Fractal Gauges for Hyperspace: One Limit Point

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree
Doctor of Philosophy
in the Graduate School of The Ohio State University

By
Na Peng, B.S.

Graduate Program in Mathematics

The Ohio State University
2010

Dissertation Committee:
Professor Gerald A. Edgar, Advisor
Professor Neil Falkner
Professor Jeffery McNeal
Abstract

We discuss the problem of finding the exact fractal gauge function for a hyperspace (the space of nonempty compact subsets) of a metric space. We consider only the simplest nontrivial example, where the metric space is a sequence with a single limit point.
I am greatly indebted to my advisor Dr. Edgar, for his continual guidance and constant encouragement and help throughout the period of my graduate studies. Also, I want to give my thanks to Dr. Pittel, without who I would hardly realize the beauty of the Probability Theory. Finally, many thanks to Wilson, for the support and help through these years.
Vita

2005 . . . B.S., Southeast University, Nanjing, China
2005-10 Graduate Teaching Associate,
Department of Mathematics,
The Ohio State University,
Columbus, Ohio

Fields of Study

Major Field: Mathematics
Specialization: Fractal Geometry
# Table of Contents

Abstract .................................................................................................................. ii

Acknowledgments ...................................................................................................... iii

Vita ............................................................................................................................ iv

1 Introduction ............................................................................................................ 1
  1.1 Hyperspace ......................................................................................................... 3
  1.2 Covering and Packing Measures ........................................................................ 4
  1.3 General Setting .................................................................................................. 8

2 Example $2^{-n} + 2^{-m}$ ..................................................................................... 11
  2.1 Constituents ....................................................................................................... 12
  2.2 Densities ............................................................................................................ 13
  2.3 Covering Measure ............................................................................................. 13
  2.4 Packing Measure .............................................................................................. 15

3 General $2^{-n}$ ..................................................................................................... 18
  3.1 $\rho_2(m, n) = 2^{-\min(m,n)}$ .......................................................................... 19
  3.2 $\rho_3(m, n) = |2^{-m} - 2^{-n}|$ ........................................................................ 20
  3.3 General Case ..................................................................................................... 23

4 Distance is in the form of $f(m \wedge n)$ ................................................................. 27
  4.1 First example: $f(x) = \frac{2^{-x}}{x}$ .................................................................. 28
  4.2 Three-parameter example: $f_{\alpha,\beta,\gamma}(x) = x^{\beta/\alpha}2^{-x/\alpha}(\ln x)^{\gamma/\alpha}$ .................................................................................................................. 29
  4.3 $f_\alpha(x) = x^{-\alpha}$ ........................................................................................ 30
  4.4 $f_{\alpha,s,M}(x) = M^{1/\alpha} \left(\ln x - \frac{s}{\alpha} \ln \ln x\right)^{-1/\alpha}$ ......................... 31
  4.5 The choice of $\varphi$ .......................................................................................... 31
5 The $1/n$ metric ................................................................. 35
  5.1 Classify $\mathcal{K}$ by the idea of missing number ................. 35
  5.2 The Markov Process ....................................................... 38
  5.3 Coverings ................................................................. 41

Bibliography ................................................................. 55
The notion of dimension arose naturally as humans explored the existing world. But, as the knowledge grew, sets with non-integer dimensions were introduced as a generalization of this notion. They are called fractal dimensions.

First, Hausdorff introduced the Hausdorff measure $\mathcal{H}^s$, where the corresponding gauge function (Def 1.1) is $\varphi(r) = (2r)^s$ and $s \in \mathbb{R}^+$. Later, mathematicians introduced the idea of gauge functions, replacing $\varphi(r)$ by any positive increasing function. Then, Tricot and Taylor [7][6] introduced the notion of packing measure. This opened a new window for more calculations, and we are not limited to the subsets of $\mathbb{R}^n$ any more.

Meanwhile, mathematicians started to study the hyperspace equipped with the Hausdorff metric and to look for the relation between the hyperspace and the original space. E. Boardman [8] studied the hyperspace of the unit interval $[0, 1]$. Let $f_\alpha(t) = 2^{-\alpha t^{-1}}$, $g_\alpha(t) = 2^{-t^{-\alpha}}$ for $\alpha > 0, t > 0$, then consider the Hausdorff measure generated by either functions, we have $g_\alpha(\mathbb{H}[0, 1]) = \infty$, $f_1(\mathbb{H}[0, 1]) = 0$. When $S$ is a countable subset of $[0, 1]$, this offers some insight to the problem I am working on here.

At the same time, when $S$ has the entropy dimension $\frac{1}{2}$, the guess is the possible gauge function for the hyperspace here could be $\varphi(r) = 2^{-Mr^{-1/2}}$ where $M > 0$ [2](6.2.3).
I will work on the hyperspace of a countable set with one limit point, with various metrics.

Chapter Two begins in an easy special case with a calculation of the covering measure and the packing measure. First, after defining the iterated function system, we can see that, except for a measure zero set, the hyperspace is self-similar. This gives us the fractal gauge. Then density theorems are used in a direct calculation to find the two desired measures. Then, coverings and packings are found to provide alternate proofs.

In Chapter Three, we generalize the result of Chapter Two by studying various metrics satisfying a certain property. Our goal is to find the corresponding fractal gauge to give the exact covering and packing measures as before. Due to the similarity of the metrics, the basic structure of the hyperspace is preserved and the argument in Chapter Two can be used in this case.

Things get more interesting in Chapter Four. With the help of the min function, we define several generalized types of metrics involving functions with different orders. The search for the fractal gauge heavily depends on the asymptotic behavior of the metric functions. The goal here is to make both the covering and packing measures positive and finite. Again, the basic structure of the hyperspace is the same, and our work simplifies. Next, in section 4.5, the direction changes: given a function, how can we find a corresponding metric to make these two measures positive and finite?

Finally, we come to the Chapter Five. The regular Euclidean metric on \( \mathbb{R} \) is chosen. Unfortunately, this time the basic structure of the hyperspace is different and other arguments have to be used. To get an estimate of the size of the constituents, we study the Markov Process of “no \( a \) consecutive heads in flipping \( n \) coins”. Then, we discuss the covering gauge for two different sets, a measure 1 set and the whole
space. First, the definition of the fractal measures would naturally give a bound on
the order of the gauge function. Second, we find specific covering for each sets and
direct calculation gives another bound for the order of the gauge function. Last, we
give a lower bound for the packing gauge by constructing a specific packing.

1.1 Hyperspace

We will define the Hausdorff metric and the hyperspace.

Definition 1.1. Let $X$ be a metric space with metric $\rho$, let $A, B \subseteq X$, $r > 0$. Define

$$N_r(A) = \{ x \in X : \text{there exists } y \in A \text{ such that } \rho(x, y) < r \},$$

$$\sigma(A, B) = \inf \{ r > 0 : A \subseteq N_r(B) \text{ and } B \subseteq N_r(A) \}.$$ 

Then $\sigma$ is the Hausdorff metric for closed subsets of $X$.

Definition 1.2. Let $X$ be a complete metric space. Then the hyperspace of $X$ is the
set $\mathbb{H}(X)$ of all nonempty compact subsets of $X$ equipped with the Hausdorff metric.

With the Hausdorff metric, the hyperspace shares some topological properties
with $X$.

Proposition 1.3. If $X$ is a complete metric space, then $\mathbb{H}(X)$ is a complete metric
space. If $X$ is separable, then $\mathbb{H}(X)$ is separable. If $X$ is compact, then $\mathbb{H}(X)$ is
compact.

For proof, see [2, 2.4.1, 2.4.4].
1.2 Covering and Packing Measures

We will recall definitions and theorems to be used later. To begin with, let us define two fractal measures on a metric space, in particular on this hyperspace. First, let us start with the gauge function. We will follow the definition from [1].

**Definition 1.4.** A *gauge* is a function $\varphi$ defined on an interval $(0, \delta)$ for some $\delta > 0$ satisfying the following two conditions

- $\varphi(r) > 0$ for all $r > 0$.
- if $r_1 < r_2$, then $\varphi(r_1) \leq \varphi(r_2)$.

**Definition 1.5.** A gauge $\varphi$ is called *blanketed* iff $\limsup_{r \to 0} \varphi(2r) \varphi(r) < \infty$.

We will need this later in the Density Theorems.

**Definition 1.6.** Let $X$ be a metric space, $a \in X, r > 0$. The *open ball* $\Delta(a, r)$ is defined as

$\Delta(a, r) = \{ x \in X : \rho(x, a) < r \}$.

Correspondingly, the *closed ball* $\overline{\Delta}(a, r)$ is

$\overline{\Delta}(a, r) = \{ x \in X : \rho(x, a) \leq r \}$.

**Definition 1.7.** A *constituent* in $X$ is an ordered pair $(a, r)$ where $a \in X, r > 0$.

**Definition 1.8.** Let $X$ be a metric space, let $a \in X$, let $\mu$ be a finite Borel measure on $X$, and let $\varphi$ be a gauge. The *upper $\varphi$-density* of $\mu$ at $a$ is

$$D^\varphi_\mu(a) = \limsup_{r \to 0} \frac{\mu(\Delta(a, r))}{\varphi(r)}.$$
The lower $\varphi$-density of $\mu$ at $a$ is
\[
\mathcal{D}_\mu^\varphi(a) = \liminf_{r \to 0} \frac{\mu(\Delta(a, r))}{\varphi(r)}.
\]

**Definition 1.9.** Let $X$ be a metric space, and let $A \subseteq X$. A cover or centered cover of $A$ is a set $\beta$ of constituents such that
\[
A \subseteq \bigcup_{(x,r) \in \beta} \Delta(x, r), \quad \text{and} \quad x \in A \text{ for all } (x, r) \in \beta
\]
If $\delta > 0$, then we say the cover $\beta$ is a $\delta$-fine cover provided $r < \delta$ for all $(x, r) \in \beta$.

Define
\[
C_\delta^\varphi(A) = \inf \left\{ \sum_{(x,r) \in \beta} \varphi(r) : \beta \text{ is a } \delta\text{-fine cover of } A \right\}
\]
\[
C_0^\varphi(A) = \sup_{\delta > 0} C_\delta^\varphi(A) = \lim_{\delta \to 0} C_\delta^\varphi(A)
\]
\[
\mathcal{C}^\varphi(A) = \sup \{ C_0^\varphi(E) : E \subseteq A \}
\]
Here $\mathcal{C}^\varphi$ is called the $\varphi$-covering outer measure

**Definition 1.10.** Let $X$ be a metric space, and let $A \subseteq X$. A packing of $A$ is a set $\pi$ of constituents such that $x \in A$ for all $(x, r) \in \pi$, and $\rho(x, x') > r + r'$ for all $(x, r), (x', r') \in \pi$ with $(x, r) \neq (x', r')$. If $\delta > 0$, then we say the packing $\pi$ is $\delta$-fine provided $r < \delta$ for all $(x, r) \in \pi$.

Define
\[
\mathcal{P}_\delta^\varphi(A) = \sup \left\{ \sum_{(x,r) \in \pi} \varphi(r) : \pi \text{ is a } \delta\text{-fine packing of } A \right\}
\]
\[
\mathcal{P}_0^\varphi(A) = \inf_{\delta > 0} \mathcal{P}_\delta^\varphi(A) = \lim_{\delta \to 0} \mathcal{P}_\delta^\varphi(A)
\]
Here $\mathcal{P}^\varphi$ is called the $\varphi$-packing outer measure.

**Definition 1.11.** [5] Suppose that $X$ is a metric space and $E \subset X$ is a nonempty subset, a gauge function is called a covering gauge of $E$ if $0 < C^\varphi(E) < \infty$, a packing gauge of $E$ if $0 < \mathcal{P}^\varphi(E) < \infty$.

**Definition 1.12.** Let $X$ be a metric space. Let $\mu$ be a Borel measure on $X$. Then $\mu$ has the Strong Vitali Property iff, for every Borel set $E \subseteq X$ and every fine cover $\beta$ of $E$, there exists a (countable) centered closed ball packing $\pi \subseteq \beta$ such that

$$
\mu \left( E \setminus \bigcup_{\Delta(x,r) \in \pi} \Delta(x,r) \right) = 0
$$

We say that the packing $\pi$ almost covers the set $E$.

**Theorem 1.13.** Vitali Theorem[1] Let $X$ be a metric space and $E \subseteq X$ be a subset. A fine cover of $E$ is a collection $\beta$ of constituents such that: $x \in E$ for every $(x,r) \in \beta$, and for every $x \in E$ and every $\delta > 0$, there exists $r > 0$ such that $r < \delta$ and $(x,r) \in \beta$. Then there exists either:

1. an infinite (centered closed ball) packing $\{\Delta(x_i,r_i)\} \subseteq \beta$ such that $\inf r_i > 0$, or
2. a countable (possibly finite) centered closed ball packing $\{\Delta(x_i,r_i)\} \subseteq \beta$ such that for all $n \in \mathbb{N}$,

$$
E \setminus \bigcup_{i=1}^{n} \Delta(x_i,r_i) \subseteq \bigcup_{i=n+1}^{\infty} \Delta(x_i,3r_i)
$$
Theorem 1.14. Density Theorem for the covering measure\cite{1}

Let $X$ be a metric space, let $\varphi$ be a gauge, let $\mu$ be a finite Borel measure on $X$, and let $E \subseteq X$ be a Borel set.

1. Then

$$\mu(E) \leq C^\varphi(E) \sup_{x \in E} D^\varphi_{\mu}(x)$$

except when the product is 0 times $\infty$.

2. Assume $\varphi$ is blanketed. Then

$$C^\varphi(E) \inf_{x \in E} D^\varphi_{\mu}(x) \leq \mu(E)$$

Theorem 1.15. Density Theorem for the packing measure\cite{1}

Let $X$ be a metric space, let $\varphi$ be a gauge, let $\mu$ be a finite Borel measure on $X$, and let $E \subseteq X$ be a Borel set.

1. Then

$$\overline{P}^\varphi(E) \inf_{x \in E} D^\varphi_{\mu}(x) \leq \mu(E)$$

2. If $\mu$ has the strong Vitali Property, then

$$\mu(E) \leq \overline{P}^\varphi(E) \sup_{x \in E} D^\varphi_{\mu}(x)$$

except when the product is 0 times $\infty$. 

7
1.3 General Setting

In the following, we consider the metric space $S = \{0, 1, 2, 3, \ldots \}$ equipped with various metrics $\rho$ all satisfying one condition:

0 is the only limit point and all other points are isolated.

Then, under this kind of metric, $\mathbb{H}(S) = \mathcal{K} \cup \mathcal{F}$, where

$$
\mathcal{F} = \{ A \subseteq S : 0 \notin A \text{ and } A \text{ is not nempty } \}
$$

$$
\mathcal{K} = \{ A \subseteq S : 0 \in A \}
$$

We introduce two notations here, which will be repeatedly used later.

Let $\mathcal{A}_n = \{ A \in \mathcal{K} : A \subset \{n, n+1, n+2, \ldots \} \cup \{0\}, n \in A \}$ and similarly,

let $\mathcal{B}_n = \{ B \in \mathcal{F} : B \subset \{0, 1, \ldots, n\}, n \in B \}$.

For each $n$, symmetrically, there are $2^n$ copies of $\mathcal{A}_n$, each has the format $\{(A \setminus \{n\}) \cup C_k : A \in \mathcal{A}_n \}$ with $C_k$ varies among the subsets of $\{1, \ldots, n\}$. Denote them as $\{ \mathcal{A}^k(n) \}$ with $A^1(n) = \mathcal{A}_n$.

Next, we introduce the the natural measure $\mu$ on $\mathbb{H}(S)$.

$$
\mu(\mathcal{F}) = 0, \quad \mu(\mathcal{K}) = 1 \text{ with } \mu(\mathcal{A}^k(n)) = 2^{-n} \text{ for any } n \geq 0, k \in \{1, \ldots, 2^n\}
$$

For all the $\rho$ in this paper, we will show that $\mu$ satisfies the Strong Vitali Property. This would allow us the use the second statement from Theorem 1.15.

Consider $\rho$’s where for any $\delta > 0$, there exists $N > 0$, such that $\rho(n, m) > 2\delta$, if $n, m > N$ and $\rho(n, m) < 2\delta$, if $n, m > N$.

Claim. Under such $\rho$ and the natural measure, we have $\overline{\mathbb{F}}^\rho(\mathcal{K}) = \mu(\mathcal{K})\overline{\mathbb{F}}^\rho_0(\mathcal{K})$
Proof. By definition, \( \overline{P}^\varphi(K) = \inf \left\{ \sum_{n=1}^{\infty} \overline{P}^\varphi_0(E_n) : K \subseteq \bigcup_{n=1}^{\infty} E_n \right\} \). But \( K \subseteq \bigcup_{n=1}^{\infty} E_n \), so we have \( \overline{P}^\varphi(K) \leq \overline{P}^\varphi_0(K) \).

To show the other direction of the inequality, we first look at \( A^k(n) \)'s. Fix an \( n \), for any \( \delta > 0 \) small enough, we have \( \rho(n-1,n) > 2\delta \), which implies each \( A^k(n) \) is at least \( 2\delta \) away from any other \( A^k(n) \)'s. Let \( \pi \) be an arbitrary \( \delta \)-fine packing of \( A^1(n) \), then \( \left\{ (A \setminus n \cup C_k), r \right\} : A \in A^1(n) \} \) is an \( \delta \)-fine packing of \( A^k(n) \) as well. By symmetry, we see that \( \overline{P}^\varphi_0(A^k(n)) \) are the same for each \( k \). Let \( \delta \) go to 0, we have \( \overline{P}^\varphi_0(K) = 2^n \overline{P}^\varphi_0(A^1(n)) \).

To proceed, we turn to the case where \( E \) is only measurable. Then, by the definition of \( \mu \), we see that \( E = \bigcup_{n=1}^{\infty} E_n \), where each \( E_n \) is a symmetric copy of some \( A_{m(n)} \) and they are disjoint. Here \( E_n \) is numerated according to the size of \( m(n) \). Then, given \( \delta > 0 \), there is some \( N > 0 \) such that \( \rho(N, N+1) < \delta \). This implies

\[
\overline{P}^\varphi_{2\delta}(E) = \sum_{n<N} \overline{P}^\varphi_{2\delta}(E_n) + \overline{P}^\varphi_{2\delta}(\bigcup_{n \geq N} E_n)
\]

Since each \( E_n \) is disjoint, as \( \delta \) goes to 0, we have \( N \to \infty \) and therefore \( \mu(E \setminus \bigcup_{n=1}^{N} E_i) \to 0 \). This implies

\[
\overline{P}^\varphi_0(E) = \sum_n \overline{P}^\varphi_0(E_n) = \sum_n \mu(E_n) \overline{P}^\varphi_0(K) = \mu(E) \overline{P}^\varphi_0(K)
\]

Now, for any \( \{E_n\} \) such that \( K \subseteq \bigcup_n E_n \), we group their index into two sets according based on whether there are measurable or not, demote them as \( N \) and \( M \).
Note that $\bigcup_{n \in \mathcal{M}} E_n$ is measurable, we have

\[
\sum_{n \in \mathcal{M}} \overline{P}_0^\phi(E_n) \geq \overline{P}_0^\phi(\bigcup_{n \in \mathcal{M}} E_n) = \mu\left(\bigcup_{n \in \mathcal{M}} E_n\right)\overline{P}_0^\phi(K)
\]

Therefore, taking sum with respect to $\mathcal{N}$ and $\mathcal{M}$, we get

\[
\sum_{n} \overline{P}_0^\phi(E_n) = \sum_{n \in \mathcal{N}} \overline{P}_0^\phi(E_n) + \sum_{n \in \mathcal{M}} \overline{P}_0^\phi(E_n) \geq \mu(K)\overline{P}_0^\phi(K) = \overline{P}_0^\phi(K)
\]

Since this works for any $\{E_n\}$, taking infimum, we see that $\overline{P}^\phi(K) \geq \overline{P}_0^\phi(K)$. \qed
Chapter 2

Example $2^{-n} + 2^{-m}$

In this chapter, we give $S$ the metric $\rho_1$ defined by

$$\rho_1(n, 0) = 2^{-n}, \quad \rho_1(n, m) = 2^{-n} + 2^{-m}$$

We see that under $\rho_1$, $0$ is the only limit point of $S$.

Let $\varphi(r) = 2r$, in this chapter, I will find the covering and packing measures for $\mathbb{H}(S)$.

First, let $A'_1 = K \setminus A_1, F_0 = \{B \in F : 1 \notin B\}, F_1 = \{B \in F : 1 \in B\}$, and define a function $f : S \to S$ by $f(0) = 0, f(x) = x + 1$ for $x \neq 0$. Then, for any $x \neq y$ in $S$, we have $\rho(f(x), f(y)) = \frac{1}{2} \rho(x, y)$; that is, $f$ is a similarity with ratio $\frac{1}{2}$. Then, we define two functions $g_0, g_1 : \mathbb{H}(S) \to \mathbb{H}(S)$ by:

$$g_0(E) = f[E] \cup \{1\} = \{f(x) : x \in E\} \cup \{1\}, \quad g_1(E) = f[E]$$

Then, $g_0$ maps $\mathbb{H}(S)$ onto $A'_1 \cup F_0$ and $g_1$ maps $\mathbb{H}(S)$ onto $A_1 \cup F_1$. Also, $g_1, g_2$ are similarities:

$$\sigma(g_0(E_1), g_0(E_2)) = \frac{1}{2} \sigma(E_1, E_2), \quad \sigma(g_1(E_1), g_0(E_2)) = \frac{1}{2} \sigma(E_1, E_2)$$

11
for any $E_1, E_2 \in \mathbb{H}(S)$. Therefore, the pair $(g_0, g_1)$ is an iterated function system on $\mathbb{H}(S)$ with ratio $(1/2, 1/2)$ (See [2]). Moreover, the two images $A_1 \cup F_1, A'_1 \cup F_0$ are disjoint, so this IFS satisfies the open set condition [2]. The attractor of the IFS is the set $K$, and $K$ is a self-similar Cantor set. Then we see that the Hausdorff dimension for $\mathbb{H}(S)$ is 1. [2]

\section{2.1 Constituents}

First, we calculate all the balls.

Given any $n$ and $B \in B_n$, we have

$$\Delta(B, 2^{-\ell}) = \overline{\Delta}(B, 2^{-\ell}) = \{B\}, \text{for all } \ell \geq n$$

And for any $r \in (2^{-\ell} + 2^{-m}, 2^{-\ell} + 2^{-(m-1)})$ with $m > \ell > n$, we have

$$\Delta(B, r) = \overline{\Delta}(B, r) = \Delta(B, 2^{-\ell} + 2^{-m}) = \{B\}$$

Given any $n$ and $A \in A_n$, we have

$$\Delta(A, 2^{-\ell}) \cap K = \{E \in K : E \cap \{1, \ldots, \ell\} = A \cap \{1, \ldots, \ell\}\}$$

$$\overline{\Delta}(A, 2^{-\ell}) \cap K = \{E \in K : E \cap \{1, \ldots, \ell - 1\} = A \cap \{1, \ldots, \ell - 1\}\}$$

And for any $r \in (2^{-\ell} + 2^{-m}, 2^{-\ell} + 2^{-(m-1)})$ with $m > \ell > n$, we have

$$\Delta(A, r) \cap K = \overline{\Delta}(A, r) \cap K = \{E \in K : E \cap \{1, \ldots, \ell - 1\} = A \cap \{1, \ldots, \ell - 1\}\}$$
2.2 Densities

We can find the densities with respect to the natural measure $\mu$.

Since $\mu(\mathcal{F}) = 0$ and according to the previous calculation, for $r$ small enough, $\forall n$ and $B \in \mathcal{B}_n$ $\Delta(B, r) = \Delta(B, r) = \{B\}$, thus

$$\overline{D}_\mu^\phi(B) = D_\mu^\phi(B) = 0$$

Given any $m > \ell > n$, for all $A \in \mathcal{A}_n$ and $r \in (2^{-\ell} + 2^{-m}, 2^{-\ell} + 2^{-(m-1)}],$

$$\frac{\mu(\Delta(A, 2^{-\ell}))}{\varphi(2^{-\ell})} = \frac{1}{2}, \quad \frac{\mu(\Delta(A, r))}{\varphi(r)} \leq \frac{2^{-\ell+1}}{\varphi(2^{-\ell} + 2^{-m})} = \frac{1}{1 + 2^{\ell-m}}$$

$$\frac{\mu(\Delta(A, 2^{-\ell}))}{\varphi(2^{-\ell})} = \frac{1}{2}, \quad \frac{\mu(\Delta(A, r))}{\varphi(r)} \geq \frac{2^{-\ell+1}}{\varphi(2^{-\ell} + 2^{-(m-1)})} = \frac{1}{1 + 2^{\ell-m+1}}$$

Since $\left[0, \frac{1}{2}\right] = \bigcup_{m > \ell}(2^{-\ell} + 2^{-m}, 2^{-\ell} + 2^{-(m-1)}],$ we conclude that

$$\overline{D}_\mu^\phi(A) = \limsup_{r \to 0} \frac{\mu(\Delta(A, r))}{\varphi(r)} = \lim_{m=\ell, \ell \to \infty} \frac{1}{1 + 2^{\ell-m}} = 1$$

$$\underline{D}_\mu^\phi(A) = \liminf_{r \to 0} \frac{\mu(\Delta(A, r))}{\varphi(r)} = \lim_{m=\ell+1, \ell \to \infty} \frac{1}{1 + 2^{\ell-m+1}} = \frac{1}{2}$$

2.3 Covering Measure

In this section, we will prove $C^\phi(\mathbb{H}(S)) = 1$ using two different methods.

First of all, let us do it using the densities.
On the one hand, for \( \mathcal{K} \),
\[
    C^\varphi(\mathcal{K}) \leq \frac{\mu(\mathcal{K})}{\inf_{A \in \mathcal{K}} \overline{D}_\mu(A)} = 1
\]
\[
    C^\varphi(\mathcal{K}) \geq \frac{\mu(\mathcal{K})}{\sup_{A \in \mathcal{K}} \overline{D}_\mu(A)} = 1
\]
Therefore, noting that \( \varphi \) is blanketed, \( C^\varphi(\mathcal{K}) = 1 \).

On the other hand, for \( \mathcal{F} \), given any \( m > \ell > n \), let \( \delta = 2^{-\ell} \) and
\[
    \kappa_m = \{ \Delta(B, 2^{-m}) : B \in \mathcal{B}_n \},
\]
then \( \kappa_m \) is a \( \delta \)-fine cover of \( \mathcal{B}_n \) and
\[
    C^\varphi(\mathcal{B}_n) \leq \sum_{(x, 2^{-m}) \in \kappa_m} \varphi(2^{-m}) = 2^{n-1} \cdot 2^{-m+1} = 2^{n-m}
\]
Let \( m \to \infty \), we have \( C^\varphi_\delta = 0 \) and \( C^\varphi_0(B) = \lim_{\delta \to 0} C^\varphi_\delta = 0 \).

But \( \kappa_m \) is a \( \delta \)-fine cover for any subset of \( \mathcal{B}_n \), so we conclude that \( C^\varphi(\mathcal{B}_n) = 0 \).

Therefore, \( C^\varphi(\mathcal{F}) \leq \sum_{n=1}^{\infty} C^\varphi(\mathcal{B}_n) = 0 \) and \( C^\varphi(\mathcal{H}(S)) \leq C^\varphi(\mathcal{F}) + C^\varphi(\mathcal{K}) = 1 \).

Also, \( C^\varphi(\mathcal{H}(S)) \geq \frac{\mu(\mathcal{H}(S))}{\sup_{A \in \mathcal{H}(S)} \overline{D}_\mu(A)} = 1 \).

Consequently, \( C^\varphi(\mathcal{H}(S)) = 1 \).

Secondly, let us do it by the original definition, i.e., by the coverings.

Given any \( \ell > n \) and \( A \in \mathcal{A}_n \), we have
\[
    \mu(\Delta(A, r)) = \begin{cases} 
    r, & r = 2^{-\ell} \\
    2^{-\ell+1} \leq 2r, & r \in (2^{-\ell}, 2^{-(\ell-1)})
    \end{cases}
\]
In other words, \( \mu(\Delta(A, r)) \leq 2r \).

Hence, \( \forall \delta \in [2^{-\ell}, 2^{-\ell+1}) \),
\[
    \sum_{\Delta(A, r) \in \kappa} \varphi(r) = 2 \sum_{\Delta(A, r)} r \geq \sum_{\Delta(A, r)} \mu(\Delta(A, r)) \geq 1,
\]
where \( \kappa \) is any \( \delta \)-fine cover of \( \mathcal{K} \). Thus, \( C^\varphi(\mathcal{H}(S)) \geq C^\varphi_0(\mathcal{K}) \geq 1 \).
At the same time, let us denote $2^{\{1,2,\ldots,\ell-1\}}$ as $\{G(i)\}$, then $\forall \; \delta \in [2^{-\ell}, 2^{-(\ell+1)}]$, and $\forall \; m > \ell$, consider the $\delta$-fine cover $\kappa = \left\{ (G(i) \cup \{0\}, 2^{-\ell} + 2^{-m}) \right\}$, then

$$\sum_{\kappa} \varphi(r) = 2 \cdot 2^{\ell-1} \cdot (2^{-\ell} + 2^{-m}) \to 1, \text{ as } m \to \infty$$

Now, given any $E \subseteq \mathbb{H}(S)$, we can rewrite $E = \bigcup_{i=1}^{2^{n-1}} E_i$. Here $E_i = \{E \in \mathcal{E} : E \cap \{1,2,\ldots,n-1\} \text{ is the same}\}$.

Case I: If $E_i \cap K \neq \emptyset$, then $E_i$ can be covered by $2^{\ell-n}$ copies of $\{(G(i) \cup \{0\}, 2^{-\ell} + 2^{-m})\}$;

Case II: If $E_i \cap K = \emptyset$, then $E_i$ can be covered by $2^{\ell-n}$ copies of $\{(G(i), 2^{-\ell} + 2^{-m})\}$.

Therefore, we can find a $\delta$-fine cover of $\mathcal{E}$ consisting of either $(G(i), 2^{-\ell} + 2^{-m})$ or $G(i)$ according to the above 2 cases. Then, $C_{C_{\mathcal{E}}}^\varphi(\mathcal{E}) = \lim_{\delta \to 0} C_{C_{\mathcal{E}}}^\varphi(\mathcal{E}) \leq 1$ and $C^\varphi(\mathbb{H}(S)) \leq C_{C_{\mathcal{E}}}^\varphi(\mathcal{E}) = 1$.

Combine the two, we conclude that $C^\varphi(\mathcal{E}) = 1$.

2.4 Packing Measure

Similarly, we will prove that $\mathcal{P}^\varphi(\mathbb{H}(S)) = 2$, again using two methods.

Firstly, we will do it using densities. But before that

**Claim.** Under $\rho_1$, $\mu$ satisfies the Strong Vitali Property.

**Proof.** Due to the structure of the balls in $\rho_1$, we see that given any $0 < r < 1$, there exists $n$ such that $2^{-n-1} \leq r < 2^{-n}$. This implies that there are at most $2^{n+1}$ disjoint balls with radius more than $r$. Therefore, according to the Vitali Theorem, given any subset $E$ of $\mathbb{H}(S)$ and any fine cover $\beta$ of $E$, we can find a countable centered closed
ball packing \( \{\Delta(x_i, r_i)\} \subseteq \beta \) such that, for all \( n \),

\[
E \setminus \bigcup_{i=1}^{n} \Delta(x_i, r_i) \subseteq \bigcup_{i=n+1}^{\infty} \Delta(x_i, 3r_i)
\]

Note that all these \( \Delta(x_i, r_i) \) are disjoint, we have \( \sum \mu(\Delta(x_i, r_i)) \leq 1 \). On other hand, due to the metric, to cover a ball with radius \( 3r_i \), we just need 4 balls of radius \( r_i \). Therefore, \( \sum \mu(\Delta(x_i, 3r_i)) \leq 4 \sum \mu(\Delta(x_i, r_i)) \leq 4 \). Take \( n \) to infinity, we have \( \mu(E \setminus \bigcup_{i=1}^{\infty} \Delta(x_i, r_i)) = 0 \). \( \square \)

Consider the set \( D_n = K \cup B_n \), since \( B_n \not\rightarrow \mathcal{F} \), we have \( D_n \not\rightarrow K \cup \mathcal{F} = \mathbb{H}(S) \).

For \( \overline{\mathcal{P}}^\phi(K) \), due to the above claim, now we can apply Theorem 1.14 and it shows that

\[
2 = \frac{\mu(K)}{\sup_{A \in K} D_{\mu}^\phi(A)} \leq \overline{\mathcal{P}}^\phi(K) \leq \frac{\mu(K)}{\inf_{A \in K} D_{\mu}^\phi(A)} = 2
\]

Moreover, note that \( \text{dist}(K, B_n) \geq 2^{-n} \), since \( \overline{\mathcal{P}}^\phi \) is a metric outer measure and \( B_n \) is a finite set thus has packing measure 0, we see that

\[
\overline{\mathcal{P}}^\phi(D_n) = \overline{\mathcal{P}}^\phi(K) + \overline{\mathcal{P}}^\phi(B_n) = 2
\]

Therefore, \( \overline{\mathcal{P}}^\phi(\mathbb{H}(S)) = \lim_{n \to \infty} \overline{\mathcal{P}}^\phi(D_n) = 2 \).

Secondly, we will do it using the packings.

According to above, we have already proved that \( \overline{\mathcal{P}}^\phi(\mathbb{H}(S)) = \overline{\mathcal{P}}^\phi(K) \), so it is enough to show \( \overline{\mathcal{P}}^\phi_0(K) = 2 \). The claim from 1.3 will guarantee \( \overline{\mathcal{P}}^\phi(K) = \mu(K)\overline{\mathcal{P}}^\phi_0(K) \).

For any \( \ell > n \) and \( A \in \mathcal{A}_n \), \( \mu(\Delta(A, r)) = \begin{cases} 2r, & r = 2^{-\ell} \\ 2^{-\ell+1} + r, & r \in (2^{-\ell}, 2^{-(\ell-1)}) \end{cases} \)

Thus, \( r < \mu(\Delta(A, r)) \).
On the one hand, \( \forall \delta > 0, \exists \ell, \text{ s.t } 2^{-\ell} \leq \delta < 2^{-(\ell-1)} \).

Then, given any \( \delta \)-fine packing \( \Pi \) of \( \mathcal{K} \),

\[
\sum_{\Delta(A,r) \in \Pi} 2r < 2 \sum_{\Delta(A,r) \in \Pi} \mu(\Delta(A,r)) \leq 2\mu(\mathcal{K}) = 2
\]

Moreover, for the \( \delta \) same as above, let \( r_m = 2^{-\ell-1} + 2^{-\ell+2} + \ldots + 2^{-m}, m > \ell^2 \), then \( r_m \not\rightarrow 2^{-\ell} \). To show the equality holds, we consider the packing \( \Pi_m = \{(B \cup \{0\}, r_m) : B \in \mathcal{B}_\ell\} \), then there are only \( 2^\ell \) disjoint balls there and

\[
\mathcal{P}_\delta^\varphi(\mathcal{K}) \geq \sum_{\Pi_m} \varphi(r_m) = 2r_m * 2^\ell = 1 + \frac{1}{2} + \ldots + 2^{\ell+1-m} \rightarrow 2
\]

Therefore, we proved that \( \overline{\mathcal{P}}_\delta^\varphi(\mathcal{K}) = 2 \).
We will move our focus to other metrics, with \( \rho(n, o) = 2^{-n} \), and our goal is to find the corresponding \( \varphi(r) \) for each metric such that the result \( \mathcal{C}_\mu^{\varphi} = 1 \) and \( \mathcal{D}_\mu^{\varphi} = 2 \) held. For the discussion below, we will reach this goal by showing \( \mathcal{D}_\mu^{\varphi}(A) = 1 \), \( \mathcal{D}_\mu^{\varphi}(A) = \frac{1}{2} \) a.e.

Before introducing those metrics, let us spend some time studying the general structure of the elements of \( \mathcal{A} \), which would give some insight for later calculation.

For all \( A \in \mathcal{K} \) satisfying:

There is a \( N \) such that \( A \cap \{N, N + 1, \ldots\} = \emptyset \) or \( \{N, N + 1, \ldots\} \subseteq A \)

We see that there are only countably many of them.

Therefore, almost everywhere with respect to \( \mu \), for any \( A \in \mathcal{A}_n \), there is a subsequence of natural numbers, say \( \ell_k \)'s, such that \( \ell_k \nearrow \infty \) and \( \{\ell_k\} \subseteq A \). Also, almost everywhere with respect to \( \mu \), for any \( A \in \mathcal{K} \), there is another subsequence of natural numbers, say \( \ell'_k \), such that \( \ell_k \nearrow \infty \) and \( \{\ell'_k, \ldots, \ell'_k + \frac{1}{2} \ln(2\ell'_k)\} \cap A = \emptyset \).

Even more, this implies the existence of some \( \{\ell''_k\} \) such that \( \ell''_k \nearrow \infty \), \( \{\ell''_k\} \cap A = \emptyset \) and \( \{1 + \ell''_k\} \subseteq A \).

Also, by slightly changing the proof, we can see that \( \mu \) satisfies the Strong Vitali Property under any such metrics. Therefore, we can apply the full version of the Theorem 1.14.
One more thing to notice, for each of the following cases, by an argument similar to the beginning of chapter 2, we see that the corresponding Hausdorff dimension is still 1. But, to get $C^\varphi = 1$ and $\overline{P}^\varphi = 2$, we might need some changes on $\varphi$.

3.1 $\rho_2(m, n) = 2^{-\min(m,n)}$

We introduce $\rho_2(m, n) = 2^{-\min(m,n)}, m \neq n \neq 0; \quad \rho_2(m, 0) = 2^{-m}$. Then, under $\rho_2$, 0 is still the only limit point for $S$.

First of all, let us find all the constituents. Given $\ell > n$, for any $A \in \mathcal{A}_n$ and for any $r \in (2^{-\ell}, 2^{-\ell+1})$, we have

$$\mathcal{K} \cap \Delta(A, 2^{-\ell}) = \{ E : E \cap \{1, \cdots, \ell\} = A \cap \{1, \cdots, \ell\} \}$$

$$\mathcal{K} \cap \Delta(A, r) = \mathcal{K} \cap \Delta(A, 2^{-\ell}) = \mathcal{K} \cap \Delta(A, r) = \{ E : E \cap \{1, \cdots, \ell-1\} = A \cap \{1, \cdots, \ell-1\} \}$$

Since they have the same structure as the constituents from chapter 1, we see that under $\rho_2$, we have the same densities as $\rho_1$.

Naturally, let us generalize this metric by introducing

for all $\alpha > 0, \rho^{(\alpha)}(m, n) = (\rho_2(m, n))^{1/\alpha}, \forall \ m \leq 0, n \leq 0$

This time, we let $\varphi(r) = 2 \cdot 2^\alpha$. Then, follow the same calculation as above, for $\ell > n$ and $\forall A \in \mathcal{A}_n, r \in \left(2^{-\ell/\alpha}, 2^{-(\ell-1)/\alpha}\right)$, we have

$$\mathcal{K} \cap \Delta(A, 2^{-\ell/\alpha}) = \{ E : E \cap \{1, \cdots, \ell\} = A \cap \{1, \cdots, \ell\} \}$$

$$\mathcal{K} \cap \Delta(A, r) = \mathcal{K} \cap \Delta(A, r) = \mathcal{K} \cap \Delta(A, 2^{-\ell/\alpha}) = \{ E : E \cap \{1, \cdots, \ell-1\} = A \cap \{1, \cdots, \ell-1\} \}$$

19
Note that \( \mu(\Delta(A, r)) \frac{2^{-\ell}}{\varphi(2^{-\ell/\alpha})} < 1 \), and it approaches this supremum when \( r \to 2^{-\ell/\alpha +} \). Therefore, we have

\[
\overline{D}_\mu(\mathcal{K} \cap A) = \limsup_{r \to 0} \frac{\mu(\mathcal{K} \cap \Delta(A, r))}{\varphi(r)} = \lim_{\ell \to \infty} \frac{\mu(\mathcal{K} \cap \Delta(A, 2^{-\ell/\alpha +}))}{\varphi(2^{-\ell/\alpha +})} = 1
\]

Similarly, note that \( \frac{\mu(\Delta(A, r))}{\varphi(r)} > \frac{2^{-\ell}}{\varphi(2^{-\ell/\alpha})} = \frac{1}{2} \), and it approaches this infimum when \( r \to 2^{-\ell/\alpha -} \). Therefore, we have

\[
\underline{D}_\mu(\mathcal{K} \cap A) = \liminf_{r \to 0} \frac{\mu(\mathcal{K} \cap \Delta(A, r))}{\varphi(r)} = \lim_{\ell \to \infty} \frac{\mu(\mathcal{K} \cap \Delta(A, 2^{-\ell/\alpha -}))}{\varphi(2^{-\ell/\alpha -})} = \frac{1}{2}
\]

### 3.2 \( \rho_3(m, n) = |2^{-m} - 2^{-n}| \)

Now, let us define another metric

\[
\rho_3(m, n) = |2^{-m} - 2^{-n}|, m \neq n \neq 0; \quad \rho_3(m, 0) = 2^{-m}
\]

Again, 0 is the only limit point under \( \rho_3 \).

Moreover, let \( \tilde{\varphi}(r) = 2 \cdot 2^{-\ell} \), for all \( r \in (2^{-(\ell+1)}, 2^{-\ell}] \).

Again, we try to find the structure of the constituents first.

This time, the structure of the constituents does depend of the structure of \( A \). Given any \( n + 1 < \ell < m - 1 \), for all \( A \in \mathcal{A}_n, \forall r \in (2^{-\ell} - 2^{-(m-1)}, 2^{-\ell} - 2^{-m}) \), we
start with the constituents first.

\[ \mathcal{K} \cap \Delta(A, 2^{-\ell}) = \bigcap \left\{ \begin{array}{l}
{E : E \cap \{1, \ldots, \ell - 1\} = A \cap \{1, \ldots, \ell - 1\}} \\
{E : E \cap \{\ell, \ldots\} \neq \emptyset}, \ \ell \in A \\
\mathcal{K}, \ \ell \notin A, A \cap \{\ell + 1, \ldots\} \neq \emptyset \\
{E : \ell \notin E}, \ A \cap \{\ell, \ldots\} = \emptyset 
\end{array} \right. \]

Therefore, \( \mu(\mathcal{K} \cap \Delta(A, 2^{-\ell})) = \left\{ \begin{array}{l}
2^{-\ell+1}, \ \ell \in A \\
2^{-\ell+1}, \ \ell \notin A, A \cap \{\ell + 1, \ldots\} \neq \emptyset \\
2^{-\ell}, \ A \cap \{\ell, \ldots\} = \emptyset 
\end{array} \right. \)

\[ \mathcal{K} \cap \Delta(A, 2^{-\ell}) = \bigcap \left\{ \begin{array}{l}
{E : E \cap \{1, \ldots, \ell - 2\} = A \cap \{1, \ldots, \ell - 2\}} \\
{E : E \cap \{\ell - 1, \ell\} \neq \emptyset}, \ \ell - 1 \in A \\
\mathcal{K}, \ \ell - 1 \notin A, \ell \in A \\
{E : \ell - 1 \notin E}, \ \ell - 1 \notin A, \ell \notin A 
\end{array} \right. \]
Therefore, \( \mu(K \cap \bar{\Delta}(A, 2^{-\ell})) = \begin{cases} 
3 \cdot 2^{-\ell}, & \ell - 1 \in A \\
2^{-\ell+2}, & \ell - 1 \notin A, \ell \in A \\
2^{-\ell+1}, & \ell - 1 \notin A, \ell \notin A 
\end{cases} \)

\[
K \cap \Delta(A, 2^{-\ell} - 2^{-m}) = \{E : E \cap \{1, \ldots, \ell - 1\} = A \cap \{1, \ldots, \ell - 1\}\} \\
K \cap \Delta(A, r) = \bigcap K, \quad \ell \notin A, A \cap \{\ell, \ldots, m - 1\} \neq \emptyset \\
K \cap \bar{\Delta}(A, r) = \{E : \ell \notin E\}, \quad A \cap \{\ell, \ldots, m - 1\} = \emptyset 
\]

\[
\mu(K \cap \Delta(A, 2^{-\ell} - 2^{-m})) = \begin{cases} 
2 \cdot (2^{-\ell} - 2^{-m}), & \ell \in A \\
2^{-\ell+1}, & \ell \notin A, A \cap \{\ell, \ldots, m - 1\} \neq \emptyset \\
2^{-\ell}, & A \cap \{\ell, \ldots, m - 1\} = \emptyset 
\end{cases} \]

\[
K \cap \bar{\Delta}(A, 2^{-\ell} - 2^{-m}) = \bigcap K, \quad \ell \notin A, A \cap \{\ell, \ldots, m\} \neq \emptyset \\
\{E : \ell \notin E\}, \quad A \cap \{\ell, \ldots, m\} = \emptyset 
\]

Then, we have \( \mu(K \cap \bar{\Delta}(A, 2^{-\ell} - 2^{-m})) = \begin{cases} 
2^{-(\ell-1)} - 2^{-m}, & \ell \in A \\
2^{-\ell+1}, & \ell \notin A, A \cap \{\ell, \ldots, m\} \neq \emptyset \\
2^{-\ell}, & A \cap \{\ell, \ldots, m\} \neq \emptyset 
\end{cases} \)
According to the discussion at the beginning of this chapter, except for a measure zero set, for any $A \in \mathcal{K}$, there exist sequences of indices $\ell_k'$ and $\{\ell_k''\}$.

First, let us use the $\{\ell_k''\}$ to find $\overline{D}_\mu^{\delta}(A)$. Judging by the size of $\mu(\Delta(A, r))$ corresponding to $\ell$, we see that such maximum would occur in the interval $r \in (2^{-(\ell_k''+1)}, 2^{\ell_k''}]$. But, in this interval $\frac{\mu(\Delta(A, r))}{\varphi(r)} = \frac{2^{\ell_k''+1}}{2^{\ell_k''+1}} = 1$.

Therefore, we conclude that $\overline{D}_\mu^{\delta}(A) = \limsup_{r \to 0} \frac{\mu(\Delta(A, r))}{\varphi(r)} = 1$.

Second, we will use $\{\ell_k'\}$ to find $\underline{D}_\mu^{\delta}(A)$. Again, judging by the size of $\mu(\Delta(A, r))$ corresponding to $\ell$, we see that such minimum would occur in the interval $r \in (2^{-(\ell_k'+1)}, 2^{\ell_k'})$. But, in this interval $\frac{\mu(\Delta(A, r))}{\varphi(r)} \geq \frac{2^{\ell_k'}}{2^{\ell_k'+1}} = \frac{1}{2}$, where the equality holds when $r \neq 2^{\ell_k'}$.

Therefore, we have $\underline{D}_\mu^{\delta}(A) = \liminf_{r \to 0} \frac{\mu(\Delta(A, r))}{\varphi(r)} = \frac{1}{2}$.

### 3.3 General Case

In this section, we will find some generalized results.

Given any metric $\rho$ with $\rho(n, 0)$, let $\varphi(r)$ be the corresponding gauge such that $C_\mu^\varphi = 1$ and $\overline{D}_\mu^\varphi = 2$.

Claim. If there is $N > 0$ big enough such that $\rho(m, n) \geq 2^{-m\wedge n} = \rho_2(m, n)$, $\forall m \neq n > N$, then $\varphi(r) = 2r$.

Proof. Thanks to the Triangle Inequality, we have $\rho_2 \leq \rho \leq \rho_1$ besides finite many integers. Then, for any $r \in (0, 1)$ and any $A \in \mathcal{K}$, we have $\Delta_1(A, r) \subseteq \Delta(A, r) \subseteq \Delta_2(A, r)$, same for the closed balls. Here $\Delta_i(A, r)$ stand for $\Delta(A, r)$ under the metric $\rho_i$. Note that we chose the same $\varphi$ for both $\rho_1$ and $\rho_2$, then stick to the same choice,
we will still get $\overline{D}_\mu^\varphi(K \cap A) = 1$, $D_\mu^\varphi(K \cap A) = 0.5$. Therefore, $\varphi(r) = 2r$ is the desired gauge.

Next, we turn to another kind of $\rho$ where $\rho_3(m, n) < \rho(m, n) < \rho_2(m, n)$ if $m \neq n > N$ for some $N$ big enough.

**Claim.** In this case, the desire gauge is $\tilde{\varphi}(r) = 2 \cdot 2^{-\ell}$, for all $r \in (2^{-(\ell+1)}, 2^{-\ell}]$, as in the previous section.

**Proof.** Thanks to the Triangle Inequality, we see that

$$\rho(n + 1, 0) \leq \rho(n, n + 1) \leq \rho(n, n + 2) \leq \cdots < \rho(n, 0), \forall \ n > N$$

So, given any $n + 1 < \ell < m$, for all $A \in \mathcal{A}_n$ and $r \in (\rho(\ell, m), \rho(\ell, m + 1))$

$$\mathcal{K} \cap \Delta(A, 2^{-\ell}) = \bigcap_{K, \ell \not\in A, \ A \cap \{\ell + 1, \cdots\} \neq \emptyset} \left\{ E : E \cap \{1, \cdots, \ell - 1\} = A \cap \{1, \cdots, \ell - 1\} \right\}$$

Therefore, $\mu(K \cap \Delta(A, 2^{-\ell})) = \begin{cases} 2^{-\ell+1}, & \ell \in A \\ 2^{-\ell+1}, & \ell \not\in A, A \cap \{\ell + 1, \cdots\} \neq \emptyset \\ 2^{-\ell}, & A \cap \{\ell, \cdots\} = \emptyset \end{cases}$

Since $\rho(\ell - 1, \ell) > \rho_3(\ell - 1, \ell) = 2^{-\ell}$,

$$\mathcal{K} \cap \overline{\Delta}(A, 2^{-\ell}) = \left\{ E : E \cap \{1, \cdots, \ell - 1\} = A \cap \{1, \cdots, \ell - 1\} \right\}$$
Therefore, \( \mu(K \cap \overline{\Delta}(A, 2^{-\ell})) = 2^{-\ell+1} \).

Moreover, since \( \rho(\ell + 1, \ell) > \rho(\ell + 1, 0) \), for any \( r \in (\rho(\ell + 1, 0), \rho(\ell, \ell + 1)) \)

\[
K \cap \Delta(A, \rho(\ell, \ell + 1)) = K \cap \Delta(A, r) = K \cap \overline{\Delta}(A, r) = \{ E : E \cap \{1, \cdots, \ell \} = A \cap \{1, \cdots, \ell \} \}
\]

So, \( \mu(\Delta(A, \rho(\ell, \ell + 1))) = \mu(\Delta(A, r)) = \mu(\overline{\Delta}(A, r)) = 2^{-\ell} \).

\[
K \cap \Delta(A, \rho(\ell, m + 1)) = \{ E : E \cap \{1, \cdots, \ell - 1\} = A \cap \{1, \cdots, \ell - 1\} \}
\]

\[
K \cap \Delta(A, r) = \bigcap \left\{ \begin{array}{ll}
E : E \cap \{\ell, \cdots, m\} \neq \emptyset, & \ell \in A \\
K, & \ell \notin A, A \cap \{\ell, \cdots, m\} \neq \emptyset \\
\{ E : \ell \notin E \}, & A \cap \{\ell, \cdots, m\} = \emptyset
\end{array} \right\}
\]

\[
\mu(K \cap \Delta(A, \rho(\ell, m + 1))) = \mu(K \cap \Delta(A, r)) = \mu(K \cap \overline{\Delta}(A, r)) = \left\{ \begin{array}{ll}
2^{-\ell+1} - 2^{-m}, & \ell \in A \\
2^{-\ell+1}, & \ell \notin A, A \cap \{\ell, \cdots, m\} \neq \emptyset \\
2^{-\ell}, & A \cap \{\ell, \cdots, m\} = \emptyset
\end{array} \right\}
\]

\[
K \cap \overline{\Delta}(A, \rho(\ell, m)) = \{ E : E \cap \{1, \cdots, \ell - 1\} = A \cap \{1, \cdots, \ell - 1\} \}
\]

\[
K \cap \Delta(A, \rho(\ell, m + 1)) = \bigcap \left\{ \begin{array}{ll}
E : E \cap \{\ell, \cdots, m\} \neq \emptyset, & \ell \in A \\
K, & \ell \notin A, A \cap \{\ell, \cdots, m\} \neq \emptyset \\
\{ E : \ell \notin E \}, & A \cap \{\ell, \cdots, m\} = \emptyset
\end{array} \right\}
\]
Then, we have $\mu\left(\mathcal{K} \cap \Delta(A, \rho(\ell, m))\right) = \begin{cases} 2^{-(\ell-1)} - 2^{-m}, & \ell \in A \\ 2^{-\ell+1}, & \ell \notin A, A \cap \{\ell, \ldots, m\} \neq \emptyset \\ 2^{-\ell}, & A \cap \{\ell, \ldots, m\} = \emptyset \end{cases}$

Let us use $\tilde{\varphi}$. Since the structure of the constituents in this case is quite similar as in the previous case, by a similar argument, we see that

$$\mathcal{D}_{\mu}^\varphi(A) = \limsup_{r \to 0} \frac{\mu(\Delta(A, r))}{\varphi(r)} = 1, \quad \mathcal{D}_{\mu}^\varphi(A) = \liminf_{r \to 0} \frac{\mu(\Delta(A, r))}{\varphi(r)} = \frac{1}{2}$$

Then, what happen to other metrics $\rho$ where no such $N$ could be found? Out of these two gauges, can we just settle with one? The answer is yes.

**Claim.** The gauge $\tilde{\varphi}(r)$ works here as well, if and only if, there are $\ell''_k \nearrow \infty$ such that $\rho(\ell''_k, m) < \rho_2(\ell''_k, m)$ for any $m > \ell''_k$ and any $k > 0$.

**Proof.** By the argument in the beginning of this chapter, we see that, almost everywhere on $\mathcal{K}$, $\{\ell''_k, \ell''_k + 1, \ldots\} \cap A \neq \emptyset$ for any $k > 0$ and $A \in \mathcal{K}$. Therefore, for any $r \in \left(\rho(\ell''_k, m), (\rho''_k, m + 1)\right)$ where $m > \ell''_k$, we have $\mu(A, r) = 2^{-\ell''_k+1}$. Therefore, to ensure $\mathcal{D}^\tilde{\varphi}_\mu(A) = 1$, we have to choose $\tilde{\varphi}(r)$. For $\mathcal{D}^\varphi_\mu(A)$, since $\tilde{\varphi}(r) \geq 2r$, and 0.5 can be reaches in $(2^{-\ell''_k-1}, 2^{-\ell''_k})$, we will not have any issues here.

At the same time, if not such $\ell''_k$ can be found, then for $n$ big enough, we have $\rho(n, m) \geq \rho_2(n, m)$ for all $m > n$. This falls into the category of the first claim of this section.\[\Box\]
Chapter 4

Distance is in the form of $f(m \wedge n)$

In this chapter, we will introduce a set of metrics with the following properties:

$$\rho(m, n) = f(m \wedge n) \text{ and } \rho(m, 0) = f(m) \text{ for } m \neq n > N$$

where $\forall \ x > N, 0 < f(x) \leq \frac{1}{2}, f'(x) < 0.$

Now, our goal is to find the corresponding $\varphi(r)$ such that both $C_\mu^\varphi$ and $\overline{P}_\mu^\varphi$ are positive and finite.

First of all, let us show the validity of the Triangle Inequality. On the one hand, note that $f$ is decreasing, then given any $k, n, m > N$, if $k < n \wedge m$, $\rho(m, n) = f(n \wedge m) < f(k) = \rho(k, n) < \rho(k, n) + \rho(k, m)$. Also, if $m \wedge n < k$, then $\rho(m, n) = f(n \wedge m) = f(n \wedge k) \vee f(m \wedge k) < \rho(n, k) + \rho(m, k)$. On the other hand, say $k < m \wedge n$ and $k < N$, then $\rho(m, n) \leq \frac{1}{2} = \rho(k, m) < \rho(k, m) + \rho(k, n)$. Also, of $m \wedge n < k$ and $m \wedge n < N$, then $\rho(m, n) = \frac{1}{2} = \rho(m, k) \vee \rho(n, k) < \rho(m, k) + \rho(n, k)$.

Secondly, one thing to notice, given any such metric $\rho$, for any $n > 0$ and any $A \in \mathcal{A}_n$, for $\ell > n \vee N$ and any $r \in \left( f(\ell), f(\ell - 1) \right)$ we have

$$\mathcal{K} \cap \Delta(A, f(\ell)) \subseteq \{ E : E \cap \{1, \cdots, \ell\} = A \cap \{1, \cdots, \ell\} \}$$

$$\mathcal{K} \cap \Delta(A, r) = \mathcal{K} \cap \overline{\Delta}(A, r) = \mathcal{K} \cap \overline{\Delta}(A, f(\ell)) \subseteq \{ E : E \cap \{1, \cdots, \ell - 1\} = A \cap \{1, \cdots, \ell - 1\} \}$$

Therefore, \( \frac{\mu(\Delta(A, r))}{\varphi(r)} \leq \frac{2^{-(\ell-1)}}{\varphi(f(\ell)^+)} \), as well as \( \frac{\mu(\overline{\Delta}(A, r))}{\varphi(r)} \geq \frac{2^{-(\ell-1)}}{\varphi(f(\ell - 1)^-)} \).
4.1 First example: \( f(x) = \frac{2^{-x}}{x} \)

Because \( f(x) \) satisfies all the prerequisites with \( N = 1 \), we define \( \rho_5(m, n) = \frac{2^{-(m \land n)}}{m \land n} \), with \( \rho_5(m, 0) = f(m) = \frac{2^{-m}}{m} \). According to the discussion in the beginning of this chapter, we see that \( \rho \) is indeed a metric.

Then, let \( \varphi(r) = \frac{2}{\ln 2} \cdot r |\ln r| \), then \( \varphi(r) \) is an increasing function with \( \varphi(r) \to 0 \) as \( r \to 0 \). So, it satisfies the prerequisite for a gauge function. Furthermore,

\[
\lim_{r \to 0} \frac{\varphi(2r)}{\varphi(r)} = \lim_{r \to 0} \frac{2}{r |\ln r|} = 2
\]

So, \( \varphi \) is blanketed and Density Theorem could be used for both cases.

According to the discussion at the beginning of this chapter, we have

\[
\mathcal{D}_\mu^\varphi (\mathcal{K} \cap A) = \lim_{\ell \to \infty} \frac{\mu(\mathcal{K} \cap \Delta(A, f(\ell)^+))}{\varphi(f(\ell)^+)} = \lim_{\ell \to \infty} \frac{\ell \ln 2}{\ell \ln 2 + \ln \ell} = 1
\]

\[
\mathcal{D}_\mu^\varphi (\mathcal{K} \cap A_n) = \lim_{\ell \to \infty} \frac{\mu(\mathcal{K} \cap \Delta(A, f(\ell - 1)^-))}{\varphi(f(\ell - 1)^-)} = \lim_{\ell \to \infty} \frac{(\ell - 1) \ln 2}{2 \cdot [((\ell - 1) \ln 2 + \ln(\ell - 1)]} = \frac{1}{2}
\]

Then, using the same arguments as in 2.3 and 2.4, we have both \( C_\mu^\varphi \) and \( \mathcal{P}_\mu^\varphi \) are positive and finite.

Let us generalize this metric a little bit to a two-parameter example. Pick any \( \alpha, \beta > 0 \) and let

\( f_{\alpha,\beta}(x) = 2^{-x/\alpha} x^{-\beta/\alpha} \), here \( f_{\alpha,\beta}(x) \) is decreasing with \( \lim_{x \to \infty} = 0 \). Then we define a metric by \( \rho(m, n) = f_{\alpha,\beta}(m \land n), \forall m \neq n \neq 0 \) and \( f(m, 0) = f_{\alpha,\beta}(m) \).

Next, let \( \varphi(r) = 2\alpha^\beta (\ln 2)^{-\beta \alpha} r |\ln r|^\beta \). Here \( \varphi(r) \) is an increasing function with \( \varphi(r) \to 0 \) as \( r \to 0 \). So, it satisfies the prerequisite for a gauge function as well. The
argument above shows $\varphi$ is blanket. So, the Density Theorem shows both measures are positive and finite.

Similarly, we have

$$
D_{\mu}^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell)^{+}))}{\varphi(f(\ell)^{+})} = \lim_{\ell \to \infty} \left(1 + \frac{\beta \ln \ell}{\ell \ln 2}\right)^{-\beta} = 1
$$

$$
D_{\mu}^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell - 1)^{-}))}{\varphi(f(\ell - 1)^{-})} = \lim_{\ell \to \infty} \frac{1}{2} \left(1 + \frac{\beta \ln(\ell - 1)}{(\ell - 1) \ln 2}\right)^{-\beta} = \frac{1}{2}
$$

4.2 Three-parameter example: $f_{\alpha, \beta, \gamma}(x) = x^{\beta/\alpha}2^{-x/\alpha}(\ln x)^{\gamma/\alpha}$

This time, for any $\alpha > 0, \beta, \gamma \geq 0$, we introduce $f_{\alpha, \beta, \gamma}(x)$ as stated. Note that

$$
f'_{\alpha, \beta, \gamma}(x) = x^{\beta/\alpha - 1}2^{-x/\alpha}(\ln x)^{\gamma/\alpha - 1} - x^{\beta/\alpha - 1}2^{-x/\alpha}(\ln x)^{\gamma/\alpha - 1} \frac{\ln x}{x \ln 2} + \gamma
$$

Then for some $N'$ large enough, $f'_{\alpha, \beta, \gamma}(x) < 0$ if $x \geq N'$. Therefore, there is some $N > N'$ such that $f(x) \leq \frac{1}{2}$.

Now, let us introduce the metric $\rho(m, n) = f_{\alpha, \beta, \gamma}(m \land n) \land n > M$, and $\rho(m, 0) = f_{\alpha, \beta, \gamma}(m)$. Also, for any $m$, and $n \leq M$, we let $\rho(m, n) = \frac{1}{2}$.

Next, let us introduce $\varphi(r) = 2(\ln 2)^{\beta}\alpha^{-\beta}r^{\alpha}(\frac{1}{r})^{\ln \frac{1}{r}} - \beta(\ln \ln \frac{1}{r})^{\gamma}$, then $\varphi(r)$ is an increasing function on $(0, 1)$ with $\varphi(r) \to 0$ as $r \to 0$. Furthermore,

$$
\lim_{r \to 0} \frac{\varphi(2r)}{\varphi(r)} = \lim_{r \to 0} 2^\alpha \cdot \left|1 + \frac{\ln 2}{\ln r}\right|^{\beta} \left(\frac{\ln \ln 2r}{\ln \ln r}\right)^{\gamma} = 2^\alpha
$$

So, $\varphi$ is blanket and Density Theorem could be used for both cases.
Then, again, we have

\[
\mathcal{D}_\mu^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell)\plus))}{\varphi(f(\ell)\plus)} = \lim_{\ell \to \infty} \left( 1 - \frac{\beta \ln \ell}{\ell \ln 2} - \frac{\gamma \ln \ln \ell}{\ell \ln 2} \right)^{-\beta} \left( 1 + O\left( \frac{1}{\ln \ell} \right) \right)^{\gamma} = 1
\]

\[
\mathcal{D}_\mu^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell)\minus))}{\varphi(f(\ell)\minus)}
\]

\[
= \lim_{\ell \to \infty} \frac{1}{2} \left( 1 - \frac{\beta \ln(\ell - 1)}{(\ell - 1) \ln 2} - \frac{\gamma \ln \ln(\ell - 1)}{(\ell - 1) \ln 2} \right)^{-\beta} \left( 1 + O\left( \frac{1}{\ln(\ell - 1)} \right) \right)^{\gamma}
\]

\[
= \frac{1}{2}
\]

### 4.3 \( f_\alpha(x) = x^{-\alpha} \)

In the previous two sections, \( f(x) = O(a^x) \) as \( x \to \infty \) for some \( 0 < a < 1 \). But, here we do not need \( f \) to go to zero exponentially due to the fact \( \rho(m, n) = f(m \land n) \).

This time, let us introduce \( \rho(m, n) = f_\alpha(m \land n) = (m \land n)^{-\alpha} \), then \( \rho \) is a metric.

Next, let \( \varphi(r) = 2^{-r^{1/\alpha}} \). Note that \( \varphi(r) \) is increasing with \( \varphi(r) \to 0 \), it satisfies the prerequisite of the gauge. But \( \frac{\varphi(2r)}{\varphi(r)} \to \infty \) as \( r \to 0 \), which leads to the situation that \( \varphi(r) \) is not blanketed. Then, different from the previous sections, when it comes to \( \mathcal{C}_\varphi(K) \), the density theorem can only give an lower bound. But, the upper bound part could be done by referring to last part of 2.3, where we use a special covering \( \kappa \).

\[
\mathcal{D}_\mu^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell)\plus))}{\varphi(f(\ell)\plus)} = \lim_{\ell \to \infty} 2^{-(\ell-1)+\ell} = 2
\]

\[
\mathcal{D}_\mu^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell)\minus))}{\varphi(f(\ell)\minus)}
\]

\[
= \lim_{\ell \to \infty} 2^{-(\ell-1)+\ell-1} = 1
\]
4.4 \( f_{\alpha,s,M}(x) = M^{1/\alpha} (\ln x - \frac{s}{\alpha} \ln \ln x)^{-1/\alpha} \)

This time for any \( \alpha > 0, M \geq 0, s \), we introduce \( g_{\alpha,s} = \ln x - \frac{s}{\alpha} \ln \ln x \) and let \( f_{\alpha,s,M}(x) = M^{1/\alpha} g_{\alpha,s}^{-1/\alpha} \). Note that \( g'_{\alpha,s} = \frac{1}{x} (1 - \frac{s}{\alpha} \ln x) \), then for some \( N \) large enough, if \( x > N \), then both \( g_{\alpha,s}(x) > 0 \) and \( g'_{\alpha,s}(x) > 0 \), which gives \( f'_{\alpha,s,M}(x) < 0 \). Therefore, there is some \( N' > N \) such that \( f_{\alpha,s,M}(x) \leq \frac{1}{2} \).

Next, let us introduce metric \( \rho(m,n) = f_{\alpha,s,M}(m \wedge n) \) if \( m \wedge n > N' \), and \( \rho(m,0) = f_{\alpha,s,M}(m) \). Also, for any \( m \), and \( n \leq N' \), we let \( \rho(m,n) = \frac{1}{2} \). The argument for the Triangle Inequality is the same as in the previous case.

Next, let us introduce \( \varphi(r) = 2^{1 - M^{s/\alpha} r^{-s} \exp(M r^{-\alpha})} \), then \( \varphi(r) \) is an increasing function on \((0,1)\) with \( \varphi(r) \to 0 \) as \( r \to 0 \). But \( \varphi(r) \) is not blanketed either.

Thus,

\[
\overline{D}_\mu^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell)^+))}{\varphi(f(\ell)^+)} = \lim_{\ell \to \infty} 2^{-\ell + \ell (1 - (s \ln \ln \ell)/\alpha \ln \ell)^{s/\alpha}} = 1
\]

\[
D_\mu^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell - 1)^-))}{\varphi(f(\ell - 1)^-)} = \lim_{\ell \to \infty} 2^{-\ell + (\ell - 1)[1 - O(\ln \ln \ell/\ln \ell)]^{s/\alpha}} = \frac{1}{2}
\]

Same as in the previous section, we are still able to show that both \( C_\mu^\varphi \) and \( \overline{D}_\mu^\varphi \) are positive and finite by 2.3 and 2.4

4.5 The choice of \( \varphi \)

At this stage, a question occurs. What are the reasonable conditions so that \( \varphi(r) \) is a gauge function of \( H(S) \) with nontrivial measure? Also, if it is, what is the choice of the metric \( \rho \)?
What we want to do is to define $\rho$ as in the beginning of this section, so when it comes to the choice of $f(x)$, we only need to know what are $\{f(\ell)\}_\ell$.

First of all, if $\varphi(r)$ is strictly monotone in $[0, \delta)$ for some $\delta > 0$ and $\varphi(0) = 0$, taking the absolute value if necessary, we get an positive increasing function, still call it $\varphi$. Thus, $\varphi^{-1}(r)$ exists and is also positive increasing in $[0, \delta)$. Define $f(x) = \varphi^{-1}(2^{-x})$, then $f$ is positive, decreasing, with $f(x) \to 0$ as $x \to \infty$.

Then, we define $\rho(m, n) = f(m \wedge n)$ and $\rho(m, 0) = f(m)$, by argument similar to section 4.1, we see that $\rho$ is indeed a metric.

If in this case, $\varphi$ is continuous, then we have $\forall \ell, \varphi(f(\ell)^-) = \varphi(f(\ell)) = \varphi(f(\ell)^+)$, and

$$
\mathcal{D}_\mu^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell)^+))}{\varphi(f(\ell)^+)} = \lim_{\ell \to \infty} 2^{-(\ell-1)+\ell} = 2
$$

$$
\mathcal{D}_\mu^\varphi(A) = \lim_{\ell \to \infty} \frac{\mu(K \cap \Delta(A, f(\ell-1)^-))}{\varphi(f(\ell-1)^-)} = \lim_{\ell \to \infty} 2^{-(\ell-1)+\ell-1} = 1
$$

Moreover, given any $\tilde{f}(x)$ defined on $[1, \infty)$, if $\tilde{f}$ is positive, decreasing with

$$
0 \leq a = \liminf_{x \to \infty} \frac{\tilde{f}(x)}{f(x)} \leq \limsup_{x \to \infty} \frac{\tilde{f}(x)}{f(x)} = b < \infty
$$

Then, $\tilde{f}$ could replace $f$ to give nontrivial measures in both cases if $\varphi$ is blanketed.

We could check, for sections from this chapter, the choices of $f$ all fell into this case.

If, $\varphi$ is not continuous, then $\varphi(f(\ell)^-) \leq \varphi(f(\ell)) \leq \varphi(f(\ell)^+)$, but the equality does not necessarily hold for each $\ell$. In this case, $f$ still gives nontrivial measures.

And, when we are looking for the alternative $\tilde{f}$ for $f$, the above argument works if $\varphi$ is again blanketed.

Secondly, if $\varphi(r)$ is monotone but not strictly monotone, then let $\{(a_k, b_k)\}_k$ be
the collection of intervals contained in $(0, \delta)$ where $\varphi(r) = \varphi(a_k), \forall r \in (a_k, b_k], a_{k+1} < b_{k+1} < a_k < b_k$ and $\varphi(r_1) \neq \varphi(r_2)$ if $r_1 \neq r_2 \notin (a_k, b_k]$ for any $k$. Then, we can define $\varphi^{-1}$ and the image of it does not fell into any such $(a_k, b_k]$. Everything else follows the previous discussion.

Thirdly, if $\varphi(r)$ is not monotone in any $[0, \delta)$, but $\varphi(0) = 0$, then we are looking for $\varphi_1$ and $\varphi_2$ such that $\mathcal{C}^\varphi = \mathcal{C}^{\varphi_1}$ and $\mathcal{P}^\varphi = \mathcal{P}^{\varphi_2}$.

On the one hand, let $\varphi_1$ be nondecreasing and $\varphi_1 \leq \varphi$ by replacing all the bumping intervals of $\varphi$ inside $[0, 1)$ by a horizontal line connecting the endpoints. Bumping interval is defined as the interval where function changes from decreasing to increasing at the right endpoints and the left end point is picked such that both endpoints correspond to the same $\varphi$ value. One thing to notice, they are infinitely many such intervals. Moreover, we see that $\varphi_1(r) \leq \varphi(r)$ where the equality holds when $r$ does not belong to any bumping intervals.

Let $\{(a_k, b_k)\}$ be the collection of all bumping intervals, with $a_{k+1} < b_{k+1} < a_k < b_k$. Let $\delta_k = b_k + \varepsilon_k$, where $\varepsilon_k$ is chosen to be small and $\varphi$ is increasing in $(b_k, b_k + \varepsilon_k)$. Given any $\mathcal{A} \subseteq \mathcal{H}(S)$ and $\beta_m$, any $\delta_m$–fine covering of $\mathcal{A}$, if for some $\Delta(E, r) \in \beta_m, r \in (a_k, b_k)$ for some $k$, then let $r' = b_k$, otherwise, $r' = r$. Things to notice, $\varphi_1(r) = \varphi(r')$ and $\beta'_m = \{\Delta(E, r') : \Delta(E, r) \in \beta_m\}$ is another $\delta_m$–fine covering of $\mathcal{A}$. Therefore

$$\sum_{\Delta(E, r) \in \beta_m} \varphi(r) \geq \sum_{\Delta(E, r) \in \beta_m} \varphi_1(r) = \sum_{\Delta(E, r') \in \beta'_m} \varphi(r')$$

By taking infimum, we see that $\mathcal{C}^\varphi_{\delta_m}(\mathcal{A}) = \mathcal{C}^{\varphi_1}_{\delta_m}(\mathcal{A})$. Since $\mathcal{A}$ is arbitrarily chosen, the conclusion follows.
Similarly, let $\varphi_2$ be nondecreasing and $\varphi \leq \varphi_2$ by replacing all the left invisible intervals of $\varphi$ by a horizontal line connecting the endpoints. Then, we see that $\varphi \leq \varphi_2$.

Given any set $A$ and $\pi$, a $\delta$–fine parking of $A$, if for some $\Delta(E, r) \in \pi$, $r \in (a_r, b_r)$, where $(a_r, b_r)$ is a maximal left invisible interval, which means not contained in any other left invisible interval, then we see that $\Delta(x, a_r) \subset \Delta(x, r)$. Let $r' = a_r$ in this case, otherwise, $r' = r$. Define $\pi' = \{\Delta(x, r') : \Delta(x, r) \in \pi\}$, we get

$$
\sum_{\Delta(x, r) \in \pi} \varphi(r) \leq \sum_{\Delta(x, r) \in \pi} \varphi_2(r) = \sum_{\Delta(x, r') \in \pi'} \varphi(r')
$$

By taking supreme, we see that $\overline{P}_\delta^\varphi = \overline{P}_\delta^{\varphi_2}$. Therefore, the conclusion follows.

Since both $\varphi_1$ and $\varphi_2$ are monotone, we can use the method mentioned before to find the desired metric.
In this chapter, we are dealing with the following metric

\[ \rho(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right|, \quad m \neq n \neq 0, \quad \rho(m, 0) = \frac{1}{m} \]

We will show that the covering gauge is \( \varphi(r) = 2^{-\alpha r^{-1/2}} + g(r) \) with \( g(r) = o(r^{-1/2}), \alpha < 2.4 \).

Pick any \( A \in \mathcal{K} \), then for \( E \in \Delta(A, r) \), let \( \xi(r) = \left[ \sqrt{\frac{1}{4} + \frac{1}{r} - \frac{1}{2}} \right] \). We have

\[ A \cap \{0, 1, \cdots, \xi(r)\} = E \cap \{0, 1, \cdots, \xi(r)\} \]

and no requirement for \( E \cap \{[r^{-1}]+1, [r^{-1}]+2, \cdots\} \). So the only portion we are interested in, with respect to a given \( r \), is \( [r^{-1/2}, r^{-1}] \cap E \).

### 5.1 Classify \( \mathcal{K} \) by the idea of missing number

It will be useful to represent \( \mathbb{H}(S) \) as a subspace of the infinite binary sequences.

Let’s introduce the idea of “missing number”. For \( A = \{a_n\} \in \mathcal{K} \), we say \( A \) misses the number \( m \) if \( \rho(m, n) \geq r \) for all \( n \) such that \( n \in A \), i.e., \( a_n = 1 \). We will explore \( \Delta(A, r) \) first.
Due to the structure of the metric $\rho$, $r$ is not easy to handle in this case. So, first let us go back to the space $S$. Let

$$\alpha_k = r^{-1/2} \sqrt{\frac{1}{4} k^2 r + k - \frac{k}{2}}, \beta_k = r^{-1/2} \sqrt{\frac{1}{4} k^2 r + k + \frac{k}{2}}$$

For any $n$, if $n \in (\alpha_k, \alpha_{k+1})$, then $\rho(n, n-k) < r < \rho(n, n-k-1)$. Similarly, if $n \in (\beta_k, \beta_{k+1})$, then $\rho(n, n+k) < r < \rho(n, n+k+1)$. For $r$ small enough and $k \leq r^{-1/3}/2$, we would always have $\alpha_k < \beta_k < \alpha_{k+1}$. Let $I_k = [\beta_k, \alpha_{k+1}]$, then $|I_k| \sim \frac{1}{2 \sqrt{k}} r^{-1/2}$, and for any $n \in I_k$, we have $\{m : \rho(m, n) < r\} = [n-k, n+k]$.

Thus, instead of looking at $r$, we only need to watch the pattern of missing numbers in each $I_k$. One more thing to notice, between $I_{k-1}$ and $I_k$, there are only $k$ slots.

Next, let us proceed to $\mathbb{H}(S)$. Fix certain $k$ as in the above, I want to classify all $A$’s by solely looking at $a_n$’s with $n \in I_k$. Let $N_k = |I_k|$, and to simplify writing, we will omit all $k$ subindex below.

First of all, since we are in $I$, then to miss certain number, say $m_1$, we need $a_n = 0$ for $|n - m_1| \leq k$. Otherwise, $m_1 \in \Delta(n, r)$, contradiction.

Secondly, given $m_1 < m_2$, if $A$ misses only these two numbers out of the interval $[m_1 - 1, m_2 + 1]$, then $m_2 - m_1 \geq 2k + 2$ with $a_{m_1-k-1} = a_{m_1+k+1} = a_{m_2-k-1} = a_{m_2+k+1} = 1$.

Thirdly, given $m_1 < m_2$ and $m_2 - m_1 \leq 2k + 1$, if $A$ misses both, then $A$ misses all the numbers on $[m_1, m_2]$.

Now, let us look inside $I$. There are many closed subintervals of missed numbers, disjoint in the sense that no left endpoint is the immediate successor of some right endpoint. Then, for any two such consecutive intervals, if there is only one number in between, say $n$, then $n \in A$. If there are more than 1 number in between, then we
only need the immediate follower of the first interval and the immediate predecessor of the second interval to be included in $A$. Let $\ell$ count how many numbers are missed by $A$, then we have $\sum_j j\vartheta(j) = \ell$, where $\vartheta(j)$ is the number of such subintervals with exactly $j$ integers inside. Denote $\beta = \sum_j \vartheta(j)$ and $a = 2k + 1$. Note that to miss $j$ consecutive numbers, we need at least $j + 2k$ $a_n$'s be equal to 0. So, the number of slots where freedom is allowed in the interval $I$ is less than

$$N - \sum_j (2k + j)\vartheta(j) - (\sum_j \vartheta(j) - 1) = N - a\beta - \ell + 1$$

and more than

$$N - \sum_j (2k + j)\vartheta(j) - 2(\sum_j \vartheta(j) - 1) = N - (a + 1)\beta - \ell + 2$$

Also among these slots the length of consecutive 0's is at most $2k$ as well.

One thing to notice, between $I_{k-1}$ and $I_k$, there are $k$ free slots. So, when we consider the numbers of different coverings, we should extend each $I_k$ to $[\alpha_k, \alpha_{k+1}]$ to include these free slots and $N + k$ will replace $N$ in the calculation. But, since $k = o(N)$ for $r$ small enough, this would not change the later result in section 5.4.

Given $A \in \mathcal{K}$, let $\{n_1, n_2, \ldots, n_\ell\}$ be all the numbers missed by $A$, then

$$\Delta(A, r) \cap \mathcal{K} = \{E \in \mathcal{K} : n_i \notin E, i = 1, 2, \cdots, \ell \text{ and } E \text{ does not miss } n \text{ if } n \in A\}$$

**Remark 5.1.** For $A \neq B$ with same missing numbers, denoted as $\{n_1, n_2, \ldots, n_\ell\}$.

1. Since $\sigma(A, B) < R$, we have $A \in \Delta(B, r)$ as well as $B \in \Delta(A, r)$.
2. $\Delta(A, r) \neq \Delta(B, r)$. To see this, pick any $m \in A\Delta B$, then there is some $E \in \mathcal{K}$ such that $E$ misses $m$. This implies $E \notin \Delta(A, r) \cap \Delta(B, r)$.

5.2 The Markov Process

To find $\mu(\Delta(A, r))$ for any $A, r$, we will need to count on the probability of those intervals where some numbers are missing. Note that, to miss a number $n$ in some $I_k$, $A \cap \{n - k, \ldots, n, \ldots, n + k\} = \emptyset$. Therefore, if $A$ does not miss any number from $I_k$, there are no $2k + 1$ consecutive $a_n$’s equal to 0. This corresponds to the probability of flipping a coin $m$ times without a consecutive heads. Therefore, let us detour a little bit by looking at this Markov Process.

Fix $a$, and let $p(m)$ be the above probability.

Let $M$ be the transition matrix, then $M =
\begin{pmatrix}
.5 & .5 & 0 & \cdots & 0 & 0 \\
.5 & 0 & .5 & \cdots & 0 & 0 \\
& & \ddots & & & \vdots \\
.5 & 0 & 0 & \cdots & 0 & .5 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}_{(a+1) \times (a+1)}$

Here $p(m) = 1 - M^n(1, a+1)$, where $M^n(1, a+1) = P$(there is consecutive 0’s with length $a$). Let $P(\lambda)$ be the characteristic polynomial of $M$, we have

$$P(\lambda) = \frac{(\lambda - 1)}{2\lambda - 1} \left[ (2\lambda)^{a+1} - 2(2\lambda)^a + 1 \right], \text{ let } f(z) = \frac{z^{a+1} - 2z^a + 1}{z - 1} \left( \frac{z}{2} - 1 \right)$$

We want to know the distribution of roots of $f$.

**Claim.** If $a \geq 3$, then $f$ has $a - 1$ roots inside $|z| < 1 + \frac{2}{a}$. 

Proof. Let \( g(z) = \left( \frac{z}{2} - 1 \right) (z - 2) \frac{z^a - 1}{z - 1} \), then \( f(z) - g(z) = \frac{z}{2} - 1 \).

Moreover, since both \( \left( 1 + \frac{2}{a} \right)^a \) and \( \frac{a - 2}{2(a + 1)} \) are increasing on \([3, \infty)\), for \( a \geq 5 \),

\[
\frac{|g|}{|f - g|} = \frac{|z - 2|}{|z - 1|} \geq \frac{a - 2}{2(a + 1)} \left( \left( 1 + \frac{2}{a} \right)^a - 1 \right) > 1, \quad \text{on} \quad |z| = 1 + \frac{2}{a}
\]

Then, by Rouche’s Theorem, in \(|z| < 1 + \frac{2}{a}\), both \( f \) and \( g \) have \( a - 1 \) roots.

For the case \( a = 4 \), we can use direct calculation to prove the statement. \( \square \)

Claim. \( f \) has a real root inside the interval \((2 - 2^{-a}, 2 - 2^{-a-1})\).

Proof. Since \( z = 2 \) is another root of \( f \), the only root left must be real. Denote it as \( c \). First of all, by the classical result on the stochastic matrix, we see that \(|c| < 2\). Secondly, since \( c^{a+1} - 2c^a + 1 = 0 \), we have \( 2 - c = c^{-a} < 2^{-a} \). Thirdly, if \( c \geq 2 - 2^{-a-1} \), then \( c^{a+1} - 2c^a + 1 = -(2-c)c^a - 1 \geq -(2^{-a-1}2^{-a} - 1) \geq \frac{1}{2}, \) contradiction. \( \square \)

One more thing to notice is that, if we let \( P_1(\lambda) = (2\lambda)^{a+1} - 2(2\lambda)^a + 1 \), we see that \( P_1'(\lambda) = 0 \) has only two roots: \( 0, \frac{a}{a+1} \), and neither of these is a root of \( P_1(\lambda) \) itself. Moreover, \( P_1(1) \neq 0 \). Therefore, \( P(\lambda) \) has \( a + 1 \) distinct roots, which implies that the matrix \( M \) is diagonalizable.

Denote all the eigenvalues as \( \lambda_i \) with \(|\lambda_1| < |\lambda_2| < \cdots < |\lambda_a| < \lambda_{a+1} = 1\).

Let \( A = \begin{pmatrix}
.5 & .5 & 0 & \cdots & 0 \\
.5 & 0 & .5 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
.5 & 0 & 0 & \cdots & .5 \\
.5 & 0 & 0 & 0 & 0
\end{pmatrix}_{a \times a} \) then \( M = \begin{pmatrix} A & u \end{pmatrix} \) where \( u = \begin{pmatrix} 0 \\
0 \\
\vdots \\
0 \\
.5 \end{pmatrix} \).

Direct calculation shows that \( \{\lambda_1, \lambda_1, \cdots, \lambda_a\} \) are eigenvalues of \( A \). Thus, \( A \) is also diagonalizable and we can find an orthonormal \( J \) such that \( JAJ^T = D \), where \( D \) is
a diagonal matrix with $\lambda_i$ as the $i$-th diagonal entry. The $i$-th column of $J$, denoted as $u_i$, is the corresponding eigenvector of $\lambda_i$.

Let $J_M = \begin{pmatrix} J & 1 \\ 0 & 1 \end{pmatrix}$ where $1$ is a $a \times 1$ matrix with all the entries are 1, while $0$ is a $1 \times x$ matrix with all the entries are 0. Not surprisingly, $J_M$ is the eigenvector matrix of $M$ with determinant 1. Direct calculation gives $J_M^{-1} = \begin{pmatrix} J^T & \nu \\ 0 & 1 \end{pmatrix}$, where $\nu(j) = -\sum_i u_j(i)$ and

\[
|\nu(j)| \leq \sum_i |u_j(i)|
\]

\[
= \sqrt{\sum_i |u_j(i)|^2 + \sum_{i_1 \neq i_2} |u_j(i_1)||u_j(i_2)|}
\]

\[
\leq \sqrt{\sum_i |u_j(i)|^2 + \sum_{i_1 < i_2} |u_j(i_1)|^2 + |u_j(i_2)|^2}
\]

\[
= \sqrt{a}
\]

But since $\lambda_a$ is very close to 1, more delicate calculation is need here. We use the fact that $\mu_a$ is the eigenvector of $A$ with respect to eigenvalue $\lambda_a$. First, we have $\nu(a) = \frac{1-\alpha}{2\lambda_a-1}u_a(1)$. On the other hand, we also could get

$u_a(j) = \left(\frac{1}{2\lambda_a} + \cdots + \frac{1}{(2\lambda_a)^{n-j+1}}\right) u_a(1)$ and thus $\left|\frac{u_a(1)}{2}\right| < |u_a(j)| \leq |u_a(1)|$ for any $j \neq 1$. Since $\|u_a\| = 1$, combining the previous two results, we have $-u_a(1)\nu(a) \in [1, 2]$. 

40
Note that \( \mathbf{M} = \mathbf{J}_m \begin{pmatrix} \mathbf{D} \\ 1 \end{pmatrix} \mathbf{J}_m^{-1} \).

\[
p(m) = 1 - \mathbf{M}^m(1,a + 1) = - \sum_{1 \leq i \leq a} u_i(1) \lambda_i^m \nu(i)
= \lambda_a^m \left( u_a(1) \nu(a) + \sum_{1 \leq i < a} \left| \frac{\lambda_i}{\lambda_a} \right|^m u_i(1) \nu(i) \right)
\begin{cases}
\leq (1 - 2^{-a-2})^m (2 + a^{3/2} 2^{-m}) \\
\geq (1 - 2^{-a-1})^m (2 - a^{3/2} 2^{-m})
\end{cases}
\]

### 5.3 Coverings

Let us go back to the definition of the covering measure to get some insight on \( \varphi(r) \).
We will see that \( \varphi(r) = 2^{-\alpha r^{-1/2}} + g(r) \) with \( g(r) = o(r^{-1/2}), \alpha < 2.4 \). The discussion is in some given \( I_k \).

But at first, we will evaluate a constant which will be useful later.

Let \( c_m^\beta = \sum_{\varphi(j)=\beta, \sum j \varphi(j)=m} \frac{\beta!}{\prod j \varphi(j)!} \),

**Claim.** \( c_m^\beta = \binom{m-1}{\beta-1} \).

**Proof.** Here \( c_m^\beta \) is the coefficient of \( x_m \) in \( (x + x^2 + \cdots + x^m)^\beta \).
We will prove the statement by double induction on \( \beta \leq m \).
First, when \( m = 2 \), there are only 2 cases.
If \( \beta = 1 \), then \( c_m^\beta = c_2^1 = 1 = \binom{1}{0} \). If \( \beta = 2 \), then \( c_m^\beta = c_2^2 = 1 = \binom{1}{1} \).
So, the statement holds.

Second, assume the statement is correct for all $\beta \leq m \leq m_0$, where $m_0 \geq 2$. Then for $m = m_0 + 1$, $c_{m_0 + 1}^1 = 1 = \binom{m_0}{0}$ and $c_{m_0 + 1}^2 = m_0 = \binom{m_0}{1}$. For any $3 \leq \beta \leq m_0 + 1$

$$c_{\beta + 1} = \sum_{i=\beta-1}^{m_0} c_{\beta-1} = 1 + \sum_{i=1}^{m_0-\beta+1} \binom{i+\beta-2}{i} = \binom{m_0}{m_0-\beta+1} = \binom{m_0}{\beta-1}$$

where the * holds by using the equality $\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}$ $m_0 - \beta + 1$ times. □

Next, we will deal with a measure 1 subset of $\mathcal{H}(S)$ and $\mathcal{H}(S)$ separately.

### 5.3.1 The Measure 1 set $\mathcal{G}$

First, we need this claim to restrict ourself to a much small subinterval.

**Claim.** [9] $P(\lim_n L_n/\log_2 n \leq 1) = 1$.

**Proof.** Given any $\varepsilon > 0$ and small, $P(\ell_n \geq (1 + \varepsilon) \log_2 n) \leq 2^{-[(1+\varepsilon) \log_2 n]+1} \leq n^{-(1+\varepsilon)}$.

Thus, $\sum_n P(\ell_n \geq (1 + \varepsilon) \log_2 n) \leq \sum_n n^{-(1+\varepsilon)} < \infty$, Borel-Cantelli Lemma implies $P(\ell_n \geq (1 + \varepsilon) \log_2 n$ infinitely often) = 0. Thus $P(\limsup_n \frac{L_n}{\log_2 n} \leq 1 + \varepsilon) = 1$.

On the other hand, still given any $\varepsilon > 0$ small, set $k = [(1 - \varepsilon) \log_2 n]$, then there are $\lfloor n/k \rfloor$ blocks of length k on the line of integers from 1 to n, say, $A_i$. Note that
\( A_i = \{k_{ki+1} \leq k\} \) and they are independent identically distributed.

\[
P(L_n \leq (1 - \varepsilon) \log_2 n) \leq P(A_1 \cap \cdots \cap A_{[n/k]})
\]

\[
= P(A_1)^{[n/k]}
\]

\[
= (1 - 2^{k+1})^{[n/k]}
\]

\[
\leq \exp \left( -n^{\varepsilon-1} \frac{n}{(1 - \varepsilon) \log_2 n} - 1 \right)
\]

\[
\leq \exp(-n^{\varepsilon}/ \log_2 n) < \exp(-n^{\varepsilon/2})
\]

where the last 2 inequalities hold for \( n \) big enough.

Thus

\[
\sum_{n} P(L_n \leq (1 - \varepsilon) \log_2 n) = \sum_{n} \exp(-n^{\varepsilon/2}) < \sum_{n} k! n^{-k\varepsilon/2} < \infty
\]

for well-chosen \( k \) satisfying \( k\varepsilon > 2 \).

Borel-Cantelli Lemma again tells us \( P(L_n \leq (1 - \varepsilon) \log_2 n \) infinitely often\) = 0. In other words, \( P(\lim \inf L_n/ \log_2 n \geq 1) = 1 \).

Applying the above Claim, we have

\[
\frac{\log_2 n}{\text{number of consecutive 0’s or 1’s starting from } n} \to_P 1
\]

As a result, for \( r \) small enough, except for some zero-measure set, the longest consecutive 1 string or 0 string on the interval \( n \in \left[ r^{-1/2}, r^{-1}\right] \) has a length at most \( 2 \log_2 r^{-1} \). But, for all \( n \geq (r^{-1/2} + 1) \cdot \sqrt{\log_2 r^{-1}} \), \( \rho(n, n - \log_2 r^{-1}) \geq r \). So, we are
down to the interval $\tilde{I} = \left( r^{-1/2}, (r^{-1/2} + 1) \cdot \sqrt{\log_2 r^{-1}} \right)$ and some measure 1 subset of $\mathbb{H}(S)$.

Then we need an upper bound for $\ell$. Given $\varepsilon > 0$ small, according to the Strong Law of Large Numbers, except for some zero-measure set, for any $r$ small enough and any $A = \{a_n\}$

$$\left| \frac{\# \{a_n : a_n = 0, n \in I \}}{N} - \frac{1}{2} \right| < \varepsilon, \quad \text{and} \quad \left| \frac{\# \{a_n : a_n = 1, n \in I \}}{N} - \frac{1}{2} \right| < \varepsilon$$

This implies in the entire interval $I$, $\frac{\# \{a_n : a_n = 0\}}{\# \{a_n : a_n = 1\}} < \frac{1 + 2\varepsilon}{1 - 2\varepsilon}$. We will use this ratio here and denote $\mathcal{G}$ as the subset of $\mathbb{H}(S)$ without the above two measure 0 sets.

Let $\ell$ and $\beta$ be as defined before. For those particular intervals which result in missing certain numbers, we have

$$\frac{\# \{a_n : a_n = 0\}}{\# \{a_n : a_n = 1\}} = \frac{\sum_j \vartheta(j)(2k + j)}{\beta - 1} = \frac{2k/\beta + \ell}{\beta - 1} > 2k + 1 \geq 3$$

Therefore, to assume the validity of the Strong Law of Large Numbers, we see that the number of slots with freedom in $I$ should be $O(N)$. Actually, we will see later such slots dominate $I$.

Let us say $\ell = cN$, then again, by the Strong Law of Large Numbers, we have

$$\frac{2k/\beta + \ell}{N} + \left( \frac{1}{2} - \varepsilon \right) \left[ 1 - \frac{(a + 1)\ell - 1}{N} \right] < \frac{1}{2} - \varepsilon$$

This leaves us with two extreme cases.

Case 1, those missing numbers are as spread as possible, that is $\beta = \ell$. Plug in to the above inequality, this results in $c = \frac{4\varepsilon}{a - 1 + 2(a + 1)\varepsilon} \leq \frac{4\varepsilon}{a - 1}$.
Case 2, those missing numbers are as crowded as possible. According to section 5.1, the length of consecutive 0's in $I$ can not surpass $2 \log_2 r - 1$, therefore, we see that $\beta \leq \frac{\ell}{2 \log_2 r - 1}$. Plug in again, we would still get the same estimate as in the previous case.

Consequently, we conclude that, $A$ only misses up to $\frac{4\varepsilon N}{a - 1}$ numbers for almost everywhere. Denote this upper bound by $M$.

Now we will show that, for $G$, the covering gauge is $\varphi(r) = 2^{-(1+o(1))r^{-1/2}}$.

Let us consider the $2r$-fine cover $\beta$ of $G$ consisting of $\Delta(A, r)$'s, where each $A$ has a different set of missing numbers. We have discussed at Remark 5.1 that this is indeed a covering for $G$. But, how many such balls are there in $\beta$? We first focus on the interval $\tilde{I} = \left( r^{-1/2}, (1 + r^{-1/2})\sqrt{\log_2 r^{-1/2}} \right)$.

Fix $a$, note that $N + k \sim N$, if we just consider the number of $\Delta(A, r)$'s where $A$ misses $\ell$ numbers in $I_k$, it will not change the result. We have

$$
(1 + o(1)) \sum_{\ell=1}^{M} \sum_{\beta=1}^{\ell} \sum_{\{\vartheta(j)\}} \frac{\beta!}{\prod_j \vartheta(j)!} \left( N - a\beta - \ell + 2 \right)
$$

$$
\leq (1 + o(1)) \sum_{\ell=1}^{M} \sum_{\beta=1}^{\ell} N^\beta \sum_{\{\vartheta(j)\}} \frac{1}{\prod_j \vartheta(j)!}
$$

$$
= (1 + o(1)) \sum_{\ell=1}^{M} \sum_{\beta=1}^{\ell} \left( \frac{\ell - 1}{\beta - 1} \right) \frac{N^\beta}{\beta!}
$$

$$
\leq (1 + o(1)) \sum_{\ell=1}^{M} \sum_{\beta=1}^{\ell} \frac{N^\beta \ell^{\beta - 1}}{\beta!(\beta - 1)!}.
$$

Later, We will use this sum to show that this does not change our result.

Note that $\frac{N^\beta \ell^{\beta - 1}}{\beta!(\beta - 1)!} / \frac{N^\beta - 1 \ell^{\beta - 2}}{(\beta - 1)!(\beta - 2)!} = \frac{N\ell}{\beta(\beta + 1)} \geq \frac{N}{\ell} \geq \frac{N}{M} = \frac{a - 1}{4\varepsilon}$

So, for given $\ell$, the sum is $(1 + o(1)) \frac{N^\ell \ell^{-1}}{(\ell - 1)!\ell!}$. 

45
We take sum with $\ell$, and it is $(1 + o(1)) \sum_{\ell=1}^{M} \frac{N^{\ell} \ell^{\ell-1}}{(\ell-1)! \ell!}$.

Taking the ratio again, we have

$$\frac{N^{\ell} \ell^{\ell-1}}{(\ell-1)! \ell!} \cdot \frac{N^{\ell-1} (\ell-1)^{\ell-2}}{(\ell-2)! (\ell-1)!} = \frac{N}{\ell} \left(1 + \frac{1}{\ell-1}\right)^{\ell-1} \geq \frac{N}{M} = \frac{a-1}{4\varepsilon}$$

So, in extended $I_k$ the number of variation is $(1 + o(1)) \frac{N^M M^{M-1}}{(M-1)! M!}$.

Next, we take sum with $k$

$$\prod_{k=1}^{\log_2 r^{-1}} (1 + o(1)) \frac{N^M M^{M-1}}{(M-1)! M!}$$

$$= (1 + o(1)) \exp \left( \sum_{k=1}^{\log_2 r^{-1}} \left[ \varepsilon r^{-1/2} k^{-3/2} \ln \left( \frac{e^2 \cdot k}{2\varepsilon} \right) - \ln \left( \varepsilon r^{-1/2} k^{-3/2} \right) \right] \right)$$

$$= (1 + o(1)) \exp \left( \sum_{k=1}^{\log_2 r^{-1}} \left[ \varepsilon r^{-1/2} k^{-3/2} (2 + \ln k - \ln(2\varepsilon)) - \ln \varepsilon - \ln r^{-1/2} + \frac{3}{2} \ln k \right] \right)$$

$$= (1 + o(1)) \exp \left( \varepsilon r^{-1/2} \sum_{k=1}^{\log_2 r^{-1}} \left[ 2k^{-3/2} - k^{-3/2} \ln(2\varepsilon) + k^{-3/2} \ln k \right] \right)$$

$$\leq (1 + o(1)) \exp \left( \varepsilon r^{-1/2} \left( -k^{-1/2} (8 - 2 \ln(2\varepsilon) + 2 \ln k) \right) \right)^{1+\log_2 r^{-1}}$$

$$\leq (1 + o(1)) \exp \left( -r^{-1/2} (2\varepsilon \ln(2\varepsilon) - 8\varepsilon) \right)$$

Note that we have no control over the interval $[1, r^{-1/2}]$, therefore the total number of constituents is $(1 + o(1)) \exp \left( r^{-1/2} \left( \ln 2 + 2\varepsilon \left| \ln(2\varepsilon) \right| - 8\varepsilon \right) \right)$.

If $\phi(r) = 2^{-f(r)}$ with $f(r) r^{1/2} = c + o(1)$ where $c \in (1, \infty)$, then note that $x \ln(x) \to 0$ as $x \to 0$, we can pick $\varepsilon$ small enough such that $4\varepsilon \left| \ln(2\varepsilon) \right| - 16\varepsilon \leq (c - 1) \ln 2$. The
whole discussion stands when $r$ is small enough and it results in

$$C^\phi_\delta(G) \leq (1 + o(1)) \exp \left( r^{-1/2} \left( \ln 2 + 2\varepsilon |\ln(2\varepsilon)| - 8\varepsilon \right) \right) \varphi(r)$$

$$\leq (1 + o(1)) \exp \left( \left( \frac{c + 1}{2} - f(r)r^{1/2}\right)r^{-1/2}\ln 2 \right)$$

$$\leq (1 + o(1)) \exp \left( -\frac{c - 1}{4}r^{-1/2}\ln 2 \right)$$

Taking $\delta \to 0$, we would have $C^\phi_0(G) = 0$.

For any $G' \subseteq G$, we can find a similar covering: given $\delta$, for any $r < \delta/2$, if $A \in G'$ then $\Delta(A, r)$ includes all the elements of $G'$ missing the same set of numbers. Let $\beta'$ be the collection of all such $\Delta(A, r)$'s, then $\beta'$ is the desired $\delta$-fine cover $G'$. Since $\beta'$ contains no more elements than $\beta$, a similar argument shows that $C^\phi_0(G') = 0$ as well. This implies $C^\phi(G) = 0$, and $\phi$ is too large to be the gauge we are looking for.

If here $f(r)r^{1/2} \to \infty$, then we can easily see that $C^\phi_0(G) = 0$. In other words, if the covering gauge is $\varphi(r) = 2^{-(c + o(1))r^{-1/2}}$, then we have $c \leq 1$.

To complete the argument, let us show that $c = 1$:

**Claim.** Given $c < 1$, for any $\phi_c(r) = 2^{-cr^{-1/2}}, C^\phi_0(G) = \infty$. Therefore, $\phi$ is too small to be the desired gauge.

**Proof.** Let $\beta$ be any $\delta$-fine cover of $G$.

Let $\{\Delta(A_i, r_i)\} \subset \beta$ be a minimal cover of $G_0 = \{E \in G : [1, \delta^{-1/2}] \cap E = \emptyset\}$ with $\delta > r_i \geq r_j$ if $i \leq j$. Then each $A_i \in G_0$ as well. Therefore, for any $E \notin G_0$, for any $B \in E$, $\sigma(A, B) \geq \delta$ and $E \cap \Delta(A_i, r_i) = \emptyset$. Note that $2^{-r_i^{-1}} < \mu(\Delta(A_i, r_i)) < 2^{-r_i^{-1/2}}$, we have

$$\sum_i \phi(r_i) = \sum_i 2^{-cr_i^{-1/2}} > \left( \sum_i 2^{-r_i^{-1/2}} \right)^c > \left( \sum_i \mu(\Delta(A_i, r_i)) \right)^c > \mu^c(G_0) = 2^{-c\delta^{-1/2}}$$

47
where the first inequality holds since \( c < 1 \).

Then, note that \( \mathcal{G} \) has \( 2^{\delta - 1/2} \) copies of \( \mathcal{G}_0 \), we conclude that

\[
C_\delta^2(\mathcal{G}) \geq 2^{\delta - 1/2} \cdot 2^{-c_\delta - 1/2} = 2^{(1-c)\delta - 1/2}, \quad \text{and} \quad C_0^\delta(\mathcal{G}) = \infty
\]

\( \square \)

Combining the above two arguments, we see that the desired covering gauge for this measure 1 set is \( \varphi(r) = 2^{-(1+o(1))r^{-1/2}} \).

### 5.3.2 The whole space

Now, the focus moves to the whole \( K \). Then, besides losing the upper bound for \( \ell \) in each \( I_k \), we also have to go back to the original interval \([r^{-1/2}, r^{-1}]\) rather than just \( \tilde{I} \).

In the following, we will find a specific covering, thus giving an upper bound for \( \alpha \). To reach this goal, two different constructions will be used.

First, we start with \( I_k \)'s where \( k \leq 3 \). Here, we will use the idea of missing numbers as in the previous setting. Given \( I_k \), the number of constituents in this covering is

\[
1 + \sum_{\ell=0}^N \sum_{\beta=1}^{N-(a+1)\beta+2} \frac{(\ell-1)!}{(\ell-\beta)!(\beta-1)!\beta!} \frac{(N-a\beta-\ell+2)!}{(N-(a+1)\beta-\ell+2)!}
\]

Let \( B(\ell, \beta) = \frac{(\ell-1)!}{(\ell-\beta)!(\beta-1)!\beta!} \frac{(N-a\beta-\ell+2)!}{(N-(a+1)\beta-\ell+2)!} \), to estimate the sum, let us fix \( \beta \) and then compare \( \sum_\ell B(\ell, \beta) \) first.
When $\beta = 1$, we have $B(\ell, 1) = (N - a - \ell + 2)$ for any $\ell \geq 1$. Since it decreases while $\ell$ increasing, we see that $\sum_{\ell=1}^{N-a+1} B(\ell, 1) \leq (N - a + 1)(N - a - \ell + 2)$.

When $\beta > 1$, we have $B(\ell+1, \beta) = B(\ell, \beta) = 1 - \frac{\beta}{1 - \beta/\ell}$ $\frac{1}{1 - (\beta - 1)/\ell}$. Note that this ratio decreases as $\ell$ increases, so where this ratio reaches 1, we get $\max_{\ell \geq \lambda} B(\ell, \beta)$.

This happens at $\ell = \frac{(\beta - 1)(N - a\beta + 2)}{2\beta - 1}$, and consequently $\sum_{\ell=1}^{N-(a+1)\beta+2} B(\ell, \beta) \leq (N - (a + 1)\beta + 2) B(\ell, \beta)$. There are two things worth noticing. First, $N - a\beta - \ell \beta + 2 = \frac{\beta}{\beta - 1} \ell \beta$ and $\ell \beta \sim \frac{1}{2}(N - a\beta) = O(N)$. This will be used later. Second, $\ell_{\beta+1} - \ell \beta = \frac{N + 2 - 2a\beta^2}{4\beta^2 - 1}$.

When $\beta < \sqrt{(N + 2)/2a} - 1$, let $b = \ell_{\beta+1} - \ell \beta$, then we see that $\ell \beta \sim \frac{1}{2}(N - a\beta) = O(N)$ and

$$\frac{B(\ell_{\beta+1}, \beta + 1)}{B(\ell, \beta)} = \frac{B(\ell + b, \beta + 1)}{B(\ell, \beta)}$$

$$= \frac{\ell \beta + b - 1}{\beta(\beta + 1)} \prod_{m=0}^{b-2} \left(1 + \frac{\beta - 1}{\ell \beta - \beta + 1 + m}\right) \frac{\ell \beta - \ell \beta + 1}{\ell \beta - \beta + 1 + m} \prod_{m=0}^{a+b-1} \left(1 - \frac{\beta + 1}{N - a\beta - \ell \beta - m + 2}\right)$$

$$= O(1) \frac{\ell \beta (N - 3\beta - \beta - \ell \beta + 2)}{\beta(\beta + 1)}$$

$$\geq O(N) \gg 1$$

Therefore, $B(\ell, \beta)$ increases for $\beta < \sqrt{(N + 2)/2a} - 1$ and to find the maximum among all the $\beta$, we need to turn to the case $\beta \geq \sqrt{(N + 2)/2a} - 1$.

For bigger $\beta$, now $\ell \beta$ decreases, so we let $b = \ell \beta - \ell_{\beta+1} = -\frac{N + 2 - 2a\beta^2}{4\beta^2 - 1} \leq \frac{a}{2}$.

In below, we have two estimates, one for the upper bound, while another one for the lower bound. According to these two estimates, when this ratio is close to 1, we need
\( \beta = O(\ell) \), which implies \( \beta = O(N) \).

\[
\frac{B(\ell - b, \beta + 1)}{B(\ell, \beta)} = \frac{(\ell - \beta) \cdots (\ell - \beta - b)}{(\ell - 1) \cdots (\ell - b)} \frac{1}{\beta(\beta + 1)} \cdot \frac{(N - a\beta - \beta - \ell + 2) \cdots (N - a\beta - \beta - \ell - a + b + 2)}{(N - a\beta - \ell + 2) \cdots (N - a\beta - \ell + 3 - a + b)}
\]

\[
= \frac{(\ell - \beta)(N - a\beta - \beta - \ell + 2)}{\beta(\beta + 1)} \prod_{k=1}^{b} \left(1 - \frac{\beta}{\ell - k}\right)
\]

\[
\leq 1 + o(1) \frac{(\ell - b)^2}{\beta^2}
\]

\[
\geq 1 + o(1) \frac{(\ell - b)^2}{\beta^2} \exp \left(-\frac{3\beta - 1}{2} \ln \frac{\ell - 1}{\ell - b} - \frac{3(\beta + 1)}{2} \ln \frac{N - a\beta - \ell + 2}{N - a\beta - \ell - a + b + 3}\right)
\]

\[
\geq 1 + o(1) \frac{(\ell - b)^2}{\beta^2} \exp \left(-\frac{3\beta(a - 2)}{2\ell}\right)
\]

Here inequality 2 holds because \( \ln(1 + x) \geq x/2 \) for \( x \in [0, 1] \) and \( \ell \geq \beta = O(\sqrt{N/2a}) \).

We will work on these two inequalities separately. Write \( N = c\beta \). We have

\[
\ell = \frac{(\beta - 1)(N - a\beta + 2)}{2\beta - 1} = (1 + o(1)) \frac{c - a}{2} \beta).
\]

According to the inequality 1, the ratio is \( \frac{(c - a - 2)^2}{4} \), which decreases as \( c \) decreases. Now we turn to the inequality 2, let \( f(x) = \frac{(x - a - 2)^2}{4} \exp \left(-\frac{3(a - 2)}{x - a}\right) \), we see that \( f(x) \) is also decreasing while \( x \) decreases. This implies, at some specific \( c \), this ratio reaches 1 and that is the place where \( B(\ell, \beta) \) reaches the maximum.
Let \((\ell, \beta)\) be the pair corresponding to this \(c\), then \(\ell = \frac{c-a}{2}\beta\) and

\[
B(\ell, \beta) = (1 + o(1)) \frac{\ell!(\ell - 1)!}{[(\ell - \beta)!]^2 \beta!(\beta - 1)!} \ell^{2\ell-2} \\
= (1 + o(1)) 2\pi \beta^{2\beta-2} (\ell - \beta)^{2\ell-2\beta-1} \\
= (1 + o(1)) 2\pi \beta \left(1 - \frac{2}{c-a}\right)^{2-2\ell} \left(\frac{c-a}{2} - 1\right)^{2\beta-1} \\
= (1 + o(1)) \exp \left(-2(\ell - 2) \ln \left(1 - \frac{2}{c-a}\right) + (2\beta - 1) \ln \frac{c-a-2}{2}\right) \\
= (1 + o(1)) \exp \left(\frac{r^{-1/2}}{2\sqrt{k}} \left[\frac{a-c}{c} \ln \left(1 - \frac{2}{c-a}\right) + \frac{2}{c} \ln \frac{c-a-2}{2}\right]\right)
\]

To estimate the last formula, we let

\[g_a(x) = \frac{a-x}{x} \ln \left(1 - \frac{2}{x-a}\right) + \frac{2}{x} \ln \frac{x-a-2}{2}\]

Study \(g'(x)\), we found that \(g'(x)\) is a decreasing function. Use the fact that \(\ln(1+x) < x\), we see that

\[g'_a(a+3) > 0\] while \(g'_a(5a+2) < (3a+2)^{-2}\left[\frac{1}{2} + 2\ln 2 - \ln 4a\right] < 0\)

Therefore, \(g_a(x)\) reaches its maximum in the interval \((a+3, 5a+2)\). Direct calculation gives the corresponding maximums for each \(a\) and the upper bound \(2^{(1+o(1)) \cdot r^{-1/2}}\) for the number of constituents of the covering in \(I_1 \cup I_2 \cup I_3\).

Next, we move on to the rest of \([r^{-1/2}, r^{-1}]\).

Before starting the construction, we first look at the following interval \([r^{-2/3}/\sqrt{2} - r^{-1/3}/2, r^{-1}]\), which is disjointed from all the \(I_k\)'s. Since in this interval, \(k > r^{-1/2}/2\),
we have \( \frac{r^{-1/2}}{2\sqrt{k}} < 2k \). To better study this interval, we cut it into subintervals again. Let \( \iota_k = 1 - \frac{1}{3k} \) with \( k \in \mathbb{Z}^+ \), and

\[
I'_0 = \left( \frac{r^{-2/3} - r^{-1/3}}{2}, r^{-2/3} \right], \quad I'_k = \left( \frac{r^{-i_k}}{2}, \frac{r^{-i_k+1}}{2} \right], \quad I'_\infty = \left( \frac{r^{-1}}{6}, r^{-1} \right)
\]

We just need \( r^{-i_k+1}/2 < r^{-1}/6 \), which gives \( k < \frac{\ln r}{-3 \ln 3} \).

Now, this following discussion is in both \( I_k \)'s and \( I'_k \)'s except for \( I_1, I_2, I_3 \). The idea for construction here is to find a specific \( A = \{a_n\} \), where \( \Delta(A, r) \) will play an important role. And then, we will finish this part of covering by adding some other constituents.

To construct this \( A \), we need a sequence of integers, denoted as \( \{n'\} \).

First of all, by letting \( \frac{r^{-2/3}}{\sqrt{2}} - \frac{r^{-1/3}}{2} \in A \), we guarantee \( A \) does not miss any number in \( I'_0 \). Then \( \frac{r^{-2/3}}{\sqrt{2}} - \frac{r^{-1/3}}{2} \) is the only integer in \( \{n'\} \cap I'_0 \).

Secondly, inside any \( I_k \), we construct \( A \) in the following way:

We find \( \{n'\} \) by requiring between any such two consecutive \( n \)'s, the distance is less than \( r \). A direct calculation shows that, for each \( n \in I_4 \), \( \rho(n, n + 3) < 3\zeta^{-2}r \), where \( \zeta = 1 + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{3}} \). Let this bound be \( r_1 \). Then we divide \( I_k \) and \( I'_k \) into subintervals with length \( r_1 \) and resizing each subintervals accordingly to make sure the endpoints are integers and the length is less than \( r \). This is possible since the choice of \( r_1 \) guarantees that, for any \( n \in I_k \), \( (n, r_1) \) is properly contained in \( (n, r) \) with at least 1 integer short on both side. These endpoints are in \( \{n'\} \).

Then according to remark 5.1.3, this will not only make \( A \) miss no number in \( I'_k \), but also, more importantly, for any \( B \) does not any such \( n' \), \( \sigma(A \cap I'_k, B \cap I'_k) < r \).
Last, for $I'_\infty$, we use the same idea by letting

$$\{n' \} \cap I'_0 = A \cap I'_\infty = \left\{ \frac{1}{2} r^{-1}, \frac{1}{3} r^{-1}, \frac{1}{4} r^{-1}, \frac{1}{5} r^{-1}, \frac{1}{6} r^{-1} \right\}$$

This will allow $A$ miss no number of $I'_\infty$.

Sum up, we have

$$\Delta(A, r) \ni \{ B : B \text{ does not miss } r^{-2/3}/\sqrt{2} - r^{-1/3} 2, r^{-1}/2 \text{ and any } n' \}$$

To complete the covering, we need a subcover for the other $B$’s.

Denote $N'$ as the set of $\{n'\}$ that $B$ missed, and let $A(N') = A \setminus N'$, then

1. If $n'$ is inside some $I'_k$ or $I_k$ and, in the corresponding subintervals where $n'$ served as endpoint, $B$ does not miss any other numbers, including the other two endpoints. Then because the length of either such interval is less than $r$, and both the endpoints are in $A$, we see that $A(N')$ still does not miss $n'$. But, restricted to these two consecutive subintervals, we have $\sigma(B, A(N')) < r$.

2. If $n'$ is inside some $I'_k$ or $I_k$ and $B$ misses some $m$ within the corresponding intervals where $n'$ served as endpoint. Then B will miss all the numbers in $[n', m]$ since $\rho(n', m) < r$. Let $n'_1 < n' < n'_2$ be the closet numbers to $n'$ in $N'$ such that $B$ does not miss any of them, then restricted to $(n'_1, n'_2)$,

$$\{ \text{Numbers } A(N') \text{ misses} \} \subseteq \{ \text{Numbers } B \text{ misses} \}$$

This implies, restricted to $(n'_1, n'_2)$, $\sigma(A(N'), B) < r$. 

53
Therefore, we found a covering here, consisting of \( A \cup \{ A(\mathbb{N}') \}_{N'} \). Here, the length of this entire interval is from \( r^{-1} \) to the left starting integer of \( I_4 \), that is

\[(1 + o(1))\rho \left( r^{-1}, \varsigma r^{-1/2} \right) = (1 + o(1))\varsigma^{-1} r^{1/2} - r\]

Sum up by \( k \), and include the six \( n' \) from \( I_0' \) and \( I_\infty' \), we see that the size of \( \{n'\} \) is less than

\[6 + \frac{\varsigma^{-1} r^{1/2} - r}{r_1} = (1 + o(1)) \frac{\varsigma}{3} r^{-1/2}\]

So, the new cover has \( 2^{(1 + o(1)) \frac{\varsigma}{3} r^{-1/2}} \) elements.

By similar arguments as in the previous subsection, we see that \( \alpha \leq 1 + 0.61 + \frac{\varsigma}{3} \leq 2.4. \)
Bibliography


[9] Lecture notes from Math 722 offered by Prof. Pittel in Au07