THE EFFECTS OF GOVERNMENT DEBT
WITH IMPERFECT CAPITAL MARKETS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

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* * * * *

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1990
To My Parents and To My Wife
With Love and Gratitude
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CHAPTER I

INTRODUCTION

Since Barro's (1974) revival of the "Ricardian equivalence theorem", there has been a lot of debate over the macroeconomic implications of fiscal policy. The issue is whether or not budget deficits matter. The Ricardian alternative to the standard analysis holds that a deficit-financed tax cut has no impact on aggregate demand unless the government changes the present value of government expenditures. This Ricardian conclusion emerges from the view that rational consumers fully perceive the implied future tax liabilities and react by saving more to pay for the future tax liabilities that have the same present value as the initial tax cut.¹

Although the Ricardian alternative plays a crucial role as a

¹. The Ricardian conclusions generally hinge on the following major assumptions: 1) successive generations of consumers are linked by altruistically motivated transfers, which allows their consumption decisions to be effectively planned with infinite horizons; 2) taxes are non-distortionary (lump-sum); 3) private capital markets are perfect; and 4) consumers are rational and foresighted. This assumption also rules out uncertainty about consumers' future tax liabilities. However, these major assumptions are neither necessary, nor sufficient conditions for Ricardian equivalence to hold.
benchmark from which to assess an economy’s departure from a strict Ricardian world, its underlying assumptions have been criticized as too strong. Most arguments against the Ricardian conclusions on both theoretical and empirical grounds are based on the objections to its major assumptions. In particular, the imperfection of private credit markets, one of the most compelling objections raised against assumptions on which the Ricardian conclusions are based, has been believed to cause non-neutrality as Barro (1974,1985,1989) argued. Furthermore, in light of the observation that many consumers appear to be liquidity constrained, imperfect capital markets have recently been recognized as central to the analysis of fiscal policies.

The purpose of this paper is to investigate whether government debt indeed leads to non-Ricardian results in the presence of imperfect capital markets. This paper also examines recent U.S. time series data to see whether government debt contributes to a narrowing of the spread between borrowing rates and lending rates in financial markets in the presence of liquidity constraints. As the basic framework for the analysis, we use a nonstochastic dynamic model in the context of the neoclassical paradigm for a closed economy. Two types of rational agents characterized by different subjective

2. The widely accepted definition of liquidity constraints in the literature refers to the condition where consumers either face quantity constraints on the amounts they borrow, or they have to pay a higher borrowing rate than the rate at which they could lend (see Hayashi (1987)).
discount rates are assumed to face infinite-horizon intertemporal optimization problems and lump-sum wage taxes. The model introduces an innovation in that each type’s subjective discount rate is endogenously determined by the level of utility (and thus consumption), as in Uzawa (1968), Obstfeld (1981, 1982), and Calvo and Findlay (1978) instead of being fixed. What is appealing about this innovation is that not only does a long-run steady state equilibrium exist, but it is also possible to get a broader range of quantitative outcomes in dynamic models, if the subjective discount rate is variable. Imperfect capital markets are incorporated in our model in the form of exogenous nonnegativity restrictions on the physical asset stocks, unlike the Hayashi (1987) and Yotsuzuka (1987) approaches wherein borrowing constraints endogenously arise from the problems of adverse selection. We assume for analytic convenience that only agents with high subjective discount rates (type-II agents) face exogenous borrowing constraints. The exogenous liquidity constraints we consider here have a plausible justification, in light of the observation that non-collaterlized consumer borrowing is unimportant in the U.S. mainly due to personal bankruptcy laws.

This paper proposes three analytic results which are essentially consistent with Barro’s main theoretical propositions. A key implication of our theoretical considerations holds that government

debt leads to a narrowing of the spread in the presence of liquidity constraints. Government debt is also predicted to displace private intermediation, provided the marginal cost of private intermediation increases as private loans rise in financial markets. Evidence on recent U.S. quarterly time series data generally supports this theoretical implication. Also, we find that the spread has a long run relationship with the other variables under consideration.

The rest of the paper is organized as follows. Chapter II discusses briefly a literature on the validity of the Ricardian conclusions in the presence of liquidity constraints and on the tests of the importance of liquidity constraints. Chapter III describes the structure of a closed economy in the neoclassical framework and the intertemporal optimization by two distinguishable agents characterized by different endogenous time preferences. Chapter IV derives the implications of perfect capital markets for Ricardian equivalence as a benchmark for a subsequent analysis in the presence of imperfect capital markets. Chapter V presents a model with exogenous liquidity constraints which are permanently imposed on the agents with a high subjective discount rate. Chapter VI analyzes the effects of government debt on the spread between borrowing rates and lending rates within the framework of the model with liquidity constraints. Chapter VII provides empirical tests of the theoretical implication of chapter VI, based on a vector autoregression (VAR) approach, common trends test, and co-integration test.
CHAPTER II

LITERATURE SURVEY

Since this paper focuses on the issue of whether the failure of Ricardian equivalence is attributable to liquidity constraints, it would be useful to briefly review the recent literature on the validity of the Ricardian conclusions under the prevalence of liquidity constraints and on test for the importance of liquidity constraints.

Following the lead of Barro (1974), there has been a large literature that investigates either the theoretical or empirical validity of liquidity constraints as one of the candidates to explain the rejection of Ricardian equivalence. Barro (1974, 1985, 1989) argued compellingly that an issue of government bonds may involve a net wealth effect in an imperfect capital market. The net wealth effect is presumed to arise from the under-capitalization of the future tax liabilities for the high discount group. The validity of his arguments, however, essentially depends on the assumption that the government can be better at intermediation than the private sector. In that case, Ricardian equivalence fails because of imperfect capital markets. This implication is basically in line with
that drawn from the work of Yotsuzuka (1987), although the conclusion reached by Yotsuzuka is reversed. Yotsuzuka, in an extended version of the adverse selection models of Mervyn King (1986) and Hayashi (1987), has shown that Ricardian equivalence can survive even in the presence of liquidity constraints. This neutrality result stems from the basic premise that government intermediation is a substitute for private intermediation in a model of endogenous communication where incentives for communication are explicitly considered such that the high-risk customer's information is not disclosed to the banks offering a pooled loan of limited quantity. Hayashi (1987) has provided an exhaustive survey of the literature on liquidity constraints. His theoretical framework builds on Hall's (1978) stochastic permanent income hypothesis and considers a model of asymmetric information and adverse selection, yielding a neutrality result similar to Yotsuzuka. Unlike both Hayashi and Yotsuzuka, Hubbard and Judd (1986, 1987) analyze liquidity constraints arising from a nonnegativity constraint on asset stocks. This form of exogenous liquidity constraints implies that consumers are not permitted to borrow against their future income. In addition, exogenous liquidity constraints appear to be consistent with personal bankruptcy laws in the U.S. They show that liquidity constraints may be quantitatively important in invalidating Ricardian propositions and borrowing restrictions account for a nontrivial marginal propensity to consume out of short-term tax changes.
Most recent empirical tests for liquidity constraints have focused on assessing whether the consumption of families with low net wealth is more sensitive to changes in disposable income than is predicted by the permanent income hypothesis. The excess sensitivity tests are based on the idea that likelihood of future liquidity constraints effectively shortens consumers' horizons for optimal consumption plans (see Mariger (1986,1987) and Hayashi (1987)). Zeldes (1989), using Euler equation approaches to test the permanent income hypothesis against the liquidity constraints alternative, finds the empirical result that consumption growth is negatively related to real disposable income. His empirical finding generally supports the view that liquidity constraints have an important influence on consumption. Flavin (1981,1985) has shown that the estimates of the marginal propensity to consume out of transitory income are significantly affected by the inclusion of the unemployment rate in the model as a proxy for liquidity constraints. She interprets this excess sensitivity of consumption to current income as providing some support for the role of liquidity constraints. Japelli and Pagano (1989) have found that the low levels of consumer debt observed in seven developed countries where the excess sensitivity of consumption is high can result from liquidity constraints rather than from a low demand for loans. They interpret the evidence as supporting the hypothesis that excess sensitivity may be attributed to liquidity constraints.
Furthermore, the available evidence suggests that a nontrivial fraction of the population are actually liquidity constrained. Mariger (1987) found that 19.4 percent of the population sampled are liquidity constrained using cross-sectional data of 798 U.S. families. This evidence is in rough agreement with the findings of Hayashi (1982, 1985), Hall and Mishkin (1982). Hayashi (1982) found that liquidity-constrained consumers accounted for approximately 20 percent of all consumption in post-World War II United States. Hall and Mishkin, using panel data, estimated that 20 percent of U.S. families are liquidity constrained. Hayashi (1985) used cross-sectional data to infer that 16 percent of Japanese family are liquidity constrained.

Overall, the clear implication of the literature is that liquidity constraints appear to be an important part in explaining non-Ricardian results and the observed excess sensitivity of consumption. We will focus on the implication for Ricardian equivalence in the presence of liquidity constraints below.
CHAPTER III

A BASIC ANALYTIC MODEL

Consider a closed economy inhabited by two distinguishable agents who have infinite planning horizons and perfect foresight, i.e. type-I agents and type-II agents. Suppose that a fraction $\alpha$ of the population are type-I and the rest $(1- \alpha)$ are type-II. Thus, the total population is normalized to unity. Unlike conventional intertemporal optimization frameworks, where the discount rate ($\delta_t$) is usually assumed to be constant for analytical convenience, we make a more reasonable assumption regarding each agent's instantaneous rate of time preference in this model. Namely, by introducing an innovation that each agent's subjective discount rate at which future utility is discounted to the present depends on the current level of utility, and thus of consumption (i.e., the assumption of endogenous time preference). Assume that type-I agents are characterized by a low subjective discount rate ($\delta_{1t}$) and type-II agents by a high subjective discount rate ($\delta_{2t}$), i.e. $\delta_{1t} < \delta_{2t}$ for a given level of utility. This assumption implies that type-II agents have more impatience in postponing current satisfaction for advanced timing.
of satisfaction than type-I agents.

Uzawa has shown that if consumption is made continuously over an infinite time horizon, the structure of each agent's time preference can be represented in terms of the intertemporal utility function:

\[ V = \int_0^\infty U(c_{it}) e^{-\Delta_{it}} dt, \quad (3-1) \]

where \( i = I \) and \( II \), \( c_{it} \) is the per-capita consumption of type-I agents and \( c_{2t} \) is the per-capita consumption of type-II agents in period \( t \). The instantaneous utility function \( U(.) \) is strictly concave and twice continuously differentiable with the property that \( \lim_{c_{it} \to 0} U'(c_{it}) = \infty \).

This assumption is needed to eliminate the possibility of a corner solution. That is, each agent may choose a consumption plan which allows nonnegative consumption \( (c_{it} \geq 0) \). Moreover, the specification of (3-1) states that current and future consumption are weakly separable from past consumption levels. A planner's intertemporal optimization decision at time \( t \) will not directly depend on past consumption activities, but indirectly on the past through accumulated assets (i.e., the past can matter only through the intertemporal budget constraint).

The discount factor \( \Delta_{it} \) is defined by

\[ \Delta_{it} = \int_0^t \delta_{is} [U(c_{is})] ds, \quad (3-2) \]
where \( \delta_{is} \) is the instantaneous subjective discount rate (or subjective rate of time preference) at time \( s \). Following Uzawa (1968), \( \delta_{is} \) is assumed to be a function of utility at time \( s \). So the rate of time preference \( \Delta_{it} \) depends on the time profile of a continuous utility stream \( U_t \), and is independent of the utility level beyond time \( t \).

Therefore, a change in today's consumption affects not only today's instantaneous subjective discount rate \( \delta_{is}[U(c_is)] \), but also the rate of time preference applied to utility at and after time \( s \), \( \Delta_{iv} (v > s) \)

\[
\delta_{is}[U(c_is)] \text{ is positive and satisfies } \delta''_{is}(\cdot) > 0, \delta'_{is}(\cdot) > 0,
\]

and \( \delta_{is}(\cdot) - \delta'_{is}(\cdot)U > 0. \) \hspace{1cm} (3-3)

The second assumption requires that greater consumption at time \( s \) leads the agent to raise the discount rate applied to further increments to consumption. This is needed for stability of the steady state.\(^4\) The third assumption is required to derive a continuous

\(^4\) The rationale for this property is given by Uzawa (1968) and is basically consistent with the rigorous axiomatic discussion of Koopmans, Diamond, and Williamson (1964). This property is needed for the dynamic system to have stability. If \( \delta'_{is}(\cdot) < 0 \) contrary to that property, then the dynamic system described by \( k_t \) and \( c_t \) possesses

(Footnote continues on next page)
consumption function. The last assumption requires that between two stationary consumption streams discounted at \( \delta_1 \), the one with higher level of utility is preferred (i.e., \( U(c^*_{it})/\delta_1(U^*) \) is an increasing function in \( U^* \)).

An appealing feature of Uzawa's utility structure is the introduction of the instantaneous rate of time preference which endogenously determines the level of utility in a class of recursive utility. Uzawa's specification particularly allows an increment in current consumption to change current and future rates of time preferences. This variability in the rate of time preference embodied in Uzawa's utility function generates a broader range of quantitative outcome in dynamic models as opposed to the standard specification with a constant rate of discount. More importantly, there exists a long-run stationary equilibrium in the standard neoclassical growth

(Footnote continued from previous page)

two positive characteristic roots in a two dimensional space, implying that there exists no path converging to the steady state. Intuitively, as pointed out by Fisher and Blanchard (1990), if \( \delta'_{1e}(.) < 0 \), the rich become richer and total consumption must eventually be increasing. This is inconsistent with being in steady state.

model if and only if the subjective discount rate is variable. Uzawa's utility function seems to provide a useful analytical tool for our steady state analysis.

Suppose that output in this closed economy is produced from inputs of labor and capital services with a linear homogeneous Cobb-Douglas technology. Assume that factor markets are competitive, so that the owners of labor and claims on capital receive their respective marginal products. The labor supply \((N)\), measured in efficiency units, is assumed to remain constant (i.e., labor is supplied inelastically) and normalized to unity. For simplicity, we consider the case where government taxes are lump sum and imposed only on the owners of labor as a lump-sum wage tax, \(\tau_t\). This implies that the owners of labor receive their marginal product less lump-sum wage taxes.

\[
y_t = F(1,k_t) = f(k_t) = k_t^\beta
\]

(3-4)

\[
w_t = f(k_t) - k_t f'(k_t) - \tau_t = \nu(k_t) - \tau_t = (1 - \beta)k_t^\beta - \tau_t, \quad \text{and}
\]

\[
w_t'(k) > 0
\]

(3-5)

\[
r_t = f'(k_t) = r(k_t) = \beta k_t^{\beta-1}, \quad \text{and} \quad r'(k_t) < 0,
\]

(3-6)
where $k_t$ is the capital stock and, also, the capital-labor ratio since the labor supply is normalized to unity. Conditions (3-4), (3-5), and (3-6) indicate that the path of the capital stock will imply the paths of output, wages, and the real interest rate. The output in per-capita term can be represented by

$$f(k_t) = w(k_t^*) + r(k_t)k_t$$

(3-7)

Each agent's decision at time $t$ is to allocate his per-capita disposable income ($y_{1t}$) between consumption ($c_{1t}$) and the accumulation of real assets ($a_{1t}^*$)

$$y_{1t} = v(k_t^*) - r(k_t) + r(k_t)a_{1t}^* = c_{1t} + a_{1t}^*, \text{ for type-I agents,}$$

(3-8)

$$y_{2t} = v(k_t^*) - r(k_t) + r(k_t)a_{2t}^* = c_{2t} + a_{2t}^*, \text{ for type-II agents,}$$

(3-9)

where $a_{it}$ represents the per-capita asset stock each type of agent plans to hold at time $t$. We shall assume that the wage income is equally distributed across agents, or equivalently, all agents have the same productivity, irrespective of their type. Suppose further that taxes are distributed equally across the agents.

The flow constraint linking asset accumulation to saving in per-capita terms can be written as
\[ a_{1t} = y_{1t} - c_{1t} = w(k_t) - \tau_t + r(k_t)a_{1t} - c_{1t}, \text{ for type-I agent,} \quad (3-10) \]

\[ a_{2t} = y_{2t} - c_{2t} = w(k_t) - \tau_t + r(k_t)a_{2t} - c_{2t}, \text{ for type-II agent.} \quad (3-11) \]

Suppose that aggregate per-capita asset stock \( a_t \) is ultimately divided between the aggregate per-capita capital stock \( k_t \) and the aggregate per-capita debt \( d_t \).

\[ a_t = k_t + d_t \quad (3-12) \]

\[ c_t = \alpha c_{1t} + (1-\alpha)c_{2t} \quad (3-13) \]

\[ a_t = \alpha a_{1t} + (1-\alpha)a_{2t} \quad (3-14) \]

\[ k_t = \alpha k_{1t} + (1-\alpha)k_{2t} \quad (3-15) \]

Condition (3-12) implicitly assumes that the government debt pays the same rate of return as private capital. So agents are indifferent between holding private capital and holding government debt in their portfolio. Equations (3-13), (3-14), and (3-15) define aggregate per-capita consumption \( c_t \), aggregate per-capita asset stock \( a_t \), and aggregate per-capita capital stock \( k_t \), respectively.

The government budget constraint is specified by:
\[ r_t d_t + g_t = \tau_t + \hat{d}_t, \tag{3-16} \]

where \( r_t d_t \) denotes the government’s per-capita interest payments and \( g_t \) its per capita expenditures, \( \tau_t \) represents the revenue from taxing the aggregate per-capita wage income and \( \hat{d} \) the accumulation of the government’s per-capita debt.

The differential equation of private capital accumulation, \( \dot{k}_t \), comes from the assumption of a closed economy and the assumption that the capital and consumption goods are the same. Thus, \( \alpha \) times (3-10) plus \((1-\alpha)\) times (3-11), and using (3-13), (3-14), and (3-16) yields:

\[ \dot{k}_t = f(k_t) - g_t - c_t = w(k_t) + r(k_t)k_t - g_t - c_t \tag{3-17} \]

Note further that condition (3-10) (or (3-11)) can be reduced to the following differential equation of private capital accumulation for each type of agent using the government budget constraint (3-16) and the fact that \( a_{it} = k_{it} + d_t \).

\[ \dot{k}_{1t} = w(k_{1t}) - g_t + r(k_t)k_{1t} - c_{1t}, \text{ for type-I agents} \tag{3-18} \]

\[ \dot{k}_{2t} = w(k_{2t}) - g_t + r(k_t)k_{2t} - c_{2t}, \text{ for type-II agents} \tag{3-19} \]
Essentially, neither taxes \((\tau_t)\) nor government debt \((d_t)\) appear in conditions (3-18) and (3-19). Only government expenditures \((g_t)\) matter. This result plays a key role in discussing the implications for Ricardian equivalence below.

Our first task is to incorporate the intertemporal optimization in consumption of the two different groups I and II. Each agent seeks at period \(t\) to maximize the discounted sum of future instantaneous utilities.

First, for type-I agents, their problem is to find \(\{c_{1t}\}\), maximizing lifetime welfare \(V\), subject to constraint (3-18) given the initial level of per-capita asset stock \(a_{10}\).

\[
\text{Maximize } \int_0^\infty \left[ U(c_{1t})e^{-\Delta t} / \delta_{1t}U(c_{1t}) \right] d\Delta \\
\{c_{1t}\}
\]

\[
\text{Subject to } da_{1t}/d\Delta = [w(k_t) - \tau_t + r(k_t)a_{1t} - c_{1t}! / \delta_{1t}U(c_{1t})]
\]

To simplify the optimization problem, we transformed the time variable \(t\) into a variable in which the rate of time preference \(\Delta\), became constant. Then, \(d\Delta = \delta[U(c_{1t})]dt\), is obtained by substituting (3-3)

---

6. For instance, the differential equation of per-capita capital stock for type-I agents can be found by plugging \(\tau_t = r(k_t)d_t + g_t - \delta d_t\) and \(a_{1t} = k_{1t} + d_t\) into the condition (3-10), yielding \(d\{k_{1t} + d_t\}/dt = w(k_t) - [g_t - r(k_t)d_t - \delta d_t] + r(k_t)\{ k_{1t} + d_t\} - c_{1t}\), which in turn reduces to \(k_{1t} = w(k_t) - g_t + r(k_t)k_{1t} - c_{1t}\). The same procedure is applicable for type-II agents.
into (3-2) and differentiating in terms of \( t \). Given this innovation, we can find the optimal conditions for type-I agents by applying the Pontryagin maximal principle. For each value of the discount factor \( \Delta \), \( \{c_{1t}\} \) should be chosen so that the Hamiltonian discounted at the prevailing rate of time preference \( \delta[U(c_{1t})] \), is maximized. That is:

\[
H = \frac{U(c_{1t})}{\delta[U(c_{1t})]} + P(\Delta)\{ w(k_t) - \tau_t + r(k_t)a_{1t} - c_{1t}\} / \delta[U(c_{1t})]
\]

In equation (3-20), \( P = P(\Delta) \) is the co-state (or auxiliary) variable. This is interpreted as the marginal contribution of the state variable \( a_{1t} \), to the utility function (i.e., \( \partial V / \partial a_{1t} \)), or the "shadow" price of per-capita real asset for each level of \( \Delta \). The optimal consumption \( \{c_{1t}\} \) is determined at the level at which the Hamiltonian (3-20) is maximized. The first order condition is:

\[
U'(c_{1t}) = P(\Delta) + H \{ \delta'(U) U'(c_{1t}) \}^2
\]

(3-21)

Condition (3-21) implies that for intertemporal optimization, the marginal increase in utility resulting from an increase in today's consumption must be equal to the sum of the shadow price of per-capita

---

7. To ensure that the first-order condition (3-21) yields a unique maximum for the Hamiltonian (3-20), it is sufficient to demonstrate that the Hamiltonian is strictly concave in terms of \( c_{1t} \). This is certainly true if the state variable \( a_{1t} \geq 0 \) for a given \( P(\Delta) > 0 \) (see the sufficiency theorem in Arrow and Kurz 1971 pp.43-49).
real asset and the marginal increase in the present value of the Hamiltonian.

An additional necessary condition is that the dynamic path of the shadow price of per-capita capital stock, \( P(\Delta) \), evolves according to

\[
\frac{dP}{d\Delta} = P \left( \delta_{1t}(\cdot) - r(k_t) \right) / \delta_{1t} \tag{3-22}
\]

This implies that the time derivative of \( P(\Delta) \) is given by \( \frac{dP}{dt} = P(\delta_{1t} - r_t) \). Condition (3-22) suggests that a stationary state would not occur if \( \delta_{1t} U(c_{1t}) \) was not equal to \( r(k_t) \).

The differential equation governing movements in consumption must still be derived. Using the first order condition (3-21), we can solve for the co-state variable \( P \).

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8. See Arrow and Kurz[1971 page 26-57]. This condition is readily derived by the following manipulation. Note that condition (3-20) is the current-value Hamiltonian and its discount factor \( \sigma(t) = e^{-\Delta} \). The co-state variable \( P(\Delta) \), must evolve according to the law of motion, \( \frac{\partial [\sigma(t) P]}{\partial \Delta} = - \frac{\partial [\sigma(t) H]}{\partial a_{1t}} \), or \( \frac{\partial \sigma(t)}{\partial \Delta} = - \sigma(t) \frac{\partial H}{\partial a_{1t}} \). Dividing through this equation by \( \sigma(t) \) yields

\[
\left[ \{ \frac{\partial \sigma(t)}{\partial \Delta} / \sigma(t) \} \right] P + \frac{\partial P}{\partial \Delta} = - \frac{\partial H}{\partial a_{1t}}, \quad \text{and define} \quad \rho(t) = - \left[ \{ \frac{\partial \sigma(t)}{\partial \Delta} / \sigma(t) \} \right].
\]

Then \( \frac{\partial P}{\partial \Delta} = \rho(t) P - \frac{\partial H}{\partial a_{1t}} \). Using the fact that \( \sigma(t) = e^{-\Delta} \) and \( d\Delta = \delta_{1t} dt, \rho(t) = - \frac{[d e^{-\Delta} / d\Delta] / e^{-\Delta} = - 1, \text{and} \frac{\partial H}{\partial a_{1t}} = P r(k_t) / \delta_{1t} \) from (3-20). Thus \( \frac{\partial P}{\partial \Delta} = P - P r(k_t) / \delta_{1t} \).

Therefore, \( P(\delta_{1t} - r(k_t)) / \delta_{1t} \). In addition, as pointed out by Obstfeld (1981), the co-state variable \( P \) can be shown to be strictly positive because \( P = \left[ \{ \delta_{1} - U(c_{1}) / \delta_{1} \} U \right] U \) resulting from (3-23) is always positive as long as the third condition in (3-3) holds. This justifies the argument that in the steady state \( \delta_{1t}(\cdot) = r(k_t) \).
\[ P = \frac{U'(\cdot) \delta'_{1t}[U(\cdot)] - \delta_{1t}[U(\cdot)]U'(\cdot)U(\cdot)}{\delta_{1t}[U(\cdot)] + \delta'_{1t}[U(\cdot)] U'(\cdot)\{ a_{1t}^o \}} \]  
\[ = \psi(c_{1t}^o, a_{1t}^o) \]

Differentiating \( P \) with respect to time and applying (3-22) yields

\[ \frac{dP}{dt} = \psi_c c_{1t}^o + \psi_a a_{1t}^o = P \left( \delta_{1t}[U(\cdot)] - r(k_t) \right), \]  
(3-24)

where \( \psi_c = \partial P/\partial c_{1t} \), \( \psi_a = \partial P/\partial a_{1t} \), and \( a_{1t}^o = \phi(k_t^o, a_{1t}^o, \tau_t^o, c_{1t}^o) = (w'(k_t) + r'(k_t)a_{1t}^o) k_t^o - g_t^o \) + \( r(k_t)k_{1t} - c_{1t} \). The equation \( a_{1t}^o \) can be rewritten in terms of \( k_{1t}^o \) by using \( a_{1t} = k_{1t} + d_t \) and the government budget constraint (3-16). Namely, \( k_{1t}^o = (w'(k_t) + r'(k_t)k_{1t}) k_t^o - g_t^o \) + \( r(k_t)k_{1t} - c_{1t} \). Then, condition (3-24) may be solved for the time derivative of consumption, i.e.:

\[ c_{1t}^o = \left[ \frac{P (\delta_{1t}[U(\cdot)] - r_t) - \psi_a \phi(k_t^o, g_t^o, k_{1t}^o, c_{1t}^o)}{\psi_c} \right], \]  
(3-25)

Substituting for \( \psi_a, k_{1t}^o = \phi(k_t^o, g_t^o, k_{1t}^o, c_{1t}^o) \), and \( \psi_c \) in equation (3-25) and straightforward manipulation yields:

\[ c_{1t}^o = \left[ (\delta_{1t} - r(k_t)) \delta_{1t} + \delta_{1t} U' \left( \delta_{1t} k_{1t} + \{ w'(k_t) + r'(k_t)k_{1t} \} k_t^o - g_t^o \right) \right] / \delta_{1t} \]

\[ = \left[ (\delta_{1t} - r(k_t)) \delta_{1t} + \delta_{1t} U' \delta_{1t} \{ w(k_t) - g_t + r(k_t)k_{1t} - c_{1t} \} \right] / \delta_{1t}. \]  
(3-26)
+ \delta'_t U'(w'_t + r'(k_t)k_{1t})(w(k_t) + r(k_t)k_t - g_t - c_t) - \delta'_t \hat{c}_t \right] / \Omega_1,$

where \( \hat{a}_{1t} \) and \( \hat{k}_t \) are replaced by conditions (3-10) and (3-17), and

$$\Omega_1 = \left[ \left( \frac{U''}{U'} - \delta'_t U'U'/(\delta'_t - \delta'_t U) \right) \{ \delta'_t + \delta'_t U'(\hat{k}_{1t}) \} - \left( \delta''_t U''^2 + \delta'_t U''(\hat{k}_{1t}) + \delta'_t U \right) \right]$$

Since the stationary state is defined by the conditions \( \hat{c}_t = \hat{c}_{1t} = \hat{a}_t = \hat{k}_t = \hat{k}_{1t} = 0 \), the \( \hat{c}_{1t} = \{ \delta'_t (U(c_{1t}) - r(k_t)) \delta'_t = 0 \) curve is characterized by:

$$\delta'_t [U(c_{1t}^*)] = r(k_t^*) \quad (3-27)$$

Note that the stationary-state utility \( U^* \), is determined by the equality of the marginal rate of time preference and the real interest rate \( \delta'_t [U(c_{1t}^*)] = r(k_t^*) \), as indicated in condition (3-24), i.e.

$$U(c_{1t}^*) = \delta^{-1}_t (r(k_t^*))$$

Stationary-state consumption \( c_{1t}^* \), must satisfy

$$U(c_{1t}^*) = U^*$$

and therefore must lie on the locus \( \hat{c}_{1t} = 0 \).

To see the shape of the \( \hat{c}_{1t} = 0 \) curve, differentiating

$$\delta'_t [U(c_{1t})] = r(k_t)$$

with respect to \( c_{1t} \) and \( k_t \) yields \( \delta'_t U' dc_{1t} = r'(k_t) dk_t \).
\[
\begin{align*}
\left( \frac{dc_{1t}}{dt}, \frac{dk_t}{c_{1t}} \right)^0 &= r'(k_t) / \delta^t_{1t}(.) U' < 0
\end{align*}
\]

(3-28)

For each type-II agent, the solution of the intertemporal optimization in consumption is just analogous to that of each type-I agent's. Each type-II agent's problem is to find \( c_{2t} \) by maximizing the sum of the present value of future instantaneous utilities (3-1), subject to the constraint (3-11) given the initial level of per-capita asset stock \( a_{20} \).

\[
\text{Maximize } J_\infty \{ \left[ U(c_{2t}) e^{-\Delta_2 t} / \delta_2 t[ U(c_{2t}) ] \right] d\delta \}
\]

\[
\{c_{2t}\}
\]

subject to \( da_{2t} / d\delta = [ \nu(k_t) - \tau_t + r(k_t)a_{2t} - c_{2t} ] / \delta_2 t[ U(c_{2t}) ] \)

Repeating the same procedures as we did in deriving (3-26) allows us to solve for the differential equation governing the movements in the consumption of type-II agents, \( c^*_2 t \).

\[
\begin{align*}
\dot{c}_2^t &= [ (\delta_2^t - r(k_t)) \delta_2^t + \delta^t_2 U'(\delta^t_2 k^t_{2t} + [w'(k_t) + r'(k_t) k^t_{2t} ] k^t - g^t_t) ] / \Omega_2,
\end{align*}
\]

(3-29)

\[
\begin{align*}
= [ (\delta_2^t - r(k_t)) \delta_2^t + \delta^t_2 U' \delta_2^t (w(k_t) - g_t + r(k_t) k^t_{2t} - c^t_{2t}) ] + \\
\delta^t_2 U' [(w'(k_t) + r'(k_t) k^t_{2t}) (w(k_t) + r(k_t) k^t_{2t} - g^t_t - c^t_t) - \hat{g}^t_t)] / \Omega_2,
\end{align*}
\]

where \( \hat{g}^t_t \) and \( \hat{g}^t \) are substituted from (3-19) and (3-17) respectively, and
\[\Omega_2 = \left[ \left( \frac{U''}{U'} - \delta U'/(\delta - \delta') \right) \{ \delta + \delta'U'\left( k^*_{2t} \right) \} - \left( \delta U'^2 + \delta'U'' \left( k^*_{2t} \right) + \delta'U' \right) \right] \]

The \[\delta_{2t} = \{ \delta_{2t}(U(c^*_{2t})) - r(k_t) \} \] curve is characterized by

\[\delta_{2t}(U(c^*_{2t})) = r(k^*_t) \quad (3-30)\]

Since the dynamic path of the shadow price of capital stock, \(P(\delta)\), evolves according to \(dP/d\Delta_{2t} = P \{ \delta_{2t}(U(c_{2t})) - r(k_t) \}/\delta_{2t}\), the stationary state consumption \(c^*_{2t}\), is found by setting \(\delta_{2t}\) equal to \(r(k_t)\). That is, \(U^* = U(c^*_{2t})\). This steady-state consumption then satisfies condition (3-30) (i.e., \(\delta_{2t} = 0\)). The slope of the \(\delta_{2t} = \{ \delta_{2t}(U(c_{2t})) - r(k_t) \} \delta_{2t}(.) = 0\) locus can be shown to be negative by differentiating \(\delta_{2t}(.) = r(k_t)\) in terms of \(c_{2t}\) and \(k_t\).

\[\left(\frac{dc_{2t}}{dk_t}\right)^*_c = 0 = r'(k_t)/\delta'U' < 0 \quad (3-31)\]

It is worth noting that the conditions (3-26) and (3-29) have the common \(g_t\) term. This suggests the nonneutrality of fiscal policy changes during the transition period between the two types. Furthermore, conditions (3-27) and (3-30) imply that the subjective discount rates of all agents will be equal in the steady state.
CHAPTER IV

IMPLICATIONS OF PERFECT CAPITAL MARKETS FOR RICARDIAN EQUIVALENCE

In this chapter we attempt to derive the main implications of perfect capital markets for Ricardian equivalence as a benchmark for the subsequent analysis in the presence of imperfect capital markets. Essentially, we need to note that in the presence of perfect capital market each agent is able to lend as much as he desires and to borrow any amounts against future earnings at the going real interest rate \( r(k_t) \) (i.e., there exist no differential rates between the lending rates and borrowing rates)\(^9\) In this context, we may posit that the two different rates of endogenous time preference for each type are

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\(^9\) Thus our basic model on the assumption of perfect capital markets replicates the postulate of the life-cycle hypothesis which assumes that consumers maximize the expected value of the discounted life-time (finite horizon) utility subject only to the life-time budget constraints except that our model assumes an infinite horizon. An alternative implication for perfect capital markets is proposed by Barro (1974, 1985, 1989), where under "perfect" capital markets the government debt issue and the associated capitalized tax liabilities are perfect substitutes, implying that there is no difference in the efficiency of government versus private financial intermediation for producing liquidity service and carrying out the loan process (see also Buchanan (1976)).

24
related to each other by:

\[ \delta_{1t}[U(c_{1t})] = \tau(k_t) = \delta_{2t}[U(c_{2t})]. \]

It then follows from conditions (3-26) and (3-29) that the stationary-state consumption of the system continues to be characterized by the equality of the marginal rate of time preference and the real interest rate. Thus, each agent is always allowed to stay in the steady state.

In the present setting of perfect capital markets, however, each agent has to be bounded by the additional intertemporal budget constraint on its infinite-horizon planning problem of consumption and asset accumulation. Namely, the discounted integral of lifetime consumption must be no greater than the initial per capita assets plus the capitalized value of lifetime wage income, so:

\[ \int_0^\infty c_1(t) \exp[-\int_0^t r(x)dx]dt \leq a_{10} + \int_0^\infty [w(k,t) - \tau(t)] \exp[-\int_0^t r(x)dx]dt \]

for type-I agents

\[ \int_0^\infty c_2(t) \exp[-\int_0^t r(x)dx]dt \leq a_{20} + \int_0^\infty [w(k,t) - \tau(t)] \exp[-\int_0^t r(x)dx]dt \]

for type-II agents

---

10. To derive these inequality conditions, we first need to multiply both sides of the budget constraints (3-10) and (3-11) by the integral factor \( \exp[-\int_0^t r(x)dx] \) and integrate both side of the resulting equation from 0 to \( \infty \), which yields \( \int_0^\infty a_{1t} \exp[-\int_0^t r(x)dx]dt - \)

(Footnote continues on next page)
These additional constraints clearly eliminate the possibility that an infinitely-lived agent with access to perfect capital markets can increase its consumption without limit through an ever-rising borrowing policy at the going real interest rate and thereby its

(Footnote continued from previous page)

\[ \int_0^\infty r(t)a_{1t}\exp[-\int_0^t r(x)dx]dt = \left[ \int_0^\infty \{ v(k,t) - \tau(t) \} \exp[-\int_0^t r(x)dx]dt - \int_0^\infty c_1(t)\exp[-\int_0^t r(x)dx]dt \right] \text{ for type-I agents. Since } \int_0^\infty a_{1t}\exp[-\int_0^t r(x)dx]dt - \int_0^\infty r(t)a_{1t}\exp[-\int_0^t r(x)dx]dt \text{ on the left-hand side amounts to } \int_0^\infty a_{1t}\exp[-\int_0^t r(x)dx]dt \text{ using the following integration rule, i.e., } \int \frac{d}{dt} \left( a(t)\exp[s(t)] \right) dt = \int a(t)\exp[s(t)] dt + \int a(t)\exp[s(x)] s'(x) dt, \text{ the above equality condition ends up with } \int_0^\infty a_{1t}\exp[-\int_0^t r(x)dx]dt = \lim_{t \to \infty} a_{1t}\exp[-\int_0^t r(x)dx] - a_{10} = \int_0^\infty \{ v(k,t) - \tau(t) \} \exp[-\int_0^t r(x)dx]dt - \int_0^\infty c(t)\exp[-\int_0^t r(x)dx]dt. \text{ Namely, } \lim_{t \to \infty} a_{1t}\exp[-\int_0^t r(x)dx] = a_{10} + \int_0^\infty \{ v(k,t) - \tau(t) \} \exp[-\int_0^t r(x)dx]dt - \int_0^\infty c(t)\exp[-\int_0^t r(x)dx]dt. \text{ Imposing the transversality condition at infinity } \lim_{t \to \infty} a_{1t}\exp[-\int_0^t r(x)dx] \geq 0 \text{ is equivalent to } a_{10} + \int_0^\infty \{ v(k,t) - \tau(t) \} \exp[-\int_0^t r(x)dx]dt \geq \int_0^\infty c(t)\exp[-\int_0^t r(x)dx]dt \text{ since } a_{10}, \{ v(k,t) - \tau(t) \}, \text{ and } c(t) \text{ are nonnegative, respectively. The same procedure is applicable for type-II's inequality condition.} \]
utility is unbounded. These borrowing paradoxes, however, can be removed by imposing the following feasibility condition,

\[ a_{1t} + \int_t^\infty \{ v(k,t) - \tau(t) \} \exp[-\int_t^\tau r(x)dx]dt \geq 0 \]

and

\[ a_{2t} + \int_t^\infty \{ v(k,t) - \tau(t) \} \exp[-\int_t^\tau r(x)dx]dt \geq 0, \text{ for all } t \]

The feasibility conditions imply that each agent can borrow for consumption purpose against his capitalized value of life-time wage income (i.e., human capital), but cannot exceed it.\(^{11}\) Within this framework we cannot rule out the possibility that type-II agents such as low-income households or the young might hold negative per-capita assets \(a_{2t} < 0\) if the legal, institutional, and market frameworks allow these agents to borrow any funds against future earning in excess of their per-capita asset.

\(^{11}\) Suppose the feasibility condition is false, i.e., \(a_{1t} + \int_t^\infty \{ v(k,t) - \tau(t) \} \exp[-\int_t^\tau r(x)dx]dt < 0\) for all \(t\), then this would violate the transversality condition at infinity because the capitalized value of life-time consumption is positive, so that \(\lim_{t \to \infty} a_{1t} \exp[-\int_0^t r(x)dx] < 0\). For a more sophisticated analysis of the consumers' feasibility condition in perfect capital markets, see chapter VII of Arrow and Kurz (1971).
4.1. Steady State Values

The steady state of the system occurs when all decision variables are constant, i.e., $\dot{c}_t = \dot{a}_t = \dot{k}_t = 0$. We proceed by analyzing prior conditions (3-26) and (3-29) together with (3-17) using the diagram below to determine the stationary levels of consumption and the capital stock $(c^*_t, c^*_t, k^*_t)$.

In Figure 1, the $\dot{k}_t = w(k_t) + r(k_t)k_t - g_t - c_t = 0$ curve, which is reduced to $c^*_t = (k^*_t)^\beta - g^*_t$ in the steady state, turns out to be concave in the $c_t - k_t$ plane, using conditions (3-5) and (3-6).

Conditions (3-27) and (3-30) determine the shape of the $\dot{c}_1 = 0$ and $\dot{c}_2 = 0$ loci, respectively. The $\dot{c}_1 = 0$ locus must lie above the $\dot{c}_2 = 0$ locus with its slope being larger than the latter because $\delta^*_t(.) > \delta^*_1(.)$ (i.e., $\delta^*_2(.) > \delta^*_1(.)$) for a given level of $U$ implies that $\delta^{-1}_1(.) > \delta^{-1}_2(.)$ for any given $r(k_t)$. Thus, aggregate capital stock $k^*_t$ and aggregate consumption $c^*_t$ per-capita are determined by the intersection of the $\dot{k}_t = 0$ locus and the locus $\Theta$ corresponding to aggregate per-capita consumption at the stationary state. Since the latter is a linear combination of $c^*_1$ and $c^*_2$, such that $c^*_t = \alpha c^*_1 + (1-\alpha)c^*_2$, it lies between the locus $\dot{c}_1 = 0$ and the locus $\dot{c}_2 = 0$. It
Figure 1

Steady State Values of Consumption and the Capital Stock in Perfect Capital Markets
must be the case that $c_{i_t} = a_{i_t} = k_{i_t} = 0$ on the locus $\theta$. Note that the 
steady-state consumption of type-I agents $c^*_1$, exceeds that of type-II agents $c^*_2$, at the stationary-state capital stock $k^*_t$, as illustrated in Figure 1. The intuition behind this result essentially corresponds to the fact that those with more patience (i.e., type-I agents) are rewarded by larger steady-state consumption.

An interesting aspect of the short-run adjustment is that the 
locus $\theta$ is likely to fluctuate between the $c_{1_t} = 0$ and $c_{2_t} = 0$ loci, depending on the relative magnitude of the adjustment of $k_{i_t}$ in the short run. This occurs because each locus (1) and (2) could be 
affected by the short-run dynamic behavior of the per-capita capital 
stock for the other type of agent. Thus, $k^*_t$ and each locus (1) and (2) are subject to a change during the transition period. We can rule out this possibility if we assume that each locus (1) and (2) depends 
only on $c_{i_t}$ and $k_{i_t}$.

Given the values of $c^*_t$, $k^*_t$, $c^*_{1t}$, and $c^*_{2t}$ determined above, $k^*_{1t}$ 
and $k^*_{2t}$ can be determined by conditions (3-18) and (3-19).

$$k^*_{1t} = \left\{c^*_{1t} - w(k^*_t) + g^* \right\} / r(k^*_t) \quad (4-1)$$
\[ k_{2t}^* = \left[ c_{2t}^* - \nu(k_t^*) + g^* \right] / r(k_t^*) \] (4-2)

Conditions (4-1) and (4-2) imply that \( k_{1t}^* \) exceeds \( k_{2t}^* \). Hence, the stationary per-capita asset stock for type-I agent \( a_{1t}^* (= k_{1t}^* + d_t^*) \) is also larger than that for type-II agents \( a_{2t}^* (= k_{2t}^* + d_t^*) \).

4.2. The Optimal Dynamic Path

In this section, we will examine the dynamic behavior of the system using an analysis of the roots of the system along with a graphical analysis. We limit our attention to the case in which the steady state of the system is a saddlepoint equilibrium. If a steady-state saddlepoint exists, it can be shown that all decision variables approach their respective steady-state values along a unique saddlepoint path.

To examine the dynamic characteristics of the system, we first need to linearize the disaggregated underlying system which consists of four differential equations governing movements of each type's per capita consumption \( c_{it} \) and per-capita capital stock \( k_{1t} \). The linearized four-equation system around steady state values is shown in (4-3).
\[
\begin{bmatrix}
\mathbf{c}_{1t} \\
\mathbf{c}_{2t} \\
\mathbf{k}_{1t} \\
\mathbf{k}_{2t}
\end{bmatrix} =
\begin{bmatrix}
-\alpha \mathbf{U}' \delta_1' \mathbf{x}_1 \\
\mathbf{D}_1 \mathbf{U}' \delta_1' \mathbf{x}_1 \\
\mathbf{O}^*_1 \\
(1-\alpha) \mathbf{U}' \delta_1' \mathbf{x}_1 \\
\mathbf{O}^*_1 \\
\mathbf{c}_{1t} - \mathbf{c}_{1t}^* \\
\mathbf{c}_{2t} - \mathbf{c}_{2t}^* \\
\mathbf{k}_{1t} - \mathbf{k}_{1t}^* \\
\mathbf{k}_{2t} - \mathbf{k}_{2t}^*
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{g}_t - \mathbf{g}_t^* \\
\mathbf{g}_t - \mathbf{g}_t^* \\
\mathbf{g}_t - \mathbf{g}_t^* \\
\mathbf{g}_t - \mathbf{g}_t^* \\
\mathbf{g}_t - \mathbf{g}_t^* \\
\mathbf{g}_t - \mathbf{g}_t^* \\
\mathbf{g}_t - \mathbf{g}_t^* \\
\mathbf{g}_t - \mathbf{g}_t^* \\
\mathbf{g}_t - \mathbf{g}_t^*
\end{bmatrix}
\]

where \(\mathbf{Q}^*_1 = \{ \mathbf{U}''/\mathbf{U}' - \mathbf{U}' \delta_1' / (\delta_1 + \mathbf{U}' \delta_1') \} \delta_1 + \mathbf{U}' \delta_1' \), \(\mathbf{Q}^*_2 = \{ \mathbf{U}''/\mathbf{U}' - \mathbf{U}' \delta_2' / (\delta_2 + \mathbf{U}' \delta_2') \} \delta_2 + \mathbf{U}' \delta_2'\). \(\mathbf{Q}^*_1\) and \(\mathbf{Q}^*_2\) are, respectively, evaluated in the steady state. The signs of both \(\mathbf{Q}^*_1\) and \(\mathbf{Q}^*_2\) are shown to be negative since \((\delta_1 - \delta_1' U) > 0\) and \(\delta_1' < \delta_1' U\) by condition (3-3).
\[ X_1 = w'(k_t) + r'(k_t)k_{1t}, \quad X_2 = w'(k_t) + r'(k_t)k_{2t}. \]  

These are interpreted as the marginal change in before-tax income (wage and capital income) associated with a change in the capital stock for type I and II (i.e., \( \frac{\partial y_{1t}}{\partial k_t} \)).  

\[ D_1 = -r'(k_t)\delta_1 + U'\delta_1'X_1(\delta_1 + r(k_t)) \]  
\[ D_2 = -r'(k_t)\delta_2 + U'\delta_2'X_2(\delta_2 + r(k_t)). \]  

The determinant of the system’s 4 x 4 Jacobian matrix \( \Delta \), evaluated in the steady state, turns out to be positive.  

Since the trace of this matrix is also positive, a saddlepath stability of the system requires that there be two negative eigenvalues and two positive eigenvalues. To find the real eigenvalues of the above 4x4 matrix (say matrix \( R \)), we need to compute \( \text{det}[R - \lambda I] = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)(\lambda_4 - \lambda) = 0 \). The solution of this determinant results in a polynomial \( q(\lambda) \) in \( \lambda \) (i.e., the characteristic equation of the matrix \( R \)). More specifically, \( q(\lambda) = \lambda^4 + \rho_1\lambda^3 + \rho_2\lambda^2 + \rho_3\lambda + \rho_4 \), where the relations between the roots and coefficients \( \rho_i \), of the polynomial are as follows: \( \rho_1 = -\sum_{i=1}^{4} \lambda_i \), \( \rho_2 \)

12. Notice that \( \alpha X_1 + (1-\alpha)X_2 = w'(k_t) + r'(k_t)k_t = 0 \), using condition (3-5) and (3-6).

13. Both \( X_1 \) and \( X_2 \) take a positive value since \( X_1 = w'(k_t) + r'(k_t)k_{1t} = w'(k_t) + r'(k_t)k_t + r'(k_t)(k_{1t} - k_t) \), which reduces to \( r'(k_t)(k_{1t} - k_t) > 0 \), using the fact that \( w'(k_t) + r'(k_t)k_t = 0 \).

14. Direct computation yields \( \Delta = \left( -U'\delta_1\delta_2 r'(k_t)r(k_t) \{ (1-\alpha)\delta_1 + \alpha\delta_2 \} + U^2\delta_1\delta_2 \delta_1^2 r^2(k_t) \right) / \tilde{Q}_1 \tilde{Q}_2 > 0 \).
\[ \sum_{i<j} \lambda_i \lambda_j, \quad \rho_3 = -\sum_{k>j>i} (\lambda_k \lambda_j \lambda_i), \quad \text{and} \quad \rho_4 = q(0) = \lambda_1 \lambda_k \lambda_j \lambda_i = \det R > 0. \]

Given that \( \rho_4 \) is positive, the sign of the remaining coefficients \( \rho_1, \rho_2, \rho_3 \) can be determined by direct computation of \( \det [R - \lambda I] = 0 \). The sign of \( \rho_1 \) turns out to be negative because \( \rho_1 \) equals minus the sum of the trace. The sign of \( \rho_2 \) is shown to be positive.

Although the sign of \( \rho_3 \) is somewhat indeterminate, the subsequent dynamic analyses are restricted to the case in which \( \rho_3 \) takes a positive sign so as to ensure local saddlepath stability for the system with two positive roots plus two negative roots. If \( \rho_3 < 0 \), then no roots can be negative.\(^{15}\) Therefore, on the assumption that the sign of \( \rho_3 \) is positive, the system is saddlepath stable. The system can be shown to have the two roots (one positive and one negative root) in aggregate dynamics plus the two roots of the

---

\(^{15}\) According to Descartes' rule of signs, the number of positive real roots is at most equal to the number of variations of sign in \( q(\lambda) \) and the number of negative roots equal to the number of variations of sign in \( q(-\lambda) \). Since there are two variations of sign in \( q(\lambda) \) if \( \rho_3 \) is positive such that \( q(\lambda) = \lambda^4 + p_1 \lambda^3 + p_2 \lambda^2 + p_3 \lambda + p_4 \), the given equation \( q(\lambda) \) has two positive real roots. If \( \rho_3 \) is assumed to be negative, then there are four variations of sign in \( q(\lambda) \) and so all roots are positive. Likewise, the equation \( q(\lambda) \) has two variations of sign as long as \( \rho_3 \) is positive and hence, by Descartes' rule, the equation has at most two negative real roots, i.e., \( q(-\lambda) = \lambda^4 - p_1 \lambda^3 + p_2 \lambda^2 - p_3 \lambda + p_4 \).
saddlepath-stable system for type-II agents (or type-I agents) using row and column operations to simplify the above matrix.  

Next, we need to investigate the system's dynamic properties in a neighborhood of the stationary state of aggregate per-capita consumption ($c_t$), and aggregate per-capita capital stock ($k_t$).

We first derive the dynamic system $\ddot{c}_t = \alpha \dot{c}_t + (1-\alpha)\ddot{c}_t$ that describes the behavior of $c_t$ around the steady state values. Using conditions (3-26) and (3-29), $\ddot{c}_t$ is:

$$\ddot{c}_t = \alpha \left[ (\delta_{1t} - r(k_t)) \delta_{1t} + \delta'_{1t} U' (\delta_{1t} \dot{k}_{1t} + \{w'(k_t) + r'(k_t)k_{1t}\}k_t - g_t) \right] / \Omega_1$$

$$+ (1-\alpha) \left[ (\delta_{2t} - r(k_t)) \delta_{2t} + \delta'_{2t} U' (\delta_{2t} \dot{k}_{2t} + \{w'(k_t) + r'(k_t)k_{2t}\}k_t - g_t) \right] / \Omega_2$$

$$= \alpha \left[ (\delta_{1t} - r(k_t)) \delta_{1t} + \delta'_{1t} U' \delta_1 \{w(k_t) - g_t + r(k_t)k_{1t} - c_{1t}\} + \right.$$

$$\left. \delta'_{1t} U' \{w'(k_t) + r'(k_t)k_{1t}\} \{w(k_t) + r(k_t)k_t - g_t - c_t\} - \ddot{g}_t \right] / \Omega_1$$

$$+ (1-\alpha) \left[ (\delta_{2t} - r(k_t)) \delta_{2t} + \delta'_{2t} U' \delta_2 \{w(k_t) - g_t + r(k_t)k_{2t} - c_{2t}\} \right.$$

$$+ \delta'_{2t} U' \{w'(k_t) + r'(k_t)k_{2t}\} \{w(k_t) + r(k_t)k_t - g_t - c_t\} - \ddot{g}_t \right] / \Omega_2.$$
The differential equation of \( k_t \) (equation (3-17)) can be combined to yield laws of motion for aggregate dynamics. A linear approximation to \( \dot{c}_t = \alpha \dot{c}_{1t} + (1-\alpha) \dot{c}_{2t} \) and (3-17) about \((c^*_t, k^*_t)\) yields

\[
\begin{bmatrix}
\dot{c}_t \\
\dot{k}_t
\end{bmatrix} = \begin{bmatrix}
\frac{-\alpha U' \delta_1 X_1}{\Omega_1^*} + \frac{-(1-\alpha) U' \delta_2 X_2}{\Omega_2^*} \\
\frac{\Omega_1^*}{\Omega_2^*} - 1
\end{bmatrix} \begin{bmatrix}
\xi \\
r(k_t)
\end{bmatrix} \begin{bmatrix}
c_t - c_t^* \\
k_t - k_t^*
\end{bmatrix},
\]

where \( \xi = \alpha \{-r'(k_t) \delta_1 + r' \delta_2 X_1 \} / \Omega_1^* + (1-\alpha) \{-r'(k_t) \delta_2 + U' \delta_2 X_2 \} / \Omega_2^* \). The determinant of the system's 2x2 Jacobian matrix, evaluated in steady state, is negative. It can be shown from this matrix that the sum of the two eigenvalues (the trace of the matrix) is positive while the product of the two eigenvalues (the determinant of the matrix) is negative. This implies that the system has a unique saddlepath converging to the stationary state \((c_t^*, k_t^*)\).

The saddlepath SS, sloping upward in Figure 1, is the unique convergent path.

Next, consider the subsystem that describes the dynamic behavior of \( c_{it} \) and \( k_{it} \) in the neighborhood of long-run equilibrium, \( c_{it}^* \) and \( k_{it}^* \). It follows from condition (4-3) that the phase diagram corresponding to \( \dot{c}_{it} = 0 \) and \( \dot{k}_{it} = 0 \) can be shown to be saddlepath stationary because the product of the characteristic roots (the value
of the determinant) of each 2x2 matrix in the \( c_{it} = 0 \), \( k_{it} = 0 \) system is also negative. This implies that there must be one positive and another negative eigenvalue.\(^{17}\)

4.3. Comparative Statics Analysis

We now attempt to establish the effects of an unanticipated permanent change in the government policy parameters, \( g_t^* \) and \( d_t^* \), on the initial stationary-state values of consumption and capital stock for each type of agent. As we pointed out earlier, government debt \( (d_t^*) \) and the wage tax \( (\tau_t^*) \) are eliminated entirely from the dynamic equations that govern movements of consumption and capital stock for each type of agent. Only government expenditures \( (g_t^*) \) enter into those equations. This implies the neutrality of the substitution of government debt for a lump-sum wage tax and thereby supports the Ricardian equivalence proposition of Barro (1974) in the present context of perfect capital markets. Therefore, the focus should be on the effects of government expenditures \( (g_t^*) \) on the steady state values of consumption and capital stock. Furthermore, we investigate the

\(^{17}\) The determinant of the subsystem corresponding to \( c_{1t} = 0 \), \( k_{1t} = 0 \) can be written as \( \Delta = (\alpha U' \delta'_1 X_1 (\delta_1 + \alpha X_1) - \alpha \delta_1 + U' \delta'_2 r) / \Omega_2^* \), which is negative since \( \delta_1 > \alpha X_1 \). Likewise, the determinant of the other subsystem (i.e., \( c_{2t} = k_{2t} = 0 \)) have a negative value:

\( \Delta = ((1-\alpha)U' \delta'_2 X_2 (\delta_2 - (1-\alpha)X_1) - (1-\alpha)\delta_2 + U' \delta'_2 X_2 r) / \Omega_2^* \).
quantitative effects of these policy changes on the long-run equilibrium, along with a transparent graphical analysis in order to make the quantitative analysis more convincing and instructive. For the purpose of the graphical analysis, it is worth noting that a change in $g_t^*$ affects only the $k_t^* = 0$ locus in Figure 1.

The comparative static derivatives of $c_{it}^*$ and $k_{it}^*$ with respect to $g_t^*$ are determinable, using condition (4-3). A straightforward application of Cramer's rule yields:

\[
\frac{\partial c_{1t}^*}{\partial g^*} = \frac{U' \delta_2 \delta_1 \delta_2 r' (k_t^*) r(k_t^*)}{\Delta \Omega_1^* \Omega_2^*} < 0 \quad (4-4)
\]

\[
\frac{\partial c_{2t}^*}{\partial g^*} = \frac{U' \delta_1 \delta_1 \delta_2 r' (k_t^*) r(k_t^*)}{\Delta \Omega_1^* \Omega_2^*} < 0 \quad (4-5)
\]

\[
\frac{\partial k_{1t}^*}{\partial g^*} = \frac{(1-\alpha) \left( U' \delta_1 \delta_2 r' (k_t^*) (\delta_1^* - \delta_2^*) + U' \delta_1 \delta_2 \delta_1 \delta_2 (r(k_t^*) + (1-\alpha)(x_2 - x_1)) \right)}{\Delta \Omega_1^* \Omega_2^*} > 0 \quad (4-6)
\]

\[
\frac{\partial k_{2t}^*}{\partial g^*} = \frac{\alpha \left( U' \delta_1 \delta_2 r' (k_t^*) (\delta_1^* - \delta_2^*) + U' \delta_1 \delta_2 \delta_1 \delta_2 (r(k_t^*) + \alpha(x_2 - x_1)) \right)}{\Delta \Omega_1^* \Omega_2^*} > 0 \quad (4-7)
\]

\[
\frac{\partial c_{1t}^*}{\partial d^*} = \frac{\partial k_{1t}^*}{\partial d^*} = 0 \quad \Rightarrow \quad \frac{\partial a_{1t}^*}{\partial d^*} = 1 , \quad (4-8)
\]
where $X_1 = \psi'(k_t) + r'(k_t)k_{1t}$, $X_2 = \psi'(k_t) + r'(k_t)k_{2t}$, and $\Delta (>0)$ represents the value of the determinant of the system's 4 x 4 Jacobian matrix. The sign of $\Delta \Omega_1^* \Omega_2^*$ is positive.

Conditions (4-4) and (4-5) indicate that in response to an unanticipated permanent increase in $g_t^*$, $c_{1t}^*$ will decrease by a greater amount than $c_{2t}^*$. Namely, $|dc_{1t}^*/dg_t^*| > |dc_{2t}^*/dg_t^*|$ since $\delta_2^*$ is larger than $\delta_1^*$, as was assumed. This result in fact can be confirmed graphically in Figure (1) by considering what happens to each type's initial steady-state values of consumption when $g_t^*$ increases. By an unanticipated permanent increase in $g_t^*$, the $k_t = 0$ locus moves outward, but the locus $\Theta$ is unchanged. Then, the economy is governed by the initial locus $\Theta$ and the new $k_t^* = 0$ locus with steady state at D. At the new stationary value of $k_t$ (i.e., $(k_t^*)$, $c_{1t}^*$ is unambiguously shown to be smaller than its initial steady state values and $c_{1t}^*$ ends up decreasing more than $c_{2t}^*$ in response to the increase in $g_t^*$ since the $c_{1t}^* = 0$ locus is assumed to be steeper than the $c_{2t}^* = 0$ locus.

When considering the short-run effects of an unanticipated permanent increase in $g_t^*$, the capital stock is given by the initial level $k_t^*$.
with a momentary spurt in $g_t^*$, but the consumption must jump in order to move the economy to point C on the initial capital stock $k_t^*$, and then converges monotonically along a saddlepoint path CD to the new steady state, point D. 18 Thus, in response to an unanticipated permanent increase in $g_t^*$ there is an immediate decrease in consumption because the capital stock cannot jump instantaneously. Heuristically, the unanticipated increase in $g_t^*$ is met first by a decrease in private consumption. The unanticipated increase in demand due to the increase in $g_t^*$ is a signal that the investment plans for capital must be revised upward. This raises both private consumption and capital stock during the transition periods.

Conditions (4-6) and (4-7) indicate that a permanent unanticipated increase in $g_t^*$ will raise the amount of per-capita capital stock each type of agent plans to hold in the steady state. A graphical analysis for these results is illustrated in Figure 1 by

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18. A similar graphic analysis for the short-run impacts of some particular fiscal policy (i.e., a tax cut followed with a lag by an expenditure cut and a tax cut followed with a lag by a tax increase) on equilibrium in a perfect foresight model is found in Kenneth L. Judd (1985, 1987).
the rightward shift of the $k^*_t = 0$ locus associated with the increase in $g^*_t$. This ends up raising steady state capital stock per capita and thus the stationary value of per-capita assets for each type. The short run consequences of an increase in $g^*_t$ on $k^*_t$ is governed by the saddlepath CD, in Figure 1, resulting in the long-run increase in $k^*_t$.

Condition (4-8) reveals that an unanticipated permanent increase in government debt $(d^*_t)$ will leave $c^*_it$ and $k^*_it$ unchanged. The neutrality of debt results from the government budget constraints we assumed. Namely, a permanent unanticipated increase in debt per capita $(\Delta d^*_t)$ for a given government expenditures $(g^*_t)$ should be viewed as a gift of government bonds (i.e., an increase in a net wealth) to the agents. This leads to a corresponding permanent increase in the capitalized value of per-capita wage tax equal to $\Delta \tau_t / r(k^*_t)$ because each rational agent reacts by saving more for the future tax liabilities. In this sense, government debt and taxation have equivalent effect on the economy. Hence, each agent's budget constraint (i.e., net wealth position) and thus consumption remain unchanged. It is straightforward to show this result in Figure 1. The long-run stationary values of $c^*_it$ and $k^*_it$ are actually determined independent of the level of $d^*_t$ (and $\tau^*_t$) by the conditions, $\delta^*_i[U(c^*_i)]$
result obtained in the long run holds true in the short run as well since the terms \(d_t\) and \(\tau_t\) do not appear in the loci \(c^*_t\) and \(k^*_t\). A permanent increase in each type's per capita asset \((a^*_{it})\) will come entirely from the increase in its share of the extra government debt, implying that \(\partial a^*_{it}/\partial d^*_t = 1\).

More importantly, the above results correspond to the key prediction of the Ricardian hypothesis that rational consumers base their consumption decisions on lifetime budget constraints which depend on government expenditures, but are unaffected by the way that expenditure is financed, whether by taxes or by issuing debt. Accordingly, an increase in government expenditures reduces life-time income and thereby consumption, whereas government debt is neutral. Summarizing,

**Proposition 1:** In the present setting of the model characterized by the infinite-horizon intertemporal optimization of the two types of agents with endogenous time preferences, by lump-sum wage structure, and by perfect capital markets, Ricardian equivalence does hold. An unanticipated, permanent increase in the government expenditures obviously reduces the stationary values of per-capita consumption, while raising per-capita capital stock and per-capita assets.
CHAPTER V

THE MODEL WITH LIQUIDITY CONSTRAINTS

In this chapter, we focus on the implications for whether Ricardian equivalence does hold even in the presence of imperfect capital markets. It is widely believed that the assumption of imperfect capital markets is one of the most compelling objections raised against the assumptions on which Ricardian equivalence are based. The crucial assumption of imperfect capital markets is that the loan rates available to type-II agents with high subjective discount rate are usually higher than the lending rate in the private credit markets due to the high transaction costs involved in the loan process. This implies that there exist differential rates between the borrowing rates and the lending rates. Incorporating imperfect capital markets, instead of perfect capital markets, into our basic analytic model requires constructing a model with liquidity constraints. The liquidity constraints (or borrowing constraints) considered throughout this paper take the form of exogenous nonnegativity constraints on asset stocks. If this type of liquidity constraints is present while solving the intertemporal optimization problem with budget constraints, then type-II agents should face the
additional liquidity constraints (or borrowing constraints) that they are not permitted to borrow against future income (i.e., human capital or non-traded assets), and thus their consumption is limited by current resources. They must have nonnegative asset stocks (i.e., liquid or traded assets) at all times ($a_{2t} \geq 0$). The nonnegativity constraint on asset stock is equivalent to the constraint that the type-II agent's per-capita capital stock ($k_{2t}$) cannot be less than $-d_t$.

Another important implication of liquidity constraints, as Mariger (1986, 1987) and Hayashi (1985) show, is that the likelihood of future liquidity constraints effectively shorten the consumers' horizon for the optimal consumption plan even though they face a lifetime horizon. Hence, consumption appears to be excessively sensitive to change in current income. This could render the marginal propensity to consume from short-term income change close to unity.

The incorporation of liquidity constraints enables us to presume a more realistic relationship between each type's subjective rate of time preference.

\[ \delta_{2t}(\cdot) > r(k_t) = \delta_{1t}(\cdot) \quad (5-1) \]

19. If the constraint is binding ($a_{2t} = 0$), however, then consumption can always be shifted forward in time by holding positive assets ($a_{2t} > 0$). That is, type-II agents are not constrained from saving more. This cannot be shifted backward because it would violate the nonnegative liquidity constraints.
This assumption implies that type-I agents have the same discount rate as the government and are willing to hold government debt.

For analytic convenience, we assume that type-I agents solve the intertemporal optimization without liquidity constraints. But type-II agents face liquidity constraints due to transaction costs in borrowing, or the legal enforcements such as personal bankruptcy laws. In particular, the observation that a market for non-collaterized consumer borrowing has been less developed relative to a market for collaterized loans in the U.S. appears to be consistent with personal bankruptcy laws in the U.S. (see Flavin (1985), Hubbard and Judd (1986)).

We need to modify the intertemporal budget constraint of type-II agents in condition (3-11) so as to take into account the liquidity constraints.

\[ a_{2t}^0 = w(k_t) - \tau_t + r(k_t)a_{2t} - c_{2t}, \]  

and the nonnegative conditions \( a_{2t} \geq 0 \) (i.e., \( k_{2t} \geq -d_t \))

The formulation of the Hamiltonian becomes:

\[ H = (U(c_{2t}) + \phi(\Delta)( w(k_t) - \tau_t + r(k_t)a_{2t} - c_{2t}) + \lambda(\Delta) a_{2t}) / \delta_{2t}[U(c_{2t})], \]
where $\lambda(\Delta)$ is the multiplier associated with the nonnegativity constraint on $a_{2t}$. The necessary conditions for $(c_{2t}, a_{2t})$ to be optimal are that there exist $p(\Delta)$ and $\lambda(\Delta)$ such that $(c_{2t}, a_{2t}, p(\Delta), \lambda(\Delta))$ satisfy not only (5-2) but also:

$$
\begin{align*}
\text{i) } \frac{\partial H}{\partial c_{2t}} &= 0, \quad \Rightarrow U'(c_{2t}) = P(\Delta) + H \{ \delta_{2t}^2 U'(c_{2t}) \} \\
\text{ii) } \frac{\partial p}{\partial \Delta} &= \rho(t)p(\Delta) - \frac{\partial H}{\partial k_{2t}} = p(\Delta) - \{ p(\Delta) + \lambda(\Delta) \} r(k_t) / \delta_{2t}^{\Delta}, \\
\text{and} \\
\text{iii) } \lambda(\Delta) \geq 0, \text{ if } a_{2t} > 0, \text{ for all } a_{2t} \geq 0,
\end{align*}
$$

where $p(t) = -[(\partial c(t)/\partial \Delta)/c(t)]$ and $c(t)$ represents a discount factor $e^{-\Delta}$. The first-order conditions presented in (5-4) must be altered according to the value of $a_{2t}$. Namely,

$$
(5-5)
$$

When $a_{2t} > 0$ (i.e. $k_{2t} > -d_t$), condition (iii) of (5-4) ensures that $\lambda(\Delta) = 0$. Condition (ii) of (5-4) yields $\partial p(\Delta)/\partial \Delta = p(\Delta)\{ \delta_{2t}^{\Delta - 1} - 1 \}$.

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20. This approach of handling state variable constraints of the form $a_{2t} \geq 0$ (i.e., $k_{2t} + d_t \geq 0$) is provided in Kamien and Schwartz (1982), Affow and Kurz (1971).

21. Refer to footnote 8 for a detailed procedure to derive this result. In this case, the derivative of the Hamiltonian in terms of $a_{2t}$ would be slightly different from the previous one: i.e., using (5-3), $\delta H/\partial a_{2t} = \{ p(\Delta) r(k_t) + \lambda(\Delta) r(k_t) \}/\delta_{2t}^t - r(k_t) \{ p(\Delta) + \lambda(\Delta) \}/\delta_{2t}^{t-1}$. 
Both condition (i) of (5-4) and the differential equation governing movement of assets $a_{2t}$ still hold.

\[ r(k_t) / \delta_{2t} \]

When $a_{2t} = 0$ (i.e., $k_{2t} = -d_t$), it imposes the necessary tangency condition between $a_{2t}^*$ and $a_{2t}$ so that $a_{2t}^* = 0$, and by condition (iii) of (5-4), $\lambda(\Delta) > 0$. Then, $a_{2t}^* = w(k_t) - \tau_t - c_{2t} = 0$, which in turn implies $c_{2t}^* = w(k_t^*) - \tau_t^* = w(k_t^*) - g_t^* - r(k_t^*)d_t^*$ in the steady state.

Condition (5-5) indicates that if the real assets for type-II agents are strictly positive ($a_{2t} > 0$), on an unconstrained interval, the previous conditions for the intertemporal optimization still hold. However, when the nonnegativity constraint is binding (i.e., type-II agents face permanent liquidity constraints), as condition (5-6) implies, the consumption for type-II agents is limited to current after-tax wage income. If the consumption for type-II agents at $a_{2t} = 0$ were less than \(w(k_t) - \tau_t\), their assets would grow and thus $\lambda(\Delta) = 0$. This reduces to the case on an unconstrained interval (condition (5-5)). In that case, as their assets grow without bound, their consumption would fall to zero, violating the transversality condition at infinity. It is impossible that $c_{2t} > \{w(k_t) - \tau_t\}$, at $a_{2t} = 0$, since it violates both the budget and liquidity constraints.
Therefore, $c_{2t}$ must be exactly equal to \( w(k_t) - \tau_t \) when per capita assets for a constrained type-II agent are zero.\(^{22}\)

Now consider the case in which type-I agents behave just as in chapter IV (facing perfect capital markets), while type-II agents face permanent liquidity constraints, consuming all of wage income every period and never holding assets. We consider this case to make the resulting model as simple and tractable as possible, although it might be unrealistic to assume that type-I agents always have access to perfect capital markets while type-II agents confront permanent liquidity constraints.\(^{23}\) It follows from condition (5-6) that a type-II agent's per capita assets are zero every period and thus its growth is constant. This implies that each type-II agent consumes all of his disposable wage income, i.e., his marginal propensity to consume (MPC) is unity.

Since type-I agents are assumed to have access to perfect capital markets, their aggregate consumption and their aggregate asset stocks are governed by prior conditions (3-26) and (3-10). Thus,

---

22. Hubbard and Judd(1987) provide an analysis of effecting liquidity constraints (i.e., asset stock is zero) within the context of a stochastic life-cycle model. They consider two possible dynamic paths of consumption and asset accumulation which are substantially distinguished, depending critically on whether desired initial consumption at $a_t = 0$ exceeds the initial wage income or not.

23. This approach is adapted from Paul Evans(1988), who shows using Blanchard's model that Ricardian equivalence may be a good approximation even with the existence of liquidity constraints.
\[ c_{1t} = \left[ (\delta_{1t} - r(k_t)) \delta_{1t}' + \delta_{1t}' U'(\delta_{1t}) \right] \left[ \omega(k_t) + r(k_t) k_{1t} - g_t \right] / \Omega_1 \]

and

\[ \dot{a}_{1t} = w(k_t) - \tau_t + r(k_t) a_{1t} - c_{1t} \] (5-8)

Note that this model with liquidity constraints, assumes the same basic framework as in chapter III. For convenience, we replicate the differential equation governing movements in capital stock (conditions (3-17), (3-18), and (3-19)) and the government budget constraint (condition(3-16)):

\[ \dot{k}_t = w(k_t) + r(k_t) k_t - g_t - c_t \] (5-9)

\[ \dot{k}_{i1t} = w(k_t) - g_t + r(k_t) k_{i1t} - c_{i1t}, \quad i = \text{type-I and type-II} \] (5-10)

\[ r_t d_t + g_t = \tau_t + \dot{d}_t \] (5-11)

5.1. Steady State Values

In this section, we find a somewhat remarkable result that the steady state of the model with liquidity constraints has qualitatively the same properties as the model without liquidity constraints (see chapter IV).
The stationary levels of consumption and capital stock
\((c_t^*, c_{1t}^*, k_t^*)\) can be derived by conditions (5-7), (5-9), and \(c_{2t}^* = w(k_t^*) - g_t^* - r(k_t^*)d_t^*\). Condition (5-7) implies \(\delta_{1t}[U(c_{1t}^*)] = r(k_t^*)\) in the steady state. This can be rewritten as \(U(c_{1t}^*) = \delta_{1t}^{-1}[r(k_t^*)]\).

The \(k_t = 0\) locus reduces to \(c_t^* = (k_t^*)^\beta - g_t^*\) (see chapter IV). Before determining the stationary values of \(c_{1t}\) and \(k_t\), we need to compare \(c_{1t}^*\) resulting from condition (5-10) with \(c_{2t}^* = w(k_t^*) - g_t^* - r(k_t^*)d_t^*\).

The steady state values of per capita consumption \(c_{1t}\) are given as
\[c_{1t}^* = w(k_t^*) - g_t^* + r(k_t^*)k_{1t}\] and \(c_{2t}^* = w(k_t^*) - g_t^* - r(k_t^*)d_t^*\). It must be the case that \(c_{1t}^* > c_{2t}^*\). Furthermore, we can determine \(a_t^* = \alpha a_{1t}^* = \alpha c_t^* - w(k_t^*) + g_t^* + r(k_t^*)d_t^* / r(k_t^*)\) from the locus \(a_{1t}^* = 0\), and \(a_{2t}^* = 0\) at all times because of the nonnegativity constraint on asset stock.

This implies that all stationary asset stock, consisting of \(k_t^*\) and \(d_t^*\), is held by type-I agents, while type-II agents as well as type-I agents exert their influence on the properties of the steady state because \(c_{2t}^*\) depends on \(k_t^*\).

Now we can illustrate the stationary levels of consumption and capital stock \((c_t^*, c_{1t}^*, k_t^*)\) by Figure 2. The steady state consumption \(c_t^* = \alpha c_{1t}^* + (1-\alpha)c_{2t}^*\) and capital stock \(k_t^*\) occur where the \(k_t = 0\)
locus and the locus \( \omega \) intersect. The locus \( \omega \) represents the weighted sum of the locus of each type's per-capita consumption for a given \( k_t^* \) in the steady state. Hence, the locus \( \omega \) must lie between the loci (1) and (2) and has a kink at a point at which type-II agents hit the liquidity constraints (i.e., have zero assets). The locus \( \omega \) is also shown to cross the intersecting point between the loci (1) and \( c_{2t}^* \). The slope of the stationary consumption of type-II agents (i.e., the locus (2)) turns out to be flatter than the \( k_t^* = 0 \) locus. 24

As illustrated, \( c_{1t}^* \) is larger than \( c_{2t}^* \) at a given level of \( k_t^* \). With type-II agents facing binding liquidity constraints, we need only examine \( a_{1t}^* \) in the aggregate analysis of per-capita asset stocks \( a_t^* \), since \( a_{2t}^* = 0 \). Note that the locus \( a_{1t}^* = 0 \) can be reduced to the locus \( k_{1t}^* = 0 \) as equation (3-18) implies and further to the locus \( k_t^* = 0 \) because \( k_{2t} \) is fixed. These findings regarding the steady state values have qualitatively the same properties as the model without liquidity constraints.

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24. The locus (2) is shown to be \( c_{2t}^* = (1-\beta)(k_t^*)^\beta - g_t^* - \beta(k_t^*)^{\beta-1}d_t^* = (1 - \beta - \beta d_t^* / k_t^*)(k_t^*)^\beta - g_t^* \), using condition (3-5). Thus \( c_{2t}^* \) should be drawn as concave, its slope being flatter than and its intercept being equal to those of the \( k_t^* = 0 \) locus.
Figure 2

Steady State Values of Consumption and Capital Stock in Imperfect Capital Markets
5.2. The Optimal Dynamic Path and Comparative Statics Analysis

While permanent liquidity constraints are assumed to be imposed on type-II agents, the behavior of both types of agents is important in affecting the optimal dynamic path of the system as well as the steady state. We need to examine the dynamic system that governs the movement of $c_{it}$ and $k_t$ around the steady state.

First, consider the differential equation of the consumption for type-II agents who are now permanently liquidity constrained. Since liquidity constraints are binding for type-II agents, $a_{2t} = 0$ and $c_{2t}^* = \omega(k_t^*) - g_t^* - r(k_t^*)d_t^*$. 25 The time derivative of the equation of $c_{2t}$ around the steady state is:

$$
\dot{c}_{2t} = \{\omega'(k_t) - r'(k_t)d_t\}k_t - \dot{g}_t - r(k_t)d_t
$$

$$
= \{\omega'(k_t) - r'(k_t)d_t\}k_t, \quad \text{if} \quad \dot{g}_t = \dot{d}_t = 0 \quad \text{along the transition path. Since} \quad \dot{c}_t = \alpha\dot{c}_{1t} + (1-\alpha)\dot{c}_{2t}, \quad \dot{c}_t = 0 \quad \text{can be written as:}
$$

$$
\dot{c}_t = \alpha[\delta_{1t} - r(k_t)]\delta_{1t} + \delta_{1t}U'(\delta_{1t}k_{1t} + \{\omega'(k_t) + r'(k_t)k_{1t}\})\dot{g}_t - \dot{g}_t]/\Omega_1
$$

25. When $k_{2t} = -d_t$, $k_t = \alpha k_{1t} + (1-\alpha)k_{2t} = \alpha k_{1t} - (1-\alpha)d_t$. This equation can be rearranged as $k_{1t} = k_t/\alpha + ((1-\alpha)/\alpha)d_t$. This can be reduced, using condition (3-12) and $a_{1t} = k_{1t} + d_t$, to $a_{1t} = a_t/\alpha$, which means that per-capita asset stock for type-I agent amounts to aggregate per capita asset stock divided by its population (the fraction $\alpha$).
\[ + (1-\alpha)\{ \psi'(k_t) - r'(k_t)d_t \} k_t^{o} \]
\[ = \alpha \{ (\delta_{1t} - r(k_t)) \delta_{1t} + \delta'_{1t}U' \delta_{1t} \{ \psi(k_t) + r(k_t)k_{1t} - g_t - c_{1t} \} + \delta'_{1t} U' \{ \psi'(k_t) + r'(k_t)k_{1t} \} \{ \psi(k_t) + r(k_t)k_t - g_t - c_t \} - g_t \} / \Omega_1 + \]
\[ (1-\alpha)\{ \psi'(k_t) - r'(k_t)d_t \} \{ \psi(k_t) + r(k_t)k_t - g_t - c_t \}, \]

where \(k_{1t} = k_t / \alpha + (1-\alpha)d_t / \alpha.\)

A linear approximation of the dynamic equations \(c_t^{o}\) above and

(5-9) at the steady state gives:

\[
\begin{bmatrix}
\circ c_t \\
\circ k_t
\end{bmatrix}
= \begin{bmatrix}
\begin{bmatrix}
\psi'(k_t) - r'(k_t)d_t \\
\psi'(k_t) - r'(k_t)d_t
\end{bmatrix}
\begin{bmatrix}
\circ c_t \\
\circ k_t
\end{bmatrix}
\end{bmatrix} + \begin{bmatrix}
(1-\alpha)U' \delta_{1t}X_1 - \alpha r' \delta_{1t} + \alpha \{ U' \delta_{1t}X_1 (\delta_{1t} + r) + U' \delta_{1t} \delta_{1t} (r / \alpha) \} + (1-\alpha)ru
\end{bmatrix}
\]

\[\begin{bmatrix}
\circ c_t \\
\circ k_t
\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix}
\circ c_t \\
\circ k_t
\end{bmatrix}
\end{bmatrix} \times \begin{bmatrix}
c_t - c_t^* \\
k_t - k_t^*
\end{bmatrix} \times \begin{bmatrix}
\circ c_t \\
\circ k_t
\end{bmatrix} + \begin{bmatrix}
(1-\alpha)U' \delta_{1t}X_1 - \alpha r' \delta_{1t} + \alpha \{ U' \delta_{1t}X_1 (\delta_{1t} + r) + U' \delta_{1t} \delta_{1t} (r / \alpha) \} + (1-\alpha)ru
\end{bmatrix}
\]

\[\begin{bmatrix}
\circ c_t \\
\circ k_t
\end{bmatrix} = \begin{bmatrix}
\begin{bmatrix}
\circ c_t \\
\circ k_t
\end{bmatrix}
\end{bmatrix} \times \begin{bmatrix}
c_t - c_t^* \\
k_t - k_t^*
\end{bmatrix} \times \begin{bmatrix}
\circ c_t \\
\circ k_t
\end{bmatrix} + \begin{bmatrix}
(1-\alpha)U' \delta_{1t}X_1 - \alpha r' \delta_{1t} + \alpha \{ U' \delta_{1t}X_1 (\delta_{1t} + r) + U' \delta_{1t} \delta_{1t} (r / \alpha) \} + (1-\alpha)ru
\end{bmatrix}
\]
\[ + \left[ -\alpha \delta'_1 U'(x_1 + \delta_1) \right] \frac{g_t}{\Theta_1^*} - (1-\alpha)\nu \frac{\delta_1}{\Theta_1^*} r/\Theta_1^* \left[ \begin{array}{c} g_t - g_t^* \\ d_t - d_t^* \end{array} \right] \]

where \( x_1 \) and \( \Theta_1^* \) have the same values as those in section 4.2, and \( \nu = w'(k_t) - r'(k_t)d_t > 0 \). The determinant of the system's 2x2 Jacobian matrix, evaluated in steady state, can be shown to have a negative sign. This implies that the dynamic system describing the behavior of \( c_t \) and \( k_t \) is saddlepoint stable.

The phase diagram corresponding to \( c_t^* = k_t^* = 0 \) is illustrated in Figure 2. A unique optimal dynamic path converging to the steady state will be the two trajectories originating in region \( I \) and \( III \).

With the knowledge that there is a saddlepoint equilibrium, we turn to a comparative statics analysis of the steady state in order to catch the implications for Ricardian equivalence in the presence of liquidity constraints.

Using condition (5-12), the derivatives of \( c_t^* \) and \( k_t^* \) with respect to \( g_t^* \) and \( d_t^* \) are determined. The resulting comparative statics derivatives are:

\[
\frac{\partial c^*}{\partial g^*} = \frac{\alpha r'(k_t)\delta_1 - (1-\alpha)U'\delta_1\delta_1^* r - \alpha U'\delta'_1\delta_1 x_1}{\Delta \Theta_1^*} < 0 \quad (5-13)
\]
\[
\frac{ak^*_t}{ag^*_t} = \frac{\alpha U' \delta_1 \delta_1}{\Delta \Omega_1^*} > 0 \Rightarrow \frac{ak^*_t}{ag^*_t} = \frac{U' \delta_1 \delta_1}{\Delta \Omega_1^*} > 0 \quad (5-14)
\]

\[
\frac{ac^*_t}{ad^*_t} = \frac{-(1-\alpha)U' \delta_1 \delta_1 r^2}{\Delta \Omega_1^*} < 0 \quad (5-15)
\]

\[
\frac{ak^*_t}{ad^*_t} = \frac{-(1-\alpha)U' \delta_1 \delta_1 r}{\Delta \Omega_1^*} < 0 \quad (5-16)
\]

where \( \Delta \Omega_1^* = \alpha (-r'(k_t) \delta_1 + U' \delta_1 \delta_1 (X_1 + r/\alpha)) > 0 \).

Equations (5-13) and (5-14) indicate that a permanent unanticipated rise in government expenditures \((g^*_t)\) will lead to a lower stationary level of aggregate per-capita consumption, but a higher steady-state level of aggregate per-capita capital stock. In particular, equation (5-14) comes from the fact that the comparative steady-state analysis of aggregate per-capita capital stock \((k^*_t)\) with respect to \(g^*_t\) just corresponds to that of \(k^*_{1t}\) with respect to \(g^*_t\), because \(k^*_{2t}\) is assumed to be constant. The intuition behind (5-13) is straightforward. Government expenditures have an adverse effect on per-capita consumption for a type-II agent, as condition (5-6) shows. In addition, a rising aggregate capital stock \(k^*_t\), in response to the increase in \(g^*_t\) implies a lower \(r(k^*_t)\) and thus a lower \(c^*_{1t}\) from the
condition \( r(k_t^*) = \delta \{ U(c_{1t}^*) \} \). This will lead to a negative long-run effect of a permanent unanticipated increase in \( g_t^* \) on aggregate per-capita consumption \( c_t^* \). A graphical analysis is given in Figure 3. This shows that in response to an unanticipated permanent increase in \( g_t^* \), the locus \( k_t^* = 0 \) shifts rightward and the locus \( \omega \) downward to the right of the kink. In the short run, a rise in government expenditures causes an even greater drop in aggregate private consumption, than at the long-run equilibrium, as the economy jumps from the initial equilibrium at point A to point B (i.e., government expenditures have an initial contractionary impact). This effect is reversed as aggregate per-capita capital stock \( k_t^* \), ultimately increases along the saddlepoint path BC in the long run. This adjustment leads to a lower level of aggregate per-capita consumption, but a higher level of aggregate per-capita capital stock at the long run equilibrium C.

Equation (5.15) reveals that a permanent unanticipated increase in government debt \( d_t^* \) will cause aggregate per-capita consumption \( c_t^* \) to fall. This nonneutrality result stems from a direct adverse impact of government debt on \( c_{2t}^* \) which more than offsets an opposing increase in \( c_{1t}^* \) by its indirect effect of reducing \( k_t^* \).
Equation (5-16) shows that an unanticipated permanent increase in government debt will lead to a fall in aggregate per-capita capital stock ($k_t^*$). One possible explanation of this result may be that an increase in debt raises $c_{1t}^*$ and thus induces less saving by type-I agents. The configuration shown in Figure 4 represents the qualitative effect of an unanticipated permanent increase in per-capita debt ($d_t^*$). An increase in $d_t^*$ makes only the locus $\omega$ shift downward to the right of the kink since only $c_{2t}^*$ depends on $d_t^*$. This will lead to a decrease in $c_t^*$ and $k_t^*$ at the long run equilibrium $F$. The short-run effect of an unanticipated permanent increase in government debt ($d_t^*$) is to cause the per-capita consumption to jump, at the initial capital stock $k_t^*$, in order to move the economy to point $E$ on the initial level of $k_t^*$. Then, this economy converges monotonically along a saddlepoint path $EF$ to the new steady state, point $F$. In the long run, debt ($d_t^*$) expansion crowds out the aggregate private per-capita capital stock and aggregate per-capita consumption is correspondingly lower.

Summarizing,
Figure 3

The Effect of an Increase in Government Expenditures on Aggregate Per Capita Consumption and Aggregate Per Capita Capital Stock

Figure 4

The Effect of an Increase in Government Debt on Aggregate Per Capita Consumption and Aggregate Per Capita Capital Stock
Proposition 2: In the presence of liquidity constraints, which are binding for type-II agents, Ricardian equivalence does break down in the sense that government debt causes both aggregate per-capita consumption and aggregate per-capita capital stock to decrease.
CHAPTER VI

THE EFFECT OF GOVERNMENT DEBT ON THE SPREAD

This chapter focuses on analyzing another important implication regarding the effect of government debt on the spread in the presence of liquidity constraints. We present a simple model in which government debt has the effect of narrowing the spread between the borrowing rate, implied by the subjective discount rate for type-II agents, and the lending rate representing the competitive rate of return on capital. This implication is essentially consistent with the analytic result from the inclusion of imperfect capital markets in Barro (1974, 1985, 1989).

Within the frameworks of the previous chapter V, we can show that government debt leads to an endogenous narrowing of the spread and that the government’s implicit loan tends to substitute for private financial intermediation. An increase in government debt causes \( c_{2t}^* \) to fall, but \( c_{1t}^* \) to rise. However, the decrease in \( c_{2t}^* \) outweighs the increment of \( c_{1t}^* \) and \( c_t^* \) eventually drops. It follows that the interest rate at which type-I households are willing to lend,
\( \delta_1[U(c_{1t}^*)] \), rises, while the interest rate at which type-II households are willing to borrow, \( \delta_2[U(c_{2t}^*)] \), falls. Consequently, the spread between these two interest rates narrows in response to an unanticipated permanent increase in debt. This result suggests that government debt, as an effective form of financial intermediation, tends to crowd out private financial intermediation.

Suppose type-I agents who have a lower subjective discount rate, play the role of ultimate lenders in the financial markets. Also, they are willing to lend loanable funds at the competitive rate of return on capital, which also corresponds to their subjective rate of time preferences. This assumption can be justified in our model with liquidity constraints because type-I agents, facing no liquidity constraints whatsoever, hold all assets and thus provide loanable funds to the private loan markets. Thus,

\[
\delta_1[U(c_{1t}^*)] = f'(k_t) = r_L
\]  
(6-1)

Given the present setting of imperfect capital markets, we posit that only type-II agents face liquidity constraints, due to the high transaction costs involved in loan process such as loan evaluation and enforcement. This high transaction cost is reflected in high (net-of-default risk) borrowing rates \( r_B \). We can suppose that type-II agents' subjective discount rates are equal to high borrowing rates:
\[ \delta_{2t}[U(c_{2t})] = r_B \]  

(6-2)

From conditions (6-1) and (6-2), it can be shown that \( c_{1t} \) and \( c_{2t} \) are each positively associated with \( r_L \) and \( r_B \), i.e., \( c_{1t}'(r_L) = 1/\delta_1'U' > 0 \) and \( c_{2t}'(r_B) = 1/\delta_2'U' > 0 \). Note that \( c_{1t}'(r_L) > c_{2t}'(r_B) \) because \( \delta_2' \) is assumed to be larger than \( \delta_1' \).

To see the effects of fiscal policies on the spread reflecting the rate of return per dollar for financial intermediation, we now introduce private liquid assets, such as demand deposits. These are alternative instruments for the accumulation of assets in private loanable markets. The presence of imperfect capital markets (i.e., market imperfection) and the introduction of the liquid assets in our model are necessary to generate the role for financial intermediation as the dominant vehicle for carrying out borrowing and lending to emerge.\(^{26}\) This leads to the assumption that a type-I agent’s per capita assets \( a_{1t} \) are divided between per-capita capital

\(^{26}\) It is a widely accepted notion in the banking literature that the assumption of perfectly competitive capital market is inconsistent with the reason for the existence of financial intermediaries (see, for example, Eisenbeis (1987), Kane (1987)). That is, real-world market imperfections characterized by costly and asymmetric information, transaction cost, and asset indivisibility provide a good explanation of the necessity of existing financial intermediaries. Financial intermediaries, as arbitrageurs, overcome these imperfections and thus generate their profits by such advantages as economies of scale and scope, diversification economies, and information economies.
stock \((k_t/\alpha)\), per-capita debt \((d_t/\alpha)\), and per-capita private liquid assets \((l_t/\alpha)\). This is shown as:

\[
a_{it} = a_t/\alpha = (k_t + d_t + l_t) / \alpha, \tag{6-3}
\]

Equation \(6-3\) implies that type-I agents hold all nonhuman assets available to this economy.

A type-I agent's budget constraint in the steady state can be written as:

\[
c_{1t}^* = w(k_t^*) - \tau_t^* + r_L^*(k_t^* + d_t^* + l_t^*) / \alpha, \tag{6-4}
\]

where \(l_t^*\) denotes the per-capita amount of lending by type-I agents.

As shown in chapter V, when liquidity constraints are binding, type-II agents no longer hold any nonhuman assets. Therefore, a type-II agent’s budget constraint in the steady state can be described by:

\[
c_{2t}^* = w(k_t^*) - \tau_t^* - r_B^* l_t^*/(1-\alpha), \tag{6-5}
\]

where \(-l_t^*\) represents the per-capita amount of loans available to type-II agents. The intuition for equation \(6-5\) is that when type-II agents, who would like to borrow at the lending rate \(r_L\) but who are prevented from doing so, are willing to borrow loanable funds in the market to raise current consumption in excess of their after-tax wage
income, they must incur a high borrowing rate, \( r_B^* > r_L^* \). Moreover, the loan market equilibrium condition ensures that the per-capita amount of lending by type-I agents always coincides with the per-capita amount of loans available to type-II agents. Suppose further that taxes are equally distributed across each type of agent as indicated by equations (6-4) and (6-5).

Using equations (6-4) and (6-5), and the government budget constraint \( \delta_t = r_L d_t + g_t - \tau_t \), aggregate per-capita consumption \( c_t^* = \alpha c_{1t}^* + (1-\alpha)c_{2t}^* \) can be determined.

\[
\begin{align*}
  c_t^* &= \nu(k_t^*) + r_L^* k_t - g_t^* - (r_B^* - r_L^*)l_t^* \\
  \text{(6-6)}
\end{align*}
\]

The last term in (6-6) can be interpreted as the gross rate of return for private financial intermediation.

Let the spread between the borrowing rate and the lending rate be a nondecreasing function of the per-capita amount of loans, \( h(l_t) \).

\[
\begin{align*}
  r_B - r_L &= h(l_t), \hspace{1cm} h'(l_t) \geq 0 \\
  \text{(6-7)}
\end{align*}
\]

Equation (6-7) assumes that the marginal cost of private intermediation is not decreasing with the per-capita amount of loans.

Since per-capita taxes are equally distributed across the two types of agents, the government budget constraint in per-capita terms
should be $\tau^*_t = g^*_t + r^*_L d^*_t$. Plugging $\tau^*_t = g^*_t + r^*_L d^*_t$ into equations (6-4) and (6-5) yields:

$$c^*_{1t} + g^*_t = v(k^*_t) + r^*_L (k^*_t + l^*_t + (1-\alpha)d^*_t) / \alpha$$  \hspace{1cm} (6-8)

$$c^*_{2t} + g^*_t = v(k^*_t) - \left[ r^*_B l^*_t + (1-\alpha)r^*_L d^*_t \right] / (1-\alpha)$$  \hspace{1cm} (6-9)

$$= v(k^*_t) - r^*_B l^*_t / (1-\alpha) + (r^*_B - r^*_L)d^*_t.$$  

The last term in (6-9) needs more elaboration. This term can be interpreted as the net wealth effect arising from the government induced transfer implied by its bond issue. Type-II agents do not have to incur the extra cost of this amount because government debt makes it possible for type-II agents to avoid the high transaction cost that led them, in the first place, to the high borrowing rates. This is tantamount to the assumption that government intermediation is more efficient than private intermediation for carrying out the loan process. The reason is that the government transaction cost associated with debt issue and tax collection is zero, while the private transaction cost amounts to $(r^*_B - r^*_L)$.

We now turn to the comparative steady state analysis to examine the effect of fiscal policies on the spread. Total differentiation of equations (6-8), (6-9), and (6-7) with respect to $r^*_L$, $r^*_B$, $l^*_t$, $g^*_t$, and
\(d_t\) at their initial steady state values and rearranging in matrix form gives:

\[
\begin{bmatrix}
\frac{c_1'(r_L^*) - \{k_t^* + l_t^* + (1-\alpha)\alpha_0^*\}}{\alpha} & 0 & -\frac{r_L^*}{\alpha} \\
-w'(k_t^*) + \frac{r_L^*}{\alpha} & \frac{\partial k_t^*}{\partial r_L} & \\
\frac{d_t^* - w'(k_t^*)}{\partial r_L} & c_2'(r_B^*) + \frac{l_t^*}{1-\alpha} & \frac{r_B^*}{1-\alpha} \\
1 & -1 & h'
\end{bmatrix}
\begin{bmatrix}
dr_L^* \\
dr_B^* \\
dl_t^*
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-1 & \frac{(1-\alpha)r_L^*}{\alpha} \\
0 & 0 \\
-1 & -r_L^*
\end{bmatrix}
\begin{bmatrix}
dg_t^* \\
dd_t^*
\end{bmatrix}
\]
where the determinant of the 3x3 matrix $\Delta$ is positive. 27 We can determine the comparative static derivatives of $r^*_L$, $r^*_B$, and $l^*_t$ with respect to $g^*_t$ and $d^*_t$.

\[
\frac{\partial r^*_L}{\partial g^*_t} = - \left( \frac{c'_{L} (r^*_B) h'(l^*_t) + l^*_t h'(l^*_t)/(1-\alpha) + r^*_B/(1-\alpha) + r^*_L/\alpha}{\Delta} \right) < 0, \tag{6-11}
\]

if $h'(l^*_t) \geq 0$.

\[
\frac{\partial r^*_B}{\partial g^*_t} = \frac{h'(l^*_t) \{ d^*_t - w'(k^*_t) \partial k^*_t/\partial r^*_L \} - Ah'(l^*_t) - r^*_B/(1-\alpha) - r^*_L/\alpha}{\Delta} < 0, \tag{6-12}
\]

if $h'(l^*_t) = 0$.

\[
\frac{\partial l^*_t}{\partial g^*_t} = \frac{c'_{L} (r^*_L) + l^*_t/(1-\alpha) - A + d^*_t - w'(k^*_t) \partial k^*_t/\partial r^*_L}{\Delta}, \tag{6-13}
\]

\[
\frac{\partial r^*_L}{\partial d^*_t} = \frac{(1-\alpha)r^*_L c'_{L} (r^*_L) h'(l^*_t) + l^*_t h'(l^*_t)r^*_L + r^*_L(r^*_B - r^*_L)}{\alpha \Delta} > 0. \tag{6-14}
\]

---

27. The determinant is $\Delta = \left( c'_{L} (r^*_L) - \left\{ k^*_t + l^*_t + (1-\alpha)d^*_t \right\}/\alpha - \left\{ w'(k^*_t) + r^*_L/\alpha \right\} \partial k^*_t/\partial r^*_L \right) (c'_{L} (r^*_B) h'(l^*_t) + l^*_t h'(l^*_t)/(1-\alpha) + r^*_B/(1-\alpha) + \{ d^*_t - w'(k^*_t) \partial k^*_t/\partial r^*_L \} (r^*_L/\alpha) + (r^*_L/\alpha) (c'_{L} (r^*_B) + l^*_t/(1-\alpha)) > 0$. This is certainly the case since $h'(l^*_t)$ is assumed to be nonnegative and the expression in the first parenthesis of the determinant is unambiguously positive. Note that $c'_{L} (r^*_L)$ and $c'_{L} (r^*_B)$ have positive values, but $\partial k^*_t/\partial r^*_L$ is negative as condition (6-1) indicates.
\[
\frac{\partial r_B^*}{\partial d^*} = - \frac{r_B^* h'(1_t^*) [A + ((1-\alpha)(d_t^*-W(k_t^*))\partial k_t^*/\partial r_B^*)/\alpha]}{\Delta} + \frac{r_B^*(r_B^*-r_L^*)}{\Delta} > 0, \quad \text{if } h'(1_t) = 0
\]

\[
\frac{\partial l^*}{\partial d^*} = - \frac{r_L^* - (1-\alpha)r_L^*/\alpha (c_{1t}^*(r_L^*) + l_t^*/(1-\alpha) + d_t^* - W'(k_t^*)\partial k_t^*/\partial r_L^*)}{\Delta} < 0
\]

where \(A = (c_{1t}^*(r_L^*) - k_t^* + l_t^* + (1-\alpha)d_t^*/\alpha - W'(k_t^*) + r_L^*/\alpha \partial k_t^*/\partial r_L^*)\).

Equation (6-11) indicates that an unanticipated permanent increase in government expenditures \((g_t^*)\) will lead to a fall of the lending rates if the marginal cost of private intermediation is not decreasing with the per-capita amount of loans, \(h'(1_t) > 0\). Equation (6-12) reveals that the borrowing rates will certainly fall in response to the increase in government expenditures if \(h'(1_t) = 0\).

According to equation (6-14), the lending rates at the initial steady state \((r_L^*)\) will rise in response to an unanticipated permanent increase in government debt \((d_t^*)\) as long as \(h'(1_t) \geq 0\). Equation (6-15) states that an unanticipated permanent increase in government
debt ($d^*_t$) will cause the loan rates available to type-II agents to rise if $h'(l^*_t) = 0$. This effect is shown to be somewhat ambiguous if $h'(l^*_t) > 0$. In particular, equations (6-14) and (6-15) show that $r^*_B$ falls, while $r^*_L$ rises, in response to an unanticipated permanent increase in debt as $h'(l^*_t)$ approaches infinity. This will lead to a narrowing of the spread as below.

Equation (6-16) indicates that government debt will crowd out per-capita amount of private loans ($l^*_t$) at the initial steady state unless $h'(l^*_t)$ goes to infinity. It is also easily seen from equation (6-16) that the magnitude of the displacement of private intermediation by government debt is smaller, the larger $h'(l^*_t)$ is.

However, it is not clear whether government expenditures will substitute for per-capita amount of private loans, as equation (6-13) shows.

The derivative of the spread ($r^*_L - r^*_B$) in terms of $d^*_t$ needs to be examined. This effect can be seen by subtracting (6-14) from (6-15).

\[
\frac{\partial(r^*_L - r^*_B)}{\partial d^*_t} = \frac{-r^*_L h'(l^*_t) \left[ A + (1-\alpha)(d^*_t - v(k^*_t)) \partial k^*_t / \partial r^*_L / \alpha \right]}{\Delta} - \frac{r^*_L h'(l^*_t) ((1-\alpha)c^*_L (r^*_L) + 1^*_t)}{\alpha \Delta} \leq 0
\]
\[ \frac{\partial (r_B^* - r_L^*)}{\partial g^*} = \frac{h'(l_t^*) \left[ (d_t^* - w'(k_t^*) \partial k_t^*/\partial r_t^*) - \Delta - c_t' (r_B^*) + l_t^*/(1-\alpha) \right]}{\Delta} \]  

(6-18)

The key result in equation (6-17) is that if \( h'(l_t) = 0 \), then government debt will be neutral with respect to the spread. If \( h'(l_t) > 0 \), including the case in which \( h'(l_t) \to \infty \), then, an unanticipated permanent increase in government debt will lead to a narrowing of the spread between the borrowing rate and the lending rate. However, as equation (6-18) indicates, the effect of government expenditures on the spread is ambiguous.

Summarizing,

**Proposition 3**: With imperfect capital markets, the effect of government debt on the spread depends entirely on how the marginal cost of private intermediation responds to a change in the per-capita amount of loans. If \( h'(l_t) > 0 \), including the case where \( h'(l_t) \) approaches infinity, an unanticipated permanent increase in government debt leads to a narrowing of the spread. If \( h'(l_t) = 0 \), the spread is invariant to a change in government debt. In either case, government debt displaces private intermediation \( l_t^* \). The effect is smaller, the larger \( h'(l_t) \) is.
CHAPTER VII

EMPIRICAL TEST

The purpose of this chapter is to investigate whether recent U.S. time series data provide evidence supporting the main theoretical conclusion drawn from chapter VI: government debt, and possibly short-term loans are important in explaining the fluctuations in the spread between borrowing and lending rates in financial markets.

We employ Sims (1980a, 1980b) standard vector autoregression (VAR) methodology based on the reduced form recursive identification of residuals. The standard VAR approach is motivated in our empirical test mainly because it provides a convenient framework for the empirical research associated with the short-run dynamic as well as the long-run steady state analysis. This VAR approach is designed to answer two questions. First, is there evidence that government purchases, real debt, and short-term loans are Granger-causally prior for the spread? Also, to what extent do real debt and short-term loans account for the short run as well as long run movement in the spread? As a further check for the robustness of the standard VAR analysis, we perform an alternative structural VAR analysis in which a just-identified VAR model is applied in a relatively unrestricted way,
as in Bernanke (1986), Blanchard and Watson (1985), and Blanchard (1989). While there is an obvious arbitrariness to any set of model specifications, our postulated structural model does not appear to give appreciably different results from the standard approach.

As Nelson and Plosser (1982), and other subsequent authors have shown, there is good reason to believe that many macroeconomic data follow integrated processes. From this perspective, we carry out univariate unit root tests following Dickey and Fuller (1979, 1981). The results suggest that all four time series variables under consideration are well described as difference-stationary processes. The univariate properties of these variables however cannot identify the number of stochastic trends that are common to several integrated variables in a multivariate system. To address this shortcoming of the univariate unit root tests, we apply the multivariate unit root test (common trends test) based on the Stock and Watson (1988) $q_f$ statistics to the four-variable system to test for the number of stochastic trends. This test appears to provide evidence that our four variable system contains three common stochastic trends and one co-integrating vector.

Taken as given the evidence from the common trends test, we check for whether a certain linear combination of contemporaneous values of the variables is cointegrated in the vein of Engle and Granger (1987). Using a co-integrating regression where the spread is regressed on the other integrated variables under consideration, we find evidence that the spread has a long run relationship with the
other variables. We then apply the Engle and Granger two-step method to an error-correction model as an alternative approach to testing for co-integration. This dynamic analysis provides strong support for the co-integration finding in that only the spread is significantly affected by the error correction feedback from temporary disequilibrium. Since economic theory imposing co-integration appears to be supported by the data, we also consider a VAR approach to the error correction model which is useful for analyzing an optimal forecast of the variables.

7.1. Data Descriptions

We use quarterly data for four macroeconomic time series variables for the period 1971:I - 1986:IV.\textsuperscript{28} Real government purchases, real debt, real amount of short-term loans, the spread between the borrowing rate and the lending rate are denoted as $G$, $D$, $L$, $SPR$ respectively. The variable, $G_t$, is the U.S.'s real federal purchases. Real government debt, $D_t$, is calculated by deflating the U.S.'s nominal end-of-quarter cyclically adjusted federal debt by the consumer price index for the last month of the quarter. Real federal purchases are taken from the National Income and Product Accounts (1971-82) and the Survey of Current Business (U.S. Department of

\textsuperscript{28} We have excluded pre-1971 observations because data on the amount of short-term loans and the corresponding borrowing rate are not available.
The cyclically adjusted federal debt and the consumer price index come from the Survey of Current Business. $L_t$ represents the U.S.'s short-term commercial and industrial loans made during the first full business week in the mid-month of each quarter by a sample of 340 commercial banks of all sizes divided by the mid quarter consumer price index. For the borrowing rate $r_B$, we use the weighted average rate over one business week of the mid-month of each quarter for the short-term commercial and industrial loans for 34 leading cities. The short-term commercial and industrial loans (other than construction and land development) and the interest rates charged by commercial banks on business loans come from the Federal Reserve System's E.2 release. The U.S.'s four-to-six month commercial paper rate is used as a proxy for the nominal lending rate $r_L$. This lending rate comes from the Federal Reserve System's Banking and Monetary statistics.

7.2. Standard Vector Autoregression (VAR) Analysis

In contrast to the traditional structural model, the standard vector autoregression (VAR) model treats all variables as endogenous in an unrestricted reduced form. In the standard VAR model, each endogenous variable is determined by the previous values of variables in the system and limited number of variables are included with equal lag lengths on the right side of every equation. The number of parameters estimated in the nonstructural VAR is much larger relative
to the structural model and most of the coefficients in a VAR model are insignificantly different from zero.

A crucial feature of Sims' (1980a, 1980b) standard vector autoregression methodology is its dependence on the assumption of recursive identification of the VAR innovations. That is, the method of orthogonalizing the VAR innovations via the Cholesky decomposition has the effect of assuming that the true economic model relating the VAR innovations is of a strict recursive form. Because of this unattractive assumption, the standard VAR model has been the subject of much recent criticism. While it is not exactly the case that the disturbances are truly orthogonalized in a VAR model, if they have a low correlations, then it is plausible to assume that the identification is approximately correct.

The standard VAR methodology based on Granger causality tests, impulse response functions, and variance decompositions is adopted in many recent studies on the premise that it is helpful in distinguishing a true theory from false ones. Granger causality tests in a VAR model can be used to infer multivariate exogeneity and causality. Variance decompositions (or innovation accounting) for h-period ahead forecast errors in a vector autoregressive representation determine the relative importance of each of the orthogonalized innovations in explaining each variable's forecast-error variance. The impulse response functions in a vector moving average representation shows the dynamic response of each variable to the orthogonalized innovations. Although VAR estimation is based on an
autoregressive representation, a meaningful interpretation of variance decompositions and impulse response functions in the vector autoregression system requires a vector moving average representation with orthogonalized innovations.

It also should be noted that a VAR model provides a convenient framework to explore the empirical implications of the long-run steady state, as well as short-run dynamics. This is the main reason for introducing the VAR methodology into our empirical study.

To understand the empirical evidence from the vector autoregression model, it is instructive to briefly discuss the implications for the effect of contemporaneous innovation on the long term forecast of economic variables which differ markedly from either stationary VAR's, or VAR's in first differences. If all random variables are, respectively, integrated of order one in a vector autoregressive representation (i.e., if an autoregressive unit root is considered), then a contemporaneous shock to the series has a permanent effect on the level of the future series. If each integrated element of $Y_t$ can be expressed as an invertible vector autoregression such that $A(L)\Delta Y_t = \varepsilon_t$, where $\varepsilon_t$ is a Nx1 vector of contemporaneously and serially uncorrelated innovations, then this equation can be represented by the following moving average representation: $\Delta Y_t = C(L)\varepsilon_t$. From this, we can solve for the level of $Y_t$ as $Y_t = Y_0 + C(L) \sum_{i=1}^{t} \varepsilon_{t-i}$ by backward substitution of lagged
$Y_t$, where $Y_0$ is treated as a deterministic constant. 29 Then, as Engle and Yoo (1989) have shown, the forecast of $Y_{t+h}$, for $h > 0$, can be decomposed into $Y_{t+h} = Y_0 + \sum_{i=1}^{t} \left[ \sum_{j=0}^{t+h-i} C_j \right] \epsilon_i + \sum_{i=1}^{t+h} \left[ \sum_{j=0}^{t} C_j \right] \epsilon_i$. It follows that the forecast $Y_{t+h}$ given information at time $t$ can be expressed as $Y_{t+h} = Y_0 + \sum_{i=1}^{t} \left[ \sum_{j=0}^{t+h-i} C_j \right] \epsilon_i$. It is not hard to show that the permanent effect of the shock at time $t$ on the level of the forecast $Y_{t+h}$ is equivalent to the infinite sum of the moving average coefficients of the differences of $Y_t$ since $\lim_{h \to \infty} \frac{Y_{t+h}}{\Delta Y_t} = \lim_{h \to \infty} \sum_{j=0}^{h} C_j = C(1)$. The value of $C(1)$ is one of the commonly used measures of persistence in a time series (see Campbell and Mankiw (1987)).

In contrast, if all random variables of $Y_t$ are stationary in the vector autoregressive representation, then any shock in period $t$ has only a transitory effect on the long-term forecast of the series. An invertible autoregression $A(L)Y_t = \epsilon_t$ can be written as infinite order moving average $Y_t = C(L)\epsilon_t = \sum_{j=0}^{\infty} C_j \epsilon_{t-j}$ with $\epsilon_t$ being a $N \times 1$ vector of

29. Since $Y_t = Y_0 + C(L)(1 + L + L^2 + \ldots + L^{t-1})\epsilon_t$ and $C(L) = \sum_{i=0}^{\infty} C_i L^i$, $Y_t$ can be written as $Y_t = Y_0 + C_0 \epsilon_t + (C_0 + C_1)\epsilon_{t-1} + \ldots + (C_0 + C_1 + C_2 + \ldots + C_{t-1})\epsilon_1 = Y_0 + \sum_{i=1}^{t} \left[ \sum_{j=0}^{t-i} C_j \right] \epsilon_i$. 
contemporaneously and serially uncorrelated innovations. The long term effect of the shock in period $t$ on the forecast of $Y_{t+h}$ is given by $\lim_{h \to \infty} \frac{\partial Y_{t+h}}{\partial \varepsilon_t} = \lim_{h \to \infty} C_h$. In this case, $C_h$ approaches zero for large $h$ since $\sum_{h=0}^{\infty} C_h^2$ should be finite for $Y_t$ to follow a stationary stochastic process. This implies that the level of the series shows a temporary fluctuation around a deterministic linear trend until it reverts to the trend. Thus, it seems worthwhile to perform the VAR analysis in both levels and differences.

(1) Estimation of the VARs

The focus of this subsection is on interpreting Granger causality tests, correlations among contemporaneous innovations, estimated variance decompositions, and impulse response (IR) functions within the framework of both stationary vector autoregressive model and the vector autoregression in differences. For estimation purpose, the following finite vector autoregressive (VAR) representations are considered:

$$\begin{align*}
Y_t &= \alpha + \beta t + \sum_{i=1}^{4} A_i Y_{t-i} + u_t, \\
E(u_t u'_s) &= \begin{cases} E & \text{if } t=s \\ 0 & \text{otherwise} \end{cases} \quad (7-1)
\end{align*}$$
\[ \Delta Y_t = \beta + \sum_{i=1}^{4} B_i \Delta Y_{t-i} + e_t, \quad E(e_t e'_s) = \begin{cases} \Omega & \text{if } t=s \\ 0 & \text{otherwise,} \end{cases} \tag{7-2} \]

where \( A_i \) and \( B_i \) are, respectively, a square coefficient matrix; \( u_t \) and \( e_t \) are the vectors of mean-zero, serially uncorrelated random variables (four unobservable innovations) and their contemporaneous variance-covariance matrix, \( \Sigma, \Omega \) are assumed to be positive definite symmetric; \( \alpha \) and \( \beta \) are 4x1 column vectors of parameters; the vector \( Y_t \) represents the vector of four observable random time series variables. These are real government purchases (G), real debt (D), real amount of short-term loan (L), and the spread (SPR). The first three (G, D, L) are transformed into logarithms and the other (SPR) is used in levels. Akaike’s information criterion (AIC) was used in selecting the autoregressive parameters of the above VAR system in both log levels and log differences. According to this criterion, allowing for four lags in the above vector autoregression system turns out to minimize \( \text{AIC} = [\text{RSS}_i \exp(2k_i/T)] \), where \( \text{RSS}_i \) and \( k_i \) are the residual sum of squares and the number of parameters in the ith equation, respectively (see Akaike (1974)). Note that the VAR model (7-1) uses a constant term and time trend \((\alpha, \beta t)\) whereas (7-2) uses
only a constant term ($\beta$). Prior information lead us to order the variables as G, D, L, SPR. As Gordon and King (1982) have pointed out, any contemporaneous correlation between variables might show up as a correlation between current innovations in variables, since contemporaneous right-hand variables in the log-levels or in the log-differences are omitted at the estimation stage given the above VAR frameworks. If the error terms $u_t$ and $e_t$ are respectively orthogonal to lagged values of regressors, equations (7-1) and (7-2) can be estimated consistently using ordinary least squares.

(2) **Exogeneity Test and Correlations among Contemporaneous Innovations**

We desire to test causal orderings, in the Granger sense, from the estimation of the VAR model given by (7-1). Granger causality

30. As Plosser and Schwert (1978) have shown, the vector of constant terms in the first differences version (7-2) actually corresponds to the vector of the coefficients on the time trend in the vector autoregressive representation in the levels (7-1). They also argue that while differencing does not affect the parameter estimates in a correctly specified linear model, ignoring the effects of underdifferencing can lead to a more misleading result than ignoring the effects of overdifferencing.

31. A central feature of Granger's (1969) simple causality is that X "causes" Y if we are better able to predict Y using the past history of X than simply using the past history of Y: if $\sigma^2(Y_t \mid \bar{Y}_t, \bar{X}_t) < $
test is based on the F-test of the hypothesis that four lags of a particular variable are excluded from each regression equation. The test consists of checking whether the corresponding marginal significance level for the F-test is sufficiently low. A notable feature of the pattern of exogeneity in Table 1.1 is that real government purchases, real debt, and real amount of loans are each essentially independent of all the other variables. That is, the marginal significance level of the F-test indicates that one cannot reject at the 0.20 level the hypothesis that all lagged values except for own lagged ones have zero coefficients in the first three regression equations (government purchase equation, debt equation, and loan equation). This result corresponds to what is meant by Granger causal priority of those variables – the condition that the best forecast of these variables is formed from their own lagged values alone. However, the spread is marginally predicted at the 0.05 level by real government purchases, real debt, and the real amount of short-term loans, indicating the spread exhibits substantial feedback from

(Footnote continued from previous page)

\( \sigma^2( Y_t | \bar{Y}_t, \bar{X}_t ) \), where \( \bar{Y}_t, \bar{X}_t \) are past values of \( Y_t \) and \( X_t \), that is, if the inclusion of past \( X \) in the information set upon which the prediction of \( Y \) is conditioned reduces the prediction error variance, then \( X \) Granger causes \( Y \).
Granger Causality Test

Table 1.1

<table>
<thead>
<tr>
<th>equations variables\</th>
<th>G</th>
<th>D</th>
<th>L</th>
<th>SPR</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>0.00</td>
<td>0.74</td>
<td>0.43</td>
<td>0.03</td>
</tr>
<tr>
<td>D</td>
<td>0.47</td>
<td>0.00</td>
<td>0.46</td>
<td>0.00</td>
</tr>
<tr>
<td>L</td>
<td>0.89</td>
<td>0.87</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>SPR</td>
<td>0.98</td>
<td>0.24</td>
<td>0.82</td>
<td>0.06</td>
</tr>
</tbody>
</table>

Table 1.2

<table>
<thead>
<tr>
<th>equations variables\</th>
<th>ΔG</th>
<th>ΔD</th>
<th>ΔL</th>
<th>ΔSPR</th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔG</td>
<td>0.78</td>
<td>0.44</td>
<td>0.71</td>
<td>0.14</td>
</tr>
<tr>
<td>ΔD</td>
<td>0.71</td>
<td>0.00</td>
<td>0.78</td>
<td>0.00</td>
</tr>
<tr>
<td>ΔL</td>
<td>0.95</td>
<td>0.95</td>
<td>0.14</td>
<td>0.02</td>
</tr>
<tr>
<td>ΔSPR</td>
<td>0.98</td>
<td>0.40</td>
<td>0.85</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Note: each entry in the tables exhibits the marginal significance level for the F-test on the hypothesis that four lags of each row variable can be excluded from the regression predicting column variable. Marginal significance level is the probability of obtaining as a high value of the test statistic (f) under the null hypothesis, i.e., \( F > f \).
Contemporaneous Correlations Matrix

Table 2.1

<table>
<thead>
<tr>
<th>Variables</th>
<th>G</th>
<th>D</th>
<th>L</th>
<th>SPR</th>
</tr>
</thead>
<tbody>
<tr>
<td>G</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>0.325</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>0.241</td>
<td>-0.009</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>SPR</td>
<td>0.252</td>
<td>0.201</td>
<td>-0.122</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2.2

<table>
<thead>
<tr>
<th>Variables</th>
<th>ΔG</th>
<th>ΔD</th>
<th>ΔL</th>
<th>ΔSPR</th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔG</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ΔD</td>
<td>0.381</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ΔL</td>
<td>0.232</td>
<td>0.016</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>ΔSPR</td>
<td>0.319</td>
<td>0.157</td>
<td>-0.080</td>
<td>1</td>
</tr>
</tbody>
</table>
other variables. This provides evidence that real government purchases (G), real debt (D), and the real amount of short-term loans (L) "cause" the spread in the Granger sense.

Granger causality tests for the log-differences are shown in Table 1.2. The spread reflects significant feedback from real debt, and the real amount of short-term loans with only modest feedback from real government purchases. This leads to the interpretation that real debt and the real amount of short-term loans also Granger cause the spread for the log-differences case. The results are qualitatively similar to those in the log-level version.

According to Table 2.1, the shocks from distinct variables turn out to be somewhat correlated. A conditional contemporaneous correlation of real debt with real government purchases, which is relatively higher than any other variable with real purchases, is 0.325. In particular, the contemporaneous correlation between the spread and real amount of loan is slightly negative.

(3) Variance Decompositions and Impulse Response Function

It seems plausible that the results of the above exogeneity test suggest an appropriate temporal ordering of the variables. The standard VAR models of (7-1) and (7-2) place real government purchases first in the ordering, followed by real debt, then the real amount of short-term loans. The spread is placed last in the ordering. A prior notion about the importance of autonomous government purchases suggests an ordering with government purchases first. Given this
ordering, the allocation of the variance of the spread between its own innovations and the innovations to the other causally prior variables is of interest in any investigation of the channel of influence between the spread and the other variables. However, since the components of the vector random variable \( u_t \) in (7-1) and (7-2) may be contemporaneously correlated, it is required to construct a moving average representation with orthogonalized innovations so as to decompose the variance of \( Y_t \) into the part accounted for by each orthogonalized innovation. To see this, consider a transformation of the inverted vector moving average (VMA) representation for \( Y_t \), \( Y_t = A(L)^{-1}u_t = C(L)u_t \), from a finite vector autoregression of (7-1), into a normalized one such that \( Y_t = C(L)T^{-1}T u_t = D(L)v_t \), where \( C(L)T^{-1} = D(L) \) and the vector of original disturbances, \( u_t \) and the vector of innovations, \( v_t \) are related by \( Tu_t = v_t \). \( T \) is a unique triangular Choleski factor of the \( \Sigma \) matrix with units on the main diagonal that diagonalizes the error covariance matrix: \( E(v_tv'_s) = T(Eu_tu'_t)T' = T\Sigma T' \). In that case, the orthogonalized innovations \( v_t \), are virtually equivalent to those from a system in which all lagged regressors with equal lag lengths are included in each equation and contemporaneous variables of \( Y_t \) appear on the right-hand side of regressions in a recursive fashion. From this transformation, one can obtain a decomposition of the \( h \)-period ahead forecast-error variance for the
$Y_{it}$ into the variance attributable to the jth component of $\nu_t$ by dividing the variance by the sum of the h-period ahead forecast-error variance for the $Y_{it}$. Thus, one can look at the effects of each innovation on each variable in the short, medium, and long run from estimated variance decompositions.

The results in Table 3 report the proportion of the h-period ahead forecast-error variance of row variables attributable to each innovation in column variables. Table 3 shows that the three variables G, D, L have more than 50 percent of their variances

32. To this end, consider a truncated vector moving average representation $Y_{it} = D(L)\nu_t = D_{ij}(L)\nu_t$ for $i,j = 1,..,4$. The actual value of the ith variable of $Y_{it+h}$ can be written as $E_{k=0}^{\infty} E_{j=1}^{4} D_{ijk} \nu_{t-k+h}$, which in turn reduces to $E_{k=1}^{h-1} E_{j=1}^{4} C_{ijk} \nu_{jt-k+h}$

$E_{k=h}^{\infty} E_{j=1}^{4} C_{ijk} \nu_{jt-k+h}$ where i denotes what the innovations affect; j indicates the orthogonalized innovations; k represents the lags in their dynamic effects. From this, one can derive the h-period ahead error in forecasting $Y_{it}$ as $(Y_{it} - E_{t} Y_{it+h})(Y_{it} - E_{t} Y_{it+h})'$

$E_{k=0}^{h-1} E_{j=1}^{4} (D_{ijk} \nu_{jt} \nu_{jt})' = E_{k=0}^{h-1} E_{j=1}^{4} \nu_{jt}'^2 = E_{k=0}^{h-1} D_{ijk} \nu_{jt}'^2 = E_{k=0}^{h-1} (D_{ijk} \nu_{jt}^2) = E_{k=0}^{h-1} (D_{ijk} \nu_{jt-k+h})^2$ since $E_{jt} \nu_{jt}^2 = 0$ if $t \neq k$, where $w_j^2$ represents a variance of jth component of $\nu_t$. For instance, a fraction of the h-period ahead forecast-error variance for the $Y_{it}$ contributed by the variance $\nu_{it}$ equals $\sum_{k=0}^{h-1} D_{ijk} w_1^2 / \sum_{j=1}^{4} (E_{k=0}^{h-1} D_{ijk} \nu_{jt}^2)$ (see Sargent (1979), Sims (1980b, 1981), Engle and Yoo (1987)).
Table 3

Variance Decompositions:

<table>
<thead>
<tr>
<th>Forecast Error (k quarters ahead)</th>
<th>Triangularized Innovation In:</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>G</td>
<td>D</td>
<td>L</td>
<td>SPR</td>
</tr>
<tr>
<td>1</td>
<td>100</td>
<td>0.0</td>
<td>0.0 (0.0)</td>
<td>0.0 (0.0)</td>
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<tr>
<td>3</td>
<td>97.1</td>
<td>1.2</td>
<td>1.5 (1.4)</td>
<td>0.1 (0.2)</td>
</tr>
<tr>
<td>5</td>
<td>95.3</td>
<td>2.1</td>
<td>2.1 (1.9)</td>
<td>0.5 (0.7)</td>
</tr>
<tr>
<td>9</td>
<td>95.0</td>
<td>2.2</td>
<td>2.1 (1.9)</td>
<td>0.7 (0.9)</td>
</tr>
<tr>
<td>12</td>
<td>93.7</td>
<td>3.5</td>
<td>2.2 (2.0)</td>
<td>0.7 (0.9)</td>
</tr>
<tr>
<td>20</td>
<td>90.4</td>
<td>7.0</td>
<td>1.8 (1.7)</td>
<td>0.7 (0.8)</td>
</tr>
<tr>
<td>40</td>
<td>90.2</td>
<td>7.3</td>
<td>1.8 (1.7)</td>
<td>0.7 (0.8)</td>
</tr>
<tr>
<td>1</td>
<td>12.4</td>
<td>87.6</td>
<td>0.0 (0.0)</td>
<td>0.0 (0.0)</td>
</tr>
<tr>
<td>3</td>
<td>14.0</td>
<td>84.8</td>
<td>0.7 (0.5)</td>
<td>0.5 (0.7)</td>
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<tr>
<td>5</td>
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<td>2.1 (1.7)</td>
<td>0.6 (1.0)</td>
</tr>
<tr>
<td>9</td>
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<td>67.0</td>
<td>3.3 (2.9)</td>
<td>0.3 (0.7)</td>
</tr>
<tr>
<td>12</td>
<td>35.0</td>
<td>61.8</td>
<td>2.9 (2.6)</td>
<td>0.3 (0.6)</td>
</tr>
<tr>
<td>20</td>
<td>39.8</td>
<td>55.6</td>
<td>4.4 (4.1)</td>
<td>0.3 (0.6)</td>
</tr>
<tr>
<td>40</td>
<td>40.6</td>
<td>54.7</td>
<td>4.4 (4.2)</td>
<td>0.3 (0.5)</td>
</tr>
<tr>
<td>1</td>
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<td>1.0</td>
<td>93.2 (90.0)</td>
<td>0.0 (3.2)</td>
</tr>
<tr>
<td>3</td>
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<td>83.5 (80.2)</td>
<td>0.6 (3.9)</td>
</tr>
<tr>
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<td>0.7 (3.2)</td>
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<td>0.6 (2.6)</td>
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</tr>
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<td>32.9</td>
<td>60.3 (58.4)</td>
<td>0.5 (2.5)</td>
</tr>
<tr>
<td>40</td>
<td>7.6</td>
<td>34.2</td>
<td>57.6 (55.8)</td>
<td>0.5 (2.3)</td>
</tr>
<tr>
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<td>6.4</td>
<td>1.4</td>
<td>3.1 (0.0)</td>
<td>89.1 (92.2)</td>
</tr>
<tr>
<td>3</td>
<td>7.9</td>
<td>16.0</td>
<td>7.3 (8.0)</td>
<td>68.8 (68.1)</td>
</tr>
<tr>
<td>5</td>
<td>7.3</td>
<td>20.7</td>
<td>14.2 (14.8)</td>
<td>57.8 (57.2)</td>
</tr>
<tr>
<td>9</td>
<td>15.5</td>
<td>20.0</td>
<td>12.9 (13.4)</td>
<td>51.7 (51.2)</td>
</tr>
<tr>
<td>12</td>
<td>17.7</td>
<td>19.6</td>
<td>12.6 (13.1)</td>
<td>50.2 (49.7)</td>
</tr>
<tr>
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<td>18.3</td>
<td>19.7</td>
<td>12.6 (13.2)</td>
<td>49.4 (48.8)</td>
</tr>
<tr>
<td>40</td>
<td>18.9</td>
<td>20.0</td>
<td>12.5 (13.0)</td>
<td>48.7 (48.2)</td>
</tr>
</tbody>
</table>

Note: the values in parentheses in the L and SPR columns represent the fraction of the forecast-error variance of each row variable explained by the L and SPR innovations when we reorder the innovations so that G → D → SPR → L.
accounted for by their own innovations at all time horizons. The spread has a lower long-run forecast-error variance explained by own innovations (49 percent) relative to other variables, while its variance attributed by own innovations is much larger in the short-run. This evidence suggests that interactions among variables are not so strong. Real government purchases are seen to be strictly exogenous in the sense that its own innovations account for virtually all of its contemporaneous forecast-error variance. Real debt also appears to be exogenous to short-term loans and the spread, while its variance attributable to real government purchases is fairly large at all time horizons. Of particular interest is the result that the innovations to real debt explain an appreciable fraction (about 20 percent) of variance of the spread not explained by own innovations given the ordering shown (G, D, L, SPR). Although the percent of the spread forecast variance accounted for by either real government purchases or the real amount of loan is somewhat less than the amount for real debt, it still has a non-negligible predictive content for the spread. The above decomposition results seem to be reassuring on the interpretation of Granger causality that the series G, D and L are causally prior to the spread. Another interesting aspect of Table 3 is that the qualitative properties of the decomposition of the variance of L and SPR appear not to depend on how these variables are ordered. The combined explanatory power of the two variables is not only independent of which comes first, but the decomposition of variance explained by either L, or SPR, also remains essentially
unchanged even if we consider an alternative ordering with the spread innovations ordered first, i.e., G → D → SPR → L. This may be due primarily to the weak negative contemporaneous correlation between the two variables. As Sims (1980a, 1981), Litterman and Weiss (1985) have pointed out, the ordering of variables is immaterial to the extent that the reduced form innovations in equation (7-1) are contemporaneously uncorrelated. However, the above decomposition results, which are insensitive to the ordering of variables, do not justify a causal interpretation unless the orthogonalized innovations are treated as true exogenous variables. That is, a prior predeterminedness restriction is required for the orthogonalized innovations in a VAR model (see Cooley and LeRoy (1985)).

The results about variance decompositions for the log-differences case in Table 4 are qualitatively similar to earlier findings except that the forecast-error variances of each variable attributable by its own shocks are much larger at long horizons (say 40 quarters ahead). First differencing appears to enhance the explanatory contributions of each variable to its own long-term forecast-error variance. In particular, the real amount of short-term loans and government purchases have larger and more sustained variances explained by their own innovations at all horizons. The fairly high contemporaneous correlation between real government purchases and real debt, seen from Table 2-b, seems to be reinforced by the relatively strong response of real debt to the innovations in real government purchases. Furthermore, it is worthwhile to note that the real amount of short-
### Table 4

**Variance Decompositions:**

<table>
<thead>
<tr>
<th>Forecast Error in (k quarters ahead)</th>
<th>Triangularized Innovation in:</th>
<th>AG</th>
<th>AD</th>
<th>AL</th>
<th>ASPR</th>
</tr>
</thead>
<tbody>
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<td><strong>A</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<td>96.0</td>
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<td>1.2</td>
<td>0.2</td>
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<td>93.3</td>
<td>4.9</td>
<td>1.6</td>
<td>0.3</td>
</tr>
<tr>
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<td></td>
<td>91.4</td>
<td>5.2</td>
<td>1.7</td>
<td>1.8</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>91.0</td>
<td>5.2</td>
<td>1.9</td>
<td>1.8</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>90.9</td>
<td>5.2</td>
<td>2.0</td>
<td>1.9</td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>90.9</td>
<td>5.2</td>
<td>2.0</td>
<td>1.9</td>
</tr>
<tr>
<td><strong>D</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>85.5</td>
<td>0.0</td>
<td>0.0</td>
</tr>
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<td>77.9</td>
<td>1.1</td>
<td>0.7</td>
</tr>
<tr>
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<td></td>
<td>31.6</td>
<td>65.3</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
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<td></td>
<td>34.5</td>
<td>60.4</td>
<td>3.6</td>
<td>1.4</td>
</tr>
<tr>
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<td></td>
<td>34.7</td>
<td>60.1</td>
<td>3.8</td>
<td>1.4</td>
</tr>
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<td>34.7</td>
<td>60.0</td>
<td>3.8</td>
<td>1.5</td>
</tr>
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<td></td>
<td>34.7</td>
<td>60.0</td>
<td>3.8</td>
<td>1.5</td>
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<td><strong>L</strong></td>
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<td></td>
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<td></td>
</tr>
<tr>
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<td></td>
<td>5.4</td>
<td>0.6</td>
<td>94.0</td>
<td>91.5</td>
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<td>1.8</td>
<td>91.3</td>
<td>90.2</td>
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<td>2.5</td>
<td>90.2</td>
<td>89.1</td>
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<td>2.6</td>
<td>88.9</td>
<td>87.7</td>
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<td>2.6</td>
<td>88.9</td>
<td>87.7</td>
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<td>6.3</td>
<td>2.6</td>
<td>88.9</td>
<td>87.6</td>
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<td></td>
<td>6.3</td>
<td>2.6</td>
<td>88.9</td>
<td>87.6</td>
</tr>
<tr>
<td><strong>AS</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>10.2</td>
<td>0.1</td>
<td>2.4</td>
<td>(0.0)</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>10.0</td>
<td>12.9</td>
<td>3.1</td>
<td>(4.0)</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>10.9</td>
<td>12.8</td>
<td>7.6</td>
<td>(8.2)</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>11.9</td>
<td>11.0</td>
<td>16.3</td>
<td>(18.1)</td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>11.6</td>
<td>11.0</td>
<td>18.5</td>
<td>(20.2)</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>11.6</td>
<td>11.0</td>
<td>18.6</td>
<td>(20.3)</td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>11.6</td>
<td>11.0</td>
<td>18.6</td>
<td>(20.3)</td>
</tr>
</tbody>
</table>

Note: The values in parentheses in the **AL** and **ASPR** columns represent the fraction of the forecast-error variance of each row variable explained by the **L** and **SPR** innovations when we reorder the innovations so that AG → AD → ASPR → AL.
term loans is more important for fluctuations in the spread particularly over the medium and long-term horizons when the log differences are allowed.

To construct an interpretable impulse response function in our four-variable VAR system also requires a vector moving average representation with orthogonalized innovations which was considered as $Y_t = D(L) \nu_t$ above. Since $Y_t$ is a vector of four variables, $Y_t = D(L) \nu_t$ can be written as

$$
Y_t = \begin{bmatrix}
D_{11}(L) & \ldots & D_{14}(L) \\
D_{21}(L) & D_{22}(L) & \ldots & D_{24}(L) \\
\vdots & \vdots & \ddots & \vdots \\
D_{41}(L) & \ldots & D_{44}(L)
\end{bmatrix}
\begin{bmatrix}
\nu_{1t} \\
\nu_{2t} \\
\nu_{3t} \\
\nu_{4t}
\end{bmatrix}
$$

where $D_{ij}(L) = \sum_{s=0}^{\infty} D_{ij,s} L^s$ are square summable polynomials in lag operators for $i,j = 1,2,3,4$. It follows that the $h$ period ahead impulse response of the $i$th variable of $Y_{t+h}$ to one standard deviation shock in the $j$th variable at time $s = 0$ amounts to tracing out the effect of $\nu_j$ on $Y_i$ $h$ periods ahead, i.e., the lag polynomial $\sum_{s=0}^{h} D_{ij,s} L^s$. In this case, the contemporaneous effect of $\nu_j$ on $Y_i$ is given by $D_{ij,0}$ and subsequent lag effects by $\sum_{s=1}^{h} D_{ij,s} L^s$. 
Table 5.1

Impulse Response of the Spread (SPR) to One Standard Deviation Shock in

<table>
<thead>
<tr>
<th>Innovations to</th>
<th>G</th>
<th>D</th>
<th>L</th>
<th>SPR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.19</td>
<td>0.09</td>
<td>-0.13</td>
<td>0.69</td>
</tr>
<tr>
<td>2</td>
<td>-0.18</td>
<td>-0.34</td>
<td>-0.21</td>
<td>-0.22</td>
</tr>
<tr>
<td>3</td>
<td>-0.19</td>
<td>-0.14</td>
<td>-0.04</td>
<td>-0.27</td>
</tr>
<tr>
<td>5</td>
<td>-0.09</td>
<td>-0.22</td>
<td>-0.29</td>
<td>-0.04</td>
</tr>
<tr>
<td>7</td>
<td>-0.15</td>
<td>-0.11</td>
<td>-0.03</td>
<td>-0.01</td>
</tr>
<tr>
<td>9</td>
<td>-0.13</td>
<td>-0.16</td>
<td>0.05</td>
<td>0.01</td>
</tr>
<tr>
<td>11</td>
<td>-0.11</td>
<td>0.01</td>
<td>0.02</td>
<td>0.01</td>
</tr>
<tr>
<td>13</td>
<td>-0.06</td>
<td>0.03</td>
<td>-0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>16</td>
<td>-0.03</td>
<td>0.03</td>
<td>-0.03</td>
<td>0.00</td>
</tr>
<tr>
<td>18</td>
<td>-0.02</td>
<td>0.02</td>
<td>-0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>20</td>
<td>-0.03</td>
<td>0.00</td>
<td>-0.01</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Note: each entry is multiplied by 100.

Table 5.2

Impulse Response of the Loans (L) to One Standard Deviation Shock in

<table>
<thead>
<tr>
<th>Innovations to</th>
<th>G</th>
<th>D</th>
<th>L</th>
<th>SPR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.39</td>
<td>-1.42</td>
<td>13.6</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>4.07</td>
<td>-2.58</td>
<td>7.1</td>
<td>-1.51</td>
</tr>
<tr>
<td>3</td>
<td>1.96</td>
<td>-4.74</td>
<td>9.76</td>
<td>0.55</td>
</tr>
<tr>
<td>5</td>
<td>2.27</td>
<td>-5.26</td>
<td>5.62</td>
<td>1.01</td>
</tr>
<tr>
<td>7</td>
<td>1.24</td>
<td>-5.53</td>
<td>2.46</td>
<td>0.00</td>
</tr>
<tr>
<td>9</td>
<td>0.36</td>
<td>-4.49</td>
<td>0.00</td>
<td>0.21</td>
</tr>
<tr>
<td>11</td>
<td>-0.36</td>
<td>-2.58</td>
<td>-1.03</td>
<td>0.11</td>
</tr>
<tr>
<td>13</td>
<td>0.57</td>
<td>-0.41</td>
<td>-1.46</td>
<td>0.15</td>
</tr>
<tr>
<td>16</td>
<td>0.02</td>
<td>2.07</td>
<td>-1.37</td>
<td>0.08</td>
</tr>
<tr>
<td>18</td>
<td>0.54</td>
<td>2.83</td>
<td>-1.12</td>
<td>0.07</td>
</tr>
<tr>
<td>20</td>
<td>1.04</td>
<td>2.86</td>
<td>-0.83</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Note: each entry is multiplied by 100.
The impulse response function in Table 5.1 gives the dynamic response of the spread to one standard deviation shock to each column variable in the order shown (i.e., the corresponding lag polynomial $D_{ij}(L)$ in the vector moving average representation with an orthogonalized innovations). The spread appears to respond negatively to shocks in real debt, real government purchases, and the real amount of short-term loan. More specifically, real debt is found to have a negative impact on the spread in the second through ninth quarter, with the minimum reaching $-0.34$ at the second quarter. The response of the spread to a unit increase in real government purchases is positive for the first quarter (0.19) and is negative in all subsequent quarters. The real amount of short-term loans induces a declining response in the spread in the short-run, with the minimum at $-0.29$ around fifth quarter.

As a further check for the robustness of the causal link between the spread and the real amount of short-term loans, the response of the loan to one standard deviation shocks to the four variables are shown in Table 5.2. Shocks to real debt are found to have immediate negative effects on the short-term amount of loans, with their effects remaining influential at a lag of one to about 12 quarters.

In contrast to Table 5.1 in which the loan shocks are shown to have appreciable negative effects on the spread in the first through fifth quarter, spread shocks appear to have little effect on loans. This result provides evidence that the real amount of short-term loans appears to be weakly exogenous to the spread.
### Table 6.1
Impulse Response of the Differenced Spread ($\Delta SPR$) to One Standard Deviation Shock in

<table>
<thead>
<tr>
<th>Innovations to</th>
<th>$\Delta G$</th>
<th>$\Delta D$</th>
<th>$\Delta L$</th>
<th>$\Delta SPR$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter</td>
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<td></td>
<td></td>
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<td>0.27</td>
<td>0.03</td>
<td>-0.13</td>
<td>0.78</td>
</tr>
<tr>
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<td>-0.02</td>
<td>-0.41</td>
<td>-0.10</td>
<td>-0.69</td>
</tr>
<tr>
<td>3</td>
<td>-0.23</td>
<td>0.15</td>
<td>0.13</td>
<td>-0.11</td>
</tr>
<tr>
<td>5</td>
<td>-0.08</td>
<td>-0.15</td>
<td>-0.27</td>
<td>-0.08</td>
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<tr>
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<td>0.10</td>
<td>-0.07</td>
<td>-0.17</td>
<td>-0.20</td>
</tr>
<tr>
<td>9</td>
<td>-0.07</td>
<td>0.02</td>
<td>0.02</td>
<td>0.14</td>
</tr>
<tr>
<td>11</td>
<td>-0.04</td>
<td>0.02</td>
<td>0.18</td>
<td>-0.02</td>
</tr>
<tr>
<td>13</td>
<td>0.01</td>
<td>-0.02</td>
<td>-0.02</td>
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</tr>
<tr>
<td>16</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.02</td>
<td>-0.03</td>
</tr>
<tr>
<td>18</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
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<tr>
<td>20</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
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</table>

Cumulative: -0.20 -0.33 -0.09 0.19
(20 quarters)

Note: each entry is multiplied by 100.

### Table 6.2
Impulse Response of the Differenced Loans ($\Delta L$) to One Standard Deviation Shock in

<table>
<thead>
<tr>
<th>Innovations to</th>
<th>$\Delta G$</th>
<th>$\Delta D$</th>
<th>$\Delta L$</th>
<th>$\Delta SPR$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.6</td>
<td>-1.2</td>
<td>15.00</td>
<td>0.00</td>
</tr>
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<td>2</td>
<td>1.4</td>
<td>-1.1</td>
<td>-4.64</td>
<td>-1.10</td>
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<tr>
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<td>-1.1</td>
<td>1.6</td>
<td>4.07</td>
<td>1.55</td>
</tr>
<tr>
<td>5</td>
<td>0.86</td>
<td>0.51</td>
<td>-0.66</td>
<td>0.67</td>
</tr>
<tr>
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<td>-0.41</td>
<td>0.19</td>
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<td>0.14</td>
</tr>
<tr>
<td>9</td>
<td>0.05</td>
<td>-0.04</td>
<td>-0.59</td>
<td>-0.47</td>
</tr>
<tr>
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<td>0.22</td>
<td>0.08</td>
<td>-0.32</td>
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</tr>
<tr>
<td>13</td>
<td>-0.04</td>
<td>0.02</td>
<td>0.26</td>
<td>0.18</td>
</tr>
<tr>
<td>16</td>
<td>0.05</td>
<td>0.03</td>
<td>-0.19</td>
<td>0.00</td>
</tr>
<tr>
<td>18</td>
<td>-0.04</td>
<td>0.00</td>
<td>0.10</td>
<td>0.01</td>
</tr>
<tr>
<td>20</td>
<td>0.01</td>
<td>0.00</td>
<td>0.06</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

Cumulative: 3.64 -3.87 13.93 0.25
(20 quarters)

Note: each entry is multiplied by 100.
Table 6.1 displays the estimated impulse responses of the differenced spread to one deviation standard shocks to the log-differenced variables. A notable feature of this table is that the long-term impact of real debt innovation on the spread has a strong negative response, while the innovation to short-term loans seems to have only a weak negative impact on the spread.

The results in Table 6.2 seem to be robust particularly in terms of the long-run impact of the spread shock on the differenced loans. Its long-term effect is shown to be almost zero. This finding can be taken as indicative of the causal ordering running from the loans to the spread, which is consistent with the results of Table 5.2

7.3. Structural Vector Autoregression Approach

This section presents an alternative structural VAR approach. It should be noted that the structural VAR obtains identification of the innovations via an estimated structural model, instead of an assumed recursive identification of the VAR innovations through Cholesky decomposition in the standard VAR. This way of achieving identification in the structural VAR model without being quite so restrictive has recently been explored by a number authors, such as Bernanke (1986), Blanchard and Watson (1985), Sims (1986), Blanchard (1989), and Calomiris and Hubbard (1989). Following their approaches, we consider only a just-identified structural VAR model below. We can then proceed stepwise from the reduced form to the structural model. First, estimate the unrestricted standard VAR model described in (7-1)
(or (7-2)) and form an covariance matrix \( \tilde{Q} \) (or \( \tilde{I} \)) from the estimated VAR innovations. Then, use a set of just-identifying restrictions to extract the estimates of the coefficient matrix of the contemporaneous structural model from the estimated covariance matrix \( \tilde{Q} \) (or \( \tilde{I} \)). This basic approach makes explicit identifying restrictions and gives a structural interpretation, unlike the VAR methodology. Particularly, the correlation matrix for the contemporaneous innovations is useful for obtaining identifying restrictions in the structural model. The structural VAR model we apply to the estimated VAR innovations is specified by:

\[
\begin{align*}
\text{(a)} & \quad u_g = \varepsilon_g \\
\text{(b)} & \quad u_d = a_1 u_g - a_2 u_{spr} + \varepsilon_d \\
\text{(c)} & \quad u_L = a_3 u_g - a_4 u_d + \varepsilon_L \\
\text{(d)} & \quad u_{spr} = a_5 u_d - a_6 u_L + \varepsilon_{spr},
\end{align*}
\]

where \( u_g, u_d, u_L, \) and \( u_{spr} \) represent the innovations in each variables estimated from the unrestricted VAR model allowing for a linear time trend; \( \varepsilon_g, \varepsilon_d, \varepsilon_L, \) and \( \varepsilon_{spr} \) now refer to orthogonalized structural innovations, respectively, and their covariance matrix \( \pi \) is assumed to
be diagonal. It seems plausible to specify the structural model (7-3) as a system in (ΔG, ΔD, ΔL, ΔSPR) since the Dickey-Fuller unit root test fails to reject the null that all variables under consideration has a unit root (shown in the next subsection). The four structural equations contain six coefficients and four orthogonalized innovation variances that must be estimated from the ten unique elements of the 4x4 covariance matrix Ω. Thus, the structural VAR model under consideration is just identified.

The estimated coefficients of the postulated structural model (7-3) are listed below:

(a) \( u_g = \varepsilon_g \)

(b) \( u_d = 0.263 \ u_g - 1.292 \ u_L + \varepsilon_d \)

(c) \( u_L = 2.122 \ u_g - 3.903 \ u_{spr} + \varepsilon_L \)

(d) \( u_{spr} = 0.685 \ u_d - 0.017 \ u_L + \varepsilon_{spr} \)

All estimated coefficients appear to have signs broadly consistent with the previous theoretical implications and the reduced

33. This structural model has been estimated using the computer program of Bernanke (1986), which is based on a method of moments.
form evidence except for the positive effect of debt on the spread in equation (d). As Sims (1986) argues, however, more convincing evidence of functional identification can be provided by the decompositions and impulse response functions of the orthogonalized innovations than by the above estimated coefficients. The effects of the structural innovations on the endogenous variables are shown by the set of variance decompositions, and impulse response functions below.

There are some interesting differences between Table 7 and Table 4, which uses the standard decomposition. The results in Table 7 give structural innovations to the spread a bigger role in explaining the forecast-error variance in real debt. Most striking is the finding that real debt is more important in determining the spread, explaining more than 51 percent of the variance of the spread at all horizons. This is in contrast with the result of Table 4. Moreover, the explanatory power of the innovations to short-term loans for the variance of the spread in Table 7 has slightly increased compared to the results of Table 4.

Looking at the dynamic effects of the impulse response functions in Table 8, we can see that debt shocks have a dominant effect on the spread only at the first two quarters and thereafter die quickly. But the effects of loans on the spread appear to be relatively persistent at longer horizons. This evidence might suggest that debt shocks are an important source of temporary fluctuations in the spread, while loan shocks signal the potential for long-run adjustments in the
### Table 7

**Variance Decompositions**  
(The Structural VAR)

<table>
<thead>
<tr>
<th>Forecast Error</th>
<th>( u_g )</th>
<th>( u_d )</th>
<th>( u_L )</th>
<th>( u_{spr} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k quarters ahead)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( u_g )</td>
<td>100</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>96.0</td>
<td>0.7</td>
<td>1.3</td>
<td>2.0</td>
</tr>
<tr>
<td>5</td>
<td>93.3</td>
<td>1.9</td>
<td>2.0</td>
<td>2.8</td>
</tr>
<tr>
<td>9</td>
<td>91.4</td>
<td>3.6</td>
<td>2.1</td>
<td>2.9</td>
</tr>
<tr>
<td>12</td>
<td>91.0</td>
<td>3.7</td>
<td>2.4</td>
<td>2.9</td>
</tr>
<tr>
<td>20</td>
<td>90.9</td>
<td>3.7</td>
<td>2.5</td>
<td>2.9</td>
</tr>
<tr>
<td>40</td>
<td>90.9</td>
<td>3.7</td>
<td>2.5</td>
<td>2.9</td>
</tr>
<tr>
<td>( u_d )</td>
<td>14.0</td>
<td>39.3</td>
<td>2.6</td>
<td>44.1</td>
</tr>
<tr>
<td>3</td>
<td>19.8</td>
<td>40.5</td>
<td>2.8</td>
<td>36.9</td>
</tr>
<tr>
<td>5</td>
<td>31.2</td>
<td>36.5</td>
<td>2.2</td>
<td>30.1</td>
</tr>
<tr>
<td>9</td>
<td>34.1</td>
<td>34.4</td>
<td>3.3</td>
<td>28.2</td>
</tr>
<tr>
<td>12</td>
<td>34.3</td>
<td>34.3</td>
<td>3.4</td>
<td>28.0</td>
</tr>
<tr>
<td>20</td>
<td>34.3</td>
<td>34.3</td>
<td>3.5</td>
<td>28.0</td>
</tr>
<tr>
<td>40</td>
<td>34.3</td>
<td>34.3</td>
<td>3.5</td>
<td>28.0</td>
</tr>
<tr>
<td>( u_L )</td>
<td>5.4</td>
<td>2.8</td>
<td>88.7</td>
<td>3.1</td>
</tr>
<tr>
<td>3</td>
<td>5.6</td>
<td>2.7</td>
<td>86.4</td>
<td>5.2</td>
</tr>
<tr>
<td>5</td>
<td>5.9</td>
<td>3.2</td>
<td>85.5</td>
<td>5.3</td>
</tr>
<tr>
<td>9</td>
<td>6.3</td>
<td>4.1</td>
<td>84.3</td>
<td>5.3</td>
</tr>
<tr>
<td>12</td>
<td>6.3</td>
<td>4.1</td>
<td>84.3</td>
<td>5.3</td>
</tr>
<tr>
<td>20</td>
<td>6.3</td>
<td>4.1</td>
<td>84.3</td>
<td>5.3</td>
</tr>
<tr>
<td>40</td>
<td>6.3</td>
<td>4.1</td>
<td>84.3</td>
<td>5.3</td>
</tr>
<tr>
<td>( u_{spr} )</td>
<td>10.3</td>
<td>52.6</td>
<td>2.1</td>
<td>35.1</td>
</tr>
<tr>
<td>3</td>
<td>10.1</td>
<td>64.4</td>
<td>4.7</td>
<td>20.8</td>
</tr>
<tr>
<td>5</td>
<td>10.9</td>
<td>60.6</td>
<td>9.6</td>
<td>18.8</td>
</tr>
<tr>
<td>9</td>
<td>12.0</td>
<td>52.6</td>
<td>18.3</td>
<td>17.1</td>
</tr>
<tr>
<td>12</td>
<td>11.7</td>
<td>51.1</td>
<td>20.6</td>
<td>16.6</td>
</tr>
<tr>
<td>20</td>
<td>11.7</td>
<td>51.0</td>
<td>20.8</td>
<td>16.6</td>
</tr>
<tr>
<td>40</td>
<td>11.7</td>
<td>51.0</td>
<td>20.8</td>
<td>16.6</td>
</tr>
</tbody>
</table>
Table 8

Impulse Response of the Spread Innovations ($u_{spr}$)
to One Standard Deviation Shock in

<table>
<thead>
<tr>
<th>Structural Innovations to</th>
<th>$u_g$</th>
<th>$u_d$</th>
<th>$u_L$</th>
<th>$u_{spr}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.27</td>
<td>0.61</td>
<td>-0.12</td>
<td>0.50</td>
</tr>
<tr>
<td>2</td>
<td>-0.15</td>
<td>-0.77</td>
<td>-0.17</td>
<td>-0.19</td>
</tr>
<tr>
<td>3</td>
<td>-0.23</td>
<td>0.00</td>
<td>0.16</td>
<td>-0.16</td>
</tr>
<tr>
<td>5</td>
<td>-0.08</td>
<td>-0.12</td>
<td>-0.30</td>
<td>0.02</td>
</tr>
<tr>
<td>7</td>
<td>0.10</td>
<td>-0.17</td>
<td>-0.18</td>
<td>-0.11</td>
</tr>
<tr>
<td>9</td>
<td>-0.07</td>
<td>0.12</td>
<td>0.02</td>
<td>0.08</td>
</tr>
<tr>
<td>11</td>
<td>-0.03</td>
<td>-0.02</td>
<td>0.19</td>
<td>0.00</td>
</tr>
<tr>
<td>13</td>
<td>0.01</td>
<td>0.00</td>
<td>-0.02</td>
<td>0.01</td>
</tr>
<tr>
<td>16</td>
<td>0.00</td>
<td>-0.03</td>
<td>0.05</td>
<td>-0.01</td>
</tr>
<tr>
<td>18</td>
<td>0.00</td>
<td>0.00</td>
<td>-0.02</td>
<td>0.01</td>
</tr>
<tr>
<td>20</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Note: each entry is multiplied by 100.
borrowing rates as well as lending rates in financial markets. However this implication is quite different from that of Table 6-a.

7.4. Univariate Unit Root Test

As a preliminary test for co-integration, we carry out the unit root test following Dickey and Fuller (1979, 1981) to investigate whether the sequence of above time series has integrated processes of order one. We estimated coefficients in the following autoregressive approximation (using a Said-Dickey regression) in which the first difference ($\Delta z_t$) is regressed on the lagged level ($z_{t-1}$) and lagged differences ($\Delta z_{t-i}, i=1...k$) with a time trend.\(^{34}\)

\[
Z_t - Z_{t-1} = \alpha_o + \alpha_1 t + \beta_1 z_{t-1} + \sum_{i=1}^{p} \gamma_i (Z_{t-i} - Z_{t-i-1}) + u_t,
\]

\(^{34}\) Said and Dickey (1984) develop a test for unit roots for an unknown ARIMA (p 1 q) process that can be adequately approximated by an autoregression. They also show that the coefficients in the autoregressive approximation estimated using least squares produce statistics whose limit distributions are same as those tabulated by Fuller (1976). However, the Dickey-Fuller test is not theoretically valid if the series had more than one unit root because their test is based on the assumption of at most one unit root. Furthermore, as Schwert(1987) argued, when the value of the intercept $\alpha_o$ is not known in the autoregression approximation, inclusion of a time trend is probably a prudent decision in performing unit root tests.
where $Z_t$ represents the four time series variables, real debt, real
government purchase, the real amount of short-term loans, and
the spread between the borrowing rate and lending rate. All data are
transformed to logarithms, except for the spread, and are quarterly
from 1971-I to 1986-IV. Table 9 below reports the test statistics,
$\tau(\hat{\beta}_1)$, based on the t-statistics on $\beta_1$ in the Said-Dickey regression,
and the F-test for the null hypothesis ($\alpha_1 = 0$ and $\beta_1 = 0$).

As Table 9 shows, the regression t-statistics $\tau(\hat{\beta}_1)$ range from a
low of -1.28 for real debt to a high of -3.00 for the real amount of
short term loans. These are below the 0.05 critical value of -3.50
for sample sizes of 50 (as given by a one-sided t-statistics ($t$) in
Table 8.5.2 of Fuller (1976, p373). So we cannot reject the null
hypothesis that $\beta_1$ is zero at the 5 percent significance level.
Moreover, the Box-Pierce Q-statistics provide evidence that the
regression error is white noise, so the Dickey and Fuller test based
on the assumption that residuals are white noise appear to be correct.

Next, we can construct the usual regression F-tests of the null
hypothesis $H_0$: $(\alpha_1, \beta_1) = (0, 0)$. The values of the regression
F-ratio in Table 9 are all shown to be less than the critical value
6.73 for the 5 percent significant level, except for real government
purchases whose F-ratio is below the 1 percent critical value, 9.31.
The critical values for this F-test are the test statistics $t$ in
Dickey and Fuller (1981). The evidence presented here indicates that we cannot readily reject the hypothesis that the autoregressive process (7-4) has a unit root with possible drift. What seems perhaps surprising about the results in Table 10 is that one cannot reject the null that the spread has a unit root. Accordingly, all four time series appear to be difference-stationary processes (see Nelson and Flosser (1982)).

7.5. Test for Common Trends

This section deals with tests of the number of common stochastic trends in the four-variable system. The key feature of the multivariate common trends test proposed by Stock and Watson (1988b) is that it makes it possible to determine the number of common stochastic trends in a set of multivariate integrated time series. We apply this test to our empirical investigation since the previous univariate unit root test cannot address questions concerning the interrelation among stochastic trends. Furthermore, if the number of common stochastic trends (k) is less than the number of variables (N) but each element of $Y_t$ is individually an integrated process (i.e., containing a stochastic trend), then there are $n-k$ linear combinations of contemporaneous values of $Y_t$ that are stationary. Thus, testing for the number of common stochastic trend can be thought of as testing for the number of co-integrating vectors.
Table 9

<table>
<thead>
<tr>
<th></th>
<th>$p^1$</th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\beta_1$</th>
<th>$\tau(\hat{\theta}_1)$</th>
<th>F-ratio$^2$</th>
<th>Q(21)$^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) government</td>
<td>3</td>
<td>0.620</td>
<td>-0.0014</td>
<td>-0.119</td>
<td>-2.44</td>
<td>7.95</td>
<td>13.8</td>
</tr>
<tr>
<td>purchase</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2) real debt</td>
<td>5</td>
<td>0.176</td>
<td>0.0005</td>
<td>-0.028</td>
<td>-1.28</td>
<td>3.44</td>
<td>13.4</td>
</tr>
<tr>
<td>short-term</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>lending</td>
<td>5</td>
<td>0.108</td>
<td>-0.0089</td>
<td>-0.251</td>
<td>-3.00</td>
<td>4.54</td>
<td>7.6</td>
</tr>
<tr>
<td>4) spread$^3$</td>
<td>2</td>
<td>0.007</td>
<td></td>
<td>-0.343</td>
<td>-2.29</td>
<td>2.65</td>
<td>8.9</td>
</tr>
<tr>
<td>(RB-CPR)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: 1) Akaike's information criterion (AIC) is used in selecting lag lengths ($p$) for each regression equation. As Sawa (1978) argued, the AIC tends to choose higher lag lengths than the true model for small sample case, but this bias can be negligible when $p/T < 0.1$ as it is here.
2) the F-test is for the null hypothesis $\alpha_1 = 0$, $\beta_1 = 0$ against the general alternative $\alpha_1 \neq 0$, $\beta_1 \neq 0$.
3) the spread equation is fitted to a regression equation without a time trend.
4) the Box-Pierce statistics for the first twenty one sample autocorrelations of residuals, $Q(21) = T \sum_{i=1}^{21} \rho_1^2$, is employed to test for serial autocorrelation of residuals, in which case, $Q(21)$ should be distributed as $\chi^2$ with (21- the number of parameters ) degree of freedom.
(1) Common Trends Model

As Stock and Watson (1988b, 1989), and King-Plosser-Stock-Watson (1987) have shown, if an n×1 vector, \( Y_t \), is co-integrated, then it has a "common trends representation". This representation can be derived from an n×1 cointegrated vector moving average representation in first differences,

\[
\Delta Y_t = \mu + C(L)\varepsilon_t, \tag{7-5}
\]

where \( \mathbb{E}(\varepsilon_t \varepsilon_s') = \Sigma \) for \( t \neq s \) and \( C(L) = C(1) + (1-B)C^*(L) \) with \( C(0) = I_n \). Since \( C(1) \) has a rank \( k = (n-r) \), it can be written as \( C(1) = AF \), where \( A \) is an \( n \times k \) matrix and \( F \) is an \( k \times n \) matrix. Defining \( \xi_t = \xi_{t-1} + \nu_t \) as a n×1 random walk with serially uncorrelated permanent innovations \( \nu_t = F\varepsilon_t \) and initial condition \( \xi_0 = 0 \), yields \( \xi_t = \sum_{i=1}^{t} \nu_i = F \sum_{i=1}^{t} \varepsilon_i \).

Then, successive backward substitution of (7-5) yields

\[
Y_t = Y_0 + \mu t + \left( C(1) + (1-L)C^*(L) \right) \sum_{i=1}^{t} \varepsilon_i, \tag{7-6}
\]

\[
= Y_0 + \mu t + C(1)F^{-1}\xi_t + C^*(L)\varepsilon_t
\]

\[
= Y_0 + \mu t + A\xi_t + C^*(L)\varepsilon_t,
\]

where it is assumed that \( \varepsilon_s = 0 \) for \( s \leq 0 \). This equation is called the common trends representation because the reduced rank of \( C(1) \)
(k < n) implies that there are fewer unit roots than the number of
variables and thus the trends $\xi_t$ are common. The formulation (7-6)
also represents a decomposition of the nx1 vector $Y_t$ into permanent
and transitory components just as the Beveridge and Nelson (1981)
decomposition of a univariate series into permanent and transitory
components (see Stock-Watson-King-Plosser (1987)). $A\xi_t$ can be viewed
as the permanent component of $Y_t$ (i.e., $Y_t^p$) and $C^*(L)e_t$ can be thought
of as a transitory component (i.e., $Y_t^t$). Furthermore, it is
straightforward to show that the common trends representation (7-6)
implies the existence of some nxr cointegrating vectors $\alpha$ such that
$\alpha' Y_t$ is stationary.\textsuperscript{35}

Accordingly, the common trends model (7-6) can be expressed in a
more explicit form of (7-7) consisting of both non-integrated and
integrated elements under the null hypothesis the first N-k elements
of $Y_t$ are cointegrated, whereas the final k integrated elements
represents k separate trends (see appendix A for the derivation of
equation (7-7)). As shown in (7-7), a test of the null hypothesis
that the final integrated elements of $Y_t$ have k versus m common trends

\textsuperscript{35} We can easily derive a cointegrating model with nxr cointegrating
vectors $\alpha$ from the common trend model (7-6): i.e., from (7-6), $z_t = \alpha' Y_t = \alpha' Y_0 + \alpha' C^*(L)e_t$ and $z_t$ has a finite variance on the assumption that $C^*(L)$ is absolute summable.
entails testing whether $C(1)$ has rank $k$ against the alternative, when
the intercept and time trends may be nonzero.

$$\begin{bmatrix} \alpha & \alpha^* \end{bmatrix} Y_t = \begin{bmatrix} \alpha'Y_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0_{rx1} \\ 0_{rxk} \end{bmatrix} t + \begin{bmatrix} 0_{rxk} \end{bmatrix} \xi_t + \begin{bmatrix} \alpha' & C^*(L) \end{bmatrix} \varepsilon_t$$

$$= \beta_1 + \beta_2 t + \beta_3 \xi_t + \beta_4 \varepsilon_t$$

(2) Common Trends Test Based on The Stock and Watson $q$ Statistics

A test statistic proposed by Stock and Watson (1988b) is
developed on the assumption of a finite order VAR approximation of
$\Delta W_t$:

$$\Pi(L) \Delta W_t = \eta_t,$$  \hspace{1cm} (7-8)

where $\Delta W_t = \alpha^* \Delta Y_t$, with $\alpha^*$ being an nxk matrix of constants chosen so
that $\alpha^* \alpha = 0$ and $\alpha^* \alpha^* = I_k$, and $\Pi(L)$ is a matrix lag polynomial of
known order $p$, with $\Pi(0) = I$, and $\eta_t$ is i.i.d. with mean zero and
$E(\eta_t \eta_t') = I_k$. Let $\xi_t = \Pi(L) W_t$. Then it follows from (7-8) that

36. Note that their test statistics involve a finite order vector
autoregressive approximation to the short-run relation in the series
just as the Dickey and Fuller $\tau$ statistics for testing for a unit root
in a univariate time series.
\( \Pi(L) \Delta W_t = \Delta[\Pi(L)W_t] = \eta_t \), so that under the null hypothesis the elements of \( \zeta_t = \Pi(L)W_t \) follow a random walk.

Stock and Watson's test of the hypothesis of \( k \) versus \( m \) common trends when there might be a nonzero intercept or a nonzero intercept and time trends is based on the eigenvalues of the adjusted first order sample autocorrelation matrix defined using \( \zeta_t \),

\[
\hat{\phi}_\xi^\mu = \left[ \sum \zeta_t \zeta_t' \right] \left[ \sum \zeta_{t-1} \zeta_{t-1}' \right]^{-1}
\]

and

\[
\hat{\phi}_\xi^\tau = \left[ \sum \zeta_t \zeta_t' \right] \left[ \sum \zeta_{t-1} \zeta_{t-1}' \right]^{-1},
\]

where \( \zeta_t = \Pi(L)W_t \) and \( \zeta_t = \Pi(L)\tilde{W}_t \), with \( \tilde{W}_t = \{ \hat{X}_t - T^{-1} \bar{X} \} \) and

\[
\hat{W}_t = \{ \hat{X}_t - \hat{\beta}_1 - \hat{\beta}_2 t \}, \text{ respectively, and } \hat{X}_t \text{ is generated by } \alpha^*: v_t; \text{ the autoregressive polynomial } \hat{\Pi}(L) \text{ can be estimated consistently using } \Delta W_t \text{ or } \Delta \tilde{W}_t \text{ as in (7-8)}.

The actual test statistics defined in Stock and Watson (1988b) when there are nonzero intercepts, or nonzero intercepts and time trends, are given by

\[
(7-10)
\]
\[ q_f^\mu (k, m) = T \left[ \text{real}(\hat{\lambda}_{f,m+1}^\mu) - 1 \right] \]

and

\[ q_f^\tau (k, m) = T \left[ \text{real}(\hat{\lambda}_{f,m+1}^\tau) - 1 \right], \]

where \( \hat{\lambda}_{f,m+1}^\mu \) and \( \hat{\lambda}_{f,m+1}^\tau \) are the \((m+1)\)th largest eigenvalue of \( \hat{\Phi}_f^\mu \) and \( \hat{\Phi}_f^\tau \), respectively and \( T \) is the number of observations. The critical values of these statistics from a Monte Carlo experiment are reported in Stock and Watson (1988b).

As discussed above, a test for the number of common trends in our four variable system can be performed using the Stock and Watson \( q_f \) statistics. According to the Dickey and Fuller unit root test in a univariate time series, all four time series are found to contain a unit root. This suggests that there is at least one common trend among G, D, L, and SPR. This conjecture, however, stresses the importance of performing the multivariate common trends tests based on the \( q_f \) statistics, as suggested by Stock and Watson (1988b, 1989) and King-Plosser-Stock-Watson (1987).

The multivariate common trends test performed below is based on the eigenvalues of the adjusted first order autocorrelation matrix \( \hat{\Phi}_f^\tau \) defined in (7.11). The system we estimate consists of four variables, G, D, L, and SPR. The first order correlation matrix \( \hat{\Phi}_f^\tau \) was obtained
by regressing $\zeta_t (= \hat{P}(L)\hat{W}_t)$ against $\zeta_{t-1}$, after detrending data. The $\zeta_t$ was computed using a matrix lag polynomial $\hat{P}(L)$ of order four.

$\hat{W}_t = \hat{\alpha}^* Y_t$ was generated using principal components of $Y_t$ to construct $\hat{\alpha}^*$. Namely, $\hat{\alpha}^*$ was estimated by linear combinations corresponding to the largest $k$ principal components. The real parts of the estimated eigenvalues of $\hat{\Phi}_f^T$ are ordered 0.987, 0.848, 0.819, 0.146.

The Table 10 presents the Stock and Watson $q_f^T(k,m)$ statistics. Under the null hypothesis that $Y_t$ has $k$ trends, the eigenvalues of the first order correlation matrix of the $k$-integrated components should be equal to 1.

Under the alternative that $Y_t$ has only $m < k$ trends, only $m$ of the eigenvalues should be equal to 1, which means that the $(m+1)$th largest eigenvalue, $\hat{\lambda}_{f,m+1}$, should be less than 1. As shown in Table 10, for this four-variable system, we are unable to reject the null of four independent stochastic trends ($k=4$) against the alternatives that the underlying processes have no random trend, a single random trend, two random trends (i.e. $m=0$, $m=1$, or $m=2$), since their reported test statistics are less negative than both the 5 percent and 1 percent critical values. However, test for the null of four common trends ($k=4$) against the alternative of three common trends ($m=3$) leads us to reject the null at the 1 percent level in favor of the alternative. This evidence therefore suggests that our four variable system
contains three common trends and a single co-integrating vector. The normalized co-integrating vector corresponding to the smallest eigenvalue is given by $[1 -0.072 \ 0.129 \ 0.079]$, where the variables are ordered $\text{SPR}_t$, $\text{G}_t$, $\text{D}_t$, and $\text{L}_t$. Interestingly, the estimated co-integrating vector indicates a reasonable correspondence to the estimated coefficients of the co-integrating regression.

7.6. Test for Co-integration

Here we are interested in performing the test for co-integration based on the co-integrating regression and error-correction model of Engle and Granger (1987). That there is only one co-integrating vector in our four variable system, as evidenced by the above common trends test, can be confirmed by the Engle Granger co-integration tests. Our test for co-integration examines whether there is only one unique long-run relationship among four integrated variables under consideration: that is, whether the spread depends on the other three variables in the steady state.

According to Engle and Granger (1987), a vector of non-deterministic time series variables $\text{Y}_t$ is said to be co-integrated if i) all of the elements are individually integrated processes (stationary only after differencing at least once) and ii) there are one or more linear combinations of $\alpha'\text{Y}_t$ that have a lower order of integration. The co-integrating vectors $\alpha$ describe the long-run equilibrium relation to which the variables return.
Table 10

<table>
<thead>
<tr>
<th>System</th>
<th>Test Statistics</th>
<th>Critical Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>5 %</td>
</tr>
<tr>
<td>Four Variables;</td>
<td>(a) $q_1^T(4,0) = -0.8$</td>
<td>-10.7</td>
</tr>
<tr>
<td>G,D,L,SPR</td>
<td>(b) $q_2^T(4,1) = -9.0$</td>
<td>-18.0</td>
</tr>
<tr>
<td></td>
<td>(c) $q_3^T(4,2) = -10.7$</td>
<td>-28.9</td>
</tr>
<tr>
<td></td>
<td>(d) $q_4^T(4,3) = -50.4$</td>
<td>-47.0</td>
</tr>
</tbody>
</table>

Note: the quarterly data set examined here has 59 observations (from 1972:II to 1986:IV). The critical values are from the Table 3 in Stock and Watson (1988b).
Two types of preliminary regressions can be used for estimating the unknown cointegrating vectors $\alpha$. The first is the co-integrating regression by Engle and Granger (1987) and the second is an unrestricted error-correction regression. In the first case, if each element of $Y_t$ is integrated of order one and there is a single vector in $\alpha$ such that $\alpha'Y_t$ is stationary, then the test for co-integration would be equivalent to testing whether a regression of one non-stationary series on others yields stationary residuals. In the second case, the change in one element of $Y_t$ is regressed on its own lags and the lagged levels ($Y_{t-1}$). This unrestricted error correction model will yield a nonlinear least square estimator of the co-integrating vector. Furthermore, given that there is one known co-integrating vector, the nonlinear least square estimator of $\alpha$ can be computed directly from the OLS estimator of the coefficients of the lagged levels terms in the unrestricted error correction model (see, Stock (1987)).

As Engle and Granger (1987) proved, if a vector $Y_t$ is co-integrated, then it has an error-correction representation, i.e., the estimated coefficient on $z_{t-1}$ in the error-correction representation should be non-zero. The converse is also true. This restricted error correction model is interpreted as an alternative approach to testing for co-integration based on the Engle Granger two step estimator. Also, this model should be useful to estimate the parameters
describing the short run dynamic feedback, such as the coefficients on lagged first differences of $Y_t$ and $z_{t-1}$.

An error-correction model representation with one co-integrating vector can be expressed as:

$$A^*(L)(1-L)Y_t = -\gamma z_{t-1} + u_t,$$  \hspace{1cm} (7-11)

where $u_t$ is a stationary multivariate disturbance; $A^*(L)$ is a matrix of finite polynomials, with $A^*(0) = I_N$; $z_{t-1} = \alpha'Y_{t-1}$ is an $N \times 1$ stationary purely non-deterministic time series with invertible moving average; $\gamma$ is a $N \times 1$ nonzero parameter vector. As the name error correction model suggests, the error correction term $-\gamma z_{t-1}$ is interpreted as an error that occurred at time $t-1$ but would be corrected after multivariate short-run dynamic feedback embodied in the vector of parameters on the lagged changes in $Y_t$ had worked itself out. Thus, the variables of $Y_t$ eventually regain a long-run equilibrium relationship according to (7-11).

As the prior unit root test has shown, we cannot reject the hypotheses that each of the time series, $G$, $D$, $L$, $SPR$, is integrated processes of order one. Then, as a first step, a test for the null of non co-integration with the co-integration regression Durbin-Watson test can be performed for a quick approximate result. If the Durbin-Watson statistics in the co-integrating regression approach zero, the residuals are nonstationary and thus one cannot reject the null of non
co-integration. If the Durbin and Watson is sufficiently large, we reject the null and find co-integration. With the spread as the dependent variable, the co-integrating regression is given by:

\[(7-12)\]

\[
\text{SPR}_t = -0.474 + 0.091 G_t - 0.118 D_t - 0.016 L_t + u_t^{37}
\]

\[R^2 = 0.49, \quad D.W. = 2.37\]

where the value in the parenthesis is the respective t-statistic. For this co-integrating regression, we reject the null of non co-integration at the 5 per cent level because the estimated DW is 2.37.

If co-integration holds, Stock (1987), and Engle and Granger (1987) have proved the OLS estimator of the parameters of the cointegrating vector is superconsistent (strongly consistent) because \(R^2\) asymptotically approaches one. Also, they show that the estimated coefficients of the cointegrating vector converges to a constant in probability at a rate proportional to sample size \(T\) (rather than \(T^{1/2}\))

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37. Since the series D and L have trended upward over the sample period 1971I-1986IV, I used linearly detrended data for D and L. For this, I subtracted 0.4984744 + 0.008185507t from the quarterly data for D and L. I obtained the estimated trend by regressing the logarithm of real GNP on a constant term and a trend over the sample period 1971I-1986IV.
in the standard case) whether the residuals in the regression equation are serially correlated or not. 38

At the second stage, the augmented Dickey–Fuller (ADF) test for a unit root is applied to the residuals estimated from the cointegrating regression to conduct tests on the hypothesis that all the time series are not cointegrated. Since the residuals will have a higher order serial correlation when the variables are integrated of order one, Engle and Granger recommend the Augmented Dickey Fuller test instead of the Dickey Fuller test as a way of correcting the correlations. The augmented Dickey–Fuller test includes higher order lagged differences of the residuals in the Dickey Fuller regression and is based on the t-statistics of \( \hat{\phi} \) in the following regression:

\[
\Delta u_t = -\hat{\phi} u_{t-1} + \sum_{i=1}^{p} \rho_i \Delta u_{t-i}
\]

Applying this test to the residuals in the cointegrating regression in (7-12) yields the following estimated regression:

---

38. However, as Stock (1987) proved, the estimated parameters describing the short-term dynamics of the cointegrating process in the error correction model, such as the estimated coefficients on the lagged changes in \( Y_t \) and \( \gamma \), converge at the usual \( T^{1/2} \) rate. He also showed that the two-step OLS and NLS estimators of those parameters are equivalent whereas the OLS and NLS estimators of the cointegrating vector are not.
\[ \Delta u_t = -1.520 \, u_{t-1} + 0.274 \, \Delta u_{t-1} + \varepsilon_t \]

\[ \text{(7-13)} \]

\[ R^2 = 0.62, \quad Q(21) = 10.19 \]

where one lag length is selected in the augmented Dickey-Fuller test based on the Akaike's information criterion. Notice that all of the two lagged changes are significant and the Box-Pierce Q statistic indicates no evidence of residual autocorrelation in the estimated augmented regression (7-13). The finite sample critical values for \( \phi \) in the augmented Dickey-Fuller test are obtained from 1,000 replications of a simulation experiment, where it is assumed that \( \Delta u_t \) is generated by a first-order autoregression, i.e., \( \Delta u_t = \rho \Delta u_{t-1} \), and that \( \Delta G, \Delta D, \) and \( \Delta L \) are generated by a fourth-order vector autoregression. The 5 percent critical value of the \( \phi \) calculated through the simulation is \(-3.91\) (the 1 percent and 10 critical values
are -4.80 and -3.59, respectively). Therefore, the estimated t-statistics for $\phi$, -7.87, from the augmented regression (7-13) indicates that we can reject the hypothesis of non co-integration, or accept co-integration at least at the 1 percent level.

Taken together, the above evidence suggests that a stationary long-run equilibrium for the spread exists. The cointegrating vector $\alpha$ is equal to $[1 \ -0.091 \ 0.118 \ 0.016]$, where the variables are ordered $\text{SPR}_t$, $G_t$, $D_t$, and $L_t$. The spread appears increasing in $G_t$, and decreasing in $D_t$ and $L_t$. In particular, the coefficient on $D_t$ is of the hypothesized sign and $G_t$ and $D_t$ appear to have effects consistent with the results of the standard VAR.

Since we can reject the null of non-cointegration by finding a significant t value on $\phi$ in the augmented Dickey Fuller regression, the error-correction model (7-11) can be considered as a basis for the second approach to testing for co-integration. The reason is that co-integration implies the existence of an error-correction model describing the short run dynamics and the long run equilibrium relation among variables.

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39. The 0.01, 0.05, and 0.10 critical values for the t-ratio were set equal to the tenth, fiftieth, and hundredth smallest t-ratios obtained in the simulations respectively. Note that the critical values of the t-statistics of the augmented regression obtained by our simulation are fairly similar to those for the sample size of 50 in Engle and Yoo (1987). For instance, the 5 percent critical value for the t-statistics of $\phi$ in the augmented regression calculated through the simulation is 3.91, while the critical value for this t-ratio by Engle and Yoo (1987) is 4.35.
Given the knowledge of one co-integrating vector $\hat{\alpha}$ with the normalization, $[1 \ \hat{\alpha}_g \ \hat{\alpha}_d \ \hat{\alpha}_L]$, where the variables are ordered $\text{SPR}_t$, $G_t$, $D_t$, and $L_t$, we can also obtain the nonlinear least square estimator of $\alpha$ from the unconstrained OLS estimators of the coefficients on the lagged levels in the error correction model representation. The Akaike information criterion is employed to determine the lag length in each regression. The resulting regressions obtained by estimating (7-11) using the OLS are given by:

\begin{align*}
\Delta G_t &= 0.139 - 0.041 \text{SPR}_{t-1} - 0.003 G_{t-1} - 0.057 D_{t-1} \\
&\quad + 0.007 L_{t-1} - 0.043 \Delta G_{t-1} + 0.119 \Delta D_{t-1} - 0.021 \Delta L_{t-1} \\
&\quad + 0.214 \Delta \text{SPR}_{t-1} \\
&\quad (0.27) \quad (-0.06) \quad (-0.03) \quad (-0.58) \quad (0.32) \quad (-0.27) \quad (0.41) \quad (-0.83) \quad (0.50)
\end{align*}

\begin{align*}
\Delta D_t &= -0.284 - 0.151 \text{SPR}_{t-1} + 0.067 G_{t-1} - 0.065 D_{t-1} \\
&\quad - 0.006 L_{t-1} - 0.115 \Delta G_{t-1} + 0.670 \Delta D_{t-1} + 0.006 \Delta L_{t-1} \\
&\quad + 0.287 \Delta \text{SPR}_{t-1} \\
&\quad (-1.45) \quad (-0.59) \quad (1.71) \quad (-1.72) \quad (-0.72) \quad (-1.87) \quad (5.97) \quad (0.06) \quad (1.73)
\end{align*}
\[ \Delta L_t = -4.671 + 0.896 \text{ SPR}_{t-1} + 0.879 \text{ G}_{t-1} - 0.528 \text{ D}_{t-1} \]
\[ (-1.68) \quad (0.25) \quad (1.58) \quad (-0.98) \]

\[ -0.181 \text{ L}_{t-1} + 0.434 \Delta G_{t-1} - 1.806 \Delta D_{t-1} - 0.309 \Delta L_{t-1} \]
\[ (-1.57) \quad (0.50) \quad (-1.14) \quad (-2.19) \]

\[ -1.417 \text{ SPR}_{t-1} \]
\[ (-0.61) \]

\[ \text{ASPR}_t = -0.597 - 1.516 \text{ SPR}_{t-1} + 0.150 \text{ G}_{t-1} - 0.182 \text{ D}_{t-1} \]
\[ (-3.85) \quad (-7.48) \quad (4.79) \quad (-6.05) \]

\[ -0.026 \text{ L}_{t-1} - 0.023 \Delta G_{t-1} - 0.158 \Delta D_{t-1} + 0.008 \Delta L_{t-1} \]
\[ (-4.09) \quad (-0.47) \quad (-1.78) \quad (1.01) \]

\[ + 0.269 \text{ ASPR}_{t-1}, \]
\[ (2.06) \]

where the t-values are in parentheses. Of the coefficients on the lagged levels terms in (7-14), only the lagged levels for an error correction term in the spread regression are individually significant. Furthermore, the estimator of the cointegrating vector \( \alpha \) in the spread regression is quite similar to the first stage OLS estimator of \( \alpha \). That is, the relevant parameters \( \gamma_s \alpha \) in the estimated spread regression end up with \(-1.516 [1 -0.099 \quad 0.120 \quad 0.017] \), which is roughly proportional to the \( \alpha \) in the cointegrating regression (7-13). Notice also that of all the lagged first differences, only the first
lag of change in own variable for the last three regressions of \( \Delta D_t \), 
\( \Delta L_t \), and \( \Delta SPR_t \) is significant, respectively.

Then, as a final step to carry out the two step estimation in the sense of Engle and Granger (1987), we proceed to estimate a restricted error correction model by replacing the error correction term in (7-14) with the lagged levels of estimated residuals \( \hat{u}_{t-1} \) from the co-integrating regression. The outcome is

\[
\Delta G_t = 0.005 - 0.063 \hat{u}_{t-1} + 0.220 \Delta SPR_{t-1} - 0.019 \Delta G_{t-1} \\
(1.26) \quad (-0.96)_{t-1} \quad (0.52)_{t-1} \quad (-0.13)
\]

\[+ 0.148 \Delta D_{t-1} - 0.015 \Delta L_{t-1} \]
\[\quad (0.64)_{t-1} \quad (-0.61)_{t-1}\]

\[
\Delta D_t = 0.003 - 0.188 \hat{u}_{t-1} + 0.310 \Delta SPR_{t-1} - 0.065 \Delta G_{t-1} \\
(0.22) \quad (-0.70)_{t-1} \quad (1.78)_{t-1} \quad (-1.08)_{t-1}
\]

\[+ 0.798 \Delta D_{t-1} + 0.007 \Delta L_{t-1} \]
\[\quad (8.43)_{t-1} \quad (0.75)_{t-1}\]

\[
\Delta L_t = 0.019 + 1.227 \hat{u}_{t-1} - 1.549 \Delta SPR_{t-1} + 1.176 \Delta G_{t-1} \\
(0.91) \quad (0.33)_{t-1} \quad (-0.65)_{t-1} \quad (1.43)_{t-1}
\]
\[ -1.411 \Delta D_{t-1} - 0.387 \Delta L_{t-1} \]
\[ (-1.09) \quad (-2.87) \]

\[ \Delta SPR_t = -0.001 - 1.521 \Delta u_{t-1} + 0.274 \Delta SPR_{t-1} - 0.015 \Delta G_{t-1} \]
\[ (-0.75) \quad (-7.75) \quad (2.16) \quad (-0.35) \]

\[ -0.133 \Delta D_{t-1} + 0.008 \Delta L_{t-1} \]
\[ (-1.92) \quad (1.04) \]

According to the estimated error correction system of (7-15), only the regression of \( \Delta SPR_t \) has a significant error correction term \( \hat{\Delta u}_{t-1} \) with a t-statistic of -7.75. This implies that only the spread is influenced by the error correction feedback from temporary disequilibrium. Notice that nearly significant lagged change in real debt in the spread regression reveals a modest negative correlation between the spread and real debt. Moreover, the results from the above error correction model (7-16) appear to provide evidence that government purchases, real debt, and the short-term loans are weakly exogenous since each error correction term in their regressions is not significant. These results therefore strongly support the hypothesis of co-integration between \( G_t, D_t, L_t \), and \( SPR_t \).

While the error-correction model has a good fit, the two step Engle Granger method is subject to criticism primarily because the estimate of the co-integrating vector obtained by the first step may suffer from a finite sample bias, even if they are superconsistent.
This is evidenced by the fact that the $R^2$ is actually quite small in the cointegrating regression, although the presence of cointegration implies that the $R^2$ should be asymptotically one.

(1) A VAR Approach to an Error Correction Model

Next consider a vector autoregression form of the error correction representation described above. Although the error correction model is not a vector autoregression, it is straightforward to rewrite the error correction representation (7-11) as the VAR form for $Y_t$ such that $A(L)Y_t = \varepsilon_t$, using the Granger representation theorem in Engle and Granger (1987). This VAR approach to the error correction model is particularly useful in analyzing an optimal forecast of the variables under consideration, as implied by the impulse response functions and variance decompositions. More importantly, this vector error correction representation differs from a standard VAR model in first differences since the existence of cointegration of $Y_t$ implies fewer unit roots than the number of variables. As Campbell (1987), and Campbell and Shiller (1987) have pointed out, simple first differencing of all variables can lead to the problem of over-differencing if an economic theory imposes cointegration on a set of nonstationary variables $Y_t$. One of the solution to this problem is to use a vector error-correction model instead of a pure VAR model in differences.
A vector error correction model was estimated using four lags of the first difference of G, D, L, and SPR, an intercept, and the one error correction term SPR-G-D-L. The dynamic response of the spread (ΔSPR) to one-standard deviation shock to each variable is given in Table 11.

The impact on the spread of innovations in government expenditures is seen to be immediate, followed by a modest negative effect after third quarter. The spread path also depend negatively on the debt innovation over the short-term horizon. Noticeably, its negative response to the debt innovations appears to be stronger than to the government expenditure innovations. The effect of loan innovations on the spread, which are rather modest over the very short-term, steadily declines after the fifth quarter. This feature appears to be consistent with the previous result of Table 4, although the sign of the estimated spread response to loan shocks somewhat differs between the two results. Furthermore, that the long-run response (20 quarters ahead) of the spread to all shocks considered is zero can be interpreted as the existence of the long-run equilibrium among the variables.

Interestingly, the fraction of the variance in ΔG and ΔL explained by own innovations in Table 12 is shown to be smaller over the long horizons relative to the results of Table 4 displaying the variance decompositions of the VAR model in first differences. In particular, the two factors, the spread and loan innovations, appear to be more important in explaining the movement of each series, but
the innovation in government expenditures explains a much smaller fraction of the movement of debt. The parts of the forecast error variance of the spread attributable to the debt and loan innovations are very much consistent with the results of Table 4.
Impulse Response Function and Variance Decompositions
Of the Error Correction Model

Table 11

Impulse Response of the Spread (ΔSPR )
to One Standard Deviation Shock in

<table>
<thead>
<tr>
<th>Innovations to</th>
<th>ΔG</th>
<th>ΔD</th>
<th>ΔL</th>
<th>ΔSPR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarter</td>
<td>0.22</td>
<td>0.05</td>
<td>-0.12</td>
<td>0.68</td>
</tr>
<tr>
<td>2</td>
<td>0.00</td>
<td>-0.16</td>
<td>0.15</td>
<td>0.32</td>
</tr>
<tr>
<td>3</td>
<td>-0.17</td>
<td>-0.09</td>
<td>0.25</td>
<td>0.18</td>
</tr>
<tr>
<td>5</td>
<td>-0.07</td>
<td>-0.19</td>
<td>0.09</td>
<td>-0.17</td>
</tr>
<tr>
<td>7</td>
<td>-0.07</td>
<td>-0.09</td>
<td>0.03</td>
<td>-0.17</td>
</tr>
<tr>
<td>9</td>
<td>-0.04</td>
<td>0.01</td>
<td>0.04</td>
<td>-0.07</td>
</tr>
<tr>
<td>11</td>
<td>0.01</td>
<td>0.04</td>
<td>0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>13</td>
<td>0.01</td>
<td>0.03</td>
<td>-0.02</td>
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<td>0.02</td>
</tr>
<tr>
<td>18</td>
<td>0.01</td>
<td>0.01</td>
<td>-0.01</td>
<td>0.02</td>
</tr>
<tr>
<td>20</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Note: each entry is multiplied by 100.
Table 12
Variance Decompositions (The Error Correction Model)
(U.S. 1972:II-1986:IV)

<table>
<thead>
<tr>
<th>Forecast Error</th>
<th>Triangularized Innovation in:</th>
<th>(\Delta G)</th>
<th>(\Delta D)</th>
<th>(\Delta L)</th>
<th>(\Delta S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>in (k quarters ahead)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(\Delta G)</strong></td>
<td></td>
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<tr>
<td>1</td>
<td></td>
<td>94.0</td>
<td>2.4</td>
<td>0.1</td>
<td>8.0</td>
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<td>3</td>
<td></td>
<td>84.0</td>
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<td>3.6</td>
<td>8.8</td>
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<td>5</td>
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<td>3.8</td>
<td>10.9</td>
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<tr>
<td>9</td>
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<td>80.0</td>
<td>5.4</td>
<td>3.8</td>
<td>10.9</td>
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<td></td>
<td>79.9</td>
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<td>3.8</td>
<td>10.9</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>79.9</td>
<td>5.4</td>
<td>3.8</td>
<td>10.9</td>
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<tr>
<td>40</td>
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<td>79.9</td>
<td>5.4</td>
<td>3.8</td>
<td>10.9</td>
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<tr>
<td><strong>(\Delta D)</strong></td>
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<td>0.0</td>
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<td>0.7</td>
</tr>
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<td>90.0</td>
<td>13.1</td>
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CHAPTER VIII

CONCLUSION

In this paper, we have examined whether non-Ricardian results followed from the consideration of liquidity constraints within the framework of the nonstochastic neoclassical model for a closed economy. The main theoretical conclusions we reached were consistent with those of Barro (1974, 1985, 1989). We proposed three key analytic findings. First, as may be expected in a Ricardian world, Ricardian equivalence did hold in our benchmark model characterized by an infinite-horizon intertemporal optimization of two types of agents with different endogenous time preferences, by lump-sum wage taxes, and by perfect capital markets. Second, when exogenous liquidity constraints were permanently binding for type-II agents, who had a high subjective discount rate, non-Ricardian results occurred in that government debt caused both aggregate consumption and aggregate capital stock to fall. In the presence of binding liquidity constraints for type II agents, government debt led to a narrowing of the spread, provided the marginal cost of private intermediation was an increasing function of the per capita amount of loans. In that case, government debt, as an effective form of financial
intermediation, displaced private intermediation.

In our empirical analysis, the main finding was that real debt had an appreciable negative effect on the spread in the short run dynamic as well as in the long-run steady state. Also, we found that the spread had an unique long-run relationship with the other variables considered here. This suggested that the U.S. economy may be properly approximated by the steady state. The empirical failure of Ricardian equivalence in assessing the effect of debt on the spread in the U.S. offers support for our theoretical conclusion that the role of liquidity constraints is important in explaining the non-Ricardian results. Confirmation of this is to be found in the other evidence pointing to the relative importance of liquidity constraints for accounting for the real effects of fiscal policy changes in the U.S. (for instance, Hubbard and Judd (1986,1987)).

One useful extension of this analysis is to examine the implication of non-lump-sum taxes (for example, proportional tax) for Ricardian equivalence. While we considered a lump-sum tax structure only for analytic convenience, it would be a more realistic formulation to consider the possibility that the government collected taxes in non-lump-sum manner.

It is quite conceivable to improve upon the present empirical test along two dimensions. First, since the data we use are considerably less than those typically found in macroeconomic research, it would be desirable to use a longer data set if additional data became available. This would make our test results more
reliable. Finally, as Runkle (1987) argued, we still need to provide confidence intervals for variance decompositions and impulse response functions for unrestricted VAR to give them more useful inferences.
APPENDIX A

COMMON TRENDS MODEL RELATIVE TO CHAPTER VII
COMMON TREND MODEL RELATIVE TO CHAPTER VII

This appendix presents the derivation of the common trends model (7-7). As is found in (7-6), since \( C(1) \) has rank \( k < n \), there is an \( n \times r \) matrix \( H_1 \) with rank \( r \) such that \( C(1)H_1 = 0_{n \times r} \), where \( 0_{n \times r} \) is an \( n \times r \) matrix of zeros. If \( H_2 \) is an \( n \times k \) matrix with rank \( k \) and columns orthogonal to the columns of \( H_1 \), then \( C(1)H_2 = A \) has rank \( k \). The \( n \times n \) matrix \( H = (H_1 \ H_2) \) is nonsingular and \( C(1)H = (0_{n \times r} \ A_{n \times k}) \).

Accordingly, the common trend representation (7-6) can be composed of both non-integrated and integrated components such that the first \( r \) elements of \( Y_t \) have a representation for the stationary linear combinations \( z_t = \alpha'y_t \), whereas the remaining \( k(=n-r) \) elements of \( Y_t \) are expressed in terms of the \( k \) separate common trends. To show this, consider a linear transformation of \( Y_t \), denoted by \( X_t = DY_t = [\alpha \ \alpha^*]'Y_t \), where \( \alpha^* \) is an \( n \times k \) matrix of constants chosen so that \( \alpha^*\alpha = 0 \) and \( \alpha^*\alpha^* = I_k \). Let \( W_t \) denote the final \( k \) integrated elements of \( Y_t \), \( W_t = \alpha'y_t \). Note also that \( \alpha^* \) is computed by linear combinations corresponding to the largest \( k \) principal components of \( Y_t \).
and $\alpha$ by linear combinations corresponding to the smallest $r$ components when $D$ is not known. It follows from (7-5) that

\begin{equation}
\Delta W_t = \alpha' \Delta Y_t = \Delta \alpha' \gamma_t = \alpha' \mu + \alpha' C(L) \epsilon_t.
\end{equation}

Combining $z_t = \alpha' Y_t$ and $\Delta W_t$,

\begin{equation}
\begin{bmatrix}
I_r & 0 \\
0 & \Delta I_k
\end{bmatrix}
\begin{bmatrix}
\alpha' Y_t \\
\alpha' \mu
\end{bmatrix}
= \begin{bmatrix}
\alpha' Y_0 \\
\alpha' \mu
\end{bmatrix}
+ \begin{bmatrix}
\alpha' C(L)
\end{bmatrix} \epsilon_t.
\end{equation}

Recursive substitution of (A-2) yields a representation in terms of the transformed levels of $Y_t$,

\begin{equation}
\begin{bmatrix}
\alpha & \alpha'
\end{bmatrix}
\begin{bmatrix}
\alpha' Y_t \\
\alpha' \mu
\end{bmatrix}
= \begin{bmatrix}
\alpha' Y_0 \\
\alpha' \mu
\end{bmatrix}
+ \begin{bmatrix}
0_{r \times 1}
\end{bmatrix} t + \begin{bmatrix}
0_{r \times k}
\end{bmatrix} \xi_t + \begin{bmatrix}
\alpha' C(L)
\end{bmatrix} \epsilon_t.

= \beta_1 + \beta_2 t + \beta_3 \xi_t + \beta_4 \epsilon_t,
\end{equation}

(see, Stock and Watson (1988b, 1989), and King-Plosser-Stock-Watson (1987)).
References


