L(s, \pi_1 \times \tilde{\pi}_1) = \frac{1}{1 - q^{-2s}}, \text{ and } L(1 - s, \pi_1 \times \tilde{\pi}_1) = \frac{1}{1 - q^{-2(1-s)}}.$$ 

Since for the unramified quadratic character $\eta$ of $F^\times$, $N_{E/F}(\mathcal{O}_E^\times) = \mathcal{O}_F^\times$, $\eta(\mathcal{O}_F^\times) = 1$ and $N_{E/F}(\varpi_E) = \varpi_F^2$, we have $\xi \cdot (\eta \circ N_{E/F}) = \xi$ and hence $\pi_1 \otimes \eta \cong \pi_1$, which combines with the result of [12] gives $L$-function of a pair $(\pi_1, \tilde{\pi}_1)$ as above. Note that $N = c(\pi_1) = 2c(\xi) := 2c$ as in (5.37). The goal of this section is to prove the following:

**Theorem 5.2.1.** Let $\pi_1 = \rho(E/F, \xi)$ with an unramified field extension $E/F$ and $\pi_2 =$
π1. Then the pair (π1, π2) is optimal 1-regular. And we can choose an appropriate \( \Phi \in S(F^2) \) so that

\[
I(s, W_1^0, W_2^0, \Phi) = L(s, \pi_1 \times \pi_2).
\]

Before proving this Theorem, we need some preliminary results.

**Lemma 5.2.2.** The conductor n of the pair (π1, π2) is even.

**Proof.** For \( \Phi_0 = \text{Char}(O \times O) \),

\[
I(s, W_1^0, W_2^0, \Phi_0) = \frac{V_{2c}}{1 - q^{-2s}} \sum_{k_i \in K/K_0(2c)} I_{k_i}.
\]

From \( \hat{\Phi}_0 = \Phi_0 \) by Remark 5.1.3,

\[
I(1 - s, \tilde{W}_1^0, \tilde{W}_2^0, \hat{\Phi}_0) = I(1 - s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi_0) = \frac{V_{2c}}{1 - q^{-2(1-s)}} \sum_{k_i \in K/K_0(2c)} \tilde{I}_{k_i}.
\]

Applying the Functional equation (5.1) gives

\[
\sum \tilde{I}_{k_i} = \omega_{\pi_1}(-1) A q^{ns} \sum I_{k_i},
\]

where n is the conductor of the pair (π1, π2), \( A = \epsilon(1/2, \pi_1 \times \pi_2, \psi^{-1})q^{-n/2} \) as in (5.2), \( I_{k_i} \) is from (5.12) and \( \tilde{I}_{k_i} \) is from (5.14). Note that \( \sum \tilde{I}_{k_i} \) and \( \sum I_{k_i} \) are polynomials, since \( \frac{I(s, W_1^0, W_2^0, \Phi_0)}{L(s, \pi_1 \times \pi_2)} \) and \( \frac{I(1 - s, W_1^0, \tilde{W}_2^0, \Phi_0)}{L(1 - s, \pi_1 \times \pi_2)} \) are in \( \mathbb{C}[q^s, q^{-s}] \) by the main result of [16].

On the other hands, \( W_f(g) \) (5.31) \( f(g)(1_E) = 0 \) unless \( \text{det}(g) \in N_{E/F}(E^\times) \) by the formula (5.34) for the new vector \( f \in V_\rho \). This implies that both integrals \( I_{k_i} \) and \( \tilde{I}_{k_i} \) are zero unless \( a \in N_{E/F}(E^\times) \). Since \( E/F \) is unramified, \( a \in N_{E/F}(E^\times) \) means
that \( \text{ord}(a) \) is even. Thus, both \( \sum \tilde{I}_{k_i} \) and \( \sum I_{k_i} \) are sum of even degree monomials, which implies that \( n \) should be even. \( \square \)

**Lemma 5.2.3.** For \( \Phi_1^* \) as in \([5.15]\),

\[
I(1-s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi_1^*) = V_2c q^{-2l(1-s)} \xi(-1).
\]

**Proof.** By the equation \((5.16)\), it is enough to compute \( \tilde{I}_{w_1} = \xi(-1) \).

First, consider the integrands \( W_1^0(g_1) \) and \( W_2^0(g_2) \) of \( \tilde{I}_{w_1} \) where

\[
g_1 = \begin{pmatrix}
w & \left( a^{-1} \right) \\
1 & w_1
\end{pmatrix}, \quad g_2 = \begin{pmatrix}
w & \left( -a^{-1} \right) \\
1 & w_1
\end{pmatrix}.
\]

Then \( W_1^0(g_1) \) \( \overset{[5.32]}{=} f(g_1)(1_E) \) which is nonzero only if \( \text{det}(g_1) \in N_{E/F}(E^\times) \) or \( -a^{-1} \in N_{E/F}(E^\times) \), and \( W_2^0(g_2) \) \( \overset{[5.32]}{=} \tilde{f}(g_2)(1_E) \) \( \neq 0 \) only if \( \text{det}(g_2) \in N_{E/F}(E^\times) \) or \( a^{-1} \in N_{E/F}(E^\times) \) by \((5.34)\):

For \( a \in N_{E/F}(E^\times) \), let \( N_{E/F}(i) = -1 \) \((i \in \mathcal{O}_E^\times)\) and \( N_{E/F}(b) = a^{-1} \), by the matrix multiplication, we have

\[
g_1 = w \begin{pmatrix} a^{-1} \\ 1 \end{pmatrix} w_1 = \begin{pmatrix} -a^{-1} \\ 1 \end{pmatrix} \begin{pmatrix} a \\ a^{-1} \end{pmatrix},
\]

and similarly \( g_2 = \begin{pmatrix} a^{-1} \\ 1 \end{pmatrix} \begin{pmatrix} -a \\ -a^{-1} \end{pmatrix} \).
Now we compute $W_1^0(g_1)$ and $W_2^0(g_2)$ for $a \in N_{E/F}(E^\times)$ as follows: s

$$W_1^0(g_1) = \begin{pmatrix} w_{\xi,\psi} \begin{pmatrix} -a^{-1} & a \\ 1 & a^{-1} \end{pmatrix} \varphi_0^\xi \end{pmatrix}(1_E)$$

$[5.31]$ \quad $|a^{-1}|^{1/2} \xi(b) \begin{pmatrix} a \\ a^{-1} \end{pmatrix} \varphi_0^\xi (bi)$

$[5.29]$ \quad $|a|^{1/2} \xi(bi) \chi(a) \varphi_0^\xi (abi)$. 

Since by definition of $\varphi_0^\xi$ in $(5.33)$, we have

$$\varphi_0^\xi (abi) = \begin{cases} \xi^{-1}(abi) & \text{if } abi \in O_E^\times, \\ 0 & \text{otherwise}. \end{cases}$$

And by the choice of $b$, we have $abi \in O_E^\times$ if and only if $a \in O^\times$. Thus,

$$W_1^0(g_1) = \begin{cases} |a|^{1/2} \xi^{-1}(a) \chi(a) & \text{if } a \in O^\times, \\ 0 & \text{otherwise}. \end{cases}$$

Similarly, using $\xi^{-1}$ instead of $\xi$, we have,

$$W_2^0(g_2) = \begin{cases} |a|^{1/2} \xi^{-1}(b) \chi(-a) \varphi_0^\xi (-ab) \\ |a|^{1/2} \xi(-a) \chi(a) & \text{if } a \in O^\times, \\ 0 & \text{otherwise}. \end{cases}$$

So,

$$W_1^0(g_1)W_2^0(g_2) = \begin{cases} |a|\xi(-1) & \text{if } a \in N_{E/F}(E^\times) \cap O^\times, \\ 0 & \text{otherwise}. \end{cases}$$

Thus,

$$\tilde{I}_{w_1} = \xi(-1) \int_{F^\times} \Xi_{O^\times}(a)|a|^{1-s}d^xa = \xi(-1), \quad (5.38)$$

as desired. □
Corollary 5.2.4. For $\Psi_l^0$ as in (5.15),

$$I(s, W_1^0, W_2^0, \Psi_l^0) = A V_{2e} q^{-2l(1-s)} q^{ns} \left( \frac{1 - q^{-2(1-s)}}{1 - q^{-2s}} \right).$$

Proof. Note that $\omega_x(-1) = \xi(-1)$. This follows from the Functional equation (5.2) and Lemma 5.2.3. 

Finally, we are ready to prove Theorem 5.2.1.

Proof. Let $X = q^s$. Taking $l = 0$ in (5.27) gives

$$\frac{I(s, W_1^0, W_2^0, \Phi_0^0)}{L(s, \pi_1 \times \pi_2)} = V_{2e}(1 - X^{-2}).$$

From Lemma 5.2.2, we know that the conductor of the pair $(\pi_1, \pi_2)$ is even. If we set $l = -\frac{n}{2} - 1$ for $\Psi_l^0$, then Corollary 5.2.4 gives

$$\frac{I(s, W_1^0, W_2^0, \Psi_{-\frac{n}{2}+1})}{L(s, \pi_1 \times \pi_2)} = A V_{2e} q^{n+2} (X^{-2} - q^{-2}).$$

Thus, let $\Phi' = A q^{n+2} \Phi_0^0 + \Psi_{-\left(\frac{n}{2}+1\right)}$, then the corresponding local integral becomes

$$\frac{I(s, W_1^0, W_2^0, \Phi')}{L(s, \pi_1 \times \pi_2)} = A V_{2e} q^{n+2} (1 - q^{-2}),$$

which is nonzero as desired.

Finally, if we normalize $\Phi$ by $\Phi = \frac{\Phi'}{A V_{2e} q^n (q^2 - 1)}$, then

$$I(s, W_1^0, W_2^0, \Phi) = L(s, \pi_1 \times \pi_2).$$

Therefore, the pair $(\pi_1, \pi_2)$ is optimal 1-regular. 

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5.2.3  Supercuspidal representation $\pi_1$ attached to $(\xi, E/F)$; $E/F$ ramified

If $E/F$ ramified, then from [6] we know that $\pi_1 = \rho(E/F, \xi)$ is totally ramified, that is, for any unramified quasicharacter $\mu \neq 1$ of $F^\times$, $\pi_1 \not\cong \pi_1 \otimes \mu \circ \det$. Thus, by [12], we have

$$L(s, \pi \times \tilde{\pi}) = \frac{1}{1-q^{-s}} \quad \text{and} \quad L(1-s, \pi \times \tilde{\pi}) = \frac{1}{1-q^{-1+s}}.$$

**Theorem 5.2.5.** Let $\pi_1 = \rho(E/F, \xi)$ with a ramified field extension $E/F$ and $\pi_2 = \tilde{\pi}_1$. Then the pair $(\pi_1, \pi_2)$ is optimal 1-regular if the conductor $n$ of the pair $(\pi_1, \pi_2)$ is odd. And we can choose an appropriate $\Phi \in S(F^2)$ so that

$$I(s, W_1^0, W_2^0, \Phi) = L(s, \pi_1 \times \pi_2).$$

Recall that if $q$ is odd, then $N = c(\pi_1) = c(\xi) + 1$ by (5.37).

**Proposition 5.2.6.** Let $X = q^s$ and $\mathcal{A} = e^{-(1/2, \pi_1 \times \pi_2, \psi)}q^{-n/2}$, then

$$I(s, W_1^0, W_2^0, \Psi_i^0) = \chi(-1)AV_Nq^{-2l} \frac{X^{n+2l}(1 - q^{-1}X)}{1 - X^{-1}}.$$

**Proof.** From the definition (5.15) of $\Psi_i^0 \in S(F^2)$ and the Functional equation (5.2), we have

$$I(s, W_1^0, W_2^0, \Psi_i^0) = \omega_{\pi_1}(-1)Aq^{ns}I(1-s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi_i^*).$$

Note that $\omega_{\pi_1}(-1) = \chi(-1)\xi(-1)$. By the equation (5.16),

$$I(1-s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi_i^*) = V_Nq^{-2(1-s)}I_{w_1}.$$

Now our problems reduces to compute $I_{w_1}$. 69
Let \( g_1 = w \begin{pmatrix} a^{-1} \\ 1 \end{pmatrix} w_1 \) and \( g_2 = w \begin{pmatrix} -a^{-1} \\ 1 \end{pmatrix} w_1 \). As before we compute the integrands \( W_0^0(g_1) = f(g_1)(1_E) \) and \( W_0^0(g_2) = \tilde{f}(g_2)(1_E) \) from (5.35). By definition of \( f \) and \( \tilde{f} \), we separate the case \( \det(g_1) = -a^{-1} \) and \( \det(g_2) = a^{-1} \) is in \( N_{E/F}(E^\times) \) or not. First, assume that \( -1 \in N_{E/F}(E^\times) \), that is, \( N_{E/F}(i) = -1 \) for \( i \in E^\times \). Then 

- If \( \det(g_1) \in N_{E/F}(E^\times) \), that is, \( a \in N_{E/F}(E^\times) \) such that \( N_{E/F}(b) = a^{-1} \), then same as the case \( E/F \) unramified:

\[
W_0^0(g_1) = \begin{cases} 
|a|^{1/2} \xi^{-1}(a) \chi(a) & \text{if } a \in \mathcal{O}^\times, \\
0 & \text{otherwise},
\end{cases}
\]

and

\[
W_0^0(g_2) = \begin{cases} 
|a|^{1/2} \xi(-a) \chi(a) & \text{if } a \in \mathcal{O}^\times, \\
0 & \text{otherwise},
\end{cases}
\]

So,

\[
W_0^0(g_1)W_0^0(g_2) = \begin{cases} 
|a| \xi(-1) & \text{if } a \in \mathcal{O}^\times, \\
0 & \text{otherwise}
\end{cases} = \begin{cases} 
\xi(-1) & \text{if } a \in \mathcal{O}^\times, \\
0 & \text{otherwise}.
\end{cases}
\]

- If \( \det(g_1) \not\in N_{E/F}(E^\times) \), that is, \( a \not\in N_{E/F}(E^\times) \), then

\[
W_0^0(g_1) = f(g_1)(1_E) \begin{pmatrix} x \\ 1 \end{pmatrix} (\omega_{\xi,\psi}(g_1 t^{-1}) \varphi^g_\xi)(1_E), \quad \text{where } t = \begin{pmatrix} x \\ 1 \end{pmatrix} \text{ with } x \in \mathcal{O}^\times \text{ and } x \not\in N_{E/F}(E^\times). \]

So
\[ W_1^0(g_1) = \begin{pmatrix} w_{\xi,\psi} & -x^{-1} \\ a^{-1} & \varphi_\xi^0 \end{pmatrix} \begin{pmatrix} 1_E \end{pmatrix} \]
\[ = \begin{pmatrix} w_{\xi,\psi} & -(ax)^{-1} \\ -1 & a^{-1} \end{pmatrix} \begin{pmatrix} \varphi_\xi^0 \end{pmatrix} \begin{pmatrix} 1_E \end{pmatrix}. \]

Let \( d \in E^\times \) such that \( N_{E/F}(d) = (ax)^{-1} \), then
\[ W_1^0(g_1) \mid -ax|^{-1/2}\xi(di) \begin{pmatrix} w_E & a \\ a^{-1} & \varphi_\xi^0 \end{pmatrix} (di) \]
\[ = |a|^{1/2}\xi(di)\chi(a)\varphi_\xi^0(adi). \]

Since by the definition \( \varphi_\xi^0 \) in (5.33), we have
\[ \varphi_\xi^0(adi) = \begin{cases} \xi^{-1}(adi) & \text{if } adi \in \mathcal{O}_E^\times, \\ 0 & \text{otherwise.} \end{cases} \]

And by the choice of \( d \), we have \( adi \in \mathcal{O}_E^\times \) if and only if \( a \in \mathcal{O}^\times \). Thus,
\[ W_1^0(g_1) = \begin{cases} |a|^{1/2}\xi(a^{-1})\chi(a) & \text{if } a \in \mathcal{O}^\times, \\ 0 & \text{otherwise.} \end{cases} \]

Similarly, \( \det(g_2) \notin N_{E/F}(E^\times) \), so with \( t \) as above,
\[ W_2^0(g_2) = \tilde{f}(g_2)(1_E) = (w_{\xi^{-1},\psi}(g_2t^{-1})\varphi_{\xi^{-1}}^0)(1_E). \]

So
\[ W_2^0(g_2) = \begin{cases} |a|^{1/2}\xi(-a)\chi(a) & \text{if } a \in \mathcal{O}^\times, \\ 0 & \text{otherwise.} \end{cases} \]
Then
\[ W_1^0(g_1)W_2^0(g_2) = \begin{cases} \xi(-1) & \text{if } a \in \mathcal{O}^\times, \\ 0 & \text{otherwise.} \end{cases} \]

Thus, for any \( a \in F^\times \), we have
\[ W_1^0(g_1) = \begin{cases} |a|^{1/2}\xi^{-1}(a)\chi(a) & \text{if } a \in \mathcal{O}^\times, \\ 0 & \text{otherwise,} \end{cases} \]
and
\[ W_2^0(g_2) = \begin{cases} |a|^{1/2}\xi(-a)\chi(a) & \text{if } a \in \mathcal{O}^\times, \\ 0 & \text{otherwise,} \end{cases} \]

Now if \(-1 \notin N_{E/F}(E^\times)\), then \( \det(g_1) \in N_{E/F}(E^\times) \) if and only if \( a \notin N_{E/F}(E^\times) \) if and only if \( \det(g_2) \notin N_{E/F}(E^\times) \). By the above computation, we also have
\[ W_1^0(g_1)W_2^0(g_2) = \begin{cases} \xi(-1) & \text{if } a \in \mathcal{O}^\times, \\ 0 & \text{otherwise.} \end{cases} \]

Finally, we have
\[ \tilde{I}_w_1 = \int_{E^\times} |a|^{1-s}W_1^0(g_1)W_2^0(g_2)d^\times a \]
\[ = \int_{\mathcal{O}^\times} |a|^{1-s}\xi(-1)d^\times a \]
\[ = \xi(-1). \]

Therefore,
\[ I(s, W_1^0, W_2^0, \Psi_1^0) = \chi(-1)\mathcal{A}_q^nV_Nq^{-2(1-s)} \left( \frac{1 - q^{-(1-s)}}{1 - q^{-s}} \right), \]
and by letting \( X = q^s \), we are done. \( \square \)

We are now in a position to prove the Theorem 5.2.5.
Proof. Assume that \( n = c(\pi_1 \times \pi_2) \) is odd. Let \( X = q^s \), then \( L(s, \pi_1 \times \pi_2) = \frac{1}{1 - X^{-1}} \).

Recall that \( I(s, W^0_1, W^0_2, \Phi^0) = V_N q^{-2s} \) from (5.27). Consider

\[
\Phi' = (Aq^{n+1}\chi(-1))\Phi^0 + \Psi_0^{0\left(\frac{n+1}{2}\right)}.
\]

Then we have

\[
I(s, W^0_1, W^0_2, (Aq^{n+1})\Phi^0) = \frac{AV_N q^{n+1}\chi(-1)}{1 - X^{-1}}(1 - X^{-1}),
\]

and by Proposition 5.2.6

\[
I(s, W^0_1, W^0_2, \Psi_0^{0\left(\frac{n+1}{2}\right)}) = AV_N q^{n+1}(X^{-1} - q^{-1})
\]

So

\[
I(s, W^0_1, W^0_2, \Phi') = \frac{AV_N q^n(q - 1)\chi(-1)}{1 - X^{-1}}.
\]

Thus, letting \( \Phi = \frac{\Phi'}{AV_N(q - 1)q^n\chi(-1)} \) gives

\[
I(s, W^0_1, W^0_2, \Phi) = L(s, \pi_1 \times \pi_2).
\]

Thus, the pair \( \pi_1, \pi_2 \) is optimal 1-regular if \( n = c(\pi_1 \times \pi_2) \) is odd. \( \square \)

5.3 Pair of induced representations

5.3.1 Preliminaries

We review the induced representation of \( GL(2) \) first. Let \( \pi = Ind(\chi_1 \otimes \chi_2) \) be an induced representation of \( GL(2) \), then \( f \in V_\pi \) means that for all \( g \in GL(2), a, d \in F^\times \), and \( x \in F \), a locally constant function \( f \) on \( GL(2) \) satisfies

\[
f \left( \begin{pmatrix} a & x \\ 0 & d \end{pmatrix} g \right) = \chi_1(a) \chi_2(d) |a/d|^{1/2} f(g).
\] (5.39)
From [2], $\text{Ind}(\chi_1 \otimes \chi_2)$ is irreducible if and only if $\chi_1 \chi_2^{-1} \neq \nu \pm 1$. We call this representation a \textit{full-induced representation}. If $\text{Ind}(\chi_1 \otimes \chi_2)$ is reducible, say $\chi_1 \chi_2^{-1} = \nu$, then there is unique irreducible generic quotient representation, we take $\pi$ as an irreducible generic quotient of $\text{Ind}(\chi_1 \otimes \chi_2)$ and denote by $\pi = [\chi_2, \chi_1] = \text{St}_2(\chi_2 \nu^{1/2})$. We call this representation a \textit{special representation} or twisted Steinberg representation by $\chi_2 \nu^{1/2}$ of length 2. Casselman computed the conductor $N = c(\pi_1)$ of the induced representation $\pi_1 \in \text{Irr}(\text{GL}(2))$ in [7, Section 1] as follows:

$$N = c(\pi_1) = \begin{cases} 
c(\chi_1) + c(\chi_2) & \text{if } \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \text{ with } \chi_1 \chi_2^{-1} \neq \nu \pm 1, \\
1 & \text{if } \pi_1 = \text{St}_2(\chi) \text{ and } \chi \text{ is unramified}, \\
2c(\chi) & \text{if } \pi_1 = \text{St}_2(\chi) \text{ and } \chi \text{ is ramified.} 
\end{cases} \quad (5.40)$$

For the pair $(\pi_1, \pi_2)$ of induced representations of $\text{GL}(2)$, Gebart and Jacquet [12, Proposition (1.4)] computed the $L(s, \pi_1 \times \pi_2)$ and $\epsilon$-factor as follows:

1. For $\pi_2 = \text{Ind}(\mu_1 \otimes \mu_2)$ and any $\pi_1 \in \text{Irr}(\text{GL}(2))$,

$$\begin{align*}
L(s, \pi_1 \times \pi_2) &= L(s, \pi_1 \otimes \mu_1) L(s, \pi_1 \otimes \mu_2) \\
\epsilon(s, \pi_1 \times \pi_2, \psi) &= \epsilon(s, \pi_1 \otimes \mu_1, \psi) \epsilon(s, \pi_1 \otimes \mu_2, \psi). 
\end{align*} \quad (5.41)$$

2. For $\pi_2 = [\mu_2, \mu_1]$ with $\mu_1 = \mu_2 \nu$, which is a unique irreducible quotient of $\text{Ind}(\mu_2 \otimes \mu_1)$ and $\pi_1 = [\chi_2, \chi_1]$,

$$\begin{align*}
L(s, \pi_1 \times \pi_2) &= L(s, \mu_2 \otimes \chi_1) L(s, \mu_1 \otimes \chi_1) \\
\epsilon(s, \pi_1 \times \pi_2, \psi) &= \epsilon(s, \pi_1 \otimes \mu_1, \psi) \epsilon(s, \pi_1 \otimes \mu_2, \psi). 
\end{align*} \quad (5.42)$$
Let $\pi_1 = Ind(\chi_1 \otimes \chi_2)$ be a full induced representation, that is, $\chi_1\chi_2^{-1} \neq \nu^{\pm 1}$ and $\pi_2 = \tilde{\pi}_1$, then the equation (5.41) with
\[
\pi_1 \otimes \chi_1^{-1} \cong Ind(1 \otimes \chi_2\chi_1^{-1}), \quad \text{and} \quad \pi_1 \otimes \chi_2^{-1} \cong Ind(\chi_1\chi_2^{-1} \otimes 1),
\]
gives

• if $\chi_1\chi_2^{-1} = \nu^t$ for some $t \in \mathbb{C}$ with $t \neq \pm 1$, then $L$-function of a pair is following:
\[
L(s, \pi_1 \times \pi_2) = L^2(s, 1)L(s, \nu^t)L(s, \nu^{-t}) = \frac{1}{(1-q^{-s})(1-q^{-s+t})(1-q^{-s-t})},
\]
\[
L(1-s, \pi_1 \times \pi_2) = L^2(1-s, 1)L(1-s, \nu^t)L(1-s, \nu^{-t}) = \frac{1}{(1-q^{-1+s})(1-q^{-1+s-t})(1-q^{-1+s+t})}.
\]

And the $\epsilon(s, \pi_1 \times \pi_2) = 1$ since
\[
\epsilon(s, \pi_1 \otimes \chi_1^{-1}, \psi) = \epsilon(s, \pi_1 \otimes \chi_2^{-1}, \psi) = 1.
\]

• if $\chi_1\chi_2^{-1} \neq \nu^t$ for any $t \in \mathbb{C}$,
\[
L(s, \pi_1 \times \pi_2) = \frac{1}{(1-q^{-s})^2} \quad \text{and} \quad L(1-s, \pi_1 \times \pi_2) = \frac{1}{(1-q^{-1+s})^2},
\]

And the $\epsilon(s, \pi_1 \times \pi_2) = \chi_1\chi_2^{-1}(-1)q^{-2f(s-1/2)}$ where $f = c(\chi_1\chi_2^{-1})$ since
\[
\epsilon(s, \pi_1 \otimes \chi_1^{-1}, \psi) = \epsilon(s, \chi_1^{-1}\chi_2, \psi) \quad \text{and} \quad \epsilon(s, \pi_1 \otimes \chi_2^{-1}, \psi) = \epsilon(s, \chi_1\chi_2^{-1}, \psi).
\]

Let $\pi_1 = [\chi^{\nu^{-1/2}}, \chi^{\nu^{1/2}}]$ be a special representation and $\pi_2 = \tilde{\pi}_1 = [\chi^{-1}\nu^{-1/2}, \chi^{-1}\nu^{1/2}]$, then the equation (5.42) with
\[
\pi_1 \otimes \chi^{-1}\nu^{1/2} \cong [1, \nu] \quad \text{and} \quad \pi_1 \otimes \chi^{-1}\nu^{-1/2} \cong [\nu^{-1}, 1],
\]

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gives

\[ L(s, \pi_1 \times \pi_2) = \frac{1}{(1 - q^{-s})(1 - q^{-1-s})}, \quad L(1 - s, \pi_1 \times \pi_2) = \frac{1}{(1 - q^{1-s})(1 - q^{-2+s})}. \]

And the \( \epsilon(s, \pi_1 \times \pi_2, \psi) = q^{-2(s-1/2)} \), it follows from

\[ \epsilon(s, \pi_1 \otimes \chi^{-1} \nu^{1/2}, \psi) = \epsilon(1/2, [1, \nu], \psi)q^{-(s-1/2)}, \]
\[ \epsilon(s, \pi_1 \otimes \chi^{-1} \nu^{-1/2}, \psi) = \epsilon(1/2, [\nu^{-1}, 1], \psi)q^{-(s-1/2)}, \]

and

\[ \epsilon(1/2, [1, \nu], \psi)\epsilon(1/2, [\nu^{-1}, 1], \psi) = \nu(-1) = \omega_{[1, \nu]}(-1) = 1. \]

We summarize the above observation:

**Lemma 5.3.1.** Let \( \pi_1 \) be a generic constituent of an induced representation of \( GL(2) \) with its central character \( \omega_1 \) and its conductor \( N \). Then we can compute \( \epsilon \)-factor of the pair \( (\pi_1, \tilde{\pi}_1) \). In fact, the conductor \( n = c(\pi_1 \times \tilde{\pi}_1) \) and the root number \( \epsilon(1/2, \pi_1 \times \tilde{\pi}_1, \psi) \) of the pair \( (\pi_1, \tilde{\pi}_1) \) are following:

\[
\begin{cases} 
2c(\chi_1 \chi_2^{-1}) & \text{if } \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \text{ with } \chi_1 \chi_2^{-1} \neq \nu^{\pm 1}, \\
2 & \text{if } \pi_1 = \text{St}_2(\chi) = [\chi \nu^{-1/2}, \chi \nu^{1/2}]. 
\end{cases}
\]

And

\[
\epsilon(1/2, \pi_1 \times \tilde{\pi}_1, \psi) = \begin{cases} 
\chi_1 \chi_2^{-1}(-1) & \text{if } \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \text{ with } \chi_1 \chi_2^{-1} \neq \nu^{\pm 1}, \\
1 & \text{if } \pi_1 = \text{St}_2(\chi) = [\chi \nu^{-1/2}, \chi \nu^{1/2}]. 
\end{cases}
\]

Now, for any induced representation \( \pi_1 \) and the pair \( (\pi_1, \tilde{\pi}_1) \) we know \( A = \epsilon^{-1}(1/2, \pi_1 \times \tilde{\pi}_1, \psi)q^{-n/2} \) from the functional equation (5.2) as follows:
\[
\mathcal{A} = \begin{cases} 
\chi_1^{-1}\chi_2(-1)q^{-f} & \text{if } \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \text{ with } \chi_1\chi_2^{-1} \neq \nu \pm 1, \quad f = c(\chi_1\chi_2^{-1}) , \\
q^{-1} & \text{if } \pi_1 = \text{St}_2(\chi) = [\chi\nu^{-1/2}, \chi\nu^{1/2}] .
\end{cases}
\]

(5.45)

Next let \( f \in V_{\pi_1} \) and a nontrivial additive character \( \psi \) of \( F \) with \( c(\psi) = 0 \), we derive the corresponding Whittaker function \( W_f \in W(\pi_1, \psi) \) as

\[
W_f(g) = \Lambda_{\psi}(\pi_1(g)f) = \lim_{i \to \infty} \int_{\mathbb{F}^*} f \left( \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} \right) \psi^{-1}(x) dx ,
\]

(5.46)

where \( dx \) is normalized Haar measure with \( \text{Vol}(\mathcal{O}) = 1 \).

We discussed the new vector formula in the Kirillov model in the Chapter 1. In \cite{22} Proposition 2.1.2 and Lemma 2.2.1, Schmidt computes the explicit formula for the new vector \( f \in V_{\pi} \) for \( \pi \in \text{Irr}(GL(2)) \) and corresponding \( W_f(I_2) = \Lambda_{\psi}(f) \). From this, we define the normalized new vector \( f^0 = f/\Lambda_{\psi}(f) \).

Now if we let \( \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \) with \( c(\psi) = 0 \) and \( f^0 \in V_{\pi_1} \) be its new vector, then we have the following result:

**Lemma 5.3.2.** As a function of \( F \), \( f^0 \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix} \) becomes as follows:

- If both \( \chi_1 \) and \( \chi_2 \) are ramified, then
  \[
f^0 \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix} = \begin{cases} 
q^{n_2/2} \epsilon(1/2, \chi_2, \psi)\chi_1^{-1}(x) & \text{if } \text{ord}(x) = n_2 , \\
0 & \text{otherwise}.
\end{cases}
\]

(5.47)

- If \( \chi_1 \) is unramified but \( \chi_2 \) is ramified, then
  \[
f^0 \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix} = \begin{cases} 
q^{n_2/2} \epsilon(1/2, \chi_2, \psi)\chi_1^{-1}(x^{n_2}) & \text{if } \text{ord}(x) \geq n_2 , \\
0 & \text{otherwise}.
\end{cases}
\]

(5.48)
• If \( \chi_1 \) is ramified but \( \chi_2 \) is unramified, then

\[
f^0 \begin{pmatrix} 1 & x & 1 \\ \end{pmatrix} = \begin{cases} \chi_1^{-1}(x)\chi_2(x)|x|^{-1} & \text{if} \ ord(x) \leq 0 \\ 0 & \text{otherwise} \end{cases},
\]

(5.49)

We also have the useful identity for \( \pi \in \text{Irr}(GL(2)) \) whose central character \( \omega_\pi \) is unramified.

**Lemma 5.3.3.** Let \( \pi \in \text{Irr}(GL(2)) \) be such that \( \omega_\pi \) is unramified. Then, for all \( g \in GL(2) \), we have

\[
W_0^\pi \left( g \begin{pmatrix} 1 \\ \omega^N \end{pmatrix} \right) = \lambda W_0^\pi(g),
\]

where \( \lambda^2 = \omega_\pi^N(\omega) \) and \( N = c(\pi) \).

**Proof.** Note that \( W_\pi^0(g) \) is fixed by \( K_0(N) \) and \( W_\pi^0(1) = 1 \). Let

\[
W_\pi^1(g) = W_\pi^0 \left( g \begin{pmatrix} 1 \\ \omega^N \end{pmatrix} \right),
\]

which is in \( \mathcal{W}(\pi, \psi) \) and for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(N) \), we have

\[
\begin{pmatrix} \omega^{-N} \\ 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \omega^N \end{pmatrix} = \begin{pmatrix} \omega^{-N} \\ 1 \end{pmatrix} \begin{pmatrix} b\omega^N & a \\ d\omega^N & c \end{pmatrix} = \begin{pmatrix} d & c\omega^{-N} \\ b\omega^N & a \end{pmatrix} \in K_0(N),
\]

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so $W_\pi^1(g)$ is fixed by $K_0(N)$. By the uniqueness of the new vector, $W_\pi^1(g) = \lambda W_\pi^0(g)$ for some $\lambda$ and

$$W_\pi^1 \left( g \left( \begin{array}{cc} 1 & 1 \\ \omega^N & \omega^N \end{array} \right) \right) = \lambda W_\pi^0 \left( g \left( \begin{array}{cc} 1 & 1 \\ \omega^N & \omega^N \end{array} \right) \right) = \lambda^2 W_\pi^0(g).$$

On the other hand, by the definition of $W_\pi^1$, we have

$$W_\pi^1 \left( g \left( \begin{array}{cc} 1 & 1 \\ \omega^N & \omega^N \end{array} \right) \right) = W_\pi^0 \left( g \left( \begin{array}{cc} 1 & 1 \\ \omega^N & \omega^N \end{array} \right) \right) = W_\pi^0 \left( g \left( \begin{array}{cc} 1 & 1 \\ \omega^N & \omega^N \end{array} \right) \right) = \omega_\pi^N \omega_\pi W_\pi^0(g).$$

Thus, $\lambda^2 = \omega_\pi^N(\omega)$, and we are done.

5.3.2 A pair of level one induced representations for $GL(2) \times GL(2)$

We call a representation $\pi \in \text{Irr}(GL(n))$ level one if $N = c(\pi) = 1$.

**Theorem 5.3.4.** Let $\pi_1 \in \text{Irr}(GL(2))$ be a level one induced representation and $\pi_2 = \tilde{\pi}_1$. Then the pair $(\pi_1, \pi_2)$ is optimal 1-regular and $I(s, W_1^0, W_2^0, \Phi) = L(s, \pi_1 \times \pi_2)$ for some $\Phi$.

By the formula for the conductor of $\pi_1$ in (5.40), if $c(\pi_1) = 1$ then $\pi_1$ should be either a full-induced representation $\text{Ind}(\chi_1 \otimes \chi_2)$ such that one of $\chi_i$ is unramified and the other is ramified with the conductor 1, or $\pi_1 = \text{St}_2(\chi) = [\chi^{1/2}, \chi^{1/2}]$ with unramified $\chi$. To prove Theorem 5.3.4, we will separate these two cases and use different method to show them.
**Proposition 5.3.5.** Let \( \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \) with \( \chi_1 \) unramified and \( \chi_2 \) ramified with \( c(\chi_2) = 1 \), and \( \pi_2 = \bar{\pi}_1 \). Then the pair \((\pi_1, \pi_2)\) is optimal 1-regular and \( I(s, W_1^0, W_2^0, \Phi) = L(s, \pi_1 \times \pi_2) \) for some \( \Phi \).

In this case, we have the new vectors for \( \pi_1 \) and \( \pi_2 \) in their Kirillov models as in (1.8):

\[
W_1^0 \begin{pmatrix} a \\ 1 \end{pmatrix} = \Xi_{\mathcal{O}}(a) \chi_1(a) |a|^{1/2}, \quad W_2^0 \begin{pmatrix} a \\ 1 \end{pmatrix} = \Xi_{\mathcal{O}}(a) \chi_1^{-1}(a) |a|^{1/2}.
\]

And so

\[
B(s, W_1^0, W_2^0) = \frac{1}{1 - q^{-s}},
\]

which implies that

\[
I(s, W_1^0, W_2^0, \Phi_m) = \frac{V_1 X^{-2l}}{1 - X^{-1}} \text{ where } X = q^s.
\]

Note that

\[
L(s, \pi_1 \times \pi_2) = \frac{1}{(1 - q^{-s})^2} \text{ and } L(1 - s, \pi_1 \times \pi_2) = \frac{1}{(1 - q^{-1+s})^2}.
\]

**Lemma 5.3.6.** Let \( \pi_1 \) be from Proposition 5.3.5 and \( X = q^s \),

\[
I(s, W_1^0, W_2^0, \Phi_m) = \frac{\chi_2(-1) q^{-2m} V_1 X^{2m + 1}}{(1 - X^{-1})^2}.
\]

**Proof.** The functional equation (5.2) and \( \mathcal{A} = \chi_2(-1) q^{-1} \) by (5.45) gives

\[
I(s, W_1^0, W_2^0, \Phi_m) = q^{-1} q^{2s} \left( \frac{1 - q^{-(1-s)}}{1 - q^{-s}} \right)^2 I(1 - s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi_m).
\]
And
\[ I(1 - s, \tilde{W}_1^0, \tilde{W}_2^0, \Phi_m) = V_1 \left( \frac{q^{-2m(1-s)}}{1 - q^{-2(1-s)}} \right) \sum_{k_i \in K/K_0(1)} \tilde{I}_{k_i}, \]
where \( \tilde{I}_{k_i} \) as in (5.14) and by Remark 5.1.2

\[ K/K_0(1) = \left\{ w_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \hat{\alpha} = \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix} : \alpha \in \mathcal{O}/\varpi \mathcal{O} \right\}. \]

Thus,
\[ I(s, W_1^0, W_2^0, \Psi_m) = V_1 q^{-1-2m} q^{2(1+m)s} \frac{1 - q^{-(1-s)}}{(1 - q^{-s})^2} \left( \sum_{k_i \in K/K_0(1)} \tilde{I}_{k_i} \right) \]
\[ = V_1 q^{-2m} q^{(2m+1)s} \frac{1 - q^{-1+s}}{(1 + q^{-1-s})} \sum_{k_i \in K/K_0(1)} \tilde{I}_{k_i}. \]

Now it reduces the problem to compute \( \sum_{k_i \in K/K_0(1)} \tilde{I}_{k_i} = \chi_2(-1) \frac{1 + q^{1-s}}{1 - q^{-1+s}}. \)

- For \( \tilde{I}_{w_1} \) as in (5.14):

\[ W_1^0 \left( w \left( a^{-1} \begin{array}{c} \cdot \end{array} 1 \right) w_1 \right) = W_1^0 \left( \begin{pmatrix} a^{-1} \\ \cdot \\ \cdot \end{pmatrix} \right) \left( -a \begin{array}{c} \cdot \\ \cdot \end{array} 1 \right) \]
\[ = \omega_1(a^{-1}) W_1^0 \left( -a \begin{array}{c} \cdot \\ \cdot \end{array} 1 \right) \]
\[ = \chi_2^{-1}(a) \Xi_{\mathcal{O}}(a) |a|^{1/2}. \]

And similarly,
\[ W_2^0 \left( w \left( -a^{-1} \begin{array}{c} \cdot \end{array} 1 \right) w_1 \right) = \chi_2(-a) \Xi_{\mathcal{O}}(a) |a|^{1/2}. \]
So,
\[ I_{w_1} = \int_{F^\times} \Xi_{\sigma}(a) \chi_2(-1) |a|^{1-s} d^\times a = \frac{\chi_2(-1)}{1-q^{-(1-s)}}. \]

Note that the computation of \( I_{w_1} \) is independent of the conductor of \( \chi_2 \), that is, if \( \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \) with any unramified \( \chi_1 \) and any ramified \( \chi_2 \), the result above is true.

- For \( \tilde{I}_{\tilde{\alpha}} \) as in (5.14), first we consider the integrands \( W_0^0(g_1) \) and \( W_2^0(g_1) \), where

\[
g_1 = \begin{pmatrix} 1 & a^{-1} & 1 \\ 1 & 1 & \alpha \\ 1 & 1 & 1 \end{pmatrix},
\]

and

\[
g_2 = \begin{pmatrix} 1 & -a^{-1} & 1 \\ 1 & 1 & \alpha \\ 1 & 1 & 1 \end{pmatrix}.
\]

Let \( f_0 \in V_{\pi_1} \) and \( \tilde{f}_0 \in V_{\pi_2} \) be the new vectors, then we compute the corresponding new vectors \( W_1^0(g_1) := W^0(f_0) \) and \( W_2^0(g_2) = W_{\tilde{f}_0}(g_2) \) as defined in (5.46):

\[
f^0 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix} g_1 \overset{\text{let} \ f^0(\tilde{g}_1)}{=} \tilde{f}_0(\tilde{g}_1), \quad \tilde{g}_1 = \begin{pmatrix} -a^{-1} \\ \alpha + xa^{-1} & 1 \end{pmatrix}.
\]

Since \( \tilde{g}_1 \) can be rewritten as

\[
\tilde{g}_1 = \begin{pmatrix} -a^{-1} \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ y & 1 \end{pmatrix} \quad \text{with} \quad y = \alpha + xa^{-1},
\]

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\[ f^0(\hat{g}_1) \overset{(5.39)}{=} \chi_1(-a^{-1}) - a^{-1}|1/2 f^0 \begin{pmatrix} 1 \\ y \\ 1 \end{pmatrix}. \]

By the formula (5.48) for \( f^0 \), we have

\[
f^0(\hat{g}_1) = \begin{cases} 
\chi_1^{-1}(a)|a|^{-1/2} q^{n_2/2} \epsilon(1/2, \chi_2, \psi) \chi_1^{-1}(\omega^{n_2}) & \text{if } y \in \omega \sigma, \\
0 & \text{otherwise,}
\end{cases}
\]

where \( y = \alpha + xa^{-1} \). So, for \( x = ay - a\alpha \) and \( n_2 = 1 \), we have

\[
W_1^0(g_1) = \lim_{i \to \infty} \int_{p^{-i}} f^0(\hat{g}_1) \psi^{-1}(x) dx = \chi_1^{-1} \left( a \right) |a|^{-1/2} q^{1/2} \epsilon(1/2, \chi_2, \psi) \chi_1^{-1}(\omega) \int_{\omega \sigma} \psi^{-1}(-a\alpha + ay) d(ay) = \chi_1^{-1} \left( a \right) |a|^{1/2} q^{1/2} \epsilon(1/2, \chi_2, \psi) \chi_1^{-1}(\omega) \psi(a\alpha) \int_{\omega \sigma} \psi^{-1}(ay) dy,
\]

and by (5.19),

\[
W_1^0(g_1) = \begin{cases} 
q^{-1/2} \epsilon(1/2, \chi_2, \psi) \chi_1^{-1}(\omega) \chi_1^{-1} \left( a \right) |a|^{1/2} \psi(a\alpha) & \text{if } \text{ord}(a) \geq -1, \\
0 & \text{otherwise.}
\end{cases}
\]

Similarly, let \( \tilde{f}^0 \in V_{\pi_2} \) be the new vector, then

\[
\tilde{f}^0(\hat{g}_2) = \tilde{f}^0 \begin{pmatrix} a^{-1} \\ 1 \\ z \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} \text{ with } z = \alpha - xa^{-1}.
\]

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And, let \( z = \alpha - xa^{-1} \),

\[
\tilde{f}^0 (\tilde{g}_2) = \frac{1}{|a|^{1/2} \tilde{f}^0} \begin{pmatrix} 1 \\ z \\ 1 \end{pmatrix}
\]

\[
\tilde{f}^0 (\tilde{g}_2) = \chi_2 (a) |a|^{-1/2} \chi_2 (z) \chi^{-1} |z|^{-1} \text{ if } \text{ord}(z) \leq 0,
\]

\[
0 \text{ otherwise.}
\]

So, for \( x = a\alpha - az \), we have

\[
W_2^0 (g_2) = \lim_{i \to \infty} \int \tilde{f}^0 (\tilde{g}_2) \psi^{-1} (x) dx
\]

\[
= \chi_2 (a) |a|^{-1/2} \int_{\text{ord}(z) \leq 0} \chi_2 (z) \chi^{-1} (z) |z|^{-1} \psi^{-1} (a\alpha - az) d(a\alpha - az)
\]

\[
= \chi_1 (a) |a|^{1/2} \psi (-a\alpha) \int_{\text{ord}(z) \leq 0} \chi_2 (az) \chi^{-1} (az) |az|^{-1} \psi (az) d(az)
\]

\[
= \chi_1 (a) |a|^{1/2} \psi (-a\alpha) \sum_{m=0}^{\infty} \int_{\text{ord}(z) \leq 0} \chi_1 (1) \chi_2 (z) |z|^{-1} \psi (z) d(z).
\]

By (5.20), the last integral is nonzero if and only if \(-m + \text{ord}(a) = -c(\chi_1^{-1}) = -1\), in which case it takes the value \( \epsilon (0, \chi_1^{-1}, \psi) = \chi_1 (\varpi) q^{-1/2} \epsilon (1/2, \chi_2, \psi) \).

Thus,

\[
W_2^0 (g_2) = \begin{cases} 
\chi_1 (\varpi) q^{-1/2} \epsilon (1/2, \chi_2, \psi) \chi_1 (a) |a|^{1/2} \psi (-a\alpha) & \text{if } \text{ord}(a) \geq -1, \\
0 & \text{otherwise.}
\end{cases}
\]

Finally, we have

\[
W_1^0 (g_1) W_2^0 (g_2) = \Xi_{\chi_{-1}} (a) q^{-1} \epsilon (1/2, \chi_2, \psi) \epsilon (1/2, \chi_2, \psi) |a|
\]

\[
= \Xi_{\chi_{-1}} (a) \chi_2 (-1) q^{-1} |a|.
\]

Hence for any \( \alpha \in \mathcal{O} / (\varpi) \),

\[
\tilde{I}_{\alpha} = \int_{\varpi^{-1} \mathcal{O}} \chi_2 (-1) q^{-1} |a|^{1-s} d^x a = \chi_2 (-1) q^{-1} \frac{q^{1-s}}{1 - q^{-(1-s)}}.
\]
Thus,
\[
\sum \bar{I}_\alpha = q \bar{I}_\alpha = \frac{\chi_2(-1)q^{1-s}}{1 - q^{-1-s}}.
\]
Therefore, we have
\[
\sum_{k_i \in K/K_0(1)} \bar{I}_{k_i} = \frac{\chi_2(-1)(1 + q^{1-s})}{1 - q^{-1-s}},
\]
as desired. \(\square\)

We are ready to prove Proposition 5.3.5

**Proof.** Recall that
\[
I(s, W_1^0, W_2^0, \Phi_0^0) = \frac{V_1}{1 - X^{-1}}.
\]
By Lemma 5.3.6 we have
\[
I(s, W_1^0, W_2^0, \Psi_{-1}) = \frac{\chi_2(-1)V_1q^2X^{-1}}{(1 - X^{-1})^2}.
\]
Moreover, we have \(\Psi_{-1} = q^2\Phi_1\) by Remark 5.1.3 which implies that
\[
I(s, W_1^0, W_2^0, \Phi_1) = \frac{\chi_2(-1)V_1X^{-1}}{(1 - X^{-1})^2}.
\]
Applying \(\Phi' = \chi_2(-1)\Phi_0^0 + \Phi_1\) to the local integral gives
\[
I(s, W_1^0, W_2^0, \Phi') = \frac{\chi_2(-1)V_1}{(1 - X^{-1})^2} = \chi_2(-1)q^2V_1 L(s, \pi_1 \times \pi_2).
\]
Note that \(V_1 = Vol(K_0(1)) = \frac{1}{q + 1}\). Thus, letting \(\Phi = \frac{(1 + q)\Phi'}{\chi_2(-1)}\) gives
\[
I(s, W_1^0, W_2^0, \Phi) = L(s, \pi_1 \times \pi_2).
\]

Thus, the pair \((\pi_1, \tilde{\pi}_1)\) is optimal 1-regular for a level one full induced representation \(\pi_1\). \(\square\)
Proposition 5.3.7. Let $\pi_1 = St_2(\chi)$ with an unramified $\chi$ and $\pi_2 = \tilde{\pi}_1$. Then the pair $(\pi_1, \pi_2)$ is 1-regular.

Let $\pi_1 = [\chi \nu^{-1/2}, \chi \nu^{1/2}]$, then we have the new vectors for $\pi_1$ and $\pi_2$ in their Kirillov models as in (1.8):

$$W_1^0 \begin{pmatrix} a \\ 1 \end{pmatrix} = \Xi_\mathcal{O}(a)\chi^{1/2}(a)|a|^{1/2} = \Xi_\mathcal{O}\chi|a|,$$

and

$$W_2^0 \begin{pmatrix} a \\ 1 \end{pmatrix} = \Xi_\mathcal{O}(a)\chi^{-1/2}(a)|a|^{1/2} = \Xi_\mathcal{O}\chi^{-1}|a|.$$

And so

$$B(s, W_1^0, W_2^0) \overset{2.3}{=} \int_{F^\times} \Xi_\mathcal{O}(a)|a|^{s+1}d^\times a = \frac{1}{1 - q^{-(1+s)}},$$

which implies that

$$I(s, W_1^0, W_2^0, \Phi_1^0) \overset{5.9}{=} \frac{V_1X^{-2t}}{1 - q^{-1}X^{-1}} \quad \text{with} \quad X = q^s.$$

Note that

$$L(s, \pi_1 \times \pi_2) = \frac{1}{(1 - q^{-1-s})(1 - q^{-s})}, \quad L(1 - s, \pi_1 \times \pi_2) = \frac{1}{(1 - q^{1+s})(1 - q^{2+s})}.$$  

Lemma 5.3.8. Let $\pi_1$ be from Proposition 5.3.7 and $X = q^s$,

$$I(s, W_1^0, W_2^0, \Psi_m) = \frac{V_1X^{2m-1}}{(1 - X^{-1})(1 - q^{-1}X^{-1})}.$$
Proof. First, since the central characters $\omega_1, \omega_2$ of $\pi_1, \pi_2$ are $\chi^2, \chi^{-2}$ respectively, which are unramified, applying Lemma [5.3.3] gives that, for all $g$

\[
W_1^0 \left( g \left( \begin{array}{c} 1 \\
\omega \end{array} \right) \right) = \lambda_1 W_1^0 (g) \quad \text{where } \lambda_1^2 = \chi^2(\omega),
\]

\[
W_2^0 \left( g \left( \begin{array}{c} 1 \\
\omega \end{array} \right) \right) = \lambda_2 W_2^0 (g) \quad \text{where } \lambda_2^2 = \chi^{-2}(\omega).
\]

Note that $(\lambda_1 \lambda_2)^2 = 1$.

We have $I(s, W_1^0, W_2^0, \Phi_m) \overset{(5.11)}{=} \left( \frac{V_1 q^{-2m s}}{1 - q^{-2s}} \right) \sum_{K/K_0(1)} I_{k_i}$. To compute $\sum I_{k_i}$, we consider integrands $W_1^0(g_i)$ and $W_2^0(h_i)$ of each integration $I_{k_i}$ as in (5.12), where for $k_i \in K/K_0(1)$

\[
g_i = \left( \begin{array}{c} a \\ 1 \end{array} \right) k_i \quad \text{and} \quad h_i = \left( \begin{array}{c} -a \\ 1 \end{array} \right) k_i.
\]

- For $k_i = w_1$, by the matrix multiplication

\[
\left( \begin{array}{c} a \\ 1 \end{array} \right) w_1 = \left( \begin{array}{c} -\omega^{-1} \\ -\omega^{-1} \end{array} \right) \left( \begin{array}{c} -a \omega \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ \omega \end{array} \right),
\]

we have

\[
W_1^0 \left( \left( \begin{array}{c} a \\ 1 \end{array} \right) w_1 \right) = \omega_1 (-\omega^{-1}) W_1^0 \left( \left( \begin{array}{c} -a \omega \\ 1 \end{array} \right) \left( \begin{array}{c} 1 \\ \omega \end{array} \right) \right)
\]

\[
= \omega_1 (-\omega^{-1}) \lambda_1 W_1^0 \left( \left( \begin{array}{c} -a \omega \\ 1 \end{array} \right) \right)
\]

\[
= \left\{ \begin{array}{ll}
\lambda_2^2 \lambda_1 \chi(-a \omega) & \text{if } \text{ord}(a) \geq -1, \\
0 & \text{otherwise}.
\end{array} \right.
\]

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Similarly, we have
\[
W_2^0 \left( \begin{pmatrix} -a \\ 1 \end{pmatrix} w_1 \right) = \begin{cases} \lambda_1^2 \lambda_2 \chi^{-1}(a \varpi) |a \varpi| & \text{if } \text{ord}(a) \geq -1, \\ 0 & \text{otherwise}. \end{cases}
\]

Thus,
\[
I_{w_1} = \int_{E^\times} W_1^0 \left( \begin{pmatrix} a \\ 1 \end{pmatrix} w_1 \right) W_2^0 \left( \begin{pmatrix} -a \\ 1 \end{pmatrix} w_1 \right) |a|^{s-1} d^\times a
\]
\[
= \int_{\varpi^{-1}} |\varpi|^2 |a|^{s+1} d^\times a = \sum_{n \geq -1} q^{-n(s+1)} q^{-2}
\]
\[
= \frac{q^{s-1}}{1 - q^{-1-s}}.
\]

- For \( k_i = 0 \),
  \[
  I_0 = B(s, W_1^0, W_2^0) = \frac{1}{1 - q^{-1-s}}.
  \]

- For \( k_i = \hat{\alpha} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \) with \( \alpha \neq 0 \), consider the following matrix multiplication
  \[
  \begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha 1 \end{pmatrix} = \begin{pmatrix} 1 & a/\alpha \\ \alpha 1 \\ -\alpha \end{pmatrix} \begin{pmatrix} -\alpha \\ 1 \end{pmatrix} \begin{pmatrix} a/\alpha^2 \\ 1 \end{pmatrix} w_1 \begin{pmatrix} 1 & \alpha^{-1} \\ \alpha^{-1} \\ 1 \end{pmatrix},
  \]

where \( \begin{pmatrix} 1 & \alpha^{-1} \\ \alpha^{-1} \\ 1 \end{pmatrix} \in K_0(1) \) since \( \alpha \in \mathcal{O}^\times \).

Then,
\[
W_1^0 \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} 1 & a/\alpha \\ \alpha 1 \end{pmatrix} \right) = \psi(a/\alpha) W_1^0 \left( \begin{pmatrix} a/\alpha^2 \\ 1 \end{pmatrix} w_1 \right),
\]
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which becomes
\[
W_1^0 \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \hat{\alpha} \right) = \begin{cases} 
\lambda_2^2 \lambda_1 \chi(a \varpi) \psi(a/\alpha) |a \varpi| & \text{if } \operatorname{ord}(a) \geq -1, \\
0 & \text{otherwise.}
\end{cases}
\]

And
\[
W_2^0 \left( \begin{pmatrix} -a \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \hat{\alpha} \right) = \begin{cases} 
\lambda_1^2 \lambda_2 \chi^{-1}(a \varpi) \psi(-a/\alpha) |a \varpi| & \text{if } \operatorname{ord}(a) \geq -1, \\
0 & \text{otherwise.}
\end{cases}
\]

Hence, for \( \alpha \neq 0 \), we have
\[
\int_{F^\times} W_1^0 \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \hat{\alpha} \right) W_2^0 \left( \begin{pmatrix} -a \\ 1 \end{pmatrix} \hat{\alpha} \right) |a|^{s-1} d^\times a 
\]
and so
\[
\sum_{\alpha \neq 0} I_{\hat{\alpha}} = (q - 1) I_{\hat{\alpha}} = \frac{(q - 1) q^{s-1}}{1 - q^{-1-s}}.
\]

Finally,
\[
\sum I_{k_1} = \frac{1 + q (q^{s-1})}{1 - q^{-s-1}} = \frac{1 + q^s}{1 - q^{-s-1}}.
\]

Thus,
\[
I(s, W_1^0, W_2^0, \Phi_m) = \left( \frac{V_1 q^{-2ms}}{1 - q^{-2s}} \right) \left( \frac{1 + q^s}{1 - q^{-s-1}} \right) = \left( \frac{V_1 q^{-2ms + s}}{1 - q^{-2s}} \right) \left( \frac{q^{-s} + 1}{1 - q^{-s-1}} \right)
\]
\[
= \left( \frac{V_1 q^{(1-2m)s}}{1 - q^{-s}} \right) \left( \frac{1}{1 - q^{-s-1}} \right).
\]

Now we go back to Proposition 5.3.7 for \( \pi_1 = St_2(\chi) \) with an unramified \( \chi \):
Proof. Recall that

\[ I(s, W_1^0, W_2^0, \Phi_0) = \frac{V_1}{1 - q^{-s-1}}. \]

And by Lemma 5.3.8,

\[ I(s, W_1^0, W_2^0, \Phi_1) = \frac{V_1 q^{-s}}{(1 - q^{-s})(1 - q^{-s-1})}. \]

Note that \( V_1 = Vol(K_0(1)) = \frac{1}{q + 1} \).

If we let \( \Phi = (1 + q)(\Phi_0^0 + \Phi_1) \), then

\[ I(s, W_1^0, W_2^0, \Phi) = \frac{1}{(1 - q^{-s-1})(1 - q^{-s})} = L(s, \pi_1 \times \pi_2). \]

Thus, the pair \((\pi_1, \pi_2)\) is 1-regular if \(\pi_1\) is a special representation twisted by an unramified character. □

Now we are ready to prove Theorem 5.3.4.

Proof. By Proposition 5.3.5 and Proposition 5.3.7, we know that for any level one generic constituent of an induced representation \(\pi \in Irr(GL(2))\), the pair \((\pi, \tilde{\pi})\) is optimal 1 regular. More precisely, if we take \(\Phi = \frac{\Phi_0^0 + \Phi_1}{\omega_{\pi}(-1)V_1}\), then \((W_1^0, W_2^0, \Phi)\) is a test vector for the pair \((\pi, \tilde{\pi})\).

□

5.3.3 A pair \((\pi, \tilde{\pi})\) for \(GL(2) \times GL(2)\) with \(c(\pi) \geq 2\)

Let \(\pi_1\) be a generic constituent of an induced representation with \(N = c(\pi_1) \geq 2\), that is, \(\pi_1 = Ind(\chi_1 \otimes \chi_2)\) or \(\pi_1 = St_2(\chi)\) with a ramified character \(\chi\).
Theorem 5.3.9. Let $\pi_1 = Ind(\chi_1 \otimes \chi_2)$ such that $c(\chi_1\chi_2^{-1}) \leq 1$ or $\pi_1 = St_2(\chi)$ with $\chi$ ramified. Assume that $N \geq 2$ and $\pi_2 = \tilde{\pi}_1$. Then the pair $(\pi_1, \pi_2)$ is 1-regular.

Proof. In this case, $\pi_1$ should be either a full-induced representation $Ind(\chi_1 \otimes \chi_2)$ such that both $\chi_1$ and $\chi_2$ are ramified characters with $c(\chi_1\chi_2^{-1}) \leq 1$ or a special representation twisted by a ramified character $\chi$, that is, $\pi_1 = St_2(\chi) = [\chi^{-1/2}, \chi^{1/2}]$.

Then we have either

$$\pi'_1 \cong \pi_1 \otimes \chi_2^{-1} \cong Ind(\chi_1\chi_2^{-1} \otimes 1), \quad \pi'_2 \cong \pi_2 \otimes \chi_2 \cong Ind(1 \otimes \chi_1^{-1}\chi_2),$$

or

$$\pi'_1 \cong \pi_1 \otimes \chi^{-1} \cong [\nu^{-1/2}, \nu^{1/2}], \quad \pi'_2 \cong \pi_2 \otimes \chi \cong [\nu^{-1/2}, \nu^{1/2}].$$

Notice that for any case, a pair $(\pi'_1, \pi'_2)$ is optimal 1-regular with a test vector $(W^0_{\pi'_1}, W^0_{\pi'_2}, \Phi)$ where $\Phi$ as follows:

$$\Phi = \begin{cases} 
\Phi_0 = Char(O \times O) & \text{if } \pi_1 = Ind(\chi_1 \otimes \chi_2) \text{ with } c(\chi_1\chi_2^{-1}) = 0, \\
\frac{\Phi_0 + \Phi_1}{\chi_1\chi_2^{-1}(-1)V_1} & \text{if } \pi_1 = Ind(\chi_1 \otimes \chi_2) \text{ with } c(\chi_1\chi_2^{-1}) = 1, \\
\frac{\Phi_0 + \Phi_1}{V_1} & \text{if } \pi_1 = St_2(\chi) \text{ with a ramified } \chi.
\end{cases}$$

And $L(s, \pi_1 \times \pi_2) = L(s, \pi'_1 \times \pi'_2)$ by [12, Proposition (1.4)]. Now, as shown in (3.5), we define test vectors for $\pi_1$ and $\pi_2$ respectively,

$$W^0_{1,\chi_2^{-1}}(g) = \chi_2(det(g))W^0_{\pi'_1}(g), \quad W_{2,\chi_2}(g) = \chi_2^{-1}(det(g))W^0_{\pi'_2}(g),$$

and $\Phi$ as above.
Then
\[
I(s, W^0_{1, \chi_2^{-1}}, W^0_{2}, \Phi) = \int_{N_2 \setminus G_2} W^0_{1, \chi_2^{-1}}(g) W_2 \chi_2(g) \Phi(e_2 g) |\text{det}(g)|^s dg
\]
\[
= \int_{N_2 \setminus G_2} \chi_2 (\text{det}(g)) W^0_{\pi_1'}(g) \chi_2^{-1}(g) W^0_{\pi_2'}(g) \Phi(e_2 g) |\text{det}(g)|^s dg
\]
\[
= I(s, W^0_{\pi_1'}, W^0_{\pi_2'}, \Phi) = L(s, \pi_1' \times \pi_2')
\]
\[
= L(s, \pi_1 \times \pi_2).
\]

Thus, we are done. □

From now on, we consider \( \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \) with \( c(\chi_1 \chi_2^{-1}) \geq 2 \). First, we study the case when \( \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \) with unramified \( \chi_1 \) and ramified \( \chi_2 \) with \( c(\chi_2) \geq 2 \).

**Proposition 5.3.10.** Let \( \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2) \) with unramified \( \chi_1 \) and ramified \( \chi_2 \) with \( c(\chi_2) \geq 2 \) and \( \pi_2 = \tilde{\pi}_1 \). Then the pair \( (\pi_1, \pi_2) \) is 2-regular.

**Proof.** Note that
\[
L(s, \pi_1 \times \pi_2) = \frac{1}{(1 - q^{-s})^2} \quad \text{and} \quad L(1 - s, \pi_1 \times \pi_2) = \frac{1}{(1 - q^{-1+s})^2}.
\]

By the equation (5.9), and the fact that \( B(s, W^0_1, W^0_2) = \frac{1}{1 - q^{-s}} \),
\[
I(s, W^0_1, W^0_2, \Phi') = \frac{V_N X^{-2l}}{1 - X^{-1}} \quad \text{with} \quad X = q^s.
\]

Recall that \( \Psi^0_l \) is such that \( \tilde{\Psi}^0_l = \Phi^*_l = \text{Char}(\varpi \mathcal{O}^* \times \varpi^{N+l} \mathcal{O}) \) and by the equation (5.16),
\[
I(1 - s, \tilde{W}^0_1, \tilde{W}^0_2, \Phi'_l) = V_N q^{-2l(1-s)} \tilde{I}_{w_1} = V_N q^{-2l(1-s)} \frac{\chi_2(-1)}{1 - q^{-(1-s)}},
\]

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where \( \tilde{I}_{w_1} \) is from the proof of Lemma 5.3.6. By the Functional equation (5.2), we have
\[
I(s, W_1^0, W_2^0, \Psi_0^0) = \omega_{\pi_1}(-1)Aq^{n^a}\chi_2(-1)V_Nq^{-2l(1-s)}\frac{1-q^{-(1-s)}}{(1-q^{-s})^2},
\]
where
\[
A = \chi_1^{-1}\chi_2(-1)q^{-f} = \chi_2(-1)q^{-N} \quad \text{since} \quad f = c(\chi_1\chi_2^{-1}) = c(\chi_2) = N,
\]
and \( n = c(\pi_1 \times \pi_2) = 2N \). Thus,
\[
I(s, W_1^0, W_2^0, \Psi_0^0) = V_Nq^{-(N+2l)}q^{2(N+\ell)s} \frac{1-q^{-1+s}}{(1-q^{-s})^2}.
\]
For any \( \ell \), we have
\[
I(s, W_1^0, W_2^0, \Phi_{\ell}^0) = V_N X^{-2l(1-X^{-1})} L(s, \pi_1 \times \pi_2),
\]
\[
I(s, W_1^0, W_2^0, \Psi_{\ell}^0) = V_N q^{-(N+2l)} X^{2(N+\ell)}(1-q^{-1}X) L(s, \pi_1 \times \pi_2).
\]
By [16] Theorem 2.7, we know that there is \( W_1' \in \mathcal{W}(\pi_1, \psi) \), \( W_2' \in \mathcal{W}(\pi_2, \psi^{-1}) \) and \( \Phi' \in S(F^2) \) so that \( I(s, W_1', W_2', \Phi') = 1 \).

Let \( Y = X^{-1} = q^{-s} \),
\[
P(Y) := \frac{V_N^{-1}I(s, W_1^0, W_2^0, \Phi_{\ell}^0)}{L(s, \pi_1 \times \pi_2)} = Y^{2l} - Y^{2l+1},
\]
\[
Q(Y) := \frac{V_N^{-1} q^{-N} I(s, W_1^0, W_2^0, \Psi_{\ell}^0)}{L(s, \pi_1 \times \pi_2)} = q^{2l}Y^{2l} - q^{2l-1}Y^{2l-1},
\]
\[
R(Y) := \frac{I(s, W_1', W_2', \Phi')}{L(s, \pi_1 \times \pi_2)} = 1 - 2Y + Y^2.
\]
By the simple algebra, we have
\[
q^2 R(Y) - Q(Y) - (2q^2 - q) P(Y) = -q^2 + q,
\]
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and if we let $\Upsilon_1 = \frac{(2q - 1)}{V_N(q - 1)} \Phi_1^0 + \frac{1}{V_N q^{N+1}(q-1)} \Psi_{0,1}^0$ and $\Upsilon_2 = \frac{q}{q-1} \Phi'$, then

$$I(s, W_1^0, W_2^0, \Upsilon_1) + I(s, W_1', W_2', \Upsilon_2) = L(s, \pi_1 \times \pi_2).$$

Thus, the pair $(\pi_1, \pi_2)$ is $2$-regular if $\pi_1$ is a full-induced representation from unramified $\chi_1$ and ramified $\chi_2$. □

**Theorem 5.3.11.** Let $\pi_1 = \text{Ind}(\chi_1 \otimes \chi_2)$ with $c(\chi_1 \chi_2^{-1}) \geq 2$ ramified. Then the pair $(\pi_1, \pi_2)$ is $2$-regular.

**Proof.** Let $\pi_1 \in \text{Irr}(GL(2))$ such that $N \geq 2$. We only need to take care of the case $\pi_1 = \text{Ind}(\chi_1 \otimes \chi_2)$ with both ramified $\chi_1$ and $\chi_2$ such that $c(\chi_1 \chi_2^{-1}) \geq 2$. But, if we consider $\pi'_1 \cong \pi_1 \otimes \chi_2^{-1} \cong \text{Ind}(\chi_1 \chi_2^{-1} \otimes 1)$ and $\pi'_2 \cong \pi_2 \otimes \chi_2 \cong \text{Ind}(1 \otimes \chi_1^{-1} \chi_2)$, which we already know that a pair $(\pi'_1, \pi'_2)$ is $2$-regular by Proposition 5.3.10. So again using the twisted argument as in the Section 4.3, we complete the proof. □

**Remark 5.3.12.** For any linked pair for $GL(2) \times GL(2)$ related to Lapid’s question, it is left to compute $I(s, W_1^0, W_2^0, \Phi_0)$, which is quite complicated. Once we know what the ratio is for the already proven pair $(\pi_1, \pi_2)$ as $r$-regular with $r \leq 2$, then using Lemma 5.1.4 answers if the pair $(\pi_1, \tilde{\pi}_1)$ can be optimal $1$-regular or not.
CHAPTER 6

L-FUNCTIONS OF A PAIR \((\pi_1, \pi_2)\) FOR \(GL(N) \times GL(2)\)

FOR \(N \geq 3\)

In this chapter we will use the result of the previous two chapters to study the case of the pair \((\pi_1, \pi_2)\) for \(GL(n) \times GL(2)\) \((n \geq 3)\). Since the poles of \(L\)-function of pair \((\pi_1, \pi_2)\) in \(GL(n) \times GL(m)\) are the poles of \(L\)-functions of their derivative pair \((\pi_1^{(n-k)}, \pi_2^{(m-k)})\) for \(GL(k) \times GL(k)\) by [8], it is reasonable for us to consider the pair \((\pi_1^{(n-2)}_i, \pi_2)\) in \(GL(2) \times GL(2)\) and \((\pi_1^{(n-1)}_{1,j_1} \times \pi_2^{(1)}_{2,j_2})\) in \(GL(1) \times GL(1)\) where for \(i = 1, 2\), \(\pi_i^{(k)}\) is an irreducible constituents of \(\pi_i^{(k)}\). Recall that \(\{\pi_1^{(k)}\}\) is the finite set of irreducible constituents of \(k\)-th derivative \(\pi_1^{(k)}\) of \(\pi_1\).

Throughout this chapter, we assume that \(\pi_2 \in Irr(GL(2))\) ramified and \(N\) be the conductor of \(\pi_2\). And we restrict our attention to the case \(\pi_1 \in Irr(GL(n))\) of the form \(\pi_1 = Ind(\Delta_1 \otimes \cdots \otimes \Delta_t)\) with \(\Delta_i = [\rho_i, \cdots, \rho_i, \nu_i^{l-1}]\) as in (1.4) such that \(\rho_i \neq \rho_j\) for all \(i \neq j\). And for any constituent \(\pi_1^{(n-2)}_i\) of \(\pi_1^{(n-2)}\), we assume that a pair \((\pi_1^{(n-2)}_i, \pi_2)\) for \(GL(2) \times GL(2)\) is 1-regular.

The main goal of this chapter is to realize which pair \((\pi_1, \pi_2)\) is 1-regular. To do this, we will separate the types of pairs \((\pi_1, \pi_2)\) depending on \((\pi_1^{(n-2)}_i, \pi_2)\) as a pair for \(GL(2) \times GL(2)\) as follows:

1. a pair \((\pi_1^{(n-2)}_i, \pi_2)\) for \(GL(2) \times GL(2)\) is totally non-linked for every constituent
\( \pi_{1,i}^{(n-2)} \) of \( \pi_{1}^{(n-2)} \). Call such pair \((\pi_1, \pi_2)\) for \( GL(n) \times GL(2) \) \( \mathcal{T}_0 \). Recall that for any totally non-linked pair for \( GL(2) \times GL(2) \) is (optimal) 1-regular by Theorem 4.2.1.

2. a pair \((\pi_{1,i}^{(n-2)}, \pi_2)\) for \( GL(2) \times GL(2) \) is linked 1-regular for some constituents of \( \pi_{1}^{(n-2)} \) and totally non-linked for any other constituent. Call such pair \((\pi_1, \pi_2)\) for \( GL(n) \times GL(2) \) with \( \pi_1 \in \text{Irr}(GL(n)) \) \( \mathcal{T}_1 \). We discussed the linked 1-regular pair for \( GL(2) \times GL(2) \) in the Section 5.2 for the supercuspidal pairs and Section 5.3 for other pairs.

3. pair \((\pi_{1,i}^{(n-2)}, \pi_2)\) for \( GL(2) \times GL(2) \) is partially non-linked for some \( i \) such that the pair is 1-regular. Call such pair \((\pi_1, \pi_2)\) for \( GL(n) \times GL(2) \) \( \mathcal{T}_2 \). We discussed the partially non-linked 1-regular pair in Theorem 4.3.1 when we are in the case \( \mathcal{R}_0 \). So when we consider the pair of \( \mathcal{T}_2 \), we exclude the case \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \).

6.1 A pair \((\pi_1, \pi_2)\) of \( \mathcal{T}_0 \) for \( GL(n) \times GL(2) \), \( n \geq 3 \)

**Proposition 6.1.1.** Let \((\pi_1, \pi_2)\) be a pair in \( \mathcal{T}_0 \) for \( GL(n) \times GL(2) \), \( n \geq 3 \). Then the pair \((\pi_1, \pi_2)\) is optimal 1-regular. In other words, for the fixed new vector \( W_2^0 \) of \( \pi_2 \), there is an explicit Whittaker function \( \widehat{W}_1 \) of \( \pi_1 \) such that

\[
L(s, \pi_1 \times \pi_2) = I(s, \widehat{W}_1, W_2^0).
\]

To get some idea to prove this, let us start with the following explicit example.

**Example 6.1.2.** Let \( \pi_1 = \text{Ind}(\chi_1 \otimes \chi_2 \otimes \chi_3) \in \text{Irr}(GL(3)) \) and \( \pi_2 = St_2(\mu) = [\mu^{-1/2}, \mu^{1/2}] \) with \( c(\pi_2) = 1 \) such that a pair \((\pi_1, \pi_2)\) is of \( \mathcal{T}_0 \), that is, any pair
$(\pi_{1,i}, \pi_2)$ is totally non-linked. Let $W_1^0$ and $W_2^0$ be the new vectors for $\pi_1$ and $\pi_2$ respectively. Note that $I(s, W_1^0, W_2^0) = 0$ by Proposition 2.2.2.

Let $\Phi \in S(F^2)$ such that

$$
\hat{\Phi}(x,y) = \begin{cases} 
(1 + q) & \text{if } x \in \mathcal{O}, y \in \mathcal{O}^\times \\
0 & \text{otherwise.}
\end{cases}
$$

Then define

$$
\hat{W}_1(g) = \int_{F^2} W_1^0 \left( g \left( \begin{array}{c} I_2 \\ I_2^t X \end{array} \right) \right) \Phi(X)dX,
$$

where $X = (x_1, x_2) \in F^2$ and $dX$ is the product of the Haar measures $dx_i$, normalized such that $Vol(\mathcal{O}) = 1$. Then $\hat{W}_1 \in \mathcal{W}(\pi_1, \psi)$ and for any $g \in GL(2)$, we have

$$
\hat{W}_1 \left( \begin{array}{c} g \\ 1 \end{array} \right) = \begin{cases} 
(1 + q)W_1^0 \left( \begin{array}{c} g \\ 1 \end{array} \right) & \text{if the last row of } g \text{ is in } \mathcal{O} \times \mathcal{O}^\times, \\
0 & \text{otherwise.}
\end{cases}
$$

Because if we let $g \in GL(2)$, then by the matrix multiplication

$$
\left( \begin{array}{c} g \\ 1 \end{array} \right) \left( \begin{array}{c} I_2 \\ I_2^t X \end{array} \right) = \left( \begin{array}{c} I_2 \\ I_2^t Y \end{array} \right) \left( \begin{array}{c} g \\ 1 \end{array} \right),
$$

where $Y = (y_1, y_2)$ with $y_1 = g_{11}x_1 + g_{12}x_2$ and $y_2 = g_{21}x_1 + g_{22}x_2$. 

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we have

\[
\widehat{W}_1 \begin{pmatrix} g \\ 1 \end{pmatrix} = \int_{F^2} \psi(g_2) W_1^0 \begin{pmatrix} g \\ 1 \end{pmatrix} \Phi(X) dX
\]

\[
= W_1^0 \begin{pmatrix} g \\ 1 \end{pmatrix} \int_{F} \int_{F} \Phi(x_1, x_2) \psi(g_{21}x_1 + g_{22}x_2) dx_1 dx_2
\]

\[
= W_1^0 \begin{pmatrix} g \\ 1 \end{pmatrix} \widehat{\Phi}(g_{21}, g_{22}) = W_1^0 \begin{pmatrix} g \\ 1 \end{pmatrix} \widehat{\Phi}(e_2 g).
\]

Then

\[
I(s, \widehat{W}_1, W_2^0) = \int_{N_2 \setminus G_2} \widehat{W}_1 \begin{pmatrix} g \\ 1 \end{pmatrix} W_2^0(g) |\det g|^{s-1/2} dg
\]

\[
= \int_{N_2 \setminus G_2} W_1^0 \begin{pmatrix} g \\ 1 \end{pmatrix} \widehat{\Phi}(e_2 g) W_2^0(g) |\det g|^{s-1/2} dg
\]

\[
= \int_{K_2 \setminus A_2} W_1^0 \begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} k \\ 1 \end{pmatrix} \widehat{\Phi}(e_2 ak) W_2^0(ak) |a|^{s-1/2} da dk.
\]

For \( a = \text{diag}(a_1, a_2) \) and \( k \in K_2 \), \( \widehat{\Phi}(e_2 ak) = 0 \) unless \( a_2 \in \mathcal{O} \times \) and \( k \in K_0(N) \).

Note that as a function of \( GL(2) \), both \( \widehat{W}_1 \begin{pmatrix} g \\ 1 \end{pmatrix} \) and \( W_2^0(g) \) are invariant under
\( K_0(1) \subset K_2 \) and \( V_1 = Vol(K_0(1)) = (1 + q)^{-1} \). So,

\[
I(s, \hat{W}_1, W_2^0) = \int_{K_0(1)} \int_{F^\times} W_1^0 \left( \begin{pmatrix} a \\ I_2 \end{pmatrix} \right) \left( \begin{pmatrix} k \\ 1 \end{pmatrix} \right) (1 + q) W_2^0 \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \right) |a|^{s-3/2} d^\times a d k
\]

\[
= V_1(1 + q) \int_{F^\times} W_1^0 \left( \begin{pmatrix} a \\ I_2 \end{pmatrix} \right) W_2^0 \left( \begin{pmatrix} a \\ 1 \end{pmatrix} \right) |a|^{s-3/2} d^\times a
\]

\[
= \int_{F^\times} W_1^0 \left( \begin{pmatrix} a \\ I_2 \end{pmatrix} \right) \Xi_\sigma(a) \pi_2^{(1)}(a) |a|^{s-1} d^\times a
\]

\[
= I(s, W_1^0, \pi_2^{(1)}).
\]

Note that \( \pi_2^{(1)} = \mu \nu^{1/2} \) is an unramified representation of \( GL(1) \) and by the Theorem 2.2.1 we know that the pair \((\pi_1, \pi_2^{(1)})\) is optimal 1-regular such that \( I(s, W_1^0, \pi_2^{(1)}) = L(s, \pi_1 \times \pi_2^{(1)}) \). We also know that \( L(s, \pi_1 \times \pi_2) = L(s, \pi_1 \times \pi_2^{(1)}) \) by the Corollary 1.3.3. Thus,

\[
I(s, \hat{W}_1, W_2^0) = L(s, \pi_1 \times \pi_2),
\]

that is, the pair \((\pi_1, \pi_2)\) is optimal 1-regular.

We can generalize the method we used in Example 6.1.2 to prove Proposition 6.1.1. First, we discuss the modified new vector \( \hat{W}_1 \) for a fixed ramified \( \pi_2 \in Irr(GL(2)) \) with \( N = c(\pi_2) \geq 1 \).

Fix an open compact set \( A = A_1 \times A_2 \subset F^2 \), if we let \( \Phi = \phi_1 \times \phi_2 \) where \( \phi_j \in S(F) \).
such that $\hat{\phi}_j = \text{Char}(A_j)$ and define a modified new vector $\hat{W}_1 \in \mathcal{W}(\pi_1, \psi)$ as (6.1),

$$\hat{W}_1(g) = \int W_1^0 \left( \begin{pmatrix} I_2 & ^tX & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \right) \Phi(X) dX,$$

(6.2)

Then,

$$\hat{W}_1 \left( \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} \right) = W_1^0 \left( \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} \right) \prod_{j=1}^2 \left( \int \phi_j(x_j) \psi(g_{2j}x_j) dx_j \right)$$

$$= W_1^0 \left( \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} \right) \prod_{j=1}^2 \hat{\phi}_j(g_{2j}).$$

Thus,

$$\hat{W}_1 \left( \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} \right) = \begin{cases} W_1^0 \left( \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} \right) & \text{if the last row of } g \text{ is in } A \\ 0 & \text{otherwise} \end{cases}$$

(6.3)

$$= W_1^0 \left( \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} \right) \Psi(e_2g), \quad \Psi = \hat{\phi}_1 \times \hat{\phi}_2.$$

Now, we are ready to prove Proposition 6.1.1.

**Proof.** Let $A = A_1 \times A_2 := \mathcal{O}^N \times \mathcal{O}^\times$, $\hat{\phi}_1(x) = V_N^{-1} \Xi_{A_1}(x)$ and $\hat{\phi}_2(y) = \omega_2^{-1}(y)\Xi_{A_2}(y)$. Define $\hat{W}_1$ as (6.2).

Then for $g \in \text{GL}(2)$, we have as in (6.3),

$$\hat{W}_1 \left( \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} \right) = W_1^0 \left( \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} \right) \Psi(e_2g),$$

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where

\[
\Psi(x, y) = \begin{cases} 
V^{-1}_N \omega_2^{-1}(y) & \text{if } x \in \omega^N O, y \in O^x, \\
0 & \text{otherwise.}
\end{cases}
\]

Then an argument similar to the one used in Example 6.1.2 shows that

\[I(s, \hat{W}_1, W_2^0) = L(s, \pi_1 \times \pi_2).\]

Thus, the pair \((\pi_1, \pi_2)\) is optimal 1-regular. \(\square\)

**Remark 6.1.3.** As mentioned in Remark 4.3.2, any totally non-linked pair \((\pi_1, \pi_2)\) for \(GL(n) \times GL(n)\) is optimal 1-regular. And if we consider pairs \((\pi_1, \pi_2)\) for \(GL(n) \times GL(m)\) with \(n \geq m + 1\) such that the pair \((\pi_{1,i}^{(n-m)}, \pi_2)\) for \(GL(m) \times GL(m)\) is totally non-linked for every constituent \(\pi_{1,i}^{(n-m)}\) of \(\pi_1^{(n-m)}\), then the pair \((\pi_1, \pi_2)\) for \(GL(n) \times GL(m)\) is optimal 1-regular by the same argument which is used in Proposition 6.1.1.

### 6.2 A pair \((\pi_1, \pi_2)\) of \(T_1\) or \(T_2\) for \(GL(n) \times GL(2)\), \(n \geq 3\)

From now on, we restrict our attention to the case when a pair \((\pi_1, \pi_2)\) of type either \(T_1\) (a pair \((\pi_{1,i}^{(n-2)}, \pi_2)\) is linked for some constituents of \(\pi_{1,i}^{(n-2)}\) and totally non-linked for any other constituent) or \(T_2\) (a pair \((\pi_{1,i}^{(n-2)}, \pi_2)\) is partially non-linked for some \(i\)).

First, we need to discuss pull back Whittaker functions from the Whittaker functions of derivatives. The key idea to construct such function is from [16, Section 9]: Let \(\sigma_1\) and \(\sigma_2\) be irreducible generic representations of \(GL(r_1)\) and \(GL(r_2)\) respectively. Then by Rodier [21], we know that \(Ind(\sigma_1 \otimes \sigma_2)\) has the unique Whittaker functional,
so if $\pi$ is the unique generic quotient of $\text{Ind}(\sigma_1 \otimes \sigma_2)$, then $W(\text{Ind}(\sigma_1 \otimes \sigma_2)) = W(\pi)$.

More explanations on the quotient of induced representation can be found in [20].

**Proposition 6.2.1.** [16, Proposition 9.1] Let $\pi$ be the unique generic irreducible quotient of $\text{Ind}(\sigma_1 \otimes \sigma_2)$ with $\sigma_i \in \text{Irr}(\text{GL}(r_i))$ with $n = r_1 + r_2$. For given $\Phi \in S(F^{r_2})$ and $W_2 \in W(\sigma_2, \psi)$, there is a $W \in W(\pi, \psi)$ such that

\[
W \begin{pmatrix} g & 0 \\ I_{r_1} & 1 \end{pmatrix} = W_2(g)\Phi(e_{r_2}g)\left|\det(g)\right|^{r_2/2},
\]

for all $g \in \text{GL}(r_2)$.

### 6.2.1 A pair $(\pi_1, \pi_2)$ with a supercuspidal representation $\pi_2$ for $\text{GL}(n) \times \text{GL}(2)$

Let $\pi_2$ be a supercuspidal representation of $\text{GL}(2)$.

**Proposition 6.2.2.** Let $\pi_2$ be a supercuspidal representation of $\text{GL}(2)$ and $\pi_1 \in \text{Irr}(\text{GL}(n))$ be such that a pair $(\pi_1, \pi_2)$ of type $T_1$. Assume that for any pair $(\pi_1^{(n-2)}, \pi_2)$ is optimal 1-regular. Then the pair $(\pi_1, \pi_2)$ is optimal 1-regular. In particular, we have an explicit Whittaker function $\hat{W}_1 \in W(\pi_1, \psi)$ such that $I(s, \hat{W}_1, W_2^0) = L(s, \pi_1 \times \pi_2)$.

**Proof.** If $n = 3$, then only possible $\pi_1$ such that $(\pi_1, \pi_2)$ is in $T_1$ is $\pi_1 = \text{Ind}(\chi \otimes \rho)$ with $\rho = \tilde{\pi}_2\nu_{s_0}$ for some $s_0$. Note that $\pi_1^{(1)} = \rho$. And in this case, by [8, Corollary, page 63],

\[
L(s, \pi_1 \times \pi_2) = L(s, \chi \times \pi_2)L(s, \rho \times \pi_2) = L(s, \pi_1^{(1)} \times \pi_2).
\]
From the Section 5.2, we have a test vector \((W^0_{\pi_1(1)}, W^0_2, \Phi)\) for \(L(s, \pi_1^{(1)} \times \pi_2)\). With this data, as in the equation (6.4), we have the pull back function \(W_1 \in W(\pi_1, \psi)\) from the new vector \(W^0_{\pi_1(1)}\) as

\[
W_1 \begin{pmatrix} g \\ 1 \end{pmatrix} = W^0_\rho(g)\Phi(e_2g)|det(g)|^{\frac{1}{2}}.
\]

Then

\[
I(s, W_1, W^0_2) = \int_{N_2 \setminus G_2} W_1 \begin{pmatrix} g \\ 1 \end{pmatrix} W^0_2(g)|det(g)|^{s-1/2}dg
\]

For general \(n\), let \(\pi_1 = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_t)\) with \(\Delta_i = [\rho_i, \rho_i \nu, \cdots, \rho_i \nu^{l_i-1}] \in \text{Irr}(GL(r_i))\) as in (1.4). From our assumption that a pair \((\pi_1, \pi_2)\) is in \(T_1\) and \(\pi_2\) is a super-cuspidal representation of \(GL(2)\), there is at least one \(\Delta_j = [\rho_j, \rho_j \nu, \cdots, \rho_j \nu^{l_j-1}] \in \text{Irr}(GL(r_j))\) such that \(r_j\) is a multiple of 2 and \(\rho_j \in \text{Irr}(GL(2))\) is supercuspidal such that \((\rho_j, \pi_2)\) is linked 1-regular. Now we collect all such \(\Delta_j\) and set \(J \subseteq \{1, 2, \cdots, t\}\) such that \((\rho_j, \pi_2)\) is optimal 1-regular as a pair of supercuspidal representations for \(GL(2) \times GL(2)\). Then we have

- \(\pi_1^{(n-2)}\) contains \(\Delta_j^{(r_j-2)}\) as a constituent and \(\Delta_j^{(r_j-2)} = \rho_j \nu^{l_j-1} \in \text{Irr}(GL(2))\) for \(j \in J\) and the pair \((\rho_j \nu^{l_j-1})\) is linked 1-regular.

- \(L(s, \Delta_i \times \pi_2) = 1\) for any \(i \notin J\).
\[ L(s, \Delta_j \times \pi_2) = L(s, \rho_j \nu_{j^{-1}} \times \pi_2) = \prod (1 - \beta_j q^{-s})^{-1} \text{ with the product over all } \beta_j = q^{s_0} \text{ such that } \tilde{\pi}_2 \cong \rho_j \nu_{j^{-1} + s_0} \text{ by } [12]. \]

Thus,

\[
L(s, \pi_1 \times \pi_2) = \prod_{i=1}^{t} L(s, \Delta_{1,i} \times \pi_2) = \prod_{j \in J} L(s, \Delta_{1,j} \times \pi_2)
\]

\[
= \prod_{j \in J} L(s, \Delta_{j}^{(r_{j^{-2}})} \times \pi_2)
\]

\[
= \prod_{j \in J} L(s, \rho_j \nu_{j^{-1}} \times \pi_2).
\]

For \( j_1, j_2 \in J \) with \( j_1 \neq j_2, \beta_{j_1} \neq \beta_{j_2} \) since \( \pi_1 \) is in the general position, so there are some \( \alpha_j \in \mathbb{C} \) such that

\[
\prod_{j \in J} L(s, \rho_j \nu_{j^{-1}} \times \pi_2) = \sum_{j \in J} \alpha_j L(s, \Delta_{1,j} \times \pi_2).
\]

On the other hands, from the Section 5.2 for each linked 1-regular pair \((\rho_j \nu_{j^{-1}}, \pi_2)\) of supercuspidal representations, we have a test vector \((W_0^{\rho_j \nu_{j^{-1}}}, W_2^0, \Phi_j)\) such that

\[
I(s, W_0^{\rho_j \nu_{j^{-1}}}, W_2^0, \Phi_j) = L(s, \rho_j \nu_{j^{-1}} \times \pi_2).
\]

If we let \( \sigma_j = \rho_j \nu_{j^{-1}} \) and \( \sigma_j' = [\rho_j, \rho_j \nu_1, \cdots, \rho_j \nu_{j^{-2}}] \), then \( \Delta_j \) is a generic quotient of \( \text{Ind}(\sigma_j \otimes \sigma_j') \). From Proposition [6.2.1] we can define a pull back function \( W_{\Delta_j} \in W(\Delta_j, \psi) \) from the new vector \( W_0^{\rho_j \nu_{j^{-1}}} \), as in [6.4] such that for \( g \in GL(2) \), we have

\[
W_{\Delta_j} \left( \begin{array}{c} g \\ I_{r_{j^{-2}}} \end{array} \right) = W_0^{\rho_j \nu_{j^{-1}}}(g) \Phi_j(e_2 g) |\det(g)|^{\frac{r_{j^{-2}}}{2}}.
\]

And for each \( j \in J \), if we let \( \Sigma_j = \Delta_j \) and \( \Sigma_j' = \text{Ind}(\Delta_1 \otimes \cdots \otimes \Delta_j \otimes \cdots, \Delta_i) \), then
\( \pi_1 \) is a unique generic quotient of \( \text{Ind}(\Sigma_j' \otimes \Sigma_j) \), again by (6.4), for \( h \in GL(r_j) \) we have

\[
W_{1,j} \begin{pmatrix} g \\ I_{n-r_j} \end{pmatrix} = W_{\Delta_j}(h) \Psi_j(e_{r_j}h)|\text{det}(g)|^{\frac{n-r_j}{2}},
\]

where \( \Psi_j = \text{Char}(\mathcal{O}^{r_j}) \in S(F^{r_j}) \).

Thus, for each \( j \in J \) and \( g \in GL(2) \), we have

\[
W_{1,j} \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} = W_{1,j} \begin{pmatrix} g \\ I_{r_j-2} \\ I_{n-r_j} \end{pmatrix} = W_{\Delta_j} \begin{pmatrix} g \\ I_{r_j-2} \\ I_{n-2} \end{pmatrix} |\text{det}(g)|^{\frac{n-r_j}{2}} = W_{\rho_j l_j-1}^0(g) \Phi_j(e_{2g}) |\text{det}(g)|^{\frac{n-2}{2}}.
\]

And so

\[
I(s, W_{1,j}, W_2^0) = \int_{N_2 \backslash G_2} W_{1,j} \begin{pmatrix} g \\ I_{n-2} \end{pmatrix} W_2^0(g) |\text{det}(g)|^{s-\frac{n-2}{2}} dg = \int_{N_2 \backslash G_2} W_2^0(g) |\text{det}(g)|^{s-\frac{n-2}{2}} dg = I(s, W_2^0, \Phi_j) = L(s, \rho_j l_j-1 \times \pi_2).
\]

Finally, if we choose \( W_1 = \sum_{j \in J} \alpha_j W_{1,j} \), then

\[
I(s, W_1, W_2^0) = \sum_{j \in J} \alpha_j I(s, W_{1,j}, W_2^0) = \sum \alpha_j L(s, \Delta_{1,j} \times \pi_2).
\]

Therefore, the pair \( (\pi_1, \pi_2) \) is optimal 1-regular. \( \square \)
6.2.2 A pair \((\pi_1, \pi_2)\) with a special representation \(\pi_2\) for \(GL(n) \times GL(2)\)

Let \(\pi_2 = St_2(\mu) = [\mu \nu^{-1/2}, \mu \nu^{1/2}]\) be the special representation of \(GL(2)\) and \(\pi_2^{(1)} = \mu \nu^{1/2}\).

**Proposition 6.2.3.** Let \((\pi_1, \pi_2)\) be a pair in \(T_1\) for \(GL(n) \times GL(2)\), \((n \geq 3)\). Then the pair \((\pi_1, \pi_2)\) is 1-regular.

**Proof.** We use the induction on \(n \geq 3\):

1. For \(n = 3\),
   
   (a) If \(\pi_1 = St_3(\chi) = [\chi \nu^{-1}, \chi, \chi \nu]\) with \(\chi \mu = \nu^{s_0}\) for some \(s_0\). Then \(\pi_2^{(1)} = St_2(\chi \nu^{1/2}) = [\chi, \chi \nu]\) and \(L(s, \pi_1 \times \pi_2) = L(s, \pi_1^{(1)} \times \pi_2)\). By assumption for the pair of \(T_1\), the pair \((\pi_1^{(1)}, \pi_2)\) is linked and \(\pi_2 = St_2(\mu)\). By Proposition 5.3.7 and Theorem 5.3.9, we have a test vector \((W_{\pi_1^{(1)}}, W_2, \Phi)\) for the pair \((\pi_1^{(1)}, \pi_2)\). On the other hands, since \(\pi_1\) can be thought as the generic constituent of \(Ind(\chi \nu^{-1} \otimes \pi_1^{(1)})\), by the equation (6.4), we have a pull back Whittaker function \(W_1\) of \(\pi_1\) from a test vector \(W_{\pi_1^{(1)}}\) of \(\pi_1^{(1)}\) and \(\Phi\) from a test vector for the pair \((\pi_1^{(1)}, \pi_2)\) above. Then we have

\[
I(s, W_1, W_2) = \int_{N_2 \backslash G_2} W_1 \left( \begin{array}{c} g \\ 1 \end{array} \right) W_2(g) |\det g|^{s-1/2} dg \\
= \int_{N_2 \backslash G_2} W_{\pi_1^{(1)}}(g) W_2(g) \Phi((0,1)g) |\det g|^{s} dg \\
= I(s, W_{\pi_1^{(1)}}, W_2, \Phi) = L(s, \pi_1^{(1)} \times \pi_2).
\]

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(b) If \( \pi_1 = Ind(\chi_1 \otimes St_2(\chi_2)) \) with \( \chi_2 \mu = \nu^{s_0} \) for some \( s_0 \). Note that \( \pi^{(1)} \) is glued from \( \pi^{(1)}_{1,1} = St_2(\chi_2) \) and \( \pi^{(1)}_{1,2} = Ind(\chi_1 \otimes \chi_2 \nu^{1/2}) \) and \( (\pi^{(1)}_{1,1}, \pi_2) \) is a linked pair. Moreover, by [8] we have

\[
L(s, \pi_1 \times \pi_2) = L(s, \chi_1 \times \pi_2) L(s, St_2(\chi_2) \times \pi_2).
\]

First, we consider the case when all characters \( \chi_1, \chi_2 \) and \( \mu \) are unramified. By Proposition 5.3.7, we have a test vector \( (W^0_{\pi^{(1)}_{1,1}}, W^0_2, Y_1) \) for a linked pair \( (\pi^{(1)}_{1,1}, \pi_2) \) where \( Y_1 = (1 + q)(\Phi_0^0 + \Phi_1) \). Now we have a pull back function \( \widehat{W}_1^1 \) of \( \pi_1 \) from the new vector \( W^0_{\pi^{(1)}_{1,1}} \) of \( \pi^{(1)}_{1,1} \) as

\[
\widehat{W}_1^1 \begin{pmatrix} g \\ 1 \end{pmatrix} = W^0_{\pi^{(1)}_{1,1}}(g) Y_1((0, 1)g)|\text{det}(g)|^{1/2},
\]

since \( \pi_1 = Ind(\chi_1 \otimes \pi^{(1)}_{1,1}) \). Then, we have

\[
I(s, \widehat{W}_1^1, W_2^0) = I(s, W^0_{\pi^{(1)}_{1,1}}, W^0_2, Y_1) = L(s, \pi^{(1)}_{1,1} \times \pi_2)
= L(s, \chi_2 \nu^{-1/2} \times \mu \nu^{1/2}) L(s, \chi_2 \nu^{1/2} \times \mu \nu^{1/2}).
\]

On the other hand, we have the new vector \( W_1^0 \) of \( \pi_1 \), by [1.8],

\[
W_1^0 \begin{pmatrix} g \\ 1 \end{pmatrix} = W^0_{\pi^{(1)}_{1,2}}(g) \Phi(e_2g)|\text{det}(g)|^{1/2},
\]

where \( \Phi = \text{Char}(\mathcal{O} \times \mathcal{O}) \). As in [6.3] with \( \Psi = \text{Char}(\varpi \mathcal{O} \times \mathcal{O}^\times) \), we have a Whittaker function \( \widehat{W}_1^2 \) of \( \pi_1 \) as

\[
\widehat{W}_1^2 \begin{pmatrix} g \\ 1 \end{pmatrix} = W_1^0 \begin{pmatrix} g \\ 1 \end{pmatrix} \Psi(e_2g).
\]

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Now setting $\Upsilon_2 = V_1^{-1}\Psi$ gives
\[
\hat{W}_1^2 \begin{pmatrix} g \\ 1 \end{pmatrix} = W_0^{\pi_1,2}(g) \Upsilon_2(e_2 g) |\text{det}(g)|^{1/2}.
\]
Moreover, $(W_0^{\pi_{1,2}}, W_0^{\pi_2}, \Upsilon_2)$ is a test vector for the totally non-linked pair $(\pi_{1,2}^{(1)}, \pi_2)$ as discussed in Theorem 4.2.1.
Thus,
\[
I(s, \hat{W}_1^2, W_2^0) = I(s, W_0^{\pi_1,2}, W_2^0, \Upsilon_2) = L(s, \pi_{1,2}^{(1)} \times \pi_2)
\]
\[
= L(s, \chi_2 \nu^{1/2} \times \mu \nu^{1/2}) L(s, \chi_1 \times \mu \nu^{1/2}).
\]
If we set $\alpha_1$ and $\alpha_2$ such that $\alpha_1 = \frac{\chi_1(\varpi)}{\chi_1(\varpi) - q^{1/2} \chi_2(\varpi)}$ with $\alpha_1 + \alpha_2 = 1$, where $\alpha_1$ is well-defined by the irreducibility of $\pi_1$, that is, $\chi_1 \chi_2^{-1} \nu^{1/2} \neq \nu^{\pm 1}$. And then letting
\[
\hat{W}_1 = \alpha_1 \hat{W}_1^1 + \alpha_2 \hat{W}_1^2,
\]
gives
\[
I(s, \hat{W}_1, W_2^0) = \alpha_1 I(s, W_1^1, W_2^0) + \alpha_2 I(s, W_2^1, W_2^0)
\]
\[
= L(s, \chi_1 \times \pi_2) L(s, \pi_{1,2}^{(1)} \times \pi_2)
\]
\[
= L(s, \pi_1 \times \pi_2).
\]
Next, we consider the leftover case when only $\chi_1$ is unramified, $\chi_2$ and $\mu$ are ramified, or all characters $\chi_1, \chi_2$ and $\mu$ are ramified. Since we have $L(s, \pi_1 \times \pi_2) = L(s, \pi_{1,2}^{(1)} \times \pi_2)$, take $\alpha_1 = 0$ above, i.e. $\hat{W}_1 = \hat{W}_1^2$, so
\[
I(s, \hat{W}_1^2, W_2^0) = L(s, \pi_1 \times \pi_2).
\]
2. For \( n \geq 4 \): general case

To use the similar argument for the case \( n = 3 \), we separate the pair \((\pi_1, \pi_2)\) in \( T_1 \) into \( T_{1,0} \) and \( T_{1,1} \) depending on the structure of \( \pi_1 = Ind(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t) \) with \( \Delta_i = [\rho_i, \cdots, \rho_i \nu_i^{h_i - 1}] \) as follows:

- If \( \pi_1 \) is the representation of the following form, then denote by a pair \((\pi_1, \pi_2)\) \( T_{1,0} \):
  
  \[
  \begin{align*}
  -\pi_1 &= St_n(\chi) = [\chi \nu^{-\frac{n-1}{2}}, \cdots, \chi \nu^{\frac{n-1}{2}}] \quad \text{or} \\
  -\pi_1 &= Ind(St_{l_1}(\chi) \otimes \Delta_2 \otimes \cdots \otimes \Delta_t) \quad \text{where } l_1 \geq 2, \text{ and for } i \geq 2 \\
  \Delta_i &= [\rho_i, \rho_i \nu^1, \cdots, \rho_i \nu_i^{h_i - 1}] \quad \text{with } \rho_i \in \text{Irr}(GL(m_i)) \quad \text{with } m_i \geq 2.
  \end{align*}
  \]
  (6.5)

- If \( \Delta_1, \Delta_2 \) of \( \pi_1 \) have the following form, then denote by a pair \((\pi_1, \pi_2)\) \( T_{1,1} \):
  
  \[
  \begin{align*}
  -\Delta_1 &= \chi_1, \quad \text{or} \\
  -\Delta_1 &= St_2(\chi_1), \quad \text{and} \\
  -\Delta_1 &= St_{l_2}(\chi_2), \quad l_2 \geq 3 \quad (6.6) \\
  -\Delta_1 &= St_{l_1}(\chi_1), \Delta_2 = St_2(\chi_2), \quad (6.7) \\
  -\Delta_1 &= St_{l_1}(\chi_1), \Delta_2 = St_2(\chi_2), \quad (6.8)
  \end{align*}
  \]

(a) If the pair \((\pi_1, \pi_2)\) in \( T_{1,0} \) for \( GL(n) \times GL(2) \) such that \( \pi_1 \) is the representation of the form (6.5), then \( \pi_1^{(1)} = [\chi \nu^{-\frac{n-1}{2}}, \cdots, \nu^{-\frac{n-1}{2}}, \cdots, \chi \nu^{\frac{n-1}{2}}] = St_{n-1}(\chi \nu^{1/2}) \) for the first case or \( \pi_1^{(1)} = Ind(St_{l_1-1}(\chi \nu^{1/2}) \otimes \Delta_2 \otimes \cdots \otimes \Delta_t) \) for the second case, so the pair \((\pi_1^{(1)}, \pi_2)\) for \( GL(n-1) \times GL(2) \) is in either \( T_{1,0} \) for the first case or \( T_0 \) for the second case. So we use the induction hypothesis for the pair \((\pi_1^{(1)}, \pi_2)\) in \( T_{1,0} \) and we use Proposition 6.2.3 for the pair \((\pi_1^{(1)}, \pi_2)\) in \( T_0 \) to get a test vector \((\hat{W}_{\pi_1}^{(1)}, W_2^0)\) for 1-regular pair.
\((\pi_1^{(1)}, \pi_2)\) in \(GL(n - 1) \times GL(2)\). Note that for both case \(\pi_1\) is a generic quotient of \(Ind(\chi \nu^* \otimes \pi_1^{(1)})\) with \(* = -\frac{n-1}{2}\) or \(-\frac{l_1-1}{2}\). By Proposition 6.2.1 we define \(\hat{W}_1 \in \mathcal{W}(\pi_1, \psi)\) as

\[
\hat{W}_1 \begin{pmatrix} g \\ 1 \end{pmatrix} = \hat{W}_{\pi_1^{(1)}}(g) \Phi_0(e_{n-1}g) |\text{det}(g)|^{1/2},
\]

where \(\Phi_0 = \text{Char}(O^{n-1})\). Then we have

\[
I(s, \hat{W}_1, W_0^0) = I(s, \hat{W}_{\pi_1^{(1)}}, W_2) = L(s, \pi_1^{(1)} \times \pi_2) = L(s, \pi_1 \times \pi_2).
\]

(b) If \((\pi_1, \pi_2)\) in \(\mathcal{T}_{1,1}\) for \(GL(n) \times GL(2)\) such that \(\pi_1\) is the representation of the form (6.6), then \(\pi_1^{(1)} = Ind(\chi_1 \otimes St_{l_2-1}(\chi_2 \nu^{1/2}) \otimes \Delta_3 \otimes \cdots \otimes \Delta_t)\) or \(\pi_1^{(1)} = Ind(St_{l_2}(\chi_2) \otimes \Delta_3 \otimes \cdots \otimes \Delta_t)\), so the pair \((\pi_1^{(1)}, \pi_2)\) for \(GL(n - 1) \times GL(2)\) is also in \(\mathcal{T}_{1,1}\) for the first case or \(\mathcal{T}_0\) for the second case with a test vector \((\hat{W}_1, W_2^0)\). Again for each case, we can think \(\pi_1\) as a generic quotient of \(Ind(\chi_2 \nu^{-l_2-1} \otimes \pi_1^{(1)})\) or \(Ind(\chi_1 \otimes \pi_1^{(1)})\). Thus by Proposition 6.2.1 we define

\[
\hat{W}_1 \begin{pmatrix} g \\ 1 \end{pmatrix} = \hat{W}_{\pi_1^{(1)}}(g) \Phi_0(e_{n-1}g) |\text{det}(g)|^{1/2},
\]

and we have \(I(s, \hat{W}_1, W_2^0) = L(s, \pi_1 \times \pi_2)\) as desired.

(c) If \((\pi_1, \pi_2)\) in \(\mathcal{T}_{1,1}\) such that \(\pi_1\) is the representation of the form (6.7) or (6.8), then \(\pi_1^{(1)}\) has constituents \(\pi_{1,1}^{(1)} = Ind(St_2(\chi_2) \otimes \Delta_3 \otimes \cdots \otimes \Delta_t)\) and \(\pi_{1,2}^{(1)} = Ind(\chi_1 \otimes \chi_2 \nu \otimes \Delta_3 \otimes \cdots \otimes \Delta_t)\), i.e., \((\pi_{1,1}^{(1)}, \pi_2)\) is \(\mathcal{T}_{1,0}\) and \((\pi_{1,2}^{(1)}, \pi_2)\)
is in $\mathcal{T}_0$. Or $\pi_1^{(1)}$ has $\pi_{1,1}^{(1)} = \text{Ind}(\text{St}_{t-1}(\chi_1 \nu) \otimes \Delta_2 \otimes \Delta_3 \otimes \cdots \otimes \Delta_t)$ and $\pi_{1,2}^{(1)} = \text{Ind}(\Delta_1 \otimes \text{St}_{t-2}(\chi_2 \nu) \otimes \Delta_3 \otimes \cdots \otimes \Delta_t)$. Both are in $\mathcal{T}_{1,1}$ with $\pi_{2}$. Then for each $i = 1, 2$, we have 

\[
\hat{W}_1^i \left( \begin{array}{c} g \\ 1 \end{array} \right) = \hat{W}_{1,i}(g)\Phi_0(e_{n-1}g)|\det(g)|^{1/2}
\]

where $\hat{W}_{1,i}$ is a test function of $\pi_{1,i}^{(1)}$, and $\Phi_0 = \text{Char}(\mathcal{O}^{n-1})$.

Finally, if we choose $\hat{W}_1 = \alpha_1 \hat{W}_1^1 + \alpha_2 \hat{W}_1^2$ for some $\alpha_1$ and $\alpha_2$ satisfying $\alpha_1 + \alpha_2 = 1$, then $I(s, \hat{W}_1, W_2^0) = L(s, \pi_1 \times \pi_2)$ as desired.

Thus, for general $n \geq 4$, we are done.

Therefore, the pair $(\pi_1, \pi_2)$ in $\mathcal{T}_1$ for $GL(n) \times GL(2)$, $(n \geq 3)$ is 1-regular. \[ \]

**Proposition 6.2.4.** Let $(\pi_1, \pi_2)$ be a pair in $\mathcal{T}_2$ for $GL(n) \times GL(2)$, $(n \geq 3)$. Then the pair $(\pi_1, \pi_2)$ is 1-regular.

**Proof.** If $(\pi_{1,j}^{(n-2)}, \pi_2)$ is a partially non-linked 1-regular, then for $\sigma_j = \pi_{1,j}^{(n-2)}$, the pair $(\sigma_j, \pi_2)$ is not linked but the pair $(\sigma_j^{(1)}, \pi_2^{(1)})$ is ramified linked. Thus, we only consider the case when $\mu$ is ramified. If we rewrite $\pi_1$ and $\pi_2$ as $\pi_1 = \mu^{-1} \otimes \pi_1'$ and $\pi_2 \cong \pi_2' \otimes \mu$. Then the pair $(\pi_1', \pi_2')$ is in either $\mathcal{T}_0$ or $\mathcal{T}_1$, that is, we have $I(s, \hat{W}_{\pi_1'}, W_{\pi_2'}^0) = L(s, \pi_1' \times \pi_2')$.

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Similar with the twisted new vector in the equation (3.5), we can define Whitaker functions 
\( \hat{W}_1(g) = \mu^{-1}(\det(g))\hat{W}_{\pi_1}(g) \) which is in \( \mathcal{W}(\pi_1, \psi) \) and 
\( \hat{W}_2(g) = \mu(\det(g))W_{\pi_2}^0 \) which is in \( \mathcal{W}(\pi_2, \psi^{-1}) \). Then

\[
I(s, \hat{W}_1, \hat{W}_2) = \int_{N_2 \backslash G_2} \hat{W}_1 \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \hat{W}_2(g) |\det g|^{s-1/2} dg
\]

\[
= \int_{N_2 \backslash G_2} \hat{W}_{\pi_1'} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W_{\pi_2}^0(g) |\det g|^{s-1/2} dg
\]

\[
= L(s, \pi_1' \times \pi_2') = L(s, \pi_1 \times \pi_2).
\]

\( \square \)

So we can conclude that

**Corollary 6.2.5.** For any \( \pi_1 \in \text{Irr}(GL(n)) \) with \( n \geq 3 \), the pair \( (\pi_1, \pi_2) \) is 1-regular if \( \pi_2 \) is the special representation of \( GL(2) \).

### 6.2.3 A pair \((\pi_1, \pi_2)\) with a ramified full induced representation \( \pi_2 \) for \( GL(n) \times GL(2) \)

Let \( \pi_2 = \text{Ind}(\mu_1 \otimes \mu_2) \) such that \( \pi_2^{(1)} = \mu_1 \oplus \mu_2 \) and \( \mu_2 \) ramified.

Recall that a pair \((\pi_1^{(n-2)}, \pi_2)\) is 1-regular in \( GL(2) \times GL(2) \) unless we are in cases \( \mathcal{R}_1, \mathcal{R}_2 \) in the proof of Theorem 4.3.1 or linked as Theorem 5.3.11. Mainly, we will prove the following:

**Proposition 6.2.6.** For any \( \pi_1 \in \text{Irr}(GL(n)) \) with \( n \geq 3 \), \((\pi_1, \pi_2)\) is 1-regular if \((\pi_1^{(n-2)}, \pi_2)\) is 1-regular for any \( j \).
Proof. Let $\pi_1 = Ind(\Delta_1 \otimes \Delta_2 \otimes \cdots \otimes \Delta_t)$ with $\Delta_i = [\rho_i, \rho_i \mu, \cdots, \rho_i \mu^{i-1}]$ as before. First, assume that $\mu_1$ is unramified.

- If $\pi_1 \in Irr(GL(n))$ so that for any $j$, $(\pi_1^{(n-1)}, \mu_2)$ not linked, then $(\pi_1, \pi_2)$ is in $T_0$, so we are done.

- If $(\pi_1^{(n-1)}, \mu_2)$ is linked for some $j$. If there are more than two linked pair $(\pi_1^{(1)}, \mu_2)$, then $\rho$ cannot be unramified character, that is, $\rho$ is either ramified character or supercuspidal representation of $GL(r_i), r_i \geq 2$. Such $(\pi_1, \pi_2)$ is in $T_2$ so that we can rewrite $\pi_1 = \mu_{-1} \otimes \pi'_1$ and $\pi_2 = \mu \otimes \pi'_2$, and a pair $(\pi'_1, \pi'_2)$ is in $T_0$. Thus, we can use the twisted technique like before. In other words, from the test vector $(\hat{W}_{\pi'_1}, W_{\sigma'_2}^0)$ of 1-regular pair $(\pi'_1, \pi'_2)$, if we define

$$\hat{W}_1(g) = \mu_2(det(g))\hat{W}_{\pi'_1}(g), \quad \hat{W}_2(h) = \mu_2^{-1}(det(h))\hat{W}_{\pi'_2}(h).$$

Then, $I(s, \hat{W}_1, \hat{W}_2) = L(s, \pi_1 \times \pi_2)$.

- If there is only one linked pair $(\pi_1^{(n-1)}, \mu_2)$, say $\pi_1^{(n-1)} = \chi_1$ and $\pi_1^{(n-1)} := \chi_2$ is unramified character, then $Ind(\chi_1 \otimes \chi_2)$ is a constituent of $\pi_1^{(n-2)}$. In this case, each $\rho$ is ramified with $(\rho, \mu_2)$ not linked or supercuspidal. Moreover, a pair $(Ind(\chi_1 \otimes \chi_2), \pi_2)$ must be linked in $GL(2) \times GL(2)$. If not, the above pair is the case, $R_1$ or $R_2$ in the proof of Theorem 4.3.1, which is 2-regular. Such pair $(\pi_1, \pi_2)$ is in $T_1$, so that we can use the pull back function like before.

Define

$$\hat{W}_1 \left( \begin{array}{cc} g & \mu_2 \\ I_{n-2} & \end{array} \right) = W_{Ind(\chi_1 \otimes \chi_2)}^0(g) \Phi(e_2 g) |det(g)|^\frac{n^2}{2},$$

(6.10)
where $\Phi$ is from the linked pair $(Ind(\chi_1 \otimes \chi_2), \pi_2)$ as in the Section 5.3.1., then
$I(s, \widehat{W}_1, W_2^0) = L(s, \pi_1 \times \pi_2)$.

- If there is only one linked pair $(\pi_1^{(1)}, \mu_2)$ and $\rho_i$ is not unramified character for any $i$, then such $(\pi_1, \pi_2)$ is in $T_2$ so that so that we can rewrite $\pi_1 = \mu_2^{-1} \otimes \pi'_1$ and $\pi_2 = \mu_2 \otimes \pi'_2$, and a pair $(\pi'_1, \pi'_2)$ is $T_0$ as before. So in this case, we can find a test vector $(\widehat{W}_1, \widehat{W}_2)$ of a pair $(\pi_1, \pi_2)$ as in the equation (6.9).

Now, assume that $\mu_1$ is also ramified.

- If a pair $(\pi_1, \pi_2)$ is in $T_0$, then by the Proposition $\text{6.1.1}$ we are done.

- If a pair $(\pi_1, \pi_2)$ is in $T_1$, i.e., $\pi^{(n-2)}_{1,1}$ such that $(\pi^{(n-2)}_{1,1}, \pi_2)$ linked and $(\pi^{(n-2)}_{1,j}, \pi_2)$ is totally non-linked for any other constituent $\pi^{(n-2)}_{1,j}$ of $\pi^{(n-2)}_1$. Then, we can define a test vector $(\widehat{W}_1, W_2^0)$, where $\widehat{W}_1$ as in the equation (6.10) and $\Phi$ is from the linked pair $(\pi^{(n-2)}_{1,1}, \pi_2)$ as in the Section 5.3.2.

- If a pair $(\pi_1, \pi_2)$ is in $T_2$, then again we can consider new pair $(\pi'_1, \pi'_2)$, which is in $T_0$, then we can find a test vector like (6.9).

Therefore, the pair $(\pi_1, \pi_2)$ in $T_1$ or $T_2$ for $GL(n) \times GL(2)$, $(n \geq 3)$ is 1-regular if $\pi_2 = Ind(\mu_1 \otimes \mu_2)$. $\square$
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