GLOBAL EXISTENCE FOR BUBBLES IN A HELE-SHAW CELL WITH ARBITRARY NONZERO SURFACE TENSION

DISSERTATION

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ABSTRACT

My current research concerns global existence for arbitrary nonzero surface tension of bubbles in a Hele-Shaw cell.

Without imposed pressure gradient or side walls, the circular bubble is shown to be asymptotically stable to all sufficiently smooth initial perturbation.

For the bubbles translating in the presence of a pressure, when the cell width to bubble size is sufficiently large, we show that a unique steady translating near-circular bubble symmetric about the channel centerline exists, where the bubble translation speed in the laboratory frame is found as part of the solution. We prove global existence for symmetric sufficiently smooth initial conditions close to this shape and show that the steady translating bubble solution is an attractor within this class of disturbances. In the absence of side walls, we prove stability of the steady translating circular bubble without restriction on symmetry of initial conditions. These results hold for any nonzero surface tension despite the fact that a local planar approximation near the front of the bubble would suggest Saffman-Taylor instability. An important element of the proof was the introduction of a weighted Sobolev norm that accounts for stabilization due to advection of disturbances from the front to the back of the bubble.
We exploit a boundary integral approach that is particularly suitable for analysis of nonzero viscosity ratio between fluid inside and outside the bubble.
To my son, Eric
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CHAPTER 1
INTRODUCTION

1.1 Evolution equations

Displacement of a more viscous fluid by a less viscous one in a Hele-Shaw cell (see Figure 1.1) is a canonical problem in a much wider class of Laplacian growth problems, such as dendritic crystal growth, electrochemical growth, diffusion limited aggregation, filtration combustion and tumor growth. It has attracted many physicists and mathematicians. In the recent thirty years, there are many reviews on this subject, see e.g. Saffman [33], Bensimon et al. [7], Homsy [17], Pelce [27], Kessler et al. [24], Tanveer [41] & [42], Hohlov [16], and Howison [21] & [22].

![Figure 1.1: Displacement of a viscous fluid by a less viscous fluid in a Hele-Shaw cell](image)

We consider the motion of the interface in the frame where the steady bubble
is stationary. Figure 1.2 shows the Hele-Shaw flow geometry, when viewed from the top. We define \( v \) to be the constant fluid velocity in the laboratory frame at \( \pm \infty \); this motion is forced by a uniform far-field pressure gradient, when there is one. The steady bubble translation speed in the lab frame in the case of imposed pressure gradient is given\(^1\) by \( u \). All lengths are measured in units of some length scale whose choice is different with or without pressure gradient. This choice is made more explicit later as we discuss each case separately. In terms of this length scale, the non-dimensional half width of the strip is \( \frac{\pi}{\beta} \).

The two-phase Hele-Shaw problem in the steady bubble frame is described mathematically as follows: \( \Omega_2(t) \subset \mathbb{R}^2 \) denotes the simply connected bounded domain occupied by fluid with viscosity \( \mu_2 \) at time \( t \), while a different fluid of viscosity \( \mu_1 \) occupies \( \Omega_1(t) \), where \( \Omega_1(t) \cup \Omega_2(t) \) constructs the strip which half width is \( \frac{\pi}{\beta} \), i.e., \( \{(x,y) | x \in \mathbb{R}, -\frac{\pi}{\beta} < y < \frac{\pi}{\beta}\} \) (see Figure 1.2). In the case of an imposed pressure gradient, it is convenient to define a non-dimensional parameter \( u_0 \) (which will turn out to be small for small \( \beta \)) so that

\[
\frac{1 - \frac{\mu_2}{\mu_1}}{1 - \frac{\mu_2}{\mu_1}} u + 2 = \frac{1 - \frac{\mu_2}{\mu_1}}{v - \frac{\mu_2}{\mu_1} u} u
\]

We will also consider the simpler case of no pressure gradient at \( \infty \), i.e. \( v = 0 \), in which case there is no translating for steady bubble. To present both cases simultaneously, we introduce notation \( V_\infty \) so that \( V_\infty = 0 \) correspond to nontranslating steady bubble, and \( V_\infty = 1 \) correspond to translating steady bubble. Then in the frame of reference

---

\(^1\) \( u \) is not specified, but determined as part of the steady solution.
\[ y = \frac{\pi}{\beta}, \quad \frac{\partial \phi_1}{\partial y}(x, \frac{\pi}{\beta}) = 0 \]

\[ \Omega_1 \]

\[ \Omega_2 \]

\[ C \]

\[ \Delta \phi_2 = 0 \]

\[ \phi_1 \sim -(u_0 + 1)V_\infty x + o(1) \]

\[ \Delta \phi_1 = 0 \]

\[ y = -\frac{\pi}{\beta}, \quad \frac{\partial \phi_1}{\partial y}(x, -\frac{\pi}{\beta}) = 0 \]

Figure 1.2: The Hele-Shaw cell geometry for a bubble in a frame where the steady bubble is at rest.

where the steady bubble is at rest, the velocity potential \( \phi_1 \) is related to the pressure \( p_1 \) through

\[ \phi_1 + \frac{u}{(v - \frac{\mu_2}{\mu_1}u)}V_\infty x = -\frac{b^2}{12\mu_1 l} \frac{1}{(v - \frac{\mu_2}{\mu_1}u)}p_1, \]

where \( l \) is the length scale used for non-dimensionalization, as specified later. Similarly, the velocity potential \( \phi_2 \) is related to the pressure \( p_2 \) through

\[ \phi_2 + \frac{u}{(v - \frac{\mu_2}{\mu_1}u)}V_\infty x = -\frac{b^2}{12\mu_2 l} \frac{1}{(v - \frac{\mu_2}{\mu_1}u)}p_2. \]

Hence, on the interface \( \Omega_1 \cap \Omega_2 = C \), the MacLean-Saffman [25] condition on pressure
jump equaling product of surface tension and curvature implies

\[(2 + u_0)V_\infty x + \phi_1 - \frac{\mu_2}{\mu_1} \phi_2 = B \kappa,\]  

(1.1)

where \(B = \frac{\bar{\sigma}^2 \phi_2}{12 \mu_1 (v - \frac{\mu_2}{\mu_1} u)^2}\) with \(\bar{\sigma}\) the surface tension, and \(\kappa\) is the non-dimensional curvature.

The kinematic condition that ignores 3-D effects on the interface \(C\) is given

\[\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} = U,\]

(1.2)

where \(n\) is the inward normal unit on \(C\), and \(U\) is the normal speed of the bubble.

There are two simplifications: one is \(\beta = 0\), i.e. wall effects are ignored, and the second is \(V_\infty = 0\), i.e., there is no pressure gradient to cause translation of the steady bubble. In the case when both \(\beta = 0\) and \(V_\infty = 0\), it is convenient to choose the non-dimensional initial bubble perimeter to be \(2\pi\), i.e. choose length scale \(l = \frac{1}{2\pi} \times \) initial bubble perimeter. The time scale in this case is implicit by choosing \(B = 1\) in (1.1). However, we will retain the symbol \(B\) in the calculations to make the role of surface tension more explicit. When \(\beta \neq 0\) or \(V_\infty = 1\), we choose \(l = \frac{1}{2\pi} \times \) steady bubble perimeter, i.e. choose non-dimensional steady bubble perimeter to be \(2\pi\).

The interface between the two fluids with different viscosities is described parametrically at any time \(t\) by \(z = x(\alpha, t) + iy(\alpha, t)\), where \(\alpha\) is chosen so that \(z(\alpha + 2\pi, t) = z(\alpha, t)\). Following Hou et al [18], [19], instead of using \(x, y\) as the dynamical variables, we describe the evolution in terms of \(\theta(\alpha, t)\) and \(s_\alpha(\alpha, t)\), where \(s_\alpha = \sqrt{x_\alpha^2 + y_\alpha^2}\), and \(x_\alpha + iy_\alpha = s_\alpha e^{i\pi/2 + i\alpha + i\theta}\). Note \(\frac{\pi}{2} + \alpha + \theta(\alpha, t)\) is the angle formed between the tangent
to the curve and the positive \(x\)-axis, as the boundary is traversed counter-clockwise with increasing \(\alpha\) (see Figure 1.2). Then the unit tangent vector on the interface \(t = (-\sin(\alpha + \theta), \cos(\alpha + \theta))\) and the unit normal vector pointing inward at bubble interface is \(n = (-\cos(\alpha + \theta), -\sin(\alpha + \theta))\). The interface \(z(\alpha, t) = x(\alpha, t) + iy(\alpha, t)\) is reconstructed by using

\[
z(\alpha, t) = z(0, t) + \int_0^\alpha s_\alpha(\alpha', t)e^{i\pi/2 + i\alpha' + i\theta(\alpha', t)}d\alpha',
\]

where \(z(0, t)\) is determined by from the evolution of a single point as described later.

In \(\Omega_1\) or \(\Omega_2\), we seek a vortex sheet (double layer) representation of the velocities of the two fluids \((u_1, v_1)\) or \((u_2, v_2)\) through

\[
u_1, v_2 - iv_1, v_2 = -(u_0 + 1)V_\infty + \frac{1}{2\pi i} \int_0^{2\pi} \gamma(\alpha') \mathcal{M}(z, \alpha') d\alpha',
\]

where for \(\beta = 0\),

\[
\mathcal{M}(z, \alpha') = \frac{1}{z - z(\alpha')};
\]

and for \(\beta \neq 0\),

\[
\mathcal{M}(z, \alpha') = \frac{\beta}{4} \coth \left[ \frac{\beta}{4} (z - z(\alpha')) \right] - \frac{\beta}{4} \tanh \left[ \frac{\beta}{4} (z - z^*(\alpha')) \right].
\]

It can be checked from (1.4) that \(v_1 = \frac{\partial \phi_1}{\partial \eta} = 0\) on \(y = \text{Im} z = \pm \frac{\pi}{\beta}\); and as \(x = \text{Re} z \to \pm \infty, u_1 - iv_1 \to -(u_0 + 1)\); thus both far-field and wall boundary conditions in the case \(\beta \neq 0\) are automatically satisfied.

As \(C\) is approached, (1.4) implies the following limit for sufficiently regular \(\gamma(\alpha, t)\) and \(z(\alpha, t)\):

\[
u_{1,2} - iv_{1,2} = -(u_0 + 1)V_\infty + \frac{1}{2\pi i} \text{PV} \int_0^{2\pi} \gamma(\alpha') \mathcal{R}(\alpha, \alpha') d\alpha' \pm \frac{\gamma(\alpha)}{2z_\alpha(\alpha)}.
\]
Hence, the normal velocity \( U = (u_1, v_1) \cdot \mathbf{n} = (u_2, v_2) \cdot \mathbf{n} \) is given by

\[
U = (u_0 + 1)V_\infty \cos(\alpha + \theta(\alpha)) + \text{Re} \left( \frac{z_\alpha}{2\pi i s_\alpha} \text{PV} \int_0^{2\pi} \mathcal{R}(\alpha, \alpha') \gamma(\alpha') d\alpha' \right),
\]

where for \( \beta = 0 \),

\[
\mathcal{R}(\alpha, \alpha') = \mathcal{M}(z(\alpha), \alpha') = \frac{1}{z(\alpha) - z(\alpha')};
\]

for \( \beta \neq 0 \),

\[
\mathcal{R}(\alpha, \alpha') = \mathcal{M}(z(\alpha), \alpha') = \frac{\beta}{4} \coth \left[ \frac{\beta}{4} (z(\alpha) - z(\alpha')) \right] - \frac{\beta}{4} \tanh \left[ \frac{\beta}{4} (z(\alpha) - z^*(\alpha')) \right].
\]

Also,

\[
\partial_\alpha \phi_{1,2} = \text{Re} \left( z_\alpha (u_{1,2} - iv_{1,2}) \right)
= (u_0 + 1)V_\infty s_\alpha \sin(\alpha + \theta(\alpha)) + \text{Re} \left( \frac{z_\alpha}{2\pi i} \text{PV} \int_0^{2\pi} \mathcal{R}(\alpha, \alpha') \gamma(\alpha') d\alpha' \right) \pm \frac{1}{2} \gamma(\alpha).
\]

Taking derivative with respect to \( \alpha \) on both sides of (1.1), we have

\[
(2 + u_0)V_\infty x_\alpha + \phi_{1,2} - \frac{\mu_2}{\mu_1} \phi_{2,2} = B \left( \frac{\theta_\alpha}{s_\alpha} \right)_\alpha.
\]

Using (1.9) and (1.10), we deduce the following linear integral equation for \( \gamma \):

\[
\gamma(\alpha) = -2a_\mu \text{Re} \left( \frac{z_\alpha}{2\pi i} \text{PV} \int_0^{2\pi} \mathcal{R}(\alpha, \alpha') \gamma(\alpha') d\alpha' \right)
+ 2 \left( 1 + \frac{\mu_2}{\mu_1 + \mu_2} u_0 \right) V_\infty s_\alpha \sin(\alpha + \theta(\alpha)) + \sigma \left( \frac{\theta_\alpha}{s_\alpha} \right)_\alpha,
\]

where \( a_\mu = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \) and \( \sigma = B \frac{2\mu_1}{\mu_1 + \mu_2} \).

The evolution of the interface (1.2) is equivalent to

\[
(x_t(\alpha, t), y_t(\alpha, t)) = U \mathbf{n} + T \mathbf{t},
\]
where $T$ is some arbitrarily chosen the tangent velocity of the interface. By taking derivative of (1.12) with respect to $\alpha$, and using $t_\alpha = (1 + \theta_\alpha)n$, $n_\alpha = -(1 + \theta_\alpha)t$, it follows

\[
\begin{cases}
\theta_t(\alpha, t) = \frac{1}{s_\alpha} \left[ U_\alpha(\alpha, t) + T(\alpha, t)(1 + \theta_\alpha(\alpha, t)) \right], \\
s_{\alpha,t}(\alpha, t) = T_\alpha(\alpha, t) - (1 + \theta_\alpha(\alpha, t))U(\alpha, t).
\end{cases}
\] (1.13)

Hou et al in [18] were the first to use this formulation. They also note that if the tangent velocity is chosen in accordance to

\[
T = \int_0^\alpha \left( 1 + \theta_{\alpha'}(\alpha', t) \right) U(\alpha', t) d\alpha' - \frac{\alpha}{2\pi} \int_0^{2\pi} \left( 1 + \theta_\alpha(\alpha, t) \right) U(\alpha, t) d\alpha.
\] (1.14)

Then (1.13) implies $s_\alpha = \frac{L}{2\pi}$ and

\[
\begin{cases}
\theta_t(\alpha, t) = \frac{2\pi}{L} U_\alpha(\alpha, t) + \frac{2\pi}{L} T(\alpha, t)(1 + \theta_\alpha(\alpha, t)), \\
L_t(t) = -\int_0^{2\pi} \left( 1 + \theta_\alpha(\alpha, t) \right) U(\alpha, t) d\alpha,
\end{cases}
\] (A.1)

Remark. The special choice of tangential velocity $T$ in (1.14) not only makes $s_\alpha$ independent of $\alpha$, but also ensure that the term containing the highest $\alpha$ derivative of $\theta$ is linear and has constant coefficient. We will make the same choice in the ensuing analysis.

\section*{Note 1.1.}
(1.12) and (1.14) imply that $(x_t(0, t), y_t(0, t)) = U(0, t)n(0, t)$, which allows reconstruction of $z(\alpha, t)$ from $\theta$ in accordance to (1.3) when integrated in time. Furthermore, we note in particular that

\[
y_t(0, t) = -U(0, t) \sin (\theta(0, t)).
\] (1.15)
There is a vast literature on zero surface tension $\sigma = 0$, though initial value problem in this case is ill-posed [20], [14] and not always physically relevant (See [42] for detailed discussion of this issue). With surface tension, there are rigorous local existence results for general initial conditions both for one and two phase problems [10], [12] using different approaches. Also there are some global existence and nonlinear stability results [8], [15] for one and two phase Hele-Shaw for near-circular initial shapes in the absence of any forcing such as fluid injection or pressure gradient. These have been generalized to non-Newtonian one phase fluids [11]. There are similar results available for the two phase Stefan problem [13], [29], which is mathematically close to but distinct from the two-phase Hele-Shaw (also called Muskat problem) being studied here. It is widely recognized that global existence problem with surface tension for arbitrary initial shape is a difficult open problem, though there is quite a substantial literature involving formal asymptotic and numerical computations (see cited reviews above). Even the restricted problem of stability of steadily propagating shapes such as a semi-infinite finger [45], [46] or a finite translating bubble [46] for nonzero surface tension remains an open problem for rigorous analysis. Translation causes complications in global analysis due to a less viscous fluid displacing a more viscous one – a planar front is known to be unstable [32] in this case.

In this thesis, we extend the Friedman & Tao [15] results to more general non-analytic initial conditions in the absence of imposed pressure gradient $V_\infty = 0$ and sidewalls ($\beta = 0$) or other confinements. Furthermore, we also study the motion of a bubble in a Hele-Shaw cell subjected to an external pressure gradient that causes it to translate ($V_\infty = 1$) with or without side wall effects ($\beta \neq 0$ or $\beta = 0$). Our
methodology is also different and uses a boundary integral formulation of Hou et al [18] as described already. This formulation has been widely used for numerical calculations in a wide variety of free boundary problems involving Laplace’s equation. Ambrose [3] has recently used this formulation to prove local existence for the Hele-Shaw flow of general initial shapes [3] without surface tension. Given the wide use of boundary integral methods in computations, one motivation for the present paper is to further develop the mathematical machinery associated with this method so as to be applicable to more general existence problems.

**Definition 1.2.** Let $r \geq 0$. The Sobolev space $H^r_p$ is the set of all $2\pi$-periodic function $f = \sum_{-\infty}^{\infty} \hat{f}(k)e^{ika}$ such that

$$
\|f\|_r = \left( \sum_{k=-\infty}^{\infty} |k|^{2r} |\hat{f}(k)|^2 + |\hat{f}(0)|^2 \right)^{\frac{1}{2}} < \infty.
$$

**Note 1.3.** For $f, g \in H^r_p$, the Banach Algebra property $\|fg\|_r \leq C_r\|f\|_r\|g\|_r$ for $r \geq 1$ with some constant $C_r$ depending on $r$ is easily proved and will be useful in the sequel. Also, in what follows the $\hat{\cdot}$ symbol will reserved for Fourier components.

**Definition 1.4.** The Hilbert transform, $\mathcal{H}$, of a function $f \in H^0_p$ with Fourier Series $f = \sum_{-\infty}^{\infty} \hat{f}(k)e^{ika}$ is given by

$$
\mathcal{H}[f](\alpha) = \frac{1}{2\pi}PV \int_{0}^{2\pi} f(\alpha') \cot \frac{1}{2}(\alpha - \alpha')d\alpha' = \sum_{k \neq 0} -i \text{sgn}(k)\hat{f}(k)e^{ika}.
$$

**Note 1.5.** For $f \in H^1_p$, the Hilbert transform commutes with differentiation. We will denote derivative with respect to $\alpha$, either by $D_\alpha$ or subscript $\alpha$. Also, for the sake of
brevity of notation, the time \( t \) dependence will often be omitted, except where it might cause confusion otherwise.

**Definition 1.6.** We define the operator \( \Lambda \) to be a derivative followed by the Hilbert transform: \( \Lambda = \mathcal{H}D_\alpha \). Following Ambrose [3], we also define commutator

\[ [\mathcal{H}, f]g = \mathcal{H}(fg) - f\mathcal{H}(g). \]

**Note 1.7.** It is clear that

\[
\left( \int_0^{2\pi} \left( f^2 + f\Lambda f \right) d\alpha \right)^{1/2}
\]

is equivalent to \( H_p^{1/2} \) norm of a real-valued \( 2\pi \)-periodic function \( f \). Further, note the operator \( \Lambda \) is self-adjoint in \( H_p^{1/2} \) Hilbert space.

**Definition 1.8.** We define a linear integral operator \( \mathcal{K}[z] \), depending on \( z \), as

\[
\mathcal{K}[z]f = \frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} f(\alpha') \left\{ \mathfrak{R}(\alpha,\alpha') - \frac{1}{2z_\alpha(\alpha')} \cot \frac{1}{2}(\alpha - \alpha') \right\} d\alpha'.
\]  

(1.16)

**Remark.** For \( 2\pi \)-periodic functions \( f \) and \( z \), it is clear that the upper and lower limits of the integral above can be replaced by \( a \) and \( a + 2\pi \) respectively for arbitrary \( a \).

**Definition 1.9.** We define a complex operator \( \mathcal{G}[z] \), depending on \( z \), so that

\[
\mathcal{G}[z] \gamma = z_\alpha \left[ \mathcal{H}, \frac{1}{z_\alpha} \right] \gamma + 2iz_\alpha \mathcal{K}[z] \gamma.
\]

(1.17)

It is also convenient to define a related real operator \( \mathcal{F}[z] \), depending on \( z \), so that

\[
\mathcal{F}[z] \gamma = \text{Re} \left( \frac{1}{i} \mathcal{G}[z] \gamma \right).
\]

(1.18)
Hence, the normal velocity is

\[ U(\alpha, t) = \frac{\pi}{L} \mathcal{H}[\gamma] + \frac{\pi}{L} \text{Re} \left( \mathcal{G}[z] \gamma \right) + (u_0 + 1)V_\infty \cos (\alpha + \theta(\alpha)) \]  

\( (1.19) \)

(1.11) and (1.14), with \( s_\alpha = \frac{L}{2\pi} \), can be written as by

\[
\left\{ \begin{array}{l}
\gamma(\alpha, t) = -a_p \mathcal{F}[z] \gamma(\alpha, t) + \frac{L}{\pi} \left( 1 + \frac{\mu_2}{\mu_1 + \mu_2} u_0 \right) V_\infty \sin(\alpha + \theta) + \frac{2\pi}{L} \sigma \theta_{\alpha \alpha}, \\
T(\alpha, t) = \int_0^\alpha \left( 1 + \theta_{\alpha'}(\alpha', t) \right) U(\alpha', t) d\alpha' - \frac{\alpha}{2\pi} \int_0^{2\pi} \left( 1 + \theta_{\alpha}(\alpha, t) \right) U(\alpha, t) d\alpha,
\end{array} \right. \tag{A.2}
\]

The initial condition is given by

\[ \theta(\alpha, 0) = \theta_0(\alpha), \quad L(0) = L_0. \]  

\( (1.20) \)

### 1.2 Equivalent evolution system

**Definition 1.10.** We introduce a family of projector operators \( \{ \mathcal{Q}_n \} \) by

\[
\mathcal{Q}_n f = f - \sum_{k=-n}^{n} \hat{f}(k) e^{ik\alpha}
\]

for \( f = \sum_{-\infty}^{\infty} \hat{f}(k) e^{ik\alpha} \) and \( n \in \mathbb{Z}^+ \cup \{0\} \). Henceforth, we will denote \( \tilde{\theta} = \mathcal{Q}_1 \theta \).

**Definition 1.11.** We define \( \mathcal{H}^r \) as a subspace of \( H^r_p \) containing real valued functions so that \( \phi \in \mathcal{H}^r \) implies \( \mathcal{Q}_1 \phi = \phi \). Note in this subspace, \( \| \phi \|_r = \| D^r_\alpha \phi \|_0 \) for \( r \geq 1 \).

It turns out that linearization of (A.1)-(A.2) about the steady shape gives rise to neutrally stable modes, including \( \hat{\theta}(\pm 1; t) \). Therefore, it is convenient to project
away these modes in the time evolution equation and introduce instead a suitable constraint to determine \( \hat{\theta}(\pm 1; t) \).

The equivalent system suitable for analysis is

\[
\begin{align*}
\frac{\partial \tilde{\theta}(\alpha, t)}{\partial t} &= \frac{2\pi}{L} Q_1(U_\alpha + T(1 + \theta_\alpha)), \\
\frac{dL(t)}{dt} &= -\int_0^{2\pi} (1 + \theta_\alpha) U d\alpha, \\
\frac{d\hat{\theta}(0; t)}{dt} &= \frac{1}{L} \int_0^{2\pi} T(1 + \theta_\alpha) d\alpha,
\end{align*}
\]

(B.1)

with \( U(\alpha, t) \), a function of \( \gamma \) and \( \theta \), determined by (1.6), \( \gamma(\alpha, t) \), \( T(\alpha, t) \), \( \hat{\theta}(1; t) \) and \( \hat{\theta}(-1; t) \) determined\(^2\) by

\[
\begin{align*}
\gamma(\alpha, t) &= -a_\mu F[z] \gamma(\alpha, t) + \frac{L}{\pi}(1 + \frac{\mu_2}{\mu_1 + \mu_2} u_0) V_\infty \sin(\alpha + \theta) + \frac{2\pi}{L} \sigma \theta_{\alpha\alpha}, \\
T(\alpha, t) &= \int_0^\alpha (1 + \theta_{\alpha'}(\alpha')) U(\alpha') d\alpha' - \frac{\alpha}{2\pi} \int_0^{2\pi} (1 + \theta_\alpha(\alpha)) U(\alpha) d\alpha, \\
\int_0^{2\pi} \exp \left( i\alpha + i(\hat{\theta}(-1; t) e^{-i\alpha} + \hat{\theta}(1; t) e^{i\alpha} + \tilde{\theta}(\alpha, t)) \right) d\alpha &= 0,
\end{align*}
\]

(B.3)\n
(B.4)\n
(B.5)

where \( \theta(\alpha, t) = \hat{\theta}(0; t) + \hat{\theta}(-1; t) e^{-i\alpha} + \hat{\theta}(1; t) e^{i\alpha} + \tilde{\theta}(\alpha, t) \). We use initial conditions:

\[
\hat{\theta}(\alpha, 0) = Q_1 \theta_0, \quad L(0) = L_0, \quad \hat{\theta}(0; 0) = \hat{\theta}_0(0).
\]

We prove in Chapter 2 that the evolution system (A.1)-(A.2) is equivalent to (B.1)-(B.5). Once this equivalence is determined, it is convenient to analyze (B.1)-(B.5) with the initial conditions (1.21) instead of the original system of equations.

\(^2\)Since \( \theta(\alpha, t) \) is real valued, note \( \hat{\theta}^*(1; t) = \hat{\theta}(-1, t) \).
1.3 Main results

We organize the thesis as follows. In Chapter 2, we prove several preliminary lemmas about some integral operators.

Chapter 3 considers the motion of bubble without imposed pressure gradient or side walls ($V_\infty = 0$, $\beta = 0$). We analyze the evolution of $\tilde{\theta}(\alpha, t) = \sum_{k \neq 0, \pm 1} \hat{\theta}(k; t)e^{ik\alpha}$ and $L(t)$. We first form a Galerkin approximation in a finite dimensional space. We then show that solutions to these approximate equations exist locally in time by using the standard Picard theorem for differential equations in Banach spaces. We then define the energy $E(t) = \frac{1}{2} \| D_\alpha \tilde{\theta} (\cdot, t) \|^2_0$ for $r \geq 4$. We estimate its growth and find that if $\|Q_1 \theta_0\|_r$ is small enough, then there exists a positive constant $A$, which for concreteness is chosen to be $\frac{\sigma}{18}$, so that

$$\frac{dE}{dt} \leq -AE. \quad (1.22)$$

The estimate (1.22) holds for the Galerkin approximate equations and is independent of the dimension of the Galerkin subspace. The exponential decay estimates on $E(t)$ implied by this inequality help continue solutions of the Galerkin approximations to arbitrary time. Further estimates show that $\{\tilde{\theta}_n\}_{n=2}^\infty$ form a Cauchy sequence in $\dot{H}^4$, which is used to show that $\tilde{\theta}_n$ converges to a strong solution $\tilde{\theta}$ of the original system, which also decays exponentially. The functional relation determining $\hat{\theta}(\pm 1; t)$ shows that $\theta - \hat{\theta}(0; t)$ also decays exponentially in time, which implies that circular steady shape is asymptotically stable. At the end, we determine the evolution of
\( \hat{\theta}(0; t) \), which does not affect the evolution of the boundary shapes in the absence of side-walls. It is found that \( \hat{\theta}(0; t) \) is globally bounded.

The main result in Chapter 3 is the following Theorem:

**Theorem 1.12.** There exists \( \epsilon > 0 \) such that for \( r \geq 4 \), if \( \| \mathcal{Q}_1 \theta_0 \|_r < \epsilon \), then there exists \( (\theta, L) \in C([0, \infty); H^r_p \times \mathbb{R}) \cap C^1([0, \infty); H^{r-3}_p \times \mathbb{R}) \), which satisfies (A.1)-(A.3) with the initial condition (1.20) globally. Furthermore, \( \| \hat{\theta} \|_r, \hat{\theta}(1; t) \) and \( \hat{\theta}(-1; t) \) each decay exponentially as \( t \to \infty \), \( |\hat{\theta}(0; t)| \) remains finite, while \( L \) approaches \( 2\sqrt{\pi V} \), \( V \) being the invariant bubble area. Thus a near-circular bubble is asymptotically stable for sufficiently small distortions in the \( H^r_p \) space.

In Chapters 4, 5, 6 and 7, we discuss the nonlinear stability of the translating bubble in a Hele-Shaw cell. There exists significant complications when Hele-Shaw wall effects and bubble translation are introduced, which are not present in any of the prior bubble analysis [8].

A major difficulty in the global analysis is that the solution to the linearized equation for nonzero imposed pressure gradient (i.e. \( V_\infty = 1 \)) is not easy to control for small surface tension \( \sigma \) for arbitrary wave number \( k \). To the best of our knowledge, this problem has not been tackled rigorously before even for one fluid problem or any other steady curved shape.

In Chapter 4, we introduce a system of equations (D.1)-(D.2), equivalent to (B.1)-(B.5), by replacing the evolution equation of \( L \) by a constraint that determines \( L \) in terms of the invariant area \( V \) and other variables. This is more convenient since we found it difficult to obtain suitable energy estimates for \( L \) in this case.
In Chapter 5, we consider bubble evolution in the case of an imposed pressure gradient \((V_\infty = 1)\) with no side-walls \((\beta = 0)\). We introduce a weighted Sobolev space so that we are able to control potentially unstable terms by the stabilizing effects of surface tension. This can be understood physically to be the result of advection of disturbance from the front to the side of the bubble. We note that since the normal motion along the sides or rear of the bubble is not directed towards the more viscous fluid, it is expected to be stable from local arguments.

Using contraction theorem in a suitable Banach space, the following results hold for \(\beta = 0\) and \(V_\infty = 1\), i.e., translating bubble without wall effects:

**Theorem 1.13.** For \(\sigma > 0\), there exists \(\epsilon > 0\) such that for \(r \geq 3\), if \(\|Q_1 \theta_0\|_r < \epsilon\) and \(|L_0 - 2\pi| < \epsilon \leq \frac{1}{2}\), then there exists a unique solution \((\theta, L) \in C([0, \infty), H^r_p \times \mathbb{R})\) to the Hele-Shaw problem (A.1)-(A.3) with the initial condition (1.20). Further, \(\|\hat{\theta}\|_r\) and \(|\hat{\theta}(\pm 1; t)|\) each decay exponentially as \(t \to \infty\), \(\|\hat{\theta}(0; t)\|\) remains finite, while \(L\) approaches \(2\sqrt{\pi V}\). Thus the steady circular translating bubble in the presence of an imposed pressure gradient is asymptotically stable for sufficiently small initial disturbances in the \(H^r_p\) space.

**Remark.** The proof is completed at the end of Chapter 5 (see Note 5.3). 

Let \(\theta^{(s)}\) correspond to a steady translating bubble and \(2 + u_0\) be the corresponding non-dimensional translation speed of the steady bubble in the laboratory frame. The existence and uniqueness of the steady solution \(\theta^{(s)}\) with constraint \(u_0 = u_0(\beta)\) is discussed in Chapter 6. More precisely we have the following theorem:
Theorem 1.14. For any surface tension $\sigma > 0$ and $r \geq 3$, there exist $\epsilon > 0$, $\Upsilon > 0$, two balls $O_1 = \{ \beta \in \mathbb{R} : 0 \leq \beta < \Upsilon \}$ and $O_2 = \{(u,v) \in \dot{H}^r \times \mathbb{R} \mid \|u\|_r < \epsilon, |v| < \epsilon \}$, so that for sufficiently small $\epsilon$ and $\Upsilon$, $(\tilde{\theta}^{(s)}, u_0)^T : O_1 \to O_2$ with $(\tilde{\theta}^{(s)}, u_0)$ determining the shape and velocity of the steady symmetric translating bubble for $\beta \in O_1$.

Further, $\tilde{\theta}^{(s)}$ is odd and there exists $C$ independent of $\epsilon$ and $\Upsilon$ such that

$$\|\tilde{\theta}^{(s)}\|_r + |u_0| + \|\gamma^{(s)} - 2\sin(\cdot)\|_{r-2} \leq C\beta^2.$$ 

Remark. We will prove Theorem 1.14 in Chapter 6. Note results for steady bubble and finger without restriction on $\beta$ but small $\sigma$ is available in [45], [46] and [47]. Here, there is no restriction in $\sigma > 0$, but it is held fixed as $\beta$ is made sufficiently small. Existence of at least one steady translating finger solution for $\sigma > 0$ has been proved earlier [35] using different methods.

In Chapter 7, we consider bubble motion in the presence of an imposed pressure gradient and sidewalls ($V_\infty = 1, \beta \neq 0$), but is for the class of initial shapes that are symmetric about the channel centerline. Symmetry implies $\theta$ is an odd function of $\alpha$.

Definition 1.15. The unsteady perturbation term is defined by

$$\Theta(\alpha, t) = \theta(\alpha, t) - \theta^{(s)}(\alpha). \quad (1.23)$$

We also define $\tilde{\Theta}(\alpha, t) = Q_1 \Theta(\alpha, t)$.

By employing contraction theorem in a suitable weighted Sobolev space, for $\beta \neq 0$ and $V_\infty = 1$, we prove the following result:
Theorem 1.16. For $\sigma > 0$, there exist $\epsilon, \Upsilon > 0$ such that for $r \geq 3$, if $\|\bar{\Theta}(\cdot, 0)\|_r < \epsilon$ and $|L_0 - 2\pi| < \epsilon \leq \frac{1}{2}, 0 < \beta < \Upsilon$, with $\bar{\Theta}(-\alpha, 0) = -\bar{\Theta}(\alpha, 0)$, then there exists a unique solution $(\theta, L) \in C([0, \infty), H^r_p \times \mathbb{R})$ with $\theta(-\alpha, t) = -\theta(\alpha, t)$ to the Hele-Shaw problem (A.1)-(A.3) with the initial condition (1.20). Furthermore, $\|Q_0 \Theta\|_r$ decays exponentially as $t \to \infty$, while $L$ approaches $2\sqrt{\pi \sqrt{V}}$. Thus the translating steady bubble is asymptotically stable for sufficiently small symmetric initial disturbances in the $H^r_p$ space.

Remark. This theorem is proved in Chapter 7 (See Note 7.4). Though the proof only seeks solution within the symmetric class of shapes, local uniqueness of solution to the initial value problem without a priori assumptions on symmetry is readily available through routine energy estimates. This proves that if initial shapes are symmetric, they remain symmetric later in time. \qed
CHAPTER 2
PRELIMINARY LEMMAS

Definition 2.1. We decompose coth and cot functions into a singular and nonsingular parts at the origin:

\[
\begin{align*}
\coth(w) &= \frac{1}{w} + l_1(w), \\
\cot(w) &= \frac{1}{w} + l_2(w).
\end{align*}
\]

We decompose \(K = K_1 + K_2\), where

\[
\begin{align*}
K_1[z]f &= \frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} f(\alpha') \left\{ \frac{1}{z(\alpha) - z(\alpha')} - \frac{1}{z_\alpha(\alpha')} \cot \frac{1}{2} (\alpha - \alpha') \right\} d\alpha', \\
K_2[z]f &= \frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} f(\alpha') \left\{ \frac{\beta}{4} l_1 \left[ \frac{1}{4} \beta (z(\alpha) - z(\alpha')) \right] \\
&\quad - \frac{\beta}{4} \tanh \left[ \frac{\beta}{4} (z(\alpha) - z^*(\alpha')) \right] \right\} d\alpha'.
\end{align*}
\]

Definition 2.2. Related to \(G\) and \(F\), we define operators \(G_1, F_1\) so that

\[
\begin{align*}
G_1[z]\gamma &= z_\alpha \left[ \mathcal{H}, \frac{1}{z_\alpha} \right] \gamma + 2iz_\alpha K_1[z] \gamma, \\
G_2[z]\gamma &= 2iz_\alpha K_2[z] \gamma, \\
F_1[z] \gamma &= \text{Re} \left( \frac{1}{i} G_1[z] \gamma \right). \\
\end{align*}
\]

(2.1)

Definition 2.3. We introduce functions

\[
\begin{align*}
\omega_0(\alpha) &= \int_0^\alpha e^{i\alpha'} d\alpha', \\
\omega(\alpha, t) &= \int_0^\alpha e^{i\alpha' + i\theta(1; t) e^{i\alpha'} + i\hat{\theta}(0; t) e^{-i\alpha'} + i\tilde{\theta}(\alpha', t)} d\alpha'.
\end{align*}
\]

(2.2)
Note 2.4.

\[ z(\alpha, t) = \frac{L}{2\pi} e^{i\frac{\pi}{2} + i\hat{\theta}(0, t)} \omega(\alpha, t) + z(0, t). \]  

(2.3)

Note 2.5. It is readily checked that

\[ \text{Re} \left( \frac{\omega_{0, \alpha}}{\pi} \text{PV} \int_0^{2\pi} \frac{f(\alpha')}{\omega_{0}(\alpha) - \omega_{0}(\alpha')} d\alpha' \right) = \mathcal{H}(f), \]  

(2.4)

and \( G_1[\omega_0]f = i\hat{f}(0) \) for \( f(\alpha) = \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ik\alpha}. \)

Definition 2.6. Let \( r \geq 3 \). We define the open balls:

\[ B_r^r = \left\{ f \in \dot{H}^r \| f \|_r < \epsilon \right\}; \]

\[ S_M = \left\{ y \in \mathbb{R} \| y \| < M \right\}, \]

for some \( M \) independent of \( \beta \). We also define the open ball:

\[ V_r = \left\{ (f, s) \in \dot{H}^r \times \mathbb{R} \| f \in B_r^r, |s - 2\pi| < 1 \right\}. \]

Remark. We will eventually choose \( \epsilon > 0 \) small enough for Lemma 3.5 and Theorem 1.12 to apply.

Proposition 2.7. There exists \( \epsilon_1 > 0 \) so that (B.5) implicitly defines a unique \( C^1 \) function \( G : \{ f \in \dot{H}^1 \| f \|_1 < \epsilon_1 \} \to \mathbb{R}^2 \) satisfying \( (\text{Re} \hat{\theta}(1; t), \text{Im} \hat{\theta}(1; t)) = G(\bar{\theta}(t)) \) with \( G(0) = 0 \) and \( G_{\theta}(0) = 0 \). Moreover, \( G \) satisfies the following estimates for all \( f, f_1, f_2 \in \{ f \in \dot{H}^1 \| f \|_1 < \epsilon_1 \} : \)

\[ |G(f)| \leq \frac{1}{2} \| f \|_1, \]  

(2.5)

\[ |G(f_1) - G(f_2)| \leq \frac{1}{2} \| f_1 - f_2 \|_1. \]  

(2.6)

Further, if \( \bar{\theta}(-\alpha, t) = -\bar{\theta}(\alpha, t) \), then \( \hat{\theta}^* (1; t) = -\hat{\theta}(1; t) \), is purely imaginary.
Note 2.8. Note that calculation of $\hat{\theta}(1; t)$ (and therefore of $\hat{\theta}(-1; t) = \hat{\theta}^*(1; t)$) from $\tilde{\theta}$ in Proposition 2.7 allows computation of

$$Q_0\theta = \tilde{\theta}(\alpha, t) + \hat{\theta}(1; t)e^{i\alpha} + \hat{\theta}(-1; t)e^{-i\alpha}$$

and this is an odd function of $\alpha$ for odd $\tilde{\theta}$. Also, note that having determined $\gamma$, $\hat{\theta}(1; t)$ and $\hat{\theta}(-1; t)$, (1.19) and (B.4) determine $U$ and $T$ needed in (B.1)-(B.2).

Lemma 2.9. For $r \geq 3$ and sufficiently small $\epsilon_1$ the following statements (i.) and (ii.) are equivalent:

(i.) $(\theta, L) \in C^1([0, S]; H^r_p \times \mathbb{R})$, with $\theta$ real-valued and $|L - 2\pi| < 1$, is a solution of (A.1) for $t \in [0, S]$ satisfying initial condition (1.20), where $\gamma$, $T$ and $U$ are determined by (A.2) and (1.19).

(ii.) $(\tilde{\theta}, L, \hat{\theta}(0; t)) \in C^1([0, S]; \dot{H}^r \times \mathbb{R}^2)$ with $|L - 2\pi| < 1$ and $\|\tilde{\theta}\|_1 < \epsilon_1$, be a real-valued solution of (B.1)-(B.2) for $t \in [0, S]$ satisfying initial condition (1.21), where $\gamma$, $T$, $\hat{\theta}(\pm 1; t)$ and $U$ are determined from (B.3)-(B.5), (1.19) and

$$\theta = \tilde{\theta} + \hat{\theta}(0; t) + \hat{\theta}(1; t)e^{i\alpha} + \hat{\theta}(-1; t)e^{-i\alpha}.$$ 

Proof. Let $(\theta, L) \in C^1([0, S]; H^r_p \times \mathbb{R})$ be the solution of the evolution equations (A.1) where $\gamma$, $T$ and $U$ are determined by (A.2) and (1.19) with the initial condition (1.20). Define $p(t) = \int_0^{2\pi} e^{i\alpha + i\theta(\alpha, t)} d\alpha$. We note that $p(0) = 0$, while

$$p'(t) = i \int_0^{2\pi} e^{i\alpha + i\theta(\alpha, t)} \theta_\alpha d\alpha.$$ 

Substituting for $\theta_\alpha$ from (A.1), and using the identity $(e^{i\alpha + i\theta})_\alpha = i(1 + \theta_\alpha)e^{i\alpha + i\theta}$, we have

$$p'(t) = \frac{2\pi}{L} \int_0^{2\pi} \left[iU_\alpha e^{i\alpha + i\theta} + T(e^{i\alpha + i\theta})_\alpha\right] d\alpha.$$
We integrate the last term by parts; we use (A.2) to substitute for \( T \). There is no end-point contribution since \( T \) and \( e^{i\alpha+i\theta} \) are periodic. We have
\[
p'(t) = \frac{2\pi}{L} \int_0^{2\pi} (iU_\alpha e^{i\alpha+i\theta} - (1 + \theta_\alpha)U e^{i\alpha+i\theta} - \frac{1}{2\pi} L t e^{i\alpha+i\theta}) d\alpha.
\]
Since \( iU_\alpha e^{i\alpha+i\theta} - (1 + \theta_\alpha)U e^{i\alpha+i\theta} = (iU e^{i\alpha+i\theta})_\alpha \), we obtain
\[
p' = -\frac{L t}{L} p.
\]
Note that \( L > 2\pi - 1 > 0 \). Furthermore, \( L_t \) is continuous in \([0, S]\) from (A.1). So \( p(t) = 0 \) is the unique solution to the above ordinary differential equation with \( p(0) = 0 \) for \( t \in [0, S] \). Hence
\[
e^{i\hat{\theta}(0;t)} \int_0^{2\pi} \exp \left( i\alpha + i(\hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(\alpha,t)) \right) d\alpha = 0,
\]
implying
\[
\int_0^{2\pi} \exp \left( i\alpha + i(\hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(\alpha,t)) \right) d\alpha = 0 \text{ for } t \in [0, S].
\]
Thus \( (\hat{\theta} = Q_1 \theta, L, \hat{\theta}(0;t)) \) satisfies (ii).

Conversely, suppose that \( (\hat{\theta}, L, \hat{\theta}(0;t)) \in C^1([0, S]; \hat{H}' \times \mathbb{R}^2) \) satisfies (B.1) and (B.2) with the initial condition (1.21), where \( \gamma, \, T, \, \hat{\theta}(\pm1; t) \) and \( U \) are determined by (B.3)-(B.5) and (1.19). Let \( \theta = \hat{\theta} + \hat{\theta}(0;t) + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(-1;t)e^{-i\alpha} \). We note from Proposition 2.7 that \( \hat{\theta}(\pm1) \) scale as \( \epsilon_1 \) and hence is small. We note from (B.5) that
\[
p(t) = e^{i\hat{\theta}(0;t)} \int_0^{2\pi} \exp \left( i\alpha + i(\hat{\theta}(-1;t)e^{-i\alpha} + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(\alpha;t)) \right) d\alpha = 0.
\]
It is convenient to define \( \mathcal{O}(\alpha, t) = U_\alpha + T(1 + \theta_\alpha) \). From \( p'(t) = 0 \), using (B.1), we obtain
\[
0 = \int_0^{2\pi} e^{i\alpha+i\theta} \left( \left( \hat{\theta}(1; t) - \frac{2\pi}{L} \mathcal{O}(-1; t) \right) e^{-i\alpha} + \left( \hat{\theta}(1; t) - \frac{2\pi}{L} \mathcal{O}(1; t) \right) e^{i\alpha} \right) d\alpha. \quad (2.7)
\]
Let $e^{i\alpha + i\theta} = \sum_{k=-\infty}^{\infty} \hat{c}(k)e^{ik\alpha}$. Hence for sufficiently small $\epsilon_1$, using Proposition 2.7 and Sobolev inequality $|\.|_\infty < C\|\.\|_1$,

$$|\theta - \hat{\theta}(0;t)|_\infty = |\hat{\theta}(\alpha, t) + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(-1;t)|_\infty \leq C\|\hat{\theta}\|_1$$

is small, which clearly ensures $|\hat{c}(1)| > |\hat{c}(k)|$ for $k \neq 1$. Note further that (2.7) implies

$$(\hat{\theta}_t(-1; t) - \frac{2\pi}{L} \hat{\mathcal{O}}(-1; t))\hat{c}(1) + (\hat{\theta}_t(1; t) - \frac{2\pi}{L} \hat{\mathcal{O}}(1; t))\hat{c}(-1) = 0.$$

Since $\mathcal{O}(\alpha, t)$ and $\hat{\theta}$ are real valued, $\hat{\theta}_t(-1; t) - \frac{2\pi}{L} \hat{\mathcal{O}}(-1; t)$ is the complex conjugate of $\hat{\theta}_t(1; t) - \frac{2\pi}{L} \hat{\mathcal{O}}(1; t)$. It is clear that if $|a_1| \neq |a_2|$, then the only solution to $a_1\eta + a_2\eta^* = 0$ is $\eta = 0$. Hence

$$\hat{\theta}_t(-1; t) - \frac{2\pi}{L} \hat{\mathcal{O}}(-1; t) = 0 \text{ and } \hat{\theta}_t(1; t) - \frac{2\pi}{L} \hat{\mathcal{O}}(1; t) = 0.$$

Hence, $(\theta = \hat{\theta} + \hat{\theta}(0; t) + \hat{\theta}(1;t)e^{i\alpha} + \hat{\theta}(-1;t)e^{-i\alpha}, L)$ will satisfy (i). $\square$

We will henceforth discuss global solutions $(\hat{\theta}, L, \hat{\theta}(0; t))$ of (B.1)-(B.2) with the initial conditions (1.21), where $\gamma, T, \hat{\theta}(\pm 1; t)$ and $U$ are determined by (B.3)-(B.5) and (1.19).

**Definition 2.10.** Define $\hat{\theta}(1; t) = r_1 + ir_2$. Then since $\theta$ is real valued, $\hat{\theta}(-1; t) = r_1 - ir_2$.

**Remark.** (B.5) becomes

$$\int_0^{2\pi} \exp \left( i\alpha + i((r_1 + ir_2)e^{i\alpha} + (r_1 - ir_2)e^{-i\alpha} + \sum_{k=-\infty, \neq 0, \pm 1}^{\infty} \hat{\theta}(k)e^{ik\alpha}) \right) d\alpha = 0. \quad (2.8)$$
In order to prove Proposition 2.7, we need the following lemma:

**Lemma 2.11.** *Implicit function Theorem ([30]):* Let $G_1$, $G_2$, and $G_3$ be Banach spaces and $F$ a mapping from an open subset of $G_1 \times G_2$ into $G_3$. Let $(f_0, g_0)$ be a point in $G_1 \times G_2$ satisfying:

(i) $F(f_0, g_0) = 0$;

(ii) $F$ is continuously differentiable at $(f_0, g_0)$;

(iii) the partial Fréchet derivative $D_g F(f_0, g_0)$ is invertible from $G_2$ to $G_3$.

Then, there is a neighborhood $K_1$ of $u_0$ in $G_1$ and a neighborhood $K_2$ of $v_0$ in $G_2$ and a $C^1$ map $G : K_1 \to K_2$ so that $F(f, G(f)) = 0$ for all $f \in K_1$. Further for each $f \in K_1$, $G(f)$ is the unique point $g$ in $K_2$ satisfying $F(f, g) = 0$.

**Definition 2.12.** In the bubble context, we define

$$F(f, g) = \int_0^{2\pi} \exp \left( i\alpha + i \left(2(\cos \alpha - \sin \alpha) + f \right) \right) d\alpha$$

with $g = (r_1, r_2)$.

**Remark.** Note $F : \dot{H}^1 \times \mathbb{R}^2 \to \mathbb{C}$.

**Proof of Proposition 2.7:** We now show that the Fréchet derivative of $F(f, g)$ with respect to $u$ exists in $\dot{H}^1 \times \mathbb{R}^2$. Since

$$|F(f + h, g) - F(f, g) - \int_0^{2\pi} ih(\alpha) \exp \left( i\alpha + i \left[2(\cos \alpha - \sin \alpha) + f(\alpha) \right] \right) d\alpha|$$

$$= \left| \int_0^{2\pi} \exp \left( i\alpha + i \left[2(\cos \alpha - \sin \alpha) + f(\alpha) \right] \right) \left\{ \exp [ih(\alpha)] - 1 - ih(\alpha) \right\} d\alpha \right|$$

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the Fréchet derivative of $F$ with respect to $f$ is

$$D_f F(f, g) h = \int_0^{2\pi} i h(\alpha) \exp \left( i\alpha + i\left(2(r_1 \cos \alpha - r_2 \sin \alpha) + f(\alpha)\right) \right) d\alpha,$$

for $h \in \dot{H}^1$. It is clear that $D_f F(f, g) : \dot{H}^1 \to \mathbb{C}$ is the bounded linear operator for all $(f, g) \in \dot{H}^1 \times \mathbb{R}^2$.

Similarly,

$$D_g F(f, g) \delta g = 2i \int_0^{2\pi} (\delta r_1 \cos \alpha - \delta r_2 \sin \alpha) \exp \left( i\alpha + i\left(2(r_1 \cos \alpha - r_2 \sin \alpha) + f(\alpha)\right) \right) d\alpha,$$

with $\delta g = (\delta r_1, \delta r_2) \in \mathbb{R}^2$ is a bounded linear operator for all $(f, g) \in \dot{H}^1 \times \mathbb{R}^2$, with

$$D_g F(0, 0) \delta g = 2i \int_0^{2\pi} (\delta r_1 \cos \alpha - \delta r_2 \sin \alpha) e^{i\alpha} d\alpha = 2\pi(\delta r_2 + i\delta r_1).$$

Clearly $D_g F(0, 0)$ is invertible. So by the implicit function theorem (Lemma 2.11), with $(f_0, g_0) = (0, 0)$, there exist neighborhood $K_1 = \{ f \in \dot{H}^1 : \|f\|_1 < 2\epsilon_1 \}$ of 0 in $\dot{H}^1$, and a neighborhood $K_2$ of $(0, 0)$ in $\mathbb{R}^2$, and a $C^1$ map $G : K_1 \to K_2$, so that $F(f, G(f)) = 0$ for all $f \in K_1$.

Taking the Fréchet derivative on both sides of $F(f, G(f)) = 0$ with respect to $f$ at $f = 0$, we have

$$D_f F(0, G(0)) h + D_g F(0, G(0)) D_f G(0) h = 0,$$

for $h \in \dot{H}^r$. Since $D_f F(0, 0) h = 0$, for $h \in \dot{H}^r$, $G(0) = 0$ and $D_g F(0, 0)$ is invertible,
it is easy to get $D_fG(0) = 0$. We also have from continuity $\|D_fG(f)\| \leq \frac{1}{2}$ for $f \in K_1$. Hence,

$$|G(f)| \leq \left| \int_0^1 D_fG(tf) f dt \right| \leq \frac{1}{2} \|f\|_1,$$

$$|G(f_1) - G(f_2)| \leq \left| \int_0^1 D_fG(f_1 + t(f_2 - f_1))(f_2 - f_1) dt \right| \leq \frac{1}{2} \|f_1 - f_2\|_1$$

for all $f, f_1, f_2 \in \{f \in \dot{H}^1 : \|f\|_1 < \epsilon_1\}$.

Further, using

$$\int_0^{2\pi} \exp\left( i\alpha + i\hat{\theta}(1; t)e^{i\alpha} + i\hat{\theta}(-1; t)e^{-i\alpha} + i\tilde{\theta}(\alpha, t) \right) d\alpha = 0,$$

we choose conjugate on both sides and obtain

$$\int_0^{2\pi} \exp\left( -i\alpha - i\hat{\theta}^*(1; t)e^{-i\alpha} - i\hat{\theta}^*(-1; t)e^{i\alpha} - i\tilde{\theta}(\alpha, t) \right) d\alpha = 0.$$

So we have

$$\int_0^{2\pi} \exp\left( i\alpha - i\hat{\theta}(-1; t)e^{i\alpha} - i\hat{\theta}(1; t)e^{-i\alpha} - i\tilde{\theta}(-\alpha, t) \right) d\alpha = 0.$$

Hence, if $\tilde{\theta}(\alpha)$ is odd, then we obtain

$$\int_0^{2\pi} \exp\left( i\alpha - i\hat{\theta}(-1; t)e^{i\alpha} - i\hat{\theta}(1; t)e^{-i\alpha} + i\tilde{\theta}(\alpha, t) \right) d\alpha = 0.$$

By the uniqueness of $G$ in the open set $K_1$, we have $\hat{\theta}(1; t) = -\hat{\theta}(-1; t) = -\hat{\theta}^*(1; t)$.

**Corollary 2.13.** There exists sufficiently small $\epsilon_1 > 0$ so that for $\theta \in H^{r+1}_p$ with $r \geq 0$, if $\|\hat{\theta}\|_1 < \epsilon_1$, then $\theta$ satisfying (B.5) implies $\|\theta_\alpha\|_r \leq 2\|\tilde{\theta}_\alpha\|_r$. 

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Proof. We note from the relation between $\theta$ and $\tilde{\theta}$ that
\[
\|\theta_\alpha\|_r^2 = \sum_k |k|^{2r+2}|\hat{\theta}(k)|^2 = 2|G(\tilde{\theta})|^2 + \|\tilde{\theta}_\alpha\|_r^2.
\]
The rest follows from bounds on $G(\tilde{\theta})$ in Proposition 2.7.

In the rest of this chapter, we will obtain a variety of routine estimates for integral operators and other functions in terms of $\tilde{\theta}$ and $\hat{\theta}(0; t)$. Recall tangent angle of the curve is $\frac{\pi}{2} + \alpha + \theta(\alpha) = \frac{\pi}{2} + \alpha + \tilde{\theta}(\alpha) + \hat{\theta}(0; t) + \hat{\theta}(-1; t)e^{-i\alpha} + \hat{\theta}(1; t)e^{i\alpha}$, where $\hat{\theta}(1; t)$ and $\hat{\theta}(-1; t)$ are determined by $G(\tilde{\theta})$.

The next lemma gives a bound for $\omega_\alpha$ in terms of $\tilde{\theta}$.

**Lemma 2.14.** Assume $\|\tilde{\theta}\|_1 < \epsilon_1$ where $\epsilon_1$ is small enough for Corollary 2.13 to apply.

Then $\omega$ determined by $\tilde{\theta} \in \dot{H}^r$ through (2.2) for $r \geq 1$, satisfies the following estimates:
\[
\|\omega_\alpha\|_r \leq C_1(\|\tilde{\theta}\|_r + 1) \exp\left(C_2\|\tilde{\theta}\|_{r-1}\right), \quad \frac{1}{\omega_\alpha} \leq C_1(\|\tilde{\theta}\|_r + 1) \exp\left(C_2\|\tilde{\theta}\|_{r-1}\right),
\]  
(2.9)
where constants $C_1$ and $C_2$, depend only on $r$, and particularly for $r = 1$, $C_2 = 0$.

Similarly, if $z$ determined by $(\tilde{\theta}, \tilde{\theta}(0; t), L) \in \dot{H}^r \times \mathbb{R}^2$, then for $r \geq 1$,
\[
\|z_\alpha\|_r \leq C_1L(\|\tilde{\theta}\|_r + 1) \exp\left(C_2\|\tilde{\theta}\|_{r-1}\right),
\]  
(2.10)
where constants $C_1$ and $C_2$, depend only on $r$, and particularly for $r = 1$, $C_2 = 0$.  

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Further, if $\omega^{(1)}, \omega^{(2)}$ correspond respectively to $\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^r$, where $\|\tilde{\theta}^{(1)}\|_1, \|\tilde{\theta}^{(2)}\|_1 < \epsilon_1$, then for $r \geq 1$,

$$\|\omega^{(1)}(1) - \omega^{(2)}(2)\|_r \leq C_1 \|\tilde{\theta}^{(1)}(1) - \tilde{\theta}^{(2)}(2)\|_r \exp \left( C_2 \left( \|\tilde{\theta}^{(1)}(1)\|_r + \|\tilde{\theta}^{(2)}(2)\|_r \right) \right), \quad (2.11)$$

$$\left| \frac{1}{\omega^{(1)}(1)} - \frac{1}{\omega^{(2)}(2)} \right|_r \leq C_1 \|\tilde{\theta}^{(1)}(1) - \tilde{\theta}^{(2)}(2)\|_r \exp \left( C_2 \left( \|\tilde{\theta}^{(1)}(1)\|_r + \|\tilde{\theta}^{(2)}(2)\|_r \right) \right), \quad (2.12)$$

while for $r \geq 2$,

$$\|\omega^{(1)}(1) - \omega^{(2)}(2)\|_r \leq C_1 \left( \|\tilde{\theta}^{(1)}(1) - \tilde{\theta}^{(2)}(2)\|_r + \|\tilde{\theta}^{(2)}(2)\|_r \|\tilde{\theta}^{(1)}(1) - \tilde{\theta}^{(2)}(2)\|_{r-1} \right) \times \exp \left( C_2 \left( \|\tilde{\theta}^{(1)}(1)\|_{r-1} + \|\tilde{\theta}^{(2)}(2)\|_{r-1} \right) \right), \quad (2.13)$$

$$\left| \frac{1}{\omega^{(1)}(1)} - \frac{1}{\omega^{(2)}(2)} \right|_r \leq C_1 \left( \|\tilde{\theta}^{(1)}(1) - \tilde{\theta}^{(2)}(2)\|_r + \|\tilde{\theta}^{(2)}(2)\|_r \|\tilde{\theta}^{(1)}(1) - \tilde{\theta}^{(2)}(2)\|_{r-1} \right) \times \exp \left( C_2 \left( \|\tilde{\theta}^{(1)}(1)\|_{r-1} + \|\tilde{\theta}^{(2)}(2)\|_{r-1} \right) \right), \quad (2.14)$$

where the constants $C_1$ and $C_2$ depend only on $r$.

Proof. For the formula $\omega_\alpha = e^{i\alpha + i\tilde{\theta}^{(1)}(1)} e^{i\alpha + i\tilde{\theta}^{(-1)}(1)} e^{-i\alpha + i\tilde{\theta}}$, it is easy to obtain

$$\|\omega_\alpha\|_0 \leq C.$$ 

Let us consider for $0 < k \leq r$. The chain rule gives

$$D^k_{\alpha} \omega_\alpha = \sum_{k_1 + \cdots + k_j = k, j \geq 1} C_{\beta} D^k_{\alpha} (\alpha + \theta) \cdots D^k_{\alpha} (\alpha + \theta) \omega_\alpha.$$ 

So by Sobolev embedding Theorem, $|f|_\infty \leq C \|f\|_1$, we have

$$\|D^k_{\alpha} \omega_\alpha\|_0 \leq C \|1 + \theta_\alpha\|_k - 1 + \|\theta_\alpha\|_k - 1 + \cdots + \|\theta_\alpha\|_k - 1 \leq C_1 \exp \left( C_2 \|\theta_\alpha\|_{k-1} \right), \quad (2.15)$$

where the constants, $C_1$ and $C_2$, depend only on $r$. 

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For $r = 1$, we have

$$\|D_\alpha \omega_\alpha\|_0 = \|1 + \theta_\alpha\|_0 \leq C(1 + \|\tilde{\theta}\|_1).$$

For $r \geq 2$, we note

$$D^r_\alpha \omega_\alpha = D^{r-1}_\alpha \left[i(1 + \theta_\alpha)\omega_\alpha\right].$$

Hence, by noting Banach algebra property (see Note 1.3, Corollary 2.13 and (2.15), we get

$$\|D^r_\alpha \omega_\alpha\|_0 \leq \left\|i(1 + \theta_\alpha)\omega_\alpha\right\|_{r-1} \leq C_1(\|\tilde{\theta}\|_r + 1) \exp\left(C_2\|\tilde{\theta}\|_{r-1}\right),$$

where the constants, $C_1$ and $C_2$, depend only on $r$. Since $\frac{1}{\omega_\alpha} = e^{-i\alpha - i\hat{\theta}(1; t)}e^{i\alpha - i\hat{\theta}(-1; t)}e^{-i\alpha - i\hat{\theta}}$, the preceding arguments are clearly applied to $\frac{1}{\omega_\alpha}$ as well and (2.9) follows from Corollary 2.13 for a modified constant $C_2$.

Actually, we always have

$$\|D^k_\alpha \psi_\alpha\|_0 = \frac{L}{2\pi} \|D^k_\alpha \omega_\alpha\|_0,$$

for all $k \geq 0$. So (2.10) holds.

To prove (2.11), we will use induction technique. From series representation of the exponential, by Corollary 2.13, we note that

$$\|\omega^{(1)}_\alpha - \omega^{(2)}_\alpha\|_0 \leq C\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_1,$$

$$\|D_\alpha (\omega^{(1)}_\alpha - \omega^{(2)}_\alpha)\|_0 \leq \|i(1 + \theta^{(1)}_\alpha)\omega^{(1)}_\alpha - i(1 + \theta^{(2)}_\alpha)\omega^{(2)}_\alpha\|_0 \leq C\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_1 (1 + \|\tilde{\theta}^{(1)}\|_1).$$
Hence, (2.11) holds for \( r = 1 \). Suppose for \( r = k - 1 \), (2.11) holds. Then for \( r = k \), by Corollary 2.13 and (2.9), we have

\[
\| D_k^k (\omega_1^{(1)} - \omega_2^{(2)}) \|_0 = \| D_k^{k-1} [i(1 + \theta_1^{(1)})\omega_1^{(1)} - i(1 + \theta_2^{(2)})\omega_2^{(2)}] \|_0 \\
\leq C_1 \left( \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_k + \| \tilde{\theta}^{(2)} \|_k \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_{k-1} \right) \exp \left( C_2 (\| \tilde{\theta}^{(1)} \|_{k-1} + \| \tilde{\theta}^{(2)} \|_{k-1}) \right),
\]

where the constants, \( C_1 \) and \( C_2 \), depend only on \( k \). Hence, (2.11) holds for \( r \geq 1 \).

Almost identical arguments are applied to prove (2.12).

Further, by (2.11), (2.13) follows. Almost identical arguments are applied for (2.14).

\[\square\]

**Definition 2.15.** We define operators \( \Xi_e, \Xi_s, \Xi_c \) so that

\[
\Xi_e[u](\alpha) = e^{iu(\alpha)} - 1 - iu(\alpha), \\
\Xi_s[u; a](\alpha) = \sin (u(\alpha) + \alpha + a) - \sin(\alpha + a) - u(\alpha) \cos(\alpha + a), \\
\Xi_c[u; a](\alpha) = \cos (u(\alpha) + \alpha + a) - \cos(\alpha + a) + u(\alpha) \sin(\alpha + a),
\]

for a real function \( u \in H_p^r \) with \( r \geq 1 \).

**Lemma 2.16.** If \( F \) is an entire function of order one\(^1\) with \( F(u) = \sum_{j=j_0}^{\infty} a_j u^j \) for \( j_0 = 1 \) or 2. Then for \( u \in H_p^{r+1} \) with \( r \geq 1 \), \( F(u(\alpha)) \) satisfies

\[^1\]An entire function \( f \) of order \( m \) satisfies

\[ |f(z)| \leq e^{C|z|^m}, \text{ for } z \in \mathbb{C}. \]
(i) \( j_0 = 1 \):

\[
\| F(u(\cdot)) \|_{0} \leq C_1 \exp (C_2 \| u \|_1) \| u \|_1,
\]

\[
\| F(u(\cdot)) \|_{r+1} \leq C_1 \exp (C_2 \| u \|_r) \| u \|_{r+1};
\]

(ii) \( j_0 = 2 \):

\[
\| F(u(\cdot)) \|_{0} \leq C_1 \exp (C_2 \| u \|_1) \| u \|_1^2,
\]

\[
\| F(u(\cdot)) \|_{r+1} \leq C_1 \exp (C_2 \| u \|_r) \| u \|_{r+1} \| u \|_r,
\]

where the constants \( C_1 \) and \( C_2 \) depend only on \( r \).

Further, if both \( u^{(1)} \) and \( u^{(2)} \) belong to \( H^r_p \), then for \( r \geq 1 \),

(i) \( j_0 = 1 \):

\[
\| F(u^{(1)}(\cdot)) - F(u^{(2)}(\cdot)) \|_{0} \leq C_1 \| u^{(1)} - u^{(2)} \|_1 \exp \left[ C_2 \left( \| u^{(1)} \|_1 + \| u^{(2)} \|_1 \right) \right]
\]

\[
\| F(u^{(1)}(\cdot)) - F(u^{(2)}(\cdot)) \|_{r+1} \leq C_1 \left( \| u^{(1)} - u^{(2)} \|_{r+1} + \| u^{(1)} - u^{(2)} \|_r \| u^{(2)} \|_{r+1} \right)
\]

\[
\times \exp \left[ C_2 \left( \| u^{(1)} \|_r + \| u^{(2)} \|_r \right) \right];
\]

(ii) \( j_0 = 2 \):

\[
\| F(u^{(1)}(\cdot)) - F(u^{(2)}(\cdot)) \|_{0} \leq C_1 \| u^{(1)} - u^{(2)} \|_1 \left\{ \exp \left[ C_2 \left( \| u^{(1)} \|_1 + \| u^{(2)} \|_1 \right) \right] - 1 \right\}
\]

\[
\| F(u^{(1)}(\cdot)) - F(u^{(2)}(\cdot)) \|_{r+1} \leq C_1 \left( \| u^{(1)} - u^{(2)} \|_{r+1} \| u^{(1)} \|_r + \| u^{(1)} - u^{(2)} \|_r \| u^{(2)} \|_{r+1} \right)
\]

\[
\times \exp \left[ C_2 \left( \| u^{(1)} \|_r + \| u^{(2)} \|_r \right) \right],
\]

where the constants \( C_1 \) and \( C_2 \) depend only on \( r \).
Proof. Consider $j_0 = 1$ firstly. Let $F(u) = uh(u)$. Then $h(u)$ is also an entire function of order 1.

$$
\| F(u(\cdot)) \|_\infty \leq C_1 \exp(C_2 \|u\|_\infty) \|u\|_\infty \leq C_1 \exp(C_2 \|u\|_1) \|u\|_1.
$$

(2.16)

We see

$$
\| D_\alpha F(u(\cdot)) \|_0 = \| u_\alpha D_\alpha F \|_0 \leq C_1 \exp(C_2 \|u\|_1) \|u\|_1.
$$

For $k \geq 2$, by Banach Algebra property, we also have

$$
\| D_\alpha F(u(\alpha)) \|_{k-1} \leq C \| D_\alpha u \|_{k-1} \| D_\alpha F(u(\alpha)) \|_{k-1} \leq C \|u\|_k \sum_{j=1}^\infty |a_j| |u|^{j-1}_{k-1}
$$

$$
\leq C_1 \|u\|_k \exp(C_2 \|u\|_{k-1}).
$$

(2.17)

Hence, by (2.16) and (2.17), we have for $k \geq 2$,

$$
\| F(u(\cdot)) \|_k \leq C_1 \|u\|_k \exp(C_2 \|u\|_{k-1}),
$$

(2.18)

with $C_1$ and $C_2$ depending only on $k$.

Let $F(u) = u^2 g(u)$. Then $g(u)$ is also an entire function of order 1.

$$
\| F(u(\cdot)) \|_\infty \leq C \exp(\|u\|_\infty) \|u\|_\infty^2 \leq C \exp(\|u\|_1) \|u\|_1^2.
$$

And $D_\alpha F(u)$ is the entire function of order 1 with $j_0 = 1$, so for $k \geq 2$, by Banach Algebra and (2.18), we have

$$
\| D_\alpha F(u(\cdot)) \|_{k-1} \leq C \|u_\alpha\|_{k-1} \| D_\alpha F(u(\alpha)) \|_{k-1} \leq C_1 \|u\|_k \|u\|_{k-1} \exp(C_2 \|u\|_{k-1})
$$

with $C_1$ and $C_2$ depending only on $k$. Hence, for $k \geq 2$,

$$
\| F(u(\cdot)) \|_k \leq C_1 \|u\|_{k-1} \|u\|_k \exp(C_2 \|u\|_{k-1}),
$$

(2.19)
with $C_1$ and $C_2$ depending only on $k$.

By the same technique, we obtain the difference results.

\[ \square \]

**Note 2.17.** In particular, $\Xi_e$, $\Xi_s$ and $\Xi_c$ satisfy Lemma 2.16 with $j_0 = 2$. $\sin(\alpha + a + u) - \sin(\alpha + a)$ also satisfies Lemma 2.16 with $j_0 = 1$.

**Definition 2.18.** We denote the bubble area by $V(t)$. From geometric consideration, it is readily seen

\[ V(t) = \frac{1}{2} \text{Im} \int_0^{2\pi} z_\alpha z^* d\alpha. \tag{2.20} \]

**Proposition 2.19.** For $r \geq 4$, let $(\theta(\alpha, t), L(t), y(0, t)) \in C^1([0, S], H^r_p \times \mathbb{R} \times S_M)$ with $|L - 2\pi| < \frac{1}{2}$ be a solution to the system (B.1)-(B.5) and (1.15) satisfying initial conditions (1.21) and $y(0, 0) = y_0$. Then the bubble area $V(t)$ for the corresponding $z$ is invariant with time, i.e. $V(t) = V(0) \equiv V$.

**Proof.** Taking the derivative with respect to $t$ on both sides of (2.20), we have

\[
\frac{dV(t)}{dt} = \frac{1}{2} \text{Im} \int_0^{2\pi} (z_\alpha z^*_t - z_t z^*_\alpha) d\alpha \\
= -\frac{L}{4\pi} \text{Re} \int_0^{2\pi} (ie^{i\theta(\alpha)}z_t^* + z_t (-ie^{-i\theta(\alpha)} - i\alpha)) d\alpha \\
= -\frac{L}{2\pi} \int_0^{2\pi} (x_t, y_t) \cdot n d\alpha = -\frac{L}{2\pi} \int_0^{2\pi} U d\alpha. \tag{2.21}
\]
Using (1.19), we have
\[
\frac{dV(t)}{dt} = -\text{Re} \left( \int_0^{2\pi} \frac{z_\alpha(\alpha)}{2\pi} \text{PV} \int_{\alpha-\pi}^{\alpha+\pi} \gamma(\alpha') \Re(\alpha, \alpha') \text{d}\alpha' \text{d}\alpha \right). \tag{2.22}
\]

Since we see
\[
\text{Re} \left( \text{PV} \int_0^{2\pi} \frac{z_\alpha(\alpha)}{z(\alpha) - z(\alpha')} \text{d}\alpha \right)
= \text{Re} \left( \lim_{b \to 0} \int_0^{\alpha'-b} + \int_{\alpha'+b}^{2\pi} \frac{d}{d\alpha} \log(z(\alpha) - z(\alpha')) \right)
= \log |z(2\pi) - z(\alpha')| - \log |z(0) - z(\alpha')| = 0,
\]
we get
\[
\frac{dV(t)}{dt} = 0.
\]

Hence the area of the bubble is invariant with time.

**Note 2.20.** From Definition 2.18,

\[
V = \frac{L^2}{8\pi^2} \text{Im} \int_0^{2\pi} \omega_\alpha \omega^* \text{d}\alpha.
\]

Hence, the bubble perimeter length \(L\) may be expressed as

\[
L = \sqrt{\frac{8\pi^2 V}{\text{Im} \int_0^{2\pi} \omega_\alpha \omega^* \text{d}\alpha}}, \text{ where } V = \frac{L^2}{8\pi^2} \text{Im} \left\{ \int_0^{2\pi} \omega_\alpha(\alpha,0) \omega^*(\alpha,0) \text{d}\alpha \right\} \tag{2.23}
\]

In simplifying integral operators, we will find ensuing bounds on certain divided differences useful.

**Definition 2.21.** For \(z \in H_p^r\), we define operators \(q_1\) and \(q_2\) so that

\[
q_1[z](\alpha, \alpha') = \frac{z(\alpha) - z(\alpha')}{\alpha - \alpha'} = \int_0^1 Dz(t\alpha + (1 - t)\alpha') \text{d}t,
\]
\[ q_2[z](\alpha, \alpha') = \frac{z(\alpha) - z(\alpha') - z_\alpha(\alpha)(\alpha - \alpha')}{(\alpha - \alpha')^2} = \int_0^1 (t - 1)D^2z((1 - t)\alpha + t\alpha')dt, \]

where \( D \) and \( D^2 \) denote first and second derivatives with respect to the argument.

**Proposition 2.22.** There exists \( \epsilon_1 > 0 \) so that \( \| \tilde{\theta} \|_1 \leq \epsilon_1 \) implies

\[
|q_1[z](\alpha, \alpha')| \geq \sqrt{\frac{\pi V}{24}}, \quad \text{for} \ 0 < |\alpha - \alpha'| \leq \pi, \tag{2.24}
\]

which means the curve, \( z \), is non-self-intersecting.

**Proof.** We note that

\[
q_1[\omega](\alpha, \alpha') = \frac{\int_\alpha^\alpha e^{i\eta + i\tilde{\theta}(\eta) + i\hat{\theta}(1; t)e^{\eta} + i\hat{\theta}(-1; t)e^{-\eta}}d\eta}{\alpha - \alpha'}.
\]

Further,

\[
\left| \int_\alpha^\alpha e^{i\eta + i\tilde{\theta}(\eta) + i\hat{\theta}(1; t)e^{\eta} + i\hat{\theta}(-1; t)e^{-\eta}}d\eta \right| - \left| \int_\alpha^\alpha e^{i\eta}d\eta \right| \\
\geq \left| \frac{\int_\alpha^\alpha e^{i\eta}(e^{i\tilde{\theta}(\eta)} + i\hat{\theta}(1; t)e^{\eta} + i\hat{\theta}(-1; t)e^{-\eta}) - 1)d\eta \right| \\
\leq 2\sqrt{2} \max_{\eta \in [0, 2\pi]} |\tilde{\theta}(\eta) + \hat{\theta}(1; t)e^{\eta} + \hat{\theta}(-1; t)e^{-\eta}|.
\]

This bound is a consequence of the inequality

\[ |e^{i\zeta} - e^{i\zeta'}| \leq \sqrt{2}|\zeta - \zeta'|, \quad \text{for all} \ \zeta, \zeta' \in \mathbb{R}. \]

We choose \( \epsilon_1 > 0 \) small enough so that Proposition 2.7 holds and from Sobolev embedding theorem,

\[ 2\sqrt{2} \max_{\eta \in [0, 2\pi]} |\tilde{\theta}(\eta) + \hat{\theta}(1; t)e^{\eta} + \hat{\theta}(-1; t)e^{-\eta}| \leq c\| \tilde{\theta} \|_1 \leq \frac{1}{8}; \]

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where $c$ is some constant.

It is easy to see that

$$\left| \frac{\int_{\alpha'}^{\alpha} e^{in} d\eta}{\alpha - \alpha'} \right| \geq \frac{1}{4}, \text{ for } 0 < |\alpha - \alpha'| \leq \pi.$$ 

Thus, if $\|\tilde{\theta}\|_1 \leq \epsilon_1$, we have

$$\left| \int_{\alpha'}^{\alpha} e^{in} \left[ e^{i\eta} + i\tilde{\theta}(\eta) + i\hat{\theta}(1; t)e^{it} + e^{i\hat{\theta}(-1; t)e^{-it}} \right] d\eta \right| \geq \frac{1}{8}.$$ 

By Proposition 2.22, there is $\epsilon_1 > 0$ sufficiently small so that if $\|\tilde{\theta}\|_1 \leq \epsilon_1$, we have

$$q_1[\omega](\alpha, \alpha') \geq \frac{1}{8}. \quad (2.25)$$

Furthermore, using $\text{Im} \int_0^{2\pi} e^{i\alpha} \int_0^{\alpha} e^{-i\alpha'} d\alpha' d\alpha = 2\pi$, by Sobolev’s embedding theorem and Proposition 2.7, we have

$$\left| \text{Im} \int_0^{2\pi} \omega \omega^* d\alpha - \text{Im} \int_0^{2\pi} e^{i\alpha} \int_0^{\alpha} e^{-i\alpha'} d\alpha' d\alpha \right| \leq 16\sqrt{2\pi^2} \|\theta\|_\infty \leq C\|\tilde{\theta}\|_1,$$

where $C$ is some constant. If let $C\epsilon_1 \leq \pi$, then

$$\pi \leq \text{Im} \int_0^{2\pi} \omega \omega^* d\alpha \leq 3\pi.$$

By (2.23), the lower and upper bounds for $L$ are

$$\sqrt{\frac{8\pi V}{3}} \leq L \leq \sqrt{8\pi V}. \quad (2.26)$$
Combining (2.25) and (2.26), if \( \| \tilde{\theta} \|_1 < \epsilon_1 \), then
\[
|q_1[z](\alpha, \alpha')| = \frac{L}{2\pi} |q_1[\omega](\alpha, \alpha')| \geq \sqrt{\pi V/24}, \text{ for all } 0 < |\alpha - \alpha'| \leq \pi.
\]

\[\square\]

**Lemma 2.23.** (See [1] or §A.1 for proof) Let \( z_{\alpha} \in H^k_p \) for \( k \geq 0 \). Then \( D^k_\alpha q_1, D^k_\alpha q_1 \in H^0[a, a + 2\pi] \) in both the variables \( \alpha \) or \( \alpha' \) and satisfy the bounds
\[
||D^k_\alpha q_1[z]||_0 \leq CL||\omega_\alpha||_k, \quad ||D^k_\alpha q_1[z]||_0 \leq CL||\omega_\alpha||_k
\]
with \( C \) only depending on \( k \) (in particular independent of \( a \)). Further if \( z_{\alpha a} \in H^k_p \) for \( k \geq 0 \), then \( D^k_\alpha q_2, D^k_\alpha q_2 \in H^0[a, a + 2\pi] \) in both the variables \( \alpha \) and \( \alpha' \) and satisfy
\[
||D^k_\alpha q_2[z]||_0 \leq CL||\omega_{\alpha a}||_k, \quad ||D^k_\alpha q_2[z]||_0 \leq CL||\omega_{\alpha a}||_k
\]
with \( C \) only depending on \( k \).

**Lemma 2.24.** Let \( \omega^{(1)}, \omega^{(2)} \in H^{j+1}_p \) for \( j \geq 0 \). Suppose
\[
|q_1[\omega^{(1)}](\alpha, \alpha')| \geq \frac{1}{8}, \text{ for } 0 < |\alpha - \alpha'| \leq \pi.
\]
Then for \( j = 0 \), there exists constant \( C_1 \) independent of \( \alpha \) such that
\[
\left( \int_{\alpha-\pi}^{\alpha+\pi} \left| \frac{q_2[\omega^{(2)}](\alpha, \alpha')}{q_1[\omega^{(1)}](\alpha, \alpha')} \right|^2 d\alpha' \right)^{\frac{1}{2}} \leq C_1 \|\omega^{(2)}_\alpha\|_1.
\]
Further, for \( j \geq 3 \),
\[
\left( \int_{\alpha-\pi}^{\alpha+\pi} \left| \frac{D^2_\alpha q_2[\omega^{(2)}](\alpha, \alpha')}{D^2_\alpha q_1[\omega^{(1)}](\alpha, \alpha')} \right|^2 d\alpha' \right)^{\frac{1}{2}} \leq C_2 \left( \|\omega^{(2)}_\alpha\|_{j+1} + \|\omega^{(2)}_\alpha\|_{j-1}\|\omega^{(1)}_\alpha\|_j \right) \left( \|\omega^{(1)}_\alpha\|_{j-1} + 1 \right), \quad (2.27)
\]
where \( C_2 \) depends on \( j \) alone, but not on \( \alpha \).
Proof. We note that

\[ D_j^{q_2} q_1 = \sum_{l=0}^{j} C_{j,l} D_{\alpha}^{j-l} q_2 D_{\alpha}^l \frac{1}{q_1}. \]

Using Lemma 2.23 with \( L = 2\pi \) it follows that for \( l \geq 1 \)

\[ \left\| D_{\alpha}^l \frac{1}{q_1} \right\|_0 \leq C_1 \| q_1 \|_l \left( 1 + \| q_1 \|_{l-1}^{l-1} \right) \leq C_1 \| \omega_\alpha^{(1)} \|_l \left( \| \omega_\alpha^{(1)} \|_{l-1}^{l-1} + 1 \right), \]

and

\[ \left\| D_{\alpha}^j \frac{q_2}{q_1} \right\|_0 \leq C \sum_{l=1}^{j-1} \left\| D_{\alpha}^{j-l} q_2 \right\|_0 \left\| D_{\alpha}^l \frac{1}{q_1} \right\|_\infty + C \left\| D_{\alpha}^j \frac{1}{q_1} \right\|_0 \| q_2 \|_\infty + C \left\| D_{\alpha}^j q_2 \right\|_0 \frac{1}{q_1} \|_\infty. \]

The lemma immediately follows from Lemma 2.23 on using \( \| \frac{1}{q_1} \|_\infty \leq C \) and \( \| D_{\alpha}^l \frac{1}{q_1} \|_\infty \leq C \left\| \frac{1}{q_1} \right\|_{l+1}. \)

Lemma 2.25. Assume \( \omega^{(1)}, \omega^{(2)} \in H_p^{j+1} \) for \( j \geq 0 \). Assume further that

\[ \left| q_1 [\omega^{(1)}](\alpha, \alpha') \right| \geq \frac{1}{8} \text{ for } 0 < |\alpha - \alpha'| \leq \pi. \]

Then for \( j = 0 \), there exists constant \( C_1 \) independent of \( \alpha \) such that

\[ \left( \int_0^{2\pi} \left| \frac{q_1 [\omega^{(2)}](\alpha, \alpha')}{q_1 [\omega^{(1)}](\alpha, \alpha')} \right|^2 d\alpha' \right)^{\frac{1}{2}} \leq C_1 \| \omega^{(2)} \|_0. \]

Further, for \( j \geq 3 \),

\[ \left( \int_0^{2\pi} \left| D_{\alpha}^j q_1 [\omega^{(2)}](\alpha, \alpha') \right|^2 d\alpha' \right)^{\frac{1}{2}} \leq C_2 \left( \| \omega^{(2)} \|_j + \| \omega^{(2)} \|_{j-2} \| \omega^{(1)} \|_{j-1} \right) (1 + \| \omega^{(1)} \|_{j-1}^{j-1}), \quad (2.28) \]

where \( C_2 \) depends on \( j \) only.
Proof. The proof is almost identical to that of Lemma 2.24. It uses Lemma 2.23 and the lower bound on \( q_1[\omega^{(1)}] \). We note that integrand on the left of (2.28) is 2\( \pi \)-periodic in \( \alpha' \), noting that factors of \( (\alpha - \alpha') \) in \( q_1[\omega^{(1)}] \) and \( q_1[\omega^{(2)}] \) cancel each other. We are therefore free to replace the upper and lower bound in the integral in \( \alpha' \) by \( \alpha + \pi \) and \( \alpha - \pi \) respectively for which \( |q_1| \) is bounded below as needed. 

\[ \]

Lemma 2.26. Assume \( f, g \in H^j_p \), for \( j \geq 0 \), with Fourier components \( \hat{f}(0), \hat{g}(0) = 0 \) and \( h \in H^0_p \). Suppose

\[
| \int_0^1 g(t\alpha + (1-t)\alpha') dt | \geq \frac{1}{8}, \text{ for } 0 \leq |\alpha' - \alpha| \leq \pi.\label{2.29}
\]

Then for \( j = 0 \), there exists constant \( C_1 \) independent of \( \alpha \) such that

\[
\int_0^{2\pi} \left| \frac{h(\alpha')}{\int_{\alpha'}^{\alpha} g(\tau) d\tau} \right| d\alpha' \leq C_1 \| h \|_0 \| f \|_0.
\]

Further, for \( j \geq 3 \),

\[
\int_0^{2\pi} \left| \frac{h(\alpha') D^j \int_{\alpha'}^{\alpha} f(\tau) d\tau}{\int_{\alpha'}^{\alpha} g(\tau) d\tau} \right| d\alpha' \leq C_2 \| h \|_0 (\| f \|_j + \| f \|_{j-2} \| g \|_j) (1 + \| g \|_{j-1}^{-1}),
\]

where \( C_2 \) depends on \( j \) only.

Proof. We define

\[
\omega^{(1)}(\alpha) = \int_0^\alpha g(s) ds, \quad \omega^{(2)}(\alpha) = \int_0^\alpha f(s) ds.
\]

Clearly, \( \omega^{(1)}, \omega^{(2)} \in H^{k+1}_p \) since \( \hat{g}(0) = 0 = \hat{f}(0) \). We note

\[
\frac{\int_{\alpha'}^{\alpha} f(\tau) d\tau}{\int_{\alpha'}^{\alpha} g(\tau) d\tau} = \frac{q_1[\omega^{(2)}](\alpha, \alpha')}{q_1[\omega^{(1)}](\alpha, \alpha')}.
\]
Further, we note that the given condition on lower bound involving $g$ becomes
\[ |q_1[\omega^{(1)}](\alpha, \alpha')| \geq \frac{1}{8}. \]

Using Lemma 2.25, the proof follows using Cauchy Schwartz inequality.

\[ \Box \]

**Lemma 2.27.** Assume $\omega^{(1)}, \omega^{(2)} \in H_j^{j+2}$ with $j \geq 0$. Suppose
\[ |q_1[\omega^{(1)}](\alpha, \alpha')| \geq \frac{1}{8}, \quad |q_1[\omega^{(2)}](\alpha, \alpha')| \geq \frac{1}{8} \quad \text{for } 0 < |\alpha - \alpha'| \leq \pi. \]

Then $j = 0$, for any $a \in \mathbb{R}$, there exists constant $C_1$ independent of $\alpha$ and $a$ such that
\[ \left\{ \left( \int_a^{a+2\pi} \left| \frac{q_2[\omega^{(2)}](\alpha', \alpha)}{q_1[\omega^{(2)}](\alpha', \alpha)} - \frac{q_2[\omega^{(1)}](\alpha', \alpha)}{q_1[\omega^{(1)}](\alpha', \alpha)} \right|^2 d\alpha' \right)^{1/2} \right\} \leq C_1 \|\omega^{(2)} - \omega^{(1)}\|_1 \left( 1 + \|\omega^{(1)}\|_1 \right). \]

Further, for $j \geq 3$,
\[ \left\{ \left( \int_a^{a+2\pi} \left| D^j_\alpha \left( \frac{q_2[\omega^{(2)}](\alpha', \alpha)}{q_1[\omega^{(2)}](\alpha', \alpha)} - \frac{q_2[\omega^{(1)}](\alpha', \alpha)}{q_1[\omega^{(1)}](\alpha', \alpha)} \right) \right|^2 d\alpha' \right)^{1/2} \right\} \leq C \left( \|\omega^{(2)} - \omega^{(1)}\|_{j+1} + \|\omega^{(2)} - \omega^{(1)}\|_j \|\omega^{(1)}\|_j \right) \left( 1 + \|\omega^{(1)}\|_j + \|\omega^{(2)}\|_j \right). \]

where $C$ depends on $j$ alone, but not on $a$.

**Proof.** We note from the definitions of $q_1$ and $q_2$ that the nonperiodic term \( \frac{1}{\alpha - \alpha'} \) that appears in each \( \frac{q_2}{q_1} \) in the integrand cancels each other out and we are left with integrating a $2\pi$-periodic function in $\alpha'$; hence there is no dependence on $a$, and we may choose $a = \alpha - \pi$ in the proof. The rest of the proof is similar to that of Lemma 2.24. We note that
\[ \frac{q_2[\omega^{(2)}]}{q_1[\omega^{(2)}]} = \frac{q_2[\omega^{(1)}]}{q_1[\omega^{(1)}]} = \frac{q_2[\omega^{(2)} - \omega^{(1)}]}{q_1[\omega^{(2)}]} = \frac{q_2[\omega^{(1)}]q_1[\omega^{(2)} - \omega^{(1)}]}{q_1[\omega^{(1)}]q_1[\omega^{(2)}]} \]

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and that the denominators are bounded away from zero. We use Lemmas 2.23, 2.24 and the Banach algebra property for $\| \cdot \|_j$ norms in $\alpha'$ for $j \geq 1$. For $j = 0$, the result follows from

$$\left\| \frac{q_2[\omega^{(1)}]}{q_1[\omega^{(1)}]} \right\|_{L^\infty} \leq C \left\| \frac{q_2[\omega^{(1)}]}{q_1[\omega^{(1)}]} \right\|_1,$$

where the norms are taken in $\alpha'$.

**Lemma 2.28.** For $\| \tilde{\theta} \|_1 < \epsilon$ sufficiently small, $\omega$ determined from $\tilde{\theta}$ through (2.2), then for $\tilde{\theta}, J \in H^r_p$ for $r \geq 3$ and any $a$, there exists constant $C_r$ only depending on $r$ such that

$$\left\| \frac{1}{\pi} PV \int_a^{a+2\pi} \frac{\omega_0(\alpha) J(\alpha') d\alpha'}{\omega(\alpha) - \omega(\alpha')} \right\|_r \leq C_r \left( \| J \|_r + \| J \|_0 \| \tilde{\theta} \|_{r+1} \exp \left( C_r \| \tilde{\theta} \|_r \right) \right).$$

**Proof.** We note from (2.4) that $J \in H^0_p$,

$$\frac{\omega_0}{\pi} PV \int_a^{a+2\pi} \frac{J(\alpha') d\alpha'}{\omega_0(\alpha) - \omega_0(\alpha')} = \mathcal{H}[J](\alpha) + i \tilde{J}(0)$$

and we know that $\| \mathcal{H} J \|_r = \| J \|_r$. Therefore, the integrand may be written as

$$i \tilde{J}(0) + \mathcal{H}[J](\alpha) + \frac{1}{\pi} \int_{\alpha-\pi}^{\alpha+\pi} d\alpha' J(\alpha') \left\{ \frac{q_2[\omega](\alpha,\alpha')}{q_1[\omega](\alpha,\alpha')} - \frac{q_2[\omega_0](\alpha,\alpha')}{q_1[\omega_0](\alpha,\alpha')} \right\}.$$

The proof follows from applying Lemmas 2.27, 2.14, Proposition 2.22 and using Cauchy Schwartz inequality and noting $\omega = \omega_0$, when $\tilde{\theta} = 0$. \qed

**Lemma 2.29.** Assume $\tilde{\theta} \in \dot{H}^{r+1}$, $J \in H^r_p$ and $\omega^{[3]} \in H^{r+1}_p$ for $r \geq 3$. Assume $\tilde{\theta}_1 < \epsilon$ is sufficiently small and $\omega$ is determined from $\tilde{\theta}$ through (2.2). Then for any $a$, there exists constant $C_r$ only depending on $r$ such that

$$\left\| \omega_\alpha PV \int_a^{a+2\pi} \frac{J(\alpha') q_1[\omega^{[3]}](\alpha,\alpha') d\alpha'}{\omega(\alpha) - \omega(\alpha')} \right\|_r \leq C_r \left\{ \| \omega^{[3]} \|_r \left( \| J \|_r + \| J \|_0 \| \tilde{\theta} \|_{r+1} \exp \left( C_r \| \tilde{\theta} \|_r \right) \right) + \| \omega^{[3]} \|_{r+1} \exp \left( C_r \| \tilde{\theta} \|_r \right) \right\}.$$
Proof. We note that

\[
\omega_\alpha \text{PV} \int_a^{a+2\pi} \frac{J(\alpha')q_1[\omega[3]](\alpha, \alpha')d\alpha'}{(\omega(\alpha) - \omega(\alpha')) q_1[\omega](\alpha, \alpha')} = -D_\alpha \int_a^{a+2\pi} \frac{J(\alpha')q_1[\omega[3]](\alpha, \alpha')}{q_1[\omega](\alpha, \alpha')}d\alpha' + \frac{\omega[3]}{\omega_\alpha} \text{PV} \int_a^{a+2\pi} \frac{J(\alpha')\omega_\alpha(\alpha)d\alpha'}{\omega(\alpha) - \omega(\alpha')}.
\]

We rely on Lemmas 2.25 and 2.27, as well as Cauchy Schwartz inequality, and Banach algebra property of \( \| \cdot \|_r \) norm for \( r \geq 1 \) to complete the proof. \( \Box \)

Lemma 2.30. Suppose for \( r \geq 2 \), \( z \in H_p^r \) corresponds to \( \tilde{\theta} \in \tilde{H}^{r-1} \) through (2.2) and (2.3) and \( \| \tilde{\theta} \|_1 < \epsilon_1 \), where \( \epsilon_1 \) is small enough for Propositions 2.7 and 2.22 to apply. Further assume \( |L - 2\pi| \leq \frac{1}{2} \) and \( y(0, t) \in \mathcal{S}_M \). Then there exists \( \Upsilon > 0 \) such that if \( 0 \leq \beta < \Upsilon \), then \( \mathcal{K}[z] : H_p^0 \rightarrow H_p^{r-2} \), and in particular, there are positive constants \( C_1 \) depending on \( r \) such that

\[
\| \mathcal{K}[z]f \|_{r-2} \leq C_1 \| f \|_0 (1 + \beta^2)(1 + \| \omega_\alpha \|_{r-1}^3).
\]

(2.30)

Further, \( \mathcal{K}[z] : H_p^1 \rightarrow H_p^{r-1} \), and

\[
\| \mathcal{K}[z]f \|_{r-1} \leq C_1 \| f \|_1 (1 + \beta^2)(1 + \| \omega_\alpha \|_{r-1}^3).
\]

(2.31)

Proof. We will deal with \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) separately. By Lemma 6 in [1] or §A.2, we have

\[
\| \mathcal{K}_1[z]f \|_{r-2} \leq C_1 \| f \|_0 (1 + \| \omega_\alpha \|_{r-1}^2),
\]

(2.32)

\[
\| \mathcal{K}_1[z]f \|_{r-1} \leq C_1 \| f \|_1 (1 + \| \omega_\alpha \|_{r-1}^3),
\]

(2.33)

where the positive constants \( C_1 \) both depend on \( r \).
Now consider $D^{-1}K_2[z] f$, given by:

\[
\frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} f(\alpha') D^{-1}_\alpha \left\{ \frac{\beta}{4} l_1 \left( \frac{1}{4} \beta (z(\alpha) - z(\alpha')) \right) - \frac{\beta}{4} \tanh \left[ \frac{\beta}{4} (z(\alpha) - z^*(\alpha')) \right] \right\} d\alpha' = \frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} f(\alpha') D^{-1}_\alpha \left\{ \frac{1}{4} \beta (z(\alpha) - z(\alpha')) + \frac{1}{4} \tanh \left[ \frac{1}{4} \beta (z(\alpha) - z(\alpha')) \right] \right\} d\alpha'.
\]

Equation (2.34) involves up to $r-1$ derivative of $z$. From (2.26),

\[
|z(\alpha) - z(\alpha')| = \frac{L}{2\pi} \left| \int_{\alpha}^{\alpha'} e^{i\zeta + i\theta(\zeta)} d\zeta \right| \leq \frac{L}{2} < 2\pi
\]

(2.35)

\[
z(\alpha, t) - z^*(\alpha', t) = (z(\alpha, t) - z(\alpha', t)) + 2i(y(\alpha', t) - y(0, t)) + 2iy(0, t).
\]

(2.36)

From (2.35), (2.36) and $|y(0, t)| < M$, there exists $\Upsilon > 0$ small enough so that if $0 \leq \beta < \Upsilon < 1$, then $|\beta (z(\alpha) - z(\alpha'))| \leq \pi$, and $|\beta [(z(\alpha) - z(\alpha')) + 2i(y(\alpha', t) - y(0, t)) + 2iy(0, t)]| < C\beta$. Since $l_1$ and tanh analytic, we conclude that

\[
\|K_2[z] f\|_{r-1} \leq C_1 \beta^2 \|f\|_0 \left( 1 + \|\omega_\alpha\|_{r-1}^{-1} \right),
\]

(2.37)

where $C_1$ depends only on $r$. Combining (2.32), (2.33) and (2.37), we complete the proof.

\[\square\]

**Note 2.31.** Note from (2.1) and (2.37), for $r \geq 1$ and $|L - 2\pi| < \frac{1}{2}$, by Lemma 2.14, it follows that

\[
\|G_2[z] f\|_{r-1} \leq C_1 \beta^2 \|f\|_0 \exp \left( C_2 \|\tilde{\theta}\|_{r-1} \right),
\]

(2.38)

where $C_1$ and $C_2$ depend only on $r$. 

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Lemma 2.32. If \( f \in H^1_\rho \), \( \omega^{(1)} \) and \( \omega^{(2)} \) correspond to \( \tilde{\theta}^{(1)} \) and \( \tilde{\theta}^{(2)} \), each in \( \dot{H}^1 \), respectively with \( \| \tilde{\theta}^{(1)} \|_1, \| \tilde{\theta}^{(2)} \|_1 < \epsilon_1 \), then for sufficient small \( \epsilon_1 \),

\[
\| K_1^{(1)} f - K_1^{(2)} f \|_0 \leq C_1 \| f \|_0 \exp \left( C_2 \left( \| \tilde{\theta}^{(1)} \|_1 + \| \tilde{\theta}^{(2)} \|_1 \right) \right) \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_1.
\]

Suppose \( \tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^\tau \). Then for \( r \geq 1 \),

\[
\| K_1^{(1)} f - K_1^{(2)} f \|_r \leq C_1 \exp \left( C_2 \left( \| \tilde{\theta}^{(1)} \|_r + \| \tilde{\theta}^{(2)} \|_r \right) \right) \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_r \| f \|_1,
\]

while for \( r \geq 3 \),

\[
\| K_1^{(1)} f - K_1^{(2)} f \|_r \leq C_1 \exp \left( C_2 \left( \| \tilde{\theta}^{(1)} \|_{r-1} + \| \tilde{\theta}^{(2)} \|_{r-1} \right) \right) \left( \| \tilde{\theta}^{(1)} \|_r + \| \tilde{\theta}^{(2)} \|_r \right) \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_{r-1} + \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_r \| f \|_1,
\]

where constants \( C_1 \) and \( C_2 \) depend on \( r \) only.

Proof. We note that

\[
K_1^{(1)} f - K_1^{(2)} f = -\frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} f(\alpha') \left( \frac{q_2^{(1)}}{q_1^{(1)}}(\alpha') - \frac{q_2^{(2)}}{q_1^{(2)}}(\alpha') \right) d\alpha' - \frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} f(\alpha') \left( \frac{1}{\omega_\alpha^{(1)}}(\alpha') - \frac{1}{\omega_\alpha^{(2)}}(\alpha') \right) \left( \frac{q_2^{(1)}}{q_1^{(1)}}(\alpha') - \frac{1}{2\omega_\alpha^{(1)}(\alpha')} \right) \left( \frac{1}{2\omega_\alpha^{(2)}(\alpha')} \right) l_2 \left( \frac{1}{2}(\alpha - \alpha') \right) d\alpha'.
\]

We also have

\[
\frac{q_2^{(1)}}{q_1^{(1)}} - \frac{q_2^{(2)}}{q_1^{(2)}} = \frac{q_2^{(1)} - q_2^{(2)}}{q_1^{(1)}} - \frac{q_2^{(2)}}{q_1^{(2)}} - \frac{q_2^{(2)}}{q_1^{(1)}} q_1^{(2)} - \frac{q_2^{(2)}}{q_1^{(2)}} q_1^{(1)}.
\]
Therefore, using Sobolev inequality $|.|_{\infty} \leq C|.|_1$, we obtain

$$
\|K_1[\omega^{(1)}]f - K_1[\omega^{(2)}]f\|_0 \leq C_1 \|f\|_0 \left\| \frac{1}{\omega_\alpha^{(2)}} \right\|_1 \left( \|q_2[\omega^{(1)}] - \omega^{(2)}\|_0 \right)
+ \|q_1[\omega^{(1)}] - \omega^{(2)}\|_1 \|q_2[\omega^{(2)}]_0\| + C_2 \|f\|_0 \left\| \frac{1}{\omega_\alpha^{(1)}} - \frac{1}{\omega_\alpha^{(2)}} \right\|_1 \left( \|q_2[\omega^{(2)}]\|_0 + 1 \right).
$$

The first statement follows easily from Lemmas 2.14, 2.23 and 2.24. Further, using one integration by parts, the rth derivative of $K_1[\omega]f$ is

$$
D_\alpha^r K_1[\omega]f(\alpha) = D_\alpha^{r-1} \frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} D_\alpha' \left( \frac{f(\alpha')}{\omega_\alpha'(\alpha')} \right) \left[ \frac{\omega_\alpha(\alpha)}{\omega(\alpha) - \omega(\alpha')} - \frac{1}{2} \cot \frac{1}{2}(\alpha - \alpha') \right] d\alpha'.
$$

(2.39)

Hence, we have

$$
D_\alpha^r (K_1[\omega^{(1)}]f(\alpha) - K_1[\omega^{(2)}]f(\alpha)) = \frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} \left( D_\alpha' \left( \frac{f(\alpha')}{\omega_\alpha^{(1)}(\alpha')} \right) D_\alpha^{r-1} q_2[\omega^{(1)}](\alpha, \alpha') - D_\alpha' \left( \frac{f(\alpha')}{\omega_\alpha^{(2)}(\alpha')} \right) D_\alpha^{r-1} q_1[\omega^{(2)}](\alpha, \alpha') \right) d\alpha'.
$$

(2.40)

Let us see the first part on the right side of (2.40). It can be split as

$$
\frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} D_\alpha' \left( \frac{f(\alpha')}{\omega_\alpha^{(1)}(\alpha')} \right) \left[ \frac{\omega_\alpha^{(2)}(\alpha') - \omega_\alpha^{(1)}(\alpha')}{\omega_\alpha^{(2)}(\alpha')} \right] D_\alpha^{r-1} q_2[\omega^{(1)}](\alpha, \alpha') \frac{1}{q_1[\omega^{(1)}](\alpha, \alpha')} d\alpha'
+ \frac{1}{2\pi i} \int_{\alpha-\pi}^{\alpha+\pi} D_\alpha' \left( \frac{f(\alpha')}{\omega_\alpha^{(2)}(\alpha')} \right) \left[ \frac{q_2[\omega^{(1)}](\alpha, \alpha') q_1[\omega^{(2)}](\alpha, \alpha') - q_2[\omega^{(2)}](\alpha, \alpha') q_1[\omega^{(1)}](\alpha, \alpha')} {q_1[\omega^{(2)}](\alpha, \alpha') q_1[\omega^{(1)}](\alpha, \alpha')} \right] d\alpha'.
$$

(2.41)
By Proposition 2.7, Lemma 2.14, Lemma 2.24 and Note 1.3, the $L^\infty$-norm of the first part of (2.41) is bounded by

$$C_1 \left\| \frac{f}{\omega_\alpha^{(1)}}(\omega_\alpha^{(1)} - \omega_\alpha^{(2)}) \right\|_1 \| \omega_\alpha^{(1)} \|_r \exp \left( C_2 \| \omega_\alpha^{(1)} \|_{r-1} \right)$$

$$\leq C_1 \left( \| \hat{\theta}^{(1)} \|_r + \| \hat{\theta}^{(2)} \|_r + 1 \right) \exp \left( C_2 \left( \| \hat{\theta}^{(1)} \|_{r-1} + \| \hat{\theta}^{(2)} \|_{r-1} \right)\right) \| \hat{\theta}^{(1)} - \hat{\theta}^{(2)} \|_1 \| f \|_1,$$

with $C_1$ and $C_2$ depending on $r$. For the second term in (2.41), we use the Cauchy-Schwartz inequality, Lemmas 2.14, 2.24 to obtain the bound as quoted in the Lemma.

For the third term, we apply similar argument. We note that for $0 \leq l < r - 1$,

$$\left\| D_\alpha^l \left[ \frac{q_2[\omega^{(2)}](\alpha, \alpha')}{q_1[\omega^{(2)}](\alpha, \alpha')} \right] D_\alpha^{r-1-l} \left[ \frac{q_1[\omega^{(2)}] - \omega^{(1)}[\alpha, \alpha']}{q_1[\omega^{(1)}](\alpha, \alpha')} \right] \right\|_0$$

$$\leq \left\| D_\alpha^l \left[ \frac{q_2[\omega^{(2)}](\alpha, \alpha')}{q_1[\omega^{(2)}](\alpha, \alpha')} \right] \right\| \left\| D_\alpha^{r-1-l} \left[ \frac{q_1[\omega^{(2)}] - \omega^{(1)}[\alpha, \alpha']}{q_1[\omega^{(1)}](\alpha, \alpha')} \right] \right\|_0.$$

It is readily checked that

$$D_\alpha^{\alpha'} \left[ \frac{q_2[\omega^{(2)}](\alpha, \alpha')}{q_1[\omega^{(2)}](\alpha, \alpha')} \right] = -D_\alpha \left[ \frac{q_2[\omega^{(2)}](\alpha', \alpha)}{q_1[\omega^{(2)}](\alpha', \alpha)} \right].$$

Since $\| \cdot \|_\infty \leq C \| \cdot \|_1$, it follows from Lemma 2.24 that for $l < r - 1$,

$$\left\| D_\alpha^l \left[ \frac{q_2[\omega^{(2)}](\alpha, \alpha')}{q_1[\omega^{(2)}](\alpha, \alpha')} \right] \right\|_\infty \leq C_1 \| \omega_\alpha^{(2)} \|_r \exp \left( C_2 \| \omega_\alpha^{(2)} \|_{r-1} \right)$$

with $C_1$ and $C_2$ depending on $r$. When $l = r - 1$,

$$\left\| D_\alpha^{r-1} \left[ \frac{q_2[\omega^{(2)}](\alpha, \alpha')}{q_1[\omega^{(2)}](\alpha, \alpha')} \right] \frac{q_1[\omega^{(2)}] - \omega^{(1)}(\alpha, \alpha')}{q_1[\omega^{(1)}](\alpha, \alpha')} \right\|_0$$

$$\leq \left\| \frac{q_1[\omega^{(2)}] - \omega^{(1)}(\alpha, \alpha')}{q_1[\omega^{(1)}](\alpha, \alpha')} \right\|_\infty \left\| D_\alpha^{r-1} \left[ \frac{q_2[\omega^{(2)}](\alpha, \alpha')}{q_1[\omega^{(2)}](\alpha, \alpha')} \right] \right\|_0.$$

Once again using Sobolev inequality $\| \cdot \|_\infty \leq C \| \cdot \|_1$ and using Lemmas 2.14, 2.24 we obtain the stated bounds.
Since the function $l_2$ is symmetric about $\alpha$ and $\alpha'$, it is easy to see that the stated bounds also hold for the second part on the right side of (2.40).

For $r \geq 3$, we use the more refined estimates in Lemma 2.14 to obtain the third statement.

\begin{proof}

\end{proof}

**Lemma 2.33.** Let $0 \leq \beta < \Upsilon$, where $\Upsilon$ is small enough for Lemma 2.30 to apply. Let $f \in H^1_p$, and $z_1, z_2$ correspond respectively to $(\tilde{\theta}_1, L_1(t), \dot{\theta}_1(0; t))$ and $(\tilde{\theta}_2, L_2(t), \dot{\theta}_2(0; t))$ (see (2.3)). Further, assume $\|\tilde{\theta}_1\|_1, \|\dot{\theta}_2\|_1 < \epsilon_1$, $|L_1 - 2\pi| < \frac{1}{2}$, $|L_2 - 2\pi| < \frac{1}{2}$ and $y_1(0, t) = \text{Im} z_1(0, t), y_2(0, t) = \text{Im} z_2(0, t)$ belong to $S_M$. Then for sufficient small $\epsilon_1$ and $\Upsilon$, there exists constant $C_1$ depending only on $r$ so that

$$\|G_2[z_1] f - G_2[z_2] f\|_0 \leq C_1 \beta^2 \|f\|_0 (\|\theta_1 - \theta_2\|_1 + |L_1(t) - L_2(t)| + |y_1(0, t) - y_2(0, t)|).$$

If $\tilde{\theta}_1, \tilde{\theta}_2 \in \dot{H}^r$, then for $s \geq 1$,

$$\|G_2[z_1] f - G_2[z_2] f\|_r \leq C_1 \beta^2 \|f\|_1 \exp \left( C_2 (\|\tilde{\theta}_1\|_s + \|\tilde{\theta}_2\|_s) \right) (\|\theta_1 - \theta_2\|_r$$

$$+ |L_1(t) - L_2(t)| + |y_1(0, t) - y_2(0, t)|),$$

for constants $C_1$ and $C_2$ depending on $r$ only.

Further, if $L_1$ and $L_2$ correspond to the same area $V$, then

$$|L_1(t) - L_2(t)| \leq C \|\tilde{\theta}_1 - \tilde{\theta}_2\|_1,$$

with $C$ depending on area $V$ alone.

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Proof. From (2.1), we have

$$\|G[z_1]f - G[z_2]f\|_r \leq 2 \|(z_{1,\alpha} - z_{2,\alpha})K[z_1]f\|_r + 2 \|z_{2,\alpha} (K[z_1] - K[z_2]) f\|_r.$$ 

Hence, we will consider the estimate of \(\|K[z_1]f - K[z_2]f\|_r\) firstly.

We denote the \(n\)-th derivative of \(l_1\) by \(l_1^{(n)}\). \(D_k^\alpha (K[z_1]f(\alpha) - K[z_2]f(\alpha))\) for \(0 < k \leq s\) is separated into two parts. The first part involves \(l_1\):

$$\frac{\beta^2}{8\pi i} \sum_{k_1 + \cdots + k_m = k} C_k \int_{\alpha - \pi}^{\alpha + \pi} f(\alpha') \left[D_k^{k_1} z_1(\alpha) \cdots D_k^{k_m} z_1(\alpha) l_1^{(m)} \left(\frac{1}{4} \beta (z_1(\alpha) - z_1(\alpha'))\right) - D_k^{k_1} z_2(\alpha) \cdots D_k^{k_m} z_2(\alpha) l_1^{(m)} \left(\frac{1}{4} \beta (z_2(\alpha) - z_2(\alpha'))\right)\right] d\alpha'.$$

(2.43)

We perform our standard addition and subtraction, and for any arbitrary term in (2.43), for example

$$C_k \frac{\beta^2}{8\pi i} \int_{\alpha - \pi}^{\alpha + \pi} f(\alpha') \left[D_k^{k_1} (z_1(\alpha) - z_2(\alpha)) \cdots D_k^{k_m} z_1(\alpha) l_1^{(m)} \left(\frac{1}{4} \beta (z_1(\alpha) - z_1(\alpha'))\right) d\alpha' + \cdots + D_k^{k_1} z_2(\alpha) \cdots D_k^{k_m} z_2(\alpha) \left(l_1^{(m)} \left(\frac{1}{4} \beta (z_1(\alpha) - z_1(\alpha'))\right) - l_1^{(m)} \left(\frac{1}{4} \beta (z_2(\alpha) - z_2(\alpha'))\right)\right)\right] d\alpha'.$$

By (2.35), we know that for sufficient small \(\Upsilon\), we have \(\left|\beta (z(\alpha) - z(\alpha'))\right| \leq \pi\). Since \(l_1\) is analytic and \(l_1^{(m)}\) is Lipschitz continuous, and

$$z_1(\alpha) - z_1(\alpha') - z_2(\alpha) + z_2(\alpha') = \int_\alpha^\tau (z_{1,\alpha}(\tau) - z_{2,\alpha}(\tau)) d\tau,$$

by Lemma 2.14 and Equation (2.26), the \(H^0_p\) norm of the above equation is bounded.
by

\[ C_1 \beta^2 \exp \left( C_2 \left( \| \tilde{\theta}_1 \|_k + \| \tilde{\theta}_2 \|_k \right) \right) \| f \|_0 \left( C_3 \| z_{1, \alpha} - z_{2, \alpha} \|_{k-1} \right) \\
+ \sup_{|\alpha - \alpha'| \leq \pi} \left| \tan^{(m)} \left( \frac{\beta}{4} (z_1(\alpha) - z_1^*(\alpha')) \right) - \tan^{(m)} \left( \frac{\beta}{4} (z_2(\alpha) - z_2^*(\alpha')) \right) \right| \\
\leq C_1 \beta^2 \exp \left( C_2 \left( \| \tilde{\theta}_1 \|_k + \| \tilde{\theta}_2 \|_k \right) \right) \| f \|_0 \left( C_3 \| \theta_1 - \theta_2 \|_{k-1} + \| \theta_1 - \theta_2 \|_1 + |L_1 - L_2| \right), \]  

(2.44)

where \( C_1, C_2 \) and \( C_3 \) depend on \( k \). As \( k = 0 \), by Lipschitz continuous of \( l_1 \), (2.44) is also obtained where \( k = 0 \), \( C_1 \) is some constant, \( C_2 = C_3 = 0 \).

The second part of \( D^k_\alpha \left( K_2[z_1]f(\alpha) - K_2[z_2]f(\alpha) \right) \) for \( 0 < k \leq s \) is

\[ -\frac{\beta^2}{8\pi i} \sum_{k_1 + \cdots + k_m = k} C_k \int_{\alpha - \pi}^{\alpha + \pi} f(\alpha') \left[ D_{\alpha}^{k_1} z_1(\alpha) \cdots D_{\alpha}^{k_m} z_1(\alpha) \tanh^{(m)} \left( \frac{1}{4} \beta (z_1(\alpha) - z_1^*(\alpha')) \right) \right. \\
- D_{\alpha}^{k_1} z_2(\alpha) \cdots D_{\alpha}^{k_m} z_2(\alpha) \tanh^{(m)} \left( \frac{1}{4} \beta (z_2(\alpha) - z_2^*(\alpha')) \right) \left. \right] d\alpha'. \]  

(2.45)

Similar to (2.43), by Lemma 2.30, using (2.36), we know that for sufficient small \( \gamma \), by Lemmas 2.14, the \( H^0_\rho \) norm of the above equation is bounded by

\[ C_1 \beta^2 \exp \left( C_2 \left( \| \tilde{\theta}_1 \|_k + \| \tilde{\theta}_2 \|_k \right) \right) \| f \|_0 \left( C_3 \| z_{1, \alpha} - z_{2, \alpha} \|_{k-1} \right) \\
+ \sup_{|\alpha - \alpha'| \leq \pi} \left| \tanh^{(m)} \left( \frac{\beta}{4} (z_1(\alpha) - z_1^*(\alpha')) \right) - \tanh^{(m)} \left( \frac{\beta}{4} (z_2(\alpha) - z_2^*(\alpha')) \right) \right| \\
\leq C_1 \beta^2 \exp \left( C_2 \left( \| \tilde{\theta}_1 \|_k + \| \tilde{\theta}_2 \|_k \right) \right) \| f \|_0 \left( C_3 \| \theta_1 - \theta_2 \|_{k-1} + \| \theta_1 - \theta_2 \|_1 + |L_1(t) - L_2(t)| + |y_1(0, t) - y_2(0, t)| \right), \]  

(2.46)

where \( C_1, C_2 \) and \( C_3 \) depend on \( k \). As \( k = 0 \), by Lipschitz continuous of \( \tanh \), (2.46) is also obtained where \( k = 0 \), \( C_1 \) is some constant, \( C_2 = C_3 = 0 \).
Hence, using Lemma 2.14, \( \|hg\|_0 \leq |h|_\infty \|g\|_0 \leq C\|h\|_1 \|g\|_0 \), \( \|hg\|_r \leq \|h\|_s \|g\|_r \) for \( s \leq 1 \), we get the first two statements.

(2.20) means that

\[
L^2_1(t) \text{Im} \int_0^{2\pi} \omega_1,\alpha \omega^*_1 d\alpha = 8\pi^2 V,
\]

\[
L^2_2(t) \text{Im} \int_0^{2\pi} \omega_2,\alpha \omega^*_2 d\alpha = 8\pi^2 V.
\]

Subtracting the above two equations, we have

\[
|L_1(t) - L_2(t)| = \left| -\frac{L^2_2(t)}{L_1(t) + L_2(t)} \text{Im} \left( \int_0^{2\pi} \omega_1,\alpha \omega^*_1 d\alpha - \int_0^{2\pi} \omega_2,\alpha \omega^*_2 d\alpha \right) \right|
\]

\[
\leq \frac{L^2_2(t)}{L_1(t) + L_2(t)} \left| \int_0^{2\pi} \omega_1,\alpha (\omega^*_1 - \omega^*_2) d\alpha + \int_0^{2\pi} \omega^*_2 (\omega_1,\alpha - \omega_2,\alpha) d\alpha \right|
\]

\[
\leq \frac{L^2_2(t)}{L_1(t) + L_2(t)} \left( \|\omega_1,\alpha\|_0 \|\omega_1 - \omega_2\|_0 + \|\omega_2\|_0 \|\omega_1,\alpha - \omega_2,\alpha\|_0 d\alpha \right). \quad (2.47)
\]

Since \( \omega_1(0) = \omega_1(2\pi) = \omega_2(0) = \omega_2(2\pi) = 0 \), by Poincaré inequality, (2.26) and Lemma 2.14, we obtain (2.42).

\[\square\]

**Lemma 2.34.** (See [1] or §A.3 for proof) For \( \psi \in H^r_p \) with \( r \geq 1 \), the operator \([\mathcal{H}, \psi]\) is bounded from \( H^0_p \) to \( H^{-1}_p \). And we have

\[
\|[\mathcal{H}, \psi]f\|_{r-1} \leq C\|f\|_0 \|\psi\|_r,
\]

where \( C \) depends on \( r \).
Lemma 2.35. For $r > \frac{1}{2}$ and $\psi \in H^r_p$, the operator $[\mathcal{H}, \psi]$ is bounded from $H^1_p$ to $H^r_p$, and

$$\| [\mathcal{H}, \psi] f \|_r \leq C \| f \|_1 \| \psi \|_r,$$

where $C$ depends on $r$.

Proof. We know that

$$\| [\mathcal{H}, \psi] f \|_r^2 = \sum_{k \neq 0} |k|^{2r} |\mathcal{H}(\psi f)(k) - \hat{\psi} \mathcal{H} f(k)|^2 + |\mathcal{H}(\psi f)(0) - \hat{\psi} \mathcal{H} f(0)|^2.$$

Since

$$\mathcal{H}(\psi f)(k) = (-i) \text{sgn}(k) \hat{\psi} f(k) = (-i) \text{sgn}(k) \sum_{j=-\infty}^{\infty} \hat{\psi}(j) \hat{f}(k-j), \text{ for } k \neq 0,$$

and

$$\hat{\psi} \mathcal{H} f(k) = \sum_{j=-\infty}^{\infty} \hat{\psi}(j) \hat{\mathcal{H}} f(k-j) = (-i) \sum_{j \neq k} \hat{\psi}(j) \text{sgn}(k-j) \hat{f}(k-j),$$

by Cauchy’s inequality and the inequality $\| gb \|_0 \leq |b|_\infty \|g\|_0 \leq C \| b \|_1 \| g \|_0$, we have

$$\| [\mathcal{H}, \psi] f \|_r^2$$

$$= \sum_{k \neq 0} |k|^{2r} \left| -i \text{sgn}(k) \sum_{j=-\infty}^{\infty} \hat{\psi}(j) \hat{f}(k-j) + i \sum_{j \neq k} \hat{\psi}(j) \text{sgn}(k-j) \hat{f}(k-j) \right|^2$$

$$+ \left| -i \sum_{j \neq 0} \hat{\psi}(j) \text{sgn}(-j) \hat{f}(-j) \right|^2$$

$$= \sum_{k>0} |k|^{2r} \left| 2 \sum_{j>k} \hat{\psi}(j) \hat{f}(k-j) + \hat{\psi}(k) \hat{f}(0) \right|^2$$

$$+ \sum_{k<0} |k|^{2r} \left| 2 \sum_{j<k} \hat{\psi}(j) \hat{f}(k-j) + \hat{\psi}(k) \hat{f}(0) \right|^2 + \left| \sum_{j \neq 0} \hat{\psi}(j) \text{sgn}(-j) \hat{f}(-j) \right|^2.$$
Proposition 2.7 and Lemmas 2.30 and 2.33 apply, there exists constants 
\( y \)

We define
\[
\{ |j|^r \hat{\psi}(j)| \} \in \mathbb{Z} = \Psi \text{ and } \{ \hat{f}(j) \} \in \mathbb{Z} = f.
\]

By Proposition 3.1999 in [23], we know that \( \| f * \Psi \|_2 \leq \| f \|_1 \| \Psi \|_2 \). Hence we obtain the result of the lemma.

\( \square \)

**Note 2.36.** Actually, by cancellation, we have
\[
\mathcal{G}[z]f = \omega_\alpha \left[ \mathcal{H} \cdot \frac{1}{\omega_\alpha} \right] f + 2i\omega_\alpha \mathcal{K}_1[ \omega ] f + \mathcal{G}_2[z] f. \tag{2.48}
\]

**Lemma 2.37.** Assume \( 0 \leq \beta < \Upsilon, f \in H^1_p \) and let \( z^{(1)} \) and \( z^{(2)} \) correspond respectively to \( (\tilde{\theta}^{(1)}, L^{(1)}(t), \tilde{\theta}^{(1)}(0; t)) \) and \( (\tilde{\theta}^{(2)}, L^{(2)}(t), \tilde{\theta}^{(2)}(0; t)) \) (see (2.3)). Further, assume \( \| \tilde{\theta}^{(1)} \|_1, \| \tilde{\theta}^{(2)} \|_1 < \epsilon_1, \| L^{(1)} - 2\pi \| < \frac{1}{2}, \| L^{(2)} - 2\pi \| < \frac{1}{2} \) and \( y^{(1)}(0, t) = \text{Im} z^{(1)}(0, t), y^{(2)}(0, t) = \text{Im} z^{(2)}(0, t) \) belong to \( S_M \). Then for sufficient small \( \epsilon_1 \) and \( \Upsilon \) so that Proposition 2.7 and Lemmas 2.30 and 2.33 apply, there exists constants \( C_1 \) so that
\[
\| \mathcal{G}[z^{(1)}]f - \mathcal{G}[z^{(2)}]f \|_0 \leq C_1 \| f \|_0 \left\{ \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_1 + \beta^2 \left( |L^{(1)}(t) - L^{(2)}(t)| + \| \theta^{(1)} - \theta^{(2)} \|_1 \right) + |y^{(1)}(0, t) - y^{(2)}(0, t)| \right\}. 
\]
Furthermore, if $\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^r$, then for $r \geq 1$,
\[
\| G[z^{(1)}] f - G[z^{(2)}] f \|_r \leq C_1 \| f \|_1 \exp \left( C_2 \left( \| \tilde{\theta}^{(1)} \|_r + \| \tilde{\theta}^{(2)} \|_r \right) \right) \left\{ \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_r \\
+ \beta^2 \left[ |L^{(1)}(t) - L^{(2)}(t)| + \| \theta^{(1)} - \theta^{(2)} \|_r + |y^{(1)}(0, t) - y^{(2)}(0, t)| \right] \right\},
\]
where the constants $C_1$ and $C_2$ depend on $r$.

**Proof.** The proof follows from Lemmas 2.32, 2.33 and 2.35, once we note the relation (2.48).

**Proposition 2.38.** Let $0 \leq \beta < \Upsilon$, where $\Upsilon$ is small enough for Lemmas 2.30 and 2.33 to apply. Assume $z$ corresponds to $(\tilde{\theta}, L(t), \hat{\theta}(0; t))$ through (2.3) for $r \geq 3$ with $\tilde{\theta} \in \dot{H}^s$. Further assume $\| \tilde{\theta} \|_1 < \epsilon_1$, $|L - 2\pi| < \frac{1}{2}$ and $y(0, t) = \text{Im} z(0, t)$ belongs to $S_M$, and $|u_0| < 1$. Then for sufficiently small $\epsilon_1$ and $\Upsilon$, there exists a unique solution $\gamma \in \{ u \in H^{s-2}_p | \hat{u}(0) = 0 \}$ satisfying (B.3). This solution $\gamma$ satisfies the estimates
\[
\| \gamma \|_0 \leq C_0 \left( \sigma \| \tilde{\theta} \|_2 + 1 \right), \\
\| \gamma \|_1 \leq C_1 \left( \sigma \| \tilde{\theta} \|_3 + (1 + \| \tilde{\theta} \|_0) \right),
\]
where $C_1$ depends on $r$.

Let $z^{(1)}$ and $z^{(2)}$ correspond respectively to $(\tilde{\theta}^{(1)}, L^{(1)}(t), \hat{\theta}^{(1)}(0; t))$ and $(\tilde{\theta}^{(2)}, L^{(2)}(t), \hat{\theta}^{(2)}(0; t))$ (see (2.3)). Further assume $\| \tilde{\theta}^{(1)} \| < \epsilon_1$, $\| \tilde{\theta}^{(2)} \| < \epsilon_1$, $|L^{(1)} - 2\pi| < \frac{1}{2}$, $|L^{(2)} - 2\pi| < \frac{1}{2}$ and $y^{(1)}(0, t) = \text{Im} z^{(1)}(0, t)$, $y^{(2)}(0, t) = \text{Im} z^{(2)}(0, t)$ belong to $S_M$. Then for sufficient small $\epsilon_1$ and $\Upsilon$, $\gamma^{(1)} - \gamma^{(2)} \|_0 \leq C \left( \| \theta^{(1)} - \theta^{(2)} \|_2 + |L^{(1)}(t) - L^{(2)}(t)| + \beta^2 |y^{(1)}(0, t) - y^{(2)}(0, t)| \right).$ (2.49)
Further, if \( \tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^r \), then the corresponding \((\gamma^{(1)}, U^{(1)}, T^{(1)})\) and \((\gamma^{(2)}, U^{(2)}, T^{(2)})\) determined from (B.3), (1.19) and (B.4) satisfy

\[
\|\gamma^{(1)} - \gamma^{(2)}\|_{s-2} \leq C_1 \exp \left( C_2 \left( \|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r \right) \left( \|\theta^{(1)} - \theta^{(2)}\|_s + |L^{(1)}(t) - L^{(2)}(t)| + \beta^2 |y^{(1)}(0, t) - y^{(2)}(0, t)| \right) \right), \tag{2.50}
\]

\[
\|U^{(1)} - U^{(2)}\|_{s-2} \leq C_1 \exp \left( C_2 \left( \|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r \right) \left( \|\theta^{(1)} - \theta^{(2)}\|_s + |L^{(1)}(t) - L^{(2)}(t)| + \beta^2 |y^{(1)}(0, t) - y^{(2)}(0, t)| \right) \right), \tag{2.51}
\]

\[
\|T^{(1)} - T^{(2)}\|_{r-1} \leq C_1 \exp \left( C_2 \left( \|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r \right) \left( \|\theta^{(1)} - \theta^{(2)}\|_s + |L^{(1)}(t) - L^{(2)}(t)| + \beta^2 |y^{(1)}(0, t) - y^{(2)}(0, t)| \right) \right), \tag{2.52}
\]

where \(C_1\) and \(C_2\) depend on \(r\).

**Proof.** Since \( \mathcal{F}_1[\omega_0] \gamma = \dot{\gamma}(0) = 0 \) (see (2.1) and Note 2.5), (B.3) implies

\[
[I + a_\mu (\mathcal{F}[z] - \mathcal{F}_1[\omega_0])] \gamma = \frac{2\pi \sigma}{L} \theta_{\alpha\alpha} + \frac{L}{\pi} \left( 1 + \frac{\mu_2}{\mu_1 + \mu_2} \right) \sin (\alpha + \theta(\alpha)). \tag{2.53}
\]

Therefore, if \( \tilde{\theta} \in \dot{H}^2 \), then by Note 2.5, Lemmas 2.30 and 2.37 (note that Lemma 2.37 still holds for \( \mathcal{G}_1 \)) imply

\[
\|\mathcal{F}[z] \gamma - \mathcal{F}_1[\omega_0] \gamma\|_0 \leq \|2i z a K_2[z] \gamma\|_0 + \|\mathcal{G}_1[z] \gamma - \mathcal{G}_1[\omega_0] \gamma\|_0
\leq C_1 \left( \|\tilde{\theta}\|_1 + C_2 \beta \right) \|\gamma\|_0, \tag{2.54}
\]

where \(C_1\) depends on \(\epsilon_1\) and \(C_2\) depends on \(L\). So, for sufficiently small \(\epsilon_1\) and \(\Upsilon > 0\), if \(\|\tilde{\theta}\|_1 \leq \epsilon_1\) and \(0 \leq \beta < \Upsilon\), then

\[
[1 + a_\mu (\mathcal{F}[z] - \mathcal{F}_1[\omega_0])]^{-1}
\]
exists and is bounded independent of any parameters. Therefore, it follows from (2.53) that

\[ \| \gamma \|_0 \leq C_0 \left( \frac{\sigma}{L} \| \tilde{\theta} \|_2 + L \right). \]

Further, we obtain from the second part of Lemma 2.30 and Lemma 2.37,

\[ \| \mathcal{F}[\gamma] - \mathcal{F}_1[\omega_0]|_{s-2} \leq C_1 \left( \exp(C_2 \| \tilde{\theta} \|_{s-2}) \| \tilde{\theta} \|_{s-2} + \beta^2 \exp(C_2 \| \alpha \|_{s-2}) \right) \| \gamma \|_1, \]

where \( C_1 \) and \( C_2 \) depend on \( r \). Therefore, for \( s \geq 3 \), it follows from (B.3) that

\[ \| \gamma \|_{s-2} \leq \frac{2\pi \sigma}{L} \| \tilde{\theta} \|_s + CL(1 + \| \tilde{\theta} \|_{s-3}) \]

\[ + C_1 \left( \exp(C_2 \| \tilde{\theta} \|_{s-2}) \| \tilde{\theta} \|_{s-2} + \beta^2 \exp(C_2 \| \alpha \|_{s-2}) \right) \| \gamma \|_1 \]  

(2.55)

which \( C, C_1 \) and \( C_2 \) depend on \( r \), which implies for sufficiently small \( \epsilon_1 \) and \( \Upsilon \) that

\[ \| \gamma \|_1 \leq \frac{2\pi \sigma}{L} \| \tilde{\theta} \|_3 + CL(1 + \| \tilde{\theta} \|_0). \]  

(2.56)

From (B.3), we obtain

\[ \| \gamma^{(1)} - \gamma^{(2)} \|_{s-2} \leq C \left( \frac{|L^{(1)} - L^{(2)}|}{L^{(1)} L^{(2)}} \right) + \frac{1}{L^{(2)}} \left( \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_s + |\tilde{\theta}^{(1)}(0; \tau) - \tilde{\theta}^{(2)}(0; \tau)| \right) \]

\[ + \left\| \mathcal{F}[z^{(1)}] \gamma^{(1)} - \mathcal{F}[z^{(2)}] \gamma^{(2)} \right\|_{s-2}, \]

and using Lemma 2.37, we have

\[ \left\| \mathcal{F}[z^{(1)}] \gamma^{(1)} - \mathcal{F}[z^{(2)}] \gamma^{(2)} \right\|_{s-2} \leq \left\| \mathcal{F}[z^{(1)}](\gamma^{(1)} - \gamma^{(2)}) - \mathcal{F}_1[\omega_0](\gamma^{(1)} - \gamma^{(2)}) \right\|_{s-2} \]

\[ + \left\| \mathcal{F}[z^{(1)}] \gamma^{(2)} - \mathcal{F}[z^{(2)}] \gamma^{(2)} \right\|_{s-2} \]

\[ \leq C_1 \exp \left( C_2 \| \tilde{\theta}^{(1)} \|_r \right) \left( \| \tilde{\theta}^{(1)} \|_{s-2} + \beta^2 \exp(C_2 \| \alpha^{(1)} \|_{s-2}) \right) \| \gamma^{(1)} - \gamma^{(2)} \|_1 \]

\[ + \left\| \mathcal{F}[z^{(1)}] \gamma^{(2)} - \mathcal{F}[z^{(2)}] \gamma^{(2)} \right\|_{s-2} \]  

(2.57)
with $C_1$ and $C_2$ depending on $r$. Hence by Lemma 2.37 again, the fourth and fifth statements in the proposition follow.

From (1.19), it follows that

$$\|U^{(1)} - U^{(2)}\|_{s-2} = \left\| \frac{\pi}{L^{(1)}} \mathcal{H}[\gamma^{(1)}] - \frac{\pi}{L^{(2)}} \mathcal{H}[\gamma^{(2)}] \right\|_{s-2} + \left\| \frac{\pi}{L^{(1)}} \mathcal{G}[z^{(1)}] \gamma^{(1)} - \frac{\pi}{L^{(2)}} \mathcal{G}[z^{(2)}] \gamma^{(2)} \right\|_{s-2}$$

$$+ (u_0 + 1) \left\| \cos(\alpha + \theta^{(1)}(\alpha)) - \cos(\alpha + \theta^{(2)}(\alpha)) \right\|_{s-2},$$

by Lemmas 2.14 and 2.37, it is easy to obtain (2.51).

Also from (B.4), we have

$$\|T^{(1)} - T^{(2)}\|_{r-1} \leq \left\| (1 + \theta_{1,\alpha})U^{(1)} - (1 + \theta_{2,\alpha})U^{(2)} \right\|_{s-2},$$

by (2.51), we get (2.52).
CHAPTER 3
GLOBAL EXISTENCE FOR THE NON-TRANSLATING BUBBLE

Let us consider the motion of the non-translating bubble ($V_\infty = 0$) in the infinite domain ($\beta = 0$, i.e. without side walls effects) firstly. Hence, in this case, we consider the initial bubble perimeter $L_0 = 2\pi$. We also have the simple expression for $G[z]$:

$$G[z] \gamma = \omega_\alpha \left[ \mathcal{H}, \frac{1}{\omega_\alpha} \right] \gamma + 2i\omega_\alpha \mathcal{K}[\omega] \gamma = G[\omega] \gamma. \quad (3.1)$$

Remark. The equivalence holds since $\hat{\theta}(0; t)$ and $L$ cancel out.

3.1 Galerkin approximation

From the set of equations in (B.1)-(B.5), it is easily seen that $\hat{\theta}(0; t)$ does not affect the evolution of $\tilde{\theta}$ and $L$, so it is convenient to first determine solution $(\tilde{\theta}, L)$; determination of $\hat{\theta}(0; t)$ is then simply reduced to an integration of the equation (B.2). It is convenient to introduce a Galerkin approximations as described in this section.

Definition 3.1. We define a family of Galerkin projections $\{P_n\}_{n=2}^\infty$, where

$$P_n f(\alpha) = \sum_{k=-n,k\neq0,\pm1}^n \hat{f}(k)e^{ik\alpha}, \text{ for all } f = \sum_{-\infty}^\infty \hat{f}(k)e^{ik\alpha}.$$
We define the approximate solution $\tilde{\theta}_n(\alpha, t)$ of order $n$ of the problem in the following way:

$$\tilde{\theta}_n(\alpha, t) = \sum_{k=-n, k \neq 0, \pm 1}^{n} \hat{\theta}_n(k; t)e^{i k \alpha}.$$ 

The approximate equations are

$$\begin{align*}
\frac{\partial \tilde{\theta}_n(\alpha, t)}{\partial t} &= \frac{2\pi}{L_n} P_n \left( U_n, \alpha + T_n \left( 1 + \theta_n, \alpha \right) \right), \\
\frac{dL_n(t)}{dt} &= -\int_{0}^{2\pi} (1 + \theta_n, \alpha) U_n d\alpha,
\end{align*}$$

with $\gamma_n$, $T_n$ and $\hat{\theta}_n(\pm 1; t)$ (where $\hat{\theta}_n(1; t) = \hat{\theta}_n(-1; t)$ because $\theta_n$ is real) determined by

$$\begin{align*}
\left( I + a_{n, \mu} F[\omega_n] \right) \gamma_n(t) &= \frac{2\pi}{L_n} \sigma \theta_n, \\
T_n(\alpha, t) &= \int_{0}^{\alpha} (1 + \theta_n, \alpha') U_n(\alpha') d\alpha' - \frac{\alpha}{2\pi} \int_{0}^{2\pi} (1 + \theta_n, \alpha') U_n(\alpha') d\alpha', \\
\int_{0}^{2\pi} \exp \left( i\alpha + i(\hat{\theta}_n(-1; t)e^{-i\alpha} + \hat{\theta}_n(1; t)e^{i\alpha} + \tilde{\theta}_n(\alpha, t)) \right) d\alpha &= 0,
\end{align*}$$

where

$$\begin{align*}
\theta_n, \alpha &= \tilde{\theta}_n, \alpha - i\hat{\theta}_n(-1; t)e^{-i\alpha} + i\hat{\theta}_n(1; t)e^{i\alpha}, \\
\omega_n(\alpha) &= \int_{0}^{\alpha} e^{i\eta + i\tilde{\theta}_n(-1; t)e^{-i\eta} + i\hat{\theta}_n(1; t)e^{i\eta} + i\hat{\theta}_n(0)} d\eta, \\
U_n &= \frac{\pi}{L_n} \mathcal{H}[\gamma_n] + \frac{\pi}{L_n} \text{Re} \left( \mathcal{G}[\omega_n], \gamma_n \right).
\end{align*}$$
3.2 Main results

Let \( X_n = (\tilde{\theta}_n, L_n) \). The Galerkin approximate equations (C.1)-(C.2) reduce to an ODE in the Banach space \( \dot{H}^r \times \mathbb{R} \):

\[
\frac{dX_n}{dt} = F_n(X_n), \quad X_n(0) = (P_n \theta_0, 2\pi),
\]

(3.2)

where \( F_n(X_n) = (F_{n,1}(X_n), F_{n,2}(X_n)) \) are given by

\[
F_{n,1} = \frac{2\pi}{L_n} P_n (U_{n,\alpha} + T_n (1 + \theta_{n,\alpha})),
\]

(3.3)

\[
F_{n,2} = -\int_0^{2\pi} (1 + \theta_{n,\alpha}) U_n(\alpha) d\alpha.
\]

(3.4)

For the approximate equation (3.2), we have the following results:

**Proposition 3.2.** Assume \( P_n \theta_0 \in B^r_\varepsilon \) for \( r \geq 3 \). For sufficiently small ball size \( \varepsilon \) of \( B^r_\varepsilon \), there exists the unique solution \( X_n \in C^1 ([0, S_n]; V_r) \) to the ODE in Eq. (3.2), where \( S_n \) depends on \( n \), \( r \) and \( \varepsilon \).

**Remark.** We will prove this proposition in §3.5 using Picard theorem (See for instance Chapter 3 in [26]).

**Proposition 3.3.** Assume \( X_n = (\tilde{\theta}_n, L_n) \) is a solution of the initial value problem (3.2). Then there exists \( \varepsilon > 0 \) such that if \( \|P_n \theta_0\|_r < \varepsilon \) for \( r \geq 3 \), then

\[
\|\tilde{\theta}_n(\cdot, t)\|_r \leq \|P_n \theta_0(\cdot)\|_r e^{-\frac{1}{18} \sigma t}, \quad |L_n^3 - 8\pi^3| \leq C \varepsilon \left(1 - e^{-\frac{1}{18} \sigma t}\right),
\]

with a constant \( C \) independent of \( n \) for any time \( t \geq 0 \) where the solution exists.

**Remark.** We will prove the priori estimate in §3.4.
Theorem 3.4. Given the initial condition $X_n(0) \in \mathcal{V}_r$, for any $n \geq 2$ and $r \geq 3$. For sufficiently small $\epsilon$, there exists for all time a unique solution $X_n(t) \in C^1([0, \infty); \mathcal{V}_r)$ to the approximate equation (3.2).

Proof. Proposition 3.2 shows the existence and uniqueness of solutions $X_n$ locally in time. Then by continuation of an autonomous ODE on a Banach space (see [26] in Chapter 3), we know that the unique solution $X_n \in C^1([0, T_0]; \mathcal{V}_r)$ either exists globally in time or $T_0 < \infty$ and $X_n(t)$ leaves the open set $\mathcal{V}_r$ as $t \nearrow T_0$. Suppose $T_0 < \infty$. Combining Propositions 3.3 and 3.11, we know that solution remains in the open set $\mathcal{V}_r$ as $t \nearrow T_0$. Hence it shows that solution to Eq. (3.2) exists globally in time. \hfill \Box

From the solutions to the approximate equation (3.2), we will deduce the existence and uniqueness of solutions to the evolution system (B.1), (B.3)-(B.5) globally in time (Theorem 1.12) using the following lemma and proposition:

Lemma 3.5. For $r \geq 4$, there exists sufficiently small $\epsilon > 0$ such that for any $S > 0$, solutions $X_n = (\tilde{\theta}_n, L_n)$ of the approximate equation (3.2) for different $n$ form a Cauchy sequence in $C\left([0, S]; \hat{H}^1 \times \mathbb{R}\right)$. As $n \to \infty$, the limit $X = (\tilde{\theta}, L) \in C([0, S]; \mathcal{V}_r) \cap C^1\left([0, S]; \hat{H}^{r-3} \times \mathbb{R}\right)$ and is the unique classical solution to (B.1), (B.3)-(B.5) satisfying the initial condition (1.21).

Remark. The proof is given in §3.5. \hfill \Box
Proposition 3.6. Let $(\tilde{\theta}, L) \in C([0, \infty); \mathcal{V}_r)$ be a solution to the system (B.1), (B.3)-(B.5) with the initial condition (1.21) for $r \geq 4$. If $Q_1\theta_0 \in \mathcal{B}^r_\varepsilon$, then for all $t \geq 0$, we have

$$\|\tilde{\theta}(\cdot, t)\|_r \leq \|Q_1\theta_0(\cdot)\|_r e^{-\frac{1}{36}\sigma t},$$

$$|\hat{\theta}(1; t)| = |\hat{\theta}(-1; t)| \leq \frac{1}{2}\|Q_1\theta_0(\cdot)\|_r e^{-\frac{1}{36}\sigma t},$$

$$|L(t) - 2\sqrt{\pi V}| \leq C\|Q_1\theta_0\|_r e^{-\frac{1}{36}\sigma t},$$

$$|\hat{\theta}(0; t) - \hat{\theta}_0(0)| \leq C\|Q_1\theta_0\|_r,$$

where $C$ depends on the diameter of $\mathcal{B}^r_\varepsilon$.

Remark. We will prove Lemma 3.5 and Proposition 3.6 in §3.5. Further, the result above together with Proposition 2.7 shows that $\theta(\alpha, t) - \hat{\theta}(0; t)$ goes to 0 exponentially as $t \to \infty$. □

Proof of Theorem 1.12: This immediately follows from Lemma 3.5 and Proposition 3.6.

3.3 Regularity of $\gamma$

Actually, for nontranslating bubble without side walls effects, i.e. $\beta = 0$ and $V_\infty = 0$, by Proposition 2.38, we have

Proposition 3.7. Assume $\tilde{\theta} \in \dot{H}^r$ for $r \geq 3$. If $\|\tilde{\theta}\|_1 < \epsilon_1$, then for sufficiently
small \( \epsilon_1 \), there exists unique solution \( \gamma \in \{ f \in H_p^{s-2} | \hat{f}(0) = 0 \} \) satisfying (B.3). This solution \( \gamma \) satisfies the estimates

\[
\| \gamma \|_0 \leq \frac{C_0 \sigma}{L} \| \hat{\theta} \|_2,
\]
\[
\| \gamma \|_{s-2} \leq \frac{C_1 \sigma}{L} \exp(C_2 \| \hat{\theta} \|_{s-2}) \| \hat{\theta} \|_r,
\]
\[
\left\| \gamma - \frac{2\pi}{L} \sigma t_{\alpha\alpha} \right\|_s \leq \frac{C_3 \sigma}{L} \exp(C_4 \| \hat{\theta} \|_{r-1}) \| \hat{\theta} \|_s \| \hat{\theta} \|_3,
\]

where \( C_1, C_2, C_3 \) and \( C_4 \) depend on \( r \), but all are independent of \( L \). And for \( s = 3 \), \( C_2 = 0 \).

If \( \gamma^{(1)} \) and \( \gamma^{(2)} \) correspond to \( (\hat{\theta}^{(1)}, L^{(1)}) \) and \( (\hat{\theta}^{(2)}, L^{(2)}) \), each in \( \mathcal{V}_s \), then for \( 3 \leq s \leq r \),

\[
\| \gamma^{(1)} - \gamma^{(2)} \|_{s-2} \leq C \left( \| \hat{\theta}^{(1)} - \hat{\theta}^{(2)} \|_r + |L^{(1)} - L^{(2)}| \right),
\]
\[
\left\| \gamma^{(1)} - \frac{2\pi}{L} \sigma t_{\alpha\alpha}^{(1)} - \gamma^{(2)} + \frac{2\pi}{L} \sigma t_{\alpha\alpha}^{(2)} \right\|_{s-2} \leq C \left( \| \hat{\theta}^{(1)} - \hat{\theta}^{(2)} \|_{s-2} + |L^{(1)} - L^{(2)}| \right),
\]

where \( C \) depends on the diameter of \( \mathcal{V}_s \) and \( r \).

**Lemma 3.8.** Assume \( \hat{\theta} \in \dot{H}^r \) for \( r \geq 3 \). If \( \| \hat{\theta} \|_1 < \epsilon_1 \), for sufficiently small \( \epsilon_1 \), the corresponding \( U \) by (1.19) and (3.1), \( T \) by (B.4), with \( \hat{\theta}(\pm 1; t) \) determined by \( \hat{\theta} \) using (B.5), satisfies the following estimates:

\[
\left\| U - \frac{2\pi^2 \sigma}{L^2} \mathcal{H}[\theta_{\alpha\alpha}] \right\|_0 \leq \frac{C_1 \sigma}{L^2} \| \hat{\theta} \|_1 \| \hat{\theta} \|_2,
\]
\[
\left\| U - \frac{2\pi^2 \sigma}{L^2} \mathcal{H}[\theta_{\alpha\alpha}] \right\|_{s-2} \leq \frac{C_2 \sigma}{L^2} \exp(C_3 \| \hat{\theta} \|_{s-2}) \| \hat{\theta} \|_{s-2} \| \hat{\theta} \|_3,
\]
\[
\left\| U - \frac{2\pi^2 \sigma}{L^2} \mathcal{H}[\theta_{\alpha\alpha}] \right\|_s \leq \frac{C_2 \sigma}{L^2} \exp(C_3 \| \hat{\theta} \|_{r-1}) \| \hat{\theta} \|_s \| \hat{\theta} \|_3,
\]

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\[ \|U\|_{s-2} \leq \frac{C_2 \sigma}{L^2} \exp(C_3 \|\tilde{\theta}\|_{s-2}) \|\tilde{\theta}\|_r, \]
\[ \|T\|_{r-1} \leq \frac{C_2 \sigma}{L^2} \exp(C_5 \|\tilde{\theta}\|_s) \|\tilde{\theta}\|_s, \]

where \( C_1 \) depends on \( \epsilon_1 \), \( C_2 \) and \( C_3 \) depend on \( r \).

If \( U^{(1)} \) and \( U^{(2)} \) (or \( T^{(1)} \) and \( T^{(2)} \)) correspond respectively to \( (\tilde{\theta}^{(1)}, L^{(1)}) \) and \( (\tilde{\theta}^{(2)}, L^{(2)}) \), each in \( \mathcal{V}_s \), then for \( r \geq 3 \),
\[ \|U^{(1)} - U^{(2)}\|_{r-2} \leq C_4 \left( \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r + |L^{(1)} - L^{(2)}| \right), \]
\[ \|U^{(1)} - \frac{2\pi^2}{(L^{(1)})^2} \mathcal{H}[\tilde{\theta}_a^{(1)}] - U^{(2)} - \frac{2\pi^2}{(L^{(2)})^2} \mathcal{H}[\tilde{\theta}_a^{(2)}]\|_{r-2} \leq C_4 \left( \|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r \right) \left( \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_{r-2} + |L^{(1)} - L^{(2)}| \right), \]
\[ \|T^{(1)} - T^{(2)}\|_{r-1} \leq C_4 \left( \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r + |L^{(1)} - L^{(2)}| \right), \]

where \( C_4 \) depends on the diameter of \( \mathcal{V}_s \) and \( r \).

### 3.4 Energy estimate

We will use the \( H^r_p \) norm of \( \tilde{\theta}_n \) to define energy; it is defined by
\[ E_n(t) = \frac{1}{2} \int_0^{2\pi} (D^r_\alpha \tilde{\theta}_n)^2 d\alpha. \]

We first need to estimate the following terms in the evolution equations.

**Lemma 3.9.** Assume \( X_n = (\tilde{\theta}_n, L_n) \) is a solution to the initial value problem (3.2) with \( \tilde{\theta}_n \in B^r_\epsilon \) for \( r \geq 3 \). If the size \( \epsilon \) of the ball \( B^r_\epsilon \) is small enough, then \( \tilde{\theta}_n(., t) \in B^r_\epsilon \)
for all \( t \) for which solution exists. Further, the corresponding energy \( E_n \), as defined above, satisfies the inequality
\[
\frac{dE_n}{dt} \leq -\frac{\pi^2 \sigma}{L_n^2} E_n.
\]

**Proof.** For \( r \geq 3 \), taking the derivative of \( E_n(t) \) with respect to \( t \), we have
\[
\frac{d}{dt} E_n(t) = \int_0^{2\pi} (D^r_\alpha \breve{\theta}_n)(D^r_\alpha \breve{\theta}_n,t)d\alpha.
\]

Using (C.1)-(C.2), on integration by parts we find
\[
\frac{d}{dt} E_n = I_1 + I_2 + I_3 + I_4,
\]
where
\[
I_1 = -\int_0^{2\pi} D^{r+1}_\alpha \breve{\theta}_n D^r_\alpha (P_n U_n) d\alpha,
\]
\[
I_2 = \int_0^{2\pi} D^r_\alpha \breve{\theta}_n D^{r-1}_\alpha (P_n U_n) d\alpha,
\]
\[
I_3 = \int_0^{2\pi} D^r_\alpha \breve{\theta}_n D^{r-1}_\alpha P_n (\theta_n U_n) d\alpha,
\]
\[
I_4 = \int_0^{2\pi} D^r_\alpha \breve{\theta}_n D^r_\alpha P_n (T_n \theta_n U_n) d\alpha.
\]

On using \( \breve{\theta}_n = P_n \theta_n \), we can rewrite
\[
I_1 = -\frac{2\pi^2 \sigma}{L_n^2} \int_0^{2\pi} D^{r+1}_\alpha \breve{\theta}_n D^r_\alpha \mathcal{H}[\breve{\theta}_n] d\alpha - \int_0^{2\pi} D^{r+1}_\alpha \breve{\theta}_n P_n D^r_\alpha \left[ U_n - \frac{2\pi^2 \sigma}{L_n^2} \mathcal{H}[\theta_n] \right] d\alpha.
\]

Using Lemma 3.8 to bound the second term \( I_1 \), it follows from Cauchy-Schwartz inequality that
\[
I_1 \leq -\frac{2\pi^2 \sigma}{L_n^2} \|\breve{\theta}_n\|^{2}_{r+3/2} + \frac{C_1 \sigma}{L_n^2} \exp(C_2 \|\breve{\theta}_n\|_{r-1}) \|\breve{\theta}_n\|_{r} \|\breve{\theta}_n\|_{r+1} \|\breve{\theta}_n\|_{3},
\]

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where $C_1$ and $C_2$ depend on $r$. Consider $I_2$. Applying Lemma 3.8 once again, we obtain

\[
I_2 = \frac{2\pi^2\sigma}{L_n^2} \int_0^{2\pi} D_r^{r} \tilde{\theta}_n \mathcal{H}[\tilde{\theta}_n] d\alpha + \int_0^{2\pi} D_r^{r} \tilde{\theta}_n \mathcal{H}[\tilde{\theta}_n] d\alpha \leq \frac{2\pi^2\sigma}{L_n^2} \left\| \tilde{\theta}_n \right\|_{r+1/2}^2 + C \frac{\sigma}{L_n^2} \exp(C \left\| \tilde{\theta}_n \right\|_{r-1}) \left\| \tilde{\theta}_n \right\|_3^2 \left\| \tilde{\theta}_n \right\|_3,
\]

\[
I_3 = \frac{2\pi^2\sigma}{L_n^2} \int_0^{2\pi} D_r^{r} \tilde{\theta}_n \mathcal{H}[\tilde{\theta}_n] d\alpha \leq \frac{C \sigma}{L_n^2} \left\| \tilde{\theta}_n \right\|_{r+1} \left\| \tilde{\theta}_n \right\|_r \left\| \tilde{\theta}_n \right\|_{r-1} + \frac{C \sigma}{L_n^2} \exp(C \left\| \tilde{\theta}_n \right\|_{r-1}) \left\| \tilde{\theta}_n \right\|_3 \left\| \tilde{\theta}_n \right\|_3,
\]

\[
I_4 = \int_0^{2\pi} D_r^{r} \tilde{\theta}_n P_n \left( \sum_{j=0}^r C_{r,j} D_{\alpha}^{j} T_n D_{\alpha}^{r+1-j} \right) d\alpha \leq \frac{C_1 \sigma}{L_n^2} \left( \exp(C_3 \left\| \tilde{\theta}_n \right\|_3) \left\| \tilde{\theta}_n \right\|_3 \left\| \tilde{\theta}_n \right\|_{r+1} + \exp(C_2 \left\| \tilde{\theta}_n \right\|_{r-1}) \left\| \tilde{\theta}_n \right\|_r \left\| \tilde{\theta}_n \right\|_{r-1} \right),
\]

where $C_1$ and $C_2$ depend on $r$. Adding up $I_1$ through $I_4$, using $(\tilde{\theta}_n, L_n) \in \mathcal{V}_r$ and the fact that $\left\| \tilde{\theta}_n \right\|_{r+1/2} \leq \frac{1}{4} \left\| \tilde{\theta}_n \right\|_{r+3/2}$ since the Fourier 0 and ±1 modes for $\tilde{\theta}_n$ are zero, we obtain for $r = 3$,

\[
\frac{d}{dt} E_n \leq -\frac{3\pi^2\sigma}{2L_n^2} \left\| \tilde{\theta}_n \right\|_{r+3/2}^2 + \frac{C \sigma}{L_n^2} \left\| \tilde{\theta}_n \right\|_{r+1}^2 \left\| \tilde{\theta}_n \right\|_3 \leq \frac{\sigma}{L_n^2} \left\| \tilde{\theta}_n \right\|_{r+3/2}^2 \left( \frac{3}{2} \pi^2 - C \left\| \tilde{\theta}_n \right\|_3 \right) \leq -\frac{3\pi^2\sigma}{2L_n^2} E_n \left( 1 - \frac{2C}{3\pi^2} (2E_n)^{1/2} \right),
\]  

(3.5)
and for $r > 3$,

\[
\frac{d}{dt} E_n \leq -\frac{3\pi^2\sigma}{2L_n^2} \|	ilde{\theta}_n\|^2_{r+3/2} + \frac{C\sigma}{L_n^2} \|	ilde{\theta}_n\|^2_{r+1} \|	ilde{\theta}_n\|_{r-1} \\
\leq -\frac{\sigma}{L_n^2} \|	ilde{\theta}_n\|^2_{r+3/2} \left(\frac{3}{2}\pi^2 - C\|	ilde{\theta}_n\|_{r-1}\right) \leq -\frac{3\pi^2\sigma}{2L_n^2} E_n \left(1 - \frac{2C}{3\pi^2} (2E_n)^{1/2}\right), \tag{3.6}
\]

where $C = C_1 \exp(C_2 \|	ilde{\theta}\|_{r-1})$ with $C_1$ and $C_2$ depending on $r$. It immediately follows that if $1 - \frac{2C}{3\pi^2} (2E_n)^{1/2} > 0$ initially, then $E_n(t)$ decreases in time and $E_n(t) \leq E_n(0)$ for all $t$. This implies that for small enough $\epsilon$, if $\tilde{\theta}_n \in B^r_\epsilon$ initially, it remains there for any $t$ for which the solution exists. More generally, we have

\[
\frac{dE_n}{dt} \leq -\frac{\pi^2\sigma}{L_n^2} E_n.
\]

\[\square\]

**Corollary 3.10.** Assume $\left(\tilde{\theta}_n, L_n\right)$ is a solution to the initial value problem (3.2) with $\tilde{\theta}_n \in B^r_\epsilon$ with $r \geq 3$. Then for sufficiently small ball size $\epsilon$ of $B^r_\epsilon$, we have

\[
\frac{dE_n}{dt} \leq -\frac{\pi^2\sigma}{L_n^2} \|	ilde{\theta}_n\|^2_{r+3/2},
\]

\[
\frac{d\|	ilde{\theta}_n\|^2_{r+1}}{dt} \leq -\frac{\pi^2\sigma}{L_n^2} \|	ilde{\theta}_n\|^2_{r+1}.
\]

**Proof.** The proof of the first statement comes from (3.5) and (3.6).

Replacing $r$ by $r + 1$ in (3.6), we obtain

\[
\frac{d\|	ilde{\theta}_n\|^2_{r+1}}{dt} \leq -\frac{3\pi^2\sigma}{2L_n^2} \|	ilde{\theta}_n\|^2_{r+5/2} + \frac{C\sigma}{L_n^2} \|	ilde{\theta}_n\|^2_{r+2} \|	ilde{\theta}_n\|_r \\
\leq -\frac{3\pi^2\sigma}{2L_n^2} \|	ilde{\theta}_n\|^2_{r+5/2} \left(1 - \frac{2C}{3\pi^2} \|	ilde{\theta}_n\|_r\right), \tag{3.7}
\]

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where \( C = C_1 \exp \left( C_2 \| \tilde{\theta}_n \|_r \right) \) with \( C_1 \) and \( C_2 \) depending only on \( r \). Hence for small enough \( \epsilon \), if \( \tilde{\theta}_n \in \mathcal{B}^\epsilon_r \), then by (3.10), we have
\[
\frac{d\| \tilde{\theta}_n \|^2_{r+1}}{dt} \leq -\frac{\sigma \pi^2}{L_n^2} \| \tilde{\theta}_n \|^2_{r+1}.
\]

\[ \square \]

**Proposition 3.11.** Let \((\tilde{\theta}_n, L_n)\) be continuous solution to (C.1)-(C.2) with \( \tilde{\theta}_n(t) \in \mathcal{B}^\epsilon_r \), with \( r \geq 3 \) and with the initial conditions (3.2) Then for sufficiently small ball size \( \epsilon \) of \( \mathcal{B}^\epsilon_r \), as long as solution exists,
\[
E_n(t) \leq E_n(0) \exp \left[ -\frac{\sigma t}{18} \right], \quad (3.8)
\]
\[
\| \tilde{\theta}_n(\cdot, t) \|^2_{r+1} \leq \| \tilde{\theta}_n(\cdot, 0) \|^2_{r+1} \exp \left[ -\frac{\sigma t}{18} \right], \quad (3.9)
\]
\[
| L_n^3(t) - 8\pi^3 | \leq C \sqrt{E_n(0)} \left( 1 - \exp\left( -\frac{1}{18} \sigma t \right) \right), \quad (3.10)
\]
where \( C \) depends on the diameter of \( \mathcal{B}^\epsilon_r \), not on \( n \).

**Proof.** We note from the evolution equation for \( L_n \) may be rewritten as
\[
L_n^2 \frac{dL_n}{dt} = -L_n^2 \int_0^{2\pi} \left[ U_n - \frac{2\pi^2}{L_n^2} \mathcal{H}[\theta_n, \alpha] \right] d\alpha - L_n^2 \int_0^{2\pi} U_n \theta_n, d\alpha.
\]
Using Lemma 3.8, on integration, it follows that
\[
| L_n^3(t) - (2\pi)^3 | \leq C \int_0^t \| \tilde{\theta}(\cdot, t') \|^2_{3} dt' \leq C \int_0^t E_n(t') dt', \quad (3.11)
\]
where $C$ depends on the diameter of $B_r^\epsilon$. Since $E_n(t) \leq E_n(0)$, it follows that

$$|L_3^3(t)| \leq 8\pi^3 + CE_n(0)t,$$

where $C$ depends on the diameter of $B_r^\epsilon$. Using Lemma 3.9, we obtain preliminary estimates:

$$E_n(t) \leq E_n(0) \exp \left\{ -\frac{3\sigma\pi^2}{CE_n(0)} \left[ (8\pi^3 + CE_n(0)t)^{1/3} - 2\pi \right] \right\}.$$  

Going back to (3.11), it follows that for sufficiently small $E_n(0)$, for any $t$,

$$|L_3^3(t) - 8\pi^3| < 1 \quad (3.12)$$

which implies that $L_n$ cannot escape the interval $(2\pi - 1, 2\pi + 1)$. Going back to Lemma 3.9, this implies that

$$\frac{d}{dt} E_n \leq -\frac{\pi^2\sigma}{(2\pi + 1)^2} E_n \leq -\frac{\sigma}{18} E_n,$$

and therefore (3.8) follows. (3.9) follows from Corollary 3.10 once we use (3.12).

Furthermore, plugging estimates (3.8) into (3.11), we have

$$|L_3^3(t) - 8\pi^3| \leq \frac{18CE_n(0)}{\sigma} \left[ 1 - \exp \left( -\frac{\sigma t}{18} \right) \right].$$

Proof of Proposition 3.3: This follows readily from Lemma 3.9 and Proposition 3.11, since Lemma 3.9 assures that as long as solution $X_n$ to (3.2) exists, corresponding $\tilde{\theta}_n$ does not exit the ball $B_r^\epsilon$ and therefore Proposition 3.11 can be applied to obtain estimates on $E_n(t)$ and $L_n(t)$.  

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3.5 Existence of Solutions

In this section, we demonstrate existence of solutions to initial value problem (3.2). We then show that these solutions converge (as the truncation \( n \) tends to \( \infty \)) to a solution of (B.1), (B.3)-(B.5) with the initial condition (1.21). We demonstrate that this solution to (B.1), (B.3)-(B.5) with initial condition (1.21) is unique and has the same regularity as the initial data.

Definition 3.12. We define

\[
\|X\| = \|u\|_r + |v|
\]

for \( X = (u, v) \in H^r_p \times \mathbb{R} \).

Proof of Proposition 3.2: First we show that the operator \( F_n : \mathcal{V}_r \to H^r_p \times \mathbb{R} \) is bounded, i.e. \( \|F_{n,1}\|_r + |F_{n,2}| < \infty, \forall X_n \in \mathcal{V}_r \). It follows from Lemma 3.8 that

\[
\|P_n U_n,\alpha + P_n T_n (1 + \theta_{n,\alpha})\|_r \leq \|U_{n,\alpha}\|_r + \|T_n\|_r + \|T_n\|_r \|\theta_{n,\alpha}\|_r \\
\leq C \left( \|\hat{\theta}_n\|_{r+3} + \|\hat{\theta}_n\|_{r+1} + \|\hat{\theta}_n\|_{r+1}^2 \right) \leq C n^3 \|\hat{\theta}_n\|_r,
\]

\[
|F_{n,2}| \leq \|1 + \theta_{n,\alpha}\|_0 \|U_n\|_0 \leq C \|\hat{\theta}_n\|_2 \left( 1 + \|\hat{\theta}_n\|_1 \right),
\]

where \( C \) depends on \( n, r \) and the diameter of \( \mathcal{V}_r \).

Consider \( X_n^{(1)}, X_n^{(2)} \in \mathcal{V}_r \). We have

\[
\|F_{n,1}(X_n^{(1)}) - F_{n,1}(X_n^{(2)})\|_r \leq \left\| \left( \frac{2\pi}{L_n^{(1)}} - \frac{2\pi}{L_n^{(2)}} \right) P_n \left( U_n^{(1)}(1 + \theta_{n,\alpha}^{(1)}) \right) \right\|_r \\
+ \frac{2\pi}{L_n^{(2)}} \left\| P_n \left( U_n^{(1)} - U_n^{(2)} \right) \right\|_r \\
+ \frac{2\pi}{L_n^{(2)}} \left\| P_n (T_n^{(1)}(1 + \theta_{n,\alpha}^{(1)}) - T_n^{(2)}(1 + \theta_{n,\alpha}^{(2)})) \right\|_r. \tag{3.13}
\]
It follows from Lemma 3.8 that

\[
\left\| \left( \frac{2\pi}{L_n^{(1)}} - \frac{2\pi}{L_n^{(2)}} \right) P_n \left( U_n^{(1)} + T_n^{(1)} (1 + \theta_n^{(1)}) \right) \right\|_r \leq C n^3 \| L_n^{(1)} - L_n^{(2)} \| \leq c \| X_n^{(1)} - X_n^{(2)} \|,
\]

where \( c \) depends on \( n, r \) and the diameter of \( \mathcal{V}_r \). Further, using Lemma 3.8

\[
|F_{n,2}^{(2)} - F_{n,2}^{(2)}| \leq C \left( \| U_n^{(1)} - U_n^{(2)} \|_0 (1 + \| \tilde{\theta}_n^{(1)} \|_1) + \| U_n^{(2)} \|_0 \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_1 \right)
\leq C \left( |L_1 - L_2| + \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_1 \right) \leq c \| X_n^{(1)} - X_n^{(2)} \|
\]

where \( c \) depends on \( n, r \) and the diameter of \( \mathcal{V}_r \). Therefore, from ODE theory, it follows that there exists local solution \( X_n \in C^1 ([0, S_n]; \mathcal{V}_r) \) over some time interval \( S_n \) that may depend on \( n, r \) and \( \epsilon \).

\textbf{Lemma 3.13. There exists sufficiently small }\epsilon > 0 \text{ such that solutions } X_n = (\tilde{\theta}_n, L_n) \in C^1 ([0, S]; \mathcal{V}_r) \text{ of the initial value problem (3.2) form a Cauchy sequence in } C \left( [0, S]; \dot{H}^1 \times \mathbb{R} \right) \text{ for any } S > 0.\]

\textit{Proof.} We define difference energy function \( E_{mn} \) as

\[
E_{mn} = E_{mn}^1 + (L_n - L_m)^2
\]

where \( E_{mn}^1 = \frac{1}{2} \int_0^{2\pi} \left( D_\alpha (\tilde{\theta}_n - \tilde{\theta}_m) \right)^2 d\alpha \). Notice that \( E_{mn}(0) = E_{mn}^1(0) \). Without loss of generality, we assume \( m > n \) as otherwise we can switch the role of \( m \) and \( n \) in the ensuing argument.
Using the first equation in (C.1),
\[
\frac{dE_{mn}}{dt} = \int_0^{2\pi} D_\alpha (\tilde{\theta}_n - \tilde{\theta}_m) D_\alpha \left( \frac{2\pi}{L_n} P_n U_n - \frac{2\pi}{L_m} P_m U_m \right) d\alpha \\
+ \int_0^{2\pi} D_\alpha (\tilde{\theta}_n - \tilde{\theta}_m) D_\alpha \left( \frac{2\pi}{L_n} P_n \left( T_n (1 + \theta_n, \alpha) \right) - \frac{2\pi}{L_m} P_m \left( T_m (1 + \theta_m, \alpha) \right) \right) d\alpha \equiv I_1 + I_2.
\]

(3.15)

Defining \( \tilde{\theta}_{nm} = \tilde{\theta}_n - \tilde{\theta}_m \), it is clear that
\[
I_1 = -2\pi \left( \frac{1}{L_n} - \frac{1}{L_m} \right) \int_0^{2\pi} D_\alpha^2 \tilde{\theta}_{nm} P_n D_\alpha U_n d\alpha + \frac{2\pi}{L_m} \int_0^{2\pi} D_\alpha \tilde{\theta}_{nm} (P_n - P_m) D_\alpha^2 U_n \\
+ \frac{2\pi}{L_m} \int_0^{2\pi} D_\alpha \tilde{\theta}_{nm} P_m D_\alpha^2 (U_n - U_m) \equiv I_{1,1} + I_{1,2} + I_{1,3}
\]

From estimates in Lemma 3.8 and restrictions due to \( \left( \tilde{\theta}_n, L_n \right), \left( \tilde{\theta}_m, L_n \right) \in V_r \), we obtain
\[
|I_{1,1}| \leq c \epsilon E_{mn}^{1/2} \| \tilde{\theta}_{nm} \|_2,
\]
where \( c \) depends on the diameter of \( V_r \). We note that since \( P_n \theta_n = \tilde{\theta}_n \) and \( P_m \theta_n = \tilde{\theta}_n \), as \( m > n \), we can write \( I_{1,2} \)
\[
I_{1,2} = \frac{2\pi}{L_m} \int_0^{2\pi} D_\alpha \tilde{\theta}_{nm} D_\alpha^2 [P_n - P_m] \left( U_n - \frac{2\pi^2 \sigma}{L_n^2} \mathcal{H}[\theta_{n,\alpha}] \right) d\alpha.
\]

Therefore, using Lemma 3.8,
\[
|I_{1,2}| \leq \frac{c}{n} E_{mn}^{1/2} \left\| U_n - \frac{2\pi^2 \sigma}{L_n^2} \mathcal{H}[\theta_{n,\alpha}] \right\|_3 \leq \frac{C}{n} E_{mn}^{1/2},
\]
where \( C \) depends on the diameter of \( V_r \). Using \( P_m \theta_n = \tilde{\theta}_n \), \( P_m \theta_m = \tilde{\theta}_m \),
\[
I_{1,3} = \frac{2\pi}{L_m} \int_0^{2\pi} D_\alpha \tilde{\theta}_{nm} P_m D_\alpha^2 \left( U_n - \frac{2\pi^2 \sigma}{L_n^2} \mathcal{H}[\theta_{n,\alpha}] - U_m + \frac{2\pi^2 \sigma}{L_m^2} \mathcal{H}[\theta_{m,\alpha}] \right) \\
+ \frac{4\pi^3 \sigma}{L_m L_n^2} \int_0^{2\pi} D_\alpha \tilde{\theta}_{nm} D_\alpha^2 \mathcal{H}[\tilde{\theta}_{nm,\alpha}] - \frac{4\pi^3 \sigma}{L_m} \left( \frac{1}{L_n^2} - \frac{1}{L_m^2} \right) \int_0^{2\pi} D_\alpha^2 \tilde{\theta}_{nm} D \mathcal{H}[\tilde{\theta}_{m,\alpha}] d\alpha.
\]
Integrating by parts the second term in $I_{1,3}$ above and using Lemma 3.8 again, we obtain

$$|I_{1,3}| \leq -\frac{4\pi^3\sigma}{L_mL_n^2} \|\tilde{\theta}_{nm}\|_{5/2} + C\epsilon E_{mn} + C\epsilon E_{mn}^{1/2} \|\tilde{\theta}_{nm}\|_2,$$

where $C$ depends on the diameter of $V_r$. Now using Lemma 3.8, we obtain

$$\frac{d(L_n - L_m)^2}{dt} = 2(L_m - L_n) \int_0^{2\pi} [(U_n - U_m)(1 + \theta_{n,\alpha}) + U_m(\theta_{n,\alpha} - \theta_{m,\alpha})] \, d\alpha$$

$$\leq cE_{nm}^{1/2} (\epsilon \|U_n - U_m\|_0 + \|U_n - \frac{2\pi^2\sigma}{L_n^2} H[\theta_{n,\alpha}] - U_m + \frac{2\pi^2\sigma}{L_m^2} H[\theta_{m,\alpha}]\|_0$$

$$+ \|\tilde{\theta}_{nm}\|_1 \|U_m\|_0) \leq C\epsilon \left(E_{nm} + E_{nm}^{1/2} \|\tilde{\theta}_{nm}\|_2 \right),$$

$C$ depends on the diameter of $V_r$. So for $I_2$, we use the same method as we did for $I_1$ and combine all the terms. So we obtain

$$\frac{dE_{mn}}{dt} \leq -\frac{4\pi^3\sigma}{L_mL_n^2} \|\tilde{\theta}_{nm}\|_{5/2}^2 + \frac{4\pi^3\sigma}{L_mL_n^2} \|\tilde{\theta}_{nm}\|_{3/2}^2 + c\epsilon E_{mn}^{1/2} \|\tilde{\theta}_{nm}\|_2 + \frac{c}{n} E_{mn}^{1/2} + c\epsilon E_{mn}$$

$$\leq -\frac{3\pi^3\sigma}{2(2\pi + 1)^3} \|\tilde{\theta}_{nm}\|_{5/2}^2 + \frac{c}{2\epsilon} \|\tilde{\theta}_{nm}\|_2^2 + \frac{c}{n} E_{mn}^{1/2} + c_1 \epsilon E_{mn},$$

where $c$ and $c_1$ depends on the diameter of $V_r$. Since $\|\tilde{\theta}_{nm}\|_{5/2} \geq \|\tilde{\theta}_{nm}\|_2$, it follows that for $\epsilon$ sufficiently small

$$-\frac{3\pi^3\sigma}{(2\pi + 1)^3} \|\tilde{\theta}_{nm}\|_{5/2}^2 + c\epsilon \|\tilde{\theta}_{nm}\|_2^2 \leq 0.$$

So,

$$\frac{dE_{mn}}{dt} \leq cE_{mn} + \frac{c}{n} E_{mn}^{1/2}.$$

This can be restated as

$$\frac{dE_{mn}^{1/2}}{dt} \leq cE_{mn}^{1/2} + \frac{c}{n}.$$
We solve the differential inequality to see that

\[ E^{1/2}_{mn}(t) \leq E^{1/2}_{mn}(0)e^t + \frac{1}{n}(e^t - 1). \]

Since

\[ E_{mn}(0) = E^1_{mn}(0) \leq \frac{c}{n^2}||\tilde{\theta}_0||^2_r, \]

we have

\[ E^{1/2}_{mn}(t) \leq \frac{c}{n}(||\tilde{\theta}_0||_r + 1)e^t. \]

Thus, solutions do form a Cauchy sequence in \( C\left([0, S]; \hat{H}^1 \times \mathbb{R}\right). \)

\[ \square \]

**Remark.** We now know that the solutions of the initial value problem (3.2), \((\tilde{\theta}_n, L_n)\), approach a limit as \( n \to \infty \) in \( C\left([0, S]; \hat{H}^1 \times \mathbb{R}\right). \) Call this limit \( X = (\tilde{\theta}, L). \)

\[ \square \]

**Note 3.14.** By Proposition 3.3, we know that \( ||\tilde{\theta}_n(\cdot, t)||_r \leq ||Q_1\tilde{\theta}_0||_r \) for all \( t \geq 0. \)

Since \( \hat{H}_r \) is a Hilbert space, its unit ball is weakly compact. Thus, \( \tilde{\theta}_n \rightharpoonup \tilde{\theta} \) in \( \hat{H}_r. \)

Furthermore, by Fatou’s Lemma, we also have

\[ ||\tilde{\theta}||_r \leq \lim inf_{n \to \infty} ||\tilde{\theta}_n||_r \leq ||Q_1\tilde{\theta}_0||_r. \]

**Lemma 3.15.** For \( r \geq 3 \), there exists sufficiently small ball size \( \epsilon \) of \( B^r_\epsilon \) such that as \( n \to \infty \), the limit of the initial value problem (3.2) \( X = (\tilde{\theta}, L) \in C\left((0, S]; \mathcal{V}_r\right) \) for any \( S > 0. \)

**Proof.** Note that estimates in Corollary 3.10 and Proposition 3.11. Since \( L_n \in (2\pi - 1, 2\pi + 1) \), we have

\[ \frac{dE_n}{dt} \leq -\frac{\sigma}{9}||\tilde{\theta}_n||_{r+3/2}^2. \]
It implies
\[ \frac{1}{2}E_n(t) + \frac{\sigma}{9} \int_0^t \| \tilde{\theta}_n \|_{r+3/2}^2 dt \leq \frac{1}{2}E_n(0) \leq \frac{1}{2} \| Q_1 \theta_0 \|_r. \]

Hence \( \tilde{\theta}_n \) is a bounded sequence in \( L^2 \left( [0, \infty), \dot{H}^{r+3/2} \right) \). So, there exists a subsequence that converges weakly, and it is easily argued that the limit can only be \( \tilde{\theta} \). This means that for any interval \((0, S')\) there exists \( S_0 \) in that interval so that \( \| \tilde{\theta}(\cdot, S_0) \|_{r+3/2} < \infty \). Now consider the solution to (B.1), (B.3)-(B.5) with \( S_0 \) as initial time. In particular, \( \tilde{\theta}(\cdot, S_0) \in \dot{H}^{r+1} \cap B^r_{\epsilon} \). Taking \( \tilde{\theta}(\cdot, S_0) \) as initial data in \( \dot{H}^{r+1} \cap B^r_{\epsilon} \), repeating the proof of Proposition 3.2 with \( r+1 \) instead of \( r \), and by Corollary 3.10 and Proposition 3.11, we have global solutions \( \tilde{\theta}^S_{S^0} \in C^1 \left( [S_0, \infty), \dot{H}^{r+1} \cap B^r_{\epsilon} \right) \) for sufficiently small \( \epsilon \). Again, by uniqueness of solutions to the approximate equation (3.2) (Proposition 3.2), these solutions are identical to \( \tilde{\theta}_n \) in their intervals of existence. Also, by Proposition 3.11, we have
\[
\| \tilde{\theta}_n(\cdot, t) \|_{r+1} \leq \| \tilde{\theta}_n(\cdot, S_0) \|_{r+1} e^{-\frac{\sigma}{36}(t-S_0)} \leq \| \tilde{\theta}(\cdot, S_0) \|_{r+1} e^{-\frac{\sigma(t-S_0)}{36}}, \text{ for all } t \geq S_0.
\]

\( (3.16) \)

From interpolation theorem in Sobolev space, we have
\[
\| \tilde{\theta}_m - \tilde{\theta}_n \|_r \leq C \| \tilde{\theta}_m - \tilde{\theta}_n \|_{1-r+1} \| \tilde{\theta}_m - \tilde{\theta}_n \|_{r+1}. \tag{3.17}
\]

By Lemma 3.13 and (3.16), we know that the right side of (3.17) goes to zero uniformly on \([S_0, S]\), as \( n, m \to \infty \) for any \( 1 \leq s < r+1 \). This implies \( X \in C \left( [S_0, S]; \dot{H}^s \times \mathbb{R} \right) \). Since the choice of \( S' \) is arbitrarily small, it follows that \( \tilde{\theta} \in C \left( (0, S], \dot{H}^r \right) \). \( \square \)
Proposition 3.16. (continuity at $t = 0$ in $\dot{H}^r$) For $r \geq 3$, we have

$$\lim_{t \to 0^+} \|\tilde{\theta}(\cdot, t) - Q_1\theta_0\|_r = 0. \quad (3.18)$$

Proof. Replacing $r + 1$ by $r$ in (3.17), using the uniform bound of $\tilde{\theta}_n$ in $\dot{H}^r$ and $\tilde{\theta}_n \in C^1\left([0, \infty); \dot{H}^r\right)$, we find that $\tilde{\theta}_n \to \tilde{\theta}$ in $C\left([0, S]; \dot{H}^r\right)$ as $n \to \infty$ for any $S > 0$, where $1 \leq s < r$.

Let $\eta > 0$ and $\phi \in H^{-r}_p$. For any $r$ satisfying $1 \leq s < r$, choose $\varphi \in H^{-s}_p$ so that

$$\|\phi - \varphi\|_{-r} \leq \frac{\eta}{3}. \quad (3.19)$$

We know that such a $\varphi$ can be found since $H^{-s}_p$ is dense in $H^{-r}_p$. We have

$$\langle \phi, \tilde{\theta}_n \rangle - \langle \phi, \tilde{\theta} \rangle = \langle \phi - \varphi, \tilde{\theta}_n \rangle + \langle \varphi - \phi, \tilde{\theta} \rangle + \langle \varphi, \tilde{\theta}_n - \tilde{\theta} \rangle, \quad (3.20)$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing with dual spaces. The first two terms can be bounded by $\frac{\eta}{3}$ using (3.19) and uniform bounds on $\tilde{\theta}$ and $\tilde{\theta}_n$ in $\dot{H}^r$. For the third term, we choose $n$ large enough so that $\|\tilde{\theta} - \tilde{\theta}_n\|_r \leq \eta/3$. Thus, (3.20) is bounded by $\eta$. Since $\eta$ is arbitrary and these bounds are uniform in time, we conclude that $\tilde{\theta} \in C_W\left([0, S]; \dot{H}^r\right)$. To prove the lemma, it is enough to show $\lim_{t \to 0^+} \|\tilde{\theta}(\cdot, t)\|_r = \|Q_1\theta_0\|_r = 0$.

By Note 3.14, we know $\|\tilde{\theta}(\cdot, t)\|_r \leq \|Q_1\theta_0\|_r$. This means $\limsup_{t \to 0^+} \|\tilde{\theta}(\cdot, t)\|_r \leq \|Q_1\theta_0\|_r$. From the fact that $\tilde{\theta} \in C_W([0, S]; \dot{H}^r)$, we have $\liminf_{t \to 0^+} \|\tilde{\theta}(\cdot, t)\|_r \geq \|Q_1\theta_0\|_r$. Hence, (3.18) holds. This gives us strong right continuity at $t = 0$.

By Lemma 3.15 and Proposition 3.16, we have

Corollary 3.17. For $r \geq 3$, there exists sufficient small ball size $\epsilon$ of $B^r_\epsilon$ such that $X \in C\left([0, S]; V_r\right)$ for any $S > 0$.  

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Proposition 3.18. For \( r \geq 4 \), \( X \) is a classical solution to the initial value problem (B.1), (B.3)-(B.5) with the initial condition (1.21) for any \( S > 0 \), where \( \tilde{\theta} \in C^3([0, S]; \mathbb{T}[0, 2\pi]) \cap C^4([0, S]; C(\mathbb{T}[0, 2\pi])) \) and \( L \in C^1[0, S] \).

Proof. For \( r \geq 4 \), by Sobolev embedding theorem and Corollary 3.17, we know \( X \in C^3([0, S]; \mathbb{T}[0, 2\pi] \times \mathbb{R}) \) and \( \tilde{\theta}_n \to \tilde{\theta} \) as \( n \to \infty \) in \( C^3([0, S]; \mathbb{T}[0, 2\pi]) \cap C([0, S]; \dot{H}^r) \), for \( 1 \leq s < r \).

Since \( G \) is \( C^1 \) in the open ball \( \dot{H}^1 \), \( G(\tilde{\theta}_n) \to G(\tilde{\theta}) \) as \( n \to \infty \). So \( \hat{\theta}(1; t) = G(\tilde{\theta}) \) and \( \tilde{\theta} \) satisfy (B.5). By Proposition 3.7 and (2.57), we see that both \( \{\gamma_n\}_{n=2}^\infty \) and \( \{\mathcal{F}[\omega_n]\gamma_n\}_{n=2}^\infty \) are Cauchy sequences in \( C^3([0, S]; \mathcal{H}^1(\mathbb{T}[0, 2\pi])) \). Hence, it allows us to pass to the limit as \( n \to \infty \) in the equation

\[
(I + a_\mu \mathcal{F}[\omega]) \gamma_n = \frac{2\pi}{L_n} \theta_{n,\alpha\alpha},
\]

and obtain

\[
(I + a_\mu \mathcal{F}[\omega]) \gamma = \frac{2\pi}{L} \theta_{\alpha\alpha}.
\]

By Proposition 3.7 again, we have \( \gamma \in C^4([0, S]; H^{r-2}([0, S]; \dot{H}^s)) \). We also have

\[
\tilde{\theta}_n(\alpha, t) = P_n \theta_0(\alpha) + \int_0^t F_{n,1}(X_n(t')) dt'.
\]

From Lemma 3.8, it follows that \( \{F_{n,1}\}_{n=2}^\infty \) is a Cauchy sequence in \( C^4([0, S]; \dot{H}^0) \).

Replacing \( r + 1 \) by \( r - 3 \) and \( \tilde{\theta}_n \) by \( F_{n,1} \) in (3.17) with the uniform bound of \( F_{n,1} \) in \( \dot{H}^{r-3} \), we see \( \{F_{n,1}\}_{n=2}^\infty \) is a Cauchy sequence in \( C^4([0, S]; \dot{H}^s) \) for \( 0 \leq s < r - 3 \). Hence, we take the limit in (3.21), yielding

\[
\hat{\theta}(\alpha, t) = Q_1 \theta_0(\alpha) + \int_0^t F^1(X(t')) dt',
\]
where $F^1$ is the right-hand side of the first equation in (B.1). This is differentiable in time, giving $\tilde{\theta}_t = F^1(X) \in C([0, S]; C(T[0, 2\pi]))$. Similarly, $L$ satisfies the second equation of (B.1) and $L_t \in C[0, S]$. Thus, $X$ is a classical solution to (B.1), (B.3)-(B.5) with initial condition (1.21). □

**Lemma 3.19.** For $r \geq 3$, there exists sufficiently small ball size $\varepsilon$ of $B^r_\varepsilon$ such that if $X^{(1)} \in \mathcal{V}_r$ and $X^{(2)} \in \mathcal{V}_r$ are solutions to the initial value problem (B.1), (B.3)-(B.5) with the initial condition (1.21) for the interval of time $[0, S]$ with any $S > 0$, and the corresponding initial data $X^{(1)}(\alpha, 0) \in \mathcal{V}_r$ and $X^{(2)}(\alpha, 0) \in \mathcal{V}_r$, then for $0 \leq t \leq S$,

$$
\left\| \tilde{\theta}^{(1)}(\cdot, t) - \tilde{\theta}^{(2)}(\cdot, t) \right\|_1 + |L^{(1)}(t) - L^{(2)}(t)|
\leq \left( \left\| \tilde{\theta}^{(1)}(\cdot, 0) - \tilde{\theta}^{(2)}(\cdot, 0) \right\|_1 + |L^{(1)}(0) - L^{(2)}(0)| \right) \exp\{Ct\}.
$$

**Proof.** This proof is very similar to the proof of Lemma 3.13, and we re-use some notation. Define $E_d$, the energy function for the difference of two solutions, by $E^1_d + (L^{(1)} - L^{(2)})^2$. Here,

$$
E^1_d = \frac{1}{2} \int_0^{2\pi} (D_\alpha(\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}))^2 d\alpha.
$$

We now wish to estimate how this energy changes over time.

$$
\frac{dE^1_d}{dt} = \int_0^{2\pi} D_\alpha(\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}) \frac{2\pi}{L^{(1)}} U^{(1)} - \frac{2\pi}{L^{(2)}} U^{(2)} d\alpha
$$

$$
+ \int_0^{2\pi} D_\alpha(\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}) \frac{2\pi}{L^{(1)}} (T^{(1)}(1 + \theta^{(1)})) - \frac{2\pi}{L^{(2)}} (T^{(2)}(1 + \theta^{(2)})) d\alpha.
$$
Using the same estimates as that in Lemma 3.13, we have

\[
\frac{dE^i_1}{dt} \leq -\frac{4\pi^3}{(L^{(2)})^3}\sigma\left(\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|^{2}_{5/2} - \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|^{2}_{3/2}\right) + \epsilon \left(\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|^{2}_{2} + E_d\right),
\]

with \( c \) depends on the diameter of \( V_r \). We also have

\[
\frac{d(L^{(1)} - L^{(2)})^2}{dt} \leq \epsilon \left(\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|^{2}_{2} + E_d\right),
\]

with \( c \) depends on the diameter of \( V_r \). As what we did in Lemma 3.13, for sufficiently small \( \epsilon \), there exists a positive constant \( B \) such that

\[
\frac{dE_d}{dt} \leq CE_d.
\]

We solve the differential inequality to see that

\[
E_d(t) \leq E_d(0)e^{Ct}.
\]

This proves the theorem.

Hence, uniqueness follows from Lemma 3.19.

**Lemma 3.20.** For \( r \geq 3 \), there exists sufficiently small ball size \( \epsilon \) of \( B^*_r \) such that solution \( X = (\tilde{\theta}, L) \in V_r \) to (B.1) , (B.3)-(B.5) with the initial condition (1.21) is unique in \( \tilde{H}^1 \times \mathbb{R} \).

**Proof of Lemma 3.5:** This follows from Lemmas 3.13, 3.15, 3.20, Corollary 3.17 and Proposition 3.18.

**Proof of Proposition 3.6:**
Since $(\tilde{\theta}_n, L_n)$ converges to $(\tilde{\theta}, L)$ in $C\left((0, S]; \dot{H}^r \times \mathbb{R}\right)$ for any $S > 0$, by Proposition 3.3 and $\|P_n \theta_0\|_r \leq \|Q_1 \theta_0\|_r$, we have

$$\|\tilde{\theta}(\cdot, t)\|_r = \lim_{n \to \infty} \|\tilde{\theta}_n(\cdot, t)\|_r \leq \|Q_1 \theta_0\|_r e^{-\frac{1}{36} \sigma t}. \tag{3.22}$$

By (2.5) and (3.22), the statement for $\hat{\theta}(\pm 1; t)$ hold.

Since the area is invariant with time, by (2.42),

$$|L - 2\sqrt{\pi V}| \leq C\|\tilde{\theta}\|_1$$

with $C$ depending on the diameter of $B_\epsilon$. From (3.22), the result for $L$ follows.

From (B.2), using $2\pi - 1 < L < 2\pi + 1$, we have

$$\left|\hat{\theta}(0; t) - \hat{\theta}_0(0)\right| \leq C \int_0^t \|T(\cdot, t')\|_0 \left(1 + \|\tilde{\theta}(\cdot, t')\|_1\right) dt \leq C \int_0^t \|\tilde{\theta}(\cdot, t')\|_3 dt. \tag{3.23}$$

Hence, plugging estimates (3.22) into (3.23), the result for $\hat{\theta}(0; t)$ holds.
CHAPTER 4
EQUIVALENT EVOLUTION EQUATIONS FOR THE TRANSLATING BUBBLE

Let us study the translating bubble \((V_\infty = 1)\) moving in a Hele-Shaw cell with or without side walls effects \((\beta \neq 0 \text{ or } \beta = 0)\). In this chapter, we will replace the evolution system (B.1)-(B.5) with the initial condition (1.21) by an equivalent system that will be discussed, which gives a constraint for the length \(L\).

When we consider the side walls effect, the evolution of \(y(0,t)\) will affect the system (B.1)-(B.5). So the evolution system (B.1)-(B.5) and (1.15) with the initial conditions (1.21) and \(y(0,0) = y_0\) is equivalent to the following evolution system for \((\tilde{\theta}(\alpha,t), \hat{\theta}(0;t), y(0,t)) \in \dot{H}^r \times \mathbb{R} \times S_M\), where \(\theta = \hat{\theta}(0;t) + \hat{\theta}(-1; t)e^{-i\alpha} + \hat{\theta}(1; t)e^{i\alpha} + \tilde{\theta}\):

\[
\begin{align*}
\tilde{\theta}_t(\alpha, t) &= \frac{2\pi}{L} Q_1(U_\alpha + T(1 + \theta_\alpha)), \\
\frac{d\hat{\theta}(0; t)}{dt} &= \frac{1}{L} \int_0^{2\pi} T(\alpha, t)(1 + \theta_\alpha(\alpha, t)) d\alpha,
\end{align*}
\]

(D.1)

\[
y_t(0, t) = -U(0, t) \sin(\theta(0, t)),
\]

(D.2)

with \(U(\alpha, t)\), a function of \(\gamma, \theta\), determined by (1.6), \(\gamma(\alpha, t), L(t), T(\alpha, t)\) and \(\hat{\theta}(\pm 1; t)\) determined by

\[
(I + a_\mu \mathcal{F}[z]) \gamma = \frac{2\pi}{L} \sigma \theta_\alpha + \frac{L}{\pi} \left(1 + \frac{\mu_2}{\mu_1 + \mu_2} u_0\right) \sin(\alpha + \theta),
\]

(D.3)
\[ L(t) = \sqrt{\frac{8\pi^2V}{\text{Im} \int_0^{2\pi} \omega_0 \omega^* d\alpha}}, \text{ where } V = \frac{L_0^2}{8\pi^2} \text{Im} \left\{ \int_0^{2\pi} \omega_\alpha(\alpha, 0) \omega^*(\alpha, 0) d\alpha \right\}, \] (D.4)

\[ T(\alpha, t) = \int_0^\alpha (1 + \theta_{\alpha'}) U(\alpha') d\alpha' - \frac{\alpha}{2\pi} \int_0^{2\pi} (1 + \theta_\alpha) U(\alpha) d\alpha, \] (D.5)

\[ \int_0^{2\pi} \exp \left( \frac{i\pi}{2} + i\alpha + \hat{\theta}(-1; t)e^{-i\alpha} + i\hat{\theta}(1; t)e^{i\alpha} + i\tilde{\theta}(\alpha, t) \right) d\alpha = 0. \] (D.6)

With the initial condition

\[ \tilde{\theta}(\alpha, 0) = Q_1 \theta_0, \quad \hat{\theta}(0; 0) = 0 \text{ and } y(0, 0) = y_0. \] (4.1)

**Lemma 4.1.** For \( r \geq 3 \) and sufficiently small \( \epsilon_1 \) the following statements (i.) and (ii.) are equivalent:

(i.) \( (\tilde{\theta}, \hat{\theta}(0; t), L(0, t), y(0, t)) \in C^1([0, S], \hat{H}^r \times \mathbb{R}^2 \times S_M) \) satisfies (B.1)-(B.2) and (1.15) with initial conditions (1.20) and \( y(0, 0) = y_0 \), where \( \theta \) is real-valued, \( \|Q_1 \theta\|_1 < \epsilon_1 \) and \( |L - 2\pi| < \epsilon_1 < \frac{1}{2} \), while \( \gamma, T, \hat{\theta}(\pm 1; t) \) and \( U \) are determined by (B.3)-(B.5) and (1.19).

(ii.) \( (\tilde{\theta}, \hat{\theta}(0; t), y(0, t)) \in C^1([0, S], \hat{H}^r \times \mathbb{R} \times S_M) \) satisfies (D.1)-(D.2), initial conditions (4.1), with \( \|\tilde{\theta}\|_1 < \epsilon_1 \), where \( \gamma, T, \hat{\theta}(\pm 1; t), L \) and \( U \) are determined by (D.3)-(D.6), (1.19) and

\[ \theta(\alpha, t) = \tilde{\theta}(\alpha, t) + \hat{\theta}(0; t) + \hat{\theta}(1; t)e^{i\alpha} + \tilde{\theta}(-1; t)e^{-i\alpha}. \]

**Proof.** Let \( (\tilde{\theta}(\alpha, t), \hat{\theta}(0; t), L(t), y(0, t)) \in C^1([0, S]; \hat{H}^r \times \mathbb{R}^2 \times S_M) \) be the solution of the evolution equations (B.1)-(B.2) and (1.15), where \( \gamma, T, \hat{\theta}(\pm 1; t) \) and \( U \) are determined by (B.3)-(B.5) and (1.19) with initial conditions (1.21) and \( y(0, 0) = y_0 \). Using area invariant with time, we express the length of the interface in term of \( V \) and
\[ \tilde{\theta}. \] Thus \((\tilde{\theta}, \hat{\theta}(0; t), y(0, t))\) satisfies the equation (D.1)-(D.2) where \(\gamma, L, T, \hat{\theta}(\pm 1; t)\) and \(U\) are determined by (D.3)-(D.6) and (1.19) with the initial condition (4.1) for \(t \in [0, S]\).

Conversely, suppose that \((\tilde{\theta}(\alpha, t), \hat{\theta}(0; t), y(0, t))\) \(\in C^1([0, S]; H^r \times \mathbb{R} \times S_M)\) be the solution of the system (D.1)-(D.2), where \(\gamma, L, T, \hat{\theta}(\pm 1; t)\) and \(U\) are determined by (D.3)-(D.6) and (1.19) with initial condition (4.1). (D.4) can be written as

\[
\frac{1}{2} \left( \frac{L}{2\pi} \right)^2 \text{Im} \int_0^{2\pi} \omega_\alpha \omega^* d\alpha = V.
\]

Taking the derivative on both sides with respect to \(t\), we have

\[
\frac{L}{2\pi} \text{Im} \int_0^{2\pi} \omega_\alpha \omega^* d\alpha + \frac{L}{2\pi} \text{Im} \int_0^{2\pi} \omega_\alpha \omega_t^* d\alpha = 0. \tag{4.2}
\]

Using integration by parts, we also have

\[
\frac{L}{2\pi} \omega_t = \frac{Li}{2\pi} \int_0^\alpha e^{i\eta + i\theta(\eta)} \theta_t(\eta) d\eta
\]

\[
= i \int_0^\alpha e^{i\eta + i\theta(\eta)} (U_\eta + T(1 + \theta_\eta)) d\eta
\]

\[
= iU(\alpha)e^{i\alpha + i\theta(\alpha)} - iU(0) + \int_0^\alpha e^{i\eta + i\theta(\eta)} (1 + \theta_\eta) Ud\eta
\]

\[
+ T(\alpha)e^{i\alpha + i\theta(\alpha)} - \int_0^\alpha e^{i\eta + i\theta(\eta)} (1 + \theta_\eta) Ud\eta
\]

\[
+ \frac{1}{2\pi} \omega \int_0^{2\pi} (1 + \theta_\alpha) Ud\alpha
\]

\[
= (iU(\alpha) + T(\alpha)) \omega_\alpha - iU(0) + \frac{1}{2\pi} \omega \int_0^{2\pi} (1 + \theta_\alpha) Ud\alpha.
\]

Since the area of the bubble is invariant with time, \(\int_0^{2\pi} U(\alpha, t) d\alpha = 0\). Using \(\hat{T}(0; t) = 0\) and plugging the above formula into (4.2), we have

\[
\frac{L}{2\pi} \text{Im} \int_0^{2\pi} \omega_\alpha \omega^* d\alpha + \frac{1}{2\pi} \text{Im} \int_0^{2\pi} \omega_\alpha \omega^* d\alpha \int_0^{2\pi} (1 + \theta_\alpha) Ud\alpha = 0.
\]
Furthermore, if $\epsilon > 0$ is small enough, then $\text{Im} \int_{0}^{2\pi} \omega_\alpha \omega^* d\alpha \neq 0$. Hence, we obtain

$$L_t = -\int_{0}^{2\pi} (1 + \theta_\alpha) U d\alpha.$$ 

Hence, $(\tilde{\theta}(\alpha, t), \hat{\theta}(0; t), L(t), y(0, t))$ satisfies the system (B.1)-(B.2) and (1.15) for $t \in [0, S]$ with the initial conditions (1.21) and $y(0, 0) = y_0$, where $\gamma, T, \hat{\theta}(\pm 1; t)$ and $U$ are determined by (B.3)-(B.5) and (1.19).

Hence we will consider the solution to the evolution equation (D.1)-(D.2) with the initial condition (4.1) in the space $\dot{H}^r \times \mathbb{R} \times S_M$, where $\gamma, L, T, \hat{\theta}(\pm 1; t)$, and $U$ are determined by (D.3)-(D.6) and (1.19).
CHAPTER 5
GLOBAL EXISTENCE OF THE TRANSLATING BUBBLE WITHOUT WALL EFFECTS

In this section, we consider bubble solutions in the absence of side walls ($\beta = 0$) for near-circular initial shapes. It is readily checked that a time-independent solution that satisfies (D.1), (D.3)-(D.6) is $\theta = 0$, $\gamma = 2 \sin \alpha$, $u_0 = 0$, $V = \pi^1$ this describes a steady circular bubble translating along the positive $x$-axis in the laboratory frame with speed $2 + u_0 = 2$. The uniqueness of this steady state, at least locally in the neighborhood of this solution, is established in a more general context in the steady state analysis of Chapter 6 for $\beta \geq 0$. Note in that case steady bubbles are not circular and move along the positive $x$-axis in the lab frame with speed $2 + u_0(\beta)$.

However, if we overlook the equation for $\dot{\theta}(0; t)$ which only affects parametrization $\alpha$ of the boundary, the remaining equations in (D.1), (D.3)-(D.6) are seen to be satisfied even for $\theta = \theta^{(s)} \equiv \hat{\theta}(0; t)$, $\gamma = \gamma^{(s)} \equiv 2 \sin \left( \alpha + \hat{\theta}(0; t) \right)$, with $u_0 = 0$ and $V = \pi$. Geometrically, this still corresponds to the same translating steady circular bubble, despite the time dependence of $\dot{\theta}(0; t)$ does not affect the circular shape and

\footnote{This is consistent, as it must be, with our choice length scale $L = L^{(s)} = 2\pi$ as the perimeter length of a steady bubble.}
the normal speed \( U = 0 \) at the interface, as it must be in the frame of the steady bubble.

5.1 \textit{A priori} estimates

In studying the time evolution of near-circular interface, it turns out to be more convenient to use the time-dependent \( \gamma^{(s)} \) and define a perturbed vortex sheet strength

\[
\Gamma(\alpha, t) \equiv \gamma(\alpha, t) - \gamma^{(s)}(\alpha, t).
\]

Using (D.3) and the property \( F[\omega_0] \gamma^{(s)} = 0 \) (see Note 2.5), it follows that

\[
(I + a_\mu F[\omega]) \Gamma = -a_\mu [F[\omega] \gamma^{(s)} - F[\omega_0] \gamma^{(s)}] + \frac{2\pi - L}{L} \sigma_{\alpha\alpha} + \sigma_{\alpha\alpha} \nonumber
\]

\[
+ \frac{L - 2\pi}{\pi} \sin(\alpha + \theta) + 2 \left( \sin(\alpha + \theta) - \sin(\alpha + \hat{\theta}(0; t)) \right). \tag{5.1}
\]

Further, from expression for \( \gamma^{(s)} \) and property \( G_1[\omega_0] \gamma^{(s)} = 0 \) (see Note 2.5), the normal velocity \( U \) in (1.19) for \( \beta = 0 \) may be re-expressed as

\[
U = \frac{\pi}{L} \mathcal{H}[\Gamma] + \text{Re} \left[ \frac{\pi}{L} G[\omega] - \frac{1}{2} G[\omega_0] \gamma^{(s)} \right] + \cos(\alpha + \theta) - \cos(\alpha + \hat{\theta}(0; t)) \tag{5.2}
\]

\textbf{Proposition 5.1.} If \( \hat{\theta} \in \dot{H}^r \) with \( ||\hat{\theta}||_1 < \epsilon_1 \) and \( ||\hat{\theta}(0; t)|| < \infty \), then for sufficiently small \( \epsilon_1 \), there exists a unique solution \( \Gamma \in \{ u \in H^{r-2}_p|\bar{u}(0) = 0 \} \) for \( r \geq 3 \) satisfying (5.1). This solution \( \Gamma \) satisfies the estimates

\[
||\Gamma||_0 \leq C ||\bar{\theta}||_2, \tag{5.3}
\]

\[
||\Gamma||_{r-2} \leq C_1 \exp(C_2 ||\bar{\theta}||_{r-2}) ||\bar{\theta}||_r, \tag{5.4}
\]

\[
\left\| \Gamma - \sigma \frac{2\pi}{L} \theta_{\alpha\alpha} \right\|_{r-2} \leq C_1 \exp \left( C_2 ||\bar{\theta}||_{r-2} \right) ||\bar{\theta}||_{r-1}, \tag{5.5}
\]

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where $C_1$ and $C_2$ depend only on $r$.

Let $\Gamma^{(1)}$ and $\Gamma^{(2)}$ correspond to $(\tilde{\theta}^{(1)}, \hat{\theta}^{(1)}(0; t))$ and $(\tilde{\theta}^{(2)}, \hat{\theta}^{(2)}(0; t))$ respectively. Assume $\|\tilde{\theta}^{(1)}\|_1 < \epsilon_1$ and $\|\tilde{\theta}^{(2)}\|_1 < \epsilon_1$. If $\tilde{\theta}^{(1)}, \tilde{\theta}^{(2)} \in \dot{H}^r$ with $r \geq 3$, then for sufficient small $\epsilon_1$, 

$$\|\Gamma^{(1)} - \Gamma^{(2)}\|_{r-2} \leq C_1 \exp \left( C_2 (\|\tilde{\theta}^{(1)}\|_r + \|\tilde{\theta}^{(2)}\|_r) \left( \|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_r + |\hat{\theta}^{(1)}(0; t) - \hat{\theta}^{(2)}(0; t)| \right) \right),$$

(5.6)

where $C_1$ and $C_2$ depend on $r$ alone.

Proof. In statements (2.49) and (2.50) in Proposition 2.38, we take $\beta = 0$, $\gamma^{(2)} = \gamma$, $\tilde{\theta}^{(1)} = \tilde{\theta}$, $L^{(1)} = L$, $\gamma^{(2)} = \gamma^{(s)} = 2 \sin \left( \alpha + \hat{\theta}(0; t) \right)$, $\tilde{\theta}^{(2)} = 0$, $L^{(2)} = 2\pi$

and use Lemma 2.37 to obtain statements (5.3) and (5.4).

(5.1) can be written as

$$\Gamma - \sigma \frac{2\pi}{L} \theta_{\alpha\alpha} = -a_\mu [\mathcal{F}[\omega] \gamma - \mathcal{F}[\omega_0] \gamma] + \frac{L - 2\pi}{\pi} \sin (\alpha + \theta) + 2 \left( \sin (\alpha + \theta) - \sin (\alpha + \hat{\theta}(0; t)) \right).$$

Hence, by Lemma 2.37 with $\beta = 0$, Lemmas 2.33 and 2.16 (see Note 2.17), we obtain (5.5).

The statement (5.6) follows in a similar manner from (2.50).

\[\square\]

When there is no side walls effect ($\beta = 0$), it is readily checked from (D.1), (D.3)-(D.6) that $y(0, t)^2$ does not affect the evolution of $\tilde{\theta}$ or $\hat{\theta}(0; t)$. So, in this section

\[\text{We ignored in all cases } x(0, t) = \text{Re } z(0, t), \text{ which does not affect the evolution of the shape function } \theta.\]
we will ignore (D.2) all together, since translations do not affect the shape and if necessary, \( y(0, t) \) can be calculated from (D.2) at the end.

The main result in this section is the following proposition:

**Proposition 5.2.** For \( \sigma > 0 \), there exists \( \epsilon > 0 \) such that for \( r \geq 3 \), if \( \| Q_1 \theta_0 \|_r < \epsilon \), then there exists a unique solution \( (\tilde{\theta}, \hat{\theta}(0; t)) \in C([0, \infty), \dot{H}^r \times \mathbb{R}) \) to the Hele-Shaw problem (D.1), (D.3)-(D.6) satisfying initial conditions (4.1). Further, \( \| \tilde{\theta} \|_r \), \( |\dot{\theta}(\pm1; t)| \) and \( |L - 2\pi| \) each decay exponentially as \( t \to \infty \), \( |\hat{\theta}(0; t)| \) remains finite.

Thus the circular translating steady bubble is asymptotically stable for sufficiently small initial disturbances in the \( H^r \) space.

**Note 5.3.** Proof of Proposition 5.2 is given at the end of §5.4. Note also Proposition 5.2 and Lemma 2.9 imply Theorem 1.13.

Before we consider global solutions to the system (D.1), (D.3)-(D.6) for initial condition (4.1). First some additional estimates are needed for the terms that arise in the evolution equations.

**Definition 5.4.** We define operator \( \mathfrak{W} \) so that

\[
\mathfrak{W}[f](\alpha) = \frac{1}{2\pi} \int_0^{2\pi} \gamma^{(s)}(\alpha') \int_0^\alpha \frac{Q_0(f(\zeta)\omega_0(\zeta)) d\zeta}{\omega_0(\alpha) - \omega_0(\alpha')} d\alpha'.
\]

**Lemma 5.5.** For \( f \in H^k_p \), there exists constant \( C_1 \) only dependent on \( k \) so that

\[
\| \mathfrak{W}[f] \|_k \leq C_1 \| f \|_k
\]

**Proof.** We take \( \omega_{0\alpha} \) and \( Q_0[f\omega_{0\alpha}] \) to be \( g(\alpha) \) and \( f(\alpha) \) in Lemma 2.26 respectively and define \( h = \gamma^{(s)} \). Note that for this choice, the condition \( \hat{g}(0) = 0 = \hat{f}(0) \) as well
as the lower bound constraint on $g = \omega_{0,\alpha} = e^{i\alpha}$ is satisfied. The proof follows since $\| \cdot \|_{L^\infty}$ bounds in $\alpha$ on $D^j_D[\mathfrak{f}]$ imply $\| \cdot \|_j$ bounds in the Lemma statement.

From (5.1), after some algebraic manipulation, it follows that

$$\Gamma(\alpha, t) = \frac{2\pi}{L} \sigma \theta_{\alpha\alpha} + \Gamma_L(\alpha, t) + N_1(\alpha, t) + N_2(\alpha, t) + N_3(\alpha, t), \quad (5.7)$$

where

$$\Gamma_L(\alpha, t) = 2Q_0\theta(\alpha, t) \cos (\alpha + \dot{\theta}(0; t)) + \frac{L - 2\pi \pi}{\alpha} \sin (\alpha + \dot{\theta}(0; t)) - a_\mu \Re \left( \frac{\partial}{\partial \alpha} \{ \mathfrak{f}[Q_0\theta](\alpha) \} \right), \quad (5.8)$$

$$N_1 = a_\mu \Re \left( \frac{1}{i} G[\omega] \Gamma + \frac{1}{i} G[\omega_0] \Gamma \right) + \frac{L - 2\pi \pi}{\alpha}(\sin(\alpha + \theta) - \sin(\alpha + \dot{\theta}(0; t)))$$

$$+ a_\mu \Re \left( i(e^{iQ_0\theta} - 1) \left\{ \frac{\omega_{0,\alpha}}{\omega_{\alpha}} [G[\omega]\gamma^{(s)} - G[\omega_0]\gamma^{(s)}] - 2 \left( \frac{\omega_{0,\alpha}}{\omega_{\alpha}} - 1 \right) \cos (\alpha + \dot{\theta}(0; t)) \right\} \right)$$

$$+ 2\Xi_s \left[ Q_0\theta; \dot{\theta}(0; t) \right] \quad (5.9)$$

$$N_2 = -2a_\mu \Re \left( \frac{1}{i} \frac{\partial}{\partial \alpha} \{ \mathfrak{f}[\Xi_s\theta][Q_0\theta](\alpha) \} \right), \quad (5.10)$$

and

$$N_3 = \Re \left( \frac{a_\mu \omega_{0,\alpha}}{i\pi \omega_{\alpha}} \int_{0}^{\alpha + \pi} \gamma^{(s)}(\alpha') q_1[\omega - \omega_0](\alpha, \alpha') \left[ \frac{q_2[\omega](\alpha, \alpha')}{q_1[\omega](\alpha, \alpha') - q_2[\omega_0](\alpha, \alpha')} \right] \right)$$

$$+ \Re \left( \frac{a_\mu \omega_{0,\alpha}}{i\pi} \left[ \frac{1}{\omega_{\alpha}} - \frac{1}{\omega_{0,\alpha}} \right] \int_{0}^{\alpha + \pi} \frac{\Gamma^{(s)}(\alpha') q_1[\omega - \omega_0](\alpha, \alpha') \omega_{0,\alpha}(\alpha)}{q_1[\omega_0](\alpha, \alpha') \omega_{0,\alpha}(\omega_0(\alpha))} \right). \quad (5.11)$$

Further, from (1.19) it follows that normal velocity

$$U(\alpha, t) = \frac{2\pi^2}{L^2} \sigma \mathcal{H}(\theta_{\alpha\alpha})(\alpha) + U_L(\alpha, t) + \frac{1}{2} \mathcal{H} \left( N_1(\cdot) + N_2(\cdot) + N_3(\cdot) \right)(\alpha) + N_4(\alpha), \quad (5.12)$$
where

\[ U_L(\alpha, t) = \frac{1}{2} \mathcal{H}[\Gamma_L](\alpha, t) + \frac{L - 2\pi}{L} \cos (\alpha + \hat{\theta}(0; t)) - Q_0 \theta \sin (\alpha + \hat{\theta}(0; t)) \]

\[ - \text{Re} \left( \frac{1}{i} \frac{\partial}{\partial \alpha} \mathfrak{M}[Q_0 \theta](\alpha) \right), \]

and

\[ N_4(\alpha) = \text{Re} \left( \pi \mathcal{G}[\omega] \Gamma - \frac{1}{2} \mathcal{G}[\omega_0] \Gamma \right) + \frac{2\pi - L}{L} \text{Re} \left( \frac{1}{2} \left[ \mathcal{G}[\omega] \gamma^{(s)} - \mathcal{G}[\omega_0] \gamma^{(s)} \right] \right) \]

\[ + \text{Re} \left( \left( e^{i Q_0 \theta} - 1 \right) \left\{ \frac{\omega_0}{2\omega_0} \left( \mathcal{G}[\omega] \gamma^{(s)} - \mathcal{G}[\omega_0] \gamma^{(s)} \right) - \left( \frac{\omega_0}{\omega_0} - 1 \right) \cos (\alpha + \hat{\theta}(0; t)) \right\} \right) \]

\[ - \text{Re} \left( \frac{\omega_0}{2\pi} \text{PV} \int_{\alpha - \pi}^{\alpha + \pi} \gamma^{(s)}(\alpha') \frac{g_1[\omega - \omega_0](\alpha, \alpha')}{q_1[\omega_0](\alpha, \alpha')} \left( \frac{1}{\omega(\alpha) - \omega(\alpha')} - \frac{1}{\omega_0(\alpha) - \omega_0(\alpha')} \right) d\alpha' \right) \]

\[ + \frac{2\pi - L}{2L} \mathcal{H}[\Gamma - 2\pi \sigma \theta_{\alpha\alpha}] + \text{Re} \left( \frac{\partial}{\partial \alpha} \mathfrak{M}[\Xi_e[Q_0 \theta]](\alpha) \right) + \Xi_e \left[ Q_0 \theta; \hat{\theta}(0; t) \right] \quad (5.13) \]

Using (5.12) and (D.5), it follows that

\[ \hat{\theta}_t = \frac{2\pi}{L} Q_1 (U_\alpha + T(1 + \theta_\alpha)) = \mathcal{A}[\hat{\theta}](\alpha, t) + \mathcal{A}_N[\hat{\theta}, \hat{\theta}(0; \cdot), L](\alpha, t) + \mathfrak{N}[\hat{\theta}, \hat{\theta}(0; \cdot)](\alpha, t), \]

(5.14)

where the operators \( \mathcal{A} \) and \( \mathcal{A}_N \) acting on real valued functions \( \hat{\theta} \in \mathcal{H}^r \) for \( r \geq 3 \) are defined by

\[ \mathcal{A}[\hat{\theta}](\alpha, t) = \sum_{k=2}^{\infty} e^{ika} \left( -\sigma d(k) \hat{\theta}(k; t) + m(k) \hat{\theta}(k + 1; t) \right) + \text{c.c.}, \quad (5.15) \]

\[ \mathcal{A}_N[\hat{\theta}, \hat{\theta}(0; \cdot), L](\alpha, t) \]

\[ = \sum_{k=2}^{\infty} e^{ika} \left\{ \left( -\frac{8\pi^3}{L^3} + 1 \right) \sigma d(k) \hat{\theta}(k; t) + e^{-i \hat{\theta}(0; t)} \left( \frac{2\pi}{L} - 1 \right) m(k) \hat{\theta}(k + 1; t) \right\} \]

\[ + \left( e^{-i \hat{\theta}(0; t)} - 1 \right) \sum_{k=2}^{\infty} e^{ika} m(k) \hat{\theta}(k + 1; t) + \text{c.c.}, \quad (5.16) \]
where c.c. indicates complex conjugate of explicitly shown terms on the right side in each of (5.15), (5.16) and
\[ d(k) = \frac{1}{2} k(k^2 - 1), \quad m(k) = (1 + a_\mu) \frac{(k^2 - 1)(k + 1)}{k(k + 2)}, \] (5.17)

and
\[
\mathfrak{M}[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)](\alpha, t) = \frac{2\pi}{L} \mathcal{Q}_1 \left\{ \left( \frac{1}{2} \mathcal{H} \left( N_1(\cdot) + N_2(\cdot) + N_3(\cdot) \right)(\alpha) + N_4(\alpha) \right) \alpha + N_5(\alpha) \right\},
\] (5.18)

where
\[
N_5(\alpha) = \int_{\alpha}^{0} \left[ \frac{1}{2} \mathcal{H} \left( N_1(\cdot) + N_2(\cdot) + N_3(\cdot) \right)(\alpha') + N_4(\alpha') \right] d\alpha' \\
- \frac{\alpha}{2\pi} \int_{0}^{2\pi} \left[ \frac{1}{2} \mathcal{H} \left( N_1(\cdot) + N_2(\cdot) + N_3(\cdot) \right)(\alpha) + N_4(\alpha) \right] d\alpha \\
+ \int_{0}^{\alpha} \theta_\alpha(\alpha') U(\alpha') d\alpha' - \frac{\alpha}{2\pi} \int_{0}^{2\pi} \theta_\alpha(\alpha) U(\alpha) d\alpha \\
+ \left( \int_{0}^{\alpha} \theta_\alpha(\alpha') U(\alpha') d\alpha' - \frac{\alpha}{2\pi} \int_{0}^{2\pi} \theta_\alpha(\alpha) U(\alpha) d\alpha \right) \theta_\alpha(\alpha).
\] (5.19)

It is straightforward to check from (5.17) that for any \( k \geq 2 \),
\[
\frac{3}{8} k^3 \leq d(k) \leq \frac{1}{2} k^3, \quad \frac{9}{16} (1 + a_\mu) k \leq m(k) \leq (1 + a_\mu) k.
\] (5.20)

After some calculation, we also find from (D.1) that
\[
\hat{\theta}_t(0; t) = \frac{1}{L} \int_{0}^{2\pi} T(\alpha, t) \left( 1 + \theta_\alpha(\alpha, t) \right) d\alpha = \mathfrak{M}_0[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)](t),
\] (5.21)

\(^3\)Note that while \( L \) is shown as an independent argument of \( \mathcal{A}_N \), in the evolution equation (5.14), itself, \( L \) is determined from \( \tilde{\vartheta} \) through (D.4) and (2.2).
where the functional $\mathfrak{N}_0$ of real valued $(\tilde{\theta}(\alpha, t), \hat{\theta}(0; t))$ is defined by

$$
\mathfrak{N}_0[\tilde{\theta}, \hat{\theta}(0; \cdot)](t) = \int_0^{2\pi} \int_0^\alpha \left( \frac{2\pi^2}{L^3} - \frac{1}{4\pi} \right) \sigma H(\theta_{\alpha\alpha})(\alpha') + \left( \frac{1}{L} - \frac{1}{2\pi} \right) U_L(\alpha') d\alpha' d\alpha
$$

$$
- \pi \left( \frac{1}{L} - \frac{1}{2\pi} \right) \int_0^{2\pi} U_L(\alpha) d\alpha + \frac{1}{L} \int_0^{2\pi} N_5(\alpha) d\alpha + B_0[\tilde{\theta}, \hat{\theta}(0; \cdot)](t), \quad (5.22)
$$

with the functional $B_0$ defined by

$$
B_0[\tilde{\theta}, \hat{\theta}(0; \cdot)](t) = \sum_{k=1}^\infty \left( \frac{\sigma k}{2} \hat{\theta}(k; t) - e^{-i\hat{\theta}(0; t)} (1 + a_\mu) \frac{k + 1}{k(k + 2)} \hat{\theta}(k + 1; t) \right) + c.c. \quad (5.23)
$$

With respect to the functional $B_0[\tilde{\theta}(\alpha, t), \hat{\theta}(0; t)]$, the following statement readily follows.

**Lemma 5.6.** With $\tilde{\theta} \in \dot{H}_1$ and $\|\tilde{\theta}\|_2 < \epsilon$ sufficiently small, then

$$
\left| B_0[\tilde{\theta}(\alpha, t), \hat{\theta}(0; t)] \right| \leq C\|\tilde{\theta}\|_2.
$$

Further $B^{(1)}_0$ and $B^{(2)}_0$ correspond to respectively to $(\tilde{\theta}^{(1)}, \tilde{\theta}^{(1)}(0; t))$ and $(\tilde{\theta}^{(2)}, \tilde{\theta}^{(2)}(0; t))$, then

$$
|B^{(1)}_0 - B^{(2)}_0| \leq C \left\{ \|\tilde{\theta}^{(1)}(\cdot, t) - \tilde{\theta}^{(2)}(\cdot, t)\|_2 + |\hat{\theta}^{(1)}(0; t) - \hat{\theta}^{(2)}(0; t)||\tilde{\theta}^{(1)}(\cdot, t)\|_1 \right\}.
$$

**Proof.** The proof follows easily from the expression (5.23), and Proposition 2.7 relating $\hat{\theta}(1; t)$ to $\tilde{\theta}$. \qed

---

Note that the Fourier component $\hat{\theta}(1; t)$ appearing in the summation is being determined indirectly from $\tilde{\theta}$ through (D.6) (see Proposition 2.7).
Lemma 5.7. If for \( r \geq 3 \), \( \tilde{\theta} \in \dot{H}^r \) and \( \| \tilde{\theta} \|_1 < \epsilon_1 \), then for sufficiently small \( \epsilon_1 \), \( N_j \), \( j = 1, \cdots, 5 \), defined by (5.9), (5.10), (5.11), (5.13) and (5.19) satisfy
\[
\| N_j \|_{r-1} \leq C_1 \exp(C_2 \| \tilde{\theta} \|_{r-1}) \| \tilde{\theta} \|_{r-1} \| \tilde{\theta} \|_r, \tag{5.24}
\]
where \( C_1 \) and \( C_2 \) depend only on \( r \). Further let \( N_j^{(1)} \) and \( N_j^{(2)} \) correspond to \( (\tilde{\theta}^{(1)}, \hat{\theta}^{(1)}(0; t)) \) and \( (\tilde{\theta}^{(2)}, \hat{\theta}^{(2)}(0; t)) \) respectively, each in \( \dot{H}^r \times \mathbb{R} \) with \( \| \tilde{\theta}^{(1)} \|_1 \) and \( \| \tilde{\theta}^{(2)} \|_1 < \epsilon_1 \). Then for sufficiently small \( \epsilon_1 \),
\[
\left\| N_j^{(1)} - N_j^{(2)} \right\|_{r-1} \leq C_1 \exp \left( C_2 (\| \tilde{\theta}^{(1)} \|_{r-1} + \| \tilde{\theta}^{(2)} \|_{r-1}) \right) \left\{ (\| \tilde{\theta}^{(1)} \|_{r-1} + \| \tilde{\theta}^{(2)} \|_{r-1}) \right. \\
\times \left( \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_r + \| \hat{\theta}^{(1)}(0; t) - \hat{\theta}^{(2)}(0; t) \|_r \right) + (\| \tilde{\theta}^{(1)} \|_r + \| \tilde{\theta}^{(2)} \|_r) \| \tilde{\theta}^{(1)} - \tilde{\theta}^{(2)} \|_{r-1} \right\}, \tag{5.25}
\]
Proof. For estimating \( N_1 \) we use Lemmas 2.16 (see Note 2.17), 2.37, 2.33 (in particular (2.42) for \( L^{(1)} = L, L^{(2)} = 2\pi \), the latter corresponding to \( \tilde{\theta} = 0 \)) and Proposition 5.1. For \( N_2 \), we use Lemmas 2.16 (see Note 2.17) and 5.5. For \( N_3 \), we use Lemmas 2.14, 2.25, 2.27 and 2.29 together with Cauchy-Schwartz inequality to get the desired bound.

For (5.8), by Lemmas 2.33 and 5.5, we have
\[
\| \Gamma_L \|_{r-3} \leq C \| \tilde{\theta} \|_{r-3}. \tag{5.26}
\]
For \( N_4 \) we rely on (5.26), Lemmas 2.16 (see Note 2.17), 2.33 (equation (2.42) in particular), 2.37, 5.5 and Proposition 5.1. \( N_5 \) uses bounds similar to \( N_j \) for \( j = 1, \cdots, 4 \) as well as bounds on \( U \) (In Proposition 2.38, we choose \( U^{(1)} = U, U^{(2)} = 0, \tilde{\theta}^{(1)} = \hat{\theta}, \tilde{\theta}^{(2)} = 0 \), and \( L^{(1)} = L, L^{(2)} = 0 \) in (2.51) to get the bound of \( U \)).
Corollary 5.8. If for \( s \geq 3, \tilde{\theta} \in \dot{H}^r \) and \( \|\tilde{\theta}\|_1 < \varepsilon_1 \), then for sufficiently small \( \varepsilon_1 \), the function \( \mathcal{N} \), and the functional \( \mathcal{N}_0 \), defined in (5.18) and (5.22) satisfy the following estimates

\[
\|\mathcal{N}\|_{r-1} \leq C_1 \exp(C_2 \|\tilde{\theta}\|_s)\|\tilde{\theta}\|_s\|\tilde{\theta}\|_{r+1}, \tag{5.27}
\]

\[
|\mathcal{N}_0| \leq C_1 \exp(C_2 \|\tilde{\theta}(:, t)\|_3)\|\tilde{\theta}(\cdot, t)\|_3^2 + C_1\|\tilde{\theta}(\cdot, t)\|_2.
\]

where \( C_1 \) and \( C_2 \) depend only on \( r \). Further, let \( (\mathcal{N}^{(1)}, \mathcal{N}_0^{(1)}) \) and \( (\mathcal{N}^{(2)}, \mathcal{N}_0^{(2)}) \) correspond to \( (\tilde{\theta}^{(1)}, \tilde{\theta}^{(1)}(0; t)) \) and \( (\tilde{\theta}^{(2)}, \tilde{\theta}^{(2)}(0; t)) \) respectively, each in \( \dot{H}^r \times \mathbb{R} \) with \( \|\tilde{\theta}^{(1)}\|_1 \) and \( \|\tilde{\theta}^{(2)}\|_1 < \varepsilon_1 \). Then for sufficiently small \( \varepsilon_1 \),

\[
\|\mathcal{N}^{(1)} - \mathcal{N}^{(2)}\|_{r-1} \leq C_1 \exp\left(C_2\left(\|\tilde{\theta}^{(1)}\|_s + \|\tilde{\theta}^{(2)}\|_s\right)\right)\left\{\|\tilde{\theta}^{(1)}\|_s + \|\tilde{\theta}^{(2)}\|_s\right\}
\times \left(\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_{r+1} + \|\tilde{\theta}^{(1)}(0; t) - \tilde{\theta}^{(2)}(0; t)\|_3\right) + \left(\|\tilde{\theta}^{(1)}\|_{r+1} + \|\tilde{\theta}^{(2)}\|_{r+1}\right)\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_s\right\}, \tag{5.28}
\]

\[
\left|\mathcal{N}_0^{(1)} - \mathcal{N}_0^{(2)}\right| \leq C_1 \exp\left(C_2\left(\|\tilde{\theta}^{(1)}\|_3 + \|\tilde{\theta}^{(2)}\|_3\right)\right)\left\{\|\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}\|_3^2 + \|\tilde{\theta}^{(1)}\|_3\|\tilde{\theta}^{(1)}(0; t) - \tilde{\theta}^{(2)}(0; t)\|_3\right\}, \tag{5.29}
\]

where \( C_1 \) and \( C_2 \) depend on \( r \).

Proof. On using Lemmas 5.6 and 5.7, the proof follows from the expressions of \( \mathcal{N} \) and \( \mathcal{N}_0 \) in terms of \( N_1, \cdots, N_5 \). \( \square \)

5.2 Weighted Sobolev Space and Estimates

For any surface tension \( \sigma \), we choose the integer \( K \) by
(a) If $\sigma \geq 1$, then $K = 2$;

(b) If $0 < \sigma < 1$, then $K \geq \sqrt{1 + \frac{4}{\sigma}}$.

We define the weight $w(\sigma, k)$ so that

$$w(\sigma, k) = \sigma^{K-|k|} \text{ for } 2 \leq |k| \leq K(\sigma), w(\sigma, k) = 1 \text{ for } |k| > K(\sigma). \quad (5.30)$$

**Definition 5.9.** Let $r \geq 0$. We define a family of weighted Sobolev norm in $\dot{H}^r$ by

$$\|u\|_{w,r}^2 = \sum_{k=2}^{\infty} w^2(\sigma, k)|k|^{2r}|\hat{u}(k)|^2 + \sum_{k=-2}^{-\infty} w^2(\sigma, k)|k|^{2r}|\hat{u}(k)|^2, \quad (5.31)$$

and the corresponding inner-product:

$$(v, u)_{w,r} = \sum_{k=2}^{\infty} w^2(\sigma, k)|k|^{2r}\hat{v}^*(k)\hat{u}(k) + \sum_{k=-2}^{-\infty} w^2(\sigma, k)|k|^{2r}\hat{v}^*(k)\hat{u}(k). \quad (5.32)$$

**Note 5.10.** If $u$ and $v$ are real valued, the inner-product reduces to

$$(v, u)_{w,r} = 2 \text{ Re} \left[ \sum_{k=2}^{\infty} w^2(\sigma, k)|k|^{2r}\hat{v}^*(k)\hat{u}(k) \right]. \quad (5.33)$$

**Remark.** It is clear that for any fixed $\sigma > 0$, the two norms $\| \cdot \|_{w,r}$ and $\| \cdot \|_r$ are equivalent.

The following two lemmas involve useful inner product estimates involving $A$ and $A_N$:

**Lemma 5.11.** For any $r \geq 0$ and real valued $v \in \dot{H}^{r+3/2}$,

$$(v, -A[v])_{w,r} \geq \frac{15\sigma}{64} \|v\|_{w,r+3/2}^2. \quad (5.34)$$
Proof. It is convenient to define
\[
\delta = \sup_{k \geq 2} \frac{m(k)w(k, \sigma)}{\sigma d^{1/2}(k)d^{1/2}(k+1)w(k+1, \sigma)}.
\]
Since \((1 + a_\mu) \leq 2\), it is not difficult to conclude from the explicit expressions of \(d(k)\) and \(m(k)\) that in all cases, \(\delta \leq \frac{3}{8}\). Then, it follows from Cauchy Schwartz inequality that
\[
\sum_{k=2}^\infty k^{2r}w^{2k}(\sigma, k)m(k)\Re \{(\hat{v}^*(k)\hat{v}(k+1)} \leq \frac{3}{8} \sigma \sum_{k=2}^\infty k^{2r}w^{2k}(\sigma, k)d(k)\hat{v}(k)^2.
\]
It follows that
\[
(v, -A[v])_{w,r} \geq \frac{5 \sigma}{8} \sum_{k=2}^\infty k^{2r}w^{2k}(\sigma, k)d(k)\hat{v}(k)^2 \geq \frac{15 \sigma}{64} \|v\|_{w,r+3/2}^2.
\]
\[
\Box
\]

With respect to the operator \(A_N\), we have the following estimate:

**Lemma 5.12.** For \(r \geq 3\), assume real \(f, f_1, f_2 \in \mathring{H}^r\) and \(a, a_1, a_2, L, L_1, L_2\) are real numbers satisfying constraint \(|L - 2\pi| \leq \frac{1}{2}, |L_j - 2\pi| \leq \frac{1}{2}\) for \(j = 1, 2\). Then there exists constant \(C_r\) only dependent on \(r\) so that
\[
\|A_N[f, a, L]\|_{w,r-3/2} \leq C_r \sigma \left( |L - 2\pi| \|f\|_{w,r+3/2} + |a| \|f\|_{w,r-1/2} \right),
\]
\[
\|A_N[f_1, a_1, L_1] - A_N[f_2, a_2, L_2]\|_{w,r-3/2} \leq C_r \sigma \left( |L_1 - L_2| \|f_2\|_{w,r+3/2} + |a_1 - a_2| \|f_2\|_{w,r-1/2} + |L_1 - 2\pi| \|f_1 - f_2\|_{w,r+3/2} + |a_1| \|f_1 - f_2\|_{w,r-1/2} \right).
\]

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Proof. From the definition of $A_N$, it follows that

$$
\|A_N[f, a, L]\|_{w, r-3/2} \leq 2 \left( 1 - \frac{8\pi^3}{L^3} \right)^2 \sum_{k=2}^{\infty} \sigma^2 k^{2r-3} d^2(k) w^2(k, \sigma) |\hat{f}(k)|^2 \\
+ 2 \left( \left| \frac{2 \sin \frac{a}{2} }{L} \right|^2 + \left| \frac{2\pi}{L} - 1 \right|^2 \right) \sum_{k=2}^{\infty} k^{2r-3} m^2(k) w^2(k, \sigma) |\hat{f}(k + 1)|^2 \\
\leq C_r \sigma^2 \left( |L - 2\pi|^2 \|f\|_{r+3/2}^2 + (|a|^2 + |L - 2\pi|^2) \sup_{k \geq 2} \frac{m^2(k) w^2(k, \sigma)}{\sigma^2 (k + 1)^2} \|f\|_{r-1/2}^2 \right) \\
\leq C_r \left( |L - 2\pi|^2 \|f\|_{r+3/2}^2 + |a|^2 \|f\|_{r-1/2}^2 \right).
$$

Therefore, from bounds on $d(k)$ and $m(k)$, it follows that

$$
\|A_N[f_1-f_2, a_1, L_1]\|_{w, r-3/2} \leq C_r \sigma \left( |L_1 - 2\pi|^2 \|f_1 - f_2\|_{w, r+3/2} + |a_1|^2 \|f_1 - f_2\|_{w, r-1/2} \right).
$$

Further, since

$$
A_N[f, a_1, L_1] - A_N[f, a_2, L_2] = \sigma \left( \frac{8\pi^3}{L_2^3} - \frac{8\pi^3}{L_1^3} \right) \sum_{k=2}^{\infty} e^{ika} d(k) \hat{f}(k) \\
+ \left\{ \frac{2\pi}{L_1} - \frac{2\pi}{L_2} \right\} \left( e^{ia_1} - e^{ia_2} \right) \sum_{k=2}^{\infty} e^{ika} m(k) \hat{f}(k + 1),
$$

the results follow from the definition of $\| \cdot \|_{w, r}$ on using the restriction on $L_1, L_2$. \qed

5.3 Linear Evolution and space-time estimates

**Definition 5.13.** For $r \geq 3$, we define the space of real valued functions

$$
H^r_{\sigma} \equiv C([0, \infty), \dot{H}^r) \cap L^2([0, \infty), \dot{H}^{r+3/2}),
$$

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equipped with the norm $\| \cdot \|_{H^r}$ defined by

$$
\| u \|_{H^r}^2 = \sup_{t \geq 0} e^{\sigma t} \| u(\cdot, t) \|_{w, r}^2 + \frac{\sigma}{4} \int_0^\infty e^{\sigma t} \| u(\cdot, t) \|_{w, r+3/2}^2 dt.
$$

We now consider linear evolution equation

$$
v_t(\alpha, t) - A[v](\alpha, t) = f(\alpha, t) \quad \text{with} \quad v(\cdot, 0) = v_0 \in \dot{H}^r,
$$

where $f \in H_{\sigma}^{r-3}$.

**Lemma 5.14.** (A priori linear energy estimates) Suppose $r \geq 3$, $f \in H_{\sigma}^{r-3}$ and Then a solution $v(\cdot, t) \in \dot{H}^r$ to (5.34) will satisfy the following energy inequality for any $t$:

$$
e^{\sigma t} \| v(\cdot, t) \|_{w, r}^2 + \frac{\sigma}{4} \int_0^t e^{\sigma \tau} \| v(\cdot, \tau) \|_{w+3/2, r}^2 d\tau \leq \| v_0 \|_{w, r}^2 + \frac{8}{3\sigma^2} \| f \|_{H_{\sigma}^{r-3}}^2,
$$

and thus

$$
\| v \|_{H^r}^2 \leq \| v_0 \|_{w, r}^2 + \frac{8}{3\sigma^2} \| f \|_{H_{\sigma}^{r-3}}^2.
$$

**Proof.** Taking the $(\cdot, \cdot)_{w, r}$ inner-product on both sides of (5.34) with $v$, we obtain

$$
\frac{d}{dt} \| v \|_{w, r}^2 - 2 (v(\cdot, t), A[v])_{w, r} = 2 (v(\cdot, t), f(\cdot, t))_{w, r}. \tag{5.35}
$$

From Lemma 5.11, this implies

$$
\frac{d}{dt} \| v \|_{w, r}^2 + \frac{15\sigma}{32} \| v(\cdot, t) \|_{w, r+3/2}^2 \leq 2 \| v(\cdot, t) \|_{w, r+3/2} \| f(\cdot, t) \|_{w, r-3/2}.
$$

Noting that

$$
|k|^{r+3/2} \geq 2^{1/2} |k|^{r+1} \geq 2 |k|^{r+1/2} \geq 2^{3/2} |k|^r \quad \text{for} \quad k \geq 2
$$

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implies that
\[ \|v\|_{w,r+3/2} \geq 2^{1/2}\|v\|_{w,r+1} \geq 2\|v\|_{w,r+1/2} \geq 2^{3/2}\|w\|_r. \]

It follows that on using Cauchy Schwartz inequality,
\[ \frac{d}{dt}\|v\|^2_{w,r} + \sigma\|v\|^2_{w,r} + \frac{\sigma}{4}\|v(\cdot, t)\|^2_{w,r+3/2} \leq \frac{32}{3\sigma}\|f(\cdot, t)\|^2_{w,r-3/2}. \]

Integration gives the desired energy inequality. Noting that this is true for any \( t \), and using the definition of \( \| \cdot \|_{H^r} \), we obtain the given bounds on \( \|v\|_{H^r} \).

Remark. Proof of existence of a solution to the linear equation (5.34) for given real valued \( f \in H^{r-3}_\sigma \) and the initial condition \( v_0 \in \dot{H}^r \), satisfying the given conditions follows in a standard manner. Note that we can introduce a sequence of Galerkin approximants \( v_N(\alpha, t) \) containing a finite number of Fourier modes. This will satisfy the energy bounds in Lemma 5.14, independent of \( N \). These approximates clearly solve linear ODEs for which the unique solutions exist globally. In the Hilbert space \( L^2([0, S], H^{r+3/2}) \), there exists a subsequence of \( v_N \to v \) weakly. Therefore for almost all \( t \in [0, S] \), this subsequence denoted again by \( v_N(\cdot, t) \to v(\cdot, t) \) strongly in \( \dot{H}^r \).

From the energy bound, the limit \( v(\cdot, t) \) is bounded in \( \dot{H}^r \) for any \( t \in [0, S] \), and \( v \in L^2([0, S], H^{r+3/2}) \). It is also easy to check that the limiting solution satisfies (5.34) in a classical sense for sufficiently large \( r \). This proves existence of a global classical solution for any \( t \) noting that \( r \geq 3 \) since \( r \) is arbitrary. The uniqueness of this solution follows from the energy bound itself. \( \square \)

Definition 5.15. It is convenient to define a linear operator \( e^{tA} \) so that
\[ v = e^{tA}v_0 \]
is the unique solution \( v(\alpha, t) \in \dot{H}^r \) satisfying (5.34) for \( f = 0 \), with the initial condition \( v(\alpha, 0) = v_0 \).

**Note 5.16.** It is easily seen that \( e^{tA} \) is a semi-group. Further, using Duhammel principle, the solution \( v(\alpha, t) \in \dot{H}^r \) satisfying (5.34) for \( v_0 = 0 \) may be expressed as

\[
v(\alpha, t) = \int_0^t e^{(t-\tau)A} f(\alpha, \tau) d\tau.
\]

(5.36)

**Remark.** The energy bounds in Lemma 5.14 imply that

\[
\|e^{tA}v_0\|_{H^r_\sigma} \leq \|v_0\|_{W,r}, \quad \left\| \int_0^t e^{(t-\tau)A} f(\cdot, \tau) d\tau \right\|_{H^r_\sigma} \leq \frac{2\sqrt{2}}{\sqrt{3}\sigma} \|f\|_{H^{r-3}}.
\]

(5.37)

\[\square\]

### 5.4 Nonlinear evolution, contraction map and proof of Proposition 5.2

We express the evolution equation (5.14) in the integral form:

\[
\bar{\theta}(\alpha, t) = e^{tA}\bar{\theta}_0 + \int_0^t d\tau e^{(t-\tau)A} \left\{ \mathfrak{M}[\bar{\theta}(\cdot, \tau), \hat{\theta}(0; \tau)] + \mathcal{A}_N[\theta(\cdot, \tau), \hat{\theta}(0; \tau), L(\tau)] \right\}
\equiv S_1[\bar{\theta}, \hat{\theta}(0; \cdot)](\alpha, t),
\]

(5.38)

\[
\hat{\theta}(0; t) = \int_0^t \mathfrak{M}_0 \left[ S_1(\alpha, \cdot), \hat{\theta}(0; \cdot) \right](\tau) d\tau \equiv S_2[\bar{\theta}, \hat{\theta}(0; \cdot)](t),
\]

(5.39)

where \( L = L(t) \) is determined in terms of \( \bar{\theta}(\cdot, t) \) through (D.4) and (2.2).

Equations (5.38) and (5.39) will be the basis of a contraction mapping theorem for \((\bar{\theta}, \hat{\theta}(0; t))\) in an small ball in the space

\[
\mathcal{D} \equiv H^r_\sigma \times C[0, \infty)
\]

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equipped with the norm \( \| \cdot \|_{\mathcal{D}} \) so that

\[
\left\| (\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)) \right\|_{\mathcal{D}} = \|\hat{\vartheta}\|_{H^s} + |\hat{\vartheta}(0; \cdot)|_{\infty}.
\]

(5.40)

First, we define a mapping in \( \mathcal{D} \) by

\[
\mathcal{S}[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)] \equiv \left( \begin{array}{c}
\mathcal{S}_1[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)](\alpha, t) \\
\mathcal{S}_2[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)](t)
\end{array} \right).
\]

Secondly, we estimate the nonlinear terms in the space \( H^s \).

**Lemma 5.17.** For \( r \geq 3 \) and \( \sigma > 0 \), assume \((\tilde{\vartheta}(\alpha, t), \hat{\vartheta}(0; t))\) satisfy the condition

\[
\|\tilde{\vartheta}\|_{H^s} \leq \epsilon, \quad |\hat{\vartheta}(0; \cdot)|_{\infty} \leq \epsilon.
\]

(5.41)

Then for \( \mathcal{A}_N[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot), L(\cdot)](\alpha, t), \mathcal{M}[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)](\alpha, t) \) and scalar \( \mathcal{M}_0[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)](t) \), determined from (5.16), (5.18) and (5.22) respectively,

\[
\left\| \mathcal{A}_N[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot), L] + \mathcal{M}[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)] \right\|_{H^{s-3}} \leq c_1 \|\tilde{\vartheta}\|_{H^s} \left( \left\| \mathcal{M}_0[\tilde{\vartheta}, \hat{\vartheta}(0; \cdot)](\tau) \right\|_{\mathcal{D}} \right) \leq c_2 \|\tilde{\vartheta}\|_{H^s}^3.
\]

Further, if both \((\tilde{\vartheta}^{(1)}(\alpha, t), \hat{\vartheta}^{(1)}(0; t))\) and \((\tilde{\vartheta}^{(2)}(\alpha, t), \hat{\vartheta}^{(2)}(0; t))\) satisfy (5.41), then the corresponding \((\mathcal{A}_N^{(1)}, \mathcal{M}^{(1)}, \mathcal{M}_0^{(1)})\) and \((\mathcal{A}_N^{(2)}, \mathcal{M}^{(2)}, \mathcal{M}_0^{(2)})\) satisfy

\[
\left\| \mathcal{A}_N^{(1)} - \mathcal{A}_N^{(2)} \right\|_{H^{s-3}} + \left\| \mathcal{M}^{(1)} - \mathcal{M}^{(2)} \right\|_{H^{s-3}} \leq c_3 \epsilon \left( \left\| \tilde{\vartheta}^{(1)} - \tilde{\vartheta}^{(2)} \right\|_{H^s} + \left\| \hat{\vartheta}^{(1)}(0; \cdot) - \hat{\vartheta}^{(2)}(0; \cdot) \right\|_{\infty} \right),
\]

\[
\left| \int_0^t (\mathcal{M}_0^{(1)} - \mathcal{M}_0^{(2)})d\tau \right|_{\infty} \leq c_4 \left( \left\| \tilde{\vartheta}^{(1)} - \tilde{\vartheta}^{(2)} \right\|_{H^s} + \epsilon \left\| \hat{\vartheta}^{(1)}(0; \cdot) - \hat{\vartheta}^{(2)}(0; \cdot) \right\|_{\infty} \right).
\]

Note that \( \Gamma \) and \( L \) appearing in the expressions are determined in terms of \( \hat{\vartheta} \) and \( \hat{\vartheta}(0; t) \) through (5.7) and (D.4) on using (2.2) and (2.3).
Proof. We note the bounds for $A_N$, $\mathfrak{A}$ and $\mathfrak{A}_0$ in Lemma 5.12 and Corollary 5.8. It follows from the equivalence of $\| \cdot \|_r$ and $\| \cdot \|_{w,r}$ norms and the definition of $\| \cdot \|_{H^s_r}$ norm that

$$
eq \sqrt{e^{\sigma t/2}} \| A_N[\tilde{\theta}, \hat{\theta}(0; \cdot), L] + \mathfrak{A}[\tilde{\theta}, \hat{\theta}(0; \cdot)]\|_{w,r-3}
\leq C e^{\sigma t/2} (\| \tilde{\theta} \|_{w,r} \| \tilde{\theta} \|_1 + \| \tilde{\theta} \|_{w,r-1} \| \tilde{\theta} \|_{w,r-2} + |\hat{\theta}(0; \cdot)|_{\infty} \| \tilde{\theta} \|_{w,r-2})
\leq C \| \tilde{\theta} \|_{H^s_r} \left( \| \tilde{\theta} \|_{H^s_r} + |\hat{\theta}(0; \cdot)|_{\infty} \right).
$$

Further, it follows that

$$
\int_0^\infty e^{\sigma t} \| A_N[\tilde{\theta}, \hat{\theta}(0; t), L] + \mathfrak{A}[\tilde{\theta}(\cdot, t), \hat{\theta}(0; t)]\|_{w,r-3/2} \, dt
\leq C \sup_t \left[ e^{\sigma t} \| \tilde{\theta} \|_{w,r}^2 + |\hat{\theta}(0; t)|_2^2 \right] \int_0^\infty e^{\sigma t} \| \tilde{\theta} \|_{w,r+3/2}^2 \, dt
\leq C \| \tilde{\theta} \|_{H^s_r}^2 \left( \| \tilde{\theta} \|_{H^s_r}^2 + |\hat{\theta}(0; \cdot)|_\infty^2 \right).
$$

Therefore the bounds for $\| A_N + \mathfrak{A} \|_{H^s_r-3}$ follows. For $\mathfrak{A}_0$, we use Corollary 5.8 again to note

$$
\left| \int_0^t \mathfrak{A}_0[\tilde{\theta}, \hat{\theta}(0; \cdot)](\tau) \, d\tau \right| \leq C \int_0^\infty \left( \| \tilde{\theta}(\cdot, \tau) \|_{w,3}^2 + \| \tilde{\theta}(\cdot, \tau) \|_{w,2} \right) \, d\tau \leq c_2 \| \tilde{\theta} \|_{H^3_r}.
$$

The statements for the differences of $\mathfrak{A}$, $\mathfrak{A}_0$ for different $(\tilde{\theta}, \hat{\theta}(0; t))$ follow from parallel statements in Lemma 5.12 and Corollary 5.8.

We have the following contraction properties in a ball

$$
O_\epsilon \equiv \{(u, v) \in D | \| u \|_{H^s_r} \leq \epsilon, |v|_{\infty} \leq \epsilon \}.
$$
Lemma 5.18. Let $\sigma > 0$, $r \geq 3$. Assume $\left(\hat{\theta}, \hat{\theta}(0; t)\right) \in \mathcal{V}_\epsilon$ and $c_1, c_2, c_3, c_4$ are as defined in Lemma 5.17. If for sufficiently small $\epsilon$, $\|\hat{\theta}_0\|_{w,r} < \min\left\{\frac{\epsilon}{2}, \frac{\epsilon}{2c_4}\right\}$ and $\hat{\theta}(0; 0) = 0$, then

$$S[\tilde{\theta}, \hat{\theta}(0; \cdot)] \in \mathcal{V}_\epsilon.$$ Further, if each of $\left(\tilde{\theta}^{(1)}, \hat{\theta}^{(1)}(0; t)\right)$ and $\left(\tilde{\theta}^{(2)}, \hat{\theta}^{(2)}(0; t)\right)$ belongs to $\mathcal{V}_\epsilon$, then there exists $c_5$ depending on $c_1, \cdots, c_4$, such that

$$\left\|S[\tilde{\theta}^{(1)}, \hat{\theta}^{(1)}(0; \cdot)] - S[\tilde{\theta}^{(2)}, \hat{\theta}^{(2)}(0; \cdot)]\right\|_D \leq c_5 \epsilon \left\|(\tilde{\theta}^{(1)} - \tilde{\theta}^{(2)}, \hat{\theta}^{(1)}(0; \cdot) - \hat{\theta}^{(2)}(0; \cdot))\right\|_D.$$

Proof. Define $c_6 = \frac{2\sqrt{2}}{\sqrt{3}\sigma} c_1$. By (5.37) and Lemma 5.17, we have

$$\left\|S_1[\tilde{\theta}, \hat{\theta}(0; \cdot)]\right\|_{H^2_{\epsilon}} \leq \|\hat{\theta}_0\|_{w,r} + c_6 (\|\tilde{\theta}\|_{H^2_{\epsilon}} + \|\tilde{\theta}\|_{H^2_{\epsilon}} \|\hat{\theta}(0; \cdot)\|_{\infty}) \leq \epsilon,$$

if $c_6 \epsilon < \frac{1}{4}$. We also have

$$\left\|S_2[\tilde{\theta}, \hat{\theta}(0; \cdot)]\right\|_{\infty} \leq c_2 \left\|S_1[\tilde{\theta}, \hat{\theta}(0; \cdot)]\right\|_{H^3_{\epsilon}} \leq c_2 \left(\|\tilde{\theta}_0\|_{w,r} + c_6 \|\tilde{\theta}\|_{H^2_{\epsilon}} + c_6 \|\tilde{\theta}\|_{H^2_{\epsilon}} \|\hat{\theta}(0; \cdot)\|_{\infty}\right) \leq \epsilon,$$

if $c_2 c_6 \epsilon < \frac{1}{4}$.

The statements for the differences of $S$, for different $\left(\tilde{\theta}, \hat{\theta}(0; t)\right)$ follows from parallel statements in Lemma 5.17.

Note 5.19. Constants $c_1, c_2, c_3, c_4$ and $c_5$ can depend on $\sigma$.

Proof of Proposition 5.2: If $c_5 \epsilon < 1$, then it is clear that the right sides of (5.38) and (5.39) define a contraction map in a small ball $O_\epsilon$ in the Banach space $\mathcal{D}$.

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Therefore, there exists a unique solution \((\tilde{\theta}, \hat{\theta}(0; t))\) satisfying equations (5.38) and (5.39), hence (D.1). The local uniqueness of solutions (see §A.4) implies that this is the only solution. The \(e^{-\sigma t/2}\) exponential decay of \(\tilde{\theta}\) and hence of \(\theta\) implies that the steady circle is approached exponentially in time. The constraint condition (D.5) implies that \(L - 2\pi\) decays exponentially.

**Note 5.20.** It is easy to show that given any \(j\), \(\tilde{\theta}(\cdot, t) \in \dot{H}^{r+3j/2}\) for \(t \geq j\) in the following manner. Since \((\tilde{\theta}, \hat{\theta}(0; t)) \in \mathcal{V}_r\) for some \(r \geq 3\), there exists \(t_0 \in [0, 1]\) such that \(\|\tilde{\theta}(\cdot, t_0)\|_{r+3/2} < \epsilon\). So we can restart clock at \(t = t_0\) and prove global solution in \(H^{r+3/2}_\sigma\). It follows that there exists \(t_1 \in (t_0, t_0 + 1]\) so that \(\|\hat{\theta}(\cdot, t_1)\|_{r+3} < \epsilon\). By bootstrapping, we obtain \(\tilde{\theta}(\cdot, t) \in \dot{H}^{r+3j/2}\) for \(t \geq j\).

Indeed more can be shown to be true. The contraction argument in Proposition 5.2 can be carried out for arbitrary sized initial condition over small sized time interval. Through bootstrapping and using Sobolev embedding theorem, we can conclude that the solution is in \(C^\infty\) in space for \(t \in (0, S]\). The property of smoothing of initial conditions is similar to other dissipative equations like Navier-Stokes.

The property is similar to other dissipative equations like Navier-Stokes.
CHAPTER 6

STEADY TRANSLATING BUBBLE IN THE CHANNEL WITH SIDEWALLS \((\beta > 0)\)

For a steadily traveling bubble solution, in the frame of an appropriately moving bubble, we have to require the normal interface speed \(U = 0\). This would imply (A.1) is automatically satisfied for a time-independent \(\theta^{(s)}(\alpha)\) and \(L = L^{(s)} = 2\pi\), where \(z(\alpha) = z^{(s)}(\alpha)\) describes the geometry shape of the steady bubble and \(\gamma(\alpha,t) = \gamma^{(s)}(\alpha)\) is determined in terms of \(\theta\) through (A.2).

Earlier, we have shown that for the bubble with the invariant area,

\[
\int_0^{2\pi} U(\alpha) d\alpha = 0. \quad (6.1)
\]

Further, there is no loss of generality in the steady problem to choose \(\hat{\theta}^{(s)}(0) = 0\) since this corresponds to a choice of origin for \(\alpha\), and make \(\alpha = 0\) correspond to \(y^{(s)}(0) = 0\). Thus, from (1.19), the steady bubble problem reduces to

\[
\mathcal{M} \left[ \tilde{\theta}^{(s)}, u_0, \beta \right] = Q_0 \left( \frac{1}{2} \mathcal{H}(\gamma^{(s)}) + \frac{1}{2} \text{Re} \left( \gamma^{(s)}(z^{(s)}) \right) + (u_0 + 1) \cos (\alpha + \theta^{(s)}(\alpha)) \right) = 0, \quad (E.1)
\]

with vortex sheet strength \(\gamma^{(s)}\) and \(\tilde{\theta}^{(s)}(\pm1)\) determined by

\[
(I + a_\mu \mathcal{F}[z^{(s)}]) \gamma^{(s)} = 2 \left( 1 + \frac{\mu_2}{\mu_1 + \mu_2} u_0 \right) \sin \left( \alpha + \theta^{(s)}(\alpha) \right) + \sigma \theta^{(s)}_{\alpha\alpha}, \quad (E.2)
\]
\[
\int_{0}^{2\pi} \exp \left( i\alpha + i(\hat{\theta}^{(s)}(-1)e^{-i\alpha} + \bar{\theta}^{(s)}(1)e^{i\alpha} + \tilde{\theta}^{(s)}(\alpha)) \right) \, d\alpha = 0, \tag{E.3}
\]
where \( \theta^{(s)} = \tilde{\theta}^{(s)} + \hat{\theta}^{(s)}(1)e^{i\alpha} + \bar{\theta}^{(s)}(-1)e^{-i\alpha} \). Hence we seek solutions \((\tilde{\theta}^{(s)}, u_0, \beta) \in \hat{H}r \times (-1, 1) \times (-\Upsilon, \Upsilon) \) \(^1\).

Recently, [45], [46] and [47] obtained selection results for steady finger for small non-zero surface tension.

**Remark.** For \( r \geq 3 \), by Propositions 2.7 and 2.38, we know that \( \| \mathcal{U} \|_{r-2} \leq C \) with \( C \) depending on \( \Upsilon \) and the diameter of \( B^r \). Hence, \( \mathcal{U} \) maps an open set of \( H^r_p \times \mathbb{R}^2 \) into the space \( H^{r-2}_p \).

\(^1\) We choose small \( \epsilon \) and \( \Upsilon \) such that Proposition 2.7 can be applied in (E.3) and Proposition 2.38 can also be applied in (E.2).
By Lemma 5.7, the statements of the Lemma follow.

We identify

Proof.

Further, let

It will be shown that

Lemma 6.3.

For any

where

\[ \mathfrak{H}^{(s)} \theta(s), u_0, \beta) = -\mathfrak{M}[\tilde{\theta}(s), u_0, \beta] + \mathfrak{M}_{\tilde{\phi}(0)}[0, 0, 0] \tilde{\theta}(s) + \mathfrak{M}_{u_0}[0, 0, 0] u_0 + \frac{\beta^2}{2} \mathfrak{M}_{\beta \beta}[0, 0, 0] \]

\[ = \mathfrak{A}[\tilde{\theta}(s)](\alpha) + \mathfrak{B}[\tilde{\theta}(s), u_0] + \mathfrak{C}[\tilde{\theta}(s), u_0, \beta] \quad (6.4) \]

with

\[ \mathfrak{A}[\tilde{\theta}(s)](\alpha) = \mathfrak{M}[\tilde{\theta}(s), 0, 0] (\alpha) - \mathfrak{M}_{\tilde{\phi}(0)}[0, 0, 0] \tilde{\theta}(s)(\alpha), \]

\[ \mathfrak{B}[\tilde{\theta}(s), u_0] = \mathfrak{M}[\tilde{\theta}(s), u_0, 0] - \mathfrak{M}[\tilde{\theta}(s), 0, 0] - \mathfrak{M}_{u_0}[0, 0, 0] u_0, \]

\[ \mathfrak{C}[\tilde{\theta}(s), u_0, \beta] = \mathfrak{M}[\tilde{\theta}(s), u_0, \beta] - \mathfrak{M}[\tilde{\theta}(s), u_0, 0] - \frac{\beta^2}{2} \mathfrak{M}_{\beta \beta}[0, 0, 0]. \]

It will be shown that \( \mathfrak{H}^{(s)} \) is either nonlinear in \((\tilde{\theta}(s), u_0)\) or at least \(O(\beta^4)\).

**Lemma 6.2.** For any \( r \geq 3 \), let \( \|\tilde{\theta}(s)\|_r \) and \( u_0 \) sufficiently small, then there exists \( C \) independent of \( u_0 \) and \( \tilde{\theta}(s) \) so that

\[ \|\mathfrak{A}[\tilde{\theta}(s)]\|_{r-1} \leq C \|\tilde{\theta}(s)\|_{r-1} \|\tilde{\theta}(s)\|_r. \]

Further, let \( \mathfrak{A}^{(1)} \) and \( \mathfrak{A}^{(2)} \) correspond to \( \tilde{\theta}_1(s) \) and \( \tilde{\theta}_2(s) \) respectively, each in \( \dot{H}^r \). Then there exists \( C \) independent of \( \beta, u_0 \) and \( \tilde{\theta}(s) \) so that

\[ \|\mathfrak{A}^{(1)} - \mathfrak{A}^{(2)}\|_{r-1} \leq C (\|\tilde{\theta}_1^{(s)}\|_{r-1} \|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r + \|\tilde{\theta}_1^{(s)}\|_r \|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_{r-1}). \]

**Proof.** We identify \( \mathfrak{A}[\tilde{\theta}(s)] \) as the nonlinear part of normal velocity \( U \) for \( \beta = 0 \) in 5.12). By Lemma 5.7, the statements of the Lemma follow. \( \square \)

**Lemma 6.3.** For any \( r \geq 3 \), let \( \|\tilde{\theta}(s)\|_r \) and \( u_0 \) sufficiently small, then there exists \( C \) independent of \( u_0 \) and \( \tilde{\theta}(s) \) so that

\[ \|\mathfrak{B}[\tilde{\theta}(s), u_0]\|_r \leq C |u_0| \|\tilde{\theta}(s)\|_r. \]

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Further, let $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ correspond to $(\tilde{\theta}^{(s)}, u_0^{(1)})$ and $(\tilde{\theta}^{(s)}, u_0^{(2)})$ respectively, each in $\tilde{H}^r$. Then there exists $C$ independent of $\beta$, $u_0$ and $\gamma^{(s)}$ so that

$$
\| \mathcal{B}^{(1)} - \mathcal{B}^{(2)} \|_r \leq C \left( |u_0^{(1)}| \| \tilde{\theta}^{(s)}_1 - \tilde{\theta}^{(s)}_2 \|_r + \| \tilde{\theta}^{(s)}_1 \|_r |u_0^{(1)} - u_0^{(2)}| \right).
$$

**Proof.** Let $\gamma^{(u_0)}$ correspond to $(\tilde{\theta}^{(s)}, u_0, 0)$, while $\gamma^{(u_0)}$ corresponds to $(\tilde{\theta}^{(s)}, 0, 0)$. Then by (1.19), we obtain

$$
\mathcal{B}[\tilde{\theta}^{(s)}, u_0] = \frac{1}{2} \mathcal{H}[\gamma^{(u_0)} - \gamma_{0}^{(u_0)}] + \frac{1}{2} \text{Re} \left( \mathcal{G}[\gamma^{(s)}](\gamma^{(u_0)} - \gamma_0^{(u_0)}) \right)
$$

$$
+ u_0 \left( \cos (\alpha + \tilde{\theta}^{(s)}(\alpha)) - \frac{\mu_1}{\mu_1 + \mu_2} \cos \alpha \right).
$$

(6.5)

For (E.2) and the relation between $\mathcal{F}$ and $\mathcal{G}$, we also have

$$
\gamma^{(u_0)} - \gamma_0^{(u_0)} = -a_\mu \text{Re} \left( \frac{1}{i} \mathcal{G}[\gamma^{(s)}](\gamma^{(u_0)} - \gamma_0^{(u_0)}) - \frac{1}{i} \mathcal{G}[\omega_0](\gamma^{(u_0)} - \gamma_0^{(u_0)}) \right)
$$

$$
+ 2u_0 \frac{\mu_2}{\mu_1 + \mu_2} \sin (\alpha + \tilde{\theta}^{(s)}).
$$

(6.6)

By Lemma 2.37 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$), from (6.6), we have

$$
\| \gamma^{(u_0)} - \gamma_0^{(u_0)} \|_1 \leq C (\| \tilde{\theta}^{(s)} \|_r \| \gamma^{(u_0)} - \gamma_0^{(u_0)} \|_1 + |u_0|).
$$

Hence for sufficient small $\| \tilde{\theta}^{(s)} \|_r$, we have

$$
\| \gamma^{(u_0)} - \gamma_0^{(u_0)} \|_1 \leq C |u_0|.
$$

(6.7)

Plugging (6.6) into (6.5), we have

$$
\mathcal{B}[\tilde{\theta}^{(s)}, u_0] = \frac{1}{2} \mathcal{H} \left[ a_\mu \text{Re} \left( \frac{1}{i} \mathcal{G}[\gamma^{(s)}](\gamma^{(u_0)} - \gamma_0^{(u_0)}) - \frac{1}{i} \mathcal{G}[\omega_0](\gamma^{(u_0)} - \gamma_0^{(u_0)}) \right) \right]
$$

$$
+ \frac{1}{2} \text{Re} \left( \mathcal{G}[\gamma^{(s)}](\gamma^{(u_0)} - \gamma_0^{(u_0)}) - \mathcal{G}[\omega_0](\gamma^{(u_0)} - \gamma_0^{(u_0)}) \right)
$$

$$
+ u_0 \left( \cos (\alpha + \tilde{\theta}^{(s)}(\alpha)) - \cos \alpha \right) + \frac{\mu_2}{\mu_1 + \mu_2} \mathcal{H} \left[ \sin (\eta + \tilde{\theta}^{(s)}) - \sin(\eta) \right](\alpha).
$$

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Hence, by Lemmas 2.14, 2.37 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$) and (6.7), we have the first statement.

For the difference term, by Lemmas 2.14, 2.37 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$) and Proposition 2.38.

**Lemma 6.4.** For any $r \geq 3$, assume $\|\tilde{\theta}^{(s)}\|_r$, $u_0$ and $\beta$ are sufficiently small. Then there exists $C$ independent of $\beta$, $u_0$ and $\tilde{\theta}^{(s)}$ so that

$$\|C[\tilde{\theta}^{(s)}, u_0]\|_r \leq C(\beta^2|u_0| + \beta^2\|\tilde{\theta}^{(s)}\|_r + \beta).$$

Further, suppose $C^{(1)}$ and $C^{(2)}$ correspond to $(\tilde{\theta}_1^{(s)}, u_0^{(1)}, \beta)$ and $(\tilde{\theta}_2^{(s)}, u_0^{(2)}, \beta)$ respectively, each in $\tilde{H}^r$. Then there exists $C$ independent of $\beta$, $u_0$ and $\tilde{\theta}^{(s)}$ so that

$$\|C^{(1)} - C^{(2)}\|_r \leq C\beta^2(\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r + |u_0^{(1)} - u_0^{(2)}|).$$

**Proof.** Suppose $\gamma_0^{(s)}$ satisfying (E.2) corresponds to $(\tilde{\theta}^{(s)}, u_0, 0)$. Then for (1.19),

$$C[\tilde{\theta}^{(s)}, u_0, \beta] = \frac{1}{2} \mathcal{H}[\gamma^{(s)} - \gamma_0^{(s)}] + \frac{1}{2} \text{Re} \left( \mathcal{G}_1[z^{(s)}](\gamma^{(s)} - \gamma_0^{(s)}) \right) + \frac{1}{2} \text{Re} \left( \mathcal{G}_2[z^{(s)}]\gamma^{(s)} \right) - \frac{\beta^2}{6} (1 + a_\mu) \cos \alpha$$

$$= \frac{1}{2} \mathcal{H} \left[ \gamma^{(s)} - \gamma_0^{(s)} + a_\mu \frac{\beta^2}{12} \sin \eta \right] (\alpha) + \frac{1}{2} \text{Re} \left( \mathcal{G}_1[z^{(s)}] (\gamma^{(s)} - \gamma_0^{(s)}) - \mathcal{G}_1[\omega_0] (\gamma^{(s)} - \gamma_0^{(s)}) \right)$$

$$+ \frac{1}{2} \text{Re} \left( \mathcal{G}_2[z^{(s)}] \gamma_0^{(s)} - \mathcal{G}_2[i\omega_0] \gamma_0^{(s)} \right) + \frac{1}{2} \text{Re} \left( \mathcal{G}_2[i\omega_0] (\gamma_0^{(s)} - 2 \sin \eta)(\alpha) \right)$$

$$+ \frac{1}{2} \text{Re} \left( \mathcal{G}_2[i\omega_0] (2 \sin \eta)(\alpha) - \frac{\beta^2}{6} \cos \alpha \right). \quad (6.8)$$

For (E.2), we also have

$$\gamma^{(s)} - \gamma_0^{(s)} = -a_\mu \text{Re} \left( \frac{1}{i} \mathcal{G}_1[z^{(s)}] (\gamma^{(s)} - \gamma_0^{(s)}) - \frac{1}{i} \mathcal{G}_1[\omega_0] (\gamma^{(s)} - \gamma_0^{(s)}) \right) - a_\mu \text{Re} \left( \frac{1}{i} \mathcal{G}_2[z^{(s)}] \gamma^{(s)} \right). \quad (6.9)$$
Proposition 2.38 gives us
\[ \|\gamma(s)\|_1 \leq C, \|\gamma_0(s)\|_1 \leq C. \]

By Lemma 2.37 (for \( \beta = 0 \) and \( L^{(1)} = L^{(2)} = 2\pi \)), Note 2.31 and (6.9), we have
\[ \|\gamma(s) - \gamma_0(s)\|_r \leq C(\|\tilde{\theta}(s)\|_r \|\gamma(s) - \gamma_0(s)\|_1 + \beta^2). \]

Hence for sufficient small \( \|\tilde{\theta}(s)\|_r \), we have
\[ \|\gamma(s) - \gamma_0(s)\|_r \leq C\beta^2. \] (6.10)

(6.9) can be rewritten as
\[
\gamma(s) - \gamma_0(s) + a_\mu \frac{\beta^2}{6} \sin \alpha = -a_\mu \text{Re} \left( \frac{1}{i} \mathcal{G}_1[z(s)](\gamma(s) - \gamma_0(s)) - \frac{1}{i} \mathcal{G}_1[\omega_0](\gamma(s) - \gamma_0(s)) \right) \\
- a_\mu \text{Re} \left( \frac{1}{i} \mathcal{G}_2[z(s)]\gamma_0(s) - \frac{1}{i} \mathcal{G}_2[i\omega_0]\gamma_0(s) - a_\mu \text{Re} \left( \frac{1}{i} \mathcal{G}_2[i\omega_0](\gamma_0(s) - 2 \sin \eta)(\alpha) \right) \\
- a_\mu \text{Re} \left( \frac{1}{i} \mathcal{G}_2[i\omega_0](2 \sin \eta)(\alpha) - \frac{\beta^2}{6} \sin \alpha \right). \] (6.11)

We see from (E.2) that
\[
\gamma_0(s) - 2 \sin \alpha = -a_\mu \text{Re} \left( \frac{1}{i} \mathcal{G}_1[z(s)]\gamma_0(s) - \frac{1}{i} \mathcal{G}_1[\omega_0]\gamma_0(s) \right) \\
+ 2 \left( \sin(\alpha + \tilde{\theta}(s)) - 2 \sin \alpha \right) + 2u_0 \frac{\mu_2}{\mu_1 + \mu_2} \sin(\alpha + \tilde{\theta}(s)) + \sigma \tilde{\theta}(s). \]

Hence by Lemmas 2.14 and 2.37 (for \( \beta = 0 \) and \( L^{(1)} = L^{(2)} = 2\pi \)), we have from above
\[ \|\gamma_0(s) - 2 \sin(\cdot)\|_1 \leq C(\|\tilde{\theta}(s)\|_r + |u_0|). \] (6.12)

We know the first derivative of \( \mathcal{G}_2[i\omega_0](2 \sin \eta)(\alpha) \) with respect to \( \beta \) at \( \beta = 0 \) is equal to 0. On calculation,
\[
\left. \left( \mathcal{G}_2[i\omega_0](2 \sin \eta)(\alpha) \right) \right|_{\beta = 0} = \frac{e^{i\alpha}}{3}. \]
Hence for sufficiently small $\beta$, by Taylor expansion, we have

$$\left\| G_2[i\omega_0](2\sin \eta)(\alpha) - \frac{\beta^2}{6} e^{i\alpha} \right\|_r \leq C\beta^4. \quad (6.13)$$

By Lemmas 2.33, 2.37 (for $\beta = 0$ and $L^{(1)} = L^{(2)} = 2\pi$), Note 2.31, (6.12) and (6.13), from (6.11) we get

$$\left\| \gamma(s) - \gamma_0^{(s)} + a_{\mu} \frac{\beta^2}{6} \sin(\cdot) \right\|_r \leq C(\beta^2\|\tilde{\theta}(s)\|_r + \beta^2 u_0 + \beta^4). \quad (6.14)$$

Hence, by Lemma 2.37, (6.12), (6.13) and (6.14), the first statement is obtained.

The proof for the second statement follows similarly. \qed

Hence we have

**Lemma 6.5.** For any $r \geq 3$, assume $\|\tilde{\theta}\|_r$, $u_0$ and $\beta$ are sufficiently small. Then there exists $C$ independent of $\beta$, $u_0$ and $\tilde{\theta}$ so that

$$\|\mathcal{N}^{(s)}\|_{r-1} \leq C \left[ |u_0|\|\tilde{\theta}\|_r + |u_0|\beta^2 + \beta^4 + \beta^2\|\tilde{\theta}\|_r + \|\tilde{\theta}\|_r\|\tilde{\theta}\|_{r-1} \right]. \quad (6.15)$$

Further, suppose $\mathcal{N}_1^{(s)}$ and $\mathcal{N}_2^{(s)}$ correspond to $(\tilde{\theta}_1^{(s)}, u_0^{(1)}, \beta)$ and $(\tilde{\theta}_2^{(s)}, u_0^{(2)}, \beta)$ respectively, each in $\dot{H}^r$. Then there exists $C$ independent of $\beta$, $u_0$ and $\tilde{\theta}^{(s)}$ so that

$$\|\mathcal{N}_1^{(s)} - \mathcal{N}_2^{(s)}\|_{r-1} \leq C \left( \beta^2(\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r + |u_0^{(1)} - u_0^{(2)}|) + \|\tilde{\theta}_1^{(s)}\|_{r-1}\|\tilde{\theta}_1^{(s)} - \tilde{\theta}_2^{(s)}\|_r + \|\tilde{\theta}_1^{(s)}\|_r|u_0^{(1)} - u_0^{(2)}| \right).$$

**Proof.** Combining Lemmas 6.2, 6.3 and 6.4, the two statements are obtained. \qed

**Definition 6.6.** We define the linear operator $A$ on $u \in \dot{H}^r$ by

$$Au = -\frac{\sigma}{2} u_{\alpha\alpha} - \sum_{k=2}^{\infty} (1 + a_{\mu}) \frac{k + 1}{k + 2} \hat{u}(k+1) e^{ik\alpha} - \sum_{k=-\infty}^{-2} (1 + a_{\mu}) \frac{k - 1}{k - 2} \hat{u}(k-1) e^{ik\alpha}. \quad (6.16)$$
**Proposition 6.7.** For \( r \geq 3 \), the linear operator \( A : \dot{H}^r \to \dot{H}^{r-2} \), is invertible. Further, \( \| A^{-1} f \|_r \leq C_r \| f \|_{r-2} \), for any \( f \in \dot{H}^{r-2} \).

**Proof.** For any surface tension \( \sigma \), there exists the integer \( K > 2 \) such that \( n^2 \geq \frac{8}{\sigma} \) for any \( |n| \geq K \). Let us define a family of the spaces \( Z_r := \{ u \in \dot{H}^r | Q_K u = u \} \) with \( r \geq 0 \). We define the linear operator \( A_K := Q_K A \), which maps from \( Z_r \) to \( Z_{r-2} \). The corresponding bilinear mapping \( E_K : Z_1 \times Z_1 \to \mathbb{R} \) is defined by

\[
E_K[u, v] = 2 \text{Re} \left( \sum_{k=K}^\infty \left[ \frac{\sigma}{2} k^2 \hat{u}(k) - (1 + a_\mu) \left( \frac{k+1}{k+2} \right)^2 \hat{u}(k+1) \right] \hat{v}(-k) \right),
\]

for any \( u, v \in Z_1 \).

It is easy to see that there exist \( a > 0 \) such that

\[
|E_K[u, v]| \leq a \| u \|_1 \| v \|_1,
\]

and

\[
E_K[u, u] \geq \frac{\sigma}{2} \| u \|_1^2 - 3 \sum_{k=K}^\infty \hat{u}(k+1) \hat{u}(-k) + \sum_{k=-K}^{-\infty} \hat{u}(k-1) \hat{u}(-k) \geq \frac{\sigma}{4} \| u \|_1^2,
\]

the last inequality is the reason that for \( |n| \geq K \), we have \( \frac{\sigma}{4} n^2 \geq 2 \).

Hence by Lax-Milgram theorem, we see that for any \( f \in \dot{H}^{r-2} \), there exists only one \( u_K \in Z_1 \) such that \( E_K[u_K, v] = (Q_K f, v)_{L^2} \) for any \( v \in Z_1 \) and so \( A_K u_K = Q_K f \) for some \( u_K \in Z_1 \). We also have

\[
\| Q_K f \|_r^2 \geq 2 \sum_{k=K}^\infty \frac{\sigma^2}{4} k^{2r} |\hat{u}_K(k)|^2 - 4 \sum_{k=K}^\infty \frac{k^{2r-2}(k+1)^2}{(k+2)^2} |\hat{u}_K(k+1)|^2 \geq \frac{\sigma^2}{4} \| u_K \|_{r-2}^2 - 2 \| u_K \|_{r-2}^2 \geq \frac{\sigma^2}{8} \| u_K \|_r^2, \quad (6.17)
\]
for \( \frac{\sigma}{4} n^2 \geq 2 \).

Let us consider the linear operator \( A \). It can be written as

\[
Au = \sum_{k=2}^{K-1} \left( \frac{\sigma}{2} k^2 \hat{u}(k) - \left( 1 + a_\mu \right) \frac{k+1}{K} \hat{u}(k+1) \right) e^{ika} + A_K Q_k u + c.c.
\]

for \( u \in \dot{H}^r \).

For any \( f \in \dot{H}^{r-2} \), there exists only one solution \( u_K \in Z_r \) such that \( A_K u_K = Q_K f \). Then using \( u_K \), we consider the following finite linear equation system for \( (b_{K-1}, b_{K-2}, b_2, b_2, \ldots b_{-K+1})^T \)

\[
\begin{pmatrix}
\frac{\sigma}{2} (K-1)^2 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0
\
-K \frac{1}{K} (1 + a_\mu) & \frac{\sigma}{2} (K-2)^2 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0
\
0 & -\left( 1 + a_\mu \right) \frac{K-1}{K} & \frac{\sigma}{2} (K-3)^2 & 0 & \ldots & 0 & \ldots & 0 & 0 & 0
\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots
\end{pmatrix}
\begin{pmatrix}
b_{K-1}
\vdots
b_{K-2}
\vdots
b_{-K+1}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\hat{f}(K-1) + (1 + a_\mu) \frac{K}{K+1} \hat{u}_K(K)
\hat{f}(K-2)
\vdots
\hat{f}(-K+1) + (1 + a_\mu) \frac{K}{-K+1} \hat{u}_K(-K)
\end{pmatrix}
\]

(6.18)

It is easy to from the triangle structure see that there exists only one solution \( (b_{K-1}, \ldots, b_2, b_2, \ldots, b_{-K+1}) \). Then we choose \( u = \sum_{k=2}^{K-1} b_k e^{ika} + \sum_{k=-2}^{-K+1} b_k e^{ika} + u_K \) and \( Au = f \). Since \( u_K \in H^r_p \), we induce \( u \in H^r_p \).
Hence, for any \( f \in \dot{H}^{r-2} \), there is only one \( u = A^{-1}f \in \dot{H}^r \). By (6.17), \( \|A^{-1}f\|_r \leq C_r\|f\|_{r-2} \).

**Proposition 6.8.** For any surface tension \( \sigma > 0 \), \( r \geq 3 \), and sufficiently small \( \epsilon \), there exists a neighborhood \( O \) of \((0,0)\) and a ball \( B^r_\epsilon \subset \dot{H}^r \) such that \( \tilde{\theta}(s) : O \rightarrow B^r_\epsilon \) with \( Q_1 \mathfrak{U}[\tilde{\theta}(s)(u_0, \beta), u_0, \beta] = 0 \). Further, \( \tilde{\theta}(s)(u_0, \beta; \alpha) \) is odd with respect to \( \alpha \) for any \((u_0, \beta) \in O\).

**Proof.** We define the operator \( T \) by

\[
T \tilde{\theta}(s) \equiv A^{-1}Q_1 \mathfrak{U}[\tilde{\theta}(s), u_0, \beta].
\]

By Lemma 6.5 and Proposition 6.7, for sufficient small \( \epsilon \), there exists a neighborhood \( O \) of \((0,0)\), such that the operator \( T \) is the contraction map in the ball \( B^r_\epsilon \) for \((u_0, \beta) \in O\).

Hence, by contraction mapping theorem, there exist open sets \( O \subset \mathbb{R}^2 \) such that \( \tilde{\theta}(s) = T \tilde{\theta}(s) \) in the ball \( B^r_\epsilon \subset \dot{H}^r \) for \((u_0, \beta) \in O\). By (6.3), we have

\[
Q_1 \mathfrak{U}[\tilde{\theta}(s)(u_0, \beta; \alpha), u_0, \beta] = 0.
\]

For any \((u_0, \beta) \in O\), we define \( \eta(\alpha) = -\tilde{\theta}(s)(u_0, \beta; -\alpha) - \tilde{\theta}(s)(1)e^{i\alpha} - \tilde{\theta}(s)(1)e^{-i\alpha}, \)
\( v(\alpha) = -(z(s)(-\alpha))^*, \) and \( \xi(\alpha) = -\gamma(s)(-\alpha). \) Then it is easy to check that

\[
\text{Re} \left( \frac{z^{(s)}(-\alpha)}{2\pi i} \right) \text{PV} \int_0^{2\pi} \gamma^{(s)}(\alpha')K(-\alpha, \alpha')d\alpha' \\
= -\text{Re} \left( \frac{v^{(s)}(\alpha)}{2\pi i} \right) \text{PV} \int_0^{2\pi} \xi(\alpha') \left\{ \frac{\beta}{4} \text{coth} \left[ \frac{\beta}{4}(v(\alpha)-v(\alpha')) \right] - \frac{\beta}{4} \tanh \left[ \frac{\beta}{4}(v(\alpha)-v^*(\alpha')) \right] \right\} d\alpha'.
\]

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Hence, $Q_1\mathcal{U}[Q_1\eta(\alpha), u_0, \beta] = Q_1\mathcal{U}[\tilde{\theta}^{(s)}(\alpha), u_0, \beta] = 0$ with $\xi(\alpha)$ satisfying (E.2).

Also by uniqueness, we have $\tilde{\theta}^{(s)}(u_0, \beta; \alpha) = Q_1\eta(\alpha) \equiv -\tilde{\theta}^{(s)}(u_0, \beta; -\alpha)$. \qed

**Note 6.9.** Note that the $\tilde{\theta}^{(s)}$ of Proposition 6.8 is not the steady state since we only required $Q_1\mathcal{U} = 0$ instead of $Q_0\mathcal{U} = \mathcal{U} = 0$. Here $u_0$ is arbitrary. The additional condition $(Q_0 - Q_1)\mathcal{U} = 0$ can be satisfied by constraining $u_0$ appropriately. The usefulness of this Proposition is to show any steady solution $\theta^{(s)}$ that actually satisfies $Q_0\mathcal{U} = 0$ must be an odd function since this is true for any sufficient small $u_0$.

**Definition 6.10.** We define a family of Banach spaces $\{X_r\}_{r \geq 0}$ by

$$X_r = \{ u \in \dot{H}^r | u(-\alpha) = -u(\alpha) \},$$

$$Y_r = \{ u \in H^r_p | Q_0 u = u, u(-\alpha) = u(\alpha) \}.$$

**Remark.** Proposition 6.8 shows us that the shape of the steady bubble must be symmetric with the center of the channel. Also $\mathcal{U} : X_r \times \mathbb{R}^2 \to Y_{r-2}$. Hence it is reasonable to consider the solution $(\tilde{\theta}^{(s)}, u_0, \beta)$ to (C.1)-(C.3) in the space $X_r \times \mathbb{R}^2$.

**Proof of Theorem 1.14:** Let $f = \mathcal{M}^{(s)}[\tilde{\theta}^{(s)}, u_0, \beta] - \frac{\partial^2}{2} Q_0 \mathcal{U}[0, 0, 0]$ and $g = A^{-1}\mathcal{H}(Q_1 f)$. Actually it is easy to check that $f(-\alpha) = f(\alpha)$ and $g(-\alpha) = -g(\alpha)$ for $\tilde{\theta}^{(s)} \in X_r$.

We define an operator $\mathcal{T}$ in $X_r \times \mathbb{R}$ by

$$\mathcal{T}[\tilde{\theta}^{(s)}, u_0] = \left( A^{-1}\mathcal{H}(Q_1 f), 2\hat{f}(1) + \frac{\mu_1 + \mu_2}{\mu_1} \left( \frac{4}{3} + \frac{4}{3} a_\mu \right) i\hat{g}(2) \right)^T.$$
By Lemma 6.5 and Proposition 6.7, for sufficient small \( \epsilon \), there exist an open set \( O_1 \subset \mathbb{R} \) and a ball \( O_2 \subset X_r \times \mathbb{R} \) such that \( \mathcal{T} \) is the contraction map in the ball \( O_2 \) for any \( \beta \in O_1 \). Hence, by contraction mapping theorem, we have \( (\hat{\theta}^{(s)}, u_0)^T = \mathcal{T}[\hat{\theta}^{(s)}, u_0] \) for any \( \beta \in O_1 \). By (6.3), we have

\[
\mathcal{Q}_0\mathcal{U}[\hat{\theta}^{(s)}(\beta; \alpha), u_0(\beta), \beta] = 0.
\]

By Lemma 6.5 and Proposition 6.7, for sufficiently small \( \epsilon \) and \( \Upsilon \), there exists \( C \) independent of \( \epsilon \) and \( \Upsilon \), such that

\[
\|\hat{\theta}^{(s)}\|_r + |u_0| \leq C \beta^2.
\]

By (E.2) that

\[
\gamma^{(s)}(\alpha)
= 2 \sin \alpha - a_\mu \Re \left( \frac{z^{(s)}}{\pi i} \PV \int_{\alpha-\pi}^{\alpha+\pi} \frac{\gamma^{(s)}(\alpha')}{z^{(s)}(\alpha') - z^{(s)}(\alpha')} d\alpha' \right)
- a_\mu \Re \left( \frac{\mu_2}{2\pi i} \sum_{n=1}^{\infty} \frac{2B_{2n}}{(2n)!} (-1)^n \beta^{2n} \int_0^{2\pi} \gamma^{(s)}(\alpha') (z^{(s)}(\alpha) - z^{(s)}(\alpha'))^{2n-1} d\alpha' \right)
+ \frac{2}{\mu_1 + \mu_2} u_0 \sin (\alpha + \theta^{(s)}(\alpha)),
\]

where \( B_n \) is \( n \)th Bernoulli number. By (6.19) and Lemma 2.37, we have

\[
\|\gamma^{(s)} - 2 \sin \alpha\|_{r-2} \leq C \beta^2,
\]

where \( C \) depends on \( \epsilon \) and \( \Upsilon \).

**Remark.** Since we consider the steady solution in \( \dot{H}^r \) for \( r \geq 3 \), where \( r \) is arbitrary, by uniqueness shown in Theorem 1.14, the steady solution is in \( H^\infty \), and hence in
$C^\infty$. The result is consistent with analyticity results for arbitrary channel width in the small $\sigma$ limit [47].

\[\square\]

**Note 6.11.** Actually for $\mu_2 = 0$, by formal expansion in correspondence to earlier calculation using conformal mapping [36], we have

\[\theta^{(s)}(\alpha) = \beta^4\left(\frac{1}{54\sigma} \sin(3\alpha) + \frac{1}{72\sigma^2} \sin(2\alpha)\right) + O(\beta^6),\]

\[u_0 = -\frac{\beta^2}{6} + \beta^4\left(\frac{7}{180} + \frac{1}{216\sigma^2}\right) + O(\beta^6),\]

\[\gamma^{(s)}(\alpha) = 2\sin\alpha - \frac{\beta^2}{6} \sin\alpha + \beta^4\left(-\frac{19}{120} + \frac{1}{72\sigma^2}\right) \sin(3\alpha) + \left(\frac{1}{72} + \frac{7}{216\sigma^2}\right) \sin\alpha\]

\[+ \frac{1}{54\sigma} \sin(4\alpha) - \frac{1}{54\sigma} \sin(2\alpha)\right) + O(\beta^6).\]

For steady states, two fluid flows can be related to one fluid flow by transform variables [39].
CHAPTER 7
EVOLUTION OF SYMMETRIC BUBBLE WITH SIDEWALLS (β > 0)

Lemma 7.1. If initial conditions satisfy the symmetry condition

\[ \theta_0(-\alpha) = -\theta_0(\alpha), \quad y_0 = 0, \]

then the corresponding solution \((\theta(\alpha, t), L(t), y(0, t))\) in \(H^1_p \times C^1 \times C^1\) to (D.1)-(D.2) satisfy symmetry condition for all time, i.e. \(\theta(-\alpha, t) = -\theta(\alpha, t)\) and \(y(0, t) = 0\).

The corresponding vortex sheet strength \(\gamma(\alpha, t)\), determined from (D.3) also obeys the symmetry condition \(\gamma(-\alpha, t) = -\gamma(\alpha, t)\) and the bubble shape is symmetric about the channel centerline (x-axis).

Proof. If \(\theta_0\) is odd and \(y(0,0) = 0\), it follows from (2.2) and (2.3), that \(z^*(\alpha, 0) = z(-\alpha, 0)\) and we have a symmetric bubble to start with. The corresponding vortex sheet strength determined from (D.3) \(\gamma(\alpha, t)\) is easily to be odd. Again, it is readily checked that that if \((\theta(\alpha, t), \gamma(\alpha, t), L(t), y(0, t))\) solve (D.1)-(D.4), then so does \((-\theta(-\alpha, t), -\gamma(-\alpha, t), L(t), -y(0, t))\). Since the initial condition is symmetric, it follows from local uniqueness of solution (see §A.4) that symmetry is preserved in time.

From the geometric relation

\[ z(\alpha, t) = \frac{iL(t)}{2\pi} \int_0^\alpha e^{i\alpha + i\theta(\alpha, t)} d\alpha + z(0, t), \]

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symmetry about the $x$ ($Re \, z$) axis follows.

**Remark.** Symmetry implies $\hat{\theta}(0; t) = 0 = y(0, t)$. and so the evolution equation for $\hat{\theta}(0; t)$ in (D.1) and $y(0, t)$ in (1.15) can be ignored. For the symmetry bubble, we also have

$$R(\alpha, \alpha') = \frac{\beta}{2} \coth \left[ \frac{\beta}{2} (z(\alpha) - z(\alpha')) \right].$$

Proposition 6.8 implies that the steady bubble solution $(\theta(s)(\alpha), \gamma(s)(\alpha))$ are also odd functions of time.

**7.1 Main results for the translating bubble in the strip**

In this section, we first state the main results for the translating bubble.

It is convenient to define

**Definition 7.2.**

$$\Gamma(\alpha, t) = \gamma(\alpha, t) - \gamma(s)(\alpha),$$

$$\theta(\alpha, t) = \tilde{\Theta}(\alpha, t) + \tilde{\theta}(s)(\alpha) + \tilde{\theta}(-1; t)e^{-i\alpha} + \tilde{\theta}(1; t)e^{i\alpha}. \quad (7.1)$$

In this section, we will find solutions $\tilde{\Theta}$ which satisfy (D.1) with the initial condition with the initial condition

$$\tilde{\Theta}(\alpha, 0) = \tilde{\Theta}_0(\alpha) \equiv Q\left[ \theta_0 - \theta(s) \right](\alpha). \quad (7.2)$$
We will also consider the motion of the interface with small symmetric perturbation around the steady bubble. Since the bubble area is invariant with time, we take \( V \) to be the steady bubble area, i.e.

\[
V = \frac{1}{2} \text{Im} \int_0^{2\pi} z^{(s)}(z^{(s)})^* d\alpha. \tag{7.3}
\]

The main result in this section is the following proposition:

**Proposition 7.3.** For \( \sigma > 0 \), there exist \( \epsilon, \Upsilon > 0 \) such that for \( r \geq 3 \), if \( \| \tilde{\Theta}(\cdot, 0) \|_r < \epsilon, 0 < \beta < \Upsilon \), then for initial shape symmetric about channel centerline, i.e. \( \tilde{\Theta}(-\alpha, 0) = -\tilde{\Theta}(\alpha, 0) \), there exists a global solution \( \tilde{\Theta} \in \dot{H}_r \) to the Hele-Shaw initial value problem with the initial condition (7.2). Furthermore, \( \| Q_0 \Theta \|_r \) decays exponentially as \( t \to \infty \). Thus the translating steady bubble is asymptotically stable for sufficiently small symmetric initial disturbances in the \( H^r_p \) space.

**Note 7.4.** Proposition 7.3 and Lemma 2.9 imply Theorem 1.16.

### 7.2 Evolution equation in integral form

It is readily checked that \( \Gamma(\alpha, t) \) satisfies

\[
\left( I + a_\mu F[z] \right) \Gamma = -a_\mu F[z] \gamma^{(s)} + a_\mu F[z^{(s)}] \gamma^{(s)} + \frac{2\pi - L}{L} \sigma \theta_{\alpha\alpha} + \sigma(\theta - \theta^{(s)})_{\alpha\alpha} + \frac{L - 2\pi}{\pi} \left( 1 + \frac{\mu_2}{\mu_1 + \mu_2} u_0 \right) \sin(\alpha + \theta) + 2 \left( 1 + \frac{\mu_2}{\mu_1 + \mu_2} u_0 \right) \left( \sin(\alpha + \theta) - \sin(\alpha + \theta^{(s)}) \right). \tag{7.4}
\]

Hence, we have
Proposition 7.5. If $\tilde{\Theta} \in \dot{H}^r$ with $\|\tilde{\Theta}\|_1 < \epsilon_1$, and $0 \leq \beta < \Upsilon$ then for sufficiently small $\varepsilon_1$ and $\Upsilon$, there exists a unique solution $\Gamma \in \{u \in H^{s-2}_p|\dot{u}(0) = 0\}$ for $r \geq 3$ satisfying (7.4). This solution $\Gamma$ satisfies the estimates

$$
\|\Gamma\|_0 \leq C\|\tilde{\Theta}\|_2,
$$

$$
\|\Gamma\|_{s-2} \leq C_1 \exp(C_2\|\tilde{\Theta}\|_{s-2})\|\tilde{\Theta}\|_r,
$$

where $C_1$ and $C_2$ depend on $r$.

Let $\Gamma^{(1)}$ and $\Gamma^{(2)}$ correspond to $\tilde{\Theta}^{(1)}$ and $\tilde{\Theta}^{(2)}$ respectively. Assume $\|\tilde{\Theta}^{(1)}\|_1 < \epsilon_1$ and $\|\tilde{\Theta}^{(2)}\|_1 < \epsilon_1$. If $\tilde{\Theta}^{(1)}, \tilde{\Theta}^{(2)} \in \dot{H}^r$ with $r \geq 3$, then for sufficient small $\epsilon_1$,

$$
\|\Gamma^{(1)} - \Gamma^{(2)}\|_{s-2} \leq C_1 \exp(C_2(\|\tilde{\Theta}^{(1)}\|_r + \|\tilde{\Theta}^{(2)}\|_r))\|\tilde{\Theta}^{(1)} - \tilde{\Theta}^{(2)}\|_r
$$

(7.5)

where $C_1$ and $C_2$ depend on $r$ alone.

Proof. In Proposition 2.38, we take $\gamma^{(2)} = \gamma, \tilde{\theta}^{(1)} = \tilde{\theta}, L^{(1)} = L$,

$$
\gamma^{(2)} = \gamma^{(s)}, \quad \tilde{\theta}^{(2)} = \tilde{\theta}^{(s)}, \quad L^{(2)} = 2\pi
$$

and use Lemma 2.37 to obtain the first two statements. The statement (7.5) follows in a similar manner from (2.50).

\[\square\]

Definition 7.6. We define the function

$$
\omega_s(\alpha) = \int_{0}^{\alpha} e^{i\tau + i\theta^{(s)}(\tau)} d\tau.
$$

And we also define a complex valued linear operator $\mathcal{F}[f]$ by

$$
\mathcal{F}[f](\alpha) = \frac{1}{2\pi} \int_{0}^{2\pi} \sin(\alpha') \int_{0}^{\alpha} \frac{Q_0(f(\eta)\omega_s(\eta))d\eta}{\omega_s(\alpha) - \omega_s(\alpha')} d\alpha'.
$$
Then the evolution equation (D.1) translates into the following equation for $\Theta$:

$$
\tilde{\Theta}_t(\alpha, t) = \frac{2\pi}{L} Q_1(U_\alpha + T(1 + \theta_\alpha)) = A[\tilde{\Theta}] + \mathcal{L}_\beta[\tilde{\Theta}] + N[\tilde{\Theta}].
$$

(7.6)

where

$$
\mathcal{L}_\beta[\tilde{\Theta}](\alpha, t) = Q_1\left\{ \left( \frac{1}{2} \mathcal{H}(L_1[\tilde{\Theta}]) \right)(\alpha, t) + \mathcal{L}_{\beta_1}[\tilde{\Theta}](\alpha, t) \right\}, \quad (7.7)
$$

$$
N[\tilde{\Theta}](\alpha, t) = \frac{2\pi}{L} Q_1\left\{ \left( \frac{1}{2} \mathcal{H}(N_1[\tilde{\Theta}]) \right)(\alpha, t) + N_2[\tilde{\Theta}](\alpha, t) \right\}
+ \frac{2\pi - L}{L} \left\{ \sum_{k=2}^{\infty} (1 + a_\mu) \frac{(k^2 - 1)(k + 1)}{k(k + 2)} \tilde{\Theta}(k + 1)e^{ik\alpha} \right\}
- \sum_{k=-2}^{-\infty} (1 + a_\mu) \frac{(k^2 - 1)(k - 1)}{k(k - 2)} \tilde{\Theta}(k - 1)e^{ik\alpha} + \mathcal{L}_\beta[\tilde{\Theta}](\alpha, t), \quad (7.8)
$$

with

$$
\mathcal{L}_{\beta_1}[\tilde{\Theta}](\alpha) = a_\mu \text{Re} \left( -\frac{1}{i} \mathcal{G}[z^{(s)}]\Gamma \right) + a_\mu \text{Re} \left( -\frac{1}{i} \mathcal{G}_1[z^{(s)}](\gamma^{(s)} - 2 \sin \alpha) + \frac{1}{i} \mathcal{G}_1[\omega_s](\gamma^{(s)} - 2 \sin \alpha) \right)
- a_\mu \text{Re} \left( z_\alpha \mathcal{K}_2(z^{(s)}) (\gamma^{(s)}(\alpha) - i \omega_\alpha \mathcal{K}_2(z^{(s)}) \gamma^{(s)}(\alpha)) \right)
- 4a_\mu D_\alpha \text{Re} \left( \mathfrak{F}[\Theta](\alpha) - \mathfrak{W}[\Theta](\alpha) \right)
+ \frac{L - 2\pi}{\pi} (\sin(\alpha + \theta^{(s)}) - \sin \alpha) + 2\Theta(\cos(\alpha + \theta^{(s)}) - \cos \alpha) + \frac{2\pi - L}{L} \sigma_{\theta^{(s)}}
+ \frac{L - 2\pi}{\pi} \frac{\mu_2}{\mu_1 + \mu_2} u_0 \sin(\alpha + \theta) + 2 \frac{\mu_2}{\mu_1 + \mu_2} u_0 \left( \sin(\alpha + \theta) - \sin(\alpha + \theta^{(s)}) \right),
$$

$$
\mathcal{L}_{\beta_2}[\tilde{\Theta}] = \frac{2\pi - L}{2L} \mathcal{H}[\gamma^{(s)} - 2 \sin \alpha'] + \text{Re} \left( \frac{1}{2} \mathcal{G}_1[z^{(s)}]) \Gamma \right) + \text{Re} \left( \frac{\pi}{L} \mathcal{G}_1[z^{(s)}] (\gamma^{(s)}(\alpha) - 2 \sin \alpha) \right)
- \frac{1}{2} \mathcal{G}_1[\omega_s] (\gamma^{(s)}(\alpha) - 2 \sin \alpha) + 2 \text{Re} \left( -\omega_\alpha \mathcal{K}_2[z^{(s)}] \gamma^{(s)}(\alpha) + \omega_\alpha \mathcal{K}_2[z^{(s)}] \gamma^{(s)}(\alpha) \right)
+ \frac{2\pi - L}{L} \text{Re} \left( \mathcal{G}_1[\omega_s] \sin \alpha \right) + \text{Re} \left( 2i \frac{\partial}{\partial \alpha} \mathfrak{F}[\Theta](\alpha) - 2i \frac{\partial}{\partial \alpha} \mathfrak{W}[\Theta](\alpha) \right)
+ u_0 \left[ \cos(\alpha + \theta(\alpha)) - \cos(\alpha + \theta^{(s)}(\alpha)) \right],
$$

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\[ L_{\beta_3}[\tilde{\Theta}] = \left( \int_0^\alpha \theta_\alpha^{(s)}(\alpha') \left( \frac{2\pi^2}{L^2} \sigma \mathcal{H}(\Theta_{\alpha\alpha})(\alpha') + L[\tilde{\Theta}](\alpha') \right) d\alpha' \right) \]
\[-\frac{\alpha}{2\pi} \int_0^\alpha \theta_\alpha^{(s)}(\alpha) \left( \frac{2\pi^2}{L^2} \sigma \mathcal{H}(\Theta_{\alpha\alpha})(\alpha') + L[\tilde{\Theta}](\alpha') \right) d\alpha \right) (1 + \theta_\alpha^{(s)}) \]
\[+ \theta_\alpha^{(s)} \left( \int_0^\alpha \theta_\alpha^{(s)}(\alpha) \left( \frac{2\pi^2}{L^2} \sigma \mathcal{H}(\Theta_{\alpha\alpha})(\alpha') + L[\tilde{\Theta}](\alpha') \right) d\alpha' - \frac{\alpha}{2\pi} \int_0^\pi L[\tilde{\Theta}](\alpha) d\alpha \right) \]
\[+ \left\{ \int_0^\alpha (1 + \theta_\alpha^{(s)}(\alpha')) \left[ \frac{1}{2} \mathcal{H}\left( \mathcal{L}_{\beta_1}[\tilde{\Theta}] \right)(\alpha) + \mathcal{L}_{\beta_2}[\tilde{\Theta}](\alpha) \right] d\alpha' \right\} (1 + \theta_\alpha^{(s)}), \]

\[ N_1[\tilde{\Theta}] = a_\mu \left( - \frac{1}{L} \mathcal{G}[z] \Gamma + \frac{1}{L^2} \mathcal{G}[z^{(s)}] \Gamma \right) + \frac{L - 2\pi}{2\pi} \left( \sin(\alpha + \theta) - \sin(\alpha + \theta_\alpha^{(s)}(\alpha)) \right) \]
\[+ 2 \left( \sin(\alpha + \theta) - \sin(\alpha + \theta_\alpha^{(s)}(\alpha)) - \Theta \cos(\alpha + \theta_\alpha^{(s)}(\alpha)) \right) - 2a_\mu \left( \frac{1}{L} \frac{\partial}{\partial \alpha} \left\{ \mathcal{G}[\Xi_\alpha^e[\Theta]](\alpha) \right\} \right) \]
\[+ 2a_\mu \left( \frac{\omega_\alpha^{(s)}}{\pi i} \int_{-\pi}^{\alpha + \pi} \sin(\alpha') \left( \frac{\mathcal{G}_1[\omega] \sin(\alpha) - \cos(\alpha)}{q_1[\omega_\alpha](\alpha, \alpha')} \right) \left( \frac{1}{\omega(\alpha) - \omega(\alpha')} - \frac{1}{\omega_\alpha(\alpha) - \omega_\alpha(\alpha')} \right) d\alpha' \right), \]

\[ N_2[\tilde{\Theta}] = \frac{2\pi - L}{2L} \mathcal{H}[\Gamma - \frac{2\pi}{L} \sigma \Theta_{\alpha\alpha}] + \left( \frac{\pi}{L} \frac{\mathcal{G}[z] \Gamma - \frac{1}{2} \mathcal{G}[z^{(s)}] \Gamma}{L} \right) \]
\[+ \frac{2\pi}{L} \left( \mathcal{G}_1[\omega] \sin(\alpha) - \mathcal{G}_1[\omega_\alpha] \sin(\alpha) \right) + \left( \frac{\partial}{\partial \alpha} \left\{ \mathcal{G}[\Xi_\alpha^e[\Theta]](\alpha) \right\} \right) \]
\[+ \left( \frac{\omega_\alpha^{(s)}}{\omega_\alpha} \right) \left( \mathcal{G}_1[\omega] \sin(\alpha) - \cos(\alpha) \right) - \left( \mathcal{G}_1[\omega_\alpha] \sin(\alpha) - \cos(\alpha) \right) \]
\[+ \left( \cos(\alpha + \theta(\alpha)) - \cos(\alpha + \theta_\alpha^{(s)}(\alpha)) \right) + \Theta \sin(\alpha + \theta_\alpha^{(s)}) \right) \]
\[-\left( \frac{\omega_\alpha^{(s)}}{\pi} \int_{\alpha - \pi}^{\alpha + \pi} \sin(\alpha') \left( \frac{q_1[\omega - \omega_\alpha](\alpha, \alpha')}{q_1[\omega_\alpha](\alpha, \alpha')} \right) \left( \frac{1}{\omega(\alpha) - \omega(\alpha')} - \frac{1}{\omega_\alpha(\alpha) - \omega_\alpha(\alpha')} \right) d\alpha' \right), \]
\[ N_\beta[\tilde{\Theta}] = \left\{ \int_0^\alpha (1 + \theta^{(s)}_\alpha(\alpha')) \left[ \frac{1}{2} \mathcal{H}\left( N_1[\tilde{\Theta}] (\cdot) \right)(\alpha') + N_2[\tilde{\Theta}](\alpha') \right] d\alpha' \right. \\
- \frac{\alpha}{2\pi} \int_0^{2\pi} (1 + \theta^{(s)}_\alpha(\alpha)) \left[ \frac{1}{2} \mathcal{H}\left( N_1[\tilde{\Theta}] (\cdot) \right)(\alpha) + N_2[\tilde{\Theta}](\alpha) \right] d\alpha \\
+ \int_0^\alpha \Theta_\alpha(\alpha') U(\alpha') d\alpha' - \frac{\alpha}{2\pi} \int_0^{2\pi} \Theta_\alpha(\alpha) U(\alpha) d\alpha \left\{ 1 + \theta^{(s)}_\alpha \right. \\
+ \left( \int_0^\alpha \Theta_\alpha(\alpha') U(\alpha') d\alpha' - \frac{\alpha}{2\pi} \Theta_\alpha(\alpha) U(\alpha) d\alpha \right) \Theta_\alpha(\alpha), \]

where

\[ \mathcal{L}[\tilde{\Theta}] = \frac{1}{2} \mathcal{H}[\mathcal{L}_1](\alpha, t) + \frac{L - 2\pi}{L} \cos \alpha - Q_0 \theta \sin \alpha + \text{Re} \left( i \frac{\partial}{\partial \alpha} \mathcal{M}[\Theta](\alpha) \right), \]

\[ \mathcal{L}_1[\tilde{\Theta}](\alpha) = 2\Theta(\alpha, t) \cos \alpha + \frac{L - 2\pi}{\pi} \sin \alpha - 4a_\mu \text{Re} \left( \frac{\partial}{\partial \alpha} \mathcal{M}[\Theta](\alpha) \right). \]

We can integrate the evolution equation (7.6) and rewrite it as the following integral equation:

\[ \tilde{\Theta}(\alpha, t) = e^{tA}\tilde{\Theta}_0 + \int_0^t e^{(t-\tau)A} \left( \mathcal{L}_1[\tilde{\Theta}] + N[\tilde{\Theta}] \right)(\alpha, \tau) d\tau \equiv \mathcal{R}[\tilde{\Theta}](\alpha, t). \quad (7.9) \]

We will eventually show that \( \mathcal{R} \) defines a contraction in a sufficiently small ball in the \( X_r \) space for \( r \geq 3 \). For that purpose we need some properties.

**Proposition 7.7.** If for \( s \geq 3 \), \( \tilde{\Theta} \in \dot{H}^r \) with \( \| \tilde{\Theta} \|_1 < \varepsilon_1 \), and \( 0 \leq \beta < \Upsilon \), then for sufficiently small \( \varepsilon_1 \) and \( \Upsilon \), the functions \( \mathcal{L}_\beta \) and \( \mathcal{N} \), defined in (7.7) and (7.8), satisfy the following estimates

\[ \left\| \mathcal{L}_\beta \right\|_{r-1} \leq C_1 \beta^2 \exp \left( C_2 \| \tilde{\Theta} \|_r \right) \| \tilde{\Theta} \|_r, \]

\[ \left\| \mathcal{N} \right\|_{r-1} \leq C_1 \exp \left( C_2 \| \tilde{\Theta} \|_r \right) \| \tilde{\Theta} \|_r \| \tilde{\Theta} \|_{r+1}, \]
where $C_1$ and $C_2$ depend only on $r$. Further, let $(\mathcal{L}_\beta^{(1)}, \mathcal{N}^{(1)})$ and $(\mathcal{L}_\beta^{(2)}, \mathcal{N}^{(2)})$ correspond to $\tilde{\Theta}^{(1)}$ and $\tilde{\Theta}^{(2)}$ respectively, each in $\dot{H}^r$ with $\|\tilde{\Theta}^{(1)}\|_1$ and $\|\tilde{\Theta}^{(2)}\|_1 < \varepsilon_1$. Then for sufficiently small $\varepsilon_1$,

$$
\|L^{(1)} - L^{(2)}\|_{r-1} \leq C_1\beta^2 \exp \left( C_2 \left( \|\tilde{\Theta}^{(1)}\|_s + \|\tilde{\Theta}^{(2)}\|_s \right) \right) \|\tilde{\Theta}^{(1)} - \tilde{\Theta}^{(2)}\|_s,
$$

$$
\|\mathcal{N}^{(1)} - \mathcal{N}^{(2)}\|_{r-1} \leq C_1 \exp \left( C_2 \left( \|\tilde{\Theta}^{(1)}\|_s + \|\tilde{\Theta}^{(2)}\|_s \right) \right) \} \left\{ \|\tilde{\Theta}^{(1)}\|_r + \|\tilde{\Theta}^{(2)}\|_r \right\} \|\tilde{\Theta}^{(1)} - \tilde{\Theta}^{(2)}\|_s,
$$

where $C_1$ and $C_2$ depend on $r$.

**Proof.** On using Lemmas 2.16 (see Note 2.17), 2.25, 2.27, 2.29, 2.37, 2.33 and Proposition 7.5, the proof follows from the expressions of $L_\beta$ and $\mathcal{N}$.

**Remark.** It is easily to check that $(L_\beta[\tilde{\Theta}] + \mathcal{N}[\tilde{\Theta}])(-\alpha) = -(L_\beta[\tilde{\Theta}] + \mathcal{N}[\tilde{\Theta}])(\alpha)$. □

### 7.3 Contraction properties of $\mathcal{R}$ and global existence for symmetric disturbances

**Note 7.8.** For the linear evolution equation (5.34), if $f$ and $v_0$ are odd with respect to $\alpha$, then by uniqueness of the linear equation (5.34), $v(-\alpha, t) = -v(\alpha, t)$.

First, by Proposition 7.7, we have

**Lemma 7.9.** Assume $0 \leq \beta < \Upsilon$. Suppose for $r \geq 3$ $\tilde{\Theta}(\alpha, t) \in X_r$ satisfy the condition $\|\tilde{\Theta}\|_{H^r} \leq \varepsilon$. Then for $L_\beta[\tilde{\Theta}](\alpha, t)$ and $\mathcal{N}[\tilde{\Theta}](\alpha, t)$ determined from (7.7) and (7.8), as $\varepsilon$ and $\Upsilon$ are small enough, we have

$$
\|L_\beta[\tilde{\Theta}] + \mathcal{N}[\tilde{\Theta}]\|_{H^{r-3}} \leq C\|\tilde{\Theta}\|_{H^r} \left( \|\tilde{\Theta}\|_{H^r} + \beta^2 \right).
$$
Further, if both $\tilde{\Theta}^{(1)}(\alpha, t)$ and $\tilde{\Theta}^{(2)}(\alpha, t)$ satisfy (7.9), then the corresponding $(L^{(1)}_\beta, N^{(1)})$ and $(L^{(2)}_\beta, N^{(2)})$ satisfy
\[
\|L^{(1)}_\beta - L^{(2)}_\beta\|_{H^r_{\sigma^{-3}}} + \|N^{(1)} - N^{(2)}\|_{H^r_{\sigma^{-3}}} \leq C(\epsilon + \beta^2)\|\tilde{\Theta}^{(1)} - \tilde{\Theta}^{(2)}\|_{H^r_{\sigma}}.
\]

Hence, by Lemmas 5.14 and 7.9, we have

**Lemma 7.10.** Assume $0 \leq \beta < \Upsilon$. Let $r \geq 3$, $\|\tilde{\Theta}_0\|_{w,r} < \frac{\epsilon}{2}$ and $\tilde{\Theta} \in X_r$ with $\|\tilde{\Theta}\|_{H^r_{\sigma}} \leq \epsilon$. For sufficiently small $\epsilon$ and $\Upsilon$, the operator $R$ defined in (7.9) satisfies the following estimate:
\[
\|R[\tilde{\Theta}]\|_{H^r_{\sigma}} \leq C\epsilon.
\]

Further, if $\|\tilde{\Theta}^{(1)}\|_{H^r_{\sigma}} \leq \epsilon$ and $\|\tilde{\Theta}^{(2)}\|_{H^r_{\sigma}} \leq \epsilon$, then
\[
\|R[\tilde{\Theta}^{(1)}] - R[\tilde{\Theta}^{(2)}]\|_{H^r_{\sigma}} \leq C\epsilon\|\tilde{\Theta}^{(1)} - \tilde{\Theta}^{(2)}\|_{H^r_{\sigma}}.
\]

Further, $R[\tilde{\Theta}](\alpha) = -R[\tilde{\Theta}](\alpha)$.

**Proof of Proposition 7.3:** If $C\epsilon < 1$, then it is clear that the right side of (7.9) define a contraction map in an $\epsilon$ ball in the Banach space $X_r \cap H^r_{\sigma}$. Therefore, there exists a unique solution $\tilde{\Theta}$ satisfying the equation (7.9), hence (D.1). The local uniqueness of solutions (see §A.4) implies that this is the only solution. The $e^{-\sigma t/2}$ exponential decay of $\tilde{\Theta}$ and hence of $\Theta$ implies that the steady symmetric translating bubble is approached exponentially in time. The constraint condition (D.5) shows that $L - 2\pi$ decays exponentially.
Appendix A

APPENDIX

A.1 Proof of Lemma 2.23 ([1]):

Proof. We note that

\[ D^k_{\alpha}q_1[\omega] = \int_0^1 t^k D^k_{\alpha}(t\alpha + (1-t)\alpha')dt, \quad D^k_{\alpha'}q_1[\omega] = \int_0^1 (1-t)^k D^k_{\alpha}(t\alpha + (1-t)\alpha')dt. \]

Then, using 2π periodicity of \( D^k_{\alpha}\), we obtain

\[
\int_a^{a+2\pi} \left| \int_0^1 t^k D^k_{\alpha}(t\alpha + (1-t)\alpha')dt \right|^2 d\alpha' \\
\leq \int_a^{a+2\pi} \left( \int_0^1 |D^k_{\alpha}(t\alpha + (1-t)\alpha')(1-t)^{1/4}|^2 dt \right)\left( \int_0^1 t^{2k}(1-t)^{-1/2} dt \right) d\alpha' \\
\leq C \int_0^1 \int_a^{a+2\pi} |D^k_{\alpha}(t\alpha + (1-t)\alpha')(1-t)^{1/4}|^2 d\alpha' dt \\
\leq C \int_0^1 \int_a^{(a+2\pi)(1-t)+\alpha} |D^k_{\alpha}(u)|^2 (1-t)^{-1/2} du \, dt \\
\leq C \int_0^1 (1-t)^{-1/2} dt \int_0^{2\pi} |D^k_{\alpha}(u)|^2 du \leq C \|D^k_{\alpha}\|_0^2.
\]

So \( D^k_{\alpha}q_1 \in H^k[a, a+2\pi] \) in variable \( \alpha' \) and \( \|D^k_{\alpha}q_1[\omega]\|_0 \leq C \|\omega\|_k \) with \( C \) only dependent on \( k \). Again

\[
\int_a^{a+2\pi} \left| \int_0^1 (1-t)^k D^k_{\alpha}(t\alpha + (1-t)\alpha')dt \right|^2 d\alpha' \\
\leq \int_a^{a+2\pi} \left( \int_0^1 |D^k_{\alpha}(t\alpha + (1-t)\alpha')(1-t)^{1/4}|^2 dt \right)\left( \int_0^1 (1-t)^{2k-1/2} dt \right) d\alpha'
\]
\[
\leq C \int_0^1 \int_a^{a+2\pi} |D^k_\alpha \omega_\alpha(t\alpha + (1-t)\alpha')(1-t)^{1/4}|^2 d\alpha' dt
\]
\[
\leq C \int_0^1 \int_a^{(a+2\pi)(1-t)+\alpha} |D^k_\alpha \omega_\alpha(u)|^2 (1-t)^{-1/2} du dt \leq C \|D^k_\alpha \omega_\alpha\|^2_0.
\]

So \( D^k_{\alpha'} q_1 \in H^k[a, a + 2\pi] \) in variable \( \alpha' \) and \( \|D^k_{\alpha'} q_1[\omega]\|_0 \leq C \|\omega_\alpha\|_k \) with \( C \) only dependent on \( k \).

We note that for \( k \geq 0 \)
\[
D^k_{\alpha'} q_2[\omega] = - \int_0^1 t^k (1-t)D^k_\alpha \omega_{\alpha\alpha}(t\alpha + (1-t)\alpha') dt,
\]
\[
D^k_{\alpha'} q_2[\omega] = - \int_0^1 (1-t)^{k+1}D^k_\alpha \omega_{\alpha\alpha}(t\alpha + (1-t)\alpha') dt.
\]

Similar arguments as above leads to the stated bounds for \( q_2 \).

From symmetry of \( q_1, q_2 \) in \( \alpha \) and \( \alpha' \), clearly the same results hold with respect to \( \alpha \) instead of \( \alpha' \) integration. \( \square \)

### A.2 Proof of Lemma 2.30 for \( K_1 ([1]) \):

**Proof.** We begin by taking \( r - 2 \) derivatives of \( K_1[\omega] f \).

\[
D^{r-2}_\alpha K_1[\omega] f(\alpha) = D^{r-2}_\alpha \frac{1}{2\pi i} \int_{-\pi}^{\pi} f(\alpha') \left[ \frac{1}{\omega(\alpha) - z(\alpha')} - \frac{1}{2\omega_\alpha(\alpha')} \cot \frac{1}{2}(\alpha - \alpha') \right] d\alpha'
\]
\[
= \frac{1}{2\pi i} \int_{-\pi}^{\pi} f(\alpha') D^{r-2}_\alpha \left[ \frac{1}{\omega(\alpha) - \omega(\alpha')} - \frac{1}{2\omega_\alpha(\alpha')} \cot \frac{1}{2}(\alpha - \alpha') \right] d\alpha'
\]
\[
= \frac{1}{2\pi i} \int_{-\pi}^{\pi} f(\alpha') D^{r-2}_\alpha \left[ \frac{1}{\omega(\alpha) - \omega(\alpha')} - \frac{1}{\omega_\alpha(\alpha')(\alpha - \alpha')} \right] d\alpha'
\]
\[
- \frac{1}{2\pi i} \int_{-\pi}^{\pi} f(\alpha') \frac{1}{2\omega_\alpha(\alpha')} D^{r-2}_\alpha l_2 \left( \frac{1}{2}(\alpha - \alpha') \right) d\alpha'
\]
\[
= P_1 + P_2.
\]
Since the function \( l_2(\beta) \) is analytical for \(-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}\), it is easy to have

\[
\|P_2\|_0 \leq \frac{C}{L}\|f\|_0, \quad \text{where } C \text{ depends on } r.
\]

Let us see \( P_1 \).

\[
P_1 = \frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} \frac{f(\alpha')}{\omega(\alpha')D_{\alpha}^{-2}} \left( \frac{\omega(\alpha)}{\omega(\alpha) - \omega(\alpha')} - \frac{1}{\alpha - \alpha'} \right) d\alpha'
\]

\[
= \frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} \frac{f(\alpha')}{\omega(\alpha')D_{\alpha}^{-2}} \left( \frac{q_2[\omega](\alpha', \alpha)}{q_1[\omega](\alpha', \alpha)} \right) d\alpha'.
\]

(2.25) implies that \(|q_1[\omega](\alpha, \alpha')| \geq \frac{1}{8}\). So by Lemma 2.24, we have

\[
\|P_1\|_0 \leq \frac{C_1}{L}\|f\|_0 \exp \left( \frac{C_2}{L}\|\omega\|_{r-1} \right).
\]

Hence first result follows. Taking \( \alpha \)-derivative \( r - 1 \) times \( K[\omega]f \) and integrating by parts once,

\[
D_{\alpha}^{r-1}K[\omega]f(\alpha) = D_{\alpha}^{r-2} \frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} D_{\alpha'} \left( \frac{f(\alpha')}{\omega(\alpha')} \right) \left[ \frac{\omega(\alpha)}{\omega(\alpha) - \omega(\alpha')} - \frac{1}{2} \cot \frac{1}{2}(\alpha - \alpha') \right] d\alpha'
\]

\[
= \frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} D_{\alpha'} \left( \frac{f(\alpha')}{\omega(\alpha')} \right) D_{\alpha}^{r-2} \left[ \frac{\omega(\alpha)}{\omega(\alpha) - \omega(\alpha')} - \frac{1}{2} \cot \frac{1}{2}(\alpha - \alpha') \right] d\alpha'
\]

\[
= -\frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} D_{\alpha'} \left( \frac{f(\alpha')}{\omega(\alpha')} \right) D_{\alpha}^{r-2} \left( \frac{q_2[\omega](\alpha', \alpha)}{q_1[\omega](\alpha', \alpha')} \right) d\alpha'
\]

\[
-\frac{1}{2\pi i} \int_{\alpha - \pi}^{\alpha + \pi} D_{\alpha'} \left( \frac{f(\alpha')}{2\omega(\alpha')} \right) D_{\alpha}^{r-2} \left( \frac{1}{2}(\alpha - \alpha') \right) d\alpha'.
\]

Using Lemma 2.24, the the second inequality follows from Cauchy-Schwartz inequality after noting that \( \|D \left( \frac{f}{\omega} \right) \|_0 \leq C\|f\|_1\|\omega_1\|_1 \)
A.3 Proof of Lemma 2.34 ([1]):

Proof. We begin by writing \([\mathcal{H}, \psi]\) as an integral operator:

\[
[\mathcal{H}, \psi]f(\alpha) = \frac{1}{2\pi} \int_{\alpha-\pi}^{\alpha+\pi} f(\alpha') (\psi(\alpha') - \psi(\alpha)) \cot \left( \frac{1}{2}(\alpha - \alpha') \right) d\alpha'.
\]

We can write the kernel as

\[
\left( \frac{\psi(\alpha') - \psi(\alpha)}{\alpha - \alpha'} \right) \left( (\alpha - \alpha') \cot \left( \frac{1}{2}(\alpha - \alpha') \right) \right).
\]

The first part of this product is a divided difference, and the second part is an analytic function on the domain \([-\frac{\pi}{2}, \frac{\pi}{2}]\). The lemma now follows from the Generalized Young’s Inequality. \(\square\)

A.4 Local uniqueness of Hele-Shaw bubble solutions:

Theorem A.1. Let \(0 \leq \beta < \Upsilon\) and \(|u_0| < 1\), where \(\Upsilon\) is small enough for Lemmas 2.30, 2.37 and Proposition 2.38 to apply. Let \((\tilde{\theta}_1(\alpha, t), \hat{\theta}_1(0; t), y_1(0, t))\) and \((\tilde{\theta}_2(\alpha, t), \hat{\theta}_2(0; t), y_2(0, t))\) be solutions of the system \((D.1)-(D.6)\) with the same initial condition \((4.1)\) in the space \(C([0, S], B^r \times \mathbb{R} \times S_M)\) with \(r \geq 4\). Suppose \(\|\tilde{\theta}_1\|_1 < \epsilon_1\) and \(\|\tilde{\theta}_2\|_1 < \epsilon_2\) such that \(|L_1 - 2\pi| < \frac{1}{2}\) and \(|L_2 - 2\pi| < \frac{1}{2}\) by \((2.42)\). Then for sufficient small \(\epsilon_1\) and \(\Upsilon\), the two solutions are the same in \(\dot{H}^2 \times \mathbb{R} \times S_M\).

Proof. We define the energy function \(E^d(t)\) for the difference of two solutions by

\[
E^d(t) = \frac{1}{2} \int_0^{2\pi} (D_\alpha^2 \tilde{\theta}_1 - D_\alpha^2 \tilde{\theta}_2)^2 d\alpha + \frac{1}{2} (\hat{\theta}_1(0; t) - \hat{\theta}_2(0; t))^2 + \frac{1}{2} (y_1(0, t) - y_2(0, t))^2.
\]
Taking derivatives on both sides with respect to $t$, and using (D.1)-(D.6), we have

\[
\begin{align*}
\frac{dE^d(t)}{dt} &= \int_0^{2\pi} D^2_\alpha(\tilde{\theta}_1 - \tilde{\theta}_2) D^3_\alpha \mathcal{Q}_1 \left( \frac{2\pi}{L_1} U_1 - \frac{2\pi}{L_2} U_2 \right) d\alpha \\
&\quad + \int_0^{2\pi} D^2_\alpha(\tilde{\theta}_1 - \tilde{\theta}_2) D_\alpha \mathcal{Q}_1 \left( \frac{2\pi}{L_1} (1 + \theta_{1,\alpha}) U_1 - \frac{2\pi}{L_2} (1 + \theta_{2,\alpha}) U_2 \right) d\alpha \\
&\quad + \int_0^{2\pi} D^2_\alpha(\tilde{\theta}_1 - \tilde{\theta}_2) D^2_\alpha \mathcal{Q}_1 \left( \frac{2\pi}{L_1} T_1 \theta_{1,\alpha} - \frac{2\pi}{L_2} T_2 \theta_{2,\alpha} \right) d\alpha \\
&\quad + \left( \dot{\theta}_1(0; t) - \dot{\theta}_2(0; t) \right) \int_0^{2\pi} \left[ \frac{2\pi}{L_1} T_1 (1 + \theta_{1,\alpha}) - \frac{2\pi}{L_2} T_2 (1 + \theta_{2,\alpha}) \right] d\alpha \\
&\quad + \left( y_1(0, t) - y_2(0, t) \right) \left[ - U_1(0, t) \sin \left( \theta_1(0, t) \right) + U_2(0, t) \sin \left( \theta_2(0, t) \right) \right] \right] = I_1 + I_2 + I_3 + I_4 + I_5.
\end{align*}
\]

By (1.19), we have

\[
\begin{align*}
I_1 &= \int_0^{2\pi} D^2_\alpha(\tilde{\theta}_1 - \tilde{\theta}_2) (D^3_\alpha + D_\alpha) \mathcal{Q}_1 \left( \frac{2\pi^2}{L_1^2} \mathcal{H}[\gamma_1] - \frac{2\pi^2}{L_2^2} \mathcal{H}[\gamma_2] \right) d\alpha \\
&\quad + \int_0^{2\pi} D^2_\alpha(\tilde{\theta}_1 - \tilde{\theta}_2) (D^3_\alpha + D_\alpha) \mathcal{Q}_1 \left( \frac{2\pi^2}{L_1^2} \Re (\mathcal{G}[z_1] \gamma_1) - \frac{2\pi^2}{L_2^2} \Re (\mathcal{G}[z_2] \gamma_2) \right) d\alpha \\
&\quad + (u_0 + 1) \int_0^{2\pi} D^2_\alpha(\tilde{\theta}_1 - \tilde{\theta}_2) (D^3_\alpha + D_\alpha) \mathcal{Q}_1 \left( \frac{2\pi}{L_1} \cos (\alpha + \theta(\alpha)) - \frac{2\pi}{L_2} \cos (\alpha + \theta(\alpha)) \right) d\alpha.
\end{align*}
\]
Using (D.3) and by Lemma 2.37 and Proposition 2.38, we have

\[
I_1 = -\sigma \int_0^{2\pi} D_\alpha^3(\tilde{\theta}_1 - \tilde{\theta}_2) \Lambda D_\alpha^2\left(\frac{4\pi^3}{L_1^3} \theta_1 - \frac{4\pi^3}{L_2^3} \theta_2\right) d\alpha \\
+ \sigma \int_0^{2\pi} D_\alpha^2(\tilde{\theta}_1 - \tilde{\theta}_2) \Lambda D_\alpha^2\left(\frac{4\pi^3}{L_1^3} \theta_1 - \frac{4\pi^3}{L_2^3} \theta_2\right) d\alpha \\
+ (1 + \frac{\mu_2}{\mu_1 + \mu_2} u_0) \int_0^{2\pi} D_\alpha^2(\tilde{\theta}_1 - \tilde{\theta}_2) \Lambda D_\alpha Q_1\left(\frac{2\pi}{L_1} \sin (\alpha + \theta_1(\alpha)) - \frac{2\pi}{L_2} \sin (\alpha + \theta_2(\alpha))\right) d\alpha \\
- a_\mu \int_0^{2\pi} D_\alpha^2(\tilde{\theta}_1 - \tilde{\theta}_2) \Lambda D_\alpha Q_1\left(\frac{2\pi^2}{L_1^2} \mathcal{F}[z_1]\gamma_1 - \frac{2\pi^2}{L_2^2} \mathcal{F}[z_2]\gamma_2\right) d\alpha \\
+ (1 + \frac{\mu_2}{\mu_1 + \mu_2} u_0) \int_0^{2\pi} D_\alpha^2(\tilde{\theta}_1 - \tilde{\theta}_2) \Lambda D_\alpha Q_1\left(\frac{2\pi}{L_1} \sin (\alpha + \theta_1(\alpha)) - \frac{2\pi}{L_2} \sin (\alpha + \theta_2(\alpha))\right) d\alpha \\
- a_\mu \int_0^{2\pi} D_\alpha^2(\tilde{\theta}_1 - \tilde{\theta}_2) \Lambda Q_1\left(\frac{2\pi^2}{L_1^2} \mathcal{F}[z_1]\gamma_1 - \frac{2\pi^2}{L_2^2} \mathcal{F}[z_2]\gamma_2\right) d\alpha \\
+ \int_0^{2\pi} D_\alpha^2(\tilde{\theta}_1 - \tilde{\theta}_2)(D_\alpha^2 + D_\alpha) Q_1\left(\frac{2\pi}{L_1} \Re (\mathcal{G}[z_1]\gamma_1) - \frac{2\pi}{L_2} \Re (\mathcal{G}[z_2]\gamma_2)\right) d\alpha \\
+ (u_0 + 1) \int_0^{2\pi} D_\alpha^2(\tilde{\theta}_1 - \tilde{\theta}_2)(D_\alpha^2 + D_\alpha) Q_1\left(\frac{2\pi}{L_1} \cos (\alpha + \theta_1(\alpha)) - \frac{2\pi}{L_2} \cos (\alpha + \theta_2(\alpha))\right) d\alpha \\
\leq -\sigma \int_0^{2\pi} D_\alpha^3(\tilde{\theta}_1 - \tilde{\theta}_2) \Lambda D_\alpha^3\left(\frac{4\pi^3}{L_1^3} \theta_1 - \frac{4\pi^3}{L_2^3} \theta_2\right) d\alpha \\
+ C\|\tilde{\theta}_1 - \tilde{\theta}_2\|_2 \left(\|\theta_1 - \theta_2\|_3 + \beta |y_1(0,t) - y_2(0,t)|\right),
\]

where \(C\) depends on \(\epsilon\). For \(I_2, I_3, I_4\) and \(I_5\), by (2.51) and (2.52) in Proposition 2.38,
we obtain

\[ I_2 + I_3 + I_4 + I_5 \leq C \| \tilde{\theta}_1 - \tilde{\theta}_2 \|_2 \left( \| \theta_1 - \theta_2 \|_3 + \beta | y_1(0, t) - y_2(0, t) | \right) \]

\[ + C | \hat{\theta}_1(0; t) - \hat{\theta}_2(0; t) | \left( \| \theta_1 - \theta_2 \|_3 + \beta | y_1(0, t) - y_2(0, t) | \right) \]

\[ + | y_1(0, t) - y_2(0, t) | \| U_1(\cdot, t) \sin ( \cdot + \theta_1(\cdot, t) ) - U_2(\cdot, t) \sin ( \cdot + \theta_2(\cdot, t) ) \|_1 \]

\[ \leq C \| \tilde{\theta}_1 - \tilde{\theta}_2 \|_2 \left( \| \theta_1 - \theta_2 \|_3 + \beta | y_1(0, t) - y_2(0, t) | \right) \]

\[ + C | \hat{\theta}_1(0; t) - \hat{\theta}_2(0; t) | \left( \| \theta_1 - \theta_2 \|_3 + \beta | y_1(0, t) - y_2(0, t) | \right) \]

\[ + | y_1(0, t) - y_2(0, t) | \| U_1(\cdot, t) \sin ( \cdot + \theta_1(\cdot, t) ) - U_2(\cdot, t) \sin ( \cdot + \theta_2(\cdot, t) ) \|_1, \]

where \( C \) depends on \( \epsilon \). Actually, combining the estimates for \( I_1, I_2, I_3, I_4 \) and \( I_5 \), by Cauchy inequality, we have

\[ \frac{dE^d(t)}{dt} \leq CE^d(t). \]

That is

\[ E^d(t) \leq E^d(0)e^{Ct}. \]

Hence, \( E^d(t) = 0 \) if \( E^d(0) = 0 \).

\[ \Box \]

A.5 The Fréchet derivative \( \mathcal{U}_{\hat{\theta}^{(s)}}[0, 0, 0] \) in Chapter 6

From (E.2), \( \gamma^{(s)} \) is the result of an operator acting on \( (\hat{\theta}^{(s)}, u_0, \beta) \). From substituting \( \hat{\theta}^{(s)} = \epsilon h \) and taking the \( \epsilon \) derivative at \( \epsilon = 0 \) and using Proposition 2.7, we have

\[ \mathcal{U}_{\hat{\theta}^{(s)}}[0, 0, 0] h = \frac{1}{2} \mathcal{H}[\gamma_{\hat{\theta}^{(s)}}[0, 0, 0] h](\alpha) + i \sum_{k=1}^{\infty} \frac{1}{k + 2} \hat{h}(k + 1) e^{i k \alpha} - h(\alpha) \sin \alpha + c.c.. \]

(A.1)
From (E.2) and Proposition 2.7, we have

\[
\gamma_{\tilde{g}(s)}[0, 0, 0]h(\alpha) = 2(1 + a_\mu)h(\alpha) \cos \alpha + \sigma h_{\alpha\alpha}(\alpha) + 2a_\mu \mathcal{H}[h \sin \alpha](\alpha) \\
- a_\mu \sum_{k=1}^{\infty} \frac{2}{k + 2} \hat{h}(k + 1)e^{ik\alpha} + c.c.. \quad (A.2)
\]

Hence, combining (A.1) and (A.2), using \( \mathcal{H}^2 = -I \), we obtain

\[
\Omega_{\tilde{g}(s)}[0, 0, 0]h = \frac{\sigma}{2} \mathcal{H}[h_{\alpha\alpha}](\alpha) + (1 + a_\mu)i \sum_{k=1}^{\infty} \frac{1}{k + 2} \hat{h}(k + 1)e^{ik\alpha} \\
- (1 + a_\mu)h(\alpha) \sin \alpha + (1 + a_\mu) \mathcal{H}[h \cos \alpha](\alpha) + c.c. \\
= \frac{\sigma}{2} \mathcal{H}[h_{\alpha\alpha}](\alpha) - i(1 + a_\mu) \sum_{k=1}^{\infty} \frac{k + 1}{k + 2} \hat{h}(k + 1)e^{ik\alpha} + c.c..
\]
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