BOND DURATION AND LIQUIDITY PREMIA:
A STUDY IN THE TERM STRUCTURE OF INTEREST RATES

DISSERTATION

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FIELDS OF STUDY

Monetary Economics
Econometrics
Economic Theory
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CHAPTER I

INTRODUCTION

The relationship between long-term rates of interest and short-term rates of interest has been of interest to economists from both microeconomic and macroeconomic points of view. On the microeconomic side, the behavior of financial intermediaries as profit-maximizing firms is influenced by the difference between long-term and short-term rates of interest. In addition, the corporation's internal decisions about how to finance capital investment projects is also dependent upon the relationship between long-term and short-term rates. On the macroeconomic side, the government's decisions about how to finance government spending, and the usefulness of debt-management policy as a whole, is tied to the issue of how long-term rates are, or are not, affected by short-term rates. The attempt in the 1960's to keep long-term rates low to encourage investment in plant and equipment while keeping short-term rates high to improve the U.S. balance-of-payments problems (called "Operation Twist" or "Operation Nudge") is a good example of how, from a policy viewpoint, economists
must have a good understanding of the relationship between long- and short-term rates of interest.

The area of economics which deals specifically with this subject is the term structure of interest rates. Currently there are three major competing theories of the term structure: the pure-expectations theory, the liquidity-preference theory, and the preferred-habitat theory. Term structure theories seek to explain the structure of interest rates by testing hypotheses derived from these theories against data from observed yield curves. Yield curves graph the yields-to-maturity of a set of bonds against their terms-to-maturity, where the bonds in the set have similar characteristics other than their terms-to-maturity. In particular, default risk is supposedly held constant. The yield curves used in current research are the Durand yield curves for high-grade corporate bonds, and the Treasury yield curves for U.S. Treasury securities. 1/

A major drawback to using these yield curves is that, except for very short-term bonds, they are derived from data based on coupon bonds, while the various theories of the term structure assume that all bonds are accumulation (single-payment) bonds. That
is, the term-structure theories assume that bonds have no intermediate payments so that the only payment occurs on the maturity date. Thus, the relevant time period between the purchase of the bond and the payment of principal and interest is the bond's term-to-maturity, which is shown by each yield curve. However, when a bond has intermediate payments, partial payments of principal and interest are made at regular intervals between the purchase of the bond and its maturity date. The time structure of payments of interest and principal are no longer as simple as is assumed by the competing theories of the term structure.

It was concern over the time structure of loan payments that led Frederick R. Macaulay to develop the concept of "duration" in 1938.²

For a study of the relations of long and short time interest rates, it would seem highly desirable to have some adequate measure of 'longness'. Let us use the word 'duration' to signify the essence of the time element in a loan. If one loan is essentially a longer term loan than another we shall speak of it as having greater 'duration'.³

Macaulay recognized that coupon bonds have special problems in terms of measuring their "duration," since payments are made in more than one future period.
It is clear that 'number of years to maturity' is a most inadequate measure of 'duration'. We must remember that the 'maturity' of a loan is the date of the last and final payment only. It tells us nothing about the sizes of any other payments or the dates on which they are to be made... 'Duration' is a reality of which 'maturity' is only one factor. 4/

Whether one bond represents an essentially shorter or an essentially longer term loan than another bond depends not only upon the respective 'maturities' of the two bonds but also upon their respective 'coupon rates' -- and, under certain circumstances, on their respective 'yields'. Only if maturities, coupon rates, and yields are identical can we say, without calculation, that the 'durations' of two bonds are the same. 5/

Macaulay proceeded to formulate a measure of the "duration" of a bond by treating a coupon bond as a series of individual loans. For example, a bond with 2 years to maturity, a coupon rate of 4%, and a face value of $1000, would be treated as a 6-month loan of $20, a 1-year loan of $20, a 1 1/2-year loan of $20, and a 2-year loan of $1020 (where coupon payments are made semi-annually). Macaulay then calculated "duration" as a weighted average of the "maturities" of these individual loans.

It would seem almost natural to assume that the 'duration' of any loan involving more than one future payment should be some sort of a weighted average of the maturities of the individual loans that correspond to each future payment. 6/
Now, if present value weighting be used, the 'duration' of a bond is an average of the durations of the separate single payment loans into which the bond may be broken up. To calculate this average the duration of each individual single payment loan must be weighted in proportion to the size of the individual loan; in other words, by the ratio of the present value of the individual future payment to the sum of all the present values, which is, of course, the price paid for the bond. 7/  

For an n-period bond having coupon payments $C$ in all periods, with a yield-to-maturity $R$, and with face value $M$, the duration $D$ of this bond is defined to be:

$$D = \sum_{t=1}^{n} v_t t$$  

where $v_t = \frac{1}{(1+R)^t} \left( \frac{C}{p} \right)$ (t = 1, 2, ..., n-1)  

$$v_n = \frac{1}{(1+R)^n} \left( \frac{C+M}{p} \right)$$  

and $P = \sum_{t=1}^{n} \frac{C}{(1+R)^t} + \frac{M}{(1+R)^n}$  

The duration of a bond is a measure of the time structure of the bond which takes into account both the size and the timing of the coupon payments, as well as the final (face value) payment. It is expressed as a
weighted average of the dates of the coupon and final payments of the bond, where the weights are the present value of the payments made at each date as a percentage of the price of the bond. Bonds with no coupons (accumulation bonds) have durations identical to their terms-to-maturity, but bonds with coupon payments have durations which are always less than their terms-to-maturity. Furthermore, the larger the coupon payments of a bond, the shorter is its duration, ceteris paribus.

There is no simple relationship between duration and term-to-maturity when coupon rates, yields-to-maturity, and terms-to-maturity all vary for a group of bonds. Bonds with the same term-to-maturity but different coupons have different durations, while bonds with different terms-to-maturity and different coupons may have the same duration.

This dissertation investigates the liquidity-preference theory of the term structure of interest rates under the conditions that coupon bonds are present in the yield curve data. It is shown that the hypotheses of the liquidity-preference theory can be altered to take into account the presence of coupon bonds in yield curve data.
To extend the liquidity-preference theory to include coupon bonds, this research uses Macaulay's concept of the duration of a bond. The duration of a bond is used in obtaining estimates of liquidity premia from data on U.S. Treasury securities. The hypotheses derived from the liquidity-preference theory are then tested.
NOTES TO CHAPTER I


3/ Macaulay, Some Theoretical Issues, p. 44.

4/ Ibid., pp. 44-45.

5/ Ibid., p. 45.

6/ Ibid., p. 46.

7/ Ibid., p. 48.
CHAPTER II
AN OUTLINE OF THE LIQUIDITY-PREFERENCE THEORY

The liquidity-preference theory of the term structure of interest rates is an extension of the pure-expectations theory. Developed by John Hicks, the liquidity-preference theory is an attempt to take into account interest-rate risk and the presumed risk aversion of investors in explaining the term structure of interest rates.\(^1\)

According to Hicks, the rate of return on a longer-term bond will include a risk premium (called a liquidity premium) to compensate long-term bondholders for bearing the uncertainty associated with the possibility of a capital loss if market interest rates rise.\(^2\) Since bond investors are assumed to be risk averse, the greater this interest-rate risk for any given bond, the larger the liquidity premium required to compensate investors for holding this bond. It is this general hypothesis that we examine in detail.\(^3\)

Hicks developed the liquidity-preference theory out of dissatisfaction with the pure-expectations theory. Basically, he modified the conditions of the pure-expectations theory to include liquidity premia.
In the literature on the term structure, the liquidity-preference theory has come to be stated in very specific terms as an extension of the pure-expectations theory.

According to the pure-expectations theory, a long-term rate of interest can be expressed as a geometric average of the current one-period spot rate and the expected one-period spot rates for future periods.\(^4/\)

Equation (2.1) expresses this relationship for an n-period bond as of time \(t\).

\[
(2.1) \quad (1 + R_{tn}) = [(1 + R_{t1})(1 + E_{t+1,t}) \ldots (1 + E_{t+n-1,t})]^{1/n}
\]

where \(R_{tn}\) = the rate of interest on an n-period bond as of time \(t\)

\(R_{t1}\) = the current one-period spot rate

\(E_{t+j,t}\) = the one-period spot rate expected to prevail in period \(t+j\) as of time \(t\); \((j = 1, 2, \ldots, n-1)\)

Wicks analyzed the bond market as a futures market in money, using analogies to commodity futures markets.\(^5/\) The pure-expectations theory implies that the expected spot rates in equation (2.1) are equal to
one-period forward rates. That is,

\[ t+j^F l_t, t = t+j^E l_t, t \quad (j = 1, 2, \ldots, n-1) \]

where \( t+j^F l_t \) is the forward rate of interest, as of time \( t \), on a one-period loan to begin in period \( t+j \).

Since a forward market in bonds does not generally exist,\(^6\) these forward rates cannot be observed directly. However, they can be computed from the term structure of interest rates, and since bonds of different maturities are available to investors, investors can implicitly buy and sell forward loans.\(^7\)

Using his analysis of the bond market as a futures market in money, Hicks expressed the liquidity (risk) premium on long-term bonds as the amount by which a forward rate exceeds the corresponding expected spot rate. That is, the forward rate was the sum of an expected spot rate and a liquidity premium. The liquidity-preference theory therefore implies that equation (2.2) should be reinterpreted as:

\[ t+j^F l_t, t = t+j^E l_t, t + t+j^L l_t, t \quad (j = 1, 2, \ldots, n-1) \]

where \( t+j^L l_t \) is the liquidity premium on a one-period loan beginning in period \( t+j \) as of period \( t \).
Liquidity premia can then be expressed as the difference between the forward rate and the expected spot rate, and from Hicks' theory we expect these liquidity premia to be positive.\(^8\)

\begin{align}
(2.4) \quad t+j^L_1,t &= t+j^F_1,t - t+j^E_1,t \quad (j = 1, 2, \ldots, n-1) \\
(2.5) \quad t+j^L_1,t &> 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad\n

The Hicksian liquidity-preference theory of the term structure expresses the rate of interest on an n-period bond as the geometric average of the current spot rate and the one-period forward rates, where the forward rates are given in equation (2.3).\(^9\)

\begin{align}
(2.6) \quad (1+R_{t,n}) &= [(1+R_{t,1})(1+t+1^F_1,t)\ldots(1+t+n-1^F_1,t)]^{1/n}

\text{Substituting from equation (2.3), we get:}

(2.7) \quad (1+R_{t,n}) &= [(1+R_{t,1})(1+t+1^E_1,t+1^L_1,t)\ldots

\ldots(1+t+n-1^E_1,t+t+n-1^L_1,t)]^{1/n}
\end{align}
The liquidity-preference theory developed by Hicks suggested that the liquidity premium is larger, the greater the interest-rate risk associated with the bond.\textsuperscript{10} Hicks maintained that the longer the term-to-maturity of a bond, the larger the change in the bond's price for a given change in the market interest rate, i.e., the longer the term-to-maturity, the greater the bond-price volatility.\textsuperscript{11} This implies that bonds with longer terms-to-maturity will have greater interest-rate risk, and hence will have larger liquidity premia. That is,

\begin{equation}
0 < \frac{L_{t+1}}{L_{t}} < \frac{L_{t+2}}{L_{t}} < \ldots < \frac{L_{t+n-1}}{L_{t}}
\end{equation}

where \( \frac{L_{t+1}}{L_{t}} \) through \( \frac{L_{t+n-1}}{L_{t}} \) are the market liquidity premia, as of period \( t \), on one-period loans beginning in periods \( t+1 \) through \( t+n-1 \), respectively. A major result of this pattern of the liquidity premia with respect to term-to-maturity is that even if the expected future spot rates are all equal, i.e.,

\begin{equation}
\frac{E_{t+1}}{L_{t}} = \frac{E_{t+2}}{L_{t}} = \ldots = \frac{E_{t+n-1}}{L_{t}}
\end{equation}
the forward rates of interest in equation (2.3) will still be increasing with term-to-maturity. Since the rates of interest on bonds of different maturities are geometric averages of these forward rates, the term structure of interest rates will be monotonically increasing in this case:

\[(2.9) \quad t_1^R < t_2^R < t_3^R < \ldots < t_n^R\]

Thus, the yield curve will be upward-sloping even though the expected future spot rates are all equal, which the pure-expectations theory would translate into a flat (horizontal) yield curve. The existence of liquidity premia therefore imparts a positive bias to the slope of observed yield curves.\(^{12}\) This approach was used by Kessel to explain his finding that an average of Durand's yield curves for corporate securities over a period of years was positively sloped.\(^{13}\)

Burton Malkiel has added a further restriction to the pattern of liquidity premia with respect to term-to-maturity. He contended that bond-price volatility, measured either in absolute or percentage terms,
increases with term-to-maturity, but at a decreasing rate.\textsuperscript{14/} From the standpoint of the liquidity-preference theory, this implied that the interest-rate risk of bonds increases at a decreasing rate with respect to term-to-maturity, and therefore the liquidity premia increase at a decreasing rate with respect to term-to-maturity. Hicks only specified that liquidity premia increase monotonically with term-to-maturity (see equation (2.8)). Malkiel's work also implied that as term-to-maturity approaches infinity, the liquidity premia are bounded by an asymptote.\textsuperscript{15/}

There have been many criticisms of the liquidity-preference theory of the term structure. Some modifications of the theory have been proposed to replace the original Hicksian version - just as the liquidity-preference theory was a modification of the pure-expectations theory.\textsuperscript{16/} These criticisms and revisions of the liquidity-preference theory are not the major concern of this paper, and are not discussed here. The next section examines the current state of our knowledge about the relationship between term-to-maturity and bond-price volatility, and the implications of this knowledge for the three restrictions placed on liquidity premia which we have discussed above:
1) Liquidity premia are positive (equation (2.5)).

2) Liquidity premia increase monotonically with term-to-maturity (equation (2.8)).

3) Liquidity premia increase with term-to-maturity at a decreasing rate; and as term-to-maturity approaches infinity, the liquidity premia are bounded by an asymptote.
NOTES TO CHAPTER II

1/ Hicks, Value and Capital, pp. 141-70.

2/ Ibid., p. 166.

3/ There are many aspects of the liquidity-preference theory that will not be explicitly dealt with in this paper, such as transactions costs and the institutional structure of the bond market. Although a complete discussion of the liquidity-preference theory would include these other aspects, it would only repeat the work of others. The contribution of this paper is in exploring the relationship between liquidity premia, interest-rate risk, and bond-price volatility. As a result, this outline of the liquidity-preference theory is intentionally brief. For a complete discussion, see Hicks, Value and Capital; Burton G. Malkiel, The Term Structure of Interest Rates: Expectations and Behavior Patterns (Princeton: Princeton University Press, 1966); J. Huston McCulloch, "An Estimate of the Liquidity Premium" (Ph. D. dissertation, University of Chicago, 1973); or Reuben A. Kessel, The Cyclical Behavior of the Term Structure of Interest Rates, occasional paper 91 (New York: National Bureau of Economic Research, 1965).


5/ Hicks, Value and Capital, pp. 141-47.

For example, by selling a zero-coupon n-period bond and purchasing a zero-coupon k-period bond, where \( k > n \), a forward \( (k-n) \)-period loan may be made that will begin \( n \) periods from the present. For further discussion of this point, see Charles R. Nelson, *The Term Structure of Interest Rates* (New York: Basic Books, 1972), pp. 6-9.


This is stated as Theorem 2 by Malkiel. See Malkiel, *The Term Structure of Interest Rates*, p. 54. However, Malkiel noted that the case of deep discount bonds was an exception to this theorem.

Ibid., pp. 24-26; or Kessel, *Cyclical Behavior of the Term Structure*, pp. 17-19.

See Kessel, *Cyclical Behavior of the Term Structure*, p. 18.


Ibid., p. 69.

CHAPTER III
LIQUIDITY PREMIA, BOND-PRICE VOLATILITY,
AND COUPON-BONDS

Coupon Bonds and Bond-Price Volatility

Hicks maintained that for a given change in yields, the prices of short-term bonds would change less than the prices of long-term bonds, \( \frac{1}{r} \) i.e., prices of short-term bonds are less volatile than the prices of long-term bonds. This proposition was stated as a theorem by Malkiel, \( \frac{2}{r} \) and has generally been presumed in all the early investigations of the liquidity-preference theory.

The relationship between bond prices and yields can be easily derived from the present-value formula for the price of a bond. The price of a bond in the current period, \( t \), is the discounted value of all future receipts.

\[
(3.1) \quad p = \frac{C_1}{(1 + R)^n} + \frac{C_2}{(1 + R)^2} + \ldots + \frac{C_n + M}{(1 + R)^n}
\]

where \( C_j \) = dollar value of the \( j \)th coupon payment \( (j=1, \ldots, n) \)

\( M = \) face value of the bond

19
\[ n = \text{term-to-maturity of the bond} \]

\[ R = \text{yield-to-maturity of the n-period bond in period } t \text{ (the internal rate of return)} \]

\[ P = \text{price of an n-period bond in period } t \]

Differentiating equation (3.1) with respect to \( R \), we obtain:

\[ (3.2) \quad \frac{d_P}{d_R} = - \left[ \frac{C_1}{(1+R)^2} + \frac{2C_2}{(1+R)^3} + \ldots + \frac{n(C+n)}{(1+R)^{n+1}} \right] \]

Since all the terms within the bracket on the right-hand side of equation (3.2) are positive, \( \frac{d_P}{d_R} < 0 \). Bond prices vary inversely with changes in yields: decreases (increases) in yields are equivalent to increases (decreases) in the prices of bonds.

The liquidity-preference theory states that liquidity premia are monotonically increasing with term-to-maturity (restriction (2) of Chapter II) on the grounds that bond-price changes are greater, the longer the term-to-maturity of the bond. But the implicit assumption of the liquidity-preference theory is that these bonds are accumulation bonds. In this case, the
price of an n-period bond can be expressed as:

\[ (3.3) \quad t^n_p = \frac{M}{(1 + R_n)^n} \]

and equation (3.2) can be rewritten as:

\[ (3.4) \quad \frac{d t^n_p}{d t^n R} = -\frac{nM}{(1 + R_n)^{n+1}} < 0 \]

Equation (3.4) can also be expressed as:

\[ (3.5) \quad \frac{d t^n_p}{t^n P} = -n \frac{d t^n R}{t^n P} \]

which says that the percentage change in the price of an n-period bond, for a given percentage change in its return \((1 + R_n)\), is inversely related to its term-to-maturity \(n\). As a result, the longer a bond's term-to-maturity, the greater its percentage price change (bond price volatility) for a given percentage change in its return.

Hicks argued that the liquidity premium of a longer-term bond would be greater than that of a shorter-term bond since the bond-price volatility of a longer-term bond is greater than that of a shorter-term bond. Equation (3.5) for accumulation bonds underlies this hypothesis.

But it has been known for some time that this bond-price volatility relationship may not hold for
coupon-bonds selling at discounts. Malkiel noted this in 1965, stating it as an exception to his general theorem of bond-price changes. More recently, Hopewell and Kaufman have reworked the mathematics of bond-price volatility, and have stated a more general theorem that applies to both coupon-bonds and accumulation bonds.

To demonstrate the distinctions between coupon and accumulation bonds in this regard, consider deriving an equation such as (3.5) for coupon-bonds. Equation (3.2) can be rewritten as

\[
\frac{d}{t_n} \left( \frac{P}{t_n} \right) = - \frac{1}{t_n} \sum_{k=1}^{n-1} \frac{k C_k}{(1 + R/t_n)^k} + \frac{n(C + M)}{(1 + R/t_n)^n} \frac{d}{t_n} \left( \frac{R}{t_n} \right)
\]

or, using equation (1.1), we have

\[
\frac{d}{t_n} \left( \frac{P}{t_n} \right) = - D_n \frac{d}{t_n} \left( \frac{R}{t_n} \right)
\]

where \( D_n \) is the duration of this \( n \)-period bond.

Equation (3.7) says that for coupon-bonds, the percentage change in the price of the bond, for a given percentage change in its return \( (1 + R/t_n) \), is inversely related to its duration \( D_n \).
This means that bond-price volatility will be greater, the greater is a bond's duration. In this case, we might expect liquidity premia to increase with duration rather than with term-to-maturity. To explain the implications of this result more fully, we must digress for a detailed discussion of the duration measure. Then we shall return to its implications for the behavior of liquidity premia.

**Duration**

As discussed in Chapter I, "duration" is a concept developed by Macaulay to measure the time structure of coupon bonds as an alternative to term-to-maturity. The duration of a bond is a measure of the time structure of the bond which takes into account both the size and the timing of the coupon payments, as well as the final (face value) payment, and is expressed as a weighted average of the dates of the coupon and final payments of the bond. The weights are the present value of the payments made at each date as a percentage of the price of the bond. For an n-period bond having coupon payments C in all periods, with a
yield-to-maturity $R$, and with face value $M$, the duration $D$ of this bond was defined in equation (1.1)

Duration can also be expressed as:

$$\begin{align*}
(3.8) \quad D &= \sum_{t=1}^{n} \frac{tc}{(1+R)^t} + \frac{nM}{(1+R)^n} \\
&= \sum_{t=1}^{n} \frac{tc}{(1+R)^t} + \frac{nM}{(1+R)^n} \\
&= \frac{\sum_{t=1}^{n} \frac{tc}{(1+R)^t} + \frac{nM}{(1+R)^n}}{P}
\end{align*}$$

Bonds with no coupons (accumulation bonds) have durations identical to their terms-to-maturity, but bonds with coupon payments have durations which are always less than their terms-to-maturity.

For example, the duration of a single-payment (accumulation) bond paying a face value of $1000 in 20 years and having a yield-to-maturity of 6 percent is 20 years.

$$D = \frac{(1000)(20)}{(1.06)^{20}} = 20$$

But consider a 20-year coupon bond paying a face value of $1000 and annual coupons of $60 (i.e., the coupon rate is 6 percent) and having a yield-to-maturity of 6 percent. The duration of this coupon bond is 11.9 years.
\[ D = \sum_{t=1}^{19} \frac{(60)t}{(1.06)^t} + \frac{(1000+60)(20)}{(1.06)^{20}} = 11.9 \]

If the above coupon bond had a yield-to-maturity of 8 percent, its duration would be 11.2 years, and if it had a coupon rate of 2 percent, its duration would be 15 years.

Consol bonds have no maturity date, so their term-to-maturity can be considered to be infinity. But the duration of a consol will be finite provided certain reasonable restrictions are met which assure convergence of the present value of the series of coupon payments. Specifically, the numerator and the denominator of the duration equation (3.8) must converge, which is true provided that \[ \left| \frac{1}{(1+R)} \right| < 1. \] When two infinite power series both converge to finite values for \[ \left| \frac{1}{(1+R)} \right| < 1, \] then the ratio of the two power series converges to the ratio of their values for \[ \left| \frac{1}{(1+R)} \right| < 1.8. \] For example, a consol bond with a coupon rate of 4 percent which is priced to yield 6 percent has a duration of 17.2 years -- considerably less than its infinite term-to-maturity.
Since the duration of a bond is a function of the coupon rate, term-to-maturity, and yield-to-maturity, two coupon bonds with very different terms-to-maturity may have approximately the same duration. For example, a 50-year 8 percent coupon bond yielding 6 percent has a duration of 15.8 years, while a 20-year 2 percent coupon bond yielding 6 percent has a duration of 15 years.\footnote{9} Furthermore, the duration of an accumulation bond having 15 years to maturity is 15 years; approximately the same as the durations of both these coupon bonds.

Duration varies inversely with the coupon rate,\footnote{10} i.e., the larger the coupon, the smaller the bond's duration, \textit{ceteris paribus}. Duration also varies inversely with the yield-to-maturity, \textit{ceteris paribus}.\footnote{11} But the relationship between duration and term-to-maturity is not so clear-cut, and it is this relationship which is important to our examination of liquidity premia.

The theorem of bond-price volatility which Hopewell and Kaufman develop states that:

For a given basis point change in market yield, percentage changes in bond prices vary proportionately with duration and are greater, the greater the duration of the bond.\footnote{12}
Mathematically stated:

\[(3.9) \quad \frac{dP}{P} = -D \frac{dR}{(1+R)}\]

For continuous discounting, \( R \) approaches zero and equation (3.9) reduces to:

\[(3.10) \quad \frac{dP}{P} = -D \, dR\]

Equation (3.10) indicates that the percentage change in bond price for a given change in yield is a linear function of the duration of the bond. Therefore, the relationship between the change in bond price and term-to-maturity will depend on the relationship between duration and term-to-maturity.

According to the liquidity-preference theory, liquidity premia are a positive function of interest-rate risk, and interest-rate risk is a positive function of bond-price volatility, i.e., the larger the change in price of the bond for a given change in yield, the larger the associated interest-rate risk of that bond. Most researchers have maintained that bond-price volatility is a monotonically increasing function of term-to-maturity. However, Malkiel noted that for deep discount bonds this relationship may not hold,\(^ {14/}\) and Hopewell and Kaufman have shown that bond-price volatility is a monotonically increasing function of duration.\(^ {15/}\) It seems reasonable to conclude, therefore, that liquidity premia should be monotonically increasing
with respect to duration rather than term-to-maturity, which is a restatement of restriction (2) in Chapter II above.

If there were a one-to-one correspondence between duration and term-to-maturity for any group of bonds, a restatement of restriction (2) would be unnecessary. But this is not the case. Hopewell and Kaufman show that the relationship between term-to-maturity and duration depends heavily on the price of the bond. For bonds selling at premium or at par, duration increases monotonically with term-to-maturity, although at a decreasing rate, ceteris paribus. For coupon bonds selling at a discount, duration increases with term-to-maturity up to a point, peaks at that point, and declines, ceteris paribus. Consider a series of coupon bonds having various maturities ranging from 1 to \( n \) where all the bonds have the same coupon rate and are priced to sell at the same yield-to-maturity. If the maximum duration for this group of bonds occurs when the term-to-maturity is \( n^* \), \( 1 < n^* < n \), bond-price volatility will increase for bonds with terms-to-maturity between 1 and \( n^* \), and decrease for bonds with terms-to-maturity between \( n^* \) and \( n \). If liquidity premia are positively related to bond-price volatility, as hypothesized by the liquidity-preference
theory, then liquidity premia should increase monotonically for bonds with maturities between 1 and \( n^* \), and decrease monotonically for bonds with maturities between \( n^* \) and \( n \). Note that this holds for a group of coupon bonds all of which have the same coupon rate and sell at the same yield-to-maturity; only their terms-to-maturity vary.

For example, consider a group of coupon bonds with a coupon rate of 2 percent and with terms-to-maturity ranging from 1 to 100 years, all of which are priced to sell at a yield-to-maturity of 6 percent. These coupon bonds are selling at a discount since the market yield (6%) exceeds their coupon rate (2%). Table 3-1 shows the durations of some selected bonds of this group having various terms-to-maturity.\(^{18}\)

Note that the duration of the 100-year 2 percent coupon bond is lower than the duration of the 50-year 2 percent bond coupon bond in this case. It turns out that the maximum duration, \( n^* \), for this group of bonds occurs approximately when the term-to-maturity of a bond is 45.5 years.\(^{19}\) For this group of bonds, then, bond-price volatility increases with term-to-maturity over the range of 1-year bonds to 45.5-year bonds, and decreases with term-to-maturity for bonds after that point. Liquidity premia should
<table>
<thead>
<tr>
<th>Years to Maturity</th>
<th>Coupon Rate 2%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.99</td>
</tr>
<tr>
<td>5</td>
<td>4.75</td>
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<td>10</td>
<td>8.89</td>
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</tr>
<tr>
<td>50</td>
<td>19.45</td>
</tr>
<tr>
<td>100</td>
<td>17.56</td>
</tr>
</tbody>
</table>

* Rates and yields in percentage per annum, compounded semianually.

therefore be increasing monotonically with term-to-maturity for bonds with terms-to-maturity between 1 year and 45.5 years, and should be decreasing monotonically with term-to-maturity for bonds with terms-to-maturity greater than 45.5 years.

Figures 3-1, 3-2, and 3-3 depict the relationships between duration and term-to-maturity for coupon bonds and non-coupon bonds with various conditions on their selling prices.

If the liquidity premia are monotonically increasing functions of bond-price volatility, then they should be monotonically increasing functions of bond-duration, whether the bond is an accumulation bond or a coupon bond. But liquidity premia will only be a monotonically increasing function of term-to-maturity if the bonds in the data set are accumulation bonds, or meet the conditions noted above.

Figures 3-4 and 3-5 depict the relationship between the size of the liquidity premium and term-to-maturity for a series of coupon bonds having identical coupon rates and yields-to-maturity when these bonds are selling above and below par, respectively, while Figure 3-6 depicts the relationship between the size of the liquidity premium and duration.
FIGURE 3-1  DURATION AND TERM-TO-MATURITY: ACCUMULATION (ZERO-COUPON) BONDS

FIGURE 3-2  DURATION AND TERM-TO-MATURITY: COUPON BONDS SELLING AT OR ABOVE PAR
FIGURE 3-3 DURATION AND TERM-TO-MATURITY: COUPON BONDS SELLING AT A DISCOUNT
FIGURE 3-4 LIQUIDITY PREMIUM AND TERM-TO-MATURITY: COUPON BONDS SELLING AT OR ABOVE PAR

FIGURE 3-5 LIQUIDITY PREMIUM AND TERM-TO-MATURITY: COUPON BONDS SELLING AT A DISCOUNT
FIGURE 3-6 LIQUIDITY PREMIUM AND DURATION
It is important to point out that for a group of coupon bonds which have identical coupon rates but have different yields-to-maturity as well as different terms-to-maturity, there is no simple relationship between duration and term-to-maturity. For example, a 20-year 4-percent coupon bond with a market yield of 4 percent has a duration of 13.9 years, while a 50-year 4-percent coupon bond with a market yield of 6 percent has a duration of 17.1 years. However, a 50-year 4-percent coupon bond with a market yield of 8 percent has a duration of 13.5 years. Is duration increasing or decreasing with term-to-maturity for 4-percent coupon bonds? This rhetorical question points out the problem encountered when yield-to-maturity is not constant.

If coupon rates, yields-to-maturity, and terms-to-maturity all vary for a group of bonds, there is no simple relationship between duration and term-to-maturity. This makes the identification of a consistent pattern of liquidity premia with respect to term-to-maturity very difficult indeed for coupon bonds. The importance of the above observations for the estimation of liquidity premia from presently available yield curve data, which includes bonds with different coupon rates, will be examined in more detail in Chapter IV.
Table 3-2 replicates a table of durations for various coupon rates, terms-to-maturity, and yields-to-maturity which was prepared by Lawrence Fisher and Roman Weil.\textsuperscript{20/}

\textbf{Portfolio Theory and Duration}

To say that liquidity premia are monotonically increasing with bond-duration implies that the "riskiness" of bonds increases with bond-duration. The traditional approach of the liquidity-preference theory has been to identify bond-price volatility with interest-rate risk. Since the initial formation of the liquidity-preference theory, there have been major developments in the analysis of the "riskiness" of stocks and bonds. Mean-variance portfolio theory and the capital-asset-pricing model have provided a framework to assess the risk of individual assets.\textsuperscript{21/} In the area of bond-valuation theory, John Boquist\textsuperscript{22/} and Boquist, Racette, and Schlarbaum\textsuperscript{23/} have recently examined the risk of coupon-bonds using the linear market model.\textsuperscript{24/}
<table>
<thead>
<tr>
<th>Years to Maturity</th>
<th>4%</th>
<th>6%</th>
<th>8%</th>
<th>4%</th>
<th>6%</th>
<th>8%</th>
<th>4%</th>
<th>6%</th>
<th>8%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>4%</td>
<td>6%</td>
<td>8%</td>
<td>2%</td>
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<td>6%</td>
<td>8%</td>
<td>2%</td>
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<td>1</td>
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<td>0.99</td>
<td>0.98</td>
<td>0.98</td>
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<td>0.99</td>
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<td>0.98</td>
<td>0.99</td>
</tr>
<tr>
<td>5</td>
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<tr>
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<td>8.33</td>
<td>7.85</td>
<td>7.49</td>
<td>8.89</td>
<td>8.16</td>
<td>7.66</td>
<td>7.28</td>
<td>8.76</td>
</tr>
<tr>
<td>100</td>
<td>26.41</td>
<td>25.01</td>
<td>24.53</td>
<td>24.29</td>
<td>25.16</td>
<td>24.73</td>
<td>24.12</td>
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<td>23.09</td>
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<td>17.16</td>
<td>17.16</td>
<td>17.16</td>
<td>17.16</td>
<td>13.00</td>
</tr>
</tbody>
</table>

* Rates and yields in percentage per annum, compounded semiannually.

Source: Fisher and Weil, p. 418.
In mean-variance portfolio theory, the risk of a particular asset is the addition to the total risk of a diversified portfolio. In the framework of the linear market model, the total risk of an asset can be decomposed into two parts: systematic risk and unsystematic risk. Unsystematic risk can be diversified away, which means that the addition to the total risk of an efficient, diversified portfolio is only the systematic risk of the asset. This approach to assessing the risk of an asset is different from the traditional approach of the liquidity-preference theory, which looks at the interest-rate risk of each individual bond by looking at the bond-price volatility of each individual bond. There is no linkage of individual bond-price volatility with a diversified portfolio.

Our discussion of liquidity premia, duration, and bond-price volatility has followed this traditional approach of the liquidity-preference theory. The question arises as to whether portfolio theory lends
any support to the hypothesis that the liquidity (risk) premium of a bond is related to its duration.

Boquist, Racette, and Schlarbaum have shown that the measure of systematic risk for a coupon-bond can be decomposed into a number of separate terms, one of which is the duration of the bond.\(^{25/}\) In the most general case, the effect of duration on the systematic risk of a coupon-bond will depend on the correlation between a change in the bond's yield and the return on a market portfolio.\(^{26/}\) The equation given by Boquist, Racette, and Schlarbaum for the risk of a default-free bond is:

\[
(3.11) \quad \beta_{it} = \frac{-D_{it} \text{Cov}(\tilde{\delta}_{it}, \tilde{R}_{mt})}{\sigma^2(\tilde{R}_{mt})} = \frac{-D_{it} \rho(\tilde{\delta}_{it}, \tilde{R}_{mt}) \sigma(\tilde{\delta}_{it})}{\sigma^2(\tilde{R}_{mt})}
\]

where: \(D_{it}\) = the duration of the bond at time \(t\)

\(\delta_{it}\) = the change in the bond's yield-to-maturity at time \(t\)

\(R_{mt}\) = the return relative on the market portfolio at time \(t\)
and where tildes (\(\tilde{\cdot}\)) represent random variables.

It can be seen from equation (3.11) that the sign of 

\[ \rho(d\hat{r}_{it}, \hat{Y}_{mt}) \]

will determine whether \(\beta_{it}\) is positive or negative. This term may be positive, negative, or zero in the general case. However, Boquist, Racette, and Schlarbaum note that for default-free bonds, such as U.S. Treasury securities, this correlation is expected to be negative.\(^{22}\) This gives the effect of duration on systematic risk an unambiguous sign: the greater is the duration of a coupon-bond, the larger is the systematic risk of that bond. Hence, bonds with longer durations will have greater systematic risk, ceteris paribus.

It is interesting to note that the above relationship will hold for default-free bonds such as U.S. Treasury securities, but not for the general group of corporate bonds, which are not default-free. The study undertaken here investigates the liquidity-preference theory using data on U.S. Treasury securities. Any extension of the approach taken here to corporate bonds, such as those included in the Durand yield curves, would have to investigate more fully the correlation between changes in bond yields and the return on the market portfolio. However, the duration of a bond will still be an important factor in the risk of that bond.
Since the duration of a bond is a measure of bond-price volatility, it appears that bond-price volatility will indeed be an important factor in assessing the "riskiness" of a bond. Hence, for default-free bonds, it does not appear to be unreasonable to expect risk premia (liquidity premia) to be positively related to bond-price volatility, as is done in the traditional approach of the liquidity preference theory.

Restrictions on Liquidity Premia

If we accept the general hypothesis of the liquidity preference theory that liquidity premia are positively related to bond-price volatility, and if we want to extend the liquidity preference theory to coupon-bonds, then we must restate the restrictions placed on liquidity premia at the end of Chapter II.

For coupon-bonds, we have shown above that restriction (2) should be restated as:

2) Liquidity premia are monotonically increasing with duration.

Restriction (3) on the liquidity premia—that liquidity premia increase with term-to-maturity
at a decreasing rate and are bounded as term-to-maturity approaches infinity — suffers from the same problems as restriction (2). Percentage bond-price changes are linear functions of duration and, when coupon bonds are selling at a discount, the percentage price changes may attain a maximum and then decrease as term-to-maturity increases. For discount coupon-bonds, therefore, restriction (3) is not accurate.

Malkiel recognized that coupon bonds had different degrees of price fluctuation than accumulation bonds. In fact, one of his theorems stated the effect which the size of coupon payments have on the price fluctuations of bonds.

The higher the coupon carried by the bond, the smaller will be the percentage price fluctuation for a given percentage change in yield except for one-year securities and consols.28/

In the earlier part of this chapter we mentioned that the coupon rate is inversely related to duration,
ceteris paribus, which is what Malkiel essentially states in the above theorem.

The concept of duration helps to explain restriction (3) on liquidity premia. For par and premium bonds, Hopewell and Kaufman point out the consistency of duration with the restriction that liquidity premia increase with term-to-maturity at a decreasing rate:

To the extent liquidity premiums are a linear function of the price riskiness of bonds and price risk is a linear function of duration, liquidity premiums are also a linear function of duration. As, in turn, duration increases with term-to-maturity but at a decreasing rate, this explains the hypothesis that liquidity premiums increase with term-to-maturity at a decreasing rate.29/

The only difficulty here is that coupon bonds selling at discounts were not included in Hopewell and Kaufman's discussion of this particular point. They only referred to bonds selling at par or with premiums.

The other part of restriction (3) -- that the liquidity premia are bounded by an asymptote as term-to-maturity approaches infinity -- can also be
explained using the concept of duration. That liquidity premia will be bounded when term-to-maturity is infinite does not have to be imposed as a separate restriction. It follows from the fact that consol bonds have finite duration provided \[ \left| \frac{1}{(1+R)} \right| < 1. \] (See the discussion of consol bonds above.) Fisher and Weil note that for all bonds, duration is bounded at perpetuity (a term-to-maturity of infinity) by the term \( \frac{R+Q}{RQ} \), where \( R \) is the yield-to-maturity and \( Q \) is the number of times per year that interest is paid and compounded.\(^{30/} \) For bonds selling at par or with premiums, this boundary value represents maximum duration. For discount coupon-bonds, the maximum duration occurs before perpetuity at some other term-to-maturity, which we referred to above as \( n^* \). This implies that liquidity premia will be bounded as term-to-maturity approaches infinity for both premium and discount bonds.

Even though the concept of duration helps us to understand the restrictions on liquidity premia contained in restriction (3), it is not accurate for
coupon-bonds, and we would like to restate it in terms of duration. To do this we must examine the bond-price volatility equation (equation (3.10) above).

From equation (3.10) we can see that for a given change in yield (dR), the percentage change in the bond's price (dP/P) is proportional to the bond's duration (D). That is,

\( \frac{dP}{dR} = -D \)  

or:

\( \frac{dP}{dR} = D \)

The relationship between duration and the percentage price change is given by:

\( \frac{3 (\frac{dP}{dR})}{3 \frac{d}{dD}} = 1 \)  

Equation (3.14) states that increases in a bond's duration lead to proportional increases in bond-price volatility. The rate of increase is given by:

\( \frac{3^2 (\frac{dP}{dR})}{3 \frac{d}{dD^2}} = 0 \)

which implies that bond-price volatility increases with duration at a constant rate.
If interest-rate risk is a function of bond-price volatility and liquidity premia are functions of interest-rate risk, as specified by the liquidity-preference theory, then liquidity premia should increase monotonically with duration at a constant rate. This would be the restatement of restriction (3) of Chapter II that corresponds to our earlier restatement of restriction (2). Figure 3-6, noted earlier, indicates this relationship between duration and liquidity premia.

**Summary**

This chapter has explored the relationship between liquidity premia, bond-price volatility, term-to-maturity, and duration. Specifically, the three restrictions that the traditional formulation of the liquidity-preference theory places on liquidity premia have been examined. Empirical investigations designed to assess the validity of the liquidity-preference theory have relied on these restrictions, stated at the end of Chapter II, as criteria to be used for discriminating between competing theories of the term structure.
To adapt the liquidity-preference theory to coupon-bonds, the second and third restrictions should be restated.

2') Liquidity premia increase monotonically with duration.

3') Liquidity premia increase with duration at a constant rate.

The importance of the above observations becomes apparent when we examine the yield-curve data used by researchers to test alternative theories of the term structure. This is the task of the next chapter.
NOTES TO CHAPTER III

1/ Hicks, Value and Capital, pp. 166-67.

2/ Malkiel, The Term Structure of Interest Rates, p. 54.

3/ Hicks, Value and Capital, pp. 166-67.

4/ Malkiel, The Term Structure of Interest Rates, pp. 79-81.


6/ Frederick R. Macaulay, Some Theoretical Problems, pp. 48-53. The concept was also independently developed in Value and Capital by J. R. Hicks, who referred to the measure as "average period." See Hicks, Value and Capital, p. 186.

7/ When $C_t = 0$ for all $t$, then $D = n$.


Specifically, in the case of consol bonds, duration is given by:

$$D = \frac{C}{(1+R)} + \frac{2C}{(1+R)^2} + \frac{3C}{(1+R)^3} + \ldots$$

$$= \frac{C}{(1+R)} + \frac{C}{(1+R)^2} + \frac{C}{(1+R)^3} + \ldots$$

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If we let \( x = 1/(1+R) \), we get:

\[
D = \frac{C(x + 2x^2 + 3x^3 + 4x^4 + \ldots)}{C(x + x^2 + x^3 + x^4 + \ldots)} = \frac{Cx(1 + 2x + 3x^2 + 4x^3 + \ldots)}{Cx(1 + x + x^2 + x^3 + \ldots)} = \frac{(1 + 2x + 3x^2 + 4x^3 + \ldots)}{(1 + x + x^2 + x^3 + \ldots)}
\]

The numerator converges absolutely to \( 1/(1-x)^2 \) provided \( |x| < 1 \), and the denominator converges absolutely to \( 1/(1-x) \) provided \( |x| < 1 \).

According to the theorem, the ratio of the two series converges to

\[
\frac{1/(1-x)^2}{1/(1-x)} = 1/(1-x)
\]

provided \( |x| = \left| 1/(1+R) \right| < 1 \).

Hence, the duration of a consol bond converges to \( 1/(1-x) = (1+R)/R \) provided that \( \left| 1/(1+R) \right| < 1 \).

This result was checked using a method of determining the coefficients of the ratio of two power series. The same result was obtained. (For a description of this method, see Earl D. Rainville, Infinite Series (New York: Macmillan, 1967), p. 129, Theorem 51.)

9/ Hopewell and Kaufman, "Bond Price Volatility," Table 1, p. 751.

10/ Ibid., p. 751.

11/ Ibid., p. 752.

12/ Ibid., p. 749.


Note that since $\frac{dP}{dR} < 0$ from equation (3.2), then $dP = -D dR$ indicates that bond-price volatility is positively related to $D$.


Ibid., footnote 4. For discount coupon bonds, duration is a maximum when:

$$\n^* = \frac{1}{R} + \frac{1}{(R-C)} + \frac{R}{(R-C)} + \frac{(R-C)}{CR(1+R)^n}.$$  

This is Table 4 of Lawrence Fisher and Roman L. Weil, "Coping with the Risk of Interest Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies," *Journal of Business* 44 (October 1971), p. 418.

Calculated from the formula of Hopewell and Kaufman (see fn. 15 above). By a method of successive approximation, we get $n^* = 45.5$.

Fisher and Weil, "Coping with the Risk," Table 4, p. 419.

22/ John A. Boquist, "A Duration-Based Bond Valuation Model" (Ph.D. dissertation, Purdue University, 1973).


26/ Ibid.

27/ Ibid., footnote 5, p. 1365.

28/ Malkiel, *The Term Structure of Interest Rates*, Theorem 5, p. 56.


30/ Fisher and Weil, "Coping with the Risk," p. 418.

CHAPTER IV
YIELD CURVES AND LIQUIDITY PREMIA

Testing Hypotheses with Observed Yield Curves

Yield curves graph the yields-to-maturity of a set of bonds against their terms-to-maturity, where the bonds in the set have similar characteristics other than their terms-to-maturity. In particular, default risk is supposedly held constant. The yield curves used in current research are the Durand yield curves for high-grade corporate bonds, and the Treasury yield curves for U.S. Treasury securities.\(^1\) However, neither of these sets of yield curves hold the coupon rates of the bonds constant.

For example, the yield curve for U.S. Treasury securities as of June 30, 1975 includes U.S. government notes and bonds with coupon rates ranging from 3% to 9% (ignoring U.S. Treasury bills, which have no coupons).\(^2\) In addition, of two U.S. government bonds due in 1998, one has a coupon-rate of 3.5% while the other has a coupon-rate of 7%. It is interesting to note that while these two bonds differ by only half a year in their terms-to-maturity, they differ by four years in their durations.\(^3\)
Treasury yield curves are drawn free-hand through the scatter of points relating yield-to-maturity and term-to-maturity. This method essentially takes a weighted average of the yields for all coupon bonds of a given term-to-maturity. Durand's yield curves for high-grade corporate bonds are also drawn free-hand, but are drawn as an envelope of the lowest yields on issues considered to be free of special influences, such as call or convertibility features.

Hypotheses of the liquidity-preference theory of the term structure of interest rates have been tested using observed yield-curve data. Specifically, the restrictions placed on the pattern of liquidity premia with respect to term-to-maturity have been tested.

Liquidity premia, which according to the liquidity-preference theory reflect the effect of interest-rate risk on the size of the yield-to-maturity of a bond, are estimated as the difference between an implied forward rate and an expected future rate. Traditionally, forward rates and expected rates are calculated between bonds with adjacent terms-to-maturity, which ignores the role of duration in explaining the interest-rate risk of coupon bonds. Grouping bonds by term-to-maturity to
calculate liquidity premia presumes that the coupon-
bonds that make up the yield curve at any given term-
to-maturity have the same degree of interest-rate risk.
According to the analysis of Chapter III, default-free
bonds with the same duration should be grouped together,
and forward rates and expected rates should be calculated
between bonds with adjacent durations rather than
between bonds with adjacent terms-to-maturity. 5/

The nature of the estimation problem for
liquidity premia which is confronted in constructing
forward and expected rates from observed yield curves
can be illustrated using forward rates. Forward rates
have been calculated from yield curves by using the
Hicksian formula: 6/

\[
(4.1) \quad F_{t+n-1,1,t} = \frac{(1 + R)^n_{t,n}}{(1 + R)^{n-1}_{t,n-1}} - 1
\]

where \( F_{t+n-1,1,t} \) = the one-period forward rate
as of period \( t \) on a loan to
begin in period \( t+n-1 \)

In equation (4.1), the yield-to-maturity \( R \) is an
average yield-to-maturity for all coupon bonds having
term-to-maturity \( n \); and the yield-to-maturity \( R_{n-1}^{t} \) is an average yield-to-maturity for all coupon bonds having term-to-maturity \( n-1 \). This makes the one-period forward rate, \( F_{t+n-1,t}^{L} \), an average of all the yields of the coupon bonds used to construct the yields \( R_{n}^{t} \) and \( R_{n-1}^{t} \). The liquidity-preference theory says that the forward rate is equal to an expected rate plus a liquidity premium. The liquidity premium derived from forward rates given by equation (4.1) will then be a weighted average of the liquidity premia associated with all the coupon bonds used to construct \( R_{n}^{t} \) and \( R_{n-1}^{t} \).

For example, suppose that two coupon bonds have term-to-maturity \( n \), and two other coupon bonds have term-to-maturity \( n-1 \). Let the coupon rates on all four bonds be different, and let \( R_{n}^{A} \) and \( R_{n}^{B} \) be the yields-to-maturity for the \( n \)-year coupon bonds with coupon payments \( C_{A} \) and \( C_{B} \), respectively. Let \( R_{n-1}^{S} \) and \( R_{n-1}^{T} \) be the yields-to-maturity for the \( (n-1) \)-year coupon bonds with coupon payments \( C_{S} \) and \( C_{T} \), respectively.

For the sake of our illustration, we let \( R_{n}^{A} \) be greater than \( R_{n}^{B} \), and let \( R_{n-1}^{S} \) be greater than \( R_{n-1}^{T} \), as shown in Figure 4-1. Let \( R_{n}^{T} \) and \( R_{n-1}^{T} \) be...
the yields-to-maturity for bonds of term-to-maturity \( n \) and \( n-1 \), respectively, which are calculated by some method to obtain a single yield curve for the above group of bonds.\(^7\)

Let us represent \( \bar{R}^n \) and \( \bar{R}^{n-1} \) in Figure 4-1 by points \( W \) and \( V \), respectively. We can think of points \( W \) and \( V \) as linear combinations of points \( A \) and \( B \), and \( S \) and \( T \), respectively, which suggests that the yields \( \bar{R}^n \) and \( \bar{R}^{n-1} \) can be considered to be linear combinations of \( R^A \) and \( R^B \), and \( R^S \) and \( R^T \), respectively. That is, for some \( 0 \leq \delta \leq 1 \), and for some \( 0 \leq \alpha \leq 1 \), we could write \( \bar{R}^n \) and \( \bar{R}^{n-1} \) as:

\[
\begin{align*}
(4.2) \quad \bar{R}^n &= \delta R^A + (1-\delta) R^B \\
(4.3) \quad \bar{R}^{n-1} &= \alpha R^S + (1-\alpha) R^T
\end{align*}
\]

The one-period forward rate of interest for a loan to begin in period \( n-1 \) is calculated from equation (4.1) by using \( \bar{R}^n \) and \( \bar{R}^{n-1} \) as follows:
FIGURE 4-1  YIELD CURVE USING TERM-TO-MATURITY
(4.4) \[ t+n-1^F_{1,t} = \frac{(1 + \bar{R}_{t-n})^n}{(1 + \bar{R}_{t-n})^{n-1}} - 1 \]

The liquidity-preference theory maintains that forward rates are equal to the sum of the expected future spot rate plus a liquidity premium. So from equation (2.3), we can rewrite \( t+n-1^F_{1,t} \) as:

(4.5) \[ t+n-1^F_{1,t} = \frac{(1 + \bar{R}_{t-n})^n}{(1 + \bar{R}_{t-n})^{n-1}} - 1 \]

\[ = t+n-1^E_{1,t} + t+n-1^L_{1,t} \]

By substituting from equation (4.2) and (4.3) we get:

(4.6) \[ t+n-1^F_{1,t} = \frac{(1 + \delta R^A_{t-n} + (1-\delta) R^B_{t-n})^n}{(1 + \alpha R^S_{t-n-1} + (1-\alpha) R^T_{t-n-1})^{n-1}} - 1 \]

Note that \( R^A_{t-n}, R^B_{t-n}, R^S_{t-n-1}, \) and \( R^T_{t-n-1} \) are the yields-to-maturity on coupon bonds with different coupon rates. These coupon bonds are likely to have different durations, and hence the bond-price volatility for each of these bonds will be different. Since different bond-price volatility implies different degrees of interest-rate risk, then the liquidity premium associated with each bond will be different.
The implication of this is that forward rates derived from equations (4.1) and (4.6) lead to an averaging of interest-rate risk for this group of coupon bonds, which leads to estimates of liquidity premia which are averages of the liquidity premia for the various coupon bonds that make up $\bar{R}_t^n$ and $\bar{R}_{t-n-1}$.

To get the point across more forcefully, let us consider three specific bonds: a 10-year accumulation bond which is priced to yield 4 percent; a 10-year 4-percent coupon bond which is priced to sell at par; and an 8.5-year accumulation bond priced to yield 4 percent.

The durations of the two accumulation bonds are the same as their terms-to-maturity, i.e., the 10-year bond has a duration of 10 years, the 8.5-year bond has a duration of 8.5 years. The duration of the 10-year 4-percent coupon bond is 8.4 years, which is very close to the duration of the 8.5-year accumulation bond. Since, according to the liquidity-preference theory, liquidity premia are functions
of interest-rate risk, interest-rate risk is a function of bond-price volatility, and bond-price volatility is a function of duration, it is argued that liquidity premia are monotonic functions of duration. Furthermore, bonds with the same duration will have similar interest-rate risk, and therefore they will have similar liquidity premia.

This implies that when estimating liquidity premia, bonds of approximately the same duration -- such as the 8.5-year accumulation bond and the 10-year 4-percent coupon bond -- should be grouped together. Forward rates should then be calculated between the yield on the 10-year accumulation bond (with a duration of 10 years), and some average yield of the 10-year 4-percent coupon bond and the 8.5-year accumulation bond (with durations of about 8.5 years). The forward rate between the bonds given above would be a 1.5-year forward rate of interest on a loan to begin in period t+8.5 and would be calculated as follows:

\[
(4.7) \quad F_{t+8.5}^{1.5,t} = \frac{(1 + \frac{\bar{R}_{10}}{t})^{10}}{(1 + \frac{\bar{R}_{8.5}}{t})^{8.5}} - 1
\]

Since forward rates are sums of expected future rates
and liquidity premia, according to the liquidity-preference theory, liquidity premia would be estimated with the forward rates and expected future rates calculated on the basis of bonds of different durations rather than different terms-to-maturity.

Hence, one approach to estimating liquidity premia in the face of the "duration problem" presented by coupon-bonds is to collect data on bonds which include bond prices, yields-to-maturity, terms-to-maturity, and coupon rates. From this information, the durations of these bonds are calculated using equation (3.8). Bonds are then grouped together on the basis of their having similar durations rather than similar terms-to-maturity, and forward rates and expected rates are calculated between bonds with adjacent durations. Estimation may be accomplished using any one of a number of techniques employed by other investigators. Estimates of liquidity premia can then be tested for consistency with the liquidity-preference theory of the term structure of interest rates. Specifically,

1) Are the estimated liquidity premia positive?
2) Are the estimated liquidity premia monotonically increasing with duration?

3) Do the estimated liquidity premia increase with duration at a constant rate?

A Digression on an Alternative Approach

The problem of the presence of coupon-bonds in observed yield-curve data has been recognized for a long time, and there have been other approaches to dealing with it.\(^{10/}\) There have been a number of attempts in recent years to calculate yield curves which are more accurate than the free-hand yield curves which were developed for corporate bonds by David Durand and for Treasury bonds by the Treasury Department.\(^{11/}\) These more sophisticated calculations attempt to construct a "corrected" yield curve which is free of any coupon-effects. That is, the "corrected" yield curve attempts to capture the relationship between term-to-maturity and "true" yields, which is the term structure of interest rates.

To the extent that these "corrected" yield curves can, in fact, represent a series of accumulation
(zero-coupon) bonds that reflect the term structure, the use of the duration concept to adjust for coupon-effects is unnecessary. However, the methods used so far for correcting yield curves are difficult and time-consuming, even with computers. Working with the duration concept is relatively easy in comparison.

Furthermore, the approach discussed above can be considered a complementary approach to those investigators who try to correct yield-curve data for coupon-effects. The duration approach attempts to utilize information on coupon-rates as well as information on prices, yields and terms-to-maturity in order to get better estimates of liquidity premia which can be used to discriminate among the various term-structure theories. The approaches taken by others to correct observed yield curves for coupon-effects have the same objectives.
NOTES TO CHAPTER IV


3/ The 3.5% coupon bond due in November, 1998 has a term-to-maturity of 23 years, 4 months, with a yield of 4.92%. The 7% coupon bond due in May, 1998 has a term-to-maturity of 22 years, 10 months, with a yield of 7.85%. The duration of the first bond is 14.99 years, while the duration of the second is 10.91 years.


5/ Recall that this hypothesis applies specifically to default-free Treasury bonds, and not necessarily to corporate bonds.
6) See Hicks, Value and Capital, pp. 145-46; Meiselman, The Term Structure of Interest Rates, pp. 4-5; and Malkiel, The Term Structure of Interest Rates, p. 22.

7) The method by which the yield curve is calculated is immaterial at this point of our discussion, provided it is some weighted average of the yields by term-to-maturity.

8) It is a 1.5-year forward rate since the bonds which are being compared are bonds with 10-year and 8.5-year durations. If we compared bonds with 10-year and 5-year durations, we would have a 5-year forward rate.


A thorough discussion of some of the problems presented by coupon bonds was presented by Malkiel, The Term Structure of Interest Rates.


12/ McCulloch, in "Measuring the Term Structure," uses a method of formulating a "discount function" to estimate prices of zero-coupon bonds for various maturities, and then fits yield curves based on these prices using quadratic splines. Other researchers mentioned in footnote 10 above use more standard regression techniques to estimate coupon-free yield curves.
CHAPTER V
DURATION-BASED YIELD CURVES

The analysis of Chapters III and IV suggest that one approach to estimating liquidity premia for default-free coupon-bonds is to calculate forward rates and expected rates between bonds with adjacent durations rather than between bonds with adjacent terms-to-maturity. In order to do this, we must have a set of bonds grouped according to their durations. This amounts to formulating new yield curves which relate the yields-to-maturity of a group of bonds to their durations. These duration-based yield curves can then be used to estimate liquidity premia for this group of bonds.

To implement this approach, the durations of U.S. Treasury notes and bonds have been calculated on a monthly basis from April 1954 through August 1975. Treasury notes, bonds, and bills were then grouped according to their durations.
The Data

The data set consists of fully taxable U.S. Treasury notes, bonds, and bills from April 1954 through August 1975, as published in the Treasury Bulletin. The Treasury Bulletin figures are over-the-counter closing bid quotations in the New York market for the last trading day of the month, as reported to the Treasury by the Federal Reserve Bank of New York. These figures are based on the average of the quotations given by four or five bond dealers. Notes and bonds having one month or more to maturity were used to determine the durations of the notes and bonds that make up the monthly Treasury yield curve. However, the 1-1/2% 5-year Treasury notes denoted E0 and EA are not included in the data set because they are issuable only in exchange for convertible 2-3/4% Investment Series "B" bonds, which are non-marketable. The amounts of these notes outstanding are relatively small.

A number of typographical errors are in the published Treasury Bulletins which are quite obvious
when one goes through them on a month-to-month, year-to-year basis. A list of these is available from the author on request.

**Durations of Notes and Bonds**

The duration of each Treasury note and bond was calculated for each month in the data set using equation (3.8). To do this, and to achieve a high degree of accuracy in grouping bonds by their durations, the numerator of duration is calculated on a monthly basis. That is, instead of having "n" in equation (3.8) represent the total number of years-to-maturity, "n" represents the total number of months-to-maturity. Equation (3.8) then requires that the yield (R) and the coupon payment (C) be on a monthly basis as well. After the duration is calculated, it is then converted to a yearly basis for purposes of comparison.

For six years of this sample, the duration of each outstanding Treasury note and bond, as of the last trading day of January, is given in Appendix A. For each of these months, the durations of the notes
and bonds were plotted against their yields. For means of comparison, the yields of the notes and bonds were also plotted against their terms-to-maturity, which represents the published yield curve. The two sets of points for each month are plotted in Appendix B.

The scatter of points representing the duration-based yield curve tends to coincide with the regular yield curve for the notes and bonds with shorter terms-to-maturity. But for notes and bonds with longer terms-to-maturity, the scatter of points representing "duration versus yield" is further away from the regular yield curve. This is as expected, as noted by Hopewell and Kaufman.

For all bonds, differences between duration and maturity are small for short maturities but increase as maturity increases.3/

The above-noted relationship was graphically depicted by Figures 3-2 and 3-3 of Chapter III for the case where all bonds had the same coupon rates and
yield-to-maturity. These figures show duration plotted against term-to-maturity for such a hypothetical case.

For each month of January, from 1955 through 1975, the relationship between duration and term-to-maturity was plotted for the outstanding notes and bonds. These graphs are presented in Appendix C for selected months. Although neither yields nor coupon rates are held constant for each monthly group of notes and bonds, we find similar relationships as those plotted in Figures 3-2 and 3-3. For example, the data for January 31, 1957 shows the same pattern for duration versus term-to-maturity as that shown in Figure 3-2.

The pattern shown by Figure 3-3 is approximated by the data for January 31, 1973 and for January 31, 1975. That is, there are a number of cases where duration does not increase monotonically with term-to-maturity. The durations of some notes and bonds may actually be smaller than the durations of other bonds having the same term-to-maturity. An examination
of the data in Appendix A reveals a number of such instances.

The data on the durations of outstanding Treasury notes and bonds follow the patterns discussed in Chapter III. That is, for coupon bonds, there is no simple relationship between term-to-maturity and duration.

A comparison of the durations and terms-to-maturity of U.S. Treasury notes and bonds was also made using two simple regressions. First, for each of the 257 months of yield curve data from April 1954 through August 1975, the durations of the Treasury notes and bonds were regressed against their terms-to-maturity and a constant term:

\[ D = a + bM + u \]

where \( D \) = duration

\( a, b \) are coefficients

\( M \) = term-to-maturity

\( u \) = an error term

Tests were then made of whether the durations and terms-to-maturity were essentially identical for any of the 257 months. If \( D \) were equal to \( M \), we would expect
to find that $\hat{a} = 0$ and $\hat{b} = 1$ in the above regressions. The results of the tests were that in each of the 257 regressions of equation (5.1), the constant term, $\hat{a}$, was always significantly different from zero, ranging between $0.5$ and $1.5$. The coefficient of $m$, $\hat{b}$, was always significantly different from one, ranging between $0.5$ and $0.7$.

A second regression was run of the durations against $M$ and $M$-squared:

$$D = a + b_1 M + b_2 M^2 + u$$

For each of the 257 months, the coefficient of $M^2$, $b_2$, was always significantly different from zero, and was negative. The values of $b_2$ were small, ranging from $-0.009$ to $-0.03$. However, the addition of $M^2$ led to a reduction in the size of the constant term and a reduction in its $t$-value in all cases. In all but a few of the 257 cases, the constant term was no longer significantly different from zero.

From this comparison we can conclude that the observed durations of notes and bonds are significantly different from their terms-to-maturity, and that the
relationship between observed durations and term-to-maturity is nonlinear (which we expected from our plots of duration vs. term-to-maturity - see Appendix B). Specifically, the fact that \( b_2 \) is significantly negative indicates that, for this set of data, the durations of the bonds cannot be assumed to be a monotonic transformation of term-to-maturity. Thus, our concern over the difference between the two measures is not merely an idle one with no counterpart in the real world.

**Missing Data Points**

One of the major difficulties in testing term-structure theories with yield-curve data is that there are generally large "gaps" in the maturity structure of outstanding bonds. For example, there may not be any bonds with term-to-maturity of, say, 22 to 28 years. The problem becomes to fill in these missing data points. This is generally accomplished by fitting a curve through the available data, and then using the fitted curve to estimate the missing data points.
Using duration instead of term-to-maturity does not eliminate this problem, but it does reduce the number of "missing data points" to be estimated. Since duration is always less than term-to-maturity for coupon bonds, the duration-based yield curve compacts the observed data points to the left on the time axis. The data points are moved closer together. In calculating the durations of notes and bonds over the sample period, we found no bonds with durations longer than 25 years, but only a small number of them with durations between 15 and 25 years.

In order to derive forward rates at equally spaced intervals, the scatter of points matching the yields and durations of Treasury notes, bills, and bonds (which make up the monthly duration-based yield curves), must be fitted by some curve. A glance at the general shapes of these plots (see Appendix B) suggests that the curve must have a nonlinear form. Also, a variety of possible shapes of these yield curves must be allowed for.

Nonlinear forms of a regression equation that would allow for a variety of these yield-curve shapes
involve regressing yields on various powers of duration or on the log of duration. The problem, then, is to find some nonlinear form of a regression equation that fits all of the 257 duration-based yield curves fairly well.

A number of different regression equations involving powers of duration (e.g., $D^{1/2}$, $D^2$, $D^3$, etc.) and the log of duration were fit to the data for January of each year, 1955-1975. The shapes of the scatter of yields and durations varied considerably for these months. A comparison of the results indicated that the following equation did as well or better than any other specification for each of these months:\(^5\)

\[(5.3) \quad R = a + b_1D + b_2D^2 + b_3D^3 + b_4(\ln D) + b_5D^{1/2} + u\]

R-squares for these regressions were generally above 60%, with many in the 80-95% range. A few of the months had very poor fits, with R-squares below 50%.

Whenever the R-square for equation (5.3) was below 90%, scatter diagrams of yields vs. durations
for that month were examined, along with plots of actual and fitted values, to attempt to determine if some improvement might be made in the estimating equation. It was found that no alternative specification improved the R-square compared to that of equation (5.3). However, there were cases where the scatter diagrams indicated that a splitting of the sample into long- and short-duration groups might improve the results for the longer-duration bonds. Since the longer-duration range of bonds include more "missing data points", performing a regression on the longer-duration sample would hopefully improve the estimates of these missing points.

In the cases of sample-splitting, the scatter diagrams generally indicated that a split could be made at a duration of 5 years. Regressions on the longer-duration samples did improve the fits for various months, especially for those months which originally had R-squares below 50%.

The F-tests for whether the estimated relationship was significantly different from the null hypothesis were passed at the 1% level in all but six
of the 257 months. For five of these six months, the relationship was significant at the 5% level, but the F-value for the remaining month (November, 1957), was not significantly different from zero even at the 20% level.

The estimated duration-based yield curve was used to obtain fitted values of yields at 6-month intervals for durations running from 6 months to 15 years. Forward rates were then calculated from these fitted yields using equation (4.4), where one period is equal to 6 months. This provided 257 sets of forward rates, one for each month in the sample period.
NOTES TO CHAPTER V

1/ Plotting duration-based yield curves was hinted at by Macaulay, Some Theoretical Problems, pp. 44, 50-52, 54-62, 68; and also by Durand and Winn, Basic Yields of Bonds, pp. 38-39. The suggestion was explicitly voiced by Hopewell and Kaufman, "Bond Price Volatility," p. 752.

2/ Prior to April, 1954, reported yields of notes and bonds were based on the mean of the bid and asked prices instead of the bid price alone. However, bid prices were reported prior to April, 1954, and a consistent series could be calculated as far back as 1942 without a great deal of difficulty. Earlier Treasury Bulletins did not include market quotations in tabular form. Yield curves were plotted from which the data might be calculated, and, of course, other sources could be used to obtain data for the period before 1942.


4/ These tests were done at the 1% level of significance using the t-distribution.

5/ A fifth-degree polynomial in D also worked well.

6/ Only polynomial and log combinations were attempted in this study. Alternative approaches include the use of cubic or quadratic splines, and the inclusion of variables other than duration in the estimating equation. In all cases, the choice of the form of estimating equation is highly arbitrary.

7/ In two cases the split was made at four years.
CHAPTER VI

ESTIMATION METHODS

Chapter IV suggested that liquidity premia could be estimated by calculating forward and expected rates between securities with adjacent durations. Chapter V outlined how forward rates were calculated for the sample period being used. The remaining problem is how to deal with the unobservable expected rates.

Traditional Distributed Lags

The general approach to estimating expected variables is to use some expectations-formation mechanism involving observable variables as a proxy for the expectations variable. In much of the price-expectations literature this involves using distributed lags in the past history of prices. Analogously, interest-rate expectations would involve using the past history of interest rates.
The recent growth of the rational-expectations literature suggests that information other than the past history of interest rates should be included in the formation of expectations.\(^2\) Thus we might want to include a distributed lag in the level or rate of growth of the money supply, or in the level or rate of growth of government spending, as a proxy for nonautoregressive elements in interest rate expectations. Such distributed lags could be estimated using geometric, Pascal, or polynomial lag techniques.\(^3\)

The regression would have the general form:

\[ j_{F,t} = a_j + \sum_{i=1}^{k} w_i s_{1,t-i} + \sum_{i=1}^{k} h_{i} z_{t-i} + u_{j,t} \]

where \( j_{F,t} = \) the \( j \)-period forward rate. (The one-period rate of interest that would prevail on a loan to begin \( j \) periods from the current period.)

\( s_{1,t-i} = \) the one-period spot rate observed in past periods.

\( z_{t-i} = \) the past history of other information such as the level of the money supply or government spending.
\[ w_{1,t}, h_{1,t} \] = the coefficients of the lagged independent variables.
\[ a_j \] = a constant term.
\[ u_{j,t} \] = an error term.
\[ t \] = the index over the sample period: monthly data from April, 1954 through August, 1975.

If the distributed lags in \( s_{1,t-1} \) and \( z_{t-1} \) capture the expected one-period spot rate to prevail \( j \) periods from the current period, then the constant term, \( a_j \), can be considered the liquidity premium associated with \( j \)-period forward rates \( (L_{j,t}) \). Using \( \hat{a}_j \), as an estimate of the liquidity premium of all one-period forward loans beginning \( j \) periods from a current period implies that this premium is constant throughout the sample period:

\[ L_{j,t} = \hat{a}_j = \bar{L}_j. \]

The liquidity premium associated with a \( j \)-period forward rate might be allowed to vary by estimating it as the sum of the constant term and some element embodied in the estimated residuals of
the regression, \( \hat{u}_{j,t} \). That is, some exogeneous
factors not explicitly included as independent
variables, but which affect the liquidity premia,
might be approximated as a function of the residuals
\( f_j(\hat{u}_{j,t}) \). Then \( L_{j,t} = \hat{a}_j + f_j(\hat{u}_{j,t}) \), and the
liquidity premium would vary over the sample period.

If the \( u_{j,t} \) exhibited positive autocorrelation
\( u_{j,t} = \rho u_{j,t-1} + e_{j,t} \); the above approach could be
implemented by associating the systematic changes in
the \( u_{j,t} \) (the term \( \rho u_{j,t-1} \)) with variations in the
liquidity premium. In this case, the estimate of \( L_{j,t} \n\)
is \( \hat{a}_j + \hat{\rho} u_{j,t-1} \). These estimates could be obtained
using either the Cochrane-Orcutt or the Hildreth-Lu
estimation technique which adjust for serial correlation.

A Polynomial Distributed Lag Approach

The estimation of equation (6.1) using
polynomial distributed lags (PDL's) is attractive for
a number of reasons. First, a PDL does not require a
strong a priori specification as to the behavior of the
lagged coefficients, as is the case with geometric or
Pascal distributed lags. All that is required is
that the lag weights approximate a polynomial of a given degree. Second, a PDL provides consistent estimates of the lag coefficients while avoiding the large loss in degrees of freedom that occurs when using ordinary least squares.\footnote{4}{4}

However, the major difficulty with using a PDL is that the researcher must choose the degree of polynomial to be used. In addition, as is the case with any distributed lag, the researcher must choose the length of the lag. There are no absolute criteria that can be used to make either choice, and, in fact, the choices are not necessarily independent.\footnote{5}{5}

For example, suppose the basis for choosing among various specifications of a PDL is the correlation between actual and predicted values of the dependent variable; the PDL giving the highest such correlation is chosen as the "best" specification. One search technique using this criterion would be to specify the length of the lag, and then search over various degrees of a polynomial. However, changing the length of the lag may also change the results, so
that one could hold the degree of polynomial constant, and then search over various lag lengths. The problem, then, is to choose a combination of lag length and degree of polynomial that yield a "best" specification according to some criterion.

Dhrymes has suggested choosing a PDL on the basis of maximizing the correlation coefficient between the actual and predicted values of the dependent variable.\(^6\) This approach was attempted for the sample period using equation (6.1), with \(Z_{t-i}\) being the level of the money supply, defined as currency plus demand deposits. This makes expected future short rates a distributed lag in past short rates and the past money supply; a specification which follows Rutledge's work on rational expectations of inflation.\(^7\)

Equation (6.1) was estimated using this double polynomial lag in short rates and money for lag lengths between 36 months to 120 months, and for degrees of the polynomial lag structure ranging from a 2nd degree through a 7th degree polynomial. On the basis of this search procedure, three or four combinations of lag lengths and degrees of polynomials could have
been chosen as having similar correlation coefficients between actual and predicted values.

However, the empirical implications of these various specifications were vastly different. Specifically, the estimated constant terms in the various specifications, which are measured in percentage points, ranged from significant negative values on the order of -1.0 to -2.0 percent, to significant positive values on the order of 28.0 to 35.0 percent. These latter $\hat{\alpha}_j$ are so high as to be uninterpretable as measures of liquidity premia.

Choosing a specific PDL among these alternative specifications in order to estimate liquidity premia would require imposing an arbitrary "plausibility" criterion: that the estimated constants be small and positive, or small and negative, as expected by the liquidity-preference theory. This is a poor research technique since it presupposes the hypothesis being tested. As a result, the polynomial distributed lag approach was abandoned as a method for testing the liquidity premium theory.
Roll and McCulloch's Approach

One approach to estimating liquidity premia, which circumvents the actual estimation of expected rates, involves a technique which has been applied by Richard Roll and J. Huston McCulloch. Estimates of liquidity premia are calculated by a method of summing forward rate differences, which is dependent on the principle that the current forecast of a future variable is an unbiased estimator of all future forecasts of that variable. Thus, the one-period spot rate expected as of time $t$ to prevail in period $t+j$ is an unbiased estimator of the one-period spot rate expected as of time $t+k$, $k < j$, to prevail in period $t+j$.

In the notation of Chapter II:

$$\text{(6.2)} \quad t+j \mathbb{E}_{1,t} = \mathbb{E}_t \{ t+j \mathbb{E}_{1,t+k} \}$$

Under the assumption that the liquidity premia for one-period loans beginning $j-1$ periods from the current period are all equal (that is, $t+j-1 \mathbb{L}_{1,t} = t+j \mathbb{L}_{1,t+1} = \mathbb{L}_j$ for all $j = 2, \ldots, n$), Roll shows that
an estimate of the liquidity premium \( L_j \) associated with the forward rates \( t+j-1, l_t, t+j l_{t+1} \), and so on for all \( j = 2, \ldots, n \) can be estimated by the sum of average forward rate differences over the sample period as follows:

\[
(6.3) \quad L_j = \sum_{i=2}^{j} \left( \frac{F_{t+i-1, l_t} - F_{t+i-1, l_{t+1}}}{F_{t+i-1, l_t}} \right)
\]

where \( F_{t+i-1, l_t} = F_{t+i-1, l_{t+1}} \), the one-period spot rate in period \( t+i \), and where the long bar (\( \bar{\quad} \)) over the term on the right-hand side of equation (6.3) represents an average over the sample period \( (t=1, \ldots, 256) \).

For example, consider the case where \( j=3 \) in equation (6.3). Then we have (suppressing the long bar over the right-hand side):

\[
\sum_{i=2}^{3} \left( \frac{F_{t+i-1, l_t} - F_{t+i-1, l_{t+1}}}{F_{t+i-1, l_t}} \right)
\]

\[
= \left( \frac{F_{t+2, l_{t+1}}}{F_{t+1, l_t}} \right)
\]

\[
+ \left( \frac{F_{t+2, l_{t+1}} - F_{t+2, l_{t+1}}}{F_{t+1, l_{t+1}}} \right)
\]
\[
= t + 1 L_{t, t}^E + t + 1 L_{t, t} - t + 1 R_{t, t}^L
\]

\[
+ t + 2 L_{t, t}^E + t + 2 L_{t, t} - t + 2 E_{t, t} + t + 1 - t + 2 L_{t, t}^E + t + 1
\]

\[
= t + 1 L_{t, t}^E + L_2 - t + 1 R_{t, t}^L + t + 2 L_{t, t}^E + L_3 - t + 2 E_{t, t}^E - L_2
\]

Now, by appealing to equation (6.2) we assume that \( t + 2 L_{t, t}^E \) is approximately equal to \( t + 2 E_{t, t}^E \), and that \( t + 1 L_{t, t}^E \) is approximately equal to \( t + 1 R_{t, t}^L \), we have:

\[
3 \sum_{i=2}^{3} \left( \frac{t+i-1 F_{t, t} - t+i-1 F_{t, t}^E}{t+i-1 F_{t, t} - t+i-1 F_{t, t}^E} \right) = L_3
\]

The same line of reasoning applies for \( j = 2, \ldots, n \).

Equation (6.3) was used by Roll and McCulloch to obtain estimates of liquidity premia. Roll's estimates were based on weekly U.S. Treasury bill data, which have no coupons. Therefore the problem of duration vs. term-to-maturity does not arise. Roll found that average liquidity premia were small and positive, but that they were not necessarily monotonically increasing.
McCulloch's estimates were based on monthly U.S. Treasury notes, bills, and bonds from December, 1946, through March, 1966. As noted in Chapter IV, McCulloch used an alternative approach to eliminate the duration problem posed by the presence of coupon bonds in the yield curve data. McCulloch found that he could not reject the hypotheses that the estimated liquidity premia were positive and monotonically increasing.

Using the forward rates calculated from the fitted values of the yields for the duration-based yield curves of Chapter V, equation (6.3) was used to estimate liquidity premia. The results of this approach are reported in Chapter VII.\textsuperscript{11}\

\textbf{Other Estimation Approaches}

A number of researchers working with price expectations or interest rate expectations have used linear stochastic processes and spectral analysis to derive expected variables from the past history of the variable itself and the past history of other information.\textsuperscript{12} This approach was not attempted here.
NOTES TO CHAPTER VI


5/ Ibid.

6/ Dhrymes, Distributed Lags, pp. 338-44.

7/ See Rutledge, A Monetarist Model, pp. 47-54.


The fitted values were used, rather than a combination of actual and fitted values, in order to base the estimates on smoother yield curves. Because of the small number of bonds with long durations, using a combination of actual and fitted values allows large fluctuations in forward rates associated with long-duration bonds. Other problems involved in the long-duration estimates are discussed in Chapter VII.

See, for example, Nelson, The Term Structure of Interest Rates; or Rutledge, A Monetarist Model.
CHAPTER VII
ESTIMATES OF LIQUIDITY PREMIA

From the forward rates derived from the estimated duration-based yield curves of Chapter V, liquidity premia were estimated using equation (6.3). Forward rates were calculated at 6-month intervals from j = 6 months through j = 15 years, so that the estimated liquidity premia are associated with one-period (6-month) forward rates beginning in periods t+j-1, where j = 2, ..., 30, and t is the month in which the observation on the yield curve occurs. Thus, liquidity premia were estimated for durations ranging from 6 months to 15 years.

The above technique yielded estimates of L_j (j = 2, ..., 30) as averages of 256 observations on the forward rate differences (t = 1, ..., 256), and are presented in Table 7-1. Since adjacent observations of forward rate differences both depend on the same information, we can expect that the measurement errors of the forward rates are not
<table>
<thead>
<tr>
<th>Table 7-1</th>
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<td>ESTIMATES OF LIQUIDITY PREMIA:</td>
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<tr>
<td>BASED ON ALL OBSERVATIONS AND ON ALTERNATE OBSERVATIONS</td>
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<tr>
<td>FOR TOTAL SAMPLE PERIOD (I-III)</td>
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<tr>
<td>N=256</td>
</tr>
<tr>
<td>L2</td>
</tr>
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<td>.406</td>
<td>-.128</td>
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<td>(.303)</td>
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<td>$L_{18}$</td>
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<td>(.222)</td>
<td>(.318)</td>
<td>(.309)</td>
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<td>$L_{19}$</td>
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<td>.148</td>
<td>-.399</td>
<td>(.228)</td>
<td>(.331)</td>
<td>(.314)</td>
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<td>$L_{20}$</td>
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<td>.036</td>
<td>-.508</td>
<td>(.232)</td>
<td>(.343)</td>
<td>(.315)</td>
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<td>$L_{21}$</td>
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<td>-.640a</td>
<td>(.239)</td>
<td>(.356)</td>
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<td>$L_{22}$</td>
<td>-.498a</td>
<td>-.226</td>
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<td>(.369)</td>
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<td>$L_{23}$</td>
<td>-.644a</td>
<td>-.370</td>
<td>-.918a</td>
<td>(.254)</td>
<td>(.383)</td>
<td>(.325)</td>
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<td>$L_{24}$</td>
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<td>-.500</td>
<td>-1.06a</td>
<td>(.263)</td>
<td>(.397)</td>
<td>(.305)</td>
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<td>$L_{25}$</td>
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<td>-.669</td>
<td>-1.25a</td>
<td>(.273)</td>
<td>(.412)</td>
<td>(.359)</td>
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<tr>
<td>$L_{26}$</td>
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<td>-.854</td>
<td>-1.47a</td>
<td>(.285)</td>
<td>(.426)</td>
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<td>$L_{27}$</td>
<td>-1.30a</td>
<td>-.106a</td>
<td>-1.71a</td>
<td>(.298)</td>
<td>(.444)</td>
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<tr>
<td>$L_{28}$</td>
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<td>-.35a</td>
<td>-2.03a</td>
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<td>(.462)</td>
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<tr>
<td>$L_{29}$</td>
<td>-2.04a</td>
<td>-.69a</td>
<td>-2.38a</td>
<td>(.331)</td>
<td>(.482)</td>
<td>(.453)</td>
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<tr>
<td>$L_{30}$</td>
<td>-2.46a</td>
<td>-2.13a</td>
<td>-2.79a</td>
<td>(.354)</td>
<td>(.507)</td>
<td>(.494)</td>
<td></td>
</tr>
</tbody>
</table>

* Significantly different from zero at the 5% level. Standard errors in parentheses. (.05 = .01%)
serially independent.\textsuperscript{1} McCulloch handled this problem by discarding alternate observations of forward rate differences when estimating the $L_j$.\textsuperscript{2}

This approach was attempted here, and the estimated liquidity premia using only even (or odd) observations (128 observations in all) are presented in Table 7-1. The difficulty is that the even and odd estimates are not very similar. While the estimates are generally not significantly different from each other, there are some cases where the estimate based on the odd observations is significantly positive while the estimate based on the even observations is not significantly different from zero. This lack of similarity between the estimates based on the even and odd estimates leads to problems of reporting and interpreting the results of the empirical tests if only one set of alternate observations is used. Although it is desirable to adjust for the serial dependence of the forward rate differences which are used to estimate the liquidity premia, it is not reasonable to use only the odd observations or only the even observations as estimates in this case, since the
interpretation of the results would be significantly different. Therefore, further examination of the liquidity premia will be based on the estimates from the total sample period (i.e., using all 256 observations).

**Subsamples**

McCulloch divided his sample period into four subperiods, two of which are covered by the sample period used here. McCulloch found that the period December, 1946 to February, 1951 (the pre-Accord period) had a different mean and variance from the three subsamples included in the post-Accord period, April, 1951 to March, 1966. He also found that the three later periods had the same mean but different variances. (These three periods were: April 1951 to December 1955; January 1955 to December 1960; and January 1961 to March 1966.)

To examine whether or not the duration-based estimates of liquidity premia behaved similarly to those estimated by McCulloch, we divided the sample period into three post-Accord periods of approximately equal length:
Period I: April 1954 to December 1960 (82 observations)

Period II: January 1961 to December 1967 (86 observations)

Period III: January 1968 to August 1975 (88 observations)

The first two periods are slightly larger than those of McCulloch, but include the same months used in his third and fourth periods.

Estimates of liquidity premia ($l_j$, $j = 2, ..., 30$) were derived using equation (6.3) for each of periods I, II, and III (See Table 7-2). The means of the sum of the forward rate differences (which are the estimated liquidity premia) and their variances were then tested for equality among these periods using the following hypotheses.

$H_1$: A common mean and variance for all three subperiods.

$H_2$: A common mean and variance for periods I and II, and a separate mean and variance for period III.

$H_3$: A common mean and variance for periods II and III, and a separate mean and variance for period I.

$H_4$: A different mean and variance for each period.
These hypotheses were compared to one another using asymptotic \( \chi^2 \) statistics for each set of estimated liquidity premia \( L_j, j = 2, \ldots, 30 \), with the following results:

(A) \( H_1 \) was rejected in favor of \( H_2 \) in all cases.

(B) \( H_3 \) was rejected in favor of \( H_4 \) in all cases.

(C) \( H_2 \) could not be rejected in favor of \( H_4 \) in most cases. Exceptions were \( L_2 \) through \( L_7 \), and \( L_{27} \) through \( L_{30} \).

The results of these tests implied that the estimated means and variances of the sum of the forward rate differences were relatively homogeneous for \( L_8 \) through \( L_{26} \) in periods I and II, while the means and variances were not homogeneous between periods I and III, and between periods II and III.

The equality of the variances of the forward rate differences between periods I and II was tested separately using an F-test with 81 and 85 degrees of freedom.\(^3\) The hypothesis that the variances were equal could not be rejected (at the 1% level) in all but nine cases.\(^4\)
The equality of the means of the forward rate differences (the estimated liquidity premia) was tested between all of the periods: I and II, II and III, and I and III. Between periods I and II, the hypothesis that the liquidity premia were equal ($L^I_j = L^{II}_j$) could not be rejected at the 5% level for $L_7$ through $L_{30}$, but could not be accepted for $L_2$ through $L_6$. Between periods II and III, the hypothesis that the liquidity premia are equal could not be accepted for $L_{22}$ through $L_{30}$. Between period I and III, the hypothesis of equal liquidity premia could not be rejected for $L_2$ through $L_{30}$.

In summary, the variances of the sum of the forward rate differences in period III are significantly different from the variances in periods I and II. But the variances in periods I and II are generally not significantly different from each other. The liquidity premia in periods I, II, and III are not significantly different from each other for $L_7 - L_{19}$. In the case of period III, this result is apparently due to the large variances of the sum of the forward rate differences in this subsample.
For $L_2 - L_6$, the liquidity premia of periods II and III were not significantly different from each other, as was the case for periods I and III. But these premia were significantly different from each other for periods I and II. For $L_{20} - L_{30}$, the premia in periods I and II were not significantly different from each other, but there was a significant difference between periods II and III and between periods I and III.

The results of these tests on the equality of the estimated liquidity premia across periods are mixed. The general assumption that liquidity premia are constant over time is given support by the results for $L_7 - L_{19}$, but is not given support by the results for the shorter- or longer-duration estimates. The possibility that liquidity premia change over time due to changes in market factors, such as the level of interest rates, or due to changing attitudes towards interest-rate risk on the part of investors, cannot be refuted by this evidence. However, a full investigation of the influences on the size of these liquidity premia is beyond the scope of the present work.

For periods I and II, the results reported here are similar to those found by McCulloch for his third and fourth periods (which correspond to periods I
and II), except that he reported different variances in all periods. Also, McCulloch found no significant difference between any of the estimated $L_j$ among his post-Accord periods.

Since the above tests indicate some homogeneity of the estimates between periods for various $L_j$, Table 7-2 presents estimates of the liquidity premia for periods I, II, and III, separately; and for periods I and II, II and III, and I through III, combined. Standard errors are given in parentheses.\(^6\)

The Pattern of Liquidity Premia

For $L_2 - L_6$ (the liquidity premia associated with forward rates one year through three years hence)\(^2\) the period I estimates are all significantly positive, and increase with duration from $L_2 = .414$ to $L_6 = .692$. (See Table 7-2: .1 is equal to .1%.) The $L_2 - L_6$ estimates for the combined period II-III are also significantly positive, although they are smaller than the premia in period I. The period II-III estimates also are increasing with duration, from $L_2 = .235$ to $L_6 = .307$. 
<table>
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<th>II</th>
<th>III</th>
<th>II-III</th>
<th>I-II</th>
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<td>L₂</td>
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<td>0.414&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.219&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.250&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.336&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.314&lt;sup&gt;a&lt;/sup&gt;</td>
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<td></td>
<td>(0.031)</td>
<td>(0.038)</td>
<td>(0.026)</td>
<td>(0.079)</td>
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<td>0.569&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.270&lt;sup&gt;a&lt;/sup&gt;</td>
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<td></td>
<td>(0.054)</td>
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<td>0.639&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.294&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.312&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.303&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.462&lt;sup&gt;a&lt;/sup&gt;</td>
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<tr>
<td></td>
<td>(0.074)</td>
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<td>(0.067)</td>
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<td>L₅</td>
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<td>0.308&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.305&lt;sup&gt;a&lt;/sup&gt;</td>
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<td>(0.092)</td>
<td>(0.118)</td>
<td>(0.089)</td>
<td>(0.226)</td>
<td>(0.122)</td>
<td>(0.074)</td>
</tr>
<tr>
<td>L₆</td>
<td>0.432&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.692&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.310&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.297&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.307&lt;sup&gt;a&lt;/sup&gt;</td>
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<td>0.290&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.314&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.502&lt;sup&gt;a&lt;/sup&gt;</td>
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<td>0.178&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.294&lt;sup&gt;a&lt;/sup&gt;</td>
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<td>(0.215)</td>
<td>(0.381)</td>
<td>(0.219)</td>
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<td>0.638&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.250&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.181&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.215&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.439&lt;sup&gt;a&lt;/sup&gt;</td>
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<td>(0.170)</td>
<td>(0.234)</td>
<td>(0.189)</td>
<td>(0.405)</td>
<td>(0.224)</td>
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<td>0.638&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.476&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.243&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.358&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.555&lt;sup&gt;a&lt;/sup&gt;</td>
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<td>0.266&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.266&lt;sup&gt;a&lt;/sup&gt;</td>
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<td>(0.203)</td>
<td>(0.426)</td>
<td>(0.237)</td>
<td>(0.167)</td>
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<tr>
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<td>0.605&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.516&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.210&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.358&lt;sup&gt;a&lt;/sup&gt;</td>
<td>0.566&lt;sup&gt;a&lt;/sup&gt;</td>
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<tr>
<td></td>
<td>(0.192)</td>
<td>(0.284)</td>
<td>(0.226)</td>
<td>(0.440)</td>
<td>(0.248)</td>
<td>(0.180)</td>
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<td>L₁₅</td>
<td>0.357</td>
<td>0.579</td>
<td>0.416</td>
<td>0.093</td>
<td>0.253</td>
<td>0.496&lt;sup&gt;a&lt;/sup&gt;</td>
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<tr>
<td></td>
<td>(0.200)</td>
<td>(0.297)</td>
<td>(0.295)</td>
<td>(0.455)</td>
<td>(0.259)</td>
<td>(0.191)</td>
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<td>L₁₆</td>
<td>0.254</td>
<td>0.552</td>
<td>0.299</td>
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<td>0.114</td>
<td>0.423&lt;sup&gt;a&lt;/sup&gt;</td>
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<td></td>
<td>(0.208)</td>
<td>(0.306)</td>
<td>(0.261)</td>
<td>(0.470)</td>
<td>(0.270)</td>
<td>(0.201)</td>
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TABLE 7-2 (Continued)

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<th>I-II</th>
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<td>L17</td>
<td>.133 ( .215)</td>
<td>.525</td>
<td>.181</td>
<td>-2.261</td>
<td>-0.642</td>
<td>.249</td>
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<tr>
<td>L18</td>
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<td>.075</td>
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<td>-0.208</td>
<td>.283</td>
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<tr>
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<td>.477</td>
<td>-.044</td>
<td>-.767</td>
<td>-4.405</td>
<td>.210</td>
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<tr>
<td>L20</td>
<td>-.236 ( .233)</td>
<td>.461</td>
<td>-.080</td>
<td>-1.03</td>
<td>-3.564</td>
<td>.184</td>
</tr>
<tr>
<td>L21</td>
<td>-.366 ( .239)</td>
<td>.446</td>
<td>-.124</td>
<td>-1.35*</td>
<td>-7.53*</td>
<td>.154</td>
</tr>
<tr>
<td>L22</td>
<td>-.496* ( .246)</td>
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<td>-.131</td>
<td>-1.72*</td>
<td>-9.38*</td>
<td>.145</td>
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<tr>
<td>L23</td>
<td>-.614* ( .254)</td>
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<td>-.134</td>
<td>-2.34*</td>
<td>-1.15*</td>
<td>.141</td>
</tr>
<tr>
<td>L24</td>
<td>-.782 ( .263)</td>
<td>.427</td>
<td>-.085</td>
<td>-2.59*</td>
<td>-1.35*</td>
<td>.165</td>
</tr>
<tr>
<td>L25</td>
<td>-.963* ( .275)</td>
<td>.426</td>
<td>-.072</td>
<td>-3.29*</td>
<td>-1.61*</td>
<td>.171</td>
</tr>
<tr>
<td>L26</td>
<td>-1.116* ( .285)</td>
<td>.426</td>
<td>-.046</td>
<td>-3.73*</td>
<td>-1.91*</td>
<td>.184</td>
</tr>
<tr>
<td>L27</td>
<td>-1.39* ( .298)</td>
<td>.430</td>
<td>-.027</td>
<td>-4.42*</td>
<td>-2.25*</td>
<td>.196</td>
</tr>
<tr>
<td>L28</td>
<td>-1.69* ( .313)</td>
<td>.429</td>
<td>-.030</td>
<td>-5.25*</td>
<td>-2.69*</td>
<td>.168</td>
</tr>
<tr>
<td>L29</td>
<td>-2.01* ( .333)</td>
<td>.427</td>
<td>-.154</td>
<td>-6.10*</td>
<td>-3.20*</td>
<td>.130</td>
</tr>
<tr>
<td>L30</td>
<td>-2.46* ( .354)</td>
<td>.421</td>
<td>-.300</td>
<td>-7.26*</td>
<td>-9.82*</td>
<td>.052</td>
</tr>
</tbody>
</table>

* Significantly different from zero at the 5% level. Standard errors in parentheses. (.01 = .01%)
For $L_7 - L_{19}$, the estimates for periods I, II, and III were not significantly different from each other. The combined period I-III estimates of $L_7 - L_{14}$ are significantly positive, ranging from .351 to .475, but they show no tendency either to increase or decrease with duration. The period I-III estimates of $L_{15} - L_{18}$ are positive, but not significantly different from zero. The estimate of $L_{19}$ is negative, but is also not significantly different from zero.

Looking at the estimates of $L_7 - L_{19}$ for the combined period I-II and for period III separately, we find that the period III estimates are always smaller. In fact, none of the period III estimates of $L_7 - L_{19}$ are significantly different from zero. They are positive for $L_7 - L_{15}$, and negative for $L_{16} - L_{19}$.

The period I-II estimates, on the other hand, are significantly positive for $L_7 - L_{16}$, and are still positive, but not significantly different from zero, for $L_{17} - L_{19}$. In neither case is there an obvious tendency for the premia to increase or decrease with duration. (Duration increases from 3 1/2 years to 9 1/2 years for $L_7 - L_{19}$.)
The liquidity premia associated with durations of 10 to 15 years ($L_{20} - L_{30}$) were found to be significantly different between period III and the combined period I-II. The period I-II estimates of $L_{20} - L_{30}$ are positive but not significantly different from zero, and they show no general pattern of either increasing or decreasing with duration. The period III estimates are all negative, and are significantly so for $L_{21} - L_{30}$. However, it is at this point that a serious shortcoming in the estimation of the longer-duration liquidity premia must be discussed.

**Biases in Estimating Longer-Duration Liquidity Premia**

Most long-term U.S. Treasury bonds have certain provisions which lead to an understatement of their yields, and an overstatement of their durations, relative to long-term bonds that do not have such provisions. Specifically, many of the outstanding long-term bonds during the period 1954-1975 were acceptable at par in payment of the estate taxes of the owner upon his death. As a result, if the bond were selling at a discount, the estate would
only pay the capital-gains tax rate on the difference between the price of the bond and its par value. These estate bonds (also called "flower bonds") were issued prior to 1963 when market interest rates were low, and had coupon rates ranging from 2 1/2 to 4 1/4%. Furthermore, there was a statutory provision that all Treasury bonds having coupons higher than 4% would have the flower-bond provisions.\(^8\)

Another factor involved here is that there was a statutory limitation on the size of the coupon rate that could be paid on Treasury bonds; the limit was 4 1/4%.\(^9\) This limitation, which was partially relaxed in 1971, prevented coupon rates on Treasury bonds from rising with market interest rates during the late 1960's and early 1970's. Since a bond whose coupon rate is less than its yield-to-maturity must sell at a discount, the increasing difference between the statutory ceiling and market yields led to these Treasury bonds being sold at deep discounts, and discouraged the Treasury from issuing new bonds.

Thus, the long-term bonds making up the Treasury yield curve were predominately flower bonds
which had the potential for special estate tax advantages. As market interest rates rose during the late 1960's, the potential tax advantage increased. If investors mainly purchase these lower bonds for these tax advantages, we could expect that their prices (yields) would be higher (lower) than they would be in the absence of these tax advantages, and that the effect would have increased during the late 1960's and early 1970's (which corresponds to period III).  

The above two effects (tax advantages and ceiling coupon rates) tend to make these "lower bonds" the longer-duration bonds during our sample period. In Chapter III we noted that the coupon rate on a bond and its duration are inversely related. Thus, the smaller the coupon on the bond, the larger is its duration relative to other bonds. Due to the 4 1/4% ceiling rate, the coupon rates of Treasury bonds remained lower than those on Treasury notes, which were not subject to the ceiling. In addition, when the ceiling was relaxed and new long-term bonds were issued, the new bonds carried coupons of 6 1/4 to 7 1/2%, which made them shorter in duration than the
older issues having the same term-to-maturity. For example, of two bonds quoted on January 31, 1975, that were both due in 1998, one had a coupon of 3 1/2% and a duration of 15 years, while the other had a coupon of 7% and a duration of 11 1/2 years.

We also noted in Chapter III that a bond's yield-to-maturity is inversely related to its duration. Thus, the relatively low before-tax yields of the outstanding flower bonds (due to their tax advantages) tend to make them the longest-duration bonds in our sample for any given month. Both of these effects, then, have the result of biasing downward the yield calculations for longer-duration bonds.

The estimated duration-based yield curve is biased for longer-duration bonds in that the curve tends to slope sharply downward after a point. (See Appendix B.) This affects the estimated forward rates used to estimate liquidity premia for these longer durations. The forward rate differences in these ranges of duration will be biased downward, and will lead to an underestimation of the liquidity premia associated with these forward rates. Since the estimates of liquidity premia for even longer
durations are also dependent on these biased forward rate differences, all of the longer-duration premia are affected.

The durations of these flower bonds generally range between ten to fifteen years during 1954-1975 (see Appendix A), which would affect our estimates of $L_{20} - L_{30}$. There may also be smaller effects on the intermediate-duration bonds since some of the flower bonds come closer to maturity during the sample period. This may especially be true during period III, due to the rise in market interest rates during this period. As market interest rates rise, the tax advantages associated with the low-coupon flower bonds increases. The difference between the reported yield and the after-tax yield would widen for such bonds, which may bias estimates for even shorter-duration premia. However, the bias would not be as large as for longer-duration premia.

We cannot, therefore, have a great deal of confidence in our longer-duration estimates of liquidity premia; specifically, $L_{20} - L_{30}$. Furthermore, the period III estimates of the liquidity
premia are especially subject to biases, and even estimates of premia associated with durations less than ten years (i.e., before L_{20}) may be affected.

A solution to these problems of tax bias in the longer-duration bonds would be to adjust all of the yields (for bills, notes, and bonds) to an after-tax basis. Such an undertaking in the context of duration-based yield curves must be left for future research. However, it should be noted that J. Huston McCulloch has worked on this problem, using his work on estimating the coupon-free term structure as a foundation.\textsuperscript{11/}

\section*{Shorter-Duration Liquidity Premia}

If we concentrate on the shorter-duration liquidity premia, from L_2 - L_{19}, we described in a previous section that the estimates are either significantly positive, or are not significantly different from zero. The period I-II estimates, in which we have the most confidence, are significantly positive up to durations of 8 1/2 years (L_{17}), and then are positive, but not significant, for L_{17} - L_{19}. These estimates tend to support McCulloch's
finding over a similar sample period. McCulloch found no significantly negative premia, although he reported an insignificant negative premium for maturities of 10, 15, and 30 years. (The standard errors of the estimated $L_j$ tend to increase with duration; which McCulloch also noted with respect to term-to-maturity.) Our shorter-duration estimates indicate significantly positive premia for durations up to eight years, which is an even stronger result than McCulloch obtained. McCulloch reported positive premia, but not significantly so, for maturities between one and ten years. Only his estimates for maturities of one month through one year were significantly positive.

Although we cannot have as much confidence in our period III estimates, we must recall that the premia for periods I, II, and III were not significantly different from each other for $L_7$ - $L_{19}$; and that the estimates of $L_2$ - $L_6$ for periods II and III were also not significantly different from each other. Taken as part of a combined estimate over period I-III or period II-III, the estimates again
tend to be significantly positive. Only the period I-III estimates of \( L_{15} - L_{19} \) are not significantly different from zero.

These results indicate that the majority of the shorter-duration liquidity premia are positive, which tends to support the liquidity premium theory.

**Monotonicity**

To test the monotonicity of the liquidity premia with respect to duration, we first looked at the increment in the liquidity premium as duration increases for shorter-duration premia, \( L_2 - L_{19} \). (This technique was used by McCulloch with respect to maturity.)

\[
\Delta L_j = L_j - L_{j-1}
\]

These increments are shown in Table 7-3 along with their standard errors for various periods.

In all of the periods (or combinations of periods), we observe both positive and negative values for these terms, but they are rarely
TABLE 7-3

INCENTIVES IN LIQUIDITY PRIMIA FOR VARIOUS PERIODS
(Related to All Observations)

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<th>III</th>
<th>II-III</th>
<th>I-II</th>
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<td>( \Delta L_3 )</td>
<td>0.064g</td>
<td>0.149g</td>
<td>0.050g</td>
<td>0.055</td>
<td>0.053</td>
<td>0.008g</td>
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<td>(0.027)</td>
<td>(0.033)</td>
<td>(0.023)</td>
<td>(0.060)</td>
<td>(0.032)</td>
<td>(0.020)</td>
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<td>( \Delta L_4 )</td>
<td>0.034</td>
<td>0.075g</td>
<td>0.073</td>
<td>0.006</td>
<td>0.015</td>
<td>0.049g</td>
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<tr>
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<td>(0.022)</td>
<td>(0.038)</td>
<td>(0.023)</td>
<td>(0.054)</td>
<td>(0.029)</td>
<td>(0.019)</td>
</tr>
<tr>
<td>( \Delta L_5 )</td>
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<td>0.038</td>
<td>0.014</td>
<td>-0.007</td>
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<td>0.025</td>
</tr>
<tr>
<td></td>
<td>(0.020)</td>
<td>(0.028)</td>
<td>(0.022)</td>
<td>(0.049)</td>
<td>(0.027)</td>
<td>(0.018)</td>
</tr>
<tr>
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<td>0.015</td>
<td>0.010</td>
<td>-0.007</td>
<td>0.001</td>
<td>0.012</td>
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<tr>
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<td>(0.020)</td>
<td>(0.018)</td>
<td>(0.024)</td>
<td>(0.016)</td>
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<td>0.001</td>
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</tr>
<tr>
<td></td>
<td>(0.017)</td>
<td>(0.025)</td>
<td>(0.018)</td>
<td>(0.039)</td>
<td>(0.022)</td>
<td>(0.015)</td>
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<tr>
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<td>-0.001</td>
<td>0.002</td>
<td>-0.002</td>
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<tr>
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<td>-0.0009</td>
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<td>(0.058)</td>
<td>(0.025)</td>
<td>(0.151)</td>
<td>(0.082)</td>
<td>(0.085)</td>
<td>(0.078)</td>
</tr>
<tr>
<td>( \Delta L_{11} )</td>
<td>-0.010</td>
<td>0.012</td>
<td>0.071</td>
<td>-0.113</td>
<td>-0.021</td>
<td>0.043</td>
</tr>
<tr>
<td></td>
<td>(0.071)</td>
<td>(0.027)</td>
<td>(0.178)</td>
<td>(0.109)</td>
<td>(0.104)</td>
<td>(0.092)</td>
</tr>
<tr>
<td>( \Delta L_{12} )</td>
<td>0.097g</td>
<td>0.004</td>
<td>0.275g</td>
<td>-0.062</td>
<td>0.142g</td>
<td>0.115g</td>
</tr>
<tr>
<td></td>
<td>(0.039)</td>
<td>(0.025)</td>
<td>(0.085)</td>
<td>(0.068)</td>
<td>(0.054)</td>
<td>(0.066)</td>
</tr>
<tr>
<td>( \Delta L_{13} )</td>
<td>0.027</td>
<td>-0.012</td>
<td>0.068</td>
<td>0.023</td>
<td>0.045</td>
<td>0.029</td>
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<td></td>
<td>(0.024)</td>
<td>(0.022)</td>
<td>(0.051)</td>
<td>(0.048)</td>
<td>(0.034)</td>
<td>(0.028)</td>
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<td>( \Delta L_{14} )</td>
<td>-0.037</td>
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<td>-0.034</td>
<td>-0.056</td>
<td>-0.045</td>
<td>-0.028</td>
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<td></td>
<td>(0.018)</td>
<td>(0.020)</td>
<td>(0.039)</td>
<td>(0.032)</td>
<td>(0.025)</td>
<td>(0.022)</td>
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<td>( \Delta L_{15} )</td>
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<td>-0.025</td>
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<td>-0.116g</td>
<td>-0.054g</td>
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<td>(0.018)</td>
<td>(0.036)</td>
<td>(0.036)</td>
<td>(0.025)</td>
<td>(0.020)</td>
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<td>-0.072g</td>
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<td>(0.018)</td>
<td>(0.016)</td>
<td>(0.034)</td>
<td>(0.039)</td>
<td>(0.026)</td>
<td>(0.013)</td>
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<td>( \Delta L_{17} )</td>
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<td>-0.118g</td>
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<td>-0.157g</td>
<td>-0.073g</td>
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<td>(0.018)</td>
<td>(0.014)</td>
<td>(0.031)</td>
<td>(0.041)</td>
<td>(0.026)</td>
<td>(0.017)</td>
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<tr>
<td></td>
<td>I-III</td>
<td>I</td>
<td>II</td>
<td>III</td>
<td>II-III</td>
<td>I-II</td>
</tr>
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<td>------</td>
<td>------</td>
<td>------</td>
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</tr>
<tr>
<td>$\Delta L_{18}$</td>
<td>-.121*</td>
<td>-.023</td>
<td>-.106*</td>
<td>-.226*</td>
<td>-.165*</td>
<td>-.066*</td>
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<tr>
<td></td>
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<td>(.014)</td>
<td>(.029)</td>
<td>(.042)</td>
<td>(.026)</td>
<td>(.015)</td>
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<td>$\Delta L_{19}$</td>
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<td>-.024</td>
<td>-.118*</td>
<td>-.279*</td>
<td>-.200*</td>
<td>-.073*</td>
</tr>
<tr>
<td></td>
<td>(.020)</td>
<td>(.015)</td>
<td>(.032)</td>
<td>(.045)</td>
<td>(.028)</td>
<td>(.018)</td>
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</table>

* Significantly different from zero at the 5% level. Standard errors in parentheses. (.01 = .01%)
significantly different from zero. The cases where the increments are significantly positive occur for $\Delta L_2$, $\Delta L_4$, and $\Delta L_{12}$; the cases where they are significantly negative occur for $\Delta L_{15} - \Delta L_{19}$. From this information, it is difficult to accept the hypothesis that the liquidity premia are monotonically increasing with duration.

However, using these increments as tests is a fairly demanding test for establishing monotonicity. A less demanding test is to determine whether the scatter of points of the estimated liquidity premia and their associated durations can be fit by a positively-sloped regression line. That is, we use a regression equation of the form:

$$L_j = a + bD_j + u$$

where $D_j$ is the duration associated with liquidity premium $L_j$; $a$ and $b$ are coefficients, and $u$ is an error term. Our hypothesis of monotonicity would be supported if $b$ is positive.

In addition, the hypothesis that these premia increase monotonically with duration at a
constant rate can be tested using a regression of the form:

\[ I_j = a + b_1 D_j + b_2 D_j^2 + u \]  

If the hypothesis is true, we would expect to find \( \hat{b}_2 \) to be close to zero, and \( \hat{b}_1 \) greater than zero.

Regressions of equations (7.2) and (7.3) were performed for various subperiods, and the results are reported in Tables 7-4 and 7-5. For equation (7.2), the slope coefficient, \( b \), was negative in all cases except one (period II), but was not significantly different from zero for periods I or II or period I-II combined. In none of the periods was there strong evidence of monotonicity of the premia with respect to duration.

For equation (7.5), \( \hat{b}_1 \) is positive and \( \hat{b}_2 \) is negative. The coefficients are significantly different from zero in all cases. Thus, the addition of the \( D_j^2 \) term makes the coefficient of \( D_j \) significantly positive in all cases. The pattern of liquidity premia with respect to duration that is suggested by these results are as follows: The premia increase with duration at a decreasing rate
TABLE 7-4:

RESULTS OF EQUATION (7.2)*

\[ L_j = a + b D_j + u \]

<table>
<thead>
<tr>
<th>Period</th>
<th>( \hat{a} )</th>
<th>( \hat{b} )</th>
<th>( R^2 )</th>
</tr>
</thead>
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<td>-.024</td>
<td>.26</td>
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<tr>
<td>I</td>
<td>.63</td>
<td>-.006</td>
<td>.04</td>
</tr>
<tr>
<td></td>
<td>(14.8)</td>
<td>(-.79)</td>
<td></td>
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<tr>
<td>II</td>
<td>.30</td>
<td>.002</td>
<td>.001</td>
</tr>
<tr>
<td></td>
<td>(4.37)</td>
<td>(.163)</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>.50</td>
<td>-.067</td>
<td>.56</td>
</tr>
<tr>
<td></td>
<td>(5.8)</td>
<td>(-4.3)</td>
<td></td>
</tr>
<tr>
<td>II-III</td>
<td>.80</td>
<td>-.06</td>
<td>.28</td>
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<td></td>
<td>(5.4)</td>
<td>(-2.4)</td>
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<tr>
<td>I-II</td>
<td>.94</td>
<td>-.004</td>
<td>.003</td>
</tr>
<tr>
<td></td>
<td>(9.8)</td>
<td>(-.023)</td>
<td></td>
</tr>
</tbody>
</table>

* t-values in parentheses.
TABLE 7-5:
RESULTS OF EQUATION (7.3)*
\[ L_j = a + b_1D_j + b_2D_j^2 + u \]

<table>
<thead>
<tr>
<th>Period</th>
<th>( \hat{a} )</th>
<th>( \hat{b}_1 )</th>
<th>( \hat{b}_2 )</th>
<th>( \hat{\rho}^2 )</th>
</tr>
</thead>
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<tr>
<td>I-III</td>
<td>.179 (2.8)</td>
<td>.133 (4.7)</td>
<td>-.015 (-5.7)</td>
<td>.77</td>
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<tr>
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<td>.41 (9.4)</td>
<td>.110 (5.48)</td>
<td>-.0116 (-5.9)</td>
<td>.71</td>
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<td>-.08 (.71)</td>
<td>.119 (2.32)</td>
<td>-.0117 (-2.33)</td>
<td>.28</td>
</tr>
<tr>
<td>III</td>
<td>.052 (.632)</td>
<td>.170 (4.49)</td>
<td>-.023 (-6.44)</td>
<td>.98</td>
</tr>
<tr>
<td>II-III</td>
<td>.133 (.711)</td>
<td>.289 (3.39)</td>
<td>-.035 (-4.26)</td>
<td>.69</td>
</tr>
<tr>
<td>I-II</td>
<td>.498 (4.16)</td>
<td>.229 (4.21)</td>
<td>-.023 (-4.39)</td>
<td>.58</td>
</tr>
</tbody>
</table>

* t-values in parentheses.
over a range of duration, but at some point may start to decrease.

Summary

Although the pattern of liquidity premia described above is not entirely consistent with the hypotheses set forth in Chapter IV (that liquidity premia increase monotonically with duration at a constant rate), it is interesting to note that we have been able to identify shorter-duration liquidity premia that do increase monotonically with duration at least over some range. However, it must be admitted that of the three hypotheses set forth in Chapter IV, we have produced strong evidence in favor of only one of them: positive liquidity premia.
NOTES TO CHAPTER VII

1/ Von Neumann ratios for the $L_j$ based on the total sample confirmed this.

2/ This section repeatedly refers to the work of McCulloch, "An Estimate of the Liquidity Premium" in the Journal of Political Economy.

3/ The critical value was 1.673, interpolated from a table of the F-distribution in George W. Snedecor, Statistical Methods, 5th ed. (Ames, Iowa: Iowa State University Press, 1956).

4/ These nine were for $L_2$, $L_3$, $L_4$, $L_{12}$, and $L_{26} - L_{30}$.

5/ Using a two-tailed test.

6/ These standard errors are the standard deviations of the means of the forward rate differences, divided by the square root of the sample size.

7/ Recall that the $L_j$ are for 6-month intervals; i.e., one period equals 6 months.


9/ U.S. Treasury bonds were defined as securities with 5 or more years to maturity until 1971, when they were changed to 7 or more years to maturity. (Treasury notes were then changed from 1 to 5 years to 1 to 7 years.)


This section again relies heavily on McCulloch's "An Estimate of the Liquidity Premium" in the *Journal of Political Economy*.


These negative values may also indicate the presence of tax problems discussed earlier with respect to the estimate of $L_{20} - L_{30}$.

McCulloch reported three significantly positive increments, which were for maturities of one, two, and three months. The values for other maturities were both positive and negative, but not significant. However, since there were no significantly negative increments, he did not reject the hypothesis of the monotonicity of the premia with respect to maturity.
CHAPTER VIII
CONCLUSIONS

The liquidity-preference theory of the term structure maintains that the size of the liquidity (risk) premium of a bond is related to the interest-rate risk of that bond. The general hypotheses of this theory are that these premia are positive, and are monotonically increasing with maturity at a decreasing rate.

This study utilizes the fact that for coupon bonds, interest-rate risk is a function of a bond's duration rather than a bond's term-to-maturity. In light of this, the hypotheses of the liquidity-preference theory are restated to the effect that liquidity premia are positive, and are monotonically increasing with duration at a constant rate.

To test these hypotheses, the duration of U.S. Treasury securities were calculated for a sample period from April, 1954, through August, 1975. Monthly duration-based yield curves were estimated from this data to fill in missing observations on yields and durations. Forward rates were then calculated for securities with adjacent durations, at 6-month intervals.
Estimates of liquidity premia were obtained using a method of forward rate differences suggested by the works of Roll and McCulloch. These estimates were tested for equality among three subperiods of our sample, and combined estimates were then examined depending on the results of these tests.

It was found that the longer-duration estimates of the liquidity premia were biased due to the effects of flower bonds and the coupon rate ceiling. Thus, very little confidence could be placed in the estimated premia for durations between ten and fifteen years.

For bonds with durations between 6 months and ten years, however, the estimated premia were generally positive and significantly different from zero. None of the premia for this range of durations was greater than seven-tenths of one percent (0.7%). This evidence tends to support the hypothesis that liquidity premia are positive.

However, no strong evidence was found to support the second and third hypotheses, that the estimated premia increase monotonically with duration at a constant rate, even though the tests were done only for the shorter-duration premia. Regression results suggest that the premia increase at a decreasing rate up to some point, and then may decrease with duration.
The fact that the empirical results for periods I-II tended to be similar to those found by McCulloch is interesting in that McCulloch also adjusted the observed yield curve data for the presence of coupon bonds. The duration-based approach used here may be thought of as an alternative to McCulloch's approach of estimating the "true" term structure by means of a discount function.\(^2\)

The use of the concept of duration in the term structure literature has only recently had many advocates. There are undoubtedly a large number of ways that this measure might be improved upon as a measure of the "time structure" of coupon bonds. This paper only explores one application of Macaulay's duration to term structure problems.

In doing so, we have ignored many other difficulties that arise in working with the term structure of interest rates, such as the accurate calculation of forward rates and the problems of tax biases. The solving of all these difficulties at one time represents an ideal experiment that awaits future investigation.
NOTES TO CHAPTER VIII

1/ Roll, The Behavior of Interest Rates; McCulloch, "An Estimate of the Liquidity Premium."

2/ McCulloch, "An Estimate of the Liquidity Premium."
APPENDIX A

DURATIONS OF NOTES AND BONDS
FOR SELECTED MONTHS

The following tables give the durations of Treasury notes and bonds, outstanding as of the last trading day of January, for selected years.
PLEASE NOTE:

Pages 129-158 are a computer print-out. Type is broken and indistinct. Best available copy. Filmed as received.

UNIVERSITY MICROFILMS
<table>
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<tr>
<th>YEARS TO MATURITY</th>
<th>DURATION IN YEARS</th>
<th>COUPON</th>
<th>MATURITY DATE</th>
<th>TERMS TO MATURITY</th>
<th>PRICE</th>
<th>YIELD</th>
<th>QUOTATION DATE</th>
</tr>
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<td>0.083333333333</td>
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The number of observations = 28
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<th>YEARS</th>
<th>MONTHS</th>
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<th>PRICE</th>
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APPENDIX B

GRAPHS OF YIELD VERSUS DURATION AND YIELD VERSUS TERM-TO-MATURITY

The following graphs depict the relationship between the yields and durations, and between the yields and term-to-maturity, of the U.S. Treasury notes and bonds outstanding, as of the last trading day of January, for selected years. The two graphs for each month are shown consecutively, with the yield vs. duration curve shown first, followed by the yield vs. term-to-maturity curve.
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**DATA FOR 1/31/57**

**CMNITAB** (PLOTTING YIELD CURVES FOR DURATION AND MATURITY)

**DURATION** vs. **YIELD** graph

- Points plotted: 28
- Points not plotted because they fall outside of bounds: 0

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**Y:** 0.0

**X:** 1.3333E 01
**Y:** 2.6667E 01

**X:** 4.0000E 01

---

---
DATA FOR 1/31/57

CYRITAB (PLOTTING YIELD CURVES FOR DURATION AND MATURITY)

- AXES: MATURITY: LOG: YIELD
- TOTAL NO. OF PTS. PLOTTED IS 28 AND ALL NOT PLOTTED BECAUSE THEY FALL OUTSIDE OF BOUNDS IS 0

1. CCOCOE 01

2. CCCCE CC

3. CCCCE CC

4. CCCCE CC

5. CCCCE CC

6. CCCOE CQ

7. CCCOE CQ

8. CCCOE CQ

TERM TO MATURITY

0.0 1.3333E 01 2.6667E 01 4.0000E 01
DATA FOR 1/30/60

CMNITAB "(PLOTTING YIELD CURVES FOR DURATION AND MATURITY)

TOTAL NO. OF PTS. PLOTTED IS 35 AND AC. NOT PLOTTED, BECAUSE THEY FALL OUTSIDE OF BOUNDS IS 0

1.0000E 01

2.0000E 01

3.0000E 01

4.0000E 01

5.0000E 01

6.0000E 01

7.0000E 01

8.0000E 01

YIELD

1.3333E 01

2.6667E 01

4.0000E 01

DURATION

0.0

0.0

1.3333E 01

2.6667E 01

4.0000E 01
DATA FOR 1/29/60

CYTITAB' (PLOTTING-YIELD CURVES FOR DURATION AND MATURITY)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Total No. of Pts. Plotted Is 39</th>
<th>NOT PLOTTED BECAUSE THEY FALL OUTSIDE OF BOUNDS IS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>C00</td>
<td></td>
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DATA FOR 1/31/63

CYRITAB (PLOTTING YIELD CURVES FOR DURATION AND MATURITY)

ABS-DURATION YIELD

TOTAL NO. OF PTS. PLOTTED IS 48 AND NO. NOT PLOTTED BECAUSE THEY FALL OUTSIDE OF BOUNDS IS 0

1. CODE O1

2. CODE O2

3. CODE O3

4. CODE O4

5. CODE O5

6. CODE O6

7. CODE O7

8. CODE O8

9. CODE O9

10. CODE O10

11. CODE O11

12. CODE O12

13. CODE O13

14. CODE O14

15. CODE O15

16. CODE O16

17. CODE O17

18. CODE O18

19. CODE O19

20. CODE O20

21. CODE O21

22. CODE O22

23. CODE O23

24. CODE O24

25. CODE O25

26. CODE O26

27. CODE O27

28. CODE O28

29. CODE O29

30. CODE O30

31. CODE O31

32. CODE O32

33. CODE O33

34. CODE O34

35. CODE O35

36. CODE O36

37. CODE O37

38. CODE O38

39. CODE O39

40. CODE O40

41. CODE O41

42. CODE O42

43. CODE O43

44. CODE O44

45. CODE O45

46. CODE O46

47. CODE O47

48. CODE O48

DURATION 1.2333E 01 2.6667E 01 4.0000E 01
<table>
<thead>
<tr>
<th>Maturity</th>
<th>Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.0</td>
<td>1.333E+01</td>
</tr>
<tr>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>1.333E+01</td>
<td>1.333E+01</td>
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</tbody>
</table>

DATA FOR 1/31/63

CMNITAB (PLOTTING YIELD CURVES FOR DURATION AND MATURITY)

TOTAL NO. OF PTS. PLOTTED IS 49 AND NO. NOT PLOTTED BECAUSE THEY FALL OUTSIDE OF BOUNDS IS 0
DATA FOR 1/31/88

CMITAB (PLOTTING YIELD CURVES FOR DURATION AND MATURITY)

<table>
<thead>
<tr>
<th>DURATION</th>
<th>TOTAL NO. OF PTS.</th>
<th>YIELD</th>
<th>FLOTTED IS</th>
<th>ALL. NOT FLOTTED</th>
<th>BECAUSE THEY FALL OUTSIDE OF BOUNDS IS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000</td>
<td>6.0000</td>
<td>3.20</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
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<td>4.0000</td>
<td>3.20</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>3.0000</td>
<td>2.0000</td>
<td>3.20</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>4.0000</td>
<td>1.0000</td>
<td>3.20</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
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</table>

0.0 1.3333E 01 2.1667E 01 4.0000E 01
CMNITAB - (PLOTTING YIELD CURVES FOR DURATION AND MATURITY)

DATA FOR 1/29/71:

ACC - DURATION - YIELD
TOTAL NO. OF PTS. PLOTTED IS 46 AND NO. NOT PLOTTED BECAUSE THEY FALL OUTSIDE OF BOUNDS IS 0

1.000E 01

2.000E 01

3.000E 01

4.000E 01

5.000E 01

6.000E 01

7.000E 01

8.000E 01

9.000E 01

1.000E 02

F00E 03

F00E 04

DURATION 1.3333E 01 2.6667E 01 4.0000E 01
CIRITAB (PLOTTING YIELD CURVES FOR DURATION AND MATURITY)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Yield</th>
<th>Pts. Plotted</th>
<th>Not Plotted</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.0</td>
<td>3.724</td>
<td>47</td>
<td>0</td>
</tr>
<tr>
<td>4.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>1.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Data for 1/31/73**

**Total No. of Pts. Plotted is 47 and No. Not Plotted Because They Fall Outside of Bounds is 0**

**Term to Maturity**

0.0 - 1.3333E+01

4.0000E+01
<table>
<thead>
<tr>
<th>DATA FOR 1/31/75</th>
<th>CUNITAB (PLOTTING YIELD CURVES FOR DURATION AND MATURITY)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABS. DURATION</td>
<td>5.20E-01 YIELD</td>
</tr>
<tr>
<td>TOTAL NO. OF PTS. PLOTTED IS 35 AND NO. NOT PLOTTED BECAUSE THEY FALI OUTSIDE OF BOUNDS IS 0</td>
<td></td>
</tr>
</tbody>
</table>

| 1.00E+00 C1      |                                             |
| 2.00E+00 C1      |                                             |
| 3.00E+00 C1      |                                             |
| 4.00E+00 C1      |                                             |
| 5.00E+00 C1      |                                             |
| 6.00E+00 C1      |                                             |
| 7.00E+00 C1      |                                             |
| 8.00E+00 C1      |                                             |

YIELD

| 1.00E+00 C1      | 2.32 3 52 |
| 2.00E+00 C1      | 3.32 52  |
| 3.00E+00 C1      | 4.32 52  |
| 4.00E+00 C1      | 5.32 52  |

X

X

0.0 1.3333E 01 2.6667E 01 4.0000E 01
APPENDIX C

GRAPHS OF TERM-TO-MATURITY VERSUS DURATION

The following graphs depict the relationship between the durations and terms-to-maturity of the U.S. Treasury notes and bonds that make up the Treasury yield curve, as of the last trading day of January, for selected years. The points on the dashed line (also represented by the period marks) represent a 45-degree line where duration and term-to-maturity would be identical. The asterisk marks (*) show the actual relationship between duration and term-to-maturity for each month. The plotting of more than one point at the same location on the graph is represented by a numeral (2, 3, 4, and so on).
DATA FOR 1/29/71

ARS- MATURITY 100- MATURITY (*) DURATION (*)

TOTAL NO. OF PTS. PLOTTED IS 42 AND NO. NOT PLOTTED BECAUSE THEY FALL OUTSIDE OF BOUNDS IS 0

4.0000E 01

3.2000E 01

2.4000E 01

1.6000E 01

7.5999E 00

5.0000E 00

0.0

0.0 1.3333E 01 2.6667E 01 4.0000E 01

TERM TO MATURITY
(DATA FOR 1/31/73)

TOTAL NO. OF PTS. PLOTTED IS 94 AND NO. NOT PLOTTED BECAUSE THEY FALL OUTSIDE OF BOUNDS IS 0

4.0000E 01

2.4000E 01

2.2000E 01

1.6000E 01

1.5999E 00

C.0

0.0

1.3333E 01

2.6667E 01

4.0000E 01

TERM TO MATURITY
BIBLIOGRAPHY


