THE EFFECTS OF DIFFERING TECHNOLOGICAL APPROACHES TO CALCULUS ON STUDENTS' USE AND UNDERSTANDING OF MULTIPLE REPRESENTATIONS WHEN SOLVING PROBLEMS

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

The Ohio State University

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College of Education
To my loving wife Lois. Without her support,
this would not have been possible.
ACKNOWLEDGEMENTS

This work marks the completion of a dream for me. To all my family, friends, and teachers who made this possible, I can only say thank you.

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FIELDS OF STUDY

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CHAPTER I

THE PROBLEM

During the last decade there has been a great deal of criticism concerning the undergraduate calculus curriculum. This criticism actually began much earlier and resulted in the 1983 Sloan Foundation-sponsored conference held at Williams College in Williamstown, Massachusetts. At this conference the status of calculus in the core undergraduate mathematics curriculum was challenged by supporters of discrete mathematics. In response to this challenge, another Sloan Foundation-sponsored conference, called the Tulane Conference, was organized in an attempt to gather input on how to revitalize calculus and reaffirm its importance in the college mathematics curriculum.

At the Tulane Conference, the participants in the Methods Workshop (Schoenfeld, et al., 1986) suggested specific goals for calculus instruction, including emphasizing conceptual understanding and exposing students to a wide variety of problems and problem situations.

Instruction should be aimed at conceptual understanding, and at the developing in students the ability to apply the subject matter they have studied with flexibility and resourcefulness. (p. xvi)
The Methods Workshop participants advocated the (appropriate) use of relevant technologies, in particular, the use of calculators and computers, in calculus instruction in order to introduce students to the contemporary mathematical tools that they will or could be using. This view of instruction is reflected in the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989), and *Everybody Counts: A Report to the Nation on the Future of Mathematics Education* (National Research Council, 1989), both of which call for an emphasis on conceptual understanding in the mathematics instruction and on making appropriate use of technology. As Tucker (1990, p. 8) states, "The Curriculum and Evaluation Standards for School Mathematics ... is in the same spirit as calculus reform."

**Statement of the Problem**

Since the Tulane Conference, the calculus reform movement has been fueled by concern over issues such as the difficulties experienced by students with some of the ideas of calculus (Davis & Vinner, 1986; Ferrini-Mundy & Graham, 1991a; Selden, Mason, & Selden, 1989; Tall, 1990, 1992; Tall & Vinner, 1981; Williams, 1991), the high failure rate of calculus students (Steen, 1987b), the role of calculus in filtering students out of, rather than pumping them into, science and engineering fields (White, 1987). One of the more
critical issues has been determining the role of technology in the revision and revitalization of the calculus curriculum (Porter, 1988; Shumway, 1990; Small & Hosack, 1986). Hughes-Hallet (1992) may have best described technology’s role in calculus reform, stating that:

At the moment we have a golden opportunity for reform, and we should use it to refocus our courses on mathematics - on mathematical thinking rather than on mindless computation. This should be the central mission of calculus reform. Looking at our educational system from this point of view suggests that technology is neither the problem nor the cure for our problems. But technology is an important tool in attacking these problems. (pg. 61)

Another crucial issue that has impacted not just revision of calculus curriculum and instruction but mathematics curriculum and instruction in general has been deciding how to best make use of multiple representations of mathematical concepts. An emerging theoretical view on mathematical learning that has been growing in significance is that multiple representations of concepts can be utilized to help students develop deeper, more flexible understandings (Hiebert & Carpenter, 1992; Kaput, 1989a; Skemp, 1987). In addition, the opportunity to provide students greater access to multiple representations of mathematical concepts has increased with advancements in technology (Fey, 1989; Goldenberg, 1987, Kaput, 1992). The philosophy
behind the Harvard Project, known as the "Rule of Three," is one that emphasizes multiple representations. Hughes-Hallet (1991) states:

This [philosophy] is based on the belief that in order to understand an idea, students need to see it from several points of view, and to build a web of connections between the different viewpoints. I believe that in calculus most of the ideas should be presented in three ways: graphically and numerically, as well as in the traditional algebraic way. Technology is invaluable here. (p. 33)

Instructional practices that involve the use of multiple representations are not employed simply because technology now makes multiple representations more readily accessible, but because of the potential benefits associated with their use. Dufour-Janvier, Bednarz, and Belanger (1987) point out that multiple representations are inherent in mathematics, provide multiple concretizations of a concept, may be used to mitigate difficulties students have in accomplishing certain tasks, and may make mathematics more interesting to students. Kaput (1989b) indicates that multiple representations allow for the suppression of some aspects of complex concepts and the accentuation of other aspects thus helping to facilitate the cognitive linking of representations.

One major concern with the use of multiple representations in teaching mathematics is exactly how to utilize them in order to reap the greatest benefits. Dufour-Janvier et al. (1987) point out that the use of representations
in current instructional practices often does not lead to the attainment of desired objectives such as having students perceive representations as mathematical tools for solving problems and helping students in the "construction" of a concept by viewing common properties and differences between representations of the concept. They go on to state:

If we want representations to be really useful in learning mathematical concepts, a number of questions will have to be considered: Which representations should be retained? Are there representations that are more appropriate than others for developing a concept? How should the representations be used? In which contexts? [italics added] What are the difficulties and the children's conceptions that need to be taken into account when a representation is used? Are there representations that are more appropriate to the level of development of the child and to where he is in regard to the learning of mathematics. (Dufour-Janvier et al., 1987, p. 116)

One use of multiple representations advocated by some of the current calculus reform projects (Tucker, 1990) involves having students interpret and solve sequences of related problems which are designed to introduce students to new material and provided them with a context for investigation and discussion by exploring different possible representations of the situations and concepts described in the problems. Such is the case with the Calculus & Mathematica course being developed at the University of Illinois and the Ohio
State University. The course is being designed according to the philosophy that the best way for students to become involved in calculus is to immerse them in a collection of well chosen problems (Brown, Porta, & Uhl, 1990). This instructional philosophy is in keeping with the recommendations made by the NRC (1989), which include a call for increased focus on:

- Seeking solutions, not just memorizing procedures;
- Exploring patterns, not just learning formulas;
- Formulating conjectures, not just doing exercises. (p. 84)

Concerns about the use of technology, the use of multiple representations, and numerous other issues has generated great interest in the revamping of both the calculus curriculum and calculus instruction (Douglas, 1986d; Shumway, 1989; Steen, 1987a) and has led to the creation of numerous calculus reform projects (Tucker, 1990). The majority of these projects are based on calculus instruction emphasizing conceptual understanding and problem-solving ability development rather than skill development and memorization. Many of the projects utilize computer software and calculators with graphic and symbol manipulation capabilities, stress the use of multiple representations of concepts, or have students interpret/solve specific types of problems. The present study investigated how students in Calculus & Mathematica, a course where technology is used extensively and instruction emphasizes multiple
representations of concepts and the interpreting and solving of problems, differ from (a) students in a traditional calculus class and (b) students in a calculus course where graphics calculators were used extensively to emphasize graphical representations of concepts in their abilities to use and understand multiple representations when solving calculus problems.

Need for Study

Many current calculus reform projects (Tucker, 1990) make use of technology, emphasize multiple representations of concepts, or have students interpret/solve specific types of related problems, but there is little, if any, empirical evidence of the effectiveness of these approaches. There is a growing body of research on the effects of using different forms of technology in calculus instruction (e.g., Beckmann, 1988; Crocker, 1991; Emese, 1993; Heid, 1985; Palmiter, 1986; Stout, 1991; Tall, 1986). Much of this research has focused on using technology to increase the emphasis on conceptual understanding while decreasing the emphasis on routine skills (e.g., Hawker, 1987; Heid, 1985; Judson, 1989, Palmiter, 1986; Schrock, 1990).

Little of the research on calculus instruction has studied programs like Calculus & Mathematica (Crocker, 1991; Park, 1993) which emphasize technology, multiple representations, and solving/interpreting problems, or
focused on the relationship between use of technology and the use of multiple representations (Beckmann, 1988; Hart, 1991) or technology use in conjunction with students solving and interpreting problems. This study sought to gather empirical evidence of the effectiveness of calculus instruction that emphasizes multiple representations of concepts, interpreting and solving of problems, and appropriate use of technology, in developing students' abilities to use and understand multiple representations, or multiple techniques, for solving calculus problems. Such evidence lends support to the type of calculus curriculum and instruction advocated by many calculus reform projects, in particular, those project following a philosophy similar to that of Calculus & Mathematica.

Research Questions

This study investigated the following research questions among college students for three instructional approaches to calculus, (a) the traditional approach, (b) the Calculus & Mathematica approach, and (c) a somewhat traditional approach that also emphasizes graphical representation through extensive use of graphics calculators:

1. What is the relationship between the instructional approach that students experience and any change in their initial preference for different representation when solving calculus problems?
2. What is the relationship between the instructional approach that students experience and their abilities to use graphical, numerical, and symbolic representations when solving calculus problems?

3. What is the relationship between the instructional approach that students experience and their abilities to see, or make, connections between graphical, numerical, and symbolic representations in the context of problem situations?

**Importance of the Study**

Numerous recent studies on the effects of using technology in calculus instruction (Crocker, 1991; Cunningham, 1991; Estes, 1990; Heid, 1985; Judson, 1989; Palmiter, 1986; Park, 1993; Schrock, 1990) have provided evidence that students' conceptual understanding can be improved by using technology to place more emphasis on conceptual development and less emphasis on computational skills. What these studies did not address was if and how the use of technology as a regular part of instruction made students think about calculus differently. By focusing on students' use and understanding of multiple representations when solving calculus problems, this study attempts to address the issue of how the use of technology influences the way students think about calculus.
Organization of this Dissertation

This chapter introduced the statement of the problem, the need for the present study, and the research questions to be investigated. Chapter II presents a theoretical rationale for this study and reviews the literature related to this research. Chapter III describes the methodology and procedures used in the study. Chapter IV relates the different environments for the three calculus courses that participated in this research. Chapter V contains the quantitative and qualitative results of the study and a discussion of these results. Chapter VI includes a discussion of implications, recommendations, limitations and directions for further research.
CHAPTER II

REVIEW OF LITERATURE AND THEORETICAL FRAMEWORKS

This research was designed to investigate how students in calculus classes where technology was used extensively and instruction emphasized multiple representations of concepts and the interpreting and solving of problems differ from students in a traditional calculus class in their abilities to use and understand multiple representations, or multiple techniques, for solving calculus problems. This chapter reviews literature relevant to this study and presents a theoretical framework for the research.

The chapter is divided into six sections. The first section presents some historical background on the current calculus reform movement, including a characterization of the role of technology in this reform movement. The second section describes research on the use of technology in calculus instruction. The third section presents research related to students’ use and understanding of multiple representations in mathematics. The fourth section describes research related to the influence of instruction on the use and understanding of representations in mathematics. The fifth section contains theoretical frameworks supporting calculus instruction that uses multiple
representations (Hiebert & Carpenter, 1992) and that has students interpret and solve specific types of problems (Dubinsky, 1991). The final section relates the theoretical frameworks to the three research questions of this study.

**The Calculus Reform Movement**

In 1983, at a Sloan Foundation-sponsored conference held at Williams College in Williamstown, MA, the status of calculus in the core undergraduate mathematics curriculum was challenged by supporters of discrete mathematics. In response to this challenge, another Sloan Foundation-sponsored conference was organized to gather input on how to revitalize calculus and reaffirm its importance in the college mathematics curriculum. The Tulane Conference, held in 1986, marked the beginning of the current calculus reform movement.

**The Tulane Conference and Calculus Reform**

In his proposal for holding the Tulane conference, Douglas (1986c) specified the following problems with calculus instruction: (a) The calculus curriculum has not been rethought for over two decades and contains too many topics; (b) the mix of calculus students is different from when the curriculum and teaching methods were devised; (c) the level of expectations, lowered in the sixties to accommodate the poorly prepared and motivated students of that time, is too low for today's students; and (d) calculus enrollment has grown
substantially while resources, for the most part, have shrunk, and the resulting larger classes have resulted in less interaction between students and instructor. Three issues became central to the reform movement. These issues were determining (a) what should be taught in a typical calculus course, (b) what methods should be used to teach calculus, and (c) what should be the role of technology in calculus curriculum and instruction.

**What Should be Taught in a Calculus Course**

The first central issue concerned reducing the number of topics and determining which topics should be in a typical calculus course (Epp, 1986; Rodi, 1986; Steen, 1986; Tucker et al., 1986). In developing a syllabus for the first semester of a calculus course (Tucker et al., 1986), the Content Workshop participants came to the consensus that such topics as l'Hospital's rule, related rates, and the epsilon-delta definition of limit should be excluded, limits and continuity should received reduced emphasis, and graphical and numerical treatment of functions, Newton's method, numerical integration, and the use of technology should be included. Steen (1986) asked whether the content of calculus should change to reflect the way it is now used in the scientific and engineering communities, and whether tradition or usefulness in an advanced mathematics course were still adequate reasons for keeping a topic in the calculus curriculum. Rodi (1986) argued for caution in removing topics,
particularly computational skills, from the curriculum since these topics often serve more than one purpose. As an example, he points out that work on methods of integration does more than develop a student's mechanical skills, it also can develop the students' ability to analyze and recognize patterns and to choose the proper tools for attacking problems. Epp (1986) may have best expressed the feelings of many of the conference participants when she said

If it comes to a choice, will we settle for superficial knowledge of a lot or deeper understanding of less? Perhaps less is more" (p. 58).

Related to the issue of determining the topics that should be presented in a typical calculus course was the issue of determining goals for a calculus course. The Content Workshop participants suggested the following as possible goals, or competencies for a first year calculus course (Tucker, et al., 1986):

Among the competencies should be the ability to give a coherent mathematical argument; students must be able not only to give answers but also to justify them. Calculus should teach students to apply mathematics in different contexts, to abstract and generalize, to analyze quantitatively and qualitatively. Students should learn to read mathematics on their own. And, of course, calculus must also teach mechanical skills, both by hand and by machine. As for things to know, students must understand the fundamental concepts of calculus: change and stasis, behavior at an instant and behavior in the average, approximation and error. Students must also know the vocabulary of calculus used to
describe these concepts, and they should feel comfortable with the vocabulary when it is used in other disciplines. (pp. viii-ix)

What Methods Should be Used to Teach Calculus

The second central issue of the Tulane conference concerned methods of teaching. Douglas (1986a) sums up the essence of this issue by stating,

The principal problem with calculus today is the way it is taught in most American colleges and universities. Calculus is usually taught in large lectures, with limited interaction between students and teachers. ... The demoralizing effect this has on both faculty and students cannot be overstated. Calculus cannot be learned passively. (p. 5)

Methods Workshop participants offered numerous suggestions for teaching (Schoenfeld, et al., 1986) including the following: (a) use "real world" problems to serve as a context for doing mathematics, and for introducing mathematical concepts or ideas, (b) use theoretical problems to help students develop an understanding of theoretical ideas, (c) have students construct examples, and (d) assign multiple-step problems and problems requiring a longer time, such as two or three weeks, to solve. In addition to these suggested changes in teaching, changes were proposed in evaluation and testing that included the use of open-ended exams, oral exams or presentations, take-home exams, test on reading assignments, and student portfolios of their "best work" during the course.
What Should be the Role of Technology in Calculus

The third central issue of the Tulane conference concerned determining the role of technology in the revision and revitalization of calculus. Research into the effects of various uses of technology for teaching calculus (Bell, 1970; DeBoer, 1974, Holloein, 1971) has been going on since the early 70’s. The *Agenda for Action* (National Council of Teachers of Mathematics, 1980) stated: "Mathematics programs must take full advantage of the power of calculators and computers at all grade levels" (p. 8). Douglas (1986b) may have best described technology’s possible effect on the calculus curriculum by saying,

> Technology is not going to let calculus instruction stay the same!
> Anyone who has seen hand-held calculators which output the graph of an equation visually realizes that we can and, indeed, we will have to change what we ask students to learn and what we test them on. (p. v)

Porter (1988) agreed with Douglas stating: "The undergraduate mathematics curriculum must be changed to take into account the availability of significant computational facilities" (p. x).

The push for technology-based curriculum reform has grown rapidly since the Tulane conference. Shumway (1988) gave a strong indication of this push for greater incorporation of technology into mathematics when he drew the
following conclusion based on his participation in and observations of the Sixth International Congress on Mathematical Education:

1. Calculators that have symbolic manipulators, extensive graphics capabilities, and higher-order programming languages must be required for all teaching, homework, and testing in mathematics.

2. Because of the computational power available to everyone, concepts and proofs should now be considered basic mathematics and emphasized in the mathematics curriculum.

3. Substantial changes to the curriculum must be made in order to de-emphasize numeric and symbolic computations and emphasize earlier, deeper, conceptual learning.

4. Teaching strategies should focus on examples, nonexamples, and proofs, and not on drill-and-practice.

One type of technology often connected with the calculus reform movement is the computer algebra system. Birkoff (1972) first noted the immense impact that computer algebra systems could have on undergraduate mathematics education. Since then much has been written concerning the use of computer algebra systems.

Small, Hosack, and Lane (1986) raised many questions about the possible effects of computer algebra systems for undergraduate teaching, and stated that some of the potential advantages to using computer algebra systems were that students would develop a more positive attitude toward mathematics, become
more actively involved with mathematics, and spend more time organizing their thoughts when attempting to solve problems. Hosack (1988) identified six areas in which the use of a computer algebra system could effect major changes in undergraduate mathematics: (1) student perception of what is important in mathematics, (2) the role of approximation and error bound analysis, (3) integration of problem solving methods, (4) development of problem solving skills, (5) development of an exploratory approach to learning mathematics, and (6) exercises and tests.

One highly attractive feature of technology like the computer algebra systems is that it reduces the need to perform tedious and repetitive algebraic procedures (Boyce, 1988). Small and Hosack (1986) suggested that using computer algebra systems would free students from carrying out routine algorithmic manipulations, allowing more time to be spent on enhancing students’ conceptual understanding of calculus, without seriously decreasing their ability to recall and implement standard algorithms. Zorn (1986) stated that symbolic manipulation programs would help to remove computational obstacles that force the almost exclusive use of elementary functions and exact methods, thus allowing for a wider variety of exercises, examples, and applications as well as more use of approximate method. Cipra (1988) thought that the use of computer algebra system would allow for the inclusion of more
realistic and complex applications instead of problems that had been artificially revised for simplicity so that they could be solve by paper-and-pencil computations. In a report on the current status of calculus reformation, Johnson (1994) described the potential benefits of using technology to aid in the understanding of calculus concepts as follows:

The use of technology, i.e. graphics calculators, computers, and sophisticated software, ..., permits the consideration of more realistic problems, ..., allows one to spend more time on a concept since not as many topics need to be covered as in the traditional approach, and ..., eliminates some of the algebraic manipulations which many of today's students seem to do so poorly. (p. 53)

One difficulty with incorporating technology into the classroom is altering the course to make proper use of the technology, as evidenced by the recent situation at the University of Pennsylvania (DeLoughry, 1993) where there was "[a]n angry outcry against the use of a computer program [Maple V] in calculus classes" (p. A17). Officials at Penn had decided to begin using computers in introductory calculus classes with the intent of de-emphasizing the traditional calculations involved in calculus so that students could concentrate more on the concepts behind the calculations. However, when the new classes were implemented, problems arose that caused a widespread revolt by the students against the use of the computer software. One of their main
complaints was that the instructors were including Maple assignments in the coursework even though the instructors did not know how to use the software. Since many of the faculty members were ill-prepared [because they had not attended workshops on using the Maple software], few made the effort to integrate their class materials and the Maple assignments ... . The result ... was that students saw the computer exercises as irrelevant to the classroom instruction and considered them a waste of time. (DeLoughry, 1993, p. A17)

Hughes-Hallet (1992) may have best summed up the concerns of many calculus reformers about the incorporation of technology into the calculus curriculum, stating that:

A [calculus] course that focuses on the same topics and gives the same types of problems, but with technology tacked on, will be no better than what we have now because it still won’t encourage students to think. Indeed it might be worse than the current course, as it will be more lumbering. It is not enough to keep the topics the same but allow students to use the calculator or computer. (p. 61)

**Calculus Reform since the Tulane Conference**

Three major steps toward calculus reform came out of the Tulane conference. The most widely known step took place in Washington, DC a little over 21 months after the Tulane Conference when the National Academy of Sciences and the National Academy of Engineering sponsored the Calculus for
a New Century colloquium. The areas of calculus reform addressed at Calculus for a New Century colloquium included: (a) incorporating technology as a tool in calculus, (b) promoting student understanding of calculus, (c) developing innovative textbooks, (d) considering the needs of the "client disciplines," and (e) changing evaluation and testing practices.

The second major step, which came out of the Tulane conference recommendations (Crocker 1990; Douglas, 1986b), was the creation of small, experimental calculus projects that were to serve as models and bring about development of curriculum materials, new teaching methods, and revised textbooks. Since the Tulane conference, numerous calculus projects (Tucker, 1990) involving new or revised syllabi, the use of innovative teaching techniques and methodologies, and the incorporation of various types of technology have been instituted at various American colleges and universities, providing evidence that this step toward calculus reform is well underway and moving toward completion.

The third major step was a call for more research and discussion on calculus curriculum and instruction. In the next section, I will look at some of the research, focusing on research involving the use of different forms of technology in calculus instruction.
Research on the Use of Technology in Calculus Instruction

In this section, 13 recent research studies on the effects of using technology as an integral part of calculus instruction are reviewed. Six of the studies (Cunningham; 1991; Heid, 1985, 1988; Judson, 1989; Melin-Conejeros, 1992; Palmiter, 1986, 1991; Schrock, 1990) involved the use of computer algebra systems, five (Beckmann, 1988; Estes, 1990; Hart, 1991; Melin, 1990; Stout, 1991) involved the use of graphics software, graphics calculators, or "supercalculators" like the Hewlett-Packard 28S, and the last two studies (Crocker, 1991; Park, 1993) looked at students enrolled in Calculus & Mathematica classes.

Computer Algebra Systems

Heid (1985, 1988) used a computer algebra system (muMATH) and other computer programs as part of an experimental business calculus class where skills and concepts were presented in a different order than in a traditional calculus classes. The 39 students in her two experimental classes spent the first 12 weeks of the course semester learning only concepts and theory followed by 3 weeks of paper-and-pencil computations and skills practice. Homework assignments and quizzes given in the experimental classes tested for understanding of concepts, rather than computational proficiency. The computer algebra system and other computer programs were used to decrease
the time and attention spent on skill development that is typical in a traditional calculus class.

Heid collected various types of qualitative and quantitative data in order to compare her experimental class to a traditional lecture class that served as a control group. Analysis of the qualitative data indicated that the main difference between the experimental and control groups was that the students in the experimental group were better able to verbalize descriptions of concepts. The experimental groups scored higher than the control group on a common conceptual examination while the control group scored slightly higher on the final examination, which consisted of mostly paper-and-pencil computations. Heid indicates that other factors, such as the different classroom and learning environments of the experimental and control groups, may have contributed to the high scores for the experimental groups on the conceptual examination.

Based on her results, Heid concluded that using the computer helped students by freeing up time from doing computations, allowing them to focus on exploring new problem situations rather than on performing the computations correctly. She also felt that the results suggested that concepts can be learned without concomitant skill development, just as skills are typically learned in the absence of conceptual understanding in a typical calculus class.
Palmiter (1986, 1991) studied the impact of computer algebra system use on students in an integral calculus class. Skills and concepts were resequenced in her experimental class, but not to the same extent as they were in the Heid study. This class spent the first five weeks of the course studying concepts while using the computer algebra system MACSYMA to do computations and then spent the remaining five weeks learning paper-and-pencil techniques of integration. The 40 students in the experimental class covered in five weeks the same conceptual material that the 41 students in the two control groups covered in ten weeks. The control groups were two traditional calculus classes that differed only in that the one class was given optional supplemental presentations and materials on numerical approximation techniques using programmable calculators and the other was not. All three groups were taught in lecture-recitation format.

At the end of five weeks, the experimental group was given two examinations. The first was a computational examination that included two problems that were beyond the scope of a traditional integral calculus course and two problems where the integral in question had to first be set up by students before it could be solved on the computer. The second was a conceptual examination that included no computations involving special techniques of integration. The conceptual examination was given to the first
control group and the computational examination was given to the second control group at the end of the 10th week. The experimental group was allowed to use MACSYMA on the computational examination but were given half the time - 45 minutes as compared to 90 minutes - as the control groups to complete this examination.

Analysis of the results of these examinations indicated that the experimental group scored significantly better than either of the control groups on the respective examinations and that students in the experimental group could solve integration problems, with the help of the computer algebra system, that were beyond the scope of a traditional, integral calculus course. An analysis of final examination scores of the experimental group, the first control group, and three other lecture sections not associated with the study indicated that the experimental group did at least as well or, in three of four cases, much better than the other traditional classes on the final. A further analysis of the scores of the experimental group and first control group on just the conceptual portion of the final examination indicated that there was no significant difference between the scores of the two groups. Palmier felt this result indicates that learning computational techniques along with concepts has a negative effect on learning the concepts. She also believed that the results support the findings of Heid study, that concepts can be mastered without
concomitant skill development and that the use of computer refocused students’
thinking away from computations.

Judson (1989) did a study similar to Palmiter’s in an introductory
business calculus class. Skills, concepts, and applications were resequenced so
that her two experimental class studied concepts and applications of derivatives
during the first half of the course and then studied computational techniques
and the rules of differentiation during the second half. The same materials
were presented in the opposite order in the two control groups. A computer
algebra system, Maple, was used in the experimental groups to facilitate the de-
emphasis of arithmetic and computational skills and the emphasis on problem
solving and conceptual understanding. All four groups received the same
instruction except for some Maple demonstrations and one lecture on the use of
Maple given to the experimental groups during the first half of the course in a
different classroom equipped for computer demonstrations.

At the end of the course, the four groups took a common final. The
problems in this examination were identified as testing for skills, concepts, or
application, with some of the problems identified as testing for both concepts
and applications. Multiple linear regression was used to study the relationship
between variables representing scores on the skills, concepts, and applications
subtests of the final, SAT scores, the instructor, the treatment, and the
interaction of instructor and treatment. An analysis of the regression model indicated that the only significant effects were between the SAT scores and the concepts subtest scores and between the SAT scores and the applications subtest scores. No significant effects were found between the treatment and any of the other variables studied, even when the teacher variable and the interaction variable were each eliminated from the model.

Even with the lack of significant treatment effects, there were still differences between the mean scores of the experimental and control groups on the three subtest of the final. The mean scores of the experimental groups were greater than those of the control groups on both the concepts and applications subtests and about the same on the skills subtests. This result led Judson to conclude that use of a computer algebra system has a positive effect on the teaching of concepts and, like Heid and Palmiter, that skill development is not prerequisite to conceptual understanding.

Schrock (1990) conducted a study on the effects of the use of the computer algebra system Maple on skill development, conceptual understanding, and problem solving ability for students in a differential calculus class. The experimental group and two control groups in this study covered all the same material but the experimental group received instruction that emphasized conceptual understanding, not skill development, and used Maple
extensively while the control group received instruction that emphasized skill
development. The control groups were two traditional calculus classes that
differed only in that one was taught by the investigator and the other was
taught by another instructor. The three groups used in the study comprised all
the differential calculus students at a two-year college.

At the beginning of the course, half of the students in each group took a
calculus pretest, as part of a pretest-posttest design to measure the students’
computational skills, while the other half of the group took an algebra
proficiency test. At the end of the course, all of the students took a common
final that served as the posttest, as well as two other examinations, one that
tested conceptual understanding and one that problem solving ability on
applications problems. Problems from the 1985 Advanced Placement AB
Calculus Exam were used to create the pretest and the posttest. Since the
posttest also served as the final for the course, it included problems that tested
for conceptual understanding and problem solving ability, though almost 75%
of the problems tested for computational skills. Only the computational
problems on the final were used as the posttest.

On all three examinations, analysis of the scores indicated that there were
significant differences between the experimental group and combined control
groups and between the experimental group and each control group, but not
between the two control groups. In addition, there was a testing effect for the computational skills pretest-posttest instrument, but no group effect or group by test interaction effect. An analysis of the scores of the three groups on the final examination indicated there were significant differences between the experimental group and combined control groups, between the experimental group and the control group not taught by the investigator, and between the two control groups, but not between the experimental group and the control group taught by the investigator. Based on these results, Schrock concluded that the use of a computer algebra system can help to enhance conceptual understanding prior to the development of extensive computational skills.

Cunningham (1991) investigated the effects of a computer software package (Calculus) with symbolic manipulation capabilities on calculus achievement. The software package was used for classroom demonstrations for both the experimental group and control group in this study were given. Both groups covered all the same material but the experimental group relied extensively on the software to perform computer-generated symbolic manipulations such as the evaluation of limits, differentiation, and indefinite integration while the control group relied on traditional paper-and-pencil techniques to do all of the symbolic manipulations. Two similar posttests were developed by the researcher to measure calculus achievement in terms of
conceptual understanding and symbolic manipulation skills. The students in the experimental group were divided into two subgroups for the posttests. One subgroup was given access to the software for the first posttest but not the second, and the second subgroup was given access to the software for the second posttest but not the first.

On the symbolic manipulation portion of both posttests, the subgroup in the experimental group with access to the software obtained a significantly higher mean score than the control group while the subgroup without access to the software obtained a similar mean score to that of the control group. On the conceptual understanding portion of the first posttest, the subgroup in the experimental group without access to the software for that posttest obtained a significantly higher mean score than the control group. The subgroup with access to the software for that posttest obtained a higher, but not significantly higher, mean score than the control group. On the conceptual understanding portion of the second posttest, both subgroups in the experimental group obtained similar mean scores to that of the control group. Based on these results, Cunningham believed that the use of the symbolic manipulation software improved conceptual understanding and did not cause negative effects when access to the software was denied. He also indicated that use of the software by the instructor during classroom instruction in concert with
extensive use by the students outside the classroom and during examinations may have been a factor in students’ success.

Melin-Conejeros (1992) investigated the effects of doing calculus homework assignments in a mathematics laboratory equipped with a computer algebra system (Derive) on students’ achievement. Unlike the other studies previously mentioned, the computer algebra system was not used as part of in-class instruction, but was only available for use on homework assignments. For the study, the 12 students in the experimental group were to do their homework in the computer laboratory using the computer algebra system. The 16 students in the control group were given similar homework assignments to be done using standard paper-and-pencil techniques.

All participants in the study were given two midterms and a final examination. No calculator or computer use was allowed on any of these examinations. Problem solving interviews with six students from each treatment group were conducted near the end of the course in order to obtain additional information on their concept attainment. The examination scores were submitted to a MANCOVA, using ACT scores as the covariate. The results of this data analysis indicated that there were no differences between treatment groups on overall achievement, on skills achievement, or on concept achievement. The data from the interviews suggested that the students in the
experimental group had a better understanding of certain concepts such as asymptotes, concavity of graphs of functions, continuity, increasing and decreasing functions, and limits of functions. Based on these results, Melin-Conejeros concluded that computer algebra systems should not be used just for homework, but made an integral part of calculus instruction, and that homework problems should be designed to take advantage of the capabilities of the computer algebra system.

Two common characteristics shared by the previous six studies are that their results suggest that students’ conceptual understanding can be improved by using technology to place more emphasis on conceptual development with less emphasis on computational skills and that skill acquisition need not be a prerequisite for conceptual understanding. However, none of this research examined how the use of multiple representations in calculus instruction, along with the emphasis on technology, may have impacted students’ construction and understanding of concepts or their techniques for solving calculus problems. Also, this research did not examine calculus courses where the problems given to students were specifically designed to make the best use of the technology available, as Melin-Conejeros (1992) recommended. This study sought to examine the impact of calculus instruction that emphasized the use of technology and multiple representations, and that had students solve/interpret
problems specifically designed to take advantage of the capabilities of the available technology, on students’ thinking when solving calculus problems.

**Graphics Calculators and "Supercalculators"**

Melin (1990) conducted a study to compare the achievement of students who used a graphics calculator (Casio fx-7000G) to those who did not have access to this technology. Two groups of 24 students were exposed to the same topics (critical points, existence of derivatives, continuity of a function, vertical and horizontal asymptotes, cusps, and vertical tangents), examples, and problems, and used the same textbook. The main difference between the two groups was that the students in the calculator group were asked to produce their own graphs of functions being discussed in class whereas the students in the control group were shown similar graphs that were drawn by the instructor on the chalkboard. Other differences included students in the calculator group being shown more graphical examples than students in the non-calculator group and the researcher teaching only the calculator group and not the control group.

Two departmental examinations not specifically design for the experiment were used to compare the two groups. The experimental instruction transpired between the two examinations, and students in either group were not allowed to use calculators on the second examination. An analysis of the scores indicated a significant difference favoring the calculator group (adjusted mean of 116.55
out of 150 points) over the non-calculator group (adjusted mean of 97.49 out of 150 points) on the second examination. In conclusion, Melin suggested that a textbook specifically designed to encourage and incorporate the use of graphics calculators might have produced even better results on the part of the students in the calculator group.

Estes (1990) examined the effects of implementing graphics calculator and computer technologies as instructional tools in an applied calculus class. One main component of this research was to determine the impact of the technology on conceptual and procedural achievement. A pretest, treatment, posttest research design was employed. Group equivalence was measured on the pretest by the CLEP Test of College Algebra.

During the semester, the experimental and control groups covered basically the same material. The main difference between the groups was that calculator and computer technology were used extensively for instruction and in the assignments of the experimental group while a traditional approach to calculus instruction was used with the control group. The students from both groups were given conceptual questions on each unit test, and at the end of the semester, they were all given the same exit exam, used as the posttest, which contained both procedural and conceptual questions. A comparison of the two groups on conceptual and procedural achievement gains from the pretest to the
posttest indicated a significant difference in conceptual achievement favoring
the experimental group with no significant difference between the two groups
on procedural achievement.

Stout (1991) performed a two-day study examining the effects of using a
"supercalculator" on students’ understanding of the concept of derivative. A
supercalculator is a hand-held calculator, like the Hewlett-Packard (HP) 48SX
or the HP28S used in this study, with both graphics and symbolic manipulation
capabilities. The nine students in the experimental group studied the concept
of derivative by using the supercalculator to graph a polynomial, find and
graph its derivative, and then analyze relationships between the two graphs.
The ten students in the control group studied the concept of derivative in a
traditional manner using a standard calculus text without the aid of any
graphics or symbolic manipulation technology.

At the end of the second day of the study, two five-item tests were
administered to both groups. Students were not allowed to use calculators on
either of the tests. For the first test, students were presented with the graphs of
functions and asked to sketch the graph of the corresponding derivatives. On
the second test, students were presented with the graphs of derivatives of
functions and asked to sketch the graph of the corresponding function.
Analysis of the test scores indicated a significant difference in favor of the
experimental group on the first test and no significant difference between the groups on the second test.

Beckmann (1988) examined students' understanding of certain calculus concepts as developed through the use of computer-generated and non-computer-generated graphical representation systems. She divided 163 calculus students enrolled in first-semester calculus into four groups with each group receiving different exposure to graphical representations. The first group studied a computer-graphically developed conceptual course where concepts were presented through real-world applications and were first represented graphically before then being translated to a symbolic representation. The second group studied the same course as the first group, but were also given access to a computer graphics software package and related supplemental assignments. The third group studied a similar course as the first two groups except that the graphically development of concepts was done without the aid of a computer. Finally, the fourth group studied what would be considered a traditional skills-oriented calculus course.

The students in the first two groups were compared on, among other things, their ability to use graphs to describing calculus concepts. No significant differences were found, the students in the second group did score slightly higher than the ones in the first group. The students in all four groups
were compared on, among other things, their performance on the departmental final examination, and on routine applied, routine symbolic, and nonroutine symbolic questions. Significant differences were found favoring the students in the first group over the ones in the fourth group on non-routine symbolic questions and on both types of routine questions, though Beckmann indicates the later result was questionable. Nonroutine symbolic questions, like the one shown below, were ones for which students had no answer or previously established routine procedure.

Indicate which of the graphs below correspond to a function f that has the following properties:

\[ f(c) > 0, \quad f'(c) > 0, \quad f''(c) > 0 \]

at the point \( x = c \).

Briefly, support your answer. (Students were shown five different graphs.) (Beckmann, 1988, p. 393)

Based on the results, Beckmann believed that developing calculus concepts through the use of a graphical representation system, particularly a computer-generated system, could positively affect student understanding without necessarily having a negative influence on skill acquisition.

Hart (1991) investigated the impact of the use of and access to a supercalculator on how students in differential and integral calculus make use of and deal with graphical, numerical, and symbolic representations of functions. The experimental group for this study consisted of 324 students
from 12 institutions participating in the Oregon State University Calculus Project. The students in the experimental group studied an experimental curriculum that emphasized multiple representations (symbolic, graphical, numerical). In particular, these students were taught graphical and numerical methods for analyzing and solving problems with the aid of a supercalculator along with traditional methods of calculus. The experimental group also used a textbook that emphasized multiple representations and how to use technology to work with these representations. The control group for this study consisted of students from various institutions who were taught calculus in a traditional manner. Students in this group were allowed to use calculator on quizzes and exams but were required to show all of their work symbolically and to present answers in exact form. Data for this study were gathered through the use of paper-and-pencil tests and task-based interviews.

An analysis of the data indicated that the students from the experimental group were better able to use graphical and numerical representations, exhibited better connections between different forms of representations, and displayed less compartmentalization than the students from the traditional group. The results also suggested that individual students have definite preferences in their use of particular representations, but the factors that influence their choices are varied. Some of the factors Hart determined as having an influence on
students' choices of representation were: students' confidence in their symbolic manipulation skills, the complexity of the symbolic information, the experiences of the student with each form of representation, and the students' perception of the viability of each form of representation as an option.

As was the case with the studies involving computer algebra systems, the results of these studies on graphics software, graphics calculators, and "supercalculators" suggest that students' conceptual understanding can be improved by using technology to place more emphasis on conceptual development with less emphasis on computational skills. However, except for Beckmann's (1988) and Hart's (1991) research, none of this research examined the relationship between the use of multiple representations in calculus instruction and students' construction and understanding of concepts. Also, none of this research focused on the relationship between the use of multiple representations in calculus instruction and how students use multiple representations when solving calculus problems. The fact that previous studies have not considered the impact of these relationships contributes to the importance of the present study.

Calculus & Mathematica

Crocker (1991) did a qualitative study of students at the Ohio State University enrolled in an experimental calculus sequence, called Calculus &
*Mathematica*, designed specifically to use the computer algebra system

*Mathematica*. The intent of the study was to examine the interactions, concept development, and problem-solving skills of students in this environment via classroom observations, student interviews, and a student questionnaire. She conducted a series of four interviews with nine students chosen from 36 students who volunteered to participate in the study. Students were chosen based on a purposive sampling process that used maximum variation in perceived ability as the criteria for generating the sample.

The interviews were done over a 5-month period while students were enrolled in the first two quarters of the calculus sequence. The first interview took place near the end of the first quarter of the sequence, the second at the beginning of the second quarter, and subsequent interviews every ten days to two weeks thereafter. In all of the interviews, students were given problems and asked to solve and discuss each problem. Students were given access to *Mathematica* for solving problems in all of the interviews except the first. During the first interview, students were also asked to talk about the new approach to teaching calculus used in this course and the course itself. After the final interview, a written questionnaire was administered to obtain additional background information on the students and information on their attitude toward the course similar to that obtained during the first interview.
In analyzing the data from the interviews, Crocker observed that students exhibited an improvement in the understanding of the concept of derivative over the two quarters of calculus. This finding supports the conclusions of the earlier research that conceptual understanding is improved or enhanced when, by minimizing paper-and-pencil computations through the use of a computer algebra system, concepts and not computational skills are made the focus of the calculus course. She also found that the students developed a strong connection between the concept of derivative and slope.

Park (1993) conducted a study to evaluate the Calculus & Mathematica course at the University of Illinois at Urbana-Champaign, and to compare the mathematical achievement and attitudes of the Calculus & Mathematica students to those of students in a traditional calculus course. For the achievement portion of the study, Park administered an achievement test to all of the students and designed two analysis methods to compare the students' concept maps. The results of the achievement test indicated that the Calculus & Mathematica students showed better conceptual understanding than the traditional calculus students, without any significant loss in computational proficiency. The results from both analysis methods on students' concept maps were favorable toward the Calculus & Mathematica students.
The results of these two studies support the findings of the previously cited research that conceptual understanding is improved or enhanced when, with the help of technology such as computer algebra systems, graphics calculators, or "supercalculators," concepts and not computational skills are made the focus of the calculus course. What is not addressed in this research, but is addressed in the present study, is how the technology and the instructional approach for that technology impacts students' use and understanding of multiple representations when solving calculus problems.

Students' Use and Understanding of Multiple Representations

Some researchers of students' understanding of calculus concepts such as function, limit, derivative, and integral have noted implications for instruction that call for teachers and students to make greater use of multiple representations. As Goldenberg (1987) states,

Each well-chosen representation views a function from a particular perspective that captures some aspect of the function well, but leaves another less clear; taken together multiple representations should improve the fidelity of the whole message. (p. 97)

Some calculus reform projects (Harvard Project, Oregon State University Project) and one precalculus curriculum (Schwarz, Dreyfus, & Bruckheimer, 1990) call for instruction that emphasizes the use of multiple representation.
The *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) calls for similar instruction in grades 9-12. However, as the following studies suggest, students can work with more than one representation but may not fully understand the connections between various representations.

In a study on students' understanding of the concept of function, Markovits, Eylon, and Bruckheimer (1986) found that most students comprehend that a function can have more than one representation. Approximately 50% of the students in the study were able to identify two representations of the same function, one algebraic and the other graphical. The difficulty for the remaining students appeared to be that they ignored the domain and range of the functions in question. About two-thirds of the students had trouble determining an algebraic representation for a linear function given in graphical form. It also appeared that students had much more difficulty transferring from algebraic to graphical representations, and vice versa, with unfamiliar functions than with functions that were similar to ones they had seen in class.

Monk (1989) suggested that the representation used to describe a function in a contextualized setting plays a large role in whether or not students can answer questions pertaining to the function. He gave the following problem as an example:
A car moves along a track, and its position at various times is described by the following function (given by a graph or formula). Give the time at which the car is 10 miles from the start. (p. 3)

Monk believed that most students in a beginning calculus class would be able to correctly solve this problem if the function was described using a graphical representation but not if it was described using a symbolic representation.

In their study on student understanding of central concepts in calculus, Ferrini-Mundy and Graham (1991b) found that students tend to view algebraic and graphical representations of the same concept as being independent and that the ability to make connections between the two forms of representation may differ depending on the concept in question. For example, one student from the study was able to connect information provided by a symbolic representation of a function to information about its graphical representation when discussing the concept of continuity, but had difficulty making the connection when discussing the concept of derivative. Ferrini-Mundy and Graham concluded that

> graphical contexts and algebraic contexts may function for students as separate worlds, ... competing, conflicting conceptions and conclusions are held quite comfortably and routinely in the development of calculus concepts. (pp. 17-20)
In a study done by Dreyfus and Eisenberg (1988), three groups of calculus students were each given a different problem involving a function represented in symbolic form. While most of these students were able to draw appropriate sketches to represent the information presented in each problem, they were not able to relate graphical information from the sketch when describing the problem algebraically. According to Dreyfus and Eisenberg, this result suggests that students have difficulties making connections between algebraic and graphical representations of the same problem.

Dufour-Janvier et al. (1987) noted that students often do not see the same task when different representations of the same problem are given, but instead believe that there as many problems as there are representations. They present the following as an example:

A child resolves a problem using a representation. We then show him the same problem resolved by someone else using a different representation. When we deliberately show him the answer of this other child (the answer happens to be incorrect); a number of children are not at all disturbed and find this quite natural because in their view the first problem was done one way and the second done in another way. (p. 114)

Dufour-Janvier et al. point out that this example indicates that students do not see all of the representations accompanying a single task as different ways for tackling the same situation.
As the literature reviewed in this section shows, students seem to be able to work with different forms of representations, but seem unable to relate similar information provided by different representations. As Lesh, Post, and Behr (1987) point out, the ability to make connections between representations is important for solving problems.

Good problem solvers tend to be sufficiently flexible in their use of a variety of relevant representational systems that they instinctively switch to the most convenient representation to emphasize at any given point in the solution process. (p. 38)

This study seeks to investigate the relationship between different instructional approaches to calculus and students’ use of, and understanding of connections between, multiple representations when solving calculus problems.

**Influence of Instruction on the Use and Understanding of Representations**

For years, mathematics educators have advocated using more than one type of representation to present mathematical concepts to students. As Kaput (1992) suggests, using more than one representation, or notation, system helps to present a clearer picture of a concept or idea.

Complex ideas are seldom adequately represented using a single notation system. For example, differential calculus has several different facets that must be expressed in different types of notation systems. Rate of change can be seen in notations that include instantaneous change, in slopes of
coordinate graphs, in formal algebraic derivatives, and in numerical difference quotients, to name a few.... Each notation system reveals more clearly than its companion some aspect of the idea while hiding some other aspect. The ability to link different representations helps reveal the different facets of a complex idea explicitly and dynamically. (p. 542)

Perhaps because of the "obvious" benefits of using more than one representation to present concepts, little research has been done to investigate how using different representations in instruction influences students' use and understanding of various representations of complex concepts like those encountered in a differential calculus course.

Much of the research concerning the use of different representations in instruction has dealt with the use of concrete materials or manipulatives (e.g., Hart, 1989; Nesher, 1989; Raphael & Wahlstrom, 1989; Sowell, 1989), often in conjunction with written symbols, for teaching arithmetic or algebra concepts. The results of this research has little bearing on the present study since the investigations did not involve the type of complex concepts, or the types of symbolic, graphical, or numerical representations, typically encountered in a differential calculus course. Some research has examined the effects of spatial visualization training on spatial ability. This research is of interest since spatial visualization training involves work with graphical representations and since students' use and understanding of graphical representations can be considered
a component of their spatial ability. A few of these studies are discussed in the following section.

**Influence of Instruction on Spatial Visualization Ability**

Moses (1977) investigated the effects of instruction in perceiving spatial figures and using visual solution processes on spatial visualization ability and problem solving performance for fifth graders. Her findings indicated that instruction did affect spatial visualization ability but did not significantly affect problem solving performance. In a review of research on visualization in mathematics education, Bishop (1989) claimed that there was little evidence showing success of training in visual processing, i.e. the use of graphical or figural information. This claim was supported by Clements (1982a, 1982b). Despite the lack of evidence, Bishop still felt that the research indicated it was possible to teach students the skills necessary for interpreting graphical or figural information. These findings suggest instruction that makes extensive use of graphical representations may influence students’ use and understanding of graphical information and representations, but their applicability to the present study is somewhat limited since the research did not involve the type of complex concepts encountered in a differential calculus course.

Shoaf-Grubbs (1992) investigated the effect of the graphics calculator on students’ spatial visualization ability in an elementary college algebra course.
In the experimental group for this study, graphic calculators were used as an aid in the learning of three highly visual algebra topics: linear equations, systems of equations and inequalities involving linear functions with two variables, and (vertical) parabola. The use of the graphics calculators constituted the only difference between the experimental and control groups. The students in both groups were pre- and post-tested for spatial skills level in general elementary algebra concepts and for spatial skills level in the three specific algebra topics. The results indicated that the use of the graphics calculator had a positive effect upon both the general and specific spatial skills, which suggests that the additional graphical representations provided by the graphics calculator had some impact on students’ use and understanding of graphical representations of elementary algebra concepts. The results of this study are of some interest since graphics calculators were used to increase the emphasis on and usage of graphical representation, which was similar to the way graphics calculators were utilized in the graphics calculator calculus course observed in the present study.

Ferrini-Mundy (1987) examined the effects of a spatial-training program on college students’ calculus achievement, spatial visualization ability, and the use of visualization in solving solid-of-revolution problems. The treatment groups of the study participated in a spatial-training program that included
work on (a) two-dimensional spatial tasks, (b) three-dimensional spatial tasks, (c) rotations from two- into three-dimensional space, (d) area estimation, and (e) the development of three-dimensional images from two-dimensional representations. Spatial ability was assessed using the Space Relations Subtest, Form T, of the Differential Aptitude Test as both the pretest and posttest. Use of visualization in solving solid-of-revolution problems was assessed using a solid of revolution problem and a five-item instrument designed to assess the ability to correctly identify and sketch solids of revolution given the plane figure to be rotated. The findings indicated that the spatial-training program was not effective in improving spatial ability scores, though the scores on the posttest did favor the treatment groups, but was effective in improving the scores on the solid-of-revolution problems. Ferrini-Mundy contented that the lack of improvement in spatial visualization scores may have been a result of the spatial ability test being too narrowly focus to assess the wide range of spatial tasks included in the training. The results suggest that instruction emphasizing graphical representation may affect students’ use of graphical representations with certain calculus concepts.

Influence of Instruction on Use of Representations in Calculus

What little research there has been on the use of multiple representations in calculus instruction has focused mostly on how the instruction affected
students' achievement or conceptual understanding and not on how the use of multiple representations in instruction influenced students' use and understanding of representations. One exception is the previously mentioned research by Hart (1991) on the impact of supercalculators on students' use and understanding of graphical, numerical, and symbolic representations of functions in differential and integral calculus. Hart's results indicated that teaching students graphical and numerical methods, with the aid of supercalculators, in conjunction with the traditional symbolic methods of calculus, improved the students' abilities to use graphical and numerical representations and to understand the connections between different forms of representations. These findings suggest the use of multiple representations in instruction has some impact on students' use of, and understanding of connections between, multiple representations when solving calculus problems.

Like Hart's research, the present study examines the impact of calculus instruction that emphasized the use of technology and multiple representations on students' use and understanding of graphical, numerical, and symbolic representations of calculus concepts. Hiebert and Carpenter (1992) contend that such research, where the focus is not just on which instructional practices produces better achievement or conceptual understanding, but also on how the
instructional practices impact on students' thinking, is the only type of research that will contribute to our understanding of students' understanding.

We would argue that there is little value in research that pit one instructional treatment against another without providing a detailed account of the effects of the instruction on students' learning and thinking. Without such accounts, the information does little to increase our own understanding of the teaching-learning process. (p. 92)

The fact that the present study does seek to investigate how calculus instruction that emphasized the use of technology and multiple representations, and that had students solve/interpret problems specifically designed to take advantage of the capabilities of the available technology, influences students' thinking when solving calculus problems contributes to its importance.

**Theoretical Framework for the Research**

In this section, theoretical frameworks that form the basis for the theoretical framework of this research study are presented. Framework looks at instructional influence on students' use of different forms of representations. The final two frameworks support calculus instruction that makes use of multiple representations and that has students solve and interpret specific types of problems.
Theoretical Basis for Using Multiple Representations

The theoretical framework that supports the use of multiple representations in calculus instruction is built from a framework for learning mathematics with understanding proposed by Hiebert and Carpenter (1992) based on the notion that understanding can be described in terms of internal knowledge structures. This framework was also valuable in formulating the interview guide questions used in the study, which are described in the next chapter.

A Framework for Defining Mathematical Understanding

The term *representation* has heretofore referred to an "external" as opposed to an "internal" representation. Dufour-Janvier et al. (1987) make the following distinction between the two types of representation:

> [Internal representations] concern most particularly mental images corresponding to internal formulations we construct of reality (we are here in the domain of the signified). [External representations] refer to all external symbolic organizations (symbol, schema, diagram, etc.) that have as their objective to represent externally a certain mathematical "reality." (p. 109)

External representations are the means by which mathematical ideas can be communicated and take the form of physical objects, pictures, spoken language, or written symbols (Lesh, Post, & Behr, 1987). It is when a mathematical
idea or concept can be represented in any one form or in all forms of representation that the use of multiple (external) representations is possible.

Hiebert and Carpenter (1992) propose a theoretical framework for understanding based on the premise that knowledge is represented internally, or mentally, in some structured form. An effective way of describing understanding is in terms of the structure of an individual's internal representations. The framework is built on two assumptions drawn from cognitive science. The first is that some relationship exists between external and internal representations. The second is that internal representations can be connected or related. A final assumption for the framework is that connections between a students' internal representations can be induced or constructed by having the student build connections between corresponding external representations. The connections between external representations are between not only distinct representations of the same concept, but also related ideas within one type of external representation.

Hiebert and Carpenter suggest that when connections between internal representations of concepts are constructed, these representations and connections form networks of knowledge. They hypothesize these networks may be structured as vertical hierarchies, as webs, or as some combination of the two structures. These networks of connected internal representations are
similar to what Tall and Vinner (1981) referred to as "concept images." The idea that students build these structures of represented knowledge is supported by constructivist learning theory since constructivist theory proposes that people try to make sense out of new information by evaluating, connecting, and organizing the information relative to their own prior experiences.

Hiebert and Carpenter go on to apply their framework of structures of represented knowledge to define understanding.

A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of the connections. A mathematical idea, or procedure, or fact is understood thoroughly if it is linked to existing networks with stronger and more numerous connections. (Hiebert & Carpenter, 1992, p. 67)

This definition of understanding in mathematics is supported by many of the more recent discussions concerning representations and understanding in mathematics (Davis, 1984; Hiebert, 1986; Janvier, 1987; Wagner & Kieran, 1989). Hiebert and Carpenter go on to describe a process for building understanding based on their definition.

Networks of mental representations are built gradually as new information is connected to existing networks or as new relationships are constructed between previously disconnected
information. Understanding grows as the networks become larger and more organized ... and as relationships become strengthened with reinforcing experiences and tighter network structuring.

(Hiebert & Carpenter, 1992, p. 69)

Under this framework, a student's ability to use, or understand the connections between, multiple representation when solving a problem would depend on the student's network(s) of internally represented knowledge corresponding to the problem. For example, an individual might have an internal network based on (a) what types of representations can be applied to a particular problem, (b) what problem-solving techniques can be applied to a particular type of representation, or (c) what types of representations and problem-solving techniques can be used on particular types of problems. This internal network might actual consist of different disjoint networks each containing various bits of information concerning representations or problem-solving techniques, or it might be one larger, well-connected network where various representations and problem-solving techniques are connected. As with the case of understanding, the student's ability to use multiple representations, or multiple techniques, when solving a problem is determined by the number and strength of the connections within the corresponding internal network(s).
Based on this definition of understanding, we could say that a primary goal of mathematics instruction is to help students to construct large, well-connected internal networks of representations. It follows that mathematics education research can help to provide better understanding of how to best help students construct these well-structured networks.

**Internal Knowledge Structures and Mathematical Understanding**

The theoretical framework for examining issues of understanding proposed by Hiebert and Carpenter (1992) is based on the assumption that knowledge is represented internally, and that these internal representations are structured. This subsection focuses on literature related to the notion that understanding can be defined in terms of the way information is mentally represented and structured.

In order to think about and understand mathematical ideas, facts, or procedures, it is necessary to represent them internally in such a way that the mind can somehow operate on them. For years, the associationist perspective in psychology (Thorndike, 1914; Skinner, 1953) ruled out any discussion of internal, or mental, representation since these representations cannot be observed which meant any discussion of how ideas are represented internally would be based on high degrees of inference. However, recent work in cognitive science, where a strong emphasis has been placed on finding ways to
precisely modeling mental representations (Gardner, 1985), has revitalized the study of mental representations. As Shepard (1982) points out there has been a shift in psychological theorizing "from a preoccupation with discrete sequential processes presumed to mediate observed behavior to a concern with structurally dense parallel or analogical representations that are operated upon by those process and that, in turn, guide and constrain those processes" (p. 49).

Two assumption from the work in cognitive science form the basis for Hiebert and Carpenter’s (1992) framework.

First, we assume some relationship exists between external and internal representations. Second, we assume that internal representations can be related or connected to one another in useful ways. (p. 66)

There is some debate concerning if and how external and internal representations are related. One school of thought (Shepard, 1982) is that the form of a mental representation mimics in some way the external object or event being represented. A second school of thought (Newell, 1980) is that there is a common form of mental representation used to represent all information. A final school of thought (Greeno, 1988; Kaput, 1988) which Hiebert and Carpenter use in their framework and indicate is supported by evidence from a variety of task situations (Gonzalez & Kolers, 1982; Stigler,
1984) is that "the nature of the internal representation is influenced and constrained by the external situation being represented" (p. 66).

As Hiebert and Carpenter point out, the idea that understanding in mathematics is making connections between ideas, facts, and procedures is not new. Recently, this theme has emerged frequently in discussions on representations and understanding mathematics (Davis, 1984; Janvier, 1987; Skemp, 1987). Skemp’s (1987) description of understanding in terms of schema, "[t]o understand something means to assimilate it into an appropriate schema" (p. 29), closely parallels Hiebert and Carpenter’s (1992) notion that "[a] mathematical idea or procedure is understood if it is part of an internal network [of representations]" (p. 67). Here, Skemp views a schema as a conceptual structure where individual concepts are connected to other concepts via hierarchies and cross-linkages. Generally, schema are thought of as relatively stable internal networks constructed at a relatively high level of abstraction so they can serve as templates for interpreting specific events. Thus, while Hiebert and Carpenter’s internal networks of represented knowledge can be structured as vertical hierarchies, as webs, or, as some combination of the two structures, schema must have some sort of hierarchical structure. Another important difference between schema and internal networks is that schema are abstract representations to which specific situations are
connected as special cases whereas internal networks are collections of representations to which specific situations are connected if some aspect or part of the situation has something in common with the representations in the collection. It is these similarities and differences between schema and internal networks of represented knowledge that make both useful for both describing and comprehending mathematical understanding.

**Summary of Internal Knowledge Structures**

The notion that internal representations and connections between internal representations form internal knowledge structures provides a useful way of thinking about mathematical understanding, for several reasons.

First, it provides a level of analysis that makes contact with both theoretical cognitive issues and practical educational issues. The discussion of understanding can proceed in a way that places it in a larger theoretical context, while at the same time providing sufficient specificity to draw substantive instructional implications. Second, it provides a coherent framework for connecting a variety of issues in mathematics teaching and learning, both past and present.... Third, it suggests interpretations of students' learning that help to explain their successes and failures, both in and out of school.

(Hiebert & Carpenter, 1992, p. 67)

For the purposes of this study, the theoretical framework for examining issues of understanding proposed by Hiebert and Carpenter will be useful for
discussing and analyzing relationships between calculus instruction that emphasizes multiple representations and students' ability to use multiple representations for solving calculus problems.

**Theoretical Basis for Solving and Interpreting Problems**

The final theoretical framework, which supports calculus instruction that has students solve and interpret specific types of problems designed to reinforce or produce concepts and connections between concepts, is based on a theory of mathematical knowledge and its construction proposed by Dubinsky (1991) that applies the Piagetian notion of reflective abstraction to advanced mathematical thinking. A discussion on why this theoretical framework and the Hiebert and Carpenter framework are both necessary for this research is presented at the end of this section.

**ReFlective Abstraction and Mathematical Understanding**

The major part of every meaningful life is the solution of [mathematical] problems ... . It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts. (Halmos, 1980, p. 519)

In the Mathematical Association of America’s report, *Problem Solving in the Mathematics Curriculum* (Schoenfeld, 1983), the recommendation was made to encourage the use of a "problem based approach" for teaching
mathematics. Such an approach is meant to actively engage students in the
process of doing mathematics, foster an alert and questioning attitude in
students, promote the participation of students in discussing, solving, and
presenting their solutions to problems, and help students to employ ideas rather
than simply regurgitate them. Many current calculus reform projects have
adopted this approach to instruction.

One possible use of multiple representations in conjunction with the
"problem based approach" for teaching mathematics is to have students
investigate and explore different possible problem solutions by looking at
various representations of the situations and concepts described in the
problems. Based on the Hiebert and Carpenter (1992) framework, this
instructional approach should serve to increase students' understanding of
concepts described in the problems by helping them increase the number and
strength of the connections between corresponding representations. Hiebert and
Carpenter's belief that mathematical understanding is derived from the
connections between mental representations is similar to Kaput's (1989b) belief
that the connections that can be made between distinct versions of the same
type of representation or between different types of representations are
epistemological sources of mathematical meaning. Kaput offers the following
as the epistemological sources of mathematical meaning:
a. transformations within and operations on a particular representation system;
b. translations across mathematical representation systems;
c. translations between non-mathematically described situations and mathematical representation systems;
d. consolidation and reification of actions, procedures, or webs of related concepts into phenomenological objects that can then serve as the bases of new actions, procedures, and concepts at a higher level of organization. (p. 106)

Kaput's first three sources of mathematical meaning correspond to the various types of connections that can be made between distinct versions of the same type of representation and between different types of representations.

Kaput's fourth source of mathematical meaning can be thought of as the abstraction of actions or objects at one level of organization for use at a higher level of organization. This type of abstraction is similar to what Piagetian called *reflective abstraction*. According to Piaget, reflective abstraction consists of drawing properties from mental or physical actions at a particular level of thought.

[R]eflective abstraction consists in deriving from a system of actions or operations at a lower level, certain characteristics whose reflection ... upon actions or operations of a higher level it guarantees; for it is only to be conscious of the process of an earlier construction through a reconstruction on a new plane. (Beth & Piaget, 1966, p. 189)
Whatever is thus "abstracted" through reflective abstraction is then projected onto a higher plane of thought where other actions may also be present (Piaget, 1985), which in turn can lead to the construction of new actions by a conjunction of abstractions. Reflective abstraction forms the basis for a theory proposes by Dubinsky (1991) that supports having students interpret and solve various related problems as a means of constructing or acquiring mathematical knowledge.

Dubinsky applies the Piaget's notion of reflective abstraction to advanced mathematical thinking and then uses this notion to form a theory of mathematical knowledge and its construction or acquisition. To do this, he makes two assumptions with regards to mathematical knowledge and its construction. The first assumption is that mathematical knowledge consists of a collection of schemas, which he describes as follows.

A schema is a more or less coherent collection of objects and processes. A subject's tendency to invoke a schema to understand, deal with, organize, or make sense out of a perceived problem situation is her or his knowledge of an individual concept in mathematics. Thus an individual will have a vast array of schema ... [and] these schema must be interrelated in a large complex organization. (Dubinsky, 1991, p. 102)
The second assumption is that one means by which students construct new mathematical knowledge is through the acts of recognizing and solving problems, asking new questions, and creating new problems.

According to Dubinsky, reflective abstraction is the construction of mental objects and mental actions on these objects, and reflective abstraction occurs when students are constructing new knowledge by solving and interpreting problems.

[A] subject will have a propensity for responding to certain kinds of problems in a relatively (but far from totally) consistent way which we can ... describe in terms of schema. When the subject is successful, we say that the problem has been assimilated by the schema. When the subject is not successful then, in favorable conditions, her or his existing schema may be accommodated to handle the new phenomenon. This is the constructive aspect of reflective abstraction. (Dubinsky, 1991, p. 103)

This notion of reflective abstraction forms the basis for Dubinsky’s hypothesis that learning involves the application of reflective abstraction to existing schemas in order to construct new schemas for understanding concepts. This hypothesis can be supported by constructivist learning theory since the notion of applying reflective abstraction to an existing schema is similar to the notion of evaluating, connecting, and organizing new information relative to prior experiences. The hypothesis is also supported by Hiebert and Carpenter’s
theory of internal networks of representations since applying reflective abstraction to an existing schema is analogous to building connections between external representations in order to stimulate the construction of connections between corresponding internal representations.

For the purposes of this study, the theory of mathematical knowledge and its construction through reflective abstraction proposed by Dubinsky provides a framework for discussing and analyzing differences in students’ use and understanding of multiple representations when solving calculus problems. This framework focuses on whether or not students were given problems to interpret and solve that provided them with the opportunity to construct mathematical knowledge, possibly through the use of reflective abstraction.

Why Two Frameworks to Analyze Understanding of Multiple Representations

In the previous section, it was noted that Dubinsky’s (1991) theory of mathematical knowledge and its construction through reflective abstraction was supported by Hiebert and Carpenter’s (1992) theory of internal networks of representations. However, there are two critical differences between the theories which make both important to this research.

The first difference between the theories involves the amount of emphasis placed on having students interpret and solve problems as a means of constructing or acquiring concepts. For Dubinsky, the interpreting and solving
of problems are necessary if students are to construct or acquire concepts through reflective abstraction. For Hiebert and Carpenter, interpreting and solving of problems are just means of increasing the number or strength of connections between representations. The second difference between the theories involves how schemas and networks of represented knowledge are altered in order to construct mathematical knowledge. For Dubinsky, existing schema are altered through reflective abstraction on the actions taken on mathematical objects, not on the objects themselves, which creates new actions that are added to the schema. For Hiebert and Carpenter, networks of represented knowledge are altered by making new connections between representations already in the network (which may or may not involve actions on the representations) and by making connections to new representations.

The differences between the theories of Dubinsky and Hiebert & Carpenter make both useful for explaining differences in students' use and understanding of multiple representations when solving calculus problems. As such, both are valuable as theoretical rationales and theoretical frameworks for this research.
Theoretical Frameworks and Research Questions

In this section, the three research questions for this study will be discussed in terms of the two theoretical frameworks. The research questions are restated below.

1. What is the relationship between the instructional approach that students experience and any change in their initial preference for different representation when solving calculus problems?

2. What is the relationship between the instructional approach that students experience and their abilities to use graphical, numerical, and symbolic representations when solving calculus problems?

3. What is the relationship between the instructional approach that students experience and their abilities to see, or make, connections between graphical, numerical, and symbolic representations in the context of problem situations?

The items addressed in these research questions - changes in students’ initial preference for different representations, students’ abilities to use graphical, numerical, and symbolic representations when solving calculus problems, and students’ abilities to see, or make, connections between graphical, numerical, and symbolic representations in the context of problem situations - can each be related to the networks of represented knowledge induced by the instructional approach experienced by the students. Suppose students experience an
instructional approach that emphasizes one particular form of representation for presenting concepts or solving problems. Based on the Hiebert and Carpenter theoretical framework, the researcher would hypothesize that these students are likely to develop disjoint or weakly connected networks of represented knowledge associated with symbolic, graphical, and numerical representations.

With this structuring of internal networks, the students would tend to:

- have a significant initial preference for, and be better at using, the emphasized representation since it is likely their internal network corresponding to that representation is larger and more well-connected than their internal networks corresponding to other representations,
- have more difficulty using other forms of representations since it is likely their internal networks corresponding to the other representations are smaller and not as well-connected as the internal network corresponding to the emphasized representation, and
- have difficulty seeing, or making, connections between different forms of representations since it is likely their internal networks corresponding to the different representations are disjoint or weakly connected.

Now suppose students experience an instructional approach that emphasizes multiple representation for presenting concepts or solving problems. Based on the Hiebert and Carpenter theoretical framework, the researcher would hypothesize that these students may develop well-connected networks of represented knowledge associated with symbolic, graphical, and numerical representations, but not necessarily, especially if connections between different forms of representations are not made readily apparent to the students. Based
on the Dubinsky theoretical framework, the researcher also would hypothesize that the students' internal networks are more likely to have a greater number of stronger connections if, in addition, the instructional approach has students interpret and solve various related problems as a means of establishing connections between different representations of the same concept. With this structuring of internal networks, the students would tend to:

- have no significant initial preference for a particular representation since it is unlikely any of their internal networks corresponding to a particular representation would be significantly larger or more well-connected than the other networks.

- be adept at using more than one form of representation since it is likely they would have constructed large, well-connected internal networks for more than one form of representation, and

- have less difficulty seeing, or making, connections between different forms of representations since it is likely their internal networks corresponding to the different representations will be well-connected and strongly connected.

As these two examples illustrate, the theoretical frameworks for this study can be used to explain how different instructional approaches to calculus may be related to changes in students' initial preference for different representation and to students' use of, and understanding of connections between, graphical, symbolic, and numerical representations when solving calculus problems.
CHAPTER III

THE METHOD

The purpose of this study was to investigate how students in calculus classes where technology was used extensively, and instruction emphasized multiple representations of concepts and problem solving, differ from students in a traditional class in their abilities to use and understand multiple representations, or techniques, when solving calculus problems. Both quantitative and qualitative data were collected in order to make comparisons among the students from the three calculus courses participating in the study. In this chapter, the methodology used in the study will be described.

Subjects

Participants for this study were undergraduate students enrolled in three different calculus courses at a large midwestern university during Autumn Quarter 1993. Each course was the first in a four-quarter calculus sequence designed primarily for mathematics, science, and engineering students. One course, Math 151, was taught as a traditional calculus course. The second course, Math 151G, was similar in content to Math 151 but the instruction and
assignments for this course stressed graphical representations through the use of
graphics calculators. The final course, Math 151C, was the electronic calculus
course, Calculus & Mathematica, as initially designed by Brown, Porta, and
Uhl (1990) and later revised by Davis, Porta, and Uhl (1994). A detailed
description of the three course environments is presented in the next chapter.

Students from two intact Math 151 recitation sections from the same
lecture, two intact Math 151G recitation section from the same lecture, and two
intact Math 151C classes participated in the study. It should be noted that the
Math 151 lecturer has a reputation for being one of the better calculus
instructors at his university, the Math 151G lecturer is considered one of this
country’s experts on the use of graphics calculators in instruction, and one of
the two Math 151C instructors is regarded as a leader in the area of calculus
reform and was a co-author of the materials used in the Calculus &
Mathematica course at The Ohio State University.

Instrumentation

In order to conduct this study, instruments needed to be constructed to
look for any impact that the instructional approaches may have on students’
initial choice of representation when solving problems or on their use and
understanding of multiple representations when solving calculus problems.
This required the researcher to develop means for measuring students’ preferences in initial choice of representation when solving problems before and after completing their calculus courses, and means for measuring, or assessing, students’ use and understanding of multiple representations, or techniques, for solving calculus problems after they had completed their calculus courses.

The 18-item Representations Test (see Appendix A) designed by the researcher was used to determine whether or not students have preferences, prior to beginning their calculus course, in their initial choice of representation when solving problems. This test consists of nine problems. For each problem, graphical, numerical, and analytical representations of the problem were given, and students were asked to choose which representation they would be most likely to use to solve the problem and which representation they would be least likely to use. The problems were designed so that an answer could not be read directly from the table that served as the numerical representation since the results from an earlier pilot study suggested that students’ responses were biased toward the numerical representation when the answer was readily apparent in the table.

The four-item Calculus Representations Test designed by the researcher was used to measure students’ use and understanding of multiple
representations when solving calculus problems and as a measure of students' preferences, after completing their calculus course, in their initial choice of representation when solving problems. Content validity was established by a panel of experts prior to the administration of the instrument. One-on-one interviews with individual students were used to further assess and examine their use and understanding of multiple representations and multiple techniques for solving the problems given in the Calculus Representations Test.

**Development of the Calculus Representations Test**

One of the major problems of research that compares students in a traditional calculus course to students in a calculus course that utilizes some type of technology-rich environment is determining how to do the comparison in a way equitable to students in either course. The difficulty lies in the types of problems typically done by students in these course since these problems tend to reflect the different emphases, purposes, and goals of these two distinctly different types of calculus instruction. In his discussion of final examinations from different traditional calculus courses, Steen (1987b) succinctly describes the types of problems that students in these courses are typically expected to solve:
Most questions [on the examinations] asked for straightforward calculations or posed template problems that are taught over and over again in the course and that are in the textbook in nice boxed examples. Anybody who is wide awake and pays attention ought to be able to figure out how to do these types of problems. (p. 11)

Expectations are quite different in the Calculus & Mathematica course where the philosophy, as described by Brown, Porta and Uhl (1990), emphasizes the application of skills rather than simply learning skills:

Our view is that calculus is nothing more or less than a course on how to use the tools of differentiation, integration and approximations to make precise measurements. ... Our goal is to elicit the proper mathematical response to a given situation. (p. 104)

One solution to this obstacle of finding common ground on which to compare students in these two types of calculus courses involved finding problems that any calculus students, no matter which course they completed, should be able to solve.

An added criterion on the choice of problems to be used in the Calculus Representations Test was that each problem could be represented and (possibly) solved analytically, geometrically, or numerically. This criterion was necessary in order to allow the researcher to use the Hiebert and Carpenter (1992) framework for structures of represented knowledge to explore students’
understanding of calculus. Having problems that could be solved using
different representations made it possible for the researcher to discuss multiple
representations and multiple techniques for solving the problems with students
during the interviews and thus explore a student’s understanding by looking at
the number and strength of the student’s connections between representations.

A preliminary set of questions was constructed and then pilot tested twice
at the participating university. These questions were adapt from some
problems found in the Harvard Project textbook, *Calculus: Preliminary Edition*
(Hughes Hallet et al., 1992). The pilot tests were conducted during Autumn
Quarter 1992 and Winter Quarter 1993 using volunteers from Math 151 and
Math 151C courses - three or four students from each course each quarter. The
questions were also distributed to different mathematicians and mathematics
educators around the United States. Based on the results of the pilot tests, and
the suggestions and comments from the panel of experts, the four problems
presented in Figures 1 and 2 on the following two pages were chosen to be
used in the study.
1. The population of a herd of deer is given by the function

\[ P(t) = 4000 \cdot 500(\cos 2\pi t) \]

where \( t \) is measured in years and \( t = 0 \) corresponds to January 1.

a. When in the year is the population at its maximum? What is that maximum?

b. When in the year is the population at its minimum? What is that minimum?

c. When in the year is the population increasing the fastest? When in the year is the population decreasing the fastest?

d. Approximately how fast the population is changing on the first of July?

e. When you first started to solve this problem, which type of representation did you use? (circle one)
   symbolic graphical numerical

f. Which type of representation did you eventually use to solve this problem? (circle one)
   symbolic graphical numerical

2. Suppose \( N \), the total number of people who have contracted a disease \( t \) days after its outbreak, is given by the formula

\[ N = \frac{1,000,000}{1 + 5000e^{-0.1t}} \]

a. In the long run, how many people will contract the disease?

b. Is there a maximum number of people who will eventually contract the disease? Explain.

c. Is there any day on which more than a million people fall sick? Half a million? Quarter of a million? (Note: You do not need to determine on what days these things happen.)

d. When you first started to solve this problem, which type of representation did you use? (circle one)
   symbolic graphical numerical

e. Which type of representation did you eventually use to solve this problem? (circle one)
   symbolic graphical numerical

Figure 1: Problems 1 and 2 of the Calculus Representations Test
3. The table below gives U.S. population figures between 1790 and 1980.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population (in millions)</th>
<th>Year</th>
<th>Population (in millions)</th>
<th>Year</th>
<th>Population (in millions)</th>
<th>Year</th>
<th>Population (in millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1790</td>
<td>3.9</td>
<td>1840</td>
<td>17.1</td>
<td>1890</td>
<td>62.9</td>
<td>1940</td>
<td>131.7</td>
</tr>
<tr>
<td>1800</td>
<td>5.3</td>
<td>1850</td>
<td>23.1</td>
<td>1900</td>
<td>76.0</td>
<td>1950</td>
<td>150.7</td>
</tr>
<tr>
<td>1810</td>
<td>7.2</td>
<td>1860</td>
<td>31.4</td>
<td>1910</td>
<td>92.0</td>
<td>1960</td>
<td>179.0</td>
</tr>
<tr>
<td>1820</td>
<td>9.6</td>
<td>1870</td>
<td>38.6</td>
<td>1920</td>
<td>105.7</td>
<td>1970</td>
<td>205.0</td>
</tr>
<tr>
<td>1830</td>
<td>12.9</td>
<td>1880</td>
<td>50.2</td>
<td>1930</td>
<td>122.8</td>
<td>1980</td>
<td>226.5</td>
</tr>
</tbody>
</table>

a. Approximately how fast was the population changing in the years 1900, 1945, and 1980?

b. During what year(s) does it appear that the population growth was the greatest? Explain.

c. Based on the data from the table, what population would you predict for the 1990 census?

d. When you first started to solve this problem, which type of representation did you use? (circle one)
   symbolic          graphical          numerical

e. Which type of representation did you eventually use to solve this problem? (circle one)
   symbolic          graphical          numerical

4. a. Show that $x > 2 \ln x$ for all $x > 0$.
   (Note: This is equivalent to showing that $e^x > x^2$ for all $x > 0$.)

b. Is it true that $x > 3 \ln x$ for all $x > 0$?
   If not, estimate $M$ such that $x > 3 \ln x$ for all $x > M$.

c. What would you predict is the largest value of $a$ for which $x > a \ln x$ for all $x > 0$? (Note: This is equivalent to predicting the largest value of $a$ for which $e^x > x^e$ for all $x > 0$.)

d. When you first started to solve this problem, which type of representation did you use? (circle one)
   symbolic          graphical          numerical

e. Which type of representation did you eventually use to solve this problem? (circle one)
   symbolic          graphical          numerical

Figure 2: Problems 3 and 4 of the Calculus Representations Test
Development of Interview Questions

The purpose of the interviews was to better assess students' understanding of their solutions to the problems, to determine if students could solve the problems using different representations, either analytical, geometrical, or numerical, and to examine the connections students may or may not have between made different representations of the same problem. To do this, the interview questions had to probe students' responses to questions concerning their solutions to these problems and assess whether or not they could solve (or describe how to solve) the problems using another method. Thus, the interview questions had to be designed in such a way that the interviewer would be sure to cover all the important issues during the interviews and would have the flexibility to ask questions that might evolve from students' responses during the interview. These requirements precluded the use of a standardized set of questions for all of the interviews and motivated the use of the general interview guide approach (Patton, 1990) for collecting qualitative data. The fact that this approach fits the needs of this proposed study is readily apparent from Patton's (1990) description:
The interview guide simply serves as a basic checklist during the interview to make sure that all relevant topics are covered. The interview guide presumes that there is common information that should be obtained from each person interviewed, but no set of standardized questions are written in advance. The interviewer is thus required to adapt both the wording and the sequence of the questions to specific respondents in the context of the actual interview. (p. 280)

During the interviews, students were asked to discuss their solutions to each problem on the Calculus Representations Test. Then they were asked to solve the problems using techniques or representations different from the ones they had already used. It should be noted that as a time-saving measure, whenever students mentioned using a graphical representation to solve a problem, the interviewer would provide them with an appropriate graph using either a TI-81 graphics calculator or Mathematica graphics, and whenever students mentioned using a derivative and had not already computed that derivative, the interviewer would either provide the derivative or help the student compute it. The interview guide shown in Figure 3 was used to help direct the interviewer’s choice and wording of questions.
1. How did the student "solve" the problem? What steps were taken and why? What type of representation or representations were used?

2. Why did the student use the particular technique chosen to solve the problem? Is the only technique the student has ever used, or knows how to use, to solve this type of problem?

3. Did the student verify the solution? If so, was a different technique used?

4. Can the student solve, or describe how to solve, the problem using another representation? If not, what about after
   a. asking if the student has ever seen this type of problem, or a similar one, solved using a different representation?
   b. discussing some aspect of one representation in order to get students to consider how a different representation might be used on the problem, e.g., Why do you set the first derivative equal to zero when trying to locate extrema?
   c. suggesting that the student use some part of a previous solution in another way, e.g., Could you use that graph to also determine where the population is increasing the fastest?
   d. suggesting that an appropriate graphical, numerical, or algebraic representation could be used to solve the problem?
   e. asking the student solve a similar, simpler problem using an appropriate graphical, numerical, or algebraic representation?

**Figure 3: Interview Guide Questions**

**Procedures**

Permission to conduct the study was obtained from the university’s Mathematics Department. This permission allowed the researcher to administer the Representations Test and Calculus Representation Test to the students in
the respective calculus courses as part of normal class meetings and to ask for volunteers to participate in the interview portion of the study.

Students attending their Math 151 (n = 51) and Math 151G (n = 26) recitation sections on the second day during the second week of classes or attending their Math 151C classes (n = 52) on the third day during the second week were administered the Representations Test. Since this instrument was to assess any preferences students might have in their initial choice of representation when solving problems prior to beginning their calculus course, the test was given early in the quarter in order to minimize the confounding effects of the instructional methods encountered in the different calculus courses. The researcher did not attempt to locate students registered for each class but not in attendance on the specified days.

Students attending their Math 151 (n = 40) and Math 151G (n = 24) recitation sections on second day during the eleventh and final week of classes or attending their Math 151C classes (n = 36) on the fourth day during the final week were administered the Calculus Representations Test. Because of time limitations, each student was asked to solve only two of the four problems on this test using whatever means they normally use to solve homework problems. All students were asked to solve problem 1 of the test. The other problem was assigned randomly so that approximately the same number of
students in each of the six participating classes attempted problems 2, 3, and 4. The decline in the number of students taking the Calculus Representations Test from those taking the Representations Test was due mostly to withdrawals from the respective courses, though there were also some students registered for the courses who were not in attendance on the days that they instrument was administered. The researcher did not attempt to locate these students.

Classroom Observations

During the time between the administration of the Representations Test and the Calculus Representations Test, the researcher made weekly observations in each of the three courses. The purpose of these observations was to document the use of multiple representations in each of the different courses by both the instructors and the students. For the 151C classes, observations were made in each class once per week. For the Math 151 and 151G classes, observations were made during one lecture and one recitation class each week. Thus, two observations were made per course each week.

To make the observations, the researcher first divided the last 44 minutes of each class period into 11 four-minute chunks of time. Then, four of these time-chunks were randomly select. During these times, the use of graphical, symbolic, and numerical representations were tabulated. The researcher counted how many times during each of the four time-chunks a certain type of
representation was used by the instructor or, in the case of the Math 151C classes, by the students on their computer screen, to depict a particular problem or situation. For this tabulation, different types of representations used for different problems or problem situations were distinguished from different renderings of the same type of representation used on the same problem or problem situation.

In addition to the classroom observations, the researcher tabulated the number of times each of the different forms of representation was used as an initial representation for the homework problems in each of the three courses. This measure, along with the one from the classroom observations, provided a means for assessing and documenting the use of multiple representations in each class.

**Individual Student Interviews**

The interview portion of the study was conducted during the first four weeks of Winter Quarter 1994 in order to assure that all of the students had been out of their Math 151 course for approximately the same amount of time. Of the 40 Math 151 students, 36 Math 151C students, and 24 Math 151G students who were given the Calculus Representations Test, 21, 22, and 14 from each class, respectively, volunteered to participate in the interviews. From this group, 36 students, 12 from each of the different courses, were
chosen using stratified random sampling. The strata for this sampling were the three possible versions of the Calculus Representations Test. These strata were chosen to assure that the 12 students from each course consisted of four students who had worked on Problem 2, four who had worked on Problem 3 and four who had worked on Problem 4. Once volunteers from each course were divided into the three strata, student gender and initial preference for the different representations, as indicated on the Representations Test, were used to develop a rank ordering for making phone calls to schedule the interviews.

Students were called during the first week of Winter Quarter 1994 and scheduled for interviews according to the rank ordering, except when the researcher was unable to contact a student, in which case the next student in the ordering was contacted. An additional constraint on the choice of students was that they be enrolled in the next course in the calculus sequence at the university. The small number of available volunteers and changes in students’ schedule after interviews had been scheduled forced the use of five students who did not meet this constraint, although three of these students stated that they planned to take the next course in the sequence during Spring Quarter 1994. The interviews lasted between 25 to 45 minutes with the majority lasting about 40 minutes. Students were paid for participating in the interviews since interviews were conducted during students’ free time.
Research Design

The first research question for this study was:

1. What is the relationship between the instructional approach that students experience (Calculus & Mathematica, traditional, or graphics calculators) and any changes in their initial preference for different representation when solving calculus problems?

The results from the Representations Test were compared to students’ initial preference for different representations on Problem 1 of the Calculus Representations Test to investigate whether or not there had been any change in the students’ initial preferences.

The other two research questions were as follows:

2. What is the relationship between the instructional approach that students experience and their abilities to use graphical, numerical, and symbolic representations when solving calculus problems?

3. What is the relationship between the instructional approach that students experience and their abilities to see, or make, connections between graphical, numerical, and symbolic representations in the context of problem situations?

An analysis of students’ solutions to the problems in the Calculus Representations Test and their responses to questions asked during the individual interviews was used to investigate these research questions. Part of the analysis included charting the different types of representation and
techniques each student used for solving the problems during the Calculus Representations Test and during the individual interviews. The amount of prompting used by the researcher during the interviews in order to elicit the use of specific representations or techniques when solving problems were also incorporated in the analysis.

This choice of experimental design was made not only to look for possible differences in students' understanding of multiple representations among the different calculus courses, but also to provide information that could be used to explain other differences among the students. The use of the Representations Test and Calculus Representations Test together provides a means for assessing any possible changes in choice of initial representation when solving problems that may be attributable to the different classroom environments and instructional methods. The Calculus Representations Test only provides limited insight into students' understanding of multiple representations, which is why the follow-up interviews were included as the final part of the design. Only so much information can be gleaned by looking at how the students solved the problems on paper or at the computer. The interview process will provide additional information on how students use and understand different representations when solving problems that might not be discerned by simply looking at their solutions to the problems.
CHAPTER IV
THE ENVIRONMENT

The instructional environments of the three calculus courses are a factor in this research since the study investigates how the instructional approaches of the courses impact students' use and understanding of multiple representations when solving calculus problems. A comparison of the courses on their use of multiple representations during classroom instruction and in homework assignments will show that multiple representations were used more frequently in the graphics calculator and Calculus & Mathematica courses. A comparison of students' initial preference for different representations, using the results of the Representations Test, will show that the students were equivalent in their initial preferences at the beginning of each course.

Math 151

The Math 151 course at the Ohio State University is considered a traditional differential calculus like those typically taught at most universities and colleges. The course is taught in a lecture-recitation format with 48-minute lectures run by the course instructor on Mondays, Wednesdays, and Fridays and 48-minute recitation run by graduate teaching assistants on Tuesdays and
Thursdays. All new material is presented during the lecture. During recitation, problems from homework assignments and examinations are discussed, with the teaching assistant typically presenting the students' solutions to the problems, and occasional quizzes are given. Between the administration of the Representations and Calculus Representations Tests, three midterm examinations were taken by the students. About 13% of the students enrolled in the course dropped out during the time period of this study.

The course textbook, *Calculus* (Finney & Thomas, 1991), is used by many other schools for similar calculus courses. The topic outline for Math 151 (see Figure 4) indicates extent of coverage of standard topics from differential calculus. One topic not covered that had an impact on this study was exponential and logarithmic functions, in particular $e^x$ and $\ln x$.

During Autumn Quarter 1994, Math 151 students were allowed to bring graphics calculators to class and to use them to do their homework, but these calculators could not be used during quizzes or examinations. Graphics calculators were not used for course instruction in either lectures or recitations during the quarter.
Chapter 2 Limits and Continuity

Limits
The Sandwich Theorem and $\frac{\sin \theta}{\theta}$
Limits Involving Infinity
Continuous Functions
Defining Limits Formally with Epsilons and Deltas

Chapter 3 Derivatives

Slopes, Tangent Lines, and Derivatives
Differentiation Rules
Velocity, Speed, and Other Rates of Change
Derivatives of Trigonometric Functions
The Chain Rule
Implicit Differentiation and Fractional Powers
Linear Approximations and Differentials
Newton's Method for Approximating Solutions of Equations

Chapter 4 Applications of Derivatives

Related Rates of Change
Maxima, Minima, and the Mean Value Theorem
Curve Sketching with $y'$ and $y''$
Graphing Rational Functions - Asymptotes and Dominant Terms
Optimization
Antiderivatives, Initial Value Problems, and Mathematical Modeling

Figure 4: Topic Outline for Math 151

Math 151G

What makes the Math 151G course different from Math 151 is that the course content and instruction in Math 151G are designed to incorporate and make extensive use of graphical representations through the use of a graphics calculator. Whenever possible, concepts are presented in the textbook and during lectures using both symbolic and graphical representation. Students are
required to have a graphics calculator and use it during class, on homework assignments, and during examinations. It should be noted that the graduate teaching assistant for this course did not show nearly the same enthusiasm, or depth of understanding, for using the graphics calculator as the course instructor. The teaching assistant did not use the graphics calculator as often as the instructor, nor did he use the overhead projection attachment that allows the calculator display to be seen on a projection screen. It is unclear what, if any, impact this difference in graphics calculator use between the instructor and assistant had on students’ use of the graphics calculator.

Except for the required work with the graphics calculator and the additional use of graphical representations, Math 151G was very much like Math 151, as the topic outline for the course indicates (see Figure 5). The course was taught using the same five meetings per week lecture-recitation format as Math 151, and three midterm examinations were given between the administration of the Representations and Calculus Representations Tests. As was the case with Math 151, about 13% of the students enrolled in Math 151G dropped out during the time period of this study.

The course textbook, *Calculus: A Graphing Approach, Volume 1* (Finney, Thomas, Demana, & Waits, 1993), was similar in content and presentation to the textbook, *Calculus* (Finney & Thomas, 1991), used for the Math 151. A
Figure 5: Topic Outline for Math 151G

topic-by-topic comparison shows that parts of many lessons and many exercise sets from the Math 151 text are copied verbatim in the Math 151G text. For example, both texts use exactly the same exercises in the lesson on the chain
rule. Also, one of the requirements placed on the course instructor by the Ohio State University was that 75% of the problems on each examination in the course had to be equivalent to those used in the traditional course. This meant that the problems were to be solvable using only symbolic methods.

Two main differences distinguish the texts for Math 151G and Math 151. The first was that the Math 151G text included additional sections in most lessons where graphical representations of concepts were presented and discussed. Some of these new sections replaced similar material found in the Math 151 text. The second difference was that the exercise sets in the Math 151G text require additional work either verifying or solving problems graphically. The exercise sets reflect the three following approaches to solving problems introduced to students in Math 151G.

1. Solve analytically, and support the results graphically.
2. Solve graphically, and confirm the results analytically.
3. Solve using a combination of graphical and analytic techniques.

(F. Demana, personal communication, April 18, 1994)

The purpose of introducing the first two approaches listed above is to help students understand the role of analytical solution techniques in calculus and realize that a graph by itself does not constitute a proof.
Math 151C

Math 151C, Calculus & Mathematica, is an electronic calculus course designed around the computer software Mathematica which can perform a myriad of mathematical functions, including arithmetic computations, symbolic manipulations, two- and three-dimensional graphs, all the hand calculations normally associated with calculus, and, more importantly, word processing. The word processing capabilities allowed the course designers to create something they called "Mathematica notebooks" which are live electronic documents consisting of a mixture of static text and active programs. Porta and Uhl (1990) described their vision of the notebooks as follows:

Imagine a calculus book in which every example can be modified on the spot to become infinitely different examples, a calculus book that can plot any calculus function, a calculus book that can differentiate, integrate, find roots, expand power series - in short a calculus book that can teach the students and act as a slave for them. (p. 74)

The notebooks provide an interactive environment where students can enact portions of the notebook to create graphics, see problems solved, and experiment with the mathematics. For most problems, students worked out a few concrete examples and then embarked on their own mathematical explorations in order to find ways to interpret and solve the problem(s). Because virtually all of the course materials are contained in the Mathematica
notebooks, lecturing was kept to a minimum, though the instructor would spend time in front of the class discussing a problem when it appears that are significant number of students were having difficulties with the problem.

Math 151C was taught using Mathematica notebooks originally written by Brown, Porta, and Uhl (1991) and later revised for use at the Ohio State University by Davis, Porta, and Uhl (1994). The course was divided into nine "Lessons" with each "Lesson" consisting of four distinct Mathematica notebooks. The Basics and Tutorial notebooks covered all the essential material for that lesson and include solutions to all of the questions and problems. The GiveItaTry notebook contained problems for the student to solve and interpret and included tips for solving some problems but did not include any solutions. The Literacy Sheet notebook contained more problems, covering what were considered to be the important concepts and skills in that lesson that were to be solved using paper-and-pencil, though the computer could occasionally be used. A textbook (Davis, Porta, & Uhl, 1994) containing a hardcopy of the notebooks was available to students at the time of this study.

For each Lesson, students were required to turn in an individual assignment, group assignment, and the Literacy Sheet notebook. Individual and group assignments were taken from the problems in the GiveItaTry notebook and had to be turned into the instructor electronically via the network
electronic-mail system, while the Literacy Sheet notebook was to be printed out, completed using paper and pencil, and turned in. All members of a group, which consisted of three to five students, received the same grade on each group assignment, no matter who worked on the assignment and who did not. A separate grade was assigned to each group member based on the instructor perception of how well each student performed within the group structure. No midterm examinations were administered during the quarter, but weekly quizzes were given over the materials in the current Literacy Sheet notebook. Between the administration of the Representations and Calculus Representations Test, about 19% of the students enrolled in the course dropped out.

**Course Instruction and Role of the Instructor**

Cooperative groups were an essential part of instruction in Math 151C. All group assignments were to be done using a cooperative group structure with the students in each group assigning themselves to the roles of recorder, facilitator, and so on. Two or three days per week were set aside by the instructors for cooperative group work; no individual work was allowed on those days. The instructors also suggested that students also complete their Literacy Sheets during cooperative group work, though this suggestion was not always heeded. This structure was used in the Math 151C course because it is the Calculus & *Mathematica* authors’ belief that cooperative groups provide the
best means for getting students to embarked on the mathematical explorations necessary in order to interpret and solve the problems and thus learn calculus.

The instructional philosophy of Calculus & Mathematica forces a drastic change in the role of the instructor from the role usually assumed by an instructor in a beginning calculus course at the university level. Crocker (1991, pp. 15-16) observed the following primary changes in the instructor’s role:

a. facilitating learning activities rather than presenting information
b. stimulating discussion and group work
c. raising questions rather than providing answers
d. troubleshooting problems with materials as well as equipment

Crocker gives the following description of learning environment in the Calculus & Mathematica classroom, based on her own observation.

The professor and teaching assistants worked as a support mechanism for the learning taking place and not as the providers of knowledge. The relaxed and open atmosphere of the classroom promoted students working together. The professor and teaching assistants for the course worked as participants with the students. The students were relaxed and open to discuss the mathematics with each other as well as the professor and teaching assistants. (Crocker, 1991, p. 16)

This role for the instructor is similar to the role of a teacher presented by The Mathematics Sciences Education Board (1990) which is that a teacher should be a role model, consultant, moderator, interlocutor, and questioner.
Comparing Math 151C to Math 151 and Math 151G

By design, much of the material covered in Math 151C was different from that covered in Math 151 or 151G, as the topic outline suggests (see Figure 6). Similar topics were usually presented in a manner quite unlike that used in Math 151 or 151G. Brown, Porta, and Uhl (1990) listed the following changes in content incorporated into the Calculus & Mathematica course:

Feeling for limits and convergence is set up through plots showing the curves \([f(x + h) - f(x)]/h\) crawl onto \(f'(x)\) as \(*h*\) gets small.

The chain rule forms the keystone of our treatment of differentiation. ...

Continuity and limits do not sit at the front of the course but emerge in a natural way throughout the course. Students work on continuity by being able to report how many accurate decimals of \(x\) are needed to calculate, say, eight accurate decimals of \(f(x)\).

The Mean Value Theorem is studied as a consequence of something we call the Race Track Principle. This principle says that if \(f(a) = g(a)\) and \(f'(x) \geq g'(x)\) for \(x \geq a\), then \(f(x) \geq g(x)\) for \(x \geq a\). Another version of the principle says that if \(f(a) = g(a)\) and \(f'(x)\) is close to \(g'(x)\) for \(x\) near \(a\), then \(f(x)\) is close to \(g(x)\) for \(x\) near \(a\). The Mean Value Theorem is a corollary of the Race Track Principle.

Early in the course we confront the problem of plotting \(f(x)\) given only \(f'(x)\) and one value of \(f(x)\). Differential equations appear liberally in this course, but maybe not so much as in some other calculus revision projects.

Our course is not in the business of defining slopes, arcs, arc lengths, and volumes. We are in the business of measuring these quantities. Newton never heard of a Riemann sum, but Newton did teach us via the fundamental theorem ... that we can measure any quantity once we calculate its derivative. Finding the derivative of area, volume, arc length, and the like is good geometric mathematics. (pp. 104-105)
Lesson 1  Growth
Growth of (a) line functions \( f(x) = ax + b \), (b) power functions
\( f(x) = ax^k \), and (c) exponential functions \( f(x) = ae^{bx} \)
Dominance in the global scale and percentage growth

Lesson 2  Natural Logs and Exponentials
The natural base e, the natural logarithm, and unnatural bases
Percentage growth of exponential functions: Doubling

Lesson 3  Instantaneous Growth Rates
Average growth rate versus instantaneous growth rates
The instantaneous growth rate of (a) \( x^k \) is measured by \( kx^{k-1} \),
(b) \( \sin x \) is measured by \( \cos x \), (b) \( \cos x \) is measured by
-\( \sin x \), (d) \( \log x \) is measured by \( 1/x \), and (e) \( e^x \) is measured by \( e^x \).

Lesson 4  Rules of the Derivative
Derivatives and instantaneous growth rates
The chain rule and other rules for taking derivatives
Using the logarithm to calculational advantage
Instantaneous percentage growth rate
Exponential growth dominates power growth and power growth
dominates logarithmic growth

Lesson 5  Using the Tools
Using the derivative to (a) find maximum values and minimum values,
(b) help get a good representational plot, and (c) fit data by curves

Lesson 6  The Differential Equations of Calculus
The most important of all differential equations: \( y'(x) = r y(x) \)
The logistic differential equation: \( y''(x) = r y(x)(1 - y(x)/b) \)
Logistic growth is controlled growth.
The differential equation: \( y''(x) = r y(x) + b \)

Lesson 7  The Race Track Principle
The Race Track Principle
The Race Track Principle and differential equations,
The Race Track Principle and Euler’s method of faking the plot of the
solution of a differential equation

Lesson 8  More Differential Equations
Euler’s faker and Mathematica’s faker
Simultaneous differential equations: The predator-prey model

Lesson 9  Parametric Plotting
Parametric plots in two dimensions
Parametric plots of curves and surfaces in three dimensions
Derivatives for curves given parametrically

Figure 6: Topic Outline for Math 151C
In the preface to their text, Davis, Porta, and Uhl (1994) offer students a list of significant changes in content and emphasis from a traditional calculus course that includes the following items:

Your writing, plotting, and experimenting is the stock and trade in the course.

[You experience] a course in measurements heavily intertwined with other parts of science and the world.

Your emphasis is on linear and exponential growth from the beginning, before calculus begins. Linear functions are those with constant growth rates; exponential are those with constant percentage growth rates.

You learn at the very beginning that exponential growth dominates power growth without appeal to the mysticism of L'Hospital's rule or any other calculus ideas.

You study functions not for their own sake, but rather for the measurements they make.

You learn about the derivative as a measurement of the instantaneous growth rate. As a result, the idea that functions with positive derivatives are increasing functions is available to you immediately without waiting for the Mean Value Theorem. The interpretation of the derivative as the slope of the tangent line is delayed.

You work with and analyze real world data on applications important to you. ...

You learn the meaning of the derivative as a measurement at the same time you're learning to calculate derivatives. This idea is reinforced by many plots that you produce and analyses of the graphs of f(x) and f'(x) on the same axes. ...

Following Poincare, you do differentiation of functions of two variables with respect to each variable without much fanfare.
You do serious work with mathematical models involving derivatives. The benefits are twofold: Working with the models reinforces the idea of what a derivative is, and you can experience the tenacles of calculus outside the traditional calculus classroom. ...

Parametric plots in two and three dimensions are studied in the first course [of the calculus sequence] because they provide you with the needed plotting freedom for what’s to come. (pp. iv-v)

**Use of Representations During Course Instruction**

The researcher made twice-weekly observations for each course in order to document the use of multiple representations by both the instructors and the students. For the observations, the use of graphical, symbolic, and numerical representations were tabulated during four randomly chosen four-minute time-chunks during each class. For this tabulation, different types of representations used on different problems or situations were distinguished from different renderings of one type of representation used on the same problem or situation. The results are presented in Table 1.

Given the nature of differential calculus, it was not surprising that symbolic representations dominated all three courses and that numerical representations were used infrequently. It was also not surprising that nearly 90% of the representations used in the traditional course were symbolic, or that the smallest difference in use of symbolic and graphical representations occurred in the Calculus & *Mathematica* course. What was surprising was that
Table 1: Use of Representations During Class by Course

<table>
<thead>
<tr>
<th>Representation</th>
<th>151</th>
<th>151G</th>
<th>151C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent of Different Problem Representations of the Following Type</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Graphical</td>
<td>9.3</td>
<td>23.0</td>
<td>32.3</td>
</tr>
<tr>
<td>Numerical</td>
<td>4.3</td>
<td>9.3</td>
<td>6.0</td>
</tr>
<tr>
<td>Symbolic</td>
<td>86.4</td>
<td>67.7</td>
<td>61.7</td>
</tr>
<tr>
<td>Percent of Total Representation Used of the Following Type</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Graphical</td>
<td>7.3</td>
<td>21.4</td>
<td>35.1</td>
</tr>
<tr>
<td>Numerical</td>
<td>3.7</td>
<td>7.3</td>
<td>6.8</td>
</tr>
<tr>
<td>Symbolic</td>
<td>89.0</td>
<td>71.3</td>
<td>58.1</td>
</tr>
</tbody>
</table>

symbolic representations were used as often as they were in the graphics
calculator course, about three times as often as graphical representations were used. One possible explanation for this result is the similarity of content in the
traditional and graphics calculator courses, as previously discussed. There was
a strong emphasis on symbolic representations in the graphics calculator course,
even with its added emphasis on graphical representations, since its content was
adapted from the content of the traditional course, where a heavy emphasis is
placed on symbolic representations.

In addition to the classroom observations, the researcher tabulated
different uses of graphical, numerical, and symbolic representations on the
assigned homework problems for each course. These results are presented in
Table 2. In this table, the entry "Only symbolic representations were used"
Table 2: Use of Representations in Problem Assignments by Course

<table>
<thead>
<tr>
<th>How representations were used in problems</th>
<th>Percentage of problems by Course</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>151</td>
</tr>
<tr>
<td>Numerical representation could be used to solve problem or was an initial representation.</td>
<td>4.2</td>
</tr>
<tr>
<td>Graphical representation could be used to solve problem or was an initial representation.</td>
<td>9.5</td>
</tr>
<tr>
<td>Graphical representation was somehow used.</td>
<td>20.7</td>
</tr>
<tr>
<td>Only symbolic representations were used.</td>
<td>76.8</td>
</tr>
</tbody>
</table>

refers to problems where symbolic representations were (a) the only representations presented initially and (b) expected to be the only type of representation used by students to solve the problem.

As expected, symbolic representations were emphasized in the traditional course, graphical representations were emphasized in the graphics calculator course, and graphical and symbolic representations received about the same emphasis in the Calculus & Mathematica course. The infrequent use of numerical representations followed the same pattern found during the classroom observations. This lack of emphasis in the Calculus & Mathematica course was not expected considering the authors' claims the students would be working with and analyzing real-world data. A likely explanation for this
occurrence is that the course instructors did not assign many problems that involved the use of numerical representations. This would also explain why the usage of numerical representations was low during classroom observations since the observations in Math 151C consisted, for the most part, of watching the students work on assigned problems.

Tables 1 and 2 suggest that there were difference in the degree of use of graphical and symbolic representations in the three courses. Symbolic representations were used much more often than graphical representations in the traditional course, graphical representations received much more emphasis in the graphics calculator course than in the traditional course, and graphical and symbolic representations received about the same emphasis in the Calculus & Mathematica course.

**Students’ Initial Preferences for Representations at Beginning of Course**

One purpose of the Representations Test was to determine whether or not the students from each course were equivalent in their initial preference for different representations. Each student was classified as having a preference initially determined by the maximal chosen representation. For example, a student who chose a graphical initial representation on five problems, a symbolic initial representation on three problems, and a numerical initial
representation on one problem was classified as having a preference initially for graphical representations. Students who had more than one maximal chosen representation - four graphical, four symbolic, and one numerical, for example, were classified as having no preference initially. Initial preferences were classified as significant if the chi-square test for equi-probable means (Mendenhall, Sheaffer, & Wackerly, 1986) had a p-value less than 0.05. The percentages of students’ initial preference for each type of representation are presented in Table 3.

Table 3: Students’ Preferences Initially for Different Representations on the Representation Test by Course

<table>
<thead>
<tr>
<th>Course</th>
<th>Percent Preferring</th>
<th>No Preference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Symbol</td>
<td>Graph</td>
</tr>
<tr>
<td>151</td>
<td>Overall</td>
<td>45.10</td>
</tr>
<tr>
<td></td>
<td>Significant</td>
<td>5.88</td>
</tr>
<tr>
<td>151G</td>
<td>Overall</td>
<td>42.31</td>
</tr>
<tr>
<td></td>
<td>Significant</td>
<td>15.38</td>
</tr>
<tr>
<td>151C</td>
<td>Overall</td>
<td>46.15</td>
</tr>
<tr>
<td></td>
<td>Significant</td>
<td>23.08</td>
</tr>
<tr>
<td>All Courses</td>
<td>Overall</td>
<td>44.96</td>
</tr>
<tr>
<td></td>
<td>Significant</td>
<td>14.73</td>
</tr>
</tbody>
</table>
The results presented in Table 3 are interpreted as follows. For Calculus & Mathematica, 46.15% of the students had an initial preference for symbolic representations, and 23.08% had a significant initial preference for symbolic representations. Thus, 50% of these students (23.08% out of a possible 46.15%) with an initial preference for symbolic representations had a significant initial preference. Similarly, 33.3% of the Calculus & Mathematica students (11.54% out of a possible 34.62%) with an initial preference for graphical representations and 14.3% (1.92% out of a possible 13.46%) with an initial preference for numerical representations had significant initial preferences. Finally, 38.78% of the Calculus & Mathematica students (36.54% out of a possible 94.23%) with an initial preference for a particular representation had a significant initial preference for that representation.

All of the students had about the same preference initially for symbolic representations. The students from the traditional and Calculus & Mathematica courses had similar initial preferences for graphical and numerical representations. The students from the graphics calculator course had a higher preference initially for graphical representations and a lower preference initially for numerical representations than the students from the other courses. A substantially higher percentage of the graphics calculator students exhibited a significant initial preference for graphical representations.
One explanation for the difference in initial preference for graphical representations is the self-selection of the students in the three courses. It is reasonable to assume that students who chose the graphics calculator calculus course would originally have a stronger preference for graphical representations than students in the other courses, since the Math 151G students knew from the course description that they would be required to do extensive work with graphs. Students in the other courses would not be expected to have a stronger preference for graphical representations since the other course descriptions made no specific mention of work with graphs, though the description for Math 151C did mention that work on a computer was required.

Based on this comparison of initial preferences for different representations at the beginning of the course, the researcher believes it is likely that the students from the three courses were equivalent in their preference initially for symbolic representations, and that the students from the traditional and the Calculus & Mathematica course were equivalent in their preference initially for graphical and numerical representations.

**Summary of Results**

Based on the results from classroom observations and the Representations Test, the actual behavior of the instructors and students for each of the three
calculus courses were as expected. In the traditional course, the course content and the instructors emphasized symbolic representations, and the students showed a preference for using symbolic representations. In the graphics calculator course, the course content and the instructors placed a great deal more emphasis on graphical representations than in the traditional course, and the students showed a decided preference for graphical representations. In the Calculus & Mathematica course, the course content and instructors placed close-to-equal emphasis on graphical and symbolic representations, and the students showed only a slight greater preference for symbolic representation over graphical representations. This alignment of expectations versus actual outcomes for the course materials, instructors, and students makes it possible to look for relationships between the different technological approaches to calculus instruction and students' use and understanding of multiple representations when solving calculus problems.
CHAPTER V

RESULTS AND DISCUSSION

This chapter contains analyses of the results from the Calculus Representations Test and analyses and comparisons of student interviews for each problem on the Calculus Representations Test. The results of the Calculus Representations Test are presented first. The analysis of results includes a comparison of students’ initial preferences for representations, as determined by Problem 1 of the Calculus Representations Test, and a comparison of these student preferences to the ones from the Representations Tests to determine if there was any change in students’ initial preference for representations that might be attributed to the different instructional approaches of the calculus courses. A summary and comparison of students interviews are presented next. Students from the different courses are compared on the amount of prompting used by the researcher during the interview to obtain appropriate descriptions on how to use different forms of representations to solve each problem. An analysis of the interviews also provides a comparison of students’ understanding of different relationships and connections between certain types of representations of calculus concepts.
Students' Initial Preferences for Representations at End of Course

The results from Problem 1 of the Calculus Representations Test were analyzed to investigate whether or not students from the three calculus courses had different preferences initially for representations and to determine whether or not there had been a change in students' initial preference for different representations since the administration of the Representations Test. Problem 1 was used in this investigation since it was the only problem given to all the students who took the Calculus Representations Test. For the analysis, Problem 1 (see Figure 7) was broken into the following four parts.

a. Locating the extrema of population function $P(t)$.
b. Deciding which extrema of $P(t)$ are maxima and which are minima.
c. Determining when the population is increasing/decreasing the fastest.
d. Approximating the rate of population change on July 1.

The population of a herd of deer is given by the function

$$P(t) = 4000 - 500(\cos 2\pi t)$$

where $t$ is measured in years and $t = 0$ corresponds to January 1.

a. When in the year is the population at its maximum? What is that maximum?
b. When in the year is the population at its minimum? What is that minimum?
c. When in the year is the population increasing the fastest? When in the year is the population decreasing the fastest?
d. Approximately how fast is the population changing on the first of July?

Figure 7: Problem 1 of the Calculus Representations Test
Students' First Choice of Representation

On each part of Problem 1, the students' first choice of representation was determined by looking at their solutions to that problem, by looking at their answers to the multiple-choice question concerning the type of representation used when they first started solving the problem, and, whenever possible, by analyzing their responses to interview questions. Unlike the Representations Test, it was possible for students to choose "combination" representations, i.e. representations that combines graphical, symbolic, or numerical representations. For example, if a student chose initially to substitute values of \( t \) into \( P'(t) \) that were just less than and just greater than the value of \( t \) for which the function had an extremum, then the student's first choice of representation was classified as symbolic/numerical since the student calculated a derivative (symbolic) and then evaluated the derivative at more than one point (numerical). If a student chose initially to trace along the graph of \( P(t) \), then the student's first choice of representation was classified as graphical/numerical since the student generated a graph (graphical) and then determined values of \( x- \) and \( y- \) coordinates of points on the graph (numerical). Finally, if a student chose initially to use a graph of \( P'(t) \), then the student's first choice of representation was classified as graphical/symbolic since the student calculated a derivative (symbolic) and then generated its graph (graphical).
The results of the analysis of students' first choice of representation for each part of Problem 1 are presented in Table 4.

**Table 4:** Comparison of Students’ First Choice of Representation on Each Part of Problem 1 of the Calculus Representations Test

<table>
<thead>
<tr>
<th>Different Parts of Problem 1</th>
<th>Course</th>
<th>Percent Using Given Representation First</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>G</td>
</tr>
<tr>
<td>Locate extrema</td>
<td>151</td>
<td>10.0</td>
</tr>
<tr>
<td></td>
<td>151G</td>
<td>79.1</td>
</tr>
<tr>
<td></td>
<td>151C</td>
<td>27.7</td>
</tr>
<tr>
<td>Decide if extremum is a maximum or minimum</td>
<td>151</td>
<td>12.5</td>
</tr>
<tr>
<td></td>
<td>151G</td>
<td>58.3</td>
</tr>
<tr>
<td></td>
<td>151C</td>
<td>36.1</td>
</tr>
<tr>
<td>Determine when population is increasing or decreasing fastest</td>
<td>151</td>
<td>27.5</td>
</tr>
<tr>
<td></td>
<td>151G</td>
<td>62.5</td>
</tr>
<tr>
<td></td>
<td>151C</td>
<td>33.3</td>
</tr>
<tr>
<td>Approximate rate of population change on July 1</td>
<td>151</td>
<td>2.5</td>
</tr>
<tr>
<td></td>
<td>151G</td>
<td>41.7</td>
</tr>
<tr>
<td></td>
<td>151C</td>
<td>16.7</td>
</tr>
</tbody>
</table>

**Abbreviations for Representations in Table 4:**

- G - Graphical
- S - Symbolic
- N - Numerical
- G/N - Graphical/Numerical
- G/S - Graphical/Symbolic
- S/N - Symbolic/Numerical
The following observations about the students' first choice of representations can be made based on these results:

1. Traditional students chose symbolic representations first most often, especially when the symbolic representation involved use of the first derivative. They chose numerical representations first next most often and graphical representations least often.

2. Graphics calculator students chose graphical representations first most often by a wide margin. When "combination" representations were taken into account, numerical representations were chosen first next most often, followed closely by symbolic representations.

3. Calculus & Mathematica students chose symbolic representations first slightly more often than they chose graphical representations, followed fairly closely by numerical representations.

4. Calculus & Mathematica students' first choices of representation was the most evenly distributed across the three main forms of representations for the students from the three courses. The graphics calculator students had the most uneven distribution of first choices.

5. Calculus & Mathematica students chose "combination" representations first slightly more often than the students from the other two courses.

6. All the students had difficulties initially on part d, and all but the Calculus & Mathematica students had difficulties initially on part c.
One possible explanation for the high number of students who did not initially attempt to solve part c (20%) and part d (30%) was that the wording, or context, of these questions was unfamiliar to the students, making it difficult for them to determine exactly how to go about solving these problems. During the interviews, some students commented that they had not understood exactly what the problems were asking for until the interviewer had had a chance to explain them.

In summary, the results of this analysis suggest that Calculus & Mathematica students are more flexible in their first choice of representation than the students from the other courses, particularly those from the graphics calculator course. For example, about 22% of the Calculus & Mathematica students answered parts a and b initially by using the first derivative to locate the extrema and then using a graph to determine maxima and minima. Only about 4% of the graphics calculator students and 7% of the traditional students used this method. Meanwhile, about 75% of the graphics calculator students selected some form of graphical representation first, and about 35% of the traditional students selected some form of symbolic representation first on both part a and part b. Such flexibility in first choice of representation may indicate a better understanding of how to use the different types of representations associated with the concepts presented in this problem.
Students' Initial Preference for Representations

Students were classified as having an initial preference for different representations based on maximal chosen representation of those first chosen when solving each part of Problem 1. For example, a student who chose a graphical representation first on two parts, a symbolic representation first on one part, and a numerical representation first on one part was classified as having a preference initially for graphical representations. For "combination" representations, each type of representation was counted once so that a student that chose a graphical, a graphical/numerical, a symbolic/graphical, and a symbolic representations first would also be classified as having a preference initially for graphical representations. Students who had more than one maximal chosen representation — two graphical and two symbolic, or one graphical, one numerical, and two with no first choice, for example — were classified as having no preference initially. As was the case with the Representations Test, initial preferences were classified significant if the chi-square test for equi-probability means (Mendenhall, Sheaffer, & Wackerly, 1986) had a $p$-value less than 0.05.

The percent of students who had initial preferences on Problem 1 for each type of representation are presented in Table 5. A comparison of these results to the results from the Representations Test are presented in Table 6.
Table 5: Comparison of Students’ Preferences Initially for Different Representations on Problem 1 of the Calculus Representations Test

<table>
<thead>
<tr>
<th>Calculus Course</th>
<th>Percent Preferring</th>
<th>No Preference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Symbol</td>
<td>Graph</td>
</tr>
<tr>
<td>151</td>
<td>Overall</td>
<td>40.00</td>
</tr>
<tr>
<td></td>
<td>Significant</td>
<td>0.50</td>
</tr>
<tr>
<td>151G</td>
<td>Overall</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>Significant</td>
<td>0.00</td>
</tr>
<tr>
<td>151C</td>
<td>Overall</td>
<td>33.33</td>
</tr>
<tr>
<td></td>
<td>Significant</td>
<td>2.78</td>
</tr>
<tr>
<td>All Courses</td>
<td>Overall</td>
<td>28.00</td>
</tr>
<tr>
<td></td>
<td>Significant</td>
<td>3.00</td>
</tr>
</tbody>
</table>

Table 6: Comparison of Students’ Preference Initially for Different Representations on the Two Representations Test

<table>
<thead>
<tr>
<th>Calculus Course</th>
<th>Percent Preferring</th>
<th>No Preference</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Symbol</td>
<td>Graph</td>
</tr>
<tr>
<td>151</td>
<td>Rep Test</td>
<td>45.10</td>
</tr>
<tr>
<td></td>
<td>CRep Test</td>
<td>40.00</td>
</tr>
<tr>
<td>151G</td>
<td>Rep Test</td>
<td>42.31</td>
</tr>
<tr>
<td></td>
<td>CRep Test</td>
<td>0.00</td>
</tr>
<tr>
<td>151C</td>
<td>Rep Test</td>
<td>46.15</td>
</tr>
<tr>
<td></td>
<td>CRep Test</td>
<td>33.33</td>
</tr>
<tr>
<td>All Courses</td>
<td>Rep Test</td>
<td>44.96</td>
</tr>
<tr>
<td></td>
<td>CRep Test</td>
<td>28.00</td>
</tr>
</tbody>
</table>
The comparison of the results from the Representations and Calculus Representations Tests indicates that there were shifts in students' initial preferences for representations. The most dramatic shift occurred in the graphics calculator course where there was a sharp increase in initial preference for graphical representations and an even sharper decrease in initial preference for symbolic representations. In each course, there was an increase in the number of students with no initial preference for a representation, which may be attributable to the low number of questions (n = 4) used to determine preferences on the Calculus Representations Test.

In the Calculus & Mathematica and traditional courses, the preference initially for symbolic representations decreased — only slightly in the case of the traditional course — and the preference initially for numerical representations increased slightly. Also, the preference initially for graphical representations decreased in the traditional course. Comments by students suggest that they used numerical representations initially more often because they were not always sure what was being asked for in the problems and thus were unsure what symbolic or graphical methods to use on these problems.

The comparison of the Representations and Calculus Representations Tests indicates the instructional approach that students experienced had an impact on their initial preference for different representations when solving
problems. In the graphics calculator course, students’ initial preference for graphical representations increased noticeably. In the traditional course, students’ initial preference for symbolic representations decreased slightly, but increased in comparison to the two other representations. In the Calculus & Mathematica course, students’ initial preference for symbolic representations decreased, making their initial preferences for symbolic and graphical representations equivalent. These shifts in initial preferences suggest that the Calculus & Mathematica approach to calculus instruction may be better than the other approaches at helping students recognize how to best use different representations of the same concept when initially solving calculus problems.

Summary and Comparison of Student Interviews

This section contains summaries and comparisons of the student interviews for each problem on the Calculus Representations Test. The section has a main subsection for each problem. Each subsection begins with a comparison by course of the amount of prompting used by the interviewer to obtain from students the appropriate use of, or description of the use of different representations when discussing problems during the interviews. This is followed by a summary and discussion of the interviews for each part of the problem in question that focuses on students’ abilities to use, and understand
connections between different representations of the same concept. In the interview excerpts, the interviewer’s comments are shown in italics. Each subsection ends with a summary of the results along with comparisons and observations about the students from each course on the problem in question.

Coding the Amount of Prompting Used during Student Interviews

The fourth question of interview guide (see Figure 8), previously described in Chapter 3, was used as a guide for coding the amount of prompting used by the interviewer to obtain from students a description of an appropriate use of different representations when discussing each problem. The amount of prompting was coded as (a) correct/none if the interviewer obtained an acceptable response immediately after asking for a response or after reaching part 4a of the interview guide, (b) correct/some if the interviewer obtained an acceptable response after reaching either part 4b or 4c of the interview guide, (c) correct/much if the interviewer obtained an acceptable response after reaching either part 4d or 4e, (d) incorrect/some if the interviewer did not obtain an acceptable response after reaching any part of the interview guide, and (e) N/A if the interviewer did not attempt to obtain a response for that particular representation. For "combination" representations, prompting was used to obtain an acceptable response only after students had described a use of the representation that was not acceptable.
Can the student solve, or describe how to solve, the problem using another representation? If not, what about after

a. asking if the student has ever seen similar problems solved using a different representation?

b. discussing one representation to get students to consider how it relates to a different representation, e.g., Why do you set f’ equal to 0 when trying to locate extrema?

c. suggesting that the student use some part of a previous solution in another way, e.g., Could you use that graph to also determine where the population is increasing the fastest?

d. suggesting that an appropriate graphical/numerical/algebraic representation could be used to solve the problem?

e. asking the student solve a similar, simpler problem using an appropriate graphical/numerical/algebraic representation?

Figure 8: Question 4 from Interview Guide

In each subsection, the coding results for the corresponding problem are presented in tabular form, with the last category, N/A, not included. The percent totals for some of the "combination" representations exceed 100% because, on a particular problem, it was possible for students to give two or more different responses that would all be classified as the same type of "combination" representation. For example, on one problem, if one student described two solution methods, one using the graph of the first derivative and the other using the graph of the second derivative, then that student was classified as having used two different graphical/symbolic representations.
Discussion of Problem 1

The first problem dealt with determining (a) when a trigonometric population function attain its maximum and minimum values, (b) what these maximum and minimum values were, (c) when the function was increasing and decreasing the fastest (i.e., when did the function attain its maximum and minimum rate of change), and (d) what the rate of change of the function was at a particular time (see Figure 7). Interview coding results are presented in Tables 7 and 8.

Use of Symbolic Representation

In working with symbolic representations, the Calculus & Mathematica students did markedly better than both the traditional and the graphics calculator students while the traditional and graphics calculator students did about the same. Unexpectedly, the traditional students had some difficulty explaining how to distinguish between maxima and minima, and even more difficulty explaining how to determine when the population increased or decreased the fastest. None of the traditional students said, without prompting, that the time of greatest increase or decrease in population might be discovered by setting the second derivative equal to 0 and solving for t. Two-thirds of these students could not describe this technique even with prompting.
Table 7: Comparing Students on Prompting Needed to Obtain Acceptable Use of Representations on Parts a and b of Problem 1

<table>
<thead>
<tr>
<th>Representation</th>
<th>Amount of Prompting Coding for Different Representations</th>
<th>% Describing Acceptable Use of Given Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Locate Extrema</td>
<td>Decide if Extremum is a Maximum or Minimum</td>
</tr>
<tr>
<td></td>
<td>151</td>
<td>151G</td>
</tr>
<tr>
<td><strong>Graph</strong></td>
<td>Correct/None</td>
<td>58.3</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>41.7</td>
</tr>
<tr>
<td></td>
<td>Correct/Much</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Incorrect/Some</td>
<td>0.0</td>
</tr>
<tr>
<td><strong>Number</strong></td>
<td>Correct/None</td>
<td>33.3</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>25.0</td>
</tr>
<tr>
<td></td>
<td>Correct/Much</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Incorrect/Some</td>
<td>8.3</td>
</tr>
<tr>
<td><strong>Symbol</strong></td>
<td>Correct/None</td>
<td>58.3</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>16.7</td>
</tr>
<tr>
<td></td>
<td>Correct/Much</td>
<td>8.3</td>
</tr>
<tr>
<td></td>
<td>Incorrect/Some</td>
<td>16.7</td>
</tr>
<tr>
<td><strong>GS</strong></td>
<td>Correct/None</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>0.0</td>
</tr>
<tr>
<td><strong>GN</strong></td>
<td>Correct/None</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>0.0</td>
</tr>
<tr>
<td><strong>SN</strong></td>
<td>Correct/None</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>0.0</td>
</tr>
<tr>
<td>Repr.</td>
<td>Amount of Promoting Coding for Different Representations</td>
<td>% Providing Acceptable Use of Given Representation</td>
</tr>
<tr>
<td>-------</td>
<td>-----------------------------------------------------------</td>
<td>--------------------------------------------------</td>
</tr>
<tr>
<td>Graph</td>
<td>Correct/None</td>
<td>41.7</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>25.0</td>
</tr>
<tr>
<td></td>
<td>Correct/Much</td>
<td>8.3</td>
</tr>
<tr>
<td></td>
<td>Incorrect/Some</td>
<td>25.0</td>
</tr>
<tr>
<td>Num.</td>
<td>Correct/None</td>
<td>25.0</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Correct/Much</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Incorrect/Some</td>
<td>0.0</td>
</tr>
<tr>
<td>Sym.</td>
<td>Correct/None</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>16.7</td>
</tr>
<tr>
<td></td>
<td>Correct/Much</td>
<td>16.7</td>
</tr>
<tr>
<td></td>
<td>Incorrect/Some</td>
<td>66.7</td>
</tr>
<tr>
<td>G/S</td>
<td>Correct/None</td>
<td>16.7</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>16.7</td>
</tr>
<tr>
<td>G/N</td>
<td>Correct/None</td>
<td>8.3</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>0.0</td>
</tr>
<tr>
<td>S/N</td>
<td>Correct/None</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>25.0</td>
</tr>
</tbody>
</table>
Even more expected was the difficulty that the graphics calculator students had describing how to locate extrema symbolically. Only 25% of these students stated, without prompting, that extrema might be located by setting the first derivative equal to 0 and solving for \( t \). Two graphics calculator students and two traditional students could not describe this technique even with prompting, which is surprising, considering the emphasis placed on this technique for locating local maxima and minima in each course.

Both the traditional and graphics calculator students had some trouble describing how to find the rate of population change on July 1st use symbolic representations, which is done by substituting the value of \( t \) corresponding to July 1st — about one half — into the first derivative. Only one-third of the students from each course described this technique without prompting.

The technique for distinguishing maxima and minima using only symbolic representations, the so-called Second Derivative Test, was not discussed with the Calculus & Mathematica students since that technique was not presented in the course. Instead, various "combination" representations that might be used to differentiate between maxima and minima were discussed.

Use of Graphical and Numerical Representations

In working with graphical representations, the graphics calculator students did slightly better than Calculus & Mathematica students who did noticeably
better than the traditional students. The traditional students needed more prompting on how to use or interpret graphical representations than either the graphics calculator or Calculus & Mathematica students. The students from all three courses had the most trouble locating the points on the graph of the population function were the population was increasing or decreasing the fastest. A number of students from each course also had difficulty describing how to use the graph of the function to determine the rate of population change on July 1st. The graph of the population function and its first derivative used with the Calculus & Mathematica students are shown in Figures 9 and 10. Similar graphs done on a TI-81 graphics calculator were used with the traditional and graphics calculator students.

**Figure 9:** Graph of function $P(t) = 4090 - 500 \cos 2\pi t$ shown to students during interviews
Figure 10: Graph of first derivative of function $P(t) = 4000 - 500 \cos 2\pi t$ shown to students during interviews

Use of Numerical Representations

In working with numerical representations, Calculus & Mathematica students did somewhat better than both traditional and graphing calculator students with the latter two groups being about equal. During interviews, not much time was spent discussing numerical representations in order to devote more time for discussing and exploring students' use and understanding of symbolic, graphical, and "combination" representations.

Use of "Combination" Representations

The following "combination" representations were mentioned most often by students when describing how to solve the different parts of Problem 1.
1. Use the graph of \( P(t) \). (graphical/symbolic)
2. Use the graph of \( P''(t) \). (graphical/symbolic)
3. Evaluate \( P'(t) \) for different values of \( t \). (symbolic/numerical)
4. Graph \( P(t) \) using a graphics calculator or Mathematica and then trace along the graph. (graphical/numerical)
5. Make use of the fact that the Sine function has maximum and minimum values of 1 and -1, respectively. (symbolic/numerical)

As the results in Tables 7 and 8 indicate, Calculus & Mathematica students used "combination" representations, particularly graphical/symbolic and symbolic/numerical representations, more often than the other students. Graphics calculator students used "combination" representation slightly more often than traditional students, particularly on part \( b \) where many suggested using graphical/numerical representations, such as using the TRACE feature on a graphics calculator to determine specific values of a function from its graph.

**Traditional Student Interviews**

As you read the interview excerpts that follow, look for symptoms concerning these behaviors in the descriptions given by the traditional students:

- recognition of the connection between the slope of a function and its rate of change
- recall of what information about a function and its graph was provided by its first and second derivatives
- recall of what information about a first derivative and its graph was provided by its second derivative
- recognition of how to obtain information about rates of change from a graphical representation
• recognition of how the technique for locating the extrema of a function using its derivative could be used to locate the extrema of any derivative of the function

• recognition that points where the concavity of the graph of a function changes could also be where the function attains its maximum or minimum slope, or where it attained its greatest or least rate of change

• use of "combination" representations

• recognition of how to use representations in ways different from those typically taught in the course

• relating of unfamiliar problem situations or descriptions to familiar uses of representations

**Parts a and b.** Traditional students had little difficulty describing ways of using numerical or graphical representations to locate maxima and minima, but had some trouble using symbolic representations, particularly when trying to determine if an extrema was a maximum or a minimum. Students were often unable to recall either which function or derivative was used, or how a function or derivative was used, with a particular symbolic technique. The inability to remember the correct techniques suggests that students may have tried to memorize procedures for locating maxima and minima using derivatives without really knowing why the procedures worked.

**Use of symbolic representations to locate extrema.** Most traditional students had no trouble describing the standard symbolic technique of setting the first derivative equal to zero and solving in order to locate extrema symbolically, though not all remembered the technique initially.
Subject T12:

Might be able to take derivatives.... Take derivative of a - I don’t know if you could take the derivative. 
That’s the derivative. What could I do with that to solve the problem? 
Um - set $t$ equal 0, $t$ equal 1. 
Okay. Why necessarily $t$ equal 0 and $t$ equal 1? 
Because when $t$ equal 0, it’s January 1st. That’s why I put $t$ equal 1 at December 31st. 
Okay, but would that necessarily give me maximum and minimum? 
I can’t really remember. I did really bad on that chapter.... 
Is there anything special about the slope of a function at its maximum or at its minimum? 
Should be 0. 
Okay. So does that tell you ... something to do with the derivative? 
Put $t$ equal 0 and - 
Put $t$ equal to 0? 
Right - or the derivative equal to 0.

Subject T9:

If I remember correctly, you take the derivative ... and solve the derivative ... and then plug that in to the original equation. 
Okay. Solve - what do you mean by solve the derivative? 
Find the value of $x$ when you take the derivative. 
How will I find a value of $x$?.... What would you do with the derivative? 
Set it equal to 0.

Subject T6:

I could - ah - try to find the point where it’s at the maximum.... Whatever the point is where $y$ - where the $P(t)$, the function, cause that’s $y$ - is the greatest for this.... You can find $x$ and $y$ and you’d have the maximum. 
What about the second part?.. Would you do that for the minimum also? 
Uh. You could use the second derivative, or the first derivative - my fault - set it to 0 and actually find out where the minimum is - 
Can you do that with the maximum? 
I probably could. Yeah.... 
I was just wondering what you meant by use the derivative? 
Set it equal to 0. Yeah. Zero.
At first, Subject T12 did not seem to remember, or possibly realize, how to make use of the derivative to solve this problem. Subject T6 did remember the standard technique for locating extrema symbolically, but only thought to use it to locate minima while using another technique to locate maxima numerically before agreeing that the symbolic technique could be used to find both type of extrema. This lack of recognition on how to make suitable use of symbolic representations was a recurring problem with the traditional students and suggests that they did not understand or know what information is actually provided by the first derivative of a function.

Two subjects were unable to describe the standard method for locating extrema symbolically, though both did remember that it had something to do with derivatives, until after the interviewer had asked them if they remembered something about the minima or maxima of a function occurring when the first derivative was equal to zero. One of these two subjects could not remember this technique even after the interviewer had, as the subject had suggested, evaluated the first derivative at 0, ½, and 1, which are the values of $t$ at which the maxima and minima of the function occur, and at ¼, where the first derivative is not equal to zero, to illustrate that the value of the first derivative was zero only at extrema.
Use of symbolic representation to distinguish maxima and minima.

Traditional students had more difficulty working with symbolic representations to determine if an extrema was a maxima and minima once the values of \( t \) for the extrema had been established. Only 25% of the students described the technique for using the second derivative to distinguish maxima and minima, the Second Derivative Test, without prompting, while 25% could not recall the technique even after prompting. Subjects T8 and T1 were two of the subjects who described the Second Derivative Test without prompting, though Subject T8’s description did occur during a discussion on solving part \( c \) symbolically.

Subject T8

In the first question, I asked you to find a maximum. What did you do?
You knew when it was - well, you would get - um - set equal to 0. Find any particular points that you would get.
Okay. What did you set equal to zero in that first one?
The derivative. The first derivative is set equal to 0.
Okay... I've got the first derivative .... What might I do?
You could take the second derivative and find out when that equals - Okay. Take the second derivative and when - In the first derivative, when you got your zeros, you could plug that into the equation and if it came out to be a negative answer that meant it was a maximum and vice versa.

Subject T1:

The easiest way would just be to find where the first derivative is 0.... And that’s your max or your min. You have to decide which it is.
You mentioned about plugging the values into the second derivative and - when you plug it into the second derivative, how do you determine if its a max or a min?
Um. I think if it's, if the - if you get a number greater than 0 then it's a min, if remember correctly. If it's less, then its the max.
Both subjects managed to recite the Second Derivative Test correctly, though
either did so with a great deal of confidence or certainty. Other subjects also
knew that they had to somehow use the second derivative to differentiate
between maxima and minima, but could not recall exactly how to use it.

Subject T6:

*So let's say I've set [the first derivative equal to 0 and I find some values
where it's equal to 0. How do I decide they're max and mins?*
Ah - take the second derivative and if it's greater than 0, it's going to be -
you know, fill in with your answer you got into the first equation. You
know, set the second derivatives, set equal 0.... Get a number out and if
you fill it in the first equation and its greater than 0, it's going to be a
minimum and if it's less than 0, it's going to be a max.

Subject T3:

*Now what do I do to decide which of these is maximum and which of
these is minimum?*
Set em - set em up in, like a graphical. Well, set em up in a table....
You could take the second derivative and check concavity.... Trying to
think about.... Oh, what do you call it? Ah - a table that tells you like -
when you multiply each, it gives you a positive or a negative.... You can
take that with the first and second derivative. And when you put those
together, that'll give you - that'll give you the concavity. And where each
one changes - cause these are just the points of inflection.

*OK. These, these places here [where P'(t) equals 0] are points of
inflection?*
No, no. These are max and mins. I'm sorry. I can't remember whether it's
the first or second derivative that you take that for.

*Do you remember anything about taking these values and plugging them
into the second derivative and looking for something?*
I don't remember that.

Subject T6's decision to substitute the values of t where the second derivative
was equal to 0 into the original function, rather than substitute the values of t
where the first derivative was equal to 0 into the second derivative indicated confusion over, or an inability to recall, which values should be substituted into which function for the Second Derivative Test. For Subject T3, the confusion seemed to be over which derivative to use to set up a sign chart the subject had been taught to use for this type of problem. For this chart, which was used by one other traditional student, the sign of the first derivative for values of $t$ just less than and just greater than the values of $t$ for each extrema are written in a chart and then used to decide whether the extrema are maxima, minima, or neither. Subject T3 correctly described how to use this symbolic/numerical representation after being helped to recognize the correct derivative to use.

Subject T11 also could not recall exactly how to distinguish maxima and minima symbolically but thought the technique was related to a process where derivatives were used to help accurately sketch the graphs of functions.

Subject T11:

*I’ve found out that these three values are places where the derivative’s 0, and I’m interested in figuring out which ones give me the maximum and - The same way you go about graphing, though. You could try, you know, the same process. You could ... figure out the max and the minimums. By plugging in values?*

No. I forget exactly what test it’s called, but you take the, you know, the first derivative, second derivative, and then the third derivative. It’s the same process you going through when you’re trying to graph something yourself but - ah - it’s got the different tests where you figure out what the local maximum and what the local minimum is for each... It’s just a
process you go through when you’re trying to graph something without doing the final step which is graphing itself.

*What does the second derivative give me?*

Well, I think the first gives you what [the instructor] called the local extremes. And then I think the second tells you which is which... I'm not positive... cause I think the first one just basically tells you... what the possibilities are and with the second one - is when you identify each.

Subject T11 was trying to solve the problem by describing a process for curve sketching that is taught in most traditional calculus courses where the first derivative is used to locate local maxima and minima and intervals on which the function is increasing or decreasing and the second derivative is used to determine the inflection points and intervals on which the function is concave up or concave down. The last sentence of the excerpt suggests that the subject understood the roles of the first two derivatives in this process; however, at no time during the interview did the subject describe correctly what to do with each derivative in order to obtain the desired information.

*Use of graphical representations.* Traditional students had little difficulty describing how to locate maxima or minima graphically. For students who needed to be prompted before describing correctly how to solve the problem graphically, that prompting consisted of asking if they remembered discussing in class any techniques for visually locating extrema. The following excerpt exemplifies the typical response given by traditional students when asked to locate maxima and minima graphically.
Subject T1:

You could graph it.

Okay. *What would you do if you graphed it... Let's say that we put together a graph. Something like this. How would you use it to solve the problem?*

Um. Is this the picture of it?

*That's a reasonable approximation.*

Figure out the highest point

Okay... *What about [the minimum]?*

The lowest point.

All traditional students were able to point out the correct maximum and minimum points given the graph of the function, though 25% of them did not suggest using a graph until after being asked if a graph could be used to distinguish maxima and minima.

*Use of numerical representations.* Traditional students also had little difficulty describing how to locate extrema numerically. For students who needed to be prompted before describing how to solve the problem numerically, that prompting consisted of asking them if there was another way of solving the problem given that only an approximate solution was required.

Seven of the 12 traditional students described correctly how to use numerical representations to determine if an extrema was a maxima or minima. Four students did so without prompting. A few suggested evaluating the population function at various values of $t$ and then taking the maximum and
minimum of these values to be the maximum and minimum populations.

Others suggested looking at values of the function near the extrema.

Subject T2:

Is there some way you can distinguish between the maximums and minimums without the graph?
Yeah, I can't remember it but there's a way. Has to do with checking to see if it's high and low on each side of it.

Subject T9:

How do I determine ... whether [the point where t is equal to \( \frac{1}{2} \)] is a maximum or a minimum?
Plug it into the original formula.
But once I get the value how do I know that's a maximum or minimum?
Oh gosh. I forget how to do this - some Algebra II. You approach - you take the limit from either side - approach it from the left and the right. So I'd take the values approaching \( \frac{1}{2} \) and plug it into the equation.

Both these subjects wanted to observe the values of the function on either side of the extrema values in order to decide if the extrema were maxima or minima, though neither appeared to be completely clear on this notion. Subject T9's comments suggest that the subject was trying to adapt or utilize the limiting process used for defining the derivative at a point as a means for determining whether the point is a maximum or a minimum.

Use of "combination" representations. Two "combination" representations were mentioned by the traditional students when describing ways to differentiate between maxima and minima. The previously mentioned symbolic/numerical technique of creating a sign chart using the first derivative
to determine whether the function was increasing or decreasing on either side of an extrema was described by 2 of 12 students, and the graphical/numerical technique of graphing the function on a graphics calculator and then using the TRACE key was suggested by 3 of 12 students. In each case, the students described, with little or no prompting, an appropriate technique for using these representations.

Part c. This part of Problem 1 presented the most difficulty for the traditional students. Typically, about 5 to 10 minutes of each interview was spent discussing this problem. Much of this discussion focused on soliciting descriptions of uses of symbolic representations to solve the problem. The difficulty appeared to be that the students did not understand, or remember, how the first derivative was related to the slope, or rate of change, of a function or what information about the first derivative can be determined using the second derivative. This severely hampered students' abilities to make use of symbolic representations, and to a lesser extent graphical representations, to determine when the population was increasing fastest or decreasing fastest. Students' difficulty remembering the connection between the first derivative, slope, and rate of change was somewhat unexpected considering the first derivative was defined in terms of rate of change and slope in the traditional students' course textbook.
Derivatives are the functions we use to measure the rates at which functions change (p. 127).... We should define the rate at which the value of the function \( y = f(x) \) is changing with respect to \( x \) at any particular value \( x = x_1 \) to be the slope of the tangent to the curve \( y = f(x) \) at \( x = x_1 \) (p. 130).... When the number \( f'(x) \) exists it is called the slope of the curve \( y = f(x) \) at \( x \). The line through the point \((x, f(x))\) with slope \( f'(x) \) is the tangent to the curve at \( x \) (Finney & Thomas, 1991, p. 133).

Another difficulty on this problem appeared to be that many students were not sure, at first, what to look for when trying to determine when the population was increasing the fastest or decreasing the fastest. Many students stated that they could determine where the function was increasing or decreasing, but not where it was increasing or decreasing the fastest. In all, 10 of 12 initial solutions to this problem were stated in terms of intervals where the function was increasing or decreasing rather than single points where the function increased or decreased fastest. This misconception also hindered the discussion of symbolic and graphical techniques for solving this problem.

**Use of graphical representations.** In order to solve this problem graphically, students needed to understand that where the graph of the function had its greatest positive (negative) slope was where the function was increasing (decreasing) the fastest. Only 4 of the 12 students interviewed recognized this relationship without prompting. Subjects T3 and T6 were two such students.
Subject T3:

Well, it's going to be decreasing the fastest probably right around in here. You'd have to take a look at the tangent, where it - it goes down the most which probably is going to be right around here, about .75. The other one's going to be about .25 where is increasing the most on the tangent. *Okay. So you'd look at the tangent and what would you be looking for in the tangent when you did that?* Um - whatever one has the highest slope. Yeah. The highest rise over run.

Subject T6:

What I was probably thinking at the time is just look at the slope of the graph. And when the slope is the greatest positive, it would be increasing the fastest, and when it was the greatest negative, it would be decreasing the fastest.

In all, 9 of 12 students correctly identified the points where the function increased and decreased fastest. Two students who did not identify the correct points appeared to have similar misconceptions about where these points were located on the graph.

Subject T5:

Take the slope of the graph and when the slope is greatest, that's when it's increasing the fastest.... If the graph is pretty clear, you can sometimes tell [where these points are]. *I've got the graph in front of me. Could you point out where it appears that the population is increasing the fastest or decreasing the fastest?* Probably decreasing the fastest up here. [point to right of maximum in middle of year; see Figure 11] *Okay. And where do you think it might be increasing the fastest?* Probably on the bottom. [point to right of minimum at beginning of year; see Figure 11]
Figure 11: Points where population was increasing fastest and decreasing fastest as identified by Subjects T5 and T8

Subject T8:
I guess the slope would be - um - the steepest at that point.... Whatever point the slope is the greatest would be increasing the fastest. Where do you think it would be increasing the fastest on that graph? I guess it'd be down here because that's more of a steep line. [point to right of minimum at beginning of year; see Figure 11] Okay. Where do you think it would be decreasing the fastest then? Right at the top. Near zero. Or, no - decreasing. It would be negative. Down here then. [point to left of minimum at end of year; see Figure 11]

Neither subject gave any further explanation during their interviews as to why they chose those particular points for where the function was increasing and decreasing the fastest. Their choice of points (see Figure 11) and comments suggest they thought that the function was increasing fastest where the graph
went from gradually to sharply rising and that the function was decreasing fastest where the graph went from gradually to sharply falling, or vice versa.

*Use of symbolic representations.* No traditional student described how to determine when the population was increasing or decreasing the fastest symbolically without some prompting, and two-thirds of the students never adequately described how to solve the problem. Much of the prompting on this problem focused on first getting students to equate greatest slope to maximum slope and then having them make the connection between using the first derivative to find the maximum and minimum values of a function and using the second derivative to find the maximum and minimum slopes of the function, which are the maximum and minimum values of its first derivative.

The following suggests both Subject T3 and Subject T1 made this connection.

**Subject T3:**

*What information does the first derivative give me?*

The first derivative gives you the max - maximum and minimum. *The maximum and minimum of the function, but that's only when I set it equal to 0. What does it give me in general?*

It gives you the slope - the slope of your - the slope of the equation. *Okay. So right here I've got an equation... What I'm looking for is way to figure out when this equation has a maximum.*

So you take the second derivative. That’ll give you the slope of that [the first derivative] right here. *Okay. So if I take the second derivative and, and go through this process [set it equal to 0 and solve], will that give me maximum slopes and minimum slopes?*

Yeah. It could.
Subject T1:

_Can you tell me anything about what's happening to the slope in those areas [where the graph is going almost straight up]?_  
The slope is almost - I don't know. For it to be straight up and down that be undefined. The slope's getting greater, vertical, when its increasing...  
So it is the first derivative. Well, the first derivative is getting greater.  
_So what do you think would happen at the place where it was increasing the fastest? Would that - how might that effect the first derivative._  
It'd be the largest.  
_What about where it was decreasing the fastest. Where would that be in terms of the first derivative._  
It would be ... the smallest.  
_Let's go back to what you said about the second derivative. How could I put the second derivative to use to find out those places. We'll focus on the increasing._  
Take the second derivative and set it equal to 0 and then that'll give you the max of the first - it'll give you the extremes.  
_Of what?_  
Of the first derivative.

Subject T3 and Subject T1, who gave one of the best explanations on this problem, both seemed to understand that the process for using the first derivative to find extrema of a function could be utilized with the second derivative to find extrema of the first derivative. Other students did not make this connection, as the following excerpts suggest.

Subject T4:

_Okay.... In this case, we talked about population being at a maximum or a minimum and now ... I'm asking you where the slope is at a maximum or a minimum.... Because isn't this the place - you said the slope was the steepest here. So the slope would have to have a maximum value there. So would you take the place where [the first derivative] equals zero like in [part a] and just plug it into the -_  
_Okay. Well, you already set that equal to 0 and -_
Didn’t I like, use that in the original equation though so wouldn’t you use that in the derivative equation.

*Well, if I took this value into the derivative equation, wouldn’t -*

Oh, that would be 0.

*What process might I use to find out where the slope is a maximum?*

I don’t know.

Subject T6:

*Back here [in part a], what did you find?*

Ahh. Okay. I found - ah - when that was a maximum. The slope - that point was a maximum.

*Okay, but was it the slope that was the maximum there? What did you find that was the maximum on this?*

I said I found when - ah - the population was a maximum.

*When you found the maximum population, what did you suggest we do?*

Take the derivative, set it to zero.

*Okay. Can I do anything like that down here?*

Set it to a maximum value. Get x.

*What do you mean by set it to a maximum value?*

[no response given]

No subject recognized how the process for finding maxima and minima of the original function could be applied to find maxima and minima of the first derivative. Subject T4 wanted to use the values of the first derivative that were obtained when determining the extrema of the original function rather than use the second derivative to obtain values where the first derivative has its extrema. Subject T6 seemed to understand that the maximum value of the first derivative needed to be determined but did not know how to accomplish this.
Like Subject T3, Subject T11 first related the first derivative to maxima and minima, but unlike Subject T3, was never able to recognize, or remember, the connection between slope and the first derivative.

Subject T11:

*What information does this first derivative give me?*
It represents the local maximum, local minimum.

*Why is the first derivative 0 at the local maximums and local minimums?*
If you were just kinda following the - following the graph along, at those points it really has - it doesn’t have a slope.

*So the first derivative gives me slope?*
I mean, if you look at the local max and minimum - I think. If you look at the maximum, then the graph completely changes and goes in a different direction. Then, at a certain point on there, it doesn’t have - I mean the slope is 0. It has a slope but the slope is 0. That’s another way to look saying that, you know, that’s where the slope is 0.

*So where the slope is 0 - where maxima or minima occur - Is the first derivative - is it giving me slope of the function? Is that what it does?*
I don’t want to exactly say that it gives you what it is cause I’m not really sure what you plug in to give you the exact - I mean it tells you where it’s zero.

*Well, it told where it was zero because we set it equal to zero. But on this one back here. When I put in the .25, I got ... about 3000 and something.*

*Is that telling me that the slope at .25 is 3000 and something.*
No. Um - then could you go about - if your looking at when the population increases the fastest, can you take like the - I was thinking like, somehow taking the maximum and - ah -

*If I put 0 in the first derivative, or 1/2 in the first derivative, or 1 in the first derivative, which is - which is what I did before, I got zeros in those cases. So what I’m asking is, when 1 - when I plug a value into the first derivative, does that give me the slope of the function at that point?*
No. I think what it gives you is when you plugged in the 0 - when you set it equal to 0, what it gave you is, I guess, it gave you the y-coordinate at 0. It gave you an accurate number for the y-coordinate at 0. And then, I mean, I guess if you plug in 1/2, it’d give you - it’d give you - again, it’d give you the y-coordinate.
The last comment suggests Subject T11 was somehow confusing the values obtained from the first derivative with those obtained from the original function. It also appeared that this subject understood how to use the technique of setting the first derivative equal to zero to locate extrema of a function, but did not understand why the technique worked. The subject recognized the slope of the function was zero at a maximum but did not make the connection between the zero slope and setting the derivative equal to zero, and thus, never made the connection between the first derivative and slope.

Like Subjects T1 and T3, Subject T9 described the correct technique for solving this problem symbolically, but only after recalling, or realizing, that the first derivative provided information about the slope of the function.

Subject T9:

*If I put a number in the first derivative, say I put in - I don’t know - 1/4, what does that tell me? What’s that number represent? If it’s less than the maximum value for the function, tells you that, on that interval where 1/4 is, that the function is increasing. What about the specific number?... 1000π.... Does it have any specific meaning?... Or do we just look at the sign - whether it’s positive or negative and do that? I mean, does that number have any meaning at all? Yes, but I can’t remember at the moment. I want to say that it’s the value of the function in the interval, but you’d have to plug it back into the original function. Why is that to find that maximum value, you told me to take the first derivative and set it equal to 0.... Why are we setting it equal to 0?... Is there anything special about that maximum point ... in terms of the way the graph is shaped near it? Starts to flatten out at that point.*
What's a particular ... attribute of a flat line?
It has a slope of 0. First derivative is the slope. Gives the slope of a function.
Now, can I somehow relate slope to this notion of increasing and decreasing the fastest?
A negative slope would be a decreasing function. Positive slope would be an increasing function.
What about increasing the fastest? Can I relate that somehow to slope? If I'm looking for the greatest slope, aren't I looking for a maximum slope? Um hm.
Okay. Can you think of a way - based on what we just talked about here using the first derivative ... I might be able to find the maximum slope using the function - maybe the first derivative - maybe something else? Take the second derivative. Set that equal to zero and then plug the answer into the first derivative?

Subject T9 initially associated the sign of the first derivative to the intervals where the graph of the function was increasing or decreasing. Even after being helped to recognize the connection between the slope of the function and its first derivative, the subject maintained the previous association by relating the sign of the slope to increasing or decreasing functions. It was only after the interviewer suggested that greatest slope also meant maximum slope that the subject realized how the second derivative could be used to solve the problem.

The following excerpt from the conversation with Subject T2 further illustrates the traditional students' difficulty recognizing that the first derivative supplied information about the slope of a function.
Subject T2:

Can you think of any way I might ..., without using the graph, figure out where this function has its - its graph has its greatest slope? Not really. I know the slope's the change in $x$ and $y$ things there, but I - Is there a function that you use at all that equates to slope? That's related to slope in any way? Is there any function you work with that - related to [the function], perhaps - that somehow gives us the slope of it? Ahh - yeah, but I can't remember it. I remember hearing something about it but I can't -

Well, you said here, in this, that you took the first derivative and set it equal to 0 and that's where you would look for maximums and minimums. Why necessarily zero? Why would I want to set the first derivative equal to zero to determine maximums and minimums?

Cause that's where it changes its direction. Charges its direction. Okay. What changes its direction?

Well, that's where the graph was either going up or down was at the derivative zero mark.

Okay... I understand that it goes from ... increasing to decreasing, but what does zero have to do with that. Is there - I mean you're saying it changes direction and zero's where it changes direction, but - I mean, what is it exactly that we're looking at that we're saying it's changing direction. What value is that first derivative representing?

I thought it was the slope but I don't know.

Well it sounds fine. So again I ask then, is there a function related to this that tells me something about its slope?

Not that I can think of. I can't - I think I remember hearing something, people telling me about it Didn't you just say the first derivative has something to do with slope.

Yeah.

Okay. The first derivative of this describes its slope, doesn't it? The first derivative tells you the slope at particular places. That - that's the function I was referring to ...

So the first derivative of any graphed function is its slope?

If I put a number in the first derivative, it gives me the slope at that point. You remember discussing that at all in class?

Yeah. A little bit... Yeah. Some of it is starting to come back, but it's - I didn't realize that the derivative was - ahh - the actual slope of it. Okay. It's - it's used to solve for the slope.
The excerpt indicates Subject T2 appeared to understand that the slope of the function was zero at a maximum or minimum of the graph, but did not recognize the connection of this to setting the first derivative equal to zero. Even after stating that the value of the first derivative represented slope, the subject did not realize that the function that provided information about the slope of the original function was the first derivative. The subject’s last comment about not realizing that the first derivative was used to find slopes reflected the unclear recollections of several traditional students whom did not seem to remember studying this aspect of the derivative during the course.

Subject T7 was another traditional student who first related the first derivative to intervals where the function was increasing and decreasing. Subject T7 also recognized that where the function increased or decreased fastest was somehow related to the location of the inflection points of its graph, but was unable to adequately explain the connection.

Subject T7:

*What I'm interested in is at what time during the year is it increasing the fastest.... Is there any way I can figure that out?*

You mean like when you - when you take the derivative. Is that what you mean? When you find when it’s increasing and decreasing.

*Okay. When you take the derivative, that tells you where the graphs increasing and where it's decreasing, but what I'm interested in is where is it increasing the fastest? I understand that if I -*  
Like inflection points, is that what -

*Is - is an inflection point where it's increasing the fastest?*
No, that’s where it’s changing.

*What do you mean? If - if - what happens at inflection points?*

It changes direction. Is that what it is? I mean like (inaudible) -

*All right. What changes direction?*

That’s concavity, but -

*Okay. It changes concavity at inflection points?*

(inaudible) use the inflection points and then do the concavity, but that’s why you would - that’s not increasing or decreasing. (inaudible)

Subject T7 was on the right track with the inflection points but like other traditional students, seemed unable to envision how the point of fastest increase or decrease could be related to inflection points and concavity, and thus, never recognized that the inflection points were the solution. The inability to see this connection was likely because, as one comment in the excerpt above suggests, the subject’s experiences with inflection points in the calculus class probably dealt almost exclusively with determining concavity.

Subject T10 thought the solution to this problem might be found by setting the second derivative equal to zero and solving for $t$. This subject remembered that these values of $t$ gave the inflection points of the graph, but, like Subject T7, then had difficulty associating inflection points to the points where the population was increasing or decreasing the fastest.

Subject T10:

*So what I’m really looking for - when I talk about the steepest slope, I’m talking about the maximum slope. Can you think of any way I might be able to figure out the maximum slope of the function?*
Could you find the maximum of - of it? When the year - ah - find out -
ah - when the year’s at its maximum and then take the derivative of - of
it at that point?

Here’s where the year’s at the maximum, Okay. Is that going to be the
place that’s going to have the maximum slope, right here?
Um - would you - could you take the derivative of the - well it’d be the
slope, the tangent line, then plug your maximum point in for the
derivative?

I’m not sure I’m following you. You’d find the slope of the tangent line?
Or the tangent line is the slope.

When I talk about the slope of a function, I talk about the slope of the
tangent line at that point.... Again, I’m think of how can I find the
maximum slope? Can you think of anything - I mean, involving
derivatives, perhaps, where I could come up with a maximum slope?
Would you set it equal to 0, or -

What would you set equal to zero?
The first derivative?

Well, didn’t we do that to find -
Ah - the second derivative. Wouldn’t that give you your inflection points,
right?

Would inflection points be where - ah - the slope would be the greatest or
the least, or the - the greatest increasing or greatest decreasing?
Um - let’s see. Inflection points are where it changes from - ah - max,
from min to max or -
At that inflection point, would that be where the slope would be maximal,
as it goes from increasing to decreasing?
I want to say yes but I’m not sure.

At first, Subject T10 wanted to find the maximum slope by taking the
derivative and evaluating it where the population was at its maximum, not
realizing that the value of the derivative was zero at that point. After
discussion, the subject suggested setting the second derivative equal to zero to
determine the maximum slope of the function, and remembered that this
process was used to locate inflection points. As with Subject T7, this
realization seemed to confuse Subject T10, who, as the last comment indicates, wanted to believe the inflection point was where the function attained its maximum slope, but had difficulty doing so, possibly because the subject had only viewed inflection points in terms of concavity and not in terms of slopes.

The following excerpt from the conversation with Subject T2 further illustrates the traditional students’ difficulty viewing inflection points as points where the graph of the function attains its maximum and minimum slope, rather than just as points where concavity of the graph changes.

Subject T2:

*Can you think of a way that I could do something to the first derivative to help me find where the greatest slope occurs? Is there any process I could use, beyond just looking at the graph, where I could find the greatest slope?*

If you had - ahh - maximum or minimum value.

*Okay. And how - how could I find that maximum or minimum value? Would that be the second derivative? A point of inflection?*

Okay. *What would you do with the second derivative?*

Set that one equal to zero. That would tell you - I thought that told you where supposedly it changed its direction or something like that?

*Changes.*

I can’t remember.

*Concavity?*

Yeah.

*Okay.... When it changes concavity, ... from say concave up to concave down. Does that tell me anything about what’s happening to the slope? It’s changing from either a positive to a negative? Well, it’s still going to be positive. I mean somewhere in here is a point of inflection where the concavity changes. But in that whole area, the slope’s still positive. But the point of [inflection] tells you something else. It tells you where the slope goes -*
To zero, doesn’t it?
*No. The slope’s not zero there... We’ll look at the graph for now to get it across, but here’s the slope. What’s it doing as I move along the curve? Slowly increasing.*

Okay. And what happens at the point of [inflection]? What’s it going to start doing?
Slowly decrease.
Okay. So at the point of [inflection], it has stopped increasing and started decreasing.
So it’s reached its highest point.
*It’s been increasing all this time now and at its point of inflection, it switches to decreasing. So at that point of inflection, it must have been at its - the slope must have been at its maximum value.*
So that’s what the, the point of inflection is? The change of concavity is...
I didn’t know that.

This subject, like many others, seemed genuinely surprised that the inflection point could be where the slope attained its maximum value. Many students only seem able to think of inflection points in terms of changes in concavity.

It seemed that previous experiences with inflection points were clouding the students’ insights into what was happening to the graph at its inflection points.

*Use of numerical representations.* The amount of time spent discussing techniques for solving this problem symbolically and graphically left little time for the discussion of numerical techniques with the students. Only a few students mentioned possible ways, like the one given below, of using numerical representation to determine when the population was increasing or decreasing the fastest.
Subject T12:
I took .25, .5, 1, .75, and 0 ... and used those values to compare them. And then I just did numbers around them, just to make sure I was right. Okay. Well, how did you decide then that at .25 the population was INCREASING the fastest? I mean, what did you do to determine that? See I put in .25 and I checked how much the - the population was and then I put .26 and it was a greater increase ... than .26 to .27.

Subject T12 determined the time when the population was increasing the fastest by evaluating the function at particular values and then looking for the greatest increase between $y$-values. The subject stated that the decision to evaluate the function near .25 was made based on the fact that the maximum value of the function had been at .5 and a minimum value had been at 0.

*Use of "combination" representations.* Some traditional students mentioned using "combination" representations involving the first derivative to determine when the population increased and decreased fastest. They found the extrema of the first derivative either by evaluating the derivative at several values of $t$ (symbolic/numerical) or by using its graph (graphical/symbolic).

Subject T8:

*I want to know where that function is increasing the fastest and you told me that's basically where the slope is the greatest.*

Right.

*How could I somehow put the first derivative to use to figure out where -* If you take the first derivative and put your $x$ points into the equation and get the highest value?

*Okay. You put different $x$ points in and you get different values for the first derivative and you'd look for a maximum?*

Yeah.
Subject T4:

*Let's say I had a graph of the first derivative. Could I use that in any way on that question there about when its increasing the fastest or decreasing the fastest?*

So like all the parts above are where it’s increasing, so .. I guess where the max point would be would be where it’s increasing the fastest.

*So where the max point of the first derivative is is where its increasing? Is that right?*

That’s what I’m asking you. Does that sound right?

Yeah.

*What about - okay. And then what about the min point? What would that give you?*

Where it’s decreasing the fastest.

It should be noted that Subject T8 did not suggest this method for solving the problem until realizing, after some prompting by the interviewer, that the first derivative provided information about the slope of the function. When first asked about the problem, Subject T8 suggested that the answer might be determined by finding the maximum value of the second derivative.

Subject T8:

*Can you think of any way to find out, you know, how I might approach finding out when it increases the fastest or decreases the fastest?*

Would that be the same as acceleration? You might be able to take the second derivative, and - I’m not sure what. Increasing the fastest? I think, maybe, that you might be able to take the second derivative ... at different time periods during the year, like ½, 1, and -

Okay. And what would you be looking for in the second derivative?

I guess which - when you keep plugging in numbers, whichever one is the highest will be increasing the fastest? Cause since the second derivative is like taking acceleration.
Subject T8 seemed to be confusing acceleration with maximum velocity, which is equivalent to fastest rate of increase. It was only after recognizing that the rate of increase was related to slope that this subject suggested determining the maximum value of the first derivative and not the second derivative.

**Part d.** One of the more unanticipated results from Problem 1 was that so many traditional students did not make an initial attempt to solve this problem. Fifteen of the forty students did not write a response to the question when they took the Calculus Representations Test. Originally, the researcher had expected students would have little difficulty recognizing that determining how fast the population was changing on July 1st, which is about halfway through the year, was equivalent to determining \( P'(\frac{1}{2}) \). During the interviews, it became clear to the researcher that some students did not recognize, or realize, that how fast the population was changing on one day corresponded to the rate of change, i.e., the value of the first derivative, for that day.

*Use of symbolic representations.* As was the case with parts b and c, students’ difficulties describing how to use symbolic representations on part d appeared to be related to their inability to recognize, or recall, what information about the function was provided by the first derivative, which made it difficult to figure out correct techniques for solving the problem symbolically or graphically, as the following excerpt suggests.
Subject T2:

How might I determine how fast the population’s changing on July 1st? I know it has to do something with derivatives, but I don’t know what....

Okay. Again, if I’m talking about how fast the population is changing, that’s rate of change, and what - what function do we use to describe - That’s the first derivative.

What if I take t equal to ½? Is there something I could do with that value in the first derivative to figure out an answer there?

Plug it into the, the original equation.

Okay. If - if I plug ½ into the original equation, what will that give me?

[no response]

It gives me the population on July 1st.... But I’m interested in how fast the population’s changing on July 1st. Is there any function that might give me that information?

I’m sure there is but I don’t know -

You were just telling me that the first derivative is rate of change.

Yeah, but that just gives a day. Oh, is that what - if you -

If I plug in that value into the first derivative, what -

Is that the rate of change then?

On that day. Remember we were just discussing that [in part c]? Put in a value in the first derivative, that tells me the slope ... at that point. But the slope is the rate of change.

So for whatever - so whatever day on the first derivative, if I plugged into this, that would be the rate of change for that day.

Subject T2 initially stated that the first derivative was used to describe the rate of change of a function, but then did not seem to realize that the value of the first derivative when \( t \) was equal to \( \frac{1}{2} \) was a measure of the rate of change, or slope, of the function for the time of the year corresponding to \( \frac{1}{2} \). The subject’s eventual recognition of this relationship may have been due, at least in part, to a lengthy discussion on the relationship between the first derivative and slope that took place while working on part c.
Like Subject T2, Subject T7 also did not seem to know what information about a function was provided by its first derivative. This subject’s troubles appeared to stem from viewing first derivatives in terms of maxima, minima, and intervals where the function is increasing or decreasing instead of in terms of rate of change or slope.

Subject T7:

For this one, I want to figure out how fast the population is changing on July 1st ... How might I do that?
You could plug \( \frac{1}{2} \) into the equation and see where it is on there because you already know where it’s increasing fastest and decreasing fastest.
You’d put the \( \frac{1}{2} \) in here?
Wait. Put it in the second, or the first derivative.
Is that what the first derivative tells me is, is rate of change? How fast it’s changing?
Um - I don’t know. I totally forget. Um -
What information does the first derivative give me? What does it tell me?
The max, min, and the intervals where the function’s increasing and decreasing, and - I don’t know.
Put in one-half, you get zero. What does, what does that tell me?
That obviously gives you the ... max but - So you use second derivative. What I’m wondering is what does that zero tell me? When I put in \( \frac{1}{2} \), the derivative gave me zero. What does that zero tell me about the graph or about the function?
Um. I don’t know. I feel stupid.

This subject’s inability to recognize the connection between the rate of change and slope of a function and its first derivative precluded any chance of the subject recognizing how to solve the problem using the first derivative.
Many traditional students were able to figure out that this problem could be solved by determining the value of the first derivative when \( t \) was equal to one half. Below are some examples of these solutions.

Subject T9:

First derivative is acceleration.... If you could apply the concept of speed to this, like that the first derivative on the first of July or at one-half. 
Okay. And what would that tell me. If I take the first derivative and put in one-half, what does that tell me? 
Zero. That’s how the population is changing on the first of July.

Subject T1:

Could you like, find out what point this is at one-half and then - um - where the second, first derivative is there, and then - um - do you need the second derivative? 
Again, it’s asking how fast is it changing. Which part of the function gives us how fast things are changing, the function itself, first derivative, second derivative? What talks about how fast it’s changing? 
Um - first derivative? 
All right. We’ll say the first derivative is correct. What would you do? If we had the first derivative in front of us, what would you do? 
Stick in one-half for \( t \) and find out the value.

Subject T4:

Well, at that point, though, that’s the max where it’s not really changing at all, is it? 
You said that it was at a max, so you mean the population is at a max. 
And then you said that the population wasn’t changing. Now why did you say the population wasn’t changing? 
Because - um - the slope is zero at that time. 
The slope is zero. And how did you find out that the slope was zero at that time. What did you do to find that? 
By - um - putting the - one-half of the year and put it into the equation of the derivative.
Each of these subjects did describe the correct way to solve the problem symbolically, though all except Subject T4 had some problems with their descriptions. Subject T1 was not sure, at least initially, whether the first derivative or second derivative provided the correct information for solving the problem. Subject T9 appeared to come up with the correct solution after incorrectly associating rate of population change to acceleration and acceleration to the first derivative.

Subject T4 correctly solved the problem by first recognizing that the function was at a maximum at that time, which meant the function would not be changing, and then associating the lack of change at that point to obtaining a zero for the value of the derivative. This subject was one of the few traditional students who seemed to recognize that determining how fast the population was changing on July 1st was equivalent to determining the slope of the graph of the function in the middle of the year.

Subject T4:
If you had a graph and you were at like where half the year was.
Would you use the graph of the function?
Yeah.
Okay... How would you use that to answer the question about how, approximately how fast the population's changing at the halfway point? You'd look at half of the graph and, uh, see what the slope is.
Subject T8 also suggested solving the problem by finding the slope of the graph of a function in the middle of the year, but ended up finding the slope of the wrong function.

Subject T8:

If you graphed it and if you found - um - on your graph where the half - the halfway mark, ... you could find the slope of that number.  
What graph would you use? Would you use the graph of the first derivative or the graph of the function?  
The graph of the function. No, you'd get the slope. Because you need the slope, you'd do the first derivative. And then find the halfway mark of your function and that would - if you got slope at that point, that would be the change.

At first, Subject T8 seemed to recognize that the slope of the graph of the function at the halfway point provided a solution to the problem. Then, after the interviewer asked about using the graph of the first derivative, the subject switched to wanting to determine the slope of that graph, or at least appeared to make this switch since it was never made clear during the interview what the subject meant by the phrase "slope at that point." The last comment indicates that the subject thought determining how fast the population was changing meant determining how fast the first derivative, or rate of change of population, was changing. This misconception suggests that the subject may have been unclear on how to interpret the phrase "how fast is the population changing" in terms of the function and its derivatives, which in turn suggests
that the student may not understand what information about a function is provided by its first derivative.

*Use of non-symbolic representations.* Since so much time was spent discussing the previous parts of Problem 1 and discussing techniques for solving this problem symbolically, little time was left for discussing the use of graphical, numerical, or "combination" representations when determining how fast the population was changing on July 1st. The use of graphical representations to determine the rate of change of the population on July 1st was discussed with only seven traditional students. Five students were able to describe correctly how to solve the problem graphically, and four did so without prompting. One student determined how fast the population was changing on July 1st by calculating the value of the first derivative on July 1st from its graph (graphical/symbolic). Five students described methods for approximating the solution to the problem numerically. All five estimated the change in population by either determining the slope between pairs of points near July 1st on the graph of the function or by somehow comparing values of the function on or near July 1st, as illustrated by the following excerpt.

Subject T9:

Take the values on either side of July 1st and ...subtract the largest from the - subtract the smallest from the largest which should give you an idea of the change in population between the two days.
Graphics Calculator Student Interviews

As you read the interview excerpts that follow, look for symptoms concerning these behaviors in the descriptions given by the graphics calculator students:

- recall of how to use the first derivative to locate extrema
- recognition of the connection between the slope of a function and its rate of change
- recall of what information about a function and its graph was provided by its first and second derivatives
- recall of what information about a first derivative and its graph was provided by its second derivative
- recognition of how to obtain information about rates of change from a graphical representation
- recognition that points where the concavity of the graph of a function changes could also be where the function attains its maximum or minimum slope, or where it attained its greatest or least rate of change
- use of "combination" representations, particularly graphical/numerical representations
- recognition of how to use representations in ways different from those typically taught in the course
- relating of unfamiliar problem situations or descriptions to familiar uses of representations

Parts a and b. Graphics calculator students had little difficulty using graphical representations to locate maxima and minima, which should be expected considering the emphasis on using and interpreting graphs in their course. Few students had difficulty describing how to use numerical
representations or "combination" representations. Many students determined the maximum and minimum populations by using a graphics calculator's TRACE feature, on the graph of the population function to determine the coordinates of specific points on the graph (graphical/numerical representation).

Graphics calculator students did have significant difficulties describing how to use symbolic representations, both to determine if an extrema was a maximum or a minimum and to locate extrema. Many students did not remember being taught techniques for locating extrema symbolically. Of the 12 students interviewed, four had to be reminded about having used derivatives to locate extrema, four had to be reminded about locating extrema by using the technique of setting the first derivative equal to zero, and two had to be reminded about both having used derivatives and about using the aforementioned technique for locating extrema. The inability to remember these essential symbolic concepts and techniques suggests that students may have focused their attention on the graphical components of this particular problem to the point of virtually ignoring the symbolic components.

*Use of graphical representations.* No graphics calculator students needed prompting in order to describe how to use graphical representations to locate maxima and minima. An unexpected outcome from the work with graphical representations was that 3 of the 12 students chose to graph the function using
the parametric mode of their graphics calculator. In each instance, the reason
given for choosing parametric mode was that the variable in the population
function was \( t \), which represented time, which meant use parametrics.

Subject G1:

Cause they had it in - it looked like parametrics here which is why I went
to do that.

*Well, \( t \) is used - it's just because the variable was time as opposed to - .
Yeah, that's parametrics. Time is the variable in parametric mode.*

Subject G6:

*Was it because of the \( t \) that you decided to do it parametrically?*

Right. Because it's time.

*All right. So anytime you see a \( t \) in an equation, you - you tend to think of it as doing it parametrically?*

Right.... Cause with like \( P(x) \), I think of a function.

Each subject was using a technique taught in the graphics calculator course
where the parametric equations \( x(t) = t \) and \( y(t) = P(t) \) are graphed, producing
a vertical line which simulates the "motion" of the function \( P(t) \) over time. By
using the TRACE feature of the calculator, the students could then view the
changes in the population throughout the time period graphed and see when
and where the maximum and minimum values of the population occurred by
watching how the cursor moved between points on the vertical line.

Rather than use the TRACE feature, Subject G1 graphed the function
using the calculator's DOT mode to in order to view the changes in population
and determine the maximum and minimum values.
Subject G1:

Yeah, I put it on the DOT mode so you could see that, where the dots were close together, you know, it was going fast and decreased and decreased until it hit a maximum at 4500.

The notion of using the spacing between dots as a means for estimating the rate of population change mentioned by this subject will be discussed further when the results of the interviews on part c of Problem 1 are presented.

Of the 12 students interviewed, only six initially chose to determine whether an extremum was a maximum and minimum graphically, down from the 11 who initially chose to locate the extrema graphically. Some students appeared hesitant to use the graph on this problem because they thought an exact answer was desired and did not seem to think they could come up with an exact answer just using the graph. A number of students stated that they would used the graph as a guide for deciding upon values of t to substitute into the function in order to determine exact maximum and minimum values, rather than just read the values off the graph, as the following excerpt indicates.

Subject G5:

Well, I started out as - uh - thinking of it as a graph and reading the numbers from the graph.... But I ended up having - having to just - um - put in. I started out with - uh - the middle of the year as in - um - putting in - um. What - what did I put in. I don't remember. But I - um - plugged in a number.
Though it is not entirely clear from this excerpt exactly what Subject G5 did to solve the problem, it appears the subject used the graph to decide that the maxima and minima occurred when \( t \) was equal to \( \frac{1}{2} \) and 0, respectively, and then substituted these values into the original function to obtain the exact values of the maxima and minima. It appears that this subject, along with a few others, did not accept the results obtained from a graph as being accurate enough to answer the question. Since only three students mentioned using the graphics calculator's ZOOM-IN feature, or the ZOOM-In and TRACE features together, it may be that these students had forgotten that the coordinates of any point on a graph can be ascertained to whatever decimal place accuracy is available on the calculator by using the ZOOM-IN feature repeatedly. It is also possible students did not understand the connection between the different graphical representations of the same function used to estimate, with increasing accuracy, the coordinates of the maxima and minima of the function.

*Use of "combination" representations.* Along with graphical representations, graphical/numerical representations were used quite often to locate maxima and minima. Seven students suggested using the TRACE feature on the graph of the population function to determine when and where the maximum and minimum values occurred by looking at and comparing the coordinates of different points on the graph. Three of these students made this
their first choice of representation when they solved the problem. Two other students used the MAX/MIN key on the TI-85 calculator, which gives the coordinates of the maximum and minimum points of the graph shown on the graphics display of the calculator. The only other "combination" representation used by graphics calculator students to locate maxima and minima was a symbolic/numerical representation that involved evaluating the first derivative at points on either side of an extremum to determine if the slope of the function changes from positive to negative or vice versa at the extremum. This technique was mentioned by 4 of the 12 students interviewed.

Use of numerical representations. For the most part, graphics calculator students had little difficulty describing how to locate the extrema of the population function numerically. As was the case with the traditional students, the only prompting used to obtain a description of how to solve the problem numerically was to ask if the subject could think of another method to solve the problem given that only an approximate solution was required. One student, Subject G4, chose to make use of the numerical properties of the Cosine function, rather than just calculate a multitude of values for the population function, in order to determine when and where the maximum and minimum values occurred.
Subject G4:

What I'd look at is the cosine. What's the biggest value for cosine because - um. I would look for what the least - what cosine could be the least of because we're subtracting here.... So I'd say cosine at - the least cosine could be would be probably -1. So where cosine is -1 here, I would, you know, multiply by - but now looking at it even more is that I'd look at for where it would equal 0 cause then I would have just 4000. *Do you think that would be a maximum or a minimum there?*
That would be the - I see. Where - where cosine equals 1, that would be the minimum. Where cosine equals -1, that would be the maximum.

Initially, Subject G4 was on the right track noting that the maximum value of the function would occur when the cosine was at its least because the cosine term was being subtracted. It is unclear why the subject then decided to consider the value of 4000 when t was equal to 0, but in the end, the subject was able to identify correctly the values of cosine corresponding to the maximum and minimum population.

After graphical and graphical/numerical representations, the next most used, and seemingly most understood form of representation, mentioned by students for determining if an extremum was a maximum or minimum was the numerical representation. In working with numerical representations, students chose to either check the value of the function on either side of the extremum, like Subject G2, or observe the values of the function at the extremum, like Subject G11.
Subject G2:

Take some points on either side of [the extrema] and see whether they go up or down.

Subject G11:

I'd find where [the first derivative] is equal to zero and that should give me - uh - that should give me the max and then that max would be the - would be the x-value and I'd enter that into the original equation which would give me the number of years. Those are three places where it gives me zero. Are those all maxs?

Those aren't all maxs. However, if you enter all them in, whichever gives - gives you the highest \( P(\cdot) \) value, that's the max.

What about the other ones?

They would - they could be local maxs or mins. Uh - it's whatever gives you the highest y-value, that's the ... max.

Okay. Would I do anything different to locate where the minimum was?

The minimum. You just find - if you had these three, it would give you - your lowest y-value would be the - uh - would be the minimum.

Subject G11 was the only student of the seven who mentioned using a

numerical representation to distinguish between maxima and minima who needed any prompting.

Use of symbolic representations to locate extrema. Probably the most unexpected result from the interviews on Problem 1 was the graphics calculator students' inability to remember the standard symbolic technique of setting the first derivative equal to zero and then solving for the variable in order to determine the maxima and minima of a function. Only one student chose this representation initially to solve the problem. Of the remaining students interviewed, seven were unable to describe how to use the technique without
some prompting, and two were unable to describe how to use the technique even with prompting. Only six students remembered that the first derivative could be used to locate extrema, and many of these students could not recall exactly how to use the first derivative. This result was unexpected considering that this technique for locating extrema symbolically was stressed in the course textbook and by the course instructor. Subject G2 was one of the students who remembered the technique of taking the first derivative and setting it equal to zero, but for some reason, this subject though that this technique was used only to locate maxima.

Subject G2:

[Take] the derivative and then set it equal to 0 to find the max.
Okay. Is there another way - something else I do to find the minimum?
Find the minimum is - uh - set \([P(t)]\) equal to 0 just as it is. ... Set it equal to 0 and then solve for \(t\), and then \(t\) would equal - 
If I put 0 in for \(t\), ... wouldn’t that mean \(8 = \cos 2\pi t\)...
Is there a value I can put in there so the Cosine would be 8?
Ummm - you could probably find one, I suppose, but it’d be something pretty hairy. ... Wait. Cosine’s based on unit circle.
Um hm. So is there any value I could get that would produce an 8?
Not if you take the cosine of it.
So if I ... take the first derivative and set it equal to 0, ... does that find only maxs and not mins?
The second derivative. And it would find minimums.
I take the second derivative and set it equal to 0? Is that how I find minimums?
Yeah. That should work.
It was unclear why this student thought there were different techniques for
determining maxima and minima symbolically, but this misconception suggests
that the student did not understand why the first derivative is set equal to zero
in order to locate extrema.

A few students were able to remember how to locate extrema by setting
the first derivative equal to zero after being asked to determine the first
derivative of a linear function and then relate the value of that derivative to the
te slope of the function, as the following excerpts illustrate.

Subject G1:
I know how to find derivatives but I’m not sure exactly what they mean.
Suppose we take ... y = 2x + 1. What’s that derivative.
Two
Okay. Can you tell me anything about the slope of that line.
There we go. All right. I’m coming back now. All right. Yeah, through
this and the - uh - slope of the tangent point.
Right. Okay. Now, does that jog your memory in terms of using the
derivative on how to - uh - find maximums and minimums?
Where the slope of the tangent’d be zero would be your maximum point.
Um - well, wait a minute. On this graph, you see you could have zero at
two different points, couldn’t you? So one would be a max and one
would be a min.

Subject G4:
You can also look at the derivatives. Like the first would be the - I’d
actually look where the roots of ... maybe the second derivative are.
Okay. The roots of the second derivative?... What information does the
first derivative give you?
The first derivative gives you if it’s concave up or concave down?... Let’s
see. First derivative would give you - thinking about it here.
What’s the derivative of y = 2x + 1?
The derivative of that is - um - two.
Okay. Can you tell me anything about the slope of that line?
Slope of that line is - is zero.
The slope of THIS line?
Oh, this - okay. The slope of this line is two and y-intercept is one.
Okay. And its derivative was two.
Was two.... The first derivative's two.
And the slope of the line is 2.
That's - I can't believe I forgot that. Yeah. That's right. So, that would, that would give me the slope.
The first derivative gives me slope. So, if I want to find these maximums or minimums, I take the first derivative and do what with it?
Find where the slope equals zero.

Both subjects were able to describe more or less how to locate extrema symbolically after working the simpler problem which appeared to help them remember the connection between the first derivative of a function and its slope. Other prompts designed to help students recognize what information was provided by the first derivative proved effective in assisting other graphics calculator students in remembering how to locate extrema symbolically.

Subject G9:

Probably set it equal to 0.
Set the function equal to 0?
Um hm. That's what I would have tried first, but I don't know if that would work or not. Maybe take a derivative, the derivative of it.
Okay. What would you do with the derivative?
Hmm: That would - sheesh. I don't know.
What information does the derivative give me?
I don't know. It would give you -
For example, let's put in one-half for t.... [Then the value of function] is zero. Does that zero tell me anything?
At one-half, ... it's zero.
So at one-half, I get a zero. And what seems to be happening at one-half? The graph - that's a max at one-half.... Yeah, that's the max. Okay. Yeah. All right. I see what I'm doing now. When you take the first derivative and set it to zero, that would give you your max and mins.

Subject G12:

Find the derivatives.

*Do you recall anything about finding derivatives and locating maximums?*

Yeah.... I wasn't very familiar with doing it in parametric mode though.

Well, I'd probably just find the derivative without - See. I was always thinking of this in parametric mode so - um - I would probably do that. I would probably find the derivative.

*Okay. And then do what?*

And then just find ... that equal to zero. Which would give you your max.

*So you set the derivative equal to zero?*

I think so.

*Why zero?*

Cause that where it would cross the x-axis.

*Why am I interested in that?... Why does it tell me a maximum occurs?*

Uh - cause that's just the way derivatives work.

*What information is the derivative giving me, so that when I find this maximum ... the derivative's zero?... Is there any way that maybe the derivative is somehow connect with slope? Does that ring a bell?*

No.

*Up here at the maximum, what's the slope of the function?*

It's zero. Okay. Okay. ... Yeah. Yeah. it brought me back.

Subject G9 remembered how to locate extrema by setting the first derivative equal to zero after recognizing that the value of t where the first derivative was zero corresponded to a local maximum on the graph of the function, the first derivative has a value of zero. Subject G12 remembered this symbolic technique but did not remember why the first derivative was set equal to zero to locate maxima and minima - "that's just the way derivatives work" - until
after recognizing that the graph of the function had a slope of zero at a local maximum. In each case, discussing what happened to the graph of the function at a maximum seemed to help the subject remember the derivative provided information about the slope of the function.

As the previous excerpt suggests, thinking of the original function in terms of parametrics hindered Subject G12’s ability to use symbolic representations to locate extrema. The other students who had used the graphics calculator’s PARAMETRIC mode, Subjects G1 and G6 were at first unable to describe how to solve this problem symbolically. Both were then able to recall a correct symbolic technique after the interviewer suggested rewriting the function \( P(t) \) using the variable \( x \) in place of \( t \). It appeared these subjects thought that since the variable in the original function was \( t \) and not \( x \), the standard symbolic techniques for solving the problem could not be applied and other techniques had to be utilized, as the following excerpt suggests.

Subject G6:

*You mentioned confirm analytically. How would, what would you do to solve it analytically, or confirm it?*

Well, I had a problem with that last quarter. I had a problem with this section. Um - I really didn’t know how to set up a parametric equation. I knew how to put it in the calculator.

*I’ll just change \([t]\) to \(x\) so that we don’t have to think of it parametrically.*

For the maximum, I’d try - I’d probably find the first derivative…. Solve it. I’d find the first derivative probably and solve it, make it equal 0.
Why in particular did you choose setting the first derivative equal to zero in order to find the maximums and minimums? I mean, why - why zero? Why zero? That’s the way I was taught - told to do it.

... Could you have done the same thing if I had kept that variable as a $t$ instead of switching it to $x$? Did the - did the variable matter? Probably not.

Just ... when you saw $t$, you thought in terms of parametrics.

Well, yeah. I thought there was a different way of solving it.

At first, Subject G6 seemed to think that because $t$ as the variable, parametrics were somehow involved which meant that it was not possible to use standard symbolic technique of taking the first derivative and setting it equal to zero.

When $t$ was replaced by $x$, the subject was able to describe how to locate the extrema by setting the first derivative equal to zero, though like Subject G12, this subject did not seem to know why this technique worked - "Why zero? That’s the way I was taught..." After their interviews, all three students who mentioned parametrics seemed to understand that use of the variable $t$ did not automatically mean parametrics were involved and that techniques for solving a problem symbolically were not affected by the choice of variable.

Two students were unable to describe how to use the standard symbolic technique until after the interviewer mentioned it to them. Subject G8 could not remember how to use the technique on the simpler problem of determining the minimum of the function $f(x) = x^2$. Subject G5 did not even remember being taught this technique during the graphics calculator calculus course.
Subject G5:

Taking the first derivative of the \( P(t) \) would give - uh. It should - it should give me the minimum and the maximum.

Okay. Well, what would you do with the first derivative... Do you remember any techniques or something you used to -

Well, we never - uh - solved for a max - minimum and maximum with - uh - trigonometric functions in precalculus or high school, but we would - normally would have solved for \( x \). But in this case, it would be \( t \).

This is the [derivative] you get. What equation would you solve?

Yeah, I know. The equation’d be \( \dot{y} \) equals this for the graph.

Do you recall discussing ... some equation you created with the first derivative so that you could determine if something was a max or a min?

Not really off the top of my head. I don’t remember. I was -

Okay. How about something like - do you remember doing this? Taking the first derivative and setting it equal to zero. And then solving for \( t \). And solve for \( t \)? Uh - I don’t remember from last quarter, but we did that. That’s what we did in precalc in high school. I remember doing that.

Okay.... So in your calculus class, you don’t remember solving maximum and minimum problems other than looking at them graphically.

No.... Not from last quarter.

This subject was not the only student who had difficulty remembering having been taught how to locate maxima and minima symbolically during the graphics calculator calculus course. Four other students stated that they thought they had solved problems like these but couldn’t remember for sure.

Possible reasons for the graphics calculator students’ lack of recollection of symbolic techniques will be discussed further in Chapter 6.

Use of symbolic representations to distinguish maxima and minima.

Graphics calculator students had trouble - even more trouble than the traditional students - using symbolic representations to determine if an extrema
was a maxima or minima. Only 4 of the 12 students interviewed described a
technique for using the second derivative to distinguish maxima and minima,
the Second Derivative Test, one without prompting, while five could not recall
the technique even after prompting. In most cases, the students were not sure
how to use the second derivative to determine if an extrema was a maxima or
minima, as the following excerpts illustrate.

Subject G4:

When you’re taking the max, you want to find the points where the points
before it and after it are - uh. That’s where I’d go to the second
derivative.
Okay. So now I take the second derivative and what do I do with that?
Okay. When you take the second derivative, you’d look at the - the roots.
I mean where it equals zero. Where the second derivative equals zero.
And by looking at that, you - if it’s - if it’s positive, I mean -
We know that ... the first derivative is zero when $t$ equals 0, $t$ equals $\frac{1}{2}$,
and $t$ equals 1.
What I’d do is put these into [the second derivative]. I’d sketch this and
put the roots. Let’s say I have $t$ equal 0, $t$ equals $\frac{1}{2}$, and $t$ equal 1. Now,
I’d probably put a point before that and if it - if it was maybe say -1 or -
I mean you know it’s going to be zero at these points. Now, if its -
Which is going to be zero at those points? The first derivative or second
derivative? Because it looks like you’re saying you’re going to put the
values in the second derivative.
Well, I’d put these values in the second derivative. In the second
derivative, ... it’s basically giving you the slope of [the first derivative]....
So you’re looking for where the slope is negative in between the two. So
I mean, for example, if that were to be a maximum point, you would have
a positive slope, and then the max, and then negative slope.
And then if it was a minimum, you’d have a negative then a positive.
Right.
And you’d do this from the second derivative?
Yes.
Subject G10:

I would have to use the Second Derivative Test, I believe. And that way I would get the answer. 
*Do you remember what the Second Derivative Test was?*

Let me think here.... Test the second derivative by imputing a value to the left and right of \( t = \frac{1}{2} \) and one's - let's see. From the left side, it's positive and on the right side, it's negative. Um - then it'd be increasing and decreasing so you get a maximum.

These subjects appeared to be confusing the Second Derivative Test with the symbolic/numerical technique for determining if an extremum is a maximum, a minimum, or neither by evaluating the first derivative at points on either side of an extremum to determine if slope of the function changes from positive to negative or vice versa. Both students correctly recognized the slope of the graph went from positive to zero to negative at a maximum. Subject G4 recognized that the second derivative "gave" the slope of the first derivative, but did not seem to understand, or realize, it was the slope of the function and not its first derivative that was being used in this technique. Subject G10 seemed to think that the sign of the second derivative, not the first derivative, described where the function was increasing or decreasing and because of this, described an incorrect technique that the subject thought was the Second Derivative Test. Subject G10 was not the only student to have this misconception about what the sign of the second derivative represented, as the following excerpt suggests.
Subject G9:

_How do I distinguish, without looking at the graph, which are max and which are mins?_

You’d have to take the second derivative ... to find out which is which. _Okay. So ... I take the second derivative and then what?_

And then set that equal to 0. _Do I use these [values of t where P'(t) = 0] at all?_

Those. Yeah. You use those after you get the - let me think.... Put those numbers in for t and it would give you negatives or positives, I believe. _Okay. And what do the negatives or positives tell me._

When it’s negative, it would be decreasing, and when it’s positive it would be increasing. _So does that tell me when I get maxs or mins? For example, if I put in \( \frac{1}{2} \) here. ... I end up with \(-2000\pi^2\)._ 

Okay. So that’d be negative. So that’d means it’d be decreasing on the interval from - when it gets to - goes to \( \frac{1}{2} \). As the limit approaches \( \frac{1}{2} \). _Okay. So does that tell me it’s a max or a min?_

That would tell you it’d be a min, I think? _[But] I know that’s a max._

With prompting, Subject G9 remembered that the second derivative had to be evaluated at values of \( t \) where the first derivative was equal to zero, but like Subject G10, thought that the sign of the second derivative described where the function was increasing or decreasing. Hence, Subject G9 was confused when, for \( t \) equal to one-half, the sign of the second derivative indicated a local minimum when it should have indicated a local maximum. Most graphics calculator students were like Subjects G9 and G10 in that they did not seem to know, or remember, the information provided by the second derivative.

It should be noted that less interview time was spent discussing how to distinguish between maxima and minima than in discussing how to locate
extrema because the graphics calculator students had difficulty remembering how to solve the latter problem symbolically. During the discussion on how to determine if an extrema was a maximum or minimum, symbolic representations were usually considered last, if at all, particularly if the student had previously described how to solve this problem using a numerical representation or symbolic/numerical technique described previously.

Part c. Graphics calculator students had a relatively easier time than the traditional students on this part of Problem 1, though they did have some difficulties. Students seemed to have a somewhat better understanding of how the first derivative related to the slope, or rate of change, of a function or of what information about the first derivative could be provided by the second derivative. This may explain why they did better than traditional students describing how to use symbolic and graphical representations to determine when the population was increasing the fastest or decreasing the fastest. However, as with traditional students, the difficulties that the graphics calculator students had remembering the connection between the first derivative, slope, and rate of change were somewhat unexpected considering the first derivative was defined in terms of rate of change and slope in the traditional students' course textbook.
Derivatives are the functions we use to measure the rates at which [functions] change (p. 3-1). We should define the rate at which the value of the function \( y = f(x) \) is changing with respect to \( x \) at any particular value \( x = x_1 \) to be the slope of the tangent to the curve \( y = f(x) \) at \( x = x_1 \) (p. 3-4). When the number \( f'(x) \) exists it is called the slope of the curve \( y = f(x) \) at \( x \). The line through the point \((x, f(x))\) with slope \( f'(x) \) is the tangent to the curve at \( x \) (Finney, Thomas, Demana, & Waits, 1993, p. 3-8).

Like traditional students, graphics calculator students seemed unsure what to look for in order to determine when the population increased or decreased fastest. Six of the 12 initial solutions to this problem were stated in terms of intervals where the function was increasing or decreasing rather than single points where the function increased or decreased fastest. This uncertainty did not prove as much a hindrance during the discussion of symbolic and graphical solution techniques as it did during the traditional students interviews.

*Use of graphical representations.* Six of the 12 students interviewed recognized, without prompting, that where the graph of the function had its greatest positive (negative) slope was where the function was increasing (decreasing) the fastest. Below are excerpts from interviews with four students who recognized this relationship.

Subject G10:

Well you could first graph the first derivative. And then take the slope of, you know, any given point and wherever the slope was the greatest. *What does the first derivative give me again?* I think that would give you a rate of population in years.
And so then if I wanted to figure out when the population was changing the fastest, I would look at the slope of the first derivative?
Slope of the tangent is actual the second derivative. So, no. I’d think you’d look at the slope of the function itself at the given point,... Where it’s increasing the fastest would be where - um - on the regular function itself would have the sharpest slope.

Subject G4:
You want to find where the slope is the highest.... You’d probably want where it has the greatest - more vertical of a line. The greatest slope.

Subject G6:
Find the slopes of certain years, and the higher the slope, the faster the increase.... I’d graph the equation and find the areas of steepest slope.

Subject G11:
It’s where your slope is steepest, either positively or negatively. What - whatever gives the greatest value for the slope, either positive or negative.

In all, 11 of the 12 students correctly identified the points where the function was increasing and decreasing the fastest. The one student who chose incorrect points appeared to have a similar misconception about where these points were on the graph as one traditional student, Subject T8.

Subject G3:
I’m interested in when in the year is it increasing the fastest.
On the graph it looks about to be the end of January, beginning of February. [point to right of minimum at beginning of year; see Figure 12] What about decreasing the fastest then?
Near the beginning of December. [point to left of minimum at end of year; see Figure 12]
How are you deciding that those are the places. What are you looking for there to decide that where it’s increasing the fastest or decreasing.
Um - how concave the curve is at that point.
Figure 12: Points where population is increasing fastest and decreasing fastest as selected by Subject G3

The choice of points (see Figure 12) and comments suggest that the subject equated increasing and decreasing fastest to having the "greatest" concavity and that the "greatest" concavity occurred when the graph went from gradually to sharply rising or from sharply to gradually falling. Thus, it appears Subjects G3 and T8 chose the same incorrect points for essentially the same reasons.

Not all students determined when the population increased or decreased fastest by estimating where the graph had its greatest and least slope. Some students used the symmetry of the graph, as the following excerpts illustrate.
Subject G5:

At the beginning of the year, the population started out low and increased up toward the middle of the year which is the summer, and it started out slow in the beginning of the winter and when it was becoming summer, it's - it was again slowing down, reaching the maximum. So, it should be increasing the fastest in the spring, and then, for the same reason, it should be decreasing, or decreasing the fastest, in the fall.

Subject G12:

When I just kinda analyzed the graph, I realized that at this point, or at halfway through, it was at its maximum. Its minimum was at zero. And because the graph is, you know, pretty much symmetric through there, then I just estimated it to be right there in the center [between 0 and .5].

Subject G9:

I would say probably right in the middle [of where it's rising], it’d be increasing the fastest.... Whatever the top is.... Basically 4500, almost. And then 0. So take 4500, divided by 2, and I would say that’d be where it’s increasing the fastest. And then substitute that in for your y-coordinate and find the x. And that should give you two numbers. And the first one would be increasing and the second one would be decreasing.

All three noticed that the graph had a maximum in the middle of the year and minimums at the beginning and end of the year and inferred for this that the graph was increasing fastest halfway between the first minimum and the maximum and that the graph was decreasing fastest halfway between the maximum and second minimum. Subject G9 made an error in presuming that the minimum value of the function was 0 and not 3500, but otherwise, the subject’s technique for determining when the population was increasing the fastest and decreasing the fastest was correct.
As mentioned in the section on parts a and b of Problem 1, Subject G1 graphed the population function $P(t)$ using the graphics calculator’s DOT mode. In order to identify when the function was increasing and decreasing the fastest, the subject looked for where the dots on the graphics screen got farther apart because, as the subject stated, "Far apart means faster growth and close together means slower growth." Subject G1 used the TRACE feature on the calculator to help locate when and where the dots on the screen were farthest apart. Subject G8 did not graph the population using the DOT mode, but did use the TRACE feature to locate where the difference between $y$-values of consecutive dots, or points, was greatest in order to determine when the population was increasing fastest.

**Subject G8:**

I watched the numbers and the higher they increased - ... The slope is increasing vertically, but, where you’re tracing, each dot has a - the difference between that dot and the last dot is higher.

Subjects G1 and G8 were the only two students to use the TRACE feature in this way (graphical/numerical representation) to determine when the population was increasing the fastest or decreasing the fastest. This is in stark contrast to the previous problem where nine students used this type of graphical/numerical representation, involving the TRACE feature on the calculator, to determine the maximum and minimum populations.
Use of symbolic representations. Graphics calculator students were better at describing how to use symbolic representations to determine when the population was increasing or decreasing the fastest than the traditional students. Five students were able to describe a strictly symbolic technique for solving this problem without prompting and only two students were unable to describe adequately such a technique even with prompting. This result was somewhat unexpected considering all the difficulties the graphics calculator students had describing how to locate maxima and minima symbolically.

One of the better description on how to determine when the population was increasing the fastest and decreasing the fastest was given Subject G10.

Subject G10:

We had the original equation, which is population. And then if you take the derivative of that, ... and you set that equal to zero, you get the maximum and minimum of the population.... So $P'(t)$ - um - is actually a rate ... and if you take the derivative of that then and set that equal to zero, then you would get - uh - where the rate is at its maximum or minimum. And so that's how I got it increasing and decreasing.

Later in the discussion of this problem, Subject G10 mentioned that the points where the function was increasing the fastest or decreasing the fastest were points of inflection. Unlike the traditional students, most graphics calculator students seemed to understand that at an inflection point, a graph might not only change concavity but also attain its maximum positive, or negative, slope.
which would mean that the graph was increasing the fastest or decreasing the
fastest at these points.

Subject G1:
Wherever the slope is the most or the least... What's that called? The
inflection point. The point where the slope changes from positive to
negative would be the - uh - point at which it's increasing or decreasing
at its fastest rate.

Subject G5:
Now that I think about it, it would most likely be the inflection points on
the graph.
Okay. Why would you say the inflection points?
Because - um - up to the inflection point - um - before the maximum
value, the population is increasing and then after the inflection point, it's
starting to decrease as we approach the maximum.

Most graphics calculator students recognized that the points where the
population increased, or decreased, the fastest could be located by determining
where the second derivative was equal to zero. Most also knew these were the
inflection points of the graph.

Subject G3:
Go through derivative again and find out when the derivative changes
over and usually right around that point when it's -
Okay. What do you mean by changes over?
From concave into convex.
And you find those [places] using the first derivative?
Umm - I think it's second derivative. Yeah, second derivative.
So I've got the second derivative. What do I do with it?
Uhh - If it comes out to be - uh - like zero, or something like that.
Why necessarily do you think the concavity - where the concavity changes
is where it's increasing the fastest, say. We'll focus on that.
Well, as long as it keeps - um - the concave going up, it means the - it's still increasing faster and faster and faster, and soon as it gets to points where it starts decreasing a little bit, or not increasing as fast, then that's the point where it'd start changing over to the other way.

Subject G6:

You could - um - I guess could find inflection points? Inflection points - usually the slopes steeper at those points.

Okay. What is an inflection point exactly?

It's where - it's the point where the concavity of the curve changes.

How do I find this?

Second derivative.

And what do I do with it?

Set it equal to zero.

The previous excerpts illustrate that the graphics calculator students did not appear to have the same difficulty as the traditional students associating inflection points, or the points where the second derivative was equal to zero, to the points where the population was increasing or decreasing the fastest.

A few graphics calculator students who had difficulty describing how to determine where the population was increasing or decreasing the fastest initially wanted to set the first derivative equal to zero in order to locate these points before realizing they actually needed to use the second derivative.

Subject G4:

Set [the first derivative] equal to zero.

Would that give me the maximum of it? If I set it -

For either the max or the min of the slope.

What I want is the maximum of the slope. I would assume that where it would be equal to zero wouldn't be the maximum.

Right. That's right. To get the - to get the highest slope -
You told me ... to get the maximum population, I would take the first derivative and set it equal to zero. Would something like this work here? For the max, yeah. You would take the derivative of the derivative, which is the second derivative, and you find the zero of that.

Subject G2:

Looked at the second derivative and then - I'm trying to remember if I had the root program then. For the [TI-81, Cause then I would've taken the root of where it was.]

Take the root of the second derivative?

Of the first derivative. Where it would be flattest.

Didn't we say that the root of the first derivative was where the function itself was at a maximum?

Right. So then in the first derivative, the max and the min would be found by the root - but I can't remember if I had the root program then or not. Because if I did, I'd do it that way.

Okay. To answer this one, you'd find the root of which? The first derivative? Second derivative? What?

First.

Okay. Are those the places? Remember what I'm asking here is where's the population increasing the fastest, where it's decreasing the fastest. It'd be where the slope was most positive and most negative.

You said .25 [was where the population increased fastest] ... You're saying you got that .25 by looking for where the first derivative was 0?

No. Where the first derivative is zero is where [the graph of the function is] flat... Where the derivative was at its max, it's increasing and where the first derivative was at its min, it's going to be decreasing.

What would you be finding the root of to determine where the first derivative was flattest?

Where it crossed the x-axis but it doesn't. I couldn't have used the [root] program.

What information does the second derivative give me?

It tells when the first derivative is flat, the second derivative is zero. When the first derivative is steeped, or crosses the zero, or the x-axis, the second derivative has a hump there. Either positive or negative, depending on which way the first derivative was sloping at the time. ... [Where] the second derivative is zero, the first derivative is sloping the most - no - the first derivative is flat the most and the first problem is sloping the most.
With both of these subjects, the interviewer had to point out that where the first derivative was equal to zero was where the function attained its maximum or minimum value, not its maximum or minimum slope. This prompting, along with a hint about adapting the technique for determining when the maximum population occurred, was enough to help Subject G4 recognize that the maximum or minimum values of the first derivative could be located by setting the second derivative equal to zero. Helping Subject G2 recognize this relationship was more difficult because the subject chose to describe everything in terms of the shapes of the respective graphs and had become fixated with whether or not a root program was available on his graphics calculator at the time of the Calculus Representations Test. The excerpt above suggests that Subject G2 understood that the population was increasing fastest or decreasing fastest when the slope of the graph was the most positive or negative and that the second derivative would be zero at the places where the maximum or minimum values of the first derivative occurred or, according to the subject, where the graph of the first derivative was flat.

Two students, Subjects G1 and G5, were unable to provide acceptable descriptions of how to determine symbolically when the population was increasing or decreasing the fastest. Subject G1 mentioned using the second derivative to solve the problem, but was not exactly sure how to use it.
Subject G1:

*Does the second derivative give me any information?*
Yeah, that was the inflection point, wasn’t it?

*How do I use the second derivative to find the inflection point?*
Do you set it to zero? Is that right? I can just not remember this stuff. I think I’ve repressed all this information.... I don’t remember how to find the inflection points. I do remember you deal with the second derivative.

Like this subject, Subject G5 suggested that the inflection points might be where the population increased or decreased fastest and that these points were located where the second derivative was equal to zero. However, neither subject was able to explain why where the second derivative was equal to zero might also be where the population increased or decreasing fastest.

*Use of numerical representations.* As was the case with the traditional students, few graphics calculator students described techniques for using numerical representations to determine when the population was increasing fastest or decreasing fastest. Each of these students described techniques similar to the graphical/numerical technique of using the TRACE feature on the graphics calculator to locate where the difference between y-values of consecutive dots, or points, was greatest positive or negative. Instead of checking the difference in y-values, these students wanted to check slopes between different pairs of points in order to locate where the slope of the graph of the function was maximal in either the positive or negative direction.
Use of "combination representations." Graphics calculator students made slightly more use of "combination" representations than traditional students to determine when the population was increasing and decreasing the fastest. Half the students interviewed mentioned looking at either the graph of the first derivative (graphical/symbolic) to determine where it attained its maximum and minimum values, or the graph of the second derivative to determine where it crossed the x-axis and attained a value of zero. Other students, like Subject G11, preferred to locate these extrema by evaluating the first derivative at different values of t (symbolic/numerical).

Subject G11:
You could test - uh - you could just take y' again and test values, say at, at .1, at .2, at different intervals and then you could - If you set from .1 all the way to .9, you could figure out where the greatest values are. You'd just have to enter - enter them in the - uh. You should enter those values into the first derivative equation.... Whatever ends up being the biggest value is the greater slope. So, that's going to be the steeper the curve is.... Whatever's the greatest value. Whatever's the furthest - furthest value from zero, either most negative or most positive, those'd be the steepest and the fastest increase or decrease.

This excerpt again illustrates how, at least on this particular problem, the graphics calculator students seemed to have a better understanding of the relationship between the slope of a function and its first derivative than the traditional students.
Part d. Many graphics calculator students did not make an initial attempt to solve this problem. Six of the 24 students did not write a response to the question when they took the Calculus Representations Test. As previously mentioned, this result was unexpected since it had been presumed students would recognize that the value of the first derivative when \( t \) is equal to \( \frac{1}{2} \) - the time corresponding to July 1st - could be used to represent how fast the population was changing on July 1st. This problem also presented the most difficulties for the graphics calculator students when trying to use graphical representations, as illustrated by the fact that 25\% of them were unable to describe correctly how to use a graphical representation to determine how fast the population was changing on July 1st.

Use of symbolic representations. As was the case with traditional students, it appeared that graphics calculator students did not recognize, or realize, that how fast the population was changing on one day corresponded to the rate of change, i.e., the value of the first derivative, for that day, as the following excerpt suggests.

Subject G1:

*The first derivative represented what?*

The - um - the maximum extrema, the local extrema.

*I use that to find the local extrema, but what do -*

The slope of the tangent line.

*Does that have anything to do with how fast the population’s changing?*
Yeah. If it’s - the slope of the tangent lines is at zero.
Okay, but how do I know that? You said it from the graph. Is there any way I could find it from the function? from the first derivative?
Analytically? Take the first derivative. Set it to zero.
But I don’t want to know where it’s zero. I want to know what happens - in July.
In July. Or halfway through the year.... Let’s say I wanted to find out what the population was on July 1st. What would you do?
On July 1st... Without graphing. I’d take the derivative of the equation. That would give me the population?
No. Set that to zero.
What does that give me? If I take the derivative and set it equal to zero, that’s not giving me the population, that’s giving me maximums and minimums.... How might I figure out what the population is?
Could you set \( P(t) \) equal to ... \( \frac{1}{2} \)?... What about if you set \( t \) to that? ...
You could put it equal to \( t \). Set \( t \) to \( \frac{1}{2} \) and then solve your equation.
Okay. So that would tell me what the population is halfway through the year.... Now the question is though, I don’t want to know what the population is, I want to know how fast the populations changing. Would you plug \( \frac{1}{2} \) into the first derivative?

Subject G1 remember that the first derivative gave information about the slope of the tangent line of the function, but did not comprehend how the slope of the function at a point could be related to how fast the function was changing at that point. The excerpt indicates the subject was not sure at first how to determine either the rate of change of the population or the population itself, but eventually recognized that the function represented the population and the first derivative represented the rate of change, or the slope. During the interview, Subject G1 suggested that both the rate of change of the population and the population itself could be calculated by setting the first derivative equal
to zero. Many graphics calculator students seemed to default to this technique when they encountered a problem, particularly one involving rate of change, that they were unsure how to solve.

In the excerpt above, Subject G1 mentioned that the first derivative was used to determine the slope of the tangent line. Many other graphics calculator students recognized this relationship, some with and some without prompting, but not all recognized that the slope of the population function on a particular day was a measure of how fast the population was changing on that day.

Subject G8:

*What am I asking for when I ask how fast the population's changing?*
How fast the numbers are changing. Your output.
*Does any function give me that information?*
Any function. You mean any derivative?
*Does the derivative give me that information?*
I don't think so.
*For that matter, what does the derivative give me? When I talk about the derivative of this function, ... what information does it give me?*
Gives you the max and min.
*Does it give me anything else. For example, say I put .25 in [P'(t)]. I'll end up getting about 1000π. What does that number represent?*
The number of deer.
*If I put .25 in [the original function], that would give me the - uh - number of deer, but I'm putting it in the first derivative. What information does that give me?... What does that 1000π represent?... Here's the equation of a line [y = 2x + 1]. What's the derivative of that?*
The derivative of that? Two.
*Since this a line, can you tell me what the slope is?*
It's two.
Okay. *Does that ring any bells relating the derivative to - Slope. Yeah. The first derivative shows the slope of that.*
...Is there any way I can use the first derivative here?
Yeah. See where it’s zero. Wait a minute. Yeah. See where it’s zero.
*Does that tell me how fast the population’s changing on July 1st?*
I’m not sure. Just take - okay. Wait a minute. July 1st. That’s halfway through the year? Then you just put - uh - .5 in the first derivative. that’ll give you the slope there.
*Okay. Does that tell me how fast it’s changing then? If I know the slope. You can make an estimate, but -*
Okay. *So, if I know the slope, that can give - I can then estimate how fast it’s changing there?*
Yeah, but - let’s see. Okay. I have the slope for it. Then could you take the slope and - uh - multiply it by -
*So once you know the slope, you want to multiply it by something. Would that give me how fast it’s changing?*
[no response]

Subject G8 was yet another student who initially did not think the first derivative provided information about the rate of change of a function but provided information about maxima and minima. After comparing the first derivative of a linear function to the slope of that function, the subject remembered that the first derivative provided information about the slope of a function. However, Subject G8 never realized that slope measured how fast the function was changing. The subject thought the slope of the function on July 1st, which was the same as the value of the first derivative at 0.5, was only an estimate of how fast the population was changing on July 1st, and could not relate this estimate to a more precise value for the rate of population change on July 1st. Notice that like Subject G1, Subject G8 first thought to solve this problem by determining when the first derivative was equal to zero.
A few graphics calculator students tried to relate determining how fast the population was changing to determining velocity or acceleration. These students initially thought that rate of population change was equivalent to acceleration, as the following illustrate.

Subject G7:

Would you use the original equation, wouldn’t you, maybe?
*If I put \( \frac{1}{2} \) in there, would it give me how fast the population’s changing?*
No. That’d give the population. The amount at that time.
*Is there an equation I have that I could plug a value into find out how fast it’s changing?*
Use the second derivative? That’s giving - cause that gives the acceleration. Put \( \frac{1}{2} \) in.
*Does that tell me how fast it’s changing? The acceleration?*
Okay. First derivative.... You’d look at where - this is equivalent to .5 or \( \frac{1}{2} \). And then find the y-value for that on the second derivative - on the first derivative, I mean.

Subject G6:

You just plug \( [\frac{1}{2}] \) in for \( t \) [in] the original [equation].
*If I plug in \( \frac{1}{2} \) for \( t \) in the original, doesn’t that give me the population?*
Um - that’s true. How fast is it changing so that’d be ... like the second derivative. Cause I think of like - um - acceleration, distance and velocity, and those three.
*Okay. And then first derivative’s velocity, second derivative’s acceleration. All right, but here, if I’m asking how fast it’s changing, aren’t I asking for a rate of change, which is which derivative?*
That’d probably be the - uh - the second derivative.
*If I’m talking about ... driving a car at 55 miles an hour, isn’t that the rate my car is -*
True. Well, yeah. that’s right. I was thinking the change....
*So would I put \( u \) in -*
The first derivative. Yeah, the first.
Both Subjects G7 and G6 initially suggested evaluating the population function \( P(t) \) at one-half to determine how fast the population was changing on July 1st and then switched to evaluating the second derivative when it was pointed out that \( P(\frac{1}{2}) \) represented the population on July 1st. These students did not recognize the connection between how fast the population was changing and rate of change of population until prompted about the meanings of velocity and acceleration. The difficulty, as was often the case with both the traditional and graphics calculator students, was that students did not seem to relate how fast is the population changing to the population function and its derivatives or how to associate the rate of change of a function to its first derivative.

Some graphics calculator students had no difficulty relating the rate of change of population to the first derivative, as the following excerpts illustrate.

Subject G10:

If you were to put .5 in the, you know, the function itself, it would just give you the population of deer at the first of July. So then the first derivative, as I said, was rate. I think it’d be like an instantaneous rate at any, you know, point that you would plug in there. So I plugged in .5 and I got the instantaneous rate at exactly on the first of July, which I assumed was halfway, and that’s what I got was zero. That it’s not changing at all.

Subject G3:

Actually on the first of July, it shouldn’t be changing at all… If that’s the very middle of the year, … [then] it’s when it’s changing from - um - increase to decrease in population. If that - the very middle of the year is where its derivative is zero so it’s not changing at all at that point.
These two students, along with two others of the 12 interviewed, were able to explain correctly, without any prompting, how the value of the first derivative, when $t$ was equal to 0.5 or $\frac{1}{2}$, represented how fast the population was changing on July 1st.

*Use of non-symbolic representations.* As was the case with traditional students, since so much time was spent discussing the previous parts of Problem 1 and discussing techniques solving this problem symbolically, little time was left for discussing how to use graphical, numerical, or "combination" representations to determine how fast the population was changing on July 1st. Graphics calculator students had some difficulty solving this problem graphically. Three students were unable to describe correctly how to use graphical representation to determine the rate of change of the population on July 1st. The eight students who were able to describe how to solve the problem graphically gave descriptions similar to the one given by Subject G8.

**Subject G8:**

Find out if like - if it goes up and then it comes back down.... The first of July’s is gonna be near the top, and I know that’s it gonna be - it’s gonna flatten out before it comes back down ... [so] it’s not going to be that great of a change.

Subject G8 seemed to understand that the flat shape of the graph near July 1st meant that the population would not be changing very quickly around that time,
but never converted that knowledge into a numerical estimate for how fast the population was changing. The interviewer considered descriptions like these appropriate use of a graphical representation when solving the problem because, in most cases, there was not enough time left during the interviews to further explore students’ understanding of how to use the graphical representation to solve the problem.

Six students described numerical methods for estimating how fast the population was changing on July 1st. All approximated the solution either by comparing values of the function on or near July 1st or by determining the slope between pairs of points near July 1st on the graph of the function, as illustrated by the following excerpt.

Subject G7:

Well, you can take two points and use the slope. Um - what’s it called. The - like y minus y₁ equals slope x minus x₁. That equation. And just put the points in that way.

All right. What points would you use cause I’m interested in figuring out what it is at - where t is ½... What would you do for the other point?

If you put like ½ in the original equation. What that equals. The y-value for that.

Okay. So we’ve got the ½ and the y-value that corresponded to ½.

Then you use the same slope as you found in the first derivative.

Okay. Now remember, we don’t have the first derivative.

Oh. That’s right... Oh, you pick two points and you - um. It’d be like y₁ minus y₂ over ... x₁ minus x₂ ... to find slope.
A few graphics calculator students, like Subject G3, suggested using the TRACE feature on the calculator (graphical/numerical representation) to observe what happened to the value of the function on or near July 1st.

Subject G3:
Well, you can look at the graph and kinda estimate.... I'd use the TRACE function and just go through to about where .5 is on the x-axis and go back and forth a little bit just to see how it's changing. And since it's increasing on one side and decreasing on the other side, so it's not changing that much at that point.

This approach to determining the rate of change of population on July 1st is the same as the numerical method of comparing values of the function on or near July 1st except that the values of the function are computed automatically by the TRACE feature on the calculator instead of by the student. Other than this graphical/numerical representation of using the TRACE feature, the only other "combination" representation on this problem mentioned by the graphics calculator students was to estimate the value of the first derivative on July 1st from its graph (graphical/symbolic).

Calculus & Mathematica Student Interviews.

As you read the interview excerpts that follow, look for symptoms concerning these behaviors in the descriptions given by the Calculus & Mathematica students:
• recognition of the connection between the slope of a function and its rate of change
• recall of what information about a function and its graph was provided by its first and second derivatives
• recall of what information about a first derivative and its graph was provided by its second derivative
• recognition of how to obtain information about rates of change from a graphical representation
• use of symbolic representations when not in conjunction with other representations
• recognition that points where the concavity of the graph of a function changes could also be where the function attains its maximum or minimum slope, or where it attained its greatest or least rate of change
• use of "combination" representations, particularly graphical/symbolic and symbolic/numerical representations
• recognition of how to use representations in ways different from those typically taught in the course
• relating of unfamiliar problem situations or descriptions to familiar uses of representations

**Parts a and b.** Calculus & Mathematica students had little trouble describing how to use any of the different forms of representations to locate maxima and minima. The only exception was determining symbolically if extrema were maxima or minima, which was not discussed in the interviews since the Second Derivative Test was not taught in the course. In working with symbolic representations, Calculus & Mathematica students did not have the same difficulty as students from other courses recognizing what information about a function and its graph could be provided by the first derivative.
Many Calculus & Mathematica students mentioned ways for using different "combination" representations both to locate extrema and to determine if an extremum was a maximum or minimum. They made much more use of graphical/symbolic and symbolic/numerical representations than students from other courses. Ten of the 12 Calculus & Mathematica student described how to use at least one form of "combination" representation to determine the maximum and minimum populations. This extensive use of "combination" representations suggests that the Calculus & Mathematica students had a better, more integrated understanding of the connections between the different representations of concepts used in this particular problem.

Use of graphical representations. Calculus & Mathematica students had few problems using graphical representations to solve parts a and b. This was to be expected since students were asked to work with and interpret many different types of graphs in order to solve problems assigned in the course. Only 3 of 12 Calculus & Mathematica students had to be prompted about using a graphical representation to locate the extrema of the population function. This prompting consisted of asking whether or not they remembered discussing techniques for visually locating maxima and minima. One student who needed prompting, Subject C1, mentioned using a graph but did not believe the graph could be used to locate the extrema accurately.
Subject C1:

If you have the graph ... and not the derivative, ... you could look like at maximum and minimum points.

So if you wanted to, you could look at the graph.... Let's say I've created a graph.... Can I answer the question without ... the derivative?

Just with [the graph]?... Not really.

Okay, Why not? Why don't you think so?

You'd almost have to know that exact point. I don't - it'd be hard to find without the derivative.... I think it'd be tough to find out exactly. You could know, like, a point. If you had, like, the y and the x. If you had those axes labeled, you could find a general point by just eyeballing it, but I mean, you can't really -

Here's a graph.... Would that be enough for me to come up with an answer or do I need to use the derivative?

I don't know any other way, other than the derivative.

Based on what you see here, ... what do you think the answer to the first part of this is? When in the year is the population at its maximum?

According to this graph?... Um - about halfway through the year.

Okay. Didn't you just give me an answer without the derivative?

True, but I mean, it's not a very GOOD answer, it's just halfway. ... It's approximate. You can get an approximation.

Subject C1, like some graphics calculator students, seemed hesitant to use the graph because, according to the subject, the graph could only produce an approximate answer, not an exact answer, to the problem. It appeared that this subject had forgotten, or did not realize, that the coordinates of any point on a graph can be ascertained to 9-place decimal accuracy by simply altering the range parameters of the Mathematica code used to graph the function. This feature of Mathematica is the manual equivalent of the ZOOM-IN feature on a graphics calculator. Somewhat surprisingly, few Calculus & Mathematica
students, like Subject C12, mentioned "zooming-in" like this in order to more accurately determine the coordinates of the maxima and minima for the graph of population function.

Subject C12:

You could zero in on it, using - changing the range a little, if you wanted to. You can see that it's almost - about .5, so you can go like this [subject altered range parameters in Mathematica code, producing new graph].... Now you just keep doing that until you see that it's exactly at .5.

Subject C12 was one of the few students from any of the courses who seemed to recognize that graphical representations could be altered to provide as accurate a solution to a problem as desired, and could in some cases, indicate the exact solution to the problem. It was unclear if the students did not remember that such manipulations of graphs were possible, and, in fact, simple using graphics calculators or graphics software like Mathematica, or if they perhaps did not understand the connection between different graphical representations of the same function used to estimate more accurately the coordinates of the maxima and minima of the function.

One student, Subject C6, seemed to understand how to locate extrema graphically, but had difficulty explaining how to interpret the graph of the population function, as the following excerpt illustrates.
Subject C6:

Well, I knew that the max would be - um - somewhere in July from what the graph was stating. Um - and you know, it might be just exactly in July or whatever, just that it was in the center. And then with the carrying capacity, as always learned - um - once you hit max, you taper off, down, and then you’ll go below the standard which was given. Then you’ll find min and then you’ll start leveling off somewhere, or going back up, into - you know, depending on what species, and stuff like that.

*Any particular reason you thought that the max was 4000?*

I just, you know, kinda looked at the graph and that.... Yeah, it’s pretty close to 4000. It’s almost 5000, basically.... I just took a stab at it because of the height difference and stuff like that.

*Why did you think it was, the, the population was zero in December.*

Now when you thinking of this, you have a brood stock that never - um - cause since I’m in Natural Resources, you know that the population has a brood stock and then the fluctuation above it and then when zero is stated, it means that either it dips down into the brood stock, or it comes straight, almost to it. I know it wasn’t in the problem but -

*Okay. So when you mean 0, are you meaning there’s no deer left or there’s -*

There’s just a brood stock left.... And in December - um - that’s the coldest month, and it’s like that’s when it will drop the most. And it’ll go as close to zero as possible. Close to the brood stock as possible.... The population that comes after the brood stock would be down to zero.... Basically all this was guessing.

Subject C6 correctly located on the graph when the maximum and minimum population occurred but misinterpreted, or misidentified, the y-coordinates of the points on the graph corresponding to the maximum and minimum populations. As the comments suggest, the difficulty was that the subject tried to interpret this situation in terms of concepts pertaining to animal populations that had been taught in courses from the Natural Resources Department. For
example, the subject seemed to interpret the \( y \)-values from the graph of the population function as representing the number of animals in the population above the brood stock, which seems to represent some type of base population from which the entire population is generated, and not the number of animals in the entire population. This may explain why the subject thought the minimum value of the graph had to be 0 and not 3500. As we will see later, Subject C6’s interpretation of this problem made it difficult for the subject to describe how to use various types of representations to solve the different parts of Problem 1 and for the researcher to determine if the subject truly understood how to use these representations.

**Use of numerical representations.** Ten of the 12 Calculus & Mathematica students described correctly ways for locating extrema numerically, seven without any prompting. As before, students who had to be prompted before describing how to locate extrema numerically were asked if they could think of another way to solving the problem given that only an approximate solution was required. Four students overall - two initially - made use of the numerical properties of the Cosine function, rather than just calculate values of the function to determine the maximum and minimum populations and when they occurred. Below are two subjects’ descriptions for how to determine the maximum population using the properties of the Cosine function.
Subject C11:
I figured that out by - ah - I guess just looking at the idea of the Cosine ... and - ah - trying to figure out - and knowing that Cosine could vary from 1 to -1.... I realized that at -1, it’d then be 4000 + 500, and at that point, that’s going to be as high as it will ever get. And I, you know, figured out what value would make Cosine of $2\pi t$ equal to -1 so it be at its maximum and - ah - since the $t$ was multiplied by $2\pi$ ... I knew I’d have to cancel out the 2 cause cosine of $\pi$ is -1.

Subject C12:
The max has to be when 500 times $\cos 2\pi t$ is at a min since it is being subtracted from 4000. The lowest $\cos 2\pi t$ can be is when it is -1. Cosine equals -1 at $\pi$, so $\cos 2\pi t$ is -1 when $t$ is $\frac{1}{2}$ which is the middle of the year. Plug this into the original equation to get ... $P(t)$ equal to 4500.

All the subjects who mentioned this technique recognized that the minimum population occurred when $\cos 2\pi t$ attained its maximum value and the maximum population occurred when $\cos 2\pi t$ attained its minimum value.

Ten of the 12 students interviewed suggested using numerical representations to distinguish maxima and minima. As with the students from the other courses, the Calculus & Mathematica students mentioned either checking the value of the function on either side of the extrema or observing the values of the function at the extrema. One student, Subject C4, had an intriguing misconception about where the maxima and minima of a function were located, as the following excerpt illustrates.
Subject C4:

I thought that it would go in order of - I didn’t realize it would come out that way because - I guess maybe I’m remembering a different lesson or something, but ... I though that you could go in the order, like go by the magnitude of the number ... to figure out which was the lowest point and which was the highest point.... Because this is an x, that’s way down at 1. This x would be at 1 and that x would be at 10, so I was thinking - 

Don’t I need y-values though to decide that? Wouldn’t I need the y-values that correspond to these to figure out which was which rather than - Yeah, you would but I’m not altogether sure how you can get the y-value. Well, what could I do with this so I could figure out the y-value. Well I was thinking you could plug it back into the equation, but if you plug it back into the equation, you get - You’re plugging it back in to f’. Oh. That’s true. Where would you want to plug it in? Back into the f(x). ... That’ll give you the y-values and then you’ll have points and you can look at the points to decide which.

Subject C4 at first thought that the largest value of x corresponded to the maximum value of the function and the smallest value of x corresponded to the minimum value. Thus the subject was somewhat surprised when shown an example of a function - a quartic polynomial - where the maxima and minima did not correspond the largest and smallest values of x, respectively. As the comments above indicate, Subject C4 did eventually recognize how to determine which extrema were maxima and which were minima numerically.

Use of symbolic representations to locate extrema. Calculus & Mathematica students had an easier time than students from other courses working with symbolic representations to determine the maximum and
minimum populations. Every student was able to describe correctly the
standard technique of setting the first derivative equal to zero and solving, and
all but two did so without prompting. One of the two students who needed
prompting, Subject C6, had to be reminded about the connection between the
first derivative and the slope of a function before remember, or realizing, that
the value of the first derivative was zero at a maxima or minima.

Subject C6:

That's the derivative. If I put ... 5 in there, I get 0 out.
Um hm. As the min.
No. I just get - that's what the - if I put .5 in for t, in the derivative, the
answer I get is zero. Do you know what that zero represents when I get
that value out of a derivative? I mean, what that zero's representing?
Um - not off the top of my head, no. I knew before, but I'm like - um.
Do you remember discussing the idea of the slope of the function? ... Is
that anyway related to the derivative?
That's probably the slope at one point. You know, finding the derivative
is - defines slopes or whatever. And with that being zero, there was no
slope at that point. So when you put .5 and get 0, you're not at - okay!
That tells you, there you are not increasing and you're not decreasing. So
you're just there, basically, where would be a max or a min. You
wouldn't really know whether it's a max or a min, basically, but that
states that you're not increasing or decreasing.

This subject at first could not recall how to use derivatives to locate the
extrema of a function and, as the comments above suggest, did not remember
what a value of zero for the first derivative represented. After recognizing that
the value of the first derivative represented the slope of the function at a point,
Subject C6 suddenly seemed to realize that where the slope of the function was
zero, it was not increasing or decreasing, which meant the function could be at a maxima or minima. At this point in the interview, the subject did mention that one way to locate the extrema might be to determine where the first derivative was equal to zero.

Subject C4 was the other student who needed prompting before remember the technique of setting the first derivative equal to zero to locate the extrema of a function. This prompting consisted of providing, at the subject request, the graphs of both the population function and its first derivative so that the subject could look at what was happening to the first derivative at the extrema of the population function.

Subject C4:

Where it crosses the x-axis that’s where the function is its highest and when it’s - At the peaks and troughs show where it crosses. Okay. It crosses - the peaks and troughs are where it crosses the x-axis. And where it crosses the x-axis, that’s the highest point. Where it crosses the x, when x is equal to zero - um. When f - when the derivative is equal to zero, you have the highest point.

*Highest point. What about lowest points?*

Well, they would be your peaks and troughs. Your highest - no, that would be where it crosses the x-axis. Your peaks and troughs are where it crosses the x-axis. Well, your extreme values are where it crosses the x, the x-axis.

*Okay. So that’s where the derivative is equal to what? It crosses the - where the derivative crosses the x-axis, what does that tell you about the value of the derivative?*

It’s equal to zero.

*It’s equal to zero. Now does that only give me maximum points ... ?*

No, it’ll give you every extreme point, the highs and the lows.
Subject C4 mentioned using the derivative to locate maxima and minima but could not recall how to use it. The subject then asked to look at the first derivative visually along with the graph of the function. While viewing both these graphs, Subject C4 recognized that the peaks and troughs of the graph of the population function - this is terminology used in the Calculus & Mathematica course - occurred when the graph of the first derivative was crossing the x-axis, which corresponded to the first derivative having a value of zero. It is important to note here that this student used a graphical/symbolic representation - the graph of the first derivative - to help understand how to use a symbolic representation to solve the problem.

*Use of symbolic representations to distinguish maxima and minima.* The use of symbolic representations to determine if an extrema was a maxima or minima was not discussed during the interviews with the Calculus & Mathematica students since the Second Derivative Test was not taught in the Calculus & Mathematica course. Instead, more time was spent discussing ways to use for "combination" representations to solve this problem.

*Use of "combination" representations.* By far, Calculus & Mathematica students made more use of "combination" representations, particularly graphical/symbolic and symbolic/numerical representations, to locate maxima and minima than students from other courses. Five students, including Subject
C4 as previously mentioned, suggested using the graph of the first derivative (graphical/symbolic representation) to locate the extrema of the population function. Four of these students also understood how the graph of the first derivative could be used to distinguish between maxima and minima. Below are excerpts for two of these students.

Subject C7:

You could look at the first derivative curve and see where it was positive and where it was negative. And where it went from positive through zero to negative would be a maximum and where it came up from negative to positive would - and zero would be a minimum. You could find a value right below zero or right above [the] zero ... and see which one was positive and which was negative.

Subject C5:

Depending on which direction the function is heading will determine whether it’s a maximum or a minimum.

*What do you mean by the direction the derivative's heading?*

If it’s going from, let’s see, decreasing to increasing, it'd be a minimum. So there’s a positive slope, it’d be a minimum. If it’s a negative slope then it’d be at a maximum because it’s going from increasing to decreasing.

*Okay. Now do you mean, ... let me see if I follow this. Um - when you say positive slope, it'll be a minimum, what are you checking to see where this positive slope is?*

Just, if you think of the graph [of the first derivative] in your mind.

Both Subject C7 and Subject C5 gave superior explanations on how to use the graph of the first derivative to distinguish maxima and minima. Each recognized that when the first derivative went from negative to positive, the population function went from decreasing to increasing and a local minimum
occurred and when the first derivative went from positive to negative, the population function went from increasing to decreasing and a local maximum occurred. It should be noted that no student from the other courses suggested using this technique for determine if an extrema was a maxima or minima.

In the last sentence of Subject C7’s comments above, the subject mentions finding the value of the derivative just before and just after the zero of the derivative to determine which is positive and which is negative. The subject was referring to the symbolic/numerical technique for distinguishing maxima and minima where the first derivative is evaluated on either side of the extrema to determine if the original function goes from decreasing to increasing (minima) or increasing to decreasing (maxima). Eight Calculus & Mathematica students, including Subjects C10 and C1, suggested using this technique to solve this part of Problem 1.

Subject C10:
If it’s a - then you solve it for - if it’s a greater than zero. If the derivative’s greater than zero. 
Greater than zero where?
At that point, I mean, to the left of that point. To the left of that point, if it’s greater than zero, then the function’s increasing. And if it’s less than zero - ah - to the right of that point, you know that it’s decreasing. So, if something is increasing and then decreasing, the only way that can happen is if - if it’s a maximum.... With a minimum, one side would be decreasing. The - ah - left side would be decreasing while the right side would be increasing.
Subject C1:

You could find, like, the next point over, ... and see if it’s negative. That means the point before it would have been a maximum cause it’d have to go down afterwards. And if it was a minimum, it would be positive.

Students who used "combination" representation to distinguish maxima and minima appeared to understand that the first derivative provided information about the slope of a function. They also recognized what changes in slope to look for at maxima and minima. It should be noted that Subject C1 later acknowledged that a point before the extremum had to be checked to verify that the value of the derivative was different on each side of the extremum.

Unlike the graphics calculator students, few Calculus & Mathematica students suggested determining when and where the maximum and minimum values of the population occurred by looking at and comparing the coordinates of different points on the graph of the population function (graphical/numerical representation). This result was somewhat unexpected since the Mathematica graphics software has a feature much like the TRACE feature on a graphics calculator that can be used to determine the coordinates of points on a graph.

More Calculus & Mathematica students than students from other courses mentioned using the first derivative to locate extrema and then using the graph of the function to determine if the extrema was a maximum or minimum.
Subject C5:

I usually use a combination. I’ll usually find the derivative and then, if the graph is simple enough, I won’t even plot it, but I then sometimes’ll plot it and see which ones are maxes and which ones are mins ... and then just calculate the exact value.

Calculus & Mathematica students extensive use of "combination" representation and their use of symbolic representations coupled with graphical representations suggests that they may have a better understanding of the connections between the different representations of the concepts used in this particular problem of locating maximum and minima.

Part e. Calculus & Mathematica students had the easiest time of all the students working on this part of Problem 1, though they had some difficulty and did not do as well as on the previous two parts. They seemed to have the best understanding of the relationships between the first derivative and the slope of a function and of the information about the first derivative provided by the second derivative. Almost all the Calculus & Mathematica students recognized where the population increased or decreased fastest corresponded to where the first derivative attained its maximum or minimum value. A few also recognized these points to be the inflection points of the graph of the population function, but most did not make this connection, possibly because very little was spent discussing inflection points during the course.
Calculus & Mathematica students made extensive use of "combination" representations, particularly graphical/symbolic representations. Every student described how to use at least one form of "combination" representation to determine when the population was increasing and decreasing fastest, and all but one described how to use a graphical/symbolic representation.

Unlike the traditional or graphics calculator students, the Calculus & Mathematica students seemed to understand what to look for when trying to determine when the population was increasing the fastest or decreasing the fastest. Only 2 of the 12 initial solutions to this problem were stated in terms of intervals where the function was increasing or decreasing rather than single points where the function was increasing, or decreasing, the fastest.

Use of graphical representations. Eight of 12 students recognized, without prompting, that where the graph of the function had its greatest positive (negative) slope was where the function was increasing (decreasing) the fastest. Below are excerpts from interviews with five students who recognized this relationship.

Subject C12:
You can look right there and you can see, along with slope where it is. It's increasing, where slope's the largest.

Subject C7:
You could graph the original function and look when it looked steepest.
Subject C2:
Look at, I guess, where it grows fastest.... I mean, I would assume that that would be like where its slope is like the - I mean the highest. Where it seems to be the highest.... where I have like, the least change in $x$ where, you know, the most change in $y$.

Subject C8:
You could look at the original function and look where it looks like it’s sloping up fastest.

Subject C5:
What are you looking for in that graph, to figure out ... where the slope is steepest.... Then if can find one, you’ve got the other because it’s going to be the same distance from 0 and 1 just because it’s symmetrical.

In all, 11 of the 12 students correctly identified the points where the function was increasing and decreasing the fastest. Four of these students mentioned using the symmetry of the graph, as Subject C5 did in the excerpt above, to locate these points. The 12th student, Subject C4, never even attempted to identify the points on the graph of the original function where the population was increasing fastest or decreasing fastest.

Subject C4:
You know, when it’s increasing the fastest, ... when the sign changes. *What sign’s changing?*
Okay. It’s still increasing here [points to where the graph is approaching the local maximum when $t = .5$], still increasing cause your $x$-values are getting - I mean your $y$-values are getting bigger.... Now when the $y$-value starts to decrease, it’s no longer growing so that’s point when it’s - *So right here [local maximum when $t = .5$] is where the population’s increasing the fastest?*
No. It’s about to stop there.
So where's it increasing the fastest? You could probably do it with derivatives. See I need to look at it - I can't figure it out from this. I don't - I didn't remember the point like when \( f(x) \) is equal to this, it's increasing the fastest. I always do it visually so if I look at the derivative -

Subject C4 never attempted to identify the points on the graph of the population function or on the graph of its first derivative where the population was increasing the fastest or decreasing the fastest because the subject was not able to visualize what these points represented on the respective graphs. As will be shown later, the subject did identify these points on the graph of the second derivative.

**Use of symbolic representations.** Calculus & *Mathematica* students did better than graphics calculator students, and substantially better than traditional students, describing how to determine symbolically when the population was increasing or decreasing the fastest. Seven students described strictly symbolic techniques for solving this problem without prompting. Only one student was unable to adequately describe such a technique even with prompting. As the following excerpts suggest, Calculus & *Mathematica* students, like graphics calculator students, seemed to recognize that where the second derivative was equal to zero was also where the first derivative had its maxima and minima and the original function was increasing, or decreasing, fastest.
Subject C5:
You can find the second derivative which should give the maximum. Yeah, it’ll give you the maximum increasing and the maximum decreasing.
What do you do with the second derivative? The second derivative measures the growth of the derivative, so you’re trying to find out when it’s growing the fastest. You find out where it’s zero and that gives the maximum, again, the minimum. In the same way, you’re just taking another step ... because you’re doing the growth rate this time instead of the standard function.

Subject C2:
The derivative of the derivative would solve this. The max of the growth rate would be when the population increases the fastest.... Set the second derivative equal to 0 and solve.

Subject C7:
You could take the second derivative and see where it was zero. And when the second derivative was - let’s see. When the second derivative was zero, then you’d have a maximum or minimum in the first derivative.

Subject C8:
The population is increasing the fastest for when \( P'(t) \) is greatest ... [and] the population is decreasing the fastest for when \( P'(t) \) is smallest, because \( P'(t) \) represents the instantaneous growth rate of the function.... I know that the derivative would have to be at its lowest and highest points in there. I could take the second derivative ... and find where those points are zero, ... finding where the maximum and minimum values for [the first derivative] are. Then I would know where the exact values for [the points where population is increasing fastest or decreasing fastest] are.... I would take the second derivative and find out when it was zero.

None of these students mentioned that the points where second derivative was equal to zero were the inflection points on the graph of the function. In the Calculus & Mathematica course, points of inflection were not stressed nearly as
much as in either the traditional or graphics calculator courses. For the most part, the students who recognized that the points where the second derivative was equal to zero were inflection points also recognized that the inflection points could correspond to where the population was increasing the fastest or decreasing the fastest, as the following illustrates.

Subject C11:

You could start off, you know, with the actual graph.... And when - the graph of the derivative if you wanted to ... and - ah - you could see, like, when it's at its - when it's at a peak or at a dip, that would be a point where it'd be increasing its fastest or decreasing its fastest cause it's measuring instantaneous growth rate ... and if you took the normal function itself ... and just graphed it. I would assume that it would be at a point approximately here [at t = .25] cause, you know, it's right where changes - its inflection point.

Is that where the steepest - is that where it would be increasing the fastest? At the inflection points?

I'm not sure, but that would mean - that'd just be where I'd, like, start off is, like, right in between the dip and the valley, but I'm not - I don't even know. I'm not sure that it would be -

Well, what's happening at the inflection point.

Okay. That is - the inflection point would be - Okay, right here, the [second] derivative is zero, and, it might be where the derivative is at its highest. The only way you could figure that out for sure is if you took the derivative - the second derivative of the function ... and - ah - where it is zero - cause that shows where - When the second derivative is 0, that shows where the actual - the first derivative is at its peak or valley and that would show where [the function] is growing at its fastest.... I would say at the inflection points, that would be where its increasing the fastest.

Subject C11 started by identifying correctly the points on the graph of the first derivative where the function was increasing fastest and decreasing fastest.
The subject then used these points to locate the correct points on the graph of the function, which were identified as inflection points. Finally, after some initial uncertainty, Subject C11 recognized that the inflection points represented the points where the first derivative had a maximum or minimum, which in turn represented the points where the function was increasing fastest and decreasing fastest. This excerpt illustrates how the Calculus & Mathematica students understood the connections between different forms of representations, particularly graphical and symbolic, used to solve Problem 1.

Subject C10 stated that the inflection points of the graph of the population function were the places where the population is increasing the fastest and decreasing the fastest, but the subject also indicated that inflection points occurred when both the first and second derivative were equal to zero.

Subject C10:

_You’re say these are inflection points. Well, what are inflection points?_ Inflection points are where the - ah - concavity changes. For example, a min and a max is where the derivative is equal to 0. An inflection point is also where the derivative is equal to zero. Now the difference is that if it’s a - you talked about before then. If it’s increasing on one side and decreasing on the other. That would mean that it’s a maximum. An inflection point would be where it is - it’s increasing here but it’s concave upward, and that you can tell by the second derivative.... Then to the - to the left of that point, if it’s - if it’s increasing ... and it’s concave upward and then to the right of the point, if it’s increasing but it’s concave downward, that’s an inflection point.
I was wondering. If you think of your description, if I’m interested in where the population increasing the fastest, would that be where the derivative is zero?

It’d be where the second derivative is zero. The second derivative, not the first. So, most likely, if you take the derivative of THAT function [first derivative], then ... your inflection point would be at zero.

Subject C10 identified correctly the point on the graph of the population function, at \( t = 0.25 \), where the population was increasing the fastest. The subject also indicated that the first derivative would be equal to zero at this point. It was only after being shown the graphs of the population function and its derivative simultaneously that Subject C10 recognized that the first derivative was not necessarily zero at an inflection. It is possibly that this subject’s misconception about inflection points was prompted by work with the Second Derivative Test in an earlier precalculus course. In most textbooks, the Second Derivative Test includes a statement to the effect that when \( f'(c) \) is equal to zero, if \( f''(c) \) is also equal to zero, then the point \( (c, f(c)) \) is an inflection point of the graph of \( f(x) \).

Subject C6 was the only Calculus & Mathematica student unable to describe correctly how to use symbolic representations to determine when the population increased fastest and decreased fastest. The subject knew the solution involved the first derivative but did not seem sure how to use it, as the following excerpt suggests.
Subject C6:

You can do it maybe with derivatives, you know.... Plug in points, maybe. 
*And then what? Look for the -*
The most drastic difference between two, maybe, I don’t know.
*If I’m plugging into the derivative, I’d want to compare values?*
Um - you can do it that way.
*Well, what am I - when I plug into the derivative, what am I looking for since I’m saying I want to know when it’s increasing the fastest?*
You’re looking for - um - the highest number?
*Okay, And for decreasing the fastest, I’d be looking for what?*
Um - still the highest number. Um - you know, cause the lowest number is going to be - um - where there’s no slope - where there’s less activity.
*Couldn’t I have a negative number?*
Okay. You can have a negative number. But derivatives can’t give you negative numbers, right? You can’t have negative numbers in derivatives.

At first, Subject C6 wanted to locate where the population was increasing or decreasing fastest by determining when the greatest difference between values of the first derivative, not the population function, occurred. Then the subject appeared to recognize that the solution called for determining when the maximum value of the first derivative occurred. However, comments made later in the interview indicated that Subject C6 was not sure the largest positive and negative values of the first derivative corresponded to the points where the population increased fastest and decreased fastest, but thought it was probably correct. Subject C6 eventually recognized that the value of a derivative could be negative after determining the derivative of the function \( y = -2x + 1 \).
After pointing out that the greatest value of the first derivative was the same as its maximum value, the researcher asked Subject C6 how the maximum value of the first derivative could be determined symbolically. As before, the subject suggested using the correct derivative but did not know what to do with it.

Subject C6:
Find the double derivative?... Plug it back into the first equation for \( t \)? I don't know. The second derivative.
Okay, I have the second derivative. Now, what do you mean. Plug this back in for \( t \)? Or - I'm not sure I'm following.
Yeah, neither am I. Um - you do something with two of the derivatives or with the original equation, ... you know, where they're put together in some way. Put one answer in the other.

Subject C6 never recognized that the maxima and minima of the first derivative could be located by determining where the second derivative was equal to zero.

Use of "combination" representations. Every Calculus & Mathematica student interviewed described how to use some form of "combination" representation to determine when the population was increasing fastest or decreasing fastest. A few students suggested using the graph of the second derivative (graphical/symbolic) simply as means for determining when the second derivative was equal to zero. Subject C4 seemed to understand why the problem could be solved by determining where the graph of the second derivative crossed the \( x \)-axis, as the following suggests.
Subject C4:

I’m interested in where the population’s increasing, or decreasing, fastest. I’d say where the second derivative crosses the x-axis.

Okay. Why do you think that?

Because, well the reason that I rationalize it, I know that if - um. Okay. On the first derivative, where it crosses the x-axis, you have your extreme points. I just remembered that. On the second - I may be remembering incorrectly, but I remember on the second, if you take the second derivative, you can find out where it’s increasing the fastest because where it crosses the x-axis will give you the places.

Subject C4 appeared to recognize that the points where the second derivative crossed the x-axis - where the second derivative was equal to zero - were the extrema of the first derivative which were also the points where the population was increasing, or decreasing, fastest.

Ten students mentioned using graphical/symbolic representations involving the graph of the first derivative to solve the problem. This result was not unexpected considering most Calculus & Mathematica students appeared to recognize that the points where the population was increasing or decreasing the fastest were also the points where the first derivative attained its maximum or minimum value. Below are examples of the students’ description of how to use the graph of the first derivative to solve this problem.

Subject C11:

You could start off, you know, with ... the graph of the derivative if you wanted to ... and - ah - you could see, like, when it’s at its - when it’s at a peak or at a dip, that would a point where it’d be increasing its fastest or decreasing its fastest cause it’s measuring instantaneous growth rate.
Subject C7:
You could graph the derivative to see when the high point and the low point of the derivative were. The high point would be increasing the fastest, low point would be decreasing the fastest.

Subject C9:
I would just graph the derivative and see where it is farthest above and farthest below 0.... It would be increasing the fastest at the top of this curve and it'd be decreasing the fastest at the bottom.

Subject C10:
What would you be looking for there [in the graph of $P'(t)$]?
You'd be looking for the maximum because - ah - because for an inflection point, the first derivative must be equal to zero. The inflection point is found by the second derivative.
Um hmm. I though we decided that it was where the second derivative was equal to zero -
Right. It is.
But not necessarily where the first is equal to zero.
Right. Oh, sorry. That's what I meant to say.

Subject C10 was the student mentioned previously who thought inflection points occurred only when the first and second derivatives were equal to zero.

The subject still seemed to be having difficulty accepting that the first derivative did not have to be equal to zero at an inflection point, even after the earlier discussion and after being shown graphical evidence to the contrary - the graph of the first derivative indicated that the first derivative was not zero at the maximum point of its graph.
One student who mentioned using the graph of the first derivative to determine when the population was increasing fastest and decreasing fastest did not appear to understand how values of the first derivative were related to the population function, as the following suggests.

Subject C3:
I would graph $P'(t)$ and see when the biggest growth is. I would also look for the biggest on the graph and see when in the year they occur.  

*Here’s a graph of the first derivative. What would you do with that if I’m interested in when it’s increasing fastest and when it’s decreasing fastest?*

Well, it’s at zero here. Actual, that’s the highest point. Then you’ve got these zeros here. These are when the derivative is equal to zero.

*Okay. So in terms of when the population is increasing the fastest, where would that be on this graph?*

I would think it’s at the beginning [point corresponding to January 1st] where the graph of the derivative crosses the x-axis.

*What does the first derivative represent? I’ve got a graph of the first derivative. What do all these values I’ve got plotted here ... represent?*

I think you can tell where it’s at zero or the highest and lowest point.

*Okay, but what do they represent in terms of this function? What are they describing? Like you said, here the derivative’s zero. We’ve got a maximum. Here the derivatives are zero. I’ve got minimums. Here, you know, at 4, the derivative’s ... about 2000. Okay. What does that 2000 tell me about this graph [of the function] at 4. What information is that? Uhh - I’m not sure. I don’t remember.*

At first, Subject C3 indicated that the points where the graph of the first derivative crossed the x-axis were where the population was increasing fastest and decreasing fastest. These points were actually where the graph of the first derivative was increasing fastest and decreasing fastest. After being reminded that the first derivative was related to the rate of change of a function and that
the fastest increase in population corresponded to the maximum population
growth, Subject C3 recognized that the population growth would not be
maximal when the derivative was equal to zero. The subject then went on to
identify the correct points on the graph of the first derivative, and on the
population function, where the population was increasing fastest and decreasing
fastest. It was unclear, even after the subject correctly solved the problem, if
the subject understood what the values of the first derivative represented.

Six Calculus & Mathematica students suggested using some form of
symbolic/numerical representation involving the first derivative to determine
when the population was increasing fastest and decreasing fastest. Most of
these consisted of evaluating the first derivative for different values of $t$ until
the maximum and minimum values were determined. Two students, Subjects
C7 and C12, recognized that the first derivative was a Sine function and
proceeded to use properties of the Sine function to determine the maximum and
minimum values of the first derivative.

Subject C7:
Well, the question is where it’s increasing the fastest.... It’s increasing the
fastest when the slope is greatest positive number which would be - um -
in this case, when $t$ is $\frac{1}{4}$, since we’re dealing with a year.
I’m just trying to figure out why you came up with the - you know, how
you came up with the $\frac{1}{4}$. I see the derivative there -
Oh because sine is the greatest at $\pi/2$. 
Subject C12:

Population is increasing fastest when $P'(t)$, or $1000\pi \sin 2\pi t$, is at its largest. The largest $\sin 2\pi t$ can ever get is 1. $\sin x$ equals 1 when $x$ is $\pi/2$. Therefore, $\sin 2\pi t$ equals 1 when $t$ is $\frac{1}{4}$. When $t$ is $\frac{1}{4}$, we are in the end of March. Population is decreasing fastest when $1000\pi \sin 2\pi t$ is at its smallest. $\sin x$ can go no smaller than -1. $\sin x$ equals -1 at $3\pi/2$. This means that $\sin 2\pi t$ equals 1 when $t$ is $\frac{3}{4}$. When $t$ is $\frac{3}{4}$, it is the beginning of October.

No Calculus & Mathematica student mentioned using any type of graphical/numerical representation to determine when the population was increasing or decreasing fastest.

Use of numerical representations. As was the case with students from other courses, few Calculus & Mathematica students described techniques for using numerical representations to determine when the population was increasing fastest or decreasing fastest. These students checked values of the function in order to locate the greatest positive or negative difference between values, checked slopes between different pairs of points in order to locate where the slope of the graph of the function was maximal in either the positive or negative direction, or both, as in the case of Subject C6.

Subject C6:

You could just plug in points to find, you know, differences. Or find the slope and when the slope is - um - drastically changed from one point to the next.
After making this comment, Subject C6 was then able to describe correctly what to look for in the differences and the slopes in order to approximate when the population was increasing fastest or decreasing fastest.

**Part d.** Like the students from the other courses, many Calculus & *Mathematica* students did not make an initial attempt to solve this problem. Nine of the 36 students did not write a response to the question when they took the Calculus Representations Test. However, unlike the students from the other courses, most Calculus & *Mathematica* students seemed to recognize that how fast the population was changing on one day corresponded to the rate of change, or the value of the first derivative, for that day.

*Use of symbolic representations.* As with the previous problems, Calculus & *Mathematica* students had the easiest time of all the students working with symbolic representations. Eight of the 12 students interviewed described how to use a symbolic representation to approximate how fast the population was changing on July 1st with prompting and only one student was unable to describe a correct symbolic technique even with prompting. As the following excerpts suggest, Calculus & *Mathematica* students recognized the value of the first derivative of the population function represented how the population was changing on July 1st.
Subject C3:

You could find out the - um - equation and you go and you find July
being - um - some month.

*July 1st is about halfway through the year.*

Halfway. Yeah, you can get halfway. You can find - just go out and
guess about what - you know, what halfway would be numerical and plug
that into the equation.

*Of the first derivative or of the population?*

The population.

*Will that give me how fast the population's changing?*

Actually, no. You wanted to put it into the derivative. You have to find
the change, and not the actual population.... So. you'd put it into the
derivative.

Subject C2:

*Any particular reason you [substituted ½] in the first derivative as
opposed to the function ... to find how fast the population's changing?*

Um - that was for - what was that one?... So this is like, the rate of
change and so if I get from this [derivative], I guess, the value will give
me the rate of change.

Subject C7:

*What if I said August 1st, which we'll call, say, seven-twelfth through the
year? How, how could I figure out -*

You could take the - um - derivative at - well, you find the derivative
then plug in 7/12 for t.

Subject C9:

*What if I was interested in some other time, say March 31st, which is .25,
I mean, how would I figure that out.*

Probably be best to look at the derivative.... Since the derivative measures
the growth rate, then you could see at .25 the growth rate is - oh -
whatever y is,... the value of the derivative is.

Each student seemed to understand that determining how fast the population
was changing meant determining a rate of change, or growth rate, which was
given by the first derivative. Subjects C7 and C9 were asked to solve the problem for a different time of the year since both had recognized the rate of change on July 1st would be zero based on the result from part a.

One student, Subject C6, who did recognize that the value of the first derivative approximated how fast the population was changing, did not appear to understand exactly how the values of the first derivative related to the graph of the function, as the following suggests.

Subject C6:

*Wouldn’t the first derivative give me the rate of change of population?*
It could, if you manipulated it correctly.
*What do you mean by manipulated it correctly?*
Use the correct way of plugging in the right numbers or whatever.
*So if I want to know how fast the population’s changing [on July 1st], could I take that value for the middle of the year, say .5, and put it in the derivative? Would that give me how fast the population’s changing?*
That might. We could try.
*Okay. Well, how would we - how would we verify that?*
Um - it depends on what the answer gives you.
*It gave me zero. Does that seem reasonable that it - the first of July this is - if it’s zero that says the population’s not changing at all.*
If it’s halfway through and that’s the max - um - it tapers off for awhile and the changing rate could be zero at that point. Now let’s say if you would a point where x is equal to .3 and see what that gives you. If it gives you, you know, a zero also, then you’re basically out of luck.
*Why do you say that?*
It would state that at .3, ... where there is changing, and it says it’s zero, then you can’t use that way, but probably at .3, it might give you something else.
*If I put .3 in ... the derivative, what type of value should I get?*
You should have a high value because it’s changing dramatically at that point.
Subject C6’s statement that the value of the first derivative at 0.3 might be zero indicates that the subject may not understand the connection between the value of the first derivative and the slope of the graph of the original at a particular point. Though the subject did recognize that the value of the first derivative should be greater when \( t \) was equal to 0.3 since the graph was increasing sharply at that point, it was not apparent if the subject understood what the value of the first derivative represented.

*Use of non-symbolic representations.* Since little interview time with the traditional and graphics calculator students was spent discussing how to use graphical, numerical, or "combination" representations to determine how fast the population was changing on July 1st, the same was done with the Calculus & *Mathematica* students to keep the comparisons of the students’ performance on this problem equitable. Calculus & *Mathematica* students had some difficulty solving this problem graphically. Only 5 of the 12 interviewed described correctly, without prompting, how to use graphical representation to determine the rate of change of the population on July 1st. One student, Subject C6, apparently did not understand how to approximate the rate of change of population from the graph of the population function, as the following excerpt suggests.
Subject C6:

With the graph you could see that in July [it] has a high point and changes from increasing to decreasing. *What does that mean?... You're saying it's a high point and it's changing from increasing to decreasing. So how fast is the population changing? Um - how fast is the population changing? Um - it depends on how long it has no slope.*

*Well, I'm looking at right on July 1st, which is about halfway through the year. So, based on what you're seeing in the graph, what do you think it would be there? You mentioned something about the slope being how long the slope was zero or the change was zero.*

Yeah. Um - pretty fast. I don’t know. What? You want an actual number? In seconds? I don’t know. So many - um. I really don’t know how fast it’s changing.

These comments, along with Subject C6’s previous comments, suggest that the subject did not understand the relationship between the slope of the graph of a function, the instantaneous rate of change of the function, and the value of the first derivative at a particular value of the variable.

Other Calculus & *Mathematica* students had difficulties trying to use graphical representations to determine how fast the population was changing on July 1st, but for the most part, the students seemed able to interpret the rate of change of population from the graph of the function, as the following suggest.

Subject C8:

Just look at the original plot, and find the first of July. ... and - ah - well, July 1st is roughly half, so - umm - looks like it be about - right in the middle there, it’s won’t be changing much... I’d have to say it looks like it’s going to be not zero, because - Wait. It might be just around zero because it’s July first.
Subject C5:

The answer’s zero. How’d you get it.
Um - because it’s neither increasing or decreasing and the, uh, slope is zero at that point, or the instantaneous growth is zero which means the slope is zero.
Do you remember at all how you found it? Did you use derivative? Did you use graphs?
I went back to the graph .. and it went across at the midway point and that’s where [the slope] is zero and increasing to decreasing. So it was neither increasing nor decreasing at that point and it was simply stagnant.

Unlike Subject C6, Subject C5 recognized that when the graph went from increasing to decreasing, or when the slope of the graph was zero, the rate of change of population at that point was zero.

Five students did describe numerical techniques for estimating how fast the population was changing on July 1st, all without prompting. These students approximated the solution either by comparing values of the function on or near July 1st or by determining the slope between pairs of points near July 1st on the graph of the function, as illustrated by the following excerpts.

Subject C7:

You could find a slope from the original function. Like you could find the value a little bit above 7/12 and the value a little bit below 7/12 and take the slope and that would give you, like, a less than instantaneous rate of change.

Subject C1:

You could look at - it’d be a rough estimate again though, but you could look at like, .5 then and like, .51, and see the difference between them. .. That’d be - that might give you an estimation of the slope.
Subject C11:

If you, like, put in - plugged in various values in the normal function ... and then manipulated them and - okay, what do I want to do? Ah - and compare ... the different values, I guess, you know, around that July 1st date and see how they relate to each other as far as.... Yeah, I would say you'd just, you know, plug it in and compare values from around there.

Subject C7, like many students from all three courses, used a value of \( t \) of 7/12 to approximate July 1st since July was the seventh month of the year.

The only "combination" representation mentioned by the Calculus & Mathematica students to approximate how fast the population was changing on July 1st was to use the graph of the first derivative (graphical/symbolic).

Seven of the 12 students suggested using this graph to estimate the value of the first derivative on July 1st, or to verify the value was zero, as Subject C9 did.

Subject C9:

Well, just looking at the graph of - the first graph, you can see that it’s - that it’s not changing but to be sure, we could graph the derivative and see that the derivative crosses the x-axis at that point.

Okay, So what your saying then is I look at the derivative when it’s .5? You look at the derivative and the derivative - where the derivative crosses the x-axis is at .5.

Well then, why don’t I look at, I mean -
To see if it’s at .5.

Okay, what a minute. I’m interested in when - the first of July, which is .5. Now, I agree that it crosses zero at .5.
So that means that it’s not changing.

Okay, So what I was asking is ... were [you] looking at the value of the derivative when \( t \) was .5? Is that what you’re doing here or are you just looking for where the derivative was 0?
Well, I was looking to find out where it was 0 and hope that it was at .5 since, looking at the graph, I thought that it was going to be there.
Subject C9 initially thought the population was not changing, based on the shape of the graph of the population function on July 1st. The subject then suggested verifying this supposition by checking to see if the graph of the first derivative crossed the x-axis on July 1st, which would confirm that the value of the derivative was zero.

**Summary of Problem 1**

In this section, the results of the Calculus Representations Test and student interviews for Problem 1 are summarized by discussing and comparing students’ (a) use of representations, (b) recognition and recall of different calculus concepts and solution techniques, and (c) interpretation of problem situations and recognition of connections between representations.

*Use of representations.* The percents of students in each course using the different forms of representations for each part of Problem 1 are summarized in Tables 9 and 10.
Table 9: Comparing Students on Use of Representations on Parts a and b of Problem 1 by Course

<table>
<thead>
<tr>
<th>Representation</th>
<th>Use of Different Representations</th>
<th>% Describing Acceptable Use of Given Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Locate Extrema</td>
<td>Decide if Extremum is a Maximum or Minimum</td>
</tr>
<tr>
<td></td>
<td>151</td>
<td>151G</td>
</tr>
<tr>
<td>Graph</td>
<td>Correct</td>
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</tr>
<tr>
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</tr>
<tr>
<td></td>
<td>Not Used</td>
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</tr>
<tr>
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</tr>
<tr>
<td></td>
<td>Incorrect</td>
<td>8.3</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
<td>33.3</td>
</tr>
<tr>
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</tr>
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</tr>
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Table 10: Comparing Students on Use of Representations on Parts c and d of Problem 1 by Course

<table>
<thead>
<tr>
<th>Rep.</th>
<th>Use of Different Representations</th>
<th>% Providing Acceptable Use of Given Representation</th>
<th>Determine when Population is Increasing or Decreasing Fastest</th>
<th>Approximate the Rate of Population Change on July 1</th>
</tr>
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<tbody>
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<td>Graph</td>
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<td>151 151G i51C</td>
<td>151 151G i51C</td>
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</tr>
<tr>
<td></td>
<td>Not Used</td>
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<td>100.0 75.0 100.0</td>
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</tr>
<tr>
<td>S/N</td>
<td>Correct</td>
<td>25.0 33.3 50.0</td>
<td>0.0 0.0 0.0</td>
<td>0.0 0.0 0.0</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
<td>75.0 66.7 50.0</td>
<td>100.0 100.0 100.0</td>
<td>100.0 100.0 100.0</td>
</tr>
</tbody>
</table>
Based on the results presented in Tables 9 and 10, the following observation can be made.

- Calculus & \textit{Mathematica} students correctly used symbolic representations more often than either traditional students or graphics calculator students.

- Graphics calculator students correctly used symbolic representations involving the second derivative more often, and correctly used symbolic representations involving the first derivative slightly less often, than traditional students.

- Calculus & \textit{Mathematica} students correctly used graphical representations about as often as graphics calculator students and more often than traditional students.

- Calculus & \textit{Mathematica} students correctly used numerical representations just slightly more often than either traditional students or graphics calculator students.

- Calculus & \textit{Mathematica} students correctly used graphical/symbolic and symbolic/numerical representations significantly more often than graphics calculator students who used these forms of representations slightly more often than traditional students.

- Graphics calculator students correctly used graphical/numerical representations significantly more often than traditional students who used this form of representation slightly more often than Calculus & \textit{Mathematica} students.

\textbf{Recognition and recall.} Based on the results of the Calculus Representations Test and student interviews, the following observations can be made concerning students’ recognition and recall of different calculus concepts and solution techniques.

- Traditional and graphics calculator students did not seem to recall, or recognize, that the first derivative of a function provided information about its slope or its rate of change.
• Graphics calculator students had difficulty recalling the symbolic technique of setting the first derivative equal to zero to locate the extrema of a function.

• Graphics calculator students did not seem to remember that the sign of the first derivative, not the second derivative, indicated whether the graph of a function is increasing or decreasing.

• Traditional and graphics calculator students had difficulty recalling the symbolic technique that uses the second derivative to determine if an extrema is a maxima or minima - the Second Derivative Test - or confused the technique with other techniques that use the first derivative.

• Calculus & Mathematica students did not know how to use the Second Derivative Test to determine if an extrema is a maxima or minima because that symbolic technique is not taught in their course.

• Traditional students and, to a lesser extent, graphics calculator students did not seem to recall, or recognize, that the second derivative provided information about the slope or rate of change of the first derivative.

• Traditional students had a great deal of difficulty recognizing the symbolic technique of setting the second derivative equal to zero to determine when a function is increasing fastest and decreasing fastest.

• Traditional and graphics calculator students did not seem to recognize that how fast a function is changing on a particular day corresponded to the value of the derivative of the population function on that day.

• Calculus & Mathematica students did the best job of recognizing correct techniques, particularly symbolic, graphical/symbolic, and symbolic/numerical techniques, for solving the problems.
Interpretation and recognition of connections. Based on the results of the Calculus Representations Test and student interviews, the following observations can be made concerning students’ interpretation of problem situations and recognition of connections between representations.

- Traditional and graphics calculator students did not seem to recognize the connection between the slope of the graph of a function and the function’s rate of change.

- Traditional students had difficulty interpreting the points where a function is increasing fastest or decreasing fastest as the points where the first derivative of the function attains its maximum or minimum value.

- Many traditional and graphics calculator students interpreted where a function is increasing fastest or decreasing fastest as intervals, rather than as single points, on the graph.

- Students from all three courses had some difficulty locating the points on the graph of a function where the function is increasing fastest or decreasing fastest.

- Traditional students did not seem to recognize the connection between using the first derivative to find the maximum and minimum values of a function and using the second derivative to find the maximum and minimum values of its first derivative.

- Traditional students had difficulty interpreting inflection points not just as points where concavity of the graph of a function changes, but also as points where the graph of the function attains its maximum slope or minimum slope.

- Traditional and graphics calculator students did not seem to recognize the connection between how fast a function is changing on a particular day and its rate of change, or the slope of its graph, on that day.

- Calculus & Mathematica students’ extensive use of graphical/symbolic and symbolic/numerical representations suggest that they recognize more of the connections between different forms of representations than the students from the other courses.
Discussion of Problem 2

The second problem used a logistics function to approximate the total number of people who had contracted a disease \( t \) days after its outbreak. This problem dealt with determining (a) how many people contracted the disease in the long run (i.e., what was the limit of the function as \( t \) approached infinity), (b) if there was a maximum number of people who contracted the disease, and (c) if there was a day on which more than one million, more than a half-million, or more than a quarter-million people fell sick (see Figure 13).

Fourteen traditional students, eight graphics calculator students, and eleven Calculus & Mathematica students were assigned Problem 2. Four students from each course participated in the interviews. Only about 15 minutes of each interview was spent discussing Problem 2. Interview coding results are presented in Table 11.

2. Suppose \( N \), the total number of people who have contracted a disease \( t \) days after its outbreak, is given by the formula \[
N = \frac{-1,000,000}{1 + 5000e^{-0.1t}}.
\]

a. In the long run, how many people will contract the disease?

b. Is there a maximum number of people who will eventually contract the disease? Explain.

c. Is there any day on which more than a million people fall sick? Half a million? Quarter of a million? (Note: You do not need to determine on what days these things happen.)

Figure 13: Problem 2 of the Calculus Representations Test
Table 11: Comparing Students on Prompting Needed to Obtain Acceptable Use of Representations on Problem 2

<table>
<thead>
<tr>
<th>Rep.</th>
<th>Amount of Prompting Coding for Different Representations</th>
<th>% Providing Acceptable Use of Given Representation</th>
<th></th>
</tr>
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<tr>
<td></td>
<td></td>
<td>Determine how many get sick in the long run</td>
<td>Determine maximum # of people who fall sick</td>
</tr>
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<td>Graph</td>
<td>Correct/None</td>
<td>75 75 75</td>
<td>50 75 75</td>
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<td></td>
<td>Correct/Much</td>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
<tr>
<td></td>
<td>Incorrect/Some</td>
<td>0 0 0</td>
<td>25 0 0</td>
</tr>
<tr>
<td>Number</td>
<td>Correct/None</td>
<td>75 25 50</td>
<td>25 0 50</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>0 25 50</td>
<td>0 25 0</td>
</tr>
<tr>
<td></td>
<td>Correct/Much</td>
<td>0 25 0</td>
<td>0 0 0</td>
</tr>
<tr>
<td></td>
<td>Incorrect/Some</td>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
<tr>
<td>Symbol</td>
<td>Correct/None</td>
<td>25 25 100</td>
<td>50 75 50</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>25 50 0</td>
<td>25 25 50</td>
</tr>
<tr>
<td></td>
<td>Correct/Much</td>
<td>25 25 0</td>
<td>0 0 0</td>
</tr>
<tr>
<td></td>
<td>Incorrect/Some</td>
<td>25 0 0</td>
<td>25 0 0</td>
</tr>
<tr>
<td>G/S</td>
<td>Correct/None</td>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
</tbody>
</table>
Use of Symbolic Representations

In working with symbolic representations, Calculus & Mathematica students did slightly better than graphics calculator students who did better than traditional students. All the students had difficulty recognizing one method for determining if some number of people fell sick on one day was to determine if the value of the first derivative of the function used to model the problem situation ever exceeded the number. Traditional and graphics calculator students had difficulty recognizing that determining how many people contracted the disease in the long run was equivalent to determining the limit of the logistics function as time approached infinity. Only 25% of these students described this technique without prompting. One traditional student was unable to describe correctly the technique even with prompting.

Use of Graphical Representations

In working with graphical representations, Calculus & Mathematica students did as well as graphics calculator students and just slightly better than traditional students. Students from all three courses had the most trouble using the graph of the logistic function to determine if there was one day on which more than one million, more than a half-million, or more than a quarter-million people fell sick. The graphs of the logistics function and its first derivative used with Calculus & Mathematica students are shown in Figures 14 and 15.
Figure 14: Graph of function $N = 1,000,000/(1 + 5000e^{-0.1x})$ shown to students during interviews

Figure 15: Graph of derivative of function $N = 1,000,000/(1 + 5000e^{-0.1x})$ shown to students during interviews
Similar graphs to those in Figures 14 and 15 done on a TI-81 graphics calculator were used with traditional and graphics calculator students.

Use of Numerical Representations

In working with numerical representations, Calculus & Mathematica students did slightly better than traditional students who did slightly better than graphics calculator students. Somewhat unexpectedly, only two Calculus & Mathematica students and one graphics calculator students suggested, without prompting, that an approximation for the number of people who contracted the disease in the long run could be obtained by substituting large values of $t$ into the logistics function. It should be noted that little time was spent discussing numerical representations, particularly on part $b$, since so much time had been spent discussing Problem 1.

Use of "Combination" Representations

Few students suggested using "combination" representations to solve the different parts of Problem 2. The only "combination" representation mentioned by any student was the graph of the first derivative of the logistics function (graphical-symbolic), which was used for part $b$ to determine if the logistics function attained a maximum value and for part $c$ to determine the maximum slope of this function. As Table 9 indicates, Calculus & Mathematica students made more use of this "combination" representation than the other students.
Traditional Student Interviews

As you read the interview excerpts that follow, look for evidence that indicates these behaviors in the descriptions given by the traditional students:

- recognition of the connection between the slope of a function and its rate of change
- recall of what information about a function and its graph was provided by its first derivative
- recall of what information about a first derivative and its graph was provided by its second derivative
- recognition of how to obtain information about rates of change from a graphical representation
- use of graphical/symbolic "combination" representation
- recognition of how to use representations in ways different from those typically taught in the course
- relating of real-life situations described in problems to the representation(s) being used to model the problem and interpreting results in terms of the problem situation
- relating of unfamiliar problem situations or descriptions to familiar uses of representations

**Part a.** Traditional students had little difficulty using graphical or numerical representations to determine how many people contracted the disease in the long run, but had the most difficulty of all using symbolic representations to solve the problem. Some students did not recognize that determining how many people would contract the disease in the long run was equivalent to determining the limit of the logistics function as \( t \) approached
infinity. These students seemed to believe that the logistic function increased without bound and did not have an upper limit. No traditional student mentioned using a "combination" representation to solve this problem.

One possible reason for traditional students’ difficulty solve this problem symbolically may have been lack of experience with the exponential function. Three of the four students interviewed stated that they had not worked with the number \( e \) previously and did not know what it represented. For these students, the interviewer explained that \( e \) represented a transcendental number between 2 and 3, and asked them to solve the problem given a revised function where \( e \) was approximated by the number 2.

*Use of symbolic representations.* Only one of four students interviewed recognized without prompting that determining how many people would contract the disease in the long run was equivalent to determining the limit of the logistics function as \( t \) approached infinity. Two students did not recognize that the function had an upper bound, as the following excerpts suggest.

*Subject T1:*

**Why in particular were you interested in the local maximum?**
That would give you - uh - the most people that would have contracted it. *That statement there - "in the long run" - what did that mean to you?*
Um - I can’t see how ... you could come up with a number. I mean, just from your wording of the question that there’s a disease and it gets greater ever - more people contract it ever day, how can there be a specific number of people who’re going to get it?
Subject T2:

I guess I'm kind a confused. I thought this one would never ... end. It would keep going on and on and on. It's not a repeating function. But could it - does it necessarily have to keep growing? Could it stop growing or slow down growing. When I look at the function - Well it's going to have to come sometime because sooner or - if [the denominator] equals zero at any time then you'll know that [the function] has an asymptote and will never reach a certain number. So if it has an asymptote, that would kinda be a cutoff, wouldn't it?

Yeah.

The difficulty appeared to be that these subjects were letting their beliefs about the real-life situation described in the problem influence their understanding of the symbolic representation used to model the situation. Since both students believed that the number of people who contracted the disease must grow without end, they could not accept the possibility that the function used to determine the number of people who contracted the disease had an upper bound. These students' difficulty in interpreting the mathematical model describing this real-life situation may be attributable to lack of experience with mathematical modeling of this type or with logistics functions. Subject T2's comment about the function not repeating suggests the subject may not have seen, or remembered seeing, functions other than periodic functions that had upper bounds. In the end, only Subject T3, did not recognize that the number of people would contract the disease in the long run could be determined by evaluating the limit of the logistics function as $t$ approached infinity.
Subject T2 mentioned that when a function had an asymptote, it never reached a certain number. This comment suggests that the subject understood what a horizontal asymptote represented. However, the following excerpt, along with Subject T2’s comment about asymptotes occurring when the denominator was zero, suggests that the subject did not understand, or realize, the distinction between vertical and horizontal asymptotes.

Subject T2:

*Is there any way I might figure out if [the function] has an asymptote?*
Set the denominator equal to zero and solve for it.
*Set the denominator equal to zero and solve for it. Why that?*
Because that way if it equals zero, then you know it has an asymptote.
*That’s one type of asymptote where the ... denominator is [zero], but if you think of functions like 1/x, okay. The graph of 1/x looks like this.*
Now, there’s an asymptote where x is equal to 0, but there’s also this asymptote out here. That has nothing to do with x equal to zero.
Yeah.
*Can you think of any way to determine [if] that has an asymptote?*
I’m learning about it, but I can’t - It has to do with - ahh - I think derivatives or something like that. We went over it in 151 class, but I can’t remember it.

It appeared that Subject T2 either did not remember, or know, that the different techniques for determining the horizontal and vertical asymptotes of the function or did not understand, or know, the difference between horizontal and vertical asymptotes. Early comments suggest that Subject T2 did understand what a horizontal asymptote represented.
Use of graphical and numerical representations. Traditional students had little difficulty using graphical or numerical representations to determine how many people contracted the disease in the long run. Students were able to interpret correctly the graph of the logistic function, as the following suggest.

Subject T1:
You'd look for the highest point but in this case it would just keep going. You would just look for what it was coming close to.

Subject T3:
Like the way that I did it. If I - cause we didn't have to give an exact answer, I drew a graph.... And basically, worked it off of that. Consider the time of the long run would consider on how large it would be. Because it would - it would keep increasing. It would probably level off.

Subject T2:
*Can I figure out an answer to that based on that graph?*
I guess so. I don't - cause you know that it slowly [increasing] but, you said its around a million. then you know it will never pass a million. *So what might that tell me about how many people will contract the disease in the long run?*
Only, not more than a million people.

Three students who solved this problem numerically mentioned substituting large values of \( t \) in the logistics function. Subject T1 recognized that the value of the function would stay about same if the value of \( t \) was large enough.

Subject T1:
If your going to put in points, you know - if your going to try numbers, you're going to try really, really big numbers for \( t \) and you're going to get the same thing.... After you use a number that's so big, you'll always get a million.
All three described a correct numerical technique for solving the problem - substitute large values of \( t \) into the logistics function to obtain an approximation for the number of people contracting the disease in the long run - without prompting.

**Part b.** Traditional students had more difficulty then other students using graphical or symbolic representations to determine if there was a maximum number of people who contracted the disease. One student never described how to solve the problem symbolically or graphically, even with prompting. Other students were able to solve the problem with little prompting. Only one student recognized the symbolic and graphical techniques used to solve part a could be used on this problem. As was the case with part a, no traditional student mentioned using a "combination" representation to solve this problem.

**Use of graphical representations.** With graphical representations, some students had difficulty interpreting what the upper bound of the logistics function, one million, could represent in this particular problem.

Subject T3:

I did exactly what this graph says but I felt that it didn’t have an end. It would continue going. It would never reach a million. *But I guess the question is does that mean there’s a maximum number?* It will never reach a million but it will come really close. *Okay. Now only - I guess my question there would be when you’re talking about people.* Yeah. There will be eventually a maximum.
Subject T3 appeared to have trouble accepting that the maximum number of people who contract the disease could be one million since the graph of the function never reached one million. The difficulty for this subject, and one other traditional student who made similar comments, seemed to be that they were not taking the problem situation into account when interpreting the graph. These students did not recognize that the upper bound of the graph could be interpreted as the maximum number of people who contract the disease, although the graph of the function never reached its upper bound.

*Use of symbolic representations.* Subjects T1 and T4 initially chose to solve the problem by determining when the first derivative of the function was equal to zero. As the following suggests, this symbolic technique was chosen because the problem called for determination of a maximum value.

Subject T4:

*Now you said to find N' and set it equal to zero. Why did you suggest that for this case.*
I guess that just how I'm use to finding maxs and mins.

*Okay, so whenever you see a max/min, your first thought is go out and find N' - find the derivative and go from there.*
Yeah.

These students did not seem to recognize that the maximum number of people who would contract the disease did not have to occur at a local maximum of the logistics function. It was unclear whether these students understood that a
function can have local maxima without having an absolute maximum or can have an upper bound without having local maxima. Subject T1 did seem to understand how to interpret the situation if the first derivative never was zero.

Subject T1:

*Is it always possible to solve* $N'$ *equal to 0? Does that always have an answer?*

No.

*What if it doesn’t have an answer?*

Well that might tell you that there’s like a - that there’s just asymptotes so it doesn’t ever - if never curves down or curves up

*What would you say if you couldn’t find a value where* $N'$ *was equal to 0.*

*How would you interpret that?*

I would say that there is no specific maximum. It’s just approaching it.

Subject T4 also recognized if the first derivative was never equal to zero, the function would had no local maxima but could have an upper bound.

**Part c.** Traditional students had the most difficulty with this part of Problem 2, particularly when trying to describe how to use symbolic representations to determine if there was one day on which more than one million, more than one-half million, or more than one-quarter million people fell sick. Two of the four students interviewed, and seven of the fourteen students assigned this problem on the Calculus Representations Test, did not give an initial response to this question. This was also the only problem on which more than one student needed prompting before describing correctly how to use graphical or numerical representations.
Use of symbolic representations. One reason for students’ initial difficulties solving this problem was that they misunderstood the problem. Three of the students interviewed thought the problem called for them to determine when the total number of people who fell sick, not the number who fell sick on one particular day, was greater than the given numbers. These students all suggested that the problem could be solved symbolically using the incorrect method of determining when the function was equal to one million, one-half million, or one-quarter million. After the interviewer explained the problem to students, most then described correctly how to solve the problem symbolically, as the following suggest.

Subject T2:
See if the first derivative - if there’s a change of over a million. 
Okay. So you’d take the first derivative and then do what?
Set it equal to whatever number you wanted to and solve for it.
Okay. And then what? If you get - let’s say I get an answer for t. What does that tell me?
On those days, that many people [fall sick].
Okay. What if I don’t get an answer for t?
There was no day that that many people got sick.

Subject T1:
If you can find where the slope is a half-million.
What might you do to find out where the slope is a half-million?
Just take the first derivative and set it equal to a half-million. Then that’ll tell you.
What if you can’t find a value.
Then there probably isn’t one.... Probably would say no to that question.
Subject T3:

You’d have to find out what the $x$ value would be to plug it in so you could plug it in because the $y$ value is how many people fall sick on that day.... And you’d want to find out - on the day where the half-million, so you’d set a half-million equal to the first derivative and solve for that.

*If you could find a value of $t$ that worked, ... would that be the day then?*

Well, yeah.

*Okay. And what if you couldn't find a value*

Then, a half-million people wouldn’t fall sick.

These students recognized that if for some value of $t$, the value of the first derivative was greater than a given number, then there would be a day on which that number of people fell sick. They also recognized that if no such value of $t$ existed, then there would not be a day on which the given number of people fell sick. Of these students, only Subject T1 described correctly, without prompting, how to solve the problem using symbolic representations.

One student, Subject T4, was unable to correctly solve this problem symbolically, even with prompting. After discussing how the slope of the graph of the function related to rate at which people fell sick, the subject seemed to recognize the solution involved the first derivative, but did not recognize how it was used, as the following suggests.

Subject T4:

Would it be where you would find ... if the slope was like a half-million. *So now does that suggest anything I might do with the equation.*

I guess find $N'$. *Okay, And then what would you do. Once I had $N'$, what could you do? Um. I don’t know.*
The subject did not recognize, or realize, that determining where the slope of the graph was about one-half million was equivalent to determining when the value of the first derivative was equal to one-half million. Subject T4 had the same difficulty remembering the connection between the slope of a function and its first derivative when working on Problem 1.

*Use of graphical and numerical representations.* All the traditional students described correctly how to use graphical representations to determine if there was one day on which more than one million, more than one-half million, or more than one-quarter million people fell sick. Three students described how to solve the problem numerically using techniques similar to the one suggested by Subject T1 in the following excerpt.

**Subject T1:**

The only way I could think is if you just keep trying numbers. I mean, well - you have to find - first find one that gives you an \( N \) that’s greater than a half-million and then do what I said - like take the one before and find how many caught it on that day.

All three subjects mentioned determining the value of the logistic function on various pairs of consecutive days and then subtracting to ascertain if the any of the differences were larger than the given numbers.

Subject T4 applied this idea of determining the difference between values of the function on consecutive days to the graph of the logistics function.
Subject T4:

I guess ... maybe if you graphed the equation.... The only thing I could think of is, like if these [axes] were marked out in the days. *Okay, so if we had days here like this, what would you do.*
I guess you solve the difference of where it was for this day, and this - between these two days [to see if it] was over a half-million.

Subject T4 used the graph to determine the value of the logistic function on consecutive days and then looked at the difference in these values to see if it was greater than a half-million. Subjects T1 and T2 suggested solving the problem using a similar graphical technique where the slope of the function on particular days was estimated from the graph to determine if the slope was ever greater than a half-million or a quarter-million. These subjects recognized that if the slope of the function was ever greater than a given number, then there had to be a day on which that many people would fall sick.

Subject T3 also used the slope of the graph of the function, but in a different manner.

Subject T3:

I believe I said like, basically took the point that had the biggest slope - the largest slope - which would be at essentially the middle of the graph [and] estimated that the largest slope wasn't big enough.

Instead of finding points where the slope was greater than the given numbers, Subject T3 looked for the point where the slope was greatest. This subject recognized that if the greatest slope attained by the graph of the function was
greater than any of the given numbers, then there had to be a day on which that number of people fell sick. Otherwise, there was no day on which more one-million, more than one-half million, or more than one-quarter million people fell sick. Subject T3 was the only traditional student to suggest using this graphical technique, exploiting the fact that students were not required to determine the days on which the specified events occurred.

**Use of "combination" representations.** Only one traditional student, Subject T1, mentioned using a "combination" representation to answer any of the questions in Problem 2. This subject suggested using the graph of the first derivative (graphical/symbolic representation) to determine if there was one day on which more than one-half million people fell sick.

**Subject T1:**

If you had a graph of the first derivative, and then looked for a half-million.

Subject T1 used the graph of the first derivative to determine if the value of the first derivative ever exceeded a half-million. This subject recognized that if the value of the first derivative never exceeded a half-million, then there would not be a day on which a half-million people fell sick.
Graphics Calculator Student Interviews

As you read the interview excerpts that follow, look for evidence that indicates these behaviors in the descriptions given by the graphics calculator students:

- recognition of the connection between the slope of a function and its rate of change
- recall of what information about a function and its graph was provided by its first derivative
- recall of what information about a first derivative and its graph was provided by its second derivative
- recognition of how to obtain information about rates of change from a graphical representation
- use of graphical/symbolic "combination" representation
- recognition of how to use representations in ways different from those typically taught in the course
- relating of real-life situations described in problems to the representation(s) being used to model the problem and interpreting results in terms of the problem situation
- relating of unfamiliar problem situations or descriptions to familiar uses of representations

Part a. Like traditional students, graphics calculator students had little difficulty using graphical or numerical representations to determine how many people contracted the disease in the long run. They did have some difficulty, but not as much as the traditional students, using symbolic representations to solve the problem. Some graphics calculator students were unable to relate the
real-life situation described in the problem to the logistics function used to model the problem. These students did not seem to recognize that determining how many people would contract the disease in the long run was equivalent to determining the limit of the logistics function as \( t \) approached infinity possibly because they could not envision the number of people who contracted the disease, and thus the logistic function used to determine this number, having an upper limit. No graphics calculator student mentioned using a "combination" representation to solve this problem.

Graphics calculator students did not have as much difficulty as traditional students working with the number \( e \). Only one of the four students interviewed stated that they did not remember working with this number previously and did not know what it represented. These students’ experiences with the number \( e \) likely occurred during a previous course since the exponential function was not discussed in any detail during this course.

*Use of symbolic representations.* Three graphics calculator students needed prompting before recognizing that determining how many people would contract the disease in the long run was equivalent to determining the limit of the logistics function as \( t \) approached infinity. As was the case with traditional students, the difficulty for these students was trying to relate the real-life situation described in the problem to the logistics function used to model the
problem. Subject G1 had the most difficulty of all graphics calculator students associating the function and its graph to the problem situation.

Subject G1:

*Based on this graph, how many people does it look like’ll contract the disease?*
All of them.
*All of them?*
Isn’t there a million people altogether?
*No... This is a model for population.... What that’s saying is a million of the people are going to catch that disease.*
Okay.
*The total number of people is going to be a million based on what we saw here.*
After t days.... So maybe I was right with my infinite number.
*Well, is a million an infinite number?*
No. Is a million the number it levels off at?
*What does it look like from the graph?*
Yeah, but are you defining a million as a set number here or are you defining it as a variable?
*Well, a million’s right here on the graph.... This [equation] models how many total number of people will catch the disease.*
That doesn’t make in sense, why a million people would catch the disease and then have it stop like that though.

Subject G1’s beliefs about the real-life situation described in the problem seemed to influence the subject’s interpretation of the graph of the logistics function used to model the situation. The subject did not appear willing to accept that there could be an upper limit to the number of people who contracted the disease. In addition, Subject G1 may have been influenced by the graph of the function done by the subject on a graphics calculator during
the Calculus Representation Test. This graph was drawn using too small a
range, making it appear as if the graph of the function, and thus the number of
people who contracted the disease, grew without bound.

After some discussion of diseases like small pox that had essentially been
eliminated, Subject G1 recognized that there could be an upper bound on the
number of people who contract a disease and that determining how many
people would contract the disease in the long run was equivalent to
determining the limit of the function as \( t \) approached infinity.

Subject G1:

Trying to set \( t \) to infinity, but that's not going to do a whole lot though. I
don't know. Could set that \( t \) to infinity but you'll have some trouble
manipulating your numbers.

After being reminded that taking the limit as \( t \) approached infinity did not
require substituting infinity for \( t \) in the equation, Subject G1 was then able to
evaluated the limit correctly.

Like Subject G1, Subject G3 seemed to think that the number of people
who contracted the disease would always be increasing and have no upper
limit, as the following excerpt suggests.

Subject G3:

Well, if you keep it in one specific formula - if it's increasing at a steady
rate - same rate every year. If a disease is around long enough and it's
spreading at this rate every year, pretty soon -
Won't that depend - if I'm modeling the disease with a particular formula, won't that depend on the formula? I mean, is it true that all formulas are just gonna keep going.
No, I guess not.

The subject's comment about the disease spreading at the same rate each year may be related to previous work involving exponential equations modeling problems where the disease, or other growth like population or bacteria, grew at a same rate each year. After prompting, Subject G3 realized that the number of people would contract the disease in the long run could be determined by evaluating the limit of the logistics function as $t$ approached infinity.

Subject G4 was the only graphics calculator student to suggest solving the problem by determining when the first derivative was equal to zero.

Subject G4:

What the graph's telling is that - um - the disease, you know, starts out very slowly. And then it breaks out, you know, into the - because of the high slope. And then it stops.
So, can I answer that first question now?... How might I answer that one? Where the first derivative is zero.... Because you want - when the slope is zero - uh - you know that's a point where there's no more growth.

Subject G4's last comment suggests the subject thought the logistics function had to stop growing in order for there to be an upper limit on the number of people who contracted the disease. The subject was shown, using a graph of the first derivative, that the first derivative was never equal to zero but did start to approach zero as the value of $t$ increased. After some discussion, Subject
G4 recognized that determining how many people would contract the disease in the long run was like looking for a limit.

Subject G4:
Like your looking for a limit to infinity or something.... The limit as $t$ goes to infinity for $N$ - for that function.

All graphics calculator students interviewed eventually recognized that determining how many people contracted the disease in the long run was equivalent to determining the limit of the function as $t$ approached infinity.

*Use of graphical and numerical representations.* Graphics calculator students had little difficulty using graphical representations to determine how many people contracted the disease in the long run. Students interpreted correctly the graph of the logistic function, as the following suggest.

Subject G2:
Cause it went off then flattened out and held steady at a million. Cause I decided that people were getting the disease, lots of people were getting the disease, and then people started getting cured of it. And then, and then no one got the disease anymore.

Subject G4:
It seems like when this outbreak occurs, that there’s going to be a number - a number of people are going to contract the disease and then the number of people that contract is going to slow down but there’s still going to be contraction...

*What does it look like might be the case in the long run? How many people do you think will contract the disease?*
Um - I guess a million. if you set an asymptote to it, you’d find what that number’s trying to get to.
Only Subject G1, who, as previously noted, had difficulty relating the logistic function and its graph to the problem situation, needed prompting before interpreting the graph correctly.

Three graphics calculator students described correctly how to solve this problem using a numerical representation, though only one did so without prompting. Each described a technique similar to the one given by Subject G4 in the following excerpt.

Subject G4:
You could also put large values - uh - for \( t \) and see - at some points, like if you put like, 2 million, or two million, you’re going to see there’s going to be a very small difference.

The three students each recognized that for large \( t \), the value of the function approximated the number of people contracting the disease in the long run.

**Part b.** Graphics calculator students had little difficulty using either graphical or symbolic representations to determine if there was a maximum number of people who contracted the disease. All students solved the problem with little or no prompting. Two students recognized that the symbolic and graphical techniques used to solve part a could be used on this problem. As was the case with part a, no graphics calculator student mentioned using a "combination" representation to solve this problem.
Use of symbolic representations. Unlike traditional students, no graphics calculator student initially suggested the problem could be solved by determining when the first derivative of the logistics function was equal to zero. Subject G2 offered the following explanation of why this symbolic technique would not work.

Subject G2:

In the first problem, you talked about using the derivative to find a maximum. Would that work here?
No, because in that one, they wanted rate and this just wants the maximum number.

This comment was unexpected considering that the textbook and instructor in the graphics calculator course emphasized using derivatives to locate maxima and minima of functions and that the subject had just used derivatives to determine the maximum and minimum populations in Problem 1. The comment suggests that the student did not completely understand the relationship between the rate of change of a function and its first derivative.

Part c. Graphics calculator students had the most difficulty with this part of Problem 2, particularly when describing how to use symbolic representations to determine if there was one day on which more than one million, more than one-half million, or more than one-quarter million people fell sick. They had some difficulty recognizing the relationship between the slope of the graph of
the function and the rate at which people fell sick and some difficulty recalling
that the second derivative, not the first derivative, was used to determine when
the maximum rate of change occurred. One student was unable to describe
correctly how to solve the problem symbolically because the student did not
appear to completely understand how the rate at which people contacted the
disease was related to the first derivative of the function. This was also the
only problem on which more than one student needed prompting before
describing correctly how to use graphical representations.

*Use of symbolic representations.* Graphics calculator students did slightly
better than the other students correctly interpreting this problem. Only two
students thought the problem called for them to determine when the total
number of people who fell sick, not the number of people who fell sick on one
particular day, was greater than the given numbers. Subject G1 was the only
student who described correctly, without prompting, how to solve the problem.

Subject G1:

We could find the point which is increasing the fastest. Or take the
inflection point where the second derivative is zero.... Find the inflection
point and find out exactly how many will get sick on that day.... Less
than half-million, then you're in trouble.

*And if it's over a half-million?*

Well, then you'd be ok for the next one. For the next, a quarter-million.

*And if it was below a half-million, I could maybe look at a quarter
million.*

And if it - if it wasn't then, either, then it'd be no for all three questions.
Subject G1 recognized that the rate of change of the function being greatest at its inflection point meant the day corresponding to the inflection point would be the day on which the most people fell sick. The subject recognized that if the number of people who fell sick on this day was not greater than a half-million, or a quarter-million, then there could be no day on which that many people fell sick. Subject G1 described correctly how to use the second derivative to locate the inflection point and how to use the logistics function to determine the number of people who fell sick on that day.

After prompting, Subject G3 described how to use the first derivative to determine if there was one day on which more than one of the three given numbers of people fell sick.

Subject G3:

Maybe take the derivative at a point? Cause it’s that instantaneous frame in time.
Okay. So, let’s say I take the derivative.... What would I do with that? Um. I really don’t know. I’m really not sure.
Since the function itself is talking about total number of people, when I talk about how many people are getting sick on a particular day, I’m talking about a rate. Because that’s the rate of change, the total number of people getting sick can be thought of as how many are getting sick on a particular day. So, if I’m talking about rate of change, as you said, derivative, ... what other thing is related to rate of change?
Um - slope. Which is the derivative at a point.... You can find out the rate of change at a certain point to find out if a half-million people will get sick that day. I mean, how many days it takes for - or how many people get sick on one day would be the slope.
So, if I want to find out ... whether or not half a million people got sick on a particular day, how could I bring the slope into play there? Or the derivative? What might I look for?

Um - you're looking for a rate of change of a half-million over one, so you find out where that would be on the graph. The derivative would be - you need to find a derivative that's equal to that slope - 500,000.

If I find a value that works, what does that tell me?

It tells me there is a day, at least one day.

Okay. And if I don't?

Then it tells you there isn't a day during this time that that many people fall sick.

At first, Subject G3 did not connect the slope of the graph of the function and the rate at which people were falling sick. Once this connection was made, the subject recognized that if the value of the first derivative was ever greater than a given number, then there would be a day on which that number of people fell sick and if the value of the first derivative was always less than a given number, then there would be no day on which that number of people fell sick.

One graphics calculator student who initially misinterpreted the problem, Subject G2, eventually described how to solve the problem symbolically.

Subject G2:

Take the limit of the first derivative as - ah. Oh, no, no, no, no. Find the first derivative at zero. Find where the first derivative is at zero.

Okay. If I find where the first derivative is zero, doesn't that give me the maximum of this function?

Maximum rate of the function.

If I want the maximum rate of the function, do I want to look at the first derivative and set it equal to zero?

Oh yeah. First derivative, it gives the hump. Second derivative, it gives - would give the - would give the cross.
So I find where the second derivative's equal to zero and what do I do with that?
You take that value and - where that would be zero, you plug that into the first derivative and that would give you the rate which was the max.
If I'm gonna answer this, what do I need to know about that maximum?
Then you look at this and see which one of these rates it matched.

Like Subject G1, Subject G2 determined when the rate of change of the logistics function was greatest — when the second derivative equaled zero — and then checked to see if the rate of change at that time — the value of the first derivative — was greater than any of the given numbers. The only difficulty this subject had was remembering that the maximum rate of change occurred when the second derivative, not the first derivative, was equal to zero.

Subject G2 made the same error when describing how to solve part c of Problem 1 where students were asked to determine when the population function increased fastest or decreased fastest.

The other graphics calculator student who initially misinterpreted the problem, Subject G4, was unable to describe how to solve the problem symbolically, even though the subject did at first suggest a correct solution technique, as the following illustrates.

Subject G4:
What I'd do is take the first derivative. It would tell you the slope. It would tell - it's the rate at which this is occurring. Set it equal to maybe - um - one million. I would probably get a slope of zero. By the - by what we just proved, the limit is a million as N goes to infinity.
Right. The limit of the original function is a million. But if you take the derivative and set it equal to a million, aren’t you trying to find the place where the slope of the graph is million? If I take the derivative -
Okay.... So you’re looking for the total number of days so maybe - No. I’m just looking on any particular day. If there’s some day on which a million people get sick, or a half-million ... or a quarter-million.... You said take the derivative ... and set it equal to a million.
No. I wouldn’t say that.
Okay. What would you do then?
I’d look at the - maybe the original graph ... and since t is a day, you’d look where t, or where N - where y equals a million.
Okay. But that would ... say that after that many days, a total of a million people have gotten sick. Remember, that gives me total. I want to know if there’s one particular day ... [on which] a million people get sick. How could I figure that out?
If the first derivative is giving me the slope and the rate at which this is occurring - okay. If the instantaneous spot - let’s say - cause I really can’t think of it cause I can’t think of the slope as great. If the slope is maybe a million, wouldn’t that mean that a million people will get that that day?

At this point, Subject G4 did not suggest checking to see if the value of the first derivative was ever equal to one million, but instead suggested checking various pairs of y-values from the graph of the function to determine if their difference was greater than one million. Considering that the subject’s initial comments suggest that the subject understood the connection between the slope of the function, the rate at which people were falling sick, and the first derivative, it was somewhat surprising that the subject did not agree that the problem could be solved by determining if the first derivative was ever equal to any of the given numbers.
Use of graphical and numerical representations. All graphics calculator students described correctly graphical techniques for solving this problem, two with prompting and two without. Three of the four students suggested using the graph of the function to estimate the slope of the function on various days in order to determine if the slope was ever greater than the given numbers. These subjects recognized that if the slope of the function was ever greater than a given number, then there was at least one day on which that number of people would fall sick. The fourth student, Subject G4, used the slope of the graph of the function in a slightly different manner.

Subject G4:

If you look at just the slopes, where the slope is maybe the greatest, that would - that's your most probable place where there would be a greater number of people contracting the disease.... Estimate the slope there to see if it's a half-million.

Subject G4 only estimated the slope of the graph for the day when the slope appeared to be greatest, rather than estimate the slope on several days. This subject recognized that if the greatest slope attained by the graph of the function was greater than any of the given numbers, then there had to be a day on which that number of people fell sick. Otherwise, there could be no day on which more one-million, more than one-half million, or more than one-quarter million people fell sick. Subject G4 was the only graphics calculator student to
use this graphical technique, exploiting the fact that students were not asked to
determine the days on which the specified events happened.

Three graphics calculator students described how to use a numerical
representation to determine if there was one day on which more than one
million, more than one-half million, or more than one-quarter million people
fell sick. These students recognized that the solution involved determining the
difference between values of the logistic function on consecutive days.

Subject G3:
You could take the - um - answers at two different consecutive points and
as one day - like day by day - and subtract the first day from the second
day and you get how many people.

Subject G4:
You’d want to find a place where the y would equal - if you take two
points, let’s say - you’d want to find where the difference would equal
maybe a million.

All three appeared to understand that if the difference between the values of
the logistic function on one pair of consecutive days was greater than the given
numbers, then there would be one day on which that number of people would
fall sick.

Use of "combination" representations. As with the traditional students,
only one graphics calculator student, Subject G2, mentioned using a
"combination" representation to answer any part of Problem 2. This subject
used the graph of the first derivative (graphical/symbolic representation) to determine if there was one day on which more than one million, more than one-half million, or more than one-quarter million people fell sick.

Subject G2:

This is when I - this one’s rate. This is when I looked at the derivative, the first derivative.... I think I did look at the first derivative cause the max was only at 25.

How did you look - what did you do to look at the first derivative. Probably numerical derivative.

You somehow used the numerical derivative to figure out what?
To figure out where the curve was at its max.

Oh, you had it graph the numerical derivative.

Yeah.

Subject G2 used the graph of the first derivative to determine the maximum value of the first derivative. The subject recognized that the maximum value of the first derivative had to be greater than one-million, one-half-million, or one-quarter million if there was to be a day on which that number of people fell sick. When Subject G2 graphed the first derivative, using the NDERV feature on the graphics calculator, during the Calculus Representations Test, it appeared to the subject the maximum value of the derivative was 250,000, but further inspection of this graph during the interview showed that the maximum value was only 25,000. From this, Subject G4 surmised that there was no day on which more than one-million, more than one-half-million, or more than one-quarter million people fell sick.
Calculus & Mathematica Student Interviews

As you read the interview excerpts that follow, look for evidence that indicates these behaviors in the descriptions given by the Calculus & Mathematica students:

- recognition of the connection between the slope of a function and its rate of change
- recall of what information about a function and its graph was provided by its first derivative
- recall of what information about a first derivative and its graph was provided by its second derivative
- recognition of how to obtain information about rates of change from a graphical representation
- use of graphical/symbolic "combination" representation
- recognition of how to use representations in ways different from those typically taught in the course
- relating of unfamiliar problem situations or descriptions to familiar uses of representations

Part a. Calculus & Mathematica students had little difficulty using the different forms of representations to determine how many people contracted the disease in the long run. They did not have the same difficulties as the other students relating the real-life situation described in the problem to the logistics function used to model the problem. All the students recognize that determining how many people would contract the disease in the long run was equivalent to determining the limit of the logistics function as $t$ approached
infinity. As with the other students, no Calculus & Mathematica student mentioned using a "combination" representation to solve this problem.

Unlike traditional and graphics calculator students, Calculus & Mathematica students had little or no trouble working with the number $e$ and the logistics function. This was to expected considering the amount of emphasis placed on learning about and solving problems involving the exponential and logistic functions in the Calculus & Mathematica course.

*Use of symbolic representations.* All four Calculus & Mathematica students recognizing that determining how many people would contract the disease in the long run was equivalent to determining the limit of the logistics function as $t$ approached infinity without prompting.

Subject C4:

Well, it's a logistic function. If you'd figure out what the limit was of a logistic function, you'd would know where its going to taper off.

Each described a method for solving the problem symbolically similar to the one given by Subject C1 in the following excerpt.

Subject C1:

*So how many people will contract the disease in the long run?*

One million. After a while, $5000e^{-0.1t}$ will get so small, it won't be much of a factor at all. It's exponential decay. Thus, you would basically have 1,000,000 over 1 when $t$ is really large.
Calculus & Mathematica students did not have the same difficulty as students from other courses relating the real-life situation described in the problem to the logistics function used to model the problem. The most likely reason for this is that, throughout the course, Calculus & Mathematica students are given numerous different types of examples and homework problems that make use of mathematical modeling.

Use of graphical and numerical representations. Calculus & Mathematica students had little difficulty using graphical or numerical representations to determine how many people contracted the disease in the long run. Students were able to interpret correctly the graph of the logistic function, as the following suggest.

Subject C4:
I'd say it seems like it's approaching its limit. I'd say roughly a million.

Subject C1:
I think you probably could - you could probably see the y is close to a million. And it's pretty much slowing down and it doesn't look like it's going to increase much more. If you move the x up a little bit more, like a lot more, you could see if it ever starts to go up again. And after like, if it keeps going straight for quite awhile, you could well that it probably will keep going straight.

Only one student, Subject C2, needed prompting before interpreting the graph correctly.
All Calculus & Mathematica students described correctly how to solve this problem using a numerical representation, though only two did so without prompting. The following suggests the students recognized that the value of the function would have to change only slightly for large values of $t$ if there was to be an upper limit on the number of people who contracted the disease in the long run.

Subject C1:

You could set $t$ real large.... Maybe 500 and see what happens. Then 600, see if it changes much.

Subject C2:

Like plug in values and see where it goes.... So we’ll just plug in, like, huge numbers and see ... what we get. And then we’ll see what happens, like, you know, as we change the value.... I guess we would have to, like, look at - I mean the change of like, the values that we plug in and see how much, you know, our results are changing. And then if we see the results are changing less and less, then we know, probably, we are getting close to the limit.

The students all recognized that for large $t$, the value of logistics function approximated the number of people contracting the disease in the long run.

**Part b.** Calculus & Mathematica students had little difficulty using graphical or symbolic representations to determine if there was a maximum number of people who contracted the disease. Only one student needed prompting before describing how to solve the problem graphically and two students needed prompting before describing how to solve the problem...
symbolically. Three students recognized the symbolic or graphical techniques used to solve part a could also be used on this problem.

Subject C1:
As $t$ approached infinity, $5000e^{-0.1t}$ gets so small that it is almost nonexistent. However, it is never zero or negative. So basically you will have one million over 1 or one million will contract the disease in an infinity amount of time.

Subject C2:
Well, I guess, I mean since it has, like, this limit thing ... and it's like it would be a million.

Subject C3:
Well, you can see from this graph here that it's almost - that it's just about leveled off here. It's almost going in a straight, straight line.... You know because it's - it's going kinda slow then all of a sudden, it shoots up right around here and then it just levels off. So you know it's not gonna get much higher and, if - and you might just say that it gonna level off right there which would be the maximum number of people.

The description given by Subject C1 is virtually identical to the one given by the subject for determining how many people contracted the disease in the long run using symbolic representations. The description given by Subject C3 is similar to the one given by the subject, and other Calculus & Mathematica students, for determining how many people contracted the disease in the long run using graphical representations.

*Use of symbolic representations.* Subjects C2 and C3 mentioned solving the problem by determining when the first derivative of the function was equal
to zero. Both students stated that they suggested using this technique because the problem called for the determination of a maximum value. It was unclear if the students recognized that the maximum number of people who would contract the disease did not have to occur at a local maximum of the function, but, as the following excerpt suggests, Subject C2 seemed to recognize that this technique might not produce a solution to the problem.

Subject C2:

*What if you couldn't produce a zero? Maybe the derivative never equaled zero. Would that -*

Yeah, that's what I thought reading the problem. That's why it wouldn't work, [what] I was trying to do.

Subject C2 and C3 both seemed to understand if the first derivative was never equal to zero, the logistic function would have no local maxima but could have an upper bound, which meant there would be a maximum number of people who contracted the disease.

*Use of "combination" representations.* Unlike the other students, Calculus \& Mathematica students mentioned using "combination" representations to help determine if there was a maximum number of people who contracted the disease. Subjects C2 and C3 both mentioned using the graph of the first derivative (graphical/symbolic representation) to determine if the first derivative was ever equal to zero. The interviewer used this graph to show the students
that the symbolic technique of locating the maxima of a function by determining when the first derivative was equal to zero could not be used to solve this problem since the first derivative was never equal to zero.

**Part c.** Calculus & Mathematica students had the most difficulty with this part of Problem 2, particularly when trying to describe how to use symbolic representation to determine if there was one day on which more than one million, more than one-half million, or more than one-quarter million people fell sick. The three Calculus & Mathematica students who were assigned this problem on the Calculus Representations Test but did not participate in the interviews did not give initial responses to the question. As with the other students, this was the only problem on which more than one student needed prompting before describing correctly how to use graphical representations.

*Use of symbolic representations.* One reason for students' initial difficulties solving this problem was that they, like the traditional students, misunderstood the problem. All four students interviewed thought the problem called for determining when the total number of people who fell sick, not the number who fell sick on one particular day, was greater than the given numbers. Students initially chose to solve the problem symbolically using the incorrect method of determining when the logistics function was equal to one million, one-half million, or one-quarter million.
Subject C2:
Set the given equation equal to these values.

After the interviewer explained the problem to the students, most were able to correctly solve the problem. Subject C2 was the only student who needed no further prompting before realizing how to solve the problem symbolically.

Subject C2:
I guess that’s the change of how many people.... We can get from the derivative thing. Like the change of - like how many people get sick over a certain period of time.... Is this going to be then, I mean - I don’t, I’m not sure about it but would this be the derivative. I mean, equate this, the derivative, to a half-million.... And then we’ll get a value of a day when we have ... a half-million people. But it probably won’t work because, I mean, a half-million one day -

*And if it didn’t work, what does that tell us? Say I took the derivative, set it equal to a half-million.*

Then there couldn’t be a half-million people sick on one certain day. I mean like - like just catch this on that day.

Subject C2 seemed to understand that if for some value of $t$, the value of the first derivative was greater than a half-million, then there would be a day on which a half-million people fell sick. The subject recognized that if no such value of $t$ existed, then there would not be a day on which a half-million people fell sick.

Subject C1 suggested using derivatives in a slightly different manner to determine if there was one day on which a half-million people fell sick.
Subject C1:
I'm not sure how to find that one. I might - umm - I could probably, maybe look at the derivative and look and see the biggest slope. - I don't, I'm not really sure how to do it. The biggest slope would probably be - well, half a million people would be a lot of people.... So if you look at the slope, it'd probably be where the greatest slope is or pretty close to it.

Subject C1 seemed to be suggesting that the day on which a half-million people fell sick would have to occur when the function had its greatest slope. After prompting, the subject recognized that by if the greatest slope of the function exceed a half-million, then there would be a day on which a half-million people fell sick, otherwise there would be no such day. Subject C1 described correctly how to use the second derivative to determine the day when the logistics function attained its greatest slope and how to use the logistics function, and its first derivative, to determine the number of people who fell sick on that day.

One Calculus & Mathematica student, Subject C3, was unable to describe correctly how to solve this problem symbolically, even with prompting. The subject recognized that the rate at which people contracted the disease was related to the slope of the graph of the logistic function. After prompting, the subject remembered that the slope of the function was related to its first derivative, but did not recognize how to use the first derivative to solve the problem, as the following excerpt suggests.
Subject C3:

When you talk about taking the first derivative and setting it equal to zero to max or mins, why are you taking the first derivative and setting it equal to zero?
Well, when that's zero, that's, that's the highest point, so ..
Just because this is zero, why should that be a highest point. What's that zero representing? What's the slope, right there, of the function?
Uh - the slope there is zero.
So what information does the first derivative -
It tells you - it tells you the slope.
So based on that, can the derivative somehow help me with that problem?
Yeah, you could find the derivative and see if it gets to the point that you needed. The slope.
Okay. So basically what I'm trying to do is look at the first derivative and see if it ever is a half-million or a quarter-million or a million? Would that answer this?
I don't think that would do it completely.

After the interviewer pointed out that the largest value of the first derivative was 25,000, the subject mentioned this would mean it was impossible for more than a quarter-million, or a half-million people, to fall sick on one day. This suggests Subject C3 may have understood the connection of the slope and the rate at which people fell sick to the first derivative but was unable to piece this all together and come up with the symbolic solution technique of determining if the first derivative was ever equal to any of the given numbers.

Use of graphical and numerical representations. All Calculus & Mathematica students described correctly graphical techniques for solving this problem, two with prompting and two without. One student, Subject C4, who
needed prompting initially chose to solve the problem graphically using the incorrect method of determining when the logistics function was equal to one million, one-half million, or one-quarter million.

Subject C4:
I'd graph this [equation] and see if there one million or one-half-million or one-quarter million on the graph.

Three students, including Subject C4, suggested using the graph of the function to estimate the slope of the function on various days in order to determine if the slope was ever greater than the given numbers. These subjects recognized that if the slope of the function was ever greater than a given number, then there was at least one day on which that number of people would fall sick.

Subject C1 estimated the slope by determining the value of the logistics function on consecutive days.

Subject C1:
I know you'd have to find a point and then the next point over would be - have to be double what that first point was.
Okay, why do you say it would have to be double?
It wouldn't have to be double, I guess. It'd have to be half a million more up the y-axis.

This technique is the graphical equivalent of the numerical technique described previously of evaluating the logistic function on various pairs of consecutive days and then subtracting to determine if any of the differences were larger than the given numbers.
Subject C1 also suggested using this technique of estimating the slope on the point of the graph where the function attained its greatest slope.

Subject C1:
So if you look at the slope, it'd probably, like, be where the greatest slope is or pretty close to it. I think I might find a point near it and the next point over and see if that is a half-million.

One other student suggested using this technique of estimating the slope of the graph for the day where the slope appeared to be greatest in order to determine if the slope was ever greater than the given numbers. These students recognized that if the greatest slope attained by the graph of the function was greater than any of the given numbers, then there had to be a day on which that number of people fell sick. This use of graphical representations exploits the fact that students were not asked to determine the days on which the specified events happened.

Two Calculus & Mathematica students described how to use a numerical representation to determine if there was a day on which more than one million, more than one-half million, or more than one-quarter million people fell sick. Each described techniques for solving the problem similar to the one given by Subject C3 in the following excerpt.

Subject C3:
Well, you can look and see if you can get anything like a - between two days where it jumps like a half-million or something.
Both recognized the difference between the values of the function on at least one pair of consecutive days had to be greater than a half-million if there was to be one day on which a half-million people would fall sick.

*Use of "combination" representations.* Three Calculus & Mathematica students described how to use "combination" representations to determine if there was one day on which more than one million, more than one-half million, or more than one-quarter million people fell sick. All suggested using the graph of the first derivative (graphical/symbolic representation) to determine the maximum value of the first derivative. These students recognized that the maximum value of the first derivative had to be greater than one-million, one-half-million, or one-quarter million if there was to be a day on which that number of people fell sick, as the following suggests.

**Subject C2:**

I guess you could look at the graph of the derivative to see if its ever greater than a half-million or whatever.

After being shown the graph of the first derivative, each student noted that since the maximum value of the first derivative was only 25,000, there was no day on which more than one-million, more than one-half-million, or more than one-quarter million people fell sick.
One student, Subject C4, asked to see the graph of the second derivative during the interview and mentioned how the second derivative could be used to determine where the function had its greatest rate of change.

Subject C4:
Where the graph of the second derivative equals zero is where the greatest change occurs.

Subject C4 was never able to explain how this information, or anything else about the graph of the second derivative, could be used to solve the problem.

Summary of Problem 2

In this section, the results of the Calculus Representations Test and student interviews for Problem 2 are summarized by discussing and comparing students’ (a) use of representations, and (b) interpretation of problem situations and recognition of different calculus concepts, solution techniques, and connections between representations.

Use of representations. The percents of students in each course using the different forms of representations for each part of Problem 2 are summarized in Table 12.
<table>
<thead>
<tr>
<th>R ep r.</th>
<th>Use of Different Representations</th>
<th>% Providing Acceptable Use of Given Representation</th>
<th></th>
<th></th>
<th></th>
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</tr>
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<tbody>
<tr>
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<tr>
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<td>75</td>
<td>100</td>
<td>25</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
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<td>0</td>
<td>75</td>
<td>75</td>
<td>50</td>
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<tr>
<td>S y m b o l</td>
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<td>75</td>
<td>100</td>
<td>100</td>
<td>75</td>
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<td>Not Used</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>G / S</td>
<td>Correct</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>50</td>
</tr>
</tbody>
</table>
Based on the results presented in Table 12, the following observation can be made.

- Calculus & Mathematica students correctly used symbolic representations just slightly more often than traditional students and as often as graphics calculator students.

- Calculus & Mathematica students correctly used numerical representations just slightly more often than graphics calculator students who used this form of representation as often as traditional students.

- Calculus & Mathematica students correctly used graphical representations just slightly more often than traditional students and as often as graphics calculator students.

- Students from all three courses had difficulty describing correctly how to use symbolic representations to determine if there was a day on which some given number of people fell sick.

- Calculus & Mathematica students correctly used graphical-symbolic representations more often than graphics calculator students who used this form of representation as often as traditional students.

- No students suggested using either graphical-numerical or symbolic-numerical representations.

**Interpretation and recognition.** Based on the results of the Calculus Representations Test and student interviews, the following observations can be made concerning students’ interpretation of problem situations and their recognition of different calculus concepts, solution techniques, and connections between representations.
• Traditional students and, to a lesser extent, graphics calculator students had difficulty recognizing that determining how many people would contract the disease in the long run was equivalent to determining the limit, as \( t \) approached infinity, of the logistics function used to approximate the total number of people who had contracted a disease \( t \) days after its outbreak.

• Traditional and graphics calculator students had difficulty relating the real-life situation described in the problem to the logistics function used to model the problem. In particular, they did not appear to realize that the function, and thus the number of people who contract the disease, had an upper bound.

• Traditional students and, to a lesser extent, Calculus & Mathematica students did not recognize that the maximum number of people who contracted the disease did not necessarily correspond to a local maximum of the logistics function.

• Many students from each course recognized that techniques used to determine how many people contracted the disease in the long run could also be used to determine the maximum number of people who contracted the disease.

• Students from all three course had difficulty interpreted the problem of determining if there was a day on which some given number of people fell sick. Most students thought the problem called for them to determine when the total number of people who fell sick, not the number of people who fell sick on one particular day, was greater than the given numbers.

• Some students from each course had difficulty describing how to determine if there was a day on which some given number of people fell sick because they did not appear to completely understand the connection between the rate at which people fell sick, the slope of a logistics function, and its first derivative.

• Calculus & Mathematica students’ slightly better performance with different representations and their greater use of graphical/symbolic representations suggest that they recognized more of the connections between different forms of representations for this particular problem than the students from the other courses.
Discussion of Problem 3

The third problem dealt with using data on the U.S. population from 1790 to 1980 to estimate (a) how fast the population was changing in 1900, 1945, and 1980, (b) when the population growth appeared to be greatest, and (c) the 1990 population (see Figure 16). Thirteen traditional students, eight graphics calculator students, and twelve Calculus & Mathematica students were assigned this problem. Four students from each course were chosen to participate in the interviews. Only about 15 minutes of each interview was spent discussing Problem 3. Interview coding results are presented in Table 13.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population (in millions)</th>
<th>Year</th>
<th>Population (in millions)</th>
<th>Year</th>
<th>Population (in millions)</th>
<th>Year</th>
<th>Population (in millions)</th>
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<td>62.9</td>
<td>1940</td>
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<td>5.3</td>
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<td>23.1</td>
<td>1900</td>
<td>76.0</td>
<td>1950</td>
<td>150.7</td>
</tr>
<tr>
<td>1810</td>
<td>7.2</td>
<td>1860</td>
<td>31.4</td>
<td>1910</td>
<td>92.0</td>
<td>1960</td>
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<td>1820</td>
<td>9.6</td>
<td>1870</td>
<td>38.6</td>
<td>1920</td>
<td>105.7</td>
<td>1970</td>
<td>205.0</td>
</tr>
<tr>
<td>1830</td>
<td>12.9</td>
<td>1880</td>
<td>50.2</td>
<td>1930</td>
<td>122.8</td>
<td>1980</td>
<td>226.5</td>
</tr>
</tbody>
</table>

a. Approximately how fast was the population changing in the years 1900, 1945, and 1980?

b. During what year(s) does it appear that the population growth was the greatest? Explain.

c. Based on the data from the table, what population would you predict for the 1990 census?

Figure 16: Problem 3 of the Calculus Representations Test
<table>
<thead>
<tr>
<th>Representations</th>
<th>Amount of Prompting Coding for Different Representations</th>
<th>% Providing Acceptable Use of Given Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate rate of population change for given years</td>
<td>Estimate when population growth was at its greatest</td>
</tr>
<tr>
<td></td>
<td>151</td>
<td>151</td>
</tr>
<tr>
<td>Graph</td>
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</tr>
<tr>
<td></td>
<td>Correct/Some</td>
<td>50</td>
</tr>
<tr>
<td></td>
<td>Correct/Much</td>
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<td></td>
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<td>Number</td>
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<td>Correct/Some</td>
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<tr>
<td></td>
<td>Correct/Much</td>
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</tr>
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<td>Symbol</td>
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<td></td>
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<tr>
<td></td>
<td>Correct/Much</td>
<td>0</td>
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<tr>
<td></td>
<td>Incorrect/Some</td>
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<tr>
<td>G/S</td>
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<tr>
<td></td>
<td>Correct/Some</td>
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</tr>
</tbody>
</table>
Use of Symbolic Representations

In working with symbolic representations, Calculus & Mathematica students did much better than graphics calculator students who did better than traditional students. Some traditional and graphics calculator students had difficulty recognizing that, given a population function, the value of its first derivative in the years 1900, 1945, and 1980 approximated how fast the population was changing in those years. One traditional student did not recognize that the greatest population growth corresponded to the maximum value of the first derivative, but instead thought it corresponded to the maximum value of the second derivative. All students recognized that the value of a population function in 1990 would be an estimate for the actual population in 1990. It should be noted that on part b, the case of the second derivative of the population function never being equal to zero was not considered in order to expedite the discussion of this problem.

Evaluating students’ use of symbolic representations was more difficult on this problem than on others since no symbolic representation was provided. Some traditional and graphics calculator students had difficulty describing how to use symbolic representations that did not actually exist on paper. In a few cases, use of symbolic representations on parts a and b was not discussed since students did not mention the possibility of deriving a function from the data.
Use of Graphical Representations

In working with graphical representations, Calculus & Mathematica students did better than traditional students who did slightly better than graphics calculator students. Traditional and graphics calculator students exhibited difficulties recognizing how to use the scatter plot of the data to obtain the appropriate slopes for solving parts a and b. Graphics calculator students had difficulty deciding how to extend the scatter plot of the data in order to estimate the 1990 population. Traditional students had difficulty obtaining graphical representations from the data because they did not seem to recognize that the data could also represent coordinates of points.

The scatter plot of the data used with the Calculus & Mathematica students is shown in Figure 17. A similar plot done on a TI-81 graphics calculator was used with the traditional and graphics calculator students. It should be noted that only one graphics calculator student knew that it was possible to obtain a scatter plot on a TI-81 graphics calculator by using the features under its STAT menu.
Figure 17: Scatter plot of the data on U.S. populations from 1790 to 1980, population on vertical axis and years on horizontal axis, shown to students during interviews

Use of Numerical Representations

In working with numerical representations, Calculus & Mathematica students did better than traditional students who did slightly better than graphing calculator students. Traditional and graphics calculator students had difficulty recognizing how to obtain the appropriate slopes, or differences, for solving parts a and b directly from the data. All the students seemed to recognize that the graphical technique of calculating slopes of lines could be used as a numerical technique by calculating the slopes directly from the data.
It should be noted that with traditional and graphics calculator students, approximately 60% of the time spent on Problem 3 was used to discuss part a because of the difficulties students had recognizing how to obtain, or use, graphical and numerical representations from the data. This left little time to discuss parts b or c. For this reason, the excerpts from the discussion of parts b and c lack the depth and detail of the ones from part a.

Use of "Combination" Representations

A new form of "combination" representations was defined for Problem 3. Since no function had been provided to represent the U.S. population, students who suggested using the data, or a scatter plot of the data, to derive a function were classified as having used a symbolic/numerical, or graphical/symbolic representation, respectively. These "combination" representations were the only ones mentioned by traditional and graphics calculator students, except for one graphics calculator student who mentioned using the graphics calculator's TRACE feature (graphical/numerical representation). All Calculus & Mathematica students mentioned the new form of graphical/symbolic representation. In addition, some used graphical/symbolic representations and symbolic/numerical representations involving the first derivative. As the results in Table 13 indicate, the Calculus & Mathematica made more use of "combination" representations than the students from the other courses.
Traditional student interviews

As you read the interview excerpts that follow, look for symptoms that indicate these behaviors in the descriptions given by the traditional students:

- recognition of the connection between slope and rate of change
- recall of what information about a function was provided by its first and second derivative
- recognition of how to obtain information about rates of change from a graphical representation
- recognition of how to use graphical or numerical representations in the absence of a corresponding symbolic representation
- use of symbolic representation when that representation was not actually given but only hypothesized
- use of "combination" representations
- recognition of how to use representations in ways different from those typically taught in the course
- relating of real-life situations described in problems to the representation(s) being used to model the problem and interpreting results in terms of the problem situation
- relating of unfamiliar problem situations or descriptions to familiar uses of representations

Part a. Traditional students exhibited difficulties using graphical and numerical representation to approximate how fast the population was changing in 1900, 1945, and 1980. Students did not recognize, or remember, how to generate appropriate graphical or numerical representations corresponding to the data provided. Some students did not seem to recognize that the data could also represent the coordinates of points. Because of this, students had
difficulty explaining how to obtain, or use, a scatter plot of the data and had difficulty recognizing how to obtain slopes from the data.

Traditional students also had difficulties using symbolic representations to solve this problem. Part of the difficulty may have been that students could not work with symbolic representations that did not actually exist on paper but were only hypothesized. Most students who suggested using a hypothetical function derived from the data described correctly how to solve the problem.

Use of graphical representations. Though all the traditional students interviewed described correctly how to use graphical representations to approximate how fast the population was changing in 1900, 1945, and 1980, only one did so without prompting. Some students, like Subject T5, had difficulty recognizing how to obtain graphical representations from the data.

Subject T5:
Just have like the x-axis be the population and the y-axis be the year.
Graph it up like that.
Okay. And then what would you do then?
Try to plug in points in between them. Like the five years. Like 1945.
Okay. How would you come up with those values?
Probably not. Probably not.
Could you put together a graph just based on what you have there?
Not a line graph anyway.
Okay. What type of graph could you put together?
Bar graph
Ah - I guess I'm hinting at just plotting points.
Yeah, that's what I though after I said the line graph wouldn't work, I think it would.
Subject T5 was one of three students who did not at first recognize that the data could be plotted as points in a scatter plot. The difficulty for these students may have been that they did not recognize that data presented in this manner could be viewed as the coordinates of points. Subject T5 described how to plot the data immediately after the interviewer suggested thinking of individual years as y-coordinates and individual populations as x-coordinates.

When students realized that the data could represent the coordinates of points, they recognized that the slope of the line passing through points near the years in question approximated the rate of population change in those years.

Subject T5:

*What would you do with this plot?*

Take the slope of that possibly.

*Okay. How would you - how would you compute a slope there?*

You could draw the line of the slope. Then you have the x- and y-coordinate, and you can get the, the, where the x-intercept is.

*How are you determining what the line of the slope is?*

If you connect all the coordinates, then you can roughly estimate it, where the graph would be.

Subject T6:

What I probably would do is plot a graph of all these points and look at the slope.

*What would you do with [the plot]?*

Your slope should be the rate of change and so that should be how fast it’s increasing or decreasing.

*Okay. What would you do to figure the slope here?*

Slope is y minus y over x minus x. You could always - um - use that by picking another y and another x.
With students like Subject T6, it was not always clear what points were being chosen in order to determine the slope for the years 1900, 1945, and 1980. Some seemed to recognize that they would get a more accurate approximation by choosing points near the years in question. For example, Subject T8 suggested approximating how fast the population was changing in 1900 by finding the slope of the line through the points corresponding to the years 1900 and 1910. Subject T5 suggested approximating this rate of change by finding the slope of the line through the points corresponding to 1890 and 1910. Other students had difficulty deciding which points to use to approximate how fast the population was changing in the given years. Subject T7 suggested approximating the rate of change of the population in 1900 using the slope of the line through the point corresponding to 1900 and the origin.

Subject T7:

_Is there any way I can somehow figure out the slope here [at 1900]?
Well, you could take - I guess you just want how fast it’s changing from the beginning right here?
What I’m interested in is how fast is it changing right here in the year 1900.... There’s the point right there - what’s the slope there? Can I somehow figure that out?
You could just figure out rise over run, I guess. How you used to do it.
If I’m going to figure out rise over run, don’t I need two points.
Well, you can figure out what that triangle is right there. Well, that’s from changing at the beginning, right?
So you’d do it from here [the origin]. You’d go from here to there and you’d look at that triangle and figure out what that is.
But that’d be from the beginning so, I don’t know._
After prompting, Subject T7 seemed to understand that the approximation would be more accurate if both points chosen were closer to the year 1900. The subject recognized that an even better approximation could be produced using average, as the following excerpt suggests.

Subject T7:

*Could I take, say that point there [1900] and that point there [1910], and figure out a slope between them.*

Yeah.

Okay. *Would - since that point's 1900, does that give me an approximate value for how fast it's changing?*

Yeah, probably. Or you could do like, that one [1890] and that one [1900], and that one [1900] and that one [1910] and then see what like the average is.... I guess that’d probably be the better.

Subject T7 was the only traditional student to suggest using averaging to obtain a better approximation. All recognized that the slope of the line through the points corresponding to 1940 and 1950 approximated how fast the population was changing in 1945, though some needed prompting.

No traditional students suggested solving this problem by sketching a curve through the points of the scatter plot and then estimating the slopes of the tangents to the curve at the points corresponding to 1900, 1945, and 1980. Subject T6 did mention a tangent point, referring to the point where the tangent would be drawn if a graph existed, but never suggested making use of a tangent to solve the problem.
Use of numerical representations. With numerical representations, all the
traditional students recognized that differences in population could be used to
approximate how fast the population was changing in 1900, 1945, and 1980.
For the year 1900, most students approximated the rate of change of the
population with the difference in population between 1890 to 1900 or between
1900 to 1910. Subject T5 suggested solving the problem by calculating a
percentage increase. For the year 1900, the subject divided the change in
population from 1900 to 1910 by the population in 1900 to derive a percent
increase that approximated how fast the population was changing that year.
For the year 1945, the subject performed a similar calculation using the data
for the years 1940 and 1950. Subject T5 was one of only two students from
any course to suggest this numerical technique for solving the problem.

Students who solved the problem graphically and then numerically
recognized, with little prompting, that the slope of the line passing through two
points near the given years could be calculated directly from the data without
first generating a scatter plot. Some students, like Subject T8, who solved the
problem numerically and then graphically had difficulty recognizing how to
obtain the appropriate slopes from the data.
Subject T8:

*How fast was the population changing in 1900? Can you think of how I might approach that? Again, I’m talking about how fast it’s changing so what am I actually talking about?*

The slope of the equation, but I thought you’d need an equation here so you could graph it, maybe.

*Okay. Do I need to have an equation to figure out slopes?... Is there any other way I can get slopes, based on that table of values?*

I’m trying to see if dividing 76 by 62.9 would give you -

*Here’s a line. How do I figure out the slope of that line?*

I think you’d need two points.

*You’re telling me to figure out slope I need two points.*

Right.

*Can I apply that somehow to that problem? If I was interested in the change in population in the year 1900, is there any way I could use what we just talked about finding slopes to figure that out? Can’t think of it.*

*What if I put down x, y and [some numbers for each]?*

This would be one coordinate of points. This would be another coordinate of points....

*Is this any different than [what’s in the table]?*

I didn’t think you could subtract years like that.

*Well, I guess I’m asking can’t I picture that as - an x and y.... If that was just like an x and this was like a y, you could go - um - 76 minus 62.9 divided by 1900 minus 1890 to get the slope of that, if that was just like an x- and y-coordinate.*

At first, Subject T8 did not realize that the data from the table could be viewed as the coordinates of points and used to determine slopes. When the interviewer suggested thinking of the individual years as y-coordinates and the individual populations as x-coordinates, the subject had no difficulty describing correctly how to calculate the slope of a line through two points that would approximate how fast the population was changing in 1900.
Subject T8's comment about not thinking years could be subtracted suggests the students' beliefs about the real-life data influenced how the student thought the problem could be solved. This difficulty was similar to the one encountered with traditional and graphics calculator students on Problem 2 and may be attributable to lack of experience solving problems involving only real-life data instead of equations or graphs. Subject T8's comment about needing an equation suggests the subject did not expect to be given only a data from which to determine rates of change or slopes, which are typically calculated using graphs or first derivatives. On a few occasions, students who did not finish explanations of graphical or numerical solutions would mention using the first derivative of an equation that did not exist. This dependence on equations may explain some difficulties students had on the problem.

Use of symbolic and "combination" representations. Three students suggested that if a function could be derived from the data such that population was a function of years, then the first derivative of this hypothetical function could be used to solve the problem. Two students recognized that the value of the first derivative of this function corresponding to the years 1900, 1945, and 1980 would approximate how fast the population was changing in those years. The third student never explained how the first derivative could be used to solve the problem. It was unclear if the student did not recall the relationship
between the first derivative and the rate of change of the function or if the
student could not describe how to solve the problem because the function was
only hypothetical, and not an actual function that the subject could manipulate.

The two traditional students who described correctly how to solve this
problem symbolically suggested that the hypothetical function might be derived
from the scatter plot. Utilizing the graph to derive the function was classified
as use of a graphical/symbolic representation. This was the only use of a
"combination" representation by the traditional students on part \( a \).

**Part b.** Traditional students had little difficulty using numerical or
graphical representations to estimate when the population growth was greatest,
but had some difficulty using symbolic representations. Apparently, after
discussing graphical and numerical solutions for part \( a \), students had a better
understanding of how to obtain and use graphical and numerical representations
from the data to solve this problem. With symbolic representations, one
student did not recognize that population growth was greatest when the first
derivative attained its maximum value.

**Use of numerical representations.** With numerical representations,
traditional students recognized one way of determining when the greatest
population growth occurred was to determine which ten-year period had the
greatest increase in population.
Subject T7:
Look between two points to see which one changes the most.

Subject T6:
I thought, originally when the question was asked, you know, when did the population jump [most] between the two [years].... During the last few years ... where the greatest deviation occurred.

Subject T8:
1950 to 1960 had the biggest difference in the ten year span than any of than any of the other ten year spans had.

Subject T5 suggested estimating when the greatest population growth occurred by determining which ten-year period had the greatest percentage change (increase in population over ten-year period divided by population at beginning of ten-year period).

*Use of graphical representations.* With graphical representations, traditional students recognized that the greatest population growth corresponded to where the scatter plot had its largest slope between consecutive points.

Subject T5:
Looking on the graph, the bigger slope would mean a greater increase.

Subject T6:
Once again, you can look at the graph and maybe - ah - approximate when the slope is going to be increasing the most.

Subject T8:
I guess whichever had the steepest slope would be the greatest.
Traditional students did not as much difficulty working with graphical representations on this problem as they did on part a since, after the discussions for part a, they now seemed to have a better understanding of how the scatter plot was generated from the data. It was unclear from the comments whether Subject T6 was approximating when the slope was increasing most from the scatter plot or from a sketch of a curve through the points of the scatter plot. No student suggested that the problem could be solved by sketching a curve through the points of the scatter plot and then estimating where this graph attained its greatest slope.

*Use of symbolic and "combination" representations.* Three students suggested that if a population function could be derived from the data then either the first, or second, derivative of this hypothetical function could be used to estimate when population growth was greatest. Two of these students observed correctly that the greatest population growth corresponded to the maximum value of the first derivative of this function, which corresponded to when the second derivative was equal to zero. The third student, Subject T8, thought the greatest population growth corresponded to the maximum value of the second derivative, not the first derivative, as the following suggests.
Subject T8:

Just like the other problem you could use a second derivative.... If you've got the equation, and if you've got the derivative and the second derivative and if you found where they had the max, the second derivative had, had its max, you could find what year span that it was the greatest.

This subject had made the same error on part c of Problem 1 when describing how to determine when population increased or decreased fastest. Subject T8 was the only traditional student to not describe correctly how to solve this problem symbolically.

Of the traditional students who described how to solve this problem symbolically, only Subject T6 suggested deriving the hypothetical function from the scatter plot (graphical/symbolic representation). This was the only use of a "combination" representation by the traditional students on part b.

Part c. Traditional students had little difficulty describing how to use numerical, graphical, or symbolic representations to estimate the population in 1990. No "combination" representations were mentioned by students. Unlike parts a and b, no student suggested deriving a function to fit the data.

Use of numerical representations. With numerical representations, traditional students recognized that this problem could be solved by calculating the increase in population for a few of the ten-year intervals just prior to 1990 and then use these differences to estimate the population in 1990.
Subject T5:

Maybe compare the changes between like 1960 and 70, and 70 and 80, and then maybe - uh - see if they’re steady. Then just estimate that it’ll be steady in the next ten years.

Subject T6:

See the pattern - like how much the deviation is between like - you go by ten years increments or a decade. Find out the pattern of the increase of the increment and estimate from that.

Subject T7:

Well, you might want to start like here, in 1940, and then see how much each one of them [population during 10 year intervals] changes and then, like, try to, like, estimate.

Subject T8:

I think I just subtracted like these two [populations in 1960 and 1970] and found the difference and then these two [populations in 1970 and 1980] and knew that - since these are all increasing, the differences, that difference would have to be greater than the previous difference.

*Use of graphical representations.* With graphical representations, students recognized the 1990 population could be estimated from a graph of data if the graph could be extended beyond the year 1980 to include the year 1990.

Subject T7:

Plot the numbers on a graph and then it would be easier to see how the population changed between every ten years. Then you could extend the graph to the year 1990 to see the estimation for the population that year.

Subject T6:

If you just had the graph, you can take the - ah - the curve and make a smooth curve and end up with your x and y.
Subject T8:

You could um, like extrapolate the graph and find - if you had like 980 and you had equal spaces or whatever, you could find 990 and see where that graph would extend to at that point, at those coordinates.

On this problem, unlike parts a and b, students mentioned using the shape of the points in the scatter plot to sketch a curve through the points and thus extend the graph so that population in 1990 could be estimated.

Use of symbolic and "combination" representations. All students interviewed recognized that if a function could be derived for this data, then the value of that function in 1990 would be an estimate for the actual U.S. population in 1990. The three students who received prompting were all asked how they might solve the problem if a function could be derived from the data.

No traditional student was classified as having used one of the new forms of "combination" representations to solve this problem since no one mentioned using the data or its scatter plot to derive a population function. Subject T5 mentioned trying to derive an exponential equation to fit the data, since, according to the subject, the shape of the scatter plot looked like the graph of an exponential function, but the subject could not describe a means for generating this equation.
Graphics Calculator Student Interviews

As you read the interview excerpts that follow, look for symptoms that indicate these behaviors in the descriptions given by the graphics calculator students:

• recognition of the connection between slope and rate of change
• recall of what information about a function was provided by its first and second derivative
• recognition of how to obtain information about rates of change from a graphical representation
• recognition of how to use graphical or numerical representations in the absence of a corresponding symbolic representation
• use of symbolic representation when that representation was not actually given but only hypothesized
• use of "combination" representations
• recognition of how to use representations in ways different from those typically taught in the course
• relating of real-life situations described in problems to the representation(s) being used to model the problem and interpreting results in terms of the problem situation
• relating of unfamiliar problem situations or descriptions to familiar uses of representations

Part a. Graphics calculator students had about as much difficulty as traditional students using graphical and numerical representation to approximate how fast the population was changing 1900, 1945, and 1980. Like traditional students, they had difficulty explaining how to obtain the appropriate slopes for solving the problem from the data or a scatter plot of the data. Some graphics
calculator students initially did not recognize that differences in population could be used to approximate the rate of population change for the given years.

Graphics calculator students also had difficulties using symbolic representations to solve this problem. As with the traditional students, the difficulty may have been that students could not work with symbolic representations that did not actually exist on paper but were only hypothesized. In addition, one student did not recognize that the slope of a hypothetical population function at the points corresponding to the given years approximated how fast the population was changing in those years.

Use of graphical representations. All the graphics calculator students interviewed described correctly how to use graphical representations to determine how fast the population was changing in 1900, 1945, and 1980, but only one without prompting. Students recognized that slopes of lines through points near the years in question approximated how fast the population was changing in those years, but, as the following suggest, students had difficulty recognizing how to use the scatter plot of the data to obtain appropriate slopes.

Subject G8:

Is there any way I could get that same information [about slopes] without necessarily coming up with an equation?
What? With just a graph?... Just trace along it.
Okay. What do you mean by trace along it?
Connect all the dots and then trace along that, whatever line it gives you.
Subject G6:

Find the slopes of these years, 1900, 1945, 1980, to find the rate of change.
I've got [points] plotted. Now how would you find the slopes from these?
Well, you just go point to point.
Okay. So 1900 is about right there. What would you do to find the slope there at 1900?
Oh, there. I'd a - hmmm. At that point. And there's no line - there's like no line, right?
Not right now. ... Could I put a line in? I've got all these points here.
Yeah, you could put a line on it.
Okay. Well, even without the line, is there someway I can figure out the slope at that point? I mean at the point here, is there something I could do to figure out the slope there? What do I need to figure out the slope?
You need to have two points to figure out the slope or you can an equation, function equation.
Do I have two points I could use?
Um hm.

Both subjects did not recognize that slope could be determined using only the coordinates of the points. Before mentioning that two points were needed to determine a slope, Subject G6 seemed to think that there had to be a line through the points in order to calculate the slope. Subject G8 also wanted to have lines through pairs of points so it would be possible to trace along these line, using the graphics calculator's TRACE feature, and determine the amount of change between different pairs of points. This subject never mentioned calculating slopes of lines through different pairs of points to approximate how the population was changing. Subject G8's use of the graphics calculator's TRACE feature was classified as use of a graphical/numerical representation on
this problem, even though the graphics calculator did not have the capacity to trace along a line drawn between two points on a scatter plot.

The one student, Subject G7, who described how to solve this problem graphically without prompting recognized that slopes of lines passing through points near the years 1900, 1945, and 1980 approximated how fast the population was changing in those years, but could not remember the correct formula for calculating the slope of a line given two points on the line.

Subject G7:

*What am I interested in here?*

The slope.

*Okay. To find the slope, what can I do? I don’t have an equation but I’ve got all those points of data. How can I find a slope?*

I guess you could graph it, the points. Like if you took the year as like your x-values and the population as your y-values.

*Okay. I’ve got the points plotted. Now what would you do?*

Pick the year before it, I guess, and, um, subtract the two y-values and divide by 2 and subtract the two x-values and divide by two and that’ll give you. Wait a minute. That’s not how you do this. I’m trying to think what I did. There’s an equation for it.

Subject G7 made this same error during the discussion of part *d* of Problem 1 when the subject was describing how to use numerical representations to estimate the rate of change of the population on July 1st.

Most graphics calculator students recognized that they would obtain a more accurate estimate for how fast the population was changing in 1900, 1945, and 1980 by determining slopes using points near these years. Students
usually suggested approximating the rate of change of the population for 1900 by finding either the slope of the line through the points corresponding to 1900 and 1910 or the slope of the line through the points corresponding to 1890 and 1900. No student mentioned averaging these slopes, as one traditional student had suggested. All recognized, some after prompting, that the slope of the line through the points corresponding to 1940 and 1950 provided an approximation for how fast the population was changing in 1945.

Subject G6 initially chose to approximate how fast the population was changing in 1900 using the slope of the line through the points corresponding to 1790 and 1900.

Subject G6:

*What two points might I use if I ... want to know approximately how fast it's changing in 1900.*

Put the first year, 1790, to 1900.

*Could I get maybe a better approximation if I used something other than 1790?... Can you think of two points I might use that would give me a little better idea of what might be happening around 1900?*

With that graph, not really.

*What if I used 1890 and 1900?*

Probably could.

*Maybe 1900 and 1910. Would that - would those give me a little better approximation of what is happening around 1900?*

It could or it couldn’t.

*Okay. When wouldn’t it?*

With a - um - curving line you can’t - these slopes aren’t always the same, point to point. You usually have to find a tangent to get the slope of the point. It be better if you could find a tangent at that point than to find, you know, the point before or after.
But as I remember, isn’t the tangent defined - ... Let’s say I’ve got a curve. And there’s the point and I want to find a tangent there. Isn’t one of the ways we define a tangent is by looking at secants? And then seeing as they get closer and closer.
True. Um hm.
So, which do you think would give me a better approximation, the one over here at 1790 or the one over here at 1890?
1890, probably.... That was another point, secants, that I didn’t quite understand.

Even though Subject G6 never recognized, without prompting, that the slope of a line through points near the year 1900 provided a better approximation for the rate of change of the population for 1900, the subject seemed to understand that the best approximation for the rate of change would be the slope of the tangent at the point corresponding to 1900. Subject G6’s comments suggest some understanding of the relationship between the slope of a curve at a point and the tangent to the curve at that point, but did not completely understanding of the mathematical process by which this relationship was established.

Subject G6 was one of two graphics calculator students who suggested solving this problem by sketching a curve through the points of the scatter plot and then estimating the rate of change of the population at the points corresponding to 1900. Subject G6 was the only one to mention using the slopes of tangents to this curve to approximate how fast the population was changing in the given years. The other student, Subject G5, was unable to describe how to estimate the rate of change of the population for the year 1900
from this curve, which suggests the subject did not remember, or understand, the relationship between rate of change at a point and the slope of the tangent to the curve at that point. These students were the only ones to mention using this graphical method for solving the problem.

*Use of numerical representations.* With numerical representations, all the students recognized that differences in population could be used to approximate how fast the population was changing in 1900, 1945, and 1980, but only one did so with no prompting. As with the traditional students, most graphics calculator students approximated the rate of change of the population in 1900 with the increase in population from 1890 to 1900 or from 1900 to 1910. No student mentioned approximating the rate of change using some form of percent increase, as one traditional student had suggested. Some students recognized that the graphical technique of calculating slopes of lines through points near the years in question could be used as a numerical technique for solving the problem by calculating the slopes directly from the data.

One student, Subject G6, who solved the problem graphically and then numerically, initially did not recognize that slopes could be calculated directly from the data without first generating a scatter plot. After prompting, the subject described how to calculate a slope from the data to approximate how fast the population was changing in 1900. Subject G6 was the only graphics
calculator student not to mention using differences in population to approximate the rate of change of the population for the given years.

Use of symbolic and "combination" representations. Three students suggested that if a function could be derived from the data such that population was a function of years, then the first derivative of this population function could be used to solve the problem. Two students recognized that the value of the first derivative of the derived function corresponding to the years 1900, 1945, and 1980 would approximate how fast the population was changing in those years. One of these students, Subject G5, suggested estimating an equation based on the points in the scatter plot, which was classified as use of a graphical/symbolic representation. This subject was the only graphics calculator student to use of this form of "combination" representation on part a.

The third student, Subject G8, who suggested using the first derivative of a hypothetical function to solve this problem, recognized that the value of the first derivative of this function corresponding to the years 1960, 1945, and 1980 was the slope of the function at the points corresponding to these years. What this subject did not recognize was that these slopes approximated how fast the population changed in those years.
Subject G8:

Let's say you came up with an equation.... How could I use that to help me figure out how fast the population's changing?

Just take the - uh - first derivative and then you - uh. Yeah, but you're asking for certain years. Oh, yeah. Just plug in - wait a minute.... What number do I take? You take the - uh. Let me think. Did you - um. On that graph, which one was on the x-coordinate and which one was on y?

The x-coordinates the year and y-coordinates the population.

Then you just take this year and you plug it in.

Okay. Would that - does that tell me how fast it's changing?

That'll give you the slope.

Uh huh. Does the slope tell me how fast it's changing?

No, not really.

Subject G8 had made a similar claim during the discussion of part $d$ of Problem 1 when the subject suggested that the slope of a function at a point did not represent the rate of change of the function at that point, but was only an estimate for it. It was not until the interviewer pointed out that the subject's graphical and numerical solutions for this problem had used slopes to approximate how fast the population was changing that Subject G8 agreed that slope was also a measure of the rate of change.

Subject G7 suggested the exponential equation $Y = Pe^t$ could be used to determine how fast the population was changing in 1900, 1945, and 1980. This choice of equation was based on the subject's previous experiences with population and interest problems similar to this problem. For the year 1900, Subject G7 determined a value of $r$ such that the graph of the equation $Y = Pe^t$
would pass through the points corresponding to 1790 and 1900. This value of 
$r$, for $P$ equal 3.9, the population in 1790, $Y$ equal 76, the population in 1900, 
and $t$ equal 110, the number of years between 1790 and 1900, approximated 
how fast the population was changing in 1900. The same calculation was done 
for the year 1980 with the point corresponding to 1980 replaced by the point 
Corresponding to 1900. Subject G7 did not calculate a value of $r$ for the year 
1945 since no population was given for that year, and did not suggest another 
method for approximating the rate of change of the population for that year.

It was unclear if the subject recognized that this method produced two 
different equations to fit the data, and not one equation. This suggests the 
subject did not understand how to apply this technique to the problem at hand 
but was just attempting to use a memorized procedure for finding rates of 
increase using exponential equations. Even though this method of deriving 
equations to fit different pairs of data points did not make use of all the data, it 
produced a viable approximation, though perhaps not a good approximation, for 
how fast the population changed in the given years, as the problem requested.

The technique used by Subject G7 to derive an equation from the given 
data was classified as use of a symbolic/numerical representation. This subject 
was the only graphics calculator student to make use of this form of 
"combination" representation on part $a$. 
Part b. Graphics calculator students had little difficulty using graphical or numerical representations to estimate when the population growth was greatest, but had some difficulty using symbolic representations. Apparently, they, like the traditional students, had a better understanding of how to obtain and use graphical and numerical representations from the data to solve this problem after discussing graphical and numerical solutions for part a. With symbolic representations, two student did not recognize that greatest population growth occurred when the second derivative was equal to zero. No "combination" representations were mentioned by students. Unlike part a, no student suggested deriving a population function to fit the data.

Use of graphical representations. With graphical representations, graphics calculator students recognized where the population growth was greatest corresponded to where the scatter plot of the data had its largest slope between consecutive points.

Subject G7:
You can look and see where it's - the graph is the steepest - between which years and that'd tell you.

Subject G5:
What might I look for ... to determine where the population growth was the greatest? Is there some characteristic of the graph I could look for? Um - where the - uh - graph jumps on the population axis. The greater the jump there, the greater the increase.... The slope at that part of the graph is greater than before it or after it, at any point along the graph.
Subject G8:

Um - where the slope is the greatest.
Okay. Now what would I do to find out where the slope was the greatest?
Well, if you had the graph, then you just see where it's most vertical....
You just find out where your highest slope is and that'll give you where
the population growth is the greatest. Cause the higher the slope, then the
more, the bigger the population increases.

Subject G6:

Find the highest slope.
So I look at the graph and try to figure out where the highest slope is?
That would be probably like maxima/minima. I probably - I might be
wrong with that.
Okay. Maximum/minimum of?
Like of the graph. Uh. I don't know. Maybe it is the slopes cause that's
basically velocity.

Subject G6 was confused by whether the problem asked for maximum
population, which would be the maximum value of the graph, or maximum
population growth, which would be the maximum value of the slope of the
graph. After being reminded that the problem asked for when the population
growth was greatest, the subject agreed the solution occurred when the graph
had its highest slope.

Graphics calculator students did not have as much difficulty working with
graphical representations on part b as they did on part a since discussions for
part a helped them recognize how to obtain slopes from the scatter plot. As
with traditional students, no graphics calculator student suggested the problem
could be solved by sketching a curve through the points of the scatter plot and
then estimating where this graph attained its greatest slope. Subject G5 did
describe this method, after prompting, when asked how to solve the problem
given a smooth curve fitted to the data rather than a scatter plot of the data.

*Use of numerical representations.* Like traditional students, graphics
calculator students recognized that one method for estimating numerically when
population growth was greatest was to determine which of the ten-year periods
had the greatest increase in population.

**Subject G5:**

Between 1950 and 1960, a 28.3 million increase was the largest compared
to the changes between other ten-year periods

**Subject G7:**

1950 to 1960 had the greatest increase of 28.3 million people

The only student to need prompting on this problem, Subject G6, did not
recognize that slopes of lines through pairs of points could be calculated from
the data without first generating a scatter plot. Only after it was pointed out
that slopes between pairs of data points were similar to differences between
populations from the table of data did Subject G6 recognize the highest slope
corresponded to greatest difference in populations over a ten-year period.

*Use of symbolic and "combination" representations.* Three students
suggested that if a function could be derived from the data such that population
was a function of years, then either the first or second derivative of this
hypothetical function could be used to estimate when population growth was greatest. As the following excerpt suggests, Subject G8 recognized that the greatest population growth corresponded to when the second derivative of the hypothetical function was equal to zero, which corresponded to the maximum value of the first derivative.

Subject G8:

What would you do ... to find out where ... the highest slope occurred? I think we can take the second derivative to find that. What: would I do to the second derivative? Make it equal to zero. Okay. Would that give me where the first derivative’s maximum? It should give you cause the first one gives you the max and min - max and min of the first equation and the second derivative should give you the max.... So, wherever it was maximum, that’d be the greatest change.

Subject G8 was the only graphics calculator students to describe correctly how to solve this problem symbolically without prompting.

Subjects G6 and G7 initially wanted to set the first derivative equal to zero in order to determine when the greatest population growth occurred.

Subject G6:

You’d find the first derivative. Set it equal to zero and find the maxima and minima from that. Okay. Wouldn’t that give me the maximum population rather than the population growth?... if I took the first derivative and set it equal to zero, wouldn’t that give me the maximum of the population rather than the maximum of - rather than the greatest population growth? Um hm. It goes back to the question before, part c. Where you could probably find like a - um - like the second derivative. Find the inflection point where you can find the highest - the fastest whatever growing.
Subject G7:

If you had an equation and you took the derivative and you found its max. I don’t know.

*Okay. I’m interested where the population growth was the greatest.*

That’d be where the rate of increase was the greatest.

*Okay. So what would I do then?*

Um - you’d put that back in the original equation. I guess that value.

*Which value?*

Um - that you’d get from the second derivative, I mean the first derivative.

*Okay. I’ve taken the first derivative and what have I done?*

Set it to zero and found the max.

*Okay. Does that give me where the growth is the greatest or the population is the greatest?*

That would give you where the growth is the greatest.

*If I take the first derivative and set it equal to zero?*

Okay. Population, I guess. Um - so then that would mean you’d take the second derivative and set that equal to zero and that would give you where the growth is the greatest.

Both subjects recognized the greatest population growth corresponded to when the second derivative, not the first derivative, was equal to zero after the interviewer reminded them that setting the first derivative equal to zero would locate the maximum population.

No graphics calculator was classified as having used one of the new forms of "combination" representations to solve this problem since, unlike the discussions for part *a*, no student mentioned using the data or a scatter plot to derive a function to fit the data.
Part c. Graphics calculator students had little difficulty describing how to use numerical and symbolic representations to estimate the population in 1990, but did have difficulty using graphical representations. Students recognized the 1990 population could be estimated from a graph of data if the graph could somehow be extended beyond the year 1980 to include the year 1990, but had difficulty devising ways to extend the scatter plot beyond the year 1980.

Use of numerical representations. With numerical representations, traditional students recognized this problem could be solved by calculating, or estimating, the increase in population for a few ten-year intervals just prior to 1990 and then using these differences to estimate the 1990 population.

Subject G5:
Starting from 1950, the increase has been more than 20 million so I would estimate that the population would be around 246 million or more, about 20 million more than the population in 1970.

Subject G8:
[I] took the increase between - from 1970 to 1980 and added it on.

Subject G6:
After I find the average growths during those - every ten years and then from that estimated calculation, I’d find it for the next year, 1990.

Use of graphical representations. Graphics calculator students had difficulties explaining how to use graphical representations to estimate the population in 1990. All recognized the 1990 population could be estimated
from a scatter plot if the plot could be extended beyond 1980 to include 1990, but none did so without prompting. Prompting usually consisted of discussing possible ways to extend the scatter plot to include a point corresponding to 1990. Subjects G8 and G6 chose to extend the plot using a line.

Subject G8:

_How would you estimate it of that graph? What would you do?_
Uh - on that graph. Try to predict where the next point might be.... If you drew a line but that's not - it doesn't. If you drew a line and then just followed the way the graph is doing.

Subject G6:

_Could I somehow just do something with the graph to give me that information._
You could probably estimate it but it wouldn't be a really good estimate. _How would estimate it using the graph? What might you do?_
I would probably follow the line up.

Both subjects wanted to use a line to extend the scatter plot; however, neither student made it clear what line they were referring to during these discussions. Both subjects seemed to be talking about drawing a line that would fit the shape of the points in the scatter plot, rather than a line that passed through the points near the year 1990, such as the points corresponding to 1970 and 1980.

Subject G6 mentioned estimating the 1990 population by using an equation of a line to extend the scatter plot.
Subject G6:

*What could I do with that [the graph] to help me figure out ... 1990.*
Um - I would try to find the equation of the line. And then, once I have the equation, to graph it and find the point for - like the time of 1990, and then compare it to the year.

Again, it was unclear what line Subject G6 was referring to during this discussion. It appeared the subject wanted to determine the equation of a line that fit the shape of the points in the scatter plot, but it is possible the subject was referring to the line passing the points corresponding to 1970 and 1980.

Subject G5 was the only graphics calculator student to suggest drawing a curve to fit the data and then extending the curve to include the year 1990.

Subject G5:

*If you have the smooth curve and if your graph goes out to 1900, if the - uh - x-axis reaches 1990, then the curve should extend up to 1990 and you can take an approximation from there.*

Subject G7 was the only other student who discussed solving this problem by drawing a curve to fit the data. This subject did not mention drawing the curve but did describe how to predict the 1990 population if such a curve existed.

*Use of symbolic and "combination" representations.* All four students recognized that if a function could be derived to fit this data, then the value of that function for 1990 would be an estimate for the actual U.S. population in 1990. The two students who received prompting on this problem were asked how they might solve the problem if a function could be derived from the data.
Before recognizing the value of the hypothetical function in 1990 approximated the population for that year, Subject G7 tried to solve this problem using the exponential equation \( Y = Pe^r \) the subject had used when attempting to solve part \( a \). In this case, the subject sought to calculate a value of \( Y \) corresponding to 1990 by using the value of \( r \) calculated in part \( a \) that estimated the rate of change of the population in 1980 and then letting \( P \) equal 226.5, the population in 1980, and \( t \) equal to 10, the number of years between 1980 and 1990. Subject G7 did not recognize the values of \( P \) and \( t \) for the point corresponding to 1790, not the values of \( P \) and \( t \) chosen, should have been used with the chosen value of \( r \) in order to derive an exponential equation that would provide a rough approximation of the population in 1990. Again it appeared the subject did not understand how to apply this technique to the problem at hand but was just attempting to use memorized procedures concerning exponential equations.

Subject G7's technique for deriving an equation from the data was classified as use of a symbolic/numerical representation. The only other "combination" representation mentioned on part \( c \) was the graphic/symbolic representation suggested by Subject G6. This subject first used the scatter plot to derive a linear equation to fit the data, then graphed the derived equation and used this graph to estimate the 1990 population.
Calculus & Mathematica Student Interviews

As you read the interview excerpts that follow, look for symptoms that indicate these behaviors in the descriptions given by the Calculus & Mathematica students:

- recognition of the connection between slope and rate of change
- recall of what information about a function was provided by its first and second derivative
- recognition of how to obtain information about rates of change from a graphical representation
- recognition of how to use graphical or numerical representations in the absence of a corresponding symbolic representation
- use of symbolic representation when that representation was not actually given but only hypothesized
- use of "combination" representations
- recognition of how to use representations in ways different from those typically taught in the course
- relating of real-life situations described in problems to the representation(s) being used to model the problem and interpreting results in terms of the problem situation
- relating of unfamiliar problem situations or descriptions to familiar uses of representations

Part a. All but one Calculus & Mathematica students interviewed had little difficulty describing how to use graphical or numerical representations to estimate how fast the population was changing in 1900, 1945, and 1980. It appeared that the one student either did not understand what the question "how fast was the population change in the years 1900, 1945, and 1980" meant or
did not understand how to approximate an instantaneous rate of change using
the information provided for the problem.

Calculus & Mathematica students had little difficulty using symbolic
representations to solve this problem. Unlike the other students, Calculus &
Mathematica students seemed able to work with symbolic representations that
did not actually exist on paper but were only hypothesized. Calculus &
Mathematica students made more use of "combination" representations than
other students. All suggested using Mathematica to generate a scatter plot of
the data and then derive a best-fit function from the points in the scatter plot.

Use of graphical representations. Most Calculus & Mathematica students
recognized the slope of lines through points near the years in question
approximated how fast the population was changing in those years.

Subject C5:

You could take the nearest points around it ... and then find out the
growth between those and that'd give you pretty close.... For 1900, you'd
use 1890 and 1910 ... find out the growth between 1890 and 1900, the
slope there ... 1900 and 1910 and find the slope there.... And then just
average them.

Subject C8:

Take 1900 and just take it over a small slope from like here [1900] to
here [1910]. You can approximate it that way.... That what I did here
cause I said 1900 to 1910 ... and I found, like, a slope at that point.
These comments suggest students recognized their estimates for how fast the population was changing in the years 1900, 1945, and 1980 were more accurate if the points chosen were near the years in question. Subject C5 was the only Calculus & Mathematica student who suggested using the method of averaging slopes, which was also mentioned by one traditional student, to approximate the rate of change of the population.

No Calculus & Mathematica student suggested solving this problem by sketching a curve through the points of the scatter plot and then estimating the slopes of the tangents to the curve at the points corresponding to 1900, 1945, and 1980. The students who had little difficulty with this problem all recognized that the slope of the line through the points corresponding to 1940 and 1950 approximated how fast the population was changing in 1945.

One student, Subject C6, had difficulty working with graphical representations to determine how fast the population was changing in 1900, 1945, and 1980. To begin with, the subject did not recognize that this problem was unrelated to Problem 1 discussed earlier, as the following suggests.

Subject C6:

Um - well you can take the numeric values from the population for those years and take it and put it into an equation. 
Okay. What equation might you use? 
Ohh - population equation. I don't know. Maybe the one that we were just working on.
When Subject C6 was first shown a scatter plot of the data, it appeared that the subject did not understand how the points in the scatter plot had been obtained.

Subject C6:

*You mentioned the idea of slope with this. Is there any way just based on these data points I can figure out a slope?*

Well, slope is the - um - change in y over the change in x. Which is *m*. But using the equation of a line, maybe you can, you know, get it, maybe not. Okay. Um - but the change in y over the change in x. You don’t really have an *x* or a *y* distinct in here, do we?

After prompting, Subject C7 realized how the scatter plot had been generated from the data. Then, when asked to solve this problem using the scatter plot, Subject G7 proceeded to calculate the slope of the line through the points corresponding to 1900 and 1980.

Subject C6:

*Okay. So you’re looking at the slope between these two points right here. Yeah. The change in y over the change in x.*

*And you’d use that to approximate what? The change in 1900 or the change in 1980 or the change in - what would that be your estimate of?* Just estimating a change from 1900 to 1980.... Basically here, what I would’ve wanted to do is take a rate of change from [1790] to [1900] and a rate of change from [1900] to [1980]. And you’d get two values here and take the average of the two values. Then you would have a rate of change from [1900] to [1980]. Maybe. I don’t know. Sounds good.

It appeared the subject was attempting to calculate an average of change of the population between 1900 to 1980, though this could also be construed as a rough estimate of the rate of change in 1900. This suggests that Subject C6 either did not understand what the question "how fast was the population
change in the years 1900, 1945, and 1980" meant or did not recognize how to estimate the rate of change of the population in the given years using appropriate slopes derived from the scatter plot of the data.

*Use of numerical representations.* All Calculus & *Mathematica* students recognized the graphical technique of calculating slopes of lines through points near the years in question could be used as a numerical technique for solving the problem by calculating slopes directly from the data. They also recognized differences in population could be used to approximate how fast the population was changing in 1900, 1945, and 1980. Subject C7 used the average change in population over the 20-year time period between 1890 and 1910, which is equivalent to slope of the line through the points corresponding to 1890 and 1910, to approximate how fast the population was changing in the year 1900.

Subject C7: All I did was just take the amount the year before that was given, and the year after that, so like between ... 1890 and 1910 ... then I divided that by, what 20 years.... Yeah, so 30 million divided by 20 years.

When the interviewer pointed out that a calculation using the data for the years 1890 and 1900, or the years 1900 and 1910, might provide a better estimate, particularly if the change in population from 1890 to 1900 was dramatically different from the change in population from 1900 to 1910, Subject C7 gave the following response.
Subject C7:

Maybe if it had looked like - if had been like 76, then 90, then 91, I probably wouldn't have used the same method. But since it looked like they were almost equal, it seemed better.... I'm a zoology major and we always do everything from the midpoints. So that seemed like 1900 would then be the midpoint, so you could take the points outside of that.

This comment suggests the subject understood that the approximation could be improved if the average change was calculated over a shorter time period. It is likely that Subject G7 chose to solve the problem graphically using the similar technique of calculating the slope of the line through the points corresponding to 1890 and 1910 to solve this problem, as opposed to using the points corresponding to 1890 and 1900 or 1900 and 1910, for much the same reason.

Subject C8 suggested solving the problem by calculating a percentage increase. For the year 1900, the subject made a rough estimate of the increase in population from 1900 to 1910 and then divided this number by the population in 1900 to derive a percent increase that approximated how fast the population was changing for that year. For the year 1945, Subject G8 performed a similar calculation using the data from 1940 and 1950, but instead of dividing by the population in 1940, the subject divided by the average of the 1940 and 1950 populations. Subject C8, along with Subject T5, were the only students to suggest using this numerical technique to estimate how fast the population was changing in 1900, 1945, and 1980.
Subject C6 was the only Calculus & Mathematica student to have difficulty using numerical representation to solve this problem. During the interview, it was unclear if the subject understood what was being asked for in this problem. As the following suggests, Subject C6 may have thought the problem asked for a rate of change of the population between the given years.

Subject C6:

You could take the one 1900 data point and - um - subtract it from the 1945 data point.

There's no 1945 data point.

Well, 1940. Close enough. And then - um - get a value. And then take - um - 1940 value, then subtract it from 1980 value and just see how much of a difference between those two numbers. An average of them, or something like that.

And what would that give me then? If I took the 1900 to 1940 and then the 1940 to 1980 and then you say average it, which one of these would it approximate? Does it approximate all of them or one of them or what? Um - well not really average. You know, you’ll have a number from here and then you’ll have a number from here, and you’ll see there’s a big jump in the difference. And you ... might be able to figure out how fast the population was changing by the average rate of change.

Subject C6 seemed to understand the differences in populations between 1900 and 1940 and between 1940 and 1980 represented rates of change of the population. The subject appeared to be determining an average of change of the population between 1900 to 1980 and not rates of change for 1900 and 1980. Subject C6’s comment about the population in 1940 being close enough to use for the unknown population in 1945 supports the supposition that the
subject either did not understand what the problem was asking, or possibly did not understand how to approximate an instantaneous rate of change using just the information provided. Still, previous comment suggest Subject C6 may have understood that slopes and differences in populations represented measures of rates of change that could be used to estimate how fast the population was changing in the given years.

*Use of symbolic and "combination" representations.* All four students suggested that if a function could be derived from the data, then the values of first derivative of this hypothetical function corresponding to 1900, 1945, and 1980 would approximate how fast the population was changing in those years.

Subject C5:

At 1900, find the value of the derivative which will give you value of the change.

Subject C6:

*Mathematica* could find an equation and then - um - by using that equation, maybe find the derivative and then find the rate of change. *Let's say I've got a derivative of some function ... and I want to know how fast the population's changing for 1900. What would you do?* I would plug in the y - um - for 1900 into the derivative. *The y?* Or the x. Well, I'd plug in 1900.

Subject C7:

You could take the derivative at 1900... You could take the derivative of the function. And then you could take - Yeah, you could find - Well, you could find the derivative at various points.
Subject C8:

Since the question is how fast the population's changing in those years, so I'm think how fast is the population changing is like, has to do with the derivative. So I look at the derivative at those years, 1900, 1945, 1980 and I find exactly what the number was that was changing those years.

Each student suggested the hypothetical population function might be obtained by using *Mathematica* to generate a scatter plot of the data and derive a best-fit function from the points in the scatter plot. Utilizing the graph to derive the function was classified as use of a graphical/symbolic representation, even though *Mathematica* actually derived the function from the graph for the students. This was the only use of a "combination" representation by the Calculus & *Mathematica* students on part a.

**Part b.** Calculus & *Mathematica* students had little difficulty describing how to use any of the different forms of representations to estimate when population growth was greatest. In addition, they were the only students to suggest using "combination" representations, other than the ones involving the derivation of a function from the data or its scatter plot, to solve the problem.

*Use of graphical representations.* With graphical representations, all students recognized that the greatest population growth occurred when the scatter plot had its largest slope between consecutive points.
Subject C5:

Go between points and find which point has the greatest slope. I mean, look at first and see where it looks greatest, then work around there until you find the greatest slope, find the greatest slope.

Subject C6:

About right here it's changing faster.

*Okay. So you're looking for the fastest change?*

Yeah. The steepest slope. Basically, find the slope of different points in the year. You can do it this way and find the change in y and the change of x and find the greatest slope.... You can do it directly from the points.

Subject C7:

You could just look where it looks the steepest.

Subject C8:

Well, graphically I'd look at and see how fast it would change. I mean, I'd look for the point where it was like, more - ah - more horizontal to, or vertical to horizontal.

*Okay. So you're looking for what?*

Where it's steepest.

No Calculus & *Mathematica* student suggested that this problem could be solved by sketching a curve through the points of the scatter plot and then estimating where this graph attained its greatest slope.

**Use of numerical representations.** In the excerpt above, Subject C6 noted that the calculation for determining the greatest slope could be done directly from the data. Subject C5 made the same observation.
Subject C5:

Can I do something like that without necessarily looking at - without having a plot? Could I just get that from the data somehow? Yeah. You could go through the data point and look for the greatest slope between points.... [At] the end it seems to have the greatest differences. Maybe work back and see what it does. And it happens to be getting a greater slope, then I'd find a spot in the middle and work around and see where they start - which direction they had greater or smaller until finally you hit a point where it goes greater and then starts going smaller again.

All Calculus & *Mathematica* students recognized that slopes of lines through pairs of points could be calculated directly from the data without first generating a scatter plot.

Somewhat unexpectedly, only one Calculus & *Mathematica* student, Subject C8, mentioned solving this problem by determining which ten-year period had the greatest increase in population. This student also estimated when population growth was greatest by determining which ten-year period had the greatest percentage increase in population. Subject C7 also used increase in population to estimate when the greatest population growth, but this subject looked for the greatest increase over a 40-year time period, rather than a 10-year time period.

*Use of symbolic and "combination" representations.* All four students suggested that if a function could be derived from the data, then either the first or second derivative of this hypothetical function could be used to estimate
when population growth was greatest. As the following excerpts suggest, all
the students recognized that the greatest population growth corresponded to the
maximum value of the first derivative of the hypothetical function, and most
recognized the maximum value of the first derivative corresponded to when the
second derivative was equal to zero.

Subject C5:
I would solve the second derivative for 0 because that is where the
growth rate is greatest or least.

Subject C6:
Or find the derivative.... You’d be looking for - um - where the highest
derivative was.

Subject C7:
Well, this one clearly could have been second derivative. Or no. Yeah, second derivative would tell you when the first derivative was the
greatest. Yeah. Which is what you were looking for.

Subject C8:
Well, if you found the derivative, you could take where the derivative
was - um - was greatest because were looking for when the population
growth is greatest, not the population but the population growth.... You
can find that by either graphing it, looking at it and saying, hey, that
right, right there is where it’s the biggest.... Or taking the second
derivative and looking where that’s equal zero and looking for where
max/min values would be - derivative would be highest.

As was the case in part a, each student suggested obtaining this hypothetical
function by using Mathematica to generate a scatter plot and derive a best-fit
function from the points in the scatter plot (graphical/symbolic representation).
Along with aforementioned graphical/symbolic representations, Calculus & Mathematica students used other "combination" representations to solve this problem. Subject C8 estimated when population growth was greatest by using the graph of the first derivative to determine when the derivative attained its maximum value (graphical/symbolic representation). Subjects C5 and C7 ascertained when the greatest population growth occurred by evaluating the first derivative for various years to determine when the derivative attained its maximum value (symbolic/numerical representation). No students from other courses mentioned using these types of "combination" representations to solve the problem.

**Part c.** Calculus & Mathematica students had the least difficulty of all students describing how to use the different representations to estimate the population in 1990. As was the case with part b, they were the only students to suggest using "combination" representation, other than the ones involving the derivation of a function from the data or its scatter plot, to solve the problem.

*Use of graphical and numerical representations.* With graphical and numerical representations, most students recognized, without prompting, that the 1990 population could be estimated if the scatter plot, or the data, could somehow be extended beyond the year 1980 to include the year 1990. Subject C5 suggested using slopes to extend the graph.
Subject C5:
You could take the slopes you’re getting and find out difference between
them, how they’re changing. And have another slope with that same rate
of change and then get that on and just take the value you get in 1990.
[Or] continue the curve you think you have and then just bring it down
and approximate.

Subject C5 recognized this graphical technique could be used as a numerical
technique for solving the problem by calculating slopes directly from the data.

In the excerpt, the subject mentioned using the "curve you think you have" to
estimate the population in 1990. Subject C5 was the only Calculus &

Mathematica student to suggest solving the problem by drawing a curve that fit
the shape of the points in the scatter plot and extended to include 1990.

Subject C6 suggested using slopes calculated directly from the data to
estimate the population in 1990.

Subject C6:

You have all these years and you can estimate from the past years of
what 1990 will be like by taking how the slope of the line is changing
from each year to each year to each year and you can estimate how much
more it will change.... From 1900, you have a certain change and then
going up to like 1920, you have certain change ... and each year, it’ll
change by so much.... You can just take the estimated values that you
calculated through the change in y over the change in x and then estimate.

This subject recognized that this numerical technique could also be used as a

graphical technique for solving the problem by calculating the slopes from the
points in the scatter plot.
Some students, such as Subject C7, recognized the problem could be solved by calculating the increase in population for a few ten-year intervals just prior to 1990 and then use these differences to estimate the population in 1990.

Subject C7:

*What population would you predict for the 1990 census?*

245 million [since] the population jumps seemed to be getting a little smaller each ten-year period.... Like here it jumped 26 million [from 1960 to 1970] and then here it jumped 21 million. So I'm saying well next time it'll jump 19 million.

Subject C8 used this method to solve the problem, but instead of determining the exact change in population for different ten-year periods, this subject made rough estimates of these changes and used them to predict the 1990 population.

*Use of symbolic and "combination" representations.* All students interviewed recognized that if a function could be derived for this data, then the value of that function for 1990 would be an estimate for the actual U.S. population in 1990. Subject C6 noted that such a function, or equation, could be used to estimate the population for any year.

Subject C6:

You can find an equation and plug in any year. It could be 1990, 2025 or whatever, and find what the population would be from that equation.

Each student suggested obtaining this hypothetical population function by using Mathematica to generate a scatter plot of the data and derive a best-fit function from the points in the scatter plot (graphical/symbolic representation).
Subjects C7 and C8 suggested that the first derivative of the hypothetical function might be used to determine an estimate for the 1990 population.

Subject C7:
If you had the first derivative at 1980, then you could use that to figure out where it would by 1990.

Subject C8:
Taking the derivative probably wouldn’t be much. Well it could be much - it could be a little bit of help. Because if I found - if I then these points and I knew the derivative like, after those points, then I could figure out how much it was going to grow from that point and then do an extension.

Both students recognized that the first derivative could be used to calculate the slope at 1980 and that this slope could be used to estimate how much to add to the 1980 population to obtain the 1990 population. This method for solving the problem was classified as use of a symbolic/numerical representation. No other students from any course mentioned using the first derivative to estimate the 1990 population.

**Summary of Problem 3**

In this section, the results of the Calculus Representations Test and student interviews for Problem 3 are summarized by discussing and comparing students’ (a) use of representations, and (b) interpretation of problem situations and recognition of different calculus concepts, solution techniques, and connections between representations.
Use of Representations. The percents of students in each course using the different forms of representations for each part of Problem 3 are summarized in Table 14. Based on these results, the following observation can be made.

- Calculus & Mathematica students correctly used symbolic representations more often than traditional students and graphics calculator students.

- Calculus & Mathematica students correctly used graphical and numerical representations about as often as traditional students and graphics calculator students.

- Traditional and graphical calculator students had some difficulty describing correctly how to use symbolic representations to estimate how fast the population was changing in the years 1900, 1945, and 1980.

- Traditional students had some difficulty describing correctly how to use symbolic representations to estimate when the population growth was greatest.

- Calculus & Mathematica students mentioned using the new form of graphical/symbolic representation, where a function was derived from the points in a scatter plot, more often than traditional students who mentioned this form of representation as often as graphics calculator students.

- Calculus & Mathematica students correctly used graphical/symbolic and symbolic/numerical representations more often than graphics calculator students and traditional students.
Table 14: Comparing Students on Use of Representations on Problem 3 by Course

<table>
<thead>
<tr>
<th>R ep r.</th>
<th>Use of Different Representations</th>
<th>% Providing Acceptable Use of Given Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate rate of population change for given years</td>
<td>Estimate when population growth was at its greatest</td>
</tr>
<tr>
<td></td>
<td>151 G</td>
<td>151 C</td>
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<td>Graph</td>
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<td></td>
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<tr>
<td></td>
<td>Not Used</td>
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<tr>
<td>Number</td>
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<tr>
<td></td>
<td>Incorrect</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
<td>0</td>
</tr>
<tr>
<td>Symbol</td>
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</tr>
<tr>
<td></td>
<td>Incorrect</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
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</tr>
<tr>
<td>G/ S</td>
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<td>50</td>
</tr>
<tr>
<td></td>
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</tr>
<tr>
<td>G/ N</td>
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<tr>
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<tr>
<td></td>
<td>Not Used</td>
<td>100</td>
</tr>
</tbody>
</table>
Interpretation and Recognition. Based on the results of the Calculus Representations Test and student interviews, the following observations can be made concerning students' interpretation of problem situations and their recognition of different calculus concepts, solution techniques, and connections between representations.

- Traditional and graphics calculator students had some difficulty recognizing how to interpret the data and the scatter plot of the data. In particular, they had difficulty obtaining the appropriate slopes from the data and from the scatter plot.

- Traditional students had some difficulty recognizing how to obtain the scatter plot from the set of data.

- Some traditional and graphics calculator students did not recognize the value of the first derivative of a population function derived to fit the data approximated how fast the population was changing at a particular time.

- Most students recognized the graphical technique of calculating slopes from points in the scatter plot could be used as a numerical technique by calculating slopes directly from the data, and vice versa.

- Most students recognized the greatest population growth corresponded to the greatest difference between consecutive points in the scatter plot or between consecutive values of the population from the data.

- Calculus & Mathematica students seemed to have less difficulty than students from other courses solving problems where only data was provided and not any functions, equations, or graphs.

- Traditional and graphics calculator students appeared to be dependent upon having symbolic representations to solve problems.

- Calculus & Mathematica students' better performance with different representations and their greater use of "combination" representations suggest that they recognized more of the connections between different forms of representations for this particular problem than the students from the other courses.
Discussion of Problem 4

The fourth problem dealt with (a) showing that $x > 2 \ln x$ (or $e^x > x^2$) for all $x > 0$, (b) showing that $x > 3 \ln x$ (or $e^x > x^3$) for all $x > 0$, and then estimating the value of $M$ such that $x > 3 \ln x$ (or $e^x > x^3$) for all $x > M$, and (c) predicting the largest value of $a$ for which $x > a \ln x$ (or $e^x > x^a$) for all $x > 0$ (see Figure 18). Thirteen traditional students, eight graphics calculator students, and thirteen Calculus & Mathematica students were assigned Problem 4. Four students from each course were chosen to participate in the interviews. Only about 15 to 20 minutes of each interview was spent discussing Problem 4. Interview coding results are presented in Tables 15 and 16.

4.  
   a. Show that $x > 2 \ln x$ for all $x > 0$.
      (Note: This is equivalent to showing that $e^x > x^2$ for all $x > 0$.)
   b. Is it true that $x > 3 \ln x$ for all $x > 0$?
      If not, estimate $M$ such that $x > 3 \ln x$ for all $x > M$.
   c. What would you predict is the largest value of $a$ for which $x > a \ln x$ for all $x > 0$? (Note: This is equivalent to predicting the largest value of $a$ for which $e^x > x^a$ for all $x > 0$.)

Figure 18: Problem 4 of the Calculus Representations Test
Table 15: Comparing Students on Prompting Needed to Obtain Acceptable Use of Representations on Parts a and c of Problem 4

<table>
<thead>
<tr>
<th>Rep</th>
<th>Amount of Prompting Coding for Different Representations</th>
<th>% Providing Acceptable Use of Given Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Show that $x &gt; 2 \ln x$ for all $x &gt; 0$</td>
<td>Predict $a$ such that $x &gt; a \ln x$ for all $x &gt; 0$</td>
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<td></td>
<td>Correct/Some</td>
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</tr>
<tr>
<td></td>
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<tr>
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<td>Correct/Much</td>
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<tr>
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Table 16: Comparing Students on Prompting Needed to Obtain Acceptable Use of Representations on Part b of Problem 4

<table>
<thead>
<tr>
<th>Repr.</th>
<th>Amount of Prompting Coding for Different Representations</th>
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</thead>
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<td>Estimate $M$ such that $x &gt; 3 \ln x$ for all $x &gt; M$</td>
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</table>
Use of Symbolic Representations

In working with symbolic representations, Calculus & *Mathematica* students did slightly better than either graphics calculator students or traditional students, though they all had trouble using symbolic representations on the different parts of Problem 4. The main difficulty was that the symbolic method for proving or disproving the inequalities in these problems was not readily apparent to students. For example, on part a, students were first expected to recognize that the inequality \( x > 2 \ln x \) could be rewritten as \( x - 2 \ln x > 0 \). Next, they were expected to recognize that the latter inequality could be proved true by using the first derivative to determine the minimum value of the function \( y = x - 2 \ln x \) and then showing this minimum value was greater than zero. Students were suppose to use similar versions of the technique on parts b and c. Virtually all students needed considerable prompting about using this technique, particular on part a when they were first introduced to the technique. Most recognized the inequalities could be proved or disproved by showing that the minimum values of the corresponding functions were greater than zero or less than zero, respectively, but had difficulty recognizing how to use the information provided by the graphs and the derivatives of the functions to solve the problems. Most students’ ability to use symbolic representations to prove or disprove inequalities improved after discussing how to use them on part a.
Some of the difficulties students had using symbolic representations may be attributable to lack of experience working with logarithmic functions and, in the case of part c, working with equations or inequalities of more than one variable. Many traditional and graphics calculator students mentioned they had not worked with natural logarithms in their courses, though some mentioned studying them in a previous course. In an attempt to make things equal for all students, the interviewer computed derivatives and calculated values involving natural logarithms for students who indicated they could not do so themselves.

Use of Graphical Representations

In working with graphical representations, Calculus & Mathematica students did better than graphics calculator students who did slightly better than traditional students. Some traditional students and one graphics calculator student had difficulty applying the information provided by different graphs to prove, or disprove, inequalities. For example, many students did not recognize that the inequality \( x > 2 \ln x \) had to be true if the graph of the function \( y = x - 2 \ln x \) was always above the \( x \)-axis. The graphs of the functions used with the Calculus & Mathematica students are shown in the following figures: graphs of \( y = x \) and \( y = 2 \ln x \) are shown in Figure 19, graphs of \( y = x \) and \( y = 3 \ln x \) are shown in Figure 20, the graph of \( y = x - 2 \ln x \) is shown in Figure 21 and the graph of \( y = x - 3 \ln x \) is shown in Figure 22.
Figure 19: Graph of $y = x$ and $y = 2 \ln x$ shown to students during interviews.

Figure 20: Graph of $y = x$ and $y = 3 \ln x$ shown to students during interviews.
Figure 21: Graph of $y = x - 2 \ln x$ shown to students during interviews

Figure 22: Graph of $y = x - 3 \ln x$ shown to students during interviews
Similar graphs done on a TI-81 graphics calculator were used with the traditional and graphics calculator students.

**Use of Numerical Representations**

Few students made use of numerical representations, except on part a when all the students suggested solving the problem numerically. Students apparently recognized showing an inequality true, or disproving an inequality, by substituting various values into that inequality would be laborious and might not work since it was possible to miss values where the inequality was false.

**Use of "Combination" Representations**

Graphics calculator students used "combination" representations much more often than Calculus & Mathematica students who used them more often than traditional students. Graphics calculator students made extensive use of graphical/numerical representations. These representation almost always involved using the TRACE feature on a graphics calculator to trace along graphs either to determine $x$-coordinates of points of intersection or to check whether $x$-coordinates of the points on the graph were always greater than their corresponding $y$-coordinates. Calculus & Mathematica students used only one form of "combination" representations, a graphical/symbolic representation involving what was referred to in their course as the Race Track Principle. In order to prove inequalities using this principle, students needed to compute
derivatives and compare growth rates of functions, which they did using graphs of the derivatives. Only two traditional students mentioned "combination" representations. One suggested using the TRACE feature on the graph of $y = x - 3 \ln x$ (graphical/symbolic representation) to determine the value of $M$ such that $x > 3 \ln x$ for all $x > M$. The other suggested using the first derivative of $y = x - a \ln x$ for different values of $a$ (symbolic/numerical representation) to predict the largest value of $a$ for which $x > a \ln x$.

Traditional Student Interviews

As you read the interview excerpts that follow, look for evidence relating to these behaviors in the descriptions given by the traditional students:

- recall of what information about a function and its graph was provided by its first derivative
- use of "combination" representations
- recognition of how to use representations in ways different from those typically taught in the course
- recognition of connections between one representation and a different representation of an alternate form of the first representation
- relating of unfamiliar problem situations or descriptions to familiar uses of representations

**Part a.** Traditional students had the most difficulty of all the students using graphical representations to solve this problem. Most had difficulty applying the information provided by different graphs to prove that $x > 2 \ln x$ for all $x > 0$. In particularly, students did not seem to make the connection
between the graph of the function \( y = x - 2 \ln x \) always being above the \( x \)-axis and the inequality \( x > 2 \ln x \) having to be true. Their difficulties may have been the result of lack of experience with the natural log function and its graph, or of not understanding how the graph of \( y = x - 2 \ln x \) was related to the inequality \( x > 2 \ln x \).

Traditional students also had difficulty working with symbolic representations on part \( a \). This was not unexpected considering that the technique of showing the inequality \( x > 2 \ln x \) is true by verifying the minimum value of the corresponding function \( y = x - 2 \ln x \) is greater than zero was not an easily recognizable method for solving this problem. Each student mentioned using numerical representations to solve this problem. Most also recognized that verifying an inequality for numerous values of the variable did not constitute a proof of the inequality. No traditional student used any form of "combination" representation to prove the inequality \( x > 2 \ln x \).

Use of graphical representations. Only one of four traditional students interviewed described correctly, without prompting, how to show graphically that \( x > 2 \ln x \) for all \( x > 0 \). Subject T9 confirmed that the graph of \( y = x \) was always above the graph of \( y = 2 \ln x \) using a graphics calculator. Subject T10 tried to prove the inequality using the graph of \( y = 2 \ln x - x \), but had difficulties interpreting this graph.
Subject T10:

Oh. I know what I was trying to do. I was trying to see if it was above the x-axis and it wasn’t, so I didn’t know what to do after that. *What if it’s not above the x-axis? Does that mean that the statement’s true?*

I think it’s false, but -

*Okay. Well, what did you write here?*

I thought it had to be true. I didn’t know.

Subject T10 appeared to use a technique for solving inequalities where the inequality is rewritten in the form (algebraic expression) > 0 and then the algebraic expression is graphed to determine where it crosses the x-axis. The subject did not realize that since the inequality $x > 2 \ln x$ had been rewritten as $2 \ln x - x < 0$, and not $x - 2 \ln x > 0$, the graph of the function $y = 2 \ln x - x$ would be below the x-axis when the inequality was true. This is likely a case where the subject was trying to apply a previously learned technique without completely understanding the technique.

Subjects T11 and T12 both interpreted correctly that since the graph of $y = x - 2 \ln x$ was above the x-axis, the inequality $x > 2 \ln x$ was true, though Subject T11 was not at first sure a graph could help solve the problem.

Subject T11:

*Might a picture help me?*

I’m not really sure.

*What if I draw a picture of this ... $x - 2 \ln x$. Does that picture help me at all?*

I can’t really tell what’s going on. That goes up so I don’t think - It doesn’t look like $x$ is going to get any smaller than 0.
Subject T12 seemed a bit unsure how the information from the graph of 
\( y = x - 2 \ln x \) could be related to the inequality \( x > 2 \ln x \), but after prompting 
was able to interpret the graph.

*Use of numerical representations.* All the students suggested showing the 
inequality was true numerically by substituting several values of \( x \) into the 
inequality to verify that \( x \) was always greater than \( 2 \ln x \) for all \( x > 0 \). The 
one student who used this method initially, Subject T11, only verified the 
inequality for one value of \( x \). This subject did not seem to realize that showing 
the inequality true for one value of the variable did not mean that it was true 
for all values. After discussing the example of the inequality \( x^2 > x \), the 
subject recognized it was necessary to verify the inequality for more than one 
value of the variable. All the subjects recognized that using numerical 
representations to show the inequality was true might not always work since it 
was possible to miss values of the variable where the inequality was not true.

*Use of symbolic representations.* In discussing how to prove the 
inequality symbolically, one traditional student, Subject T9, appeared to 
recognize how the symbolic technique of showing the minimum value of 
\( y = x - 2 \ln x \) was greater than zero proved the inequality \( x > 2 \ln x \) was true. 
In the following excerpt, the subject had just been asked what might be done 
with the derivative of the function \( y = x - 2 \ln x \).
Subject T9:

Set it equal to zero again.... Set the function equal to zero.
*All right. So we'll set it equal to zero.... So x is equal to 2.*
Then plug that back into the original function. But the first derivative is positive which should mean it's increasing.
*What did you say? Plug this back into the original?*
Yeah.
*Okay. So I plug it [2] back into that.... It's a positive number.... What does that give me?*
It's the relationship between x and the 2 ln x values.
*When you set this equal to zero, what were you looking for?... What does that give you?*
Gives you the slope of the function?
*Tell's me the slope of the function is zero there which tells me what's occurring at that point?*
It's a flat line.... Maximums [or] minimums.
*Here's a minimum when x is 2 and at that point, x is 2 and y is .6137. Is that enough to show that it's true. Does that supply me with enough information.*
Yes, because here the zero is the y ... and if it's .6137 and it's 2, then it's greater.

Subject T9 recognized that since the minimum value of the function
\[ y = x - 2 \ln x \]
was 0.6137, then the function was always greater than 0.6137, which meant \( x - 2 \ln x \) was always greater than zero or \( x \) was always greater than 2 ln x.

Subjects T10 and T11 seemed to understand how the results of the symbolic technique could be used to prove the inequality. With Subject T10, the function being discussed is \( y = 2 \ln x - x \) and not \( y = x - 2 \ln x \).
Subject T10:

So I’ve found the maximum. And then as you said before, I can take value and plug it back into my original.... It’s -.6137.... Does that give me any information toward proving this?
It tells you it’s less than zero.
Okay. And that’s the maximum value.... Does that tell me anything about the function at all?
It says the statement’s true.
Why do you say that?
I don’t know. Well, when - when you bring this over, it just show that this maximum value can’t be any higher or greater than that - is less than zero, so -

Subject T11:

At its minimum, y is positive.... If that’s the minimum value of the function, is it possible for the function to ever go less than zero?
No it isn’t
Is this enough to prove that statement is true, \( x - 2 \ln x > 0? \)
If you figure out its minimum, its lowest point - its lowest point is a positive number, then, I mean, its highest point not going to be a negative number.

It appeared that Subject T10 recognized that since the maximum value of \( y = 2 \ln x - x \) was less than zero, that meant \( 2 \ln x - x \) was always less than zero, or \( x \) was always greater than \( 2 \ln x \). Subject T11’s comment suggests the subject understood that since the lowest value of \( y = x - 2 \ln x \) was greater than zero, all the value of the function would be greater than zero. Unlike Subject T9, neither of these students recognized to how to execute this symbolic technique during the discussion of the procedure.
Subject T12 had the most difficulty of the traditional students interviewed working with symbolic representations. To begin with, the subject suggested taking the derivative of each side of the inequality $x > 2 \ln x$.

Subject T12:

Take the derivative?

*Take the derivative of what?*

Um - everything. Like $2 \ln x$. Um - it be 0 is greater than 2 natural log. Zero? Why zero?

It wouldn’t work out.

Subject T12 did not realize that the inequality had to be rewritten in the form (algebraic expression) $> 0$ in order to take a derivative or that it was not possible to take the derivative of each side of an inequality. It was not unexpected that the subject might think the derivative of $2 \ln x$ was $2 \ln$ considering that the subject had not previously studied the derivative of the logarithmic function.

After prompting, Subject T12 rewrote the inequality $x > 2 \ln x$ in the form $x - 2 \ln x > 0$ to get the function $y = x - 2 \ln x$. The subject was then asked what might be done with the derivative of the function in order to prove the inequality.

Subject T12:

Set it equal to 0. Get the slope.

*So, x equals 2. What does that give me?*

Slope.

*Slope of what?*
Of the graph, the function $x - 2 \ln x$.

I get - you mean the slope's always 2?

[long pause by subject]

When I set a first derivative equal to zero, what am I ... finding?

The max slope

The max slope?

I mean - ah - the maximum value of the graph.

So what do I do with that 2?

Plug it into the original equation.

That gives me about .614.... Does that help me?

Isn't it showing that there's none less than zero? If it's a minimum value?... See I don't really know what you're asking here?

Subject T12 was unable to explain why the inequality $x > 2 \ln x$ must be true because the minimum value of the function $y = x - 2 \ln x$ was greater than zero. The difficulty appeared to be that the subject did not understand how the information provided by the graph and the derivative of the function $y = x - 2 \ln x$ could be used to prove the inequality $x > 2 \ln x$. This may explain why the subject had difficulties going through the process of determining the minima of the function since the subject did really understand the purpose of the procedure. It should be noted that for Subject T12, the symbolic method of proving the inequality was discussed before the graphical method, which might explain why the subject had difficulty recognizing that the inequality $x > 2 \ln x$ was true since the minimum value of the graph of $y = x - 2 \ln x$ was greater than zero.
Part b. Traditional students did somewhat better describing how to use
g graphical and symbolic representations to show that $x \neq 3 \ln x$ for all $x > 0$.
This improvement is likely the result of having discussed graphical and
symbolic techniques for proving that $x > 2 \ln x$ for all $x > 0$ for part a. One
student mentioned using numerical representations to solve this problem,
although this student did initially believe the inequality $x > 3 \ln x$ was true for
all $x > 0$. As with part a, no student mentioned using any form of
"combination" representation to show that $x \neq 3 \ln x$ for all $x > 0$.

Use of graphical representations. All four students described correctly
how to solve this problem graphically. Subject T9 demonstrated that $x \neq 3 \ln x$
for all $x > 0$ by showing that the graph of $y = x$ crossed the graph of $y = 3 \ln
x$. The following excerpt suggests the subject understood how this result would
show up on the graph of the function $y = x - 3 \ln x$.

Subject T9:

Looking at the graphs, there’s that space where it [graph of $y = x$] goes
over [graph of $y = x - 3 \ln x$]. And when you combine the two graphs,
that’s space is going to be negative, I guess.

Subjects T11 and T12 recognized the inequality $x > 3 \ln x$ was not true for all
$x > 0$ since the graph of $y = x - 3 \ln x$ went below the $x$-axis. Subject T10 first
attempting to use the graph of $y = \ln x - \frac{1}{3}x - 1$, which the subject had derived
from the inequality $3 \ln x - x > 3$, to prove or disprove that $x > 3 \ln x$ for all
$x > 0$. It was unclear from the discussion with the student why the new inequality had been derived. After identifying the correct function, Subject T10 was then able to explain correctly why the inequality $x > 3 \ln x$ was not true for all $x > 0$ since the graph of $y = 3 \ln x - x$ went above the $x$-axis.

*Use of numerical representations.* Only one student, Subject T11, used numerical representations to determine whether the inequality $x > 3 \ln x$ was true for all $x > 0$. Initially, Subject T11 thought the inequality was true since the subject had only checked the inequality for one value of $x$, as the subject had done on part $a$, and it had been true for that value. After discussing part $a$, the subject recognized that showing the inequality true for one value of the variable was not enough to show it true for all values. After testing a few more values of $x$, Subject T11 determined that $x \neq 3 \ln x$ for all $x > 0$. No other traditional student mentioned solving this problem numerically, and the interviewer did not ask students to discuss numerical solution methods in order to save time for discussion of graphical and symbolic methods.

*Use of symbolic representations.* Many traditional students appeared to understand how to adapt the symbolic technique used for part $a$ in order to solve this problem. Subjects T9 and T12 both seemed to recognize that since the minimum value of $y = x - 3 \ln x$ was not greater than zero, the inequality $x > 3 \ln x$ was not true for all $x > 0$. 

Subject T9:

Since this is less than zero, then this function is not always going to be greater than zero which means that one of these is false.... Since this is the negative number, this $[3 \ln x]$ would have to be greater than that $[x]$.

Subject T12:

*I get negative, in this case. With the three, it’s -.296.*

So that would be false.

*Why?*

Cause it’s less than 0.

Okay. And we want it to be -

Greater than 0.

Both subjects appeared to recognize that for the inequality $x - 3 \ln x > 0$ to be true, the minimum value of the function $y = x - 3 \ln x$ had to be greater than zero, and since the minimum value was -0.296, $x - 3 \ln x$ was not always greater than zero, which meant $x \neq 3 \ln x$ for all $x > 0$.

Subject T10 applied the symbolic technique to the function $y = 3 \ln x - x$ and not $y = x - 3 \ln x$.

Subject T10:

Find the max with the derivative set equal to zero.

*And then what? Say I found the max.*

The max would be found greater than zero... so then the statement would be false.

This subject recognized that since the maximum value of $y = 3 \ln x - x$ was greater than zero, $3 \ln x - x$ was not always less than zero, which meant that $x$ was not always greater than $3 \ln x$. 
Subject T11 seemed to have more difficulty than the other traditional students comprehending why obtaining a negative number for the minimum value of the function \( y = x - 3 \ln x \) meant that \( x \neq 3 \ln x \).

Subject T11:

I got a negative number there for y.... In this one it was positive. And now over here it's negative. Does that matter? Is it still going to be true?

I mean - I can't say. If you plug it in and everything, it'd just be a different equation cause it's not 2 \( \ln x \), it's 3 \( \ln x \).

Does the fact that when I checked the minimum and it came up with a negative number instead of a positive affect whether this is true or not. Guess not cause it's a negative number.

Subject T11's first comment suggests the subject did not understand how comparing minimum values of the functions \( y = x - 2 \ln x \) and \( y = x - 3 \ln x \) could provide a clue as to whether or not \( x > 3 \ln x \) for all \( x > 0 \). After further discussion, Subject T11 seemed to understand why obtaining a negative number for the minimum value of \( y = x - 3 \ln x \) meant that \( x \neq 3 \ln x \) for all \( x > 0 \).

Determining the value of \( M \). Traditional students mentioned a number of different ways for determining the value of \( M \) such that \( x > 3 \ln x \) for all \( x > M \). Three students determined where the graph of \( y = x - 3 \ln x \) crossed the \( x \)-axis. One of them initially suggested solving the equation \( x = 3 \ln x \) analytically before recognizing that was not possible. The fourth student graphed \( y = x - 3 \ln x \) on a graphics calculator and then used the TRACE feature to determine the largest \( x \)-intercept (graphical/numerical representation).
This was the only use of a "combination" representation by traditional students on this problem.

**Part c.** Traditional students had more difficulty with this problem than any other. Two of four students interviewed, and ten of thirteen students assigned the problem on the Calculus Representations Test, did not give an initial response to this question. One possible reason for this was that students did not know what to do about the second variable, $a$, particularly when using symbolic representations. Students did not realize the variable $a$ could be treated as a constant when taking a derivative with respect to $x$, which was not unexpected, considering the students’ lack of experience taking derivatives of function of more than one variable. Both traditional students who attempted to solve this problem symbolically seemed perplexed when informed the minimum value of the function $y = x - a \ln x$ occurred when $x$ was equal to $a$.

**Use of graphical representations.** All four students described correct graphical techniques for predicting the largest value of $a$ for which $x > a \ln x$ for all $x > 0$. Three students mentioned graphing the function $y = x - a \ln x$ for different values of $a$ and then determining the largest value of $a$ for which the graph did not cross the $x$-axis. One student, Subject T11, tried to estimate the value of $a$ such that the graph of $y = x - a \ln x$ was tangent to the $x$-axis by looking at the difference of the minimum values of the graphs of $y = x - 2 \ln x$
and \( y = x - 3 \ln x \) to. The fourth student, Subject T9, described basically the same technique except the subject graphed \( y = x \) and \( y = a \ln x \), rather than the function \( y = x - a \ln x \). Subject T9 was the only traditional student to surmise, without prompting, that the value of \( a \) was between 2 and 3.

*Use of numerical representations.* One traditional student, Subject T11, suggested using numerical representations to predict the largest value of \( a \) for which \( x > a \ln x \) for all \( x > 0 \). Subject T11 mentioned choosing values of \( a \), substituting values of \( x \) into the inequality \( x > a \ln x \) for each value of \( a \) in order to verify the inequality true, and then selecting the largest value of \( a \) from those for which the inequality was true. The subject did recognized the difficulty of numerically verifying the inequality \( x > a \ln x \) for each value of \( a \).

*Use of symbolic representations.* Only two students attempted to solve this problem symbolically. Neither succeeded. Subject T10 suggested looking at the maximum value of the function \( y = a \ln x - x \), but, as the following indicates, the subject did not know how to use this to predict the value of \( a \) such that \( x > a \ln x \) for all \( x > 0 \).

Subject T10:

Could you find the maximum \( x \) value and plug it into the original equation and find your \( y \) value?

*So \( x \) is equal to \( a \) [when the first derivative is equal to 0]*... So I’ve got a \( \ln a \) - \( a \) and that’s the \( y \) value of the maximum. Does that help me any?

No.
Subject T10 did not recognize that the maximum value of \( y = a \ln x - x \) had to be less than zero for the inequality to be true since if \( a \ln a - a \) was always less than zero, then \( a \ln a \) would always be less than \( a \), which meant \( \ln a \) had to be less than 1, or \( a \) had to be less than \( e \). Subject T12 had similar difficulty recognizing how to continue the solution once it was determined the minimum value of the function \( y = x - a \ln x \) was \( a - a \ln a \). Considering the traditional students' lack of experience working with exponential and logarithmic functions and with functions of more than one variable, it was not unexpected that no traditional student described how to use this symbolic technique to predict the largest value of \( a \) for which \( x > a \ln x \) for all \( x > 0 \).

**Use of "combination" representations.** Subject T9 suggested trying to predict the largest value of \( a \) for which \( x > a \ln x \) for all \( x > 0 \) by first determining the minimum value of the function \( y = x - a \ln x \) for different values of \( a \) and then determining the largest value of \( a \) for which the minimum value was greater than zero. This method was classified as use of a symbolic/numerical representation and was the only use of a "combination" representation by traditional students on part c.

**Graphics Calculator Student Interviews**

As you read the interview excerpts that follow, look for evidence relating to these behaviors in the descriptions given by graphics calculator students:
• recall of what information about a function and its graph was provided by its first derivative
• use of "combination" representations
• recognition of how to use representations in ways different from those typically taught in the course
• recognition of connections between one representation and a different representation of an alternate form of the first representation
• relating of unfamiliar problem situations or descriptions to familiar uses of representations

Part a. Graphics calculator students had little difficulty using graphical or numerical representations to solve this problem. Most seemed to recognize how to apply the information provided by different graphs to prove that $x > 2$ \ln x for all $x > 0$, though in one instance, it was unclear whether or not a student actually understood the graphical technique used to solve this problem. Most also recognized that verifying an inequality for numerous values of the variable did not constitute a proof of the inequality.

Graphics calculator students had about as much difficulty as traditional students working with symbolic representations on part a. As with traditional students, this was not unexpected considering that the technique of showing the inequality $x > 2 \ln x$ is true by verifying the minimum value of the corresponding function $y = x - 2 \ln x$ is greater than zero was not an easily recognizable method for solving this problem.
All the graphics calculator students used a graphical/numerical representation on part a. Many used the TRACE feature of a graphics calculator to trace along the graph of \( y = x - 2 \ln x \) and verify that \( x \)-coordinates were always greater than their corresponding \( y \)-coordinates.

*Use of graphical representations.* All the graphics calculator students interviewed described correctly how to show graphically that \( x > 2 \ln x \) for all \( x > 0 \). Subject G9 confirmed that the graph of \( y = e^x \) was always above the graph of \( y = x^2 \). This subject recognized that if \( e^x > x^2 \) for all \( x > 0 \), then \( x > 2 \ln x \) for all \( x > 0 \). Subject G11 confirmed that the graph of \( y = x \) was always above the graph of \( y = 2 \ln x \). Subject G10 first attempted to prove the inequality using only the graph of \( y = 2 \ln x \) before recognizing that the proof would be simpler if the function \( y = x \) was also graphed.

Subject G10:

*Since I've got the graph of 2 ln x in there, is there any other graph I might be able to add to it that would help me.*

Ohhh, let's see here.... If you graphed \( x \) ... yeah - that'll give you all your \( x \)-values on a graph and you can compare the two. And that right there would be your proof, I mean not a proof but visual, you know, proof. That way you could look at it and very easily tell that your \( x \)-values are - you know, of the function \( x \) are always greater than the \( y \)-values.

Subject G12 also attempted to prove the inequality using only the graph of the function \( y = 2 \ln x \). This subject seemed to think that the shape of the graph indicated that \( x \) would always be greater than \( 2 \ln x \).
Subject G12:

Just looking at it, it seems that the - uh - y-value isn't increasing enough for it to ever gain over the x.

Even with prompting, Subject G12 never mentioned including the graph of 
\[ y = x \] along with the graph of \[ y = x - 2 \ln x \] in order to verify that \( x \) was always greater than \( 2 \ln x \).

No graphics calculator student suggested using the graph of \( y = x - 2 \ln x \) to prove that the inequality \( x > 2 \ln x \) was true for all \( x > 0 \). After the interviewer suggested using this graph, students recognized the inequality would be true if the graph of \( y = x - 2 \ln x \) was above the x-axis, though one student, Subject G12, needed prompting. This subject seemed unsure that the graph constituted proof that the inequality was true.

Subject G12:

That's not really definitive either. 
*Is that proof?*

That's proof that it's \[ x - 2 \ln x \] greater than zero. Because it's - it never actually touches zero and it's never going to the way the graph looks.

It was unclear whether or not Subject G12 recognized that since \( x - 2 \ln x \) was always above the x-axis, the graph of \( y = x - 2 \ln x \) was always greater than 0, which meant that \( x \) would always have to be greater than \( 2 \ln x \). Later comments suggested that the subject did not understand this relationship.
Use of "combination" representations. All graphics calculator students suggested proving the inequality was true by graphing the function \( y = 2 \ln x \) and then comparing the \( x \)- and \( y \)-coordinates of this graph. Students recognized if \( x \)-coordinates were always greater than their corresponding \( y \)-coordinates, then \( x \) would always be greater than \( 2 \ln x \).

Subject G10:

So we're saying that at any \( x \)-value - uh - that \( x \)-value is greater than this function, the \( y \)-value of that function. Well, you could - the easiest way to do it probably'd be just to look at it and say - and look here on the \( x \)-axis and say, well at that point you go up to your function and you know that \( x \)-value of your function, and map that across to the \( y \) and find out.

Subject G10 suggested estimating the \( x \)- and \( y \)-coordinates of different points from the graph of the function. The other students suggested using the TRACE feature of the graphics calculator to trace along the graph of \( y = 2 \ln x \) and verify that the \( x \)-coordinates were always greater than their corresponding \( y \)-coordinates. Utilizing this technique to show the inequality \( x > 2 \ln x \) was true for all \( x > 0 \) was classified as use of a graphical/numerical representation. This was the only use of a "combination" representation by graphics calculator students on part \( a \).

Use of numerical representations. All four students suggested showing the inequality was true numerically by substituting several values of \( x \) into the inequality to verify that \( x \) was always greater than \( 2 \ln x \) for all \( x > 0 \).
Students also recognized that this method for showing the inequality was true did not always work since it was possible to miss values of x where the inequality was false. It should be noted that no graphics calculator student seemed to recognize that the numerical technique was equivalent to the graphical/numerical technique of comparing the x- and y- coordinates of the graph of \( y = 2 \ln x \).

*Use of symbolic representations.* Three of four students recognized, after prompting, how the symbolic technique of showing the minimum value of the function \( y = x - 2 \ln x \) was greater than zero proved the inequality \( x > 2 \ln x \) was true. In the following excerpts, Subjects G9 and G10 had just been asked what might be done with the derivative of the function \( y = x - 2 \ln x \).

Subject G10:

Well, if you took the first derivative and you set it equal to zero, you’d get the - uh - maximums and minimums... And you get zero and two [for \( x \)]. Now, then you could take the second derivative, if you wanted to, and you could set it equal to zero and you find out which was the maximum and which was the minimum.

*Well actually, the only number’s two. Two’s the only value.*

So the only number’s two. So then, two - if we already have the first graph, we can just look at it and, and you know, say well, two’s a minimum. Therefore, we know that that’s the lowest point, you know, over that interval, you know, greater than zero. Um - so that’d be the lowest point on that curve. So, of course it would never - uh - and, you know, plug in two and you’d get the y-value and of course, it’s greater than zero according to the graph.
Subject G9:

That would give you one messed up derivative. You get one ... minus two over x.... You set the derivative equal to 0. That would give you one plus 2 over x which equals 2. It’s still positive.

*But that - what does this tell me? x equals 2 is what?*

That’s where you slope is zero.

*Can I use that somehow?*

It’ll prove that - up here, that’ll prove this.

*Why do you say that?*

Because it’s still greater than - what would happen if you stuck this number in up here? That would give you that 2 was greater than 2 ln 2.

*I would put it in here since that’s the function. That would be ... .614. Does that help me at all.*

Hmm - did something. I don’t know what it’s helping so far. Still greater than 0. I would say yeah, that would prove it right there.

*Why?*

Because all your - if that number would be greater than 0, that’d mean your slopes would be - they wouldn’t be greater than 0. But everything would be - is greater than 0. Even this. When you put - substitute this stuff back in. That’s where your slope is 0. When you substitute that back in to here and it’s still positive. That’ll prove that. Gosh.

Both appeared to understand that since the minimum value of the function

$$y = x - 2 \ln x$$

was greater than 0, then $$x - 2 \ln x$$ was always greater than zero, which meant that $$x$$ was always greater than 2 ln $$x$$, though both had difficulty verbalizing this notion. It should be noted that Subject G9 initially suggested using the function $$y = e^x - x^2$$, and was describing how to prove the inequality using that function when both the subject and interviewer realized that the symbolic technique could not be used with that function since its graph had no local minima.
An unexpected result from the discussions with Subjects G9 and G10 on the use of symbolic representations was that both seemed to think an inequality would still hold true after taking the derivative of each side of the inequality.

Subject G9:

If you took the derivatives of these two here, right here, after you got them both equal, $x^2$ and $e^x$, you’d get $2x$ and $e^x$. You’d get $e^x$ is greater than $2x$.

Subject G10:

I was going to say take the derivative of both sides. Then you get 1 is greater than ... 2 over $x$, which, you know, you could compare those.

After discussion, both students recognized that an inequality would not necessarily remain true after taking the derivative of each side. The interviewer did not discuss with the students where they might have previously encountered this manipulation.

Subject G11 recognized how to use the symbolic technique to prove the inequality, but, unlike the others, also recognized the possibility that the graph could begin decreasing and cross the $x$-axis for some large value of $x$.

Subject G11:

You could take the first derivative of that. Set that equal to zero.... There’s some sort of value at $x$ equals 2. Uhh - by testing on either side of it, you could - you could enter in the values of say, 1.5, that would give you a negative slope and 2.5 would give you a positive slope.... So that’s a minimum. So, since that’s the minimum value for the graph, it has to be - this statement $[x - 2 \ln x > 0]$ has to be true, which makes that statement $[x > 2 \ln x]$ true.
Why does this necessarily have to be true just because that’s a minimum value. I need a little more than that.

Need more? Because that’s the minimum value. That’s the minimum - that’s the minimum value for the entire function though. So it doesn’t - that function doesn’t get any lower than 2.

Well, no. That’s the x-value though.

There’s no - Oh, yeah. Two. Okay. So, y is 2 minus 2 ln 2. So it doesn’t get any lower than .6. Okay. So it doesn’t get - so that’s the actual minimum y-value. And it doesn’t - it doesn’t get any lower than that.... That’d be the minimum for the whole graph.... One problem might be that way out, beyond the scope of the graph, it might go down again.

But then if that was true, what would you have to have out there?

A maximum, also.

Okay. But how many critical points did you have?

Only one.

So can it have a maximum out there?

No. It can’t.

Subject G11 recognized that since the minimum value of the function

\[ y = x - 2 \ln x \]

was greater than 0, the inequality \( x - 2 \ln x > 0 \) had to be true, and, thus, the inequality \( x > 2 \ln x \) was true. The excerpt indicates this subject recognized, with prompting, that since the function had only one critical point, it could not have a local maxima along with the existing local minima, and, thus, the graph could not decrease and cross the x-axis for a large value of x.

The fourth student, Subject G12, never recognized how the symbolic technique of determining the minimum value of the function \( y = x - 2 \ln x \) could be used to prove the inequality \( x > 2 \ln x \) was true. This subject never seemed to recognize why the inequality \( x > 2 \ln x \) had to be true if the minimum value of the function \( y = x - 2 \ln x \) was greater than zero. The
difficulty appeared to be that Subject G12 did not understand how the
information provided by the graph of \( y = x - 2 \ln x \) could be used to show that
the inequality \( x > 2 \ln x \) was true.

Part b. Most graphics calculator students had little difficulty using
graphical representations to show that \( x > 3 \ln x \) was not true for all \( x > 0 \). All
but one student seemed to recognize how to apply the information provided by
different graphs to or disprove the inequality. This one student recognized how
to disprove the inequality \( x > 3 \ln x \) by showing that for some points on the
graph of \( y = 3 \ln x \), the \( x \)-coordinates were less than their corresponding
\( y \)-coordinates, but was never able to describe how to use only graphical
representations to solve the problem.

Graphics calculator students did not show as much improvement as
traditional students using symbolic representations to solve part b after
discussing symbolic techniques for solving part a. Only two students were able
to describe correctly how to show symbolically that \( x \neq 3 \ln x \) for all \( x > 0 \).
Finally, only one student mentioned using either numerical representations or
"combination" representations to solve this problem.

Use of numerical and "combination" representations. Only one graphics
calculator student mentioned using either numerical or "combination"
representations to show that \( x > 3 \ln x \) was not true for all \( x > 0 \). Subject G11
suggested solving the problem numerically by substituting different values of x into the inequality until a value was found such that x was less than 3 ln x.

Subject G12 suggested using a graphical/numerical representation involving the TRACE feature of the graphics calculator to demonstrate that x ≠ 3 ln x for all x > 0. This subject traced along the graph of y = 3 ln x in order to locate x-coordinates that were not greater than their corresponding y-coordinates.

*Use of graphical representations.* Three of four students described correctly how to show graphically that x ≠ 3 ln x for all x > 0. Subject G9 determined that the graph of y = e^x crossed the graph of y = x^3. This subject recognized that if e^x ≠ x^3 for all x > 0, then x ≠ 3 ln x for all x > 0. Subject G11 established that the graph of y = x crossed the graph of y = 3 ln x. As the following suggests, Subject G10 recognized that the inequality x > 3 ln x was not true for all x > 0 since the graph y = x - 3 ln x went below the x-axis.

Subject G10:

Since we subtracted the 3 ln x, we want that to be greater than zero - uh - for all x greater than zero. And if you stare at your graph [of y = x - 3 ln x], of course it crosses the x-axis at two - you know, appears like at least two points, you know, according to the viewing window. So we know that it is not true that for all x greater than zero.

During the interview, Subject G10 kept referring to the work done with graphical representations during the discussion of part a while describing how to solve part b graphically.
Subject G12 was unable to describe correctly how to use graphical representations, and not the aforementioned graphical/numerical representation, to prove that $x \not< 3 \ln x$ for all $x > 0$. The subject never recognized that the function $y = x$ could be graphed along with the function $y = x - 3 \ln x$ in order to show that there existed values of $x$ such $3 \ln x$ was greater than $x$. Nor did Subject G12 recognize this problem could be solved by showing the graph of $y = x - 3 \ln x$ was not always above the $x$-axis. Subject G12’s difficulties suggest a misunderstanding on part $a$ of why the inequality $x > 2 \ln x$ had to be true if the graph of the function $y = x - 2 \ln x$ was always above the $x$-axis.

*Use of symbolic representations.* Subjects G10 and G11 did better describing how to use the symbolic technique utilized in part $a$ to show that $x \not< 3 \ln x$ for all $x > 0$. As with the traditional students, this improvement probably resulted from discussing this technique to show symbolically that $x > 2 \ln x$ for all $x > 0$. Both subjects recognized showing the minimum value of the function $y = x - 3 \ln x$ was less than zero proved $x > 3 \ln x$ was not true for all $x > 0$.

Subject G10:

We can take the first derivative and set it equal to zero. And then we get the minimum point... It'd be a negative value. So we know that it's less than zero which would disprove ... that statement $[x > 3 \ln x]$. 
Subject G11:

Take the derivative.... Set that equal to zero. And then that'll give you 1 is equal to 3 over x. So x equals 3.... If that's the minimum, you'd have to plug 3 back into the - into the first equation, right here. And if that's greater than zero, then - or, no. 

Got less than zero.

Okay. So that's less than zero. So then that statement isn't true. So x is not always greater than 3 ln x.

Both subjects seemed to realize that for the inequality x - 3 ln x > 0 to be true, the minimum value of the function y = x - 3 ln x had to be greater than zero, but since this was not the case, then x ≠ 3 ln x for all x > 0.

Subject G9 attempted to solve this problem symbolically by equating the derivatives of e^x and x^3, which was equivalent to determining when the derivative of the function y = e^x - x^3 was equal to zero. The subject did not recognize that the symbolic technique could not be applied to this function since its graph did not have a local minimum. Subject G9 was unable to describe how to use the symbolic technique on the function y = x - 3 ln x to show that x ≠ 3 ln x for all x > 0.

The fourth student, Subject G12, never recognize this problem could be solved by showing the minimum value of the function y = x - 3 ln x was less than zero. This subject also made the same mistake that Subjects G9 and G10 made on part a concerning taking the derivative of each side of an inequality.
Determining the value of $M$. Graphics calculator students mentioned using two different forms of representations to determine the value of $M$ such that $x > 3 \ln x$ for all $x > M$. All four mentioned using the TRACE feature on a graphics calculator (graphical/numerical representation) to determine the value of $M$. Subjects G10 and G12 suggested using the TRACE feature on the graph of $y = x - 3 \ln x$ to determine the $x$-intercepts. Subject G11 suggested using the TRACE feature to determine the points of intersection of the graphs of $y = x$ and $y = 3 \ln x$. Subject G9 suggested using the TRACE feature to determine the points of intersection of the graphs of $y = e^x$ and $y = x^3$. Subject G10 also suggested determining the value of $M$ by locating where the graph of $y = x - 3 \ln x$ crossed the $x$-axis.

Part c. Graphics calculator students had more difficulty with this problem than any other. Two of four students interviewed, and five of eight students assigned the problem on the Calculus Representations Test, did not give an initial response to this question. As with traditional students, graphics calculator students had difficulty working with the second variable, $a$, particularly when trying to use symbolic representations, although they did have slightly less difficulty than traditional students using symbolic representation to try to predict the largest value of $a$ for which $x > a \ln x$ for all $x > 0$. Two students who attempted to solve this problem symbolically did
not complete their solutions because they were unable solve equations and inequalities involving both the variable $x$ and the variable $a$.

Another possible reason for students’ difficulties on this problem was that students did not know, or could not remember, how to work with natural logarithms, as the following excerpts suggests.

Subject G10:

See my problem is that I’m very weak as far as natural logs. I never had a teacher who stressed natural logs. So I’m not really sure - I can’t comprehend right now what natural log even is.... I’m not sure what, you know, what $e$ is. I don’t comprehend that cause I’ve never actually been taught that. I don’t understand it.

The one student, Subject G11, who recognized that the largest value of $a$ such that $x > a \ln x$ for all $x > 0$ could be determined by solving the inequality $a - a \ln a > 0$ did not appear to realize that if $\ln a > 1$, then $a > e^1$.

Use of graphical representations. All students except for Subject G12 used graphical representations to predict the largest value of $a$ for which $x > a \ln x$ for all $x > 0$. Two students mentioned graphing the function $y = x - a \ln x$ for different values of $a$ and then determining the largest value of $a$ for which the graph did not cross the $x$-axis. The third student, Subject G9, initially suggested graphing the functions $y = e^x$ and $y = x^a$ for different values of $a$, but then decided that $a$ might be equal to $e$. 
Subject G9:

Well, knowing from what I know now, I would probably say it’s e. So you’d graph x'[ and e'[... There’s x'. There’s e'[... I’d have to zoom-in a little on that one spot there... Hard to tell whether that crossed or not. [zoomed-in]

No. I don’t think it crossed.... If you kept on zooming, you’d probably get it to where they don’t - or where they’re almost exactly equal, but I don’t think you’d every find it crossed.

Use of numerical representations. One graphics calculator student,

Subject G11, suggested using numerical representations to predict the largest value of a for which x > a ln x for all x > 0. This subject mentioned choosing values for a and then evaluating the inequality x > a ln x for different values of x in order to estimate the largest value of a for which x is always greater than a ln x.

Use of symbolic representations. When discussing how to use symbolic representations to solve this problem, Subject G10 suggested looking at the minimum value of the function y = x - a ln x, but did not recognize how to use this information to predict the value of a such that x > a ln x for all x > 0.

Subject G10:

Take the derivative and set it equal to zero. ... What we want to know is where this x is greater than zero, I think. Yeah. No. No. I’m wrong.
Where the y-value of that x. Put that back in there.... I mean, I know I’m on the right track. it’s just a matter of reasoning this out cause I, these two variables threw me off.... [determined x = a when derivative was equal to zero] So x is equal to our a.
That's where the minimum occurs ... x is equal to a is where the minimum occurs. Now what needs to be true about that minimum for this statement to be true? What needs to be true about the minimum?

Okay. It needs - it needs to be greater than zero. Okay, x is equal to a.
That's where our minimum occurs. Okay. I'm starting to picture this now.
And we need that to be greater than zero in order to find out our answer.
So, where x equals a is greater than zero. I can't reason it out. I can't do it right now.

Subject G10 recognized that the minimum value of \( y = x - a \ln x \) had to be greater than zero for the inequality to be true, but did not recognize that \( a \) had to substituted for \( x \) to obtain the minimum value, \( a \ln a - a \), of the function, producing the inequality \( a \ln a > 0 \), which, when solved, showed that \( a \) had to be less than \( e \). The subject's inability to finish the problem may be attributable to confusion over what to do with the variable \( a \) when solving this problem.

Subject G11 recognized that the minimum value of the function \( y = x - a \ln x \) had to be greater than zero, but like Subject G10, never recognized how to use this information to predict the largest value of \( a \) such that \( x > a \ln x \) for all \( x > 0 \).

Subject G11:

We could just do what we've been doing, \( x - a \ln x \) ... is greater than zero. And then just take the derivative of this,... And set that equal to zero. So one equals a/x. So x, let's see. So x equals a.

All right. Then what do I do with that?... What did you do previously with that value?
With the x-value?
Yeah. When you, when you, when you found the minimum, you got a value for x and what did you do with it?
|Substituted it back in... Well, since $a$ equals $x$, your greatest value for $a$ is, can, has to be $x$.
|You don't know what $x$ is.
|It doesn't matter what $x$ is.
|Well then you mean anything would work? |Err, no. Wait a second. Uhh.
|When I substituted a in for $x$, and I got $a - a$ $\ln a$, what must be true about that for this statement to be true?
|To be greater than - it has to be greater than zero. So that means that - uh - so that means that $x$ must be greater than - So that means that for all $x$ -

After prompting, Subject G11 realized that if the minimum value of the function $y = x - a \ln x$ was greater than zero, then the inequality $a \ln a > 0$ had to be true. After additional prompting on solving logarithmic equations, the subject eventually determined that $a > e$. Subject G11 was the only graphics calculator student to describe correctly how to use symbolic representations to predict the largest value of $a$ for which $x > a \ln x$ for all $x > 0$, which was not unexpected, considering the graphics calculator students' lack of experience working with exponential and logarithmic functions and with functions of more than one variable.

use of "combination" representations. Subjects G12 and G11 were the only students to describe how to use "combination" representations to predict the largest value of $a$ for which $x > a \ln x$ for all $x > 0$. Subject G12 suggested using the TRACE feature of the graphics calculator to trace along the graph of $y = e \ln x$ and verify that the $x$-coordinates were always greater
than their corresponding \( y \)-coordinates (graphical/numerical representation). As the following suggests, Subject G12 chose \( e \) as the value of \( a \) based on prior knowledge of natural logarithms.

Subject G12:

"I knew that - um - \( e \) was the inverse of the natural log. And - um - they away I arrived at guessing it was I knew it was between 2 and 3 and I knew \( e \) was 2 point something or other, so I thought that - just the correlation helped me to guess this.

Subject G12 was the only other graphics calculator to surmise, without prompting, the relationship between the value of \( a \) and \( e \). However, as was the case with part \( b \), Subject G12 was unable to describe correctly how to use any other form of representation to solve this problem.

Subject G11 described how to solve this problem using a "combination" representation that involved the minimum value of the function \( y = x - a \ln x \).

Subject G11:

"If it's between \( 2 \) and \( 3 \), I'd say - I would say \( 2.5 - 2.5 \ln 2.5 \). And then if that was greater than zero -

\[ \text{You'd just keep plugging in values here until you -} \]

Until you find something. Until it - instead of being greater than zero, it becomes less than zero.

Yeah. And then whatever greatest value you can get that's still greater than zero would be your largest value for \( a \)... And that should be close to - I think \( e \) is 2.667 or something like that.

Subject G11 recognized that the minimum value of the function \( y = x - a \ln x \), \( a - a \ln a \), had to be greater than zero if the inequality \( x > a \ln a \) was to be
true for all $x > 0$. The subject determined that by evaluating $a \ln a$ for
different values of $a$, it was possible to estimate the largest value of $a$ for
which the inequality $a \ln a > 0$ was true. Since Subject G11 had determined
the minimum value of the function $y = x - a \ln x$ using the derivative of the
function, this technique for solving the problem was classified as use of a
symbolic/numerical representation.

Calculus & Mathematica Student Interviews

As you read the interview excerpts that follow, look for evidence relating
to these behaviors in the descriptions given by Calculus & Mathematica
students:

* recall of what information about a function and its graph was provided
  by its first derivative
* use of "combination" representations
* recognition of how to use representations in ways different from those
typically taught in the course
* recognition of connections between one representation and a different
representation of an alternate form of the first representation
* relating of unfamiliar problem situations or descriptions to familiar uses
  of representations

Part a. Like the graphics calculator students, Calculus & Mathematica
students had little difficulty using graphical or numerical representations to
solve this problem. All recognized how to apply the information provided by
different graphs to prove that $x > 2 \ln x$ for all $x > 0$. Most also recognized
that verifying an inequality numerically by checking various values of the
variable did not constitute a proof of the inequality.

Calculus & Mathematica students did slightly better than the other
students working with symbolic representations on part a. As with the other
students, they had some difficulty at first understanding the symbolic technique
for proving \( x > 2 \ln x \) by verifying that the minimum value of the
corresponding function \( y = x - 2 \ln x \) is greater than zero.

The only "combination" representation mentioned by Calculus &
Mathematica students on this problem was the graphical/symbolic
representation involving the Calculus & Mathematica Race Track Principle.
The version of this principle used by students states that, on an interval, if the
value of a function \( f \) is greater than or equal to the value of another function \( g \)
at the beginning of the interval, and if the derivative of function \( f \) is always
greater than the derivative of function \( g \) on the interval, then function \( f \) will
always be greater than function \( g \) on that interval. All students correctly
applied the Race Track Principle to show \( x > 2 \ln x \) for all \( x > 0 \). This was
classified as use of a graphical/symbolic representation since it involved the use
of derivatives and the comparison of growth rates of functions, which the
Calculus & Mathematica students did using the graphs of the derivatives.
Use of graphical representations. All Calculus & Mathematica students interviewed described correctly how to show graphically that \( x > \sqrt{2 \ln x} \) for all \( x > 0 \). One method suggested by each student was to graph the functions \( y = e^x \) and \( y = x^2 \) and confirm that the graph of \( y = e^x \) was always above the graph of \( y = x^2 \). All had recognized that if \( e^x > x^2 \) for all \( x > 0 \), then \( x > \sqrt{2 \ln x} \) for all \( x > 0 \). Subjects C9 and C11 also showed that the inequality was true by verifying that the graph of \( y = x \) was always above the graph of \( y = \sqrt{2 \ln x} \). No student suggested using the graph of \( y = x - 2 \ln x \) to prove that the inequality \( x > 2 \ln x \) was true for all \( x > 0 \). After the interviewer suggested using this graph, students recognized that the inequality would be true if the graph of \( y = x - 2 \ln x \) was above the x-axis.

Subject C9:

Right. Yeah. It would then cause you're just rewriting it and it says that it's always greater than zero and the graph says that's true.

Subject C10:

Yeah, I think so. Because if you plug some points in, you see that any point will be greater than 0.

Okay. Well, doesn't the graph show that.

Right. It does show that.

Subject C11:

Yeah, cause it doesn't cross - doesn't go below zero, and it says here that it's got to be greater than zero.
Subject C10's comments suggested the subject did not initially recognize the graph alone could be used to determine if \( x - 2 \ln x \) was always greater than zero without having to substitute values into the inequality \( x - 2 \ln x > 0 \) to verify it. Later discussion indicated the subject knew how to determine from the graph only if the inequality was true.

**Use of numerical representations.** All students suggested showing the inequality was true numerically by verifying, for several values of \( x \), that \( x \) was always greater than \( 2 \ln x \). One student, Subject C12, mentioned constructing a table of values as a means for showing the inequality was true. All recognized that this method for showing the inequality was true did not always work since it was possible to miss values of \( x \) where the inequality was false.

**Use of "combination" representations.** All Calculus & Mathematica students mentioned using the Race Track Principle to prove \( x > 2 \ln x \) for all \( x > 0 \). Three students applied the Race Track Principle to the functions \( y = e^x \) and \( y = x^2 \) for \( x > 0 \) to show that \( x > 2 \ln x \) for all \( x > 0 \).

Subject C12:

Take \( e \) to both sides to get \( e^x \) is greater than \( x^2 \). The derivatives of both are \( e^x \) and \( 2x \). \( e^x \) is always larger than \( 2x \), so \( e^x \) is always growing faster than \( x^2 \). So \( e^x \) is greater than \( x^2 \). Log both sides to get \( x \) is greater than \( 2 \ln x \). I forgot to ... figure out the starting points. That's the Race Track Principle.... The faster horse is starting - you have the faster horse starting ahead of the slower horse so it will always stay ahead.
Subject C10:

Look at the derivatives, $e^x$ and $2x$. $e^x$ is exponential which is more, more dominate over power functions.

*Isn’t there a little something else I need in there, not just that it dominates it, but - does the starting place have anything to do with it?*

Yeah, but if you look at their starting places, they start very close together [and] $e^x$ starts on top.

Subject C11:

Exponential growth dominates power growth.... When you look at the derivative of each, the derivative of $e^x$ is $e^x$. The derivative $x^2$ is $2x$ and so therefore, they’re starting off at points where $e^x$ is ahead ... and it’s instantaneous growth rate is gonna to clearly out run that of $x^2$ because it’s only $2x$ and $e^x$ is - since it already dominates power growth, it’s definitely going to dominate linear growth. So, since it already starts out ahead, the Race Track Principle says if you start out ahead and you’re a faster horse, then you’re going to go ahead and pull ahead and win by even more.

Subjects C11 and C12’s reference to horses comes from another version of the Race Track Principle. For this problem, the appropriate version of the Race Track Principle states if two horses start a race from the same point and one horse always run faster than the other, then the faster horse will win the race.

The fourth Calculus & *Mathematica* applied the Race Track Principle to the functions $y = x$ and $y = 2 \ln x$ for $x > 0$.

Subject C9:

The growth rate of $\log$ of $x$ isn’t as fast as $x$, so it’s like the Race Track Principle.... Even if they start kinda in the same area or - I don’t know. If they start kinda close to each other, like between 0 and 4 or whatever that was, then since $\log$ of $x$ is growing slower than $x$, that means $x$ is greater.... It’s not Race Track Principle the way they start at the same point, but they kinda look a lot closer at 2 then they do out farther.
The difficulty with applying the Race Track Principle to the functions \( y = x \) and \( y = 2 \ln x \) for \( x > 0 \) is that the growth rate of \( y = 2 \ln x \) is greater than the growth rate of \( y = x \) for \( 0 < x < 2 \) since \( 2/x > 1 \) for \( 0 < x < 1 \). When this was pointed out, Subject C9 recognized that since the Race Track Principle could be applied to the functions for \( x > 2 \) and since \( x \) was always greater than \( 2 \ln x \) for \( 0 < x < 2 \), then it was true that \( x > 2 \ln x \) for all \( x > 0 \).

Use of symbolic representations. Calculus & Mathematica students seemed to recognize, after prompting, how the symbolic technique of showing the minimum value of \( y = x - 2 \ln x \) was greater than zero proved that the inequality \( x > 2 \ln x \) was true. In the following excerpts, Subjects G9 and G10 had just been asked how to determine the minimum value of the graph of the function \( y = x - 2 \ln x \).

Subject C11:

You could say, just by looking at this right here, it appears the minimum is at 2. And so then you could just like plug that back in.

Without using the graph, could we find the minimum?

If you took the derivative.... Get the derivative and -

We found the derivative and it agrees with you that 2, \( x \) equals 2 is the minimum. What am I going to do with that?

Since you know that it’s a minimum, plug that back into the original equation that you had. You know, the one that’s greater than zero, and then if it’s greater than zero, then you know, cause that’s the very lowest point that it could be and that proves it’s going to be greater - or, you know, greater than zero. That’s as low as it can go. If it’s going to greater than zero when you plug it back in, then - ah - there aren’t any other points that could go below zero.
Subject C12:

You can take the derivative of both sides and find the minimum. If it’s greater than 0 then -

*If it’s greater than zero, this \(x - 2 \ln x > 0\) will be true then?*

Yes.

*And if it’s less than 0, what would happen?*

Then it’s not true. So then, the derivative of that is 1 minus 2 over x. And then set that equal to 0. One equals 2 over x. Two equals x.... So then you plug that in to the equation and then you have 2 - 2 \ln 2. I don’t know what that is but it’s got to be less than 1. It’s less than 1 because it’s e to the blank equals 2.... So it’s true.

Subjects C11 and C12 recognized that since the minimum value of the function \(y = x - 2 \ln x\) was greater than 0, the inequality \(x - 2 \ln x > 0\) had to be true.

These subjects, along with Subject C9, recognized proving \(x - 2 \ln x > 0\) for all \(x > 0\) was equivalent to proving \(x > 2 \ln x\) for all \(x > 0\).

As the following excerpt suggests, Subject C10 did not seem to understand how the information provided by the graph of \(y = x - 2 \ln x\) could be used to show that the inequality \(x > 2 \ln x\) was true.

Subject C10:

You could - uh - take the derivative.... And set it equal to 0.

*Okay. That solves out to x equals 2. That gives me the x-value of the minimum. What do I need to do to get the y-value?*

Just plug it back in.

*Okay. So I get ... about .614. Does this information help me prove that?*

It helps you prove that if you have a minimum, that it’s gonna to decrease and then it’s going to go back up. So it’s gonna increase. And since we - we put the two functions together, it shows that one function is gonna be lower than the other function. Well, one function is going to dominate - dominate the other function.
Subject C10 did not recognize that the minimum value of the function 
\[ y = x - 2 \ln x \] had to be greater than 0 in order for the inequality \( x - 2 \ln x > 0 \) to be true. However, when asked if the inequality \( x > 2 \ln x \) would have been true had the minimum value not been greater than zero, the subject asserted that it would not be true. This suggested Subject C10 might have understood the minimum value had to be greater than zero for the inequality to be true.

**Part b.** Most Calculus & *Mathematica* students had little difficulty using either graphical or numerical representations to show that \( x > 3 \ln x \) was not true for all \( x > 0 \), though only two students mentioned solving the problem numerically. They also did better working with symbolic representations. This improvement probably resulted from discussing how to use symbolic representation to prove \( x > 2 \ln x \) for all \( x > 0 \) for part a.

As was the case on part a, the only "combination" representation mentioned by Calculus & *Mathematica* students on this problem was the graphical/symbolic representation involving the Calculus & *Mathematica* Race Track Principle. Two students suggested using the Race Track Principle to show \( x > 3 \ln x \) for all \( x > 0 \), not realizing that the inequality was not true for all \( x > 0 \). These students thought that the growth rates of the functions they used for this problem, \( y = e^x \) and \( y = x^3 \), fit the criteria of the Race Track
Principle — they did not — because they assumed exponential growth was always greater than power growth.

Use of graphical representations. All the Calculus & Mathematica students interviewed described correctly how to show graphically that \( x \neq 3 \ln x \) for all \( x > 0 \). Subjects C9 and C11 determined that the graph of \( y = x \) crossed the graph of \( y = 3 \ln x \). Subjects C10, C11, and C12 determined that the graph of \( y = e^x \) crossed the graph of \( y = x^3 \), recognizing that \( x \neq 3 \ln x \) for all \( x > 0 \) if \( e^x \neq x^3 \) for all \( x > 0 \).

When the interviewer mentioned using the function \( y = x - 3 \ln x \) to solve this problem, students suggested graphing the function to determine if it ever went below the \( x \)-axis.

Subject C9:
If we look at this, then it says that it \([x - 3 \ln x]\) is not always greater than zero... because the graph goes below zero.

Subject C11:
If this \( 3 \ln x \) is ever greater than \( x \), that means it \([\text{graph of } y = x - 3 \ln x]\) is going to dip below the zero line and it dipped below the zero line.

Subject C12:
You can take the equation \( x - 3 \ln x \) and see if its always greater than zero, and you would see that it wasn’t. That the graph went below zero at the crossing point.
These excerpts suggested recognition that the inequality $x > 3 \ln x$ was not true since the graph of $y = x - 3 \ln x$ went below the $x$-axis.

**Use of numerical representations.** Two students suggested using numerical representations to determine at least one value of $x$ for which the inequality $x > 3 \ln x$ was not true. One student, Subject C12, mentioned constructing a table of values as a means for determining values of $x$ for which the inequality was not true. Both students recognized that this method for disproving the inequality might not work since it was possible to miss values of $x$ where the inequality was false.

**Use of "combination" representations.** Subject C10 and C12 initially thought the inequality $x > 3 \ln x$ was true for all $x > 0$ and attempted to prove this using the Race Track Principle. Both students applied the Race Track Principle to the functions $y = e^x$ and $y = x^3$ for $x > 0$, thinking the growth rate of $e^x$ was always greater than the growth rate of $x^3$, since, as Subject C10 stated, "Exponential growth is always more dominate than power growth."

What these students did not seem to realize was that exponential growth only dominates power growth on a global scale, i.e., for large values of the variable. Near the origin, it is possible for power growth to be greater than exponential growth, as it was in this case since $x^3$ is growing at a slightly faster rate than $e^x$ for $x$ between about 0.91 and 3.73. Both students recognized this shortcoming
in their arguments after being shown the graphs of the derivatives of the functions \( y = e^x \) and \( y = x^3 \) so they could see that the growth rates of the functions would prevent them from applying the Race Track Principle to this problem. Utilizing the Race Track Principle in this way still was classified as use of a graphical/symbolic representation even though the representation was applied incorrectly.

*Use of symbolic representations.* Calculus \& Mathematica students described correctly how to use the symbolic technique utilized in part a to show that \( x > 3 \ln x \) for all \( x > 0 \). As was the case with the other students, it appeared that the discussion of how to use this technique to show symbolically that \( x > 2 \ln x \) for all \( x > 0 \) helped students to apply the technique to this problem. As the following excerpts suggest, students seemed to recognize that showing the minimum value of the function \( y = x - 3 \ln x \) was less than zero proved that the inequality \( x > 3 \ln x \) was not true for all \( x > 0 \).

**Subject C9:**

What would happen if you did what we did on the problem before?... You could find the minimum value of this \( y = x - 3 \ln x \) by finding where the derivative crosses the \( x \)-axis. And see what that is. Since it’s less than zero, \( x - 3 \ln x \) is not always greater than 0.

**Subject C10:**

You could see where the minimum falls. If the minimum falls below the \( x \)-axis then you know that it \( [x - 3 \ln x > 0] \) is false.
Subject C11:

Taking the derivative, you could find the minimum point. And, you know, since the minimum point would then be negative, you could show it \( x - 3 \ln x > 0 \) was false that way.

Students appeared to understand that for the inequality \( x - 3 \ln x > 0 \) to be true, the minimum value of the function \( y = x - 3 \ln x \) had to be greater than zero, and since this was not the case, then it must be that \( x \neq 3 \ln x \) for all \( x > 0 \).

_Determining the value of M_. Calculus & Mathematica students mentioned using two different forms of representations to determine the value of \( M \) such that \( x > 3 \ln x \) for all \( x > M \). All four students suggested locating \( M \) graphically by determining where the graph of \( y = x - 3 \ln x \) crossed the \( x \)-axis or where the graph of \( y = e^x \) crossed the graph of \( y = x^3 \). One student,

Subject C10, initially misidentified the point on the graph of \( y = x - 3 \ln x \) whose \( x \)-coordinate represented \( M \).

Subject C10:

Most likely, it’d be this minimum.
_Well, is it true that if I start at this minimum, will x - 3 ln x be greater than zero starting at that at that minimum?_
Yeah.
_Right there. It looks like it’s negative right there…. Is it greater than zero down -_
No, it’s not.
_When does it start -_
Around 4.
It is possible Subject C10 was thinking all values had to be greater than \( M \) when the choice was made to use the minimum value of the graph of 
\[ y = x - 3 \ln x \] as the value of \( M \).

Two students suggested determining the value of \( M \) by solve the equation 
\[ x = 3 \ln x \] analytically. One student, Subject C12, attempted to solve the equation analytically using Mathematica and received an error message that basically stating that the equation could not be solved using standard analytical method. Both students eventually determined the value of \( M \) graphically.

**Part c.** Like traditional and graphics calculator students, Calculus & Mathematica students had more difficulty with this problem than any other. Two of four students interviewed, and seven of thirteen students assigned the problem on the Calculus Representations Test, did not initially respond to this question. As with the other students, Calculus & Mathematica students had some difficulty working with the second variable, \( a \), particularly when trying to use symbolic representations, although they did have slightly less difficulty than the other students predicting the largest value of \( a \) for which \( x > a \ln x \) for all \( x > 0 \) using symbolic representation. Some apparently did not realize that \( a \) could be treated as a constant when taking a derivative with respect to \( x \). This was somewhat expected with Calculus & Mathematica student, considering that they were asked to calculate derivatives of function of more than one variable
and considering that *Mathematica* has the capability to calculate such a
derivative for them. The one student who was unable to describe how to solve
this problem symbolically had been able to solve parts *a* and *b* symbolically
but could not adapt the solution methods to work with the variable *a* rather
than the numbers 2 and 3.

*Use of graphical representations.* All Calculus & *Mathematica* students
used graphical representations to predict the largest value of *a* for which \( x > a \)
\( \ln x \) for all \( x > 0 \). Subject C9 suggested graphing \( y = x \) and \( y = a \ln x \) for
different values of *a*.

Subject C9:

I would graph them, \( x \) and \( a \ln x \), changing *a* until I got \( x \) greater than *a*
\( \ln x \) [where] *a* is largest.

Subject C11 mentioned graphing the function \( y = x - a \ln x \) for different values
of *a* and then determining the largest value of *a* for which the graph did not
cross the *x*-axis.

Subject C11:

Well, by looking at that one graph, you could take \( x - a \ln x \) and, you
know, you could just like go down with different values of *a*. Knowing
that at 3, it doesn’t work, so it’d be between 2 and 3.

Subjects C10 and C12 mentioned graphing the functions \( y = e^x \) and \( y = x^e \) for
different values of *a*, but only Subject C12 suggested the first choice for a
value of *a* should be the number *e*. 
Subject C12:

Don't ask me why I would do this, but I would try \( e \) first because it works for 2 and it doesn't work for 3. And \( e \) - since it - since we use \( e \) enough, I would just guess it's that first and see if it would work.

Based on the graphs of \( y = e^x \) and \( y = x^e \), Subject C12 concluded the value of \( a \) had to be strictly greater than \( e \) since the two graphs had a point of tangency.

*Use of numerical representations.* Two students suggested using numerical representations to predict the largest value of \( a \) for which \( x > a \ln x \) for all \( x > 0 \). They mentioned choosing values for \( a \) and then evaluating the inequality \( x > a \ln x \) for different values of \( x \) in order to estimate the largest value of \( a \) for which \( x \) is always greater than \( a \ln x \). Both agreed that this method would at best produce a rough approximation for the actual value of \( a \).

*Use of "combination" representations.* Somewhat expectedly, no Calculus & Mathematica student suggested using a "combination" representation to predict the largest value of \( a \) for which \( x > a \ln x \) for all \( x > 0 \). This was one of only two problems on the Calculus Representations Test where Calculus & Mathematica students did not describe a solution technique involving some form of "combination" representation.

*Use of symbolic representations.* Subject C12 gave by far the best description of all students on how to use symbolic representations to predict the largest value of \( a \) for which \( x > a \ln x \) for all \( x > 0 \).
Subject C12:

Well, if we did the same thing as before, we want the minimum of that $[y = x - a \ln x]$ greater than zero. So take the derivative. [Had Mathematica compute the derivative]. The derivative of that is $1$ minus $a$ over $x$. And set that equal to zero. One equals $a$ over $x$. So $a$ equals $x$. Then what do I do with that?

You plug that back into the equation. You get $a - a \ln a$.

When I substituted $a$ in for $x$ and I got $a - a \ln a$. What must be true about that for the inequality to be true?

It has to be greater than zero. So $a - a \ln a$ is greater than $0$.

Can I solve that?

We can have Mathematica solve it. $a$ is equal to $e$ like I thought.

Drawing on the work from parts $a$ and $b$, the subject recognized the inequality $x > a \ln x$ would be true if the minimum value of the function $y = x - a \ln x$ was always greater than zero. Subject C12 was one of the few students who recognized when the minimum value of function $y = x - a \ln x$ was greater than zero, the inequality $a \ln a > 0$ was true. Subject C12 was also the only student to use Mathematica to calculate the derivative of $y = x - a \ln x$ and to solve the equation $a \ln a = 0$ as a means of solving the inequality $a \ln a > 0$.

Subject C10 was the only other Calculus & Mathematica student to describe correctly how to solve this problem symbolically but was only able to do so with a substantial amount of prompting. This subject recognized that the minimum value of the function $y = x - a \ln x$ had to be greater than zero for the inequality $x > a \ln x$ to be true for all $x > 0$. The amount of prompting given to Subject C10 while solving the problem made it difficult to determine
if the subject actually understood how to put this information to use to solve the problem. During the discussion, it appeared that Subject C10 knew how to determine the minimum value of the function, $a \ln a$, using the derivative. However, it is unclear if the subject recognized the correct inequality, $a \ln a > 0$, based on this minimum value, or could solve this inequality and determine the value of $a$ for which $x > a \ln x$ for all $x > 0$.

One student, Subject C11, never mentioned using symbolic representations to solve this problem. The fourth student, Subject C9, was unable to describe correctly how to use the symbolic technique to predict the largest value of $a$ for which $x > a \ln x$ for all $x > 0$, though, as the following excerpt suggests, the subject did recognize that the inequality $x > a \ln x$ would be true if the minimum value of the function $y = x - a \ln x$ was always greater than zero.

Subject C9:

If the [inequality] was going to be true, what had to be true about the minimum [of the function $y = x - a \ln x$]? Where does it have to be?

It depends on - it’s suppose to be greater or less than zero. If you want it to be, but if you want it to be true, it’s has to be greater than zero.

Summary of Problem 4

In this section, the results of the Calculus Representations Test and student interviews for Problem 4 are summarized by discussing and comparing students’ (a) use of representations, and (b) interpretation of problem situations
and recognition of different calculus concepts, solution techniques, and connections between representations

Use of Representations. The percents of students in each course using the different representations on each part of Problem 4 are summarized in Tables 17 and 18. Based on these results, the following observation can be made.

- Calculus & Mathematica students correctly used symbolic representations more often than traditional students who correctly used this form of representation slightly more often than graphics calculator students.

- Students from all three courses had difficulty describing correctly how to use symbolic representations to show that \( x > 2 \ln x \) for all \( x > 0 \), to show that \( x \neq 3 \ln x \) for all \( x > 0 \), and to predict the largest value of \( a \) for which \( x > a \ln x \) for all \( x > 0 \).

- Traditional students were the only ones to have any real difficulty describing correctly how to use graphical representations to show that \( x > 2 \ln x \) for all \( x > 0 \).

- Calculus & Mathematica students correctly used graphical representations slightly more often than traditional students who correctly used this form of representation slightly more often than graphics calculator students.

- Students from the three courses correctly used numerical representations about equally often, though this form of representation was not used often by students from any courses.

- Calculus & Mathematica students were the only students to use graphical/symbolic representations.

- Graphics calculator students were the only students to use graphical/numerical representations, excepting for one traditional student.

- Only one traditional and one graphics calculator student made use of symbolic/numerical representations.
Table 17: Comparing Students on Use of Representations on Parts a and c of Problem 4 by Course

<table>
<thead>
<tr>
<th>Repr.</th>
<th>Use of Different Representations</th>
<th>% Providing</th>
<th>Acceptable Use of Given Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Show that $x &gt; 2 \ln x$ for all $x &gt; 0$</td>
<td>Predict $a$ such that $x &gt; a \ln x$ for all $x &gt; 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>151</td>
<td>151G</td>
</tr>
<tr>
<td>Graph</td>
<td>Correct</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>Incorrect</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Number</td>
<td>Correct</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>Incorrect</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Symbol</td>
<td>Correct</td>
<td>75</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>Incorrect</td>
<td>25</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>G/S</td>
<td>Correct</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>S/N</td>
<td>Correct</td>
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<td>0</td>
</tr>
<tr>
<td></td>
<td>Not Used</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>G/N</td>
<td>Correct</td>
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</tr>
<tr>
<td></td>
<td>Not Used</td>
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<td>0</td>
</tr>
</tbody>
</table>
Table 18: Comparing Students on Use of Representations on Part b of Problem 4 by Course

<table>
<thead>
<tr>
<th>Rep Pr</th>
<th>Use of Different Representations</th>
<th>% Providing Acceptable Use of Given Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Show that $x &lt; 3 \ln x$ for all $x &gt; 0$</td>
<td>Estimate $M$ such that $x &gt; 3 \ln x$ for all $x &gt; M$</td>
</tr>
<tr>
<td>Graph</td>
<td>Correct</td>
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</tr>
<tr>
<td></td>
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</tr>
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<td></td>
<td>Not Used</td>
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</tr>
<tr>
<td>Number</td>
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<tr>
<td></td>
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</tr>
<tr>
<td></td>
<td>Not Used</td>
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<td>Symbol</td>
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<tr>
<td></td>
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<tr>
<td>G/S</td>
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<tr>
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<tr>
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<td></td>
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</tr>
</tbody>
</table>
**Interpretation and Recognition.** Based on the results of the Calculus Representations Test and student interviews, the following observations can be made concerning students’ interpretation of problem situations and their recognition of different calculus concepts, solution techniques, and connections between representations.

- All students had difficulty recognizing how to use symbolic representations to prove that \( x > 2 \ln x (x \neq 3 \ln x) \) for all \( x > 0 \) by showing that the minimum value of the function \( y = x - 2 \ln x \) (\( y = x - 3 \ln x \)) is always above (crosses) the \( x \)-axis.

- All students had difficulty recognizing how to use symbolic representations to predict the largest value of \( a \) for which \( x > a \ln x \) for all \( x > 0 \) by determining the largest value of for which the minimum value of the function \( y = x - a \ln x \) is always above the \( x \)-axis.

- Traditional students, and to a lesser extent graphics calculator students, had some difficulty recognizing how information provided by the graphs of the functions \( y = x - 2 \ln x \) and \( y = x - 3 \ln x \) could be used to prove or disprove the inequalities \( x > 2 \ln x \) and \( x > 3 \ln x \), respectively.

- Most students from each course seemed to recognize that if the minimum value of some function \( f(x) \) was always greater than zero, then the corresponding inequality \( f(x) > 0 \) had to be true.

- Students from all three courses had difficulty working with the inequalities \( x > a \ln x \) and \( x - a \ln x > 0 \) and the related function \( y = x - a \ln x \) because they did seem to know how to work with the second variable, \( a \), in these inequalities and function. In particular, students did not appear to realize that the variable \( a \) could be treated as a constant when taking a derivative with respect to \( x \).

- Students from each course did better working with symbolic representations after having discussed how to use this form of representation to solve part \( a \).
• Some traditional students did better working with graphical representations after having discussed how to use this form of representation to solve part a.

• Graphics calculator students recognized how the TRACE feature of a graphics calculator could be used to help determine whether or not an inequality of the form \( x > f(x) \) was true by tracing along the graph of \( y = f(x) \) and checking whether or not the \( x \)-coordinates were always greater than their corresponding \( y \)-coordinates.

• Some Calculus & Mathematica students did not realize that even though exponential growth dominates power growth, this does not necessarily mean that the value of an exponential function will always be greater than the corresponding value of power function, especially near the origin.

• Calculus & Mathematica students’ slightly better performance with different representations and their greater use of graphical/symbolic representations suggest that they recognized more of the connections between different forms of representations for this particular problem than the students from the other courses.
CHAPTER VI

DISCUSSION AND RECOMMENDATIONS

This research examined how students in Calculus & Mathematica differ from students in a traditional calculus course and students in a calculus course where graphics calculators were used extensively in order to emphasize graphical representations of concepts in their abilities to use and understand multiple representations, or multiple techniques, when solving calculus problems. Two sets of classes from each course participated in the research. Twelve students from each course were selected for interviews concerning their use and understanding of multiple representations of concepts when solving calculus problems. Three research questions were investigated in this study:

1. What is the relationship between the instructional approach that students experience and any change in their initial preference for different representation when solving calculus problems?

2. What is the relationship between the instructional approach that students experience and their abilities to use graphical, numerical, and symbolic representations when solving calculus problems?

3. What is the relationship between the instructional approach that students experience and their abilities to see, or make, connections between graphical, numerical, and symbolic representations in the context of problem situations?
The purpose of this chapter is to discuss answers to these research questions. First, a summary of the findings from the study are presented. These findings provide support for using both the theoretical frameworks and the findings as a basis for proposed answers to the research questions. Significance of the study, limitations of the study, conclusions and recommendations, and suggestions for further research conclude the chapter.

Summary of Findings

This section is divided into five subsections. First, observed differences in the use of representations in the three calculus courses are discussed. The next two subsections present findings concerning students' initial preferences for, and use of, different representations when solving calculus problems. The final two subsections describe findings supported by the data collected during the study concerning students' recognition and recall of various calculus concepts and solution techniques, and their interpretation of problem situations and recognition of connections between representations.

Use of Representations in Course Instruction

In the traditional calculus course, Math 151, almost 90% of the total number of representations used during instruction were symbolic. Close to 80% of the assigned homework problems used only symbolic representations.
Both of these were the highest totals of usage of symbolic representations for the three courses. Little use was made of graphical, numerical, or any form of "combination" representations either in instruction or on homework problems. The observed emphasis of course instruction was on having students (a) practice and perfect techniques for manipulating symbolic representations, in particular, those techniques used to determine derivatives of various functions, and (b) use symbolic representations to solve certain types of problems usually involving the determination of maximum or minimum values of a function.

In the graphics calculator calculus course, Math 151G, even though about 70% of the total number of representations used during instruction were symbolic, almost 25% were graphical representations, which was about three times the number used in the traditional course. During lecture, the instructor made extensive use of the graphics calculator to represent problems graphically, and students were expected to do so also on their own graphics calculator, though this was often not the case. A little over 60% of the assigned homework problems made some use of graphical representations, which was the highest total of the three courses, and only about 30% of the problems used only symbolic representations, which was the lowest total of the three courses. However, few homework problems were specifically designed to have students interpret connections between different representations of the same concept.
In the graphics calculator course, more use was made of numerical and "combination" representations, particularly graphical/numerical representations and, to a lesser extent, graphical/symbolic representations, in instruction and on homework problems than in the traditional course. The observed emphasis of course instruction was similar to the traditional course’s with the exception that emphasis was also placed on having students (a) view graphical, as well as symbolic, representations of concepts and problems and (b) use graphical representations to confirm results found analytical and to solve problems, either alone or in conjunction with symbolic or numerical representations.

In the Calculus & Mathematica (C&M) course, Math 151C, close to 35% the total number of representations used during instruction were graphical — the highest total of the three courses — while only about 60% were symbolic — the lowest total of the three courses. It is important to remember that instruction in this course did not often involve having material presented by an instructor to the entire class, but typically involved having material presented in "Mathematica notebooks" that students were to complete before doing assigned homework problems. About 50% of the assigned homework problems made some use of graphical representations, and about 40% used only symbolic representations. Many homework problems were specifically designed to have students discuss connections between different forms of representations.
In the Calculus & Mathematica course, numerical representations were used in instruction and on homework problems about as often as in the graphics calculator course. However, "combination" representations, particularly graphical/symbolic and symbolic/numerical representations, were used in instruction and on homework problems much more often than in either of the other courses. Numerous homework problems required students to use different representations in conjunction in order to arrive at a solution. Unlike the other courses, the observed emphasis of course instruction in Calculus & Mathematica was on having students solve a variety of problems designed to have students (a) view and discuss connections between different forms of representations of the same concept or (b) use different representations, alone or in conjunction with other representations, to solve the problems.

**Students' Initial Preferences for Representations**

The following observations concerning students' initial preferences for representations are supported by the data collected during the study.

- Traditional students had a noticeable preference initially for symbolic representations when first solving problems.
- Graphics calculator students had a significant preference initially for graphical representations when first solving problems.
- Calculus & Mathematica students had equal preferences initially for symbolic and graphical representations when first solving problems.
• Traditional students showed a noticeable decline in initial preference for graphical representations from the beginning to the end of the course.

• Graphics calculator students showed a significant rise in initial preference for graphical representations and a significant decline in initial preference for symbolic representations from the beginning to the end of the course.

• Calculus & Mathematica students showed a decline in initial preference for symbolic representations and a slight rise in initial preference for numerical representations from the beginning to the end of the course.

Students’ Use of Representations

The following observations concerning students’ use of different representations are supported by the data collected during the study. Remember that students were compared on their abilities to describe correctly how to use different representation to solve problems.

• In working with symbolic representations, Calculus & Mathematica students did noticeably better than graphics calculator students, who did about as well as traditional students.

• In working with graphical representations, Calculus & Mathematica students did just slightly better than graphics calculator, who did better than traditional students.

• In working with numerical representations, Calculus & Mathematica students did better than traditional students, who did just slightly better than graphics calculator students.

• In working with graphical-symbolic representations, Calculus & Mathematica students did significantly better than graphics calculator students, who did better than traditional students, though the latter two groups did not often use either of these forms of representations.

• In working with symbolic-numerical representations, Calculus & Mathematica students did noticeably better than graphics calculator students, who did better than traditional students, though the latter two groups did not often use either of these forms of representations.
In working with graphical/numerical representations, graphics calculator students did significantly better than traditional students, who did better than Calculus & Mathematica students, though the latter two groups did not often use this form of representation.

Recall that graphical/symbolic, symbolic/numerical, and graphical/numerical representations are types of "combination" representations, that is, representations that are combinations of graphical, numerical, or symbolic representations, such as the graph of the first derivative of a function (graphical/symbolic). The findings concerning students’ use of different representations are summarized in Table 15. These findings were based on comparisons of the prompting needed to obtain acceptable use of different representations by students from each course on each of the four problems on the Calculus Representations Test (see Tables 7, 8, 11, 13, 15, and 16 in Chapter 5).

The findings presented in Table 15 indicate that Calculus & Mathematica students were better able to use the different forms of representations, except for graphical/numerical representations, than the students from the other courses. Recall that for this study, use of a representation refers to describing how that representation might be used to solve a problem, and not to performing manipulations on the representation, such as calculating a first derivative or sketching a graph.
Table 15: Comparing Students’ Performance with Different Forms of Representations by Course

<table>
<thead>
<tr>
<th>Representation</th>
<th>Observed Performance with Different Representations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>151</td>
</tr>
<tr>
<td>Graphical</td>
<td>6</td>
</tr>
<tr>
<td>Numerical</td>
<td>7</td>
</tr>
<tr>
<td>Symbolic</td>
<td>6</td>
</tr>
<tr>
<td>Graphical/Symbolic</td>
<td>2*</td>
</tr>
<tr>
<td>Symbolic/ Numerical</td>
<td>4*</td>
</tr>
<tr>
<td>Graphical/ Numerical</td>
<td>5*</td>
</tr>
</tbody>
</table>

* - indicates that students in the course did not often use this form of representation

Coding for Table 15:
Students’ usage of representations was rated on a scale of 1 to 10 with a score of 10 representing the best usage of all students participating in the interviews. Comparisons were based on the prompting needed to obtain acceptable use of different representations on each of the four problems on the Calculus Representations Test (see Tables 7, 8, 11, 13, 15, and 16 in Chapter 5). Recall that use of a representation refers to describing how to solve problems using that representation, and not to performing manipulations on the representation.
Students’ Recognition and Recall of Concepts and Techniques

The following observations concerning students’ recognition and recall of different calculus concepts and solution techniques are supported by the data collected during the study. These observations are listed by course.

Traditional students exhibited:

- difficulty recalling what information about a function and its graph was provided by its first and second derivatives, e.g., the first derivative provides information about a function’s slope and rate of change,
- difficulty recalling what information about a first derivative and its graph was provided by its second derivative,
- difficulty recognizing how to obtain information about rates of change from a graphical representation,
- difficulty recognizing how to use graphical or numerical representations in the absence of a corresponding symbolic representation,
- some difficulty explaining how to use a symbolic representation when that representation was not actually given but only hypothesized, and
- difficulty recognizing how to use representations in ways different from those typically taught in the course, e.g., showing symbolically that the minimum value of the function \( y = x - f(x) \) is greater than zero to prove that the inequality \( x > f(x) \) is always true.

Graphics calculator students exhibited:

- difficulty recalling how to use the first derivative of a function to locate its extrema,
- difficulty recalling what information about a function and its graph was provided by its first and second derivatives,
- some difficulty recalling what information about a first derivative and its graph was provided by its second derivative, though to a lesser extent than traditional students,
• some difficulty recognizing how to obtain information about rates of change from a graphical representation,

• some difficulty recognizing how to use graphical or numerical representations in the absence of a corresponding symbolic representation,

• some difficulty explaining how to use a symbolic representation when that representation was not actually given but only hypothesized, though to a slightly lesser extent than traditional students, and

• difficulty recognizing how to use representations in ways different from those typically taught in the course.

*Calculus & Mathematica students exhibited:*

• little difficulty recalling what information about a function and its graph was provided by its first and second derivatives,

• little difficulty recalling what information about a first derivative and its graph was provided by its second derivative,

• some difficulty recognizing how to obtain information about rates of change from a graphical representation, though to a lesser extent than the other students,

• some difficulty working solely with symbolic representations, i.e. not being allowed to look at a corresponding graphical representation while working with a symbolic representation,

• some difficulty recognizing how to use graphical or numerical representations in the absence of a corresponding symbolic representation, though to a lesser extent than the other students,

• little difficulty explaining how to use a symbolic representation when that representation was not actually given but only hypothesized, and

• some difficulty recognizing how to use representations in ways different from those typically taught in the course, though to a lesser extent than the other students.
Students' Interpretation of Problems and Recognition of Connections

The following observations concerning students’ interpretation of problem situations and recognition of connections between representations are supported by the data collected during the study. These observations are listed by course.

*Traditional students exhibited:*

- difficulty recognizing the connection between the slope of a function and its rate of change,
- some difficulty recognizing how the technique for locating the extrema of a function using its derivative could be adapted in order to locate the extrema of any derivative of the function,
- difficulty recognizing that points where the concavity of the graph of a function changes could also be where the function attains its maximum or minimum slope, or where it attained its greatest or least rate of change,
- difficulty relating real-life situations described in problems to the representation(s) being used to model the problem and interpreting results in terms of the problem situation,
- difficulty recognizing connections between one representation and a different representation of an alternate form of the first representation, e.g., using the graph of \( y = x - 2 \ln x \) to prove the inequality \( x > 2 \ln x \),
- less aptitude for using "combination" representations than students from other courses, and
- difficulty relating unfamiliar problem situations or descriptions to familiar uses of representations, e.g., recognizing that determining how fast the population is changing at a specific time is equivalent to determining the rate of change of the corresponding population function, which is the value of its first derivative.

*Graphics calculator students exhibited:*

- difficulty recognizing the connection between the slope of a function and its rate of change,
• some difficulty recognizing how the technique for locating the extrema of a function using its derivative could be adapted in order to locate the extrema of any derivative of the function, though to a lesser extent than traditional students,

• little difficulty recognizing that points where the concavity of the graph of a function changes could also be where the function attains its maximum or minimum slope, or where it attained its greatest or least rate of change,

• difficulty relating real-life situations described in problems to the representation(s) being used to model the problem and interpreting results in terms of the problem situation,

• some difficulty recognizing connections between one representation and a different representation of an alternate form of the first representation,

• better abilities than the other students at making appropriate use of graphical/numerical representations, particular when the representation made use of the TRACE feature of a graphics calculator, and

• difficulty relating unfamiliar problem situations or descriptions to familiar uses of representations.

Calculus & Mathematica students exhibited:

• no difficulty recognizing the connection between the slope of a function and its rate of change,

• no difficulty recognizing how the technique for locating the extrema of a function using its derivative could be adapted in order to locate the extrema of any derivative of the function,

• little difficulty recognizing that points where the concavity of the graph of a function changes could also be where the function attains its maximum or minimum slope, or where it attained its greatest or least rate of change,

• slight difficulty relating real-life situations described in problems to the representation(s) being used to model the problem and interpreting results in terms of the problem situation,

• slight difficulty recognizing connections between one representation and a different representation of an alternate form of the first representation,
• significantly better abilities than the other students at making appropriate use of graphical/symbolic and symbolic/numerical representations, and

• some difficulty relating unfamiliar problem situations or descriptions to familiar uses of representations, though to a lesser extent than the other students.

Discussion of Research Questions

In this section, answers to each research question are proposed. First, the findings for students from each course are related to the theoretical frameworks to illustrate the correlation between these findings and the findings expected based on the theoretical frameworks and provide support for using the frameworks, along with the findings, as a basis for the proposed answers.

Relating Findings to the Theoretical Frameworks

In this subsection, the theoretical frameworks for the study are applied to the findings on instructional approaches that students from each course experienced to determine what findings concerning changes in students’ initial preferences for representations and their abilities to use, and see connections between, representations would be expected based on the theoretical frameworks. Expected and actual findings are compared to assess how well the actual findings fit the theoretical frameworks. The purpose of the comparison is to provide support for using the theoretical frameworks as a basis for the answers to the research questions.
Traditional Students

The instruction approach experienced by traditional students emphasized using symbolic representations to present concepts and solve problems. Little use was made of graphical, numerical, or any form of "combination" representations. Based on the Hiebert and Carpenter (1992) theoretical framework, the researcher hypothesized that these students were likely to develop a larger, more well-connected internal network of represented knowledge associated with symbolic representations and smaller, less well-connected internal networks associated with graphical and numerical representations, and that students’ internal networks associated with symbolic, graphical, and numerical representations would be disjoint or weakly connected. With this structuring of internal networks, students would presumably tend to have:

- a greater initial preference for, and be better able to use, symbolic representations since their internal network corresponding to symbolic representations is larger and more well-connected than the ones corresponding to other representations,
- more difficulty working with non-symbolic representations since their internal networks corresponding to these representations are smaller and not as well-connected as the one corresponding to symbolic representations, and
- difficulty seeing, or making, connections between different representations or different forms of the same non-symbolic representation since their internal networks corresponding to different representations are disjoint or weakly connected.
The actual findings for traditional students seem consistent with these anticipated findings. Traditional students' initial preference for symbolic representations did increase relative to their initial preferences for graphical or numerical representations from the beginning to the end of the course. They had difficulty working with graphical representations and to a lesser extent, numerical representations, and made little use of "combination" representations. Finally, traditional students had difficulty making connections between different representations, such as recognizing how slope, rate of change, and the first derivative were related, recognizing how to use representations in ways different from those typically taught in the course, and relating unfamiliar problem situations or descriptions to familiar uses of representations.

The correlation of the actual results to the anticipated results based on the Hiebert and Carpenter theoretical framework provide support for using this framework as a basis for the answers to the research questions.

**Graphics Calculator Students**

The instruction approach experienced by graphics calculator students emphasized using both graphical and symbolic representations to present concepts and solve problems. Some use was made of numerical and "combination" representations, particularly graphical/numerical and graphical-symbolic representations. Students were expected to use graphical
representations to confirm results found analytical and to solve problems, either alone or in conjunction with symbolic or numerical representations, but were given few problems specifically designed to have them discuss, or interpret, connections between different representations of the same concept. Based on the Hiebert and Carpenter theoretical framework for this study, the researcher hypothesized that these students were likely to develop larger, more well-connected internal networks of represented knowledge associated with symbolic and symbolic representations along with a smaller, less well-connected internal network associated with numerical representations. Based on Dubinsky (1991) theoretical framework for this study, the researcher hypothesized that students' internal networks associated with symbolic and numerical representations were likely to be disjoint or weakly connected and their internal networks associated with symbolic and graphical representations and with graphical and numerical representations might or might not be strongly connected since the problems solved by students did have them try to relate graphical representations to symbolic and numerical representations, but did not have them explore or establish connections between these forms of representations. With this structuring of internal networks, students would presumably tend to have:
greater initial preferences for, and be better able to use, graphical or symbolic representations, or both, since their internal networks corresponding to these representations are larger and more well-connected than the ones corresponding to other representations, and

less difficulty seeing, or making, connections between graphical and symbolic representations, and between graphical and numerical representations, since more connections, or stronger connections, may have been formed between each pair of corresponding internal networks.

Some of the actual findings for graphics calculator students, in particular, those concerning symbolic representations, do not seem consistent with these anticipated findings. To see why this occurred, it is necessary to re-evaluate the hypothesized structuring of graphics calculator students’ internal networks in light of students’ attitudes toward graphical representation that were present when entering the course, or developed during the course. It is the researcher’s belief that these attitudes influenced the development of students’ internal networks associated with graphical and symbolic representations.

Data collected during the study provided evidence that graphics calculator students had a greater preference initially for graphical representations than the other students when entering their respective courses. This result was not unexpected considering the course description and title suggested that course instruction would emphasize using graphical representations through the use of a graphics calculator. Observations done during the study indicated that there was added emphasis placed on understanding how to make appropriate use of
graphical representation by the course lecturer and textbook, though both did also emphasize using symbolic, as well as graphical, representations. Taking these findings into consideration, the researcher hypothesizes that graphics calculator students developed, or entered the course with, the attitude that the main focus of the course was on using graphical representations of calculus concepts and not on using or making connections between graphical and symbolic representations of calculus concepts. The following excerpts from students’ interviews tend to support this hypothesis.

Subject G1:
We did a lot with the graph, basically, as far as that goes, finding maxima and minima. I'm think we covered it algebraically.

Subject G3:
Analytically but I was never very good in calculus doing the analytical problems.... Most of the problems he asked, they asked like this {finding maxima and minima}, I would do graphically.

Subject G8:
Yeah, I remember doing max/mins. We had to find - um - I don't know how without my graphing calculator.

The main focus for these students seemed to be on solving problems graphically. Further evidence of this focus was the number of students who were able to locate the maxima and minima of the function in Problem 1 graphically, but could not recall how to locate them symbolically.
Under these circumstances, it is reasonable to assume that most graphics calculator students perceived the instructional approach they experienced to be one that emphasized graphical representations and not both graphical and symbolic representations. Based on the theoretical frameworks for this study, the researcher hypothesizes that these students were likely to develop a larger, more well-connected internal network of represented knowledge associated with graphical representations and smaller, less well-connected internal networks associated with symbolic and numerical representations, and that students' internal networks associated with symbolic, graphical, and numerical representations would be disjoint or weakly connected. With this revised structuring of internal networks, students would presumably tend to have:

- a greater initial preference for, and be better able to use, graphical representations since their internal network corresponding to graphical representations is larger and more well-connected than the ones corresponding to other representations,

- more difficulty working with non-graphical representations since their internal networks corresponding to these representations are smaller and not as well-connected as the one corresponding to graphical representations, and

- difficulty seeing, or making, connections between different representations or different forms of the same non-graphical representation since their internal networks corresponding to different representations are disjoint or weakly connected.

Most the actual findings for graphics calculator students seem consistent with these revised anticipated findings. Graphics calculator students' initial
preference for graphical representations did increase significantly from the beginning to the end of the course while their initial preference for symbolic representations decreased significantly. They had more difficulty working with symbolic representations and to a lesser extent, numerical representations, and made little use of graphical/symbolic or symbolic/numerical representations.

Finally, like the traditional students, graphics calculator students had difficulty making connections between different representations, such as recognizing how slope, rate of change, and the first derivative were related, recognizing how to use representations in ways different from those typically taught in the course, and relating unfamiliar problem situations or descriptions to familiar uses of representations.

The correlation of the actual results to the anticipated results based on both theoretical frameworks for this study support using both frameworks as a basis for answering the research questions. In particular, graphics calculator students’ difficulties seeing, or making, connections between graphical and symbolic representation, even after solving problems where they were required to use both these representations, suggests that students may need to solve problems specifically designed to explore or establish a connection between representations in order to see, or make, that connection, as Dubinsky (1991) hypothesized.
It should be noted that graphics calculator students’ extensive use of graphical/numerical representations is likely related to their propensity for using the TRACE feature on the graphics calculator to determine coordinates of points on the graph of a function, which was classified as use of a graphical/numerical representation. Use of the TRACE feature is one of the most emphasized problem-solving methods in courses using graphics calculators and is easy to learn on the calculator.

**Calculus & Mathematica Students**

The instruction approach experienced by Calculus & Mathematica students emphasized using multiple representations, including different forms of "combination" representations, to present concepts and solve problems. Students were required to interpret and solve various related problems specifically designed to examine, explore, or establish connections between different representations of the same concept. Based on both theoretical frameworks for this study, the researcher hypothesized that these students were likely to develop large, well-connected internal network of represented knowledge associated with each different form of representation, and that students’ internal networks associated with symbolic, graphical, and numerical representations were likely to be strongly connected. With this structuring of internal networks, students would presumably tend to:
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have no notable initial preference for a particular representation since their internal networks corresponding to different representations should all be approximately the same size and equally well-connected,

be adept at using more than one form of representation since their internal networks corresponding to different representations should all be relatively large and well-connected, and

have less difficulty seeing, or making, connections between different representations or different forms of the same representation since there should be more connections, or stronger connections, between their internal networks corresponding to different representations.

The actual findings for Calculus & Mathematica students seem consistent with these anticipated findings. Calculus & Mathematica students' initial preference for symbolic representations did decrease so that at the end of the course, their initial preference for graphical and for symbolic representations were about the same. They were better at using symbolic, graphical, numerical, graphical/symbolic, and symbolic/numerical representations than the other students. Finally, Calculus & Mathematica students had the least difficulty of all students seeing, or making, connections between different representations, such as recognizing how slope, rate of change, and the first derivative were related, recognizing how to use representations in ways different from those typically taught in the course, and relating unfamiliar problem situations or descriptions to familiar uses of representations.

The correlation of the actual results to the anticipated results based on both theoretical frameworks for this study provide further support for using
these frameworks as a basis for the answers to the research questions. In particular, the finding that Calculus & Mathematica students had the least difficulty of all students seeing, or making, connections between different representation further supports Dubinsky (1991) hypothesis that students need to solve problems specifically designed to examine, explore, or establish connections between representations in order to see, or make, the connections.

**Proposed Answers to Research Questions**

Discussions of proposed answers to the research questions investigated in this study follow.

**First Research Question**

What is the relationship between the instructional approach that students experience and any change in their initial preference for different representation when solving calculus problems?

In the traditional course, symbolic representations were emphasized in course instruction, and students’ initial preference for symbolic representations increased relative to their initial preferences for other representations from the beginning to the end of the course. In the graphics calculator course, graphical and symbolic representations were emphasized in course instruction, but students apparently considered the main emphasis to be on graphical representations. Students’ initial preference for graphical representations increased significantly and their initial preference for symbolic representations
decreased significantly from the beginning to the end of the course. In the Calculus & Mathematica course, multiple representations - mostly symbolic and graphical - were emphasized in course instruction, and students' initial preference for symbolic representations decreased from the beginning to the end of the course so that their initial preferences for symbolic and graphical representation were about the same.

These findings from the research suggest that students' initial preferences for different representations gravitate toward the emphasized representation(s) of the instructional approach that they experienced. This conjecture is supported by the Hiebert and Carpenter theoretical framework.

According to the Hiebert and Carpenter framework, students are likely to develop a larger, more well-connected internal networks of represented knowledge associated with any emphasized representation(s) and smaller, less well-connected internal networks associated with any non-emphasized representation(s). With this structuring of internal networks, students would tend to have greater initial preferences for emphasized representations since their internal networks corresponding to these representations should be larger and more well-connected than the ones corresponding to any non-emphasized representation(s).
Second Research Question

What is the relationship between the instructional approach that students experience and their abilities to use graphical, numerical, and symbolic representations when solving calculus problems?

In the traditional course, symbolic representations were emphasized in course instruction, and traditional students had some difficulty working with all the different forms of representations, including symbolic representations. Their difficulty with symbolic representations appeared to relate to their lack of experience or familiarity working with symbolic representations to solve the types of problems used in this study. In the graphics calculator course, graphical and symbolic representations were emphasized in course instruction, but students apparently considered the main emphasis to be on graphical representations. Students had little difficulty working with graphical and graphical/numerical representations and some difficulty working with the forms of representations, including symbolic representations. In the Calculus & Mathematica course, multiple representations — mostly symbolic and graphical — were emphasized in course instruction. Calculus & Mathematica students were better than the other students working with all the forms of representations, including symbolic representations but excluding graphical/numerical representations, which they hardly ever used.
These findings from the research suggest that when solving calculus problems, students are better able to use a representation when the representation is emphasized in the instructional approach that they experienced but only to the extent that the problems being solved are ones with which they have had some previous experience. Taken together, the Hiebert and Carpenter and the Dubinsky theoretical frameworks provide support for this conjecture.

According to the Hiebert and Carpenter framework, students are likely to develop a larger, more well-connected internal networks of represented knowledge associated with any emphasized representation(s) and smaller, less well-connected internal networks associated with any non-emphasized representation(s). With this structuring of internal networks, students would tend to be better able to use emphasized representations when solving problems since their internal networks corresponding to these representations should be larger and more well-connected than the ones corresponding to any non-emphasized representation(s). However, according to the Dubinsky framework, if the problem being solved is one with which students have had little previous experience, then there may be difficulties the first few times students attempt to solve the problem as the problem is assimilated into their existing schema when the attempt is successful, or existing schema are revised when the attempt
is unsuccessful. Thus, based on the two frameworks, students are better able to use the emphasized representation(s) of the instructional approach they experienced when solving calculus problems with which they have had some previous experience but not necessarily when solving problems with which they have had little previous experience.

Third Research Question

What is the relationship between the instructional approach that students experience and their abilities to see, or make, connections between graphical, numerical, and symbolic representations in the context of problem situations?

In the traditional course, symbolic representations were emphasized in course instruction, and traditional students had the most difficulty of all the students recognizing connections between different representations and different forms of the same non-symbolic representation. They made the least use of "combination" representations of all the students.

In the graphics calculator course, graphical and symbolic representations were emphasized in course instruction, but students apparently considered the main emphasis to be on graphical representations. In addition, graphics calculator students were expected to use graphical representations to confirm results found analytical and to solve problems, either alone or in conjunction with symbolic or numerical representations, but were given few problems
specifically designed to have them discuss, or interpret, connections between
different representations of the same concept. These students had difficulty
recognizing connections between different representations, including between
graphical and symbolic representations, and different forms of the same non-
graphical representation. They made little use of any "combination"
representations except for graphical/numerical representations.

In the Calculus & Mathematica course, multiple representations - mostly
symbolic and graphical - were emphasized in course instruction. In addition,
Calculus & Mathematica students were required to interpret and solve various
related problems specifically designed to explore or establish connections
between different representations of the same concept. These students were
better than the other students at recognizing connections between different
representations and different forms of the same representation, and made the
most use of graphical/symbolic and symbolic/numerical representations.

These findings from suggest that *in the context of problem situations,*
students are better able to see, or make, a connection between different
representations when one or more of the representations is emphasized in the
instructional approach that they experienced and when then instructional
approaches includes having students solve problems specifically designed to
explore or establish the connection(s) between the representations. As was the
case with the second research question, the theoretical frameworks for this study, used together, provide support for this conjecture.

According to the Hiebert and Carpenter framework, students develop connections between the internal network of represented knowledge associated with different representations and are likely to develop a larger, more well-connected internal networks of represented knowledge associated with any emphasized representation(s) and smaller, less well-connected internal networks associated with any non-emphasized representation(s). With this structuring, students may develop more connections, or stronger connections between internal networks when one or more of the internal networks is associated with an emphasized representation, but not necessarily, especially if the connections between different representations are not made readily apparent to students. This observation is supported by the findings for the graphics calculator students who had difficulty recognizing connections between graphical and symbolic representations even though both of these representations were emphasized, both individually and in combination, in course instruction.

According to the Dubinsky framework, students are more likely to develop a greater number of stronger connections between internal networks if students interpret and solve problems specifically designed to have them explore and establish connections between different representations of the same
concept. Thus, based on the two frameworks, students may recognize connections between emphasized representations of the instructional approach they experienced, but are more likely to recognize the connections when, in addition, they interpret and solve various related problems designed to help them establish, and not just observe, the connections.

**Significance of the Study**

A theoretical contribution of this study is the theoretical framework developed and used in this research. Using internal networks of represented knowledge, and connections between these networks, to discuss and support conjectures concerning students use of, and understanding of connection between, different representations proved to be successful. In addition, while similar versions of this theoretical framework have been used for research related to understanding in school mathematics (e.g. Chi & Koeske, 1983, Kieren, 1988; Putnam, Lampert, & Peterson, 1990; Riley, Greeno, & Heller, 1983), this framework has apparently not be used previously for research on undergraduate mathematics. It is also apparently the first application of Dubinsky’s (1991) theory of mathematical knowledge and its construction or acquisition to research on the use and understanding of multiple representations of concepts.
One significant contribution of this research is the finding that solving and interpreting problems facilitates the development of connections between different representations of the same concept, as evidenced by the difference in the abilities of the graphics calculator and Calculus & *Mathematica* students to recognize connections between representations. Both groups of students were provided with numerous opportunities to view graphical and symbolic representations of the same concept. The main difference between the groups was that Calculus & *Mathematica* students discussed the connections amongst themselves while solving problems designed to have them explore or establish the connections. While graphics calculator students heard their lecturer point out and discuss the connections, and only solved problems designed to have them look at more examples of the connections but not explore or establish the connections for themselves. The findings provide evidence that the approach used in Calculus & *Mathematica* was adequate, and the approach used in the graphics calculator students was inadequate, for promoting students’ understanding, or recognition, of connections between different representations.

A second contribution of this research is the methodology used in the study. Focusing on students’ descriptions of different techniques for solving problems proved to be successful as a means for exploring their abilities to use, and their understanding of, different representations of the same concepts.
A third contribution is that the research provides further evidence that a 
calculus course is not improved by simply adding a technological component to 
an existing course, even when the instructor for the course makes ever effort to 
properly incorporate the technology into the course. As mentioned in 
discussion of course environments in Chapter IV, the textbook used in the 
graphics calculator was essentially a traditional calculus textbook with new 
material on graphical representations and on the use of graphics calculators 
tacked on, and with minor alterations made to the problem sets so that 
additional work with graphical representations done on the graphics calculator 
was included. In the researcher's opinion, this textbook was a primary 
contributing factor behind students' difficulties working with, and 
understanding connections between, different representations. The textbook 
provided insufficient reinforcement of the connections between different 
representation, in particular, graphical and symbolic representations, 
established, examined, and discussed by the course instructor during lectures, 
and as such, helped to negate any progress toward improving students' 
understanding of connections between representations made by the instructor 
during lecture. This circumstance provides further support for Hughes-Hallet's 
(1991) claim that "we should first shape the mathematics in our courses and 
then see where the technology fits in" (p. 33).
A fourth contribution of this research is that it provides evidence that students "behave" as they are taught. The main emphasis of instruction in the traditional course was on using symbolic representations, and traditional students preferred using them when solving the problems on the Calculus Representations Test. The main emphasis of instruction in the graphics calculator course, as far the students were concerned, was on using graphical representations, and graphics calculator students preferred using them when solving the problems on the Calculus Representations Test. The main emphasis of instruction in the Calculus & Mathematica course was on using multiple representations, and Calculus & Mathematica students displayed preferences for using different representations, particularly symbolic and graphical, when solving the problems on the Calculus Representations Test. These findings suggest instruction that emphasizes using multiple representations may help students understand how to use different representations when solving problems.

A fifth contribution of this research is that it provides evidence in support of electronic calculus courses, like Calculus & Mathematica, different from the evidence provided by previous research comparing students from traditional calculus courses to students from calculus courses that made use of technology, such as computer algebra systems like Mathematica (e.g., Heid, 1985; Palmiter,
Like most of the earlier research, this research did a comparison of students' conceptual understandings. Unlike the earlier research, this research did not focus on the correctness of students' answers to test items related to conceptual understanding, but instead on the correctness of students' descriptions of solution techniques that make use of different representations and their understanding of these solution techniques and the corresponding representations. As such, this research provides insight into the similarities and differences in the conceptual, and procedural, knowledge of students from the different courses, and helps build a more compelling argument in favor of the type of technology use advocated in the Calculus & Mathematica course since the methodology used in this research differs from that previously used in this type of research.

A sixth contribution is that the findings provide empirical evidence of the effectiveness of calculus instruction that emphasizes appropriate use of technology, multiple representations of concepts, and interpreting and solving problems in developing students' abilities to use, and understand connections between, multiple representations when solving calculus problems. This evidence lends support to the type of calculus curriculum and instruction advocated by many calculus reform projects, in particular, those projects that adhere to a philosophy similar to that of Calculus & Mathematica.
Limitations of the Study

Limitations of this study include the number of students interviewed and methods of data collection. In the case of the number of students interviewed, since only four students from each course were assigned Problem 2, four were assigned Problem 3, and four were assigned Problem 4, the findings may not provide a true indication of how students from each course use, or understand connections between different representations when solving these problems.

Limitations related to data collection concerned two components, quantitative instruments and interviews. For quantitative instruments, it was not possible to calculate a measure of reliability for the Representations Test. As Keller and Hirsch (1992) point out, there is currently no accepted method for determining the reliability of an instrument used to collect polychoomous preference data, like the Representations Test constructed for this study. Reliability estimates such as the Kuder-Richardson or the Cronbach alpha statistics can be thought of as measures of the ability of the instrument to distinguish among students. Such a statistic is inappropriate for an instrument in which it is expected that groups of students will be similar.

Data collection for the other quantitative instrument used in this research, the Calculus Representations Test, was limited by wording of problems, time constraints, and identification of initial choice of representation. In the case of
problem wording, some students were unclear on, or did not understand what was being asked for in some problems, such as part c of Problem 2. During the administration of the Calculus Representations Test, such students either did not attempt to solve the problem, or solved what they thought was the problem. In either instance, the data may have provided an incomplete or inaccurate assessment of how students initially preferred to solve the actual problem.

Time limited the Calculus Representations Test results. Only 25 minutes was allotted for this test since it was administered during scheduled class time. For some students, the time was inadequate, even though, during the last ten minutes of the test, they were instructed to describe how they would solve the problems. A third limitation related to the Calculus Representations Test was students’ identification of their initial choice of representation. A few students apparently were unclear what was meant by a symbolic, numerical, or graphical representation and may have misidentified their initial choice of representation.

In the case of interviews, a primary limitation was time. Interviews lasted up to 45 minutes, which typically was not enough time to discuss the use of all the different forms of representations on the two problems on the Calculus Representations. Lack of time limited consistency of questioning across interviews. Every effort to adhere to the interview guide constructed for the study, but it was not possible to ask the same questions of all students or to
spend comparable time per problem, especially when many spent the majority of the interview discussing different parts of Problem 1.

Other limitations concerning interviews related to the prompting. It can be argued that prompting on one problem might have helped students recognize techniques for solving other problems that might not have been considered otherwise. Prompting on one problem also impacted on accurately assigning an amount of prompting to students’ correct descriptions of how to use a representation to solve a different problem. Finally, consistent classification of the prompting given students proved difficult, even using the interview guide as a basis for classification, since it was not always clear when, or if, a student had described correctly how to use a particular representation.

Graphs furnished during interviews were a limitation to the study. The researcher provided students with specific graphs, rather than have students generate their own graphs, when students mentioned making use of graphical representations to solve a problem. This method for graphical representations was used since students’ ability to generate graphs was not being tested in this study and since it conserved time during interviews. However, it was not always clear if students would have generated the same graph as the one provided. Thus, the data on students’ use of graphical representations may not be a completely accurate assessment of actual ability to use this representation.
Finally, although this study provided a better picture of students’ understanding of different representations of calculus concepts, of how they are used to solve problems, and of how they are connected, it is only a first rendering that needs more refinement before it can be considered a complete picture portraying students’ use and understanding of multiple representations.

**Conclusions and Recommendations**

This research suggests that differing technological approaches to calculus instruction have an impact on students’ initial choice of representations and their use of, and understanding of connections between different representation when solving calculus problems. Traditional and graphing calculator students exhibited preferences for using symbolic and graphical representations, respectively, when solving problems. Both groups of students had difficulty making connections between different representations, such as recognizing how the slope of a function, its rate of change, and its first derivative were related. Calculus & Mathematica students had equivalent initial preferences for graphical and symbolic representations. They exhibited no particular preference for using any one representation when solving problems and made connections between corresponding graphical and symbolic representations better than students from other courses. These differences in students’ use and
understanding of multiple representations are related to the instructional approach of the different courses in all probability.

In Calculus & Mathematica, technology was an aid for emphasizing multiple representations of concepts and provided students the means to interpret and solve a variety of problems designed to explore, establish, and reinforce the connections between different representations. In traditional calculus, multiple representation were (and are) not typically emphasized. Instead, an emphasis is placed on having students develop proficiency with paper-and-pencil symbolic manipulations and solution techniques. In graphics calculator calculus, both graphical and symbolic representations were emphasized, but students did not work on problems designed to help develop connections between these representations. Instead, the problems assigned were similar to those used in traditional calculus, though students were asked to use graphs to verify results and solve problems. These differences in the instructional approaches offer a possible explanation as to why the Calculus & Mathematica students appear to have a better understanding of how to use, and make connections between multiple representations when solving problems.

Recommendations

Several recommendations arise from findings of this research in the spirit of the Curriculum and Evaluation Standards for School Mathematics (NCTM,
1989), the *Professional Standards for Teaching Mathematics* (NCTM, 1991), and *A Call for Change: Recommendations for the Mathematical Preparation of Teachers of Mathematics* (Leitzel, 1991). First, the NCTM *Standards* (1989) calls for a mathematics curriculum that "emphasizes conceptual understanding, multiple representations and connections, mathematical modeling, and mathematical problem solving" (p. 125). Based on this study, it seems apparent that parts of the curriculum must be taught in concert, not in isolation. Emphasis on multiple representations and connections alone appears to help but *not* necessarily promote understanding of connections between representations. Emphasis on multiple representations and connections established in a problem-solving setting designed for students' exploration of connections does appear to promote understanding of connections between representations.

The first recommendation forms the basis for another recommendation. The two *Standards* (NCTM, 1989, 1991) and *A Call for Change* (Leitzel, 1991) all recognize the importance of the teacher in accomplishing the type of reform to mathematics curriculum suggested in the former document. Therefore, teachers need to be prepared to help students make connections between different representations of the same concept, not by simply showing them the connections, but working problems with them that explore, establish, and reinforce the connections. Teachers also need to avoid the novice teacher
"trap" of not making connections themselves (Leinhardt, 1983, 1984) by being aware of, and understanding connections between representations so they will not miss opportunities to make these connections explicit for students.

Teachers need to remember that what students know is just as important as what they do not know. This research focused on the difficulties students had using and making connection between representations, but it also demonstrates that there are many things about representations that students do know and understand. It is important that teachers strive to increase what students know about mathematics, such as multiple representations and connections, rather than being overly concerned about what they do not know.

A fourth recommendation focuses on another distributer of knowledge, the textbook. This study indicates that textbooks can help or hinder students’ development of connections between representations. It appears that work on connections done by the instructor in the classroom needs to be supported and reinforced by the problems and topics in the textbook in order for connections to be maintained by students. With this in mind, new calculus textbooks must be written so that multiple representations, connections, and technology are not simply tacked onto the topics and problems from existing textbooks, but are woven into a set of new topics and problems that emphasize multiple representations and connections, and make appropriate use of technology.
Suggestions for Further Research

Numerous possibilities for further research are implicit in this study. One possibility is to do a similar study to the present research using a modified methodology where the administration of the calculus problems and interviews occur at the same time. This would provide better insight into students’ initial preferences for different representations and avoid the difficulty of retention of knowledge between the time of the testing and the time of the interviews. Also, a longitudinal study with the same students over the entire calculus sequence could provide a more complete picture of the impact of differing technological approaches to calculus instruction on students’ use of, and understanding of connections between different representation when solving calculus problems because students will have had longer exposure to the particular approaches to instruction.

The findings from this study suggest that students approach contextual problems differently than they do non-contextual ones. With symbolic representations, most students had little difficulty describing how to determine the slope of a given function at a given value, but had much more difficulty describing how to determine how fast a population was changing on a given day when provided a function that modeled the population. Further research is needed to examine differences in students’ use and understanding of multiple
representations when solving contextual and non-contextual problems. This research would extend previous research done by Keller and Hirsch (1992) that looked at students’ preferences for different representations of functions in contextual and non-contextual settings.

Finally, recent technological advancements have lead to the emergence of software that allows for multiple, dynamically linked representations (see Kaput, 1992, 1993, for example). This study suggests that research is needed to determine how students develop connections between representations that are "linked" by the software. Will students view these connections as if they were being presented by an instructor, or will they explore and establish the connections themselves as they might if the connections were not made explicit by the software? Research will be needed to determine what type of instruction and problems are needed to make the best use of the dynamically linked representations. Instruction and problems designed to help students explore and establish connections between different representations will have to change in light of software that will establish the connections for students.
APPENDIX A

REPRESENTATIONS TEST
REPRESENTATIONS TEST

Name:

Date:

Course:

Instructor:

Instructions: For each problem on the following pages, answer the given multiple-choice questions. Please do not solve the problems. For the questions below, circle the most accurate response.

Prior to this quarter, had you ever taken a mathematics course in high school or college where a graphing calculator or mathematics computer software was used as a regular part of the course? Yes No

Prior to this quarter, had you ever used a graphing calculator or mathematics computer software to do any of the work for a mathematics classes? Yes No

Prior to this quarter, did you ever own a graphing calculator or some type of mathematics computer software? Yes No
A certain bacteria doubles its population every 20 hours. You are to determine how many bacteria are present after the fifth day. The number of bacteria present is represented by the table, equation, and graph shown below.

**Table Representation:**

<table>
<thead>
<tr>
<th>Time (Days)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Bacteria</td>
<td>100</td>
<td>152</td>
<td>230</td>
<td>348</td>
<td>528</td>
<td>800</td>
<td>1213</td>
<td>1838</td>
<td>2786</td>
</tr>
</tbody>
</table>

**Equation Representation:**

\[ N = 100(2^{1.2t}) \]

**Graph Representation:**

1. Which representation are you most likely to use to solve this problem?
   a. the table of values  b. the equation  c. the graph

2. Which representation are you least likely to use to solve this problem?
   a. the table of values  b. the equation  c. the graph
A jet aircraft begins a steady climb of 15°. You are to determine its change in altitude after flying for four miles. The aircraft’s altitude is represented by the table, equation, and graph shown below.

*Table Representation:*

<table>
<thead>
<tr>
<th>Distance Traveled (Miles)</th>
<th>0</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1</th>
<th>1.25</th>
<th>1.50</th>
<th>1.75</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Altitude (Feet)</td>
<td>25000</td>
<td>25354</td>
<td>25707</td>
<td>26061</td>
<td>26415</td>
<td>26769</td>
<td>27122</td>
<td>27476</td>
<td>27830</td>
</tr>
</tbody>
</table>

*Equation Representation:*

\[ A = 25000 + 5280d \tan 15° \]

*Graph Representation:*

3. Which representation are you most likely to use to solve this problem?
   a. the table of values       b. the equation       c. the graph

4. Which representation are you least likely to use to solve this problem?
   a. the table of values       b. the equation       c. the graph
You are asked to determine at what times during the day the depth of the water in Boston Harbor is greater than 3 feet. The depth of the water in the harbor is represented by the table, equation, and graph shown below.

**Table Representation:**

<table>
<thead>
<tr>
<th>Time since Midnight (Hours)</th>
<th>0</th>
<th>1.5</th>
<th>3</th>
<th>4.5</th>
<th>6</th>
<th>7.5</th>
<th>9</th>
<th>10.5</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Depth (Feet)</td>
<td>9.90</td>
<td>8.46</td>
<td>5.00</td>
<td>1.53</td>
<td>0.10</td>
<td>1.53</td>
<td>5.00</td>
<td>8.46</td>
<td>9.90</td>
</tr>
</tbody>
</table>

**Equation Representation:**

\[ A = 5 + 4.9 \cos \frac{\pi}{6} t \]

**Graph Representation:**

- **Depth (Feet)**

5. Which representation are you most likely to use to solve this problem?
   - a. the table of values
   - b. the equation
   - c. the graph

6. Which representation are you least likely to use to solve this problem?
   - a. the table of values
   - b. the equation
   - c. the graph
You are asked to estimate the optimum exercise heart rate of a 22-year-old classmate. The optimum exercise heart rate is represented by the table and graph shown below.

**Table Representation:**

<table>
<thead>
<tr>
<th>Age (Years)</th>
<th>31</th>
<th>42</th>
<th>18</th>
<th>24</th>
<th>19</th>
<th>55</th>
<th>61</th>
<th>44</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heart Rate</td>
<td>117</td>
<td>107</td>
<td>127</td>
<td>122</td>
<td>126</td>
<td>104</td>
<td>99</td>
<td>109</td>
<td>127</td>
</tr>
</tbody>
</table>

**Graph Representation:**

7. Which representation are you most likely to use to solve this problem?
   a. the table of values  
   b. the equation  
   c. the graph

8. Which representation are you least likely to use to solve this problem?
   a. the table of values  
   b. the equation  
   c. the graph
Rafael tosses a rock off the edge of a 10-meter high cliff. You are to determine when the rock will strike the ground. The height of the rock above the ground is represented by the table, equation, and graph shown below.

**Table Representation:**

<table>
<thead>
<tr>
<th>Time (Seconds)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (Meters)</td>
<td>10.0</td>
<td>16.3</td>
<td>20.1</td>
<td>21.5</td>
<td>20.4</td>
<td>16.9</td>
<td>10.9</td>
<td>2.5</td>
<td>-8.4</td>
</tr>
</tbody>
</table>

**Equation Representation:**

\[ H = -4.9t^2 + 15t + 10 \]

**Graph Representation:**

9. Which representation are you most likely to use to solve this problem?
   a. the table of values  
   b. the equation  
   c. the graph

10. Which representation are you least likely to use to solve this problem?
   a. the table of values  
   b. the equation  
   c. the graph
The Jeromes are planning to plant a large rectangular vegetable garden in their backyard. They want the garden to have an area of at least 575 square feet, but they only have 100 feet of wired mesh fence to use for fencing. You are to determine the possible dimensions for the Jerome’s garden given that they plan to use all of the fencing and that they plan for the sides of the garden to have whole number lengths. The area of the garden is represented by the table, equation, and graph shown below.

*Table Representation:*

<table>
<thead>
<tr>
<th>Length (Feet)</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width (Feet)</td>
<td>45</td>
<td>40</td>
<td>35</td>
<td>30</td>
<td>25</td>
<td>20</td>
<td>15</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>Area (Square Feet)</td>
<td>225</td>
<td>400</td>
<td>525</td>
<td>600</td>
<td>625</td>
<td>600</td>
<td>525</td>
<td>400</td>
<td>225</td>
</tr>
</tbody>
</table>

*Equation Representation:*

$L(50 - L) > 575$

*Graph Representation:*

11. Which representation are you most likely to use to solve this problem?
   a. the table of values    b. the equation    c. the graph

12. Which representation are you least likely to use to solve this problem?
   a. the table of values    b. the equation    c. the graph
You are asked to determine what the temperature, in degrees Fahrenheit, of a can of soda will be two days after it is placed in an old refrigerator. The temperature of the can of soda is represented by the table, equation, and graph shown below.

**Table Representation:**

<table>
<thead>
<tr>
<th>Time (Hours)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature (°F)</td>
<td>70</td>
<td>55</td>
<td>47.5</td>
<td>43.75</td>
<td>41.88</td>
<td>40.94</td>
<td>40.47</td>
<td>40.23</td>
<td>40.12</td>
</tr>
</tbody>
</table>

**Equation Representation:**

\[ N = 40 + 30(2^{-0.5t}) \]

**Graph Representation:**

![Graph showing temperature over time](image)

13. Which representation are you most likely to use to solve this problem?
   a. the table of values    b. the equation    c. the graph

14. Which representation are you least likely to use to solve this problem?
   a. the table of values    b. the equation    c. the graph
Theresa replaced her old car, which averaged approximately 20 miles per gallon, with a new car that should average 35 miles per gallon. You are asked to determine how many miles she must drive in her new car in one year in order to spend at least $275 less on gasoline than she would have if she was still driving her old car given that gasoline is about $1.20 per gallon. Her annual cost for gasoline is represented by the table, equation, and graph shown below.

**Table Representation:**

<table>
<thead>
<tr>
<th>Miles Driven</th>
<th>2000</th>
<th>4000</th>
<th>6000</th>
<th>8000</th>
<th>10000</th>
<th>12000</th>
<th>14000</th>
<th>16000</th>
<th>18000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Old Annual Cost</td>
<td>120</td>
<td>240</td>
<td>360</td>
<td>480</td>
<td>600</td>
<td>720</td>
<td>840</td>
<td>960</td>
<td>1080</td>
</tr>
<tr>
<td>New (Dollars)</td>
<td>68.57</td>
<td>137.14</td>
<td>205.71</td>
<td>274.29</td>
<td>342.86</td>
<td>411.43</td>
<td>480.00</td>
<td>548.57</td>
<td>617.14</td>
</tr>
</tbody>
</table>

**Equation Representation:**

\[ C = 1.20 \left( \frac{8}{20} - \frac{5}{35} \right) \]

**Graph Representation:**

15. Which representation are you most likely to use to solve this problem?
   a. the table of values    b. the equation    c. the graph

16. Which representation are you least likely to use to solve this problem?
   a. the table of values    b. the equation    c. the graph
You are asked to estimate the U.S. population for the year 2000. The U.S. population between 1800 and 1980 is represented by the table and graph shown below.

**Table Representation:**

<table>
<thead>
<tr>
<th>Year</th>
<th>1800</th>
<th>1820</th>
<th>1840</th>
<th>1860</th>
<th>1880</th>
<th>1900</th>
<th>1920</th>
<th>1940</th>
<th>1960</th>
<th>1980</th>
</tr>
</thead>
<tbody>
<tr>
<td>Population</td>
<td>5.3</td>
<td>9.6</td>
<td>17.1</td>
<td>31.4</td>
<td>50.2</td>
<td>76.0</td>
<td>105.7</td>
<td>131.7</td>
<td>179.0</td>
<td>226.5</td>
</tr>
</tbody>
</table>

(in millions)

**Graph Representation:**

Population (in millions)

17. Which representation are you most likely to use to solve this problem?
   a. the table of values       b. the equation       c. the graph

18. Which representation are you least likely to use to solve this problem?
   a. the table of values       b. the equation       c. the graph
LIST OF REFERENCES


Melin, J. (1990, November). *The enhancement of concept development in calculus through the use of the graphing calculator*. Paper presented at the Third Annual Conference on Technology in Collegiate Mathematics, Columbus, OH.


