THE DETERMINANTS OF INFORMATION VALUE: AN INQUIRY IN THE
CONTEXT OF THE COST-VOLUME-PROFIT DECISION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Ronald Walter Hilton, B. S., M. S.

* * * * *

The Ohio State University
1977

Reading Committee:
John V. Baumler
Thomas J. Burns
J. Stephen Henderson
James C. Kinard
William T. Morris
Clark A. Mount-Campbell

Approved by

Thomas J. Brunia
Advisor
Faculty of Accounting
ACKNOWLEDGMENTS

To the many people who have contributed to the successful completion of this study, I would like to express my sincere gratitude. Special thanks are due to my reading committee, Professors John V. Bauml, Thomas J. Burns, J. Stephen Henderson, James C. Kinard, William T. Morris, and Clark A. Mount-Campbell. The substantive suggestions and constructive criticism of the committee have been most helpful throughout the research. Comments from the members of The Ohio State University Accounting Research Colloquium were also helpful in the early stages of the study. In particular, I want to thank my advisor and dissertation chairman, Thomas J. Burns, for his invaluable guidance throughout my graduate program. His assistance went beyond the duty of any dissertation advisor.

I am grateful to Ernst and Ernst for helping to financially support my research through a dissertation fellowship, and to Gloria Irwin for her masterful job of typing of the dissertation.

The support of my family and my wife's family has been an important source of encouragement in reaching this goal. To my parents, I owe a considerable debt of gratitude for their continual support of my educational development over a number of years.

Above all, I owe my deepest appreciation to my wife, Meg. For her devotion, understanding, and patience throughout this graduate program, she shares equally in the attainment of our goal.
VITA

April 5, 1949 . . . . . . Born - Montrose, Pennsylvania

1971. . . . . . . . . . . . B.S., The Pennsylvania State University,
University Park, Pennsylvania

1973. . . . . . . . . . . . M.S., The Pennsylvania State University,
University Park, Pennsylvania

1974-1977 . . . . . . . Teaching Associate, Faculty of Accounting,
The Ohio State University, Columbus, Ohio

FIELDS OF STUDY

Major Field: Accounting

Studies in Economic Theory

Studies in Industrial and Systems Engineering
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>ix</td>
</tr>
<tr>
<td><strong>PART I. INTRODUCTION AND THEORETICAL DEVELOPMENT</strong></td>
<td></td>
</tr>
<tr>
<td>CHAPTER 1. INTRODUCTION</td>
<td></td>
</tr>
<tr>
<td>1.1 Alternative Information Theories and the Accountant's Role as Information Evaluator</td>
<td>2</td>
</tr>
<tr>
<td>1.2 Plan of this Study</td>
<td>7</td>
</tr>
<tr>
<td>CHAPTER 2. THEORY AND ANALYTICAL FRAMEWORK</td>
<td></td>
</tr>
<tr>
<td>2.1 The Value of Information Defined</td>
<td>11</td>
</tr>
<tr>
<td>2.2 The Determinants of Information Value</td>
<td>15</td>
</tr>
<tr>
<td>2.2.1 Structure of Outcome and Utility Functions.</td>
<td>19</td>
</tr>
<tr>
<td>2.2.2 Action Flexibility</td>
<td>25</td>
</tr>
<tr>
<td>2.2.3 Initial Uncertainty</td>
<td>30</td>
</tr>
<tr>
<td>2.2.4 Information System</td>
<td>41</td>
</tr>
<tr>
<td>2.2.5 Structure of the State Space</td>
<td>50</td>
</tr>
<tr>
<td>2.3 Summary</td>
<td>51</td>
</tr>
</tbody>
</table>

iv
PART II. INVESTIGATION OF THE DETERMINANTS OF INFORMATION VALUE IN THE CONTEXT OF THE COST-VOLUME-PROFIT DECISION

CHAPTER 3. THE COST-VOLUME-PROFIT MODEL, ANALYSIS OF STOCHASTIC FIXED COST, AND ANALYSIS OF STOCHASTIC UNIT VARIABLE COST WITH UNRESTRICTED PRIOR

3.1 Introduction

3.2 The Cost-Volume-Profit Model

3.3 Uncertainty About Fixed Costs

3.4 Uncertainty About Unit Variable Cost: Results Based on Unrestricted Prior

3.4.1 Generalized Information System

3.4.2 Perfect Information System

3.5 Summary

CHAPTER 4. ANALYSIS OF STOCHASTIC UNIT VARIABLE COST WITH UNRESTRICTED PRIOR

4.1 Generalized Information System

4.2 Perfect Information System

4.3 Noisy Information System

4.4 Noiseless Imperfect Information System

4.5 Summary

CHAPTER 5. ANALYSES OF STOCHASTIC QUADRATIC COST PARAMETER AND ACTION FLEXIBILITY GIVEN STOCHASTIC UNIT VARIABLE COST

5.1 Introduction

5.2 Uncertainty About the Quadratic Cost Parameter

5.2.1 Generalized Information System

5.2.2 Perfect Information System

5.3 Investigation of Action Flexibility Given Uncertainty About Unit Variable Cost
7.1.2 Extra-system Parameters Associated With Action Flexibility ............ 201
7.1.3 Initial Uncertainty ............... 203
7.1.4 Information System .............. 205
7.2 Performance Evaluation .......... 216
7.2.1 Traditional and Ex Post Variance Analysis Systems ................. 218
7.2.2 Managerial Performance Evaluation ............... 226
7.2.3 Information System Implementation Performance Evaluation ......... 228
7.2.4 Information System Design Performance ... 232
7.3 Summary .................................. 233

CHAPTER 8. SUMMARY AND CONCLUSION ........................................ 235
8.1 Review and Summary of Results .......... 235
8.2 Limitations of the Study .............. 241
8.2.1 General Limitations .............. 241
8.2.2 Specific Limitations .............. 242
8.3 Suggested Extensions ................ 244

APPENDIX A. SOME FUNDAMENTALS OF INFORMATION ECONOMICS ............ 248
APPENDIX B. LIST OF NOTATION AND DEFINITIONS ........................ 254
APPENDIX C. MAXIMIZATION OF INFORMATION VALUE WITH RESPECT TO DISCRETE PRIOR STATE PROBABILITIES ................. 259
APPENDIX D. PROOF OF COROLLARY 6-5.2, SECTION 6.3.3 ............... 266
BIBLIOGRAPHY ............................................. 272
**LIST OF TABLES**

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Selected Values for ( m(\phi,t) )</td>
<td>124</td>
</tr>
<tr>
<td>2</td>
<td>( t_1^<em>, t_2^</em>, t_3^* ) for Selected Values of ( \delta )</td>
<td>180</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>1</td>
<td>The Set X for Discrete Valued $x_j$</td>
<td>28</td>
</tr>
<tr>
<td>2</td>
<td>First-order and Second-order Stochastic Dominance</td>
<td>31</td>
</tr>
<tr>
<td>3</td>
<td>Cubic Cost Function</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>Marginal Cost Curve for Cubic Cost Function</td>
<td>60</td>
</tr>
<tr>
<td>5</td>
<td>Quadratic Cost Function for $\tau = \hat{\tau}$</td>
<td>60</td>
</tr>
<tr>
<td>6</td>
<td>Quadratic Cost Function for $\pi = \hat{\pi}$</td>
<td>64</td>
</tr>
<tr>
<td>7</td>
<td>Effect of Convexity of the Cost Function</td>
<td>66</td>
</tr>
<tr>
<td>8</td>
<td>$U(h^b_t)$ as a Function of Quadratic Cost Parameter $a$</td>
<td>77</td>
</tr>
<tr>
<td>9</td>
<td>Effect of Quadratic Cost Parameter on Optimal and Suboptimal Profit Levels</td>
<td>79</td>
</tr>
<tr>
<td>10</td>
<td>Economic Interpretation of Effect of Parameter $a$ on $U(h^b_t)$</td>
<td>81</td>
</tr>
<tr>
<td>11</td>
<td>Interaction of Information Timing and the Quadratic Cost Parameter</td>
<td>83</td>
</tr>
<tr>
<td>12</td>
<td>$U(h^b_{\infty,t})$ as a Function of Unit Price, $r$</td>
<td>87</td>
</tr>
<tr>
<td>13</td>
<td>Economic Interpretation of Effect of Unit Price on $U(h^b_{\infty,t})$</td>
<td>89</td>
</tr>
<tr>
<td>14</td>
<td>Interaction of Information Time and Unit Price</td>
<td>91</td>
</tr>
<tr>
<td>15</td>
<td>Interaction of the Quadratic Cost Parameter and Unit Price</td>
<td>91</td>
</tr>
<tr>
<td>16</td>
<td>Interpretation of Functional Dependence of $U(h^b_{\infty,t'},$ on $\text{var}(b)$</td>
<td>102</td>
</tr>
</tbody>
</table>
Interaction of Initial Uncertainty and the Quadratic Cost Parameter ........................................ 103

Interpretation of Interaction Between Initial Uncertainty and the Quadratic Cost Parameter .......... 105

Interaction of Initial Uncertainty and Information Timing ......................................................... 106

Interpretation of Interaction Between Initial Uncertainty and Information Timing ...................... 108

Expected Posterior Variance as a Function of Information Time (A) and Accuracy Responsiveness (B) ...... 110

Accuracy of $h_{t,\delta}^b$ as a Function of Information Time (A) and Accuracy Responsiveness (B) .......... 111

$U(h_{t,\delta}^b)$ as a Function of Information Time ................................................................. 116

$U(h_{t,\delta}^b)$ as a Product of Outcome Magnitude and Information Accuracy Factors ................. 118

$U(h_{t,\delta}^b)$ as a Function of $t$ for Several Values of $\delta$ ................................................... 119

Economic Interpretation of Effect of Accuracy Parameter $m$ on $U(h_{t,m}^b)$ .................................. 122

$U(h_{\phi,t}^b)$ as a Product of $\hat{O}(t)$ and $\hat{A}(t)$ ................................................................. 125

$U(h_{t}^a)$ as a Function of Unit Price ..................................................................................... 132

Economic Interpretation of Effect of Unit Price on $U(h_{t}^a)$ ...................................................... 133

$U(h_{t}^a)$ as a Function of Unit Variable Cost ........................................................................... 134

Economic Interpretation of Effect of Unit Variable Cost on $U(h_{t}^a)$ ..................................... 135

$U(h_{\omega,t}^b)$ as a Function of the Upper Bound on the Production Rate .................................... 144

Two Maximal Expectations Composing $U(h_{\omega,t}^b)$ .............................................................. 146

Economic Interpretation of Effect of Upper Bound on $U(h_{\omega,t}^b)$ ............................................ 146

$E[w(x,b)|h_{0,\lambda}]$ as a Function of $\lambda$ ............................................................................. 149
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Functional Dependence of $U(h_{w,0}^b)$ on $\lambda$</td>
<td>151</td>
</tr>
<tr>
<td>Inverse Demand, Marginal Revenue and Total Revenue Functions Under Monopoly</td>
<td>156</td>
</tr>
<tr>
<td>$U(h_{t}^b)$ as a Function of Demand Function Parameter $q$</td>
<td>158</td>
</tr>
<tr>
<td>Economic Interpretation of Effect of Parameter $q$ on $U(h_{t}^b)$</td>
<td>159</td>
</tr>
<tr>
<td>Economic Interpretation of Interaction Between Demand Function Parameter $q$ and Quadratic Cost Parameter $a$</td>
<td>162</td>
</tr>
<tr>
<td>Economic Interpretation of Interaction Between Demand Function Parameter $q$ and Initial Uncertainty</td>
<td>164</td>
</tr>
<tr>
<td>Economic Interpretation of Effect of Demand Function Parameter $q$ on $U(h_{t}^b)$</td>
<td>165</td>
</tr>
<tr>
<td>Expected Posterior Variance After Receipt of $i$th Signal as a Function of $i$th Signal Time ($A$) and Accuracy Responsiveness ($B$)</td>
<td>174</td>
</tr>
<tr>
<td>$i$th Component of Accuracy Vector as a Function of $i$th Signal Time ($A$) and Accuracy Responsiveness ($B$)</td>
<td>175</td>
</tr>
<tr>
<td>$U(h_{t,\delta}^b)$ as a Function of the $i$th Signal Time</td>
<td>179</td>
</tr>
<tr>
<td>Effect of Random Information Timing</td>
<td>214</td>
</tr>
</tbody>
</table>
PART I

INTRODUCTION AND THEORETICAL DEVELOPMENT

1.
CHAPTER 1
INTRODUCTION

1.1 Alternative Information Theories and the Accountant's Role as Information Evaluator

The recognition by accountants of their role as producers of information has evolved over a considerable period of time. A distinctly information-oriented view can be found in the writings of some early accounting theorists.¹ This view has been emphasized to a greater and greater extent throughout the development of accounting theory. Vatter, for example, writing in the 1950's, stressed the role of accounting as the collection, preservation, and reporting of information that is useful for management.²

The development in the early 1960's of general postulates and principles of accounting theory exhibited a continuing recognition of the information production role of accounting. Studies by Moonitz³ and Sprouse and Moonitz⁴ recognized the role of accounting in providing data

¹See, for example, J. Maurice Clark, Studies in the Economics of Overhead Costs (Chicago, Illinois: The University of Chicago Press, 1923), especially Chapters IX and XII.
as a basis for making choices from available alternatives, and supplying "comprehensive and dependable information that management needs to control and administer the resources in its charge efficiently and productively."6

In research of the latter 1960's, ASOBAT described accounting as "the process of identifying, measuring, and communicating economic information to permit informed judgments and decisions by users of the information."7 This process was characterized as the realization of a general information theory. "Essentially accounting is an information system. More precisely, it is an application of a general theory of information to the problem of efficient economic operations."8 Ijiri described the role of accounting as communication, via quantitative information, of an entity's economic events.9

More recently, the Accounting Principles Board of the AICPA described the accounting function as the provision of "quantitative information, primarily financial in nature, about economic entities that is intended to be useful in making economic decisions."10 Taking the

---

8Committee to Prepare a Statement of Basic Accounting Theory, A Statement, p. 64.
same view, the Trueblood Committee reported that "the basic objective of financial statements is to provide information useful for making economic decisions."\(^{11}\)

This information-production function view has been consistently expressed by theorists in both managerial and financial accounting, and it has spawned considerable research, in recent years, into the nature of information and its generation. Such research has emphasized the accountant's role as an information evaluator. Given an array of alternative information structures, the accountant's task is to select the optimal one for the users of the information.

General information theories may be categorized as either descriptive or normative. Descriptive theories are oriented toward explaining the choice and utilization of information systems by users of the information. Examples of descriptive information theory research in the accounting literature include Driver and Mock,\(^{12}\) Ashton\(^{13}\) and Libby.\(^{14}\)


Two general normative theories of information have received attention in the accounting literature. The first, known as information theory, was derived from mathematical communication theory and relies on the concept of entropy to establish the value of information systems and a preference ordering over systems. Treatments in the accounting and management literature include Bedford and Onsi, Lev, and Ronen and Falk. The entropy concept is not, however, in general, a valid measure of information value, nor is it generally a valid normative criterion for selection among alternative information systems. This point was made relatively early in the development of information theory by Howard; it has subsequently been reinforced by Marschak and Radner as well as others. Itami has demonstrated that only under

15 Other normative theories have been proposed for the accounting information systems choice decision, but these are not general normative theories of information system selection. They are specifically tailored to the selection of a particular type of accounting system. For example, securities price association tests have been suggested as a normative criterion for selection of an external reporting system. For a review of this and other normative approaches to external accounting system selection, see Committee on Concepts and Standards for External Financial Reports, Statement on Accounting Theory and Theory Acceptance (Sarasota, Florida: American Accounting Association, 1977).


very restrictive conditions does the entropy concept provide a valid measure for the value of information.\textsuperscript{23}

The second general normative theory of information to have received attention in the accounting literature is the theory of information economics.\textsuperscript{24} This theory is based on economic and statistical decision theory\textsuperscript{25}, and is the most comprehensive normative theory of information conceived to date. Information economics is designed to facilitate selection of the optimal information system, in a cost/benefit sense, by explicitly recognizing the decision maker's preferences\textsuperscript{26} and beliefs.\textsuperscript{27}

A major concept in information economics is the value of an information system. This value is, by definition, equivalent to the value of

\begin{itemize}
\item \textsuperscript{24}For example, see Gerald A. Feltham, Information Evaluation (Sarasota, Florida: American Accounting Association, 1972).
\item \textsuperscript{26}The decision maker is assumed to be rational in the sense that he possesses a utility function and orders actions, with regard to preference, on the basis of expected utility maximization. For an axiomatic development of such a utility theory see J. von Neumann and Oskar Morgenstern, Theory of Games and Economic Behavior (Princeton, New Jersey: Princeton University Press, 1953). See also Leonard J. Savage, The Foundations of Statistics (New York: John Wiley and Sons, 1954).
\end{itemize}
the information provided by the system. The subject of this research is the normative theory of information value; specifically it is concerned with the determinants of information value. The study identifies and characterizes the factors which jointly determine the value of an information system.

Increased understanding of these factors, and the role they play in determining information value, should be useful to accountants in their role as information evaluators. Understanding the relationship of various information system characteristics, as well as environmental and technological factors, to the usefulness of the output from an accounting information system should be helpful. Such an increased understanding should facilitate decisions about the acquisition and utilization of accounting information systems, as well as their alteration to meet changing environmental and technological conditions. In addition, insight may be gained into how factors which are external to an information system might be adjusted to improve the usefulness of the data provided by the system. At the present time, very little research exists which addresses these issues.

1.2 Plan of this Study

The theory of information economics was selected as the basis for this study since it provides a comprehensive normative model of information analysis for the rational economic decision maker. In Chapter 2 of the study, the general information evaluation model from information economics is presented, and alternative measures of information value are discussed. The determinants of information value are identified, and various general characterizations of the determinants are discussed.
In some cases, intuitive speculation is provided with regard to the role of a particular factor as a determinant of information value. Finally, a review is provided of the literature which addresses these information value determinants.

In Part II of the study, an analytical investigation is conducted with regard to the value determinants of several accounting information systems in the context of a cost-volume-profit decision. The problem situation considered is a single-decider problem, where the information system is either imposed on the decision maker, or is selected with respect to his beliefs and preferences. An internal reporting context (i.e., cost-volume-profit analysis), as opposed to either an external reporting setting\textsuperscript{28} or a market setting\textsuperscript{29}, was chosen to partially


justify this simplification.

Chapter 3 presents the particular cost-volume-profit model to be investigated. The model is characterized by a linear total revenue function and a quadratic cost function with uncertain parameters. The roles played by the value determinants of several cost information systems are then addressed under a rather general set of assumptions. First, mathematical statements are derived to precisely characterize the value of a particular information system and its functional dependence on various determinants. Sensitivity analysis is conducted to investigate the direct and interaction effects of each of several factors which jointly determine the value of the information system. Second, the mathematical statements are restated in intuitive economic terms, and economic theory is used to explain, with varying levels of precision, the phenomena which have been mathematically characterized.

This same general approach (i.e., precise mathematical analysis followed by restatement in intuitive economic terms) is utilized throughout Part II, so that the value determinants of several cost information systems are addressed under a variety of environmental and technological conditions. Chapter 4 investigates the same information systems considered in Chapter 3, but with a mild restriction on the probability distribution over the uncertain cost function parameter. This allows the derivation of results which are not possible under the rather nonrestrictive assumptions of Chapter 3.

Chapters 5 and 6 extend the analysis in several ways. For example, restrictions are placed on the feasible production quantities in the cost-volume-profit decision model, and the assumption of perfect
competition (adopted in Chapters 3 and 4) is relaxed.

In Part III of the study, Chapter 7 draws some inferences from the analytical development of Part II to provide normative implications for managerial behavior regarding the design and use of a cost information system for the cost-volume-profit decision. First, various general implications are discussed. Second, attention is directed to some implications of the analysis in Part II for performance evaluation of production management and the cost information system. Finally, Chapter 8 summarizes the findings of the study. A discussion is also provided of the limitations and some possible extensions of the research.
CHAPTER 2
THEORY AND ANALYTICAL FRAMEWORK

2.1 The Value of Information Defined

The theory of information economics is a normative framework for evaluation of alternative information systems which have the potential to reduce uncertainty and alter the perceived optimal action in decision problems which face uncertain environments. Central to this theory is the concept of information value. (Appendix A provides a short review of some fundamental concepts in information economics.) Three definitions for the value of information\(^1\) may be found in the literature. The first definition is given below, where the value of information system \(h\), \(U(h)\), is given in units of utility.\(^2\) The decision maker and information evaluator are assumed to be the same individual or share identical beliefs and preferences.

\[
U(h) = \max_{x \in X} \max_{s \in S} E \{u(w(x,s))|y\} - \max_{x \in X} \max_{s \in S} E \{u(w(x,s))\}
\]

where: \(h = H\) = set of information systems
\(\{y\} = Y\) = set of signals for system \(h\)

\(^1\)The phrase "value of information" will be used in this study to refer to some notion of worth for the information which is measured in unspecified units. The same concept is referred to in some literature as the "expected value of information," and no distinction is being made here. The value of information is, of course, by definition an expectation.

\(^2\)A complete listing of the notation used in this study is provided in Appendix B.
\( \{s\} = S = \text{set of uncertain states of nature} \) (i.e., \( S \) is a random variable)

\( \{x\} = X = \text{set of actions} \)

\( w \) denotes the outcome function mapping act-state pairs into outcomes, \( z \in Z \) (i.e., \( z = w(x,s) \))

\( u \) denotes the utility function mapping outcomes into utility levels

\( E \) denotes the expectation operator

Assuming continuous spaces for \( Y, S \) and \( X^3 \), \( U(h) \) can be reexpressed as follows:

\[
U(h) = \int_{y \in Y} \max_{x \in X} \int_{s \in S} u(w(x,s)) p(s|y,h) p(y|h) ds \, dy = \max_{x \in X} \int_{s \in S} u(w(x,s)) p(s|y,h) ds
\]

where: \( p(s) \) denotes the prior probability density function (pdf) over states

\( p(s|y,h) \) denotes the posterior pdf over states given signal \( y \) from information system \( h \)

\( p(y|h) \) denotes the prior pdf over signals from system \( h \)

\( U(h) \) will be called the utility of information system \( h \). This formulation of information value is given by Feltham\(^5\), Marschak and Radner\(^6\), and Demski\(^7\). It measures the expected incremental utility that the decision maker will enjoy if the system \( h \) is adopted, above that which is expected if no information system is employed.

\(^3\)This assumption will be maintained throughout the study. It is not a restrictive assumption, and an analogous theory exists for discrete valued signal, state, and action spaces.

\(^4\)\( p \) will be used to denote several distinct pdf's. In each case, the argument of \( p \) will make it clear which density is involved.

\(^5\)Feltham, Information.

\(^6\)Marschak and Radner, Economic, pp. 87-90.

U(h) is always nonnegative as one would expect. By definition of the maximization operator:

$$\max_{x \in X} \int_{s \in S} u(w(x,s)) p(s|y,h) ds \geq \int_{s \in S} u(w(x,s)) p(s|y,h) ds$$

for every $y \in Y$ and every $x \in X$.

Since the above weak inequality holds for every $y$, taking an expectation over $\{y\} = Y$ preserves the inequality.

$$\int_{Y} \left( \max_{x \in X} \int_{s \in S} u(w(x,s)) p(s|y,h) ds \right) p(y|h) dy \geq \int_{Y} \int_{s \in S} u(w(x,s)) p(s|y,h) ds p(y|h) dy$$

for every $x \in X$.

The right-hand side of the last inequality may be reexpressed as

$$\int_{S} u(w(x,s)) p(s) ds$$. Substituting this expression into the above inequality yields the following:

$$\int_{Y} \left( \max_{x \in X} \int_{s \in S} u(w(x,s)) p(s|y,h) ds \right) p(y|h) dy \geq \int_{X} u(w(x,s)) p(s) ds$$

for every $x \in X$.

Since the last inequality is true for any $x \in X$, it is true for the $x$ which maximizes the right-hand side of the inequality.

(2.2) $\int_{Y} \max_{x \in X} \int_{s \in S} u(w(x,s)) p(s|y,h) p(y|h) ds dy \geq \max_{x \in X} \int_{s \in S} u(w(x,s)) p(s) ds$

Finally, this implies the following:

$$\int_{Y} \max_{x \in X} \int_{s \in S} u(w(x,s)) p(s|y,h) p(y|h) ds dy - \int_{X} u(w(x,s)) p(s) ds \leq 0$$

This is the definition of $U(h)$ given by equation (2.1); thus, $U(h)$ is nonnegative.

Alternatively, the decision maker (DM) may express the value of information in the same units as the first-level outcome, $w(x,s)$. This would be monetary units in many decision problems, but is not restricted to this. Two such information value measures may be specified.
The demand value of information system \( h \), \( F(h) \), is given implicitly by equation (2.3) below. This expression represents the maximum amount, measured in units of \( w(x,s) \), that the DM would be willing to exchange for information system \( h \), given that he now employs the null system.

\[
(2.3) \quad \int \max_{y \in Y_e} \int_{x \in X_e \in S} u(w(x,s) - F(h)) p(s|y,h) p(y|h) dY_e ds = \max_{y \in Y_e} \int_{x \in X_e \in S} u(w(x,s)) p(s) ds
\]

This formulation of information value is given by Marschak and Radner\(^8\) and Ohlson.\(^9\) \( F(h) \) is always nonnegative. In inequality (2.2), any quantity subtracted from \( w(x,s) \), in the expression on the left-hand side, must be nonnegative in order to drive the weak inequality to a strict equality. This is true since \( u(z) \) is increasing in its argument.

The supply value of information system \( h \), \( G(h) \) is given implicitly by equation (2.4) below. This expression represents the minimum amount, measured in units of \( w(x,s) \), that the DM would be willing to accept in exchange for information system \( h \), which he currently employs. The assumption is made that the null information system will be employed if system \( h \) is exchanged.

\[
(2.4) \quad \int \max_{y \in Y_e} \int_{x \in X_e \in S} u(w(x,s) - G(h)) p(s|y,h) p(y|h) dY_e ds = \max_{y \in Y_e} \int_{x \in X_e \in S} u(w(x,s) + G(h)) p(s) ds
\]

This formulation of information value is given by Ohlson.\(^10\) \( G(h) \) is always nonnegative. In inequality (1.2), any quantity added to \( w(x,s) \), in the expression on the right-hand side, must be nonnegative in order to drive the weak inequality to a strict equality. Once again, this is

---

\(^8\) Marschak and Radner, Economic, pp. 87-90.


true since \( u(z) \) is increasing in its argument.

If the utility function is linear, \( U(h) \), \( F(h) \) and \( G(h) \) give the same ordering of information systems with respect to value. Let

\[ u(z) = \alpha + \beta(z), \text{ where } \beta > 0, \]

represent the linear utility function. From

equation (2.1):

\[
(2.5) \quad U(h) = \int_{Y \times X \times E \times S} \max_{y \in Y} \left\{ \alpha + \beta \left[ w(x,s) \right] \right\} p(s \mid y,h) p(y \mid h) ds dy = \max_{x \in X \times E \times S} \int_{x \in X} \left\{ \alpha + \beta \left[ w(x,s) \right] \right\} p(s \mid y,h) ds dy
\]

\[
(2.6) \quad \Leftrightarrow \int_{Y \times X \times E \times S} \max_{y \in Y} \frac{1 - \beta(h)}{\beta} p(s \mid y,h) p(y \mid h) ds dy = \max_{x \in X \times E \times S} \int_{x \in X} \left\{ \alpha + \beta \left[ w(x,s) \right] \right\} p(s \mid y,h) ds dy
\]

\[
(2.7) \quad \Leftrightarrow \int_{Y \times X \times E \times S} \max_{y \in Y} \left\{ \alpha + \beta \left[ w(x,s) \right] \right\} p(s \mid y,h) p(y \mid h) ds dy = \max_{x \in X \times E \times S} \int_{x \in X} \left\{ \alpha + \beta \left[ w(x,s) + \frac{1}{\beta} U(h) \right] \right\} p(s \mid y,h) ds dy
\]

Comparing (2.6) with (2.3) and (2.7) with (2.4), \( F(h) = \frac{1}{\beta} U(h) = G(h) \). Since \( \beta > 0 \), \( F(h) > F(h') \Leftrightarrow G(h) > G(h') \Leftrightarrow U(h') > U(h') \). Therefore, when the DM has a linear utility function, all three value measures yield the same ordering with respect to value of information systems \( h \in H \). Moreover, if \( u(z) = \alpha + z \) (i.e., \( \beta = 1 \)), \( U(h) = F(h) = G(h) \). In this case, the three value measures for information system \( h \) are identical.

The value of information as defined by \( U(h) \), \( F(h) \) or \( G(h) \), is a "gross" concept of value rather than a "net" concept since the cost of information system \( h \) has not been subtracted from these measures.

Clearly, an information system selection requires balancing the system's gross value against its cost. This study, however, is concerned with the gross value of information and its determinants.

2.2 The Determinants of Information Value

Turning more directly to the subject of this study, what factors determine information value, as measured by \( U(h) \), \( F(h) \) or \( G(h) \)? Five determinants may be identified as follows:
(i) Structure of outcome function, w, and utility function, u.

(ii) Structure of action set, X.

(iii) Degree of uncertainty implicit in the prior, p(s).

(iv) Nature of the information system mapping, R, which maps S onto Y. Equivalently, the likelihood function p(y|s), which reflects the DM's perception of R. p(s) and p(y|s) are combined in Bayesian fashion to form p(s|y), which is the explicit representation, in equations (2.1), (2.3) and (2.4), of this factor.

(v) Structure of state space, S.

The identification of (iii) and (v) as separate determinants of information value is to some extent arbitrary. They could have been combined into the single determinant, p(s), which is the DM's prior subjective pdf over states of nature. p(s) captures both the structure of the state space, S, as it is perceived by the DM, and the degree of uncertainty in p(s). There is some justification for considering the two separately, however, since each has a convenient intuitive interpretation, and the degree of uncertainty is a rather important factor in determining information value. These assertions will be made clear later in the study.

The five factors listed above will be called the direct determinants of information value, since they appear explicitly in the general expressions for U(h), F(h) and G(h). The original identification of determinants (i) through (iv) is due to Itami, who also included as a determinant the timing of a signal received by the DM.

---

The expressions for \( U(h) \), \( F(h) \) and \( G(h) \) given above are general in nature; they are valid for any conceivable decision situation. Given a particular decision context, the direct determinants, which are the components of these general expressions, take on specific form. For example, the utility function might be negative exponential or logarithmic, the action set might be an interval of the real line, and the prior pdf over states might be normal. The form of each of the direct determinants, for a particular decision setting, is characterized by one or more parameters, which distinguish a specific form for each direct determinant from all other forms. It is these parameters which transform the general expressions for information value, given by equations (2.1), (2.3) and (2.4), into expressions for the value of a particular information system for a particular decision problem.

The parameters which characterize the direct determinants in a given decision setting will be called indirect determinants of information value. The functional parameters of the outcome and utility functions (e.g., coefficients, exponents, etc.), the parameters which determine the set of feasible actions, the parameters of the prior and likelihood pdf's, and the parameters which describe potential states of nature are all indirect determinants of information value.

Information timing, listed by Itami as an information value determinant\(^{12}\), could be an example of an indirect determinant of information value. Information timing could be a parameter of the outcome function if action is taken after information occurs, and the outcome resulting from an act-state pair depends on the timing of the action. One example

\(^{12}\text{Itami, "Evaluation," p. 88.}\)
of this situation is an investment decision where the timing of the
investment action determines the length of time during which the invested
funds earn a return and the factor used to discount returns to their
present value. Information timing could be a parameter of the action
space structure if, again, an action is taken only after information is
received, and the set of feasible actions changes over time. Similarly,
information timing could be a parameter of the information system likeli-
hood function if the DM's perceived relationship between signals and
states is dependent on signal timing. In a particular situation, informa-
tion timing could be a parameter of any or all of these direct deter-
minants. An indirect determinant which is a parameter of more than one
direct determinant will be called a spanning indirect determinant of
information value.

Another example of a parameter which could potentially be a span-
ning indirect determinant is the DM's wealth. Wealth may, for instance,
affect the outcome function, the utility function, or the action set.
In research where the DM's resources were implicitly treated as an indi-
rect (but not spanning) determinant, Ohlson found that information value
is increasing in the endowment of an investor facing a single-period
portfolio selection decision.13 Kihlstrom, investigating the demand
function for information about product quality, found that information
value is increasing in the consumer's resources.14

14Richard Kihlstrom, "A Bayesian Model of Demand for Information
About Product Quality," International Economic Review. February, 1974,
413-439.
Attention will now be turned to each of the direct determinants, associated indirect determinants, and a review of the literature that deals with these concepts.

2.2.1 Structure of Outcome and Utility Functions

The value of information will clearly be affected by the outcome function, \( w \), and the utility function, \( u \). \( w \) and \( u \) are listed as a single direct determinant of information value since they can be, and in some studies are (for example, Marschak and Radner\(^{15} \)), combined into a single mapping from the Cartesian product of the action and state spaces to utility space. The following are equivalent expressions:

(i) \( z = w(x,s) \) and utility = \( u(z) \)
(ii) utility = \( u(w(x,s)) \)
(iii) utility = \( \zeta(x,s) \) where \( \zeta = u \circ w \), and \( \circ \) is the functional composition operator.

\( \zeta \) is often called the payoff function. In this study, \( u \) and \( w \) will be used in order to maintain the individual identities of the outcome and utility functions.

The structure of \( w \) is the DM's perception of a complex system (physical, financial, etc.) which transforms act-state pairs into outcomes. There are an infinite number of possibilities for this structure, and there is little that can be said, in general, about properties that the function might have. In a particular decision setting, \( w \) is wholly determined by its parameters; these parameters would be instrumental in indirectly determining information value. In some cases, \( w \) may exhibit somewhat regular behavior; such structural properties as increasing, decreasing, or constant returns with respect to the action variable \( x \) (with \( s \)

\(^{15}\)Marschak and Radner, Economic, p. 43.
held constant), may be present. In any case, the form of \( w \) is an important determinant of information value.

The properties of the utility function, \( u \), depend on the DM's relative preference for outcomes, or, equivalently, his attitude toward risk. Assuming that \( u \) is a twice continuously differentiable function and that more of the outcome, \( z \), is preferred to less, the DM's risk attitude is reflected by the sign of \( u'' \). (Prime and double prime denote the first and second derivative, respectively.) The DM exhibits risk \( \{ \text{aversion, neutrality, affinity} \} \) at \( z \) if \( u''(z) > 0 \). Since empirical evidence indicates that most DM's are predominantly risk averse, an interesting property of the utility function is the DM's degree of risk aversion.

As Arrow points out\(^{16}\), there might be a temptation to use \( u''(z) \) as a measure of the degree of risk aversion at the point \( z \). This measure, however, suffers from a severe defect, since the utility function is merely a representation of a preference ordering, and it is this preference ordering which has behavioral significance, not the utility function itself. Since the utility function is determined only up to a positive linear transformation\(^ {17}\), multiplication of the function by a positive constant does not change the preference ordering. It does, however change \( u''(z) \).\(^ {18}\) Therefore "the quantity \( u''(z) \), [notation changed to be consistent with that used in this study], is not itself a suitable measure of risk aversion, since it depends on the units in which utility

\(^{17}\)Marschak and Radner, Economic, p. 39.
\(^{18}\)Arrow, "The Theory," p. 94.
is measured. Similarly, the curvature of \( u(z) \), measured at the point \( z \) (i.e., reciprocal of the signed radius of the circle which is tangent to \( u(z) \) at \( z \)), is altered when \( u \) is multiplied by a positive constant. It too, then, is unsuitable as a measure of risk aversion.

Arrow\(^{21}\) and Pratt\(^{22}\) each independently developed two measures of risk aversion which avoid the difficulty mentioned above. The absolute risk aversion at the point \( z \), denoted by \( \kappa(z) \), is defined by \(-u''(z)/u'(z)\). Utility function \( u_1 \) displays greater absolute risk aversion than utility function \( u_2 \) in the interval, \([a, \beta]\), if \( \kappa_1(z) > \kappa_2(z) \) for all \( z \in [a, \beta] \). The function exhibits increasing constant decreasing absolute risk aversion at \( z \) as \( \kappa'(z) \leq 0 \).

Thus \( \kappa(z) \) contains a great deal of information about the structure of \( u \) at the point \( z \).

The intuitive interpretation of \( \kappa(z) \) concerns the risk premium that a risk averse DM would be willing to pay in terms of expected outcome, in order to avoid a risky situation (with positive expected outcome) in favor of a certain outcome. The risk premium is defined as the expected outcome of a lottery minus its certainty equivalent. This difference is always positive given a lottery with positive expectation and a risk averse DM. This risk premium has been shown by Menezes and Hanson\(^{23}\) to be invariant under a positive linear transformation of the utility.

---
function, and consequently, they have interpreted it as a measure of risk aversion. It is intuitively appealing that the greater the risk premium the DM is willing to pay, the greater the degree of risk aversion. Pratt\textsuperscript{24} has shown that, under suitable assumptions about the utility function and lottery involved, the risk premium is an increasing function of $\kappa(z)$. Therefore, as $\kappa(z)$ increases, the risk premium increases, and the degree of risk aversion increases. $\kappa(z)$ is, then, an appropriate measure of risk aversion for risky situations where outcomes are additive transformations of wealth.

For a risky situation where outcomes are multiplicative transformations (proportions) of wealth, Arrow\textsuperscript{25} and Pratt\textsuperscript{26} have developed the concept of relative risk aversion, defined as $-zu''(z)/u'(z)$. The relative risk aversion function has an analogous intuitive interpretation to that of absolute risk aversion, only in the context of a lottery where outcomes are proportions of wealth.

As Menezes and Hanson point out, the Arrow-Pratt absolute risk aversion function "arises when considering an individual's aversion to risk as wealth is varied but the risk remains unchanged," while the relative risk aversion function "becomes relevant when wealth and the risk are changed in the same proportion."\textsuperscript{27} As another measure of risk aversion, Menezes and Hanson develop the partial relative risk aversion function, $-zu''(z+W)/u'(z+W)$, which becomes relevant "when the risk is varied but wealth, $W$, remains fixed."\textsuperscript{28} [Notation changed to be consistent with

\textsuperscript{24}Pratt, "Risk Aversion," pp. 124-126.  
\textsuperscript{25}Arrow, "The Theory," pp. 90-120.  
\textsuperscript{26}Pratt, "Risk Aversion," pp. 133-135.  
\textsuperscript{27}Menezes and Hanson, "On the Theory," p. 481.  
\textsuperscript{28}Menezes and Hanson, "On the Theory," p. 481.
that used in this study.] This measure of risk aversion is also related to the concept of a risk premium for those individuals who:

...would be willing to pay a larger (risk) premium as a percentage of the fair actuarial value of an unfavorable simple lottery, the greater is the magnitude of the loss included in the lottery. For example, if such an individual would be willing to pay $150 to insure against a .1 probability of a loss of $1000, he would pay more than $260 to insure against a loss of $2000 with the same probability. In some sense, then, these individuals are more averse to risk, the greater is the size of the potential loss.²⁹

Each of the three measures of risk aversion (i.e., absolute, relative, and partial relative) is an appropriate measure for a particular type of risky situation confronted by a risk averter. Degree of risk aversion is a major structural property of the risk averter's utility function, and is an important indirect determinant of information value. Ohlson, in an investigation of the investor's decision is a single-period portfolio selection problem, found evidence that the value of private information is relatively greater for a risk-tolerant (i.e., low risk aversion) investor.³⁰

In a given decision setting, the various parameters which characterize a specific utility function, and its associated risk aversion functions, are indirect determinants of interest. The DM's wealth is one example, as has been mentioned earlier, of such a parameter that might be an indirect determinant of information value, since it affects the absolute and relative risk aversion functions.


The precise roles of the degree of risk aversion or the various parameters of specific utility functions in determining information value are unknown. There is no research to show, for example, whether the value of a risk averter's information system is, in general, increasing or decreasing, or even monotonic, in the degree of risk aversion. One reason for the lack of such research is the extreme difficulty of obtaining such results in the general case. Recall that the utility of information system h, U(h), is defined as the difference between two maximal expectations (see equation 2.1). In establishing a result of monotonicity of U(h) with respect to some structural parameter of u or w, it is not sufficient to demonstrate monotonicity of each of the maximal expectations in the parameter of interest. If both \( E(\max_{x \in X} E[u(w(x,s)) | y]) \) \( \max_{x \in X} E[u(w(x,s))] \) are shown to be decreasing in a given parameter of u or w, it may be that the marginal rate of decrease of the former maximal expectation is everywhere lower than that of the latter maximal expectation. In this case, U(h) would be monotonically increasing in the given parameter. Alternatively, such may not be the case. It is this complexity which makes a general analysis of the determinants of information value very difficult and general results relatively sparse.\(^{31}\) The problem is even greater when information value is defined as F(h) or G(h) and u is nonlinear. In those cases, the value of information is not even a linear combination of the two expectations (see equations 2.3 and 2.4). The relationships are more complex. This problem does not exist in

\(^{31}\) Could pointed this out with regard to another determinant of information value, the degree of uncertainty in p(s), which will be addressed in section 2.2.3. See John P. Could, "Risk, Stochastic Preference, and the Value of Information," *Journal of Economic Theory*, 1974, p. 79.
addressing the effect of risk aversion (or other decision problem parameters) in determining rational behavior in simpler settings, which do not involve more than one mathematical expectation, as do the expressions for the value of information. The concept of risk aversion, for example, has been quite useful in explaining rational behavior in insurance buying\textsuperscript{32} and liquidity preference.\textsuperscript{33}

In any case, the structural properties of $u$ and $w$ are determinants of information value, and there is a need for further investigation of their complex role in this matter.

\subsection*{2.2.2 Action Flexibility}

Another important determinant of information value is the structure of $X$, the set of possible actions. Each $x \in X$ may have several components. Therefore, an action may be regarded as a vector (denoted by $x$), each component of which is a particular aspect of the composite act, $x$. Let $x_j$ denote the $j$th component of the composite act, $x$. If all $x_j$ are continuous real variables, $X$ may be considered an $n$-dimensional vector space, where $n$ is the dimension of each $x \in X$. In this case, the structure of $X$ may be characterized by the bounds on the space $X$. The bounds on $X$ may be represented by the $m$-dimensional vector function, $C(x) \leq 0$, where $0$ denotes the $m$-dimensional zero vector. $C$ is a constraint function which defines the bounds on the set $X$.

The nature of $C$ plays an important role in determining information value. If $C$ imposes tight bounds on $X$, or allows for minimal


flexibility\textsuperscript{34} in Itami's terminology,\textsuperscript{35} the value of information may be reduced. This would result from the DM's inability to capitalize on new information by taking the appropriate action.

In a particular decision setting, the parameters of $C$ would be indirect determinants of information value. Wealth might be such a parameter in some problems, since one would expect greater flexibility for a wealthy DM than for a less wealthy DM. For example, a large firm with considerable assets would probably have easier access to borrowed funds for the purpose of plant expansion (i.e., greater flexibility) than would a small business.

Action timing might be another parameter of the constraint function, $C$, since in many decision situations action flexibility changes over time. Itami investigates such a decision problem in which a decision maker desires to satisfy exactly an unknown, single-period demand quantity.\textsuperscript{36} The costs of shortage or surplus are known and linear in the amount of shortage or surplus, respectively. The production process operates at a constant rate over a finite period of time, but the rate can be costlessly altered at any time during the production period. There is an upper bound constraint on the amount that can be produced during a given fraction of the period. Perfect information about demand is forthcoming at some point during the period.

\textsuperscript{34}There are numerous concepts of decision flexibility in the literature. The term is used here to refer to a notion of the size of, or number of feasible points in, the action set. It is a concept quite different from some others appearing in the literature. For an alternative definition, see for example, Stephen R. Heinemann and Edward J. Lusk, "Decision Flexibility: An Alternative Evaluation Criterion," The Accounting Review, January, 1976, pp. 51-64.

\textsuperscript{35}Itami, "Evaluation," p. 12.

\textsuperscript{36}Itami, "Evaluation."
Itami derives an expression for the value of perfect demand information as a function of the time when it is received. The function is piecewise linear concave in information timing (delay). Information time, in this case, is an indirect determinant of information value because of its effect on the direct determinant, action flexibility. As information is received later, the set of actions available during the production sub-period after receipt of the information becomes smaller. This results from the upper bound on the amount that can be produced during a given fraction of the period. When perfect demand information is received, one of three possible situations may occur. (1) The DM has already produced a quantity which does not exceed the actual demand, but which is great enough that any deficiency can still be produced without exceeding the upper bound for the remaining time. (2) The DM has already produced a quantity in excess of the actual demand, and will incur a surplus cost even after shutting down for the remainder of the period. (3) The DM has produced a quantity so small that he cannot produce the deficiency in the remaining time without exceeding the upper bound, and will, therefore, incur a shortage cost. As time progresses, without the DM having the benefit of perfect demand information, situations (2) and (3) become more likely relative to situation (1). There is a critical time, which depends on the demand distribution, before which situation (1) will definitely occur. Prior to this time, there is no loss in information value due to delay. After this critical time, the expected cost increases due to the increased likelihood that situation (2) or situation (3) will occur, and the value of the information declines linearly. This phenomenon results in the piecewise linear concave nature of the
dependence of information value on its timing.

Turning now to a discrete valued $x_j$, consider the structure of $X$ in this setting. Suppose for example, that $x_j \in \Lambda_j$, where
\[ \Lambda_j = \{0, \lambda_j, 2\lambda_j, 3\lambda_j, \ldots\}. \]
Then $X = \{ x | C(x) \neq 0$ and $x_j \in \Lambda_j, j = 1, 2, \ldots, n \}$. Figure 1 depicts the set $X$ when $n=2$ and $C(x)$ is given by: $0 \leq x_1 \leq \alpha$, $0 \leq x_2 \leq \beta$.

![Figure 1: The Set X for Discrete Valued x_j](image)

The size of $\lambda_j$, ($j = 1, 2, \ldots, n$), in addition to $C(x)$, now affects the flexibility of the DM. As $\lambda_j \to 0$, the DM approaches perfect flexibility with respect to the $j$th action component within the bounds defined by $C(x)$. This flexibility will, in general, affect information value.

A recent study which addresses flexibility as a determinant of information value is due to Merkhofer, who investigates two decision problems. In the first problem, an entrepreneur must decide upon a price and quantity for his product. The entrepreneur is defined to be "flexible"
with respect to price (or quantity) if he is free to select that variable rather than being irrevocably committed to a prior decision. He faces uncertain unit production costs and uncertain demand. Adopting specific functional forms for the production cost and demand functions and a linear utility function, Merkhofer derives the value of perfect information in each of nine situations. The nine situations result from considering all combinations of three information structures and three decision flexibility structures. The information structures include perfect information about (1) production cost, (2) demand, and (3) both production cost and demand. The flexibility structures include flexibility on (1) price, (2) quantity, and (3) both price and quantity. The derived value of perfect information is compared under the nine cases, and conclusions are drawn with respect to the effect of the various flexibility structures on the value of perfect information. No statements can be made with regard to properties of the functional dependence of information value on action flexibility, however, since no expression was derived for information value as a function of the degree of flexibility. Only the extreme points of the flexibility dimension were considered, i.e., only perfect and null flexibility were allowed. As a result, any conclusions which can be drawn with regard to the role of action flexibility as a determinant of information value are limited.

Merkhofer also performs essentially the same type of analysis in the setting of a more general quadratic program. Once again, only the extreme points of the flexibility dimension are considered, thereby eliminating any possibility of deriving an expression for information value with the degree of flexibility as an argument. The analysis does,
however, provide an interesting result concerning the additivity of information values over mutually exclusive sets of decision variables upon which flexibility is maintained.

...the first order additivity or nonadditivity of the value of information is determined by state variable correlation. To a first order approximation, if two pieces of information are uncorrelated, then the value of obtaining that information simultaneously equals the sum of the values of obtaining each item of information by itself. Similarly, the first order determinant of the additivity or nonadditivity of the value of flexibility is decision variable interaction. If the value function is a sum of functions of mutually exclusive decision vectors, then, to a first order, the value of simultaneously obtaining flexibility on both decision vectors will equal the sum of the values of obtaining flexibility on each vector individually.38

Itami, in his analysis of a firm's single-period production decision given uncertain demand, investigated the effect of action flexibility on the firm's expected profit.39 However, no attention was given to the role of flexibility in determining information value.

Further research is necessary to define various types of action flexibility, and to characterize its role as a determinant of information value.

2.2.3. Initial Uncertainty

The third direct determinant of information value to be discussed is the degree of uncertainty in p(s), that is, the prior uncertainty over states of nature. This determinant may simply be referred to as the DM's initial uncertainty. One might intuitively expect the value of information to be an increasing function of the initial uncertainty inherent in

a given decision situation. Higher order relationships between the two, however, are difficult to predict on an intuitive basis.

The concept of degree of uncertainty in a pdf is a complex issue which has been approached in a number of ways by different authors. The most general criterion for ordering uncertain outcomes is that of stochastic dominance. The pdf $p_1(z)$ is said to be at least as large as pdf $p_2(z)$, in the sense of first-order stochastic dominance, if $p_1(z) \geq p_2(z)$ for all $z \in Z$, where $P_1$ denotes the cumulative distribution function (cdf) associated with the pdf, $p_1$, and $(z) \in Z$ denotes the random variable over which the two pdf's are specified. The pdf $p_1(z)$ is said to be at least as large as pdf $p_2(z)$, in the sense of second-order stochastic dominance, if and only if $\int_{-\infty}^{z} p_1(a) da \geq \int_{-\infty}^{z} p_2(a) da$ for all $z \in Z$.\(^{40}\)

\[ \begin{align*}
\text{(A)} & \quad \begin{array}{c}
\text{Figure 2: First-order and Second-order Stochastic Dominance}
\end{array} \\
\text{(B)} & \quad \begin{array}{c}
\end{array}
\end{align*} \]

In Figure 2A, \( p_1 \) stochastically dominates \( p_2 \), in the first-order sense, since \( P_1(z) \) is everywhere at least as large as \( P_2(z) \). In Figure 2B, \( p_1 \) stochastically dominates \( p_2 \), in the second-order sense, since the summation of the areas between \( P_1(z) \) and \( P_2(z) \), where \( P_1(z) > P_2(z) \), is not less than the summation of the areas between \( P_1(z) \) and \( P_2(z) \), where \( P_1(z) < P_2(z) \).

Hadar and Russell showed that (1) \( p_1 \) stochastically dominates \( p_2 \), in the first order sense, if and only if \( p_2 \) is not preferred to \( p_1 \) by a DM with any once continuously differentiable utility function, and (2) \( p_1 \) stochastically dominates \( p_2 \), in the second-order sense, if and only if \( p_2 \) is not preferred to \( p_1 \) by any DM with a once continuously differentiable utility function where marginal utility is everywhere nonincreasing.\(^{41}\) The latter condition is equivalent to requiring that the DM's risk attitude is one of either aversion or neutrality.

If either type of stochastic dominance is present between two pdf's, the mean of the dominant pdf is at least as large as that of the other distribution.\(^{42}\) This fact seems to confound the issue of ordering uncertain prospects with respect to degree of uncertainty since one distribution may be preferred to another in part because it provides a larger expected outcome. If one agrees that risk averse individuals prefer pdf's with less rather than more inherent uncertainty, it seems reasonable to require that any criterion for ranking pdf's with respect to degree of uncertainty assign greater uncertainty to those pdf's which are less preferred by risk averse individuals. Moreover, this preference should result

\(^{41}\)Hadar and Russell, "Rules for Ordering," pp. 25-34.
strictly from greater uncertainty in the distributions, not from a lower expected outcome. To insure that a distribution is not preferred simply for its greater expectation, the criterion must require equality of the expectations of ordered distributions. This, of course, results in a partial ordering over pdf's with respect to degree of uncertainty, since pdf's with unequal expectation cannot be ordered.

Rothschild and Stiglitz have developed three such criteria which give equivalent partial orderings over pdf's with respect to the degree of uncertainty. Specifically, $p_1(z_1)$ exhibits greater uncertainty than $p_2(z_2)$ if $E(Z_1)=E(Z_2)$, and the following three equivalent statements hold:

(i) All risk averers prefer $Z_2$ to $Z_1$.
(ii) $Z_1$ is equal to $Z_2$ plus some noise.
(iii) $Z_1$ has more weight in the tails than $Z_2$.

The meaning of (i) is clear. To understand (ii) consider the random variable, $Z$, which has the property that $E(Z|Z_2)=0$ for all $Z_2$. Now let $p_1(z_1)=p_1(z_2+Z)$. This does not mean that $Z_1=Z_2+Z$, but only that $Z_1$ has the same pdf as the sum of $Z_2$ and $Z$. The additional uncertainty of $Z$ added to that of $Z_2$ yields a pdf for $Z_1$ with greater uncertainty than that for $Z_2$.

Statement (iii) means that $p_1(z_1)$ was formed by taking some weight from the center of $p_2(z_2)$ and adding it to the tails of $p_2(z_2)$ in such a way that the overall distribution has greater uncertainty.

---

a way as to leave the mean of $Z_2$ unchanged. The concept is made rigorous by introducing the "mean preserving spread," which is a function that performs the above described transformation on $p_2(z_2)$. The reader may consult Rothschild and Stiglitz for details. The stochastic dominance criterion, when applied to pdf's having the same mean, yields the same criterion as that developed by Rothschild and Stiglitz (R-S). "What the R-S definition shows is that we can think of stochastic dominance as greater riskiness when comparing random variables that have the same mean."  

Diamond and Stiglitz have developed a slightly different measure of the degree of uncertainty in a pdf. The criterion follows a similar pattern of development to that of Rothschild and Stiglitz, but instead of ordering pdf's with the same mean, the criterion orders pdf's that yield the same expected utility. Meyer has developed still another measure of a pdf's degree of uncertainty. Meyer defines cdf $P_1(z)$ to be at least as uncertain (risky) as cdf $P_2(z)$ if there exists some agent with a strictly increasing utility function, $u(z)$, such that for all agents more risk averse than he, $P_1(z)$ is not preferred to $P_2(z)$. Meyer demonstrates a relationship between this definition of degree of riskiness and the Arrow-Pratt measures of risk aversion, and shows that the definition is equivalent to the min-max definition of risk.

---

47 The min-max definition of risk implies that a risk averter chooses the pdf which minimizes the maximum loss. This represents the most extreme form of risk aversion. See Meyer, "Increasing Risk," pp. 130, 131.
Richter\textsuperscript{48} and Markowitz\textsuperscript{49} have demonstrated that if the mapping from the random state to utility is an Nth degree polynomial, only the first N moments of the utility function need be considered when comparing pdf's with respect to degree of uncertainty. This is because the expected utility, in this case, can be expressed as a function of the first N moments of the pdf over the state variable. Of course, if all moments of the pdf of order M+1 or higher can be expressed as a function of the first M moments, only the first M moments must be considered in comparing the degree of uncertainty among pdf's, when the mapping from the state variable to utility is an Nth degree polynomial, \((N>M)\). These results give rise to the familiar use of a pdf's variance as a measure of degree of uncertainty when the payoff function is quadratic or the pdf is a two-parameter distribution such as the normal. It has often been pointed out in the literature that such a criterion is severely limited in usefulness because of the restrictive assumptions necessary to insure its validity.\textsuperscript{50}

Gould has shown that, in the general case, there is no monotonic relationship between information value and the degree of uncertainty in the prior.\textsuperscript{51} Gould summarized his results as follows (Gould's \(\alpha\) denotes the state variable):

\textsuperscript{49}\textsuperscript{Harry M. Markowitz, Portfolio Selection (New Haven, Connecticut: Yale University Press, 1959).  
\textsuperscript{50}\textsuperscript{See, for example, Rothschild and Stiglitz, "Increasing Risk: 1," pp. 241-242.  
Specifically, we find that the following propositions do not hold in general:

1. The larger the number of values of \( \alpha \) that have nonzero probabilities, the more valuable is information.
2. As more and more probability is concentrated on a single value of \( \alpha \), the less valuable is information.
3. The easier it is to forecast \( \alpha \) (in terms of a minimum mean squared error criterion), the less valuable is information.
4. Riskier distributions of \( \alpha \) (in the Rothschild-Stiglitz sense of a mean-preserving spread of probability) increase the value of information.

We are able to establish the following propositions:

5. Given a finite set of values of \( \alpha \{a_1, \ldots, a_n\} \) and a scalar decision variable \( x \), the maximum value of information can be achieved by assigning nonzero probabilities to exactly two of these values of \( \alpha \).
6. Given any finite set of values of \( \alpha \{a_1, \ldots, a_n\} \) and a \( k \)-component vector of decision variables, the maximum value of information can be achieved by assigning nonzero probability to at most \( k+1 \) of these values.
7. When the payoff function is linear in \( \alpha \), then increased risk in the Rothschild-Stiglitz sense leads to an increase in the value of information.
8. Even though an increase in risk in the Rothschild-Stiglitz definition implies a mean-preserving increase in variance, it is not true that an increase in variance implies an increase in the value of information even when the payoff function is linear in \( \alpha \). 52

Three observations need to be made at this point. First, Gould proved all of his results for perfect information only, although many or all of them may apply equally to imperfect information. 53 Second, the negative results given above apply to the general case; they do not preclude the existence of a given result in a specific case. Consequently it may be possible to establish certain classes of problems in

---

52 Gould, Risk, ” p. 65.
53Perfect and imperfect information will be precisely defined in section 2.2.4. Perfect information is that which reveals, with probability one, the exact state of nature. All other information is imperfect.
which some of the results hold, even though Gould has ruled them out in the general case. Third, as Gould points out, one reason for the lack of strong results regarding the role of the degree of initial uncertainty in determining information value is that, even in the simplest case of \( U(h) \), information value is the difference between two maximal expectations. Consequently, if a determinant affects each expectation in the same direction, the marginal effect on one expectation must exceed that on the other, in general, in order for a general result to be established.\(^{54}\)

Several of Gould's conclusions can be extended to the more general case of imperfect information. Suppose (following Gould) that state variable \( s \) is distributed discretely with \( p(s_i) = \pi_i, \) \( i = 1, \ldots, n. \) Let \( \zeta_i(x) \) denote \( \zeta(x, s_i) \), the utility (payoff) in state \( i \) given action vector \( x \), and assume that \( \zeta_i(x) \) is concave in \( x \) for all \( i \). Let \( x^*_i = (x^*_1, \ldots, x^*_m) \) denote the ex ante optimal solution to the DM's problem given no information.

Suppose that the DM's information system consists of \( N \) potential signals, \( \{y_1, \ldots, y_N\} = Y \), and let \( p(y_j | s_i) = \theta_{ij}, \) \( j = 1, \ldots, N. \) Let \( x^*_j = (x^*_1, \ldots, x^*_m) \) denote the optimal action vector given that signal \( y_j \) is received.

The extension of Gould's results to the case of imperfect information is facilitated by the following two theorems, each of which is analogous to a result proven by Gould for perfect information.\(^{55}\) Both theorem's are proven in Appendix C of this study.

**Theorem 2-1:** Given that the payoff function, \( \zeta_i \), is concave in action vector \( x \), for \( i = 1, \ldots, n \), when the value \( [\text{using the utility measure,} \]

\(^{54}\)Gould, "Risk," pp. 79, 80.

\(^{55}\)Gould, "Risk," pp. 70, 73.
U(h]) of the DM's (perfect or imperfect) information system is at its maximum, with respect to the DM's prior probabilities over states of nature, \( (\pi_1, \ldots, \pi_n) \), the following relationship holds for all states \( k \) and \( \ell \) which are assigned nonzero probabilities.

\[
(2.8) \quad \sum_{j=1}^{N} [\theta_{jk} \zeta_k(x_j^*)] - \zeta_k(x^*) = \sum_{j=1}^{N} [\theta_{j\ell} \zeta_{\ell}(x_j^*)] - \zeta_{\ell}(x^*)
\]

Theorem 2-1 asserts that when the value of the information system is at its maximum, with respect to the prior probabilities over states, the following difference is equal across all states which are assigned a nonzero prior probability: the expected payoff in a particular state, given that the optimal action is taken for the received signal, minus the payoff in that particular state given the ex ante optimal action and no information. Gould calls this difference the conditional value of information (conditional on the occurrence of the particular state).

If the information system happens to be perfect, the number of signals, \( N \), is equal to the number of states, \( n \), and the likelihood probabilities satisfy the following requirement:

\[
(2.9) \quad \theta_{jk} = \begin{cases} 1, & j=k, j, k \in \{1, \ldots, n\} \\ 0, & \text{otherwise} \end{cases}
\]

In this case, condition (2.8) reduces to that given by (2.10) below, which is the result obtained by Gould for perfect information.\(^{56}\)

\[
(2.10) \quad \zeta_k(x_k^*) - \zeta_k(x^*) = \zeta_{\ell}(x_{\ell}^*) - \zeta_{\ell}(x^*) \quad \text{for } k, \ell \in \{1, \ldots, n\} | \pi_k, \pi_{\ell} > 0
\]

Here \( x_k^* \) is the optimal action vector given that state \( k \) occurs, which is equivalent to the receipt of signal \( y_k \).

**Theorem 2-2:** Given that the payoff function, \( \zeta_i \), is concave in action vector, \( x_i = (x_1, \ldots, x_m) \), for \( i = 1, \ldots, n \), the number of nonzero probabilities needed to maximize the value of the information system is in the

\(^{56}\) Gould, "Risk," p. 70.
set \{2, \ldots, m+1\}.

Theorem 2-2 asserts that a set of prior probabilities can be found which maximizes the value of information and which includes no more than one more nonzero probability than the number of components in the action vector.

Theorems 2-1 and 2-2 imply that many of Could's conclusions for perfect information, which were listed in this study, also apply to imperfect information. For example, the value of imperfect information does not, in general, increase with the number of states or decrease with concentration of probability on a single state, nor does the maximum information value involve equiprobable states of nature.

Given discrete state probabilities and a payoff function which is concave in a scalar decision variable, the maximum value of imperfect information is attained by letting exactly two states have nonzero probability. More generally, if the payoff function is concave in an \(m\)-component action vector, the maximum value of imperfect information can be achieved by assigning nonzero probability to at most \(m+1\) states of nature.

Could's remaining conclusions may also apply to imperfect information. It may, for example, be possible to show that the value of imperfect information is not directly related to the Rothschild-Stiglitz measure of degree of uncertainty or the ease in forecasting the state variable, in terms of a minimum mean squared error criterion. These issues, however, will not be pursued in this study.
Marschak investigated a simple market speculation problem, where the source of uncertainty is the one-day price change in a security, and the change is distributed by a rectangular pdf with range $\Delta$. Marschak showed the value of perfect information to be less valuable the smaller the range, $\Delta$. Equivalently, information value was shown to be increasing in the indirect determinant, $\Delta$, which can, in this case, be a measure of the degree of initial uncertainty.

Itami investigated the effects of adaptive behavior resulting from intra-period information about the parameters of a linear program which is used as a firm's single-period planning model. He found, under certain conditions imposed on the prior pdf over the objective function coefficient vector, that the value of perfect information about the objective function coefficients is non-decreasing in the degree of uncertainty in the prior. Itami measured the degree of uncertainty in the prior by the covariance matrix, which is appropriate given the restrictions placed on the prior. Itami found that the value of perfect information about the vector of resource constraints does not exhibit any unambiguous relationship with the degree of uncertainty in the prior pdf over the that vector.

In summary, the degree of initial uncertainty is an important factor in decision problems, and it exhibits a rather subtle and complex relationship with the value of information. Further research is necessary.

---

58 Itami, "Evaluation," pp. 52-84.
60 Itami, "Evaluation," pp. 95, 96.
to (1) develop a generally accepted definition and measure of the degree of uncertainty in a pdf, and (2) investigate the effect of the degree of initial uncertainty on information value.

2.2.4 Information System

An obvious determinant of information value is the nature of the information system. Many researchers in accounting have pointed out various desirable attributes of accounting information, but these comments have too often been largely superficial. For instance, many accounting studies mention the importance of timeliness of accounting information.61 No one would argue with such a general position. But exactly how does timeliness affect information value? When is information timely? What is the nature of the trade-off between timeliness and accuracy? Researchers have only begun to address such questions.

Gregory and Atwater discussed the relationship between the value of management information and the delay and reporting interval of the information system.62 Although the relationships were hypothesized, rather analytically or empirically derived, this work represents the first attempt in the accounting literature to systematically and comprehensively address the issue of information timeliness. Feltham presented a

---


multi-period information evaluation model, which was based on information economics, in the accounting literature in 1968.\textsuperscript{63} Relevance, timeliness, and accuracy of information were discussed in relation to the rigorous model. However, there was no attempt to characterize the functional relationship between these factors and information value.

Feltham's monograph expands on the earlier discussion, but does not focus on the determinants of information value.\textsuperscript{64} Others have presented similar, rather general, discussions of these attributes of information systems, but these works do not attempt a rigorous investigation of the functional dependence of information value on such information system characteristics, nor do they address the role of interactions among the characteristics in affecting information value.\textsuperscript{65}

The precise characterization of the nature of the information system, as an information value determinant, is the mapping, $R$, from states to signals. The relevant aspect of $R$ is the degree of identifiability of states from signals in the inverse correspondence, $R^{-1}$.\textsuperscript{66} $R$ may be deterministic or stochastic. If deterministic, it may be one-to-one or many-to-one. A one-to-one mapping from $S$ to $Y$ gives rise to perfect or complete information. In this case, $p(y|s) = \begin{cases} 1, & y = s \\ 0, & \text{elsewhere} \end{cases}$. If $R$ is many-to-one, the information system is said to be imperfect, but

\begin{footnotesize}
\begin{itemize}
\item \textsuperscript{63}Gerald A. Feltham, "The Value of Information," \textit{The Accounting Review}, October, 1968, pp. 684-696.
\item \textsuperscript{64}Feltham, \textit{Information Evaluation}, pp. 78-137.
\item \textsuperscript{66}Itami, "Evaluation," p. 87.
\end{itemize}
\end{footnotesize}
noiseless, and \( p(y_i|s) = \begin{cases} 1, & \text{if } s \in S_i \\ 0, & \text{elsewhere} \end{cases} \), where \( S_i \) is the \( i \)th cell of a partition on \( S \). For a noiseless information system, the set of signals, \( Y \), defines a partition on the state space, \( S \), through the inverse correspondence, \( R^{-1} \). Two noiseless information systems may be compared with respect to the fineness of the partitions induced on \( S \). Let \( R^i \) denote a partition on \( S \) composed of the cells \( S_1^i, S_2^i, \ldots, S_m^i \). A partition \( R^i \) is said to be as fine as a partition \( R^j \) if every cell, \( S_k^i \), is contained in some cell, \( S_j^j \). This criterion yields a partial ordering over partitions, since some partitions cannot be compared with respect to fineness.

The concept of fineness for noiseless information systems has been useful in developing criteria for comparing information systems with respect to value. Marschak and Radner show that if \( h_1 \) and \( h_2 \) are distinct information systems, \( h_1 \) is at least as valuable as \( h_2 \) for every pdf and every payoff function if any only if \( h_1 \) as a fine as \( h_2 \).\(^{67}\)

A positive first-order effect for fineness, as a determinant of the value of a noiseless information system, can be deduced from this theorem. Marschak applied this result to noiseless information systems in a simple market speculation context.\(^{68}\) Marschak and Radner applied the result in market speculation and production decision contexts.\(^{69}\) LaValle extends the results on partition fineness to decisions about purchasing

---

\(^{67}\) Marschak and Radner, *Economic*, p. 54.


incremental information. Butterworth applied the concept of partition fineness to evaluation of financial accounting information systems. A setting was investigated in which the level of disaggregation of the accounting system was equivalent to information system fineness. The positive first-order effect of fineness on system value was used to show that the value of the accounting system was increasing in the level of disaggregation.

Wilson investigated noiseless information systems in several decision contexts, including choice of production technique (machine setting), choice of factor proportions, and timing of deliveries. The partitions induced on the convex set of states of nature consisted of n cells of equal length. Wilson showed that the value of the noiseless information system, in the decision problems considered, was a concave decreasing function of cell length in the induced partition. In this case, cell length is an indirect determinant of information value. Note that the operative concept here differs from the concept of fineness. The two noiseless information systems which induce partitions of n and n+1 equal cells on the state space are not comparable with respect to fineness if n is greater than 1. In Part II of this study, the same type of information system will be investigated in the context of a cost-volume-profit decision problem.

---

If $R$ is stochastic, the only interesting case is many-to-many.\textsuperscript{73}

In this case, the information system is said to be noisy, and the likelihood function, $p(y|s)$ is such that (1) for at least one state $s \in S$, $p(y|s) > 0$ for at least two signals, $y \in Y$, and (2) for at least one signal $y \in Y$, $p(y|s) > 0$ for at least two states, $s \in S$.

A payoff-adequate partition of the state space is defined such that for every cell in the partition and every action, $x \in X$,

$u[w(x,s_i)] = u[w(x,s_j)]$ for every pair of states, $s_i$ and $s_j$, in the partition cell. This simply partitions the state space in a way such that, for any feasible action, all of the states in a particular cell of the partition yield the same utility level (payoff).\textsuperscript{74} If, in a decision problem, the payoff-adequate partition of the state space is fixed,\textsuperscript{75} Marschak and Radner show that "one can obtain a sharper condition for comparing the values of information structures than the fineness condition..."\textsuperscript{76} The theorem, adapted by Marschak and Radner\textsuperscript{77} from Blackwell\textsuperscript{78}, can be stated as follows:

\textsuperscript{73}The case of $R$ stochastic and one-to-many is excluded since the set of signals mapped from a particular $s$ may be collapsed into a single unique signal. Hence, $R$ may be considered one-to-one.

\textsuperscript{74}Marschak and Rader, Economic, p. 64. For further discussion, see Jacob Marschak, "The Payoff-Relevant Description of States and Acts," Econometrica, October, 1963, pp. 719-725.

\textsuperscript{75}Note that this reduces the class of admissible payoff functions (and, by implication outcome and utility functions) to a subset of the class of all such functions, such that all payoff functions in the class yield the given payoff-relevant partition of the state space.

\textsuperscript{76}Marschak and Radner, Economic, p. 64.

\textsuperscript{77}Marschak and Radner, Economic, pp. 64-67.

Theorem: Information system \( h \) is at least as valuable as system \( h' \), given the payoff adequate partition, \( S \), if and only if there are nonnegative numbers, \( \beta_{y'y} \), such that:

\[
p(y'|s) = \sum_{y \in Y'} \beta_{y'y} p(y|s) \quad \text{for every } s \in S, \quad y' \in Y', \quad \sum_{y' \in Y'} \beta_{y'y} = 1
\]

for every \( y \in Y \), where \( \{s\} \) is the set of cells in partition \( S \), and \( \{y'\} = Y' \) is the set of signals associated with system \( h' \).

In this case, information system \( h \) is said to be sufficient for system \( h' \).

The configuration of the numbers, \( \beta_{y'y} \), as well as \( p(y|s) \), for the two information systems \( h \) and \( h' \), determine the relative value of the two systems. These components are, therefore, indirect determinants of information value. Marschak\(^{79}\), McGuire\(^{80}\), and Marschak and Miyasawa\(^{81}\) have used the concept of sufficiency to develop a theory of comparison of information systems. Kihlstrom uses the sufficiency concept to define "increases in information," and then proceeds to derive a demand function for information about product quality.\(^{82}\) Miyasawa uses the concept to compare information systems for stochastic programming problems.\(^{83}\)

There is, however, no research which characterizes the precise role of the indirect determinants, \( \{\beta_{y'y}\} \) and \( p(y|s) \) in affecting the value of information system \( h' \).

---


Some information systems rely on some type of sampling procedure to observe some elements of a statistical population in order to draw inferences about general characteristics of the population. The sample size is, in this type of system, an indirect determinant of information value. The value of a sampling information system is obviously increasing in the sample size. The second order effect, however, is not clear for the general case.

Several researchers have investigated sampling information systems in various specific decision problems. Raiffa and Schlaifer derive an expression for the value of sample information as a function of the sample size for several problems. These include a two-action problem with normally distributed sample observations and known variance and a two-action problem with binomial sampling. The value of sample information, in each case, exhibits decreasing marginal returns with respect to the sample size. Morris develops expressions for the value of sample information, as a function of sample size, for inventory and rectifying inspection (quality control) decision problems. The optimal sample size is calculated. Wilson, in an investigation of a choice of production technique (machine setting) decision problem, shows that, when the output scale decision is independent of the sample size, the value of sample information exhibits decreasing marginal returns with respect to sample size. McCall investigates optimal stopping rules for the acquisition of

additional sample information in equipment replacement and investment decision problems.\textsuperscript{87}

The issue which prevails throughout the preceding discussion may be loosely termed information accuracy. An information system's accuracy is an indirect determinant of its value. This term may be made precise by stating the nature of \( R \), and then applying such concepts as fineness, number of equal cells in the partition induced on the state space, degree of uncertainty, sufficiency, or sample size. Other aspects of \( R \) may be relevant to the accuracy issue, depending on the context. In general, one would expect information value to be a nondecreasing function of accuracy, \textit{ceteris paribus}. Second order and interaction effects are not obvious and require further research.

Ijiri and Itami investigated information accuracy in the context of a single-period, quadratic cost-volume decision.\textsuperscript{88} In the Ijiri-Itami problem, the decision maker desires to satisfy exactly an unknown demand quantity for his product. The cost function for the daily production quantity is quadratic with known parameters, and imperfect demand information is forthcoming after \( n \) days of the production period have expired. The posterior variance of the demand distribution is assumed to increase proportionately with the number of days remaining in the production period after receipt of the information. In other words, information accuracy (measured inversely by the posterior variance) is increasing over time. The result is an intra-period optimal information time, since


the value of the information is increasing in accuracy, with time held constant, but decreasing over time, with accuracy held constant. Information accuracy and timing are, in this case, indirect determinants of information value.

LaValle and Rappaport, in an analysis of a simple two state-two act-two signal, discrete decision problem, investigated the functional dependence of an information system's value on the reliability of the system. In this study, the ith signal consists of the message "state i will be observed." The system's reliability is then defined as the probability of signal i given state i, a constant for i=1,2. The value of the information system, expressed as a function of reliability, was found to be high for extreme values of the reliability parameter (i.e., values close to zero or one), and low for central values of the parameter (i.e., close to one half). The term reliability seems to be somewhat of a misnomer here, since a system which is predictably unreliable is just as valuable as a reliable system. The diagnosticity of the information is the same in both cases. The real issue involved is information accuracy.

One characteristic of an accounting information system which is, in most cases, likely to affect the system's accuracy is the system's level of aggregation. Feltham and Denski and Feltham address the aggregation of cost information from an information economic perspective, and calculate the loss in profit, in a production decision context, resulting

from aggregation. Further research is needed to characterize the effect of information aggregation as a determinant of information value. 92

2.2.5. Structure of the State Space

The structure of the state space is clearly a determinant of information value. This structure can be intuitively thought of as one component of the technology and/or environment of the DM. Description of this structure would be similar to that suggested for the action space, X, viz., bounds on the state variables and interval size in the grid for a discrete valued state variable. The point has been previously made that such structural properties are also properties of the DM's prior subjective pdf over states, p(s). It is the DM's perception, or beliefs about, the state space that is relevant in a decision theoretic model, where the optimality criterion is based on consistency with the DM's beliefs and preferences. Consequently, separation of the structure of S from the degree of uncertainty is p(s), as two direct determinants of information value, is somewhat arbitrary in that both factors are properties of p(s). This separation has been justified in this study on the grounds that each of the two decision model components has a convenient intuitive interpretation, and the degree of uncertainty appears to be a very important determinant.

---

2.3 Summary

The theory of information economics provides a comprehensive normative structure for the evaluation and ordering of information systems, with respect to desirability, in a manner which is consistent with the information evaluator's beliefs and preferences. Central to this theory is the concept of the value of an information system, which is equivalent, by definition, to the value of the information provided by the system. Information economics suggests three possible definitions for information value: (1) the expected incremental utility from adoption of the information system, over the expected utility with the null system, (2) the maximum amount, in terms of the first-level outcome units of the decision problem (e.g., money), that the DM would be willing to give up in exchange for the information system, and (3) the minimum amount, in terms of the same units in which the decision problem's outcome is measured, that the DM would accept in exchange for giving up the information system and using the null system.

Five factors are instrumental in jointly determining the value of information regardless of which value definition is used. These five factors, termed the direct determinants of information value, are the following:

(i) Structure of outcome and utility functions. Equivalently, the character of the DM's environment and technology and his relative preference for outcomes.

(ii) Structure of the action space. Equivalently, the DM's flexibility.

(iii) Degree of uncertainty in the prior subjective pdf over states of nature. Equivalently, the DM's initial uncertainty.
(iv) Nature of the mapping from states to signals. Equivalently, the nature of the information system.

(v) Structure of the state space. Equivalently, the DM's perception of some aspects of his technology and/or environment.

These five factors are called direct determinants of information value, since they appear explicitly in the expressions for the value of information. In a particular decision context, these direct determinants are characterized by a set of parameters, which distinguish the given decision problem from other problems. These parameters are termed indirect determinants of information value. Parameters which affect more than one direct determinant are called spanning indirect determinants.

While there has been a considerable amount of research in information economics, there are relatively few studies which address the determinants of information value and their precise role in interacting to determine the value of a system. The Itami dissertation\textsuperscript{93} represents the current state of the art in such research. Itami's results were restricted by assumptions of perfect information, linearity of outcome and utility functions, infinite divisibility of production, and others. Furthermore, the Itami study dealt with demand information, which is more precisely a marketing problem than an accounting problem.

The limitations of the Itami study, and the fact that very few decision contexts have been investigated, leave a considerable void in understanding by accountants of the factors which determine the value of

\textsuperscript{93}Itami, "Evaluation."
systems they are routinely evaluating. With a view toward partial elimination of this void, Part II of this study presents an investigation of the determinants of cost information value in a short-run, cost-volume-profit decision problem.
PART II

INVESTIGATION OF THE DETERMINANTS
OF INFORMATION VALUE IN THE
CONTEXT OF THE COST-VOLUME-PROFILE DECISION
CHAPTER 3

THE COST-VOLUME-PROFIT MODEL, ANALYSIS OF STOCHASTIC FIXED COST, AND ANALYSIS OF STOCHASTIC UNIT VARIABLE COST WITH UNRESTRICTED PRIOR

3.1 Introduction

The purpose of this and the following three chapters is to investigate the role played by the determinants of information value in the context of a cost-volume-profit (CVP) decision, a familiar one in managerial accounting. The CVP decision-maker is a producer who seeks to maximize short-run profit by selecting the optimal production quantity subject to capacity constraints which are unalterable in the short run. The importance of CVP analysis in managerial accounting is evidenced by the prominence it receives in managerial accounting texts, the large amount of research which addresses the issue, and the purported applications it has in various short-run resource allocation decisions.

CVP analysis is a comprehensive tool in that it provides "a sweeping overview of the planning process. Managers are constantly faced with decisions about selling prices, variable costs, and fixed costs. Basically, managers must decide how to acquire and utilize economic resources in light of some objective. Unless they can make reasonably accurate predictions about cost and revenue levels, their decisions may yield undesirable or even disastrous results. These decisions are usually short-run. ... However, long-run decisions such as buying
plant and equipment also hinge on predictions of the resulting cost-volume-profit relationships.\(^1\)

CVP models range in complexity from the simple deterministic linear models dealt with extensively in most managerial accounting texts\(^2\) to those incorporating production constraints, nonlinearities, uncertainty, multiple products, or information.\(^3\) The first significant way in which the simple models were extended was to recognize the role of scarce resources in profit planning. Research by Jaedicke\(^4\), Dopuch\(^5\), Charnes, Cooper and Ijiri\(^6\) and Feltham\(^7\) contributed to the understanding that activity should be devoted to products which provide the maximum contribution margin per unit of scarce resource. Various deterministic CVP models which account for nonlinearities in the revenue or


cost functions were developed by Ijiri\textsuperscript{8}, Morrison and Kaczka\textsuperscript{9}, and Groves, Mane and Sorenson.\textsuperscript{10}

The second major improvement of CVP models involved the introduction of uncertainty into the analysis by Jaedicke and Robichek\textsuperscript{11} and Bierman.\textsuperscript{12} These models were extended to multiproduct firms by Johnson and Simik.\textsuperscript{13} Each of these analyses assumed a normal distribution for the various unknown parameters. These efforts were followed by various challenges to the distributional assumptions and investigations of the

\textsuperscript{8}Yuji Ijiri, Management Goals and Accounting for Control (Chicago: Rand McNally, 1965), Chapter 2.


CVP model under alternative distributional assumptions. Bierman and Dyckman assumed a normal distribution for sales and proceeded to develop an expression for riskiness in a breakeven context with linear utility. Recently a model was developed by Adar, Barnea and Lev which focuses on expected utility maximization rather than expected profit maximization. Along the same line Chen has specified a cost-volume-value model, the objective function of which is maximization of firm value.


Ijiri and Itami\(^{18}\) explicitly incorporate the prospect of information into CVP analysis. They consider a cost-volume decision problem in which the cost function for daily production is known and quadratic. The producer desires to satisfy exactly an unknown monthly demand quantity. Perfect information about demand quantity is forthcoming several days into the month. The authors address such issues as the value of the information and appropriate methods for performance evaluation and cost-variance analysis.

In addition to the accounting literature mentioned above, the economic theory of the firm under uncertainty provides a generalization of uncertainty embracing price and output decision analysis.\(^{19}\)

3.2 The Cost-Volume-Profit Model

The CVP models mentioned above are based on the microeconomic theory of the firm. One approach to the theory of the firm is to assume that the problem of optimal input combinations has been solved. The analysis of the firm is then conducted in terms of its revenues and costs expressed as functions of output. The producer's decision problem is to select an output at which firm profits are maximized.\(^{20}\)

The shape of the firm's cost function will vary depending upon the technology employed. A cost function that exhibits properties often assumed by economists is depicted in Figure 3, where total cost (TC) is a cubic function of output.

---

\(^{18}\)Ijiri and Itami, "Quadratic," pp. 724-737.


Figure 3. Cubic Cost Function

Figure 4 shows the associated marginal cost (MC) function which is a quadratic function of output.

Figure 4. Marginal Cost Curve for Cubic Cost Function
By way of contrast, much of the CVP analysis found in the accounting literature is based on a linear cost function. Although this function lends simplicity to the analysis, it does not reflect the "familiar property of cost behavior that marginal cost eventually increases as the volume is increased beyond an efficiency point."\(^{21}\)

To avoid this undesirable characteristic of linear cost functions as well as the mathematical difficulties associated with the cubic function, the CVP model analyzed here incorporates the following quadratic total cost function:

\[
(3.1) \quad TC = \tau a\left(\frac{x}{\tau}\right)^2 + \tau b\left(\frac{x}{\tau}\right) + \tau c = a\left(\frac{x}{\tau}\right)^2 + bx + \tau c; \quad a, b, c, \tau, x \geq 0
\]

Here \(\tau\) denotes the length of time in the production period measured in time units of some arbitrary length. \(x\) denotes the quantity produced during the production period of length \(\tau\). Equivalently, \(x/\tau\) denotes the production rate in effect throughout the production period. \(c\) denotes the fixed cost incurred even at zero production for each of the time units in which \(\tau\) is measured. Thus \(\tau c\) is the total fixed cost incurred during the production period of length \(\tau\). \(b\) denotes unit variable cost, where "variable" refers to costs that vary in direct proportion to the number of units produced\(^{22}\) (e.g., some direct materials). Thus \(b(x/\tau)\) is the rate at which variable costs are incurred, and \(\tau b(x/\tau) = bx\) is the total variable cost incurred during the production period of length \(\tau\). Therefore, \(bx + \tau c\) is the

\(^{21}\)Ijiri and Itami, "Quadratic," p. 275.

\(^{22}\)This use of the term "variable cost" has some precedent in literature which allows a nonlinear cost function. For example, see Ijiri and Itami, "Quadratic," p. 276. Also see Ijiri, Management.
summation of two types of costs incurred by the firm. \( \tau c \) is the total of all costs that increase in direct proportion to the length of the production period, and \( bx \) is the total of all costs that increase in direct proportion to the quantity produced.

A third type of cost is represented by \( \tau a(x/\tau)^2 = ax^2/\tau \). This is the total of all costs that increase at an increasing rate in the production rate, \( x/\tau \). The production rate can be increased by either (1) increasing \( x \), the quantity produced in the period of length \( \tau \), or (2) decreasing \( \tau \), the time during which the quantity \( x \) is produced. The parameter \( a \) measures the rate at which total costs of this type increase in the production rate. Finally, multiplying \( a(x/\tau)^2 \) by \( \tau \) accounts for the length of time during which production rate \( x/\tau \) is in effect. The result is that total cost of this type (called quadratic cost), given by \( ax^2/\tau \), increases at an increasing rate as \( x \) (\( \tau \)) is increased (decreased). Increasing the production rate (\( x/\tau \)), over the same period, increases costs more than proportionately.

The total cost function (3.1) is graphed in Figure 5A for \( \tau = \hat{\tau} \).
Figure 5. Quadratic Cost Function for $\hat{\tau}$

The marginal cost (MC) function given $\tau = \hat{\tau}$ is depicted in Figure 5B. Note that marginal cost is increasing at a constant rate throughout the entire production range. Although the quadratic cost function does not have a region of decreasing marginal cost, it is a reasonable approximation to the cubic function at all but extremely low or high levels of activity.\(^{23}\)

Figure 6 depicts total cost function (3.1) for $x$ fixed at $\hat{x}$. Total costs are convex \{increasing\} in $\tau$ as $\tau > x(a/c)^{\frac{1}{2}}$.

---

\( (3.2) \frac{\partial^2 TC}{\partial \tau^2} = \frac{a x^2}{\tau^2} + c \leq 0 \) as \( \tau \leq x(a/c)^{1/2} \), \( \frac{\partial^2 TC}{\partial \tau^2} = \frac{2a x^2}{\tau^3} > 0 \)

Figure 6. Quadratic Cost Function for \( x=x^{\hat{x}} \)

As Figure 6 shows, a producer wishing to produce \( \hat{x} \) units, and able to choose the production period, would find the optimal period to be \( \hat{x}(a/c)^{1/2} \), with the assumed cost function. Throughout this analysis, the simplifying assumption will be made that the firm's product is immediately sold as it is produced, thus eliminating the holding function. If the firm faces a perfectly competitive market, its total revenue (TR) function is linear in the quantity sold.

\( (3.3) \ TR = r x \)

where \( r \) denotes the unit price and \( x \) denotes the quantity produced and sold. In Chapter 5 the assumption of perfect competition will be relaxed.

Without loss of generality, let time 0 denote the beginning of the production period and time 1 denote the end. Suppose that the production rate is costlessly altered at time \( t \), \( t \in [0,1] \), when the fraction \( t \)
of the period has expired. The cost function then becomes:

\[(3.4) \ a \left( \frac{x_1^2}{t} + \frac{x_2^2}{(1-t)} \right) + b(x_1x_2) + c\]

where \(x_1\) denotes the quantity produced in the interval from time 0 to time \(t\) and \(x_2\) denotes the quantity produced between times \(t\) and 1. The outcome (profit) function is the following:

\[(3.5) \ w(x,s) = r(x_1 + x_2) - a \left( \frac{x_1^2}{t} + \frac{x_2^2}{(1-t)} \right) - b(x_1x_2) - c\]

where \(x\) denotes the production vector \((x_1, x_2)\).

Assuming the producer has the linear utility function \(u(z) = z\) and assuming certainty about \((r, a, b, c)\), the decision problem is

\[
\max_x u(w(x,s)) = \max_x w(x,s).
\]

This maximization yields the following constant optimal production rate:

\[(3.6) \ \frac{x_1^*}{t} = \frac{x_2^*}{(1-t)} = \begin{cases} 
0, & (r-b)/2a < 0 \iff b > r \\
(r-b)/2a, & \text{otherwise}
\end{cases}
\]

where \((x_1^*, x_2^*)\) denotes the optimal quantities of production in each of the two subperiods. The maximum profit is given by:

\[(3.7) \ w(x^*, s) = \begin{cases} 
[(r-b)^2/4a] - c, & b \leq r \\
-c, & \text{otherwise}
\end{cases}
\]

where \(x^* = (x_1^*, x_2^*)\).

The equality of the optimal production rates in the two subperiods implies that a total of \(x_1 + x_2\) can be produced optimally by producing at a constant rate throughout the period. This result is induced by

\[24\text{There is no loss in generality from simplifying the general linear utility function, } u(z) = \alpha + \beta z \text{ to the identity function, since the choice of scale and origin is arbitrary.}\]
the cost function. Suppose that a total of \((x_1^* + x_2^*) = (x_1^0 + x_2^0)\) is produced by operating at a rate of \((x_1^0 / t) < (x_1^* / t)\) during the first time period \([0, t]\), and operating at a rate of \([x_2^0 / (1-t)] > [x_2^* / (1-t)]\) during the second time period \([t, 1]\). Total variable cost plus total fixed cost for the two production subperiods is given by \(b(x_1^0 + x_2^0) + c\), and is invariant with respect to the production rates used in the two time intervals. The total quadratic cost for the two production subperiods is given by \(ta(x_1 / t)^2 + (1-t) a[x_2 / (1-t)]^2\), which is a convex linear combination of \(a(x_1 / t)^2\) and \(a[x_2 / (1-t)]^2\). Figure 7. depicts the function \(a(x_i / \tau_i)^2\) as a function of production rate \(x_i / \tau_i\), where \(i \in \{1, 2\}\), \(\tau_1 = t\) and \(\tau_2 = (1-t)\).

![Figure 7. Effect of Convexity of the Cost Function](image-url)
The quadratic cost for the production period \([0,t]\) is given by \(t\) times the magnitude \(A\), and that for the period \([t,1]\) is given by \((1-t)\) times the magnitude \(B\). The total quadratic cost for both production subperiods is \(tA + (1-t)B\). This is a convex linear combination of \(A\) and \(B\) since \(t \in [0,1]\), and is shown by the magnitude \(C\) in Figure 7.

The constant optimal production rate is a convex linear combination of the other two rates assumed above:

\[
(3.8) \quad \frac{x_1^*}{t} = \frac{x_2^*}{(1-t)} = \frac{x_1^* + x_2^*}{1} = \frac{x_1^0 + x_2^0}{1} = t \frac{x_1^0}{t} + (1-t) \frac{x_2^0}{(1-t)}
\]

The total quadratic cost from producing at this rate during both time periods is \(tD + (1-t)D = D\). By the definition of a convex function\(^{25}\), \(D\) is less than \(C\), as indicated in Figure 7. Since the total of variable and fixed costs for both production subperiods is invariant with respect to the production rates used, the total cost is lower when a constant production rate is employed.\(^{26}\)

Given certainty about \((r,a,b,c)\), the producer will optimally produce at a constant rate throughout the period. The certainty assumption is an important one here. If one or more of the outcome function parameters is unknown at the outset, and the possibility exists of receiving information about some subset of those parameters, the producer may find it desirable to alter the production rate upon receipt of the information.

\(^{25}\)\(\Omega(a)\) is convex over the interval \([a',a'']\) if \(\beta\Omega(a_1) + (1-\beta)\Omega(a_2) > 2[\beta a_1 + (1-\beta)a_2]\) for all \(a_1, a_2 \in [a', a'']\) and \(\beta \in [0,1]\).

\(^{26}\)This represents a generalization of an analysis in a similar setting in Ijiri and Itami, "Quadratic," pp. 726, 727.
3.3 Uncertainty About Fixed Costs

Suppose that fixed cost (c) for the production period is unknown, and that all other parameters (r, a, b) are certain. In this case the state, s, is equal to c. Let p(c) denote the producer's subjective pdf on c. The producer's maximum expected profit is given by:

\[(3.9) \max_{x \geq 0} \int (r x - ax^2 - bx - c) p(c) dc = \begin{cases} \frac{(r-b)^2}{4a} - E(c), & r \geq b \\ -E(c), & \text{otherwise} \end{cases}\]

The optimal production rate is again:

\[(3.10) x^* = \begin{cases} \frac{(r-b)}{2a}, & r \geq b \\ 0, & \text{otherwise} \end{cases}\]

Suppose that some type of system is available which provides information about c when the fraction t of the period has expired. Denote this information system by h_t^C.

**Proposition 5-1:** \( U(h_t^C) = 0 \).

**Proof:** Let \( \{y\} \) denote the set of potential signals from h_t^C, and \( p(c|y) \) the producer's posterior pdf over c given signal y. Recalling that \( u(z) = z \), and relying on equation (3.5) the expected utility of information system h_t^C is given by:

---

27Uncertainty about fixed cost could result from a variety of factors such as impending labor negotiations for non-direct labor personnel or forthcoming changes in insurance rates or property taxes.

28Throughout the entire analysis of the CVP problem the linear utility function \( u(z) = z \) is assumed. All statements regarding information system value will be given in terms of \( U(h) \). As was pointed out in Chapter 2, \( U(h) = F(h) = G(h) \) when \( u(z) = z \). There is no loss in generality as a result of simplifying from the general linear utility function \( u(z) = \alpha + \beta z \) since in this case \( U(h) = F(h) = G(h) \). Therefore, the signs of partial derivatives of \( F(h), G(h) \) and \( U(h) \) are identical, and the direction of effects of information value determinants will be identical for all three measures.
\begin{align*}
(3.11) \quad U(h_t^c) &= \max_{x_1 \geq 0} \max_{y, x_2 \geq 0} \int \left[ r(x_1 + x_2) - \frac{a}{c} x_1 - \frac{a}{(1-t)} x_2^2 - x_1 x_2 \right] \cdot p(c | y) p(y) \cdot dc \\
&= \int_{x_1 \geq 0} \int_{x_2 \geq 0} \left[ r(x_1 + x_2) - \frac{a}{c} x_1 - \frac{a}{(1-t)} x_2^2 - x_1 x_2 \right] \cdot c \cdot p(c) \cdot dc
\end{align*}

The first term in (3.11) is the producer's expected utility given that he employs information system $h_t^c$ and the second term is his expected utility given use of the null information system. Completing the maximization and expectation operations in (3.11):

\begin{align*}
(3.12) \quad (x_1^*/t) &= \{ [x_2^*/(1-t)] | y \} = \begin{cases} 
(r-b/2a, b \leq r) \\
0 & \text{otherwise}
\end{cases}
\end{align*}

and

\begin{align*}
U(h_t^c) &= \begin{cases} 
[(r-b)^2/4a] - E(c) \cdot \{ [(r-b)^2/4a] - E(c) \}, b \leq r \\
E(c) - (-E(c)) & \text{otherwise}
\end{cases} \\
&= 0 \quad \text{Q.E.D.}
\end{align*}

Information about fixed costs is valueless in the short run. This phenomenon could have been foreseen by inspection of (3.10) which gives the optimal production rate under conditions of uncertainty about $c$. The optimal rate is not a function of $c$, and hence information about $c$ does not have the potential to influence the optimal action of the producer. This illustrates the necessary condition for an information system to have positive value, \textit{viz.}, the information system must have the potential to generate at least one signal which will, if received, change the decision maker's perceived ex ante optimal action to some action other than the ex ante optimal action given no information.

3.4 Uncertainty About Unit Variable Cost: Results Based on Unrestricted Prior

3.4.1 Generalized Information System

Suppose now that the variable (i.e., proportional) cost per unit,
b, is uncertain while the parameters r, a and c are known. In this case the state s is equal to b. Uncertainty about b could result from uncertainty about labor rate negotiations, raw material prices, or various manufacturing overhead items which vary more or less proportionately with production (e.g., some indirect materials and labor, utilities, etc.). In the absence of any information about b, the producer's decision model is:

\[
(3.13) \text{max } \{\text{max } f[r(x_1+x_2)-\frac{a}{t} x_1^2-\frac{a}{t} x_2-b(x_1+x_2)-c]p(b)db}\}
\]

where \(p(b)\) denotes the producer's subjective pdf over b. Taking the expectation w.r.t. b and then differentiating w.r.t. \(x_1\) and \(x_2\) yields the following optimal decision rule:

\[(5.14) x_1^* = \begin{cases} t[r-E(b)]/2a, & r \geq E(b) \\ 0, & \text{otherwise} \end{cases}, \quad x_2^* = \begin{cases} (1-t)[r-E(b)]/2a, & r \geq E(b) \\ 0, & \text{otherwise} \end{cases}\]

This implies a constant optimal production rate:

\[\frac{x_1^*}{t} = \frac{x_2^*}{(1-t)} = \begin{cases} [r-E(b)]/2a, & r \geq E(b) \\ 0, & \text{otherwise} \end{cases}\]

Substituting \((x_1^*, x_2^*)\) into \(w(x, s)\) yields the following maximum expected profit:

\[(5.15) E[w(x^*, b)]h_0 = \begin{cases} [r-E(b)]^2/4a, & r \geq E(b) \\ -c, & \text{otherwise} \end{cases}\]

Here \(h_0\) denotes the null information system.

Suppose now that a system is employed which provides information about b at time t, \(t \in [0, 1]\). It will be assumed throughout the analysis that there is no expectation on the producer's part of learning about
b via any means other than the information system under explicit consideration. This is equivalent to requiring that any observations on the production process or any other phenomenon which have the potential to provide information about b are incorporated into the information system under consideration. The direct determinants of the value of such an information system are the following:

(i.) Nature of the outcome and utility function;

\[ w(x,s) = w(x,b) = r(x_1 + x_2) - \frac{a}{t} x_1^2 - \frac{a}{(1-t)} x_2^2 - b(x_1 + x_2) - c; \quad u(z) = z \]

(ii.) Structure of the action space:

\[ X = [0, \infty) \]

(iii.) Degree of uncertainty in p(b).

(iv.) Nature of the information system, or the nature of

\[ R_t: S \rightarrow Y_t. \quad \text{Equivalently, the form of } p(y_t | b). \]

(v.) Structure of the state space, \( S = [0, \infty) \)

\( Y_t = \{ y_t \} \) denotes the set of signals which may potentially be generated by the information system at time t. This signal set may vary with respect to signal time t. \( p(y_t | b) \) is the likelihood function and again, may vary with the signal time t. \( R_t \), the mapping from S to \( Y_t \), will in general depend on t. Note that t is a spanning indirect determinant of information value since it appears in more than one direct determinant. Eliminating those elements which are already fixed by the decision problem specification, this list may be condensed to the following:

(i) price, \( r \)

(ii) quadratic cost parameter, \( a \)

(iii) fixed cost, \( c \)

(iv) information timing, \( t \)
(v) degree of uncertainty in \( p(b) \)

(vi) the nature of \( p(y_t | b) \)

(v) and (vi) are direct determinants, (i), (ii) and (iii) are indirect determinants, and (iv) is a spanning indirect determinant.

A linguistic note is appropriate at this point. It is premature to state, as above, that these variables are determinants of information value. More precisely, the determinants of information value are a subset (proper or otherwise) of these variables. As will shortly become apparent, fixed cost is, in fact, not a determinant of the value of information about \( b \).

Let \( h^b_t \) denote the system which provides information about \( b \) at time \( t \).

**Proposition 3-2:** For \( u(z) = z \),

\[
U(h^b_t) = \begin{cases} 
\frac{(1-t)}{4\alpha} \int_{y_e}^{y_r} \int_{y_e}^{y_r} [r - E(b | y)]^2 p(y) dy - [r - E(b)]^2, & E(b) \leq r \\
\frac{(1-t)}{4\alpha} \int_{y_e}^{y_r} \int_{y_e}^{y_r} [r - E(b | y)]^2 p(y) dy, & \text{otherwise}
\end{cases}
\]

where \( \{y\} = \hat{Y} \) denotes the set of potential signals from \( h^b_t \) and \( \hat{Y} = \{y_e \leq E(b | y) \leq y_r \} \). [The subscript \( t \) on \( y_t \) is suppressed].

**Proof:** The producer's maximum expected utility given that information system \( h^b_t \) is employed is given by:

\[
E[w(x^*, b) | h^b_t] = \max_{x_1 \geq 0} \int_{x_2 \geq 0} \int_{x_1 \geq 0} \int_{x_2 \geq 0} w(x, b) p(b) db \\
= \max_{x_1 \geq 0} \int_{x_2 \geq 0} \int_{x_1 \geq 0} \int_{x_2 \geq 0} \left[ r(x_1 + x_2) - \frac{a}{t} x_1^2 - \frac{a}{(1-t)} x_2^2 - b(x_1 + x_2) - c \right] p(b | y) db p(y) dy
\]

Note that \( w(x, b) \) is linearly separable w.r.t. \( x_1 \) and \( x_2 \). Taking expectations as indicated, and recalling that \( E[ E(b | y) ] = E(b) \), where \( E \) denotes the expectation operation w.r.t. \( p(y) \), (3.16) can be rewritten as:
(3.17) \( E[w(x^*, b) | h^b_t] = \max\{ (E-r(b)) x_1 - \frac{a}{t} x_1^2 + \max\{ (E-r(b)) y x_2 \} x_2 \}
- \frac{a}{(1-t)} x_2^2 p(y) dy - c \)

Noting that \((E-r(b)) x_2 - [a/(1-t)] x_2^2\) is concave in \(x_2\), setting its first derivative equal to zero, and recalling the constraint \(x_2 \geq 0\):

(3.18) \( x_2^* = \begin{cases} \frac{(1-t)[r-E(b)|y]}{2a}, & \text{if } E(b)|y| \leq r \\ 0, & \text{otherwise} \end{cases} \)

The unconstrained optimum for \(x_2\) is \((1-t)(E-r(b))/2a\) and the concavity of the maximand insures that the constrained optimum will be equal to the lower bound when the unconstrained optimum is negative, i.e., when \([r-E(b)|y|]<0 \iff E(b)|y| > r\).

Maximization of (3.17) w.r.t. \(x_1\) yields:

(3.19) \( x_1^* = \begin{cases} \frac{t[E-b]}{2a}, & \text{if } E(b) \leq r \\ 0, & \text{otherwise} \end{cases} \)

Substitution of \((x_1^*, x_2^*|y)\) into (3.17) yields

(3.20) \( E[w(x^*, b) | h^b_t] = \begin{cases} \frac{t[E-b]^2}{4a} + \frac{(1-t)}{4a} \int_{y \in Y} [r-E(b)|y|]^2 p(y) dy - c, & \text{if } E(b) \leq r \\ \frac{(1-t)}{4a} \int_{y \in Y} [r-E(b)|y|]^2 p(y) dy - c, & \text{otherwise} \end{cases} \)

where \(Y\) is defined above. The producer's maximum expected profit given that the null information system is employed is given by equation (3.15). Subtracting this from (3.20) yields the following:

(3.21) \( U(h^b_t) = \begin{cases} \frac{(1-t)}{4a} \int_{y \in Y} [r-E(b)|y|]^2 p(y) dy - [r-E(b)]^2, & \text{if } E(b) \leq r \\ \frac{(1-t)}{4a} \int_{y \in Y} [r-E(b)|y|]^2 p(y) dy, & \text{otherwise} \end{cases} \)

Q.E.D.
Equation (3.17) illustrates well the effect of the information system on the producer's decision making process. The optimal production quantity during the time period \([t, 1]\) given by \(x_2\) is determined after a signal \(y \in Y\) has been received. Prior to receipt of the signal, however, the specific signal which will be received is unknown. Hence, an expectation is taken w.r.t. the potential signals which may be received. The optimal production quantity during the time period \([0, t]\) must be determined at time 0, prior to the receipt of the signal from information system \(h^b_t\). Consequently, the maximization w.r.t. \(x_1\) occurs outside of the expectation over signals \(y \in Y\). Comparing the optimal production rates from (3.18) and (3.19):

\[
\frac{x^*_1}{t} = \begin{cases} \frac{[r-E(b)]}{2a}, & \text{if } E(b) \leq r \\ 0, & \text{otherwise} \end{cases}
\]

\[
\frac{(x^*_2|y)}{(1-t)} = \begin{cases} \frac{[r-E(b|y)]}{2a}, & \text{if } E(b|y) \leq r \\ 0, & \text{otherwise} \end{cases}
\]

\[
\frac{x^*_1}{t} = \frac{(x^*_2|y)}{(1-t)} \quad \Rightarrow \quad \begin{cases} E(b|y) = E(b) \\ \text{or} \\ E(b|y) > r \text{ and } E(b) > r \end{cases}
\]

The optimal production rates in the two time periods will, in general, no longer be equal. This, of course, results from adjustment of the rate by the producer to incorporate the information about unit variable cost received at time \(t\). The optimal rates will be equal only if (1) \(E(b|y) = E(b)\) meaning that the signal from \(h^b_t\) confirmed the producer's initial expectation regarding unit variable cost, or (2) \(E(b|y) > r\) and \(E(b) > r\) and
receiving the signal from information system $h_t^b$ the producer expects unit variable cost to exceed price. This, of course, indicates a short-run shut-down condition in both production subperiods.

Recalling the list of six direct and indirect determinants of information value given above, all but one of these is represented either implicitly or explicitly in the expression for $U(h_t^b)$ given by Proposition 3.2. As was previously indicated, fixed cost (c) is not an indirect determinant of the value of $h_t^b$. Price (r), the quadratic cost parameter (a), and information timing parameter (t) all appear as explicit arguments of $U(h_t^b)$. The degree of uncertainty in p(b) and the nature of $p(y|b)$ appear implicitly since they are implicit arguments of

$$
\int_{Y} [r-E(b|y)]^2 p(y) dy.
$$

The functional relationship between the value of information system $h_t^b$ and some of its indirect determinants can be addressed via the result in Proposition 3.2.

**Corollary 3.2.1:** For $t < 1$ and $Y \neq \emptyset$, the expected value of information system $h_t^b$ is linearly decreasing in signal time $t$ and strictly convex decreasing in the quadratic cost parameter $a$.

**Proof:** In Chapter 2, the expected value of an information system was shown to be nonnegative. Let $U[E(b), r]$ denote the quantity in braces in equation (3.21). Multiplying $U(h_t^b)$ by $4a/(1-t)$ and noting that $a > 0$ and $(1-t) > 0$, $L[E(b), r] \geq 0$.

Partially differentiating equation (3.21):

$$
\frac{\partial U(h_t^b)}{\partial t} = -\frac{L[E(b), r]}{4a} < 0; \frac{\partial^2 U(h_t^b)}{\partial t^2} = 0
$$

$$
\frac{\partial U(h_t^b)}{\partial a} = -\frac{(1-t) L[E(b), r]}{4a^2} < 0; \frac{\partial^2 U(h_t^b)}{\partial a^2} = \frac{(1-t) L[E(b), r]}{2a^3} > 0
$$

Q.E.D.
From Corollary 5-2.1, an obvious conclusion is that early information about unit variable cost is better than late information, ceteris paribus.\textsuperscript{29} Equivalently, the value of the information is increasing in its timeliness.\textsuperscript{30} This is, of course, an intuitively satisfying result. The later the producer receives information about unit variable cost, the shorter the period of time during which he will produce at the optimal rate given that information. One of three situations may occur. First, the signal may be such that $E(b) < E(b|y) < r$ which implies that $\{[x_2^*(1-t)]|y\} < x_1^*/t$. In this case the producer has operated at a faster production rate during the time period $[0,t]$ than was optimal given the signal from system $h_t^b$. The increased sales during this period were more than offset by the greater cost incurred by the high production rate.

Second, the signal from $h_t^b$ may indicate that $E(b|y) < E(b) < r$ which implies that $\{[x_2^*(1-t)]|y\} > x_1^*/t$. Here the producer has operated at a slower rate during the time period $[0,t]$ than was optimal in light of the signal from $h_t^b$. The lower cost from producing at a low rate was more than offset by the decrease in sales.

Finally, the signal from $h_t^b$ may indicate that $E(b|y) = E(b)$ or $E(b|y)$ and $E(b)$ are both greater than $r$. This implies that $\{[x_2^*(1-t)]|y\} = x_1^*/t$, and the producer has operated during the period $[0,t]$ at the optimal rate given the signal from system $h_t^b$.

\textsuperscript{29} All of the results throughout the analysis are ceteris paribus arguments regarding variables other than those with respect to which differentiation is conducted. Repeated use of the term, however, will be avoided.

\textsuperscript{30} $(1-t)$ may be considered a measure of timeliness.
If either of the first two situations occurs, the ex post value of the information from \( h^b_t \) will decline as the signal is received later. If the third situation occurs, the ex post value of the information from system \( h^b_t \) will be invariant with respect to signal timing. Consequently, the expected or ex ante value of information system \( h^b_t \) is decreasing in signal time \( t \) or increasing in information timeliness. From Proposition 4-2 the extreme case is given by \( U(h^b_t) = 0 \). At the period's end, information is valueless in the short run since it is too late to be acted upon.

From Corollary 3-2.1 the value of information about unit variable cost is convex decreasing in the quadratic cost parameter \( a \).

![U(h^b_t) as a Function of Quadratic Cost Parameter a](image.png)

Figure 8. \( U(h^b_t) \) as a Function of Quadratic Cost Parameter \( a \)

Recall from the discussion in Chapter 2 that \( U(h) \) is the difference between two maximal expected utility levels:

\[
(3.22) \quad U(h) = \mathbb{E} \left( \max_{y \in Y} \mathbb{E} [u(w(x,s)) | y] - \max_{x \in X} u(w(x,s)) \right)
\]

Moreover, \( U(h) \geq 0 \) which implies that the first term on the right hand side (RHS) of (3.22) is not less than the second. For information system \( h^b_t \) the two RHS terms of (3.22) are given by (3.20) and (3.15).
Rewrite (3.20) as \( E[w(x^*, b) | h^b_t] = \{f[E(b), r] / a\} - c \) and (3.15) as \( E[w(x^*, b) | h_o] = \{g[E(b), r] / a\} - c \), where \( f[\cdot] \geq 0 \) and \( g[\cdot] \geq 0 \). Hence \( U(h^b_t) = f[E(b), r] - g[E(b), r] \).

Partial differentiation of \( E[w(x^*, b) | h^b_t] \) yields:

\[
\frac{\partial E[w(x^*, b) | h^b_t]}{\partial a} = \frac{-f[E(b), r]}{a^2} < 0; \quad \frac{\partial^2 E[w(x^*, b) | h^b_t]}{\partial f[E(b), r] \partial a} = \frac{-1}{a^2} > 0
\]

\[
\frac{\partial^2 E[w(x^*, b) | h^b_t]}{\partial a^2} = \frac{2f[E(b), r]}{a^3} > 0; \quad \frac{\partial^3 E[w(x^*, b) | h^b_t]}{\partial f[E(b), r] \partial a^2} = \frac{2}{a^3} > 0
\]

Each of the corresponding partial derivatives of \( E[w(x^*, b) | h_o] \) is of the same sign as its counterpart above. These partial derivatives indicate that the marginal decrements \(^{31}\) in \( E[w(x^*, b) | h^b_t] \) and \( E[w(x^*, b) | h_o] \) with respect to \( a \) are increasing in \( f[E(b), r] \) and \( g[E(b), r] \), respectively. Since \( f[E(b), r] \geq g[E(b), r], E[w(x^*, b) | h^b_t] \) is decreasing in \( a \) at a faster rate than is \( E[w(x^*, b) | h_o] \). This phenomenon gives rise to the decreasing property of \( U(h^b_t) \) in \( a \). Moreover, the rate of decrease in the marginal decrements in \( E[w(x^*, b) | h^b_t] \) and \( E[w(x^*, b) | h_o] \) with respect to \( a \) is increasing in \( f[E(b), r] \) and \( g[E(b), r] \), respectively. \( f[E(b), r] \geq g[E(b), r] \) then implies that the rate of decrease in the marginal decrements of \( E[w(x^*, b) | h^b_t] \) with respect to \( a \) exceeds that for \( E[w(x^*, b) | h_o] \). This fact gives rise to the convexity property of \( U(h^b_t) \).

An increase in the parameter \( a \) has a depressing effect on profit since this implies that marginal cost is increasing in the production.

\(^{31}\)A decrement is the positive amount by which a variable is decreased.
rate at a faster pace. Moreover, this depressing effect is greater when production is optimally adjusted than when it is not. That is, the profit level which is maximal with respect to adjustment of the production rate is decreasing in a at a faster pace than is a profit level which is suboptimal with respect to the production rate. Figure 9. demonstrates this property.

Figure 9. Effect of Quadratic Cost Parameter on Optimal and Suboptimal Profit Levels

The decrease in the profit rate is less per unit increase in a when \( x/\tau \) is equal to either of the two suboptimal rates than when \( x/\tau \) is set at the optimal rate. When the producer employs the null information system, the optimal production rate is a function of \( E(b) \), and will, in general, be suboptimal in light of the information, which would be provided
by system $h_t^b$. In other words, $E[w(x^*, b) | h_0]$ is a suboptimal profit level based on a suboptimal production rate when compared to $E[w(x^*, b) | h_t^b]$.

Some additional intuition can be provided for the effect of the quadratic cost parameter $(a)$ on $U(h_t^b)$ by appealing to the traditional approach to profit maximization used in microeconomic theory. It must be emphasized at the outset, however, that this approach does not provide a precise interpretation of the phenomenon proved in Corollary 3.2.1. The phenomenon given in the corollary and precisely interpreted above is not identical to that exhibited in the following discussion. However, some of the factors which give rise to the precise result are also at work in driving the following result. Hopefully the discussion to follow will be a useful heuristic in aiding the intuition.\footnote{The heuristic discussion moves from a theoretical plane to another level of interpretation which is more intuitive. Some rigor and precision must be sacrificed in making this transition. Such a sacrifice is not uncommon in research, and is justified on the grounds that it is only supplemental to the rigorous interpretation.}

In the theory of the firm, profit is maximized by equating marginal cost (MC) to marginal revenue (MR) which is equivalent to the unit price under perfect competition.\footnote{E. Malinvaud, Lectures on Microeconomic Theory (London: North Holland Publishing Company Ltd., 1972), p. 66.} In this CVP problem, unit price is $r$ and marginal cost for the second production subperiod $[t, 1]$\footnote{The second subperiod is the only one affected by receipt of the information, and is therefore the relevant one in the ensuing discussion.} is given by

\[ \frac{\partial}{\partial x_2} \left( ax_2^2/(1-t) + bx_2 + c \right) = 2ax_2/(1-t) + b. \]

Figure 10A depicts the marginal revenue and marginal cost curves. $x_2^*$ is indicated by the
intersection of the marginal curves.

![Diagram showing MR, MC, and the bands of uncertainty](image)

**Figure 10. Economic Interpretation of Effect of Parameter $a$ on $U(h_t)$**

Since unit variable cost $b$ is uncertain, however, there is a band of uncertainty for the vertical axis intercept of the marginal cost curve. This is shown in Figure 10B. The result is a band of uncertainty for the marginal cost curve, which is shown in Figure 10B for two values of $a$, $a'$ and $a''$ ($a'' > a'$). This band of uncertainty about $b$ results in a range of uncertainty for the point of intersection of the MC and MR curves, which in turn projects onto the $x_2$ axis as a band of uncertainty for $x_2^*$. The range of uncertainty about $x_2^*$ is wider for $a'$ than for $a''$.

---

35 The band of uncertainty about $b$ is shown here for graphing convenience with $b < r$. This restriction is not necessary to the result.
This can also be demonstrated algebraically. Let $b^1$ and $b^2$ denote the lower and upper bounds, respectively, of the band of uncertainty about $b$. Equating marginal revenue to marginal cost for each of these values of $b$ yields the associated optimal actions, $x^*_2$ and $x^*_2$, where:

$$x^*_2 = \frac{(r-b^1)(1-t)}{2a} \quad \text{and} \quad x^*_2 < x^*_2$$

The width of the range of uncertainty about $x^*_2$ is given by:

$$x^*_2 - x^*_2 = \frac{(r-b^1)(1-t)}{2a} - \frac{(r-b^2)(1-t)}{2a} = \frac{(1-t)(b^2-b^1)}{2a}$$

The band of uncertainty about $x^*_2$ is decreasing in $a$ since $\frac{\partial (x^*_2 - x^*_2)}{\partial a} < 0$.

Since the range of uncertainty about $x^*_2$ is larger for $a'$, the value of information about $b$ is higher. Conversely, information about $b$ is of less value when there is less uncertainty about $x^*_2$, which is the case when $a$ is larger. As the quadratic cost parameter becomes larger, it dominates the producer's decision process to a greater extent and admits a lesser influence to unit variable cost. Therefore, one would expect information about $b$ to be of lesser value when $a$ increases.  

**Corollary 3.2.2:** For $t<1$ the marginal increments in $U(h^b_t)$ w.r.t. signal time $t$ are strictly convex decreasing in $a$, and the marginal decrements in $U(h^b_t)$ w.r.t. the quadratic cost parameter $a$ are linearly decreasing in $t$.

---

This approach cannot be a precise interpretation of the result in Corollary 3.2.1. It only considers the effect of $a$ on $x^*_2$ and ignores its effect on $E[w(x^*_2,b)|h^b_t]$ and $E[w(x^*_2,b)|h_0]$.
Proof: Recalling the definition of $L[E(b),r]$ given in the proof of Corollary 3-2.1, and partially differentiating equation (3.21):

$$\frac{\partial^2 U(h^b_t)}{\partial t \partial a} = \frac{L[E(b),r]}{4a^2} > 0; \quad \frac{\partial^3 U(h^b_t)}{\partial a \partial t} = 0$$

$$\frac{\partial^3 U(h^b_t)}{\partial a \partial t} = -\frac{L[E(b),r]}{2a^3} < 0$$

Q.E.D.

![Figure 11. Interaction of Information Time and the Quadratic Cost Parameter](image)

As $t$ increases the value of information system $h^b_t$ declines because the producer has less time during which to take advantage of the information about unit variable cost. $E[w(x^*,b)|h^b_t]$ converges linearly to $E[w(x^*,b)|h_0]$, or equivalently, $f[E(b),r]$ converges linearly to $g[E(b),r]$.\(^{37}\) It was shown previously that the influence of $a$ on $E[w(x^*,b)|h^b_t]$ and $E[w(x^*,b)|h_0]$ depends on $f[E(b),r]$ and $g[E(b),r]$, respectively. Therefore, as $t$ increases and $f[E(b),r]$ approaches $g[E(b),r]$, the marginal decrements with respect to $a$ in $E[w(x^*,b)|h^b_t]$

\(^{37}\)Recall that $E[w(x^*,b)|h^b] = (f[E(b),r]/a) - c$ and $E[w(x^*,b)|h_0] = (g[E(b),r]/a) - c$. 

and \(E[w(x^*,b)|h^b_t]\) converge. Since the marginal effect of a \(t\) on \(U(h^b_t)\) is equal to the difference of the marginal effects of a \(t\) on \(E[w(x^*,b)|h^b_t]\) and \(E[w(x^*,b)|h^0]\), the marginal effect of a \(t\) on \(U(h^b_t)\) will decline as the two expectations converge. In simplistic terms, as a \(t\) increases, the value of information about unit variable cost declines. At the same time, the effect of any variable which influences \(U(h^b_t)\) independently of a \(t\) also declines.

Unit price, \(r\), appears explicitly in the expression for \(U(h^b_t)\) given by Proposition 3-2. It is also represented implicitly, however, since the structure of \(\hat{Y} = \{y \in Y|E(b|y) \leq r\}\) is a function of \(r\). As a consequence, it is not possible to address the precise nature of \(r\)'s role as an indirect determinant without lending some specificity to the nature of \(h^b_t\) and its associated \(\hat{Y}\).

### 3.4.2 Perfect Information System

The results obtained for the information system \(h^b_t\) are perfectly general with respect to particular characteristics of the system (e.g., perfect, noisy, noiseless). In the event that the producer employs the system \(h^b_{\omega,t}\) which provides perfect information about \(b\) at time \(t\), the value of the system is given by Proposition 3-3:

**Proposition 3-3:** For \(u(z) = z\),

\[
(3.23) \quad U(h^b_{\omega,t}) = \begin{cases} 
\frac{(1-t)}{4a} \{\text{var}(b) - \int_{r-b}^{\infty} (r-b)^2 p(b) db\}, & \text{if } E(b) \leq r \\
\frac{(1-t)}{4a} \{\int_{0}^{r} (r-b)^2 p(b) db\}, & \text{otherwise}
\end{cases}
\]

where \(\text{var}(b)\) denotes the variance of \(p(b)\).

**Proof:** The proof follows directly from Proposition 3-2. If \(h^b_t\) is a perfect information system: \(y=b, Y=[0,\infty), E(b|y)=E(b|b)=b\), and
\( \hat{Y} = \{ y \in Y | E(b|y) \leq r \} = \{ b | b \leq r \} = [0,r] \).

Substituting from above into \( U(h^b_t) \):

\[
U(h^b_t) = \begin{cases} 
\frac{(1-t)}{4a} \int_0^r (r-b)^2 p(b) db - [r-E(b)]^2, & \text{if } E(b) \leq r \\
\frac{(1-t)}{4a} \int_0^r (r-b)^2 p(b) db, & \text{otherwise}
\end{cases}
\]

(3.24) Rearranging terms in the top line of the RHS of (3.24) and recalling that \( \text{var}(b) = E(b^2) - E(b)^2 \):

\[
\frac{(1-t)}{4a} \int_0^r (r-b)^2 p(b) db - [r-E(b)]^2 = \frac{(1-t)}{4a} \int_0^r (r-b)^2 p(b) db \\
- [r^2 - 2rE(b) + E(b^2) - \text{var}(b)]
\]

\[
\frac{(1-t)}{4a} \int_0^r (r-b)^2 p(b) db - \int_0^\infty (r-b)^2 p(b) db = \frac{(1-t)}{4a} \text{var}(b) \\
- \int_0^r (r-b)^2 p(b) db
\]

Q.E.D.

The nonnegativity of \( U(h^b_{t,0}) \) is easy to establish. \( (1-t) \), \( r \), and \( a \) are clearly nonnegative. Also:

\[
\int_0^\infty (r-b)^2 p(b) db = \int_0^\infty (b-E(b))^2 p(b) db - \int_0^r (b-r)^2 p(b) db
\]

For \( E(b) \leq r \), \( [b-E(b)]^2 \leq (b-r)^2 \) and \( \cup_{0}^{\infty} \emptyset [r,\infty) \). The expression in parentheses is also nonnegative. The results of Corollaries 3.2.1 and 3.2.2 are, of course, valid in interpreting the effects of \( a \) and \( t \) on \( U(h^b_{t,0}) \).

The role of the unit price, \( r \), can now be characterized as a determinant of the value of information system \( h^b_{t,0} \). This is possible because \( \hat{Y} \) is known precisely to be the interval \([0,r] \) when \( h^b_{t} \) is the perfect system.

**Corollary 3.3.1:** For \( t < 1 \) and \( r \epsilon [b^L, b^U] \), the expected value of information system \( h^b_{t,0} \) is strictly convex increasing in the unit price \( r \).

\(^{38}E^2(b) \) denotes \([E(b)]^2\).
for $r \leq E(b)$, and strictly concave increasing in $r$ for $E(b) < r$. Here $b^L(b^U)$ is the producer's subjective lower (upper) bound, if any, on the random variable $b$.

Proof: Partially differentiating\textsuperscript{39} equation (3.23):

$$\frac{\partial^2 U(h^b_{\infty, t})}{\partial r^2} = \begin{cases} 
\frac{(1-t)}{4a} \int_r^\infty \{ -\int_2(r-b)p(b) \} \, db > 0, & \text{if } E(b) \leq r \leq b^U \\
\frac{(1-t)}{4a} \int_0^r \{ \int_2(r-b)p(b) \} \, db > 0, & \text{if } b^L < r < E(b) 
\end{cases}$$

For $r > b^U$, $\frac{\partial U(h^b_{\infty, t})}{\partial r} = \frac{\partial^2 U(h^b_{\infty, t})}{\partial r^2} = 0$

Q.E.D.

The functional dependence of $U(h^b_{\infty, t})$ on $r$ is depicted in Figure 12.

\textsuperscript{39}The general expression for differentiation w.r.t. a limit of integration is given by:

$$Q(a, \beta) = \int g(\alpha) q(\alpha, \beta) \, d\alpha = \int \frac{h(\alpha)}{g(\alpha)} \, q(\alpha, \beta) \, d\beta + q(\alpha, h(\alpha)) \frac{\partial h(\alpha)}{\partial \alpha} - q(\alpha, g(\alpha)) \frac{\partial g(\alpha)}{\partial \alpha}$$

where $a, \beta, h$, and $q$ as used here do not correspond to their definitions outside of this footnote. See Walter C. Giffin, Introduction to Operations Engineering (Homewood, Illinois: Richard D. Irwin, Inc., 1971), pp. 415, 416.
The value of information system \( h_{\omega, t} \) is increasing in the unit price throughout the range over which the producer subjectively assigns positive probability to unit variable cost. The sensitivity of this value to \( r \) is increasing in \( r \) given that \( r < E(b) \), which indicates that the producer will temporarily shut down in the subperiod \( [0, t] \). However, the sensitivity is decreasing in \( r \) if \( r \geq E(b) \), which indicates that production will take place in the subperiod \( [0, t] \).

Recall that \( U(h_{\omega, t}^b) = E[w(\tilde{x}^*, b)h_{\omega, t}^b] - E[w(\tilde{x}^*, b)h_0] \). Equation (3.20) gives \( E[w(\tilde{x}^*, b)h_{t}^b] \). Taking note that \( \tilde{Y} = [0, r] \), \( y=b \), and \( E(b|y) = b \) for the case of perfect information, (3.20) reduces to:

\[
E[w(\tilde{x}^*, b)|h_{\omega, t}^b] = \begin{cases} 
\frac{t[r-E(b)]^2}{4a} + \frac{(1-t)}{4a} \int_0^r (r-b)^2p(b)db - c, & \text{if } E(b) < r \\
\frac{(1-t)}{4a} \int_0^r (r-b)^2p(b)db - c, & \text{otherwise}
\end{cases}
\]

(3.25)

\( E[w(\tilde{x}^*, b)|h_0] \) is given by (3.15).

First take the case of \( r \leq E(b) \). If the producer employs the null information system he shuts down for the entire production period yielding \( E[w(\tilde{x}^*, b)|h_0] = -c \) (see 3.15). Naturally, expected profit is invariant with respect to the unit price. If the producer employs system \( h_{\omega, t}^b \),
however, expected profit is convex increasing in unit price.

\[
\frac{\partial E[w(x^*,b) | h_{o,t}]}{\partial r} = \frac{(1-t)}{4a} \int_0^2 (r-b)p(b)db > 0, \quad r \leq E(b)
\]

\[
\frac{\partial^2 E[w(x^*,b) | h_{o,t}]}{\partial r^2} = \frac{(1-t)}{2a} \int_0^r p(b)db > 0 \text{ for all } r
\]

Even though the producer shuts down temporarily in the subperiod \([0,t]\), there is a possibility that production may resume in the subperiod \([t,1]\). If production takes place, profit will be increasing at an increasing rate in unit price. Since \(U(h_{o,t}^b)\) is the difference between two maximal expectations, the marginal increment in \(U(h_{o,t}^b)\) with respect to \(r\) is the difference between the corresponding marginal increments in the two expectations. When \(r \leq E(b)\), zero increments are subtracted from increasing increments to yield the result observed in Figure 12.

Now take the case of \(r > E(b)\). If the producer employs the null information system he operates for the entire production period yielding \(E[w(x^*,b) | h_{o,t}] = \{(r-E(b))^2/4a\} - c\) (see 3.15). Here expected profit is convex increasing in the unit price.

\[
\frac{\partial E[w(x^*,b) | h_o]}{\partial r} = \frac{[r-E(b)]}{2a} > 0; \quad \frac{\partial^2 E[w(x^*,b) | h_o]}{\partial r^2} = \frac{1}{2a} > 0
\]

Similarly, if the producer employs system \(h_{o,t}^b\) expected profit is convex increasing in unit price.

\[
\frac{\partial E[w(x^*,b) | h_{o,t}^b]}{\partial r} = \frac{t[r-E(b)]}{2a} + \frac{(1-t)}{2a} \int_0^r (r-b)p(b)db > 0;
\]

\[
\frac{\partial^2 E[w(x^*,b) | h_{o,t}^b]}{\partial r^2} = \frac{t}{2a} + \frac{(1-t)}{2a} \int_0^r p(b)db > 0, \quad r > E(b)
\]

In this situation the producer operates during the subperiod \([0,t]\) and may or may not operate during the subperiod \([t,1]\). Since production is taking place for at least part and possibly all of the period, profit is again increasing at an increasing rate in the price.
Again viewing the marginal increase in $U(h_{b, t}^b)$ as the difference between the marginal increments in the two maximal expectations, note that $\partial E[w(x^*, b) | h_{b, t}^b] / \partial r$ exceeds $E[w(x^*, b) | h_0] / \partial r$ by $-\frac{(1-t)^\omega}{r} \int (r-b)p(b)db/2 > 0$. This guarantees that $U(h_{b, t}^b)$ is increasing in $r$. An increase in price is of greater marginal value to the producer when he employs system $h_{b, t}^b$ and operates at an optimal rate in the second subperiod, than when he runs the risk of operating suboptimally during that subperiod.

The convexity property of $U(h_{b, t}^b)$ in $r$ results because $\partial^2 E[w(x^*, b) | h_{b, t}^b] / \partial r^2$ is less than $\partial^2 E[w(x^*, b) | h_0]$. The sensitivity of $E[w(x^*, b) | h_{b, t}^b]$ to $r$ is increasing more slowly in $r$ than is that of $E[w(x^*, b) | h_0]$.

Appealing to the less precise but intuitively appealing theory of the firm approach to interpretation, note from Figure 13 that the range of uncertainty about $x_2^*$ is increasing in $r$ as $r$ increases in the interval $[b^L, b^U]$.

![Figure 13. Economic Interpretation of Effect of Unit Price on $U(h_{b, t}^b)$](image-url)
Corollary 3.4.2: For \( t < 1 \): (i) the marginal decrements in \( U(h_{\infty}^b, t) \) w.r.t. information time \( t \) are strictly convex increasing in the unit price \( r \) for \( r \leq E(b) \), and strictly concave increasing in \( r \) for \( r > E(b) \); (ii) the marginal increments in \( U(h_{\infty}^b, t) \) w.r.t. the unit price \( r \) are linearly decreasing in information time \( t \); (iii) the marginal decrements in \( U(h_{\infty}^b, t) \) w.r.t. the quadratic cost parameter \( a \) are strictly convex increasing in unit price \( r \) for \( r \leq E(b) \), and strictly concave increasing in \( r \) for \( r > E(b) \); (iv) the marginal increments in \( U(h_{\infty}^b, t) \) w.r.t. unit price \( r \) are strictly convex decreasing in the quadratic cost parameter \( a \).

Proof: Partially differentiating equation (3.23):

\[
\frac{\partial^2 U(h_{\infty}^b, t)}{\partial r \partial t} = \begin{cases} 
\frac{1}{4a} \int r (2(r-b)p(b)db) < 0, & \text{if } E(b) \leq r \\
-\frac{1}{4a} \int r (2(r-b)p(b)db) < 0, & \text{otherwise}
\end{cases}
\]

\[
\frac{\partial^3 U(h_{\infty}^b, t)}{\partial r^2 \partial t} = \begin{cases} 
\frac{1}{4a} \int r (2p(b)db) > 0, & \text{if } E(b) \leq r \\
-\frac{1}{4a} \int r (2p(b)db) < 0, & \text{otherwise}
\end{cases}
\]

\[
\frac{\partial^4 U(h_{\infty}^b, t)}{\partial t^2 \partial r} = 0
\]

\[
\frac{\partial^2 U(h_{\infty}^b, t)}{\partial r^2 \partial a} = \begin{cases} 
\frac{(1-t)}{4a^2} \int r (2(r-b)p(b)db) < 0, & \text{if } E(b) \leq r \\
-\frac{(1-t)}{4a^2} \int r (2(r-b)p(b)db) < 0, & \text{otherwise}
\end{cases}
\]

\[
\frac{\partial^3 U(h_{\infty}^b, t)}{\partial r^2 \partial a} = \begin{cases} 
\frac{(1-t)}{4a^2} \int r (2p(b)db) > 0, & \text{if } E(b) \leq r \\
-\frac{(1-t)}{4a^2} \int r (2p(b)db) < 0, & \text{otherwise}
\end{cases}
\]
\[
\frac{a^5 U(b_{\infty}, t)}{2a^3} = \begin{cases} 
\frac{(1-t)}{2a^3} \left\{ \int_0^\infty (2(r-b)p(b)db) \right\} > 0, \text{ if } E(b) = r \\
\frac{(1-t)}{2a^3} \left\{ \int_0^\infty (r-b)p(b)db \right\} > 0, \text{ otherwise} 
\end{cases}
\]

Q.E.D.

**Figure 14. Interaction of Information Time and Unit Price**

**Figure 15. Interaction of the Quadratic Cost Parameter and Unit Price**
The sensitivity of $U(h^b_{\omega, t})$ to each of the determinants, information timeliness, unit price, and the quadratic cost parameter, is increasing in the other two.

3.5 Summary

This chapter is the beginning of an investigation into the role played by various determinants of information value. The analysis is conducted in the context of a short-run cost-volume-profit decision model. CVP analysis in managerial accounting is important. It represents a rather comprehensive overview of short-run planning, and its concepts are applicable to many types of short-run resource allocation decisions.

The particular CVP model investigated is based on a linear total revenue function, which implies a perfectly competitive setting, and a cost function which is quadratic increasing in the production rate. The quadratic cost function is a reasonable compromise between the unrealistic linear function and the mathematically complex cubic function. Other assumptions include the possibility of costless alteration of the production rate, instantaneous sales at the same rate as production, and unbounded and infinitely divisible production quantities.

The value of systems providing various types of information about fixed cost or unit variable cost is expressed as a function of several determinants. A generalized fixed cost reporting system and generalized and perfect variable cost reporting systems are considered. Unit price, the quadratic cost parameter, information timing, initial uncertainty, and the nature of the information system are identified as either explicit or implicit determinants. Sensitivity analysis is applied
to determine the effects of changes in the determinants on information system value. First and second order effects, as well as interaction effects, of the determinants are addressed.

The mathematical statements of information system value and sensitivity to determinants are restated in intuitive economic terms, and the results are given economic interpretation with various levels of precision. No attempt is made to draw managerial implications from the results; this task is reserved for a later chapter. This will permit results obtained for diverse settings and information systems to be compared, contrasted, and summarized. The result will be a greater power to generalize with respect to the normative inferences drawn regarding managerial behavior.

The results obtained for systems which provide unit variable cost information at time $t$ are completely general with respect to the producer's prior probability distribution over unit variable cost. In Chapter 4, stronger results are obtained by placing a rather mild restriction on this prior distribution.
CHAPTER 4
ANALYSIS OF STOCHASTIC UNIT VARIABLE COST
WITH RESTRICTED PRIOR

In this chapter the investigation of information systems for the
CVP decision problem continues, and unit variable cost remains the only
source of uncertainty. The condition is imposed on \( p(b) \) that \( P(r) = 1. \)
This is a relatively mild restriction requiring that the producer places
an upper bound on the values of \( b \) to which he subjectively assigns
positive probability. Moreover, the upper bound must not be greater
than \( r \). In other words, the producer does not believe it is possible
for unit variable cost in the current production period to exceed the
unit price. This restriction effectively precludes an expectation by
the producer of a short-run shut-down during either of the production
subperiods.

4.1 Generalized Information System

Proposition 4-1 gives the value of system \( h^b_t \) which provides infor-
mation at time \( t \) about unit variable cost.

\[
\text{Proposition 4-1: For } u(z)=z \text{ and } P(r)=1, \ U(h^b_t) = \frac{(1-t)(\text{var}(b) -
E[\text{var}(b|y)])}{4a}, \text{where } E \text{ denotes the expectation w.r.t. } p(y) \text{ and } \text{var}(b|y)
\]
denotes the variance of \( p(b|y) \).

\[\text{1}\]

\[P(r) = \int_{-\infty}^{r} p(b) \, db\]

94.
Proof: The proof follows from Proposition 3-2. Since P(r)=1 and F(b)=r, the top line only of the expression for U(h^b), given in Proposition 3-2, applies.

\[
(4.1) \quad U(h^b) = \frac{(1-t)}{4a} \{ \int_{Y} [r-E(b|y)]^2 p(y)dy - [r-E(b)]^2 \}
\]

Since the producer's prior subjective pdf on b assigns zero probability to values of b greater than r, it follows that p(b|y) is zero for b>r.\(^2\)

This implies that E(b|y)≤r for all y∈Y. Hence \(Y=\{y∈Y|E(b|y)≤r\} \rightarrow Y\). Equation (4.1) may then be rewritten as follows:

\[
(4.2) \quad U(h^b) = \frac{(1-t)}{4a} \{ \int_{Y} [r-E(b|y)]^2 p(y)dy - [r-E(b)]^2 \}
\]

\[
= \frac{(1-t)}{4a} \{ -2r[E(b|y)] + E[2(b|y)] + 2rE(b) - E^2(b) \}
\]

where \(E^2(b|y) \) denotes \([E(b|y)]^2\). Note that \(\text{var}(b)=E(b^2)-E^2(b)\), \(\text{var}(b|y)\)

\(=E(b^2|y)-E^2(b|y)\), \(E[E(b|y)]=E(b)\), and \(E[E(b^2|y)]=E(b^2)\).\(^3\) Given these identities, (4.2) reduces to:

\[
(4.3) \quad U(h^b) = \frac{(1-t)}{4a} \{ \text{var}(b) - E[\text{var}(b|y)] \}
\]

Q.E.D.

Under the condition P(r)=1, (3.18) and (3.19) reduce to the following optimal actions:

\[x_1^* = \frac{t[r-E(b)]}{2a}, \quad x_2^* = \frac{(1-t)[r-E(b|y)]}{2a}\]

\(^2\)From Bayes' Theorem:

\[p(b|y) = \frac{p(y|b)p(b)}{p(y)} = \frac{p(y|b) \cdot 0}{p(y)} = 0, \text{ for } b > r.\]

Also, (3.20) reduces to the following:

\[ E[w(x^*, b) | h_t^b] = \frac{t[r-E(b)]^2}{4a} + \frac{(1-t)}{4a} \int_{y \in Y} [r-E(b|y)]^2 p(y) dy - c \]

The characteristics of the functional dependence of \( U(h_t^b) \) on the indirect determinants \( a \) and \( t \) were fully described by Corollaries 3-2.1 and 3-2.2 developed for the unrestricted prior on \( b \). Those results remain valid. The unit price, \( r \), is no longer an indirect determinant of information value. Its role as a determinant, given the unrestricted prior on \( b \), resulted from its function in the producer's decision rule for the short-run shut-down decision. Since this action will no longer be optimal when \( P(r) = 1 \), it no longer affects the value of information about unit variable cost.

The value of information system \( h_t^b \) depends on the producer's expectation of the amount by which the variance of \( p(b) \) will be reduced by system \( h_t^b \). The expectation of this reduction in the variance is given by the prior variance minus the expectation of the posterior variance, or \( \text{var}(b) - E[\text{var}(b|y)] \). Two direct determinants of information value are involved here: (1) the degree of uncertainty in \( p(b) \), and (2) the nature of \( h_t^b \) as characterized by the likelihood function \( p(y|b) \). These determinants were designated in Chapter 2 as initial uncertainty and the information system, respectively. Analysis of the role played by these determinants is facilitated by more specificity about the information system. This approach will be taken here, and the next three propositions yield results for special characterizations of information system \( h_t^b \).
4.2 Perfect Information System

Let \( h^b_{\infty, t} \) denote a system which provides perfect information about unit variable cost at time \( t \). The likelihood function for this system is given by:

\[
p(y|b) = \begin{cases} 
1, & y = b \\ 
0, & \text{elsewhere}
\end{cases}
\]

**Proposition 4-2:** For \( u(z) = z \) and \( P(r) = 1 \),

\[
(4.8) \ U(h^b_{\infty, t}) = \frac{(1-t)\text{var}(b)}{4a}
\]

**Proof:** The proof follows from Proposition 4-1. For perfect information the posterior pdf on \( b \) is given by:

\[
p(b|y) = \begin{cases} 
1, & b = y \\ 
0, & \text{elsewhere}
\end{cases}
\]

This implies that \( \text{var}(b|y) = 0 \) for all \( y \in \mathcal{Y} \), and therefore \( E[\text{var}(b|y)] = 0 \). Hence, \( U(h^b_{\infty, t}) = \frac{(1-t)\text{var}(b)}{4a} \).

There are no new insights into the role of information timing alone or the quadratic cost parameter alone as indirect determinants of information value. The results of Corollaries 3-2.1 and 3-2.2 still apply here. It is possible, however, to be more precise about the role of the producer's initial uncertainty in determining the value of information system \( h^b_{\infty, t} \).

This analysis is facilitated by recasting the producer's decision problem in a slightly different (but equivalent) form. First, assume that no information system is employed, and recall the producer's decision model given previously in equation (3.13):
\[
(4.4) \max_{x_1 \geq 0, x_2 \geq 0} \left\{ (r(x_1 + x_2) - \frac{a}{t} x_1^2 - \frac{a}{(1-t)} x_2^2 - b(x_1 + x_2) - c) p(b) db \right\}
\]
and the associated optimal action vector given previously by (3.14):
\[
(4.5) \quad \frac{x_1^*}{t} = \frac{x_2^*}{(1-t)} = \frac{[r-E(b)]/2a, \text{ if } E(b) \leq r}{0, \text{ otherwise}}
\]
Given the currently imposed restriction on \(p(b)\) that \(P(r) = 1\), (4.5) reduces to:
\[
(4.6) \quad \frac{x_1^*}{t} = \frac{x_2^*}{(1-t)} = \frac{[r-E(b)]}{2a}
\]
Now the producer's outcome function is
\[
w(x, b) = (r(x_1 + x_2) - \frac{a}{t} x_1^2 - \frac{a}{(1-t)} x_2^2 - b(x_1 + x_2) - c).
\]
Substitution of (4.6) into \(w(x, b)\) provides the outcome given optimal action and conditional on the occurrence of state \(b\):
\[
(4.7) \quad [w(x, b) | h_o] = (r-b)t\frac{[r-E(b)]}{4a} + (r-b)(1-t)\frac{[r-E(b)]}{4a} - c
\]
\[
= \frac{(r-b)[r-E(b)]}{4a} - c
\]
Let \(\psi(b) = \frac{w(x^*, b)}{h_o}\) denote the payoff function for this transformed problem. \(\psi\) has no action variable argument since the action variable has been eliminated by the maximization operation. When \(P(r) = 1\), if the producer faces the decision problem given in (4.4), this is equivalent to facing the gamble\(^4\) \(\psi(b)\). Note that \(\psi(b) = \frac{(r-b)[r-E(b)]}{4a} - c\) is a first degree polynomial in the state variable \(b\). If the payoff function

\(^4\)A gamble is defined for the purposes of this discussion as a mapping from the state space to utility space. That is, \(\psi\) maps the state \(b\) into a utility level, \(\psi(b)\). It is analogous to the payoff function, but has no action argument.
is an Nth degree polynomial in the state variable, the first N moments of the pdf over the state variable are needed to determine the expected utility. Consequently, the second through Nth moments are needed to determine the R-S degree of uncertainty. In the case of \( \psi \), then, only the first moment of \( p(b) \) is required to determine expected utility, and no moments are needed to characterize degree of uncertainty. In the absence of information about unit variable cost, the producer is indifferent with respect to his level of uncertainty about that cost.

Now assume that system \( h_{\omega,t}^b \) provides the producer with perfect information. The decision problem is now given by:

\[
(4.9) \max \max_{x_1, x_2} \int \left[ r(x_1, x_2) - \frac{a}{t} x_1^2 - \frac{a}{1-t} x_2^2 - b(x_1 + x_2) - c \right] p(b) db \\
= \frac{x_1^*}{t} = \frac{[r - E(b)]}{2a}, \quad \frac{x_2^*}{b} = \frac{(r - b)}{2a} \quad \text{since } P(r) = 1. 
\]

Substitution of \( [x_1^*, (x_2^*)/b] \) into (4.9) yields:

\[
[w(x_1^*, b)|h_{\omega,t}^b] = (r - b) t \frac{[r - E(b)]}{2a} + (r - b)(1-t) \frac{(r - b)}{2a} - c = (r - b) t \frac{[r - E(b)]}{2a} + (1-t) \frac{(r - b)^2}{2a} - c 
\]

Let \( \hat{\psi}(b) = u[w(x_1^*, b)|h_{\omega,t}^b] = [w(x_1^*, b)|h_{\omega,t}^b] \) denote the payoff function for this transformed problem. Again \( \psi \) has no action variable argument since the action variable has been eliminated by the maximization operation. When \( P(r) = 1 \), if the producer faces the decision problem given in

\[ 5 \text{See Chapter 2 for a discussion of these results by Markowitz, Hadar and Russell, and Rothschild and Stiglitz.} \]

\[ 6 \text{E}[\psi(b)] = \{E(r-b)[r-E(b)]/4a\} - c = \{[r-E(b)]^2/4a\} - c \]
(4.8) this is equivalent to saying he faces the gamble \( \hat{\psi}(b) \). Note that \( \hat{\psi}(b) \) is a second degree polynomial in the state variable \( b \). The first two moments of \( p(b) \) are required to determine expected utility\(^7\), and the second moment is required to determine degree of uncertainty.

The expected utility of information system \( h^b_{\infty,t} \) is given by \( U(h^b_{\infty,t}) = E[\hat{\psi}(b)] - E[\psi(b)] \). Since \( E[\hat{\psi}(b)] \) depends on the first two moments of \( p(b) \) and \( E[\psi(b)] \) depends on the first moment only, \( U(h^b_{\infty,t}) \) depends on the first two moments. Since the degree of uncertainty for the gamble \( \hat{\psi}(b) \) depends on the second moment of \( p(b) \) and the degree of uncertainty for the gamble \( \psi(b) \) is not relevant, the effect of the degree of initial uncertainty on \( U(h^b_{\infty,t}) \) is fully characterized by the second moment of \( p(b) \).

This conclusion is consistent with the result in Proposition 4-2, viz., \( U(h^b_{\infty,t}) = (1-t)\text{var}(b)/4a \). Equivalently, \( U(h^b_{\infty,t}) = (1-t)[E(b^2) - E^2(b)]/4a \). Since the R-S degree of uncertainty is defined for fixed expectation, the degree of initial uncertainty as a determinant of \( U(h^b_{\infty,t}) \) is fully described by the second moment of \( p(b) \), \( E(b^2) \). Therefore, it may be said that the degree of initial uncertainty in \( p(b) \) is in some sense equivalent to \( E(b^2) \). Since \( E(b^2) \) is a monotonic function of \( \text{var}(b) \) for fixed \( E(b) \), and since the R-S degree of uncertainty is defined for fixed expectation, the degree of initial uncertainty may be characterized as either \( E(b^2) \) or \( \text{var}(b) \). The direction of the influence of each on \( U(h^b_{\infty,t}) \) will be the same; the only difference lies in the

\[
\begin{align*}
7E[\hat{\psi}(b)] &= E(r-b)t \left[ \frac{r-E(b)}{2a} \right] + E((r-b)^2) \left[ \frac{(1-t)}{2a} \right] = t \left[ \frac{r-E(b)}{2a} \right]^2 \\
&+ \frac{(1-t)}{2a} [r^2-2rE(b)+E(b^2)]
\end{align*}
\]
"origin" from which the degree of uncertainty is measured.

Returning briefly to Proposition 4-1, the value of the general information system $h^b_t$ was shown to be $(1-t)\{\text{var}(b) - E[\text{var}(b|y)]\}/4a$. In this case also, the degree of initial uncertainty can be characterized by $E(b^2)$. However, $E(b^2)$ is an implicit argument of $E[\text{var}(b|y)]$ along with various aspects of $h^b_t$ as characterized by $p(y|b)$. The precise role of $E(b^2)$ as a determinant of $U(h^b_t)$ cannot be established without lending greater specificity to $h^b_t$. This was the approach taken above in investigating $h^b_{\infty,t}$.

**Corollary 4-2.1:** For $P(r)=1$ and $t<1$, the expected value of information system $h^b_{\infty,t}$ is linearly increasing in $\text{var}(b)$.

**Proof:** Partially differentiating equation (4.9):

$$\frac{\partial U(h^b_{\infty,t})}{\partial \text{var}(b)} = \frac{(1-t)}{4a} > 0; \quad \frac{\partial^2 U(h^b_{\infty,t})}{\partial [\text{var}(b)]^2} = 0$$

Q.E.D.

It is intuitively that the value of information about unit variable cost is more valuable to the producer as his initial uncertainty about this cost increases. Moreover, in the extreme case of complete certainty about $b$, $\text{var}(b)=0$ and the value of information system $h^b_{\infty,t}$ is zero.

Once again consider the theory of the firm approach to solving the producer's problem, viz., equating marginal revenue with marginal cost. While this does not provide a precise interpretation of the result, it may be a useful heuristic approach. Under perfect competition marginal revenue (MR) equals the unit price, $r$. Marginal cost in the second production subperiod $[t,1]$ is given by $[2ax_2/(1-t)]b$. Figure 16 depicts this process for different widths for the band of uncertainty about unit variable cost, $b$. 
Each of the alternative bands of uncertainty for b has an upper bound less than or equal to unit price (r) reflecting the restriction that \( P(r) = 1 \). As the band of uncertainty for b becomes wider, so does the range of uncertainty for \( x^*_2 \). Since there is more uncertainty about the optimal decision, one might expect the value of information to be greater. There is not, of course, an unambiguous relationship between the degree of initial uncertainty, as measured in this case by \( \text{var}(b) \), and the range of the prior (width of the band of uncertainty). For this reason and others the approach is not a precise interpretation of the result in Corollary 4-2.1.

**Corollary 4-2.2:** For \( P(r) = 1 \) and \( t < 1 \): (i) the marginal increments in \( U(b, t) \) w.r.t. \( \text{var}(b) \) are strictly convex decreasing in the quadratic cost parameter, \( a \), and linearly decreasing in information time, \( t \); (ii)
the marginal decrements in $U(h_{\infty}^b, t)$ w.r.t. the quadratic cost parameter 
(a) are linearly increasing in var(b); (iii) the marginal decrements in 
$U(h_{\infty}^b, t)$ w.r.t. information time (t) are linearly increasing in var(b).

Proof: Partially differentiating equation (4.8):

$$\frac{\partial^2 U(h_{\infty}^b, t)}{\partial a \partial \text{var}(b)} = -\frac{(1-t)}{4a^2} < 0; \quad \frac{\partial^3 U(h_{\infty}^b, t)}{\partial a^2 \partial \text{var}(b)} = \frac{(1-t)}{2a^3} > 0$$

$$\frac{\partial^2 U(h_{\infty}^b, t)}{\partial a \partial [\text{var}(b)]^2} = 0$$

$$\frac{\partial^2 U(h_{\infty}^b, t)}{\partial t \partial \text{var}(b)} = \frac{-1}{4a} < 0; \quad \frac{\partial^3 U(h_{\infty}^b, t)}{\partial t \partial \text{var}(b)} = \frac{3a^2 U(h_{\infty}^b, t)}{\partial t^2 \partial \text{var}(b)} = 0$$

Q.E.D.

The interaction of var(b) and a in determining $U(h_{\infty}^b, t)$ is depicted 
graphically in Figure 17.

![Figure 17. Interaction of Initial Uncertainty and the Quadratic Cost Parameter](image-url)
In Corollary 3-2.1, the value of the general information system \( h_t^b \), given an unrestricted prior on \( b \), was shown to diminish in the quadratic cost parameter, \( a \). Corollary 4-2.2 shows that when \( h_t^b \) is the perfect information system and the producer is certain that unit variable cost will not exceed unit price, the rate at which the value of the system diminishes in \( a \) is increasing in initial uncertainty. The more uncertain the producer is initially about \( b \), the more valuable is the information system for any given value of \( a \), and the more sensitive is that value to changes in \( a \).

Appealing again to the concept of equating marginal revenue to marginal cost, and keeping in mind the limitations of this interpretive approach, some intuition can be provided for this phenomenon. Figure 18A depicts the change in the producer's range of uncertainty about \( x_2^* \) as the quadratic cost parameter changes from \( a' \) to \( a'' \) (\( a'' > a' \)) with the range of uncertainty about \( b \) fixed at \( \Delta_1 \). Figure 18B shows the change in the producer's range of uncertainty about \( x_2^* \) as the quadratic cost parameter changes from \( a' \) to \( a'' \) with the range of uncertainty about \( b \) fixed at \( \Delta_2 \) (\( \Delta_2 > \Delta_1 \)). The absolute change in the width of the band of uncertainty about \( x_2^* \), caused by the change from \( a' \) to \( a'' \), is clearly greater for the case with the larger band of uncertainty about \( b \). Thus one might expect a greater change in the value of information about \( b \), per unit change in \( a \), when the initial uncertainty is greater.

In Corollary 4-2.1 the value of information system \( h_{\infty, t}^b \), given the restricted prior on \( b \), was shown to increase in \( \text{var}(b) \). Corollary 4-2.2

---

\(^8\)Again the caveat must be given that the range of \( p(b) \) is not necessarily a function of the degree of uncertainty (\( \text{var}(b) \) here).
Figure 18. Interpretation of Interaction Between Initial Uncertainty and the Quadratic Cost Parameter
shows that the rate of this increase is decreasing at a decreasing rate in $a$. The lower the quadratic cost parameter, the more valuable is the information system for any given level of initial uncertainty, and the more sensitive is that value to changes in initial uncertainty. This completes the analysis of the interaction effect of $\text{var}(b)$ and $a$ in determining $U(h^b_\omega, t)$.

Turning to the interaction of $\text{var}(b)$ and $t$, the joint effect of these determinants on $U(h^b_\omega, t)$ is depicted in Figure 19.

![Figure 19. Interaction of Initial Uncertainty and Information Timing](image)

In Corollary 3-2.1, the value of the general information system $h^b_t$, given an unrestricted prior on $b$, was shown to diminish in the information timing parameter, $t$. In Corollary 4-2.2, where $h^b_t$ is the perfect information system and the producer is certain that the unit variable cost will not exceed unit price, the rate at which the value of the system diminishes in $t$ is increasing in initial uncertainty. The more
uncertain the producer is initially about \( b \), the more valuable is the information at any given time, \( t \), and the more sensitive is that value to changes in \( t \).

The equating of marginal revenue to marginal cost can again be used (with some loss of rigor) to provide some intuition for this interaction effect. Figure 20A depicts the change in the producer's range of uncertainty about \( x_2^* \) as the information time changes from \( t' \) to \( t'' \) (\( t'' > t' \)) with the range of uncertainty about \( b \) fixed at \( \Delta_1 \). Figure 20B shows the change in the producer's range of uncertainty about \( x_2^* \) as the information time changes from \( t' \) to \( t'' \) with the range of uncertainty about \( b \) fixed at \( \Delta_2 \) (\( \Delta_2 > \Delta_1 \)). The absolute change in the width of the band of uncertainty about \( x_2^* \) caused by the change from \( t' \) to \( t'' \) is clearly greater for the case with the larger band of uncertainty about \( b \). This is consistent with the point of Corollary 4-2.2 with regard to the influence of \( \text{var}(b) \) on the marginal effect of \( t \) on \( U(h^b_{o,t}) \).

In Corollary 4-2.1 the value of information system \( h^b_{o,t} \), given the restricted prior on \( b \), was shown to increase in \( \text{var}(b) \). Corollary 4-2.2 shows that the rate of this increase is decreasing at a constant rate in information time, \( t \). The earlier the information about \( b \) is provided, the more valuable is the information system for any given level of initial uncertainty, and the more sensitive is that value to changes in initial uncertainty.

4.3 Noisy Information System

Proposition 4-1 states that given \( u(z) = z \) and the restriction on \( p(b) \) that \( P(r) = 1 \), \( U(h^b_{t}) = (1 - t) \left\{ \text{var}(b) - E[\text{var}(b|y)] \right\} / 4a \), where \( h^b_{t} \) denotes any system which provides information at time \( t \) about unit variable cost. If
Figure 20. Interpretation of Interaction Between Initial Uncertainty and Information Timing
$h^b_t$ is such that $\mathbb{E}[\text{var}(b|y)] = \text{var}(b)$, then $U(h^b_t)=0$. In this case the information system is not expected to reduce the uncertainty about $b$ from its original magnitude of $\text{var}(b)$, and is, therefore, equivalent to the null system. If $h^b_t$ is such that $\mathbb{E}[\text{var}(b|y)] = 0$, the system eliminates all uncertainty and is equivalent to the perfect information system. Since information at the period's end is valueless in the short run, $U(h^b_T)=0$. Since early information is better than late information, $U(h^b_t)$ is decreasing in information time $t$.

In many information systems the accuracy of the system improves with time. In the CVP context an information system which incorporates observations on the production process would likely facilitate, as time progresses, more accurate cost function estimates. Suppose that $h^b_t$ is imperfect and noisy, and that the expected posterior variance of $b$, given a signal from the system, declines over time according to (4.10) below:

$$(4.10) \quad \mathbb{E}[\text{var}(b|y)] = (1-t^\delta)\text{var}(b), \; \delta > 0$$

Denote this system by $h^b_{t,\delta}$. At $t=0$ $\mathbb{E}[\text{var}(b|y)] = \text{var}(b)$ and $h^b_{t,\delta}$ is equivalent to the null system; at $t=1$ $\mathbb{E}[\text{var}(b|y)] = 0$, and $h^b_{1,\delta}$ is equivalent to the perfect system.

Property 4-1: $\mathbb{E}[\text{var}(b|y)]$ is

\begin{align*}
\{\text{strictly concave} \} & \quad \text{decreasing in } t \text{ as } \delta > 1.
\{\text{linearly convex} \}
\end{align*}

Proof: Partially differentiating equation (4.10):

\begin{align*}
\frac{\partial \mathbb{E}[\text{var}(b|y)]}{\partial t} &= -\delta t^{\delta-1}\text{var}(b) < 0; \quad \frac{\partial^2 \mathbb{E}[\text{var}(b|y)]}{\partial t^2} = -\delta(\delta-1)t^{\delta-2}\text{var}(b) < 0 \quad \text{as } \delta > 1.
\end{align*}

Q.E.D.
Property 4-2: $\mathbb{E}[\text{var}(b|y)]$ is strictly concave increasing in $\delta$ for a given $t \in (0,1)$.

Proof: Partially differentiating equation (4.10):

$$\frac{\partial \mathbb{E}[\text{var}(b|y)]}{\partial \delta} = -t^\delta \ln(t) \text{var}(b) > 0 \text{ since } t \in (0,1)$$

$$\frac{\partial^2 \mathbb{E}[\text{var}(b|y)]}{\partial \delta^2} = -t^\delta [\ln(t)]^2 \text{var}(b) < 0$$

Q.E.D.

Figure 21. Expected Posterior Variance as a Function of Information Time (A) and Accuracy Responsiveness (B)

The "accuracy" of information system $h_{t, \delta}^b$ might be characterized by $\{\text{var}(b) - \mathbb{E}[\text{var}(b|y)]^2\}$, i.e., the expected reduction in the variance of $b$ due to the system. Figure 22A shows the relationship between system accuracy and time, while Figure 22B shows the relationship between system accuracy and the accuracy responsiveness parameter, $\delta$.

The "accuracy" of an information system, while often mentioned in the accounting literature, seldom has been made precise, as was done here, by a rigorous definition. The same statement may be made with
Figure 22. Accuracy of $h_{t, \delta}^b$ as a Function of Information Time (A) and Accuracy Responsiveness (B)

regard to improvement in the accuracy of an information system over time.

In Figure 22, there is a finite maximum for the accuracy of an information system as measured by the expected reduction in uncertainty due to the system. This reduction cannot exceed the magnitude of initial uncertainty which in this case can be measured by $\text{var}(b)$. Initial uncertainty is a determinant of the maximal accuracy an information system can achieve. If system accuracy were measured in some other manner, this concept would remain valid. A perfect information system can always be conceived theoretically in any decision setting. The accuracy of this perfect system, however measured, establishes an upper bound on the accuracy of any conceivable information system for that decision problem.\footnote{This concept is analogous, although not identical, to Demski's result on information improvement bounds. See Joel Demski, "Information Improvement Bounds," Journal of Accounting Research, Spring, 1972, pp. 58-76.}

The nature of the relationship between system accuracy and information timing, established indirectly by Property 4-1 and graphed in
Figure 22A, depends on the responsiveness parameter, δ. The use of δ allows one to vary the nature of this relationship considerably. For δ=1, there is a linear relationship between system accuracy and information time. The accuracy of the producer's information system about unit variable cost increases proportionally with the delay of information collection. Equivalently, the accuracy of the system declines proportionately with information timeliness. If δ<1, the accuracy of $h^b_{t,δ}$ improves very rapidly with delay of information collection early in the period, but the rate of improvement declines as time progresses. As δ approaches zero, the uppermost curve in Figure 22A approaches a rectangular form, indicating that a very small delay in gathering information at the outset of the period will provide nearly perfect information. Similarly, if δ>1 the system improves in accuracy very slowly at first, but the rate of improvement increases throughout the period. As δ approaches infinity, the lower curve in Figure 22A approaches a rectangular form, indicating that the system will provide only very poor quality information until almost the very end of the period, at which time its accuracy will improve very quickly. As Figure 22B indicates, the accuracy of $h^b_{t,δ}$ decreases at a decreasing rate in the parameter δ.

Given a particular time for the collection of information about unit variable cost, the accuracy of the information will be greater the faster it has been improving with the passage of time.

**Proposition 4-5:** For $u(z)=z$ and $P(r)=1$,

\[ (4.11) \quad U(h^b_{t,δ}) = \frac{(t^δ - t^{δ+1})\var{b}}{4a} \]

**Proof:** The proof is immediate by substitution of (4.10) into the expression given for $U(h^b_{t})$ in Proposition 4-1. Q.E.D.
The role of the quadratic cost parameter \( (a) \), as an indirect determinant of \( U(h_t^b) \), was fully characterized by Corollary 3.2.1 and those results remain valid for system \( h_t^b \). The results given in Corollaries 3.2.1 and 3.2.2 concerning the role of \( t \) in determining \( U(h_t^b) \) are not valid for \( h_t^b \), since \( t \) was not assumed earlier to be an implicit argument of the posterior degree of uncertainty. The role of \( \text{var}(b) \) as a determinant of information value was not addressed for system \( h_t^b \).

**Corollary 4.3.1:** For \( P(r)=1 \) and \( t \in (0,1) \): (i) \( U(h_t^b, \delta) \) is strictly convex decreasing in the accuracy responsiveness parameter \( \delta \); (ii) \( U(h_t^b, \delta) \) is \( \text{pseudoconcave} \) in signal time \( (t) \) for \( \delta > 1 \); (iii) \( U(h_t^b, \delta) \) is linearly increasing in \( \text{var}(b) \).

**Proof:** Partially differentiating equation (4.11):

\[
\begin{align*}
(i) \quad \frac{\partial U(h_t^b, \delta)}{\partial \delta} &= \frac{\ln(t)[t^\delta - t^{\delta+1}]\text{var}(b)}{4a} < 0 \\
(ii) \quad \frac{\partial^2 U(h_t^b, \delta)}{\partial \delta^2} &= \frac{[\ln(t)]^2[t^\delta - t^{\delta+1}]\text{var}(b)}{4a} > 0
\end{align*}
\]

\[
(iii) \quad \frac{\partial U(h_t^b, \delta)}{\partial t} = \frac{t^{\delta-1}[\delta - (\delta+1)t]\text{var}(b)}{4a} \quad \forall \delta > 0 \quad \text{for } t \leq \delta/(\delta+1)
\]

Let \( t^0 = \delta/(\delta+1) \), \( t' \in [0, t^0] \), \( t'' \in [t^0, 1] \), and let \( \nabla(t) \) denote \( \partial U(h_t^b, \delta)/\partial t \) evaluated at \( t = \hat{t} \).

\[
\nabla(t')[t-t'] \leq 0 \quad \text{for } t \leq t' \quad \text{since } \nabla(t') \leq 0 \quad \text{for all } t' \in [0, t^0].
\]

Moreover, \( U(h_{t', \delta}^b) \leq U(h_{t, \delta}^b) \) for \( t \leq t' \) and \( t' \in [0, t^0] \), since \( U \) is increasing on \( [0, t^0] \).

Also, \( \nabla(t'')[t-t''] \geq 0 \quad \text{for } t \leq t'' \quad \text{since } \nabla(t'') \geq 0 \quad \text{for all } t'' \in [t^0, 1].
\]

Moreover, \( U(h_{t', \delta}^b) \geq U(h_{t, \delta}^b) \) for \( t \leq t' \) and \( t' \in [t^0, 1] \), since \( U \) is decreasing on \( (t^0, 1] \). Therefore:
\[ v(t)(t-t^\delta) \leq 0 \Rightarrow U(h^b_{t,\delta}) \leq U(h^b_{t^\delta,\delta}) \text{ for all } t, t^\delta \in [0,1]. \]

This is the definition of pseudoconcavity\(^{10}\) of \(U(h^b_{t,\delta})\) w.r.t. \(t \in [0,1]\).

Therefore, the pseudoconcavity of \(U(h^b_{t,\delta})\) w.r.t. \(t\) is established for all \(\delta > 1\). The strict concavity of \(U(h^b_{t,\delta})\) w.r.t. \(t\) for \(\delta \geq 1\) is established by the following:

\[
\frac{\partial^2 U(h^b_{t,\delta})}{\partial t^2} = \frac{\delta t^{-2}[(\delta-1)-t] \text{var}(b)}{4a} > 0 \text{ if } \begin{cases} \delta > 1 \text{ and } 0 < t < (\delta-1)/(\delta+1) \\ \delta \geq 1 \text{ and } t \in (0,(\delta-1)/(\delta+1)) \end{cases} \text{ otherwise}
\]

(iii) \[
\frac{\partial U(h^b_{t,\delta})}{\partial \text{var}(b)} = \frac{t^{\delta-1}[(\delta-1) + t] \text{var}(b)}{4a} > 0; \quad \frac{\partial^2 U(h^b_{t,\delta})}{\partial [\text{var}(b)]^2} = 0
\]

Q.E.D.

The value of information system \(h^b_{t,\delta}\) decreases at a decreasing rate in the accuracy responsiveness parameter, \(\delta\). Moreover, the sensitivity of system value to \(\delta\) is greater for highly responsive systems than for systems whose accuracy responds slowly to time. This relationship is similar to that depicted in Figure 22B for the accuracy of \(h^b_{t,\delta}\) as a function of \(\delta\). The role of initial uncertainty as a determinant of \(U(h^b_{t,\delta})\) is the same as it was for system \(h^b_{\omega,t}\) as discussed subsequent to Corollary 4-2.1. No new insights can be provided here.

Corollary 4-3.2. The optimal information time for system \(h^b_{t,\delta}\) is \(t^* = \delta / (\delta+1)\). \(t^*\) is strictly concave increasing in \(\delta\).

Proof: The first order condition for the unconstrained maximization of \(U(h^b_{t,\delta})\) w.r.t. \(t\) is given by:

\[
\frac{\partial U(h^b_{t,\delta})}{\partial t} = \frac{\delta t^{\delta-1}[(\delta-1) + t] \text{var}(b)}{4a} = 0 \Rightarrow t^* = \frac{\delta}{\delta+1}
\]

\(^{10}\)The numerical function \(\Omega\) is said to be pseudoconcave at \(\alpha\) if it is differentiable at \(\alpha\) and \(\forall \Omega(\alpha) \neq 0 \Rightarrow \Omega(\alpha) = \Omega(\alpha) \leq 0\), where \(\forall\) denotes the gradient. \(\Omega\) is said to be pseudoconcave on its domain if it is pseudoconcave at each point \(\alpha\) in its domain. See Olvi L. Mangasarian, Nonlinear Programming (New York: McGraw-Hill, 1969), p. 141.
From Corollary 3-6.1, the second order condition is satisfied by psuedoconcavity or concavity. Since \( \delta/(\delta+1) \in (0,1) \), \( t^* \) is also the constrained optimum.

\[
\frac{dt^*}{d\delta} = \frac{1}{(\delta+1)^2} > 0; \quad \frac{d^2t^*}{d\delta^2} = -\frac{2}{(\delta+1)^3} < 0
\]

Q.E.D.

**Corollary 4-3.3:** The value of system \( h_{t,\delta}^b \) given optimal information timing is given by

\[
U(h_{t,\delta}^b, \delta) = \left( -\frac{\delta}{(\delta+1)^2} \right) \left( \frac{var(b)}{4a} \right).
\]

**Proof:** The proof is immediate upon substitution of \( t^* \) into \( U(h_{t,\delta}^b) \) given by Proposition 4-3.

Q.E.D.

From Proposition 4-3, \( U(h_{0,\delta}^b) = U(h_{1,\delta}^b) = 0 \). Information at the beginning of the period is valueless. The system is equivalent to the null system since it is not expected to reduce the producer's degree of uncertainty about unit variable costs. Moreover, even perfect information at the end of the period is valueless in the short run; it is too late to be acted upon. Note from the proof of Corollary 4-3.1 that \( \delta^2 U(h_{t,\delta}^b)/\delta t^2 \) changes sign at \( (\delta-1)/(\delta+1) \) if \( \delta>1 \), indicating an inflection point. At this point, \( U(h_{t,\delta}^b) \) is increasing in \( t \) since

\[
\delta U(h_{t,\delta}^b)/\delta t > 0 \quad \text{for} \quad t < \delta/(\delta+1) \quad \text{and} \quad (\delta-1)/(\delta+1) < \delta/(\delta+1).
\]

The relationship between \( U(h_{t,\delta}^b) \) and information time \( t \) provided by Corollaries 4-3.1 through 4-3.3 is depicted in Figure 23.

Several observations can be made about the role of information time as an indirect determinant of \( U(h_{t,\delta}^b) \). The optimal system timing, \( t^* \), is always interior to the production period since \( t^* = \delta/(\delta+1) \in (0,1) \), for \( \delta>0 \). Moreover, for \( \delta>1 \), \( t^* \) is always in the second half of the period, and for \( \delta<1 \), \( t^* \) is always in the first half of the period. As Corollary 4-3.2 indicates, \( t^* \) increases in \( \delta \). As improvement in system accuracy
Figure 23. $U(h_{t,\delta}^b)$ as a Function of Information Time

becomes less responsive to time, the producer should delay longer in collecting the information. The sensitivity of optimal information timing is greater for systems whose accuracy is highly responsive to time (i.e., $\delta$ small) than for systems which are less responsive. It is interesting that optimal information timing is determined solely by the
accuracy responsiveness parameter.

From Proposition 4-3 $U(h_{t,d}^b)$ can be written as the product of two factors:

\[(4.12) \quad U(h_{t,d}^b) = \frac{(t^\delta - t^{\delta + 1}) \text{var}(b)}{4a} \cdot \frac{(1-t)}{4a} \cdot (t^{\delta \text{var}(b)})\]

The first term on the RHS of (4.12) might be called the "outcome magnitude factor," [denoted $O(t)$], and the second term the "information accuracy factor," [denoted $A(t)$].

**Corollary 4-3.4:** $O(t)$ is linearly decreasing in information time $t$;

$A(t)$ is

\[
\begin{cases}
\text{strictly concave} & \text{increasing in } t \text{ for } \delta \geq 1.
\end{cases}
\]

**Proof:** Partially differentiating (4.12)

\[
\frac{\partial O(t)}{\partial t} = \frac{-1}{4a} < 0; \quad \frac{\partial^2 O(t)}{\partial t^2} = 0
\]

\[
\frac{\partial A(t)}{\partial t} = \delta t^\delta - 1 \text{var}(b) > 0; \quad \frac{\partial^2 A(t)}{\partial t^2} = \delta(\delta - 1)t^\delta - 2 \text{var}(b) \geq 0 \text{ as } \delta \geq 1 \text{ Q.E.D.}
\]

Expressing $U(h_{t,d}^b)$ as the product of these two factors highlights the trade-off nature of the information timing decision. The outcome magnitude factor is decreasing in $t$ reflecting the penalty the producer incurs by receiving information later. The later the information time, the longer he will produce in the period $[0,t]$ at a suboptimal rate with respect to the signal from $h_{t,d}^b$. The information accuracy factor is increasing in $t$ reflecting the advantage to the producer from waiting for more accurate information. Taking the product of these two factors induces the concavity or pseudoconcavity of $U(h_{t,d}^b)$ in $t$, and drives the result that $t^*$ is interior to the production period. Figure 24 shows
$U(h_{t, \delta}^b)$ as a product of $O(t)$ and $A(t)$ for $\delta \leq 1$.

![Graph](Image)

Figure 24. $U(h_{t, \delta}^b)$ as a Product of Outcome Magnitude and Information Accuracy Factors

*This graph assumes a certain relationship between $a$ and $\text{var}(b)$. The choice of the relationship is arbitrary, but the qualitative properties of the graph are general in nature. $\delta = 1$ in the graph.

**Corollary 4-3.5:** The value of system $h_{t, \delta}^b$ given optimal information timing is strictly convex decreasing in accuracy responsiveness parameter $\delta$.

**Proof:** Partially differentiating $U(h_{t, \delta}^b)$ given by Corollary 4-3.3:

$$\frac{\partial U(h_{t, \delta}^b)}{\partial \delta} = \frac{\text{var}(b) \delta^2 [\ln \delta - \ln(\delta+1)]}{4a(\delta+1)^2} < 0$$

$$\frac{\partial^2 U(h_{t, \delta}^b)}{\partial \delta^2} = \frac{\text{var}(b)}{4a} \left( \frac{\delta^2}{(\delta+1)^2} [\ln \delta - \ln(\delta+1)]^2 + \frac{\delta-1}{(\delta+1)^2} \right) > 0$$

Q.E.D.

Using this result and the pseudoconcavity or concavity of $U(h_{t, \delta}^b)$ w.r.t. $t$ from Corollary 4-3.1, Figure 25 depicts $U(h_{t, \delta}^b)$ as a function
of $t$ for several values of $\delta$. In the graph, $\delta_i < \delta_{i+1}$, $i=1, \ldots, 6$

![Graph showing $U(h_{t*, \delta}^b)$ for various values of $\delta$.]

Figure 25. $U(h_{t*, \delta}^b)$ as a Function of $t$ for Several Values of $\delta$

This graph summarizes many of the results in Corollaries 4-3.1 through 4-3.5. $U(h_{t*, \delta}^b)$ is concave or pseudoconcave in $t$, $t^*$ is increasing in $\delta$, and the value of $h_{t*, \delta}^b$ is decreasing in $\delta$. For a given $t$, $U(h_{t*, \delta}^b)$ is decreasing in $\delta$. As $\delta$ increases, the curves become flatter indicating that the value of $h_{t*, \delta}^b$ is less sensitive to information timing. The upper bound for $U(h_{t*, \delta}^b)$ indicated on the graph is $U(h_{\infty, \delta}^b)$. No system can surpass that which provides perfect information at the beginning of the period.

4.4 Noiseless Imperfect Information System

Suppose now that a system is employed which provides noiseless imperfect information at time $t$ about unit variable cost. Let $h_{m,t}^b$ denote a system which induces a partition of $m$ equal cells on the real interval over which $p(b)>0$. Denote this interval by $(a, \beta)$, where $a=0$ and $\beta=r$. 
since \( P(r)=1 \). Let \( \Delta=8-\alpha \). The set of signals for \( h_{m,t}^b \), denoted by \( y_{m,t}^b \), is such that \( P(y_i|b\in[a+(i-1)\Delta/m, a+i\Delta/m])=1, i=1,2,\ldots,m \). Each \( y_i \) is associated with a unique subinterval of \([a,\beta]\) of length \( \Delta/m \). \( m \) is an indirect determinant of information value. \( m \) is not a measure of fineness for \( h_{m,t}^b \), since not all of the members of \( \{h_{m,t}^b\} \) are comparable with respect to the fineness criterion. For example, the partitions induced on \([a,\beta]\) by \( h_{2,t}^b \) and \( h_{3,t}^b \) are not comparable with respect to fineness.\(^\text{11}\) \( m \) may, however, be interpreted as a measure of the accuracy of system \( h_{m,t}^b \).

Suppose that \( p(b) \) is rectangular on \([a,\beta]\), \( a\geq 0, \beta \leq r \), i.e.,

\[
p(b) = \begin{cases} 
\frac{1}{\Delta}, & b \in [a, \beta] \\
0, & \text{elsewhere}
\end{cases}
\]

**Proposition 4-4:** For \( u(z) = z \) and \( p(b) \) rectangular on \([a,\beta]\) \( \leq [0, r] \),

\[
U(h_{m,t}^b) = \frac{1-t}{4am^2} \\frac{\left(m^2-1\right)\text{var}(b)}{4a m^2}.
\]

**Proof:** Since \( p(b) \) is rectangular on \([a,\beta]\) with \( \Delta=8-\alpha \), \( \text{var}(b)=\Delta^2/12 \). By Bayes Theorem \( p(b|y_i) \) is rectangular on \([a+(i-1)\Delta/m, a+i\Delta/m], i=1,2,\ldots,m \).

The variance of \( p(b|y_i) \), denoted \( \text{var}(b|y_i) \), is \( \Delta^2/12m^2 \). Therefore,

\[
E[\text{var}(b|y_i)] = \Delta^2/12m^2.
\]

From Proposition 4-1, when \( P(r)=1 \), \( U(h_{m,t}^b) = (1-t)\{\text{var}(b)-E[\text{var}(b|y)]\}/4a \).

Therefore,

\[
U(h_{m,t}^b) = \frac{(1-t)}{4a} \left( \frac{\Delta^2}{12} - \frac{\Delta^2}{12m^2} \right) = \frac{(1-t)(m^2-1)\Delta^2}{48am^2} = \frac{(1-t)(m^2-1)\text{var}(b)}{4am^2}
\]

Q.E.D.

\(^{11}\)See Chapter 2 for the condition under which partitions are comparable with respect to fineness.
Clearly, \( U(h_{m,t}^b) \) is nonnegative. As one would expect, \( U(h_{l,t}^b) = 0 \), the value of the null system. The partition induced on \([\alpha, \beta]\) by \( h_{l,t}^b \) is equal to \([\alpha, \beta]\). Similarly, as \( m \) gets large, \( U(h_{m,t}^b) \) approaches \((1-t)\text{var}(b)/4a\), the value of perfect information as given by Proposition 4-2. The roles of \( t \), \( a \) and \( \text{var}(b) \) as determinants of \( U(h_{m,t}^b) \) are the same as those indicated by Corollaries 3-2.1 and 4-2.1.

**Corollary 4-4.1:** For \( t < 1 \), \( U(h_{m,t}^b) \) is increasing in \( m \) by decreasing marginal increments as \( m \) increases in \([1, 2, 3, \ldots]\).

**Proof:** Partially differentiating equation (4.13):

\[
\frac{\partial U(h_{m,t}^b)}{\partial m} = \frac{(1-t)\text{var}(b)}{2m^3a} > 0;
\]

\[
\frac{\partial^2 U(h_{m,t}^b)}{\partial m^2} = -\frac{3(1-t)\text{var}(b)}{2m^4a} < 0
\]

Q.E.D.

As the accuracy of the noiseless system \( h_{m,t}^b \) increases, the value of the system increases. The marginal improvement, however, is diminishing in \( m \). A heuristic economic interpretation\(^{12}\) of this result concerns the band of uncertainty about the optimal action still remaining after receipt of the information. As \( m \) increases, the producer has a narrower expected posterior band of uncertainty about \( b \), and hence a narrower expected posterior band of uncertainty about \( x_2^* \), the optimal production quantity in the period \([t, 1]\). The narrower the posterior band of uncertainty about \( x_2^* \), the greater the value of the information. Figure 26 depicts this relationship for \( m \in (m', m'') \), where \( m' < m'' \).

\(^{12}\)See section 3.4.1 for the limitations of this approach to interpreting the results.
Corollary 4.4.2: (i) The marginal increments in $U(h_{m,t}^b)$ w.r.t $m$ are linearly increasing in var(b) and strictly convex decreasing in $a$ for $t < 1$, and linearly decreasing in $t$; (ii) for $t < 1$, the marginal increments [decrements] in $U(h_{m,t}^b)$ w.r.t. var(b) [t] are increasing in $m$ by decreasing marginal amounts.

Proof: Partially differentiating equation (4.13):

$$\frac{\partial^2 U(h_{m,t}^b)}{\partial m \partial a} = -\frac{\text{var}(b)}{2m^3a} < 0; \frac{\partial^3 U(h_{m,t}^b)}{\partial a \partial m^2} = 0; \frac{\partial^3 U(h_{m,t}^b)}{\partial a \partial m^2} = \frac{3\text{var}(b)}{2m^4a} > 0$$

$$\frac{\partial^2 U(h_{m,t}^b)}{\partial \text{var}(b) \partial m} = \frac{(1-t)}{2m^3a} > 0; \frac{\partial^3 U(h_{m,t}^b)}{\partial \text{var}(b) \partial m} = 0; \frac{\partial^3 U(h_{m,t}^b)}{\partial \text{var}(b) \partial m} = \frac{3(1-t)}{2m^4a} < 0$$
\[
\frac{\gamma^2 U(h^b_{m,t}) - (1-t)\text{var}(b)}{a a m} = \frac{\gamma^3 U(h^b_{m,t}) (1-t)\text{var}(b)}{a^2 a m} = \frac{\gamma^3 U(h^b_{m,t})}{a a m^2} = \frac{3(1-t)\text{var}(b)}{m^2 a^2} > 0
\]

Q.E.D.

The sensitivity of \(U(h^b_{m,t})\) to its accuracy parameter \(m\) declines as the information is delayed, the initial uncertainty is reduced, or the quadratic cost parameter is increased. As the information system becomes more accurate, the producer has a greater incentive to control, or at least monitor, other determinants. Similarly, the sensitivity of \(U(h^b_{m,t})\) to information timeliness, initial uncertainty and the quadratic cost parameter increases with system accuracy.

A noiseless imperfect information system may improve in accuracy over time. Such an improvement might be characterized by an increase in the number of equal cells in the partition induced on \([\alpha, \beta]\). Let \(h^b_{\phi,t}\) denote the system \(h^b_{m,t}\) where \(m\) is a function of \(\phi\) and \(t\) as follows:

\[
(4.14) \quad m(\phi,t) = \text{int}[t/\phi(1-t)] + 1, \quad \text{where } \phi > 0 \text{ and } \text{int}[\gamma] = \text{the integer part of } \gamma.
\]

\(\phi\) is the "accuracy responsiveness parameter" for system \(h^b_{\phi,t}\) analogous to \(\delta\) for the noisy system \(h^b_{t,\delta}\). \(\phi\) is an indirect determinant of information value. The parameter affects the speed with which system accuracy improves over time. For small values of \(\phi\) the number of cells in the partition induced on \([\alpha, \beta]\) by \(h^b_{\phi,t}\) increases more quickly as time passes. Note that for \(t=0\), \(m(\phi,t)=1\) and \(h^b_{\phi,0}\) is equivalent to the null system. As \(t\) approaches \(1\), \(m(\phi,t)\) approaches infinity and \(h^b_{\phi,t}\) is equivalent to the perfect system. Table 1 gives several values of \(m\) for selected \(\phi\) and \(t\).
Table 1

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>0</th>
<th>$\frac{1}{10}$</th>
<th>$\frac{1}{2}$</th>
<th>1</th>
<th>2</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1/2</td>
<td>$\infty$</td>
<td>11</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2/3</td>
<td>$\infty$</td>
<td>21</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3/4</td>
<td>$\infty$</td>
<td>31</td>
<td>7</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Proposition 4.5: For $u(z)=z$ and $p(b)$ rectangular on $[a, b] \subseteq [0, r],$

\[(4.15) \quad U(h^b_{\phi,t}) = \frac{(1-t)}{4a} \cdot \left(\frac{\text{int} \left[ \frac{t}{\phi(1-t)}+1 \right]}{\phi(1-t)+1} \right)^2 \cdot \frac{\text{var}(b)}{m^2} \]

Proof: The proof is immediate upon substitution of $m(\phi, t)$ given by (4.14) into equation (4.13). Q.E.D.

Note that $U(h^b_{\phi,t})$ is the product of two factors each of which is a function of $t$. The first will be termed the "outcome magnitude factor" and denoted by $O(t)$; the second will be termed the "information accuracy factor," denoted by $A(t)$. These two factors are analogous to $O(t)$ and $A(t)$ for system $h^b_{t, \delta}$.

Corollary 4.5: $O(t)$ is linearly decreasing in $t$; $A(t)$ is stepwise increasing in $t$, and stepwise decreasing in $\phi$.

Proof: From equation (4.15), $O(t) = \frac{(1-t)}{4a}$ and $A(t) = (m^2 - 1)\text{var}(b)/m^2$ where $m$ is given by (4.14). $\frac{\partial O}{\partial t} = -\frac{1}{4a} < 0$ and $\frac{\partial^2 O}{\partial t^2} = 0$. 


\(\alpha s/\omega m = 2\text{var}(b)/m^3 > 0\). By equation (4.14) \(m\) is increasing in 
\(\text{int}[t/\phi(1-t)]\). Now \(\text{int} [\cdot]\) is stepwise increasing in its argument. 
Finally, \(\alpha [t/\phi(1-t)]/\partial t = 1/\phi(1-t)^2 > 0\) and 
\(\partial [t/\phi(1-t)]/\partial \phi = -t/\phi^2(1-t) < 0\).
Q.E.D.

The interpretation of \(\hat{O}(t)\) and \(\hat{A}(t)\) is analogous to that given previously for \(O(t)\) and \(A(t)\). \(\hat{O}(t)\) reflects the penalty to the producer from getting delayed information and having produced at a suboptimal rate for a longer period of time. \(A(t)\) reflects the advantage of getting information later since it comes from a more accurate system. Viewing \(U(h_{\phi}^b, t)\) as the product of the two factors highlights the trade-off nature of the timeliness issue. \(\hat{O}(t), \hat{A}(t)\) and \(U(h_{\phi}^b, t)\) are graphed in Figure 27 for the case\(^{13}\) of \(\phi = 1, a = 1/16\) and \text{var}(b) = 4.

---
\(\hat{O}(t); \hat{A}(t); U(h_{\phi}^b, t)\)

Figure 27. \(U(h_{\phi}^b, t)\) as a Product of \(\hat{O}(t)\) and \(\hat{A}(t)\)

\(^{13}\)Some specific relationship must be assumed for \(\phi, a\) and \text{var}(b) 
in order to draw the graph. The choice was arbitrary, and similar results obtain for other selections.
The sawtooth pattern of $U(h^b_{\phi}, t)$ is caused by the stepwise increasing property of $\hat{A}(t)$ with respect to information time. At several discrete points in time, the system improves instantaneously in accuracy since the partition induced on $[\alpha, \beta]$ by $h^b_{\phi}, t$ admits one additional cell. This abruptly reduces the size of each equally large cell. This type of phenomenon would probably exist only in unusual circumstances. It could possibly characterize an information system about costs in a production process which permitted observation only at certain discrete points in time.

4.5 Summary

The investigation of value determinants for unit variable cost information systems is continued in this chapter by placing a mild restriction on the prior probability distribution over that cost parameter. The restriction is that no values of unit variable cost to which positive probability is assigned may be greater than the unit price. This effectively precludes the producer from having a prior expectation of a temporary shut-down at any time during the single production period.

Expressions are developed for the value of generalized, perfect, noisy, and noiseless imperfect unit variable cost information systems. Unit price is no longer a determinant of system value, while the quadratic cost parameter, information timing, initial uncertainty, and the nature of the information system remain as determinants. Sensitivity analysis characterizes the first and second order effects, as well as interaction effects, of the determinants in affecting information value.

Noisy and noiseless imperfect systems which improve in accuracy over time are specified and investigated. The roles of system accuracy,
report timeliness, and the responsiveness of accuracy to report timing are described. The trade-off between accuracy and timeliness is investigated.

Restatement of the mathematical results in economic terms, and various levels of economic interpretation round out the analysis. Managerial implications are reserved for a later chapter, in order to allow comparison and summarization of results for a variety of situations and information systems. In the analysis completed, the quadratic cost parameter was assumed to be known with certainty, and the production rate was unconstrained. In Chapter 5 these assumptions will be relaxed.
CHAPTER 5

ANALYSES OF STOCHASTIC QUADRATIC COST PARAMETER AND ACTION FLEXIBILITY GIVEN STOCHASTIC UNIT VARIABLE COST

5.1 Introduction

In this chapter the investigation of information systems for the CVP decision setting will be extended in two ways. In section 5.2 the quadratic cost parameter will be the only source of uncertainty to the producer, and the value determinants of various information systems will be investigated.

In section 5.3 unit variable cost is again assumed to be the producer's only source of uncertainty. Two types of production constraints will be imposed on the producer's action set, and action flexibility will be investigated as a value determinant for a system which provides perfect information about unit variable cost. Except for the departures noted above, all other previous assumptions about the CVP problem remain operative.

5.2 Uncertainty About the Quadratic Cost Parameter

If the quadratic cost parameter \( a \) is uncertain while the parameters \( r, b \) and \( c \) are known, the state \( s \) is equal to \( a \). Uncertainty could result from a myriad of sources such as uncertainty about the precise way in which production scheduling problems or employee fatigue affect the rate at which marginal costs increase. In the absence of any information about \( a \), the producer's decision model is:

128.
(5.1) \[
\max_{x_1 \geq 0, x_2 \geq 0} \{ \max \left[ \frac{r(x_1+x_2)-a}{t} + \frac{x_1^2}{1-t} - \frac{a}{(1-t)} x_2^2 - b(x_1+x_2) - c \right] p(b) db \}
\]

Performing the expectation and maximization operations in (5.1) yields the following constant optimal production rate:

(5.2) \[
\frac{x_1^*}{t} = \frac{x_2^*}{(1-t)} = \begin{cases} \frac{(r-b)/2E(a)}{r} & \text{if } r \geq b \\ 0 & \text{otherwise} \end{cases}
\]

Substitution of \((x_1^*, x_2^*)\) into \(w(x, s)\) yields the following maximum expected profit:

(5.3) \[
E[w(x^*, a|h_0)] = \begin{cases} \frac{(r-b)^2}{4E(a)} - c & \text{if } r \geq b \\ -c & \text{otherwise} \end{cases}
\]

where \(h_0\) denotes the null information system.

5.2.1 Generalized Information System

A system may be employed which provides information about \(a\) at time \(t, t \in [0,1]\). If again all information the producer receives is provided by the system under explicit consideration, the direct determinants of the value of this system are the same as those listed in section 3.4.1 for systems which provide information about unit variable cost, except \(b\) in that list is replaced by \(a\). The condensed list of potential determinants is:

(i) price, \(r\)

(ii) unit variable cost, \(b\)

(iii) fixed cost, \(c\)

(iv) information timing, \(t\)

(v) degree of uncertainty in \(p(a)\)

(vi) the nature of \(p(y_t|a)\)

where \(p(a)\) is the producer's prior on \(a\) and \(p(y_t|a)\) is the likelihood function for the information system. (v) and (vi) are direct determinants,
(i), (ii) and (iii) are indirect determinants, and (iv) is a spanning indirect determinant.

Let $h_t^a$ denote a system which provides information at time $t$ about the quadratic cost parameter.

**Proposition 5.1**: For $u(z)=z$,

$$
(5.4) \quad U(h_t^a) = \begin{cases} 
\frac{(1-t)(r-b)}{4} \left\{ \frac{1}{E[a|y]} \left( \frac{1}{E(a)} \right) - \frac{1}{E(a)} \right\}, & b \le r \\
0, & \text{otherwise}
\end{cases}
$$

**Proof**: The producer's expected utility given that information system $h_t^a$ is employed is given by:

$$
(5.5) \quad E[w(x^*,a)|h_t^a] = \max_{x_1 \ge 0} \int \max_{x_2 \ge 0} \int \int w(x,a)p(a)da 
$$

$$
= \max_{x_1 \ge 0} \left[ \frac{(r-b)x_1}{t} + \frac{x_1^2}{2} - \frac{a}{t} \frac{x_1^2}{2} - b(x_1+x_2) - c \right] p(b|y)dbp(y)dy 
$$

$$
= \max_{x_1 \ge 0} \left[ \frac{(r-b)x_1}{t} + \frac{x_1^2}{2} - \frac{a}{t} \frac{x_1^2}{2} - b(x_1+x_2) - c \right] p(b|y)dbp(y)dy - c
$$

Noting that $((r-b)x_2 - [E(a|y)/(1-t)]x_2^2)$ is concave in $x_2$, setting its first derivative equal to zero, and recalling the constraint $x_2 \ge 0$:

$$
(5.6) \quad \begin{cases} 
(x_2^*) = \frac{(1-t)(r-b)2}{2E(a|y)}, & b \le r \\
0, & \text{otherwise}
\end{cases}
$$

Similarly, maximization of (5.5) w.r.t. $x_1$ yields:

$$
(5.7) \quad x_1^* = \begin{cases} 
t(r-b)/2E(a), & b \le r \\
0, & \text{otherwise}
\end{cases}
$$

Substitution of $(x_1^*, (x_2^*)|y))$ into (5.5) yields:

$$
(5.8) \quad E[w(x^*,a)|h_t^a] = \begin{cases} 
\frac{t(r-b)^2}{4E(a)} + \frac{(1-t)(r-b)^2}{4E(a)} \frac{1}{E(a|y)} \int p(y)dy - c, & b \le r \\
c, & \text{otherwise}
\end{cases}
$$

Subtracting the producer's maximum expected profit given the null system...
[equation 5.3] from $E[w(x^*, a)|h^a_t]$ yields the following:

$$U(h^a_t) = \begin{cases} 
\frac{(1-t)(r-b)^2}{4} \left( \frac{1}{E[a|y]} - \frac{1}{E[a]} \right), & b \leq r \\
0, & \text{otherwise}
\end{cases}$$

As expected $U(h^a_t) = 0$. Information about the quadratic cost parameter at the period's end, since it is too late to be acted upon, is valueless in the short run. Recalling the list of six direct or indirect determinants of information value given above, all but one of these is represented either implicitly or explicitly in the expression for $U(h^a_t)$.

Fixed cost (c) is not an indirect determinant of $U(h^a_t)$ just as it was not for $U(h^b_t)$. Price (r), unit variable cost (b), and the information timing parameter (t) all appear explicitly in (5.4). The degree of uncertainty in $p(a)$ and the nature of $p(y_t|a)$ appear implicitly since they are implicit arguments of $\{E[1/E(a|y)] - [1/E(a)]\}$.

Information system $h^a_t$ is valueless when unit variable cost is known to exceed the unit price. In this case the producer will shut down for the production period precluding any usefulness for information about the quadratic cost parameter. Throughout the remainder of Chapter 5, the assumption is made that $b < r$.

**Corollary 5-1.1**: For $b < r$ and $t < 1$, $U(h^a_t)$ is linearly decreasing in the information time (t), strictly convex increasing in unit price (r), and strictly convex decreasing in unit variable cost (b).

**Proof**: Partially differentiating equation (5.4):

$$\frac{\partial U(h^a_t)}{\partial t} = -(r-b)^2 \left( \frac{1}{E[a|y]} - \frac{1}{E[a]} \right)/4 < 0; \frac{\partial^2 U(h^a_t)}{\partial t^2} = 0$$

$$\frac{\partial U(h^a_t)}{\partial r} = -\frac{\partial U(h^a_t)}{\partial b} = (1-t)(r-b) \left( E[1/E(a|y)] - [1/E(a)] \right)/2 > 0$$

$$\frac{\partial^2 U(h^a_t)}{\partial r^2} = \frac{\partial^2 U(h^a_t)}{\partial b^2} = (1-t) \left( E[1/E(a|y)] - [1/E(a)] \right)/2 > 0$$

Q.E.D.
As with system $h^b_t$, $U(h^a_t)$ declines proportionately with the length of the information delay. The economic interpretation of this result is analogous to that given for system $h^b_t$ (see section 3.4.1).

The value of information about the quadratic cost parameter is increasing at an increasing rate in the unit price. Recall that $U(h^a_t)$ is the difference between two maximal expected utilities as follows:

\[(5.9) \quad U(h^a_t) = E[w(x^*,a)|h^a_t] - E[w(x^*,a)|h_0] \]

Each of the expectations in (5.9) is convex increasing in unit price ($r$). (See equations (5.8) and (5.3) to verify this assertion.) The result given by Corollary 5-1.1 for $r$ implies that the marginal increments in $E[w(x^*,a)|h^a_t]$ with respect to $r$ are greater than those in $E[w(x^*,a)|h_0]$. Moreover, the rate of increase in these marginal increments is greater for $E[w(x^*,a)|h^a_t]$ than for $E[w(x^*,a)|h_0]$. Equivalently, the producer's expected maximal profit given system $h^a_t$ is more sensitive to unit price than is the maximal expected profit given no information about $a$. These relationships are depicted in Figure 28.

\[
\begin{align*}
U(h^a_t) &:= E[w(x^*,a)|h^a_t], \\
E[w(x^*,a)|h^a_t] &> E[w(x^*,a)|h_0]
\end{align*}
\]

![Figure 28. $U(h^a_t)$ as a Function of Unit Price](image-url)
Given \( b < r \), \( U(h_t^a) \) is convex increasing in \( r \) while \( U(h_{\omega,t}^b) \) is concave increasing in \( r \) for \( E(b) < r \) (see Corollary 3-3.1). The role of unit price as an indirect determinant of information value is dependent on the object of uncertainty.

Appealing again to the theory of the firm\(^1\) the producer's problem is to equate marginal revenue (equal to \( r \)) with marginal cost, given by \( 2a/(1-t) \) for the production subperiod \([t,1] \). Figure 29 shows how the unit price affects the band of uncertainty about \( x_2^* \) when \( a \) is unknown.

Since the band of uncertainty about \( x_2^* \) is greater when unit price is greater, one might expect the value of information about the quadratic cost parameter to be higher in this case.

\(^1\)This approach does not provide a precise interpretation of the result, but may provide some intuitive appeal. See section 3.4.1 for the limitations of this approach.
Analogous arguments may be used to explain the role of unit variable cost as a determinant of \( U(h^a_t) \). Corollary 6-1.1 implies that the marginal decrements in \( E[w(x^*,a)|h^a_t] \) with respect to \( b \) are greater than those in \( E[w(x^*,a)|h^a_t] \). Moreover, the rate of decrease in these marginal decrements is greater for \( E[w(x^*,a)|h^a_t] \) than for \( E[w(x^*,a)|h^a_t] \). Equivalently, the producer's expected maximal profit given system \( h^a_t \) is more sensitive to unit variable costs than is the maximal expected profit given no information about \( a \). These relationships are depicted in Figure 30.

![Figure 30. \( U(h^a_t) \) as a Function of Unit Variable Cost](image)

The theory of the firm interpretation for this result is depicted in Figure 31. Since the band of uncertainty about \( x_2^* \) is greater when unit variable cost is lower, one might expect the value of information about the quadratic cost parameter to be greater.
Corollary 5.1.2: For $b < r$ and $t < 1$: (i) The marginal decrements in $U(h_t^a)$ w.r.t. $t$ are strictly convex increasing in $r$ and strictly convex decreasing in $b$; (ii) the marginal increments in $U(h_t^a)$ w.r.t. $r$ are linearly decreasing in both $t$ and $b$; (iii) the marginal decrements in $U(h_t^a)$ w.r.t. $b$ are linearly decreasing in $t$ and linearly increasing in $r$.

Proof: Partially differentiating equation (5.4):

1. $\frac{\partial^2 U(h_t^a)}{\partial t \partial r} = -\frac{\partial^2 U(h_t^a)}{\partial b \partial t} - (r - b) \left( \frac{\partial}{\partial t} \left[ \frac{1}{E(y | a \in y) - 1/E(a)} \right] \right) / 2 < 0$

2. $\frac{\partial^3 U(h_t^a)}{\partial r^2 \partial t} = \frac{\partial^3 U(h_t^a)}{\partial b^2 \partial t} = 2 \left( \frac{\partial}{\partial t} \left[ \frac{1}{E(y | a \in y) - 1/E(a)} \right] \right) / 2 < 0$;

3. $\frac{\partial^3 U(h_t^a)}{\partial b \partial r^2} = 0,

4. $\frac{\partial^2 U(h_t^a)}{\partial b \partial r} = -(1 - t) \left( \frac{\partial}{\partial t} \left[ \frac{1}{E(y | a \in y) - 1/E(a)} \right] \right) / 2 < 0$; $\frac{\partial^3 U(h_t^a)}{\partial b^2 \partial r} = 0$ Q.E.D.

As unit price increases or unit variable cost decreases, $U(h_t^a)$ becomes more sensitive to information timeliness at an increasing rate. Hence, the timeliness issue takes on greater importance when the system
provides information about the quadratic cost parameter for a high priced or low variable cost product. Similarly, $U(h^a_t)$ is more sensitive to unit price (unit variable cost) as the information is provided earlier or unit variable cost (unit price) decreases (increases).

5.2.2. Perfect Information System

Let $h^a_{\omega,t}$ denote the system which at time $t$ provides perfect information about $a$.

Proposition 5-2: For $u(z)=z$,

$$(5.10) \quad U(h^a_{\omega,t}) = \begin{cases} \frac{(1-t)(r-b)^2}{4} \left\{ E \left[ \frac{1}{a} \right] - \frac{1}{E(a)} \right\}, & b \leq r \\ 0, & \text{otherwise} \end{cases}$$

Proof: The proof is immediate from Proposition 5-1. When $h^a_t$ represents the perfect information system $Y=a$, $y=a$, and $E(a|y) = E(a|a)=a$. Q.E.D.

Propositions 5-1 and 5-2 indicate that the degree of initial uncertainty about $a$ is not an explicit argument of either $U(h^a_t)$ or $U(h^a_{\omega,t})$.

Some insight into this condition is gained by reexpressing the producer's decision problem in a manner analogous to that used in Section 3.4.2 for the case where $b$ was uncertain. First, assuming that no information system is employed, the producer's decision problem and optimal action vector are given by (5.1) and (5.2), respectively. Assuming $b \leq r$, and substituting $(x_1^*, x_2^*)$ into $w(x,a) = r(x_1^*+x_2^*) - \frac{a}{t} x_1^2 - \frac{a}{(1-t)} x_2^2 + b(x_1^*+x_2^*) - c$ yields the following outcome given optimal action and the occurrence of state $a$:

$$(5.11) \quad [w(x^*,a)|h_0] = \frac{t[2E(a)-a](r-b)^2}{4E^2(a)} + (1-t)\frac{[2E(a)-a](r-b)^2}{4E^2(a)} - c$$

$$= \frac{[2E(a)-a](r-b)^2}{4E^2(a)} - c$$
Let \( \psi(a) = u[w(x^*, a) | h_0] \) denote the payoff function for this transformed problem. Since the action variable has been eliminated by the maximization operation, \( \psi \) has no action variable argument. Where the producer faces the decision problem given in (5.11) he faces the gamble \( \psi(a) \).

Note that \( \psi(a) = ([2E(a) - a](r-b)^2/4E^2(a))c \) is a first degree polynomial in the state variable \( a \). Using the Markowitz-Richter result regarding \( N \)-degree polynomial payoff functions, and the Rothschild-Stiglitz results on the degree of uncertainty\(^2\), none of the moments of \( p(a) \) are needed to characterize the degree of uncertainty in \( p(a) \) for the gamble \( \psi(a) \). In the absence of information about the quadratic cost parameter, the producer is indifferent with respect to his level of uncertainty about that parameter.

If system \( h_{\infty,t}^b \) provides the producer with perfect information, the decision problem is now given by:

\[
(5.12) \max \max_{x_1 \geq 0, x_2 \geq 0} \left[ r(x_1 + x_2) - \frac{a}{t} x_1^2 \right] \ x_2 \ \frac{a}{(1-t)} \ x_2^2 - b(x_1 + x_2 - a)p(a) \Rightarrow \frac{x_1^*}{t} = \frac{(r-b)}{2E(a)}; \]

\[
\frac{(x_2^*|a)}{(1-t)} = \frac{(r-b)}{2a}
\]

Substitution of \( [x_1^*, (x_2^*|a)] \) into (5.12) yields:

\[
(5.13) \ [w(x^*, a) | h_{\infty,t}^a] = \frac{t[2E(a) - a](r-b)^2}{4E^2(a)} + \frac{(1-t)(r-b)^2}{4a} - c
\]

Let \( \hat{\psi}(a) = u[w(x^*, a) | h_{\infty,t}^b] \) denote the payoff function for this transformed problem. Since it has been eliminated by the maximization operation, \( \hat{\psi} \) has no action variable argument. Where the producer faces the decision problem given by (5.12), he faces the gamble \( \hat{\psi}(a) \).

\(^2\)See Chapter 2 for a discussion of these results.
A Taylor series expansion about $E(a)$ [denoted by $\bar{a}$] of $\hat{\psi}(a)$ yields the following:

\begin{equation}
(5.14) \quad \hat{\psi}(a) = \hat{\psi}(\bar{a}) + \sum_{i=1}^{j} \frac{\hat{\psi}^{(i)}(\bar{a})}{i!} (a-\bar{a})^i + \frac{\hat{\psi}^{(j+1)}(\xi)}{(j+1)!} (a-\bar{a})^{j+1}
\end{equation}

where $\hat{\psi}^{(i)}(\bar{a})$ denotes the $i$th order derivative of $\hat{\psi}$ evaluated at $\bar{a}$ and $\xi$ is between $a$ and $\bar{a}$. Equation (5.14) provides a $j$th order Taylor polynomial approximation to $\hat{\psi}(a)$. Since $\hat{\psi}^{(1)}(\bar{a}) > 0$ for all $i \in \{1, 2, 3, \ldots\}$, $j$ must approach infinity in order for the Taylor polynomial to approach $\hat{\psi}(a)$ precisely. Therefore, $\hat{\psi}(a)$ cannot be precisely expressed as a finite-order polynomial in the state variable $a$. By the Markowitz-Richter result on $N$-degree polynomial payoff functions, the R-S degree of uncertainty in $p(a)$ cannot be expressed as a function of a finite number of moments of $p(a)$ for the payoff function $\hat{\psi}(a)$. Since $U(h^a_{\infty}, t) = E[\hat{\psi}(a)] - E[\hat{\psi}(a)]$, the degree of uncertainty in $p(a)$, as a determinant of $U(h^a_{\infty}, t)$, cannot be expressed as a function of a finite number of moments of $p(a)$.

One important qualification must be made to this statement. When $p(a)$ happens to be a particular probability distribution for which all moments of order greater than $M$ can be expressed as a function of moments of order $M$ or lower, the degree of uncertainty can be expressed as a function of these $M$ moments. \(^3\) For example, suppose that $p(a) \sim$ gamma $(\alpha, \beta)$, $\alpha > 1$, $\beta > 0$. Since its range is the positive reals and its shape can take on diverse forms given different values for the parameters $\alpha$ and $\beta$, this is a reasonable prior for $a$.

Proposition 5-3: For \( u(z) = z \) and \( p(a) \sim \text{gamma} (\alpha, \beta) \),

\[
(5.15) \quad U(h^a_w, t) = \frac{(1-t)(r-b)^2}{4} \left( \frac{\sigma^2}{\mu^2 - \mu_0^2} \right)
\]

where \( u = E(a) \) and \( \sigma^2 = \text{var}(a) \).

Proof:

\[
p(a) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} a^{\alpha-1} e^{-\beta a}, & a > 0 \\ 0 & \text{elsewhere} \end{cases}
\]

(5.16) \( E(1/a) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{1}{a} a^{\alpha-1} e^{-\beta a} da = \frac{\beta}{\Gamma(\alpha-1)} \left( \frac{\beta}{\alpha-1} \right) \int_0^\infty \frac{\alpha^{\alpha-1} e^{-\beta a} da}{a} \)

where \( \alpha = \alpha - 1 \), since for \( \alpha > 1 \), \( \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \).\(^5\) In equation (5.6), the expression in braces is a proper gamma density integrated over its range. Consequently \( E(1/a) = \beta / (\alpha - 1) \). Substituting this result into \( U(h^a_w, t) \) given by Proposition 5-2, and noting that \( E(a) = \alpha / \beta \), the following is obtained:

\[
(5.17) \quad U(h^a_w, t) = \frac{(1-t)(r-b)^2}{4} \left( \frac{\beta}{\alpha - 1} - \frac{\beta}{\alpha} \right) = \frac{(1-t)(r-b)^2}{4} \left( \frac{\beta}{\alpha(\alpha - 1)} \right)
\]

Now \( u = E(a) = \alpha / \beta \) and \( \sigma^2 = \text{var}(a) = \alpha / \beta^2 \), which implies that \( \alpha = \mu^2 / \sigma^2 \) and \( \beta = \mu / \sigma^2 \). When these substitutions are made in (5.17) and after simplifying, the desired result is achieved.

Q.E.D.

The gamma distribution is a two-parameter distribution and all of its moments are determined by \((\alpha, \beta)\) or its transformation \((\mu, \sigma^2)\). The degree of uncertainty in \( p(a) \) as a determinant of \( U(h^a_w, t) \) is fully characterized by \( \sigma^2 \).

\(^4\) \( \Gamma(a) = \int_0^\infty e^{-\gamma} \gamma^{a-1} \, dy, \ a > 0. \)

Corollary 5-3.1: For t<1, b<r and p(a) ~ gamma (α, β), U(hₐ₀, t) is increasing in \( \text{var}(a) = \sigma^2 \).

Proof: Partially differentiating equation (5.15):

\[
(5.18) \quad \frac{\partial U(h_{a_0}, t)}{\partial \sigma^2} = \frac{(1-t)(r-b)^2}{4} \cdot \frac{\mu^3}{(\mu^3 - \mu_0^2)^2} > 0 \quad \text{Q.E.D.}
\]

As the degree of initial uncertainty about the quadratic cost parameter increases, the value of perfect information about the parameter increases.

Corollary 5-3.2: For b<r and t<1: (i) the marginal increments (decrements) in \( U(h_{a_0}, t) \) w.r.t. unit price (unit variable cost) are increasing in \( \text{var}(a) \); (ii) the marginal increments in \( U(h_{a_0}, t) \) w.r.t. \( \text{var}(a) \) are strictly convex increasing (decreasing) in unit price, r (unit variable cost, b).

Proof: Partially differentiating equation (5.15):

\[
(5.19) \quad \frac{\partial^2 U(h_{a_0}, t)}{\partial a \partial \sigma^2} = \frac{\partial^2 U(h_{a_0}, t)}{\partial a^2} = \frac{(1-t)(r-b)}{2} \cdot \frac{\mu^3}{(\mu^3 - \mu_0^2)^2} > 0
\]

\[
(5.20) \quad \frac{\partial^3 U(h_{a_0}, t)}{\partial a^3} = \frac{\partial^3 U(h_{a_0}, t)}{\partial a^2 \partial \sigma^2} = \frac{(1-t)}{2} \cdot \frac{\mu^3}{(\mu^3 - \mu_0^2)^2} > 0 \quad \text{Q.E.D.}
\]

With respect to information timeliness, unit price, and unit variable cost, the sensitivity of \( U(h_{a_0}, t) \) increases in the degree of initial uncertainty. As unit price (unit variable cost) increases (decreases) or information is provided earlier, the influence of initial uncertainty, as a determinant, becomes more important.

5.3 Investigation of Action Flexibility Given Uncertainty About Unit Variable Cost

One of the direct determinants of information value discussed in Chapter 2 is the structure of \( X \), the set of actions available to the
decision maker. This determinant was labeled action flexibility. Flexibility has been largely ignored in this analysis since $X$ has been defined as the set of nonnegative real numbers. That is, production has been infinitely divisible and unbounded. In this section $b$ is assumed to be the only uncertain parameter and two types of restrictions on $X$ will be considered. The first is an upper bound on the production rate, $x/\tau$; the second is a hatch production restriction which requires that $x$ be produced in established incremental amounts.

5.3.1 Upper Bounded Production Rate

The CVP problem may be considerably enriched by inclusion of an upper bound constraint on the production rate. The upper bound (denoted by $D$) may be viewed as a capacity constraint which is fixed in the short run (e.g., machine capacity). The quadratic cost function is then viewed as reflecting other technological capacity conditions which result in increasing marginal costs over feasible production actions (e.g., overtime resulting from labor capacity or increased scrap resulting from faster production). Under these conditions, Proposition 5-4 gives the value of perfect, time $t$ information about parameter $b$ given the restriction that $P(r)=1$.

**Proposition 5-4:** For $u(z)=z$ and $P(r)=1$,

$$
U(b, t) = \begin{cases} 
(1-t) \frac{\text{var}(b)}{4a}, & \text{if } r-2aD < 0 \\
\frac{(1-t)\text{var}(b)-\frac{r-2aD}{0}(r-b-2aD)^2 p(b)db}{4a}, & \text{if } 0 \leq r-2aD \leq E(b) \\
\frac{(1-t)\int_{r-2aD}^{r} (r-b-2aD)^2 p(b)db}{4a}, & \text{if } E(b) < r-2aD 
\end{cases}
$$

**Proof:** The analysis is facilitated by distinguishing the three cases identified above: (i) $r-2aD < 0$, (ii) $0 \leq r-2aD \leq E(b)$, and (iii) $E(b) < r-2aD$. 

For case (i):

\[(5.22) \ r-2aD < 0 \iff \frac{(1-t)r/2a}{(1-t)} < D \text{ and } \frac{tr/2a}{t} < D\]

Recall from the proof for Proposition 3-2 that when \(x_1/t\) and \(x_2/(1-t)\) had no upper bound \((x_2^*|b) = (1-t)(r-b)/2a\) and \(x_1^* = t[r-E(b)]/2a\). For any \(b \in [0, r]\), \((5.22)\) implies that \((x_2^*|b)/(1-t) < D\) and \(x_1^*/t < D\). Hence \(D\) is not binding either at time 0 or at time \(t\), and Proposition 3-3 gives \(U(h_{\infty,t}^b)\) for case (i).

For case (ii) \(0 \leq r-2aD \leq E(b)\). The producer's decision problem given that he employs system \(h_{\infty,t}^b\) is:

\[(5.23) \ \max_{0 \leq x_1 \leq D} \max_{0 \leq x_2 \leq D} \frac{[(r-b)(x_1+x_2)-a(x_1^2/t)-a(x_2^2/(1-t))-c]p(b)}{t} db\]

Maximization w.r.t. \(x_2\) of \((5.23)\) yields the decision rule:

\[(5.24) \ (x_2^*|b) = \begin{cases} 
(1-t)D, & \text{if } \frac{(1-t)(r-b)/2a}{(1-t)} \geq D \iff r-2aD \geq b \\
(1-t)(r-b)/2a, & \text{otherwise}
\end{cases}\]

The decision rule is dependent on unit variable cost \(b\). For low variable cost \((\leq r-2aD)\), the unconstrained optimal production rate is high and exceeds the upper bound. The optimal rate is equal to the upper bound \(D\). For high variable cost \((> r-2aD)\), the unconstrained optimal production rate is low and the constraint is not binding.

Maximizing \((5.23)\) w.r.t. \(x_1\) yields \(x_1^* = t[r-E(b)]/2a\) since this is less than \(tD\) by the case (ii) condition. Variable cost is expected to be high enough that the unconstrained optimal production rate in the sub-period \([0, t]\) is below the upper bound. Substituting \(x_1^*\) and \((x_2^*|b)\) into \((5.23)\) and rearranging terms:
(5.25) \[ E[w(x^*_b, b)|h_{\omega, t}, 0 \leq r - 2aD \leq E(b)] = \int_0^{2aD} \frac{\varphi([-1-t](r-b))^2}{4a} + (r-b)(1-t)D \\
- a(1-t)D^2 p(b) \, db \\
+ E[(1-t)(r-b)^2/4a] + t[r-E(b)]^2/4a - c \]

The case (ii) condition, \( r - 2aD \leq E(b) \), implies that \( D^2 [r-E(b)]/2a \). If the producer employs the null system, the constraint is not binding in either subperiod and (3.15) gives:

\[ (5.26) \ E[w(x^*_b, b)|h_{\omega}, 0 \leq r - 2aD \leq E(b)] = \frac{(r-E(b))^2}{4a} - c \]

Subtracting (5.26) from (5.25), recalling that \( \var{b} = E(b^2) - E^2(b) \) and rearranging terms yields \( U(h^b_{\omega, t}) \) for case (ii) as given in the proposition.

For case (iii), \( E(b) < r - 2aD \). The producer's decision problem, given that he employs system \( h^b_{\omega, t} \), is still given by (5.23) and the decision rule \( (x^*_2|b) \) by (5.24). The explanation of \( (x^*_2|b) \) is the same as in case (ii). Maximizing (5.23) w.r.t. \( x_1 \) yields \( x_1^* = \min \{ tD, t[r-E(b)]/2a \} \).

The case (iii) condition, \( E(b) < r - 2aD \), implies that \( \min \{ \cdot \} = tD = x_1^* \). Now unit variable cost is expected to be low enough that the unconstrained optimal production rate in the first subperiod exceeds the upper bound. The constraint is binding. Substituting \( (x^*_2|b) \) and \( x_1^* \) into (5.23) and rearranging terms yields:

\[ (5.27) \ E[w(x^*_b, b)|h^b_{\omega, t}, E(b) < r - 2aD] = \int_{r - 2aD}^r \frac{(1-t)(r-b)^2}{4a} - (1-t)(r-b)D \\
+ a(1-t)D^2 p(b) \, db \\
+ E[(r-b)D - ab^2 - c] \]

The case (iii) condition implies that the constraint is binding in both subperiods if the producer employs the null information system. Hence \( x_1^*/t = x_2^*/(1-t) = D \), and:
(5.28) \( E[w(x^*,b)|h_0, E(b)<r-2aD] = E[(r-b)D-aD^2-c] \)

Subtracting (5.28) from (5.27) and rearranging terms, the expression given in the proposition for \( U(h_0, t) \) is obtained under case (iii). Q.E.D.

**Corollary 5.4.1:** For \( u(z)=z \) and \( P(r)=1 \): \( U(h_0, t) \) is invariant w.r.t. the upper bound \( D \) if \( D > r/2a \); for \( D<r/2a \), \( U(h_0, t) \) is strictly \( \{ \text{concave} \} \)

increasing in \( D \) if \( D \geq [r-E(b)]/2a \) and \( t<1 \).

**Proof:** Partially differentiating equation (5.21):

\[
\frac{\partial U(h_0, t)}{\partial D} = \begin{cases} 
0, & \text{if } r-2aD<0 \\
(1-t) \frac{r-2aD}{0} (r-b-2aD)p(b)db > 0, & \text{if } 0 \leq r-2aD \leq E(b) \\
(1-t) \frac{r-2aD}{r-2aD} (r-b-2aD)p(b)db > 0, & \text{if } E(b) < r-2aD 
\end{cases}
\]

\[
\frac{\partial^2 U(h_0, t)}{\partial D^2} = \begin{cases} 
0, & \text{if } r-2aD<0 \\
-2a(1-t) \frac{r-2aD}{0} p(b)db < 0; & \text{if } 0 \leq r-2aD \leq E(b) \\
2a(1-t) \frac{r}{r-2aD} p(b)db > 0, & \text{if } E(b) < r-2aD 
\end{cases}
\]

Q.E.D.

Figure 32 depicts the relationship of \( U(h_0, t) \) to the upper bound.

\[ U(h_0, t) \]

\[
(1-t)\text{var}(b) \\
4a
\]

\[ D \]

**Case (iii)**

\[
\frac{r-E(b)}{2a}
\]

**Case (ii)**

\[
\frac{r}{2a}
\]

**Case (i)**

Figure 32. \( U(h_0, t) \) As a Function of the Upper Bound on the Production Rate
Naturally, if the upper bound is zero, any information system is valueless since there is no way to take advantage of the information. If the upper bound is sufficiently large, the system attains its maximum value equal to \((1-t)\text{var}(b)/4a\) (see Proposition 4-2). If \(D \in [0,r/2a]\), information is of greater value the more flexible is the producer to take advantage of it. Greater flexibility comes with a looser constraint. However, the concavity or convexity of this relationship depends on the relative magnitudes of \(D\) and \([r-E(b)]/2a\). The latter, it will be recalled, is the optimal production rate given no information about \(b\). If \(D\) is less than this rate, \(U(h^b_{\infty,t})\) exhibits increasing marginal increments with respect to \(D\). If \(D\) is greater than \([r-E(b)]/2a\), the marginal increments are decreasing in \(D\).

Again recall that \(U(h^b_{\infty,t})\) is the difference between the two maximal expectations \(E[w(x^*_b,b)|h^b_{\infty,t}]\) and \(E[w(x^*_b,b)|h_0]\). Partial differentiation w.r.t. \(D\) of these two expectations under the various conditions of cases (i), (ii) and (iii) (see the proof of Proposition 5-4) yield the following results:

Case (i) \(\partial E[w(x^*_b,b)|h^b_{\infty,t}] / \partial D = \partial E[w(x^*_b,b)|h_0] / \partial D = 0\)

Case (ii) \(\partial^2 E[w(x^*_b,b)|h^b_{\infty,t}] / \partial D^2 < \partial^2 E[w(x^*_b,b)|h_0] / \partial D^2 = 0\)

Case (iii) \(\partial E[w(x^*_b,b)|h^b_{\infty,t}] / \partial D = \partial E[w(x^*_b,b)|h_0] / \partial D > 0\)

\(\partial^2 E[w(x^*_b,b)|h^b_{\infty,t}] / \partial D^2 > \partial^2 E[w(x^*_b,b)|h_0] / \partial D^2 > 0\)

The signs of the second derivatives induce the concavity and convexity exhibited in Figure 32. Figure 33 shows the two expectations as functions of \(D\).
The less precise but intuitive theory of the firm approach can again be used to lend some intuition to the result of Corollary 5.4.1. Figure 34 depicts the process of equating marginal revenue and marginal cost given a band of uncertainty about unit variable cost.
Note that as $D$ increases from $D'$ to $D''$, the band of uncertainty about $x^*_2$ increases and the value of information about unit variable cost increases. The same caveat must be given here as before for this intuitive approach. The phenomenon observed in Figure 34 is not equivalent to that exhibited in Figures 32 and 33. It is, however, a result of some of the same factors which cause the earlier result.

**Corollary 5-4.2:** For $u(z)=z$ and $P(r)=1$: (i) the marginal increments in $U(h_{o,t}^b)$ w.r.t. $D$ are linearly decreasing in $t$; (ii) the marginal decrements in $U(h_{o,t}^b)$ w.r.t. $t$ are strictly convex increasing in $D$ for $D \leq [r-E(b)]/2a$ and $t<1$.

**Proof:** The proof is immediate from inspection of the partial derivatives in the proof of Corollary 5-4.2. Q.E.D.

The sensitivity of $U(h_{o,t}^b)$ to timeliness (action flexibility) is increasing in action flexibility (timeliness). While $a$ and $r$ are also determinants of $U(h_{o,t}^b)$ in the constrained case, their precise role cannot be established since it is not possible, in the general case, to place signs on the partial derivatives of $U(h_{o,t}^b)$ with respect to these two parameters.

**5.3.2 Batch Production**

In this section unit variable cost is assumed to be the only source of uncertainty for the producer, the restriction $P(r)=1$ is assumed for $p(b)$, and there is no upper bound on the production rate. Suppose that the producer's output is produced in batches of $\lambda$ units, $\lambda>0$. Let $h_{o,t}^b$ denote the system which provides perfect information about $b$. In this case $\lambda$ is an indirect determinant of information value, but there is in general no monotonic relationship between information value and batch.
size $\lambda$. This is made clear by the following proposition and discussion.

**Proposition 5-5:** For $u(z) = z$ and $P(r) = 1$,

\begin{equation}
(5.29) \quad [U(h_{\omega,0}^b)^{\lambda}] = E[w(\tilde{x},b) | h_{\omega,0}^b, \lambda] - E[w(\tilde{x},b) | h_{\omega}^b, \lambda]
\end{equation}

where $\tilde{x}$ denotes the constrained optimal production quantity during the entire period $[0,1]$ given $\lambda$, and:

\begin{equation}
(5.30) \quad E[w(\tilde{x},b) | h_{\omega,0}^b, \lambda] = \begin{cases} 
\int_0^{r-cp(b)db} i=0 \\
\frac{r-\lambda}{r} \int_0^{r-cp(b)db + \sum_{i=1}^{(2i-1)a} (-a(i\lambda)^2+(r-b)i\lambda-c)p(b)db} \quad \text{for } \hat{i} \in \{2,3,4,\ldots\}
\end{cases}
\end{equation}

where $\hat{i} = \text{int}[\frac{r}{2a}]$ and int $[\cdot]$ is the largest integer not greater than its argument.

\begin{equation}
(5.31) \quad E[w(\tilde{x},b) | h_{\omega}^b] = \begin{cases} 
-a(j\lambda)^2+(r-b)j\lambda-c \quad \text{for } j \in \{1,2,3,\ldots\} \\
-c, \text{ otherwise}
\end{cases}
\end{equation}

**Proof:** Letting $x$ denote the quantity produced during the entire production period $[0,1]$, the producer's decision problem given no information about $b$ is:

\begin{equation}
(5.32) \quad \max_{x \in \Lambda} \{-ax^2+(r-b)x - c\} = \max_{x \in \Lambda} \{-ax^2+(r-b)x - c\}
\end{equation}

where $\Lambda = \{0, \lambda, 2\lambda, 3\lambda, \ldots\}$.

The producer's unconstrained optimal action is $x^* = \frac{r-E(b)}{2a}$ and the unconstrained maximal expected profit is
\( E[w(x^*,b)|h_o] = \{[r-E(b)]^2/4a\} - c. \) Since (5.32) is quadratic in \( x \), it is symmetric about its unconstrained maximum, \( x^* \). Therefore, the constrained maximum of (5.32) is the feasible production level closest to \( x^* \). This point, denoted by \( \tilde{x} \), will always be one of two points \( k\lambda \) or \((k+1)\lambda \) where \( x^* \in [k\lambda,(k+1)\lambda] \). \( \tilde{x} \) is given by the following:

\[
(5.33) \tilde{x} = \begin{cases} 
\lambda, & \lambda \in \left[\frac{r-E(b)}{(2j+1)a}, \frac{r-E(b)}{(2j-1)a}\right], j \in \{1, 2, \ldots\} \\
0, & \text{otherwise}
\end{cases}
\]

Substituting \( \tilde{x} \) into (5.29) yields \( E[w(\tilde{x},b)|h_o] \) given by equation (5.31). Figure 35. depicts \( E[w(\tilde{x},b)|h_o,\lambda] \) as a function of \( \lambda \).

Note that \( E[w(\tilde{x},b)|h_o,\lambda] \) attains its maximum, \( E[w(x^*,b)|h_o,\lambda] \), when \( x^* \) is an integral multiple of \( \lambda \), since then \( \tilde{x} = x^* \). If \( \lambda \) is greater than \( 2x^* \), the producer will shut down and incur a loss of \(-c\). As \( \lambda \) decreases in the range \([0,2x^*]\), \( E[w(\tilde{x},b)|h_o,\lambda] \) alternately rises and falls reflecting...
the fact that $\hat{x}$ alternately becomes closer to and further from $x^*$. If the producer employs system $h_{m,0}^b$, the constrained optimal decision rule is given by

$$\{i\lambda, \lambda \in \frac{r-b}{(2i+1)a}, \frac{r-b}{(2i-1)a}, i \in \{1, 2, \ldots\}\}$$

(5.34) $(\tilde{x}|b) = \begin{cases} 
0, & \text{otherwise} 
\end{cases}$

Note that the condition on $\lambda$ given in (5.34) is equivalent to the following:

(5.35) $\lambda \in \left[\frac{r-b}{(2i+1)a}, \frac{r-b}{(2i-1)a}\right] \iff b \in [r-\lambda(2i+1)a, r-\lambda(2i-1)a]$ \hspace{1cm} (5.36) $\lambda \in (r-b)/a, \infty) \iff b \in (r-a\lambda, \infty)$

Under the assumptions of this analysis, $b \in [0, r]$. Therefore condition (5.36) becomes $b \in [r-a\lambda, r]$ and the index $i$ in condition (5.35) has a maximum of $\hat{i} = \text{int}[\frac{1}{2}+(r/2\lambda a)]$. The latter restriction is required since for $i>\hat{i}$, (5.35) would imply that $b$ is negative. This is a violation to an assumption. Decision rule (5.34) can now be restated as follows:

(5.37) $(\tilde{x}|b) = \begin{cases} 
\hat{i}\lambda, b \in [0, r-\lambda(2i-1)a] 
\end{cases}$

Given that system $h_{m,0}^b$ is employed, the producer's conditional profit is:

(5.38) $[w(\tilde{x}, b)|h_{m,0}^b, \lambda, b] = \begin{cases} 
-a(i\lambda)^2+(r-b)i\lambda-c, & b \in [0, r-\lambda(2i-1)a] 
-a(i\lambda)^2+(r-b)i\lambda-c, & b \in [r-\lambda(2i+1)a, r-\lambda(2i-1)a], 
0, & b \in (r-a\lambda, r) 
\end{cases}$

$\lambda \geq r-\lambda(2i-1)a$. For $b$ to be nonnegative, $r-\lambda(2i-1)a$ must be nonnegative. $r-\lambda(2i-1)a \geq 0 \iff i \leq \text{int}[\frac{1}{2}+r/2\lambda a]$. Since $i$ is an integer, $i = \text{int}[\frac{1}{2}+r/2\lambda a]$, and $\hat{i}$ is maximum feasible $i = \text{int}[\frac{1}{2}+r/2\lambda a]$. 

---

6Condition (5.35) states that $b \geq r-\lambda(2i-1)a$. For $b$ to be nonnegative, $r-\lambda(2i-1)a$ must be nonnegative. $r-\lambda(2i-1)a \geq 0 \iff i \leq \text{int}[\frac{1}{2}+r/2\lambda a]$. Since $i$ is an integer, $i = \text{int}[\frac{1}{2}+r/2\lambda a]$, and $\hat{i}$ is maximum feasible $i = \text{int}[\frac{1}{2}+r/2\lambda a]$. 

---
Taking the expectation of (5.38) w.r.t. \( b \) yields \( E[w(\tilde{x}, b) | h_{\omega,0}^{b}, \lambda] \) as given by equation (5.30) in the proposition.

\[
[U(h_{\omega,0}^{b}) | \lambda] = E[w(\tilde{x}, b) | h_{\omega,0}^{b}, \lambda] - E[w(\tilde{x}, b) | h_{0}, \lambda]
\]

by definition. Q.E.D.

For \( u(z) = z \) and \( p(b) \rightarrow \text{rectangular on } [0, r] \), the functional dependence of \( U(h_{\omega,0}^{b}) \) on \( \lambda \), over part of the range of \( \lambda \), is shown in Figure 36.

![Figure 36. Functional Dependence of U(h_{\omega,0}^{b}) on \lambda](image)

The slopes depicted in the graph are determined by substituting \( p(b) = \begin{cases} 1/r, b \in [0, r] \\ 0, \text{ elsewhere} \end{cases} \) into equation (5.30) for \( p(b) \), substituting \( r/2 \) into equation (5.31) for \( E(b) \), and twice differentiating, with respect to \( \lambda \), the resulting expression for \( U(h_{\omega,0}^{b}) \), given by equation (5.29).
For example, when $\lambda \in (r/a, \infty)$, $\hat{\chi} = \hat{x} = 0$, and $U(h_{\omega,0}^b) = -c = -(-c) = 0$.

When $\lambda \in [r/2a, r/a)$, $\hat{\chi} = 1$ and $\hat{x} = 0$, which implies the following:

$$(5.39) \quad U(h_{\omega,0}^b) = \int_0^{r/a} \frac{-c}{r-a} \lambda(p(b)db + \int_0^{r-a} \frac{-a}{r-b\lambda - c} \lambda p(b)db - (-c)}$$

For $b \sim \text{rectangular on } [0, r]$, equation (5.39) reduces to the following:

$$(5.40) \quad U(h_{\omega,0}^b) = (a^2 \lambda^3 - 2ar\lambda^2 + r^2\lambda) / 2$$

Partially differentiating equation (5.40):

$$\frac{\partial U(h_{\omega,0}^b)}{\partial \lambda} = (3a^2 \lambda^2 - 4ar\lambda + r^2) / 2; \quad \frac{\partial^2 U(h_{\omega,0}^b)}{\partial \lambda^2} = 6a^2 \lambda - 4ar; \quad \frac{\partial^3 U(h_{\omega,0}^b)}{\partial \lambda^3} = 6a^2 > 0$$

The third partial is positive which implies that the second partial is increasing throughout. The second partial is equal to zero for $\lambda = \frac{2r}{3}a$. Therefore, the second partial is negative (positive) for $\lambda$ less (greater) than $\frac{2r}{3}a$. The first partial, evaluated at $\lambda = r/2a$, is equal to $-r^2/8 < 0$.

Since the first partial is decreasing for $\lambda \in [r/2a, 2r/3a)$, it is negative on that interval. Therefore, $U(h_{\omega,0}^b)$ is concave decreasing for $\lambda \in [r/2a, 2r/3a)$. The first partial, evaluated at $\lambda = r/a$, is equal to zero. Since the first partial is increasing for $\lambda \in (2r/3a, r/a)$, it is negative on that interval. Therefore, $U(h_{\omega,0}^b)$ is convex decreasing for $\lambda \in (2r/3a, r/a)$. $(2r/3a)$ is an inflection point. This analysis confirms the slopes depicted in Figure 36 for $\lambda \in (r/2a, \infty)$. The remainder of the graph may be determined through similar reasoning.

Figure 36 supports the assertion that there is, in general, no monotonic relationship between batch size and the value of perfect unit variable cost information. Even though a smaller batch size implies more feasible production rates below a given level, and a decrease in the
maximum difference between the closest feasible rate and the optimal rate, information value is not decreasing in λ. This result stems from the fact that $E[w(\tilde{x},b)|h_o,\lambda]$ and $E[w(\tilde{x},b)|h_o^b,0,\lambda]$ are not monotonically decreasing in λ (see Figure 35). While λ is an indirect determinant of information value, the direction of its effect is contingent on factors specific to the decision setting.

5.4 Summary

This chapter presents a further development of the CVP analysis. First, expressions are derived for the value of generalized and perfect systems providing information about the quadratic cost parameter. Unit price, unit variable cost, information timing, and initial uncertainty are identified as determinants of system value. First and second order effects and interaction effects of the determinants are characterized via sensitivity analysis, and are interpreted in economic terms.

The second development entails a relaxation of the assumption (made in Chapters 3 and 4) of an unconstrained production rate. This allows an investigation of flexibility as an information value determinant. Two types of constraints are independently imposed, and the value of unit variable cost information is derived. An upper bound on the production rate is found to be an indirect determinant in which system value is nondecreasing. Second order effects and interaction effects of the upper bound with other determinants are derived and economically interpreted.

A batch production constraint is imposed on the CVP decision, essentially making it an integer programming problem. The optimization problem is solved and the value of unit variable cost information is
expressed as a function of batch size. Although batch size is an indirect determinant of information value, no general monotonic relationship is exhibited.

In Chapter 6, the investigation of the CVP problem will be generalized along several dimensions. This will conclude the analysis of the determinants of information value in the CVP setting. In Part III some managerial implications of the analysis will be discussed.
CHAPTER 6

GENERALIZATION OF THE COST-VOLUME-PROFIT DECISION ANALYSIS: MONOPOLY, MULTIPLE INFORMATION TIME SYSTEMS AND MULTIPLE STOCHASTIC PARAMETERS

6.1 Introduction

The purpose of this chapter is to generalize the investigation of the CVP decision in three ways. First, the assumption of perfect competition will be relaxed. The determinants of information value will be addressed in a monopoly setting. Second, the analysis of the single information time system, $h^b_{t,1}$, will be generalized to that of the multiple information time system, $h^b_{t,6}$. Finally, attention will be directed to systems which provide information about more than one uncertain cost function parameter.

In the analysis to follow, all other assumptions made in Chapters 3, 4 and 5 remain operative. Specifically, the decision horizon is a single period, the cost function is quadratic in the production rate, sales occur instantaneously at the same rate as production, the production rate may be costlessly altered during the production period, and the production quantity is unbounded and infinitely divisible.

6.2 Monopoly

Suppose the producer is a monopolist. The inverse demand function states price as a function of the quantity sold during a given period.
of time.  

\[(6.1) \text{ price } = q \frac{x}{t} + r; \quad q < 0, \quad r > 0\]

The producer's total revenue (TR) function is then given by:

\[(6.2) \text{ TR } = (q \frac{x}{t} + r) x = \frac{qx^2}{t} + rx\]

The producer's inverse demand function and marginal revenue (MR) function are shown in Figure 37A. Figure 37B shows the total revenue function.

![Diagram](A) ![Diagram](B)

**Figure 37. Inverse Demand, Marginal Revenue and Total Revenue Functions Under Monopoly**

Again let times 0 and 1 denote the beginning and end of the production-sales period, and assume that sales take place at the same rate as production. Assuming that the production rate may be costlessly altered at time \( t \in [0,1] \), the producer's outcome (profit) function is given by:

\[\text{price} = \min(r, \frac{q(r-b)}{2(a-q)} + r) \geq 0.\]

---


2Optimization of (6.3) under certainty yields the constant optimal production rate, \( (x^*/t) = x^*/(1-t) = \max [0, (r-b)/2(a-q)]. \) Substitution of these optimal actions into inverse demand function (6.1) yields a nonnegative price: price = \( \min(r, \frac{q(r-b)}{2(a-q)} + r) = \min(r, \frac{2ar-q(r+b)}{2(a-q)}). \)
\( (6.3) \ w(x,s) = q \left( \frac{x_1^2 + x_2^2}{t} \right) + r(x_1 + x_2) - a \left( \frac{x_1^2 + x_2^2}{t} \right) - b(x_1 + x_2) - c \)

\[
= (q-a) \left( \frac{x_1^2 + x_2^2}{t} \right) + (r-b)(x_1 + x_2) - c
\]

6.2.1 Uncertainty About Unit Variable Cost

Suppose that unit variable cost is the only uncertain parameter, and that the producer employs system \( h_t^b \) which provides information about \( b \) at time \( t \).

Proposition 6-1: For \( u(z) = z \),

\[
U(h_t^b) = \begin{cases} 
\frac{(1-t)}{4(a-q)} \int_{Y \in y_1} [r-E(\partial y)]^2 p(y) dy - [r-E(b)]^2, & E(b) \neq r \\
\frac{(1-t)}{4(a-q)} \int_{Y \in y_1} [r-E(b)]^2 p(y) dy, & \text{otherwise}
\end{cases}
\]

where \( Y \supseteq \hat{Y} = \{ y \in Y | E(b) \neq r \} \).

Note that outcome function (6.3) is identical to outcome function (3.7), given in Chapter 3 for the perfectly competitive case, except that -a in (3.7) is replaced by \( q-a \) in (6.3). Therefore the proof of Proposition 6-1 proceeds in exactly the same way as that for Proposition 3-2, which gives \( U(h_t^b) \) under perfect competition, simply by replacing \( a \) by \( a-q \). In fact, the analysis of the information systems under perfect competition (given in Chapter 3) can be viewed as a special case of the analysis given here for monopoly, simply by letting \( q = 0 \).

The discussion in Chapter 3 regarding determinants, \( r, a, t \), degree of initial uncertainty, and nature of \( p(y_t | b) \) remains valid for the monopoly case. This is because the signs of the partial derivatives of \( U(h_t^b) \) with respect to these variables is the same whether \( q \) is zero or negative. The graphical analyses in Chapter 3 showing the process of equating marginal revenue to marginal cost would be altered slightly to
reflect the negative sloping inverse demand function under monopoly. The intuitive economic interpretations represented by those graphs, however, remains essentially the same.

**Corollary 6.1.1:** For $t < 1$ and $\bar{v} \neq 0$, the expected value of information system $h^b_t$ is strictly convex increasing in demand function parameter $q$.

**Proof:** It was shown in Chapter 2 that the expected value of an information system is nonnegative. Let $J[E(b), r]$ denote the quantity in braces in equation (6.4). Multiplying $U(h^b_t)$ by $4(a-q)/(1-t)$ and noting that $(a-q) > 0$ and $(1-t) > 0$, it is clear that $J[E(b), r] > 0$.

Partially differentiating equation (6.4):

$$\frac{\partial U(h^b_t)}{\partial q} = \frac{(1-t) J[E(b), r]}{4(a-q)^2} > 0; \quad \frac{\partial^2 U(h^b_t)}{\partial q^2} = \frac{(1-t) J[E(b), r]}{2(a-q)^3} > 0 \quad \text{Q.E.D.}$$

![Graph](image_url)  
**Figure 38.** $U(h^b_t)$ as a Function of Demand Function Parameter $q$

Recall that $U(h^b_t) = E[w(x^*, b) | h^b_t] - E[w(x^*, b) | h_0]$. Each of these maximal expectations is increasing at an increasing rate in $q$. Corollary 6.1.1 implies that $E[w(x^*, b) | h^b_t]$ is increasing in $q$ at a faster rate than

---

3This approach to explanation of the results is not a precise interpretation; it is presented only as an aid to the intuition. See section 3.4.1 for the limitations of the approach.
$E[w(x^*,b)|h_0]$. Moreover, the marginal increments in $E[w(x^*,b)|h^b_t]$ with respect to $q$ are increasing faster in $q$ than are those of $E[w(x^*,b)|h_0]$.

Taking the less precise but intuitive theory of the firm approach, in the second production subperiod $[t,1]$, the producer equates marginal revenue $(2qx_2/(1-t)+r)$ to marginal cost $(2ax_2/(1-t)+b)$. Figure 39A depicts this process under certainty. Figure 39B depicts the process for two values of $q$ ($q''>q'$), when there is a band of uncertainty about unit variable cost.\(^4\)

![Diagram](image)

Figure 39. Economic Interpretation of Effect of Parameter $q$ on $U(h^b_t)$

The range of uncertainty about $x_2^*$ is larger when $q$ is larger. It is intuitive that unit variable cost information is more valuable.

\(^4\)This is the only subperiod affected by the information.

\(^5\)The graph depicts the band of uncertainty such that $b<r$. This is not necessary to the result.
As the demand function parameter $q$ becomes larger (smaller in absolute value), it dominates the producer's decision to a lesser extent, thus admitting a greater influence to unit variable cost.

This can also be demonstrated algebraically. Let $b^1$ and $b^2$ denote the lower and upper bounds, respectively, of the band of uncertainty about $b$. Equating marginal revenue to marginal cost for each of these values of $b$ yields the associated optimal actions, $x'_2$ and $x'_2$, where:

$$x'_2 = \frac{(r-b^i)(1-t)}{2(a-q)} \quad \text{and} \quad x'_2 < x'_2$$

The length of the range of uncertainty about $x'_2$ is given by:

$$x'_2 - x'_2 = \frac{(r-b^1)(1-t)}{2(a-q)} - \frac{(r-b^2)(1-t)}{2(a-q)}$$

$$= \frac{(1-t)(b^2-b^1)}{2(a-q)}$$

The band of uncertainty about $x'_2$ is increasing in $q$ since $x'_2 - x'_2 / \partial q > 0$.

**Corollary 6.1.2**: For $t<1$: (i) the marginal decrements in $U(h_t^b)$ w.r.t. $t$ and $a$ are strictly convex increasing in $q$, and (ii) the marginal increments in $U(h_t^b)$ w.r.t. $q$ are linearly (strictly convex) decreasing in $t$ ($a$).

**Proof**: Recalling the definition of $J[E(b),r]$ given in the proof of Corollary 6.2.1, and partially differentiating equation (6.4):

$$\frac{\partial^2 U(h_t^b)}{\partial q \partial t} = - \frac{J[E(b),r]}{4(a-q)^2} < 0; \quad \frac{\partial^3 U(h_t^b)}{\partial q^2 \partial t} = - \frac{J[E(b),r]}{2(a-q)^3} < 0; \quad \frac{\partial^3 U(h_t^b)}{\partial q \partial t^2} = 0$$

$$\frac{\partial^2 U(h_t^b)}{\partial q \partial a} = - \frac{(1-t)J[E(b),r]}{2(a-q)^3} < 0; \quad \frac{\partial^3 U(h_t^b)}{\partial q^2 \partial a} = - \frac{\partial^3 U(h_t^b)}{\partial q \partial a^2} = - \frac{3(1-t)J[E(b),r]}{2(a-q)^4} < 0$$

Q.E.D.
The sensitivity of the value of system \( h^b \) to information timeliness is increasing at an increasing rate in the parameter \( q \). As \( q \) becomes larger (smaller in absolute value), it dominates the producer's decision about \( x_2 \) to a lesser extent, thus admitting greater influence to other parameters of the decision problem, including unit variable cost. Since unit variable cost becomes more important to the decision, timeliness of information about that cost becomes more critical.

Similarly, the sensitivity of \( U(h^b) \) to the quadratic cost parameter, \( a \), is increasing at an increasing rate in the parameter \( q \). The intuitive theory of the firm approach to interpretation of this interaction is depicted in Figure 40.

Note that the decrease in the width of the band of uncertainty about \( x^*_2 \) associated with the change in \( a \) from \( a' \) to \( a'' \) (\( > a' \)) is greater for \( q=q' \) than for \( q=q'' \) (\( < q' \))

If the restriction \( P(r)=1 \) is again placed on \( p(b) \), the following result gives the value of perfect unit variable cost information received at time \( t \).

**Proposition 6-2:** For \( u(z)=z \) and \( P(r)=1 \),

\[
(6.5) \quad U(h^b_{\infty}, t) = \frac{(1-t)\operatorname{var}(b)}{4(a-q)}.
\]

**Proof:** The proof follows from Proposition 6-1. For \( P(r)=1 \), \( \hat{Y}=Y \) and \( E(b)<r \). Given the perfect information system, \( Y=S=b \), and \( E(b|y) = E(b|b) = b \). Substituting these results into equation (6.4) and simplifying, the desired result is obtained.\(^6\)

\[^6\text{Alternatively, replace } -a \text{ by } (q-a) \text{ in the proof of Proposition 3-5.}\]
Figure 40. Economic Interpretation of Interaction Between Demand Function Parameter $q$ and Quadratic Cost Parameter $a$
Corollary 6-2: For \( P(r)=1 \) and \( t<1 \): (i) the marginal increments in \( U(h_{\omega b}^{b}, t) \) w.r.t. \( \text{var}(b) \) are strictly convex increasing in \( q \), and (ii) the marginal increments in \( U(h_{\omega b}^{b}, t) \) w.r.t. \( q \) are linearly decreasing in \( \text{var}(b) \).

**Proof:** Partially differentiating equation (6.5):

\[
\frac{\partial^2 U(h_{\omega b}^{b}, t)}{\partial q \partial \text{var}(b)} = \frac{(1-t)}{4(a-q)^2} > 0; \quad \frac{\partial^3 U(h_{\omega b}^{b}, t)}{\partial q^2 \partial \text{var}(b)} = \frac{(1-t)}{2(a-q)^3} > 0; \quad \frac{\partial^4 U(h_{\omega b}^{b}, t)}{\partial q^3 \partial \text{var}(b)} = 0 \quad \text{Q.E.D.}
\]

The sensitivity of \( U(h_{\omega b}^{b}, t) \) to initial uncertainty is increasing at an increasing rate in the demand function parameter \( q \), and the sensitivity of \( U(h_{\omega b}^{b}, t) \) to \( q \) is constantly increasing in initial uncertainty. Some intuition may be provided for this phenomenon by appealing again to the theory of the firm approach to the producer's problem. In Figure 41 note that the increase in the width of the band of uncertainty about \( x^* \) associated with the increase in \( q \) from \( q'' \) to \( q' \) is greater for the wider band of uncertainty about \( b \) than for the narrower band.

### 6.2.2 Uncertainty About the Quadratic Cost Parameter

Suppose that the quadratic cost parameter is the only source of uncertainty to the producer, and that he employs system \( h_t^a \) which provides information at time \( t \) about \( a \).

**Proposition 6-3:** For \( u(z)=z \)

\[
(6.6) \quad U(h_t^a) = \begin{cases} \frac{(1-t)(r-b)^2}{4} \left\{ E\left[ \frac{1}{E(a|y)} - q \right] - \frac{1}{E(a)-q} \right\}, & \text{for } b\geq r \\ 0, & \text{otherwise} \end{cases}
\]

The proof of Proposition 6-3 proceeds in exactly the same way as that given for Proposition 5-1 which gives \( U(h_t^a) \) under perfect competition; simply replace -a by \( (q-a) \). Therefore, the proof will not be repeated here. The results given in Corollaries 5-1.1 and 5-1.2, regarding the role of \( r, b \) and \( t \) as determinants of \( U(h_t^a) \), remain valid here.
Figure 41. Economic Interpretation of Interaction Between Demand Function Parameter $q$ and Initial Uncertainty
This result holds because the partial derivatives of equation (6.6) with respect to those variables are the same as those for equation (5.4).

**Corollary 6-3.1:** For t<1 and b<r, U(h_t^a) is convex increasing in the demand function parameter (q).

**Proof:** Let K(i) denote \[ E\left\{ \frac{1}{\left[ E(a|y)\right] - q}\right\} \cdot \frac{1}{\left[ E(a) - q\right]} \].

Partially differentiating equation (6.6) for b<r:

\[ \frac{\partial U(h_t^a)}{\partial q} = \frac{(1-t)(r-b)^2}{4} K(2) > 0; \]

\[ \frac{\partial^2 U(h_t^a)}{\partial q^2} = \frac{(1-t)(r-b)^2}{2} K(3) > 0 \]  
Q.E.D.

The sensitivity of the value of system h_t^a to q is increasing in q. An intuitive economic interpretation of this result is facilitated by Figure 42. Note that the band of uncertainty about x^*_2 associated with a given band of uncertainty about a is greater for q=q' than for q=q'' (< q'). In Figure 42, a''>a'.

![Figure 42. Economic Interpretation of Effect of Demand Function Parameter q on U(h_t^a)](image)
Since the $x^*_2$ band of uncertainty is greater when $q$ is greater, it is intuitively appealing that the value of information about $a$ is higher. As $q$ increases (decreases in absolute value), it becomes less dominant in influencing the producer's decision and, therefore, admits greater importance to other parameters including the quadratic cost parameter.

**Corollary 6.3.2:** For $t<1$ and $b<r$: (i) the marginal increments (decrements) in $U(h_t^a)$ w.r.t. $r$ ($b$ or $t$) are convex increasing in $q$; (ii) the marginal increments in $U(h_t^a)$ w.r.t. $q$ are linearly decreasing in $t$ and convex increasing (decreasing) in $r$ ($b$).

**Proof:** Recalling the definition of $K(i)$, and partially differentiating equation (6.6) for $b<r$:

$$\frac{\partial^2 U(h_t^a)}{\partial q \partial t} = -\frac{(r-b)^2}{4}K(2)<0; \quad \frac{\partial^3 U(h_t^a)}{\partial q^3} = -\frac{(r-b)^2}{4}K(3)<0; \quad \frac{\partial^3 U(h_t^a)}{\partial q^2 \partial t^2} = 0$$

$$\frac{\partial^2 U(h_t^a)}{\partial q \partial r} = -\frac{\partial^2 U(h_t^a)}{\partial q \partial b} = \frac{(1-t)(r-b)}{2}K(2)>0; \quad \frac{\partial^3 U(h_t^a)}{\partial q^2 \partial r} = -\frac{\partial^3 U(h_t^a)}{\partial q^2 \partial b} = \frac{(1-t)(r-b)}{2}K(3)>0$$

$$\frac{\partial^3 U(h_t^a)}{\partial q \partial r^2} = \frac{\partial^3 U(h_t^a)}{\partial q \partial b^2} = \frac{(1-t)}{2}K(2)>0 \quad Q.E.D.$$

The sensitivity of $U(h_t^a)$ to information timeliness, unit variable cost, and unit price is increasing at an increasing rate in $q$. As $q$ decreases in absolute value, it becomes less critical in the producer's decision model relative to other parameters. Consequently, the importance of the other parameters, as determinants of information value, is increased.
6.3 Multiple Information Time Systems

6.3.1 Generalized Multiple Information Time System

The purpose of this section is to generalize the results in Section 4.3 concerning the system $h_t^b$ which provides unit variable cost information at time $t$. System $h_t^b$ provides information at a single point in time. The generalization developed here is from the single information time system to a multiple information time system. Since the analysis of a perfectly competitive firm can be viewed as a special case of that for the monopolist by letting $q=0$, the analysis in this section adopts the assumption of monopoly.

Let $h_t^b_L$ denote a system which provides unit variable cost information at each of the times $t_i$, ($t_i \in [0,1]$ and $t_i < t_{i+1}$, $i = 1, 2, \ldots, n$). $\mathbf{t}$ denotes the information time vector $(t_1, t_2, \ldots, t_n)$. Suppose that the producer can costlessly alter the production rate any number of times and at any time during the production period. Relying on equations (3.1) and (6.2), the producer's new outcome (profit) function is given by:

$$(6.7) \quad w(x, b) = (q-a) \sum_{j=1}^{n+1} \frac{x_j^2}{(t_j - t_{j-1})} + (r-b) \sum_{j=1}^{n+1} x_j \cdot c, \quad \text{where} \quad t_0 = 0 \quad \text{and} \quad t_{n+1} = 1$$

$x_i$ denotes the production quantity in the production subperiod $[t_{i-1}, t_i]$, $x$ denotes the production vector $(x_1, x_2, \ldots, x_{n+1})$, and $x_i/(t_i - t_{i-1})$ denotes the production rate in the subperiod $[t_{i-1}, t_i]$.

**Proposition 6-4:** For $u(z) = z$ and $P(r) = 1$,

$$(6.8) \quad U(h_t^b_L) = \frac{1}{4(a-q)[1-t_1]} \left[ \text{var}(b) - \mathbb{E}_{y_1} \text{var}(b | y_1) \right]$$

$$+ \sum_{j=2}^{n} \frac{1}{(t_{j-1} - t_1)} \left[ \text{var}(b | y_{j-2}) - \mathbb{E}_{y_{j-1}} \text{var}(b | y_{j-1}) \right] \mathbb{E}_{y_1} \text{var}(b | y_1)$$
where $y_i$ denotes the signal received at time $t_i$, $y_i$ denotes the signal vector $(y_1, y_2, \ldots, y_i)$, $\text{var}(b|y_i)$ denotes the variance of $p(b|y_i)$, and $E \cdot E [\text{var}(b|y_i)]$ denotes the following iterative expectation:

$$E \cdot E \cdot E \cdot E [\text{var}(b|y_i)] = E \cdot E [\text{var}(b|y_i)] = E [\text{var}(b|y_i)]$$

$E_j$ denotes the expectation operation w.r.t. $p(y_j)$.

**Proof:** The producer's maximal expected utility (same as expected profit for $u(z)=z$) given information system $h_t^b$ is:

$$(6.9) \ E[w(x_s,b)|h_t^b] = \max \{ \int_0^t \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^2} \int_{\mathbb{R}} p(b|y_n) p(y_n|x_{n-1}) dy_n \cdots p(y_2|y_1) dy_2 p(y_1) dy_1 \}$$

Taking the expectations as indicated, and noting the linear separability of $w(x_s)$ in the $x_i$'s, (6.9) simplifies to:

$$(6.10) \ E[w(x_s,b)|h_t^b] = \max \{ \int_0^t \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^2} \int_{\mathbb{R}} p(b|y_n) p(y_n|x_{n-1}) dy_n \cdots p(y_2|y_1) dy_2 p(y_1) dy_1 \}$$

Maximization of (6.10) yields the decision rule

$$x_i^* = \frac{[r-E(b|x_{i-1})]}{2(a-q)} \text{ for } i \in \{2, 3, \ldots, n+1\}$$

and

$$x_1^* = \frac{[r-E(b)]}{2(a-q)} \text{. Substitution of this decision rule into (6.10) and}$$
simplification yields the following:

$$\text{(6.11) } E[w(x^*,b)|h_t^b] = t_1 \frac{[r-E(b)]^2}{4(a-q)}$$

$$+ \sum_{i=2}^{n+1} E \left\{ E \left\{ E \left\{ E \left\{ \frac{(t_i-t_{i-1})^2 [r-E(b)|y_{i-2}]^2}{4(a-q)} \right\} |y_{i-2}|y_{i-3} \right\} \right\} |y_{i-3} \right\} | \cdots |y_1 \right\} - c$$

$$\text{(6.12) } E[w(x^*,b)|h_t^b] = \left\{ r^2 - 2rE(b) + t_1 E^2(b) \right\}$$

$$+ \sum_{i=2}^{n+1} E \left\{ E \left\{ E \left\{ E \left\{ \frac{(t_i-t_{i-1})^2 E^2(b|y_{i-1})|y_{i-2}|y_{i-3} \cdots |y_1)}{4(a-q)} \right\} \right\} \right\} / (4(a-q)) - c$$

The producer's maximal expected utility with no information, given previously by equation (3.15)\textsuperscript{7}, is $$E[w(x^*,b)|h_0] = \left\{ \frac{r-E(b)}{2}\right\}/(4(a-q)) - c$$.

Subtracting $$E[w(x^*,b)|h_0]$$ from $$E[w(x^*,b)|h_t^b]$$, noting that

$$\text{var}(b|y_i) = E(b^2|y_i) - E^2(b|y_i),$$

and simplifying yields the following:

$$\text{(6.13) } U(h_t^b) = \left\{ -t_1 \text{var}(b) + \text{var}(b) \right\}$$

$$- \sum_{i=2}^{n+1} E \left\{ E \left\{ E \left\{ E \left\{ \frac{(t_i-t_{i-1}) \text{var}(b|y_{i-1})|y_{i-2}|y_{i-3} \cdots |y_1)}{4(a-q)} \right\} \right\} \right\} \right\} / (4(a-q))$$

This can be rewritten in the form given by equation (6.8). \text{Q.E.D.}

If q is equal to 0 and t is equal to t, expression (6.8) for $$U(h_t^b)$$ collapses to that given by Proposition 4-1 for the single information time system (given perfect competition and the restriction that $$P(r)=1$$).

Note from Proposition 6-4 that $$E_{y_1} \cdots E_{y_{i-1}} \text{var}(b|y_{i-1})$$ is the expected posterior variance of the conditional pdf on b, given the first i-1 signals from information system $$h_t^b$$. Hence $$E_{y_1} \cdots E_{y_{i-1}} \text{var}(b|y_{i-1})$$ represents the expected reduction in the posterior variance.

\textsuperscript{7}Replace a in (3.15) by (a-q) to reflect the monopoly assumption in the current analysis.
variance which results from the receipt of signal $y_i$ at time $t_i$. This expected reduction in the posterior variance is then multiplied by $(1-t_i)$, which is the period of time during which the producer may operate with the lower degree of uncertainty about $b$ (as measured by the posterior variance). The value of information system $h_t^b$ is the summation of the incremental reductions in the expected posterior uncertainty about $b$, each multiplied by the length of production time remaining after the expected reduction occurs.

The role of parameters $a$ and $q$ is the same for system $h_t^b$ as for system $h_t^b$ and will not be discussed further. The role of the information time vector is analogous to that of $t$ in determining $U(h_t^b)$; it is given by the following Corollary.

**Corollary 6.4:** For $P(r)=1$, (i) $U(h_t^b)$ is linearly decreasing in $t_i$ where $t_i < t_{i+1}$, $i \in \{1, 2, \ldots, n\}$, and $t_{n+1}=1$; (ii) the marginal decrements in $U(h_t^b)$ w.r.t. $t_i$ are invariant w.r.t. $t_j$, $j \neq i$.

**Proof:** Partially differentiating equation (6.8):

$$\frac{\partial U(h_t^b)}{\partial t_i} = \begin{cases} -(\text{var}(b) \cdot E [\text{var}(b|y_1)]) < 0 \\ -E \cdots E [\text{var}(b|y_{i-1}, y_1)] - E \cdots E [\text{var}(b|y_i)] < 0 \end{cases}$$

$$\frac{\partial^2 U(h_t^b)}{\partial t_j \partial t_i} = 0 \text{ for all } i, j \in \{1, 2, \ldots, n\} \quad \text{Q.E.D.}$$

The value of the multiple information time system decreases at a constant rate with a delay in the $i$th signal. The economic interpretation is analogous to that given in section 3.4.1 following Corollary 3-2.1 regarding information system $h_t^b$. There is no interaction effect on $U(h_t^b)$ from the timeliness of different signals. Therefore, the effect
on $U(h^b_t)$ of a delay in the $i$th signal will be unaffected by delays in other signals. The role of the interaction between the $i$th signal time and the parameters $a$ and $q$ in affecting $U(h^b_t)$ is analogous to that given previously for the interaction of $t$ with $a$ and $q$ in affecting $U(h^b_t)$.

(See sections 3.4.1 and 6.2.)

Recall the system $h^b_{\omega,t}$ which provides perfect unit variable cost information at time $t$. Its value given by Proposition 6-2 is $U(h^b_{\omega,t}) = (1-t)\text{var}(b)/4(a-q)$. Note that system $h^b_{\omega,0}$ is more valuable than system $h^b_{t}$, assuming that the latter is not degenerate in the sense that it too provides perfect information at time 0.\(^8\) Also, $h^b_{\omega,1}$ is less valuable than system $h^b_{t}$, assuming that the latter is not degenerate in the sense that it provides no information prior to time 1. This can be seen by rewriting expression (6.8) for $U(h^b_t)$ as follows:

\begin{equation}
(6.14) \quad U(h^b_{\omega,0}) = \frac{\text{var}(b)}{4(a-q)}
\end{equation}

\begin{equation}
zU(h^b_t) = \{t^0 \text{var}(b) - \sum_{i=2}^{n+1} (t^i - t_{i-1}) \{E \cdots E \text{var}(b|y_{i-1}) \}} + \text{var}(b)\}/4(a-q)
\end{equation}

\begin{equation}
z = 0 = U(h^b_{\omega,1})
\end{equation}

Since $U(h^b_{\omega,t})$ is continuous in $t$, there exists a system $h^b_{\omega,t^0}$, with $t^0 \epsilon (0,1)$, the value of which is equivalent to $U(h^b_t)$. The value of any multiple information time system providing information about unit variable cost can be equalled by an "equivalent" delayed single time perfect information system.\(^9\)

\(^8\) This case is not excluded by the general definition of $h^b_t$.

6.3.2 Imperfect Multiple Information Time System Anchored With Perfect Information

If system $h_{\infty, t}^b$ is such that perfect information about unit variable cost is provided at the last signal time in the period, $t_n$, the system's value is given by:

\[
U(h_{\infty, t}^b) = \frac{1}{4(a-q)} \left\{ (1-t_1) \text{var}(b) - \text{E} \left\{ \text{var}(b | y_1) \right\} \right\} \\
+ \sum_{i=2}^{n-1} (1-t_i) \text{E} \left\{ \text{var}(b | y_{i-1}) \right\} - \text{E} \left\{ \text{var}(b | y_i) \right\} \\
+ (1-t_n) \text{E} \left\{ \text{var}(b | y_{n-1}) \right\}
\]

where $h_{\infty, t}^b$ denotes the described system. Note that $U(h_{\infty, t}^b) > U(h_{\infty, t_n}^b)$, where the latter is the system which provides perfect information at the single point in time $t_n$. This can be seen by recalling that $U(h_{\infty, t_n}^b) = (1-t_n) \text{var}(b)/4(a-q)$ and rearranging terms in equation (6.15) as follows:

\[
U(h_{\infty, t}^b) = \frac{1}{4(a-q)} \left\{ (1-t_n) \text{var}(b) + (t_n-t_1) \text{var}(b) - \text{E} \left\{ \text{var}(b | y_1) \right\} \right\} \\
+ \sum_{i=2}^{n-1} (t_n-t_i) \text{E} \left\{ \text{var}(b | y_{i-1}) \right\} - \text{E} \left\{ \text{var}(b | y_i) \right\} \\
+ \frac{1}{4(a-q)} (1-t_n) \text{var}(b) = U(h_{\infty, t_n}^b)
\]

System $h_{\infty, t}^b$ is preferable to system $h_{\infty, t_n}^b$ even though each provides perfect variable cost information at $t_n$. This results from the early information, though imperfect, which the producer receives from system $h_{\infty, t}^b$ at times $t_1$ through $t_{n-1}$ but not from system $h_{\infty, t_n}^b$. 
6.3.3 Noisy Information About Unit Variable Cost: General Accuracy Improvement Parameter

This section provides a generalization of the results given in Section 4.3 for information system $h^b_{t_1, \delta}$. The range of $t_i$ for system $h^b_{t_1, \delta}$ is $(t_{i-1}, t_{i+1})$, where $t_0=0$ and $t_1=1$. By Corollary 6-4, $U(h^b_{t_1, \delta})$ is declining in $t_i$. Early information is preferable to late information, ceteris paribus. However, a characteristic of many information systems is that the accuracy improves with time. Suppose the expected posterior variance of $b$ after receipt of the $i$th signal from $h^b_{t_1, \delta}$ declines over the time interval $[t_{i-1}, t_{i+1}]$ according to (6.17) below:

\[
(6.17) \begin{align*}
&\left(\frac{E \cdots E}{y_1} \left[ \frac{\text{var}(b|y)}{y_i} \right] \right) = (1-t^5) \text{var}(b), \quad \delta > 0 \\
&\Rightarrow (E \cdots E) \left[ \frac{\text{var}(b|y)}{y_1} \right] \leq (E \cdots E) \left[ \frac{\text{var}(b|y_{i+1})}{y_1} \right], \quad (E \cdots E) \left[ \frac{\text{var}(b|y)}{y_{i-1}} \right]
\end{align*}
\]

Let $h^b_{t_1, \delta}$ denote the information system which behaves in this fashion. As $t_i$ approaches $t_{i-1}$, $(E \cdots E) \left[ \frac{\text{var}(b|y_i)}{y_i} \right]$ approaches its infimum given in (6.18), and as $t_i$ approaches $t_{i+1}$, $(E \cdots E) \left[ \frac{\text{var}(b|y_i)}{y_i} \right]$ approaches its supremum. $\delta$ is interpreted as an accuracy responsiveness parameter in the same manner that was discussed in Section 4.3.

**Property 6.1**: $\left(\frac{E \cdots E}{y_1} \left[ \frac{\text{var}(b|y)}{y_i} \right] \right)$ is \(\text{strictly concave}\) decreasing in $t_i$ as $\delta \leq 1$ for $t_i \in (t_{i-1}, t_{i+1})$.

**Proof**: Partially differentiating equation (6.18):

\[
\frac{\partial}{\partial t_i} \left(\frac{E \cdots E}{y_1} \left[ \frac{\text{var}(b|y)}{y_i} \right] \right) = -\delta t_i^{\delta-1} \text{var}(b) < 0
\]

\[
\frac{\partial^2}{\partial t_i^2} \left(\frac{E \cdots E}{y_1} \left[ \frac{\text{var}(b|y)}{y_i} \right] \right) = -\delta(\delta-1)t_i^{\delta-2} \leq 0 \text{ as } \delta \leq 1
\]

Q.E.D.
Property 6-2: \( \{E \cdots E \{\text{var}(b|Y_i)\}\} \) is strictly concave increasing in \( \delta \) for a given \( t_i \in (t_{i-1}, t_{i+1}) \) and \( t_i \neq (0,1) \).

Proof: Partially differentiating equation (6.18):

\[
\frac{\partial}{\partial \delta} E \cdots E \{\text{var}(b|Y_i)\} = -t_i \ln(t_i) \text{var}(b) > 0 \text{ since } t_i \in (0,1)
\]

\[
\frac{\partial^2}{\partial \delta^2} E \cdots E \{\text{var}(b|Y_i)\} = -t_i^2 \ln(t_i)^2 \text{var}(b) < 0 \quad \text{Q.E.D.}
\]

\[\{E \cdots E \{\text{var}(b|Y_i)\}\}\]

\[\{E \cdots E \{\text{var}(b|Y_i)\}\}\]

\(0\)

\(t_{i-1}\)

\(t_i\)

\(t_{i+1}\)

\(\delta > 1\)

\(\delta = 1\)

\(\delta < 1\)

\(\text{var}(b)\)

\(\text{var}(b)\)

Figure 43. Expected Posterior Variance After Receipt of ith Signal As a Function of ith Signal Time (A) and Accuracy Responsiveness (B)

The "accuracy" of information system \( h_b \) was characterized by \( \{\text{var}(b) - E[\text{var}(b|Y)]\} \), i.e., the expected reduction in the variance of \( b \).

The "accuracy" of system \( h_b \) can be described by the accuracy vector

\[ (\text{var}(b) - E[\text{var}(b|Y_1)], \text{var}(b) - E[\text{var}(b|Y_2)], \ldots, \text{var}(b) - E[\text{var}(b|Y_n)]) \]

Figure 44A shows the relationship between the ith component of the
system accuracy vector and the time of the ith signal. Figure 44B shows that accuracy component's functional dependence on $\delta$.

$$\text{var}(b) \cdot \{E \cdot \text{var}(b|y_i)\}$$

Figure 44. ith Component of Accuracy Vector as a Function of ith Signal Time (A) and Accuracy Responsiveness (B)

$h_{t_i, \delta}^b$ is an information system which provides $n$ signals at times $t_i$ through $t_n$. The "accuracy" of signal $i$ improves as it is delayed in its range $(t_{i-1}, t_{i+1})$. The accuracy responsiveness parameter ($\delta$) determines the rate at which this improvement occurs. For $\delta$ less (greater) than 1 the accuracy improves at a declining (increasing) rate. The smaller $\delta$ is, the more "accurate" is the information provided at time $t_i$ for any $t_i \in (t_{i-1}, t_{i+1})$, $i=1, 2, \ldots, n$. Further interpretive remarks could be made concerning these issues, but they parallel those given in Section 4.3 for the system $h_{t, \delta}^b$. 
Proposition 6-5: For $u(z) = z$ and $P(r) = 1$,

\[(6.19) \ U_{h_{t, \delta}}^b = \frac{\text{var}(b)}{4(a-q)} \left( t_{n}^{\delta} \sum_{i=1}^{n} t_{i}^{\delta+1} \sum_{i=2}^{n} t_{i-1}^{\delta} \right). \]

Proof: The proof follows directly from Proposition 6-4 by substituting from equation (6.17) into equation (6.8) to get the following:

\[(6.20) \ U_{h_{t, \delta}}^b = \frac{1}{4(a-q)} \left( 1-t_1 \right) \left\{ \text{var}(b) - (1-t_1^{\delta}) \text{var}(b) \right\} \]

\[+ \sum_{i=2}^{n} (1-t_{i-1}^{\delta}) \left\{ (1-t_1^{\delta}) \text{var}(b) - (1-t_{i-1}^{\delta}) \text{var}(b) \right\} \]

Rearranging terms in (6.20) yields the desired result. Q.E.D.

By rearranging terms in equation (6.19), $U_{h_{t, \delta}}^b$ may be expressed as follows:

\[(6.21) \ U_{h_{t, \delta}}^b = \frac{\text{var}(b)}{4(a-q)} \left( t_{n}^{\delta} \sum_{i=1}^{n} t_{i}^{\delta} \sum_{i=2}^{n} t_{i-1}^{\delta} (t_{i-1} - t_{i-1}) \right) \geq 0 \]

Hence the nonnegativity of $U_{h_{t, \delta}}^b$ is clearly established. The roles of $a$, $\text{var}(b)$, $\delta$ and $q$, as determinants of $U_{h_{t, \delta}}^b$, are analogous to their roles in determining $U_{h_{t, \delta}}^b$. These determinants were dealt with in Corollaries 3-2.1, 4-2.1, 4-3.1 and 6-1.1, respectively, and no new insights can be provided here.

Corollary 6-5.1. For $P(r) = 1$ and $t \in (0,1)$: (i) $U_{h_{t, \delta}}^b$ is \{psuedoconcave strictly concave\} in the $i$th signal time, $t_i$, for all $i$, as $\delta \geq 1$; (ii) the marginal increments (decrements) in $U_{h_{t, \delta}}^b$ w.r.t. $t_i$ are linearly increasing (decreasing) in the $(i+1)$st signal time, $t_{i+1}$, $i \in \{1, \ldots, n-1\}$; (iii) the marginal increments (decrements) in $U_{h_{t, \delta}}^b$ w.r.t. $t_i$ are \{convex \{linear\} concave\} increasing (decreasing) in the $(i-1)$st signal time, $t_{i-1}$, $i \in \{2, 3, \ldots, n\}$ as $\delta \geq 1$; (iv) the marginal increments or decrements in $U_{h_{t, \delta}}^b$ w.r.t. $t_i$
are invariant with respect to the jth signal time, \( t_j \), for \( j \in \{i-1, i, i+1\} \), \( j \in \{1, 2, \ldots, n\} \).

**Proof:** Rearranging terms in equation (6.19):

\[
(6.22) \quad U(h_{t, \delta}^b) = \frac{\text{var}(b)}{4(a-q)} \left\{ \sum_{i=1}^{n+1} t_i t_{i-1}^{\delta-1} - \sum_{i=1}^{n} t_i^{\delta+1} \right\}
\]

where \( t_0 = 0 \) and \( t_{n+1} = 1 \).

(i) For \( i \in \{1, 2, \ldots, n\} \), partially differentiating (6.22):

\[
\frac{\partial U(h_{t, \delta}^b)}{\partial t_i} = \frac{\text{var}(b)}{4(a-q)} \left\{ t_i^{\delta-1} + \delta t_{i+1} - (\delta+1) t_i^\delta \right\}
\]

\[
\frac{\partial^2 U(h_{t, \delta}^b)}{\partial t_i^2} = \frac{\text{var}(b)}{4(a-q)} \delta t_i^{\delta-2} - ((\delta-1)t_{i+1} - (\delta+1)t_i)^{\delta/0} \text{ as } t_i < \frac{(\delta-1)}{(\delta+1)} t_{i+1} \text{ for } \delta > 1
\]

and < 0 for \( \delta \leq 1 \).

For \( \delta \leq 1 \), the second derivative is negative throughout the domain of \( U(h_{t, \delta}^b) \) w.r.t. \( t_i \). Hence the function is strictly concave.

For \( \delta > 1 \), \( U(h_{t, \delta}^b) \) is strictly convex w.r.t. \( t_i \), for \( t_i < [(\delta-1)/(\delta+1)] t_{i+1} \); \( U(h_{t, \delta}^b) \) is strictly concave w.r.t. \( t_i \), for \( t_i > [(\delta-1)/(\delta+1)] t_{i+1} \). Evaluating \( \frac{\partial U(h_{t, \delta}^b)}{\partial t_i} \) at \( t_i = [(\delta-1)/(\delta+1)] t_{i+1} \), \( i \in \{1, 2, \ldots, n\} \), yields the following:

\[
\frac{\partial U(h_{t, \delta}^b)}{\partial t_i} \bigg|_{t_i = \frac{(\delta-1)}{(\delta+1)} t_{i+1}} = \frac{\text{var}(b)}{4(a-q)} \left\{ \delta-1 \right\} t_{i+1}^{\delta-1} - (\delta+1) t_i^\delta \left[ t_{i+1}^\delta + t_i^\delta \right]
\]

\[
\frac{\text{var}(b)}{4(a-q)} \left\{ \frac{(\delta-1)^{\delta-1}}{(\delta+1)} t_{i+1}^\delta + t_i^\delta \right\} > 0
\]

Evaluating \( \frac{\partial U(h_{t, \delta}^b)}{\partial t_i} \) at \( t_i = t_{i-1} \) and \( t_i = t_{i+1} \), \( i \in \{1, 2, \ldots, n\} \), yields the following:

\[
\frac{\partial U(h_{t, \delta}^b)}{\partial t_i} \bigg|_{t_i = t_{i-1}} = \frac{\text{var}(b)}{4(a-q)} \left\{ \delta t_i^{\delta-1} (t_{i+1}^\delta - t_i^\delta) \right\} > 0; \quad \frac{\partial U(h_{t, \delta}^b)}{\partial t_i} \bigg|_{t_i = t_{i+1}} = \frac{\text{var}(b)}{4(a-q)} \left\{ \delta t_i^{\delta-1} (-t_i^\delta) \right\} < 0
\]
Since \( \partial U(h_{t_i, \delta}^b) / \partial t_i \) is positive at both \( t_{i-1} \) and \( ((\delta-1)/(\delta+1))t_{i+1} \), \( U(h_{t_i, \delta}^b) \) is increasing on its entire convex region, 
\((t_{i-1}, (\delta-1)/(\delta+1))t_{i+1}\). Since \( U(h_{t_i, \delta}^b) \) is concave throughout the remainder of its domain w.r.t. \( t_i \), and since \( \partial U(h_{t_i, \delta}^b) / \partial t_i \) is negative at the upper end of its domain (i.e., \( t_i=t+1 \)), \( U(h_{t_i, \delta}^b) \) is pseudoconcave. 

10

Partially differentiating equation (6.22) for \( i \in \{1, 2, \ldots, n\} \):

\[
(ii) \quad \frac{\partial^2 U(h_{t_i, \delta}^b)}{\partial t_{i+1} \partial t_i} = \frac{\partial U(h_{t_i, \delta}^b)}{\partial t_i} \cdot \frac{\partial}{\partial t_{i+1}} \frac{\partial U(h_{t_i, \delta}^b)}{\partial t_i} = 0
\]

\[
(iii) \quad \frac{\partial^2 U(h_{t_i, \delta}^b)}{\partial t_{i-1} \partial t_i} = \frac{\partial U(h_{t_i, \delta}^b)}{\partial t_i} \cdot \frac{\partial}{\partial t_{i-1}} \frac{\partial U(h_{t_i, \delta}^b)}{\partial t_i} = 0 \quad \text{as} \quad \delta \leq 1
\]

\[
(iv) \quad \frac{\partial^2 U(h_{t_i, \delta}^b)}{\partial t_j \partial t_i} = 0 \quad \text{for} \quad j \neq \{i-1, i, i+1\} \quad \text{Q.E.D.}
\]

Note from the proof of Corollary 6-5.1 that \( \partial^2 U(h_{t_i, \delta}^b) / \partial t_i \) changes sign at \( ((\delta-1)/(\delta+1))t_{i+1} \) if \( \delta > 1 \), indicating an inflection point. \( U(h_{t_i, \delta}^b) \) is increasing in \( t_i \) since \( \partial U(h_{t_i, \delta}^b) / \partial t_i \) evaluated at \( ((\delta-1)/(\delta+1))t_{i+1} \) is positive:

\[
(6.23) \quad \frac{\partial U(h_{t_i, \delta}^b)}{\partial t_i} = \frac{((\delta-1)/(\delta+1))t_{i+1} - \delta t_{i-1}}{t_{i+1} - t_{i-1}} > 0
\]

Moreover, \( U(h_{t_i, \delta}^b) \) at \( t_i=t_i+1 \) is \( U(h_{t_i, \delta}^b) \) at \( t_i=t_i+1 \).

This can be seen by substituting \( t_i=t_i+1 \) and \( t_i=t_i+1 \) into equation (6.19) and rearranging terms. Using these relationships, Figure 45 depicts the functional dependence of \( U(h_{t_i, \delta}^b) \) on \( t_i \).

---

Figure 45. $U(h_{t, \delta}^b)$ as a Function of the $i$th Signal Time$^*$

$^*$The graph assumes $t_i \notin (t_1, t_n)$.

Note that the optimal time for the $i$th signal from system $h_{t, \delta}^b$ is always interior to its range $[t_{i-1}, t_{i+1}]$, because of the concavity or pseudoconcavity of $U(h_{t, \delta}^b)$ in $t_i$. This phenomenon is a result of the trade-off nature of the timing of the $i$th signal. There is an incentive for $t_i$ to be early (i.e., near $t_{i-1}$) so that the producer will have a longer time to operate after receiving the information. However, there is also an incentive to delay the $i$th signal toward its upper bound, $t_{i+1}$, in order to take advantage of the increased accuracy of the delayed information. The result of these conflicting incentives is the pseudoconcavity or concavity shown in Figure 45 and the interior optimal time for the $i$th signal. Other remarks could be made regarding the implications of the pseudoconcavity (or concavity) property, but these parallel those given in Section 4.3 for system $h_{t, \delta}^b$. 

\[ \]
Let \( t_i^{\text{ni}} \) denote the vector of optimal information times, 
\( (t_1^{\text{ni}}, ..., t_n^{\text{ni}}) \), given \( n \) signals. A general expression for \( t_i^{\text{ni}} \), as a function of \( n \) and \( \delta \), cannot be developed. However, given a specific 
value for \( n \), \( t_i^{\text{ni}} \) can be found by dynamic programming. Corollary 6.5.2, 
which is proven in Appendix D, gives \( t_i^{\text{ni}} \) for \( n \in \{1, 2, 3\} \).

Corollary 6.5.2: For \( P(r)=1, u(z)=z \), and \( h=b^b_{1, \delta} \).

\[
\begin{align*}
\overline{t}_{1}^{\text{ni}} & = \left( \frac{\delta}{\delta + 1} \right) \\
\overline{t}_{2}^{\text{ni}} & = \left( \frac{\delta^2 (\delta + 1)^{\delta - 1} \delta (\delta + 1)^{\delta}}{(\delta + 1)^{\delta + 1 - \delta} (\delta + 1)^{\delta + 1 - \delta}} \right) \\
\overline{t}_{3}^{\text{ni}} & = \left( \frac{\delta^3 (\delta + 1)^{\delta - 1} (\delta + 1)^{\delta + 1 - \delta} \delta^{\delta - 1}}{(\delta + 1)^{\delta + 1 - \delta} (\delta + 1)^{\delta + 1 - \delta} \delta (\delta + 1)^{\delta^2}} \right)
\end{align*}
\]

Table 2 gives the values of \( t_i^{\text{ni}} \), \( n \in \{1, 2, 3\} \), for several values of \( \delta \).

**Table 2.** \( t_i^{\text{ni}} \), \( t_i^{\text{ni}} \), and \( t_i^{\text{ni}} \) for Selected Values of \( \delta \)

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( \overline{t}_{1}^{\text{ni}} )</th>
<th>( \overline{t}_{2}^{\text{ni}} )</th>
<th>( \overline{t}_{3}^{\text{ni}} )</th>
<th>( \overline{t}_{1}^{\text{ni}} )</th>
<th>( \overline{t}_{2}^{\text{ni}} )</th>
<th>( \overline{t}_{3}^{\text{ni}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>.200</td>
<td>.086</td>
<td>.430</td>
<td>.049</td>
<td>.244</td>
<td>.568</td>
</tr>
<tr>
<td>1/2</td>
<td>.333</td>
<td>.181</td>
<td>.542</td>
<td>.118</td>
<td>.355</td>
<td>.655</td>
</tr>
<tr>
<td>3/4</td>
<td>.429</td>
<td>.263</td>
<td>.615</td>
<td>.187</td>
<td>.437</td>
<td>.710</td>
</tr>
<tr>
<td>1</td>
<td>.500</td>
<td>.333</td>
<td>.667</td>
<td>.250</td>
<td>.500</td>
<td>.750</td>
</tr>
<tr>
<td>2</td>
<td>.667</td>
<td>.522</td>
<td>.783</td>
<td>.437</td>
<td>.656</td>
<td>.838</td>
</tr>
<tr>
<td>3</td>
<td>.750</td>
<td>.629</td>
<td>.838</td>
<td>.553</td>
<td>.737</td>
<td>.880</td>
</tr>
<tr>
<td>4</td>
<td>.800</td>
<td>.697</td>
<td>.871</td>
<td>.630</td>
<td>.788</td>
<td>.904</td>
</tr>
</tbody>
</table>
Note from Table 2 that $t_i^n$ is increasing in $\delta$ for $n \in \{1, 2, 3\}$. As the improvement in the accuracy vector of system $h_{t, \delta}^b$ becomes less responsive to time, the optimal time for each signal provided by the system becomes later. For $\delta$ small, the accuracy vector of system $h_{t, \delta}^b$ attains a relatively high degree of accuracy at an early time. The benefit from obtaining the ith signal relatively early in $(t_{i-1}, t_{i+1})$, and basing upon it the production rate for the period $(t_i, t_{i+1})$, outweighs the benefit of waiting for further increases in the ith signal's accuracy. For $\delta$ large, the optimal time for the ith signal from system $h_{t, \delta}^b$ becomes later.

Let $t_{n0} = (\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1})$ denote the "naive" signal vector for system $h_{t, \delta}^b$, given $n$ signals. In this timing vector, the $n$ information times are equally spaced throughout the production period with no signal at times 0 or 1. One suspects that this strategy may be the one actually used in many information system designs. $t_{n0}$ may be compared to $t_{n*}$ for $n \in \{1, 2, 3\}$ (given in Table 2 for various values of $\delta$). Only when $\delta = 1$ is the naive timing vector equal to the optimal timing vector. As $\delta$ deviates more substantially from 1, the naive and optimal timing strategies deviate to a greater extent. Since $U(h_{t, \delta}^b)$ is pseudoconcave or concave in $t_i$, the greater the deviation between $t_i$ and $t_i^*$, given $t_i$, the greater the loss in information system value.

6.3.4 Noisy Information About Unit Variable Cost: Linear Accuracy Improvement

Suppose now that $\delta$ is equal to 1, implying a constant marginal increase in the accuracy of the ith signal over its range, $(t_{i-1}, t_{i+1})$.

Proposition 6-6: For $P(r) = 1$ and $u(z) = z$: 
\[(6.24) \quad U(h_{\xi,1}^b) = \frac{\text{var}(b)^{n+1}}{4(a-q)} \left\{ \sum_{i=1}^{n} t_i t_{i-1} - \sum_{i=1}^{n} t_i^2 \right\}
\]

and \[ t_1^{n-1} = \frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1} = t_0 \]

\[(6.25) \quad U(h_{\xi,1}^b) = \frac{\text{var}(b)n}{8(a-q)(n+1)}
\]

**Proof:** The proof of (6.24) is immediate from Proposition 6.5 by letting \( \delta = 1 \). To obtain \([\text{var}(b) \delta = 1]\), it must first be shown that \( U(h_{\xi,1}^b) \) is strictly concave in \( \xi \). The ith element of the gradient vector of \( U(h_{\xi,1}^b) \) is
\[
\frac{\partial U(h_{\xi,1}^b)}{\partial t_i} = \text{var}(b) \left[ t_{i-1} t_{i+1} - 2 t_i \right]/4(a-q). \]

The Hessian matrix is given by:
\[
\begin{bmatrix}
-2 & 1 \\
1 & -2 & 1 \\
& 1 & -2 & \\
& & 1 & -2 \\
& & & \ddots & \ddots \\
& & & & 1 & -2 \\
& & & & & 1 & \\
1 & & & & & & -2 \\
\end{bmatrix}
\]

Pre- and postmultiplying the matrix by the arbitrary \( n \) vector, \([a_1, \ldots, a_n]\), yields the following:

\[(6.26) [a_1, \ldots, a_n] \times \begin{bmatrix}
-2 & 1 \\
1 & -2 & 1 \\
& 1 & -2 \\
& & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & 1 & -2 \\
1 & & & & & & -2 \\
\end{bmatrix} \times [a_1, \ldots, a_n] = -a_i^2 \left( \sum_{i=1}^{n-1} (a_i - a_{i+1})^2 - a_n^2 \right) < 0
\]
Thus the Hessian matrix is negative definite, and $U(h_{t,1}^{b})$ is strictly concave.

To solve the unconstrained problem max $\var{\sum_{i=1}^{n} t_{i} t_{i-1} - \sum_{i=1}^{n} t_{i}}^{2}/4(a-q)$, set the gradient vector equal to zero. This yields $t_{i}^{n*} = i/(n+1)$, $i \in \{1, \ldots, n\}$, which satisfies the constraint that $0 \leq t_{i-1} < t_{i} < t_{i+1} \leq 1$.

Thus $t_{i}^{n*}$ is the constrained optimum as well.

Substitution of $t_{i}^{n*}$ into (6.24) and simplification yields

$$U(h_{t,n*}^{b},1) = \var(b)n/(8(a-q)(n+1)).$$

Q.E.D.

The optimal timing vector for information system $h_{t,1}^{b}$ is the same as the naive timing vector, $t_{n}^{N0}$. When the system exhibits constant marginal improvement in each accuracy vector component over the range of the associated signal time, a constant signal time interval is best.

Note that as the number of information times gets large the value of system $h_{t,n*}^{b},1$ approaches that of perfect information at the single time, $t=\frac{1}{2}$.

$$\frac{\var(b)n}{8(a-q)(n+1)} = \frac{\var(b)}{4(a-q)} = U(h_{t,1}^{b}).$$

(See Proposition 4-2 and replace a by (a-q) to reflect the monopoly assumption.)

System $h_{t,1}^{b}$ is the "equivalent" delayed, single-time, perfect information system for $h_{t,n*}^{b},1$.

Corollary 6-6.1: $U(h_{t,n*}^{b},1)$ is increasing in the number of information points, $n$, by decreasing marginal increments.

Proof: From Proposition 6-6:

\[ \Delta U(h_{t,n*}^{b},1) = U(h_{t,(n+1)*}^{b},1) - U(h_{t,n*}^{b},1) = \frac{\var(b)n+1}{8(a-q)} \frac{n}{n+2} \frac{n}{n+1} \frac{n}{8(a-q)(n+5n+1)}>0 \]

\[ \frac{\partial U(h_{t,n*}^{b},1)}{\partial n} = -\frac{\var(b)(2n+3)}{8(a-q)(n^2+5n+1)^2} < 0 \]

Q.E.D.
The value of information system $h_{tn^1,1}^b$ increases with additional signal times, but the marginal benefit is decreasing. Therefore, adding additional report times in an infrequently reporting system will make a greater difference in system value than will additional reports from an already frequently reporting system.

Corollary 6-6.2: $\Delta U(h_{tn^1,1}^b)$ is linearly increasing in $\text{var}(b)$ and convex decreasing (increasing) in $a$ (q).

Proof: Partially differentiating equation (6.27):

$$\frac{\partial \Delta U(h_{tn^1,1}^b)}{\partial \text{var}(b)} = \frac{(2n+3)}{8(a-q)(n^2+3n+1)^2} > 0; \frac{\partial^2 \Delta U(h_{tn^1,1}^b)}{\partial [\text{var}(b)]^2} = 0$$

$$\frac{\partial \Delta U(h_{tn^1,1}^b)}{\partial a} - \frac{\partial \Delta U(h_{tn^1,1}^b)}{\partial q} = \frac{\text{var}(b)}{8(a-q)^2(n^2+3n+1)} < 0; \frac{\partial^2 \Delta U(h_{tn^1,1}^b)}{\partial a^2} = \frac{\partial^2 \Delta U(h_{tn^1,1}^b)}{\partial q^2}$$

$$= \frac{\text{var}(b)}{4(a-q)^3(n^2+3n+1)} > 0 \quad \text{Q.E.D.}$$

Interpretations of the roles of $\text{var}(b)$, $a$ and $q$, as determinants of information value, are analogous to those given earlier in simpler settings.

6.4 Multiple Sources of Uncertainty

Suppose the monopolistic producer knows that the cost function is quadratic, but he is uncertain about the magnitudes of each of the three parameters, $a$, $b$ and $c$. In other words, he is uncertain about period fixed costs, unit variable cost, and the rate of increase for nonproportionally increasing (i.e., quadratic) costs. Let $h_{t}^{a,b,c}$ denote the system which provides time $t$ information about all three types of costs.

Proposition 6-7: If $\rho(a,b,c) = 0$ for $b > r$;

$$U(h_{t}^{a,b,c}) = \frac{(1-t)}{4} E \left[ \frac{[r-E(b)]^2}{[E(a)|y] - q} \right] \left[ \frac{[r-E(b)]^2}{E(a)-q} \right]$$

$$+ \frac{(1-t)}{4} E \left[ \frac{[r-E(b)]^2}{[E(a)|y] - q} \right] \left[ \frac{[r-E(b)]^2}{E(a)-q} \right]$$

$$+ \frac{(1-t)}{4} E \left[ \frac{[r-E(b)]^2}{[E(a)|y] - q} \right] \left[ \frac{[r-E(b)]^2}{E(a)-q} \right]$$

$$+ \frac{(1-t)}{4} E \left[ \frac{[r-E(b)]^2}{[E(a)|y] - q} \right] \left[ \frac{[r-E(b)]^2}{E(a)-q} \right]$$
Proof: The producer's decision problem, given that he employs system \( h_{t}^{a,b,c} \), is given by:

\[
(6.29) \max \int \int \int \left\{ \begin{array}{c}
(\frac{q-a}{t})x_1^2 + (\frac{a-q}{t})x_2^2 + (r-b)(x_1 + x_2) - c \cdot p(a,b,c|y)p(y)dcdbdady
\end{array} \right.
\]

Maximisation of (6.29) yields the decision rule:

\[
(6.30) \frac{x_1^*}{t} = \frac{r-E(b)}{2[E(a)-q]}, \quad \frac{x_2^*}{t} = \frac{r-E(b|y)}{2[E(a|y)-q]}
\]

The producer's maximum expected profit, given that he employs system \( h_{t}^{a,b,c} \), is the following:

\[
(6.31) E[w(x^*,s)|h_{t}^{a,b,c}] = \frac{t[r-E(b)]^2}{4[E(a)-q]} + E \left[ \frac{(1-t)[r-E(b|y)]^2}{4[E(a|y)-q]} \right]
\]

where \( s = (a,b,c) \).

The producer's decision problem, given that he employs the null information system, is given by:

\[
(6.32) \max \int \int \int \left\{ \begin{array}{c}
(q-a)x^2 + (r-b)x - c \cdot p(a,b,c)dcdbdady
\end{array} \right.
\]

Maximisation of (6.32) yields the decision rule, \( x^* = \frac{r-E(b)}{2[E(a)-q]} \).

The maximum expected profit is:

\[
(6.33) E[w(x^*,s)|h_{t}^{0}] = \frac{(r-E(b))^2}{4[E(a)-q]} - E(c)
\]

\( U(h_{t}^{a,b,c}) = E[w(x^*,s)|h_{t}^{a,b,c}] - E[w(x^*,s)|h_{t}^{0}] \). Subtraction of (6.33) from (6.31), and rearrangement of terms yields the desired result. Q.E.D.

Corollary 6-7: If \( p(a,b,c) = 0 \) for \( b > r \), \( U(h_{t}^{a,b,c}) \) is linearly decreasing in information time, \( t \).

Proof: Partially differentiating equation (6.28):
\[
\frac{3U(t_{t}, a, b, c)}{\partial t} = -\frac{1}{4} \sum_{y} \frac{(r - E(b|y))^2}{[E(a|y) - q]} - \frac{(r - E(b))^2}{[E(a) - q]} \frac{3U(t_{t}, a, b, c)}{\partial t^2} = 0
\]

The value of the system which jointly provides information about fixed costs, unit variable cost, and the quadratic cost parameter is decreasing at a constant rate in the time the information is received. This result is identical to that obtained for signal timing, as an indirect information value determinant, in all other cases where no other system attributes depend on signal timing. The precise role of the other determinants of information value cannot be unambiguously determined in this general case.

6.5 Summary

In this chapter, the investigation of the determinants of information value, in the context of the CVP decision, was generalized along several dimensions. First, the assumption of perfect competition, which was maintained throughout Chapters 3 through 5, was relaxed and monopoly was allowed. A linear inverse demand function was assumed; this implies a quadratic total revenue function.

The role of unit variable cost, the quadratic cost parameter, fixed cost, revenue function parameter, r, information timeliness, and initial uncertainty, as determinants of the value of systems providing information about unit variable cost or the quadratic cost parameter, is the same under monopoly as under perfect competition. Under monopoly, the inverse demand function parameter, q, is also an indirect determinant of information value. Its role is characterized with regard to systems providing information about unit variable cost and the quadratic cost parameter. These results are interpreted, with various levels of
precision, in intuitive economic terms.

Second, the results given in Chapter 3 for unit variable cost information system $h^b_t$ are generalized to allow for multiple cost reports generated at various times during the period. A generalized multiple information time system is specified, and the roles of various determinants of its value are investigated. An "equivalent", delayed, single-time perfect information system is identified. Its value is equivalent to that for the generalized imperfect information system. A multiple, imperfect report system which is anchored by a perfect cost report is investigated. Its value is greater than that for the single perfect report system, where the perfect report is received at the same time as the perfect report in the multiple report system.

A multiple unit variable cost information time system is investigated for which the accuracy of each report improves with time. This system is a generalization of system $h^b_{t,\delta}$ (see section 4.3). The determinants of the value of this system are identified, and their impact on system value is investigated.

Finally, brief consideration is given to a system which jointly provides delayed information about period fixed costs, unit variable cost, and the quadratic cost parameter.

In Part II of this dissertation, several mathematical statements have been derived to characterize the roles played by the value determinants of various information systems, under differing situations, in a CVP decision setting. These statements were restated in intuitive economic terms, and they were interpreted, with various levels of precision, in terms of the economic phenomena which are associated with the results.
In Part III of the dissertation, various inferences will be drawn with regard to some normative managerial implications of the analytical investigation just concluded.
PART III

MANAGERIAL IMPLICATIONS

AND CONCLUSION
CHAPTER 7

SOME MANAGERIAL IMPLICATIONS OF THE INVESTIGATION OF THE COST-VOLUME-PROFIT DECISION

The purpose of this chapter is to draw upon the analysis in Part II for some normative implications regarding the effects of information value determinants in the cost-volume-profit decision setting. Two types of implications will be considered. First, some rather general inferences will be drawn regarding the optimal design, use, and adaptation of cost information systems intended for the CVP decision. Second, attention will be directed to the interface between a cost information system and performance evaluation via variance analysis.

7.1 General Implications

The purpose of this section is to draw, from the theoretical investigation of Part II, some rather general inferences regarding the effects of information value determinants in the CVP decision setting. In the theoretical development several propositions were established which characterize the functional dependence of particular information systems' value on several determinants. Sensitivity analysis was then applied to isolate the effect of individual determinants as well as their interaction effects with other determinants. These mathematical results were then restated in intuitive economic terms and interpreted, with differing levels of precision, by appealing to concepts from microeconomic theory. The investigation concluded with this type of interpretation, and no
attempt was made to draw normative implications regarding the role of
the determinants in the design, adaptation, and use of the information
systems. This step was postponed until the completion of the theoretical
investigation, in order to allow comparison and summarization of results
for a variety of information systems under various assumptions.

There is no attempt made here to exhaustively consider the norma-
tive implications of the investigation. Other implications, which are
not given here, may be drawn by managers and other researchers regarding
situations of interest to them. Naturally, the inferences drawn here,
as well as those that others may draw, are restricted by the assumptions
governing the theoretical development. These assumptions, and the limi-
tations they impose, are comprehensively described in Chapter 8. One
important assumption should, however, be noted at this point. The infor-
mation systems considered only provide input for the CVP decision. This
ignores the multiplicity of uses of production cost information (e.g.,
inventory valuation, performance evaluation, budgeting). As a result,
any prescriptive statements made regarding the design and adaptation of
cost information systems are necessarily limited.

In Part II, several generic types of information systems (e.g.,
perfect, noisy, noiseless) were specified, and expressions were derived
for the systems' gross value. These expressions invariably turned out
to be functions of several parameters of the problem. Sensitivity analy-
sis was then conducted to determine which parameters affect information
value under given circumstances and how their impact is altered as cir-
cumstances vary. The kind of general normative inferences which can be
drawn are similar to those from comparative statics in economic theory.
Statements can be made about the comparative value of information systems. Such comparisons, of course, imply the existence of two or more objects to be compared, and these objects can be generated in several ways. First, the value of a particular cost information system, for a single production process, can be compared under two different sets of assumptions about the technology of the process or the environment in which it operates. One question which can be answered by making such a comparison is: "How will the usefulness\(^1\) of the cost accounting information which is currently being generated be affected if the production technology is altered or the environment changes?" Suppose, for example, that a firm manufactures an industrial chemical by a process which pollutes the air. The government imposes a tighter restriction on the amount of pollutant that can be emitted, and the firm is forced to purchase equipment designed to reduce the level of emission. The acquisition of the equipment results in higher fixed cost in each production period, and its operation entails costs which increase more than proportionately with the production rate. Management currently employs an accounting system which provides information about unit variable cost by collecting data on prices and usage of direct materials, wage rates and direct labor usage, etc. How, if at all, will the usefulness of this system change when the new pollution control equipment is installed? Should the system be adapted to meet this environmentally imposed technological change? Should more or less resources be expended on the unit variable cost reporting system?

\(^1\)"Usefulness" may be considered the vernacular for the precise concept of "value."
Second, the value of a particular cost information system can be compared under two or more production technologies. Given the cost accounting system, will it be more useful if process A is used to produce the firm's output, or if process B is used? Since an information system is a necessary input to the production process, technology decisions made by production engineers should reflect the type of information which is available and the cost of obtaining alternative kinds of information. Suppose, for example, that two alternative production processes are being considered. The first is highly labor intensive and uses inexpensive raw materials, resulting in a relatively low unit variable cost. However, the marginal cost increases rapidly with the production rate. The second is highly automated and uses higher quality raw materials, resulting in a high unit variable cost but a slowly increasing marginal cost. The cost accounting system is geared toward reporting of unit variable cost (e.g., direct material prices, wage rates, etc.) with relatively little emphasis on reporting costs which are not proportional to the production rate. These information system characteristics should be taken into account in making the production technology decision, if the nature of the technology affects the usefulness of available data. The analysis in Part II can shed some light on the effects of technology on data usefulness. The latter, of course, affects overall profitability.

Third, the value of two cost information systems may be compared for a given production technology and environment. If one of several alternative cost information systems may be installed, or changes are to be made in the extant system, the characteristics of the production setting, as well as the information systems, may be important in making the
selection. Questions such as the following might be addressed by making such a comparison: (1) Should the cost accounting system concentrate on variable costs or other types of costs? (2) Is timeliness of information critical or not too important? (3) Is it worthwhile to increase the rate at which information accuracy improves over time (i.e., accuracy responsiveness parameter)? (4) Should cost reporting be more or less frequent, or at different times?

Finally, there may be several coexisting, independent production processes, each with its own cost information system. The values of different systems for different processes may be compared. If changes are contemplated in the resources allocated to the information systems, how should these changes be made? How can each system be tailored to best meet the demands which are made on it?

In Chapter 2, the various parameters of the decision problem and information system were termed indirect determinants of information value, since together they work through the five direct determinants to affect information value. These indirect determinants can be further subdivided into the design parameters of management information systems and extra-system parameters. The former category consists of all the parameters which describe the information system (e.g., accuracy, timeliness, accuracy responsiveness, noise, fineness, etc.) The latter group consists of all indirect determinants of information value which are not characteristics of the information system (i.e., payoff parameters, flexibility parameters, initial uncertainty parameters, and environmental parameters). One of the most important points of the study is that all of these variables, both system design parameters and extra-
system parameters, play an important role in determining the usefulness and optimal design of management information systems.

There is ample evidence that managers are aware of some of the information system design parameters and the direction of their effect on system value. There is considerably less evidence that the importance of the parameters is fully understood under varying technological and environmental circumstances. There is virtually no evidence to suggest awareness of the importance of extra-system parameters on information system value. When some event occurs to alter the technology, environment, payoff function, or initial uncertainty, the information system must be adapted to meet the altered conditions. Other parameters, however, may also be changed along with system design parameters to bring the new situation into an optimal position. Economic theory would dictate that system design parameters and extra-system parameters should each be adjusted to the point where the marginal cost of further adjustment in every parameter is equal to the marginal benefit (in terms of overall firm performance). Thus, when the government imposes new environmental standards, when a superior production technology is found, when an embargo on raw materials is threatened, thereby increasing uncertainty about costs, management should no doubt adjust various design features of the cost accounting system to meet the new situation. The usefulness of the system, however, can also be altered by changing controllable extra-system parameters. It is this latter point which has been virtually ignored to date.

The sensitivity analysis conducted in Chapters 3 through 6 provides previously unavailable insights into the role of various
information value determinants in the CVP setting under a variety of circumstances. Some of the normative inferences which can be drawn from this analysis are now explored.

7.1.1 Extra-system Parameters Associated with the Payoff Function

One of the direct determinants of information value is the payoff function, which is the composition of the outcome and utility functions. Since the utility function is assumed in this study to be the identity function, only the outcome function remains as a determinant. The associated indirect information value determinants are the revenue and cost function parameters $q$, $r$, $a$, $b$ and $c$. These are extra-system indirect determinants. While they may be partially or completely controllable by management, they are not characteristics of the information system.

Information about fixed cost\(^{4}\) was found to be valueless in the short run (section 3.3), Proposition 3-1): moreover, fixed cost did not appear as an indirect determinant of the value of any information system under any circumstance (for example, see section 3.4.1, Proposition 3-2 or section 5.2, Proposition 5-1). In designing a cost information system for the short-run volume decision, management may ignore fixed cost, both as an object of reporting and as a relevant factor in determining information needs about other types of costs. For example, if management contemplates allocating resources to the installation or alteration of a cost accounting system in one of several independent production processes, any associated fixed costs are inconsequential in making the allocation decision. Similarly, if the level of fixed cost incurred in a production process changes, ceteris paribus, due to some technological or environmental event, the change will in no way effect the usefulness

\(^{4}\)Recall the definition of fixed cost given in section 3.2. Fixed cost includes any cost which does not vary in relation to the production rate.
of cost information about the process. If the information system was worthwhile, in a cost/benefit sense, prior to the change in fixed cost, it is worthwhile afterwards.

The quadratic cost parameter, $a$, measures the rate at which "quadratic cost"\(^2\) increases in the production rate. The role of $a$ alone,\(^3\) as an indirect determinant of the value of systems providing information about variable costs, was found to be the same in all situations investigated. The value of any such system was found to be decreasing at a decreasing rate in the quadratic cost parameter (section 3.4.1, Corollary 3-2.1 or Figure 8). As the quadratic cost parameter increases, quadratic cost becomes larger, relative to variable cost, at any production rate. It becomes more critical in the manager's decision model, and unit variable cost declines in importance. If a change in technology occurs such that the quadratic cost parameter for a production process increases, the usefulness of accounting information about unit variable cost declines. After such a shift occurs, management might want to examine the benefits of the information system in order to determine if it is still worth its cost. If the quadratic cost parameter is relatively low, a given change in the parameter induces a larger change in the value of the variable cost information system than when it is large. Consequently, in a firm with rapidly increasing marginal costs, the usefulness of a variable cost information system may be rather insensitive to changes in the rate at which marginal costs increase.

---

\(^2\)Quadratic cost was defined as the total of costs which increase at an increasing rate in the production rate. See section 3.2.

\(^3\)This is in contrast to the interaction effect of $a$ with other determinants of information value.
Returning to the polluting industry example mentioned earlier, recall that the installation of the pollution control equipment will cause an increase in period fixed costs and a rise in quadratic costs. If quadratic costs are relatively low prior to installation of the control device, there may be a significant decline in the importance of generating variable cost reports after the device is placed in operation. Management might well consider placing less emphasis on the reports if a significant amount of resources are consumed in the reporting process. If, however, quadratic costs were relatively high prior to installation of the pollution equipment, the importance of variable cost information will not be greatly affected by the technological change.

Another result concerning the quadratic cost parameter, as an indirect information value determinant, prevails throughout the entire analysis. For any other indirect determinant, either a system design parameter or an extra-system parameter, the absolute magnitude of the marginal effect of the determinant on variable cost information value is decreasing at a decreasing rate in a (section 3.4., Corollary 3-2.2; section 3.4.2, Corollary 3-4.2; section 4.2, Corollary 4-2.2; section 5.3.1, Corollary 5-4.2; section 6.2, Corollary 6-1.2). An increase in the timeliness, accuracy, accuracy responsiveness, or frequency of variable cost reports, a decrease in initial uncertainty, or an increase in the upper bound on the production rate will each increase variable cost information value more as the quadratic cost parameter decreases. Where management must allocate resources to install or improve a variable cost reporting system in one of two independent processes, a greater benefit will be obtained from the information about the process with less rapidly
increasing marginal costs, other things (except fixed cost) being equal. Inequality of fixed costs would be of no consequence. While it is true, ceteris paribus, that earlier, more accurate or more frequent variable cost reporting is always desirable, managers may overestimate the importance of some of these system design parameters in production settings where nonproportionally increasing costs represent a large share of total costs. Accuracy, for instance, may actually be very unimportant in such a circumstance, and it may be that information system resources are being wasted by concentrating on high report accuracy.

If the cost accounting system concentrates on providing information about costs that increase more than proportionately with the production rate, while unit variable cost is relatively certain, the usefulness of the information declines at a decreasing rate in the unit variable cost (section 5.2, Corollary 5-1.1 or Figure 30). This assertion assumes that the unit variable cost is less than the unit price; otherwise, the firm will temporarily shut down. Suppose, for example, that the price of the direct material used in a production process is increasing over time at a relatively constant rate. The importance of generating reports about nonproportionally increasing costs is continually decreasing. The more the raw material price rises, however, the less rapid the decline in usefulness of the reports.

The absolute magnitude of the marginal effect of any other indirect determinant of nonproportional cost information value (e.g., unit price, initial uncertainty about a, information timeliness) is decreasing at a decreasing rate in the unit variable cost. The timeliness of such information, for example, may be rather unimportant if the firm manufactures
a high variable cost product.

Turning now to the revenue function parameters, the role of the inverse demand function slope parameter, \( q \) (assuming monopoly), in determining the value of systems providing information about the quadratic cost parameter, is the same as it is in determining the value of unit variable cost information systems. In each case, the value of the system increases at an increasing rate in \( q \) (section 6.2.1, Corollary 6-1.1; section 6.2.2, Corollary 6-3.1). 5 If a firm operates two similar production processes, and one product is sold in a perfectly competitive market while the other is sold in a monopolistic market, cost information about the first product will be more valuable for the volume decision than will cost information about the latter product, ceteris paribus.

The role of the revenue parameter, \( r \), as a determinant of information value, is the same in the perfectly competitive case as in the monopoly case. Perfect competition will be assumed for this discussion, since then the interpretation of \( r \) (as unit price) is clearer. The role of price as an indirect information value determinant is contingent on the object of uncertainty as well as the particular information system under consideration. For systems providing perfect information about unit variable cost, the value of the information is increasing in price, first at an increasing rate and then at a decreasing rate, depending upon whether \( r \) is below or above the prior expected unit variable cost (section 3.4.2, Corollary 3-3.1). Variable cost information is more valuable for the volume decision with respect to high priced products

5Assuming that \( b < r \) or \( E(b|y) < r \) for all signals in \( Y \).
than for low priced products. If resources can be allocated to the unit variable cost reporting system for only one of several independent production processes, the per unit gain in system value will be greater, ceteris paribus, for the higher priced product. The second order effect of price depends on its relationship to the prior expected unit variable cost. This relationship is instrumental in making the shut-down decision. When unit variable cost is known to be less than unit price, the latter ceases to be a determinant of information value (section 4.2, Proposition 4-1).

The value of an accounting system that reports on nonproportional costs also increases with the unit price (section 5.2.1, Corollary 5-1.1). In the multiple independent production process firm, the per unit gain from resources allocated to the cost accounting system will be greater for a high priced product than for a low priced product, ceteris paribus. Moreover, the greater the price, the greater the rate of gain in system value. Thus the first order effect of price is invariant with respect to the object of uncertainty, but if the price is known to exceed the unit variable cost, the second order effect depends on whether the system reports nonproportional or variable costs.

7.1.2 Extra-system Parameters Associated with Action Flexibility

Action flexibility is another direct determinant of information value. One dimension of flexibility in the CVP problem is the upper bound on the production rate. A short-run upper bound could result, for example, from such conditions as fixed plant or machine capacity, limited supplies of raw materials or skilled labor, or environmental restrictions on emission of pollutants. If the upper bound is below a certain
level, variable cost information becomes more useful to management as the upper bound increases (section 5.3.1, Corollary 5-4.1).  

Suppose management is contemplating production processes for a given product. One entails a plant expansion with higher period fixed costs and no absolute bound on the production rate. The other process can be operated with the existing plant, but then facilities impose an upper bound on the production rate. A variable cost reporting system is currently used. In selecting between the two production technologies, management should consider the increased fixed cost associated with expanded plant, financing opportunities, future use for the expanded facilities, etc. However, one factor which should be considered is the appropriateness of the existing variable cost information system for the two technologies. The information from this system will be of greater use to management if the process is selected in which no upper bound is imposed on output. This is only one factor to be considered, but it is one which has not previously received much attention.

To take another example, suppose there are multiple independent production processes. Management would derive greater benefit, ceteris paribus, by developing a variable cost reporting system for a process with a high upper bound than for one with a low upper bound. If, for some reason, the upper bound increases in any of the processes, management might find it worthwhile to allocate more resources to the unit variable cost accounting system for that process.

As the discussion in Chapter 2 points out, this first order effect is obvious. The second order effect of the upper bound is contingent

6Assuming that h<r.
on whether the upper bound is below or above the ex ante optimal production rate. If the upper bound is relatively low, increases bring about larger and larger gains in the value of the variable cost information. Otherwise, the gains decrease as the upper bound increases. Suppose there are several independent production processes, and the production rate upper bounds are increased for each one due to plant expansion. For those processes with bounds below the ex ante rate, the increase in the usefulness of unit variable cost information will be greater, ceteris paribus, for the loosely constrained processes than for the more tightly constrained ones. The opposite is true for processes with upper bounds above the ex ante optimal rate.

If production occurs by a batch process, there is no monotonic relationship between the value of variable cost information and batch size (section 5.3.2, Proposition 5-5 or Figure 36). If the batch size is altered (or if batch size differs among independent processes) the effect on the usefulness of the cost information depends on the range and direction in which batch size is changing.

7.1.3 Initial Uncertainty

As the discussion in Chapter 2 points out, initial uncertainty is a direct determinant of information value. It is an extra-system determinant. Intuitively, one might expect information value to be increasing in initial uncertainty. The results of the investigation confirm this conjecture in every case where it is possible to isolate and explicitly measure initial uncertainty (section 4.2, Corollary 4-2.1; section 4.4, Corollary 4-4.2; section 5.2.2, Corollary 5-3.2). Other things being equal, management would derive the greatest benefit from collecting cost
information (quadratic or variable) on production processes where the uncertainty about cost is greatest. For variable cost information, the gain in information value increases at a constant rate in initial uncertainty. For example, suppose the purchasing agent decides to purchase direct materials from a different vendor than previously because the price charged is, on average, lower. The price offered by the new vendor, however, fluctuates to a greater extent than that offered by the former vendor. The variable cost reporting system (especially that portion which is concerned with direct material prices) will be of greater value, _ceteris paribus_, under the new vending arrangement. The change in system value will be the same regardless of the level of variable cost prior to the change of vendors.

As another illustration, suppose management is planning to adopt a new technology for a production process for which the proportionally increasing costs, under both the old and the new technologies, are uncertain and distributed by gamma densities with equal expectation. If the nonproportionally increasing costs are subject to perfect reporting at some time during the period, and if the new technology implies greater volatility (or less familiarity) with respect to these costs, the cost reporting system will be of greater value, _ceteris paribus_, under the new technology. If there are any doubts as to whether the cost reporting system is worth its cost under the old technology, these doubts should decline under the new technology.

The change in the usefulness of perfect variable cost reports that results from a change in initial uncertainty is greater for a low
quadratic cost process\textsuperscript{7} than for one with high quadratic costs (section 4.2, Corollary 4.2.2). If, for example, technological or environmental changes alter the initial uncertainty about variable cost, management is more likely to find a need for reassessment of the resources allocated to variable cost reporting when quadratic costs are low than when they are high. Suppose, for another illustration, that legislation is proposed in Congress to freeze the price of industrial steel. This would tend to lower the uncertainty about variable cost in each of a firm's multiple, independent production processes which rely on steel as a direct material. It may be desirable to reallocate some variable cost reporting effort from high quadratic cost processes to those with low quadratic costs.

7.1.4 Information System

The nature of the information system is the most obvious determinant of a system's value. The type of the system (i.e., perfect, noisy, noiseless imperfect), its accuracy, the timeliness of the reports, the frequency of reporting, and the responsiveness of system accuracy to time, as well as other factors, are important indirect determinants of system value. These are the design parameters of management information systems. Moreover, the sensitivity of system value to the extra-system parameters, discussed in sections 7.1.1 through 7.1.3, depends on these system design parameters.

Timeliness is often mentioned in the accounting literature as a desirable attribute of information. Other things being equal, late

\textsuperscript{7}A low quadratic cost process refers to one where those costs which increase more than proportionately with the production rate increase at a relatively slow pace.
information will never be preferred to early information. If no other system attributes (e.g., accuracy) are related to the timeliness dimension of a system, it is obvious that system value will be nondecreasing in timeliness. The investigation of the CVP problem confirmed this intuitive assertion for systems which report on either variable costs (section 3.4.1, Corollary 3-2.1) or quadratic costs (section 5.2.1, Corollary 5-1.1). The investigation also demonstrated, for both kinds of systems, that the sensitivity of system value to any other determinant\(^8\) is increasing in timeliness. (See for example, section 3.4.1, Corollary 3-2.2; section 3.4.2, Corollary 3-3.2; section 4.2, Corollary 4-2.2; section 4.4, Corollary 4-4.2; section 5.2.1, Corollary 5-1.2; section 5.3.1, Corollary 5-4.2; section 6.2.1, Corollary 6-1.2). This is a potentially important result. There is a much greater incentive (or necessity) for management to control, or at least monitor, other information value determinants when information is timely than there is otherwise. The impact on cost information system value of unit price, the monopoly inverse demand function parameter, \(q\), the quadratic cost parameter, unit variable cost, the upper bound on the production rate, and initial uncertainty is greater when the system provides more timely information. If, for example, management relaxes the production rate upper bound by expanding plant and equipment, the change in the usefulness of variable cost reports is much greater if the reports are timely than otherwise. For another illustration, the difference in the usefulness of variable cost reports from processes which produce high versus

---

\(^8\) The other determinant is assumed to be independent of information timing.
low priced products is greater, under certain conditions, when the reports are early than when they are late. The converse is also true. The sensitivity of cost information value to timeliness is dependent on the other determinants. If the production rate is tightly constrained or unit price is low, for example, timeliness of cost reports may not matter very much. If substantial amounts of resources are being consumed to provide very timely information, the expenditures may not be justified. If there are multiple independent production processes, and improvements are contemplated in the timeliness of cost reports, the returns in system value will be higher, *ceteris paribus*, if priority is given to loosely constrained processes or those that produce high priced products.

The second order effects of information timeliness on system value are not as obvious as the first order effects. In the CVP decision setting, cost report value decreases in direct proportion to the report delay or, equivalently, increases in direct proportion to report timeliness (section 3.4.1, Corollary 3-2.1; section 5.2.1, Corollary 5-1.1). The change in cost information value resulting from earlier receipt of an already timely report will be identical, *ceteris paribus*, to that resulting from an equal improvement in timeliness of a relatively late report.

There is evidence in the literature that managers are aware of the direction of the first order effect of information timeliness, *viz.*, information value increases, *ceteris paribus*, with earlier reporting. For example, in an accounting symposium held in 1968, corporate executives acknowledged the importance of timing of information about sales and
discussed techniques used to insure such timely information. There is little evidence, however, that managers understand that the second order effect of timeliness is, in some cases, zero or that timeliness affects the sensitivity of information value to other determinants.

One implication of these results is that management should be indifferent between the two variable cost reporting systems, $h^b_t$ and $h^b_T$, where $T = \{0,1\}$ is a random variable and $\bar{T} = \mathbb{E}(T)$. The system $h^b_T$ has some uncertainty about the timing of the variable cost report. Such uncertainty could result from a variety of sources, including a production process which can only be observed at certain random times, random variations in the workload of those compiling cost reports, and random delays in report transmission. Uncertainty about the report timing does not detract from system value. An analogous argument reveals indifference by the producer between systems $h^a_t$ and $h^a_T$. These results are not solely driven by the producer's assumed risk neutrality. In the context of the newsboy problem, Itami has demonstrated that a risk neutral decision maker would prefer the information system which is more stable with respect to timeliness, in this case $h^b_t$ or $h^a_t$.

If the cost accounting system provides multiple variable cost reports, and no other system attributes are affected by report timing,
all of the comments made above with respect to system timeliness are valid for the timeliness of each report from the multiple report system (section 6.3.1, Corollary 6-4). Moreover, the sensitivity of the value of the variable cost reporting system to the timeliness of a given report is unaffected by the timeliness of other reports.

The accuracy of a system that provides noisy information about variable cost may be characterized by the expected reduction, due to the system, in uncertainty about variable cost. Naturally, the value of the system is increasing, ceteris paribus, in the accuracy of each report (section 4.2, Proposition 4-1; section 6.3.1, Proposition 6-4). Under certain conditions, the value of the system is proportional to the expected reduction in uncertainty. This means, for instance, that the increase in system usefulness from improvement of a relatively inaccurate system will be the same as that resulting from an equal improvement in a system which is already rather accurate.

It is interesting to note that information value, given the assumed conditions, does not exhibit decreasing marginal returns (i.e., increments) with respect to either timeliness or accuracy (in a noisy system). Both determinants, as desirable attributes for an information system, provide constant returns. This is in contrast to the nonconstant marginal returns in information value induced by improvements in extra-system indirect determinants such as the quadratic cost parameter, unit variable cost, price, and the upper bound on the production rate. Whether or not the constant returns phenomenon, with respect to some system design parameters, is a peculiar characteristic of the CVP problem cannot be determined without research in other decision contexts.
Itami found nonincreasing marginal returns in information value with respect to information timeliness in his analysis of the extended newsboy problem.\textsuperscript{11}

For a noiseless, single-report variable cost information system, where system accuracy is measured by the number of equal cells in the partition induced on a restricted interval of uncertainty, and the prior and likelihood functions are rectangular, the system's value is increasing in system accuracy with decreasing marginal returns (section 4.4, Corollary 4-1.1). Therefore, the nature of the second order effect of cost report accuracy on the information system's value is, in the GYP setting, contingent on the type of system employed and the nature of the accuracy measure. Marschak found a similar phenomenon of decreasing marginal returns to information accuracy in the context of a speculator problem.\textsuperscript{12} The main implication is that managers should not assume a "law of decreasing marginal returns" to various desirable attributes of information systems. The problem setting, various extra-system parameters, the type of the information system, and the nature of the relevant measure for a given attribute determine its second order effect on the system. This remains true in spite of an intuitively apparent direction for the first order effect.

Regarding the effects of timeliness and accuracy, the situation is considerably different if the accuracy of the report(s) from a variable cost information system improves over time. The remarks made above with regard to the impact of report timeliness and accuracy are ceteris

\textsuperscript{11}Itami, "Evaluation," p. 91. See Chapter 2 for further discussion.

\textsuperscript{12}Marschak, "Problems," pp. 62-70.
paribus statements, and do not allow for such a dependent relationship between the two system attributes. In this case, there is a trade-off implicit in the report timing issue. Early information is desirable because it provides a greater length of time during which the benefits of the information can be enjoyed. Later information, however, is more accurate. The result of this trade-off, for a variable cost reporting system, is that system value first increases and then decreases as a given report is delayed (section 4.3, Corollary 4-3.3; section 6.3.3, Corollary 6-5.1). The best time to receive the report(s) is sometime in the interior of the period, not at the earliest possible time, as discussions of timeliness in the accounting literature often suggest. The only important factor in determining when the cost report(s) should be generated is the responsiveness of system accuracy to time (section 4.3, Corollary 4-3.2; section 6.3.3, Corollary 6-5.2). In designing the cost accounting system, as far as the report timing issue is concerned, management may ignore all other system design parameters and technological and environmental factors.\(^{13}\)

Of course, determining the accuracy responsiveness parameter (\(\delta\) or \(\phi\)) may not be an easy task. The type of cost accounting system and cost function estimation procedures used will be instrumental in determining the magnitude of this parameter. The noisy system parameter, \(\delta\), would likely be less than one in most systems. In this case, the accuracy of a cost report improves quickly during the beginning of its time

\(^{13}\)The technology and environment are reflected in the outcome function parameters and the structure of the state space. The implications drawn are subject to the limitations of the analysis which include assumptions about these parameters.
range, and the marginal improvements diminish as time passes (section 4.3, equation (4.10) or Figure 22A; section 6.3.3, equation (6.17) or Figure 44A). For example, a cost function estimation system may take periodic observations on cost and volume. At the point when a cost report is to be generated, these observations are utilized in some estimation model, and an estimate is generated. If the observations are taken at equal time intervals, and if the expected accuracy of the estimation model increases at a decreasing rate in the number of observations, \( \delta \) would be less than 1. A linear regression model would be an example of such an estimating procedure. If the accuracy of the estimate is measured by the vector of standard errors of the parameters, the accuracy is increasing at a decreasing rate with the number of observations.\(^\text{14}\) If the observations are taken at regular intervals and stored until a regression is run, the accuracy of the estimate would be nondecreasing over time with decreasing marginal increments.

Recall from Part II (section 4.3, Corollary 4-3.1; section 6.3.3, Corollary 6-5.1) that the value of the variable cost reporting system is \( \begin{cases} \text{concave} \\
\text{pseudoconcave} \end{cases} \) in the timing of the report (ith report for a multiple report system) as \( \delta > 1 \).\(^\text{15}\) One interesting implication of the concavity or pseudoconcavity property concerns the producer's attitude toward a variable cost reporting system for which the timing of the report(s) is somewhat random. If the accuracy responsiveness parameter, \( \delta \), is less than or equal to one, even a risk neutral producer would prefer the


\(\text{15Assuming that the unit variable cost is less than the unit price.}\)
noisy variable cost information system $h_t^b$ over system $h_t^b$, where $T_e[0,1]$ is a random variable and $\bar{e}=E(T)$. That is, the system with single known information time is preferred to the one with single unknown information time. This results from the concavity of $U(h_t^b, e)$ in $t$ when $\delta \leq 1$. The same result holds for the single report system if $\delta > 1$ and $T \leq (\delta - 1)/(\delta + 1)\epsilon(0,1)$. This phenomenon is illustrated in Figure 46A for the case of $T_e(t', t'')$. Analogous observations may be made with regard to the producer's preference for a multiple report system for which each report time is stable, rather than one with unstable report times.

If $\delta > 1$ and $T \leq (\delta - 1)/(\delta + 1)$, the risk neutral producer would actually prefer the system with random report timing over the system with known timeliness. Figure 46B illustrates this case for $T_e(t', t'')$. In this case, the convexity of $U(h_t^b, e)$ with respect to $e[0, (\delta - 1)/(\delta + 1)]$ insures that $E[U(h_T^b, e)|T \leq (\delta - 1)/(\delta + 1)] \leq U(h_T^b, e)$.

Finally, if $\delta > 1$ and $T$ may be greater or less than $(\delta - 1)/(\delta + 1)$, the producer's preference between the two information systems depends on $T_e$. Figure 46B illustrates this case for $T_e(t', t'')$, $t' < (\delta - 1)/(\delta + 1) < t'''$. Analogous observations may be made with regard to the producer's relative preference among multiple report time systems, where the stability of report timeliness differs for one or more reports.

When $\delta$ is equal to one, the accuracy of the variable cost report(s) increases at a constant rate as time passes (section 4.3, equation (4.10) or Figure 22A; section 6.3.3, equation (6.17) or Figure 44A). For a multiple cost report system, the optimal signal timing vector places the reports at equally spaced time intervals throughout the production period (section 6.3.4). This report timing strategy probably
Figure 46. Effect of Random Information Timing
parallels, to a large extent, that used in practice. If this is true, one of two situations exists: (1) management perceives a constant report accuracy improvement rate and has selected the optimal timing vector, or (2) management has naively selected a constant cost report interval without regard to the response of system accuracy to report timeliness. If the latter situation exists, there may be serious loss in information system value if the accuracy responsiveness parameter differs significantly from one. An empirical study would be necessary to determine which phenomenon exists, and the extent to which the latter situation reduces the usefulness of cost information.

There is little evidence available regarding the way in which managers make the trade-off between accuracy and timeliness of information. Whether managers correctly perceive the information system attributes upon which trade-offs are made, and attempt to optimally design systems to account for the trade-off, is an empirical question.

In a larger context, an interesting question concerns the extent to which the designers of management information systems actually recognize the determinants of information value, and the extent to which they attempt to optimize over system design parameters. This might well be a worthwhile direction for research concerned with the integration of normative and descriptive theories of information.
7.2 Performance Evaluation

In section 7.1, some general inferences regarding managerial behavior were drawn from the investigation of the determinants of information value in the CVP decision setting. Most of these implications concerned decisions regarding the design, adaptation, or use of cost information systems. This section will examine one aspect of the interface between information system design parameters and another broad area of management, viz., the evaluation of performance.

Performance evaluation is a complex and multifarious activity oriented toward providing input for a control and incentive system, the purpose of which is to encourage activity by organizational elements which is congruent with overall organizational goals. The issues which arise in evaluating performance are diverse and include behavioral, technological and environmental considerations. The design of an optimal evaluation system must properly take into account all of these complexities, and the associated sources of uncertainty, while seeking to motivate goal congruent behavior. Because of the complexities involved, performance evaluation systems used in practice tend to be satisficing in nature. Such systems often rely on the specification of some performance standard and subsequent measurement of deviations of actual performance from that standard.

One commonly used performance evaluation technique of this general form is accounting variance analysis. Under this approach a standard is set for performance at the outset of the period when performance is to be measured. Actual performance during the period is subsequently measured and compared with the standard to form a variance. The resulting set of variances is analyzed to determine which ones are significant enough to warrant investigation of the cause. Finally, significant variances are investigated, assignable causes are identified, and appropriate corrective measures, if any, are taken. Each phase of the variance analysis procedure, from standard setting to corrective action, is a complex problem and has received considerable attention in the literature. Attention will be directed in this research to the definition of accounting variances and their disaggregation, as these issues are affected by the design parameters of information systems in the CVP context.

---

16 For explanation and discussion, see, for example, Horngren, Cost Accounting, pp. 832-865, Bierman and Dyckman, Managerial, pp. 83-123, Topuph, Birnberg, and Denski, Cost Accounting, pp. 418-513, and John Dearden, Cost Accounting and Financial Control Systems (Reading, Massachusetts: Addison-Wesley, 1973), pp. 89-117.

17 Standards may be set in a variety of ways. See Horngren, Cost Accounting, pp. 193, 194.

18 "Variance" is used in this discussion to refer both to the accounting variance (i.e., difference between standard and actual performance), and the statistical variance (i.e., expectation of the sum of squared deviations of a random variable about its mean). The context should make the usage clear.

7.2.1 Traditional and Ex Post Variance Analysis Systems

Consider the performance evaluation system for the production manager of Part II. Suppose that the quadratic cost parameter is known, unit variable cost is known to be less than price, and the information system in use is $h^b_t$, which provides unit variable cost information at time $t$. The value of this system is given by (section 4.1; Proposition 4.1):

$$U(h^b_t) = \frac{1}{4a} (1-t)(\text{var}(b) - E[\text{var}(b|y)])$$

Recall that the timeliness of the information system can be measured by $(1-t)$, [section 3.4], and the system's accuracy may be characterized by $(\text{var}(b) - E[\text{var}(b|y)])$, [section 4.3]. Thus the value of the system is the product of its timeliness, its accuracy, and a constant. In the following discussion, let $v$ denote $E[\text{var}(b|y)]$. This parameter can then be used to annotate the designation for the information system, i.e., $h^b_{t,v}$. This notation allows identification of the system by both its timeliness and accuracy.

Suppose that the information system which has been designed by management is characterized with timeliness of $(1-t)$ and accuracy of $(\text{var}(b) - \bar{v})$. When the system is operated as expected (i.e., according to the way it has been designed), these timeliness and accuracy levels will be attained. Thus $(1-t)$ and $(\text{var}(b) - \bar{v})$ can be viewed as attainable standards for information system timeliness and accuracy. System $h^b_{t,\bar{v}}$ may be called the attainable standard information system.

---

20An attainable standard is the performance level expected under normally efficient operating conditions. See Horngren, Cost Accounting, p. 193.
By way of contrast, timeliness of 1, [(1 - 0) = 1], and accuracy of \( \text{var}(b), \{\text{var}(b) - 0\} = \text{var}(b) \), may be viewed as ideal standards\(^{21}\) for system timeliness and accuracy, respectively. Thus system \( h_{0,0}^b \) may be called the ideal standard information system. Note that this system is equivalent to the perfect, time zero information system, \( h_{\infty,0}^b \).

System timeliness and accuracy may vary, during a given period, from the levels expected based on system design. If, for example, an employee who generally prepares the cost reports becomes ill or goes on vacation, others may step in and do the job on a temporary basis. Unfamiliarity with the task or work overloads may cause the timeliness and/or accuracy of the cost reports to decline to levels below those expected given the system design, under normal conditions. Let \( (1 - t^0) \) and \( \{\text{var}(b) - v^0\} \) denote the system's ex post, observed timeliness and accuracy, respectively. The observed information system is denoted by \( h_{t^0,v^0}^b \).

In traditional performance evaluation, the attainable standard which would be set for the production manager is \( E[w(x^*, b) | h_{t, v}^b] \). This is the expectation of firm profit, given optimal action, based on the information system which is supposed to be in operation, viz., the attainable standard system, \( h_{t, v}^b \). While the literature makes no explicit mention of the information system upon which an attainable standard for managerial performance would be set, the assumption is that the attainable standard is the expected level of accomplishment under normal operating conditions. Under these conditions, the information system will be

\(^{21}\)An ideal standard is a performance level that will be achieved only with perfect efficiency and ideal operating conditions. Horngren, Cost Accounting, p. 193.
operating according to its design, i.e., system \( h^{b \rightarrow 22}_{t, v} \).

Let \( x^0 \) denote the actual production quantity vector for the observed subperiods, \([0, t^0]\) and \([t^0, 1]\), and let \( b^0 \) denote the ex post, observed unit variable cost for the period. The observed profit level, then, is \( w(x^0, b^0) \). Under the traditional variance analysis, the following total variance would be computed: \( E[w(x^k, b)|h^{b}_{t, v}] - w(x^0, b^0) \).

This is the difference between the attainable standard profit level and the observed profit level. A positive variance is unfavorable; a negative variance is favorable. This total variance is generally disaggregated into several variances, each of which is associated with a component of \( b^0 \) (e.g., price and quantity of direct materials). In this analysis there is no attempt made to take into account the effect of the random nature of the uncertain parameter on the manager's attainable performance. Since evaluation of the manager's performance should be based only on factors under his control, the traditional system may penalize or reward the manager due to random fluctuations in an uncontrollable parameter. Furthermore, the traditional analysis does not account for deviations in the performance of the management information system from its expected performance.

---

\[22\text{In other words, the production manager can expect, under normal operating conditions, to receive information at time } t \text{ with accuracy of } v.\]
Demska suggested an ex post variance analysis scheme to allow for the impact of the random parameter on the manager's attainable performance. In Demski's system, the uncertainty in the random parameter is suppressed, and the decision problem is solved at the beginning of the period as if the random parameter is a known constant. One possibility is to take the parameter's expectation and solve the problem on that basis. There is no explicit recognition of an intraperiod information system or the resultant intraperiod adaptive behavior. The optimal performance level obtained from this solution to the problem is called the ex ante standard.

Adapting Demski's concepts to the decision setting and notation of this research, the ex ante standard is \( W(\alpha^*|E(b)) \), the profit level achieved by operating at the optimal production rate, assuming that unit variable cost will be equal to \( E(b) \). In order to account for the effect of the random parameter on the manager's attainable performance, Demski computes an ex post standard, equal in this setting to

---


$w[\{x^*|b^0\}, b^0]$. This is the profit level achieved by operating at the optimal production rate for the entire period given the ex post, observed unit variable cost, $b^0$. Two variances are then computed:

$$w[\{x^*|E(b), E(b)\} - w[\{x^*|b^0\}, b^0]$$
$$w[\{x^*|b^0\}, b^0] - w[x^0, b^0]$$

The first variance is the difference between the ex ante and ex post optimal results. The second is the difference between the ex post optimal and observed results.

The difference between the ex ante and ex post results is a crude measure of the firm's forecasting ability. It is the difference between what the firm planned to do during the particular period and what it should have planned to do. Similarly, the difference between the ex post and observed results is the difference between what the firm should have accomplished during the period and what it actually did accomplish. It is the opportunity cost to the firm of not using its fixed facilities to maximum advantage.\(^{25}\)

An alternative and preferable ex ante standard, which still maintains the essential features of Demski's analysis, is $E[w(x^*, b)|h_0]$. This is the expected profit given the ex ante optimal production rate with no intraperiod information. The analysis based on this ex ante standard still does not explicitly incorporate the intraperiod information system and the resultant adaptive behavior, but does avoid the conceptual difficulty of suppressing uncertainty in computing the ex ante optimal standard.

While this performance evaluation scheme is undoubtedly an improvement, conceptually, over the traditional method, it is lacking in two areas. First, it fails to explicitly account for the receipt of information during the period and the resultant adaptive behavior. Second, it

does not take into account deviations in the performance of the information system from that which was expected to be realized, based on system design.

In order to meet the first objection, Itami\textsuperscript{26} has developed an extended ex post variance analysis system. Adapting his concepts to the decision setting and notation used here, the total variance is

\[ E[w(\bar{x}^*,b)|h^b_{t,0}] - w(\bar{x}^0,b^0) \]

Here the ex ante standard is the expected profit given optimal action and the receipt of perfect information\textsuperscript{27} at the standard information time, \( \bar{t} \). Itami decomposes this total variance into three variances as follows:

\[
(7.2) \ E[w(\bar{x}^*,b)|h^b_{t,0}] - w(\bar{x}^0,b) \\
= \{E[w(\bar{x}^*,b)|h^b_{t,0}] - w(\bar{x}^*|h^b_{0,0},b^0)\} \\
+ \{w(\bar{x}^*|h^b_{0,0},b^0) - w(\bar{x}^*|h^b_{\bar{t},0},b^0)\} \\
+ \{w(\bar{x}^*|h^b_{\bar{t},0},b^0) - w(\bar{x}^0,b^0)\}
\]

\[ E[w(\bar{x}^*,b)|h^b_{t,0}] \] is the ex ante optimal standard and was explained above;

\[ w(\bar{x}^0,b^0) \], also defined above, is the observed performance.

\[ w(\bar{x}^*|h^b_{0,0},b^0) \] is the ex post optimum performance. It is the profit

\textsuperscript{26}Itami, "Evaluation," pp. 121-136, 189-192. See also, Ijiri and Itami, "Quadratic," pp. 730-732. Itami developed variance analysis systems in two different production decision settings, each of which is somewhat different from that investigated here. The variance schemes are slightly different in the two contexts investigated by Itami, reflecting the differences in the settings. The variance analysis system to be developed here parallels Itami's system for the newsboy problem more closely than that for the cost-volume decision with uncertain demand.

\textsuperscript{27}Itami's analysis of adaptive behavior is largely based on the assumption that perfect information is received. His variance analysis scheme reflects this assumption and could be extended to allow imperfect information. Itami, "Evaluation," pp. 121-136, 189-192.
level which would be achieved if the manager received perfect, time zero information about unit variable cost. Thus the first of the three disaggregate variances in (7.2) "may be called the forecasting variance or random parameter variance, because this is a variance from the ex ante standard which will result even if the manager can get perfect information about parameter values at the beginning of the period. ... The magnitude of this variance would indicate the firm's (or the manager's) ability in forecasting." \(^{28}\)

\[ w(\tilde{\lambda}^b_{x\lambda}, b^0) \] is an intraperiod standard. It is the profit level which will be achieved if perfect information about unit variable cost is received at the planned report time, \(\tilde{\tau}\. The second of the disaggregate variances in (7.2) "may be called the delay variance, because this is a variance purely due to the delay of information on parameter values. It is considered uncontrollable by the manager because the superior acknowledges that the information or adaptation may be delayed by as much as" \(^{29}\) \(\tilde{\tau}\."

The third variance in (7.2), the difference between the intraperiod standard and observed performance, is termed the adaptation variance by Itami. "This variance should be zero if the manager makes intraperiod adaptation as specified (or desired) by the superior, and may be considered completely controllable by the manager." \(^{30}\)

The Itami variance analysis system is an improvement over Demski's system in that the former explicitly accounts for receipt of intraperiod information.


\(^{29}\) Itami, "Evaluation," p. 133.

information and the resulting adaptive behavior. The inclusion of the intraperiod performance standard, \( E[w(x^*|h^b_{t,0})] \), allows evaluation of the manager's performance based on his supposed knowledge at the time. By way of contrast, in Demski's ex post system, the manager's observed performance is compared with the optimal performance based on time zero knowledge of unit variable cost. Since the manager cannot be expected to have such knowledge, his performance is being evaluated against an ideal standard rather than an attainable standard. As Itami points out, the manager can never produce a favorable variance in Demski's system, and "this persistently negative character of Demski's ex post variance might have some detrimental effect on the manager's morale."\(^{31}\)

The Itami variance analysis, however, retains some shortcomings. First, it does not explicitly address deviations in the performance of the intraperiod information system from its intended performance,\(^{32}\) given its design characteristics. Second, although Itami's delay variance considers some effects of information system design, these effects are confounded with those of the random parameter (i.e., unit variable cost). In the performance evaluation variance analysis developed below, three types of performance will be addressed in addition to the effect on attainable performance of the randomness of unit variable cost. These three types of performance are managerial performance, information system implementation performance, and information system design performance.

\(^{31}\)Itami, "Evaluation," p. 134. Support for this assertion may be found in Stedry, Budget.

\(^{32}\)This is done to a limited extent in the variance analysis system for the cost-volume decision with uncertain demand. Ijiri and Itami, "Quadratic," pp. 730-732.
7.2.2 Managerial Performance Evaluation

In evaluating managerial performance, the ex ante standard is $E[w(x^*, b)|h^b_{t, v}]$. This is the expected profit, given that (1) management receives a cost report with the accuracy and timeliness intended when the information system was designed [i.e., system $h^b_{t, v}$], and (2) management acts optimally with respect to its presumed knowledge of the production cost function at any given time [i.e., $(x^*|h^b_{t, v})$]. The total managerial performance variance (MPV), $E[w(x^*, b)|h^b_{t, v}] - w(x^0, b^0)$, is the difference between the ex ante standard and the observed level of performance (profit). This variance can be decomposed as follows:

(7.3) \[ MPV = E[w(x^*, b)|h^b_{t, v}] - w(x^0, b^0) \]
\[ = E[w(x^*, b)|h^b_{t, v}] - E[w(x^*, b) h^b_{t^0, v^0}] \]
\[ + E[w(x^*, b)|h^b_{t^0, v^0}] - w((x^*|h^b_{t^0, v^0}), b^0) \]
\[ + w((x^*|h^b_{t^0, v^0}), b^0) - w(x^0, b^0) \]

The first decomposed variance is called the information system variance (ISV), and is the difference between the ex ante standard, based on the standard information system, and the expected profit level given the observed information system. It measures the difference in the expected profit level, due to deviations in the accuracy and timeliness of the information actually received, from that of the information which would have been received if the system had worked exactly according to its design. Note that $E[w(x^*, b)|h^b_{t^0, v^0}]$ is an ex post standard with respect to the information system, but an ex ante standard with respect to the random parameter, $b$.

The ISV may be unfavorable (positive) or favorable (negative). If $t^0 > \bar{t}$ and $v^0 > \bar{v}$, then $(1-t^0) < (1-\bar{t})$ and $(\text{var}(b) - v^0) < (\text{var}(b) - \bar{v})$. Both
observed accuracy and observed timeliness are lower than standard given the information system design. In this case, the ISV will be unfavorable. Similarly, if \((1-t^0)>(1-F)\) and \((\text{var}(b)\cdot\nu^0)>(\text{var}(b)\cdot\nu)\), the variance will be favorable. If observed timeliness is higher than standard while observed accuracy is lower than standard (or vice versa). The ISV may be either unfavorable or favorable, depending on the relative extent of the deviations in timeliness and accuracy. Obviously, the production manager should not be held accountable for the ISV. The production manager's performance should be judged on the basis of his actions given the information available to him, not the information which was supposed to be available to him. There is still merit in including the ISV as a component of the disaggregate MPV, however, since it highlights the fact that deviations of the information system's implementation from its intended design are potential causes for deviations in the manager's performance from the ex ante standard.

The second decomposed variance in (7.3),
\[\{E[w(\bar{x}^*, b)|h_{t0, vo}] - w(\bar{x}^*|h_{t0, vo}, b^0)\},\]

is the difference between the expected profit given the observed information system and the ex post standard, \(w(\bar{x}^*|h_{t0, vo}, b^0)\). The latter is the profit level which will be attained if the production manager acts optimally with respect to the information he receives, given that actual unit variable cost for the period is \(b^0\). This variance measures the deviation of observed performance from the ex ante standard due solely to the influence of the random parameter, unit variable cost. It is a completely uncontrollable variance, and is not attributable to any part of the organization.
The third decomposed variance in (7.3),
\[ \{w[(x^*|h_{t0}^b,v_0),b^o] - w(x^0,b^o) \} \], is the difference between the ex post standard and the observed performance level. It is the deviation between the ex ante standard and observed performance, due to the manager’s actions, given the observed information system and observed unit variable cost. This variance is termed the managerial implementation variance (MIV) and may be unfavorable (positive) or favorable (negative).

An unfavorable variance could result from the manager’s delaying adjustment of the production rate beyond time \( t^o \), failure to adjust to the optimal rate after receipt of the intraperiod cost report, or failure to utilize of understand the cost report. A favorable variance could result if the manager, relying on instinct, experience, or luck, chooses a production rate other than the ex ante optimal rate during the sub-period \( [0,t^0] \), and that rate happens to be a good choice in an ex post sense. The MIV is controllable by the production manager, and is the variance upon which his performance is evaluated.

7.2.3 Information System Implementation Performance Evaluation

In evaluating the implementation of the information system, the ex ante standard is \( V(h_{t,v}^b) \), the expected value of the standard system, i.e., the one with standard timeliness of \( (1-t) \) and standard accuracy of \{var(b)-\overline{V}\}. This ex ante standard is the expected improvement in the profit due to using the designed information system, \( h_{t,v}^b \), over the expected profit with no information, \( E[w(x^*,b)|h_0] \). The ex post observed performance of the system is given by \( \{w[(x^*|h_{t0}^b,v_0),b^o]-w[(x^*,h_0^o),b^o]\} \). This is the actual, ex post improvement in profit, due to the use of the observed information system rather than the null system, which would
have occurred during the period if the manager had acted optimally with respect to the cost report he received, given that unit variable cost was observed to be $b^o$. In other words, it is the amount by which the optimally acting production manager would be better off during a period, when the observed unit variable cost was $b^o$, as a result of using the information from the observed system, $h^{b}_{t^o,v^o}$.

The total information system implementation performance variance (IPV) is $U(h^{b}_{b,t^o,v^o})=w((X^*|h^{b}_{t^o,v^o}),b^o)-w((X^*|h^o),b^o))$. This variance can be decomposed as follows:

\[
(7.4) \quad \text{IPV} = U(h^{b}_{b,t,v}) - w((X^*|h^{b}_{t^o,v^o}),b^o) - w((X^*|h^o),b^o)
\]

\[
= (U(h^{b}_{b,t,v}) - U(h^{b}_{t^o,v^o})))
\]

\[
+ w((X^*|h^{b}_{t^o,v^o}),b^o) - w((X^*|h^o),b^o))
\]

The first disaggregate variance is called the information system variance (ISV) and is equivalent to the ISV which is a component of the decomposed MPV (see equation 7.3). Note that $\text{ISV} = (U(h^{b}_{b,t^o,v^o}))$

\[
= E[w(X^*,b)|h^{b}_{b,t,v}] - E[w(X^*,b)|h^o] - E[w(X^*,b)|h^{b}_{t^o,v^o}] - E[w(X^*,b)|h^o]
\]

\[
= E[w(X^*,b)|h^{b}_{b,t,v}] - E[w(X^*,b)|h^{b}_{t^o,v^o}] = \text{ISV} \text{ as defined in section 7.2.2 and included in equation (7.3) as a component of the disaggregate MPV.}
\]

The ISV is that part of the deviation between the ex ante standard information system value and its ex post observed value, due to deviations in the observed timeliness and accuracy from standard timeliness and accuracy, respectively. This variance will generally be nonzero when the
information system is implemented in a way which differs from its design. Such implementation differences could result from a variety of causes including temporary substitution of cost reporting system personnel, temporary work overload for system personnel, and breakdowns in information system communication channels.

The ISV can be further decomposed as follows:

\[
\text{ISV} = \{U(h_{\pi,\nu}^b) - U(h_{\pi,\nu}^{bo})\}
\]

\[
= \{U(h_{\pi,\nu}^b) - U(h_{\pi,\nu}^b)\}
\]

\[
+ \{U(h_{\pi,\nu}^b) - U(h_{\pi,\nu}^b)\}
\]

\[
+ \{U(h_{\pi,\nu}^b - U) - U(h_{\pi,\nu}^b - U)\}
\]

The first decomposed variance in (7.5), termed the information accuracy variance (IAV), is the part of the deviation between the value of the designed information system (i.e., the ex ante standard) and that of the observed system, due to the deviation of the observed system's accuracy from standard accuracy. The second decomposed variance in (7.5), termed the information timeliness variance (ITV), is the part of the deviation between the value of the designed information system (i.e., ex ante standard) and that of the observed system, due to the deviation of the observed system's timeliness from standard. The third decomposed variance in (7.5) is the joint information accuracy/timeliness variance (IJV); it is the part of the deviation of observed system value from the ex ante standard which is due to the joint impact of

\[33\text{The variance could still be zero if timeliness and accuracy differ from their standards in opposite directions, and their effects exactly offset each other.}\]
accuracy and timeliness deviations from their respective standards. The IJV is calculated by subtracting the IAV and the ITV from the ISV as follows:

\[
(7.6) \quad IJV = ISV - IAV - ITV
\]

\[
= \{U(h^b_{t,v}) - U(h^b_{t,v_0})\} - \{U(h^b_{t,v}) - U(h^b_{t,v_0})\} - U(h^b_{t,v}) - U(h^b_{t,v_0})
\]

\[
= \{U(h^b_{t,v_0}) + U(h^b_{t,v_0})\} - U(h^b_{t,v}) - U(h^b_{t,v_0})
\]

Each of the variances IAV, ITV, and IJV may be unfavorable (positive) or favorable (negative) depending on the relationship of observed accuracy and timeliness to their respective standards. These variances are controllable by those responsible for implementing the designed cost information system. Their performance is evaluated with respect to how well they implement that system.

The second decomposed variance in equation (7.4), \(\{U(h^b_{t,v_0}) - w[(x^* | h^0_{t,v_0}), b^0]\} - w[(x^* | h^0_o), b^0]\}\), is the difference between the expected value of the observed information system and the ex post, observed value of the system, given that unit variable cost of \(b^0\) was incurred during the period, and the production manager acted optimally. This variance is termed the ex post profit improvement variance (PIV), and is solely due to the impact of the random unit variable cost. It is not controllable by anyone and is not attributed to any part of the organization. It is included in the variance analysis to highlight the fact that the ex post usefulness of any information system is contingent on random events. Managers who complain in retrospect that an information system caused them to take actions which later turned out to have adverse consequences, or those who claim, after the fact, that they would
have taken the same action with or without the information received from a system, will find that these kinds of phenomena are due to random events, and are reflected in the unattributed PIV.

7.2.4 Information System Design Performance

The design of the information system may be evaluated by assessing the loss in system value due to the selection of $(1-\overline{t})$ and $\{\text{var}(b)-\overline{v}\}$ for the timeliness and accuracy standards, respectively. The evaluation is made by comparing the expected value of the designed system with that of the ideal system, $h_{0,0}^b$, where timeliness and accuracy are perfect. The total information system design performance variance (DPV) is

\[ U(h_{0,0}^b)-U(h_{t,v}^b). \]

This variance may be decomposed as follows:

\[ \text{DPV} = \{U(h_{0,0}^b)-U(h_{t,v}^b)\} \]

\[ = \{U(h_{0,0}^b)-U(h_{0,v}^b)\} + \{U(h_{0,0}^b)-U(h_{t,0}^b)\} \]

\[ + \{U(h_{0,v}^b)-U(h_{t,0}^b)\} \]

The three disaggregate variances are named the information system design accuracy variance (DAV), the information system design timeliness variance (DTV), and the information system design joint accuracy/timeliness variance (DJV). The DAV measures the loss in standard system value due to the imperfect standard accuracy. The variance is nonfavorable (nonnegative). Similarly, the DTV measures the loss in standard system value due to the delay in standard information timing. This variance is nonfavorable (nonnegative). The DJV is due to the joint effect of standard system accuracy and timeliness, and is never unfavorable (i.e., always nonpositive).
Specification of the information system design variances would allow management to determine the extent to which expected profitability is negatively affected by designing the information system with timeliness and accuracy as given by the standards for these attributes. Making this evaluation separately permits an appropriate evaluation, via the information system implementation variances, of the way in which the designed information system is actually implemented. By not confounding these two properties of information system performance, the evaluation of both system design and system implementation will be more effective. Finally, separate evaluation of information system performance permits a more effective evaluation for the production manager. Under this system, his evaluation, via the managerial performance variances, is based on the knowledge of the uncertain parameter(s) which he was actually provided by the information system.

Thus the performance of production management, information system implementation, and information system design are evaluated independently of one another, and independently of the effects on attainable profitability of the random parameter(s). The result will be a comprehensive performance evaluation system which is more effective in motivating behavior congruent with organization objectives.

7.3 Summary

In this chapter, some normative implications were drawn from the analytical investigation of Part II. First, some rather general implications were drawn regarding the optimal design, adaptation, and use of cost information systems for the cost-volume-profit decision. These implications were developed by appealing to the sensitivity analysis,
provided in Part II of the study, of the value of various information systems under a variety of assumptions about the technology and environment. The analysis allowed the comparison of an information system's value under different sets of assumptions, and the comparison among different systems of information value for a given set of technological and environmental assumptions. Some normative insights were suggested regarding the design of management information systems in several hypothetical situations.

Second, attention was turned to the interface between the design parameters of information systems and managerial performance evaluation. An extension of traditional accounting cost variance analysis was suggested, which separates the effects of deviations in information system performance from deviations in managerial performance. The result is a performance evaluation system which is more effective in motivating managers toward behavior congruent with organizational goals.
CHAPTER 8
SUMMARY AND CONCLUSION

8.1 Review and Summary of Results

Accountants have increasingly recognized their role as producers of information. The accountant's task is that of information evaluator. Given an array of alternative information systems, the accountant must select the optimal one for the users of the information. This information production function view of accounting has stimulated research into the nature and generation of information.

The most comprehensive normative theory of information is that of information economics, which is designed to facilitate the selection of an optimal information system by explicitly taking into account the decision maker's preferences and beliefs about uncertain events.

A major concept in information economics is the value of an information system. The purpose of this study has been to identify and characterize the role played by the various factors which jointly determine the value of an information system. In Chapter 2, five direct determinants of information value were identified as follows:

(i) The nature of the decision maker's outcome and utility functions, or, equivalently, the decision maker's technology and environment and his relative preference for first-level outcomes.

235.
(ii) The structure of the action set, or, equivalently, the
decision maker's flexibility.

(iii) The degree of uncertainty in the prior probability dis-
tribution over uncertain states of nature, or equivalently,
the decision maker's initial uncertainty.

(iv) The decision maker's perception of the mapping from states
to signals, or, equivalently, the nature of the informa-
tion system.

(v) The structure of the state space, or, equivalently, the
decision maker's perception of some aspects of the tech-
nology and/or environment.

A review was provided in Chapter 2 of the literature which has dealt
rigorously with these determinants. Only a few results have been obtained
which characterize the role of a particular determinant in affecting
information value, and there has been virtually no research which addresses
the effects on information value of interactions among the determinants.
In a particular decision context, each of the direct determinants is
characterized by parameters which jointly distinguish a particular deci-
sion setting and information system from other settings and information
systems. These parameters were termed indirect determinants of informa-
tion value; they include, for example, the decision maker's wealth, risk
aversion parameters, bounds on the action set, aspects of the decision
maker's prior distribution over states of nature, and the accuracy and
timing of information. The second-order direct and interaction effects,
as well as first-order effects in some cases, of these parameters in
determining information value are complex and often defy intuition.
Consequently, research is needed to develop generally accepted measures for the various determinants (e.g., degree of uncertainty in a probability distribution) and rigorously characterize their role in determining information value in varied decision settings.

Part II of the study contains an investigation of the determinants of cost information value in a cost-volume-profit decision problem. The CVP model is an important tool in managerial accounting which provides a rather comprehensive overview of short-term planning. Chapter 3 presents the CVP model under investigation; it features a linear total revenue function and a cost function which is quadratic in the production rate. The source of uncertainty in the decision problem is one or more of the cost function parameters, and various types of systems are specified which provide information about the uncertain parameter(s).

First, fixed cost is allowed to be uncertain, and the value of a fixed cost information system is shown to be of no value in the short run. Second, unit variable cost is assumed to be the only source of uncertainty, and the value of a generalized unit variable cost information system is derived as a function the product's unit price, the coefficient of the quadratic term in the cost function, the time the information is received, the degree of uncertainty in the prior, and the nature of the information system. Sensitivity analysis is applied in order to characterize the first and second order direct and interaction effects of determinants on the value of the information system. Information system value is found, for example, to be decreasing at a constant rate in the delay of the information. The precise mathematical statements of information system value and effects of the determinants are restated in intuitive economic
terms, and then interpreted, with varying levels of precision, from an economic perspective. Third, the value of a perfect unit variable cost information system is derived and several characterizations are made of the role of various information value determinants which were not possible for the generalized system.

In Chapter 4, the investigation of a unit variable cost information system is continued with a mild restriction placed on the producer's prior probability distribution over uncertain unit variable cost. The restriction requires that all values of unit variable cost which are assigned nonzero probability be less than or equal to the unit price. This requirement has the effect of precluding the optimality of a temporary shut-down for the production process. Under this restriction, stronger results are obtained for the role of several information value determinants. Generalized, perfect, and imperfect noisy and noiseless systems are investigated. One of the determinants for the value of the imperfect systems is accuracy. Information value is found to exhibit decreasing marginal increments with respect to accuracy. If an imperfect system's accuracy improves over time, the rate at which the accuracy responds to time is found to be a major information value determinant. In this case, the optimal information time is interior to the period and is a function of only the system's accuracy responsiveness parameter. The precise mathematical statements of information system value and effects of value determinants are restated and interpreted from an economic perspective.

Chapter 5 extends the analysis of the CVP problem in two ways. First, the coefficient of the quadratic term in the cost function is assumed to
be the only source of uncertainty. The value determinants of generalized and perfect information systems are investigated, and the results are interpreted in economic terms. Second, action flexibility is investigated as a determinant of the value of a perfect unit variable cost information system. Flexibility in the CVP problem is characterized in two ways: (1) the magnitude of the upper bound on the production rate, and (2) the lot size in a batch production operation. Information value is found to be increasing in the capacity constraint, but no monotonic relationship exists, in general, between information value and production lot size.

Chapter 6 generalizes the analysis of Chapters 3 through 5 in two ways. The assumption of perfect competition, implied by the linear total revenue function, is relaxed and monopoly is allowed. The parameters of the inverse demand function are found to be determinants of unit variable cost information value. The noisy, single-report unit variable cost information system of Chapter 4 is generalized to a noisy, n-report system where the accuracy of each report increases as it is delayed. The determinants of the value of this system are investigated and various multiple report timing strategies are considered in light of their effect on information system value.

Chapter 7 provides a discussion of some of the normative managerial implications of the investigation. Two types of implications are considered. First, some rather general inferences are drawn regarding the optimal design, use, and adaptation of cost information systems intended for the CVP setting. These inferences draw upon diverse parts of the analytical investigation, and summarize results across a variety of
information systems and assumptions about the technology and environment.

Sensitivity analysis was extensively conducted in the investigation of Part II in order to determine which parameters affect information value under a given set of circumstances and how their impact is altered as circumstances vary. As a result, statements can be made about the comparative value of information systems. Two basic types of comparisons can be made as follows: (1) comparison of the value of a particular information system under differing assumptions about the technology and environment, and (2) comparison of the values of two distinct information systems under a given technology and environment.

The inferences drawn from this comparative static analysis provide insights into the way in which managers should react to changes in technology and environment to optimally design, adapt and use management information systems. Perhaps the most important point of the discussion is that factors external to the information system (i.e., extra-system parameters) may, in a given situation, be as important as, or more important than, system design parameters in determining the usefulness of an information system.

In addition to these general managerial implications, Chapter 7 provides a discussion of the interface between the design parameters of information systems and managerial performance evaluation. An extension of traditional accounting cost variance analysis is suggested, which separates the effects of deviations in information system performance from deviations in managerial performance. The result is a performance evaluation system which is more effective in motivating managers toward behavior congruent with organizational goals.
8.2 Limitations of the Study

8.2.1 General Limitations

This research is a study in the economics of information, and, as such, rests on a foundation of statistical decision theory. Two major assumptions are made in this theory and are, therefore, general limitations of the research. First, the decision maker is assumed to be rational in the sense that he possesses a utility function such that actions are ordered, with respect to preference, by the criterion of expected utility maximization. This assumption, or a similar one, is required in any normative study since a prescriptive rule requires some criterion for optimization.¹

Second, decision theory assumes costless analysis. The costs of specifying the utility and outcome functions, prior probability distribution, likelihood function and action set, as well as solving the resulting decision model, are not incorporated into a decision theoretic analysis.

Other general limitations also apply to this study. The cost-volume-profit problem is a single-person problem. There is only one decision maker to receive the information, and the decision maker is assumed to be the same individual as the information evaluator. The analysis addresses the determinants of the gross value of information systems; the cost of information has not been taken into account. The information systems investigated are assumed to provide cost information only for the purpose of the CVP decision. This, of course, ignores the multiplicity of uses.

to which cost accounting information is typically put.

Finally, the CVP analysis is conducted in terms of a simplified decision model. The model is simplified along several dimensions (e.g., single period analysis), and these simplifications are discussed individually in section 8.2.2. Analysis of a simplified model is partially justified, however, by the costliness of decision theoretic analysis. Specification and solution of decision models become more costly as the model becomes more complete (less simplified). Consequently, when the costs of analysis are considered, a simplified decision model will, in general, be optimal rather than a complete model.  

3

8.2.2 Specific Limitations

The specific limitations of the study include the various dimensions along which the CVP decision model is simplified. The most important of these limitations are listed below:

(i) Through much of the analysis, a linear total revenue function is assumed; this implies a perfectly competitive market for the firm's product. This assumption is eventually relaxed to allow a monopolistic setting.

(ii) The CVP model assumes a cost function which is quadratic

---

2See Demski and Feltham, Cost Determination for an explanation of complete versus simplified decision models.

in the production rate. This function is a reasonable approximation to the cubic cost function, often assumed by economists, in all but extreme ranges of output, and is an improvement over the linear function often assumed in accounting CVP studies.

(iii) Throughout most of the analysis, a single source of uncertainty is assumed. Only one of three cost function parameters is allowed to be uncertain at a time.

(iv) The producer is assumed to have the capability of costlessly altering the production rate one or more times.

(v) The study is a single period analysis, and does not take into account the effects of current period actions or information on outcomes and uncertainty, respectively, in future periods. Such an assumption is typical for CVP analysis, since it is largely a short-run planning tool. This limitation might be mitigated, to some extent, in some contexts due to the fact that information evaluation models may, under appropriate conditions, be decomposed into two components for the present and future time periods, respectively.\textsuperscript{4}

(vi) A linear utility function is assumed, thus implying a risk neutral decision maker.

(vii) Throughout most of the analysis, infinite divisibility of the production quantity is assumed. This assumption is relaxed in Chapter 5 and consideration is given to a

batch production process.

(viii) In much of the analysis, the unit variable cost is assumed to be less than or equal to the unit price. This requirement precludes optimality of a short-run shut-down decision.

(ix) At several points in the investigation specific probability density functions are assumed (e.g., rectangular, gamma).

(x) The usual assumptions of well developed information economics models are in force. For example, perfect implementation of actions is assumed.\(^5\)

The economic interpretations of the results in Part II, as well as the managerial implications of the results, presented in Chapter 7, are subject to the general and specific limitations mentioned above. Naturally, such limitations restrict the generality of the findings. There is a trade-off between structure for the purpose of tractability and insights and restriction of applicability of results. This trade-off exists with any research methodology, and one can only hope to strike the proper balance.

8.3 Suggested Extensions

The research reviewed in Chapter 2, along with this study, has only begun to address the many issues involving the determinants of information

---

value and their effects. Much more research is necessary before evalu-
ators of information systems will have a reasonably good understanding of the role played by the factors which are important in determining information value. There are several extensions to this study which might be fruitful areas in which to begin. Some of these are listed below.

(1) The most obvious extension of the analytical investigation in Part II of this study is to relax the assumptions mentioned in section 8.2.2. Four such assumptions appear to be the most restrictive. These are single period analysis, costless information, linear utility function, and single source of uncertainty. It might be possible to relax the first assumption via a dynamic programming approach. The effects of current period actions and information on future periods would have to be explicitly introduced into the model. The second assumption, costless information, is easily relaxed, but there does not appear to be much point in doing so until some empirical evidence is gathered to suggest particular forms for information system cost functions. Now that the determinants of information value have been identified, it may be possible to characterize such cost functions with the determinants as arguments. The third and fourth assumptions may be difficult to relax without losing analytical tractability of the problem. Simulation may be an attractive alternative if this direction is chosen.

(2) The type of analysis that has been conducted in this study on the cost-volume-profit decision can be done in other decision contexts. Examples include capital budgeting, inventory, market speculation and
forecasting models.\(^6\)

(3) It would be desirable to increase the level of generality of the analysis from the highly specific approaches taken by this study and others (e.g., Itami\(^7\)). One possible way in which to pursue such generality would be to establish bounds on results, rather than precise results, by appealing to probability inequalities.

(4) There is still a need for further development of definitions and measures for some of the determinants of information value, (e.g., flexibility, degree of uncertainty, and accuracy of information systems).

(5) It may be worthwhile to explicitly introduce into the information economics model disaggregate model components. For example, the information system might be characterized by a signal generator (i.e., that part of the information system which interacts with the environment), data bank, transmission channel, decoding component, signal receiver, and signal interpreter.\(^8\) Such a characterization might suggest other indirect determinants which would be worthwhile to investigate (e.g., length of transmission channel).

(6) One very important determinant of accounting information value is its level of aggregation. An investigation of the role of this


\(^7\) Itami, "Evaluation."

determinant in several accounting contexts would be a worthwhile direction for future research.

8.4 Conclusion

This study has identified the determinants of information value and characterized their role in a cost-volume-profit decision setting. Part I of the study justified viewing accounting as an information function, presented the general information evaluation model from the normative theory of information economics, and reviewed the literature which addresses the determinants of information value. Part II presented the analytical investigation of the CVP problem. Part III provided some managerial implications of the analytical investigation, summarized the findings of the study, suggested extensions for the research, and discussed its limitations.

Despite limitations, this research has the potential to significantly increase understanding of the role played by the factors which jointly determine the value of information systems. Hopefully, this line of inquiry will be pursued by others, in order that a more complete and generalizable theory will be developed. Development of such a theory should prove useful to accountants in their role as information evaluators.
APPENDIX A

SOME FUNDAMENTALS OF INFORMATION ECONOMICS

Statistical decision theory is a normative framework for optimization in decision problems which confront uncertain environments. The theory is constructed with four major elements: the set of actions, the set of states of nature, the outcome and utility functions, and the probability distribution on the states. In this study the set of possible actions available to the decision maker (DM) is denoted by \( \{x\} = X \), where \( x \) is a particular action. The uncertain states of nature represent the various conditions of the environment which the DM may face, and are not subject to his control or influence. The set of states is denoted by \( \{s\} = S \), where \( s \) is a particular state. The elements of both \( X \) and \( S \) are mutually exclusive and collectively exhaustive.

An important distinction exists between a state of nature, \( s \), which is an environmental condition over which the decision maker has no influence, and another type of environmental condition called an outcome. An outcome results when a particular action is taken and a particular state of nature occurs. \( \{z\} = Z \) is used to denote the set of outcomes. The elements of \( Z \) are mutually exclusive and collectively exhaustive. Each outcome \( a \) is a function, denoted by \( w \), of an action-state pairing. That is \( z = w(x,s) \). It could be said, then, that the DM and nature share in determining which \( z \in Z \) will occur.
The deterministic mapping \( w \) from \( X \times S \) (the Cartesian product of the sets \( X \) and \( S \)) to \( Z \) will be referred to as the outcome function, and is one component of the third element of the decision theoretic structure. The other component is the utility function, denoted by \( u \). The argument of \( u \) is \( z \), the outcome. \( u \) represents a preference ordering over outcomes, \( z \in Z \), and is unique only up to a positive linear transformation.

The fourth element of the decision theoretic structure is the DM's subjective probability distribution over the states of nature. \( S \) may be regarded as a random variable. Hence, \( S \) is jointly regarded as the set \( S = \{ s \} \) and the random variable \( S \), a particular value of which is \( s \). The subjective probability density function (pdf) for the random variable \( S \) is denoted by \( p(s) \).

The purpose of a decision theoretic formulation is to enable the DM to maximize expected utility, with respect to his beliefs and preferences, by selecting the optimal \( x \in X \). The expected utility for a given action is:

\[
(A1) \quad E(utility|x) = \int_{s \in S} u(w(x,s)) \ p(s) \ ds
\]

The maximum expected utility is given by:

\[
(A2) \quad E(utility|x^*) = \max_{x \in X} \ E(U|x) = \max_{x \in X} \ \int_{s \in S} u(w(x,s)) \ p(s) \ ds
\]

where \( x^* \) is the optimal action.

The decision theoretic model has been extended to incorporate information about the uncertain states of nature. This development draws upon economic theory to build a normative framework for information evaluation in a decision theoretic context. An elaborate general theory of information economics can be found in sources such as Marschak
and Radner, Feltham and Demski. Central to information economics is the concept of a signal or message received by the DM. The set of all such signals is denoted by \( \{ y \} = Y \). The elements of \( Y \) are mutually exclusive and collectively exhaustive. There exists a correspondence between \( S \) and \( Y \). The DM's perception of this correspondence can be fully characterized by his likelihood function, which is the conditional pdf \( p(y|s) \). Using Bayes' Theorem the DM can update his prior pdf on \( S \), \( p(s) \), to obtain his posterior pdf, \( p(s|y) \). This transformation is made as follows:

\[
(A3) \quad p(s|y) = \frac{p(y|s) \ p(s)}{\int_{S} p(y|s) \ p(s) ds}
\]

With this updated probability distribution on the states of nature, the DM will (normatively) optimize his expected utility by using the information contained in \( y \). For a particular action \( x \) and signal \( y \), the DM's expected utility is given by:

\[
(A4) \quad E(utility|x,y) = \int_{S} u(w(x,s)) \ p(s|y) ds
\]

The optimization problem for a given signal \( y \) is:

\[
(A5) \quad \max_{x \in X} \int_{S} u(w(x,s)) \ p(s|y) ds
\]

Since the DM is uncertain as to which \( y \in Y \) will be received before he receives a signal, the DM's maximum expected utility prior to receiving a signal is:

\[1\text{ Marschak and Radner, Economic.}
\[2\text{ Feltham, Information Evaluation.}
\[3\text{ Demski, Information Analysis.} \]
(A6) \( E(utility) = \int \left( \max_{y \in Y} \int u(w(x,s)) p(s|y) ds \right) p(y) dy \)

\[ = \int \max_{y \in Y} \int u(w(x,s)) p(s|y) p(y) ds dy \]

In this expression, \( p(y) \) is the DM's marginal pdf on \( Y \) and is given by:

\[ p(y) = \int_{s \in S} p(y|s) p(s) ds \]

The DM may have several information systems available from which to select. \( \{h\} = H \) will denote this set, where \( h \) is a particular information system. Let \( p(s|y,h) \) denote the DM's updated pdf on \( S \) given signal \( y \in Y \) from information system \( h \in H \). The DM may now compute the value of information system \( h \). There are three concepts for the value of information in the literature. The first is given by (A7) below, where the value of information system \( h \), \( U(h) \), is given in units of utility:

(A7) \[ U(h) = \int_{y \in Y} \max_{x \in X} \int u(w(x,s)) p(s|y,h) p(y|h) ds dy - \max_{x \in X} \int u(w(x,s)) p(s) ds \]

The right-hand side of (A7) was formed by replacing \( p(s|y) \) and \( p(y) \) in (A6) with \( p(s|y,h) \) and \( p(y|h) \), respectively, and subtracting (A2) from (A6). Therefore, (A7) is the DM's expected utility when optimally using information system \( h \) minus his maximum expected utility when using no information. \( U(h) \) is the expected improvement in the DM's utility from using \( h \), and is called the utility of information system \( h \). (A7) is identical to equation (2.1) in section 2.1 of this study.

Alternatively, the DM may wish to express the value of information system \( h \) in the same units as \( w(x,s) \). In many decision situations this would be monetary units. Let \( \hat{F}(h) \) denote the amount, measured in units of \( w(x,s) \), that the DM gives up in exchange for information system \( h \). For any action-state pair, \((x,s)\), the net outcome to the DM is
is \( w(x,s) - \hat{F}(h) \), and the resulting utility is \( u[w(x,s) - \hat{F}(h)] \). The expected utility using system \( h \) is given by:

\[
\text{(A8)} \quad \int \max_{y \in Y} \int u[w(x,s) - \hat{F}(h)]p(s|y,h)p(y|h)dsdy
\]

Expression (A8) was formed using reasoning similar to that used in forming (A6). The maximum amount in units of \( w(x,s) \) that the DM would be willing to give up in exchange for information system \( h \) is the amount \( \hat{F}(h) \) such that the DM's expected utility using the system just equals his expected utility without it. Denote this quantity by \( F(h) \), which is implicitly defined by equation (A9).

\[
\text{(A9)} \quad \max_{x \in X} \int u[w(x,s) - F(h)]p(s|y,h)p(y|h)dsdy = \max_{x \in X} \int u(x)ds
\]

Equation (A9) is identical to equation (2.3) in section 2.1 of this study. \( F(h) \) is called the demand value of information system \( h \).

Let \( G(h) \) denote the amount, measured in units of \( w(x,s) \), that the DM receives in exchange for giving up information system \( h \). For any action-state pair, \((x,s)\), the net outcome to the DM is \( w(x,s) + \hat{G}(h) \), and the resulting utility is \( u[w(x,s) + \hat{G}(h)] \). The expected utility, having relinquished system \( h \), is given by:

\[
\text{(A10)} \quad \max_{x \in X} \int u[w(x,s) + \hat{G}(h)]p(s)ds
\]

Equation (A10) was formed using reasoning similar to that used in forming (A1). The minimum amount in units of \( w(x,s) \) that the DM would be willing to accept in exchange for giving up information system \( h \) is the \( \hat{G}(h) \) such that the DM's expected utility using the system just equals his expected utility without it. Denote this quantity by \( G(h) \), which is implicitly defined by equation (A11).
\[(A11) \int \max_{y \in Y} \int u[w(x,s)] p(s|y,h) p(y|h) ds \, dy = \max_{x \in X} \int u[w(x,s) G(h)] p(s) ds \quad x \in X, \quad s \in S\]

Equation (A11) is identical to equation (2.4) in section 2.1 of this study. \(G(h)\) is called the supply value of information system \(h\).
APPENDIX B
NOTATION AND DEFINITIONS

a  coefficient of quadratic term in quadratic production cost function
b  unit variable cost
c  period fixed cost
f[E(b),r] \begin{cases} 
\frac{t(r-E(b))^2}{4a} + \frac{(1-t)}{4a} \int_{Y}^{} [r-E(b)|y|^2] p(y) dy, & \text{if } E(b) < r \\
\frac{(1-t)}{4a} \int_{Y}^{} [r-E(b)]^2 p(y) dy, & \text{otherwise}
\end{cases}
g[E(b),r] \begin{cases} 
\frac{(r-E(b))^2}{4a}, & r < E(b) \\
0, & \text{otherwise}
\end{cases}
h  an information system
i,j,k,l  indeces, integers
i  \text{int}\left[\frac{i}{2} + \frac{(r/2a)}{i}\right]
m(\phi,t)  \text{int}\left[t/\phi(1-t)\right]+1, \text{the number of equal cells in a partition on } [a,\beta]

\begin{cases} 
\text{Chapter 6: number of reports from multiple-report, unit variable cost information system} \\
\text{Appendix C: number of states of nature} \\
\text{Appendix C: number of states with nonzero probability}
\end{cases}
p(\cdot)  \text{probability density function (pdf) over } (\cdot)
p(s)  \text{prior pdf over states of nature}
p(y|s)  \text{likelihood function}

254.
\( p(s|y) \)  \text{posterior pdf over states of nature} \\
\( p(y) \)  \text{pdf over signals} \\
\( q \)  \text{parameter in monopolist's inverse demand function} \\
\( r \)  \begin{align*}
&\text{under perfect competition: unit price} \\
&\text{under monopoly: parameter in monopolist's inverse demand function}
\end{align*} \\
\( s \)  \text{state of nature} \\
\( s \)  \text{cell in payoff adequate partition on } S \\
\( t \)  \text{fraction of single period expired at receipt of report from single-report information system} \\
\( t_i \)  \text{fraction of single period expired at receipt of } i\text{th report from multiple-report information system} \\
\( \bar{t} \)  \text{expected report timing in random-time, single-report information system} \\
\( \bar{t} \)  \text{vector of report times} \\
\( u \)  \text{utility function; maps outcomes, } z \in \mathbb{Z}, \text{ into utility space} \\
\( w \)  \text{outcome function; maps act-state pairs, } (x,s), \text{ into outcome space} \\
\( x \)  \text{an action} \\
\( x_1 \)  \text{quantity produced in production subperiod, } [0,t] \\
\( x_2 \)  \text{quantity produced in production subperiod, } [t,1] \\
\( x_i \)  \text{quantity produced in production subperiod prior to receipt of } i\text{th report from multiple-report information system} \\
\( x \)  \text{action vector} \\
\( x^* \)  \text{feasible action} \\
\( x^* \)  \text{optimal action} \\
\( x^* \)  \text{optimal action vector} \\
\( y \)  \text{a signal} \\
\( Y_t \)  \text{signal at time } t
$\mathbf{Y}_i$  
vector of signals, $(y_1, \ldots, y_i)$

$z$  
an outcome; $z = \omega(x,s)$ (for example, $z$ is profit in Part II)

$A(t), \hat{A}(t)$  
information accuracy factor for noisy [noiseless] information system

$A,B$  
matrices

$C$  
\begin{align*}
\text{Chapter 5: vector function} \\
\text{Appendix C: matrix}
\end{align*}

$D$  
upper bound on production rate.

$E$  
expectation operator

$E^2(b)$  
$[E(b)]^2$

$E^2(b|y)$  
$[E(b|y)]^2$

$F(h)$  
demand value of information system, $h$; see equation (2.3) or Appendix A

$F_i(\cdot)$  
ith stage cumulative maximand in dynamic program

$G(h)$  
supply value of information system, $h$; see equation (2.4) or Appendix A

$H=\{h\}$  
set of information systems

$J[E(b),r]$  
\begin{align*}
\left\{ \begin{array}{l}
\int_{y \in \mathcal{Y}} [r-E(b|y)]^2 p(y)dy - [r-E(b)]^2, E(b) \leq r \\
\int_{y \in \mathcal{Y}} [r-E(b|y)]^2 p(y)dy, \text{otherwise}
\end{array} \right.
\end{align*}

$K(i)$  
\begin{align*}
\mathbb{E} \left[ \frac{1}{[E(a|y)-q]I} - \frac{1}{[E(a)-q]I} \right]
\end{align*}

$L$  
\begin{align*}
\left\{ \begin{array}{l}
\int_{y \in \mathcal{Y}} [r-E(b|y)]^2 p(y)dy - [r-E(b)]^2, E(b) \leq r \\
\int_{y \in \mathcal{Y}} [r-E(b|y)]^2 p(y)dy, \text{otherwise}
\end{array} \right.
\end{align*}

$M$  
rank of matrix, $A$

$N$  
number of potential signals

$O(t), \hat{O}(t)$  
outcome magnitude factor for noisy [noiseless] information systems
P  cumulative distribution function (cdf)
R  mapping from states to signals
R^i  ith partition on S
R  return function in dynamic program
S={s}  state space
S={A}  payoff adequate partition on S
T  random information time
U(h)  utility of information system h; see equation (2.1) or Appendix A.
X={x}  set of actions
Y={y}  set of signals
Y  \{y|E(b|y)\geq r\}
Y_t={y_t}  set of signals at time t.
Z={z}  set of outcomes
\alpha, \beta  scalars
\gamma  dummy variable
\delta  accuracy responsiveness parameter for noisy information system
\zeta  payoff function; maps act-state pairs, (x,s), into utility space
\theta_ji  p(y_j|s_i)
\kappa(z)  absolute risk aversion function; defined by \(-u''(z)/u'(z)\).
\lambda  batch size in batch production process
\mu  expectation of gamma pdf
\nu  see page 270
\xi  scalar
\pi_j  p(s_i)
\rho  ith stage state variable
\( \sigma^2 \)  
variance of gamma pdf

\( \tau \)  
length of the production period

\( \nu \)  
Kuhn-Tucker multiplier

\( \phi \)  
accuracy responsiveness parameter in noiseless information system

\( \omega \)  
Kuhn-Tucker multiplier

\( \Gamma(\alpha) \)  
\( \int_0^\infty \gamma e^{-\gamma} d\gamma, \alpha > 0 \), where e denotes the exponential function

\( \delta - \alpha \), a scalar

\( \Lambda \)  
\( \{0, \lambda, 2\lambda, 3\lambda, \ldots\} \)

\( \Pi \)  
multiplication operator

\( \Sigma \)  
summation operator

\( \psi, \hat{\psi}, \Psi, \hat{\Psi} \)  
transformed payoff functions from state space to utility space, where the action variable has been eliminated by the maximization operation

\( \Omega \)  
a numerical function

\( \nabla \)  
gradient

\( \sim \)  
has the distribution

\( \emptyset \)  
nul set

\( \vec{0} \)  
zero vector

\( \text{int}[\cdot] \)  
largest integer less than or equal to its argument

\( \ln \)  
natural logarithmic function

\( \text{var}(b) \)  
variance of \( p(b) \)

\( \text{var}(b|y) \)  
variance of \( p(b|y) \)

\( \text{var}(b|y_i) \)  
variance of \( p(b|y_1) \)

\( E_{y_1} \left[ \text{var}(b|y_1) \right] = E_{y_1} \left[ \cdots E_{y_1} \left( E \left( \text{var}(b|y_1) \right) |y_{i-2}y_{i-1}y_i \right) \right) \right) \right) \right) \cdots |y_1) \)

\( S_i \)  
ith cell in partition on \( S \)

\( S_j^i \)  
jth cell in partition \( i \) on \( S \)
APPENDIX C

MAXIMIZATION OF INFORMATION VALUE WITH RESPECT TO DISCRETE PRIOR STATE PROBABILITIES

C.1 Proof of Theorem 2.1, Section 2.2.3

Suppose that the state variable, s, is distributed discretely with p(s_i)=\pi_i, i=1,...,n. Let \zeta_i(x) denote \zeta(x,s_i), the utility (payoff) in state i given action vector x, and assume that \zeta_i(x) is concave for all i. Let x^*=(x_1^*,...,x_m^*) denote the solution to the DM's problem given no information, which is the following:

(C1) \max_{x \in \mathcal{X}} \sum_{i=1}^{n} \pi_i \zeta_i(x)

Since \zeta_i(x) is concave in x for all i, (C2) is a necessary and sufficient condition for the solution to (C1):

(C2) \frac{\partial}{\partial x_{k}} \sum_{i=1}^{n} \pi_i \zeta_i(x)|_{x=x^*} = 0, k=1,...,m

Or, equivalently:

(C3) \sum_{i=1}^{n} \pi_i \frac{\partial \zeta_i(x^*)}{\partial x_{k}} = 0, k=1,...,m

Suppose that the information system consists of N potential signals \{y_1,...,y_N\}=Y, and let p(y_j|s_i)=\theta_{ij}, j=1,...,N. Let x^{j^*}=(x_1^{j^*},...,x_m^{j^*}) denote the optimal action vector given that signal y_j is received. x^{j^*} is the solution to the following problem:

(C4) \max_{x \in \mathcal{X}} \sum_{i=1}^{n} \pi_i \zeta_i(x) \left[ \begin{array}{c} \frac{\theta_{ij}}{\sum_{k \neq j} \theta_{ik} \pi_k} \\ \frac{\theta_{ij}}{\sum_{k \neq j} \theta_{ij} \pi_k} \end{array} \right]

The term in brackets in (C4) is the application of Bayes' Theorem.
to derive the posterior probability of state $s_i$ given signal $y_j$. Since $\zeta_i$ is concave in $\bar{x}$ for all $i$, (C5) is a necessary and sufficient condition for the solution to (C4):

$$\left. \frac{n}{\sum_{i=1}^{n} \zeta_i(x)} \left[ \frac{\theta_{ij} \pi_i}{\sum_{k \neq l \neq j} \theta_{jk} \eta_{i}} \right] \right|_{x, x' = x} = 0, \; i = 1, \ldots, m$$

Or, equivalently:

$$\left. \frac{n}{\sum_{i=1}^{n} \zeta_i(x)} \left[ \frac{\theta_{ij} \pi_i}{\sum_{k \neq l \neq j} \theta_{jk} \eta_{i}} \right] \right|_{x, x' = x} = 0, \; i = 1, \ldots, m$$

**Theorem 2-1 (repeated):** Given that the payoff function, $\zeta$, is concave in action vector, $\bar{x}$, when the value [using the utility measure, $U(h)$] of the DM's information system is at its maximum, with respect to the DM's prior probabilities over states of nature, $(\pi_1, \ldots, \pi_n)$, the following relationship holds for all states $k$ and $l$ which are assigned nonzero probabilities:

$$(C7) \sum_{j=1}^{N} \left[ \frac{\theta_{jk} \zeta_k(x^*)}{\sum_{i=1}^{n} \zeta_i(x)} \right] - \zeta_k(x^*) = \sum_{j=1}^{N} \left[ \frac{\theta_{jk} \zeta_k(x^*)}{\sum_{l=1}^{n} \zeta_l(x)} \right] - \zeta_k(x^*)$$

**Proof:** The prior probabilities, $\pi_1, \ldots, \pi_n$, which maximize the utility of information are the solution to the following problem:

$$\max_{\pi_1, \ldots, \pi_n} \sum_{i=1}^{N} \left[ \frac{\theta_{ij} \pi_i}{\sum_{k \neq l \neq j} \theta_{jk} \eta_{i}} \right] \pi_i \sum_{j=1}^{N} \zeta_j(x) \pi_j$$

subject to:

$$\sum_{i=1}^{n} \pi_i = 1$$

$$\pi_i \geq 0, \; i = 1, \ldots, n$$

In (C8), $\sum_{k=1}^{n} \theta_{jk} \pi_k$ is the probability of signal $y_j$, $p(y_j)$.

---

1The proof proceeds in exactly the same way as Gould's proof of the analogous result for perfect information. See Gould, "Risk," pp. 69-70.
The objective function of nonlinear program (C8) can be restated as follows:

\[(C9) \quad \sum_{j=1}^{N} \sum_{i=1}^{n} \xi_i(x_j^*) \theta_{ji} \pi_{i1} - \sum_{i=1}^{n} \xi_i(x^*) \pi_{i} \]

\[(C10) - \sum_{j=1}^{N} \left( \sum_{i=1}^{n} \pi_{i} \left[ \theta_{ji} \xi_i(x_j^*) - \frac{\xi_i(x^*)}{n} \right] \right) \]

The entire nonlinear program may be restated as follows:

\[(C11) \quad \max_{\pi_1, \ldots, \pi_n} \sum_{i=1}^{n} \pi_{i} \left[ \theta_{ji} \xi_i(x_j^*) - \frac{\xi_i(x^*)}{n} \right] \]

subject to: \( \sum_{i=1}^{n} \pi_{i} - 1 \leq 0 \)

\[- \sum_{i=1}^{n} \pi_{i+1} \leq 0 \]

\[- \pi_{i} \leq 0, \; i=1, \ldots, n \]

The Kuhn-Tucker conditions\(^2\) for this problem are given by (C12) through (C19).

\[(C12) \quad \sum_{j=1}^{N} \left\{ \theta_{j} \xi_{j} \frac{\partial f_{j}(x^*)}{\partial x_{k}} - \frac{\xi_{k}(x^*)}{n} \right\} \]

\[+ \sum_{i=1}^{n} \pi_{i} \left[ \theta_{ji} \xi_{i} \frac{\partial f_{i}(x^*)}{\partial x_{k}} - \frac{\xi_{i}(x^*)}{n} \right] ; \; N \leq 1 \]

\[+ \omega_{i}^1 - \omega_{i}^n - u_{k} = 0, \; k=1, \ldots, n \]

\[(C13) \quad \sum_{i=1}^{n} \pi_{i} - 1 \leq 0 \]

\[(C14) \quad - \sum_{i=1}^{n} \pi_{i+1} \leq 0 \]

\[(C15) \quad - \pi_{i} \leq 0, \; i=1, \ldots, n \]

\(^2\) Mangasarian, *Nonlinear*, p. 94.
(C16) \( \omega' (\sum_{i=1}^{n} \pi_i - 1) = 0 \)

(C17) \( \omega'' (- \sum_{i=1}^{n} \pi_i + 1) = 0 \)

(C18) \( u_i (- \pi_i) = 0, \ i = 1, ..., n \)

(C19) \( \omega', \omega'', u_k \geq 0, \ k = 1, ..., n \)

\( \omega', \omega'' \), and \( u_k, (k = 1, ..., n) \), are the Kuhn-Tucker multipliers.

Condition (C12) may be restated as follows:

\[
(C20) \quad - \sum_{j=1}^{N} \left[ \left( \frac{\partial \xi_{i}^{*} (x^*)}{\partial x_{k}} \right) \frac{\partial x_{k}}{\partial \pi_{j}} \right] \]
\[
\quad + \sum_{k=1}^{m} \left( \frac{\partial \xi_{i}^{*} (x^*)}{\partial x_{k}} \right) \frac{\partial x_{k}}{\partial \pi_{j}} \left( \frac{1}{N} \sum_{i=1}^{n} \pi_{i} \frac{\partial \xi_{i}^{*} (x^*)}{\partial x_{k}} \right) \frac{\partial x_{k}}{\partial \pi_{j}} \right] \}
\]
\[\omega' - u_k = 0, \ k = 1, ..., n \]

where \( \omega = \omega' - \omega'' \). By condition (C6), for the maximization of the DM's problem given the receipt of signal \( y \), \( \left( \frac{n}{\sum_{i=1}^{n} \pi_{i} \xi_{i}^{*} (x^*)} \right) = 0 \) for all \( \pi \).

Moreover, by condition (C3), for the maximization of the DM's problem given no information, \( \left( \frac{1}{N} \sum_{i=1}^{n} \pi_{i} \frac{\partial \xi_{i}^{*} (x^*)}{\partial x_{k}} \right) = 0 \) for all \( \pi \).

Therefore, the second term on the left-hand side of equation (C20), \( \sum_{k=1}^{m} \left[ ... \right] \) is zero. Kuhn-Tucker condition (C12) then reduces to the following:

\[
(C21) \quad - \sum_{j=1}^{N} \left[ \theta_{jk} \xi_{k}^{*} (x^*) - \xi_{k} (x^*) \right] + \omega - u_k = 0, \ k = 1, ..., n
\]

For any \( \pi_k \) which is nonzero, by Kuhn-Tucker condition (C18), \( u_k \) is zero, and equation (C21) reduces to the following:

\[
(C22) \quad \sum_{j=1}^{N} \left[ \theta_{jk} \xi_{k}^{*} (x^*) \right] - \xi_{k} (x^*) = \omega, \ \text{for} \ \{k \in \{1, ..., n \} | \pi_k > 0 \}
\]

Equivalently:

\[
(C23) \quad \sum_{j=1}^{N} \left[ \theta_{jk} \xi_{k}^{*} (x^*) \right] - \xi_{k} (x^*) = \sum_{j=1}^{N} \left[ \theta_{jk} \xi_{k}^{*} (x^*) \right] - \xi_{k} (x^*), \ \text{for}
\]
C.2 Proof of Theorem 2, Section 2.2.3

Theorem 2-2 (repeated): Given that the payoff function, \( \xi_i \), is concave in \( x = (x_1, \ldots, x_m) \), for \( i = 1, \ldots, n \), the number of nonzero probabilities needed to maximize the value of the information system is in the set \( \{2, \ldots, m+1\} \).

Proof: Suppose the number of nonzero probabilities in the solution to (C11) is \( \hat{n} \). With no loss of generality, let the first \( \hat{n} \) states have nonzero probability in this solution. \( \bar{x}^* \), by condition (C3), satisfies the following system.

\[
\begin{align*}
\pi_1 \frac{\partial \xi_1(x^*)}{\partial x_1} + \pi_2 \frac{\partial \xi_2(x^*)}{\partial x_1} + \cdots + \pi_n \frac{\partial \xi_n(x^*)}{\partial x_1} &= 0 \\
\pi_1 \frac{\partial \xi_1(x^*)}{\partial x_2} + \cdots + \pi_{\hat{n}} \frac{\partial \xi_{\hat{n}}(x^*)}{\partial x_2} &= 0 \\
\vdots & \quad \vdots \\
\pi_1 \frac{\partial \xi_1(x^*)}{\partial x_m} + \cdots + \pi_{\hat{n}} \frac{\partial \xi_{\hat{n}}(x^*)}{\partial x_m} &= 0
\end{align*}
\]

(C24)

If \( \hat{n} = m+1 \), the assertion of the theorem is true. Therefore, suppose \( \hat{n} > m+1 \), and rewrite system (C24) as follows:

(C25) \( AB = C \)

where:

---

3The proof proceeds in exactly the same way as Gould's proof of the analogous theorem for perfect information. See Gould, "Risk," pp. 73-74.
\[
A = \begin{bmatrix}
\frac{\partial \xi_1(x^*)}{\partial x_1} & \cdots & \frac{\partial \xi_{n-1}(x^*)}{\partial x_1} \\
\vdots & \ddots & \vdots \\
\frac{\partial \xi_1(x^*)}{\partial x_n} & \cdots & \frac{\partial \xi_{n-1}(x^*)}{\partial x_n}
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
\pi_1^* \\
\vdots \\
\pi_{n-1}^*
\end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix}
-\frac{\partial \xi_n(x^*)}{\partial x_1} \\
\vdots \\
-\frac{\partial \xi_n(x^*)}{\partial x_n}
\end{bmatrix}
\]

The value of the maximized objective function of (C11) can be restated as follows:

\[
(C26) \quad \sum_{i=1}^{n} \pi_i^* \sum_{j=1}^{N} \left\{ \theta_{ji} \xi_i(x^*) \right\} - \frac{\xi_i(x^*)}{N}
\]

Following Gould, the assumption can be made, with no loss of generality, that \(C\) and each column of \(B\) are nonzero. If a particular column were a zero vector, the associated \(\pi_i^*\) could be made zero, its probability proportionately distributed among the other states with nonzero probability, and, by condition (C22), the optimum for (C11), given by (C26), would remain unchanged. System (C25), along with the requirement that \(B\) is nonnegative, may be viewed as a linear program, where \(B\) is a feasible solution and \(\text{rank}(A) \leq m\), since \(m \leq n-1\).\(^4\) Suppose the rank of \(A\) is \(M(M \leq m)\). By a fundamental result in the theory of linear programming, there is an alternative solution to (C25) with at most \(M\) nonzero variables.\(^5\) This

---


yields a revised system. Subtracting $C$ from both sides of the revised system yields a solution to system (C24) with at most $M+1$ nonzero variables. Since (C24) is a homogeneous system, dividing each of the new nonzero variables by their sum preserves a system of equations with at most $m+1$ nonzero state probabilities and whose sum is one. Since $\vec{x}$ is optimal for the old set of nonzero state probabilities, $(\pi_1^*, \ldots, \pi_n^*)$, since $\zeta_i(x)$ is concave in $x$ for $i=1, \ldots, n$, and since $\vec{x}^*$ satisfies the first-order condition for the maximization of (C1) under the new set of at most $m+1$ nonzero state probabilities, $\vec{x}^*$ remains optimal for the new set of probabilities. Condition (C22) implies that the new probabilities give the same value to (C26) as did the initial set of probabilities. The new set has at most $m+1$ nonzero probabilities.

There must be at least two nonzero probabilities, or the DM's problem degenerates to a deterministic one, where information is of no value.

Q.E.D.
APPENDIX D

PROOF OF COROLLARY 6-5.2, SECTION 6.3.3

Corollary 6-5.2 is restated below:
For \( P(r)=1 \), \( u(z)=z \), and \( h_{t, \delta}^b \):

\[
\begin{align*}
    t_{1*} &= \left( \frac{\delta}{\delta+1} \right) \\
    t_{2*} &= \left( \frac{\delta^2 (\delta+1)^{\delta-1}}{\delta (\delta+1)^{\delta+1-\delta}} \right) \\
    t_{3*} &= \left( \frac{\delta^3 (\delta+1)^{\delta-1} (\delta+1)^{\delta+1-\delta}}{\delta (\delta+1)^{\delta+1-\delta}} \right) \\
    t_{\infty} &= \left( \frac{\delta^2 (\delta+1)^{\delta+1-\delta}}{\delta (\delta+1)^{\delta+1-\delta}} \right)
\end{align*}
\]

Proof: From Proposition 6-5, \( U(h_{t, \delta}^b) \) will be maximized by solving the following problem:

\[
(D1) \quad \max_{t_{-i, t}} \left\{ \frac{n_1}{t_i} \frac{t_i+1}{t_i} \frac{n_2}{t_i-1} \frac{t_i}{t_i} \right\}
\]

This problem can be solved by dynamic programming if Mitten's conditions of separability and monotonicity\(^1\) can be met. Using backward recursion, and backward indexing, let \( t_i \) denote the \( i \)th decision variable, let \( \delta_i \) denote the \( i \)th state variable and define the transition function as follows:


266.
\( o_i(o_{i+1}, t_{i+1}) = t_{i+1}, \ i=1, \ldots, \ n-1 \) with \( \rho_n \equiv 1 \) and \( \rho_0 \equiv 0 \)

Let \( R_i \) denote the \( i \)th stage return function, where:

\[ R_i(\rho_i, t_i) = t_i^\delta - t_1^\delta+1 \]

Summing the return functions and substituting according to the transition function:

\[
\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} t_i^\delta \left( \sum_{i=1}^{n} t_i^{\delta+1} \right) = \sum_{i=1}^{n} t_i^{\delta+1} \left( \sum_{i=1}^{n} t_i^{-\delta+1} \right), \text{ where } t_{n+1} = 1.
\]

Therefore, the separability condition is met. Since the objective function is the summation of the stage return functions, it is increasing in the \( i \)th return function, and the monotonicity condition is met.

Let \( F_i(\rho_i) = \max_{t_i} \left( R_i(\rho_i, t_i) + F_{i-1}(\rho_{i-1}) \right), \ i=1, 2, \ldots, n, \) and \( F_0(\rho_0) = 0 \).

The first stage optimization problem is given by:

\[ (D2) \max_{0 \leq \rho_1} \left( R_1(\rho_1, t_1) + F_0(\rho_0) \right) = \max_{0 \leq \rho_1} \left( t_1^\delta - t_1^{\delta+1} \right) \]

Differentiating (D2):

\[
\frac{\partial t_1^\delta}{\partial \rho_1} = t_1^\delta \left( \delta \rho_1 - (\delta+1) t_1 \right) \leq 0 \quad \text{as } t_1^\delta \leq \rho_1
\]

\[
\frac{\partial^2 t_1^\delta}{\partial \rho_1^2} = t_1^\delta \left( \delta \rho_1 - (\delta-1) \rho_1 - (\delta+1) t_1 \right) \leq 0
\]

as \( t_1^\delta \leq \rho_1 \) for \( \delta \leq 1 \), and \( < 0 \) for \( \delta > 1 \).

If \( \delta \leq 1 \), the maximand is concave and \( t^* = \left( \frac{\delta}{\delta+1} \right) \rho_1 \in [0, \rho_1] \), since \( \delta > 0 \) and \( \rho_1 \geq 0 \). For \( \delta > 1 \), the inflection point, \( (\delta-1)/(\delta+1) \rho_1 \), is less than \( \delta/(\delta+1) \rho_1 \); therefore, the maximand is increasing on its convex portion.

The maximum is, therefore, on the concave portion and again occurs at \( t_1^* = \left( \frac{\delta}{\delta+1} \right) \rho_1 \). Recall that \( r_1(\rho_2, t_2) = t_2 \). If \( n=1 \), \( t_2 \equiv 1 \) and \( t_1 = \delta/(\delta+1) \).

This is consistent with the result of Corollary 4.3.2 in section 4.3.
Therefore:

\[ F_1(\rho_1) = t_1^{\delta} \rho_2 - t_1^{\delta+1} \]
\[ = \frac{\delta}{\delta+1} t_1^{\delta+1} \rho_1 - \frac{\delta}{\delta+1} t_1^{\delta+1} \rho_1 = \frac{\delta}{\delta+1} \rho_1 \]

If \( n>1 \), the second stage optimization problem is given by:

\[(D3) \max_{0 \leq t_2 \leq \rho_2} \{ R_2(\rho_2, t_2) + F_1(\rho_1) \} \]
\[ = \max_{0 \leq t_2 \leq \rho_2} \{ t_2^{\delta} \rho_2 - t_2^{\delta+1} + \frac{\delta}{\delta+1} t_2^{\delta+1} \} \]
since \( \rho_1(\rho_2, t_2) = t_2 \).

Differentiating (D3):

\[ \frac{\partial F_1}{\partial t_2} = t_2^{\delta-1} \{- (\delta+1) t_2^{\delta+1} + \frac{\delta}{\delta+1} t_2^{\delta+1} \} < 0 \]

as \( t_2 = t_2^{\delta} \leq (\delta+1) \frac{\delta}{\delta+1} \rho_2 \)

\[ \frac{\partial^2 F_1}{\partial t_2^2} = t_2^{\delta-2} \{- (\delta+1) t_2^{\delta+1} + \delta (\delta+1) t_2 + \frac{\delta}{\delta+1} t_2^{\delta+1} \} \]
\[ \geq 0 \text{ as } t_2 = t_2^{\delta} \leq (\frac{\delta-1}{\delta+1}) \frac{\delta}{\delta+1} \rho_2 < \frac{\delta}{\delta+1} \frac{\delta}{\delta+1-\delta} \rho_2, \]

for \( \delta>1 \), and \( < 0 \) for \( \delta<1 \).

If \( \delta<1 \), the maximand is concave and \( t_2^*=\{ - \frac{\delta (\delta+1)}{\delta+1-\delta} \rho_2 \in [0, \rho_2] \} \). The coefficient of \( \rho_2 \) in \( t_2^* \) is less than 1 since its reciprocal minus one is positive:

\[ \frac{\delta (\delta+1)}{\delta+1-\delta} - 1 > 0 \Leftrightarrow (\delta+1)(\delta+1)^{\delta} - \delta^{\delta-\delta} - (\delta+1)^{\delta} > 0 \Leftrightarrow \]
\[ (\delta+1-\delta)(\delta+1)^{\delta} - \delta^{\delta} = -(\delta+1)^{\delta} - \delta^{\delta} > 0. \]

If \( \delta>1 \), the inflection point, \( \{(\delta-1)(\delta+1)^{\delta}/[(\delta+1)^{\delta+1-\delta}]\} \rho_2 \), is less than \( (\delta(\delta+1)^{\delta}/[(\delta+1)^{\delta+1-\delta}]\) \rho_2. Therefore, the maximand is increasing on its convex portion. The maximum is, therefore, on the concave
portion and again occurs at \( t_2^* = \frac{-\delta(\delta+1)^\delta}{(\delta+1)^{\delta+1-\delta^\delta}} \rho_2 \). Recall that
\[ \rho_2(\rho_3, t_3) = t_3. \]
If \( n=2 \), \( t_3^1 = t_3 \) and \( t_2^* = \frac{-\delta(\delta+1)^\delta}{(\delta+1)^{\delta+1-\delta^\delta}} \). Since \( \rho_1(\rho_2, t_2) = t_2 \)
and \( t_1^* = \frac{-\delta}{\delta+1} \rho_1 \), \( t_1^* \) is given by:
\[
t_1^* = \left\{ \frac{\delta}{\delta+1} \right\} \left\{ \frac{-\delta(\delta+1)^\delta}{(\delta+1)^{\delta+1-\delta^\delta}} \right\} = \frac{-\delta(\delta+1)^{\delta-1}}{(\delta+1)^{\delta+1-\delta^\delta}}
\]
Regardless of the value of \( n \), \( F_2(\rho_2) \) is given by:
\[ F_2(\rho_2) = R_2(\rho_2, t_2^*) + F_1(t_2^*) \]
Substituting for \( \rho_2^* \):
\[ F_2(t_3) = R_2(t_3, t_2^*) + F_1(t_2^*) \]
If \( n>2 \), the third stage optimization problem is given by:
\[
(D4) \max_{0 \leq t_3 \leq \rho_3} R_3(\rho_3, t_3) + F_2(t_3)
\]
\[
= \max_{0 \leq t_3 \leq \rho_3} \left\{ \frac{\delta}{\delta+1} t_3^{-\delta-1} \left[ \frac{-\delta(\delta+1)^{\delta}}{(\delta+1)^{\delta+1-\delta^\delta}} \right] t_3^{\delta+1} \right\}
- \frac{-\delta(\delta+1)^{\delta}}{(\delta+1)^{\delta+1-\delta^\delta}} t_3^{\delta+1} \left[ \frac{-\delta(\delta+1)^{\delta}}{(\delta+1)^{\delta+1-\delta^\delta}} t_3^{\delta+1} \right]
\]
Differentiating \((D4)\):
\[
\frac{\partial}{\partial t_3} = \delta - 1 \left\{ \frac{\delta}{\delta+1} \right\} \left[ \frac{-\delta(\delta+1)^{\delta}}{(\delta+1)^{\delta+1-\delta^\delta}} \right] t_3^{\delta+1} \left[ \frac{-\delta(\delta+1)^{\delta}}{(\delta+1)^{\delta+1-\delta^\delta}} t_3^{\delta+1} \right] \}
\]
\[ = \delta \rho_3 \left[ \frac{-\delta(\delta+1)^{\delta}}{(\delta+1)^{\delta+1-\delta^\delta}} t_3^{\delta+1} \left[ \frac{-\delta(\delta+1)^{\delta}}{(\delta+1)^{\delta+1-\delta^\delta}} t_3^{\delta+1} \right] \}
\]
\[ = \delta (\delta+1) \left[ \frac{-\delta(\delta+1)^{\delta}}{(\delta+1)^{\delta+1-\delta^\delta}} t_3^{\delta+1} \left[ \frac{-\delta(\delta+1)^{\delta}}{(\delta+1)^{\delta+1-\delta^\delta}} t_3^{\delta+1} \right] \}
\]
= \delta \rho_3 / \nu$, where \( \nu \) denotes the denominator of the last expression above.

\[
\frac{\partial^2 \cdot}{\partial t_3^2} = \begin{cases} 
3 + \delta (\delta - \delta + \delta + 1) t_3 + \delta (\delta + 1) \left[ \frac{\delta (\delta + 1)}{(\delta + 1)(\delta + 1 - \delta)} \right] t_3 \\
- \delta (\delta + 1) \left[ \frac{\delta (\delta + 1)}{(\delta + 1)(\delta + 1 - \delta)} \right]^{\delta + 1} \end{cases}
\]

\[
\delta \geq 0 \text{ as } t_3 \leq (\delta - 1) \rho_3 / \nu < \delta \rho_3 / \nu \text{ for } \delta > 1, \text{ and } < 0 \text{ for } \delta \leq 1.
\]

If \( \delta \leq 1 \), the maximand is concave and \( t_3^* = \delta \rho_3 / \nu \)

\[
= \frac{\delta [(\delta + 1)(\delta + 1 - \delta) + \delta (\delta + 1) \delta^2]}{(\delta + 1)[(\delta + 1)(\delta + 1 - \delta) + \delta (\delta + 1) \delta^2]} \rho_3 \in [0, \rho_3].
\]

The coefficient of \( \rho_3 \) in \( t_3^* \) is less than 1 since its reciprocal minus 1 is positive:

\[
\frac{\delta [((\delta + 1)(\delta + 1 - \delta) + \delta (\delta + 1) \delta^2]}{\delta [(\delta + 1)(\delta + 1 - \delta) + \delta (\delta + 1) \delta^2]} - 1 > 0 \Leftrightarrow [(\delta + 1)(\delta + 1 - \delta) + \delta (\delta + 1) \delta^2] > \delta^2
\]

\[
\Leftrightarrow (\delta + 1)(\delta + 1 - \delta) \delta^2 > \delta (\delta + 1) \delta^2 \quad \Leftrightarrow (\delta + 1)(\delta + 1) > 0
\]

If \( \delta > 1 \), the inflection point, \( (\delta - 1) \rho_3 / \nu \), is less than \( \delta \rho_3 / \nu \).

Therefore, the maximand is increasing on its convex portion. The maximum is, therefore, on the concave portion and again occurs at

\[
t_3^* = \frac{\delta [(\delta + 1)(\delta + 1 - \delta) + \delta (\delta + 1) \delta^2]}{\delta [(\delta + 1)(\delta + 1 - \delta) + \delta (\delta + 1) \delta^2]} \rho_3.
\]

Recall that \( \rho_3 t_4 = t_4 \). If \( n = 3, t_4 = 1 \) and \( t_3^{**} = \left[ \frac{\delta [((\delta + 1)(\delta + 1 - \delta) + \delta (\delta + 1) \delta^2]}{\delta [(\delta + 1)(\delta + 1 - \delta) + \delta (\delta + 1) \delta^2]} \right] \rho_3 \).

Iterative substitution of \( t_3^{**} \) into \( t_2^* \) and \( t_1^* \), given previously, yields:

\[
2v \text{ is positive. This can be seen by reexpressing } \nu \text{ with the lowest common denominator, } (\delta + 1)(\delta + 1 - \delta) \delta^2, \text{ and simplifying.}
\]
\[
\begin{align*}
\hat{t}_2 &= \frac{\delta^2 (\delta+1)^{\delta - 1} (\delta + 1)^{\delta + 1 - \delta \delta} \delta - 1}{(\delta + 1) [(\delta + 1)^{\delta + 1 - \delta \delta} - \delta \delta (\delta + 1)^{\delta 2}]}
\end{align*}
\]

\[
\begin{align*}
\hat{t}_1 &= \frac{\delta^3 (\delta+1)^{\delta - 1} (\delta + 1)^{\delta + 1 - \delta \delta} \delta - 1}{(\delta + 1) [(\delta + 1)^{\delta + 1 - \delta \delta} - \delta \delta (\delta + 1)^{\delta 2}]}
\end{align*}
\]

Q.E.D.
BIBLIOGRAPHY


Burns, Thomas J., Editor. The Use of Accounting Data in Decision Making. Columbus, Ohio: College of Commerce and Administration, The Ohio State University, 1967.


