Valid Inequalities for The 0-1 Mixed Knapsack Polytope with Upper Bounds

Dissertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy
in the Graduate School of The Ohio State University

By

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2010

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ABSTRACT

The polyhedral structure of the convex hull of the 0-1 mixed knapsack polytope defined by a knapsack inequality with continuous and binary variables with upper bounds is investigated. This polytope arises as subpolytope of more general mixed integer problems such as network flow problems, facility location problems, and lot-sizing problems.

Two sets of valid inequalities for the polytope are developed. We derive the first set of valid inequalities by adding a new subset of variables to the flow cover inequalities. We show that, under some conditions, this set is facet defining. Computational results show the effectiveness of these inequalities.

The second set of valid inequality is generated by sequence independent lifting of the flow cover inequalities. We show that computing exact lifting coefficients is NP-hard. As a result, an approximate lifting procedure is developed. We give computational results that show the effectiveness of the valid inequalities and the lifting procedure.
To Elif Ilke Cimren
ACKNOWLEDGMENTS

I wish to express my sincere appreciation to my advisor, Professor Marc E. Posner for his excellent guidance and assistance during the course of this research, and also for his patience teaching me how to write a paper. Throughout my Ph.D. studies, he has not only guided or contributed to my life educationally, but also taught me so many aspects of life. His intellectual and emotional support, and enthusiasm made this dissertation possible.

I also wish to thank Professor Nicholas G. Hall, Professor Simge Kucukyavuz, and Professor Suvrajeet Sen for their support during the dissertation process as members of my committee.

Special thanks are also to Dr. Joseph Fiksel for his support and guidance during my study at The Ohio State University. Finally, I wish to thank my wife of her support during my years at Ohio State.
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CHAPTER 1

INTRODUCTION

In this dissertation, we study the polyhedral structure of the following 0-1 mixed knapsack polytope with upper bounds (P):

\[
\sum_{j \in N} (x_j + w_j y_j) \leq \bar{d},
\]

\[
x_j \leq m_j y_j, \ j \in N,
\]

\[
x_j \geq 0, \ j \in N,
\]

\[
y_j \in \{0, 1\}, \ j \in N,
\]

where \(m_j > 0\), \(0 < w_j \leq \bar{d}\), and \(N = \{1, \ldots, n\}\). We develop two new sets of valid inequalities for \(P\). The first set of valid inequalities is developed by adding a new subset of variables to the flow cover inequalities presented by Goemans (1987) and Wolsey (1989). Given a valid inequality, sequence independent lifting is a technique to develop strengthen the inequality by simultaneously including a set of new variables. The second set of valid inequalities is generated by sequence independent lifting of the flow cover inequalities.

In this chapter, we give an overview of the research. Section 1.1 presents our research motivation. Section 1.2 provides literature review. Section 1.3 describes our research contribution. Finally, Section 1.4 gives the organization of the dissertation.
1.1 Motivation

The polytope $P$ is used in many applications such as facility location problems (Aardal 1998), distribution problems (Salomon 1991), and network flow problems (Bienstock and Gunluk 1996). Aardal (1998) studies the cutting plane approach to solve the capacitated facility location problem. In Salomon (1991), a review and comparison of mathematical model formulations and solution procedures for the deterministic lotsizing problem are given. Bienstock and Gunluk (1996) analyze the polyhedral structure of a mixed integer formulation of capacitated network design problem and present computational results related with a cutting-plane algorithm which uses facet defining inequalities to strengthen the linear programming formulation. Even if these constraints are not present in the initial formulation, they can be generated by preprocessing. The main motivation of this dissertation is that the study of the polyhedral structure of $P$ allows us to derive valid inequalities for the polytopes of these complex problems.

Another motivation of this dissertation is the emergency evacuation planning problem, which has $P$ as a subproblem. Emergency evacuation is the movement of people to safety prior to and during disasters such as hurricanes, floods, fires, releases of hazardous or nuclear materials, and terrorist attacks. Travel during emergency evacuation is abnormal. It involves moving a large population that may grow or change, onto a highly congested and possibly damaged road network, towards destinations that are not easily determined (Barrett et al. 2000). Thus, moving huge populations to safe areas requires careful preparation and planning.

In the literature, simulation is used to model evacuation planning problem. Evacuation simulation models can be classified into two main categories: macroscopic and
microscopic. The macroscopic models NETVAC1 (Sheffi et al. 1982), DYNEV (Anon 1984), and MASSVAC (Hobeika and Jamei 1985) describe vehicular traffic as a fluid and use macroscopic measures such as flow, density, average speeds, etc. The microscopic models MITSIM (Yang and Koutsopoulos 1996), and DynaMIT (Ben-Akiva et al. 1998) represent individual driver behavior. In general, macroscopic models are not very sensitive to evacuee behavior because they do not keep track of individual vehicles. The microscopic models assume that the behavior of individual drivers is affected by nearby vehicles. These simulation models are often accompanied with labor intensive network coding and significant running time. Thus, they may be inappropriate for large evacuation scenarios.

Mathematical programming is also used to model evacuation planning problem (Sherali et al. 1991, Daganzo 1994, 1995, Yamada 1996, Cova and Johnson 2003, Kamiyama et al. 2006, and Tuydes and Ziliaskopoulos 2006). However, there is no study in the literature that proposes a mathematical programming model and solution procedure for the evacuation problem considering road availability, shelter selection, workforce availability for traffic management and shelters, and contraflow activities simultaneously. All of these decisions are needed to develop an effective real world response to a disaster.

We formulate the evacuation planning problem as a mixed integer programming model including important components of evacuation planning. People in the risk areas are warned to evacuate. The path of the disaster is known. Based on the disaster path, we determine the risk areas. The evacuation planning horizon is divided into a set of identical intervals. In each risk area, the amount of people which are ready to evacuate is known for each time interval. People can be transported by public buses,
cars and vans during the evacuation. An evacuee travels to shelters. Each shelter has a capacity, which limits the number of people that can enter. Moreover, each shelter requires workers to provide assistance the evacuees. The required number of workers for a shelter depends on the number people in a shelter. Because the total workforce for shelters is restricted, it may not be possible to open all available shelters.

Risk areas and destinations are connected by a network of roads. A road segment has a maximum capacity, which is based on the types of cars on the road segment. The availability of a road is known.

Contraflow activity can be implemented to speed up traffic flow on a road segment. Workforce such as Police, National Guard, and Department of Transportation workers are required to prepare a road for contraflow activity and manage the contraflow activity. However, the total amount of workforce for performing these activities is limited.

In the model, we assume that evacuees are transferred to safety by uniform vehicles. Let $G(N,E)$ be a directed road network where $E$ is the set of roads and $N$ is the set of road intersections. We use the following notation in the model:

**Sets**

$E_i^- = \text{Set of roads entering to road intersection } i \in N$

$E_i^+ = \text{Set of roads leaving from road intersection } i \in N$

$R = \text{Set of risk area}$

$S = \text{Set of shelters}$

**Parameters**

$C_i = \text{Capacity of shelter } i \in S$
\[ q_i = \text{Number of required workers to open shelter } i \in S \]
\[ p_i = \text{Number of workers required to serve an evacuee in shelter } i \in S \]
\[ Q = \text{Total available workforce for shelters} \]
\[ T = \text{Total time intervals in the time horizon, } t = 0, \ldots, T \]
\[ d_{jt} = \text{Number of available evacuees in risk area } j \in R \text{ at time } t \in \{0, \ldots, T\} \]
\[ r_j = \text{Travel time for the road segment } j \in E \]
\[ c_j = \text{Capacity of the road segment } j \in E \]
\[ a_{jt} = \begin{cases} 1 & \text{if road } j \in E \text{ is available at time } t = 0, \ldots, T \\ 0 & \text{otherwise} \end{cases} \]
\[ w_{jt} = \text{Number of required workers for contraflow traffic activity on road } j \in E \text{ at time } t \]
\[ W_t = \text{Total available workforce for contraflow activities at time } t \]

**Variables**

\[ v_{it} = \text{Amount of flow entering road network from the risk area } i \in R \text{ at time } t = 0, \ldots, T \]
\[ u_{jt} = \text{Amount of flow entering road } j \in E \text{ at time } t = 0, \ldots, T \text{ when there is no contraflow on } j \in E \]
\[ u_{jt} = \text{Amount of flow entering road } j \in E \text{ at time } t = 0, \ldots, T \text{ when there is a contraflow on } j \in E \]
\[ s_{it} = \text{Amount of flow entering final destination } i \in S \text{ at time } t = 0, \ldots, T \]
\[ z_{jt} = \begin{cases} 1 & \text{if there is contraflow on road } j \in E \text{ at time } t \\ 0 & \text{otherwise} \end{cases} \]
\[ \tilde{y}_i = \begin{cases} 1 & \text{shelter } i \in S \text{ is selected} \\ 0 & \text{otherwise} \end{cases} \]

We assume that \( \tau_j = 1 \) for all \( j \in E \); otherwise a road with \( \tau_j > 1 \) can be modeled by a series of \(|\tau_j|\) road segments. Also, \( C_i \) and \( \tilde{q}_i \) for \( i \in S \), \( Q \), \( \tilde{d}_{jt} \) for \( j \in R \) and \( t \in \{0, \ldots, T\} \), and \( c_j \) for \( j \in E \) are integers. We assume that \( u_{jt} = \bar{u}_{jt} = 0 \) for \( j \in E \) and \( t < 0 \).

A mixed integer programming model for the evacuation problem is

\[
\begin{align*}
\text{maximize} \quad & Z_p = \sum_{t=0}^{T} \sum_{i \in S} s_{it} \\
\text{s.t.} \quad & \sum_{t=0}^{\bar{t}} v_{it} \leq \sum_{t=0}^{\bar{t}} \tilde{d}_{it}, \quad i \in R, \quad \bar{t} = 0, \ldots, T, \\
& \sum_{t=0}^{\bar{t}} (u_{\ell,t-1} - \bar{u}_{\ell t}) - \sum_{j \in E_i^+} (u_{jt} - \bar{u}_{jt-1}) = 0, \quad i \in R, \quad t = 1, \ldots, T, \\
& \sum_{t=0}^{\bar{t}} (u_{\ell,t-1} - \bar{u}_{\ell t}) - \sum_{j \in E_i^+} (u_{jt} - \bar{u}_{jt-1}) = 0, \quad i \in N \setminus (R \cup S), \\
& \sum_{t=0}^{\bar{t}} (u_{\ell,t-1} - \bar{u}_{\ell t}) - \sum_{j \in E_i^+} (u_{jt} - \bar{u}_{jt-1}) - s_{it} = 0, \quad i \in S, \quad t = 1, \ldots, T, \\
& u_{jt} - a_{jt}c_j (1 - \tilde{z}_{jt}) \leq 0, \quad j \in E, \quad t = 0, \ldots, T, \\
& \bar{u}_{jt} - a_{jt}c_j \tilde{z}_{jt} \leq 0, \quad j \in E, \quad t = 0, \ldots, T, \\
& \sum_{j \in E} \tilde{w}_{jt} \tilde{z}_{jt} \leq W_t, \quad t = 0, \ldots, T, \\
& \sum_{t=0}^{T} s_{it} \leq C_i \tilde{y}_i, \quad i \in S, \\
& \sum_{i \in S} \tilde{u}_i (\sum_{t=0}^{T} s_{it}) + \tilde{q}_i \tilde{y}_i \leq Q, \\
& v_{it} \geq 0, \quad i \in R, \quad t = 0, \ldots, T,
\end{align*}
\]
\begin{align}
  u_{jt}, \bar{u}_{jt} & \geq 0, \quad j \in E, \ t = 0, \ldots, T, \quad (1.14) \\
  s_{it} & \geq 0, \quad i \in S, \ t = 0, \ldots, T, \quad (1.15) \\
  \tilde{z}_{jt} & \in \{0, 1\}, \quad j \in E, \ t = 0, \ldots, T, \\
  \tilde{y}_i & \in \{0, 1\}, \quad i \in S.
\end{align}

The objective of the model, (1.3), maximizes the total number of evacuees that arrive at shelters and evacuee-selected destinations by the time horizon \( T \). In (1.4), the total flow for each vehicle type departing a risk area is bounded by the number of vehicles that are available to leave the area for each time period. Flow balances at intersections are described in (1.5)–(1.7). Constraints (1.8) and (1.9) provide road capacity, contraflow activity decision, and road availability at each time period. The workforce constraint for contraflow activities appears in (1.10). Constraint (1.11) denotes shelter capacity and shelter selection. The staff availability for shelters is formulated by (1.12). Flows are assumed to be continuous as in (1.13)–(1.15).

The total number of variables in the model depends on the number of road segments, the number of candidate shelters, and the time horizon. Further, the number of constraints is related to the number of road segments, road intersections, the number of candidate shelters and self-selected destinations, and time horizon. For the evacuation problem during Hurricane Katrina, the model has approximately 14,000 variables and 15,000 constraints (Census.gov 2005) when only major roads and shelters are considered and the time horizon is defined in terms of hours. If local roads and shelters are considered and the time horizon is somewhat smaller, then the model becomes intractable. Since the problem in Rebennack et al. (2008) reduces to our model in polynomial time, the model is at least as hard as the model in Rebennack et al. (2008) which is NP-hard.
Branch-and-bound (B&B) is a general methodology for finding exact optimal solutions of optimization problems. Obtaining strong bounds is important in B&B because it reduces the number of linear programming problems required to be solved. Cutting planes have been successfully used in B&B framework for solving large scale integer problems such as network design problems (Magnanti et al. 1993, Magnanti et al. 1995, Bienstock and Gunluk 1995, Barahona 1996, Atamturk 2002, Ortega and Wolsey 2003, Belotti et al. 2007) and lot-sizing problems (Pochet and Wolsey 1991, Belvaux and Wolsey 2001). Motivated by this research, we generate valid inequalities for the evacuation model, and then use them to obtain efficient lower bounds.

Since the evacuation planning model is a large scale mixed integer program, we decompose the model into two disjoint subproblems. The first subproblem has the constraints that are related to flow of the vehicles on the roads and contraflow decisions which are (1.4)–(1.10), and variables \( v_{it}, u_{jt}, \bar{u}_{jt}, s_{kt} \) and \( \bar{z}_{jt} \in \{0,1\} \) for all \( i \in R, j \in E, \) and \( t = 0, \ldots, T. \)

The second subproblem determines the shelter locations. It has constraints (1.11) and (1.12), and variables \( s_{it} \) and \( \tilde{y}_i \in \{0,1\} \) for \( i \in S \) and \( t = 0, \ldots, T. \) Let \( x_i = \tilde{p}_i \sum_{t=0}^{T} s_{it} \) for \( i \in S. \) Then, the subproblem is

\[
\sum_{i \in S} (x_i + \tilde{q}_i \tilde{y}_i) \leq Q, \\
x_i \leq C_i \bar{y}_i, \ i \in S, \\
x_i \geq 0, \ i \in S, \\
\bar{y}_i \in \{0,1\}, \ i \in S
\]

which is equivalent to \( P. \) Thus, the polyhedral study of \( P \) allows us to develop valid inequalities which can be used to solve the evacuation planning problem. The valid
inequalities developed for the two subproblems can be incorporated into branch-and-cut framework to solve the large scale evacuation planning problem.

1.2 Literature Review

In this section, we first give a brief literature review of the special cases of $P$. Then, we present literature relevant to the sequence independent lifting using superadditive functions.

Special cases of $P$ are studied extensively. One special case is the knapsack polytope which is obtained by restricting $m_j = 0$ for $j \in N$ (see, for example, Balas 1975, Wolsey 1975 and Padberg 1979).

Another special case is the fixed charge network flow polytope where $w_j = 0$ for $j \in N$. Padberg et al. (1985) derive a class of facet defining valid inequalities, called flow cover inequalities. They also show that these inequalities describe the convex hull of the set of solutions for $\{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}^n | \sum_{j \in N}(x_j + w_j y_j) = \bar{d}, x_j \leq my_j$ for $j \in N\}$. The work by Padberg et al. (1985) is extended by Van Roy and Wolsey (1986), Goemans (1989), Gu et al. (1999) and Atamturk et al. (2001). Louveaux and Wolsey (2003) survey previously studied inequalities for the fixed charge polytope.

In the lifting procedure, we first fix the values of the new variables. The lifting function provides the maximum value of left hand side of the lifted inequality given that the new variables are fixed and the original variables are feasible. This value is then used to calculate the coefficients of the new variables. If the lifting function of the inequality is well-structured, then lifting can be performed efficiently. Wolsey (1977), Gu et al. (1999) and Atamturk (2003) show that if the lifting function is superadditive, then all lifting coefficients can be obtained by solving only one problem. Gu et al. (2000) study the sequence independent lifting of the flow cover inequalities for the fixed charge network flow polytope. They show that lifting function for this polytope is superadditive. Marchand and Wolsey (1999) investigate the polyhedral structure of the knapsack polytope with a single continuous variable. They develop a superadditive lifting function to lift cover inequalities for this polytope.

Usually, a lifting function is not superadditive. In this case, a superadditive lower approximation of the lifting function can be used to generate strong cuts (Gu et al. 1999, Gu et al. 2000, Atamturk 2003). Gu et al. (1999) consider the single node flow polytope \( \{(x, y) \in \mathbb{R}_+^n \times \mathbb{B}^n | \sum_{j \in \mathbb{N}^+} x_j - \sum_{j \in \mathbb{N}^-} x_j \leq \bar{d}, x_j \leq m_j y_j \text{ for } j \in \mathbb{N}\} \) where \( N^- \) and \( N^+ \) form a partition of \( \mathbb{N} \). They develop lifted flow cover inequalities. Gu et al. (2000) develop lifted knapsack cover inequalities for the 0-1 knapsack polytope. Atamturk (2003) also applies superadditive lower approximation for general mixed integer knapsack sets.

Shebalov and Klabjan (2006) study the mixed-integer polytope with variable upper bounds \( \{(x, y) \in \mathbb{R}_+^n \times \mathbb{B}^n | \sum_{j \in \mathbb{N}_1^+} x_j - \sum_{j \in \mathbb{N}_1^-} x_j + \sum_{j \in \mathbb{N}_2^+} w_j y_j + \sum_{j \in \mathbb{N}_2^-} w_j y_j \leq \bar{d}, x_j \leq u_j + m_j y_j \text{ for } j \in \mathbb{N}_1^+, x_j \leq u_j - m_j y_j \text{ for } j \in \mathbb{N}_2^- \} \) where \((\mathbb{N}_1^+ \cup \mathbb{N}_1^-) \subseteq (\mathbb{N}_2^+ \cup \mathbb{N}_2^-) \) and \( \mathbb{N}_2^+ \) and \( \mathbb{N}_2^- \) form a partition of \( \mathbb{N} \). They give sufficient conditions
under which a generalized flow cover inequality is facet defining. When the inequality does not define a facet, they use a superadditive approximation of the lifting function to lift the inequality. They show that it is NP-Hard to compute the lifting function. Then, they develop a lower bound function for the lifting function by using the LP relaxation.

1.3 Contribution

In this dissertation, two new sets of valid inequalities for $P$ are developed. We derive the first set of valid inequalities by adding a new subset of variables to the flow cover inequalities (Goemans 1987, Wolsey 1989). We show that, under some conditions, these inequalities are facet defining. Then, a computational study shows the effectiveness of the valid inequalities.

Given a valid inequality, sequence independent lifting is a technique to develop strengthen the inequality by simultaneously including a set of new variables. The second new set of valid inequalities is generated by sequence independent lifting of the flow cover inequalities. We show that computing a lifting function for $P$ is NP-hard. Then, we determine a lower bound function for the lifting function that is better lower bound than one generated by using the LP relaxation. Next, we develop a superadditive function which produces strong cuts. We show that the superadditive function is not dominated by any other superadditive function. Our resulting inequalities dominate the inequalities of Shebalov and Klabjan (2006). A separation procedure is developed for the lifted inequalities. We investigate conditions where the separation problem provides strong cuts. Then, because the separation problem is NP-hard, we develop a heuristic procedure based on properties of the problem.
A computational study shows that our sequential independent lifting procedure is efficient.

1.4 Organization of the Dissertation

The organization of the dissertation is as follows. In Chapter 2, we present basic polyhedral results for $P$ and provide the first set of valid inequalities. In Chapter 3, we develop the second set of valid inequalities by sequence independent lifting of the flow cover inequalities. Finally, in Chapter 4, we summarize our results, and discuss future research.
In this chapter, we give basic polyhedral results for \( P \). Then, we derive a set of valid inequalities by adding a new subset of variables to the flow cover inequalities presented by Goemans (1987) and Wolsey (1989). We show that, under some conditions, this set is facet defining. A separation procedure is developed for the set of valid inequalities. We investigate conditions where the separation problem provides strong cuts. Then, because the separation problem is NP-hard, we develop a heuristic procedure based on properties of the problem to determine a valid inequality that separates a fractional extreme point solution from the integer hull of \( P \). We provide a computational study to show that the set of valid inequalities are efficient.

### 2.1 Basic Polyhedral Results

In this section, we give basic polyhedral results for \( P \). W.l.o.g, redefine \( P \) so that 
\[
m_j = \min \{ m_j, d - w_j \}.
\]
Let 
\[
q = (x, y)^T = (x_1, \ldots, x_n, y_1, \ldots, y_n)^T
\]
be a solution for \( P \) and \( Q \) be a solution matrix where each feasible solution is a column. Let \( e_i^x \) be a
unit vector \((x, 0)\) for \(i \in N\) in \(P\) where \(x_i = 1, x_j = 0\) for all \(j \in N \setminus \{i\}\) and \(y_j = 0\) for all \(j \in N\). Define the unit vector \(e^y_i\) similar to \(e^x_i\).

**Proposition 2.1.1** \(\dim(P) = 2n\) which is full-dimensional.

**Proof** For each \(j \in N\), let \(0 < \epsilon_j \leq m_j\). Consider the following \(2n\) points in \(\text{conv}(P)\):
\[
q^1_j = \epsilon_j e^x_j + e^y_j, \quad j \in N,
\]
\[
q^2_j = e^y_j, \quad j \in N.
\]
Let \(Q = (q^1_1, \ldots, q^1_n, q^2_1, \ldots, q^2_n)\). After subtracting \(q^2_j\) from \(q^1_j\) for all \(j = 1, \ldots, n\), \(Q\) becomes a diagonal matrix. Therefore, \(\text{rank}(Q) = 2n\). \(\square\)

**Proposition 2.1.2** The inequality \(x_r \geq 0\) for \(r \in N\) generates a facet of \(\text{conv}(P)\).

**Proof** We find \(2n - 1\) linearly independent points in \(\text{conv}(P)\) which satisfy \(x_r = 0\) for \(r \in N\). Let \(0 < \epsilon_j \leq m_j\), for each \(j \in N\). Consider the following \(2n - 1\) points in \(P\):
\[
q^1_j = \epsilon_j e^x_j + e^y_j, \quad j \in N \setminus \{r\},
\]
\[
q^2_j = e^y_j, \quad j \in N.
\]
Let \(Q = (q^1_1, \ldots, q^1_{i-1}, q^1_{i+1}, \ldots, q^1_n, q^2_1, \ldots, q^2_n)\). After subtracting \(q^2_j\) from \(q^1_j\) for all \(j = 1, \ldots, n\), \(Q\) becomes a diagonal matrix which has \(\text{rank}(Q) = 2n - 1\). Therefore, the points are lid. Since \(x_r = 0\) goes through the origin, \(\{x \in \text{conv}(P) \mid x_r = 0\}\) for \(r \in N\) generates a facet of \(\text{conv}(P)\). \(\square\)

**Proposition 2.1.3** The inequality \(y_r \leq 1\) for \(r \in N\) generates a facet of \(\text{conv}(P)\) if \(d - w_r - w_j > 0\) for \(j \in N \setminus \{r\}\).
Proof Suppose that \( d - w_r - w_j > 0 \) for \( j \in N \setminus \{r\} \). We find \( 2n \) linearly independent points in \( \text{conv}(P) \) which satisfy \( y_r = 1 \) for \( r \in N \). Let \( 0 < \epsilon_r \leq m_r \), for each \( r \in N \) and \( 0 < \epsilon_j \leq \min\{m_j, d - w_j - w_r\} \) for \( j \in N \setminus \{r\} \). Consider the following \( 2n \) points in \( \text{conv}(P) \):

\[
q_j^1 = \begin{cases} 
\epsilon_j e_j^x + e_r^y, & j = r, \\
\epsilon_j e_j^x + e_j^y + e_r^y, & j \in N \setminus \{r\},
\end{cases}
\]

\[
q_j^2 = \begin{cases} 
\epsilon_j^y, & j = r, \\
\epsilon_j^y + e_r^y, & j \in N \setminus \{r\}.
\end{cases}
\]

Let \( Q = (q_1^1, \ldots, q_{2n}^1, q_1^2, \ldots, q_{2n}^2) \). After subtracting \( q_j^2 \) from \( q_j^1 \) for \( j = 1, \ldots, n \), and \( q_r^2 \) from \( q_j^1 \) for \( j = 1, \ldots, r - 1, r + 1, \ldots, n \), \( Q \) becomes a diagonal matrix which has \( \text{rank}(Q) = 2n \). Therefore, the points are linearly independent. \( \square \)

**Proposition 2.1.4** The inequality \( x_r \leq m_r y_r \) for \( r \in N \) generates a facet of \( \text{conv}(P) \).

Proof The inequality \( x_r \leq m_r y_r \), for \( r \in N \), is valid for \( P \), because \( m_r \leq d - w_r \).

We define \( 2n - 1 \) linearly independent points in \( \text{conv}(P) \) that satisfy \( x_r = m_r y_r \) for \( r \in N \). Let \( \epsilon_r = m_r \) and \( 0 < \epsilon_j \leq m_j \) for \( j \in N \setminus \{r\} \). Consider the following \( 2n - 1 \) points in \( \text{conv}(P) \):

\[
q_j^1 = \epsilon_j e_j^x + e_j^y, \quad j \in N,
\]

\[
q_j^2 = e_j^y, \quad j \in N \setminus \{r\}.
\]

Let \( Q = (q_1^1, \ldots, q_{2n}^1, q_1^2, \ldots, q_{2n}^2) \). Subtracting \( q_j^2 \) from \( q_j^1 \) for all \( j = 1, \ldots, r - 1, r + 1, \ldots, n \), establishes that the \( 2n - 1 \) points are linearly independent.

Since \( \{(x, y) \in \text{conv}(P) | x_r - m_r y_r = 0\} \) for \( r \in N \) goes through the origin, it generates a facet of \( \text{conv}(P) \). \( \square \)
Proposition 2.1.5 (Goemans, 1989) If \( \pi y \leq \pi_0 \) defines a facet of \( \text{conv}\{(x, y) \in \mathbb{R}_n^+ \times B^n \mid \sum_{j \in N} w_j y_j \leq \bar{d}\} \) if there exists \( y^k \in \{y \in B^n \mid \pi y \leq \pi_0, \sum_{j \in N} w_j y_j^k \leq \bar{d}, y_j = 1 \text{ for } j \in N\} \), then \( \pi y \leq \pi_0 \) defines a facet of \( \text{conv}(P) \).

2.2 Valid Inequalities

In this section, we develop a set of valid inequalities for \( P \) by adding a new subset of variables to the flow cover inequalities. We show that, under some conditions, this set is facet defining.

To extend the flow cover inequalities, let \( S \subseteq N \) where \( d < \sum_{j \in S \backslash \{k\}} (m_j + w_j) \) for \( k = \arg\min_{j \in S} \{m_j + w_j\} \). If \( S \) does not exist, there are only \(|N| + 1 \) possible optimal values for \( y \). As a result, any mixed integer problem associated with \( P \) can be solved in polynomial time. Consequently, we assume the existence of \( S \). Partition \( S \) into \( C \) and \( \bar{C} \). Let

\[
\hat{m}_j = \begin{cases} 
  m_j, & j \in C, \\
  0, & j \in \bar{C}.
\end{cases}
\]

Also, construct \( T_1 \subseteq S \) such that \( \sum_{j \in T_1} w_j < d < \sum_{j \in T_1} (\hat{m}_j + w_j) \). We assume that \( T_1 \) exists. Define \( \lambda > 0 \) and \( \tau > 0 \) such that

\[
\lambda = \bar{d} - \sum_{j \in T_1} w_j \quad \text{and} \quad \tau = \sum_{j \in T_1} (\hat{m}_j + w_j) - \bar{d} = \sum_{j \in T_1} \hat{m}_j - \lambda.
\]

Goemans (1987) shows that the flow cover inequalities

\[
\sum_{j \in T_1 \cap C} x_j + \sum_{j \in T_1} w_j y_j + \sum_{j \in T_1} \max\{\hat{m}_j + w_j - \tau, 0\}(1 - y_j) \leq \bar{d} \quad (2.1)
\]

are valid for \( P \). Also, Goemans (1989) shows that (2.1) defines facets of \( \text{conv}(P) \) under some conditions.
Reorder the variables such that \( \hat{m}_1 + w_1 = \max_{j \in T_1} \{ \hat{m}_j + w_j \} \). Let \( T_2 = \{ j \in S \setminus T_1 | \lambda < w_j \leq w_1 + \lambda \} \), and let \( T = T_1 \cup T_2 \). Also, let

\[
\gamma_j = \begin{cases} 
\min \{ \tau - \hat{m}_j, w_j \}, & j \in T_1, \\
\gamma_1 - \min \{ w_1 - w_j, \tau - \hat{m}_1 + \hat{m}_j \}, & j \in T_2.
\end{cases}
\]

By adding the variables in \( T_2 \) to (2.1), we have the inequality,

\[
\sum_{j \in T \cap C} x_j - \sum_{j \in T_1} \gamma_j (1 - y_j) + \sum_{j \in T_2} \gamma_j y_j \leq \lambda.
\]

(2.2)

Note that if \( T_2 = \emptyset \), then (2.2) equals to (2.1). Therefore, we assume that \( T_2 \neq \emptyset \).

We now establish that (2.2) is valid for \( P \).

**Lemma 2.2.1** If \( \gamma_1 = w_1 \), then \( \gamma_j = w_j \) for \( j \in T \).

**Proof** If \( \gamma_1 = w_1 \), then \( w_1 \leq \tau - \hat{m}_1 \). This implies that

\[
\hat{m}_j + w_j \leq \hat{m}_1 + w_1 \leq \tau
\]

for \( j \in T \). Thus, \( w_j \leq \tau - \hat{m}_j \). This implies that \( \gamma_j = w_j \) for \( j \in T_1 \).

For \( j \in T_2 \), (2.3) implies that \( w_1 - w_j \leq \tau - \hat{m}_1 + \hat{m}_j \). As a result,

\[
\gamma_j = w_1 - (w_1 - w_j) = w_j.
\]

**Lemma 2.2.2** If \( \gamma_1 < w_1 \), then \( \gamma_j = \max \{ \tau + w_j - \hat{m}_1 - w_1, -\hat{m}_j \} \) for \( j \in T_2 \).

**Proof** Suppose \( \gamma_1 < w_1 \). For \( j \in T_2 \),

\[
\gamma_j = \gamma_1 - \min \{ w_1 - w_j, \tau - \hat{m}_1 + \hat{m}_j \}
\]

\[
= (\tau - \hat{m}_1) + \max \{ -w_1 + w_j, -\tau + \hat{m}_1 - \hat{m}_j \}
\]

\[
= \max \{ \tau + w_j - \hat{m}_1 - w_1, -\hat{m}_j \}.
\]
Proposition 2.2.3 Inequality (2.2) is valid for $P$.

Proof We show that (2.2) is satisfied for any point $(x^0, y^0) \in P$. Let $\hat{T}^0 = \{ j \in T \mid y^0_j = 0 \}$ and $\hat{T}^1 = \{ j \in T \mid y^0_j = 1 \}$. Also, let $T^i_k = (\hat{T}^i \cap T_k)$ for $i = 0, 1$ and $k = 1, 2$. Inequality (2.2) can be written as

\[
\sum_{j \in T \cap C} x^0_j = \sum_{j \in \hat{T}^1 \cap C} x^0_j \leq \lambda + \sum_{j \in T^1_0} \gamma_j - \sum_{j \in T^1_2} \gamma_j. \tag{2.4}
\]

We can rewrite (1.1) as

\[
\sum_{j \in \hat{T}^1 \cap C} x^0_j \leq \bar{d} - \sum_{j \in \hat{T}^1} w_j = \lambda + \sum_{j \in T^1_1} w_j - \sum_{j \in \hat{T}^1} w_j = \lambda + \sum_{j \in T^1_0} w_j - \sum_{j \in T^1_2} w_j. \tag{2.5}
\]

Also, we can rewrite (1.2) as

\[
\sum_{j \in \hat{T}^1 \cap C} x^0_j \leq \sum_{j \in \hat{T}^1} \hat{m}_j = \sum_{j \in T^1_1} \hat{m}_j + \sum_{j \in T^1_2} \hat{m}_j = \tau + \lambda - \sum_{j \in T^1_0} \hat{m}_j - \sum_{j \in T^1_2} \hat{m}_j. \tag{2.6}
\]

To establish that (2.4) is valid, we show that $\min\{\lambda + \sum_{j \in T^1_1} w_j - \sum_{j \in T^1_2} w_j, \tau + \lambda - \sum_{j \in T^1_0} \hat{m}_j + \sum_{j \in T^1_2} \hat{m}_j, \} \leq \lambda + \sum_{j \in T^1_0} \gamma_j - \sum_{j \in T^1_2} \gamma_j$.

Suppose $\gamma_1 = w_1$. Lemma 2.2.1 implies that $\lambda + \sum_{j \in T^1_0} w_j - \sum_{j \in T^1_2} w_j = \lambda + \sum_{j \in T^1_0} \gamma_j - \sum_{j \in T^1_2} \gamma_j$. Thus, (2.5) establishes the result.

Alternatively, suppose that $\gamma_1 < w_1$. Then, from Lemma 2.2.2,

\[
\lambda + \sum_{j \in T^0_1} \gamma_j - \sum_{j \in T^0_2} \gamma_j = \lambda + \sum_{j \in T^0_1} \min\{\tau - \hat{m}_j, w_j\}
\]
We consider two cases:

Then, (2.7) can be written as

\[ \begin{align*}
- \sum_{j \in T_0^1} \max\{\tau + w_j - \hat{m}_1 - w_1, -\hat{m}_j\} \\
= \lambda - |T_2^1| \tau - \sum_{j \in T_0^0} \hat{m}_j - \sum_{j \in T_2^1} w_j + \sum_{j \in T_0^0} \min\{\tau, \hat{m}_j + w_j\} \\
+ \sum_{j \in T_2^1} \min\{\hat{m}_1 + w_1, \tau + \hat{m}_j + w_j\}. 
\end{align*} \]

(2.7)

Let \( B_1 = \{ j \in T_0^0 \mid \hat{m}_j + w_j \geq \tau \} \), and \( B_2 = \{ j \in T_2^1 \mid \hat{m}_j + w_j \leq \hat{m}_1 + w_1 - \tau \} \).

Then, (2.7) can be written as

\[ \begin{align*}
\lambda + \sum_{j \in T_0^0} \gamma_j - \sum_{j \in T_2^1} \gamma_j &= \lambda + (|B_1| + |B_2| - |T_2^1|) \tau - \sum_{j \in T_0^0} \hat{m}_j - \sum_{j \in T_2^1} w_j \\
+ \sum_{j \in T_0^0 \setminus B_1} (\hat{m}_j + w_j) + \sum_{j \in T_2^1 \setminus B_2} (\hat{m}_1 + w_1) \\
+ \sum_{j \in B_2} (\hat{m}_j + w_j). 
\end{align*} \]

(2.8)

We consider two cases: \(|B_1| + |B_2| \leq |T_2^1|\) and \(|B_1| + |B_2| > |T_2^1|\).

Case 1: \(|B_1| + |B_2| \leq |T_2^1|\). Then, (2.8) implies

\[ \begin{align*}
\lambda + \sum_{j \in T_0^0} \gamma_j - \sum_{j \in T_2^1} \gamma_j &\geq \lambda + (|B_1| + |B_2| - |T_2^1|) \tau - \sum_{j \in T_0^0} \hat{m}_j - \sum_{j \in T_2^1} w_j \\
+ \sum_{j \in T_0^0 \setminus B_1} (\hat{m}_j + w_j) + \sum_{j \in T_2^1 \setminus B_2} (\hat{m}_1 + w_1) \\
&= \lambda + (|B_1| + |B_2| - |T_2^1|) \tau - \sum_{j \in T_0^0} \hat{m}_j - \sum_{j \in T_2^1} w_j \\
+ \sum_{j \in T_0^0 \setminus B_1} (\hat{m}_j + w_j) - (|B_1| + |B_2| - |T_2^1|)(\hat{m}_1 + w_1) \\
+ \sum_{j \in B_1} (\hat{m}_1 + w_1) \\
&\geq \lambda + (|B_1| + |B_2| - |T_2^1|) \tau - \sum_{j \in T_0^0} \hat{m}_j - \sum_{j \in T_2^1} w_j \\
+ \sum_{j \in T_0^0 \setminus B_1} (\hat{m}_j + w_j) - (|B_1| + |B_2| - |T_2^1|) \tau \\
+ \sum_{j \in B_1} (\hat{m}_1 + w_1)
\end{align*} \]
\[ \lambda + \sum_{j \in T_0^1} w_j - \sum_{j \in T_1^1} w_j. \]

The second inequality follows from \( \tau \leq \hat{m}_1 + w_1 \) and \( \hat{m}_j + w_j \leq \hat{m}_1 + w_1 \) for \( j \in B_1 \). The result is established by (2.5).

Case 2: \( |B_1| + |B_2| > |T_2^1| \). Then, (2.8) implies

\[
\begin{align*}
&\lambda + \sum_{j \in T_0^1} \gamma_j - \sum_{j \in T_1^1} \gamma_j \geq \lambda + \tau - \sum_{j \in T_0^1} \hat{m}_j - \sum_{j \in T_1^1} w_j + \sum_{j \in T_1^0 \setminus B_2} (\hat{m}_j + w_j) \\
&\quad + \sum_{j \in B_2} (\hat{m}_j + w_j) \\
&= \lambda + \tau - \sum_{j \in T_0^1} \hat{m}_j + \sum_{j \in T_1^1} \hat{m}_j.
\end{align*}
\]

The inequality follows from \( \hat{m}_j + w_j \leq \hat{m}_1 + w_1 \) for \( j \in B_1 \). Thus, (2.6) establishes the result. \( \square \)

**Proposition 2.2.4** Inequality (2.2) generates a facet of \( \text{conv}(P) \) if

(i). \( T_2 \subseteq C \),

(ii). \( \max_{j \in N \setminus (T_1 \cup T_2)} \{w_j\} < \hat{m}_1 + w_1 - \tau \),

(iii). \( \min_{j \in T_2} \{\hat{m}_j + w_j\} > \hat{m}_1 + w_1 - \tau \).

**Proof** We find \( \text{dim}(P) = 2n \) linearly independent points to show that (2.2) generates a facet of \( \text{conv}(P) \). Suppose (i)-(iii) hold. Let \( k_1 = |T_1| \). Reindex the elements of \( T_1 \) so that \( m_i \geq m_{i+1} \) for \( i = 2, \ldots, k_1 - 1 \). Also, let \( k_2 = |T_2| \) and \( T_2 = \{k_1 + 1, \ldots, k_1 + k_2\} \). For \( j \in T_2 \), let

\[ \varepsilon^1_j = \min \left\{ \hat{m}_j, \bar{d} - \sum_{i \in T_1 \setminus \{1\}} w_i - w_j \right\}. \]

If \( \varepsilon^1_j = \hat{m}_j \), then let \( u_j = \arg\max \{i \in T_1 \mid \sum_{\ell=2}^i \hat{m}_\ell < \lambda + w_1 - w_j - \varepsilon^1_j \} \). Otherwise, let \( u_j = 0 \). For \( j \in T_2 \), let \( \varepsilon = 1 \setminus \sum_{i \in T_1 \cap C} m_i \).
When \( (i) \) holds and \( \hat{m}_1 + w_1 > \tau \), inequality (2.1) generates a facet of \( \text{conv}(P \cap \{ j \in T_2 \mid x_j = y_j = 0 \}) \) (Goemans 1989). Therefore, there exist \( 2n - 2|T_2| \) linearly independent points in \( \text{conv}(P \cap \{ j \in T_2 \mid x_j = y_j = 0 \}) \) which satisfy (2.2) at equality. Now, we find \( 2|T_2| \) aid points in \( P \). For each \( j \in T_2 \), we provide two points where \( y_j = 1 \) and \( y_k = 0 \) for \( k \in T_2 \setminus \{ j \} \). Consider \( 2|T_2| \) points in \( P \) for \( j \in T_2 \):

\[
q_j^1 = \varepsilon_j^1 e^x_j + \sum_{i=2}^{k_1} e^y_i + e^y_j
\]

\[
+ \begin{cases} 
\sum_{i=2}^{u_j} \hat{m}_i e^x_i + (\lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i) e^x_{u_j+1}, & \text{if } \varepsilon_j^1 = \hat{m}_j, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
q_j^2 = (\varepsilon_j^1 - \varepsilon) e^x_j + \sum_{i=2}^{k_1} e^y_i + e^y_j
\]

\[
+ \begin{cases} 
\sum_{i=2}^{u_j} \hat{m}_i e^x_i + (\lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i) e^x_{u_j+1} + \varepsilon e^x_{u_j+2}, & \text{if } \varepsilon_j^1 = \hat{m}_j \text{ and } \\
\varepsilon = \lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i = \hat{m}_{u_j+1}, \\
\sum_{i=2}^{u_j} \hat{m}_i e^x_i + (\lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i + \varepsilon) e^x_{u_j+1}, & \text{if } \varepsilon_j^1 = \hat{m}_j \text{ and } \\
\varepsilon = \lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i < \hat{m}_{u_j+1}, \\
0, & \text{otherwise.}
\end{cases}
\]

We first show that these points satisfy (2.2) at equality. From condition \( (i) \), \( \hat{m}_j = m_j \) for \( j \in T_2 \). Consider point \( q_j^1 \) for \( j \in T_2 \). If \( \varepsilon_j^1 = \hat{m}_j \), then

\[
(x_i, y_i) = \begin{cases} 
(0, 0), & i = 1, \\
(\hat{m}_i, 1), & i = 2, \ldots, u_j, \\
(\lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i, 1), & i = u_j + 1, \\
(0, 1), & i = u_j + 2, \ldots, k_1 \\
(\varepsilon_j^1, 1), & i = j \\
(0, 0), & \text{otherwise.}
\end{cases}
\]
From (i), $\gamma_1 = \tau - \hat{m}_1 < w_1$. Therefore, from Lemma 2.2.2 and condition (iii),

$$\gamma_j = \max \{ \tau + w_j - \hat{m}_1 - w_1, -\hat{m}_j \} = \tau + w_j - \hat{m}_1 - w_1$$

for $j \in T_2$. Point $q_j^1$ satisfies (2.2) at equality because

$$\sum_{j \in T_1 \cap C} x_j - \sum_{j \in T_1} \gamma_j (1 - y_j) + \sum_{j \in T_2 \cap C} x_j + \sum_{j \in T_2} \gamma_j y_j$$

$$= \sum_{i=2}^u \hat{m}_i - \gamma_1 + (\lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^u \hat{m}_i) + \varepsilon_j^1 + \gamma_j$$

$$= -\gamma_1 + \lambda + w_1 - w_j + \gamma_j$$

$$= -(\tau - \hat{m}_1) + \lambda + w_1 - w_j + \tau + w_j - \hat{m}_1 - w_1$$

$$= \lambda.$$

Alternatively suppose that $\varepsilon_j^1 = d - \sum_{i \in T_1 \setminus \{1\}} w_i - w_j$. Then

$$(x_i, y_i) = \begin{cases} (0, 1), & i = 2, \ldots, k_1, \\ (\varepsilon_j^1, 1), & i = j \\ (0, 0), & \text{otherwise.} \end{cases}$$

Point $q_j^1$ satisfies (2.2) at equality because

$$\sum_{j \in T_1 \cap C} x_j - \sum_{j \in T_1} \gamma_j (1 - y_j) + \sum_{j \in T_2 \cap C} x_j + \sum_{j \in T_2} \gamma_j y_j$$

$$= \varepsilon_j^1 + \gamma_j - \gamma_1$$

$$= d - \sum_{i \in T_1 \setminus \{1\}} w_i - w_j + \tau + w_j - \hat{m}_1 - w_1 - (\tau - \hat{m}_1)$$

$$= \lambda.$$

Similarly, we can show that the points $q_j^2$ for $j \in T_2$ satisfies (2.2) at equality.

We now show that $q_j^1$ and $q_j^2$ are linearly independent. Let $Q$ be $2n \times 2|T_2|$ matrix

where $q_j^1$ and $q_j^2$ are columns for all $j \in T_2$. For some $j \in T_2$, subtract $q_j^2$ from $q_j^1$. Let $\tilde{q}_j^2$ be the new $q_j^2$ column where
After multiplying $\hat{q}_2^j$ with $(\varepsilon_1^j \setminus \varepsilon)e_x^j$, the element corresponding $x_j$ becomes $\varepsilon_j^1$ in the new $\tilde{q}_2^j$. Let $\tilde{q}_2^j$ be the new $\tilde{q}_2^j$ column where

$$\tilde{q}_2^j = \varepsilon_j^1 e_x^j + \begin{cases} \varepsilon e_{x_{u_j+2}}, & \text{if } \varepsilon_j^1 = m_j \text{ and } \lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i = \hat{m}_{u_j+1}, \\ \varepsilon e_{x_{u_j+1}}, & \text{if } \varepsilon_j^1 = \hat{m}_j \text{ and } \lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i < \hat{m}_{u_j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Subtract $\tilde{q}_2^j$ from $q_1^j$. Let $\tilde{q}_1^j$ be the new $q_1^j$ column vector where

$$\tilde{q}_1^j = \sum_{i=2}^{k_1} e_i^y + e_j^y + \begin{cases} \sum_{i=2}^{u_j} \hat{m}_i e_i^x + (\lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i) e_{x_{u_j+1}} - \varepsilon_j^1 e_{x_{u_j+2}}, & \text{if } \varepsilon_j^1 = m_j \text{ and } \\ \lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i = \hat{m}_{u_j+1}, \\ \sum_{i=2}^{u_j} \hat{m}_i e_i^x + (\lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i - \varepsilon_j^1) e_{x_{u_j+1}}, & \text{if } \varepsilon_j^1 = \hat{m}_j \text{ and } \\ \lambda + w_1 - w_j - \varepsilon_j^1 - \sum_{i=2}^{u_j} \hat{m}_i < \hat{m}_{u_j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

In $\tilde{q}_1^j$, the element corresponding to $y_j$ is one and the elements corresponding to $y_k$ for all $k \in T_2 \setminus \{j\}$ is zero. In $\tilde{q}_2^j$, the element corresponding to $x_j$ for $j \in T_2$ is $\varepsilon$ and the elements corresponding to $x_k$ for $k \in T_2 \setminus \{j\}$ is zero. Also, the elements corresponding to $y_j$ for all $j \in T_2$ is zero. Therefore, the matrix $Q$ consisting of $\tilde{q}_1^j$ and $\tilde{q}_2^j$ for $j \in T_2$ has a rank of $2|T_2|$. Thus, $q_1^j$ and $q_2^j$ are linearly independent for $j \in T_2$. \(\square\)
2.3 Separation

In this section, we study separation problem for (2.2). Given solution \((x^0, y^0)\), the separation problem is solved for (2.2) by finding sets \(C, T_1, \) and \(T_2\). To find these sets for a given \(S \subseteq N\) where \(\sum_{j \in S \setminus \{k\}} (m_j + w_j) > \bar{d}\) and \(k = \arg\min_{j \in S} \{m_j + w_j\}\), we solve:

\[
(SP) \max_{C \subseteq S, T_1 \subseteq S, \quad T_2 \subseteq S \setminus T_1} \sum_{j \in T_1 \cap C} x_j - \sum_{j \in T_1} \gamma_j (1 - y_j) + \sum_{j \in T_1} w_j + \sum_{j \in T_2 \cap C} x_j + \sum_{j \in T_2} \gamma_j y_j
\]

s.t.

\[
\sum_{j \in T_1} w_j \leq \bar{d}, \quad (2.10)
\]

\[
\sum_{j \in T_1} (\hat{m}_j + w_j) > \bar{d}, \quad (2.11)
\]

\[
w_j + \sum_{i \in T_1} w_i > \bar{d}, \quad j \in T_2, \quad (2.12)
\]

\[
w_j - w_1 + \sum_{i \in T_1} w_i \leq \bar{d}, \quad j \in T_2, \quad (2.13)
\]

\[
\hat{m}_j + w_j \leq \hat{m}_1 + w_1, \quad j \in T_2, \quad (2.14)
\]

where \(\hat{m}_1 + w_1 = \max_{j \in T_1} \{\hat{m}_j + w_j\}\). Constraints (2.10)–(2.14) provide conditions that describe the definitions of the sets \(C, T_1, \) and \(T_2\).

Because the Knapsack Problem is a subproblem of \(SP\), \(SP\) is NP-hard. As a result, we develop a heuristic, SH, to determine a constraint of the form of (2.2) that separates a fractional extreme point solution \((x^0, y^0)\) from the integer hull of \(P\).

We develop a procedure which is used in SH for selecting elements in \(C, T_1, \) and \(T_2\) in two steps. In the first step, we assume that \(T_2 = \emptyset\). Then, we determine a procedure for constructing sets \(T_1\) and \(C\) to maximize

\[
\sum_{j \in T_1 \cap C} x_j - \sum_{j \in T_1} \gamma_j (1 - y_j) + \sum_{j \in T_1} w_j.
\]

(2.15)
We now present a rule which partially determines sets $T_1$ and $C$. Recall that
\[ \hat{m}_j = m_j \] for $j \in T_1 \cap C$ and $\hat{m}_j = 0$ for $T_1 \setminus C$. Also, $\lambda = \bar{d} - \sum_{j \in T_1} w_j$, and
\[ \tau = \sum_{j \in T_1} \hat{m}_j - \lambda. \] Let $T_1^\tau = \{ i \in T_1 \mid \gamma_i = \tau - \hat{m}_i \}$ and $T_1^w = \{ i \in T_1 \mid \gamma_i = w_i \}$. For a given $(x^0, y^0)$, (2.15) can be written as
\[
\sum_{j \in T_1 \cap C} x^0_j - \sum_{j \in T_1^\tau} (\tau - \hat{m}_j)(1 - y^0_j) - \sum_{j \in T_1^w} w_j(1 - y^0_j) + \sum_{j \in T_1} w_j
\]
\[ \geq \sum_{j \in T_1 \cap C} x^0_j + \sum_{j \in T_1^\tau} (\tau - \hat{m}_j)y^0_j + \sum_{j \in T_1^w} w_jy^0_j
\]
\[ = \sum_{j \in T_1 \cap C} x^0_j + \sum_{j \in T_1^\tau} \tau y^0_j - \sum_{j \in T_1^\tau} \hat{m}_j y^0_j + \sum_{j \in T_1^w} w_jy^0_j - \sum_{j \in T_1^w} \hat{m}_j y^0_j + \sum_{j \in T_1^w} \hat{m}_j y^0_j
\]
\[ \geq \sum_{j \in T_1 \cap C} (x^0_j - \hat{m}_j y^0_j) + \sum_{j \in T_1^\tau} \tau y^0_j + \sum_{j \in T_1^w} \tau y^0_j. \] (2.16)

From (2.16), large values for $x^0_k - \hat{m}_k y^0_k$ for $k \in T_1 \cap C$ and for $y^0_j$ for $j \in T_1$ are more likely to generate a separating hyperplane for $(x^0, y^0)$. Because $x^0_j - \hat{m}_j y^0_j \leq 0$ for $j \in T_1 \cap C$, the largest value for $x^0_j - \hat{m}_j y^0_j$ is zero. Therefore, we first construct $T_1$ selecting $j$ with the largest $y^0_j$ such that $\sum_{j \in T_1} w_j \leq \bar{d}$. Then, we construct $T_1 \cap C$ considering $j$ the largest $x^0_j - \hat{m}_j y^0_j$ such that $\sum_{j \in T_1} (\hat{m}_j + w_j) \geq \bar{d}$.

We now present the heuristic procedure SH to separate a fractional solution $(x^0, y^0)$ from the integer hull of $P$.

SH

0. Input $(x^0, y^0)$. Reindex the elements of $N$ such that $w_{j-1} \geq w_j$ for $j = 2, \ldots, n$.
   Let $\ell = 1$.

1. Set $S = \{ \ell, \ldots, n \}$. If $\sum_{j \in S \setminus \{ k \}} (m_j + w_j) \leq \bar{d}$ for $k = \arg\min_{j \in S} \{ m_j + w_j \}$, then stop.
2. Let \( s = |S| \). Reindex the elements of \( S \) such that \( w_1 = \max_{j \in S} w_j \) and \( y_{j-1}^0 \geq y_j^0 \) for \( j = 3, \ldots, s \).

Set \( T_1 = \emptyset \). For \( j = 1, \ldots, s \), if \( \sum_{k \in \mathcal{T}_1} w_k + w_j \leq \bar{d} \), then \( T_1 = \mathcal{T}_1 \cup \{j\} \).

3. \( \Phi = \{ j \in \mathcal{T}_1 \mid m_j + w_j \leq \hat{m}_1 + w_1 \} \). Let \( t = |\Phi| \). Order the elements of \( \Phi = \{ j_1, \ldots, j_t \} \) such that \( x_{j_{k-1}}^0 - \hat{m}_k y_{j_{k-1}}^0 \geq x_{j_k}^0 - \hat{m}_k y_{j_k}^0 \) for \( k = 2, \ldots, t \). Set \( j_{t_1} = \arg\min \left\{ i \in \Phi \mid \sum_{k=1}^i m_{j_k} > \bar{d} - \sum_{j \in \mathcal{T}_1} w_j \right\} \).

Set \( T_1 \cap C = \{ j_1, \ldots, j_{t_1} \} \).

4. Construct \( \mathcal{T}_2 \setminus C \) and \( \mathcal{T}_2 \cap C \) as

\[
\mathcal{T}_2 \setminus C = \{ j \in S \setminus \mathcal{T}_1 \mid \lambda < w_j \leq w_1 + \lambda, w_j \leq \hat{m}_1 + w_1, m_j + w_j > \hat{m}_1 + w_1, \text{ and } \gamma_j > 0 \}
\]

and

\[
\mathcal{T}_2 \cap C = \{ j \in S \setminus \mathcal{T}_1 \mid \lambda < w_j \leq w_1 + \lambda, m_j + w_j \leq \hat{m}_1 + w_1, \text{ and } x_j^0 + \gamma_j y_j^0 > 0 \}.
\]

5. Calculate \( \lambda, \tau, \gamma_j \) for \( j \in \mathcal{T}_1 \), and \( \gamma_j \) for \( j \in \mathcal{T}_2 \).

Substitute \( \mathcal{T}_1, \mathcal{T}_1 \cap C, \mathcal{T}_2, \mathcal{T}_2 \cap C, \lambda, \tau, \gamma_j \) for \( j \in \mathcal{T}_1 \cup \mathcal{T}_2 \) into (2.2).

If (2.2) is violated, then stop. Otherwise, set \( \ell = \ell + 1 \) and go to Step 1.

For each iteration \( \ell \geq 1 \), Steps 1–5 are repeated. Steps 1–3 determine sets \( S, \mathcal{T}_1 \) and \( \mathcal{T}_1 \cap C \), respectively. In Step 1, the set \( S \) is constructed as \( S = \{ \ell, \ldots, n \} \). In Step 2, we first reindex the elements of \( S \) such that \( w_1 = \max_{j \in S} w_j \) and \( y_{j-1}^0 \geq y_j^0 \) for \( j = 3, \ldots, s \). Then, we construct \( \mathcal{T}_1 \) such that \( \sum_{j \in \mathcal{T}_1} w_j \leq \bar{d} \) by including an element of \( S \) with the largest \( w_j \). This guarantees that (2.10) is satisfied. In Step 3, we construct \( \mathcal{T}_1 \cap C \) such that \( \sum_{j \in \mathcal{T}_1 \cap C} m_j + \sum_{j \in \mathcal{T}_1} \geq \bar{d} \) by including the elements of \( \mathcal{T}_1 \) with the largest \( x_j^0 - \hat{m}_j y_j^0 \). Step 3 guarantees that (2.11) is satisfied. The sets \( \mathcal{T}_2 \) and \( \mathcal{T}_2 \cap C \) are provided in Step 4 such that (2.12)–(2.14) are satisfied. In Step 5, the inequality (2.2) is formed. If (2.2) is violated, then SH stops. Otherwise, \( \ell \) is increased by one and SH iterates starting from Step 1.
2.4 Computational Study

Define a test problem $P_T$ as

$$
\text{max } \sum_{j \in N} (a_j x_j + c_j y_j),
$$

s.t. $(x, y) \in P$.

To test the effectiveness of (2.2) in solving $P_T$, we implement a branch-and-cut algorithm. The branch-and-cut procedure first solves an LP relaxation of $P_T$ using CPLEX 10.0. Given a fractional point, separating hyperplanes are found using heuristic SH. Then, the inequalities generated by the separating hyperplanes are added to the LP relaxation of $P_T$. If no separating hyperplane is found, then we branch on the fractional variable where the fractional portion is closest to 0.5. Also, the node with minimum bound is selected for branching. In all experiments, CPLEX cuts are disabled to isolate the impact due to inequalities discussed in this study. The computational experiments are performed on a Dell PC with 3.20GHZ Dual Core Processor, and 2 GB RAM.

Two sets of valid inequalities are considered in the computational study. These are flow cover inequalities (2.1) called $(FC)$ and lifted flow cover inequalities (2.2) called $(LFC)$.

We examine the following conjectures:

**Conjecture 1** The separation procedure where $m$ and $w$ have small variances generates valid inequalities that are stronger than the ones where $m$ and $w$ have large variances.

In the separation procedure, if $j \in T_1 \setminus C$, then we search for possible subsets where the variable $i$ can be exchanged with variable $j$ without changing the value of
If such an $i$ exists, then the valid inequality (2.2) is strengthened by adding $x_i$ and $y_i$. If $m$ and $w$ have small variances, then the possibility of finding subsets that contain $i$ is high.

**Conjecture 2** The separation procedure where $m$ and $w$ are highly correlated generates valid inequalities that are stronger than those when they are not.

Valid inequality (2.2) can be strengthened by lifting $x_j$ and $y_j$ when $w_j \leq w_1$ and $m_j + w_j \leq m_1 + w_1$. These conditions are more likely to be satisfied if $m$ and $w$ are highly correlated.

**Conjecture 3** The separation procedure where $\bar{d} \leq \sum_{j \in N} (m_j + w_j)/2$ generates stronger valid inequalities than when $\bar{d} > \sum_{j \in N} (m_j + w_j)/2$.

Variable $j$ is lifted if $\lambda = \bar{d} - \sum_{i \in T_1} w_i < w_j$. If $\bar{d}$ is large, then to satisfy this condition many variables may be included in $T_1$. Therefore, small number of variables in $N \setminus T_1$ can be lifted.

Define $U[\ell, u]$ to be the discrete uniform distribution over the interval $[\ell, u]$. To consider the effects of the variability of $m$ and $w$, let

$$m_j \sim U[1, 99] \text{ or } m_j \sim U[40, 60]$$

and

$$w_j \sim U[1, 99] \text{ or } w_j \sim U[40, 60].$$

To examine the effects of correlation between the $m$ and $w$, we let

$$m_j = \rho w_j + (\sqrt{1 - \rho^2})\eta + t \text{ where } \eta \sim U[1, 99],$$

$\rho \in \{-0.9, -0.5, 0.5, 0.9\}$ and $t \in \{0, -\rho 99\}$ (Neter et al. 2003). If $\rho < 0$, then let $t = -\rho 99$. Otherwise, $t = 0$. Note that if $\rho = 0$, then $m_j \sim U[1, 99]$. 28
Let $E(\cdot)$ be the expectation operator. The relationship of $\bar{d}$ and $\sum_{j \in N} m_j + w_j$ may influence the performance of SH. Consequently, we consider

D1. $\bar{d} \sim U[E(m_j + w_j), (n - 1)E(m_j + w_j)]$.

D2. $\bar{d} \sim U[E(m_j + w_j), (n + 1)E(m_j + w_j)/3]$.

D3. $\bar{d} \sim U[(n + 1)E(m_j + w_j)/3, (2n - 1)E(m_j + w_j)/3]$.

D4. $\bar{d} \sim U[(2n - 1)E(m_j + w_j)/3, (n - 1)E(m_j + w_j)]$.

For each of our experiments, ten instances are generated. The values of $a_j$ and $c_j$ are generated from $U[1, 99]$.

Let $Z^L$ be the objective value of the initial LP relaxation, and $Z^1$ be the value of the LP solution after cuts are added at the root node. The value of the optimal integer solution is $Z^*$. The proportional improvement of the objective value after cuts are added at the root node is $(Z^1 - Z^*)/(Z^L - Z^*)$. The “No. of Nodes” and “No. of Cuts” is the average number of branch-and-cut nodes explored and the average number of cuts added, respectively.

To test Conjectures 1, 2, and 3, the data is generated according to Table 2.1. For each of these experiments, $n = 250$.

The results of the experiments for the analysis of Conjectures 1, 2, and 3 are reported in Table 2.2. We observe that the average proportional improvements for $LFC$ in T11, T12, and T13 are smaller than in T14. Also, the number of nodes and the number of cuts for $LFC$ in T11, T12, and T13 are higher than in T14. Observe that the solution time for $LFC$ in T14 is smaller than T11, T12 and T13. Thus, Conjecture 1 is supported.

For Conjecture 2, from the experiments T21, T22, T23, T24 and T25, the average proportional improvement for $LFC$ increases as the correlation coefficient increases.
<table>
<thead>
<tr>
<th>Case Name</th>
<th>Exp. Name</th>
<th>$m$ and $w$</th>
<th>$\rho$</th>
<th>$\bar{d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conjecture 1</strong></td>
<td>T11</td>
<td>$m_j \sim U[1,99]$, $w_j \sim U[1,99]$</td>
<td></td>
<td>D1</td>
</tr>
<tr>
<td></td>
<td>T12</td>
<td>$m_j \sim U[1,99]$, $w_j \sim U[40,60]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>T13</td>
<td>$m_j \sim U[40,60]$, $w_j \sim U[1,99]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>T14</td>
<td>$m_j \sim U[40,60]$, $w_j \sim U[40,60]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Conjecture 2</strong></td>
<td>T21</td>
<td>$m_j = \rho w_j + (\sqrt{1 - \rho^2})\eta + t$, $w_j \sim U[1,99]$</td>
<td>-0.9</td>
<td>D1</td>
</tr>
<tr>
<td></td>
<td>T22</td>
<td></td>
<td>-0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>T23</td>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>T24</td>
<td></td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>T25</td>
<td></td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td><strong>Conjecture 3</strong></td>
<td>T31</td>
<td>$m_j \sim U[1,99]$, $w_j \sim U[1,99]$</td>
<td>0</td>
<td>D2</td>
</tr>
<tr>
<td></td>
<td>T32</td>
<td></td>
<td></td>
<td>D3</td>
</tr>
<tr>
<td></td>
<td>T33</td>
<td></td>
<td></td>
<td>D4</td>
</tr>
</tbody>
</table>

Also, for $LFC$, the average number of nodes, the average number of cuts, and the solution times decrease as $\rho$ increases. Thus, Conjecture 2 is supported.

From the experiments T31, T32, and T33, as the value of $\bar{d}$ increases, the average proportional improvement for $LFC$ decreases, and the average number of nodes increases. Also, more valid inequalities are used in the branch-and-cut procedure. Thus, Conjecture 3 is supported. Observe that the solution times for $LFC$ in these experiments increase as the value of $\bar{d}$ increases.

To observe the effect of varying $n$, let $n \in \{50, 100, 250, 500, 1000\}$. For this experiment, the policies are $m_j \sim U[1,99]$, $w_j \sim U[1,99]$, and $\rho = 0$ and $\bar{d}$ is generated according to D1. We solve these data sets using branch-and-bound and using branch-and-cut with the valid inequalities $FC$ and $LFC$. Table 2.3 shows that the average number of nodes and the average solution time increase as $n$ increases in the branch-and-bound procedure.
Table 2.2: Test results for Conjectures 1, 2, and 3

(a) \((Z^1 - Z^*) / (Z^{LP} - Z^*)\) No. of Cuts

<table>
<thead>
<tr>
<th></th>
<th>(LFC)</th>
<th></th>
<th>(LFC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
</tr>
<tr>
<td>T11</td>
<td>0.517</td>
<td>0.124</td>
<td>43.800</td>
</tr>
<tr>
<td>T12</td>
<td>0.609</td>
<td>0.226</td>
<td>30.400</td>
</tr>
<tr>
<td>T13</td>
<td>0.624</td>
<td>0.241</td>
<td>48.000</td>
</tr>
<tr>
<td>T14</td>
<td>0.654</td>
<td>0.057</td>
<td>29.120</td>
</tr>
<tr>
<td>T21</td>
<td>0.466</td>
<td>0.185</td>
<td>63.320</td>
</tr>
<tr>
<td>T22</td>
<td>0.486</td>
<td>0.196</td>
<td>56.298</td>
</tr>
<tr>
<td>T23</td>
<td>0.517</td>
<td>0.124</td>
<td>44.800</td>
</tr>
<tr>
<td>T24</td>
<td>0.552</td>
<td>0.124</td>
<td>18.300</td>
</tr>
<tr>
<td>T25</td>
<td>0.582</td>
<td>0.102</td>
<td>9.760</td>
</tr>
<tr>
<td>T31</td>
<td>0.552</td>
<td>0.163</td>
<td>19.488</td>
</tr>
<tr>
<td>T32</td>
<td>0.459</td>
<td>0.211</td>
<td>37.633</td>
</tr>
<tr>
<td>T33</td>
<td>0.249</td>
<td>0.240</td>
<td>55.995</td>
</tr>
</tbody>
</table>

(b) No. of Nodes Time (sec)

<table>
<thead>
<tr>
<th></th>
<th>(LFC)</th>
<th></th>
<th>(LFC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp.</td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
</tr>
<tr>
<td>T11</td>
<td>12.200</td>
<td>11.500</td>
<td>0.889</td>
</tr>
<tr>
<td>T12</td>
<td>12.400</td>
<td>10.828</td>
<td>0.859</td>
</tr>
<tr>
<td>T13</td>
<td>20.200</td>
<td>16.856</td>
<td>0.673</td>
</tr>
<tr>
<td>T14</td>
<td>11.600</td>
<td>11.578</td>
<td>0.506</td>
</tr>
<tr>
<td>T21</td>
<td>21.547</td>
<td>20.948</td>
<td>1.442</td>
</tr>
<tr>
<td>T22</td>
<td>18.337</td>
<td>16.642</td>
<td>1.015</td>
</tr>
<tr>
<td>T23</td>
<td>13.200</td>
<td>12.500</td>
<td>0.889</td>
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<tr>
<td>T24</td>
<td>6.779</td>
<td>4.915</td>
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<tr>
<td>T25</td>
<td>4.853</td>
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<tr>
<td>T31</td>
<td>15.411</td>
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<tr>
<td>T32</td>
<td>18.695</td>
<td>15.638</td>
<td>1.035</td>
</tr>
<tr>
<td>T33</td>
<td>24.337</td>
<td>26.630</td>
<td>1.498</td>
</tr>
</tbody>
</table>
From Table 2.3, because $n$ and $\bar{d}$ increase at the same rate, there is no significant relationship between $n$ and the average proportional improvement, the average number of nodes and cuts. However, the solution times increase for $LFC$ as $n$ increases.
Table 2.3 shows that $LFC$ cuts result in a larger integrality gap improvement at the root node than $FC$ cuts. Also, the number of nodes investigated and the number of cuts added for $FC$ cuts is larger than the number of nodes investigated and the number of cuts added for $LFC$ cuts. The solution time for $FC$ cuts is longer than the solution time for the branch-and-bound procedure. However, CPU time for $LFC$ cuts are less than the solution time required by the branch-and-bound procedure. Thus, $LFC$ cuts close most of the integrality gap for most instances and reduce the computational effort.

2.5 Final Remarks

In this study, the polyhedral structure of the convex hull of the set defined by an inequality with continuous and binary variables with upper bounds are investigated. This polytope arises as subpolytope of more general mixed integer problems such as lotsizing problems, facility location problems, and network flow problems. We first give basic polyhedral results for $P$. Then, we derive a set of valid inequalities by adding a new subset of variables to the flow cover inequality presented by Goemans (1987) and Wolsey (1989). We show that, under some conditions, this set is facet defining. A separation procedure is developed for the set of valid inequalities. We investigate conditions where the separation problem provides strong cuts. Then, because the separation problem is NP-hard, we develop a heuristic procedure based on properties of the problem to determine the valid inequality that separates a fractional extreme point solution from the integer hull of $P$.

To test the effectiveness of (2.2) in solving $P$, we implement branch-and-cut algorithm that incorporates these inequalities and we perform computational experiments.
Computational study indicates that (2.2) is stronger than the flow cover inequalities (2.1).

Future research could consider developing an alternative set of valid inequalities that include variables that are not in (2.2).
In this chapter, we develop a sequence independent lifting procedure to lift the flow cover inequalities for the polytope $P$. We show that computing lifting function for $P$ is NP-hard. Then, we determine a lower bound function for the lifting function that is a better lower bound than one generated by using the LP relaxation. Then, we develop a superadditive function which produces strong cuts. We prove that the superadditive function that we develop is not dominated by any other superadditive function and it is maximal. Our resulting inequalities can be used to strengthen the inequalities of Shebalov and Klabjan (2006). A separation procedure is developed for the lifted inequalities. We investigate conditions where the separation problem provides strong cuts. Then, because the separation problem is NP-hard, we develop a heuristic procedure based on properties of the problem.

In Section 3.1, we consider sequence independent lifting of the flow cover inequalities. The separation problem is studied in Section 3.2. Section 3.3 provides a computational study to show that our procedures are efficient.
3.1 Valid Inequalities

In this section, we develop a set of valid inequalities from the flow cover inequalities using sequence independent lifting. First, we present the lifting function and show that the associated computation problem is NP-hard. As a result, we develop a superadditive function that provides a lower bound. Then, we show that this function is not dominated by any other superadditive lower bound function.

3.1.1 Basic Lifting Results

We first describe the flow cover inequalities for $P$. Construct $C \subseteq N$ such that $\sum_{j \in C} w_j < \bar{d} < \sum_{j \in C} (m_j + w_j)$ and $c = |C|$. To develop a flow cover inequality, we assume that such a $C$ exists. Goemans (1987) shows that the flow cover inequality

$$\sum_{j \in C} x_j - \sum_{j \in C} \min \{ \sum_{\ell \in C \setminus \{j\}} (m_{\ell} + w_{\ell}) - \bar{d}, w_j \} (1 - y_j) \leq \bar{d} - \sum_{j \in C} w_j$$

(3.1)

is valid for $P$. Also, Goemans (1989) shows that (3.1) defines facets of $\text{conv}(P)$ under some conditions.

In our lifting procedure, we add some of the variables in $N \setminus C$ to (3.1). For $C \subset N$, let $B^\ell = \{ i \in N \setminus C \mid x_i^0 = y_i^0 = 0 \}$ and $B^u = \{ i \in N \setminus C \mid x_i^0 = m_i, y_i^0 = 1 \}$. Let $d = \bar{d} - \sum_{j \in B^u} (m_j + w_j)$. Define $\lambda > 0$ and $\tau > 0$ such that

$$\lambda = d - \sum_{j \in C} w_j \quad \text{and} \quad \tau = \sum_{j \in C} (m_j + w_j) - d = \sum_{j \in C} m_j - \lambda.$$

Consider the subset of $P$,

$$P_C = \{ (x, y) \in \mathbb{R}_+^c \times \mathbb{R}^c \mid \sum_{j \in C} (x_j + w_j y_j) \leq d, x_i \leq m_i y_i \text{ for } i \in C \}.$$

We lift (3.1) to construct valid inequalities for $P$ of the form
\[
\sum_{j \in C} x_j - \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j) + \sum_{j \in (B^\ell \cup B^u)} [\alpha_j(x_j - a_j) + \beta_j(y_j - b_j)] \leq \lambda, \tag{3.2}
\]

where \(a_j = b_j = 0\) for \(j \in B^\ell\), and \(a_j = m_j\) and \(b_j = 1\) for \(j \in B^u\). To construct (3.2), we need to determine \(\alpha \in \mathbb{R}^{(n-c)}\) and \(\beta \in \mathbb{R}^{(n-c)}\). Let \(f(z)\) be the lifting function for (3.1) where

\[
f(z) = \min_{(x,y) \in P_C} \{ \lambda - \sum_{j \in C} x_j + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j) | \sum_{j \in C} (x_j + w_jy_j) \leq d - z, \ 0 \leq x_j \leq m_jy_j \text{ and}\ y_j \in \{0,1\} \text{ for } j \in C \}. \tag{3.3}
\]

We now establish that the problem of finding \(f(z)\) is NP-hard. Observe that by setting \(m_j\) close to zero for \(j \in C\), \(f(z)\) can be reduced to a knapsack problem. Let \(\min\{\tau - m_j, w_j\} = \tau - m_j\) and \(x_j = m_jy_j\) for \(j \in C\). Then, \(f(z)\) reduces to

\[
f(z) = \min_{y \in P_C} \{ \lambda + \sum_{j \in C} (\tau - m_j) - \sum_{j \in C} \tau y_j | \sum_{j \in C} (m_j + w_j)y_j \leq d - z \text{ and}\ y_j \in \{0,1\} \text{ for } j \in C \}
\]

which is polynomially solvable. As a result, to establish the problem complexity, we use a polynomial time reduction from the following NP-complete problem (Garey and Johnson 1979).

**Cardinality Partition Problem:** Given sequence \(v_1, v_2, \ldots, v_{2r}\) such that \(v_i \in \mathbb{Z}_+\) for \(i = 1, \ldots, 2r\), does there exist a set \(V \subset \{1, 2, \ldots, 2r\}\) such that \(\sum_{i \in V} v_i = \sum_{j \notin V} v_j = \bar{b}\) and \(|V| = r|\)?

**Proposition 3.1.1** The recognition version of the problem to determine \(f(z)\) is binary NP-complete.
Proof Let
\[ f^0(z) = \max_{(x,y) \in P_C} \left\{ \sum_{j \in C} x_j + \sum_{j \in C} \min \{ \tau - m_j, \ w_j \} y_j \mid \sum_{j \in C} (x_j + w_j y_j) \leq d - z, \right. \]
\[ 0 \leq x_j \leq m_j y_j \text{ and } y_j \in \{0, 1\} \text{ for } j \in C \}. \]

Because \( f(z) \) can be written as \( f(z) = \lambda + \sum_{j \in C} \min \{ \tau - m_j, \ w_j \} - f^0(z) \), it is sufficient to show that finding \( f^0(z) \) is NP-complete.

Let \( P \) be the recognition version of the problem to determine \( f^0(z) \). Given a solution \((x^0, y^0) \in P_C\), \( f^0(z) \) can be calculated in \( O(c) \) time for \( z \in \mathbb{Z} \). Thus, \( P \in \mathcal{NP} \).

We use a reduction from the Cardinality Partition problem that is polynomial in the size of the input. For given \( v_1 \geq v_2 \geq \cdots \geq v_{2r} \), consider the following instance of \( P \):
\[
\begin{align*}
c & = 2r, \\
d & = 4\bar{b}r - v_1 + 1, \\
z & = 2\bar{b}r - \bar{b} - v_1 + 1, \\
m_1 & = v_1 + 1, \\
m_i & = v_i, \text{ for } i = 2, \ldots, 2r, \\
w_i & = 2\bar{b}, \\
f^0(z) & \geq 2\bar{b}r + \bar{b}.
\end{align*}
\]

Note that
\[
d - z = \bar{b} + 2\bar{b}r, \\
\tau = \sum_{i=1}^{c} (m_i + w_i) - d = \sum_{i=1}^{2r} (v_i + 2\bar{b} + 1) - (4\bar{b}r - v_1 + 1) = 2\bar{b} + v_1, \\
\min \{ \tau - m_1, w_1 \} = \min \{ 2\bar{b} + v_1 - (v_1 + 1), 2\bar{b} \} = 2\bar{b} - 1, \text{ and} \\
\min \{ \tau - m_i, w_i \} = \min \{ 2\bar{b} + v_1 - v_i, 2\bar{b} \} = 2\bar{b} \text{ for } i = 2, \ldots, 2r.
\]

We show that there is a solution to The Cardinality Partition problem if and only if there exists a solution to \( P \).
(⇒) Suppose there is a solution to the Cardinality Partition problem. In the solution, we let \( V \) be the set where \( 1 \notin V \). Let \( (x^0, y^0) \) be a solution to \( f^0(z) \) where \( y^0_j = 1 \) for \( j \in V \), \( y^0_j = 0 \) for \( j \in \{1, \ldots, 2r\} \setminus V \). Then,

\[
\sum_{j \in V} x^0_j = \min \{d - z, \sum_{j \in V} m_i\} = \min \{\bar{b} + 2br, \bar{b}\} = \bar{b}.
\]

Because \( 1 \notin V \) and \( \min \{\tau - m_j, w_j\} = 2\bar{b} \) for \( j = 2, \ldots, 2r \),

\[
f^0(z) = \sum_{j \in V} (x^0_j + \min \{\tau - m_j, w_j\}) = \bar{b} + 2\bar{b}r.
\]

(⇐) Suppose that there is a solution \( (x^0, y^0) \) to the instance of \( P \). Let

\[
V = \{j \mid y^0_j = 1, j = 1, \ldots, 2r\}. \text{ Then, } |V| \leq r \text{ because}
\]

\[
\sum_{j=1}^{2r} (x^0_j + 2by^0_j) \leq d - z = \bar{b} + 2\bar{b}r. \quad (3.4)
\]

Let \( |V| = r - k \leq r \) where \( 0 \leq k \leq r \). From (3.4), \( x^0_1 \leq (v_1 + 1)y^0_1 \), and \( x^0_j \leq v_jy^0_j \) for \( j = 2, \ldots, 2r \),

\[
\sum_{j \in V} x^0_j \leq \min \{d - z - \sum_{j \in V} w_jy^0_j, \sum_{j \in V} m_jy^0_j\} = \min \{2\bar{b}r + \bar{b} - 2\bar{b}(r - k), 2\bar{b} + 1\} = \min \{\bar{b} + 2\bar{b}k, 2\bar{b} + 1\}. \quad (3.5)
\]

Also,

\[
f^0(z) = \sum_{j \in V} x^0_j + \sum_{j \in V} \min \{\tau - m_j, w_j\} = \sum_{j \in V} x^0_j + \begin{cases} 2\bar{b} - 1 + 2\bar{b}(r - k - 1), & \text{if } 1 \in V, \\ 2\bar{b}(r - k), & \text{otherwise} \end{cases} = \sum_{j \in V} x^0_j + \begin{cases} 2\bar{b}(r - k) - 1, & \text{if } 1 \in V, \\ 2\bar{b}(r - k), & \text{otherwise}. \end{cases}
\]
If $k \geq 1$, then $\sum_{j \in V} x_j^0 \leq 2\bar{b} + 1$ from (3.5). Thus, $f^0(z) \leq 2\bar{b} + 1 + (2\bar{b}r - 2\bar{b}k) < \bar{b} + 2\bar{b}r$ from (3.6). This contradicts the assumption that $f^0(z) \geq \bar{b} + 2\bar{b}r$. Thus, $k = 0$. Because $\sum_{j \in V} x_j^0 \leq \bar{b}$ from (3.5), $f^0(z) \geq \bar{b} + 2\bar{b}r$ only if $\sum_{j \in V} x_j^0 = \bar{b}$ and $1 \notin V$. Therefore, the maximal solution $(x^0, y^0)$ has $y_j^0 = 1$ for $j \in V$ where $\sum_{i \in V} v_i = \sum_{j \in V} v_j$ and $|V| = r$. As a result, there is a solution to $P$ with value larger than or equal to $\bar{b} + 2\bar{b}r$ only when a cardinality partition exists. □

As a result of Proposition 3.1.1, we are not able to find a computable polynomial time expression for $f(z)$ for all values of $z$. Therefore, we find a computable function $\bar{f}(z)$ such that $\bar{f}(z) \leq f(z)$. We obtain $\bar{f}(z)$ by using $f(z)$ for $z$ values where $f(z)$ can be found, and approximating $f(z)$ for the remaining $z$ values.

We now provide some additional notation and definitions. Partition $C$ into $C^\gamma$ and $C^w$ where $C^\gamma = \{i \in C \mid \tau - m_i < w_i\}$ and $C^w = \{i \in C \mid \tau - m_i \geq w_i\}$. Reindex the variables such that $C^\gamma = \{1, \ldots, b\}$ and $C^w = \{b + 1, \ldots, c\}$ where $m_{j-1} + w_{j-1} \geq m_j + w_j$ for $j = 2, \ldots, b, b+2, \ldots, c$. Let $M_i = \sum_{j=1}^i m_j$, $W_i = \sum_{j=1}^i w_j$ for $i = 1, \ldots, c$, and $M_0 = W_0 = 0$. Also, let $w = \min\{w_1, \ldots, w_c\}$. Finally, let $\phi(x, y) = \lambda - \sum_{j \in C} x_j + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j)$.

The next lemma provides a property of an optimal solution to $f(z)$.

**Lemma 3.1.2** For $z \in \mathbb{R}$, there exists an optimal solution $(x^*, y^*) \in P_C$ to $f(z)$ where $y_j^* = 0$ for $j = 1, \ldots, i$, and $y_j^* = 1$ for $j = i+1, i+2, \ldots, b$, and $i \in \{0, 1, \ldots, b\}$.

**Proof** Suppose that for some optimal solution $(x^0, y^0) \in P_C$, $y_k^0 = 1$, $y_{\ell}^0 = 0$ where $1 \leq k < \ell \leq b$. Consider a new solution $(x^*, y^*) \in P_C$ where

$$y_j^* = \begin{cases} y_j^0, & j \neq \{k, \ell\}, \\ 0, & j = k, \\ 1, & j = \ell, \end{cases}$$
The solution \((x^*, y^*)\) is feasible because
\[
x^*_\ell + w_\ell y^*_\ell \leq (x^*_k + w_k - w_\ell) + w_\ell = x^*_k + w_k.
\]
Now,
\[
\phi(x^*, y^*) - \phi(x^0, y^0) = \left( \lambda - \sum_{j \in C} x^*_j + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y^*_j) \right) - \left( \lambda - \sum_{j \in C} x^0_j + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y^0_j) \right)
\]
\[
= - \min\{m_\ell, x^*_k + w_k - w_\ell\} + (\tau - m_k) + x^0_k - (\tau - m_\ell)
\]
\[
= \min\{x^0_k - m_k, (m_\ell + w_\ell - (m_k + w_k)\}
\]
\[
\leq 0.
\]
The inequality follows from \(x^0_k \leq m_k\) and \(m_\ell + w_\ell \leq m_k + w_k\). Thus, \((x^*, y^*)\) is optimal to \(f(z)\). \(\square\)

If \(b = c\), then an optimal solution to \(f(z)\) can be determined explicitly using Lemma 3.1.2. When \(b < c\), an optimal solution \((x^*, y^*) \in P\) to \(f(z)\) where \(x^*_j = m_j\) and \(y^*_j = 1\) for \(j = b, \ldots, c\) is determined using Lemma 3.1.2 for \(z \in (-\infty, -\tau)\), \(z \in [-\tau, 0)\), \(z \in [M_i + W_i, M_{i+1} + W_{i+1} - \tau)\) for \(i = 0, \ldots, b - 1\), \(z \in [M_b + W_b - \tau, M_c + W_b - \tau]\), and \(z \in (M_c + W_b - \tau, M_c + W_c - \tau]\). For the two intervals \(z \in (M_i + W_i - \tau, M_i + W_i)\) for \(i = 1, \ldots, b - 1\) and \(z \in (M_c + W_b - \tau, M_c + W_c - \tau - w)\), an optimal solution \((x^*, y^*) \in P\) does not always have \(x^*_j = m_j\) and \(y^*_j = 1\) for \(j = b, \ldots, c\). For these two intervals, we can not provide an explicit value for \(f(z)\). The next five lemmas describe \(f(z)\) for the remaining cases where \(f(z)\) can be explicitly defined.
Lemma 3.1.3 \( f(z) = -\tau \) for \( z \in (-\infty, -\tau) \).

**Proof** Because \( \tau = M_c + W_c - d \) and \( d = M_c + W_c - \tau \), we have that \( d - z \in (M_c + W_c, \infty) \). Notice that \( x_j^* = m_j \) and \( y_j^* = 1 \) for \( j \in C \) is the unique optimal solution to \( f(z) \) because \( \sum_{j \in C} (x_j^* + w_j y_j^*) = M_c + W_c \leq d + z \). Therefore, from \( \tau = M_c - \lambda \),

\[
\begin{align*}
f(z) & = \phi(x^*, y^*) \\ & = \lambda - \sum_{j \in C} x_j^* + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j^*) \\ & = \lambda - M_c \\ & = -\tau. \quad \square
\end{align*}
\]

Lemma 3.1.4 \( f(z) = z \) for \( z \in [-\tau, 0) \).

**Proof** Because \( z \in [-\tau, 0) \) and \( d = M_c + W_c - \tau \), we have that \( d - z \in (M_c + W_c - \tau, M_c + W_c] \). From \( \tau = M_c + W_c - d \), we show that there exists an optimal solution \((x^*, y^*) \in P_C \) where \( y_j^* = 1 \) for \( j \in C \) and

\[
\sum_{j \in C} x_j^* = d - z - W_c > 0. \tag{3.7}
\]

The inequality follows from \( z \in [-\tau, 0) \) and \( d - W_c \geq 0 \).

For this solution, from \( \lambda = d - W_c \),

\[
\begin{align*}
f(z) & = \phi(x^*, y^*) \\ & = \lambda - \sum_{j \in C} x_j^* + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j^*) \\ & = \lambda - d + z + W_c \\ & = z.
\end{align*}
\]
Consider any other feasible solution \((x^0, y^0) \in P_C\) where \(y^0_k = 0\) for \(k \in L \subseteq C\), \(y^0_j = 1\) for \(j \in C \setminus L\), and
\[
\sum_{j \in C \setminus L} x^0_j = \min \{d - z - W_c + \sum_{k \in L} w_k, M_c - \sum_{k \in L} m_k\}. \tag{3.9}
\]

From (3.7) and (3.9),
\[
\phi(x^0, y^0) - \phi(x^*, y^*) = \left( \lambda - \sum_{j \in C} x^0_j + \sum_{j \in C} \min \{\tau - m_j, w_j\} (1 - y^0_j) \right)
- \left( \lambda - \sum_{j \in C} x^*_j + \sum_{j \in C} \min \{\tau - m_j, w_j\} (1 - y^*_j) \right)
= - \sum_{j \in C \setminus L} x^0_j + \sum_{j \in L} \min \{\tau - m_j, w_j\} + \sum_{j \in C} x^*_j
= - \min \{d - z - W_c + \sum_{k \in L} w_k, M_c - \sum_{k \in L} m_k\}
+ \sum_{j \in L} \min \{\tau - m_j, w_j\} + d - z - W_c.
\]

Suppose that \(d - z - W_c + \sum_{k \in L} w_k \leq M_c - \sum_{k \in L} m_k\). Then,
\[
M_c + W_c - \tau < d - z \leq M_c + W_c - \sum_{k \in L} (m_k + w_k)
\]
where the lower bound of \(d - z\) follows from \(z > -\tau\). Hence, \(\tau > \sum_{k \in L} (m_k + w_k)\).

This inequality only holds if \(L \subseteq C^w\) because \(m_j + w_j > m_j + (\tau - m_j) = \tau\) for \(j \in C^\gamma\). Thus, \(\min \{\tau - m_k, w_k\} = w_k\) for \(k \in L\). As a result,
\[
\phi(x^0, y^0) - \phi(x^*, y^*) = -(d - z - W_c + \sum_{k \in L} w_k) + \sum_{j \in L} \min \{\tau - m_j, w_j\}
+ d - z - W_c
= \sum_{k \in L} (\min \{\tau - m_k, w_k\} - w_k)
= 0.
\]
Alternatively, suppose that $d - z - W_c + \sum_{k \in L} w_k > M_c - \sum_{k \in L} m_k$. Thus,

$$M_c + W_c - \tau - z = d - z > M_c + W_c - \sum_{k \in L} (m_k + w_k)$$

implies that $\sum_{k \in L}(m_k + w_k) > \tau + z$. Now, partition $L$ into $L^\gamma$ and $L^w$ where $L^\gamma = \{j \mid j \in L \cap C^\gamma\}$ and $L^w = \{j \mid j \in L \cap C^w\}$. Then,

$$\phi(x^0, y^0) - \phi(x^*, y^*) = -(M_c - \sum_{k \in L} m_k) + \sum_{j \in L} \min\{\tau - m_j, w_j\} + d - z - W_c$$

$$= \sum_{k \in L^\gamma} (m_k + (\tau - m_k)) + \sum_{k \in L^w} (m_k + w_k) - \tau - z$$

$$= \sum_{k \in L^\gamma} \tau + \sum_{k \in L^w} (m_k + w_k) - \tau - z. \quad (3.10)$$

Suppose that $L^\gamma = \emptyset$. Then, $L^w = L$. From $\sum_{k \in L}(m_k + w_k) > \tau + z$, (3.10) can be written as

$$\phi(x^0, y^0) - \phi(x^*, y^*) = \sum_{k \in L} (m_k + w_k) - \tau - z > 0.$$

Alternatively, suppose that $L^\gamma \neq \emptyset$. From $m_k + w_k > 0$ for $k \in L^w$ and $-z \geq 0$, (3.10) can be written as

$$\phi(x^0, y^0) - \phi(x^*, y^*) \geq \tau + \sum_{k \in L^w} (m_k + w_k) - \tau - z > 0. \quad \square$$

**Lemma 3.1.5** $f(z) = (i - 1)\tau$ for $i = 1, \ldots, b$ and $z \in [M_{i-1} + W_{i-1}, \ M_i + W_i - \tau]$.

**Proof** Because $z \in [M_{i-1} + W_{i-1}, \ M_i + W_i - \tau)$ and $d = M_c + W_c - \tau$, we have that $d - z \in (M_c + W_c - M_i - W_i, \ M_c + W_c - M_{i-1} - W_{i-1} - \tau]$ for $i = 1, \ldots, b$. We show that there exists an optimal solution $(x^*, y^*) \in P_C$ where $y^*_j = x^*_j = 0$ for $j \in \{1, \ldots, i\}$ and $x^*_j = m_j$ and $y^*_j = 1$ for $j \in \{i + 1, \ldots, c\}$. Because $\sum_{j=i+1}^c (m_j + w_j) \leq d - z$, the point $(x^*, y^*)$ is feasible. Now, from $\tau = M_c - \lambda,$
\( f(z) = \phi(x^*, y^*) \)
\[
= \lambda - \sum_{j=i+1}^{c} m_j + \sum_{j=1}^{i} \min\{\tau - m_j, w_j\}
\]
\[
= \lambda - \sum_{j=i+1}^{c} m_j + \sum_{j=1}^{i} (\tau - m_j)
\]
\[
= \lambda + (i - 1)\tau - M_c
\]
\[
= (i - 1)\tau.
\]

We now show that \((x^*, y^*)\) is optimal. Consider any other solution \((x^0, y^0) \in P_c\) where \(y_j^0 = 0\) for \(j \in L^\gamma \subseteq C^\gamma\), \(y_j^0 = 0\) for \(j \in L^w \subseteq C^w\), and \(y_j^0 = 1\) for \(j \in C \setminus (L^\gamma \cup L^w)\). From Lemma 3.1.2, we can assume that
\[
y_j^0 = \begin{cases} 0, & j \in \{1, \ldots, k - 1\} \cup L^w, \\ 1, & \text{otherwise} \end{cases}
\]

for some \(k \leq i\). Because the point \((x^0, y^0)\) is feasible, \(\sum_{j=k}^{c} x_j^0 \leq d - z - \sum_{j=k}^{c} w_j + \sum_{\ell \in L^w} w_{\ell}\). Thus,
\[
\phi(x^0, y^0) - \phi(x^*, y^*) = \left( \lambda - \sum_{j \in C} x_j^0 + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j^0) \right) -
\]
\[
\left( \lambda - \sum_{j \in C} x_j^* + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j^*) \right)
\]
\[
= -\sum_{j=k}^{c} x_j^0 + \sum_{j \in L^w} \min\{\tau - m_j, w_j\} + \sum_{j=1}^{k-1} \min\{\tau - m_j, w_j\}
\]
\[
+ \sum_{j=i+1}^{c} x_j^* - \sum_{j=1}^{i} \min\{\tau - m_j, w_j\}
\]
\[
= -\sum_{j=k}^{c} x_j^0 + \sum_{j \in L^w} \min\{\tau - m_j, w_j\} + \sum_{j=i+1}^{c} x_j^*
\]
\[- \sum_{j=k}^{i} \min\{\tau - m_j, \ w_j\}\]

\[\geq -(d - z - \sum_{j=k}^{c} w_j + \sum_{j \in L^w} w_j) + \sum_{j \in L^w} w_j\]

\[+ \sum_{j=i+1}^{c} m_j - \sum_{j=k}^{i} (\tau - m_j)\]

\[= -(d - z) + \sum_{j=i}^{c} (m_j + w_j) - \tau + \sum_{j=k}^{i-1} (m_j + w_j) - (i - k)\tau\]

\[\geq 0\]

because \(d - z \leq M_c + W_c - M_{i-1} - W_{i-1} - \tau = \sum_{j=i}^{c} (m_j + w_j) - \tau\) and \(m_j + w_j > \tau\) for \(j \in C^w\). \(\square\)

**Lemma 3.1.6** \(f(z) = z - M_b - W_b + b\tau\) for \(z \in [M_b + W_b - \tau, \ M_c + W_b - \tau]\).

**Proof** Because \(z \in [M_b + W_b - \tau, \ M_c + W_b - \tau]\) and \(d = M_c + W_c - \tau\), we have that \(d - z \in [W_c - W_b, \ M_c + W_c - M_b - W_b]\). We show that there exists an optimal solution \((x^*, y^*) \in P_C\) where \(y^*_j = 0\) for \(j \in C^w\), \(y^*_j = 1, 0 \leq x^*_j \leq m_j\) for \(j \in C^w\), and \(\sum_{j \in C} x^*_j = d - z - \sum_{j \in C} w_j y^*_j\). The point \((x^*, y^*)\) exists and is feasible because \(\sum_{j=b+1}^{c} x^*_j \leq M_c - M_b\) and \(\sum_{j=b+1}^{c} w_j = W_c - W_b \leq d - z\). Now,

\[f(z) = \phi(x^*, y^*)\]

\[= \lambda - \sum_{j \in C} x^*_j + \sum_{j \in C} \min\{\tau - m_j, \ w_j\}(1 - y^*_j)\]

\[= \lambda - \left(d - z - \sum_{j=b+1}^{c} w_j\right) + \sum_{j=1}^{b} (\tau - m_j)\]

\[= \lambda - \left(\lambda + \sum_{j=1}^{c} w_j - z - \sum_{j=b+1}^{c} w_j\right) + \sum_{j=1}^{b} (\tau - m_j)\]

\[= z - \sum_{j=1}^{b} w_j + \sum_{j=1}^{b} (\tau - m_j) + \sum_{j=1}^{b} m_j - \sum_{j=1}^{b} m_j\]
\[ = z - M_b - W_b + b\tau. \]

Consider any other feasible solution \((x^0, y^0) \in P_C\), where \(L^\gamma \subset C^\gamma, L^w \subset C^w\), 
\(y^0_k = 0\) for \(k \in L^\gamma \cup L^w\), \(y^0_j = 1\) for \(j \in C^\gamma \setminus (L^\gamma \cup L^w)\), and \(\sum_{j \in C} x^0_j \leq d - z - \sum_{j \in C} w_j y^0_j\).

Now,
\[
\phi(x^0, y^0) - \phi(x^*, y^*) = \left( \lambda - \sum_{j \in C} x_j^0 + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y^0_j) \right) \\
- \left( \lambda - \sum_{j \in C} x_j^* + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y^*_j) \right) \\
= -\sum_{j \in C} x_j^0 + \sum_{j \in L^\gamma} (\tau - m_j) + \sum_{j \in L^w} w_j + \sum_{j \in C} x_j^* - \sum_{j \in C^\gamma} (\tau - m_j) \\
\geq -\left( d - z - \sum_{j \in C} w_j y^0_j \right) + \sum_{j \in L^w} w_j + \left( d - z - \sum_{j \in C} w_j y^*_j \right) \\
- \sum_{j \in C^\gamma \setminus L^\gamma} (\tau - m_j) \\
= -\left( \sum_{j \in C^\gamma \setminus L^\gamma} w_j - \sum_{j \in C^w \setminus L^w} w_j \right) + \sum_{j \in L^w} w_j \\
- \sum_{j \in C^w} w_j - \sum_{j \in C^\gamma \setminus L^\gamma} (\tau - m_j) \\
= -\sum_{j \in C^\gamma \setminus L^\gamma} (\tau - m_j - w_j) \\
\geq 0. 
\]

The last inequality follows because \(\tau - m_k \leq w_k\) for \(k \in C^\gamma \setminus L^\gamma\). \(\square\)

**Lemma 3.1.7** \(f(z) = M_c + W_c - M_b - W_b + (b - 1)\tau\) for \(z \in (M_c + W_c - \tau - w, M_c + W_c - \tau]\).
Proof Because $z \in (M_c + W_c - \tau - w, M_c + W_c - \tau]$, we have that $d - z \in (0, w]$. Since $w = \min\{w_1, \ldots, w_c\}$, the only feasible solution is $x_j^* = y_j^* = 0$ for all $j \in C$ where

$$
 f(z) = \phi(x^*, y^*) = \phi(0, x^*, y^*) = \phi(0, 0, 0) = \phi(0).
$$

**Theorem 3.1.8**

$$
 f(z) = \begin{cases} 
 -\tau, & -\infty < z < -\tau, \\
 z, & -\tau \leq z < 0, \\
 (i-1)\tau, & M_{i-1} + W_{i-1} - w_{i-1} \leq z < M_i + W_i - \tau, \\
 z - M_b - W_b + b\tau, & M_b + W_b - \tau \leq z \leq M_c + W_c - \tau, \\
 M_c + W_c - M_b - W_b + (b-1)\tau, & M_c + W_c - \tau - w < z \leq M_c + W_c - \tau. 
\end{cases}
$$

Proof Lemmas 3.1.3–3.1.7 establish that the values for $f(z)$ are correct. It remains to be shown that the intervals exist and do not overlap.

Because $\tau - m_i < w_i$ for $i = 1, \ldots, b$,

$$
 M_{i-1} + W_{i-1} = M_i + W_i - m_{i-1} - w_{i-1} \\
 \leq M_i + W_i - m_{i-1} - (\tau - m_{i-1}) \\
 = M_i + W_i - \tau.
$$
Thus, the interval exist. Because $w \leq W_c-W_b$, we have $M_c+W_b-\tau < M_c+W_c-\tau-w$. Thus, no intervals overlap. □

Let

$$\bar{f}(z) = \begin{cases} 
-\tau, & -\infty < z < -\tau, \\
z, & -\tau \leq z < 0, \\
(i-1)\tau, & M_{i-1} + W_{i-1} \leq z < M_i + W_i - \tau, \\
 z - M_i - W_i + i\tau, & M_i + W_i - \tau \leq z < M_i + W_i, \\
 z - M_b - W_b + b\tau, & M_b + W_b - \tau \leq z \leq M_c + W_c - \tau - w, \\
 M_c + W_c - M_b - W_b + (b-1)\tau, & M_c + W_c - \tau - w < z \leq M_c + W_c - \tau.
\end{cases}$$

The function $\bar{f}(z)$ is obtained by using $f(z)$ for values of $z$ where $f(z)$ can be explicitly defined, and providing a lower bound $f(z)$ for the remaining values of $z$.

**Proposition 3.1.9** $\bar{f}(z) \leq f(z)$.

**Proof** From Theorem 3.1.8, $\bar{f}(z) = f(z)$ for $z \in (-\infty,-\tau)$, $z \in [-\tau,0)$, $[M_{i-1} + W_{i-1}, M_i + W_i - \tau)$ for $i = 1, \ldots, b$, and $z \in (M_i + W_i - \tau - w, M_c + W_c - \tau]$. We show $\bar{f}(z) \leq f(z)$ in the two cases where $z \in [M_i + W_i - \tau, M_i + W_i)$ for $i = 1, \ldots, b-1$ and $z \in (M_c + W_c - \tau - w, M_c + W_c - \tau - w]$.

Case 1: $z \in [M_i + W_i - \tau, M_i + W_i)$ for $i = 1, \ldots, b$. Thus, $d - z \in (M_c + W_c - M_i - W_i - \tau, M_c + W_c - M_i - W_i)$. Because we cannot explicitly find an optimal solution for $f(z)$, we show that $\phi(x^0, y^0) - \bar{f}(z) \geq 0$ for all $(x^0, y^0) \in P_C$.

From Lemma 3.1.2, if $(x^0, y^0)$ is optimal, then we can let $y_j^0 = 0$ for $j = 1 \ldots, k-1$ and $y_j^0 = 1$ for $j = k, \ldots, b$ for some $k \in \{1, \ldots, b\}$. For $(x^0, y^0)$, let $y_j^0 = 0$ for
\( j \in L^w \subseteq C^w \) and \( y^0_j = 1 \) for \( j \in C^w \setminus L^w \). Now, because \((x^0, y^0) \in P_C\),
\[
\sum_{j=k}^{b} x_j^0 + \sum_{j \in C^w \setminus L^w} x_j^0 \leq \min \left\{ d - z - \sum_{j=k}^{c} w_j + \sum_{j \in L^w} w_j, \sum_{j=k}^{c} m_j - \sum_{j \in L^w} m_j \right\}.
\]

Then,
\[
\phi(x^0, y^0) - \bar{f}(z) = \left( \lambda - \sum_{j \in C} x_j^0 + \sum_{j \in C} \min \{ \tau - m_j, w_j \} (1 - y_j^0) \right)
- (z - M_i - W_i + i\tau)
\]
\[
= \lambda - \left( \sum_{j=k}^{b} x_j^0 + \sum_{j \in C^w \setminus L^w} x_j^0 \right) + \sum_{j=1}^{k-1} \min \{ \tau - m_j, w_j \}
+ \sum_{j \in L^w} \min \{ \tau - m_j, w_j \} - (z - M_i - W_i + i\tau)
\]
\[
= (d - W_c) - \left( \sum_{j=k}^{b} x_j^0 + \sum_{j \in C^w \setminus L^w} x_j \right) + \sum_{j=1}^{k-1} \left( \tau - m_j \right)
+ \sum_{j \in L^w} w_j - (z - M_i - W_i + i\tau).
\]
(3.11)

First, suppose that \( k \leq i + 1 \). Because
\[
\sum_{j=k}^{b} x_j^0 + \sum_{j \in C^w \setminus L^w} x_j^0 \leq d - z - \sum_{j=k}^{c} w_j + \sum_{j \in L^w} w_j,
\]
(3.11) becomes
\[
\phi(x^0, y^0) - \bar{f}(z) \geq (d - W_c) - (d - z - \sum_{j=k}^{c} w_j + \sum_{j \in L^w} w_j)
+ \sum_{j=1}^{k-1} \left( \tau - m_j \right) + \sum_{j \in L^w} w_j - (z - M_i - W_i + i\tau)
\]
\[
= -W_c + M_i - W_i + \sum_{j=1}^{k-1} \left( \tau - m_j \right) + \sum_{j=k}^{c} w_j - i\tau
\]
\[
= M_i + W_i - \sum_{j=1}^{k-1} w_j + (k - 1)\tau - \sum_{j=1}^{k-1} m_j - i\tau
\]
\[
= \sum_{j=k}^{i} (m_j + w_j) - (i + 1 - k)\tau
\]

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The last inequality follows because \( m_i + w_i \geq \tau \) for \( j = 1, \ldots, b \).

Alternatively, suppose that \( b \geq k \geq i + 2 \). Because
\[
\sum_{j=k}^{b} x_j^0 + \sum_{j \in C \setminus L^w} x_j^0 \leq \sum_{j=k}^{c} m_j - \sum_{j \in L^w} m_j,
\]
\( z \leq M_i + W_i \) for \( i = 1, \ldots, b \), and \( k \geq i + 2 \), (3.11) becomes
\[
\begin{align*}
\phi(x^0, y^0) - \bar{f}(z) & \geq (d - W_c) - (\sum_{j=k}^{c} m_j - \sum_{j \in L^w} m_j) + \sum_{j=1}^{k-1} (\tau - m_j) + \sum_{j \in L^w} w_j \\
& \quad - (z - M_i - W_i + i\tau) \\
& = (d - z - M_c - W_c) + \sum_{j \in L^w} (m_j + w_j) + (k - 1)\tau \\
& \quad - (-M_i - W_i + i\tau) \\
& \geq (d - M_c - W_c + \tau - z + M_i + W_i) + \sum_{j \in L^w} (m_j + w_j) \\
& \geq (d - M_c - W_c + \tau) + \sum_{j \in L^w} (m_j + w_j) \\
& \geq 0.
\end{align*}
\]

Case 2: \( z \in [M_b + W_b - \tau, d - \underline{w}] \). Thus, \( d - z \in [\underline{w}, M_c + W_c - M_b - W_b] \). Suppose that there exists an optimal solution \((x^*, y^*) \in P\) to \( f(z) \). Then, from
\[
\sum_{j \in C} (x_j^* + \min\{\tau - m_j, w_j\} y_j^*) \leq \sum_{j \in C} (x_j^* + w_j y_j^*) \leq d - z
\]
and \( \lambda = d - \sum_{j \in C} w_j \),
\[
\begin{align*}
\phi(x^*, y^*) & = \lambda - \sum_{j \in C} x_j^* + \sum_{j \in C} \min\{\tau - m_j, w_j\} (1 - y_j^*) \\
& = \lambda - \sum_{j \in C} x_j^* + \sum_{j=1}^{b} (\tau - m_j) + \sum_{j=b+1}^{c} w_j - \sum_{j \in C} \min\{\tau - m_j, w_j\} y_j^*
\end{align*}
\]
\[
\begin{align*}
\geq & \quad \lambda + \sum_{j=1}^{b} (\tau - m_j) + \sum_{j=b+1}^{c} w_j - (d - z) \\
= & \quad - \sum_{j=1}^{c} w_j + b \tau - \sum_{j=1}^{b} m_j + \sum_{j=b+1}^{c} w_j + z \\
= & \quad z - M_b - W_b + b \tau \\
= & \quad \tilde{f}(z). \quad \square
\end{align*}
\]

### 3.1.2 Superadditive Lifting Functions

The function \( \tilde{f}(z) \) is not superadditive for \( z \in (-\infty, M_c + W_c - \tau] \) because

\[
\tilde{f}(-M_1 - W_1 - \tau) + \tilde{f}(M_1 + W_1) = -\tau + \tau > \tilde{f}(-\tau).
\]

Therefore, to lift (3.1), we define a superadditive lifting function \( g(z) \) such that \( g(z) \leq \tilde{f}(z) \). Let \( Z = \{0, \pm 1, \pm 2, \ldots\} \) and \( Z_+ = \{0, 1, 2, \ldots\} \). Gu et al. (1999) show that \( g_0(z) \) is a superadditive function where

\[
g_0(z) = \begin{cases} 
\quad i \tau, & M_i + W_i \leq z < M_{i+1} + W_{i+1} - \tau, \ i \in Z_+, \\
\quad z - (M_i + W_i) + i \tau, & M_i + W_i - \tau \leq z < M_i + W_i, \ i \in Z_+.
\end{cases}
\]

**Remark 3.1.10** \( \tilde{f}(z) = g_0(z) \) for \( z \in [0, M_b + W_b - \tau] \) and \( \tilde{f}(z) \geq g_0(z) \) for \( z \in (M_b + W_b - \tau, M_c + W_c - \tau] \).

The function \( g_0 \) is used in the next proposition to specify a range where \( \tilde{f}(z) \) is superadditive.

**Proposition 3.1.11** The function \( \tilde{f}(z) \) is superadditive for \( z \in [0, M_c + W_c - \tau - w] \).

**Proof** Because \( g_0(z) \) is a superadditive function, \( \tilde{f}(z) \) is superadditive for \( z \in [0, M_b + W_b - \tau] \). To establish that \( \tilde{f}(z) \) is superadditive for \( z \in [M_b + W_b - \tau, M_c + W_c - \tau - w] \),
we show that $\bar{f}(z_1) + \bar{f}(z_2) \leq \bar{f}(z_1 + z_2)$ for $z_1, z_2 \in [0, M_c + W_c - \tau - \omega]$, where $z_1 + z_2 \in [M_b + W_b - \tau, M_c + W_c - \tau - \omega]$.

If $z_1, z_2 \in [0, M_b + W_b - \tau]$, then Remark 3.1.10 establishes that $\bar{f}(z_1) + \bar{f}(z_2) = g_0(z_1) + g_0(z_2) \leq g_0(z_1 + z_2) \leq \bar{f}(z_1 + z_2)$.

Consequently, we assume that $z_1 \in (M_b + W_b - \tau, M_c + W_c - \tau - \omega]$. If $z_1 \in (M_b + W_b - \tau, M_c + W_c - \tau - \omega]$, then $\bar{f}(z_1) = z_1 - M_b - W_b + b\tau$. Because $\bar{f}(z) \leq z$ for $z \in [0, M_c + W_c - \tau]$, $z_1 + z_2 \in (M_b + W_b - \tau, M_c + W_c - \tau - \omega]$ implies that $\bar{f}(z_1) + \bar{f}(z_2) \leq \bar{f}(z_1) + z_2 = z_1 - M_b - W_b + b\tau + z_2 = \bar{f}(z_1 + z_2)$.

Thus, $\bar{f}(z)$ is superadditive for $z \in [0, M_c + W_c - \tau - \omega]$.

Now, we find a superadditive lifting function $g(z)$ such that $g(z) \leq \bar{f}(z)$. To describe $g(z)$, let $\xi = \max\{\tau, M_c + W_c - M_b - W_b\}$ and

$$g_1(z) = \begin{cases} 
\bar{f}(z), & 0 \leq z < M_c + W_c - \tau - \omega, \\
z - M_b - W_b + b\tau, & M_c + W_c - \tau - \omega \leq z < M_b + W_b - \tau + \xi.
\end{cases}$$

Then,

$$g(z) = g_1(z - k(M_b + W_b - \tau + \xi)) + k((b - 1)\tau + \xi),$$

$$k(M_b + W_b - \tau + \xi) \leq z < (k + 1)(M_b + W_b - \tau + \xi), \quad k \in \mathbb{Z}.$$ Observe that $g(z)$ consists of replications of $g_1(z)$ for $z \in [k(M_b + W_b - \tau + \xi), (k + 1)(M_b + W_b - \tau + \xi))$, $k \in \mathbb{Z}$. Figure 3.1 illustrates the functions $f$, $\bar{f}$, and $g$. To show that $g(z)$ is superadditive, we now establish some properties of $g(z)$.  

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Figure 3.1: Functions $f$, $\bar{f}$ and $g$ when $\xi = \tau$

Lemma 3.1.12 $g(k(M_b + W_b - \tau + \xi) + z) = g(k(M_b + W_b - \tau + \xi)) + g(z)$ for $k \in \mathbb{Z}$.

Proof Follows from the fact that $g(z)$ consists of replications of $g_1(z)$ for $z \in [k(M_b + W_b - \tau + \xi), (k+1)(M_b + W_b - \tau + \xi))$ and replication $k$ starts at point $k(M_b + W_b - \tau + \xi)$ for $k \in \mathbb{Z}$.

Lemma 3.1.13 $g(z) \leq z$ for $z \geq 0$.

Proof If $z \in [0, M_c + W_c - \tau - w)$, then $m_j + w_j > \tau$ for $j = 1, \ldots, b$ implies that $g_1(z) = \bar{f}(z) < z$. Alternatively, if $z \in [M_c + W_c - \tau - w, M_b + W_b - \tau + \xi)$, then $M_b + W_b \leq b\tau$ implies that $g(z) = g_1(z) \leq z$.

Observe that $g(k(M_b + W_b - \tau + \xi)) = k(b-1)\tau + k\xi \leq k(M_b + W_b - \tau + \xi)$ for $k \in \mathbb{Z}_+$ because $M_b + W_b > \tau$. Thus, from Lemma 3.1.12, for $z \in [0, M_b + W_b - \tau + \xi)$
and \( k \in \mathbb{Z}_+ \),

\[
g(k(M_b + W_b - \tau + \xi) + z) = g(k(M_b + W_b - \tau + \xi)) + g(z) \leq k(M_b + W_b - \tau + \xi) + z. \quad \Box
\]

**Lemma 3.1.14** For \( \Delta \geq 0 \) and \( z \geq 0 \), \( g(z) - g(z - \Delta) \leq g(M_b + W_b - \tau + \xi) - g(M_b + W_b - \tau + \xi - \Delta) \).

**Proof** The segment of \( g(z) \) with slope zero have length \( m_j + w_j - \tau \) for \( j = 1, \ldots, b \) where \( m_j + w_j \leq m_{j-1} + w_{j-1} \). Also, the segment of \( g(z) \) with slope one have height \( \tau \) or \( \xi \) where \( \xi \geq \tau \). Thus, if \( M_b + W_b - \tau + \xi \) and \( z \) are decreased by \( \Delta \), then \( g(M_b + W_b - \tau + \xi) \) decreases at least as much as \( g(z) \). This implies that

\[
g(z) - g(z - \Delta) \leq g(M_b + W_b - \tau + \xi) - g(M_b + W_b - \tau + \xi - \Delta). \quad \Box
\]

**Lemma 3.1.15** \( g(z_1) + g(z_2) \leq g(z_1 + z_2) \) for \( z_1, z_2 \in [0, M_b + W_b - \tau + \xi) \).

**Proof** Because \( g(z) = \bar{f}(z) \) for \( z \in [0, M_c + W_c - \tau - w) \), Proposition 3.1.11 establishes that \( g(z_1) + g(z_2) \leq g(z_1 + z_2) \) for \( (z_1 + z_2) \in [0, M_c + W_c - \tau - w) \).

Suppose that \( (z_1 + z_2) \in [M_c + W_c - \tau - w, M_b + W_b - \tau + \xi) \). If \( z_1, z_2 \in [0, M_b + W_b - \tau] \), then from Remark 3.1.10,

\[
\bar{f}(z_1) + \bar{f}(z_2) = g_0(z_1) + g_0(z_2) \\
\leq g_0(z_1 + z_2) \\
\leq \bar{f}(z_1 + z_2).
\]

Consequently, we assume that \( z_1 \in (M_b + W_b - \tau, M_c + W_c - \tau - w) \). Now, \( g(z_1) = z_1 - M_b - W_b + b\tau \) and Lemma 3.1.13 implies that

\[
g(z_1) + g(z_2) \leq g(z_1) + z_2 \\
= z_1 - M_b - W_b + b\tau + z_2 \\
= g(z_1 + z_2).
\]

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Thus, \( g(z_1) + g(z_2) \leq g(z_1 + z_2) \) for \((z_1 + z_2) \in [M_c + W_c - \tau - w, \ M_b + W_b - \tau + \xi)\).

Alternatively, suppose that \((z_1 + z_2) \in [M_b + W_b - \tau + \xi, \ 2(M_b + W_b - \tau + \xi))\).

Then, from Lemma 3.1.14,

\[
g(z_2) - g(z_1 + z_2 - (M_b + W_b - \tau + \xi)) \leq g(M_b + W_b - \tau + \xi) - g(z_1). \tag{3.12}
\]

Since \((z_1, z_2, z_1 + z_2 - (M_b + W_b - \tau + \xi)) \in (0, M_b + W_b - \tau + \xi]\), from (3.12),

\[
g(z_2) = g(z_1 + z_2 - (M_b + W_b - \tau + \xi)) + g(z_2) - g(z_1 + z_2 - (M_b + W_b - \tau + \xi))
\leq g(z_1 + z_2 - (M_b + W_b - \tau + \xi)) + g(M_b + W_b - \tau + \xi) - g(z_1)
= g(z_1 + z_2) - g(z_1).
\]

The last equality follows from Lemma 3.1.12 which is \( g(z_1 + z_2) = g(M_b + W_b - \tau + \xi) + g(z_1 + z_2 - (M_b + W_b - \tau + \xi)) \). \( \Box \)

**Proposition 3.1.16** The function \( g(z) \) is superadditive.

**Proof** We first show that \( g(z) \) is superadditive for \( z \in [0, +\infty) \) by showing that
\[
g(z_1) + g(z_2) \leq g(z_1 + z_2) \text{ for } z_1, z_2 \in [0, +\infty). \text{ From Lemma 3.1.15, } g(z_1) + g(z_2) \leq g(z_1 + z_2) \text{ for } z_1, z_2 \in [0, M_b + W_b - \tau + \xi).
\]

Consider two points in \([0, +\infty), k_1(M_b + W_b - \tau + \xi) + z_1 \text{ and } k_2(M_b + W_b - \tau + \xi) + z_2\) where \(z_1, z_2 \in [0, \ M_b + W_b - \tau + \xi)\) and \(k_1, k_2 \in \mathbb{Z}_+. \text{ From Lemmas 3.1.12 and 3.1.15, }\)

\[
g(k_1(M_b + W_b - \tau + \xi) + z_1) + g(k_2(M_b + W_b - \tau + \xi) + z_2)
= g(k_1(M_b + W_b - \tau + \xi)) + g(z_1) + g(k_2(M_b + W_b - \tau + \xi)) + g(z_2)
= g((k_1 + (M_b + W_b - \tau + \xi) + k_2(M_b + W_b - \tau + \xi)) + g(z_1) + g(z_2)
\leq g((k_1 + k_2)(M_b + W_b - \tau + \xi)) + g(z_1 + z_2)
= g((k_1 + k_2)(M_b + W_b - \tau + \xi) + z_1 + z_2).
\]
The proof that $g(z)$ is superadditive for $z \in (-\infty, 0)$ is similar to the proof for $z \in [0, +\infty)$.

To establish that $g(z)$ is superadditive, we need to show that $g(z_1) + g(z_2) \leq g(z_1 + z_2)$ for $z_1 \in (-\infty, 0)$ and $z_2 \in [0, +\infty)$. The segments of $g(z_1)$ and $g(z_2)$ with slope zero have length $m_j + w_j - \tau$ for $j = 1, \ldots, b$ where $m_j + w_j \leq m_{j-1} + w_{j-1}$ for $j = 1, \ldots, b$. Also, the segments of $g(z_1)$ and $g(z_2)$ with slope one have height $\tau$ and $\xi$ where $\xi \geq \tau$. Thus, if $z_1$ is added to $z_2$, then $g(z_2)$ decreases at most $|g(z_1)|$.

Hence, $g(z_1) + g(z_2) \leq g(z_1 + z_2)$. □

**Proposition 3.1.17** $g(z) \leq \bar{f}(z)$ for $z \in (-\infty, M_c + W_c - \tau]$.

**Proof** If $z \in (-\infty, -\tau]$, then $g(z) \leq -\tau = \bar{f}(z)$. If $z \in (-\tau, M_c + W_c - \tau - w)$, then $g(z) = \bar{f}(z)$. Finally, if $z \in [M_c + W_c - \tau - w, M_c + W_c - \tau]$, then

$$g(z) = z - M_b - W_b + b\tau$$

$$\leq M_c + W_c - M_b - W_b + (b-1)\tau$$

$$= \bar{f}(z). \quad \Box$$

### 3.1.3 Lifted Inequalities

We now find values for $(\alpha, \beta)$ in (3.2). Then, we examine the strength of the superadditive approximation. Let

$$h_i(z) = \max\{\alpha_i(x_i - a_i) + \beta_i(y_i - b_i) \mid x_i - a_i + w_i(y_i - b_i) = z, \quad 0 \leq x_i \leq m_i y_i \text{ and } y_i \in \{0, 1\}\} \text{ for } i \in N \setminus C. \quad (3.13)$$

To determine $(\alpha_i, \beta_i)$ for $i \in N \setminus C$, it is sufficient to find a superadditive function $g(z)$ such that $h_i(z) \leq g(z)$ for all $z$ (Gu et al. 1999). The next proposition provides values for $h_i(z)$ when $i \in B^\ell$.  

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Proposition 3.1.18 For \( i \in B^\ell \),
\[
h_i(z) = \begin{cases} 
0, & z = 0, \\
\alpha_i(z - w_i) + \beta_i, & w_i \leq z \leq m_i + w_i.
\end{cases}
\]

Proof Let \((x_i^*, y_i^*)\) be an optimal solution to \( h_i(z) \). Note that \( z = 0 \) or \( w_i \leq z \leq m_i + w_i \) because \( x_i^* = 0 \) implies that \( y_i^* = 0 \) and \( x_i^* > 0 \) implies that \( y_i^* = 1 \). For \( i \in B^\ell \), \( a_i = b_i = 0 \) in (3.13). Then, \( h_i(z) = \alpha_i x_i^* + \beta_i y_i^* \) and \( x_i^* + w_i y_i^* = z \). If \( z = 0 \), then \( y_i^* = x_i^* = 0 \) which implies that \( h_i(z) = 0 \). If \( w_i \leq z \leq m_i + w_i \), then \( w_i \leq x_i^* + w_i y_i^* \leq m_i + w_i \). Because \( y_i^* = 1 \) and \( x_i^* = z - w_i \), we have \( h_i(z) = \alpha_i(z - w_i) + \beta_i \).
\[\square\]

Suppose that \( i \in B^\ell \). To ensure that (3.2) is valid for \( P \), it is required that \( h_i(z) \leq g(z) \) for all \( z \in \{0\} \cup [w_i, m_i + w_i] \). Since \( h_i(0) = 0 \leq g(z) \) for \( z \geq 0 \), it suffices to consider \( z \in [w_i, m_i + w_i] \). We first define the set of possible \((\alpha_i, \beta_i)\) for \( i \in B^\ell \) for \( 0 \leq w_i \leq m_i + w_i \leq M_b + W_b - \tau + \xi \).

For some \( i \in B^\ell \) and \( v \in \{1, \ldots, b - 1\} \), let
\[
M_v + W_v - \tau \leq w_i \leq M_{v+1} + W_{v+1} - \tau.
\]

Then, there are two possible intervals for \( m_i + w_i \): \( m_i + w_i \leq M_{v+1} + W_{v+1} - \tau \), and \( M_{v+1} + W_{v+1} - \tau < m_i + w_i \leq M_b + W_b - \tau + \xi \). For each interval, we define possible \( h_i(z) \) functions which satisfy \( h_i(z) \leq g(z) \). Let \( h_i^1(z) \) and \( h_i^2(z) \) be the possible \( h_i(z) \) functions for these two intervals, respectively. Also, let \( J_i^1 \) and \( J_i^2 \) be the set of possible \((\alpha_i, \beta_i)\) for the two intervals, respectively.

The following two lemmas establish \( J_i^1 \) and \( J_i^2 \).

Lemma 3.1.19 For \( i \in B^\ell \), if \( m_i + w_i \leq M_{v+1} + W_{v+1} - \tau \), then
\[
J_i^1 = \left\{ \left( \frac{g(m_i + w_i) - g(w_i)}{m_i}, g(w_i) \right) \right\}.
\]
Proof For $i \in B^\ell$ and $z \in [w_i, m_i + w_i]$, $h_i^1(z)$ is the line passing through the points $(w_i, g(w_i))$ and $(m_i + w_i, g(m_i + w_i))$. Figure 3.2 shows an example of $h_i^1(z)$. Proposition 3.1.18 implies that

$$h_i^1(z) = \frac{g(m_i + w_i) - g(w_i)}{m_i}(z - w_i) + g(w_i), \quad (\alpha_i, \beta_i) = \left(\frac{g(m_i + w_i) - g(w_i)}{m_i}, g(w_i)\right),$$

and

$$J_i^1 = \left\{\left(\frac{g(m_i + w_i) - g(w_i)}{m_i}, g(w_i)\right)\right\}.$$

Figure 3.2: $g(z)$ and $h_i^1(z)$ functions

Lemma 3.1.20 For $i \in B^\ell$ and $j \in \mathbb{Z}_+$, suppose $M_j + W_j - \tau \leq m_i + w_i \leq M_{j+1} + W_{j+1} - \tau$ where $v + 1 \leq j \leq b - 1$ or $M_j + W_j - \tau \leq m_i + w_i \leq M_j + W_j - \tau + \xi$ where $j = b$. Then, let $J_i^2 = \{J_{i\ell}^2 \cup J_{i\ell+1}^2 \text{ for } \ell = 1, \ldots, j - v - 1\}$ for $i \in B^\ell$ such that

$$J_{i1}^2 = \{(\alpha_i, \beta_i) \mid \pi_{i1} \in [0, 1]\} \quad \text{where}$$

$$\alpha_i = \pi_{i1} \frac{v\tau - g(w_i)}{M_{v+1} + W_{v+1} - \tau - w_i} + (1 - \pi_{i1}) \frac{\tau}{m_{v+2} + w_{v+2}},$$

where

$$\tau = \frac{m_i + w_i}{m_i + w_i} - \frac{m_i + w_i}{m_i + w_i}.$$
\[ \beta_i = \pi_i g(w_i) + (1 - \pi_i) \left( v\tau - (M_{v+1} + W_{v+1} - \tau - w_i)\frac{\tau}{m_{v+2} + w_{v+2}} \right), \]

\[ J_{i\ell}^2 = \left\{ (\alpha_i, \beta_i) \mid \pi_{i\ell} \in [0, 1] \right\} \] for \( \ell = 2, \ldots, j - v - 1 \) where

\[ \alpha_i = \pi_{i\ell} \left( \frac{\tau}{m_{v+\ell} + w_{v+\ell}} \right) + (1 - \pi_{i\ell}) \left( \frac{\tau}{m_{v+\ell+1} + w_{v+\ell+1}} \right), \]

\[ \beta_i = \pi_{i\ell} \left( (v + \ell - 2)\tau - (M_{v+\ell-1} + W_{v+\ell-1} - \tau - w_i)\frac{\tau}{m_{v+\ell} + w_{v+\ell}} \right) \]
\[ + (1 - \pi_{i\ell}) \left( (v + \ell - 1)\tau - (M_{v+\ell} + W_{v+\ell} - \tau - w_i)\frac{\tau}{m_{v+\ell+1} + w_{v+\ell+1}} \right), \]

and

\[ J_{i,j-v}^2 = \left\{ (\alpha_i, \beta_i) \mid \pi_{i,j-v} \in [0, 1] \right\} \] where

\[ \alpha_i = \pi_{i,j-v} \left( \frac{\tau}{m_j + w_j} \right) + (1 - \pi_{i,j-v}) \left( \frac{g(m_i + w_i) - (j - 1)\tau}{m_i + w_i - (M_j + W_j - \tau)} \right), \]

\[ \beta_i = \pi_{i,j-v} \left( (j - 2)\tau - (M_{j-1} + W_{j-1} - \tau - w_i)\frac{\tau}{m_j + w_j} \right) \]
\[ + (1 - \pi_{i,j-v}) \left( g(m_i + w_i) - m_i \frac{g(m_i + w_i) - (j - 1)\tau}{m_i + w_i - (M_j + W_j - \tau)} \right). \]

**Proof** Let \( h_{i\ell}^2(z) \) for \( \ell = 1, \ldots, j - v + 1 \) be the possible \( h_i(z) \) functions for \( i \in B^\ell \) and \( z \in [w_i, m_i + w_i] \). The function \( h_{i\ell}^2(z) \) is the line passing through the points \((w_i, g(w_i))\) and \((M_{v+1} + W_{v+1} - \tau, g(M_{v+1} + W_{v+1} - \tau))\). Proposition 3.1.18 implies that

\[ h_{i\ell}^2(z) = \frac{v\tau - g(w_i)}{M_{v+1} + W_{v+1} - \tau - w_i}(z - w_i) + g(w_i). \]

For \( j \in \mathbb{Z}_+ \) and \( \ell = 1, \ldots, j - v - 1 \) such that \( v + 1 \leq j \leq b - 1 \), \( h_{i,\ell+1}^2(z) \) is the line passing through the points \((M_{v+\ell} + W_{v+\ell} - \tau, g(M_{v+\ell} + W_{v+\ell} - \tau))\) and \((M_{v+\ell+1} + W_{v+\ell+1} - \tau, g(M_{v+\ell+1} + W_{v+\ell+1} - \tau))\). Proposition 3.1.18 implies that

\[ h_{i,\ell+1}^2(z) = \frac{\tau}{m_{v+\ell+1} + w_{v+\ell+1}}(z - w_i) + (v + \ell - 1)\tau \]
\[ - (M_{v+\ell} + W_{v+\ell} - \tau - w_i)\frac{\tau}{m_{v+\ell+1} + w_{v+\ell+1}}. \]
The function $h_{i,j-v+1}(z)$ passes through the points $(M_j + W_j - \tau, g(M_j + W_j - \tau))$ and $(m_i + w_i, g(m_i + w_i))$. Proposition 3.1.18 implies that

$$h_{i,j-v+1}(z) = \frac{g(m_i + w_i) - (j - 1)\tau}{m_i + w_i - (M_j + W_j - \tau)}(z - w_i) + g(m_i + w_i) - m_i\frac{g(m_i + w_i) - (j - 1)\tau}{m_i + w_i - (M_j + W_j - \tau)}.$$

![Figure 3.3: $g(z)$, $h^2_{i1}(z)$, $h^2_{i2}(z)$, and $h^2_{i3}(z)$ functions when $j = v + 2$](image)

The sets of $h_i(z)$ satisfying $h_i(z) \leq g(z)$ are convex combinations of $h^2_{i,\ell}(z)$ and $h^2_{i,\ell+1}(z)$ for $\ell = 1, \ldots, j - v$. Note that if $h_i(z)$ is a convex combination of $h^2_{i,\ell}(z)$ and $h^2_{i,\ell+k}(z)$ for $k \geq 2$, then $h_i(z)$ is dominated by convex combinations of either $h^2_{i,\ell}(z)$ and $h^2_{i,\ell+1}(z)$, or $h^2_{i,\ell+k-1}(z)$ and $h^2_{i,\ell+k}(z)$.

Let $J^2_{i,\ell}$ be the sets of $(\alpha_i, \beta_i)$ for $\ell = 1, \ldots, j - v$. Then,

$$J^2_{i,\ell} = \{ (\alpha_i, \beta_i) \mid \pi_{i,\ell} \in [0, 1] \} \text{ where}$$

$$h_i(z) = \alpha_i(z - w_i) + \beta_i = \pi_{i,\ell}h^2_{i,\ell}(z) + (1 - \pi_{i,\ell})h^2_{i,\ell+1}(z).$$
From Lemma 3.1.19, if \( j = v+1 \), then \( M_{v+1} + W_{v+1} - \tau \leq m_i + w_i \leq M_{v+2} + W_{v+2} - \tau \) for \( i \in B^\ell \) and the set of \( h_i(z) \) that satisfies \( h_i(z) \leq g(z) \) is a convex combination of \( h^2_{i1}(z) \) and \( h^2_{i2}(z) \) (see Figure 3.4). Thus,

\[
J^2_{i1} = \{ (\alpha_i, \beta_i) | \pi_i \in [0, 1] \} \text{ where }
\]

\[
\alpha_i = \pi_i \left( \frac{v\tau - g(w_i)}{M_{v+1} + W_{v+1} - \tau - w_i} \right) + (1 - \pi_i) \left( \frac{g(m_i + w_i) - v\tau}{m_i + w_i - (M_{v+1} + W_{v+1} - \tau)} \right),
\]

\[
\beta_i = \pi_i g(w_i) + (1 - \pi_i) \left( g(m_i + w_i) - m_i \frac{g(m_i + w_i) - v\tau}{m_i + w_i - (M_{v+1} + W_{v+1} - \tau)} \right).
\]

Figure 3.4: \( g(z) \), \( h^2_{i1}(z) \), and \( h^2_{i2}(z) \) functions when \( j = v + 1 \)

Suppose that \( M_b + W_b - \tau \leq w_i \leq m_i + w_i \leq M_b + W_b - \tau + \xi \) for \( i \in B^\ell \). Then, let \( J^3_i \) be the set of \( (\alpha_i, \beta_i) \) for \( i \in B^\ell \) and \( z \in [w_i, m_i + w_i] \). The following lemma provides the set \( J^3_i \).

**Lemma 3.1.21** For \( i \in B^\ell \), if \( M_b + W_b - \tau < w_i \leq m_i + w_i \leq M_b + W_b - \tau + \xi \), then

\[
J^3_i = \{ (1, g(w_i)) \}.
\]
Proposition 3.1.22

Proof Let \( h_i^3(z) \) be the \( h_i(z) \) function for this interval for \( i \in B^u \) and \( z \in [w_i, m_i + w_i] \). The line segment between the points \((w_i, g(w_i))\) and \((m_i + w_i, g(m_i + w_i))\) generates \( h_i^3(z) \). Proposition 3.1.18 implies that

\[
    h_i^3(z) = (z - w_i) + g(w_i) \quad \text{and} \quad J_i^3 = \{(1, g(w_i))\}.
\]

For \( i \in B^u \), let \( J_i^4 \) be the set of \((\alpha_i, \beta_i)\) when \( m_i + w_i \geq M_b + W_b - \tau + \xi \). The set \( J_i^4 \) is found in a manner similar to as in Lemmas 3.1.19 – 3.1.21 because \( g(k(M_b + W_b - \tau + \xi) + z) = g(k(M_b + W_b - \tau + \xi)) + g(z) \) for \( k \in \mathbb{Z}_+ \) (see Lemma 3.1.12).

The next proposition provides \( h_i(z) \) for \( i \in B^u \) and \( z \leq 0 \).

Proposition 3.1.22 For \( i \in B^u \) and \( z \in [-m_i, 0] \cup \{-m_i - w_i\} \),

\[
    h_i(z) = \begin{cases} 
    \alpha_i z, & -m_i \leq z \leq 0, \\
    -\alpha_i m_i - \beta_i, & z = -m_i - w_i.
    \end{cases}
\]

Proof For \( i \in B^u \), \( a_i = m_i \) and \( b_i = 1 \) in (3.13). Let \((x_i^*, y_i^*)\) be an optimal solution to \( h_i(z) \). If \( y_i^* = 1 \), then (3.13) implies that \( x_i^* - m_i + w_i(y_i^* - 1) = x_i^* - m_i = z \). Because \( m_i \geq x_i^* \geq 0 \), we have \(-m_i \leq z \leq 0\). Thus, \( h_i(z) = \alpha_i(x_i^* - m_i) + \beta_i(y_i^* - 1) = \alpha_i z \).

Alternatively, if \( y_i^* = 0 \), then \( x_i^* = 0 \) and \( x_i^* - m_i + w_i(y_i^* - 1) = -m_i - w_i = z \). Thus, \( h_i(z) = \alpha_i(x_i^* - m_i) + \beta_i(y_i^* - 1) = -\alpha_i m_i - \beta_i \).

Suppose that \( i \in B^u \). To ensure that (3.2) is valid for \( P \), it is necessary that \( h_i(z) \leq g(z) \) for all \( z \in \{-m_i - w_i\} \cup [-m_i, 0] \). Figure 3.5 shows an example of \( h_i(z) \) for this case. The following lemma provides \((\alpha_i, \beta_i)\) for \( i \in B^u \).

Lemma 3.1.23 If \( z \in \{-m_i - w_i\} \cup [-m_i, 0] \), then \( \alpha_i = 1 \) and \( \beta_i = -m_i - g(-m_i - w_i) \).
**Proof** For \(z \in [-m_i, 0]\), \(h_i(z) = \alpha_i z = z \leq g(z)\). This implies that \(\alpha_i = 1\). Thus, for \(z = -m_i - w_i\),

\[
h_i(-m_i - w_i) = -\alpha_i m_i - \beta_i = -m_i - \beta_i \leq g(-m_i - w_i).
\]

From the last inequality, \(\beta_i\) is maximal when \(\beta_i = -m_i - g(-m_i - w_i)\). Consequently, for \(z \in \{-m_i - w_i\} \cup [-m_i, 0]\), \(\alpha_i = 1\) and \(\beta_i = -m_i - g(-m_i - w_i)\).

![Figure 3.5: Example of \(g(z)\) and \(h_i(z)\) functions](image)

The inequality lifted from (3.1) is provided in the next proposition. This inequality is valid because \((\alpha_i, \beta_i)\) for \(i \in B^\ell \cup B^u\) is selected such that \(h_i(z) \leq f(z)\) for \(z \in \mathbb{Z}\).

**Theorem 3.1.24** The inequality

\[
\sum_{j \in C} x_j - \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j) + \sum_{j \in B^\ell} \alpha_j x_j + \sum_{j \in B^\ell} \beta_j y_j + \sum_{j \in B^u} (x_j - m_j)
\]

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\[ + \sum_{j \in \mathcal{B}^u} [-m_j - g(-m_j - w_j)](y_j - 1) \leq \lambda \quad (3.14) \]

is valid for \( P \) where for \( i \in \mathcal{B}^l \),

\[
(\alpha_i, \beta_i) \in \begin{cases} 
  J_1^i, & M_v + W_v - \tau \leq w_i \leq w_i + m_i \leq M_{v+1} + W_{v+1} - \tau \\
  J_2^i, & M_v + W_v - \tau \leq w_i \leq M_{v+1} + W_{v+1} - \tau < m_i + w_i \\
  J_3^i, & M_b + W_b - \tau < w_i \leq m_i + w_i \leq M_b + W_b - \tau + \xi, \\
  J_4^i, & \text{otherwise.} 
\end{cases}
\]

**Proof** Follows from Lemmas 3.1.19 – 3.1.23. \( \square \)

We now examine the strength of the superadditive approximation.

**Definition.** (Gu et al. 1999) The function \( g(z) \) is *dominated* by another superadditive valid lifting function \( g'(z) \) if \( g(z) \leq g'(z) \) for all \( z \in \mathbb{Z} \) and \( g(z_0) < g'(z_0) \) for some \( z_0 \in \mathbb{Z} \).

**Proposition 3.1.25** The function \( g(z) \) is not dominated by any other superadditive lifting function for \( f(z) \).

**Proof** Suppose that \( g(z) \) is dominated by a superadditive lifting function \( g'(z) \) such that \( g'(z) \geq g(z) \) for all \( z \) and \( g'(z_0) > g(z_0) \) for some \( z_0 \). Because \( g(z) = f(z) \) for \( z \in [-\tau, 0) \) and \( z \in [M_{i-1} + W_{i-1}, M_i + W_i - \tau) \) for \( i = 1, \ldots, b \), there exists no \( z_0 \) such that \( g'(z_0) > g(z_0) \) for those intervals. Therefore, \( g'(z) = g(z) \) for \( z \in [-\tau, 0) \) and \( z \in [M_{i-1} + W_{i-1}, M_i + W_i - \tau) \) for \( i = 1, \ldots, b \). There are four cases for \( z_0 \):

- \( z_0 \in [M_i + W_i - \tau, M_i + W_i) \) for \( i = 1, \ldots, b-1 \),
- \( z_0 \in [M_b + W_b - \tau, M_b + W_b - \tau + \xi) \),
- \( z_0 \in (-\infty, -\tau) \), and
- \( z_0 \in [M_b + W_b - \tau + \xi, +\infty) \).
Case 1: $z_0 \in [M_i + W_i - \tau, M_i + W_i)$ for $i = 1, \ldots, b-1$. Observe that $g(z_0) = z_0 - M_i - W_i + i\tau$. Then,

$$g'(z_0) \leq g'(M_i + W_i - \tau) - g'(-z_0 + M_i + W_i - \tau)$$

(3.15)

$$= g'(M_i + W_i - \tau) - g(-z_0 + M_i + W_i - \tau)$$

$$\leq (i - 1)\tau - (-z_0 + M_i + W_i - \tau)$$

$$= z_0 - M_i - W_i + i\tau$$

$$= g(z_0).$$

The first inequality follows from the superadditivity of $g'(z)$. The first equality follows from $g'(z) = g(z)$ for $z \in [-\tau, 0)$. The second inequality follows from $g(-z) \geq -z$. Thus, $g'(z_0) = g(z_0)$.

Case 2: $z_0 \in [M_b + W_b - \tau, M_b + W_b - \tau + \xi)$. Observe that $g(z_0) = z_0 - M_b - W_b + b\tau$. When $i = b$ in (3.15), a proof similar to Case 1 establishes that $g'(z_0) = g(z_0)$.

Case 3: $z_0 \in (-\infty, -\tau)$. If $z_0 \in [-M_b - W_b + \tau - \xi, -\tau)$, then

$$g'(z_0) \leq g(M_b + W_b - \tau + \xi + z_0) - g(M_b + W_b - \tau + \xi)$$

$$= g(M_b + W_b - \tau + \xi + z_0) - g(M_b + W_b - \tau + \xi)$$

$$= g(z_0).$$

The first inequality follows from the superadditivity of $g'(z)$. The first equality follows from $g'(z_0) = g(z_0)$ for $z_0 \in [M_i + W_i - \tau, M_i + W_i)$ for $i = 1, \ldots, b-1$ and $z_0 \in [M_b + W_b - \tau, M_b + W_b - \tau + \xi)$ from Cases 1 and 2, respectively. The second equality follows from Lemma 3.1.12.

Alternatively, suppose that $z_0 \in (-\infty, -M_b - W_b + \tau - \xi)$. Then, $z \in [k(M_b + W_b - \tau + \xi), (k+1)(M_b + W_b - \tau + \xi))$ for some $k = -2, \ldots$. From the superadditivity
of $g'(z)$,

$$
g'(z_0) \leq g'(-(k + 1)(M_b + W_b - \tau + \xi) + z_0) - g'(-(k + 1)(M_b + W_b - \tau + \xi))
\leq g(-(k + 1)(M_b + W_b - \tau + \xi) + z_0) - g(-(k + 1)(M_b + W_b - \tau + \xi))
= g(z_0).
$$

The first equality follows from $g(z) = g'(z)$ for $z_0 \in [-M_b - W_b + \tau - \xi, 0)$. The second equality follows from Lemma 3.1.12.

Case 4: $z_0 \in [M_b + W_b - \tau + \xi, +\infty)$. Observe that $g(z)$ for $z \in [k(M_b + W_b - \tau + \xi), (k + 1)(M_b + W_b - \tau + \xi))$ for $k = 1, 2, \ldots$, is the replication of $g(z)$ for $z \in [0, M_b + W_b - \tau + \xi)$. Suppose that $z_0 \in [k(M_b + W_b - \tau + \xi) + M_i + W_i - \tau, k(M_b + W_b - \tau + \xi) + M_i + W_i)$ for $i = 1, \ldots, b - 1$ or $z_0 \in [k(M_b + W_b - \tau + \xi) + M_b + W_b - \tau, (k + 1)(M_b + W_b - \tau + \xi))$ for $k = 1, 2, \ldots$. Then, because $g(z) = g'(z)$ for $z_0 \in (-\infty, 0)$, the proof that $g'(z_0) = g(z_0)$ is similar to Cases 1 and 2, respectively.

Alternatively, suppose that $z_0 \in [k(M_b + W_b - \tau + \xi) + M_i + W_i - \tau, k(M_b + W_b - \tau + \xi) + M_i + W_i)$ for $i = 1, \ldots, b - 1$ and $g'(z)$ is nondecreasing in $z$.

In sequential lifting, new variables are reintroduced into a valid inequality one at a time (or one group at a time). If the variables are lifted in the right order, the sequential lifting produces strong valid inequalities (Crowder et al., 1983). Thus, the strength of the superadditive approximation can be determined by using sequential lifting which converts (3.1) into inequality (3.2) by including the variables $(x_j, y_j)$ for $j \in (B^l \cup B^u)$ one at a time. W.l.o.g, assume that the variables $(x_j, y_j)$ for
$j_i \in (B^\ell \cup B^u)$ are lifted in index order. Coefficients $\alpha_{jk}$ and $\beta_{jk}$ in (3.2) can be determined by solving the problems

$$f_k(z) = \min_{(x,y) \in P_C} \lambda - \sum_{j \in C} x_j + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j)$$

$$- \sum_{i=1}^{k-1} [\alpha_{ji}(x_{ji} - a_{ji}) + \beta_{ji}(y_{ji} - b_{ji})]$$

s.t. \( \sum_{j \in C} (x_j + w_j y_j) + \sum_{i=1}^{k-1} (x_{ji} + w_{ji} y_{ji}) \leq d - z, \)

\( 0 \leq x_j \leq m_j y_j, \)

\( x_j \geq 0, \)

\( y_j \in \{0, 1\} \text{ for } j \in C \cup \{j_1, \ldots, j_{k-1}\} \)

and

$$h_{jk}(z) = \max\{\alpha_{jk}(x_{jk} - a_{jk}) + \beta_{jk}(y_{jk} - b_{jk}) \mid (x_{jk} - a_{jk}) + w_{jk}(y_{jk} - b_{jk}) = z, \}

0 \leq x_{jk} \leq m_{jk} y_{jk} \text{ and } y_{jk} \in \{0, 1\} \}

for \( z \in \mathbb{Z}. \) Inequality (3.2) is valid for \( P \) for any choice of $\alpha_{jk}$ and $\beta_{jk}$ such that $h_{jk}(z) \leq f_k(z)$ (Gu et al. 1999).

Let $E = \{z \in \mathbb{Z} \mid f(z) = f_i(z) \text{ for } i = 1, \ldots, |B^\ell \cup B^u|\}$ which is the set of $z$ such that sequence independent lifting function is equal to the sequential lifting function for index $i$. The function $g$ is a maximal superadditive lifting function if $g(z) = f(z)$ for all $z \in E$ (Gu et al. 2000).

**Proposition 3.1.26** The function $g(z)$ is a maximal superadditive lifting function for $f(z)$.

**Proof** Let $\bar{E}$ be the set such that $\bar{E} = \{z \mid z \in (-\infty, M_c + W_c - \tau)\} \setminus E$. To show that $g(z)$ is maximal, we first find the set $\bar{E}$. We show that $\{z \mid z \in (-\infty, -\tau)\} \subseteq \bar{E}$ if $\tau < \xi$ and $\{z \mid z \in (-\infty, -m_b - w_b)\} \subseteq \bar{E}$ otherwise. Then, we also show that
\{z | z \in (M_i + W_i - \tau, M_i + W_i) \text{ for } i = 1, \ldots, b\} \cup \{z | z \in (M_b + W_b - \tau, M_c + W_c - \tau) \text{ for } i = 1, \ldots, b\} \subseteq \bar{E}.

We now show that \( \{z | z \in (-\infty, -\tau)\} \subseteq \bar{E} \) if \( \tau < \xi \) and \( \{z | z \in (-\infty, -m_b - w_b)\} \subseteq \bar{E} \) otherwise. Let \( j_1 \in B^u \) such that \( m_{j_1} = M_c + W_c - M_b - W_b \) and \( w_{j_1} = M_b + W_b - \tau \).

Consider lifting \( x_{j_1} \) from 0 to \( m_{j_1} \) and \( y_{j_1} \) from 0 to 1 for inequality (3.1). Maximal \( \alpha_{j_1} \) and \( \beta_{j_1} \) are determined by the line segment between the points \((w_{j_1}, f(w_{j_1}))\) and \((m_{j_1} + w_{j_1}, f(m_{j_1} + w_{j_1}))\). Because \( f(w_{j_1}) = (b-1)\tau \) and \( f(m_{j_1} + w_{j_1}) = M_c + W_c - M_b - W_b + (b-1)\tau \), lifting coefficients are \( \alpha_{j_1} = 1 \) and \( \beta_{j_1} = (b-1)\tau \).

Then, the lifted valid inequality (3.1) can be written as

\[
\sum_{j \in C} x_j - \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j) + x_{j_1} + (b-1)\tau y_{j_1} \leq \lambda. \tag{3.16}
\]

The lifting function \( f_2(z) \) for (3.16) is

\[
f_2(z) = \min_{(x,y) \in P} \lambda - \sum_{j \in C} x_j + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j) - x_{j_1} - (b-1)\tau y_{j_1},
\]

s.t. \[
\sum_{j \in C} (x_j + w_j y_j) + x_{j_1} + w_{j_1} y_{j_1} \leq d - z,\]

\[
0 \leq x_j \leq m_j y_j,\]

\[
x_j \geq 0,\]

\[
y_j \in \{0, 1\} \text{ for } j \in C \cup \{j_1\}.
\]

Suppose that \( \tau < \xi \) and \( z \in (-\infty, -\tau) \). Let \( z = -\tau - \varepsilon \) such that \( \varepsilon > 0 \). Because \( f \) is a non-decreasing function and \( m_{j_1} > \tau \) for \( x_{j_1} = \min\{\tau + \varepsilon, m_{j_1}\} \), \( y_{j_1} = 1 \), and \( \varepsilon > 0 \),

\[
f_2(z^0) = f_2(-\tau - \varepsilon)
\]

\[
\leq f(-\tau - \varepsilon + M_b + W_b - \tau + x_{j_1}) - x_{j_1} - (b-1)\tau
\]

\[
\leq f(M_b + W_b - \tau) - \min\{\tau + \varepsilon, m_{j_1}\} - (b-1)\tau
\]

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\[ = -\min\{\tau + \varepsilon, m_{j_1}\} \]
\[ < -\tau \]
\[ = f(-\tau - \varepsilon). \]

Thus, \( \{z \mid z \in (-\infty, -\tau)\} \subseteq \bar{E}. \)

Alternatively, suppose that \( \xi = \tau \) and \( z \in (-\infty, -m_b - w_b). \) Let \( z = -m_b - w_b - \varepsilon. \)

Because \( f \) is a non-decreasing function, for \( x_{j_1} = \min\{\varepsilon, m_{j_1}\}, y_{j_1} = 1, \) and \( \varepsilon > 0, \)
\[ f_2(z) = f_2(-m_b - w_b - \varepsilon) \]
\[ \leq f(-m_b - w_b - \varepsilon + M_b + W_b - \tau + x_{j_1}) - x_{j_1} - (b - 1)\tau \]
\[ \leq f(M_{b-1} + W_{b-1} - \tau) - \min\{\varepsilon, m_{j_1}\} - (b - 1)\tau \]
\[ = (b - 2)\tau - \min\{\varepsilon, m_{j_1}\} - (b - 1)\tau \]
\[ < -\tau \]
\[ = f(-m_b - w_b - \varepsilon). \]

Thus, \( \{z \mid z \in (-\infty, -m_b - w_b)\} \subseteq \bar{E}. \)

We now show that \( \{z \mid z \in (M_i + W_i - \tau, M_i + W_i) \text{ for } i = 1, \ldots, b\} \cup \{z \mid z \in (M_b + W_b - \tau, M_c + W_c - \tau) \text{ for } i = 1, \ldots, b\} \subseteq \bar{E}. \) For some \( i = 1, \ldots, b, \) let \( 0 < \varepsilon_2 < \tau \) if \( i < b \) and \( 0 < \varepsilon_2 \leq \xi \) if \( i = b. \) Let \( j_1 \in B^a \) such that \( m_{j_1} + w_{j_1} = m_i + w_i + \varepsilon_2. \)

Consider lifting \( x_{j_1} \) from \( m_{j_1} \) to 0 and \( y_{j_1} \) from 1 to 0 in the inequality (3.1). The lifted inequality is valid if \( \alpha_i(x_i - m_i) \leq x_i - m_i \) for \( x_i - m_i \leq -\tau, \) if \( \alpha_i(x_i - m_i) \leq -\tau \) for \( -\tau \leq x_i - m_i \leq 0, \) and \( -\alpha_i m_i - \beta_i \leq \tau. \) Therefore, maximal lifting coefficients are determined as \( \alpha_{j_1} = 1 \) and \( \beta_{j_1} = -m_{j_1} + \tau. \) Then, the lifted valid inequality (3.1) can be written as
\[
\sum_{j \in C} x_j - \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j) + (x_{j_1} - m_{j_1})
\]
\[ +(-m_{j_1} + \tau)(y_{j_1} - 1) \leq \lambda. \quad (3.17) \]

The lifting function \( f_2(z) \) for (3.17) is

\[
f_2(z) = \min_{(x,y) \in P} \lambda - \sum_{j \in C} x_j \\
+ \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j) - x_{j_1} + m_{j_1} + (-m_{j_1} + \tau)(1 - y_{j_1})
\]

s.t. \[
\sum_{j \in C} (x_j + w_jy_j) + x_{j_1} + w_{j_1}y_{j_1} \leq d + (m_i + w_i + \epsilon_2) - z, \\
0 \leq x_j \leq m_jy_j, \\
x_j \geq 0, \\
y_j \in \{0, 1\} \text{ for } j \in C \cup \{j_1\}.
\]

Let \( z = M_i + W_i - \tau + \epsilon_2 \). For \( x_{j_1} = y_{j_1} = 0 \),

\[
f_2(z) = f_2(M_i + W_i - \tau + \epsilon_2) \\
\leq f(-(m_i + w_i + \epsilon_2) + M_i + W_i - \tau + \epsilon_2) + m_{j_1} + (-m_{j_1} + \tau) \\
= f(M_{i-1} + W_{i-1} - \tau) + \tau \\
= \begin{cases} 0 & \text{if } i = 1 \\ (i - 1)\tau & \text{otherwise} \end{cases} \\
< \bar{f}(M_i + W_i - \tau + \epsilon_2) \\
\leq f(M_i + W_i - \tau + \epsilon_2).
\]

Thus, \( \{z \mid z \in (M_i + W_i - \tau, M_i + W_i) \text{ for } i = 1, \ldots, b\} \cup \{z \mid z \in (M_b + W_b - \tau, M_e + W_e - \tau) \text{ for } i = 1, \ldots, b\} \subseteq \bar{E} \).

Because \( \bar{E} \cap E = \emptyset \) and \( g(z) = f(z) \) for \( z \in [-\tau, 0) \) and \( z \in [M_{i-1} + W_{i-1}, M_i + W_i - \tau) \) for \( i = 1, \ldots, b \), we have \( g(z) = f(z) \) for \( z \in E \). \( \square \)

We now compare the superadditive function developed by Shebalov and Klabjan (2006) with \( g \). Shebalov and Klabjan (2006) study the mixed-integer polytope with variable upper bounds \( \{(x, y) \in \mathbb{R}_+^n \times \mathbb{B}^n \mid \sum_{j \in N_1^+} x_j - \sum_{j \in N_1^-} x_j + \sum_{j \in N_2^+} w_j y_j + \sum_{j \in N_2^-} w_j y_j \leq \bar{d}, x_j \leq u_j + m_j y_j \text{ for } j \in N_2^+, x_j \leq u_j - m_j y_j \text{ for } j \in N_2^- \} \) where \((N_1^+ \cup N_1^-) \subseteq (N_2^+ \cup N_2^-)\) and \( N_2^+ \) and \( N_2^- \) form a partition of \( N \). For their problem, they show that it is NP-hard to compute the lifting function. To lift the flow cover inequality, they develop a set of valid inequalities by using a superadditive approximation of the lifting function based on LP relaxation of the lifting function.

Let \( P_{LP} = \{(x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \mid \sum_{j \in N}(x_j + w_j y_j) \leq \bar{d}, x_j \leq m y_j, 0 \leq y_j \leq 1 \text{ for } j \in N\} \). Also, let \( f_{LP}(z) \) be the LP relaxation of \( f(z) \) where

\[
\begin{align*}
f_{LP}(z) &= \min_{x,y \in \mathbb{R}_+^n} \left\{ \lambda - \sum_{j \in C} x_j + \sum_{j \in C} \min\{\tau - m_j, w_j\}(1 - y_j) \mid \right. \\
&\left. \sum_{j \in C}(x_j + w_j y_j) \leq d - z, \\
&\quad 0 \leq x_j \leq m_j y_j \text{ and } y_j \leq 1 \text{ for } j \in C \right\}. \tag{3.18}
\end{align*}
\]

Let

\[
\tilde{g}_1(z) = \begin{cases} 
0, & 0 \leq z < M_1 + W_1 - \tau, \\
f_{LP}(z), & M_1 + W_1 - \tau \leq z < M_c + W_c - \tau - w, \\
z - M_b - W_b + b \tau, & M_c + W_c - \tau - w \leq z < M_b + W_b - \tau + \xi.
\end{cases}
\]

The superadditive function of Shebalov and Klabjan (2006) is

\[
\tilde{g}(z) = \tilde{g}_1(z - k(M_b + W_b - \tau + \xi)) + k((b - 1)\tau + \xi),
\]

\[
k(M_b + W_b - \tau + \xi) \leq z < (k + 1)(M_b + W_b - \tau + \xi), \quad k \in \mathbb{Z}.
\]

To show that \( g \) dominates \( \tilde{g} \), the next five lemmas provide upper bound values for \( f_{LP}(z) \).
Lemma 3.1.27  For \( z \in [0, M_c + W_c - \tau] \),
\[
f_{LP}(z) \leq z - \sum_{j=1}^{b} (m_j + w_j - \tau) + (m_\ell + w_\ell - \tau) \frac{d - z - \sum_{j=\ell+1}^{c} (m_j + w_j)}{m_\ell + w_\ell} + \sum_{j=\ell+1}^{b} (m_j + w_j - \tau).
\]

Proof  For \( z \in [0, M_c + W_c - \tau] \), let \( \ell = \max\{k \geq 1 \mid \sum_{j=k}^{c} (m_j + w_j) \geq d - z\} \). Also, let \( y_j^0 = 0 \) for \( j = 1, \ldots, \ell - 1 \), \( y_\ell^0 = (d - z - \sum_{j=\ell+1}^{c} (m_j + w_j)) / (m_\ell + w_\ell) \), \( y_j^0 = 1 \) for \( j = \ell + 1, \ldots, c \), and \( x_j^0 = m_j y_j^0 \) for \( j = 1, \ldots, c \). Then, \((x^0, y^0) \in P_{LP}\) because \( y_\ell^0 \leq 1 \) and \( \sum_{j=1}^{c} (x_j^0 + w_j y_j^0) = \sum_{j=\ell}^{c} (x_j^0 + w_j y_j^0) = d - z \).

From (3.18) and \( \sum_{j=1}^{c} x_j^0 = d - z - \sum_{j=1}^{c} w_j y_j^0 \),
\[
f_{LP}(z) \leq \lambda - \sum_{j=1}^{c} x_j^0 + \sum_{j=1}^{c} \min\{\tau - m_j, \ w_j\} (1 - y_j^0)
\]
\[
= d - \sum_{j=1}^{c} w_j - (d - z - \sum_{j=1}^{c} w_j y_j^0) + \sum_{j=1}^{b} (\tau - m_j)(1 - y_j^0)
\]
\[
+ \sum_{j=b+1}^{c} w_j (1 - y_j^0)
\]
\[
= z - \sum_{j=1}^{c} w_j + \sum_{j=1}^{c} w_j y_j^0 - \sum_{j=1}^{c} (m_j - \tau) + \sum_{j=1}^{b} (m_j - \tau) y_j^0
\]
\[
+ \sum_{j=b+1}^{c} w_j - \sum_{j=b+1}^{c} w_j y_j^0
\]
(3.19)
\[
= z - \sum_{j=1}^{\ell} (m_j + w_j - \tau) + \sum_{j=1}^{b} (m_j + w_j - \tau) y_j^0
\]
\[
= z - \sum_{j=1}^{\ell} (m_j + w_j - \tau) + \sum_{j=1}^{\ell-1} (m_j + w_j - \tau) y_j^0 + (m_\ell + w_\ell - \tau) y_\ell^0
\]
\[
+ \sum_{j=\ell+1}^{b} (m_j + w_j - \tau) y_j^0
\]
\[
= z - \sum_{j=1}^{b} (m_j + w_j - \tau) + (m_\ell + w_\ell - \tau) \frac{d - z - \sum_{j=\ell+1}^{c} (m_j + w_j)}{m_\ell + w_\ell} + \sum_{j=\ell+1}^{b} (m_j + w_j - \tau). \quad \Box
\]
Lemma 3.1.28 If $z \in [M_i + W_i - \tau, M_i + W_i)$ for $i = 1, \ldots, b - 1$, then

$$f_{LP}(z) < z - M_i - W_i + i\tau = \bar{f}(z).$$

**Proof** Because $z \in [M_i + W_i - \tau, M_i + W_i)$ for $i = 1, \ldots, b$, we have $d - z \in (M_c + W_c - M_i - W_i - \tau, M_c + W_c - M_i - W_i]$. Let $y_j^0 = 0$ for $j = 1, \ldots, i$, $y_{i+1}^0 = (d - z - \sum_{j=i+2}^c (m_j + w_j))/(m_{i+1} + w_{i+1})$, $y_j^0 = 1$ for $j = i + 2, \ldots, c$, and $x_j^0 = m_j y_j^0$ for $j = 1, \ldots, c$. Then, $(x^0, y^0) \in P_{LP}$ because $y_{i+1} \leq 1$ and $\sum_{j \in C}(x_j^0 + w_j y_j^0) = d - z$.

Note that $i + 1 = \max\{k \geq 1 \mid \sum_{j=k}^c (m_j + w_j) \geq d - z\}$. Because $y_{i+1} = (d - z - \sum_{j=i+2}^c (m_j + w_j))/(m_{i+1} + w_{i+1}) < 1$, setting $\ell = i + 1$ in Lemma 3.1.27 establishes that

$$f_{LP}(z) \leq z - \sum_{j=1}^b (m_j + w_j - \tau) + (m_{i+1} + w_{i+1} - \tau) \frac{d - z - \sum_{j=i+2}^c (m_j + w_j)}{m_{i+1} + w_{i+1}}$$

$$+ \sum_{j=i+2}^c (m_j + w_j - \tau)$$

$$< z - \sum_{j=1}^b (m_j + w_j - \tau) + (m_{i+1} + w_{i+1} - \tau) + \sum_{j=i+2}^c (m_j + w_j - \tau)$$

$$= z - \sum_{j=1}^{i+1} (m_j + w_j - \tau) + (m_{i+1} + w_{i+1} - \tau)$$

$$= z - M_i - W_i + i\tau.$$  

$$= \bar{f}(z). \Box$$

Lemma 3.1.29 If $z \in [M_i + W_i, M_{i+1} + W_{i+1} - \tau)$ for $i = 1, \ldots, b - 1$, then

$$f_{LP}(z) \leq i\tau = \bar{f}(z).$$

**Proof** Because $z \in [M_i + W_i, M_{i+1} + W_{i+1} - \tau)$ for $i = 1, \ldots, b - 1$, we have $d - z \in (M_c + W_c - M_{i+1} - W_{i+1}, M_c + W_c - M_i - W_i - \tau]$. Let $y_j^0 = 0$ for $j = 1, \ldots, i$, $y_{i+1}^0 = (d - z - \sum_{j=i+2}^c (m_j + w_j))/(m_{i+1} + w_{i+1})$, $y_j^0 = 1$ for $j =
Because (3.19), \(W_j\) for \(j = 1, \ldots, c\). Then, \((x^0, y^0) \in P_{LP}\) because \(y_{i+1} \leq 1\) and \(\sum_{j \in C}(x^j + w_j y^j) = d - z\). Note that \(i+1 = \max\{k \geq 1\mid \sum_{j=k}^c (m_j + w_j) \geq d - z\}\). Because \((m_{i+1} + w_{i+1} - \tau)/(m_{i+1} + w_{i+1}) < 1\), setting \(\ell = i + 1\) in Lemma 3.1.27 establishes that

\[
f_{LP}(z) \leq z - \sum_{j=1}^b (m_j + w_j - \tau) + (m_{i+1} + w_{i+1} - \tau) \frac{d - z - \sum_{j=i+2}^c (m_j + w_j)}{m_{i+1} + w_{i+1}}
+ \sum_{j=i+2}^b (m_j + w_j - \tau)
\leq z - \sum_{j=1}^b (m_j + w_j - \tau) + (d - z - \sum_{j=i+2}^c (m_j + w_j))
+ \sum_{j=i+2}^b (m_j + w_j - \tau)
= d - \sum_{j=1}^b (m_j + w_j - \tau) - \sum_{j=i+2}^c (m_j + w_j)
= M_c + W_c - \tau - \sum_{j=1}^c (m_j + w_j) + (i + 1)\tau
= i\tau
= \bar{f}(z). \quad \blacksquare
\]

**Lemma 3.1.30** If \(z \in [M_b + W_b - \tau, M_c + W_c - \tau - \underline{w})\), then

\[
f_{LP}(z) \leq z - M_b - W_b + b\tau = \bar{f}(z).
\]

**Proof** Because \(z \in [M_b + W_b - \tau, M_c + W_c - \tau - \underline{w})\), we have \(d - z \in (w, M_c + W_c - M_b - W_b]\). Let \(y^0_j = 0\) for \(j = 1, \ldots, \ell - 1\), \(y^0_\ell = (d - z - \sum_{j=\ell+1}^c (m_j + w_j))/(m_\ell + w_\ell)\), \(y^0_j = 1\) for \(j = \ell + 1, \ldots, c\), and \(x^0_j = m_j y^0_j\) for \(j = 1, \ldots, c\) where \(\ell = \max\{k \geq 1\mid \sum_{j=k}^c (m_j + w_j) \geq d - z\}\). Then, \((x^0, y^0) \in P_{LP}\) because \(y_\ell \leq 1\) and \(\sum_{j \in C}(x^0_j + w_j y^0_j) = d - z\).

Now, from (3.19),

\[
f_{LP}(z) \leq z - \sum_{j=1}^c w_j + \sum_{j=b+1}^c w_j y^0_j - \sum_{j=1}^b (m_j - \tau) + \sum_{j=b+1}^c w_j - \sum_{j=b+1}^c w_j y^0_j
\]

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\[ z = \sum_{j=1}^{b} (m_j + w_j - \tau) \]
\[ z = M_b - W_b + b\tau \]
\[ = \bar{f}(z). \quad \Box \]

**Theorem 3.1.31** The function \( g(z) \) dominates \( \tilde{g}(z) \).

**Proof** For \( z \in [k(M_b + W_b - \tau + \xi), (k + 1)(M_b + W_b - \tau + \xi)) \), \( k \in \mathbb{Z} \), \( g(z) \) consists of replications of \( g_1(z) \) and \( \tilde{g}(z) \) consists of replications of \( \tilde{g}_1(z) \). As a result, it is sufficient to show that \( \tilde{g}(z) \leq g(z) \) for \( z \in [0, M_b + W_b - \tau + \xi) \).

For \( z \in [0, M_1 + W_1 - \tau) \) and \( z \in [M_c + W_c - \tau - w, M_b + W_b - \tau + \xi) \), Lemmas 3.1.28 – 3.1.30 imply that \( g(z) \leq \tilde{g}(z) \) because \( g(z) = \bar{f}(z) \), \( \tilde{g}(z) = f_{LP}(z) \), and \( f_{LP}(z) \leq \bar{f}(z) \). \( \Box \)

Since \( g \) dominates \( \tilde{g} \), the lifted flow cover inequalities generated by using \( g \) dominates the lifted flow cover inequalities generated by using the superadditive function in Shebalov and Klabjan (2006).

**3.1.5 Alternative Valid Inequalities**

In this subsection, we find another superadditive function which can be used to develop valid inequalities (3.14). Let

\[ \tilde{g}(z) = \begin{cases} 
  i\tau, & i(M_1 + W_1) \leq z < (i + 1)(M_1 + W_1) - \tau, \ i \in \mathbb{Z}, \\
  z - i(M_1 + W_1) + i\tau, & i(M_1 + W_1) - \tau \leq z < i(M_1 + W_1), \ i \in \mathbb{Z}.
\end{cases} \]

**Proposition 3.1.32** \( \tilde{g}(z) \leq \bar{f}(z) \) for \( z \in (-\infty, M_c + W_c - \tau] \).

**Proof** There are three cases: \( z \in (-\infty, 0), z \in [0, M_b + W_b - \tau) \), and \( z \in [M_b + W_b - \tau, M_c + W_c - \tau] \).
Case 1: \( z \in (-\infty, 0) \). If \( z \in (-\infty, -\tau) \), then \( \bar{g}(z) \leq -\tau = \bar{f}(z) \). Alternatively, if \( z \in [-\tau, 0) \), then \( \bar{g}(z) = \bar{f}(z) = z \).

Case 2: \( z \in [0, M_b + W_b - \tau) \). Observe that \( i(M_1 + W_1) \geq M_i + W_i \) because \( m_1 + w_1 \geq m_i + w_i \) for \( i = 0, \ldots, b \). First, suppose that \( z \in [i(M_1 + W_1), (i + 1)(M_1 + W_1) - \tau) \) for \( i = 0, \ldots, b - 1 \). Then,

\[
\bar{g}(z) = \bar{f}(M_i + W_i) \leq \bar{f}(z).
\]

where the last inequality follows from \( z \geq i(M_1 + W_1) \geq M_i + W_i \).

Alternatively, suppose that \( z \in [i(M_1 + W_1) - \tau, i(M_1 + W_1)) \) for \( i = 1, \ldots, b - 1 \). Then,

\[
\bar{g}(z) = z - i(M_1 + W_1) + i\tau \leq z - (M_i + W_i) + i\tau = \bar{f}(z).
\]

Case 3: \( z \in [M_b + W_b - \tau, M_c + W_c - \tau] \). The slope of \( \bar{g}(z) \) is either zero or one. If \( z \in [M_b + W_b - \tau, M_c + W_c - \tau - w] \), then \( \bar{g}(z) \leq \bar{f}(z) \) because the slope of \( \bar{f}(z) \) is one.

Alternatively, if \( z \in [M_c + W_c - \tau - w, M_c + W_c - \tau] \), then the slope of \( \bar{f}(z) \) is zero. Because \( \bar{g}(z) \) is monotone non-decreasing, it is sufficient to show that \( \bar{g}(M_c + W_c - \tau) \leq \bar{f}(M_c + W_c - \tau) = M_c + W_c - (M_b + W_b) + (b - 1)\tau \).
If $M_c + W_c - \tau \leq b(M_1 + W_1)$, then we are done because $\bar{g}(M_c + W_c - \tau) \leq \bar{g}(b(M_1 + W_1)) = (b - 1)\tau$. Consequently, assume that $M_c + W_c - \tau > b(M_1 + W_1)$.

Let $k_0 = \max \{ i \geq b \mid i(M_1 + W_1) < M_c + W_c - \tau \}$. Then,

$$
\begin{align*}
\bar{g}(M_c + W_c - \tau) & \leq (M_c + W_c - \tau) - k_0(M_1 + W_1) + k_0 \tau \\
& \leq (M_c + W_c - \tau) - b(M_1 + W_1) + b \tau \\
& \leq (M_c + W_c - \tau) - (M_b + W_b) + b \tau \\
& = \bar{f}(M_c + W_c - \tau).
\end{align*}
$$

Figure 3.6 illustrates the functions $f$, $\bar{f}$ and $\bar{g}$.

To perform sequence independent lifting for variables in $N \setminus C$, we construct valid inequalities for $P$ of the form
\[
\sum_{j \in C} x_j - \sum_{j \in C} \min \{\tau - m_j, w_j\} (1 - y_j) + \sum_{j \in (B^c \cup B^a)} [\bar{\alpha}_j (x_j - a_j) + \bar{\beta}_j (y_j - b_j)] \leq \lambda,
\]
where \(a_j = b_j = 0\) for \(j \in B^c\), and \(a_j = m_j\) and \(b_j = 1\) for \(j \in B^a\). For some \(i \in B^c\) and \(\bar{v} \in \mathbb{Z}_+\), let

\[
\bar{v}(M_1 + W_1) - \tau \leq w_i \leq (\bar{v} + 1)(M_1 + W_1) - \tau.
\]

Then, to determine \((\bar{\alpha}_i, \bar{\beta}_i)\), we consider three possible intervals for \(m_i + w_i\): \(m_i + w_i \leq (\bar{v} + 1)(M_1 + W_1) - \tau\), \((\bar{v} + 1)(M_1 + W_1) - \tau \leq m_i + w_i \leq (\bar{v} + 2)(M_1 + W_1) - \tau\), and \(m_i + w_i \geq (\bar{v} + r)(M_1 + W_1) - \tau\) for some \(r \geq 2\) where \(r \in \mathbb{Z}_+\). Let \(\tilde{J}^1_i\), \(\tilde{J}^2_i\), and \(\tilde{J}^3_i\) be the set of possible \((\bar{\alpha}_i, \bar{\beta}_i)\) for the three intervals, respectively.

For \(i \in B^c \cup B^a\), the coefficients \((\bar{\alpha}_i, \bar{\beta}_i)\) can be determined using \(h_i(z)\) and \(\bar{g}(z)\) similar to Section 3.1.3. Then, for \(i \in B^c\),

\[
\tilde{J}^1_i = \left\{ \left( \frac{\bar{g}(m_i + w_i) - \bar{g}(w_i)}{m_i}, \bar{g}(w_i) \right) \right\}
\]
and

\[
\tilde{J}^2_i = \left\{ (\bar{\alpha}_i, \bar{\beta}_i) \mid \pi_i^2 \in [0, 1] \right\}
\]
where

\[
\bar{\alpha}_i = \pi_i^2 \left( \frac{\bar{v}\tau - \bar{g}(w_i)}{(\bar{v} + 1)(M_1 + W_1) - \tau - w_i} \right)
\]

\[
+ (1 - \pi_i^2) \left( \frac{\bar{g}(m_i + w_i) - \bar{v}\tau}{m_i + w_i - (\bar{v} + 1)(M_1 + W_1) + \tau} \right)
\]

\[
\bar{\beta}_i = \pi_i^2 \bar{g}(w_i) + (1 - \pi_i^2)\bar{v}\tau
\]

\[
- (1 - \pi_i^2) \left( [(\bar{v} + 1)(M_1 + W_1) - \tau - w_i] \frac{\bar{g}(m_i + w_i) - \bar{v}\tau}{m_i + w_i - (\bar{v} + 1)(M_1 + W_1) + \tau} \right).
\]

Also,

\[
\tilde{J}^3_i = \{ \tilde{J}^3_{i1} \cup \tilde{J}^3_{i2} \},
\]

\[
\tilde{J}^3_{i1} = \left\{ (\bar{\alpha}_i, \bar{\beta}_i) \mid \pi_{i1}^3 \in [0, 1] \right\}
\]
such that
\[ \bar{\alpha}_i = \pi_{i1}^3 \left( \frac{\bar{v}\tau - \bar{g}(w_i)}{(\bar{v} + 1)(M_1 + W_1) - \tau - w_i} \right) + (1 - \pi_{i1}^3) \left( \frac{\tau}{M_1 + W_1} \right) \]

\[ \bar{\beta}_i = \pi_{i1}^3 \bar{g}(w_i) + (1 - \pi_{i1}^3) \left( -\tau + \frac{\tau(\tau + w_i)}{M_1 + W_1} \right), \]

and

\[ J_{i2}^3 = \{ (\bar{\alpha}_i, \bar{\beta}_i) \mid \pi_{i2}^3 \in [0, 1] \} \]

such that

\[ \bar{\alpha}_i = \pi_{i2}^3 \left( \frac{\tau}{M_1 + W_1} \right) + (1 - \pi_{i2}^3) \left( \frac{\bar{g}(m_i + w_i) - (\bar{v} + r - 1)\tau}{m_i + w_i - (\bar{v} + r)(M_1 + W_1) + \tau} \right) \]

\[ \beta_i = \pi_{i2}^3 \left( \bar{v}\tau - ((\bar{v} + 1)(M_1 + W_1) - \tau - w_i) \frac{\tau}{(M_1 + W_1)} \right) + (1 - \pi_{i2}^3)(\bar{v} + r - 1)\tau \]

\[ - (1 - \pi_{i2}^3) \left[ ((\bar{v} + r)(M_1 + W_1) - \tau - w_i) \frac{\bar{g}(m_i + w_i) - (\bar{v} + r - 1)\tau}{m_i + w_i - (\bar{v} + r)(M_1 + W_1) + \tau} \right]. \]

**Theorem 3.1.33**

\[
\sum_{j \in C} x_j - \sum_{j \in C} \min \{ \tau - m_j, w_j \} (1 - y_j) + \sum_{j \in B^l(x, y)} \alpha_j x_j + \sum_{j \in B^l(x, y)} \beta_j y_j + \sum_{j \in B^u(x, y)} (x_j - m_j) + \sum_{j \in B^u(x, y)} [-m_j - \bar{g}(-m_j - w_j)](y_j - 1) \leq \lambda \quad \text{(3.20)}
\]

is valid for \( P \) where

\[
(\bar{\alpha}_i, \bar{\beta}_i) \in \begin{cases} 
J^1_i, & w_i + m_i \leq (\bar{v} + 1)(M_1 + W_1) - \tau, \\
J^2_i, & (\bar{v} + 1)(M_1 + W_1) - \tau \leq m_i + w_i \leq (\bar{v} + 2)(M_1 + W_1) - \tau, \\
J^3_i, & \text{otherwise.}
\end{cases}
\]

**Proof** Similar to Theorem 3.1.24. \( \square \)

### 3.2 Separation

In this section, we study separation problem for (3.14). First, we give the formulation of separation problem. Then, we provide a heuristic procedure.
Given solution \((x^0, y^0)\), the separation problem is solved for (3.14) by finding sets \(C\), \(B^u\), and \(B^\ell\). To find these sets, we solve:

\[
(SP) \quad \max_{C,B^u,B^\ell} \sum_{j \in C} x_j^0 - \sum_{j \in C} \min\{\tau - m_j, w_j\} (1 - y_j^0) + \sum_{j \in B^\ell} \alpha_j x_j^0 + \sum_{j \in B^u} \beta_j y_j^0 + \sum_{j \in B^u} (x_j^0 - m_j) + \sum_{j \in B^u} [-m_j - g(-m_j - w_j)](y_j^0 - 1) \\
+ \sum_{j \in B^u} (m_j + w_j) + \sum_{j \in C} w_j
\]

\[(3.21)\]

s.t.

\[
\sum_{j \in C} w_j \leq d - \sum_{j \in B^u} (m_j + w_j), \quad \text{(3.22)}
\]

\[
\sum_{j \in C} (m_j + w_j) \geq d - \sum_{j \in B^u} (m_j + w_j). \quad \text{(3.23)}
\]

Because knapsack problem is a subproblem of \(SP\), \(SP\) is NP-hard. As a result, we develop a heuristic, \(SH\), to determine a constraint of the form of (3.14) that separates a fractional extreme point solution \((x^0, y^0)\) from the integer hull of \(P\). Heuristic \(SH\) selects the elements in \(C\), \(B^u\), and \(B^\ell\). To determine \(C\), we set \(B^u = B^\ell = \emptyset\) and try to maximize

\[
\sum_{j \in C} x_j^0 - \sum_{j \in C} \min\{\tau - m_j, w_j\} (1 - y_j^0) + \sum_{j \in C} w_j.
\]

\[(3.24)\]

A heuristic procedure is considered to maximize (3.24). For a given \((x^0, y^0)\), we arrange the values of \(x_j^0 - m_j y_j^0\) in non-increasing order. For a given \((x^0, y^0)\), (3.24) can be written as

\[
\sum_{j \in C} x_j^0 - \sum_{j \in C^\gamma} (\tau - m_j)(1 - y_j^0) - \sum_{j \in C^w} w_j (1 - y_j^0) + \sum_{j \in C} w_j
\]

\[
= \sum_{j \in C} x_j^0 - \sum_{j \in C^\gamma} (\tau - m_j) + \sum_{j \in C^\gamma} (\tau - m_j) y_j^0 - \sum_{j \in C^w} w_j + \sum_{j \in C^w} w_j y_j^0 + \sum_{j \in C} w_j
\]

\[
\geq \sum_{j \in C} x_j^0 + \sum_{j \in C^\gamma} (\tau - m_j) y_j^0 + \sum_{j \in C^w} w_j y_j^0
\]
\[ \geq \sum_{j \in C} (x_j^0 - m_j y_j^0) + \sum_{j \in C^\gamma} \tau y_j^0. \]  

From (3.25), we construct \( C \) selecting \( j \) with the largest \( x_j^0 - m_j y_j^0 \) because a large value for (3.24) is more likely to generate a separating hyperplane for \((x^0, y^0)\). Ties are broken by choosing \( j \) with the largest \( y_j^0 \).

Now, we present a proposition that suggests a way to construct \( B^u \).

**Proposition 3.2.1** Let \( g^1(z) \) be a superadditive lifting function for the inequality (3.14) generated by the sets \( C \setminus \{k\} \) and \( B^u \cup \{k\} \) for some \( k \in C^\gamma \). Then, \( g^1(z) \geq g(z) \).

**Proof** There are four cases: \( z \in (-\infty, 0] \), \( z \in [M_{i+k} + W_{i+k}, M_{i+k+1} + W_{i+k+1} - \tau] \) for \( i = 0, \ldots, b - k - 1 \), \( z \in [M_{i+k} + W_{i+k} - \tau, M_{i+k} + W_{i+k}] \) for \( i = 0, \ldots, b - k - 1 \), and \( z \in [M_{i+k} + W_{i+k} - \tau, M_c + W_c - \tau - (m_k + w_k)] \).

Case 1: \( z \in (-\infty, 0] \). Let \( w^1 = \min_{j \in C \setminus \{k\}} w_j \). If \( w^1 = w \), then \( g^1(z) = g(z) \). Otherwise, because \( \max\{\tau, M_c + W_c - M_b - W_b - w\} \geq \max\{\tau, M_c + W_c - M_b - W_b - w^1\} \), \( g^1(z) \geq g(z) \).

Case 2: \( z \in [M_{i+k} + W_{i+k}, M_{i+k+1} + W_{i+k+1} - \tau] \) for \( i = 0, \ldots, b - k - 1 \). Then, \( z \geq M_{i+k} + W_{i+k} \geq M_{i+k+1} + W_{i+k+1} - m_k - w_k \) and \( g(z) = (i + k)\tau \). If \( i + k + 1 < b \), then

\[
g^1(z) \geq g^1(M_{i+k+1} + W_{i+k+1} - m_k - w_k) = (i + k)\tau = g(z).\]
Alternatively, if \( i + k + 1 = b \), then

\[
g^1(z) = (i + k - 1)\tau + z - (M_{i+k+1} + W_{i+k+1} - m_k - w_k - \tau)
\]

\[
\geq (i + k)\tau
\]

\[
= g(z).
\]

Case 3: \( z \in [M_{i+k} + W_{i+k} - \tau, M_{i+k} + W_{i+k}] \) for \( i = 0, \ldots, b - k - 1 \) or \( z \in [M_{i+k} + W_{i+k} - \tau, M_c + W_c - \tau - (m_k + w_k)] \). Then, \( z \geq M_{i+k+1} + W_{i+k+1} - (m_k + w_k) - \tau + z - (M_{i+k} + W_{i+k} - \tau) \) and \( g(z) = (i + k - 1)\tau + z - (M_{i+k} + W_{i+k} - \tau) \). If \( i + k + 1 < b \), then

\[
g^1(z) \geq g^1(M_{i+k+1} + W_{i+k+1} - (m_k + w_k) - \tau + z - (M_{i+k} + W_{i+k} - \tau))
\]

\[
= (i + k - 1)\tau + z - (M_{i+k} + W_{i+k} - \tau)
\]

\[
= g(z).
\]

Alternatively, if \( i + k + 1 = b \), then

\[
g^1(z) = (i + k - 1)\tau + z - (M_{i+k+1} + W_{i+k+1} - m_k - w_k - \tau)
\]

\[
\geq (i + k - 1)\tau + z - (M_{i+k} + W_{i+k} - \tau)
\]

\[
= g(z). \quad \Box
\]

**Proposition 3.2.2** Suppose that \((x^0, y^0)\) is the optimal LP solution to \( P \) where \( x^0_k = m_k \) and \( y^0_k = 1 \) for some \( k \in C^\gamma \). The value of (3.21) when \( k \) is included in \( B^n \) is larger than when \( k \) is included in \( C \).

**Proof** Suppose \((x^0, y^0) \in P\) is an optimal LP solution where \( x^0_k = m_k \) and \( y^0_k = 1 \) for some \( k \in N \). If \( k \in C \), then (3.21) reduces to

\[
\sigma(x^0, y^0) = \sum_{j \in C \setminus \{k\}} [x^0_j - \min\{\tau - m_j, w_j\}(1 - y^0_j)] + x^0_k - \min\{\tau - m_k, w_k\}(1 - y^0_k)
\]

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Alternatively, if \( k \in B^a \), then (3.21) reduces to

\[
\sigma^1(x^0, y^0) = \sum_{j \in C \setminus \{k\}} \left[ x_j^0 - \min\{r - m_j, w_j\}(1 - y_j^0) \right] + \sum_{j \in B^u \setminus \{k\}} (\alpha_j^0 x_j^0 + \beta_j^0 y_j^0) + \sum_{j \in B^u \setminus \{k\}} (x_j^0 - m_j) + \sum_{j \in B^u \setminus \{k\}} [m_j - g(-m_j - w_j)](y_j^0 - 1) + (x_k^0 - m_k) + [m_k - g(-m_k - w_k)](y_k^0 - 1) + \sum_{j \in B^u \setminus \{k\}} (m_j + w_j) + m_k + \sum_{j \in C \setminus \{k\}} w_j + w_k.
\]

We show that including \( k \) in \( B^a \) maximizes (3.21) by showing that \( \sigma^0(x^0, y^0) - \sigma^1(x^0, y^0) \leq 0 \). Now,

\[
\sigma^0(x^0, y^0) - \sigma^1(x^0, y^0) = \sum_{j \in B^u \setminus \{k\}} [(\alpha_j^0 - \alpha_j^1)x_j^0 + (\beta_j^0 - \beta_j^1)y_j^0].
\]
Let $g^1(z)$ be superadditive lifting function for $C \setminus \{k\}$ and $B^u \cup \{k\}$. We have $g^1(z) \geq g(z)$ from Proposition 3.2.1. This implies that $(\alpha_0^j - \alpha_1^j) \leq 0$ and $(\beta_0^j - \beta_1^j) \leq 0$ for $j \in B^\ell$. Thus, $\sigma^0(x^0, y^0) - \sigma^1(x^0, y^0) \leq 0$. 

Suppose that $(x^0, y^0)$ is an optimal LP solution to $P$ where $x_k^0 = m_k$ and $y_k^0 = 1$ for some $k \in C^\gamma$. As a result of Proposition 3.2.2, the value of (3.21) when $k$ is included in $B^u$ is larger than when $k$ is included in $C$.

We now present SH.

**SH**

0. Reindex the variables such that $x_{j-1}^0 - m_j y_{j-1}^0 \geq x_j^0 - m_j y_j^0$ for $j = 2, \ldots, n$. Ties are broken by choosing $j$ with the largest $y_j^0$.

Set $C = \emptyset$, $B^u = \emptyset$, $B^\ell = \emptyset$ and $\ell = 1$.

1. Determine $i_1 = \text{argmin}\{i \in \{\ell, \ldots, n\} \mid \sum_{j=\ell}^i (m_j + w_j) \geq d\}$.

If such $i_1$ does not exist, then stop.

2. If $\sum_{j=\ell}^{i_1} w_j > d$, then set $t = i_1 + 1$ and go to Step 3.

Otherwise set $C = \{1, \ldots, i_1\}$ and go to Step 4.

3. Let $s_j(r) = \max\{m_j + w_j + s_{j-1}(r - w_j), s_{j-1}(r)\}$ for $r = 0, \ldots, d$ and $j = \ell, \ldots, t$.

If $r < 0$, then $s_j(r) = -\infty$.

Let $L$ be the set of indices which maximize $s_t(d)$.

If $d \leq s_t(d) < \max_{j \in L}\{m_j + w_j\} + d$, then set $C = L$ and $i_1 = t$ and go to Step 4.

Otherwise, let $t = t + 1$ and go to Step 3.

4. Set $B^\ell = N \setminus C$.

If $x_k^0 = m_k$ and $y_k^0 = 1$ for $k \in C$ and $d - (m_k + w_k) \geq \max\{j \in B^\ell \mid m_j + w_j\}$, then set $B^u = B^u \cup \{k\}$ and $C = C \setminus \{k\}$.

5. Calculate $d$, $\lambda$, $\tau$, and $\min\{\tau - m_j, w_j\}$ for $j \in C$.

6. For each $i \in B^\ell$ and for some $v, i \in \{1, \ldots, b\}$,

let $M_v + W_v - \tau \leq w_i \leq M_{v+1} + W_{v+1} - \tau$.

If $w_i + m_i \leq M_{v+1} + W_{v+1} - \tau$, then $(\alpha_i, \beta_i) = J_i^1$. 

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Otherwise, let \( M_{v+1} + W_{v+1} - \tau \leq m_i + w_i \leq M_b + W_b - \tau + \xi \), then

\[
(\alpha_i, \beta_i) = \arg\max \{(\alpha_j, \beta_j) \in J_i^2 \mid \alpha_j x^0_j + \beta_j y^0_j \}.
\]

Else if \( M_b + W_b - \tau < w_i \leq m_i + w_i \leq M_b + W_b - \tau + \xi \), then

\[
(\alpha_i, \beta_i) = \arg\max \{(\alpha_j, \beta_j) \in J_i^2 \mid \alpha_j x^0_j + \beta_j y^0_j \}.
\]

Otherwise \( (\alpha_i, \beta_i) = \arg\max \{(\alpha_j, \beta_j) \in J_i^2 \mid \alpha_j x^0_j + \beta_j y^0_j \} \).

For each \( i \in B^n \), \((\alpha_i, \beta_i) = (1, -m_i - g(-m_i - w_i)) \).

7. Substitute \( C, B^\ell, B^n, d, \lambda, \tau, \min\{\tau - m_j, w_j\} \) for \( j \in C \), and \((\alpha_i, \beta_i) \) for \( i \in B^\ell \cup B^n \) into (3.14).

8. If (3.14) is violated, then stop.
Otherwise, set \( \ell = \ell + 1 \) and go to Step 1.

For each iteration \( \ell \geq 1 \), Steps 1–8 are repeated. Steps 1–3 determine set \( C \).

In Step 1, we first determine \( i_1 \) which is the index of minimum \( m_j + w_j \) such that \( \sum_{j=1}^{i_1} m_j + w_j \geq d \). This guarantees that constraint (3.22) is satisfied in \( SP_2 \). In Step 2, if \( \sum_{j=1}^{i_1} w_j < d \), then constraint (3.23) is also satisfied and set \( C \) is found. Otherwise, let \( t = i_1 + 1 \) and go to Step 3. Step 3 uses a dynamic programming approach to determine the set of indices, \( L \subseteq \{\ell, \ldots, t\} \), where \( \sum_{j \in L} (m_j + w_j) \) is maximized. If \( d \leq \sum_{j \in L} (m_j + w_j) \leq \max_{j \in L} \{m_j + w_j\} + d \), then constraints (3.22) and (3.23) are satisfied and \( C = L \). Otherwise, increase \( t \) by one and repeat Step 3.

Step 3 guarantees \( C \) because in the worst case all elements in \( N \) are investigated. If \( x_k^0 = m_k \) and \( y_k^0 = 1 \), then \( k \) is removed from \( C \) and added to \( B^n \). Step 4 is repeated until either \( x_k^0 < m_k \) or \( y_k^0 < 1 \). In Step 5, the parameters \( d, \lambda, \tau \), and \( \min\{\tau - m_j, w_j\} \) for \( j \in C \) are calculated. Step 6 determines \((\alpha_i, \beta_i) \) for \( i \in B^\ell(x^0, y^0) \cup B^n(x^0, y^0) \) by selecting \((\alpha_i, \beta_i) \) and \( \pi \) such that \( \alpha_i x_i^0 + \beta_i y_i^0 \) is maximized. Selecting maximum \( \alpha_i x_i^0 + \beta_i y_i^0 \) for \( i \in B^\ell(x^0, y^0) \cup B^n(x^0, y^0) \) increases possibility of (3.14) is violated because the right hand side of (3.14) gets larger. In Step 7, (3.14) is formed as in
Proposition 3.1.24. In Step 8, If (3.14) is violated, then SH stops. Otherwise, ℓ is increased by one and SH iterates starting from Step 1.

3.3 Computational Study

Let

\[ P_T : \max \sum_{j \in N} (a_j x_j + c_j y_j) \]
\[ \text{s.t.} \quad (x, y) \in P. \]

To test the effectiveness of (3.14) for solving \( P_T \), we implement a branch-and-cut algorithm. The branch-and-cut procedure first solves an LP relaxation of \( P_T \) using CPLEX 10.0. Given a fractional point, we find separating hyperplanes using heuristic SH (see Section 3.2). Then, the inequalities generated by the separating hyperplanes are added to the LP relaxation of \( P_T \). If no separating hyperplane is found, then we branch on the fractional variable where the fractional portion is closest to 0.5. Also, the node with minimum bound is selected for branching. In all experiments, CPLEX cuts are disabled to isolate the impact due to inequalities discussed in this study. The computational experiments are performed on a Dell PC with 3.20GHZ Dual Core Processor, and 2 GB RAM.

Three sets of valid inequalities are considered in the computational study. These are flow cover inequalities (3.1) called \((FC)\), lifted flow cover inequalities (3.14) called \((LFCg)\), and lifted flow cover inequalities (3.20) called \((LFC\bar{g})\). The superadditive functions \( g(z) \) and \( \bar{g}(z) \) are used in the lifted flow cover inequalities \((LFCg)\) and \((LFC\bar{g})\), respectively.

We examine the following conjectures:
Conjecture 1 The separation procedure where the variable \( j \) is selected for \( C \) when \( x_j^0 - m_jy_j^0 \) is large generates valid inequalities that are stronger than the ones where the variable is randomly selected.

We believe that Conjecture 1 is true because the inequality (3.25) suggests that construct \( C \) selecting \( j \) with the largest \( x_j^0 - m_jy_j^0 \) increases the value of (3.24). A large value for (3.24) is more likely to generate a separating hyperplane for \((x^0, y^0)\).

Conjecture 2 The separation procedure where \( B^u \neq \emptyset \) generates valid inequalities that are stronger than those where \( B^u = \emptyset \).

From Proposition 3.2.1, setting \( B^u \neq \emptyset \) increases the value of \( g(z) \) for all \( z \in (-\infty, \infty) \). Large \( g \) provides strong valid inequalities.

Conjecture 3 The separation procedure where \( m \) and \( w \) have small variances generates valid inequalities that are stronger than those where \( m \) and \( w \) have large variances.

For \( 0 \leq m_j + w_j \leq M_1 + W_1 - \tau \), the lifting coefficients of the variable \( j \) are zero. Moreover, if \( m_j + w_j < M_b + W_b - \tau \), then the lifting coefficients for \( j \) are close to the exact lifting coefficients. Consequently, if \( m_j + w_j \) is close to \( M_1 + W_1 \), then the possibility of \( m_j + w_j \) being in \([M_1 + W_1 - \tau, M_b + W_b - \tau]\) for \( i = 0, \ldots, b \) is greater. When \( m_j \) and \( w_j \) have small variances for all \( j \in C \), \( m_j + w_j \) has a small variance. Thus, there is high possibility that \( m_j + w_j \) is close to \( M_1 + W_1 \).

Conjecture 4 The separation procedure where \( m \) and \( w \) are highly correlated generates valid inequalities that are as strong as those when they are not.

We believe that Conjecture 4 is true because the lifting coefficients are determined based on the values \( w_j \) and \( m_j + w_j \). Thus, the strength of the valid inequalities does not depend on whether \( m_j \) and \( w_j \) are correlated.
Conjecture 5  The separation procedure where $\bar{d} \leq \sum_{j \in N}(m_j + w_j)/2$ generates stronger valid inequalities than when $\bar{d} > \sum_{j \in N}(m_j + w_j)/2$.

Because $\bar{d} - \sum_{j \in C} w_j > 0$, the expected number of variables that are in $C$ increases with $\bar{d}$. Therefore, if $\bar{d}$ is large, then few variables in $N \setminus C$ can be lifted. Alternatively, if $\bar{d}$ is small, then the number of elements in $C$ is small and more variables in $N \setminus C$ can be lifted.

Computational experiments are run based on two different policies for constructing $C$. These are

C1. Random.

C2. Select those variables where $x_{j-1}^0 - m_{j-1}y_{j-1}^0$ is large.

In the computational experiments, we also consider two different policies for constructing $B^u$. These are

B1. $B^u = \emptyset$.

B2. $B^u \neq \emptyset$.

Define $U[\ell, u]$ to be the discrete uniform distribution over the interval $[\ell, u]$. To consider the effects of the variability of $m$ and $w$, let

\[ m_j \sim U[1, 99] \text{ or } m_j \sim U[40, 60] \]

and

\[ w_j \sim U[1, 99] \text{ or } w_j \sim U[40, 60]. \]

To examine the effects of correlation between the $m$ and $w$, let

\[ m_j = \rho w_j + (\sqrt{1-\rho^2})\eta + t \text{ where } \eta \sim U[1, 99], \]

$\rho \in \{-0.9, -0.5, 0.5, 0.9\}$ and $t \in \{0, -0.99\}$ (Neter et al. 2003). If $\rho < 0$, then let $t = -0.99$. Otherwise, $t = 0$. Note that if $\rho = 0$, then $m_j \sim U[1, 99]$. 

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Let $E(\cdot)$ be the expectation operator. The relationship of $\bar{d}$ and $\sum_{j \in N} m_j + w_j$ may influence the performance of SH. Consequently, we consider

D1. $\bar{d} \sim U[E(m_j + w_j), (n - 1)E(m_j + w_j)]$.

D2. $\bar{d} \sim U[E(m_j + w_j), (n + 1)E(m_j + w_j)/3]$.

D3. $\bar{d} \sim U[(n + 1)E(m_j + w_j)/3, (2n - 1)E(m_j + w_j)/3]$.

D4. $\bar{d} \sim U[(2n - 1)E(m_j + w_j)/3, (n - 1)E(m_j + w_j)]$.

For each of our experiments, ten instances are generated. The values of $a_j$ and $c_j$ are generated from $U[1, 99]$.

Let $Z^L$ be the objective value of the initial LP relaxation and $Z^1$ be the value of the LP solution after cuts are added at the root node. The value of the optimal integer solution is $Z^*$. The proportional improvement of the objective value after cuts are added is $(Z^1 - Z^*)/(Z^L - Z^*)$. The “No. of Cuts” and “No. of Nodes” are the average number of cuts added and the average number of branch-and-cut nodes explored, respectively.

To test Conjectures 1 and 2, the data is generated according to Table 3.1. For each of these experiments, $n = 250$.

<table>
<thead>
<tr>
<th>Case Name</th>
<th>Exp. Name</th>
<th>C Policy</th>
<th>$B^a$ Policy</th>
<th>$m$ and $w$</th>
<th>$\rho$</th>
<th>$\bar{d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base Case</td>
<td>T01</td>
<td>C1</td>
<td>B1 B2</td>
<td>$m_j \sim U[1, 99], w_j \sim U[1, 99]$</td>
<td>0</td>
<td>D1</td>
</tr>
<tr>
<td></td>
<td>T02</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conjectures 1 and 2</td>
<td>T11</td>
<td>C2</td>
<td>B1 B2</td>
<td>$m_j \sim U[1, 99], w_j \sim U[1, 99]$</td>
<td>0</td>
<td>D1</td>
</tr>
<tr>
<td></td>
<td>T12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The results of the experiments, T01, T02, T11, and T12, are reported in Table 3.3. For $LFCg$ and $LFC\bar{g}$, the average proportional improvements of the objectives for the experiments in T01 and T02 are smaller than the objectives for the experiments in T11 and T12. The average number of nodes investigated in T01 and T02 is also larger than in T11 and T12. Observe that the average solution times in T01 and T02 are higher than the average solution times in T11 and T12. Thus, Conjecture 1 is supported.

For Conjecture 2, we compare the experiments T11 and T12 since C2 is the best policy for selecting variables for the set $C$ from Conjecture 1. For $LFCg$ and $LFC\bar{g}$, the average proportional improvements of the objectives in T11 are smaller than the average proportional improvements of the objectives in T12. Also, the average number of nodes investigated and the average number of cuts generated in T11 are larger than in T12. Note that the average solution times in T11 are larger than the average solution times in T12. Thus, Conjecture 2 is supported.

From Conjecture 1, because the separation procedure generates strong valid inequalities when the variable $j$ is selected for $C$ when $x_j^0 - m_j y_j^0$ is large, in the remaining experiments, we assume that set $C$ is constructed from those variables.

From Conjecture 2, the separation procedure where $B^u \neq \emptyset$ generates valid inequalities that are stronger than the ones where $B^u = \emptyset$. Thus, in the remaining experiments, we assume that $B^u \neq \emptyset$.

To test Conjectures 3, 4, and 5, the data is generated according to Table 3.3. For each of these experiments, $n = 250$.

The results of the experiments for Conjectures 3, 4, and 5 are reported in Table 3.4. We observe that the average proportional improvements for $LFCg$ and $LFG\bar{g}$ in T31,
Table 3.2: Test results for Conjectures 1 and 2

(a) 

\[
\frac{(Z^1 - Z^*)}{(Z^{LP} - Z^*)} \quad \text{No. of Cuts}
\]

<table>
<thead>
<tr>
<th>Exp.</th>
<th>(LFC_g) Mean</th>
<th>(LFC_g) Std. Dev.</th>
<th>(LFC_{\bar{g}}) Mean</th>
<th>(LFC_{\bar{g}}) Std. Dev.</th>
<th>(LFC_g) Mean</th>
<th>(LFC_g) Std. Dev.</th>
<th>(LFC_{\bar{g}}) Mean</th>
<th>(LFC_{\bar{g}}) Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T01</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>T02</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>T11</td>
<td>0.377</td>
<td>0.304</td>
<td>0.374</td>
<td>0.307</td>
<td>81.400</td>
<td>76.193</td>
<td>119.600</td>
<td>114.862</td>
</tr>
<tr>
<td>T12</td>
<td>0.903</td>
<td>0.296</td>
<td>0.820</td>
<td>0.296</td>
<td>32.000</td>
<td>27.970</td>
<td>35.400</td>
<td>30.062</td>
</tr>
</tbody>
</table>

(b) 

<table>
<thead>
<tr>
<th>Exp.</th>
<th>(LFC_g) Mean</th>
<th>(LFC_g) Std. Dev.</th>
<th>(LFC_{\bar{g}}) Mean</th>
<th>(LFC_{\bar{g}}) Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T01</td>
<td>221.000</td>
<td>225.140</td>
<td>221.000</td>
<td>225.140</td>
</tr>
<tr>
<td>T02</td>
<td>221.000</td>
<td>225.140</td>
<td>221.000</td>
<td>225.140</td>
</tr>
<tr>
<td>T11</td>
<td>25.000</td>
<td>22.591</td>
<td>31.000</td>
<td>41.061</td>
</tr>
<tr>
<td>T12</td>
<td>3.800</td>
<td>2.672</td>
<td>6.800</td>
<td>7.377</td>
</tr>
</tbody>
</table>

(c) 

<table>
<thead>
<tr>
<th>Exp.</th>
<th>Time (sec) (LFC_g) Mean</th>
<th>Time (sec) (LFC_g) Std. Dev.</th>
<th>Time (sec) (LFC_{\bar{g}}) Mean</th>
<th>Time (sec) (LFC_{\bar{g}}) Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T01</td>
<td>739.164</td>
<td>501.756</td>
<td>1023.148</td>
<td>878.181</td>
</tr>
<tr>
<td>T02</td>
<td>355.283</td>
<td>329.982</td>
<td>375.931</td>
<td>349.340</td>
</tr>
<tr>
<td>T11</td>
<td>1.651</td>
<td>1.442</td>
<td>2.434</td>
<td>2.307</td>
</tr>
<tr>
<td>T12</td>
<td>0.589</td>
<td>0.621</td>
<td>0.672</td>
<td>0.735</td>
</tr>
</tbody>
</table>

T32, and T33 are smaller than in T34. Observe that the solution times for \(LFC_g\) and \(LFG_{\bar{g}}\) in T34 are the smallest. Thus, the separation procedure where \(m\) and \(w\)
have small variances generates valid inequalities that are stronger than those where
$m$ and $w$ have large variances as in Conjecture 3.

For Conjecture 4, from the experiments T41, T42, T43, T44 and T45, as the
correlation coefficient increases, the average proportional improvements, the average
number of nodes and the average number of cuts for $LFCg$ and $LFG\bar{g}$ do not increase
or decrease. Also, observe that the solution times for $LFCg$ and $LFG\bar{g}$ in T41, T42,
T43, T44 and T45 are close. Thus, Conjecture 4 is supported.

The average proportional improvements for $LFCg$ and $LFG\bar{g}$ in T53 are smaller
than the average proportional improvements for $LFCg$ and $LFG\bar{g}$ in T51 and T52.
Also, the average proportional improvements in T51 is slightly better than the average
proportional improvements in T52. Thus, Conjecture 5 is supported. Observe that
the solution times for $LFCg$ and $LFG\bar{g}$ increase for T51, T52, and T53, respectively.
Table 3.4: Test results for Conjectures 3, 4, and 5

(a)

\[
\frac{Z^1 - Z^*}{Z^{LP} - Z^*}
\]

<table>
<thead>
<tr>
<th>Exp.</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Mean</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T31</td>
<td>0.903</td>
<td>0.296</td>
<td>0.820</td>
<td>0.296</td>
<td>32.000</td>
<td>27.970</td>
<td>35.400</td>
<td>30.062</td>
</tr>
<tr>
<td>T32</td>
<td>0.928</td>
<td>0.082</td>
<td>0.879</td>
<td>0.116</td>
<td>21.600</td>
<td>19.445</td>
<td>50.600</td>
<td>60.289</td>
</tr>
<tr>
<td>T33</td>
<td>0.926</td>
<td>0.227</td>
<td>0.874</td>
<td>0.124</td>
<td>28.600</td>
<td>33.598</td>
<td>47.800</td>
<td>59.968</td>
</tr>
<tr>
<td>T34</td>
<td>0.964</td>
<td>0.150</td>
<td>0.905</td>
<td>0.194</td>
<td>22.600</td>
<td>19.442</td>
<td>45.400</td>
<td>39.997</td>
</tr>
<tr>
<td>T41</td>
<td>0.897</td>
<td>0.202</td>
<td>0.816</td>
<td>0.200</td>
<td>34.800</td>
<td>34.571</td>
<td>39.200</td>
<td>37.201</td>
</tr>
<tr>
<td>T42</td>
<td>0.889</td>
<td>0.229</td>
<td>0.809</td>
<td>0.282</td>
<td>36.400</td>
<td>34.809</td>
<td>41.600</td>
<td>37.184</td>
</tr>
<tr>
<td>T43</td>
<td>0.903</td>
<td>0.296</td>
<td>0.820</td>
<td>0.296</td>
<td>32.000</td>
<td>27.970</td>
<td>35.400</td>
<td>30.062</td>
</tr>
<tr>
<td>T44</td>
<td>0.911</td>
<td>0.257</td>
<td>0.825</td>
<td>0.290</td>
<td>31.800</td>
<td>30.521</td>
<td>34.200</td>
<td>33.568</td>
</tr>
<tr>
<td>T45</td>
<td>0.909</td>
<td>0.245</td>
<td>0.823</td>
<td>0.253</td>
<td>31.940</td>
<td>24.956</td>
<td>34.670</td>
<td>28.385</td>
</tr>
<tr>
<td>T51</td>
<td>0.911</td>
<td>0.151</td>
<td>0.771</td>
<td>0.149</td>
<td>16.800</td>
<td>11.032</td>
<td>29.400</td>
<td>31.596</td>
</tr>
<tr>
<td>T52</td>
<td>0.854</td>
<td>0.169</td>
<td>0.722</td>
<td>0.275</td>
<td>34.800</td>
<td>20.266</td>
<td>57.200</td>
<td>31.324</td>
</tr>
<tr>
<td>T53</td>
<td>0.785</td>
<td>0.307</td>
<td>0.583</td>
<td>0.372</td>
<td>42.000</td>
<td>26.315</td>
<td>71.200</td>
<td>62.763</td>
</tr>
</tbody>
</table>

(b)

<table>
<thead>
<tr>
<th>Exp.</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Mean</th>
<th>Std. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T31</td>
<td>3.800</td>
<td>2.672</td>
<td>6.800</td>
<td>7.377</td>
<td>0.589</td>
<td>0.621</td>
<td>0.672</td>
<td>0.735</td>
</tr>
<tr>
<td>T32</td>
<td>3.400</td>
<td>2.980</td>
<td>8.200</td>
<td>7.121</td>
<td>0.359</td>
<td>0.379</td>
<td>0.495</td>
<td>0.419</td>
</tr>
<tr>
<td>T33</td>
<td>4.200</td>
<td>3.921</td>
<td>9.400</td>
<td>10.522</td>
<td>0.373</td>
<td>0.303</td>
<td>0.482</td>
<td>0.417</td>
</tr>
<tr>
<td>T34</td>
<td>6.600</td>
<td>7.286</td>
<td>12.400</td>
<td>13.578</td>
<td>0.206</td>
<td>0.087</td>
<td>0.316</td>
<td>0.278</td>
</tr>
<tr>
<td>T41</td>
<td>4.132</td>
<td>3.302</td>
<td>7.520</td>
<td>9.128</td>
<td>0.640</td>
<td>0.767</td>
<td>0.744</td>
<td>0.662</td>
</tr>
<tr>
<td>T42</td>
<td>4.322</td>
<td>3.325</td>
<td>7.990</td>
<td>9.124</td>
<td>0.669</td>
<td>0.772</td>
<td>0.789</td>
<td>0.661</td>
</tr>
<tr>
<td>T43</td>
<td>3.800</td>
<td>2.672</td>
<td>6.800</td>
<td>7.377</td>
<td>0.589</td>
<td>0.621</td>
<td>0.672</td>
<td>0.535</td>
</tr>
<tr>
<td>T44</td>
<td>3.776</td>
<td>2.915</td>
<td>6.569</td>
<td>8.237</td>
<td>0.585</td>
<td>0.677</td>
<td>0.649</td>
<td>0.597</td>
</tr>
<tr>
<td>T45</td>
<td>3.792</td>
<td>2.380</td>
<td>6.659</td>
<td>6.965</td>
<td>0.587</td>
<td>0.554</td>
<td>0.658</td>
<td>0.505</td>
</tr>
<tr>
<td>T51</td>
<td>4.800</td>
<td>3.286</td>
<td>8.600</td>
<td>5.550</td>
<td>0.259</td>
<td>0.248</td>
<td>0.854</td>
<td>0.811</td>
</tr>
<tr>
<td>T52</td>
<td>5.200</td>
<td>3.633</td>
<td>11.000</td>
<td>6.000</td>
<td>0.686</td>
<td>0.643</td>
<td>0.968</td>
<td>0.941</td>
</tr>
<tr>
<td>T53</td>
<td>5.400</td>
<td>4.561</td>
<td>11.400</td>
<td>8.173</td>
<td>0.992</td>
<td>0.948</td>
<td>1.293</td>
<td>1.002</td>
</tr>
</tbody>
</table>
To observe the effect of varying $n$, let $n \in \{50, 100, 250, 500, 1000\}$. For this experiment, the policies are C2, B2, $m_j \sim U[1, 99]$, $w_j \sim U[1, 99]$, $\rho = 0$, and $\bar{d}$ is generated according to D1. We solve this data set using branch-and-bound and using branch-and-cut with the valid inequalities $FC$, $LFCg$, and $LFC\bar{g}$. Tables 3.5 and 3.6 show the results of the experiment. As $n$ increases, the average solution time increases in the branch-and-bound procedure for $n = 50, 100, 250, 1000$.

From Tables 3.5 and 3.6, because $n$ and $\bar{d}$ increase at the same rate, we observe that there is no significant relationship between $n$ and the average proportional improvement. However, for $n = 50, 100, 250, 1000$, the average number of cuts and the solution times increase for $LFCg$ and $LFC\bar{g}$.

Tables 3.5 and 3.6 report that both $LFCg$ and $LFC\bar{g}$ perform better over all measures than $FC$. The average percentage improvement gaps for $LFCg$ cuts is approximately 8% greater than that of $LFC\bar{g}$ cuts. Further, $LFCg$ performs slightly better than $LFC\bar{g}$. The $LFCg$ cuts requires less than 10 nodes for most of the data sets. Thus, $LFCg$ cuts almost all of the integrality gap for most instances and reduces the computational effort significantly.

### 3.4 Final Remarks

This research develops a sequence independent lifting procedure to lift the flow cover inequalities for the polytope $P$. We establish that computing lifting function for $P$ is NP-hard. Thus, we determine a lower bound function for the lifting function that is better lower bound than one that can be found by using the LP relaxation. We show that the lower bound function is not superadditive. Therefore, we develop a superadditive function that is not dominated by any other superadditive function. We
Table 3.5: Test results for branch-and-bound (B&B) and branch-and-cut

(a)

\[
\frac{Z^1 - Z^*}{Z^{LP} - Z^*}
\]

<table>
<thead>
<tr>
<th>$n$</th>
<th>FC</th>
<th>LFC$_g$</th>
<th>LFC$_\bar{g}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
</tr>
<tr>
<td>50</td>
<td>0.281</td>
<td>0.146</td>
<td>0.882</td>
</tr>
<tr>
<td>100</td>
<td>0.441</td>
<td>0.424</td>
<td>0.922</td>
</tr>
<tr>
<td>250</td>
<td>0.134</td>
<td>0.044</td>
<td>0.903</td>
</tr>
<tr>
<td>500</td>
<td>0.453</td>
<td>0.338</td>
<td>0.845</td>
</tr>
<tr>
<td>1000</td>
<td>0.192</td>
<td>0.209</td>
<td>0.757</td>
</tr>
</tbody>
</table>

(b)

No. of Cuts

<table>
<thead>
<tr>
<th>$n$</th>
<th>FC</th>
<th>LFC$_g$</th>
<th>LFC$_\bar{g}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
</tr>
<tr>
<td>50</td>
<td>13.800</td>
<td>19.015</td>
<td>8.800</td>
</tr>
<tr>
<td>250</td>
<td>107.200</td>
<td>98.695</td>
<td>32.000</td>
</tr>
<tr>
<td>500</td>
<td>46.400</td>
<td>50.920</td>
<td>25.400</td>
</tr>
<tr>
<td>1000</td>
<td>265.000</td>
<td>291.412</td>
<td>91.440</td>
</tr>
</tbody>
</table>

(c)

No. of Nodes

<table>
<thead>
<tr>
<th>$n$</th>
<th>B&amp;B</th>
<th>FC</th>
<th>LFC$_g$</th>
<th>LFC$_\bar{g}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
<td>Std. Dev.</td>
</tr>
<tr>
<td>50</td>
<td>27.200</td>
<td>35.195</td>
<td>22.000</td>
<td>34.230</td>
</tr>
<tr>
<td>100</td>
<td>142.600</td>
<td>71.630</td>
<td>134.600</td>
<td>74.865</td>
</tr>
<tr>
<td>250</td>
<td>221.000</td>
<td>225.140</td>
<td>174.600</td>
<td>184.366</td>
</tr>
<tr>
<td>500</td>
<td>112.600</td>
<td>141.205</td>
<td>60.600</td>
<td>64.131</td>
</tr>
<tr>
<td>1000</td>
<td>488.600</td>
<td>400.785</td>
<td>393.400</td>
<td>465.286</td>
</tr>
</tbody>
</table>
Table 3.6: B&B and branch-and-cut solution times

<table>
<thead>
<tr>
<th>n</th>
<th>Time (sec.)</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B&amp;B</td>
<td>FC</td>
<td>LFCg</td>
<td>LFCg</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
</tr>
<tr>
<td>50</td>
<td>0.184</td>
<td>0.054</td>
<td>0.271</td>
<td>0.195</td>
<td>0.091</td>
<td>0.025</td>
<td>0.128</td>
</tr>
<tr>
<td>100</td>
<td>0.285</td>
<td>0.129</td>
<td>0.550</td>
<td>0.332</td>
<td>0.129</td>
<td>0.085</td>
<td>0.191</td>
</tr>
<tr>
<td>250</td>
<td>0.975</td>
<td>1.034</td>
<td>1.739</td>
<td>2.261</td>
<td>0.589</td>
<td>0.621</td>
<td>0.672</td>
</tr>
<tr>
<td>500</td>
<td>1.986</td>
<td>1.150</td>
<td>1.684</td>
<td>1.868</td>
<td>0.244</td>
<td>0.802</td>
<td>0.608</td>
</tr>
</tbody>
</table>

also show that the valid inequalities (3.14) dominate the valid inequalities of Shebalov and Klabjan (2006). A separation procedure is developed for the lifted inequalities. We investigate conditions where the separation problem provides strong cuts. Then, because the separation problem is NP-hard, we develop a heuristic procedure based on properties of the problem.

To test the effectiveness of (3.14) in solving \( P \), we implement branch-and-cut algorithm that incorporates these inequalities and we perform computational experiments. Computational study indicates that our sequence independent procedure produces stronger valid inequalities (3.14) than the flow cover inequalities (3.1). Also, the separation procedure is efficient.

Future research could consider applying the sequence independent lifting procedure to solve large scale mixed integer problems such as network flow problems which contain \( P \) as subproblem.
CHAPTER 4

SUMMARY AND FUTURE RESEARCH

In this dissertation, two new sets of valid inequalities for \( P \) are developed. The first set of valid inequalities is derived by adding a new subset of variables to the flow cover inequalities. We show that, under some conditions, these inequalities are facet defining. Computational study shows the effectiveness of the valid inequalities.

The second set of valid inequalities is generated by sequence independent lifting of the flow cover inequalities. Because determining a lifting function for \( P \) is NP-hard, we find a lower bound function for the lifting function that provides better lower bound than one generated by using the LP relaxation. Then, a superadditive function which produces strong cuts is developed. We show that this superadditive function is not dominated by any other superadditive function and it is maximal. Our resulting inequalities dominate the inequalities of Shebalov and Klabjan (2006). A separation procedure is developed for the lifted inequalities. Computational study shows that our lifting procedure provide considerably stronger cuts than the flow cover inequalities.

There are several directions for future research. First is to extend the existing classes of facets and to establish new classes of facets. Another direction is to extend our the sequence independent lifting procedure for the 0-1 mixed multiple knapsack
polytope to develop valid inequalities. A final direction is to solve the large scale mixed integer problems (e.g. the emergency evacuation planning problem) which contain $P$ as subproblem using the two sets of valid inequalities that we develop.
REFERENCES


