AN EFFICIENT REPRESENTATION FOR THE PLANAR MICROSTRIP GREEN'S FUNCTION

DISSERTATION

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By

Ikguen Choi

* * * * *

The Ohio State University

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Dissertation Committee:
Prof. Prabhakar H. Pathak
Prof. Benedikt A. Munk
Prof. Roger C. Rudduck

Approved by

Prabhakar H. Pathak
Adviser
Department of Electrical Engineering
DEDICATION

To my wife, Kihye, son Woojoo, another one coming,
and
my parents.
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VITA

December 26, 1950 .................... Born in Busan, Korea

1974 ................................. B.S., Seoul National University, Seoul, Korea

1976 ................................. M.S., Seoul National University, Seoul, Korea

PUBLICATIONS


FIELDS OF STUDY

Major Field: Electrical Engineering

- Studies in Electromagnetic Theory: Professor P.H. Pathak
  Professor J.H. Richmond
  Professor R.G. Kouyoumjian

- Studies in Communication Theory: Professor D.T. Davis

- Studies in Biomedical Engineering: R.M. Campbell

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CHAPTER 1

INTRODUCTION

A relatively simple and accurate closed form asymptotic approximation for the electric current point source microstrip surface dyadic Green's function that remains accurate everywhere except in the very close vicinity of the point source is developed in this work. In addition, an exact integral expression for this microstrip dyadic Green's function is also presented in a form which is very efficient for numerically evaluating the surface field when the observation point is again not in the immediate vicinity of the source point. Furthermore, a slight modification of the conventional Sommerfeld integral form of the microstrip Green's function developed elsewhere [1] is employed to efficiently calculate the surface fields in the immediate vicinity of the source point; if this conventional representation is switched to either the approximate or exact efficient representations developed here for the microstrip Green's function when the observation point lies outside the immediate neighborhood of the source point, then one is able to evaluate the surface field (at $z = 0$) of a source on a grounded dielectric slab microstrip configuration of Figure 1.1. in a highly efficient manner for an observation point at any distance from the source point. The planar dielectric material in Figure 1.1 is assumed here to be homogeneous and isotropic; however, it can be made lossy.
Figure 1.1. Planar microstrip antenna configuration.
The microstrip surface Green's function is useful for a rigorous analysis of microstrip antennas and microstrip guided wave structures. Microstrip guided wave structures are often used in modern microwave networks and also digital computer circuitry. Although the results developed here can be used to analyze microstrip guided wave structures, this dissertation will focus on the use of the microstrip Green's function to analyze planar microstrip antennas. As is well known, microstrip antennas are popular because of their low profile, light weight, low cost, high reliability, improved reproducibility and conformability for installation on many practical structures like high-speed aircraft, missiles and rockets. More significantly, all the associated circuitry (oscillators, amplifiers, mixers, switches and modulators) can be integrated on the same substrate. Thus, devices can be miniaturized and the unwanted effects of discontinuities (connectors) are suppressed. All these attractive advantages gave birth to a new antenna industry in spite of some drawbacks of microstrip antennas such as their narrow bandwidth and low gain. As will be reviewed in the following section, several approximate methods for analyzing microstrip antennas have succeeded to provide some design data such as input impedance, and resonant frequency for some simple shapes of microstrip patches. But, as one might expect, they cannot be employed to analyze arbitrarily shaped microstrip patch antennas and they also cannot be used to analyze mutual coupling in microstrip antenna arrays. In particular, the mutual coupling effects are a serious problem in practice as the modern microwave technology requires higher operating
frequencies and compact structures. It is therefore essential to study microstrip antennas via a rigorous analysis which automatically includes mutual coupling effects. Typically, such a rigorous analysis requires the use of a microstrip Green's function in the integral equation for the unknown currents on the microstrip patch which are induced by a coaxial feed probe or by a microstrip feed line; these induced surface currents can be found by numerically solving this integral equation (whose Kernel is the microstrip surface Green's function). The numerical solution for the unknown surface currents in the integral equation is based on the moment method (MM) [2] which reduces the integral equation to a system of linear simultaneous equations. However, the main drawback in the application of the above MM procedure comes from the fact that the conventional Sommerfeld type integral representation of the microstrip surface Green's function is very slowly convergent thereby making the accompanying numerical calculations very inefficient. The conventional Sommerfeld type integral exhibits rapid convergence only for vertical separation between source and field points; on the other hand, the MM solution of the microstrip problem requires a lateral rather than a vertical separation between source and observation points making that representation slowly convergent. Therefore, the present work is motivated by the need to find a simple, efficient and accurate closed form approximation for the microstrip dyadic Green's function. In addition to the development of the approximate closed form representation for the microstrip Green's function, a transformation based on the Cauchy's residue theory is also
developed, as indicated earlier, to obtain an exact but alternative representation for the improper Sommerfeld type integrals such that it is more efficient for numerical evaluation than the latter.

A brief review of the previous related work is in order. The first theoretical model developed to study microstrip antennas was the transmission line model [3]. This model was developed specifically for narrow rectangular patch antennas. Two parallel radiating slots located within dielectric substrate between the ends of the upper conductor and the ground plane are assumed to radiate in free space in this model. The advantage of this model is its simplicity, i.e., the resonant frequency and input resistance are given by simple formulas. However, it suffers from numerous disadvantages; namely, it is only useful for patches of rectangular shapes, the fringe factor must be empirically determined, it ignores field variations along the radiating edge and it is not adaptable to inclusion of the feed, etc. [4]. Several authors have extended the cavity model, previously used in the study of resonators [5,6] to study antennas [7,8]. The microstrip patch is modeled by a cavity bounded by two electric walls (upper patch and the lower ground conductor) and enclosed by magnetic side walls. Magnetic currents corresponding to only the resonant mode pertaining to the fundamental resonant frequency of the cavity were allowed to radiate into free space from the sides (where the magnetic wall was located) in that approach [7,8]. But, the input impedance obtained by that approach [7,8] failed to agree with experimental results. It is noted that the model in [7] predicts a circle having its center on the real axis of the
Smith chart. Experimental results, on the other hand, show a circle having its center in the upper part of the chart, i.e., an inductive shift of input impedance [9]. About the same time, Y.T. Lo et al. [10] expanded the cavity field in terms of a complete set of resonant cavity modes rather than just one resonant mode pertaining to the fundamental resonant frequency. This so called modal expansion cavity model leads to an accurate prediction of the input impedance, as a function of frequency, for various feed locations. Also, this model can be applied to study circular, rectangular and semicircular microstrip antennas. However, this model suffers some difficulties of its own when the dominant mode is not excited strongly enough. This discrepancy was noticed and corrected by W.F. Richards, Y.T. Lo and D.D. Harrison [11]. The surface wave launched by the antenna was also studied in an approximate way in the same article [11]; the effect of surface wave was neglected in the previous article by Y.T. Lo et al. [10]. Even if the modal expansion cavity model provides a good compromise between exactness and simplicity, it is known that more rigorous analyses are required to supply a firm basis that justifies assumptions made in the work of [10], such as neglecting the electric current flowing on the outer surface of the microstrip antenna, ignoring the surface wave, etc. Also the simple cavity model is limited in its applicability to certain regularly shaped microstrip patch antennas because the resonant cavity modes are not available analytically for arbitrarily shaped patches. But, most of all, it has not yet provided any means for analysis of the coupling effects which may cause serious problems in the microstrip
antenna arrays [12]. Thus, a rigorous moment method (MM) analysis (using the microstrip Green's function) was conducted by several authors.

E.H. Newman and P. Tulyathan [13] applied a moment method solution, which had been presented in [14] by E.H. Newman and D.M. Pozar, to the problem of radiation or scattering from geometries consisting of open or closed surfaces, wires and wire-surface junctions, to the microstrip patch antenna problem. In that paper [13], image theory was used with an integral equation to solve for the antenna patch current and the dielectric slab was taken into account by introducing a polarization current. The excitation by a coaxial cable was modeled by a thin vertical wire with constant current. However, they used an approximate microstrip Green's function and neglected the surface wave. The first exact expression for the microstrip Green's function was published by N.K. Uzunoglu et al. [15] and by J.R. Mosig and F.E. Gardiol [16] about the same time. Uzunoglu et al. [15] used the exact Fourier transform representation of Green's function to analyze the printed (or microstrip) dipole as a fundamental problem of printed (or microstrip) antennas. He assumed a sinusoidal current distributions on the printed dipole and then used the Fourier transform to compute the input impedance. Later, the dipole current was obtained in an exact way by solving Pocklington's equation numerically to calculate the input impedance [17] and also the mutual impedance between printed dipoles [18]. These articles involved the use of improper Sommerfeld type integral representation for the microstrip Green's function which was computed numerically by a real axis integration technique. Mosig and
Gaydol, in [16], developed a new efficient numerical technique for the Sommerfeld type integrals by working with an alternative integral representation which is obtained by deforming the original Sommerfeld type contour of integration. It is noted that in [16] the Sommerfeld type contour of integration, along the positive real axis of the complex plane with a Bessel function in the integrand is deformed to a contour along the positive imaginary axis in the same complex plane, but with a Hankel function of the first kind in the integrand for an assumed $\exp(j\omega t)$ time dependence. Later, in a book [19], they introduced an analysis of patch antennas in real space domain; it was noted therein that a moment method solution of the microstrip integral equation for patch current that makes use of rectangular cells in which constant currents flow provides sufficiently accurate results, besides being sufficiently versatile to study more complex structures. In retrospect, the present approach described in this work for obtaining an alternative, more efficient representation for the conventional Sommerfeld integral representation of the microstrip Green's function appears to be lead to expressions which are closely related to those given by Mosig and Gaydol in [19]; however, this more efficient representation is developed here by a quite different procedure than the one in [19] which requires the treatment of an additional fictitious integral arising from their contour deformation. E.H.Newman et al. [20] and D.M.Pozar [21] also used the moment method approach to study the mutual coupling of rectangular microstrip antennas in a manner similar to that done previously for the case of printed dipoles [15], [17] and
Both articles used the Fourier transform (or spectral) representation of the microstrip Green's function which employs the Fourier transform technique to facilitate the numerical integrations in the moment method impedance matrix by assuming that the surface currents flow in only one direction (longitudinal current) and that their amplitudes also change only in that direction. This spectral approach eliminates the usual self-term problems associated with the calculation of the diagonal terms of the moment method impedance matrix. However, this plane wave spectral method does not seem to be useful for the analysis of more complex shaped microstrip antennas. Besides, the efficiency of this plane wave spectral technique tends to diminish when the separation between the two current mode function increases since the spectral integrand oscillates more rapidly for the larger separation. So far a relatively simple and accurate closed form asymptotic approximation for the microstrip surface Green's function does not appear to have been given in the literature; J.R. Mosig and F.E. Gardiol in [19] mentioned that the asymptotic techniques cannot be used effectively for this purpose since the distance to the source for typical situations ranges from zero to several wavelengths. However, it was noticed in the two dimensional impedance surface problem [22] that the asymptotic techniques can provide a good closed form approximation provided the substrate is electrically thin. Therefore, the present study concentrates on finding a simple but accurate asymptotic solution for the surface field of a microstrip antenna. The well known saddle point integration method can be applied to asymptotically evaluate the
integral representation of the microstrip Green's function. However, this method involves a lengthy and complicated evaluation for the higher order terms in the asymptotic expansion which are required in this case (as the first order term vanishes when calculating the field on the surface). Unlike previous asymptotic integration methods [23,24] the present approach uses a simpler method (based partly on physical considerations) to obtain the complete asymptotic expansion of the integral provided \( k_1 d \sqrt{\varepsilon_r - 1} < \pi/2 \), where \( k_1 \) represents the wave number of free space, and \( d \) and \( \varepsilon_r \) denote the thickness and the relative dielectric constant of the substrate, respectively.

The format of this dissertation is as follows. In Chapter II, the electric field produced by a microstrip antenna is formulated in terms of the microstrip dyadic Green's function and the Green's function is constructed in terms of an appropriate magnetic vector potential; the latter is expressed in terms of the conventional Sommerfeld type integrals. Then, the microstrip surface Green's function is defined and approximated asymptotically in closed form by a new approach developed in this dissertation. Also, a very efficient numerical integration method is developed for an exact evaluation of the Sommerfeld type integrals by transforming them into alternative representation via Cauchy's residue theory. Finally, the far-zone microstrip Green's function is evaluated for completeness; this provides the radiation pattern of a microstrip antenna. In Chapter III, a brief review of the moment method solution for the coupled rectangular microstrip antennas is presented, and in Chapter IV, the excellent accuracy of the
approximate asymptotic closed-form representation for the microstrip surface Green's function is demonstrated by comparison with the exact numerical integration results for the Green's function even for relatively small separations (e.g. 0.1\lambda where \lambda = free space wavelength) between the source and field points. Also, numerical results for the self and mutual impedances in microstrip antenna arrays are compared with those given previously in the literature. In Chapter V, the conclusions are stated.

In the following, the time convention \exp(j \omega t) is assumed and suppressed; only here does 't' refer to time.
CHAPTER II

ANALYTICAL DEVELOPMENT

In the rigorous moment method (MM) analysis of microstrip antennas, the currents induced on the microstrip patch (or patches in case of an array) is expanded in terms of a set of entire domain or subsectional basis (expansion) functions or modes. The unknown amplitudes of these basis functions are then found via the MM using matrix inversion. The computation of the elements of the matrix to be inverted require a time consuming numerical integration of the Sommerfeld type integrals, in terms of which the microstrip surface Green's function is expressed. As a result of the slow convergence of the Sommerfeld type integrals for laterally separated source and field points, the rigorous analytical method becomes somewhat inefficient. Thus, it is necessary to find a closed form asymptotic approximation for these integrals which is simple and accurate enough to replace the exact integral representation in practical analysis. In the latter part of this chapter, an approximate closed form asymptotic representation for the microstrip surface dyadic Green's function is derived based on a new method developed in this work. However, first the microstrip dyadic Green's function is introduced symbolically and the intimate relation between the electromagnetic fields of the microstrip antenna and this Green's function is indicated in Section A. In section B, an exact integral representation of this microstrip Green's function is presented in terms of the
conventional Sommerfeld type integrals. In section C, the above microstrip dyadic Green's function is specialized to the case when the observation point is on the surface (z=0 in Figure 1.1); this case is of special interest in the MM solution of the microstrip problem. In section D, an efficient and exact but alternative representation of the surface Green's function is developed by transforming the Sommerfeld type integrals into a form which exhibits better convergence properties. Then, in section E, the Sommerfeld type integrals are approximated asymptotically based on a method developed in this work to yield an accurate closed form representation for the microstrip surface Green's function. In addition to the development of an approximate closed form representation for the microstrip Green's function, an asymptotic result for the far-zone microstrip Green's function is also presented for calculating the radiation field using the usual steepest descent method in the final section F.

A. FIELD FORMULATION IN TERMS OF MICROSTRIP DYADIC GREEN'S FUNCTION

The electric field of a tangential electric point current source of strength \( \bar{P}_e \) located at the coordinate origin \( \bar{r}=0 \) of the microstrip antenna configuration in Figure 2.1 is expressed first in terms of a microstrip dyadic Green's function as follows.

The governing vector wave equation for the electric field \( \bar{E} \) produced by an electric current density \( \bar{J} \) in the given geometry is given by [25]

\[
\nabla \times \nabla \times \bar{E}(\bar{r}) - k^2 \bar{E}(\bar{r}) = -j\omega \mu \bar{J}(\bar{r})
\]

(2.1)
Figure 2.1. Region $V_1$, bounded by an air-dielectric interface $S_d$ and a surface $S_{1\infty}$ at infinity contains an electric current source $\mathbf{J}$. Region $V_2$ is source free and bounded by an interface $S_d$, a perfect conducting surface $S_c$ and a surface $S_{2\infty}$ at infinity.

where $k^2 = \omega^2 \mu \varepsilon$ and may be replaced by $k_1^2 = \omega^2 \mu_1 \varepsilon_1$ in $V_1$ and $k_2^2 = \omega^2 \mu_2 \varepsilon_2$ in $V_2$ in which $(\mu_1, \varepsilon_1)$ and $(\mu_2, \varepsilon_2)$ denote the permeability and permittivity of the homogeneous, isotropic medium 1 and 2 respectively, as shown in Figure 2.1. The boundary conditions which must be satisfied by the field are

\[ \hat{z} \times \vec{E}_1 = \hat{z} \times \vec{E}_2 \quad \text{at } z=0 \quad (2.2) \]

\[ \hat{z} \times \frac{\mu_2}{\mu_1} (\nabla \times \vec{E}_1) = \hat{z} \times (\nabla \times \vec{E}_2) \quad \text{at } z=0 \quad (2.3) \]

and

\[ \hat{z} \times \vec{E}_2 = 0 \quad \text{at } z = -d \quad (2.4) \]

The subscripts 1 and 2 in (2.2), (2.3) and (2.4) denote that the field point is in regions $V_1$ and $V_2$, respectively. Now a dyadic Green's function $G$ is introduced such that

\[ \nabla \times \nabla \times \tilde{G}(\vec{r}; \vec{r}') - k^2 \tilde{G}(\vec{r}; \vec{r}') = \pm i \delta(\vec{r} - \vec{r}') \quad (2.5) \]
where \( \mathbf{I} \) denotes the unit dyad, \( \mathbf{r} \) and \( \mathbf{r}' \) denote the position vectors from the origin of the coordinate system to the observation and source points, respectively. The boundary conditions which \( \mathbf{G} \) must satisfy will be introduced later. Employing the vector Green's theorem [26] to the pair of functions \( \mathbf{E} \) and \( \mathbf{G} \) in the whole region consisting of \( V_1 \) and \( V_2 \), and interchanging the primed and unprimed variable leads to

\[
\mathbf{E}(\mathbf{r}) = -j\omega \int_{V_1} \mathbf{J}(r') \cdot \mathbf{G}(\mathbf{r}';\mathbf{r}) \, dv'
- \int_{S_{C}+S_{1\infty}+S_{2\infty}} \hat{z} \times \{ \nabla' \times \mathbf{E}(\mathbf{r}') \} \times \mathbf{G}(\mathbf{r}';\mathbf{r})
+ \hat{z} \times \mathbf{E}(\mathbf{r}') \times \{ \nabla' \times \mathbf{G}(\mathbf{r}';\mathbf{r}) \} \, ds'
\]  

(2.6)

The \( \nabla' \) denotes that the vector differentiation is carried out with respect to the primed coordinates. The surface integration on \( S_{1\infty} \) and \( S_{2\infty} \) in (2.6) vanish if we assume that \( \mathbf{E} \) and \( \mathbf{G} \) satisfy the radiation condition at infinity. Thus, only the integral on \( S_{C} \) remains in the second surface integral of (2.6). Now, one can choose the boundary condition for \( \mathbf{G} \) as follows;

\[
\hat{z} \times \mathbf{G}_{2} = 0 \quad \text{at } z = -d
\]  

(2.7)

which is analogous to the boundary condition (2.4) for the electric field \( \mathbf{E} \). This choice makes the surface integration over \( S_{C} \) in (2.6) vanish. Thus, the following simple formulation for the electric field \( \mathbf{E} \) is obtained

\[
\mathbf{E} = -j\omega \int \mathbf{J}(r') \cdot \mathbf{G}(r';\mathbf{r}) \, dv'
\]  

(2.8)
Using the identity

\[ \tilde{J} \cdot \tilde{G} = \tilde{G} \cdot \tilde{J} \]  \hspace{1cm} (2.9)

and the symmetry property

\[ \tilde{G}(\vec{r}'; \vec{r}) = \tilde{G}(\vec{r}; \vec{r}') \]  \hspace{1cm} (2.10)

where the symbol ~ denotes the transpose dyadic operator, gives the following expression:

\[ \tilde{E}(\vec{r}) = -j \omega \mu \int_{V_1} \tilde{G}(\vec{r}; \vec{r}') \cdot \tilde{J}(\vec{r}') d\vec{v}' \]  \hspace{1cm} (2.11)

As mentioned at the beginning of this part, we are interested in the case when the source \( \tilde{J}(\vec{r}') \) is an arbitrary horizontal electric point current source of strength \( \bar{p}_e \) located at \( \vec{r}'=0 \), i.e.,

\[ \tilde{J}(\vec{r}') = \bar{p}_e \delta(\vec{r}') \]  \hspace{1cm} (2.12)

where

\[ \bar{p}_e = \hat{x} p_{ex} + \hat{y} p_{ey} \]  \hspace{1cm} (2.13)

and \( \delta(\vec{r}') \) is the Dirac delta function. Then (2.11) reduces to

\[ \tilde{E}(\vec{r}) = -j \omega \mu \bar{G}(\vec{r};0) \cdot \bar{p}_e \]  \hspace{1cm} (2.14)

Thus, the boundary conditions at \( z=0 \) on \( \bar{G} \cdot \bar{p}_e \) must be the same as that on \( \bar{E} \) which are shown in (2.3). A matrix representation for (2.14) is, in rectangular coordinate system \((x,y,z)\),
\[
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
= -j\omega \begin{bmatrix}
G_{xx} & G_{zy} & G_{xz} \\
G_{yx} & G_{yy} & G_{yz} \\
G_{zx} & G_{zy} & G_{zz}
\end{bmatrix}
\begin{bmatrix}
p_{ex} \\
p_{ey}
\end{bmatrix}
\text{(2.15)}
\]

The quantity \( G \) in (2.14) or in (2.15) is referred to as the microstrip dyadic Green's function. In the next section (B), an exact integral representation will be developed for this microstrip Green's function.

B. AN EXACT INTEGRAL REPRESENTATION FOR THE MICROSTRIP GREEN'S FUNCTION

From (2.14) and (2.15) in previous section it is clear that
\[
j\omega G(\vec{r};0) \cdot \vec{p}_e
\]
is negative of vector electric field \( \vec{E} \) at \( \vec{r} \) produced by an electric point current source \( \vec{p}_e \delta(\vec{r}') = (\hat{x}p_{ex} + \hat{y}p_{ey})\delta(\vec{r}') \). Hence, the dyadic \( \hat{G} \) can be obtained once the field \( \vec{E} \) generated by the source is known. It is noted that the field \( \vec{E} \) can be evaluated by superposing the field produced by \( \hat{x}p_{ex} \delta(\vec{r}') \) and the field produced by \( \hat{y}p_{ey} \delta(\vec{r}') \). Also the field due to \( \hat{y}p_{ey} \delta(\vec{r}') \) can be directly obtained from the field of \( \hat{x}p_{ex} \delta(\vec{r}') \) via interchangeability of x-axis and y-axis in the planar microstrip structure as shown in Figure 2.2.

Therefore, it is important to first find the electric field generated by an \( \hat{x} \)-directed electric dipole of strength \( p_{ex} \) which is located at the coordinate origin (refer to Figure A.1 in Appendix A). The solution for this type of boundary value problem may be constructed in several ways; one of the most common approach is to express the field in terms of an appropriate magnetic vector potential [27]; whereas,
Figure 2.2. Geometrical relationship for the field.

Another commonly used approach is to construct the field in terms of both a magnetic vector potential and an electric vector potential [25]. In the former approach one may separate the field as sums of contributions from the point current source and a charge source which is related to the point current source by the continuity equation. On the other hand, the latter separates the field as sums of TE\textsubscript{z} mode and TM\textsubscript{z} mode components. In this dissertation, the electric field is expressed in terms of a magnetic vector potential [27] which has two magnetic potential components; one is parallel to the source, and the other is normal to the dielectric surface; thus,

\[ \mathbf{A} = \hat{x} A_{xx} + \hat{z} A_{zx} \quad \text{for the dipole source } p_{ex} \]  

(2.16)
\[ \vec{A} = \hat{y} A_{yy} + \hat{z} A_{zy} \quad \text{for the dipole source } p_{ex} \quad (2.17) \]

Refering to (A.18) of Appendix A, the electric field due to an \( \hat{x} \)-directed electric dipole located on the coordinate origin is, using the subscript \( p_{ex} \) to identify the source orientation, expressed as follows:

\[
\vec{E}_{p_{ex}} = \frac{p_{ex}}{j\omega \varepsilon} \left\{ \hat{x} \left( k^2 A_{xx} + \frac{\partial^2}{\partial x^2} A_{xx} + \frac{\partial}{\partial x} \frac{\partial}{\partial \zeta} A_{zx} \right) + \hat{y} \left( \frac{\partial^2}{\partial x \partial y} A_{xx} + \frac{\partial}{\partial y} \frac{\partial}{\partial \zeta} A_{zx} \right) + \hat{z} \left( k^2 A_{zx} + \frac{\partial^2}{\partial z^2} A_{zx} + \frac{\partial}{\partial \zeta} \frac{\partial}{\partial \zeta} A_{xx} \right) \right\} \quad (2.18)
\]

in which \( A_{xx} \) and \( A_{zx} \) denote the magnetic vector potential components.

As is derived in Appendix A, several alternative integral forms for these wave potentials are already available in literature [15,16,17]. Among them, following Sommerfeld type integral representations are of interest for our purpose, i.e., from (A.46, A.47, A.61, A.62) in Appendix A,

\[
A_{xx}(\vec{r}) = \frac{\mu_1}{2\pi} \int_0^{\infty} \frac{k_\rho J_0(k_\rho \rho)}{D_{TE}} e^{-j k z \zeta} d\rho \quad (2.19)
\]

\[
A_{zx}(\vec{r}) = -\frac{\mu_1}{2\pi} e^{-1} \cos \phi \int_0^{\infty} \frac{k_\rho^2 J_1(k_\rho \rho)}{D_{TE}} e^{-j k z \zeta} d\rho \quad (2.20)
\]

in Region \( V_1 \), and

\[
A_{xx}(\vec{r}) = \frac{\mu_2}{2\pi} \int_0^{\infty} \frac{k_\rho J_0(k_\rho \rho)}{D_{TE}} \frac{\sin k_\rho (z+d)}{\sin k_\rho d} d\rho \quad (2.21)
\]
\[
A_{zx}(\vec{r}) = -\frac{\mu_2}{2\pi} (\epsilon_r - 1) \cos \phi \int_0^\infty \frac{k^2 J_1(k_\rho \rho) \cos k_{z2}(z+d)}{D_{TE} \cos k_{z2}d} \, dk_\rho
\]  
\quad (2.22)

in region \( V_2 \), or
\[
A_{xx}(\vec{r}) = \frac{\mu_1}{4\pi} \int_C \frac{k_\rho H_0^{(2)}(k_\rho \rho) e^{-jk_{z1}z}}{D_{TE}} \, dk_\rho
\]  
\quad (2.23)

\[
A_{zx}(\vec{r}) = -\frac{\mu_1}{4\pi} (\epsilon_r - 1) \cos \phi \int_C \frac{k^2 H_1^{(2)}(k_\rho \rho) \cos k_{z2}(z+d)}{D_{TE} D_{TM} \cos k_{z2}d} \, dk_\rho
\]  
\quad (2.24)

in region \( V_1 \), and
\[
A_{xx}(\vec{r}) = \frac{\mu_2}{4\pi} \int_C \frac{k_\rho H_0^{(2)}(k_\rho \rho)}{D_{TE}} \frac{\sin k_{z2}(z+d)}{\sin k_{z2}d} \, dk_\rho
\]  
\quad (2.25)

\[
A_{zx}(\vec{r}) = -\frac{\mu_2}{4\pi} (\epsilon_r - 1) \cos \phi \int_C \frac{k^2 H_1^{(2)}(k_\rho \rho)}{D_{TE} D_{TM} \cos k_{z2}d} \, dk_\rho
\]  
\quad (2.26)

in region \( V_2 \) where
\[
D_{TE} = jk_{z1} + k_{z2} \cot(k_{z2}d)
\]  
\quad (2.27)
\[
D_{TM} = j\epsilon_r k_{z1} - k_{z2} \tan(k_{z2}d)
\]  
\quad (2.28)
\[
k_{z1} = \sqrt{k^2_{1\rho} - k^2}, \quad \text{Re}(k_{z1}) > 0, \quad \text{Im}(k_{z1}) < 0
\]  
\quad (2.29)
\[
k_{z2} = \sqrt{k^2_{2\rho} - k^2}, \quad \text{Re}(k_{z2}) > 0, \quad \text{Im}(k_{z2}) < 0
\]  
\quad (2.30)
\[ e_r = \frac{e_2}{e_1} \quad (2.31) \]

and

\[ \cos \phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}} \quad (2.32) \]

The contours of the integrals (2.19, 2.22, 2.23, 2.24) and (2.21, 2.22, 2.25, 2.26) are shown in Figures A.2 and A.3 in appendix A, respectively. The above definitions (2.27 - 2.32) will be employed throughout this dissertation unless indicated otherwise.

The electric field produced by a \( \hat{y} \)-directed electric dipole of strength \( p_{ey} \) can be obtained, as mentioned before, directly from the electric field of the dipole \( p_{ex} \) via interchangeability of \( x \)-axis and \( y \)-axis. Thus, direct interchange of \( x \) and \( y \) in (2.18) yields

\[
\mathbf{E}_{pey} = \frac{p_{ey}}{j\omega \epsilon} \left\{ \hat{x} \left( \frac{\partial^2}{\partial x \partial y} A_{yy} + \frac{\partial^2}{\partial x \partial y} A_{zy} \right) \right. \\
- \hat{y} \left( k^2 A_{yy} + \frac{\partial^2}{\partial y^2} A_{yy} + \frac{\partial^2}{\partial x \partial y} A_{zy} \right) \\
+ \hat{z} \left( k^2 A_{zy} + \frac{\partial^2}{\partial z} A_{zy} + \frac{\partial^2}{\partial y \partial z} A_{yy} \right) \right\} 
\quad (2.33) 
\]

Similar to the subscript \( p_{ex} \) in (2.18), the subscript \( p_{ey} \) in (2.33) is introduced to clarify that the fields are due to \( p_{ey} \). And the integral representations for the magnetic vector potential components \( A_{yy} \), \( A_{zy} \) are obtained by replacing \( \phi \) in (2.19 - 2.26) with \( \phi - \frac{\pi}{2} \); thus,
\begin{align}
A_{yy}(\vec{r}) &= A_{xx}(\vec{r}) \\
A_{zy}(\vec{r}) &= -\frac{\mu_2}{2\pi} (\epsilon_r-1) \sin \phi \int_0^\infty \frac{k^2 J_1(k_\rho \rho) e^{-jkz_1z}}{D_{TE} D_{TM}} \, dk_\rho
\end{align}
(2.34)

in region \( V_1 \), and

\begin{align}
A_{yy}(\vec{r}) &= A_{xx}(\vec{r}) \\
A_{zy}(\vec{r}) &= -\frac{\mu_2}{2\pi} (\epsilon_r-1) \sin \phi \int_0^\infty \frac{k^2 J_1(k_\rho \rho) \cos k_2(z+d)}{D_{TE} D_{TM} \cos k_2z_2d} \, dk_\rho
\end{align}
(2.35)

in region \( V_2 \), or

\begin{align}
A_{yy}(\vec{r}) &= A_{xx}(\vec{r}) \\
A_{zy}(\vec{r}) &= -\frac{\mu_2}{4\pi} (\epsilon_r-1) \sin \phi \int_C \frac{k^2 H_1^{(2)}(k_\rho \rho) \cos k_2(z+d)}{D_{TE} D_{TM} \cos k_2z_2d} \, dk_\rho
\end{align}
(2.36)

in region \( V_1 \), and

\begin{align}
A_{yy}(\vec{r}) &= A_{xx}(\vec{r}) \\
A_{zy}(\vec{r}) &= -\frac{\mu_2}{4\pi} (\epsilon_r-1) \sin \phi \int_C \frac{k^2 H_1^{(2)}(k_\rho \rho) \cos k_2(z+d)}{D_{TE} D_{TM} \cos k_2z_2d} \, dk_\rho
\end{align}
(2.37)

in region \( V_2 \), or

\begin{align}
A_{yy}(\vec{r}) &= A_{xx}(\vec{r}) \\
A_{zy}(\vec{r}) &= -\frac{\mu_2}{4\pi} (\epsilon_r-1) \sin \phi \int_C \frac{k^2 H_1^{(2)}(k_\rho \rho) \cos k_2(z+d)}{D_{TE} D_{TM} \cos k_2z_2d} \, dk_\rho
\end{align}
(2.38)

in region \( V_1 \), and

\begin{align}
A_{yy}(\vec{r}) &= A_{xx}(\vec{r}) \\
A_{zy}(\vec{r}) &= -\frac{\mu_2}{4\pi} (\epsilon_r-1) \sin \phi \int_C \frac{k^2 H_1^{(2)}(k_\rho \rho) \cos k_2(z+d)}{D_{TE} D_{TM} \cos k_2z_2d} \, dk_\rho
\end{align}
(2.39)

in region \( V_2 \), or

\begin{align}
A_{yy}(\vec{r}) &= A_{xx}(\vec{r}) \\
A_{zy}(\vec{r}) &= -\frac{\mu_2}{4\pi} (\epsilon_r-1) \sin \phi \int_C \frac{k^2 H_1^{(2)}(k_\rho \rho) \cos k_2(z+d)}{D_{TE} D_{TM} \cos k_2z_2d} \, dk_\rho
\end{align}
(2.40)

in Region \( V_2 \)
where
\[ \sin \phi = \frac{V}{\rho} = \frac{\rho}{\sqrt{x^2 + y^2}}. \]

Summing the fields \( \vec{E}_{\text{ex}} \) in (2.18) and \( \vec{E}_{\text{ey}} \) in (2.33) yields the electric field \( \vec{E} \) produced by both the \( \hat{x} \)-directed and \( \hat{y} \)-directed dipole sources; thus,

\[
\vec{E} = \frac{1}{j \omega \mu c} \left[ \hat{x} \left( p_{\text{ex}} (k^2 A_{xx} + \frac{\partial^2}{\partial x^2} A_{xx} + \frac{\partial^2}{\partial x \partial y} A_{zx}) + p_{\text{ey}} (\frac{\partial^2}{\partial x \partial y} A_{yy} + \frac{\partial^2}{\partial x \partial z} A_{zy}) \right) \\
+ \hat{y} \left( p_{\text{ex}} (\frac{\partial^2}{\partial x \partial y} A_{xx} + \frac{\partial^2}{\partial y \partial z} A_{zx}) + p_{\text{ey}} (k^2 A_{yy} + (\frac{\partial^2}{\partial y^2} A_{yy} + \frac{\partial^2}{\partial y \partial z} A_{zy}) \right) \\
+ \hat{z} \left( p_{\text{ex}} (k^2 A_{zx} + \frac{\partial^2}{\partial z^2} A_{zx} + \frac{\partial^2}{\partial x \partial z} A_{xx}) + p_{\text{ey}} (k^2 A_{zy} + \frac{\partial^2}{\partial z^2} A_{zy} + \frac{\partial^2}{\partial y \partial z} A_{yy}) \right) \right] 
\]

(2.42)

Now, comparing the electric field representation in (2.42) with (2.14) or (2.15) in previous section yields the following exact integral representation for the microstrip Green's function

\[
G_{xx} = \frac{1}{\mu k^2} \left[ (k^2 + \frac{\partial^2}{\partial x^2}) A_{xx} + \frac{\partial^2}{\partial x \partial y} A_{zx} \right] 
\]

(2.43)

\[
G_{yx} = \frac{1}{\mu k^2} \left[ \frac{\partial^2}{\partial x \partial y} A_{xx} + \frac{\partial^2}{\partial y \partial z} A_{zx} \right] 
\]

(2.44)

\[
G_{zx} = \frac{1}{\mu k^2} \left[ (k^2 + \frac{\partial^2}{\partial z^2}) A_{zx} + \frac{\partial^2}{\partial x \partial y} A_{xx} \right] 
\]

(2.45)
\[ G_{xy} = \frac{1}{\mu k^2} \left\{ \frac{\partial^2}{\partial x \partial y} A_{yy} + \frac{\partial^2}{\partial x \partial y} A_{zy} \right\} \]  
(2.46)

\[ G_{yy} = \frac{1}{\mu k^2} \left\{ (k^2 + \frac{\partial^2}{\partial x \partial y}) A_{yy} + \frac{\partial^2}{\partial x \partial y} A_{zy} \right\} \]  
(2.47)

\[ G_{zy} = \frac{1}{\mu k^2} \left\{ (k^2 + \frac{\partial^2}{\partial z^2}) A_{zy} + \frac{\partial^2}{\partial x \partial y} A_{yy} \right\} \]  
(2.48)

\[ G_{xz} = G_{yz} = G_{zz} = 0 \]  
(2.49)

where \( A_{xx}, A_{zx}, A_{xy}, \) and \( A_{zy} \) are those defined previously.

C. EXACT INTEGRAL REPRESENTATION FOR THE MICROWAVE'S FUNCTION EVALUATED ON THE SURFACE

Since the current on the microstrip patch is tangential to the surface, one does not have to be concerned with the normal \( (z) \) components of \( \mathbf{G} \). Thus, only the 4-components \( G_{xx}, G_{yx}, G_{xy} \) and \( G_{yy} \) remain to be considered.

As is defined in the previous chapter, the microstrip surface Green's function describes the fields on the air-dielectric interface where \( z=0 \). Hence, the exact integral representation for the microstrip surface Green's function can be obtained by simply substituting the value of zero for the variable \( z \) in the previously derived microstrip Green's function representation; of course, differentiation with respect to \( z \) should be performed prior to setting \( z \) equal to zero. Next, utilizing the following relations

\[ \frac{\partial}{\partial x} J_0(k \rho) = -\cos \phi \ k_\rho \ J_1(k \rho) \]  
(2.50)

\[ \frac{\partial}{\partial y} J_0(k \rho) = -\sin \phi \ k_\rho \ J_1(k \rho) \]  
(2.51)
\[
\frac{\partial}{\partial x} H_0^{(2)}(k_\rho \rho) = -\cos \phi \ k_\rho \ H_1^{(2)}(k_\rho \rho) \\
\frac{\partial}{\partial y} H_0^{(2)}(k_\rho \rho) = -\sin \phi \ k_\rho \ H_1^{(2)}(k_\rho \rho)
\]

and then interchanging differentiation and integration yields the following integral representation for the microstrip surface Green's function

\[
G_{xx}^S = \frac{1}{\mu_1 k_1^2} \left\{ \left( k_1^2 + \frac{\partial^2}{\partial x^2} \right) A_x + \frac{\partial^2}{\partial x^2} A_z \right\}
\]

\[
G_{yx}^S = \frac{1}{\mu_1 k_1^2} \left\{ \frac{\partial^2}{\partial x \partial y} A_x + \frac{\partial^2}{\partial x \partial y} A_z \right\}
\]

\[
G_{xy}^S = \frac{1}{\mu_1 k_1^2} \left\{ \frac{\partial^2}{\partial x \partial y} A_x + \frac{\partial^2}{\partial x \partial y} A_z \right\}
\]

\[
G_{yy}^S = \frac{1}{\mu_1 k_1^2} \left\{ \left( k_1^2 + \frac{\partial^2}{\partial y^2} \right) A_x + \frac{\partial^2}{\partial y^2} A_z \right\}
\]

where

\[
A_x(\rho) = \frac{\mu_1}{2\pi} \int_0^\infty \frac{k_\rho J_0(k_\rho \rho)}{D_{TE}} \, dk_\rho
\]

\[
A_z(\rho) = \frac{\mu_1}{2\pi} \left( \varepsilon - 1 \right) \int_0^\infty \frac{(-jk_{z1}) k_\rho J_0(k_\rho \rho)}{D_{TE}D_{TM}} \, dk_\rho
\]

or

\[
A_x(\rho) = \frac{\mu_1}{4\pi} \int_C \frac{k_\rho H_0^{(2)}(k_\rho \rho)}{D_{TE}} \, dk_\rho
\]
\[ A_z(\rho) = \frac{\mu_1}{4\pi} (\varepsilon_r^{-1}) \int_C \frac{(-jk_{z1}) k_\rho H_0^{(2)}(k_\rho \rho)}{D_{TE}D_{TM}} \, dk_\rho . \]  

(2.61)

The integration contours in (2.58, 2.59, 2.60, 2.61) are still the same as ones shown in Figure A.2 and Figure A.3. However, from now on, it is assumed that the following condition is always satisfied, namely:

\[ k_1 d \sqrt{\varepsilon_r^{-1}} < \frac{\pi}{2} , \]

(2.62)

which specifies that \( D_{TE} \) has no zero and \( D_{TM} \) has only one zero (see appendix A). This assumption may be supported from the fact that microstrip antennas are basically thin antennas, i.e., \( k_1 d \ll 1 \) [28]. The physical implication of the condition in (2.62) is that the frequency, dielectric substrate thickness and its electrical parameters are chosen such that only the lowest order surface wave mode is allowed to exist in that substrate.

Before going to the next section, it may be worth noting that the functions \( A_x \) and \( A_z \) in (2.58, 2.59) or (2.60, 2.61) are symmetric about the source point \( \rho = 0 \). This symmetric property, in conjunction with the fact that \( A_x \) and \( A_z \) represent outward propagating waves, provides the fundamental basis for developing an asymptotic closed form approximation for the integrals in the microstrip surface Green's function using a novel approach as described later in the following section E.

In the next section, an efficient numerical integration method based on the Cauchy's residue theory is introduced for an exact
evaluation of the Sommerfeld type integrals $A_x$ and $A_z$. This method, as
will be seen in the next section, is especially efficient for large and
even for moderately large values of $k_1 \rho$ in contrast to the usual real
axis integration method which becomes very slowly convergent as $k_1 \rho$
increases.

D. ALTERNATIVE, MORE EFFICIENT INTEGRAL REPRESENTATION FOR THE
MICROSTRIP SURFACE GREEN’S FUNCTION

Usually, Sommerfeld type integrals of $A_x$ and $A_z$ are evaluated along
the positive real axis using the integral forms in (2.58) and (2.59).
However, this real axis integration method has two main problems from a
numerical point of view. One is that the integrands oscillate over an
infinite integration interval, and the other is that a pole singularity
exists on the real axis. Although, many numerical techniques are
available in literature [29] to treat these problems efficiently,
generally they still require a very large number of integration points
for large values of $k_1 \rho$ because of the highly oscillating integrand.
Hence, it is desirable to find a different representation for the
Sommerfeld type integrals which circumvents those problems. Thus, one
begins by transforming the Sommerfeld type integrals in complex $k_\rho$-plane
into the complex $w$-plane by means of the following polar
transformations:

$$
\begin{align*}
  k_\rho &= k_1 \sin w, \\
  k_{z1} &= \sqrt{k_1^2 - \kappa^2} = k_1 \cos w
\end{align*}
$$

(2.63)

For convenience, consider the integral for $A_z$ first. Then, the
transformed Sommerfeld type integral for $A_z$ is
\[ A_z = \oint_{C_w} K_1 \frac{F(k_1 \sin w)}{D_{TE}D_{TM}} \cos w H_0^{(2)}(k_1 \rho \sin w) \, dw \] (2.64)

where

\[ F(k_1 \sin w) = \frac{\mu_1}{4 \pi (\epsilon_r - 1)} \left[ \begin{array}{c} -j k_1 k_\rho \\ D_{TE} D_{TM} \end{array} \right] k_\rho = k_1 \sin w \] (2.65)

The contour \( C_w \) in (2.64) represents the original contour \( C \) transformed into the complex \( w \)-plane and is shown in following Figure 2.3.

![Figure 2.3](image)

Figure 2.3. The complex \( w \) plane. \( C_w \) represents the original contour transformed into the complex \( w \) plane. The regions shaded with slanted lines and with horizontal lines correspond to top sheet where \( \Im(k_1^2 - k_\rho^2) < 0 \) and the bottom sheet where \( \Im(k_1^2 - k_\rho^2) > 0 \), respectively in the complex \( k_\rho \) plane.
Also, this figure shows pole and branch cut singularities which are images of corresponding singularities in the complex $k_\rho$-plane. However, images of the poles on the bottom sheet of the complex $k_\rho$ plane (where $\text{Im}(\sqrt{k_1^2-k_\rho^2})>0$) are not specified in the horizontally shaded regions, because, as will be noticed later, they do not have to be considered. Details involved in this transformation are not going to be described here since they are available later in section F of this chapter. Now, consider the integral

$$T = \int_{C_T} k_1 F (k_1 \sin w) \cos w H^{(2)}_0 (k_1 \rho \sin w) \, dw$$

(2.66)

where $C_T$ is a closed contour consisting of the original contour $C_w$ and five additional contours $C_1, C_2, C_3, C_4$ and $C_\infty$. Since the closed contour includes a pole at $w = \rho w_p$, we obtain, via the Cauchy's residue theory, the following result:

$$A_z = -2\pi j w_p$$

$$+ \left\{ \int_{C_1+C_2+C_3+C_4+C_\infty} k_1 F (k_1 \sin w) \cos w H^{(2)}_0 (k_1 \rho \sin w) \, dw \right\}$$

(2.67)

where $R$ is the residue of integrand at a pole $w=\rho w_p$ and is given by

$$R = \lim_{w \to \rho w_p} (w-\rho w_p) \left[ k_1 F (k_1 \sin w) \cos w H^{(2)}_0 (k_1 \rho \sin w) \right]$$

(2.68)

It can be easily verified that the contribution of $C_\infty$ to the integral is zero. On $C_1$ and $C_4$, the Hankel function in the integrand has an imaginary argument and thus becomes a modified Bessel function $K_\nu$ by the following relation

$$K_\nu(z) = \frac{\pi}{2} (-j)^{\nu+1} H^{(2)}_\nu (-jz)$$

(2.69)
A few algebraic modifications yields the following representation for $A_z$:

$$A_z = -2\pi j \cdot R$$

$$- \frac{4}{\pi} \int_0^\infty \text{Im} \left[ k_1 F (k_1 \nu \sin \nu) \right] K_0(k_1 \nu \sin \nu) d\nu$$

$$- 2j \int_0^{\pi/2} \text{Im} \left[ k_1 F (k_1 \nu \sin \nu) \right] H_0^{(2)}(k_\nu \sin \nu) d\nu \quad (2.70)$$

The representation (2.70) may be simplified by introducing new variables $t$ and $\lambda$ via the transformations

$$t = k_1 \nu \sin \nu \quad (2.71)$$

and

$$\lambda = k_1 \nu \sin \nu \quad (2.72)$$

for the second term and for the third term, respectively in (2.70). Finally,

$$A_z = -2\pi j \cdot R$$

$$+ \frac{4}{\pi} \int_0^\infty \text{Re} \left[ F(jt) \right] K_0(\rho t) dt$$

$$+ 2j \int_0^{\pi/2} \text{Im} \left[ F(\lambda) \right] H_0^{(2)}(\rho \lambda) d\lambda \quad (2.73)$$

where

$$R = \lim_{k \rightarrow 0} \left( k \rho - \lambda \rho \right) F(k_\rho \rho) H_0^{(2)}(k_\rho \rho)$$

$$\quad \rho \rightarrow \rho + \lambda \rho \quad (2.74)$$

As is well known, the residue term in (2.73) represents the bound (or non-radiating) surface wave field and the remaining integral terms represent the radiation field. It is interesting to notice that the
imaginary part of the radiation field, unlike the real part, appears only in the second integral term which has the integration interval $0 < \lambda < k_1$.

Following the same procedure yields a representation for $A_x$:

$$
A_x = \frac{4}{\pi} \int_0^\infty \text{Re} \left\{ G(jt) \right\} K_0(\rho t) \, dt \\
+ 2j \int_0^{k_1} \text{Im} \left\{ G(\lambda) \right\} H_0^{(2)}(\rho \lambda) \, d\lambda \tag{2.75}
$$

where

$$G(k \rho) = \frac{\mu_1}{4\pi} \frac{k \rho}{DTE} \tag{2.76}$$

Notice that $A_x$ consists of only the radiation field and its imaginary part also exists in the integration interval $0 < \lambda < k_1$ only.

The representations (2.73) and (2.75) provide an alternate way to compute the Sommerfeld type integrals $A_x$ and $A_z$. Notice that they do not have a pole singularity on the path of integration and are particularly useful for large values of $k_1 \rho$ since the first integral term over semi-infinite limits decays very rapidly due to the modified Bessel function $K_0(z)$ in the integrand which behaves like $\exp(-z)/\sqrt{z}$ for large values of $z$. Thus, in practice we may reduce the semi-infinite interval to a finite interval by ignoring the tail of the integrand.

They are also suitable representations for developing closed-form approximations for the microstrip Green's function which will be discussed in the next section.
Before finishing this section, it is worth noting that the same representation as presented here was derived in the complex $k_p$-plane by J.R. Mosig and F.E. Gardiol in [19]. Their method and the one presented here lead to very similar results; this may be expected since both methods deform the original path of integration over the imaginary axis using the Cauchy's residue theorem. However, in order to obtain a result similar to the above representation, they in [19] worked with a fictitious integral whose integrand includes a Hankel function of the first kind (despite an assumed $e^{j\omega t}$ time dependence). And the contour of integration in [19] was chosen to close the first quadrant of the complex plane $k_p$; whereas, the present approach introduces a polar transformation to arrive at a useful representation.

E. APPROXIMATE CLOSED FORM ASYMPTOTIC REPRESENTATION FOR THE MICROSTRIP SURFACE GREEN'S FUNCTION

It would be of interest to approximate the integrals for $A_x$ and $A_z$ asymptotically to arrive at a closed form approximation for the microstrip surface Green's function. Such a closed form representation would be far simpler and more efficient to use than an integral representation.

For relatively large values of $k_1\rho$, the integral field representations $A_x$ and $A_z$ may be approximated asymptotically by the saddle-point integration method after the Hankel functions in the integrands are replaced by the first-order large-argument approximation. This method is attractive particularly in the sense that these improper integrals can be approximated by evaluating contributions just from
appropriate critical points (in this case, a saddle-point and a pole). However, this technique involves a lengthy and complicated evaluation for the higher-order terms. The inclusion of higher order terms becomes necessary because the leading term in the ordinary saddle-point approximation vanishes in the case when the field point moves to the air-dielectric interface. Besides, the resulting asymptotic expansion is not for the original integral but for the one whose integrand is changed by replacing the Hankel function with its first-order large argument approximation. Thus, this conventional asymptotic (saddle-point) method is not efficient when the higher-order terms are necessary for a better asymptotic approximation of the integral.

In the following sections, the pertinent integrals are approximated asymptotically by a different approach which is largely based on the physical consideration of the outgoing wave radiation condition for \( k_1p^{\infty} \). First \( A_x \) will be approximated in part 1 of this section and then \( A_z \) will be approximated in part 2.

1. CLOSED FORM APPROXIMATION OF THE INTEGRAL FOR \( A_x \)

The integral field representation for \( A_x \) in (2.58) or (2.60) subject to the condition in (2.62) consists of only the continuous spectrum, i.e., the radiation field. Thus, the asymptotic approximation for the integral in (2.58) is expected to represent the radiation field but not the bound surface wave field which is also present on this structure; the latter is included in \( A_z \) through the presence of a surface wave pole in its integrand. When we separate the integral into
three sub-integration intervals, i.e.,

\[ 0 < k_\rho < k_1 \]  \hspace{1cm} (2.77)
\[ k_1 < k_\rho < k_2 \]  \hspace{1cm} (2.78)
\[ k_2 < k_\rho , \]  \hspace{1cm} (2.79)

it is easily seen that the imaginary part of this radiation field \( \text{Im}(A_x) \) comes mathematically from the integration interval \( 0 < k_\rho < k_1 \) and is given by

\[
\text{Im}(A_x) = \frac{u_1}{2\pi} \int_0^{k_1} \text{Im}\left(\frac{1}{D_{\text{TE}}}\right) k_\rho J_0(k_\rho \rho) \, dk_\rho
\]  \hspace{1cm} (2.80)

It is noted that the representation (2.80) is exactly the same as the imaginary part of (2.75) in the previous section. After employing the transformation

\[ k_\rho = k_1 t \]  \hspace{1cm} (2.81)

\( \text{Im}(A_x) \) is rewritten by

\[
\text{Im}(A_x) = \frac{u_1 k_1}{2\pi} \int_0^1 F_A(t) \sqrt{1-t^2} J_0(k_1 \rho t) \, dt
\]  \hspace{1cm} (2.82)

where

\[
F_A(t) = \left[ \text{Im}\left(\frac{1}{D_{\text{TE}}}\right)\right]_{k_\rho = k_1 t}
\]

\[
= - \frac{\sin^2 k_1 \sqrt{\varepsilon_r - t^2}}{(\varepsilon_r - t^2) \cos^2 k_1 \sqrt{\varepsilon_r - t^2} + (1-t^2) \sin^2 k_1 \sqrt{\varepsilon_r - t^2}}
\]  \hspace{1cm} (2.83)
Since, in microstrip antenna application, we may assume that
\( k_1d/\varepsilon_r-t^2 \ll 1 \) in the integration interval \( 0 < t < 1 \), \( F_A \) in (2.83) can be
approximated as follows:

\[
F_A(t) = - C_0 \sum_{n=0}^{2} A_n (1-t^2)^n
\]  

(2.84)

with

\[
C_0 = \frac{(k_1d)^2}{1 - (\varepsilon_r-1)(k_1d)^2}
\]  

(2.85)

\[
A_0 = 1 - \frac{1}{3} (k_1d)^2 (\varepsilon_r-1) + \frac{1}{36} (k_1d)^4 (\varepsilon_r-1)^2
\]  

(2.86)

\[
A_1 = - \frac{1}{3} (k_1d)^2 + \frac{1}{18} (k_1d)^4 (\varepsilon_r-1)
\]  

(2.87)

\[
A_2 = \frac{1}{35} (k_1d)^4
\]  

(2.88)

In obtaining the above result in (2.84) together with (2.85) through
(2.88), the following simplifications were introduced:

\[
\cos^2 k_1 \sqrt{\varepsilon_r-t^2} \; d = 1
\]  

(2.89)

\[
\sin^2 k_1 \sqrt{\varepsilon_r-t^2} \; d = (k_1d)^2 (\varepsilon_r-t^2)(1 - \frac{1}{6} (k_1d)^2 (\varepsilon_r-t^2))^2
\]  

(2.90)

In fact, as far as \( k_1d/\sqrt{\varepsilon_r-t^2} < \frac{\pi}{2} \) in the integration interval \( 0 < t < 1 \),
the function \( F_A \) can be expanded exactly in an infinite series of the
form
\[ F_A = \sum_{n=0}^{\infty} B_n (1-t^2)^n \]  \hspace{1cm} (2.91)

where the coefficients \( B_n \) are constants depending upon the parameters \( k_{1d} \) and \( \epsilon_r \) (see Appendix B). However, retaining all the terms as in (2.91) yields an infinite asymptotic series representation for \( A_x \) as explained in Appendix B thereby requiring additional consideration of where to truncate the series to obtain a useful closed form approximation for \( A_x \). On the other hand, using the above approximation (2.84) yields a finite series approximation for \( A_x \) (refer to (2.100)) as is described in the following which is quite accurate for usual microstrip antenna practice as observed from the numerical results in Chapter IV. Thus, rather than retaining all the terms as in (2.91), the approximation of (2.84) is used. Also it is important to note that the accuracy of the approximations (2.89) and (2.90) is more sensitive to changes in \( k_{1d} \) than the changes in \( \epsilon_r \), since \( \epsilon_r \) lies inside the radical in the argument of sinusoidal functions.

Now, substituting (2.84) into (2.82) and using the integration formula

\[ \int_0^1 x^{v+1} (1-x^2)^ \mu J_v(bx) = 2 \Gamma(\mu+1) b^{-\mu-1} J_{\nu+\mu+1}(b) \]  \hspace{1cm} (2.92)

\((b > 0, \text{Re } \nu > -1, \text{Re } \mu > -1)\)

yields

36
\[
\text{Im}(A_x) = -\frac{u_{1k_1}}{2\pi} C_0 \sum_{n=0}^{2} A_n(2)^{n+1/2} \Gamma(n+\frac{1}{2}+1)
\]

\[
(k_1\rho)^{-(n+1/2+1)}J_{n+1/2+1}(k_1\rho)
\]

(2.93)

In the above, \(\Gamma\) denotes the Gamma function; also note that the Bessel function in (2.93) is of fractional order. With the help of the following identity

\[
j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z),
\]

(2.94)

where \(j_n(z)\) represents the spherical Bessel function of the first kind, and the relations

\[
j_n(z) = f_n(z)\sin z + (-1)^{n+1}f_{-n-1}(z)\cos z
\]

\[
f_0(z) = z^{-1}, \quad f_1(z) = z^{-2}
\]

\[
f_{n-1}(z) + f_{n+1}(z) = (2n+1)z^{-1}f_n(z)
\]

(2.95)

\(n = 0, \pm 1, \pm 2, \ldots\)

we find the following expression for \(\text{Im}(A_x)\)

\[
\text{Im}(A_x) = \frac{u_{1k_1}}{2\pi} C_0 \left[ -\sin k_1 \rho \left\{ \frac{C_3}{(k_1\rho)^3} + \frac{C_5}{(k_1\rho)^3} + \frac{C_7}{(k_1\rho)^7} \right\} 
\]

\[
+ \cos k_1 \rho \left\{ \frac{C_2}{(k_1\rho)^2} + \frac{C_4}{(k_1\rho)^4} + \frac{C_6}{(k_1\rho)^6} \right\} \right].
\]

(2.96)

where
\[ C_2 = A_0 \]
\[ C_3 = A_0 = 3A_1 \]
\[ C_4 = 9A_1 - 15A_2 \]
\[ C_5 = 9A_1 - 90A_2 \]
\[ C_6 = 225A_2 \]
\[ C_7 = 225A_2 \]

(2.97)

It is noted that \( C_0, C_2, C_3, C_4, \ldots C_7 \) are all real if the
dielectric is lossless. For the time being let it be assumed that the
dielectric is lossless; this condition will be relaxed shortly;
consequently, since (2.96) furnishes the imaginary part of \( A_x \) and it is
known from physical considerations that \( A_x \) must satisfy the outgoing
wave radiation condition for \( k_1 \rho \to \infty \), it is therefore possible to deduce
\( A_x \) heuristically from \( \text{Im}(A_x) \) by simply replacing

\[- \sin k_1 \rho \text{ with } e^{-jk_1 \rho} \]  

(2.98)

and likewise by replacing

\[ \cos k_1 \rho \text{ with } j e^{-jk_1 \rho} \]  

(2.99)

Thus, the result for \( A_x \) is given by:

\[ A_x = \frac{\mu_1 k_1}{2\pi} C_0 e^{-jk_1 \rho} \sum_{n=2}^{7} \frac{D_n}{(k_1 \rho)^n} \]  

(2.100)

where
\[ D_2 = j C_2 = j A_0 \]
\[ D_3 = C_3 = A_0 - 3A_1 \]
\[ D_4 = j C_4 = j(9A_1 - 15A_2) \]
\[ D_5 = C_5 = 9A_1 - 90A_2 \]
\[ D_6 = j C_6 = j 225A_2 \]
\[ D_7 = C_7 = 225A_2 \]

(2.101)

The above result for \( A_x \) in (2.100) based on the \( \text{Im}(A_x) \) of (2.96) is accurate as can be seen from numerical results in Figures 4.1 through 4.12 which show comparison of the exact numerical integration result for \( A_x \) from the expression in (2.75) and the result from the closed form approximation in (2.100).

The corresponding results for the lossy dielectric case can now be obtained directly from (2.100) via analytic combination by making \( \epsilon_r \) in that expression complex to account for loss. It is seen in Chapter IV that numerical results based on (2.100) are indeed very accurate even for (\( \rho \)) as small as 0.2\( \lambda \) (\( \lambda \) = free space wavelengths) and it is also seen that (2.100) gives good results if the dielectric is lossy thereby lending confidence to the present approach.
2. CLOSED FORM APPROXIMATION OF THE INTEGRAL FOR $A_Z$

Unlike $A_x$, the Sommerfeld type integral of $A_z$ in (2.59) or (2.61) yields both a surface (or pole) wave and a continuous (or radiation field) spectrum contribution to the field (as may be seen by referring to (2.73) in the previous section). However, the solution for the surface wave is easily obtained by calculating the residue of the integrand at the pole (refer to the first term in (2.73) of the previous section). And the continuous (or radiation) spectral contribution may be approximated in a manner similar to that employed previously for $A_x$ after $D_{TM}$ in the integrand of (2.59) is approximated as follows using the small argument approximation for sinusoidal functions:

$$D_{TM} = \frac{\cos k_1 \sqrt{\varepsilon_r - t^2} \, d}{\varepsilon_r \sqrt{1-t^2} \cos k_1 \sqrt{\varepsilon_r - t^2} \, d + j \sqrt{\varepsilon_r - t^2} \sin k_1 \sqrt{\varepsilon_r - t^2} \, d}$$

$$= \frac{1}{\varepsilon_r^2} (-jk_1 + \frac{\varepsilon_r}{\sqrt{1-t^2}})$$

(2.102)

The pole and continuous spectral wave contributions can be treated distinctly only outside the "so called" surface (pole) wave transition region [30]; however, within this transition region which exists in the near field of the microstrip patch it is necessary to include the interaction between these two different waves because the pole and continuous spectral waves exhibit a coupling effect [30] within the transition region where the effects are no longer distinct. Mathematics related to the field description within the transition region is
available in literature [23]. Carrying out the necessary calculations (with relevant details in Section F) yields the following closed form asymptotic approximation for the integral

\[
A_z = -2\pi jR \\
+ \frac{1}{2\pi} \frac{C_0}{\varepsilon_r^2} (\varepsilon_r - 1) e^{-j k_1 R} \sum_{n=2}^{7} \frac{e_n}{(k_1 R)^n} \\
+ [-j H_0^{(2)}(k_1 R) K_0(b/k_1 R) + \frac{e^{-j k_1 R}}{k_1 R} \sqrt{2jR/b}] 
\]  

(2.103)

where R is the residue of the integrand at the pole \(k_\rho = \lambda_\rho\) (see (2.74)) and

\[
e_2 = j d_0 \\
+ e_3 = d_0 - 3d_1 \\
e_4 = j(9d_1 - 15d_2) \\
e_5 = 9d_1 - 90d_2 \\
e_6 = j225d_2 \\
e_7 = 225d_2 \\
d_n = \varepsilon_r A_n + b_n \quad ; \quad n=0,1,2 \\
b_o = 1 - 2/3 (k_1 d)^2 (\varepsilon_r - 1) + 2/15(k_1 d)^4 (\varepsilon_r - 1)^2 \\
b_1 = -2/3(k_1 d)^2 + 4/15(k_1 d)^4 (\varepsilon_r - 1) \\
b_2 = 2/15(k_1 d)^4
\]
\[ b = \sqrt{-j \cos \omega_p} \]
\[ w(z) = e^{-z^2} \text{erfc}(z) \]

where
\[ w_p = \frac{\pi}{2} + j \cosh^{-1}\left(\frac{\lambda_p}{k_1}\right). \]

In the above, the first term represents the non-radiating or bound surface wave, the second terminating a sum the radiation field contribution and the third one represents the transition function describing the interaction between the radiation field and the surface wave field. As briefly mentioned before, the third term was obtained partly heuristically by applying the uniform saddle point integration method to the integral representation for \( \text{Az} \) in (2.64) as discussed in Felsen and Marcuvitz [23].

Finally, substituting the approximations (2.100) and (2.103) into (2.54), (2.55), (2.56), (2.57) yields the closed form approximation for the microstrip surface Green's function.

F. MICROSTRIP FAR ZONE GREEN'S FUNCTION

The far zone microstrip Green's function refers to the evaluation of that Green's function in region 1 (see Figure 2.2) when the field point is far from the source. In this case, the Green's function never becomes singular. Hence, it is better, in practice, to have the Green's function representation without differential operators. Using the relations
\[ \frac{3}{\partial x} H_0^{(2)}(k_\rho \rho) = -k_\rho \cos \phi H_1^{(2)}(k_\rho \rho) \quad (2.104) \]

\[ \frac{3}{\partial x} H_1^{(2)}(k_\rho \rho) = -k_\rho \cos \phi (H_0^{(2)}(k_\rho \rho) - \frac{1}{k_\rho \rho} H_1^{(2)}(k_\rho \rho)) \quad (2.105) \]

\[ \frac{3}{\partial x} \cos \phi = \frac{1}{\rho} (1 - \cos^2 \phi) \quad (2.106) \]

the Green's function (2.43) to (2.49) becomes

\[ G_{xx} = k_1^2 g_0 - \cos^2 \phi g_1 + \frac{\cos^2 \phi}{\rho} g_2 \quad (2.107) \]

\[ G_{yx} = \sin 2\phi \left( \frac{1}{\rho} g_2 - \frac{1}{2} g_1 \right) \quad (2.108) \]

\[ G_{zx} = \cos \phi g_3 \quad (2.109) \]

\[ G_{xy} = -\sin 2\phi \left( \frac{1}{\rho} g_2 - \frac{1}{2} g_1 \right) \quad (2.110) \]

\[ G_{yy} = k_1 g_0 - \sin^2 \phi g_1 - \frac{\sin^2 \phi}{\rho} g_2 \quad (2.111) \]

\[ G_{zy} = \sin \phi g_3 \quad (2.112) \]

\[ G_{xz} = G_{zy} = G_{zz} = 0 \quad (2.113) \]

in which

\[ g_n(\vec{r}/0) = \int k_\rho^{n+1} H_n^{(2)}(k_\rho \rho) u_n(k_\rho) e^{-jk_\rho z_{12}} dk_\rho, \quad n=0,1,2,3 \quad (2.114) \]

with

\[ u_0(k_\rho) = \frac{1}{D_{TE}} \quad (2.115) \]
\[ u_1(k_{\rho}) = \frac{(jk_{z1} - k_{z2} \tan k_{z2} d)k_{\rho}^2}{D_{TE}D_{TM}} \]  
(2.116)

\[ u_2(k_{\rho}) = \frac{(jk_{z1} - k_{z2} \tan k_{z2} d)}{D_{TE}D_{TM}} \]  
(2.117)

\[ u_3(k_{\rho}) = \frac{-k_{z2} \tan k_{z2} d}{D_{TM}} \]  
(2.118)

The contour of integration for the above integrals is shown in Figure A.3 in appendix A. For evaluation of the far zone Green's function, it is desirable to transform the integrals in the complex \( k_{\rho} \) plane into the ones in the complex \( w \) plane via the polar transformations

\[ k_{\rho} = k_1 \sin w, \quad k_{z1} = \sqrt{k_1^2 - k_{\rho}^2} = + k_1 \cos w \]  
(2.119)

and

\[ x = \rho \cos \phi, \quad y = \rho \sin \phi \]  
(2.120)

For the transformations (2.120), one may refer to Figure A.2. The plus sign in the second one of (2.119) is chosen to make the point \( k_{\rho} = 0 \) correspond to \( w = 0 \). Then, the two Riemann sheets of \( k_{\rho} \) plane are mapped into the strip \(-\pi < \text{Re}(w) < \pi\), and the transformed integrals are, after the Hankel functions are replaced by their first-order large argument approximation,

\[ g_n(r/0) \sim \int_{C_w} F_n(w) e^{i q(w)} dw, \quad n=0,1,2,3 \]  
(2.121)
\[ F_n(w) = j^n \left( \frac{2j}{\pi k_1 \sin \theta \sin w} \right)^{1/2} (k_1 \sin w)^{n+1} k_1 \cos \omega_n (k_1 \sin w) \]  \hspace{1cm} (2.122)

\[ q(w) = -j \cos (w-\theta) \]  \hspace{1cm} (2.123)

\[ \Omega = k_1 r . \]  \hspace{1cm} (2.124)

Notice that, in obtaining the above representations, the following replacements are executed:

\[ \rho = r \sin \theta, \ z = r \cos \theta \]  \hspace{1cm} (2.125)

The transformation in (2.125) is shown in Figure A.2. The mapping of the contour of integration \( C \) in the complex \( k_\rho \) plane onto the contour \( C_w \) in the complex \( w \) plane is illustrated in Figure 2.4.

The pole at \( k_\rho = \lambda_\rho \) is now located at

\[ w = w_\rho = \frac{\pi}{2} + j \cosh^{-1} \left( \frac{\lambda_\rho}{k_1} \right) \]  \hspace{1cm} (2.126)

and the branch cuts associated with the branch points \( k_\rho = \pm k_1 \) are removed due to the transformation. However, the branch cut associated with the Hankel function remains within the complex \( w \) plane. \( w_s \) and \( C_{SDP} \) in Figure 2.5 denote the saddle point and the steepest descent path through the saddle point; the former is defined by \( q'(w_s) = 0 \), i.e. \( w_s = \theta \) and the latter is defined by \( \text{Im}(q(w)) = \text{Im}(q(w_s)) \). It is noted that the pole \( w_\rho \) is located between \( C_w \) and \( C_{SDP} \) only when \( \theta > \theta_c = \sin^{-1}(k_1/\lambda_\rho) \).

Now, in deforming \( C_w \) into the \( C_{SDP} \), we obtain, via the Cauchy's residue theory, the following results:
Figure 2.4. The complex $w$ plane: $C_w$, original contour in the complex $w$ plane; $C_{SDP}$, steepest descent path; $w_p$ and $w_s$ denote, respectively, the pole and saddle point.

\[
\int_{C_w} F_n(w) e^{\Omega q(w)} dw = \int_{C_{SDP}} F_n(w) e^{\Omega q(w)} dw - U(\theta - \theta_c) \cdot 2\pi j R_n \tag{2.127}
\]

where $U$ is the Heaviside unit step function and $R_n$ denotes the residue of corresponding integrands at $w_p$. However, $R_0$ is defined to be zero since the corresponding integrand of (2.121) does not have pole singularity. Contributions from the integrals joining the two paths $C_w$ and $C_{SDP}$ at infinity is eliminated since the term $\exp(2q)$ vanishes in these regions. A saddle point evaluation of the first term in (2.127) yields the following approximation:
\[ g_n(r_0) \sim + 2jk_1 \cos \theta \ (jk_1 \sin \theta)^n u_n(k_1 \sin \theta) e^{-jk_1 r} \]
\[ - U(\theta - \theta_c) 2\pi j R_n \]
\[ + \{ \pm j R_n \pi e^{-k_1 rb^2} \text{erfc}(\pi j b k_1 r) + \frac{\pi}{\sqrt{k_1 r}} \} e^{-jk_1 r} \]
\[ ; \ \text{Im}(b) > 0 \]

where

\[ R_0 = 0 \] (2.129)
\[ R_n = \left[ \frac{-u_n(k_p \rho)D_{TM}'(k_p \rho)H_n(2)(k_p \rho) e^{-jk_1 z}}{D_{TM}} \right]_{k = \lambda_p}^{n = 1, 2, 3} \] (2.130)
\[ b = \sqrt{q(w_s) - q(w_p)} \ ; \ w_s = \theta, \ w_p = \frac{\pi}{2} + j \cosh^{-1}(\frac{\lambda_p}{k_1}) \] (2.131)
\[ \text{erfc} z = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-x^2} \, dx \] (2.132)
\[ U(\theta - \theta_c) = \begin{cases} 1 & \text{for } \theta > \theta_c = \sin^{-1} \frac{k_1}{\lambda_p} \end{cases} \] (2.133)

and \( D_{TM}' \) denotes the first derivative of \( D_{TM} \) with respect to \( k_p \).

As mentioned in the previous section, the last term in (2.128) describes the interaction between the surface wave and space wave represented by the first term and the second term, respectively.

Finally, the microstrip far zone Green's function can be obtained by simply substituting the above approximation in (2.128) through (2.133) into (2.107) through (2.112).
CHAPTER III

APPLICATION OF THE MICROSTRIP GREEN'S FUNCTION

TO THE MM SOLUTION FOR THE COUPLED RECTANGULAR MICROSTRIP ANTENNAS

In this chapter, a brief review of the moment-method (MM) solution for the coupled rectangular microstrip antennas is presented, and the exact expressions for the impedance matrix \([Z]\) and the voltage matrix \([V]\) which occurs in the MM solution are developed in the real space domain via the use of a cylindrical wave spectral representation for the microstrip surface Green's function. Even though the coupled microstrip antennas of rectangular shape are considered in this chapter, the following discussions may be directly employed to deal with arbitrarily shaped microstrip patch antennas.

In the literature \([20,21]\), the plane wave spectral representation has been used for the microstrip surface Green's function in the MM analysis of the coupled rectangular microstrip antennas. Also, in the MM, the actual current induced on the microstrip patches has generally been expanded in terms piecewise sinusoidal or entire base sinusoidal functions. It allows us to perform the integrations involved in the expression for the MM \([Z]\) matrices in closed form by Fourier-transforming the suitably chosen current functions and the resulting integrands do not contain the usual Green's function singularity any longer. However, as is expected from the properties of the Fourier-transform pair, the Fourier transformed integrands decay
more slowly as the width of the basis functions decreases and oscillate more rapidly as the distance between two basis functions increases. Thus, this plane wave spectral method is not very efficient for the evaluation of the mutual impedance elements in $[Z]$ especially when the two expansion modes are narrow and widely separated. Also, for the evaluation of both $[Z]$ and $[V]$, very careful numerical integration is required in the vicinity of the surface wave pole singularity [15], which results in the degradation in the efficiency of this method. On the other hand, one can simply replace the Green's function by an alternative representation according to the needs, in the real space domain approach unlike that in the plane wave spectral approach; for example, the representations in section D of chapter 2 may be used to yield expressions for $[Z]$ and $[V]$ which are useful especially when two basis functions are widely separated. However, the integrands possess the Green's function singularity problem which must be solved in a somewhat analytical way when the two basis functions overlap as will be discussed later.

This chapter consists of three sections. In section A, the well known moment-method solution for the two rectangular microstrip antennas is briefly reviewed. In section B, the exact expressions for the mutual and self impedances of $[Z]$ are presented respectively in subsections B.1 and B.2. In section C, the exact expressions for the mutual and self impedances of $[V]$ are presented respectively in subsections C.1 and C.2.
A. BRIEF REVIEW OF THE MM SOLUTION FOR THE COUPLED RECTANGULAR MICROSTRIP ANTENNAS

Before deriving the real space domain expressions for the $Z$ and $V$ quantities, the well known MM solution for the coupled rectangular microstrip antennas [21,22] is briefly reviewed.

Figure 3.1 shows two microstrip patch antennas of rectangular shape located on a grounded dielectric slab.

![Diagram of two coupled microstrip antennas on a grounded dielectric slab.]

Figure 3.1. Two coupled microstrip antennas on a grounded dielectric slab.

Patch 1 is fed by an impressed current source $\tilde{J}_1$. The $\tilde{J}_S$ is the surface current density induced on the surface of the patches. Let the electric fields $\tilde{E}_S$ and $\tilde{E}_1$ denote the fields due to $\tilde{J}_S$ and $\tilde{J}_1$, respectively, in the presence of the grounded dielectric slab. Then the integral
equation of the unknown current can be obtained via the reciprocity
theorem after introducing an arbitrary test current $\vec{J}_T$, which is placed
inside the real microstrip patches and generates the field $\vec{E}_T$ in the
presence of the grounded dielectric slab, as follows:

$$- \int_S \vec{E}_T \cdot \vec{J}_S \, ds = \iiint_V \vec{E}_T \cdot \vec{J}_I \, dv$$  \hspace{1cm} (3.1)

In the above, the notation $\int_S$ is used to denote that the integration is
performed over the closed surface of the patches. However, since the
patches are so thin (open surface), we may neglect the current induced
on the side wall of the patches [31]. Then, the Equation (3.1) may be
simplified as follows:

$$- \int_S \vec{E}_T \cdot \vec{J}_S \, ds = \iiint_V \vec{E}_T \cdot \vec{J}_I \, dv$$  \hspace{1cm} (3.2)

where $\vec{J}_S$ denotes the vector sum of the surface current densities induced
on the upper and lower sides of the patches and the surface integral is
over the area on the dielectric-air interface which is occupied by the
microstrip patches. This integral equation may be solved by the
moment-method. Let the unknown current $\vec{J}_S$ be approximated by a finite
summation as

$$\vec{J}_S = \sum_{n=1}^{N} I_n \vec{J}_n$$  \hspace{1cm} (3.3)

where $\vec{J}_n$ is the nth basis function (expansion mode) and $I_n$ is its
unknown amplitude. Inserting (3.3) into (3.2) and then using the same
set of functions as test current functions (Galerkin) leads to a set of
linear algebraic equations to be solved for the unknown $I_n$:
\[
\sum_{n=1}^{N} I_n Z_{mn} = V_m \quad m=1,2,\ldots,N
\]  
(3.4)

where

\[
Z_{mn} = - \iint_{S_n} E_m(\vec{r}_{mn} + \vec{r}_n) \cdot \vec{J}_n(\vec{r}_n) \, ds
\]  
(3.5)

\[
V_m(q) = \iiint_{V} E_m(\vec{r}_m) \cdot \vec{J}_i(\vec{r}_m) \, dv
\]  
(3.6)

where \( \vec{r}_{mn} = (x_{mn}, y_{mn}, 0) \) is the displacement vector from the center of mode \( m \) to the center of mode \( n \), \( \vec{r}_m = (x_m, y_m, z) \) is the position vector referenced to the center of mode \( m \) and the \( q \) denotes that the mode \( q \) is excited by the impressed current source \( J_i \). In (3.5) and (3.6), \( S_n \) is the surface of the \( n \)th expansion mode and \( E_m \) is the electric field due to current \( J_m \) in the presence of the grounded dielectric slab which is given by, from (2.11) in previous chapter,

\[
\vec{E}_m(\vec{r}) = -j\omega \varepsilon \iiint_{V} \vec{G}(\vec{r};\vec{r'}) \cdot \vec{J}_m(\vec{r'})dv'
\]  
(3.7)

where the Green's function \( \vec{G} \) is the one defined in (2.43) to (2.49) or (2.54) to (2.57) depending on the location of observation points.

The excitation \( \vec{J}_i \) is modeled as

\[
\vec{J}_i(\vec{r}_m) = \hat{z} \delta(\vec{r}_m - \vec{r}_{mf}), \quad -\alpha < z < 0
\]  
(3.8)

where \( \vec{r}_{mf} \) is the displacement vector from the center of mode \( m \) to the feed location. Also, \( Z_{mn} = Z_{nm} \) since the same set of functions is used for the test current and the unknown current.
The convergence and accuracy of the MMM solution depends upon how well the true \( \tilde{J}_s \) is approximated by the sum of finite basis functions. Thus the choice of the basis function is a crucial step in the MMM solution. However, for the rectangular microstrip antennas, a single entire basis (EB) or a piecewise sinusoidal (PWS) expansion mode per each patch seems to yield reasonably accurate results as presented in [21]. In the next two sections, exact representations for impedances [Z] and [V] are derived based on the EB expansion mode given by

\[
\tilde{J}_n(x_n, y_n) = \begin{cases} 
\frac{1}{2w} \sin \frac{\pi (x_n + H)}{2H} & \text{for } |x_n| < H, |y_n| < W \\
0 & \text{, elsewhere}
\end{cases}
\] (3.9)

where \( 2W \) and \( 2H \) are width and length of two identical patches and \( n \) is identical to the patch number.

B. EXACT EXPRESSION OF [Z] IN THE REAL SPACE DOMAIN

1. MUTUAL IMPEDANCE \( Z_{mn}(n \neq m) \)

Substituting (3.9) into (3.7) and substituting this result into (3.5) yields

\[
Z_{mn} = \frac{j \omega w_1}{4w^2} \int \int \int \int G_s^S(\rho) \sin \frac{\pi (x_n' + H)}{2H} \sin \frac{\pi (x_n + H)}{2H} dx_n' dy_n' dx_n dy_n
\] (3.10)

where, from (2.54) in Chapter II,
\[ G_{xx}(\rho) = \frac{1}{u_1 k_1^2} \left\{ k_1^2 A_x(\rho) + \frac{\partial^2}{\partial x_n^2} (A_x(\rho) + A_z(\rho)) \right\} \]  \hspace{1cm} (3.11)

and

\[ \rho = \sqrt{(x_{mn} + x_n - x_m')^2 + (y_{mn} + y_n - y_m')^2} \]  \hspace{1cm} (3.12)

Since

\[ \frac{\partial}{\partial x_n} A_{x,z}(\rho) = - \frac{\partial}{\partial x_m} A_{x,z}(\rho) \]  \hspace{1cm} (3.13)

we can eliminate the partial differential operators in (3.11) via integration by parts technique. Then, applying the following formula

\[ \sin \alpha \sin \beta = \frac{1}{2} \left\{ \cos(\alpha - \beta) - \cos(\alpha + \beta) \right\} \] \hspace{1cm} (3.14)

\[ \cos \alpha \cos \beta = \frac{1}{2} \left\{ \cos(\alpha - \beta) + \cos(\alpha + \beta) \right\} \] \hspace{1cm} (3.15)

we now have

\[ Z_{mn} = \frac{j \omega}{8W^2} \int \int \int \left( \cos \frac{\pi (x_n - x_m')}{2H} + \cos \frac{\pi (x_n + x_m')}{2H} \right) \]

\[ A_x(\rho) \text{dx}_m \text{dy}_m \text{dx}_n \text{dy}_n \]

\[ A_x(\rho) \text{dx}_m \text{dy}_m \text{dx}_n \text{dy}_n \]

\[ - \frac{j \omega n^2}{32w^2H^2 k_1^2} \int \int \int \left( \cos \frac{\pi (x_n - x_m')}{2H} - \cos \frac{\pi (x_n + x_m')}{2H} \right) \]

\[ (A_x(\rho) + A_z(\rho)) \text{dx}_m \text{dy}_m \text{dx}_n \text{dy}_n \]

\hspace{1cm} (3.16)

The 4-folded integral representation of \( Z_{mn} \) becomes the double integral representation via the following transformations (see Appendix C).
\[ \alpha = x_n - x_m' \quad (3.17) \]

\[ \beta = x_n + x_m' \quad (3.18) \]

And

\[ \zeta = y_n - y_m' \quad (3.19) \]

\[ \eta = y_n + y_m' \quad (3.20) \]

Then, applying the transformation

\[ u = \frac{\alpha}{2H} \quad (3.21) \]

\[ v = \frac{\beta}{2W} \quad (3.22) \]

We have

\[
Z_{mn} = j \frac{\omega(2H)^2}{2} \int_0^1 \int_0^1 (1-v) \left\{ (1-u)\cos \pi u + \frac{\sin \pi u}{\pi} \right\} \]

\[
(A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4)) \, du \, dv \]

\[
- j \frac{\omega k_1^2}{2} \int_0^1 \int_0^1 (1-v) \left\{ (1-u)\cos \pi u - \frac{\sin \pi u}{\pi} \right\} \]

\[
(A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4)) \]

\[
+ A_z(\rho_1) + A_z(\rho_2) + A_z(\rho_3) + A_z(\rho_4) \) \, du \, dv \quad (3.23)\]
in which

\[
\begin{align*}
\rho_1 &= \sqrt{(x_{mn} + 2Hu)^2 + (y_{mn} + 2Wv)^2} \\
\rho_2 &= \sqrt{(x_{mn} + 2Hu)^2 + (y_{mn} - 2Wv)^2} \\
\rho_3 &= \sqrt{(x_{mn} - 2Hu)^2 + (y_{mn} + 2Wv)^2} \\
\rho_4 &= \sqrt{(x_{mn} - 2Hu)^2 + (y_{mn} - 2Wv)^2}
\end{align*}
\]  \hspace{1cm} (3.24)

2. SELF IMPEDANCE \( z_{mn} (n = m) \)

When \( n = m \), i.e., \( X_{mn} = Y_{mn} = 0 \) in (3.23), and the self impedance \( z_{mn} \) may be given by

\[
\begin{align*}
Z_{mn} &= j2\omega(2H)^2 \int \int (1-v) \left\{ (1-u)\cos \pi u + \frac{1}{\pi} \sin \pi u \right\} A_x(\rho) dudv \\
&\hspace{1cm} + \frac{2\omega^2}{k_1^2} \int \int (1-v) \left\{ (1-u)\cos \pi u - \frac{1}{\pi} \sin \pi u \right\} (A_x(\rho) + A_z(\rho)) dudv
\end{align*}
\]  \hspace{1cm} (3.25)

where

\[
\rho = \sqrt{(2Hu)^2 + (2Wv)^2} . \hspace{1cm} (3.26)
\]

It is noticed from the expression (3.26) that the Sommerfeld type integrals \( A_x \) and \( A_z \) in (3.25) become singular at \( u=v=0 \), that is, the integrands diverge. Usually, this kind of singularity problem can be solved using the extraction technique. One way of extracting the singularity is to subtract a certain function from the divergent
integrand in such a way that the resulting integrand becomes convergent [16]. However, in the present work, a certain singular function is subtracted from the integral itself. This approach is, in principle, similar to the previous one. However, the present approach, unlike the previous one, does not change the original Sommerfeld type integrals and, thus we can easily replace the integrals with alternative integral forms if it is needed.

Unlike the problem of the dipole in free space, the integrals $A_x$ and $A_z$ cannot be evaluated in exact closed form so that we must find certain singular functions which are embedded in these integrals. In order to do that, the integrals are rewritten as

$$A_x(\rho) = \frac{\mu_1}{2\pi} \int_0^\infty \frac{k_p J_0(k_p \rho)}{D_{\text{TE}}} \, dk_p$$

$$= \frac{\mu_1}{2\pi} \int_0^{k_c} \frac{k_p J_0(k_p \rho)}{D_{\text{TE}}} \, dk_p + \frac{\mu_1}{2\pi} \int_{k_c}^\infty \frac{k_p J_0(k_p \rho)}{D_{\text{TE}}} \, dk_p$$

(3.27)

$$A_z(\rho) = \frac{\mu_1}{2\pi} (\varepsilon_r - 1) \int_0^\infty \frac{-jk_z l_k}{D_{\text{TE}} D_{\text{TM}}} \, dk_p$$

$$= \frac{\mu_1}{2\pi} (\varepsilon_r - 1) \int_0^{k_c} \frac{-jk_z l_k}{D_{\text{TE}} D_{\text{TM}}} \, dk_p$$

$$+ \frac{\mu_1}{2\pi} (\varepsilon_r - 1) \int_{k_c}^\infty \frac{-jk_z l_k}{D_{\text{TE}} D_{\text{TM}}} \, dk_p$$

(3.28)

where $k_c$ can be any value as far as it satisfies the following approximations with negligible error.
\[
\coth(\sqrt{k_{\rho}^2 - \epsilon_r k_1^2}) \approx 1.0 \tag{3.29}
\]

and

\[
\sqrt{k_{\rho}^2 - \epsilon_r k_1^2} \approx \sqrt{k_{\rho}^2 - k_1^2} \approx k_{\rho} \tag{3.30}
\]

for all \( k_{\rho} > k_c \) on the path of integration since, when the conditions (3.29) and (3.30) are satisfied, we can modify the second integral terms in (3.27, 3.28) as

\[
\frac{\mu_1}{2\pi} \int_{k_c}^{\infty} \frac{k_{\rho} J_0(k_{\rho} \rho)}{D_{TE}} \, dk_{\rho} = \frac{\mu_1 k_1}{2\pi} \int_{k_c}^{\infty} \frac{1}{2} J_0(k_{\rho} \rho) \, dk_{\rho} - \frac{\mu_1 k_1}{4\pi} \int_{0}^{k_c} J_0(k_{\rho} \rho) \, dk_{\rho} \tag{3.31}
\]

\[
\frac{\mu_1}{2\pi} (\epsilon_r - 1) \int_{k_c}^{\infty} \frac{-jk_{\rho} k_{\rho} J_0(k_{\rho} \rho)}{D_{TE}D_{TM}} \, dk_{\rho} = -\frac{\mu_1 k_1}{2\pi} (\epsilon_r - 1) \int_{k_c}^{\infty} \frac{J_0(k_{\rho} \rho)}{2(\epsilon_r + 1)} \, dk_{\rho} \tag{3.32}
\]

Then, from (3.31) and (3.32), we can easily conclude that the singular functions embedded in \( A_x \) and \( A_z \) are, respectively

\[
\frac{\mu_1}{4\pi} \frac{1}{\rho}
\]

and

\[
\frac{\mu_1 (\epsilon_r - 1)}{4\pi (\epsilon_r + 1)} \frac{1}{\rho}
\]
Now, let

\[
A_x(\rho) = \frac{\mu_1}{2\pi} \left[ -j \int_0^\infty \frac{k_p J_0(k_p \rho)}{D_{TE}} \left( D_{TM} \right) \, dk_p - \frac{1}{2\rho} \right] + \frac{\mu_1}{4\pi} \frac{1}{\rho}
\]

\[
A_z(\rho) = \frac{\mu_1}{2\pi} \left[ \frac{j}{\varepsilon_r - 1} \int_0^\infty \frac{k_p J_0(k_p \rho)}{D_{TE} D_{TM}} \, dk_p + \frac{\varepsilon_r - 1}{2(\varepsilon_r + 1)} \frac{1}{\rho} \right]
\]

\[
- \frac{\mu_1(\varepsilon_r - 1)}{4\pi(\varepsilon_r + 1)} \frac{1}{\rho}
\]

Substituting (3.33) and (3.34) into (3.25) and then handling the singular problem due to the second terms in (3.33) and (3.34) (See Appendix D), we have

\[
Z_{mm} = j2\omega(2H)^2 \int \int (1-v) \{ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \} \bar{A}_x(\rho) \, du \, dv
\]

\[
- j \frac{2\omega^2}{k_1^2} \int \int (1-v) \{ (1-u) \cos \pi u - \frac{1}{\pi} \sin \pi u \} \bar{A}_x(\rho) + \bar{A}_z(\rho) \, du \, dv
\]

\[
+ j 2\omega (2H)^2 S_1
\]

\[
- j \frac{2\omega^2}{k_1^2} (S_1 + S_2)
\]

(3.35)

where

\[
\bar{A}_x(\rho) = A_x(\rho) - \frac{\mu_1}{4\pi} \frac{1}{\rho}
\]

\[
- \frac{\mu_1(\varepsilon_r - 1)^2}{4\pi(\varepsilon_r + 1)} \frac{1}{\rho}
\]

\[A_z(\rho) = A_z(\rho) + \frac{\mu_1(\varepsilon_r - 1)^2}{4\pi(\varepsilon_r + 1)} \frac{1}{\rho}
\]

(3.36)

(3.37)
\[ S_1 = \frac{2H}{\pi^2(2W)^2} - \frac{2}{\pi^2 2W} \left( 1 + 2\ln\left(\frac{2H}{2W}\right) \right) + \frac{1}{\pi 2W} S_1(\pi) \]

\[ + \frac{1}{2W} \int_0^1 \left( \frac{1}{\pi} \sin\pi u + (1-u)\cos\pi u \right) \ln\left(1 + \frac{1}{2W}\frac{H}{2W} \frac{2}{u^2} \right) - \sqrt{1+\left(\frac{2H}{2W}\right)^2 u^2} \, du \]

\[ - \frac{2}{\pi(2W)} \int_0^1 \sin\pi u \sin u \, du \]  

(3.38)

and

\[ S_2 = -\frac{2H}{\pi^2(2W)^2} - \frac{2}{\pi^2(2W)} + \frac{1}{\pi(2W)} S_1(\pi) \]

\[ + \frac{1}{2W} \int_0^1 \left( -\frac{1}{\pi} \sin\pi u + (1-u)\cos\pi u \right) \ln\left(1 + \frac{1}{2W}\frac{H}{2W} \frac{2}{u^2} \right) - \sqrt{1+\left(\frac{2H}{2W}\right)^2 u^2} \, du \]  

(3.39)

where \( S_1(\pi) \) is the sine integral of argument \( \pi \).

Note that \( A_x \) and \( A_z \) remain unchanged in the above representations for [Z]. Hence, as mentioned at the beginning of this chapter, we can easily replace them with any alternative representation depending upon its usefulness.

C. EXACT EXPRESSION OF \([\mathbf{V}]\) IN THE REAL SPACE DOMAIN

1. MUTUAL IMPEDANCE \( V_m(q) \) (\( m = q \))

Substituting (3.9) into (3.7) and then substituting this result into (3.6) together with (3.8), we have
\[ V_\text{m}(q) = -\frac{j\omega \mu_0}{2W} \int_{-d}^{d} \left[ \int_{-W}^{W} \int_{-W}^{W} G_{zx}(\vec{r}_m;\vec{r}_m') \cos \frac{\pi x_m'}{2H} \, dx_m' \, dy_m' \right] \delta(\vec{r}_m - \vec{r}_{mf}) \, dz \]  

(3.40)

where

\[ G_{zx} = \frac{1}{\mu_0 k_2^2} \left\{ \left( k_2^2 + \frac{\alpha^2}{\alpha z^2} \right) A_{zx} + \frac{\alpha^2}{\alpha x \alpha z} A_{xx} \right\} \]  

(3.41)

from (2.45) in Chapter II, and \( A_{xx} \) and \( A_{zx} \) are the ones defined by (2.21) and (2.22), respectively.

In (3.40), \( r_{mf} = (x_f, y_f) \) denote the position vector of feed point referenced to the center of patch \( q \). As derived in Appendix E, (3.41) may be expressed as follows

\[ G_{zx} = \frac{1}{4\pi k_2^2} \frac{\alpha^2}{\alpha x \alpha z} \text{VTEM}(\rho, z) \]  

(3.42)

where

\[ \text{VTEM}(\rho, z) = \int_{C_{k\rho}} F_B(k_\rho, z) k_\rho H_0^{(2)}(k_\rho) \, dk_\rho \]  

(3.43)

or

\[ \text{VTEM}(\rho, z) = 2 \int_{0}^{\infty} F_B(k_\rho, z) k_\rho J_0^{(2)}(k_\rho) \, dk_\rho \]  

(3.44)

where

\[ F_B(k_\rho, z) = \frac{\sqrt{k_1^2 - k_\rho^2} \sin \sqrt{k_2^2 - k_\rho^2} (z + d)}{\sqrt{k_2^2 - k_\rho^2} \left( e^{\sqrt{k_1^2 - k_\rho^2} \cos \sqrt{k_2^2 - k_\rho^2} d} + j \sqrt{k_2^2 - k_\rho^2} \sin \sqrt{k_2^2 - k_\rho^2} d \right)} \]  

(3.45)
and \( \rho = \sqrt{\left( x_m - x'_m \right)^2 + \left( y_m - y'_m \right)^2} \) . \hspace{1cm} (3.46)

Substituting (3.42) into (3.40) and then integrating with respect to \( x'_m \) and \( z \) gives

\[
V_m(q) = \frac{j\omega u}{4\pi k^2} \frac{\pi}{(2W)(2H)} \int \int_{-W-H}^{W+H} \text{VTEM}(\rho,0) \sin \frac{\pi x'_m}{2H} \, dx'_m \, dy'_m
\]

where

\[
\rho = \sqrt{(x_m + y'_m - x'_m)^2 + (y_m + y'_m - y'_m)^2} . \hspace{1cm} (3.48)
\]

Finally, using the transformations

\[
u = \frac{x'_m}{2H} + \frac{1}{2}
\]

\[
v = \frac{y'_m}{2W} + \frac{1}{2}
\]

we have

\[
V_m(q) = \frac{-j\omega u 2\pi}{4\pi k^2} \int \int_{0}^{1} \text{VTEM}(\rho,0) \cos \pi u \, du
\]

where

\[
\rho = \sqrt{(x_m + x_f - 2Hu + H)^2 + (y_m + y_f - 2Wv + W)^2}
\]

2. SELF IMPEDANCE \( V_m(q)(m = q) \)

When \( m = q \), i.e., \( x_{mq} = y_{mq} = 0 \), self impedance \( V_m(m) \) may be given by

\[
V_m(m) = \frac{j\omega u 2\pi}{4\pi k^2} \int \int_{0}^{1} \text{VTEM}(\rho,0) \cos \pi u \, du \, dv
\]

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where
\[ \rho = \sqrt{(x_f - 2Hu + H)^2 + (y_f - 2Wv + W)^2} \] (3.54)

Following the same procedure as that of \( Z_{mm} \), we can find the singular function embedded in VTEM which is given by
\[ \frac{2}{(\epsilon_r + 1)\rho} \]

Then, \( V_m(m) \) may be rewritten as follows
\[
V_m(m) = -\frac{j\omega u 2\pi}{4\pi k^2} \int_0^1 \int_0^1 \left[ VTEM(\rho, 0) - \frac{2}{(\epsilon_r + 1)\rho} \right] \cos \pi u \, du \, dv
\]
\[
- \frac{j\omega u 2\pi}{4\pi k^2} \frac{2}{(\epsilon_r + 1)} \int_0^1 \int_0^1 \cos \pi u \, du \, dv \] (3.55)

Performing integration with respect to \( v \) for the second integral term in (3.55), we finally have
\[
V_m(m) = -j \frac{\omega u 2\pi}{4\pi k^2} \int_0^1 \int_0^1 \left[ VTEM(\rho, 0) - \frac{2}{(\epsilon_r + 1)\rho} \right] \cos \pi u \, du \, dv
\]
\[
+ \frac{1}{2W} j \frac{\omega u 2\pi}{4\pi k^2} \frac{2}{\epsilon_r + 1} \int_0^1 \cos \pi u \ln \left[ y_f - W + \sqrt{(x_f - 2Hu + H)^2 + (y_f - W)^2} \right] du
\]
\[
- \frac{1}{2W} j \frac{\omega u 2\pi}{4\pi k^2} \frac{2}{\epsilon_r + 1} \int_0^1 \cos \pi u \ln \left[ y_f + W - \sqrt{(x_f - 2Hu + H)^2 + (y_f + W)^2} \right] du
\] (3.56)

where \( VTEM(\rho, 0) \) can be ones given by (3.43), (3.44) or
\[
V_T E_M(\rho, 0) = -2\pi j\cdot R_1
\]
\[+ \frac{4}{\pi} \int_0^\infty \text{Re}\left\{F_B(\rho, t) k_p^{\rho} \right\} K_0(\rho t) dt \]
\[+ 2j \int_0^1 \text{Im}\left\{F_B(\rho, 0) k_p^{\rho} \right\} H_0^{(2)}(\rho \lambda) d\lambda . \quad (3.57)
\]

where \( R_1 \) is the residue of the integrand of (3.43) at pole \( k_p = \lambda_p \) and it is given by
\[
R_1 = \lim_{k_p \to \lambda_p} (k_p - \lambda_p) F_B(k_p, 0) k_p H_0^{(2)}(k_p \rho) . \quad (3.58)
\]
The representation (3.57) is easily obtained following the procedure described in Section D of Chapter II.
CHAPTER IV
NUMERICAL RESULTS AND DISCUSSION

This chapter consists of two sections. In section A, the accuracy of the approximate closed-form representation for the Sommerfeld type integrals $A_x$ and $A_z$ as developed in Chapter II (Section E) is demonstrated by comparison with the exact numerical integration results of $A_x$ and $A_z$ based on the alternative representation of the Sommerfeld integrals for various values of the substrate thickness and dielectric constants. And, in section B, numerical results of the mutual impedance between identical expansion modes are presented to illustrate the accuracy of the approximate closed-form representation for the microstrip Green's function. Also the validity of the space domain representation of the mutual and self impedances derived in the previous chapter is verified by comparing the results with the data available in literature.

A. ACCURACY OF THE APPROXIMATE CLOSED-FORM REPRESENTATION OF THE SOMMERFELD TYPE INTEGRALS $A_x$ AND $A_z$

In Figure 4.1 through 4.12, the closed-form approximations of the Sommerfeld type field integral representations $Ax$ and $Az$ are compared with the exact numerical integration results for $k_{10}$ which varies between .1 and 2. First, in Figures 4.1 through 4.6, the operating frequency and the dielectric constant are chosen to be 633 MHZ and 2.56, respectively. In addition, the substrate thickness increases from 0.003175 m to 0.03175 m to see how the accuracy of the closed-form
Figure 4.1. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633$ MHz, $d = 0.003175$ m, $\varepsilon_r = 2.56$).
Figure 4.1. Continued.
Figure 4.2. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633$ MHz, $d = 0.00635$ m, $\varepsilon_r = 2.56$).
Figure 4.2. Continued.
Figure 4.3. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633$ MHz, $d = 0.0127$ m, $\varepsilon_r = 2.56$).
Figure 4.3. Continued.
Figure 4.4. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, $(f = 633 \text{ MHz}, d = 0.01905 \text{ m}, \varepsilon_r = 2.56)$. 
(b) $A_z$

Figure 4.4. Continued.
Figure 4.5. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633$ MHz, $d = 0.0254$ m, $\varepsilon_r = 2.56$).
Figure 4.5. Continued.
Figure 4.6. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_x$ from the expressions in (2.75) and (2.73), respectively, ($f = 633 \text{ MHz}, d = 0.03175 \text{ m}, \varepsilon_r = 2.56$).
Figure 4.6. Continued.
Figure 4.7. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633$ MHz, $d = 0.003175$ m, $\varepsilon_r = 5.12$).
Figure 4.7. Continued.
Figure 4.8. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633 \text{ MHz}$, $d = 0.0127 \text{ m}$, $\varepsilon_r = 5.12$).
Figure 4.8. Continued.
Figure 4.9. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633$ MHz, $d = 0.0254$ m, $\varepsilon_r = 5.12$).
Figure 4.9. Continued.
Figure 4.10. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633$ MHz, $d = 0.003175$ m, $\varepsilon_r = 9.$).
Figure 4.10. Continued.
Figure 4.11. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633$ MHz, $d = 0.0127$ m, $\varepsilon_r = 9$).
Figure 4.11. Continued.
Figure 4.12. Comparison of the results obtained from the closed form asymptotical approximation and the exact numerical integration of the Sommerfeld type integrals of $A_x$ and $A_z$ from the expressions in (2.75) and (2.73), respectively, ($f = 633$ MHz, $d = 0.0254$ m, $\varepsilon_r = 9$).
(b) $A_z$

Figure 4.12. Continued.
approximation depends on the electrical thickness of the substrate. Secondly, in Figures 4.7 through 4.9 and Figures 4.10 through 4.12, the operating frequency and the substrate thickness are kept at 633 MHz and 0.003175 m, respectively. However, the dielectric constant is chosen to be 5.12 in Figures 4.7 through 4.9 and is chosen as 9 in Figures 4.10 through 4.12 to see the effect on the accuracy of the closed-form approximation due to the changes in the dielectric constant. It is observed that the results obtained from the closed-form approximation and the exact numerical integration agree extremely well even for small values of \( k_{10} \) when the dielectric substrate is relatively thin. However, when it gets thicker, the discrepancy between these results becomes noticeable for small values of \( k_{10} \). Particularly, the discrepancy of the real part is more noticeable. This is mainly because a static term providing the main contribution to the integral for small values of \( k_{10} \) (see (2.73), (2.75) in section D of the previous chapter) was not considered in obtaining the closed-form approximation. On the other hand, the accuracy of the closed-form approximation is seen to be less sensitive to the changes in dielectric constant than to the electrical thickness of substrate. This might be because the dielectric constant always appears inside the radical in the integrands pertaining to the integral field representations; whereas, the electrical thickness of the substrate appears outside the radical sign.

B. MUTUAL IMPEDANCE BETWEEN TWO IDENTICAL EXPANSION MODES

In section A, the accuracy of the closed-form approximations for the Sommerfeld type integrals \( A_x \) and \( A_z \) are demonstrated. However, although the microstrip surface Green's function is expressed in terms
of these integrals, it may not be sufficient enough to prove that the resulting closed-form approximation of the microstrip surface Green's function is accurate. Thus, in this section, mutual impedances between identical expansion modes are evaluated using the closed-form approximation and compared with those based on the exact alternative representations in (2.73) and (2.75) for the microstrip surface Green's function.

Figures 4.13 through 4.15 and Figures 4.16 through 4.18 show the mutual impedances between two co-linear entire basis (EB) expansion modes (E-plane coupling) and two parallel EB expansion modes (H-plane coupling), respectively, as a function of the spacing relative to a free space wavelength. The geometry in these examples consist of two identical current modes of length 0.0655 m and width 0.1057 m on an infinite grounded dielectric substrate of thicknesses 0.001575 m, 0.006296 m and 0.012592 m. The operating frequency is 1417 MHz and the dielectric constant is 2.5. The separation between two modes varies in the x-direction only in Figures 4.13 through 4.15 and in the y-direction only in Figures 4.16 through 4.18. Again, the results obtained from the use of the closed-form approximation agree extremely well in cases of relatively thin substrates. The results for the H-plane coupling are very accurate even for relatively thick substrates. This may be understood when we notice, from the Green's function representation (2.54) in Chapter II, that the x-directed electric field component due to an x-directed electric current source is highly dependent on the x coordinate but it is not as sensitive with respect to the y coordinate. In Figures 4.19 through 4.21, the same data as in Figures 4.22 through
Figure 4.13. The mutual impedance \( z \) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength \( (f = 1417 \text{ MHz}, d = 0.001575 \text{ m}, \varepsilon_r = 2.5) \).
Figure 4.14. The mutual impedance ($z$) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 1417$ MHz, $d = 0.006296$ m, $\varepsilon_r = 2.5$).
Figure 4.15. The mutual impedance (z) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength (f = 1417 MHz, d = 0.012592 m, εr = 2.5).
Figure 4.16. The mutual impedance ($z$) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 1417$ MHz, $d = 0.001575$ m, $\varepsilon_r = 2.5$).
Figure 4.17. The mutual impedance \( z \) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength \( (f = 1417 \text{ MHz}, \ d = 0.006296 \text{ m}, \ \varepsilon_r = 2.5) \).
Figure 4.18. The mutual impedance (z) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength (f = 1417 MHz, d = 0.012592 m, $\varepsilon_r = 2.5$).
Figure 4.19. The mutual impedance \( z \) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength \( (f = 633 \text{ MHz}, \quad d = 0.003175 \text{ m}, \quad \varepsilon_r = 2.56) \).
Figure 4.20. The mutual impedance ($z$) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 633$ MHz, $d = 0.0127$ m, $\varepsilon_r = 2.56$).
Figure 4.21. The mutual impedance ($z$) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 633$ MHz, $d = 0.0254$ m, $\varepsilon_r = 2.56$).
Figure 4.22. The mutual impedance ($z$) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 633$ MHz, $d = 0.003175$ m, $\varepsilon_r = 2.56$).
Figure 4.23. The mutual impedance \( (z) \) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength (\( f = 633 \) MHz, \( d = 0.0127 \) m, \( \varepsilon_r = 2.56 \)).
Figure 4.24. The mutual impedance ($z$) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 633$ MHz, $d = 0.0254$ m, $\varepsilon_r = 2.56$).
current modes are given by 0.15 m and 0.075 m, respectively. Again, it is observed from these figures that the closed-form approximation is very accurate. Even though the main purpose of this chapter is to demonstrate the accuracy of the closed-form approximation of the microstrip surface Green's function, it may be worth noting from these figures that, unlike the mutual impedances between two dipoles in free space, the E-plane coupling, in planar microstrip structure, is always larger than the H-plane coupling after a certain range of separation mainly because of the surface wave being excited more strongly in the direction of source current polarization. When the two expansion modes are close each other, the shape of the modes strongly affects the coupling since the electric field component parallel to the modes is more sensitive to the coordinate corresponding to the direction of the current mode than to that which is normal to the direction of the current mode.

Table 4.1 shows the behavior of the mutual impedance in the E-plane between two identical expansion current modes of length 0.15 m and width 0.075 m on an infinite grounded dielectric slab of thickness 0.003175 m. The operating frequency, the dielectric constant and the dielectric loss tangent are chosen to be 633 MHz, 2.56 and 0.0015, respectively. The first column in the table denotes the separation between two modes and the data in the second column is from the results presented in [32], the third and fourth columns denote the mutual impedance obtained by using the exact microstrip surface Green's function and by using the closed-form approximation, respectively. The table shows excellent agreement between the data thereby ensuring validity of the space domain
TABLE 4.1
MUTUAL IMPEDANCE BETWEEN TWO IDENTICAL EXPANSION MODES IN THE E-PLANE

\[
\begin{array}{|c|c|c|}
\hline
\frac{s'}{\lambda} & \text{REFERENCE [32]} & \text{EXACT INTEGRATION} \\
\hline
0.0 & 0.092 + j0.399 & 0.092061631 + j0.3986848 \\
0.5 & 0.006 - j0.014 & 6.1424328E-03 - j1.3513822E-02 \\
0.75 & -0.011 - j0.001 & -1.0693505E-02 - j1.2478634E-03 \\
1.0 & 0.0 + j0.008 & -3.5248243E-04 + j8.4043546E-03 \\
\hline
\end{array}
\]

\text{CLOSED FORM APPROXIMATION}

\[
\begin{array}{|c|c|c|}
\hline
 & 6.1401268E-03 - j1.3535029E-02 \\
 & -1.0699167E-02 - j1.2516497E-03 \\
 & -3.5511656E-04 + j8.4058205E-03 \\
\hline
\end{array}
\]

\[s' = \text{separation between centers of each patch}\]
\[\lambda = \text{free space wavelength } = 0.474 \text{ m}\]
representation of the mutual and self impedances derived in the previous chapter. In fact, the first value in the forth column of the Table 4.1 represents the self impedance of a single piecewise sinusoidal expansion current mode on the grounded dielectric slab described above, and the other values actually represent the mutual coupling between two identical piecewise sinusoidal expansion current modes parallel to the axis of a perfectly-conducting cylinder of radius $10\lambda$ ($\lambda$=free space wavelength) which is coated with a dielectric material. However, such a comparison can be justified, since using either entire base expansion modes or piecewise sinusoidal expansion modes yields the same results with negligible difference (see reference [21]); furthermore, it is also known that the curvature of the cylindrical surface has little effect on the energy propagating in the direction parallel to the axis [32].

Figures 4.25 through 4.36 show the real and imaginary parts of the mutual impedances between two co-linear EB expansion modes and the two parallel EB expansion modes for various dielectric slab thicknesses. These results ensure that the closed form approximation for the microstrip surface Green's function in Chapter II can also be used for the lossy dielectric case.
Figure 4.25. The mutual impedance ($z$) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 633$ MHz, $d = 0.003175$ m, $\varepsilon_r = 2.56$, $\tan\delta = 0.0015$).
Figure 4.26. The mutual impedance ($z$) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 633$ MHz, $d = .00635$ m, $\varepsilon_r = 2.56$, $\tan\delta = .0015$).
Figure 4.27. The mutual impedance ($z$) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 633$ MHz, $d = .0127$ m, $\varepsilon_r = 2.56$, $\tan\delta = .0015$).
Figure 4.28. The mutual impedance (z) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength (f = 633 MHz, d = .01905 m, $\epsilon_r = 2.56$, $\tan\delta = .0015$).
Figure 4.29. The mutual impedance ($z$) between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 633$ MHz, $d = 0.0254$ m, $\varepsilon_r = 2.56$, $\tan\delta = 0.0015$).
Figure 4.30. The mutual impedance $(z)$ between two co-linear entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength $(f = 633 \text{ MHz}, d = .03175 \text{ m}, \varepsilon_r = 2.56, \tan\delta = .0015)$. 
Figure 4.31. The mutual impedance (z) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength (f = 633 MHz, d = .003175 m, $\varepsilon_r = 2.56$, tanh = .0015).
Figure 4.32. The mutual impedance $(z)$ between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength $(f = 633 \text{ MHz}, d = 0.00635 \text{ m}, \varepsilon_r = 2.56, \tan\delta = 0.0015)$. 
Figure 4.33. The mutual impedance (z) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength (f = 633 MHz, d = .0127 m, $\varepsilon_r = 2.56$, tan$\delta$ = .0015).
Figure 4.34. The mutual impedance (z) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength (f = 633 MHz, d = 0.01905 m, \( \varepsilon_r = 2.56 \), \( \tan\delta = 0.0015 \)).
Figure 4.35. The mutual impedance (z) between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength (f = 633 MHz, d = 0.0254 m, εᵣ = 2.56, tanδ = 0.0015).
Figure 4.36. The mutual impedance $z$ between two parallel entire base sinusoidal expansion modes as a function of separation relative to a free space wavelength ($f = 633$ MHz, $d = 0.03175$ m, $\varepsilon_r = 2.56$, $\tan\delta = 0.0015$).
A simple but accurate closed-form asymptotic approximation for the electric current point source microstrip surface dyadic Green's function that remains accurate everywhere except in the very close vicinity of the point source is developed by a new approach in this dissertation. This work is expected to be useful in the moment method analysis of microstrip antenna arrays. The excellent accuracy of the closed-form asymptotic approximation is demonstrated by comparison with the exact numerical integration results for both the Green's function and the mutual impedance between two parallel (and two co-linear) entire base sinusoidal expansion modes. The accuracy of the closed-form approximation is generally affected by the electrical thickness of dielectric slab \( k_1d \) and its dielectric constant \( \varepsilon_r \), i.e., for larger values of \( k_1d \) and \( \varepsilon_r \), its accuracy degrades. And it is also noted that the effect of the dielectric constant is not as significant as the electrical thickness of the dielectric slab. In addition to the closed-form approximation, a very efficient numerical integration method is developed for an exact evaluation of the Sommerfeld type integrals by transforming them into an alternative representation via Cauchy's residue theory. Also the far-zone microstrip Green's function is evaluated for completeness; this provides the radiation pattern of a microstrip antenna.
REFERENCES


APPENDIX A

THE EM FIELD PRODUCED BY AN \( \hat{x} \)-DIRECTED ELECTRIC DIPOLE ON AN INFINITE GROUNDED DIELECTRIC SLAB

In this appendix, the solution for the electromagnetic field produced by an \( \hat{x} \)-directed electric dipole source on an infinite grounded dielectric slab is derived in terms of a magnetic vector potential having two components which are along the source and normal to the interface. Then, for these components, \( z \)-transmission modal representations involving plane-wave and cylindrical-wave functions in cross-sectional domain are obtained. In addition to these \( z \)-transmission representations, we may develop another representation which is referred to as a radially transmitting representation via the Green's characteristic function technique [23]. However, it is not considered in present work.

The geometry of the problem is illustrated in Figure A.1 below.

Figure A.1. Geometry of an \( \hat{x} \)-directed electric dipole source of moment \( p_{\text{ex}} \) located on an infinite grounded dielectric slab.
The grounded dielectric slab is infinite in the $x,y$ direction with uniform thickness $d$. An $x$-directed electric dipole source of moment $p_{ex}$ is located at the coordinate origin. The source free regions 1 and 2 are characterized by $(\varepsilon_1,\mu_1)$ and $(\varepsilon_2,\mu_2)$, respectively; the first parameter and the second one in each parenthesis represent the permittivity and the permeability of the corresponding medium. Now, let $(E_1,H_1)$ and $(E_2,H_2)$ represent the electromagnetic fields in region 1 and in region 2, respectively. Then, they satisfy the following Maxwell's equations

$$\nabla \times \vec{H}_1 = j\omega \varepsilon_1 \vec{E}_1 \quad \text{in } V_1 \quad (A.1)$$

$$\nabla \times \vec{E}_1 = -j\omega \mu_1 \vec{H}_1 \quad \text{in } V_1 \quad (A.2)$$

$$\nabla \times \vec{H}_2 = j\omega \varepsilon_2 \vec{E}_2 \quad \text{in } V_2 \quad (A.3)$$

$$\nabla \times \vec{E}_2 = -j\omega \mu_2 \vec{H}_2 \quad \text{in } V_2 \quad (A.4)$$

and the boundary conditions

$$\hat{Z} \times \vec{E}_2 = 0 \quad \text{at } Z = -d \quad (A.5)$$

$$\hat{Z} \times (\vec{H}_1 - \vec{H}_2) = \hat{x} p_{ex} \delta(\vec{r}) \quad \text{at } z=0 \quad (A.6)$$

$$\hat{Z} \times (\vec{E}_1 - \vec{H}_2) = 0 \quad \text{at } z=0 \quad (A.7)$$
together with the radiation condition at infinity in both regions. In (A.6), \( \delta(\mathbf{r}) \) denotes the Dirac delta function and \( \mathbf{r} \) is the position vector from the coordinate origin to the observation points.

Using the classical theory of vector potentials [27], we can construct the electromagnetic fields in terms of magnetic vector potentials \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) as follows:

\[
\begin{align*}
H_1 &= \frac{1}{\mu_1} \nabla \times \tilde{A}_1 \quad (A.8) \\
\tilde{E}_1 &= -j\omega \tilde{A}_1 + \frac{1}{j \omega \mu_1 \epsilon_1} \nabla (\nabla \cdot \tilde{A}_1) \quad (A.9) \\
\tilde{H}_2 &= \frac{1}{\mu_1} \nabla \times \tilde{A}_2 \quad (A.10) \\
\tilde{E}_2 &= -j\omega \tilde{A}_2 + \frac{1}{j \omega \mu_2 \epsilon_2} \nabla (\nabla \cdot \tilde{A}_2) \quad (A.11)
\end{align*}
\]

in which \( \tilde{A}_1 \) and \( \tilde{A}_2 \) satisfy the following homogeneous Helmholtz equations

\[
\begin{align*}
\nabla^2 \tilde{A}_1 + k_1^2 \tilde{A}_1 &= 0 \quad \text{in } V_1 \\
\nabla^2 \tilde{A}_2 + k_2^2 \tilde{A}_2 &= 0 \quad \text{in } V_2 \quad \text{(A.13)}
\end{align*}
\]

where

\[
\begin{align*}
k_1^2 &= \omega^2 \mu_1 \epsilon_1 \quad \text{(A.14)} \\
k_2^2 &= \omega^2 \mu_2 \epsilon_2 \quad \text{(A.15)}
\end{align*}
\]
As is well known, two components of the vector potential are needed for a complete description of the horizontal current element problem in stratified media like the present case. Hence, let

\[ \vec{A}_1 = \hat{x} A_{1x} + \hat{z} A_{1z} \]  \hspace{1cm} (A.16)

and

\[ \vec{A}_2 = \hat{x} A_{2x} + \hat{z} A_{2z} \]  \hspace{1cm} (A.17)

Then, (A.8) through (A.13) may be rewritten in terms of the scalar quantities \( A_{1x}, A_{1z}, A_{2x} \) and \( A_{2z} \).

\[ \vec{E}_1 = \frac{1}{j \omega \mu_1 \epsilon_1} \left[ \hat{x} \left( k_1^2 A_{1x} + \frac{3 \hat{x}^2}{\hat{y} \hat{z}} A_{1x} + \frac{3 \hat{z}^2}{\hat{x} \hat{y}} A_{1z} \right) \right. \]

\[ + \left. \hat{y} \left( \frac{3 \hat{x}^2}{\hat{x} \hat{y}} A_{1x} + \frac{3 \hat{z}^2}{\hat{x} \hat{y}} A_{1z} \right) \right. \]

\[ + \left. \hat{z} \left( k_1^2 A_{1z} + \frac{3 \hat{y} \hat{z}}{\hat{x} \hat{z}} A_{1x} + \frac{3 \hat{y} \hat{z}}{\hat{x} \hat{z}} A_{1z} \right) \right] \]  \hspace{1cm} (A.18)

\[ \vec{H}_1 = \frac{1}{\mu_1} \left[ \hat{x} \frac{3 \hat{y}}{\hat{y} \hat{z}} A_{1z} + \hat{y} \left( \frac{3 \hat{x}}{\hat{z}} A_{1x} - \frac{3 \hat{y}}{\hat{z}} A_{1z} \right) + \hat{z} \left( \frac{3 \hat{y}}{\hat{x}} A_{1x} \right) \right] \]  \hspace{1cm} (A.19)

\[ \vec{E}_2 = \frac{1}{j \omega \epsilon_2} \left[ \hat{x} \left( k_2^2 A_{2x} + \frac{3 \hat{x}^2}{\hat{y} \hat{z}} A_{2x} + \frac{3 \hat{z}^2}{\hat{x} \hat{y}} A_{2z} \right) \right. \]

\[ + \left. \hat{y} \left( \frac{3 \hat{x}^2}{\hat{x} \hat{y}} A_{2x} + \frac{3 \hat{z}^2}{\hat{x} \hat{y}} A_{2z} \right) \right. \]

\[ + \left. \hat{z} \left( k_2^2 A_{2z} + \frac{3 \hat{y} \hat{z}}{\hat{x} \hat{z}} A_{2x} + \frac{3 \hat{y} \hat{z}}{\hat{x} \hat{z}} A_{2z} \right) \right] \]  \hspace{1cm} (A.20)
\[
\mathcal{H}_z = \frac{1}{\nu_2} \left[ \hat{x} \frac{\partial}{\partial y} A_{2z} + \hat{y} \left( \frac{\partial}{\partial z} A_{2x} - \frac{\partial}{\partial x} A_{2y} \right) + \hat{z} \left( -\frac{\partial}{\partial y} A_{2x} \right) \right] \quad (A.21)
\]

\[
(\nu^2 + k_i^2) \begin{pmatrix}
- A_{1x} \\
A_{2x} \\
- A_{1z}
\end{pmatrix} = 0 ; \quad i=1, 2 \quad (A.22)
\]

These quantities, of course, satisfy the radiation condition at infinity in both region (notice that they are solutions to the Helmholtz wave equation). Substituting (A.18, A.19, A.20, A.21) into the boundary conditions (A.5, A.6, A.7) and applying the radiation condition at infinity yields the boundary conditions in terms of these wave potentials

\[
A_{1x} = A_{2x} \quad \text{at } z=0 \quad (A.23)
\]

\[
\frac{\partial}{\partial z} A_{1x} - \frac{\partial}{\partial z} A_{2x} = -\nu_1 \rho_0 e^\delta(\hat{r}) \quad \text{at } z=0 \quad (A.24)
\]

\[
A_{2x} = 0 \quad \text{at } z=-d \quad (A.25)
\]

\[
A_{1z} = A_{2z} \quad \text{at } z=0 \quad (A.26)
\]

\[
\frac{\varepsilon_2}{\varepsilon_1} \left( \frac{\partial}{\partial x} A_{1x} + \frac{\partial}{\partial z} A_{1z} \right) = \frac{\partial}{\partial x} A_{2x} + \frac{\partial}{\partial z} A_{2z} \quad \text{at } z=0 \quad (A.27)
\]

\[
\frac{\partial}{\partial z} A_{2z} = 0 \quad \text{at } z=-d \quad (A.28)
\]

As is clear from (A.18) to (A.28), the EM fields can now be obtained by solving the scalar wave equations governing the quantities \( A_{1x}, A_{1z}, A_{2x} \) and \( A_{2z} \). Thus, the following sections are only concerned with the solutions for these quantities.

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1. 2-D FOURIER TRANSFORM REPRESENTATION

Since the geometry of this problem is unbounded in the xy-plane, the general solution for the homogeneous wave equations (A.22) can be represented in terms of their 2-D plane wave spectral representations (2-D Fourier transform) as follows:

\[
\begin{bmatrix}
A_{1x}(\vec{r}) \\
A_{1z}(\vec{r})
\end{bmatrix} = \int \int F_x(k_x,k_y) \left[ \begin{array}{c}
e^{-jk_{z1}z} \\
\frac{\sin k_{z2}(z+d)}{\sin k_{z2}d}
\end{array} \right] e^{-j(k_x x+k_y y)} dk_x dk_y
\]

\[\text{(A.29)}\]

\[
\begin{bmatrix}
A_{2x}(\vec{r}) \\
A_{2z}(\vec{r})
\end{bmatrix} = \int \int F_z(k_x,k_y) \left[ \begin{array}{c}
e^{-jk_{z1}z} \\
\frac{\cos k_{z2}(z+d)}{\cos k_{z2}d}
\end{array} \right] e^{-j(k_x x+k_y y)} dk_x dk_y
\]

\[\text{(A.30)}\]

where

\[
k_{z1} = \sqrt{k_1^2 - k_x^2 - k_y^2}, \quad R_e(k_{z1}) > 0, \quad \text{Im}(k_{z1}) < 0 \tag{A.31}
\]

\[
k_{z2} = \sqrt{k_2^2 - k_x^2 - k_y^2}, \quad R_e(k_{z2}) > 0, \quad \text{Im}(k_{z2}) < 0 \tag{A.32}
\]

and

\[
\vec{r} = \hat{x}x + \hat{y}y + \hat{z}z \tag{A.33}
\]

Notice that the choice of the second inequality in (A.31) ensures the radiation condition for the z-direction and the convergence of the integrals for a \(\exp(j\omega t)\) time dependence; however, as will be seen later, the choice of the second inequality of (A.32) is arbitrary. The
exponential function and the trigonometric functions in brackets of the above representations are introduced to satisfy the radiation condition for the z-direction and the boundary conditions (A.25, A.28), respectively. The \( \sin k_z d_2 \), \( \cos k_z d_2 \) terms are introduced to ensure continuity of \( A_x \) and \( A_z \) across the \( z=0 \) plane, i.e., (A.23, A.26).

The function \( F_x \) can be determined from the boundary condition (A.24). Introducing the 2-D Fourier transform of the Dirac delta function

\[
\delta(\vec{r}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(z) e^{-j(k_x x + k_y y)} dk_x dk_y
\]

(A.34)

and incorporating this representation and (A.29) into (A.24) gives

\[
\begin{pmatrix}
- A_{1x} (\vec{r}) \\
- A_{2x} (\vec{r})
\end{pmatrix} = \frac{\mu_1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{D_{\text{TE}}} e^{-j k_z l_2^2} \frac{\sin k_z (z+d)}{\sin k_z d_2} e^{-j(k_x x + k_y y)} dk_x dk_y
\]

(A.35)

in which

\[
D_{\text{TE}} = j k_z l_1 + k_z d_2 \cot k_z d_2
\]

(A.36)

Now that we have the exact representation for \( A_{1x} \) and \( A_{2x} \), we can determine the function \( F_z \) via substituting (A.35) and (A.30) into the coupled boundary condition (A.27). And the result is

\[
\begin{pmatrix}
- A_{1x} (\vec{r}) \\
- A_{2x} (\vec{r})
\end{pmatrix} = \frac{\mu_1}{4\pi^2} (\varepsilon_r - 1) \int_{-\infty}^{\infty} \frac{j k_x}{D_{\text{TE}} D_{\text{TM}}} \left[ \frac{e^{-j k_z l_2}}{\cos k_z (z+d)} \right] e^{-j(k_x x + k_y y)} dk_x dk_y
\]

(A.37)
in which

\[ D_{TM} = j \varepsilon_r k z_1 - k z_2 \tan k z_2 \, d \]  \hspace{1cm} (A.38)

and

\[ \varepsilon_r = \frac{\varepsilon_2}{\varepsilon_1} \]  \hspace{1cm} (A.39)

The expressions (A.35, A.37) are referred to as 2-D Fourier transform representations. In general, this 2-D Fourier transform representation is the most cumbersome since it involves a double integral. However, as is well known, this representation is useful when it is used in conjunction with Fourier-transformable (analytically) functions. In next section, a more convenient \( z \)-transmission modal representations involving cylindrical-wave mode function (referred to as Hankel transform representation) is derived.

2. THE HANKEL TRANSFORM REPRESENTATION

The Hankel transform representation can be easily obtained from the Fourier transform representation obtained in previous section via the following polar transformations

\[ k_x = k \, \rho \, \cos \alpha, \quad k_y = k \, \rho \, \sin \alpha \] \hspace{1cm} (A.40)

\[ x = \rho \, \cos \phi, \quad y = \rho \, \sin \phi \] \hspace{1cm} (A.41)

The transformation in (A.41) is shown in Figure A.2. Making use of these transformations in (A.35, A.37) gives
Figure A.2. Polar and cylindrical coordinate systems.

\[
\begin{aligned}
A_{1x}(\vec{r}) &= \frac{\mu_1}{4\pi^2} \int_{k=0}^{\infty} \frac{k_\rho}{DE} \left[ e^{-jkz_1z} \right. \\
A_{2x}(\vec{r}) &= \frac{\mu_1}{4\pi^2} \int_{k=0}^{\infty} \frac{k_\rho}{DE} \left. \frac{\cos k_2(z+d)}{\cos k_2 d} \right]
\end{aligned}
\]

\[
\begin{aligned}
2\pi \int_{\alpha=0}^{\alpha=0} e^{-jk_\rho \rho \cos(\alpha-\phi)} d\alpha \left. dk_\rho \right]
\end{aligned}
\]

(A.42)

\[
\begin{aligned}
A_{1z}(\vec{r}) &= -\frac{\mu_1}{4\pi^2} \int_{k=0}^{\infty} \frac{jk_\rho}{DE} \left[ e^{-jkz_1z} \right. \\
A_{2z}(\vec{r}) &= -\frac{\mu_1}{4\pi^2} \int_{k=0}^{\infty} \frac{jk_\rho}{DE} \left. \frac{\cos k_2(z+d)}{\cos k_2 d} \right]
\end{aligned}
\]

\[
\begin{aligned}
2\pi \int_{\alpha=0}^{\alpha=0} e^{-jk_\rho \rho \cos(\alpha-\phi)} d\alpha \left. dk_\rho \right]
\end{aligned}
\]

(A.43)
In obtaining the above expressions, the following relation has also been employed:
\[ \frac{dk_x}{\rho} \frac{dk_y}{\rho} = |J| \frac{dk}{\rho} \frac{d\alpha}{\rho} = k \frac{dk}{\rho} \frac{d\alpha}{\rho} \]  
(A.44)

in which $|J|$ denotes the absolute value of the Jacobian determinant $J$ given by
\[ J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \alpha} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \alpha} \end{vmatrix} \]  
(A.45)

In (A.42, A.43), the integration over $\alpha$ can be performed in closed form via the relationship
\[ \int_{0}^{2\pi} \begin{vmatrix} \cos n\alpha \\ \sin n\alpha \end{vmatrix} e^{-jtcos(\alpha-\phi)} d\alpha = 2\pi (-j)^n \begin{vmatrix} \cos n\phi \\ \sin n\phi \end{vmatrix} J_n(t) \]  
(A.46)

in which $J_n(t)$ is the cylindrical Bessel function of order $n$ and argument $t$. Thus, (A.44, A.45) becomes via (A.46)
\[ \begin{vmatrix} -A_{1x}(\vec{r}) \\ -A_{2x}(\vec{r}) \end{vmatrix} = \frac{\mu_1}{2\pi} \int_{0}^{\infty} \frac{k_p J_0(k_p \rho)}{\text{DTE}} \begin{vmatrix} e^{-jk_z l^2} \\ \frac{\sin k_z (z+d)}{\sin k_z d} \end{vmatrix} dk_p \]  
(A.47)
\[
\begin{bmatrix}
-A_{1z}(\vec{r}) \\
-A_{2z}(\vec{r})
\end{bmatrix} = -\frac{\mu_1}{2\pi} (e_r - 1) \cos\phi \int_{k_p=0} \frac{k_p^2 J_1(k_p^p)}{D_{TE}D_{TM}} dk_p \\
e^{-j k_{z1} z} \\
\cos k_{z2}(z+d) \\
\cos k_{z2}d
\end{bmatrix}
\]  

where

\[D_{TE} = j k_{z1} + k_{z2} \cot k_{z2} d\]  
(A.49)

\[D_{TM} = j \varepsilon_r k_{z1} - k_{z2} \tan k_{z2} d\]  
(A.50)

\[k_{z1} = \sqrt{k_1^2 - k_p^2}, \quad \text{Re}(k_{z1}) > 1, \quad \text{Im}(k_{z1}) < 0\]  
(A.51)

\[k_{z2} = \sqrt{k_2^2 - k_p^2}, \quad \text{Re}(k_{z2}) > 0, \quad \text{Im}(k_{z2}) < 0\]  
(A.52)

and

\[\cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}}.\]  
(A.53)

These Hankel transform representations are convenient for a numerical integration since it involves the integration along the positive real axis only. However, when \(k_{z1}\) and \(k_{z2}\) are treated real (lossless media) which is the usual case in microstrip antenna practice, their integrands have a branch point and pole singularities on the integration paths (they also have singularities along the negative real axis and on the improper sheet where \(\text{Im}(k_{z1}) > 0\). However, they are not
important as far as numerical integration is concerned. Hence, the disposition of the contours in (A.47, A.48) near the singularities must be clarified.

Noting that these awkward locations of singularities stem from approximating the real media lossless and, thus, taking the slight lossiness of the real space (or substrate) into account, it can be shown that they are slightly below the integration paths for an e^{j\omega t} time dependence. The poles of the integrands arise from zeros of $D_{TE}$ and $D_{TM}$. Let $N_{TE}$ and $N_{TM}$ represent the number of zeros of $D_{TE}$ and $D_{TM}$, respectively, then the graphical resolution method [33] provides the following useful information

\[
N_{TE} = \begin{cases} 
0 & \text{for } k_{1d} \sqrt{\varepsilon_{r} - 1} < \pi/2 \\
-i n & \text{for } (n-1/2)\pi < k_{1d} \sqrt{\varepsilon_{r} - 1} < (n+1/2)\pi
\end{cases} \quad (A.54)
\]

\[
N_{TM} = n+1 \quad \text{for } n\pi < k_{1d} \sqrt{\varepsilon_{r} - 1} < (n+1)\pi \quad (A.55)
\]

; $n = 1, 2, 3 \ldots$

$; n = 0, 1, 2 \ldots$

It is noted from (A.54, A.55) that a possible number of poles for a one-layered planar microstrip structure are determined by both the substrate thickness $d$ and the relative dielectric constant $\varepsilon_{r}$ and that the integrands have only one pole singularity when the following condition is satisfied:

\[
k_{1d} \sqrt{\varepsilon_{r} - 1} < \frac{\pi}{2} \quad (A.56)
\]
Figure A.3. Contour of integration $C_\lambda$ in the complex $k_\rho$ plane.

Using the informations described in this paragraph, it is clear that the integrals (A.47, A.48) must be evaluated along the contour $C_\lambda$ shown in Figure A.3. In addition to the pole singularities, Figure A.2 shows branch cut singularities denoted by the symbol "\[\ldots\]". They were introduced to assure single-valuedness of the double-valued function $k_{Z1} = \sqrt{k_1^2 - k_\rho^2}$ and were selected along the contours $\text{Im}(k_{Z1}) = 0$ in order for the inequality $\text{Im}(k_{Z1}) < 0$ in (A.51) to remain true on this entire complex $k_\rho$ plane (often termed as proper Riemann sheet or top sheet) on which integrations are performed. Of course, $\text{Im}(k_{Z1}) > 0$ on bottom sheet.

In next section, alternative representations for the corresponding Hankel transform representations (A.47, A.48) are derived. These
representations involve integration paths extending from $-\infty$ to $\infty$. Thus, they are particularly useful for theoretical treatments which usually require a closed integration path.

3. ALTERNATIVE REPRESENTATION FOR THE HANKEL TRANSFORM REPRESENTATION

It is often desirable to use an alternative form of the Hankel transform representation in which the integration over $k_\rho$ extends from $-\infty$ to $\infty$. While this kind of alternative representation may be obtained via characteristic Green's function technique [23], it is easier at this moment to derive it via the transformation as described below since the Hankel transform representation is already available. Upon introducing

$$J_q(t) = \frac{1}{2} \left[ H_q^{(1)}(t) + H_q^{(2)}(t) \right]$$

(A.57)

where $H_q^{(1,2)}$ is the Hankel function of the first (second) kind of order $q$ and argument $t$, we may write the representations (A.47, A.48) as

$$\begin{bmatrix} -A_{1x}(\vec{r}) \\ -A_{2x}(\vec{r}) \end{bmatrix} = \frac{\mu_1}{4\pi} \int_{k_\rho = 0}^{\infty} \frac{k_\rho H_0^{(2)}(k_\rho \rho)}{DTE} \left[ e^{-jk_z \frac{z^2}{2}} \frac{\sin k_z (z+d)}{\sin k_z d} \right] dk_\rho$$

(A.58)
\[
\begin{bmatrix}
-A_{1x}(\vec{r}) \\
A_{2x}(\vec{r})
\end{bmatrix}
= -\frac{\mu_1}{4\pi} (\varepsilon_r - 1) \cos \phi \int_{k_\rho = 0}^{\infty} \frac{k_\rho^2 H_1(2)(k_\rho \rho)}{DTE^D_{TM}} \begin{bmatrix}
- e^{-jk_z l^2} \\
\cos k_{z2}(z+d) \cos k_{z2} d
\end{bmatrix} dk_\rho
\]

\[\begin{bmatrix}
-A_{1x}(\vec{r}) \\
A_{2x}(\vec{r})
\end{bmatrix}
= -\frac{\mu_1}{4\pi} (\varepsilon_r - 1) \cos \phi \int_{k_\rho = 0}^{\infty} \frac{k_\rho^2 H_1(1)(k_\rho \rho)}{DTE^D_{TM}} \begin{bmatrix}
- e^{-jk_z l^2} \\
\cos k_{z2}(z+d) \cos k_{z2} d
\end{bmatrix} dk_\rho
\]

Introducing the change of variables

\[
k_\rho' = k_\rho e^{-j\pi} \tag{A.60}
\]

in the integrands of the second integrals in (A.58, A.59) and using the following relations

\[
H_q^{(1)}(te^{j\pi}) = -e^{-jq\pi} H_q^{(2)}(t) \tag{A.61}
\]

yields an alternative representation for the potentials

\[
\begin{bmatrix}
-A_{1x}(\vec{r}) \\
A_{2x}(\vec{r})
\end{bmatrix}
= \frac{\mu_1}{4\pi} \int_{-\infty}^{\infty} \frac{k_\rho H_0^{(2)}(k_\rho \rho)}{sT \sin k_{z2}(z+d)} \begin{bmatrix}
- e^{-jk_z l^2} \\
\sin k_{z2}(z+d) \sin k_{z2} d
\end{bmatrix} dk_\rho \tag{A.62}
\]

\[
\begin{bmatrix}
-A_{1x}(\vec{r}) \\
A_{2x}(\vec{r})
\end{bmatrix}
= -\frac{\mu_1}{4\pi} (\varepsilon_r - 1) \cos \phi \int_{-\infty}^{\infty} \frac{k_\rho^2 H_1(1)(k_\rho \rho)}{DTE^D_{TM}} \begin{bmatrix}
- e^{-jk_z l^2} \\
\cos k_{z2}(z+d) \cos k_{z2} d
\end{bmatrix} dk_\rho \tag{A.63}
\]
in which the lower integral limits indicate that the branch cut along the negative real axis introduced by the transformation (A.57) is avoided as shown in Figure A.4. The branch cut is introduced in order to assure that the principal values are used for the multiple-valued Hankel function. The choice of transformations involving $H_q^{(2)}$ instead of $H_q^{(1)}$ is motivated by the fact that, for a time dependence $e^{jwt}$, the former satisfies the radiation condition at $r^\to$.

Figure A.4 shows the proper Riemann sheet in which $\text{Im} \,(k_{z1}) < 0$. Obviously, all the singularities in Figure A.3 are shown again in this figure and considerations about these singularities in previous section still remain true.

Figure A.4. Contour of integration $C$ in the complex $k_\rho$ plane.
APPENDIX B

A COMPLETE ASYMPTOTIC EXPANSION FOR $A_\chi$

A complete asymptotic expansion of $A_\chi$ in Equation (2.58) or (2.60) is derived in the following manner. From Equation (2.80), the imaginary part of $A_\chi$ is

$$\text{Im}(A_\chi) = \frac{u_1}{2\pi} \int_0^1 k_1 \text{Im} \left( \frac{1}{D_{\chi T}} \right) k_0 J_0(k_\rho) \, dk_\rho \quad (2.80)$$

Let $k_\rho = k_1 t$, then

$$\text{Im}(A_\chi) = \frac{u_1 k_1}{2\pi} \int_0^1 F_A(t) J_0(k_1 pt) \, dt \quad (B.1)$$

where

$$F_A(t) = \left[ k_1 \text{Im} \left( \frac{1}{D_{\chi T}} \right) \right]_{k_\rho = k_1 t}$$

$$= - \frac{\sqrt{1-t^2} \sin^2 k_1 \sqrt{\epsilon_r - t^2} \, d}{(\epsilon_r - t^2) \cos^2 k_1 \sqrt{\epsilon_r - t^2} \, d + (1-t^2) \sin^2 k_1 \sqrt{\epsilon_r - t^2} \, d} \quad (B.2)$$

Clearly, $F_A(t)$ in Equation (B.2) may be expressed in an infinite series form as

$$F_A(t) = \sum_{n=1}^{\infty} (-1)^n (1-t^2)^{n-1/2} x^n \quad (B.3)$$
where
\[ X = (k_1 d)^2 \left( \frac{\tan k_1 \sqrt{\epsilon_r - t^2}}{k_1 \sqrt{\epsilon_r - t^2}} d \right)^2 \]  
(B.4)

Since
\[ \tan Z = Z + \frac{Z^3}{3} + \frac{27 Z^5}{15} + \frac{17 Z^7}{315} + \ldots + \frac{(-1)^n (-1) 2n (2^{2m-1}) B_{2n}}{(2n)!} Z^{2n-1} \]
\[ + \ldots \quad \text{for } |Z| < \frac{\pi}{2} \]  
(B.5)

where \( B_{2n} \) is the Bernoulli number, \( X \) in Equation (B.4) may be expressed as
\[ X = (k_1 d)^2 \sum_{m=0}^{\infty} A_m (k_1 d \sqrt{\epsilon_r - t^2})^{2m} \quad \text{for } k_1 d \sqrt{\epsilon_r - t^2} < \frac{\pi}{2} \]  
(B.6)

where coefficients \( A_m \) correspond to the coefficients in series of Equation (B.5), and \( 0 < t < 1 \) as is shown in (B.1). Now, it is noticed that
\[ (\sqrt{\epsilon_r - t^2})^{2m} = (\epsilon_r - t^2)^m = [(1-t^2) + (\epsilon_r - 1)]^m, \]

Hence, Equation (B.6) may be arranged to give
\[ X = (k_1 d)^2 \sum_{\ell=0}^{\infty} b_{\ell} (1-t^2)^{\ell} \]  
(B.7)

in which \( b_{\ell} (\ell=0, 1, 2, \ldots) \) denote corresponding coefficients to \( (1-t^2)^{\ell} \). Rearranging Equation (B.3) after \( X \) is replaced by the series (B.7), we have
\[ F_A(t) = \sum_{k=1}^{\infty} C_k (1-t^2)^{k-1/2} \]  
(B.8)
in which $C_k$ is a known coefficient in terms of $k_1$ and $\epsilon_r$. Then, substituting Equation (8.8) into Equation (8.1) yields

$$\text{Im}(A_x) = \frac{\mu_0 k_1}{2\pi} \sum_{k=1}^{\infty} C_k \int_0^1 t(1-t^2)^{k-1/2} J_0(k_1\rho t) \, dt \quad (8.9)$$

The above integral (8.9) may be evaluated term by term simply using the integration formula

$$\int_0^1 x^{\nu+1}(1-x^2)^\mu J_\nu(bx)dx = 2^\mu \Gamma(\mu+1)b^{-\nu+1}J_{\nu+1}(b)$$

; $b > 0$, $\Re \nu > -1$, $\Re \mu > -1$

to give

$$\text{Im}(A_x) = \frac{\mu_0 k_1}{2\pi} \sum_{k=1}^{\infty} C_k z^{k-1/2} \Gamma(k+1/2)(k_1\rho)^{-(k+1/2)} J_{k+1/2}(k_1\rho) \quad (8.10)$$

In the above, $\Gamma$ denotes the Gamma function; also note that the Bessel function in (8.10) is of fractional order. With the help of the following identity

$$j_n(z) = \sqrt{\frac{n}{2z}} J_{n+1/2}(z)$$

where $j_n(z)$ represents the spherical Bessel function of the first kind, and the relations

$$j_n(z) = f_n(z)\sin z + (-1)^{n+1}f_{-n-1}(z)\cos z$$

$$f_0(z) = z^{-1}, \quad f_1(z) = z^{-2}$$

$$f_{n-1}(z) + f_{n+1}(z) = (2n+1)z^{-1}f_n(z)$$

$(n=0, \pm 1, \pm 2, \ldots)$
we find the following expression

\[
\text{Im}(A_x) = \frac{\nu_1 k_1}{2\pi} \left[ \sin k_1 \rho \sum_{n=1}^{\infty} \frac{D_{2n+1}}{(k_1 \rho)^{2n+1}} + \cos k_1 \rho \sum_{n=1}^{\infty} \frac{D_{2n}}{(k_1 \rho)^{2n}} \right].
\]

(B.11)

Since the integral \( A_x \) is an integral representation of the radiation field, we may deduce the complete asymptotic expansion of \( A_x \) via physical considerations that \( A_x \) must satisfy the outgoing wave radiation condition for \( k_1 \rho \to \infty \). Therefore, simply replacing \( -\sin k_1 \rho \) with \( e^{-jk_1 \rho} \) and likewise replacing \( \cos k_1 \rho \) with \( je^{-jk_1 \rho} \), the following complete asymptotic expansion for \( A_x \) is obtained:

\[
A_x = \frac{\nu_1 k_1}{2\pi} \left[ \sum_{n=1}^{\infty} -\frac{D_{2n+1}}{(k_1 \rho)^{2n+1}} + j \sum_{n=1}^{\infty} \frac{D_{2n}}{(k_1 \rho)^{2n}} \right].
\]

(B.12)

In Equations (B.11) and (B.12), the notation "\( \infty \)" is used to explicate that these series are asymptotic (or semiconvergent) representations of \( \text{Im}(A_x) \) and \( A(x) \), respectively.
APPENDIX C

MUTUAL IMPEDANCE $Z_{mn}$

Equation (3.10) in the text may be transformed into Equation (3.23) in the following manner.

$$Z_{mn} = \frac{j \omega}{4w^2} \int \int_{-W-H} \int \int_{-W-H} \left[ \cos \frac{\pi x_m'}{2H} \cos \frac{\pi x_n'}{2H} A_x(\rho) \right] dx_m' dy_m' \right] \right] dx_n dy_n$$

$$+ \frac{j \omega}{4w^2 k_1^2} \int \int_{-W-W} \int \int_{-H-H} \left[ \cos \frac{\pi x_m'}{2H} \cos \frac{\pi x_n'}{2H} \frac{\partial^2}{\partial x_m^2} (A_z(\rho)+A_x(\rho)) \right] dx_m' dx_n$$

$$dy_m' dy_n \quad (C.1)$$

where

$$\rho = \sqrt{(x_m + x_n - x_m')^2 + (y_m + y_n - y_m)^2}$$

Since

$$\frac{\partial}{\partial x_m} A_{x,z}(\rho) = - \frac{\partial}{\partial x_n} A_{x,z}(\rho),$$

the double integral in the bracket of the second term in Equation (C.1) can easily be integrated by part twice to give
\[\begin{align*}
&\iint_{-H \leq \rho \leq H} \cos \frac{\pi x_m}{2H} \cos \frac{\pi x_n}{2H} \partial^2 \delta_{x_n^2} (A_x(\rho) + A_z(\rho)) \, dx_m \, dx_n \\
&= - \int_{-H}^{H} \cos \frac{\pi x_m}{2H} \left[ - \int_{-H}^{H} \cos \frac{\pi x_n}{2H} \partial^2 \delta_{x_n^2} (A_x(\rho) + A_z(\rho)) \, dx_n \right] \, dx_m \\
&= - \int_{-H}^{H} \cos \frac{\pi x_m}{2H} \left[ - \int_{-H}^{H} \cos \frac{\pi x_n}{2H} \partial^2 \delta_{x_n^2} (A_x(\rho) + A_z(\rho)) \, dx_n \right] \, dx_m \\
&\quad + \frac{\pi}{2H} \int_{-H}^{H} \sin \frac{\pi x_n}{2H} \partial^2 \delta_{x_n^2} (A_x(\rho) + A_z(\rho)) \, dx_n \right] \, dx_n \\
&= - \int_{-H}^{H} \sin \frac{\pi x_n}{2H} \left[ - \int_{-H}^{H} \cos \frac{\pi x_m}{2H} \partial^2 \delta_{x_n^2} (A_x(\rho) + A_z(\rho)) \, dx_n \right] \, dx_m \\
&\quad + \frac{\pi}{2H} \int_{-H}^{H} \sin \frac{\pi x_m}{2H} \partial^2 \delta_{x_n^2} (A_x(\rho) + A_z(\rho)) \, dx_n \right] \, dx_m \\
&= - \frac{\pi^2}{4H^2} \int_{-H}^{H} \sin \frac{\pi x_n}{2H} \sin \frac{\pi x_m}{2H} (A_x(\rho) + A_z(\rho)) \, dx_n \, dx_m
\end{align*}\]
Substituting Equation (8.2) into Equation (8.1), we now have

\[
Z_{mn} = \frac{j\omega}{8w^2} \int \int \int \left( \cos \frac{\pi(x_n-x'_m)}{2H} + \cos \frac{\pi(x_n+x'_m)}{2H} \right) A_x(r) \, dx'_m \, dy'_m \, dx_n \, dy_n
\]

\[
- \frac{j\omega n^2}{32w^2k_1^2} \int \int \int \left( \cos \frac{\pi(x_n-x'_m)}{2H} - \cos \frac{\pi(x_n+x'_m)}{2H} \right)
\]

\[
(A_x(r) + A_z(r)) \, dx'_m \, dy'_m \, dx_n \, dy_n ,
\]

since

\[
\cos \frac{\pi x'_m}{2H} \cos \frac{\pi x_n}{2H} = \frac{1}{2} \left( \cos \frac{\pi(x_n-x'_m)}{2H} + \cos \frac{\pi(x_n+x'_m)}{2H} \right)
\]

and

\[
\sin \frac{\pi x'_m}{2H} \sin \frac{\pi x_n}{2H} = \frac{1}{2} \left( \cos \frac{\pi(x_n-x'_m)}{2H} - \cos \frac{\pi(x_n+x'_m)}{2H} \right).
\]

Let \( \alpha = x_n - x'_m \)

\( \beta = x_n + x'_m \)

\( \zeta = y_n - y'_m \)

\( \eta = y_n + y'_m \),

then

\[
dx'_m \, dx_n = \frac{1}{2} \, d\alpha \, d\beta
\]

and

\[
dy'_m \, dy_n = \frac{1}{2} \, d\zeta \, d\eta
\]
Thus,

\[
Z_{mn} = \frac{j\omega}{8W^2} \frac{1}{4} \iint S_{\alpha \beta} S_{\xi \eta} \left( \cos \frac{\pi \alpha}{2H} + \cos \frac{\pi \beta}{2H} \right) A_x(\rho) \, d\xi d\eta d\alpha d\beta
\]

\[
- \frac{j\omega \pi^2}{32W^2 \pi^2 k_1^2} \frac{1}{4} \iint S_{\alpha \beta} S_{\xi \eta} \left( \cos \frac{\pi \alpha}{2H} - \cos \frac{\pi \beta}{2H} \right) (A_x(\rho) + A_z(\rho)) \, d\xi d\eta d\alpha d\beta
\]

(C.4)

where

\[
\rho = \sqrt{(x_{mn} + \alpha)^2 + (y_{mn} + \xi)^2}
\]

and the region of integration \( S_{\alpha \beta}, S_{\xi \eta} \) are shown in Figure C.1.

![Figure C.1](image_url)

(a) \( S_{\alpha \beta} \)  \hspace{1cm}  (b) \( S_{\xi \eta} \)

Figure C.1. The region of integration.
The first integral term in Equation (C.4) can be evaluated to give

\[
\frac{j\omega}{8W^2} \frac{1}{4} \iint \int (\cos \frac{\pi \alpha}{2H} + \cos \frac{\pi \beta}{2H}) A_x(\rho) \, d\eta d\zeta d\beta d\alpha = \]

\[
= \frac{j\omega}{32W^2} \iint (\cos \frac{\pi \alpha}{2H} + \cos \frac{\pi \beta}{2H}) \left( \int \int \int \int_{\zeta=0}^{\zeta+2W} \int_{\eta=\zeta+2W}^{\eta=\zeta-2W} A_x(\rho) \, d\eta d\zeta \right) \, d\beta d\alpha
\]

\[
= \frac{j\omega}{32W^2} \iint (\cos \frac{\pi \alpha}{2H} + \cos \frac{\pi \beta}{2H}) \left( \int_{\alpha=0}^{\alpha+2H} \int_{\beta=-\alpha-2H}^{\beta=-\alpha+2H} \int_{\zeta=0}^{\zeta+2W} \int_{\eta=\zeta+2W}^{\eta=\zeta-2W} A_x(\rho) \, d\eta d\zeta \right) \, d\beta d\alpha
\]

\[
= \frac{j\omega}{8W^2} \frac{2W}{(-\zeta+2W)} \left( \int_{\alpha=0}^{\alpha+2H} \int_{\beta=-\alpha-2H}^{\beta=-\alpha+2H} \int_{\zeta=0}^{\zeta+2H} \int_{\eta=\zeta+2W}^{\eta=\zeta-2W} (\cos \frac{\pi \alpha}{2H} + \cos \frac{\pi \beta}{2H}) A_x(\rho) \, d\eta d\zeta \right) \, d\beta d\alpha
\]

\[
= \frac{j\omega}{8W^2} \frac{2W}{(-\zeta+2W)} \left( \int_{\alpha=0}^{\alpha+2H} \int_{\beta=-\alpha-2H}^{\beta=-\alpha+2H} \int_{\zeta=0}^{\zeta+2H} \int_{\eta=\zeta+2W}^{\eta=\zeta-2W} (\cos \frac{\pi \alpha}{2H} + \cos \frac{\pi \beta}{2H}) A_x(\rho) \, d\eta d\zeta \right) \, d\beta d\alpha
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left( \int \int (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right) \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]

\[
= \frac{j\omega}{2} (2H)^2 \int \int (1-v) \left[ (1-u) \cos \pi u + \frac{1}{\pi} \sin \pi u \right] \left( A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4) \right) \, d\eta d\zeta
\]
where

\[ \rho_1 = \frac{1}{\sqrt{(x_{mn} + 2Hu)^2 + (y_{mn} + 2Wv)^2}} \]

\[ \rho_2 = \frac{1}{\sqrt{(x_{mn} + 2Hu)^2 + (y_{mn} - 2Wv)^2}} \]

\[ \rho_3 = \frac{1}{\sqrt{(x_{mn} - 2Hu)^2 + (y_{mn} + 2Wv)^2}} \]

\[ \rho_4 = \frac{1}{\sqrt{(x_{mn} - 2Hu)^2 + (y_{mn} - 2Wv)^2}} \]

In obtaining the last expression in Equation (C.5), the following transformations are used:

\[ u = \frac{\alpha}{2H} \]

\[ v = \frac{\beta}{2W} \]

Similarly, the second term in Equation (C.4) can be evaluated to yield

\[
\frac{-j \omega \pi^2}{32 W^2 H^2 k_1^2} \frac{1}{4} \int \int (\cos \frac{\pi \alpha}{2H} - \cos \frac{\pi \beta}{2H})(A_x(\rho) + A_z(\rho)) \, d\zeta \, d\eta \, d\alpha \, d\beta
\]

\[
= \frac{-j \omega \pi^2}{2k_1^2} \int \int (1 - v)(1 - u) \cos \pi u - \frac{1}{\pi} \sin \pi u \{(A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3)) + A_z(\rho_4) + A_z(\rho_1) + A_z(\rho_2) + A_z(\rho_3) + A_z(\rho_4)\} \, du \, dv
\]

Substituting Equations (C.5) and (C.6) into Equation (C.4), we have

Equation (3.23) in Chapter III of the text
\[ Z_{mn} = j \frac{\omega(2H)^2}{2} \int \int_{0}^{1} (1-v) \left\{ (1-u)\cos \pi u + \frac{\sin \pi u}{\pi} \right\} \]

\[ (A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4)) \, du \, dv \]

\[ - j \frac{\omega^2}{2k_1^2} \int \int_{0}^{1} (1-v) \left\{ (1-u)\cos \pi u - \frac{\sin \pi u}{\pi} \right\} \]

\[ (A_x(\rho_1) + A_x(\rho_2) + A_x(\rho_3) + A_x(\rho_4)) \]

\[ + A_z(\rho_1) + A_z(\rho_2) + A_z(\rho_3) + A_z(\rho_4)) \, du \, dv \]  \hspace{1cm} (C.7)

where

\[ \rho_1 = \sqrt{(x_{mn} + 2Hu)^2 + (y_{mn} + 2Wv)^2} \]

\[ \rho_2 = \sqrt{(x_{mn} + 2Hu)^2 + (y_{mn} - 2Wv)^2} \]

\[ \rho_3 = \sqrt{(x_{mn} - 2Hu)^2 + (y_{mn} + 2Wv)^2} \]

\[ \rho_4 = \sqrt{(x_{mn} - 2Hu)^2 + (y_{mn} - 2Wv)^2} \].
EVALUATION OF THE SINGULAR TERM

Equations (3.38) and (3.39) in the text may be obtained in the following manner.

\[
I_1 = \int \int \frac{1}{(2H)^2u^2 + (2W)^2v^2} \, dv \, du
\]

\[
= \frac{1}{2W} \int \frac{\sin \pi u}{\pi} \left[ \ln \left( v + \sqrt{v^2 + (\frac{2H}{2W})^2 u^2} \right) - \sqrt{v^2 + (\frac{2H}{2W})^2 u^2} \right] \bigg|_{v=0}^{1} \, du
\]

\[
= \frac{1}{2W} \int \frac{\sin \pi u}{\pi} \left[ \ln \left( 1 + \sqrt{1 + (\frac{2H}{2W})^2 u^2} \right) - \ln \left( \frac{2H}{2W} \right) u - \sqrt{1 + (\frac{2H}{2W})^2 u^2} + \frac{2H}{2W} u \right] \, du
\]

\[
= \frac{1}{2W} \left[ \int \frac{\sin \pi u}{\pi} x u \, du - \frac{2}{\pi^2} \ln \left( \frac{2H}{2W} \right) - \frac{1}{\pi} \int \frac{\sin \pi u}{\pi} x u \, du \right]
\]

\[
- \int_0^1 \frac{\sin \pi u}{u} \sqrt{1 + (\frac{2H}{2W})^2 u^2} \, du + \frac{1}{\pi^2} \left( \frac{2H}{2W} \right)^2
\]

\text{(0.1)}
\[ I_2 = \int_{u=0}^{1} \int_{v=0}^{1} \frac{(1-v)(1-u)\cos\pi u}{\sqrt{(2H)^2 u^2 + (2W)^2 v^2}} \, dv \, du \]

\[ = \frac{1}{2W} \int_{u=0}^{1} (1-u)\cos\pi u \left\{ \ln \left( 1 + \sqrt{\frac{2H}{2W}} \right) - \ln \left( \frac{2H}{2W} \right) \ln \left( 1 + \sqrt{1 + \left( \frac{2H}{2W} \right)^2 u^2} \right) - \ln \left( \frac{2H}{2W} \right) u \right\} \, du \]

\[ = \frac{1}{2W} \left[ \int_{u=0}^{1} (1-u)\cos\pi u \ln \left( 1 + \sqrt{1 + \left( \frac{2H}{2W} \right)^2 u^2} \right) \, du \right. \]

\[ + \frac{1}{\pi} S_i(\pi) - \frac{2}{\pi^2} \ln \left( \frac{2H}{2W} \right) - \frac{1}{\pi} \int_{0}^{1} \sin \pi u \ln u \, du - \frac{2}{\pi^2} \]

\[ - \int_{0}^{1} (1-u)\cos\pi u \sqrt{1 + \left( \frac{2H}{2W} \right)^2 u^2} \, du \]  

Hence,
\[ S_1 = I_1 + I_2 = \int_{u=0}^{1} \int_{v=0}^{1} \frac{(1-\nu)(1-u)\cos \nu u + \frac{1}{\pi} \sin \nu u}{\sqrt{(2H)^2 u^2 + (2W)^2 \nu^2}} \, dv \, du \]

\[ = \frac{2H}{\pi^2 (2W)^2} - \frac{2}{\pi^2 2W} (1 + 2\ln \left(\frac{2H}{2W}\right)) + \frac{1}{\pi 2W} S_i(\pi) \]

\[ + \frac{1}{2W} \int_{0}^{1} \left\{ \frac{1}{\pi} \sin \nu u + (1-u)\cos \nu u \right\} \left\{ \ln \left(1 + \sqrt{1 + \frac{2H}{2W}u^2}\right) - \sqrt{1 + \frac{2H}{2W}u^2} \right\} du \]

\[ - \frac{2}{\pi (2W)} \int_{0}^{1} \sin \nu \nu u \nu u \, du \]

(D.3)

and

\[ S_2 = -I_1 + I_2 = \int_{u=0}^{1} \int_{v=0}^{1} \frac{(1-\nu)(1-u)\cos \nu u - \frac{1}{\pi} \sin \nu u}{\sqrt{(2H)^2 u^2 + (2W)^2 \nu^2}} \, dv \, du \]

\[ = -\frac{2H}{\pi^2 (2W)^2} - \frac{2}{\pi^2 2W} + \frac{1}{\pi 2W} S_i(\pi) \]

\[ + \frac{1}{2W} \int_{0}^{1} \left\{ -\frac{1}{\pi} \sin \nu u + (1-u)\cos \nu u \right\} \left\{ \ln \left(1 + \sqrt{1 + \frac{2H}{2W}u^2}\right) - \sqrt{1 + \frac{2H}{2W}u^2} \right\} du \]

(D.4)

In Equations (D.3) and (D.4), \( S_i(t) \) denotes the sine integral of argument \( t \).
APPENDIX E

ALTERNATIVE REPRESENTATION FOR $G_{zx}$

$G_{zx}$ in Equation (3.41) in the text may be changed into Equation (3.42) in the following manner.

$$G_{zx}(\vec{r}) = \frac{1}{\mu_2 k_2^2} (k_2^2 A_{zx}(\vec{r}) + \frac{a^2}{\partial x \partial z} A_{xx}(\vec{r}) + \frac{3^2}{\partial z^2} A_{zx}(\vec{r}))$$  \hspace{1cm} (E.1)

When we replace the potentials $A_{zx}$, $A_{xx}$ by the corresponding Sommerfeld type integrals in Equations (2.23) and (2.24), respectively, we have

$$G_{zx}(\vec{r}) = \frac{1}{4\pi k_2^2} \left[ - (k_2^2 + \frac{a^2}{\partial z^2}) (\varepsilon - 1) \cos \phi \int_C \frac{k_p H_2^2(\lambda_p)}{D_{TE}D_{TM}} \right. \hspace{1cm} \text{(E.2)}$$

$$\cos(k_2^2 - k_p^2)(z + d) \quad \cos(k_2^2 - k_p^2)d \quad d k_p \quad \frac{a^2}{\partial x \partial z} \int_C \frac{k_p H_2^2(\lambda_p)}{D_{TE}} \sin(k_2^2 - k_p^2)(z + d) \quad \sin(k_2^2 - k_p^2)d \quad d k_p \quad \right]$$

Since

$$\frac{3}{\partial x} H_0^2(\lambda_p) = - \lambda \cos \phi H_1^2(\lambda_p),$$
\[ G_{x}(r) = \frac{1}{4\pi k^2} \left[ (k^2 + \frac{\partial^2}{\partial z^2})(\varepsilon - 1) \frac{\partial}{\partial y} \int_{C_{y}} \frac{k_{p} H^{(2)}_{0}(k_{p})}{D_{TE}D_{TM}} \frac{\cos k_{p}^2 - k_{p}^2 (z+d)}{\sin k_{p}^2 - k_{p}^2 d} d\lambda \right. \]

\[ + \frac{\partial^2}{\partial x \partial z} \int_{C_{x}} \frac{k_{p} H^{(2)}_{0}(k_{p})}{D_{TE}} \sin k_{p}^2 (z+d) \left. \sin k_{p}^2 - k_{p}^2 d \right] \].

\[ = \frac{1}{4\pi k^2} \left[ -k_{p}^2 (\varepsilon - 1) \frac{\partial^2}{\partial x \partial z} \int_{C} \frac{k_{p} H^{(2)}_{0}(k_{p})}{D_{TE}D_{TM}} \frac{\cos k_{p}^2 - k_{p}^2 (z+d)}{\sqrt{k_{p}^2 - k_{p}^2 d - k_{p}^2 d \cos k_{p}^2 - k_{p}^2 d}} dk_{p} \right. \]

\[ - (\varepsilon - 1) \frac{\partial^2}{\partial x \partial z} \int_{C} \frac{k_{p} H^{(2)}_{0}(k_{p})}{D_{TE}D_{TM}} \sin k_{p}^2 - k_{p}^2 (z+d) \left. \sin k_{p}^2 - k_{p}^2 d \right] \]

\[ - \frac{\partial^2}{\partial x \partial z} \int_{C} \frac{k_{p} H^{(2)}_{0}(k_{p})}{D_{TE}D_{TM}} \sin k_{p}^2 - k_{p}^2 (z+d) \right] \]

\[ = \frac{1}{4\pi k^2} \frac{\partial^2}{\partial x \partial z} \int_{C} \frac{(\varepsilon - 1)k_{p}^2}{\sqrt{k_{p}^2 - k_{p}^2 d}} \left. \frac{D_{TE}D_{TM} \cos k_{p}^2 - k_{p}^2 d}{d} + \right. \]

\[ \frac{1}{D_{TE} \sin k_{p}^2 - k_{p}^2 d} \left. \right] \cdot k_{p} \sin k_{p}^2 - k_{p}^2 (z+d) H^{(2)}_{0}(k_{p}) \right] \]

(E.3)

Expressions in the last brackets in Equation (E.3) may be modified in the following manner.
\[
\begin{align*}
& (\varepsilon_r - 1) k_p^2 \\
& \sqrt{k_2^2 - k_p^2} D_{TE} D_{TM} \cos k_2^2 - k_p^2 \, d + \frac{1}{D_{TE} \sin k_2^2 - k_p^2} \, d \\
& = (\varepsilon_r - 1) k_p^2 \sin \frac{k_2^2 - k_p^2}{k_p^2} \, d + \sqrt{k_2^2 - k_p^2} D_{TM} \cos k_2^2 - k_p^2 \, d \\
& \frac{j \varepsilon_r \sqrt{k_2^2 - k_p^2} \cos k_2^2 - k_p^2 \, d + j \sqrt{k_2^2 - k_p^2} \sin k_2^2 - k_p^2 \, d)}{j \sqrt{k_2^2 - k_p^2} (\varepsilon_r \sqrt{k_1^2 - k_p^2} \cos k_2^2 - k_p^2 + j \sqrt{k_2^2 - k_p^2} \sin k_2^2 - k_p^2 \, d)} \\
& = \frac{1}{\sqrt{k_2^2 - k_p^2} (\varepsilon_r \sqrt{k_1^2 - k_p^2} \cos k_2^2 - k_p^2 + j \sqrt{k_2^2 - k_p^2} \sin k_2^2 - k_p^2 \, d)} \\
& = \frac{1}{\sqrt{k_2^2 - k_p^2} (\varepsilon_r \sqrt{k_1^2 - k_p^2} \cos k_2^2 - k_p^2 + j \sqrt{k_2^2 - k_p^2} \sin k_2^2 - k_p^2 \, d)} \\
\end{align*}
\]

(E.4)

Substituting Equation (E.4) into Equation (E.3), we have

\[
G_{x}(\vec{r}) = \frac{1}{4\pi k_2^2} \frac{a^2}{3x3z} \int_{C} \frac{\sqrt{k_2^2 - k_p^2} \sin k_2^2 - k_p^2 (z+d) \cdot k_p H_0^2(k_p \rho)}{\sqrt{k_2^2 - k_p^2} (\varepsilon_r \sqrt{k_1^2 - k_p^2} \cos k_2^2 - k_p^2 + j \sqrt{k_2^2 - k_p^2} \sin k_2^2 - k_p^2 \, d) \, dk_p}
\]

or

\[
G_{x}(\vec{r}) = \frac{1}{2\pi k_2^2} \frac{a^2}{3x3z} \int_{0}^{\infty} \frac{\sqrt{k_1^2 - k_p^2} \sin k_1^2 - k_p^2 (z+d) k_p J_0(k_p \rho) \, dk_p}{\sqrt{k_2^2 - k_p^2} (\varepsilon_r \sqrt{k_2^2 - k_p^2} \cos k_2^2 - k_p^2 + j \sqrt{k_2^2 - k_p^2} \sin k_2^2 - k_p^2 \, d)}
\]