ON THE EDGE COLORING OF GRAPHS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of the Ohio State University

By

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2009

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ABSTRACT

In this thesis, we consider the problem posed by Gupta, Goldberg and Seymour: “When is the chromatic index $\chi'$ of a graph almost the same as its fractional chromatic index $\chi'^*$?”. While they conjectured that the answer is when “$\chi' > \Delta + 1$”, so far this problem is wide open.

The first part of this thesis improves the known results to this problem. Following the work of Tashkinov, we show that $\chi' - \chi'^* < 1$ whenever $\chi' > \frac{25}{24}\Delta + \frac{22}{24}$. We also show that $\chi' - \chi'^* < 1$ whenever $\chi' > \Delta + \frac{3}{2}\sqrt{\Delta}/2$. We also present some partial results each of which are very useful, if not essential, in attacking this well-known problem. Some even lead to equivalent conjectures.

In the second part, we show that the ideas of fan and path can be generalized to the analogous problem posed by Gupta by replacing the idea of “covering with matchings” with “packing with edge covers”. While we will not present any explicit results such as the one presented in the first part, this part provides some insight into the nature of alternating chains in this new context.

The last part of this thesis will be a tribute to a recent result by Goldberg [17]. Goldberg shows that it is not necessary to use the matching number of a graph while working on Conjecture 1.3.4. We prove the analogous result for Conjecture 1.3.5.
To my parents for their unconditional and unending love and support,

my wife, Habibe, for her continuous patience

and

to our daughter, Feride as a welcoming gift.
ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Neil Robertson, for his patience, guidance and constant encouragement which made this work possible. I am grateful for the enlightening discussions and suggestions as well as the intellectual support he has provided.

I would like to thank the Department of Mathematics for their support expressed through several teaching and research fellowships that I enjoyed.

I would like to thank Professor Michael Stiebitz for sharing the electronic version of the work [11] on Goldberg’s conjecture to help simplify the notation of the future work on this conjecture.

Last but not least, I would like to thank my family for their constant support and encouragement.
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CHAPTER 1
INTRODUCTION

Graph coloration studies the partitioning of objects in a graph $G$ into classes, subject to constraints. In the simplest forms of coloration models, an arbitrary map from either the set of vertices, the set of edges or the union of these two sets, respectively, into a set of $k$ colors represents a $k$-vertex, $k$-edge or $k$-total coloration of $G$. Under certain constraints, such a coloration becomes the focal point of the study of the theory of graph colorations. While the vertex and total colorations are of much interest for many, this thesis only focuses on edge colorations of graphs. Hence, we model the placements of objects into classes as the assignment of colors to edges. Whenever we use the term “coloration”, we mean edge colorations.

There are various areas of application of graph coloration ranging from social networks to DNA structure, from chemical bonding to computer science. In particular, edge coloration is heavily applied in Scheduling Theory. For more background in graph theory, see [7] and [47]. For more background in edge coloration theory, see [12] and [26]. For more background in combinatorial optimization, see [37]. For more on the graph coloration problems, see [42].

We believe that this thesis will provide a basis for those who would like to study further the problems presented in this chapter. However, we refer those who would like to learn more about the previous research on the subject to [11]. Most of our
terminology is exactly the same as in [11] and, hence, it should provide an easier transition for the interested reader.

Lastly, the reader should note that the results in this thesis are mostly structural and depend on alternating chain methods. When counting arguments are used, they are strictly elementary.

1.1 Basic Definitions

We define a graph $G$ to be a triple $(V(G), E(G), \partial_G)$, where $V(G)$ and $E(G)$ are disjoint finite sets, and $\partial_G$ is a function that maps every element of $E(G)$ to an unordered pair of distinct elements of $V(G)$. The elements of $V(G)$ and $E(G)$ are called vertices and edges of $G$, respectively, and $\partial_G$ is called the incidence function of $G$. For every edge $e$, the elements of $\partial_G(e)$ are called the endvertices of $e$. An edge $e$ and vertex $x$ are called incident to each other if $x \in \partial_G(e)$. Two vertices $x, y$ are called adjacent to each other if $\partial_G^{-1}\{\{x, y\}\} \neq \emptyset$. In this case, we also say $x$ and $y$ are neighbors. Notice that our graphs, sometimes called multigraphs, are finite and may have parallel edges, but no loops. A graph is said to be simple if the function $\partial_G$ is injective.

For a vertex $x \in V(G)$, denote by $E_G(x)$ the set of all edges $e$ of $G$ that are incident with $x$. Two distinct edges of $G$ incident to the same vertex will be called adjacent edges. Furthermore, if $X, Y \subseteq V(G)$, $X \cdot Y$ denotes the set of two element subsets $\{x, y\}$ of $X \cup Y$ such that $x \in X$ and $y \in Y$, and $E_G(X, Y) = \partial_G^{-1}(X \cdot Y)$. We write $E_G(x, y)$ instead of $E_G(\{x\}, \{y\})$ for vertices $x, y \in V(G)$, $E_G(X)$ instead of $E_G(X, X)$ and $E'_G(X)$ instead of $E_G(X, V(G) \setminus X)$. We define the boundary of $X$ by $E_G(X, V(G) \setminus X)$.

The degree of a vertex $x$ of $G$ is $d_G(x) = |E_G(x)|$ and the multiplicity of a pair of distinct vertices $x, y$ is $\mu_G(x, y) = |E_G(x, y)|$. Let $\Delta(G), \delta(G)$ and $\mu(G)$ denote
the maximum degree, minimum degree and maximum multiplicity of $G$, respectively. A graph $G$ is called simple if $\mu(G) \leq 1$. The subscript $G$ will be dropped from our notation when no confusion arises.

For a graph $G$ and a set $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced by $X$, that is $V(G[X]) = X$ and $E(G[X]) = E(X) = E_G(X,X)$. Similarly, for a set $S \subseteq E(G)$, $G : S$ denotes the subgraph induced by $S$, that is $V(G : S) = V(G)$ and $E(G : S) = S$. We also let $G(X)$ denote the subgraph with $V(G(X)) = X$ and $E(G(X)) = E'(X) = E_G(X, V(G))$. Further, $G - X = G[V(G) \setminus X]$. We also write $G - x$ instead of $G - \{x\}$. For $F \subseteq E(G)$, $G \setminus F$ denotes the subgraph $H$ of $G$ with $V(H) = V(G)$ and $E(H) = E(G) \setminus F$. If $F = \{e\}$, then we write $G \setminus e$ rather than $G \setminus \{e\}$.

As usual, we shall write $\lfloor x \rfloor$ for the lower integer part of $x$, and $\lceil x \rceil$ for the upper integer part of $x$.

### 1.2 Edge Colorations

Let $G$ be a graph. A $k$-coloration of $G$ is a map $\pi : E(G) \to \{1, \ldots, k\}$. For $x \in V(G)$ and $i > 0$, $\pi^i(x)$ denotes the set of colors which are assigned to at least $i$ edges incident to $x$ while $\pi^0(x)$ denotes the set of colors missing at $x$. Moreover, the set of colors incident to $x$ will be denoted by $\pi(x) = \pi^1(x)$, and $\bar{\pi}(x) = \pi^0(x) \cup \pi^2(x)$.

Given maps $f, g : V(G) \to \mathcal{R}$ with $f \leq g$, an $[f, g]$-coloring is a coloration $\pi$ for which every vertex $x \in V(G)$ satisfies $\pi^i(x) = \emptyset$ whenever $i \notin [f(x), g(x)]$. In particular, a $[0, 1]$-coloring will be called an edge coloring and a $[1, \infty]$-coloring will be called an edge-cover coloring.

Let $S \subset E(G)$. $S$ is called a matching in $G$ if, for any distinct pair of edges $e, f$, $\partial_E(e) \cap \partial_G(f)$. $S$ is called an edge cover if $\partial_G(S) = V(G)$. Note that edge coloring is a partition of the edge set into matchings while the edge-cover coloring is a partition
of the edge set into edge covers. While \([f, g]\)-colorations, in general, are studied in [31] and [25], our focus will be on edge colorings and edge-cover colorings. We denote by \(C_k(G)\) and \(C'_k(G)\) as the set of all \(k\)-edge colorings and \(k\)-edge-cover colorings of \(G\), respectively. The chromatic index \(\chi'(G)\) and the cover index \(\chi'_c(G)\) are the least integer \(k\) with \(C_k(G) \neq \emptyset\) and the largest integer \(k\) such that \(C'_k(G) \neq \emptyset\), respectively.

We finish this section by introducing two definitions that are commonly used in graph colorations:

Let \(G\) be graph. Then an edge \(e\) is said to be edge-critical if \(e \in E(G)\) and \(\chi'(G \setminus e) = \chi'(G) - 1\) and edge-cover-critical if \(e \notin E(G)\) and \(\chi'_c(G + e) = \chi'_c(G) + 1\). \(G\) is said to be edge-critical if all of its edges are edge-critical and edge-cover-critical if any new edge is edge-cover-critical.

Criticality of a graph is used heavily in the research regarding Conjecture 1.3.4 while it is not used much regarding Conjecture 1.3.5. One reason for this is that \(\chi'\) is naturally bounded below by 0 while there is no such upper bound present for \(\chi'_c\).

1.3 The Conjectures of Gupta-Goldberg-Seymour

Graph edge coloration has a plethora of applications to problems involving complex scheduling scenarios. One such example is coordinating the minimization of radio frequencies being assigned to mobile users as they place their calls. The heart of the problem is that users that are relatively close to each other must not be assigned the same radio frequency because conversation overlap would occur. On the other hand, users that are sufficiently far enough away from each other may be assigned the same radio frequency without suffering virtually any loss in quality because their frequencies would not be close enough to interfere. A simple graph coloring approach to this is to create a graph where the vertices are the mobile users, the edges between vertices are the communication request by two people and the colors are the frequency
assignments to each edge. Thus, the problem reduces to minimizing the number of frequencies in use at any one moment. Since the number of mobile users are rapidly increasing around the world, this increase is driving up the number of frequencies that are needed at any one moment. Finding the optimum number (chromatic index $\chi'$) of frequencies minimizes the cost and allows one to do more with the current available technology. For more on this topic see Gamst [13].

A trivial bound for the chromatic index is $\chi' \geq \Delta$ and for the cover index is $\chi'_c \leq \delta$.

Another bound is established by using a simple observation. Let $X$ be a vertex set with $|X| \geq 2$ in a graph $G$. Then the size of matching in $G[X]$ is at most $\left\lfloor \frac{|X|}{2} \right\rfloor$. Hence, $|E(X)| \leq \left( \frac{|X| - 1}{2} \right) \chi'$. One can add one more vertex and the number of edges will increase while the size of a matching in $G[X]$ will still be bounded by $\left\lfloor \frac{|X|}{2} \right\rfloor$. and if $X = V(G)$ has even number of vertices then one can delete a vertex of minimum degree and still preserve this bound. Hence, $X$ can be assumed to be odd with $|X| \geq 3$. Hence, we have the following less trivial lower bound for the chromatic index:

$$\chi'(G) \geq \max_{H \in \mathcal{O}(G)} \left\lfloor \frac{2|E(H)|}{|H| - 1} \right\rfloor$$

(1.3.1)

where $\mathcal{O}(G)$ denotes the subgraphs of $G$ with odd number of 3 or more vertices. 

We define $\Gamma(G) = \max_{H \in \mathcal{O}(G)} \left\lfloor \frac{2|E(H)|}{|H| - 1} \right\rfloor$ to be the density of $G$.

Similarly, we can show

$$\chi'_c(G) \leq \min_{H \in \mathcal{O}'(G)} \left\lfloor \frac{2|E(H)|}{|H| + 1} \right\rfloor$$

(1.3.2)

where $\mathcal{O}'(G)$ denotes the subgraphs of $G$ with odd number of vertices. We define $\Gamma_c(G) = \min_{H \in \mathcal{O}'(G)} \left\lfloor \frac{2|E(H)|}{|H| + 1} \right\rfloor$ to be the co-density of $G$. 

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On the other hand, one needs to find a reasonable upper bound for $\chi'$ and lower bound for $\chi'_c$ to be able to have any comprehension of $\chi'$ and $\chi'_c$. The first few of these bounds were given by Shannon [39] and later by Gupta [19] and Vizing [43]:

**Theorem 1.3.1** (Shannon [39], 1949; Gupta [19], 1967). $\chi'(G) \leq \left\lceil \frac{3\Delta}{2} \right\rceil$.

**Theorem 1.3.2** (Vizing [43], 1964; Gupta [19], 1967). $\chi'(G) \leq \Delta(G) + \mu(G)$.

**Theorem 1.3.3** (Gupta [19], 1967). (1) $\chi'_c(G) \geq \left\lfloor \frac{3\delta + 1}{4} \right\rfloor$

(2) $\chi'_c(G) \geq \delta(G) - \mu(G)$

While these bounds are sharp they were not enough since $\chi' - \Delta$ and $\delta - \chi'_c$ can be very large and, at the time, little was known about 1.3.1 and 1.3.2. It was Gupta [19] who conjectured the following bounds first. The first conjecture is also asked by Goldberg [14], Andersen [1] and Seymour [38] independently.

**Conjecture 1.3.4.** Let $G$ be a graph. $\chi'(G) \leq \max \{\Delta(G) + 1, \Gamma(G)\}$.

**Conjecture 1.3.5.** Let $G$ be a graph. $\chi'_c(G) \geq \min \{\delta(G) - 1, \Gamma_c(G)\}$.

When one thinks about the meaning of these conjectures then one will see that there are only three possibilities for $\chi'$ ($\Delta, \Delta + 1$ and $\Gamma(G)$) and $\chi'_c$ ($\delta, \delta - 1$ and $\Gamma_c(G)$). If one thinks that the well-known Petersen graph $P_{10}$ has $\Gamma(P_{10}) = 3$ and $\chi'(P_{10}) = 4$, then it will be clear that Conjecture 1.3.4 is best possible. This is in fact true for Conjecture 1.3.5 too.

### 1.4 On the Importance of Conjectures 1.3.4 and 1.3.5

In this section, we will first define $\chi'$ and $\chi'_c$ using mathematical programming. We will then make use of this definition to point out some of the important implications of Conjectures 1.3.4 and 1.3.5.
Mathematical Programming refers to the study of problems in which one seeks to minimize or maximize a real function by systematically choosing the values of real or integer variables from within an allowed set. Edge coloring problem can be stated as an Integer Programming (IP) problem as follows:

Let \( \mathcal{M}(G) \) denote the set of all matchings of \( G \) and \( \mathcal{M}_e(G) \) denote the set of all matchings of \( G \) containing \( e \in E(G) \). Then

\[
\chi'(G) = \min \sum_{M \in \mathcal{M}} y_M
\]

subject to:

1. \( \sum_{M \in \mathcal{M}_e} y_M = 1 \) for all \( e \in E(G) \);
2. \( y_M \in \{0, 1\} \) for all \( M \in \mathcal{M} \).

By replacing the integrality condition (2) by \( 0 \leq y_M \leq 1 \), we obtain the linear program (LP) whose solution is called the fractional chromatic index \( \chi'^*(G) \) of \( G \). Clearly, \( \chi' \geq \chi'^* \) and it is easy to check \( \chi'^* \geq \Delta \). Another, more combinatorial approach, to understanding \( \chi'^* \) is through Edmond [9]:

**Theorem 1.4.1** (Edmond, [9], 1965). \( \chi'^*(G) = \max \left\{ \Delta(G), \max_{H \in \mathcal{O}(G)} \frac{2|E(H)|}{|H| - 1} \right\} \)

This implies that, if \( \chi'(G) > \Delta(G) + 1 \), then \( \chi'(G) = \lceil \chi'^* \rceil \). Similarly, we can describe the edge-cover coloring problem as follows:

Let \( \mathcal{C}(G) \) denote the set of all edge covers of \( G \) and \( \mathcal{C}_e \) denote the set of all edge covers of \( G \) containing \( e \). Then

\[
\chi'_e(G) = \max \sum_{C \in \mathcal{C}} x_C
\]

subject to:

1. \( \sum_{C \in \mathcal{C}_e} x_C = 1 \) for all \( e \in E(G) \).
(2) $x_C \in \{0, 1\}$ for all $C \in \mathcal{C}$.

By replacing the integrality condition in (2) by $0 \leq x_C \leq 1$, we obtain a linear program (LP) whose solution is called the fractional cover index $\chi_c^*(G)$ of $G$. Another, more combinatorial approach, to understanding $\chi_c^*$ is again using Edmond’s method:

**Theorem 1.4.2.** $\chi_c^*(G) = \min \left\{ \delta(G), \min_{H \in \mathcal{O}(G)} \frac{2|E'(H)|}{|H| + 1} \right\}$.

This implies that, if $\chi'(G) > \Delta(G) + 1$, then $\chi'(G) = \lceil \chi'' \rceil$. Finding $\chi'$ and $\chi'_c$ is an $NP$-complete while we can find $\chi''$ and $\chi^*_c$ in polynomial time [21]. Hence, if Conjectures 1.3.4 and 1.3.5 hold then one can approximate $\chi'$ and $\chi'_c$ within an error of 1. If one thinks that the main applications of these two problems require lots of vertices, high maximum and minimum degrees, and high multiplicity, one will realize the importance of Conjectures 1.3.4 and 1.3.5. In fact, it is a good idea to restate them to stress this goal:

A graph $G$ is called $\chi'$-elementary if $\chi'(G) - \chi''(G) < 1$ or $\chi'_c$-elementary if $\chi'^*_c(G) - \chi'_c(G) < 1$. Such a graph will be called elementary if no confusion arises. Gupta is the first one who suggested the following two conjectures the first of which is known as the Goldberg Conjecture. We will call the second as the Gupta Conjecture to honor Gupta:

**Conjecture 1.4.3.** Let $G$ be a graph with $\chi'(G) > \Delta(G) + 1$. Then $G$ is $\chi'$-elementary.

**Conjecture 1.4.4.** Let $G$ be a graph with $\chi'_c(G) < \delta(G) - 1$. Then $G$ is $\chi'_c$-elementary.

Since $\chi'(G) \geq \Delta(G)$ and $\chi_c'(G) \leq \delta(G)$, we can now divide graphs into two classes: A graph $G$ is said to be Class-1 (Cover-Class-1) if $\chi'(G) = \Delta(G)$ ($\chi'_c(G) = \delta(G)$) and Class-2 (Cover-Class-2), otherwise; that is if $\chi'(G) = \max \{\Delta(G) + 1, \Gamma(G)\}$ ($\chi'_c(G) \geq \min \{\delta(G) - 1, \Gamma_c(G)\}$).
1.5 On the $r$-Graph Conjecture

There are variations of Conjectures 1.3.4 and 1.3.5 even though they are accepted as natural. The following conjectures are asked by Seymour and Goldberg:

**Conjecture 1.5.1** (Goldberg [15], 1974; Seymour [38], 1979). Let $G$ be a graph. 
\[ \chi'(G) \leq \max \{\Delta(G), \Gamma(G)\} + 1. \]

**Conjecture 1.5.2** (Goldberg [15], 1974; Seymour [38], 1979). Let $G$ be a graph. 
\[ \chi'(G) \leq \max \{\Delta(G), \Gamma(G) + 1\}. \]

Let $G$ be a graph. It is called $r$-regular if, for all $v \in V(G)$, $d_G(v) = r$ and it is said to be an $r$-graph if it is $r$-regular and for any odd set $X \subset V(G)$, $|E_G(X, V(G) \setminus X)| \geq r$.

Let $G$ be an $r$-regular graph and $X \subset V(G)$ be such that $|X|$ is odd and $|E_G(X, V(G) \setminus X)| < r$. $\chi'(G) > r$ since any perfect matching must contain an edge of $E_G(X, V(G) \setminus X)$. Hence, an $r$-graph is a natural candidate for $\chi' = r$. The following conjecture is asked by Seymour:

**Conjecture 1.5.3** (Seymour [38], 1979). Let $G$ be a planar $r$-graph. Then $\chi'(G) = r$.

Conjecture 1.5.3 is the generalization of Four Color Theorem since it is equivalent to the Tait [40] version of Four Color Theorem when $r = 3$. Since Four Color Theorem is one of the measuring sticks in Graphy Theory when it comes to importance and hardness of a problem, this problem can be seen as the edge coloring analog of Hadwiger’s Conjecture.

Conjecture 1.5.1 can be also stated as follows:

**Conjecture 1.5.4** (Seymour [38], 1979). The $r$-graph conjecture: Let $G$ be an $r$-graph. Then $\chi'(G) \leq r + 1$. 
Clearly, Conjecture 1.3.4 implies Conjecture 1.5.4. Moreover, Conjecture 1.5.4 is closely related with Rizzi’s Packing Postman Sets Conjecture [34] which is also a very hard problem. When one looks at Conjecture 1.3.4 from this perspective, it is clear why it is seen as one of the major problems in Edge Coloring Theory.

On the other hand, Conjecture 1.3.5 is seen as a harder problem, if not equivalent, since chromatic index, $\chi'(G)$, is a bounded hereditary graph property; that is, if $H$ is a subgraph of $G$, then $\chi'(H) \leq \chi'(G)$. However, this can not be said about $\chi'_e$ unless $H$ is on the same vertex set with $G$. Moreover, there is no upper bound for $\chi'_e$ unlike the natural bound 0 for $\chi'$. This being said, Conjectures 1.5.3 and 1.5.4 can still be generalized to the analogous problem on edge cover colorings. The author of this work first was introduced to the following versions by my advisor, Neil Robertson even though there might be others who asked it earlier:

**Conjecture 1.5.5** (Robertson). Let $G$ be a planar $r$-graph. Then $\chi'_e(G) = r$.

**Conjecture 1.5.6** (Robertson). Let $G$ be an $r$-graph. Then $\chi'_e(G) \geq r - 1$.

The nicest side of the $r$-graph conjecture is that it allows one relate to planar graphs a lot easier because of Conjecture 1.5.3. In fact, there are results available using the methods of Four Color Theorem for $r = 4, 5, 6, 7$ [38, 18].

### 1.6 Previous Results

While Conjecture 1.3.4 has been extensively studied, Conjecture 1.3.5 received little attention. However, Gupta not only suggested a unified approach to both conjectures but also provided the framework to work in such a setting. Unfortunately, it is, while natural, very hard to work a framework because chain structures are extremely complicated as we will see in Chapter 6.
The reader should note that the following results were proved without any use of Mathematical Programming. They follow directly from re-coloring techniques which are first used by Kempe [22] in his attempt to prove the Four Color Theorem.

**Theorem 1.6.1** (Gupta [19], 1967; Goldberg [15], 1974; Andersen [1], 1977). If $\chi'(G) > \frac{5}{4}\Delta + \frac{2}{4}$ then $G$ is elementary.

**Theorem 1.6.2** (Gupta [19], 1967; Andersen [1], 1977). If $\chi'(G) > \frac{7}{6}\Delta + \frac{4}{6}$ then $G$ is elementary.

**Theorem 1.6.3** (Goldberg [16], 1984). If $\chi'(G) > \frac{9}{8}\Delta + \frac{6}{8}$ then $G$ is elementary.

**Theorem 1.6.4** (Nishizeki, Kashiwagi [32], 1990; Tashkinov [41], 2000). If $\chi'(G) > \frac{11}{10}\Delta + \frac{8}{10}$ then $G$ is elementary.

**Theorem 1.6.5** (Favrholdt, Stiebitz, Toft [11], 2006). If $\chi'(G) > \frac{13}{12}\Delta + \frac{10}{12}$ then $G$ is elementary.

**Theorem 1.6.6** (Scheide, Stiebitz [35], 2009). If $\chi'(G) > \frac{15}{14}\Delta + \frac{12}{14}$ then $G$ is elementary.

On the other hand, Gupta [19] showed the following result for Conjecture 1.3.5:

**Theorem 1.6.7** (Gupta [19], 1967). If $\chi'_c(G) < \frac{7}{8}\delta + \frac{1}{8}$ then $G$ is elementary.

The following are the two main results of the first part of this thesis and they will be proven in Chapter 5:

**Theorem 1.6.8.** Let $G$ be a graph. If $\chi'(G) > \Delta(G) + \sqrt{\frac{\Delta(G)}{2}}$, then $G$ is elementary.

**Theorem 1.6.9.** Let $G$ be a graph. If $\chi'(G) > \frac{25}{24}\Delta(G) + \frac{22}{24}$, then $G$ is elementary.

A theorem will be called *Andersen-type* or *linear* if, for an integer $m \geq 3$, its statement is the following:

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“Let $G$ be a graph. If $\chi'(G) > \frac{m}{m-1}\Delta(G) + \frac{m-3}{m-1}$, then $G$ is elementary.”

It will be called asymptotic if, for integers $m \geq 2$ and $c \geq 1$, its statement is the following:

“Let $G$ be a graph. If $\chi'(G) > \Delta(G) + \sqrt[2]{\frac{\Delta(G)}{c}}$, then $G$ is elementary.”

Chapter 6 will be devoted to an introductory study of Conjecture 1.3.5 using Gupta’s methods. The purpose of the chapter is to describe the nature of chains in colorations and bring Conjecture 1.3.5 in line with Conjecture 1.3.4 by proving that Vizing’s fan and Kierstead’s path arguments can be generalized to colorations. These concepts are essential in the proof of results pertaining to Conjecture 1.3.4 and we believe that their generalization to colorations helps bring the two conjectures to the same level.

1.7 Notes

While there are also minor-hereditary results pertaining to Conjecture 1.3.4, they will not be analyzed here since they use Mathematical Programming. The interested reader should refer to Marcotte’s extraordinary work, [27, 28, 29]. We also suggest [37] as a general reference.
CHAPTER 2

EDGE COLORINGS: PRELIMINARIES AND BASIC RESULTS

Throughout this chapter, the definitions and theorems that are relevant to the study of edge colorings and Conjecture 1.3.4 will be introduced. While some of the definitions and results are borrowed from [11], new ones will also be introduced.

2.1 Re-coloring Technique

Let $H$ be a graph and let $C_k(H)$ denote the set of all $k$-colorings of $H$. For any $\pi \in C_k(H)$ and two colors $\alpha, \beta$, the $(\alpha, \beta)$-chain at $x \in V(H)$ with respect to $\pi$ is the connected component of $H : \pi^{-1}(\{\alpha, \beta\})$ containing $x$, which is denoted by $P_x(\alpha, \beta, \pi)$. A chain is either an even cycle or a path; and these will be called cyclic or acyclic chains. If $P$ is an acyclic $(\alpha, \beta)$-chain and $x, y$ are its end vertices, then at least one of $\alpha$ and $\beta$ is missing at each of $x$ and $y$ and, for this reason, it will be denoted as $P_x(\alpha, \beta, \pi)$ or $P_y(\alpha, \beta, \pi)$.

The coloration obtained from $\pi$ by switching or re-coloring an $(\alpha, \beta)$-chain $P$ is again a coloring and if it is denoted by $\varphi$, the relationship between $\pi$ and $\varphi$ is denoted by $\pi \rightarrow^P \varphi$.

We define a chain sequence of length $r \geq 0$ as a sequence of colorings $\pi_0, \pi_1, \ldots, \pi_{r-1}$ and chains $P_0, P_1, \ldots, P_{r-2}$ such that $P_i$ is a chain under $\pi_i$ and $\pi_i \rightarrow^{P_i} \pi_{i+1}$ for each
0 ≤ i ≤ r − 2. This will be denoted as \( \pi_0 \rightarrow^{R_b} \pi_1 \rightarrow^{R_1} \pi_2 \rightarrow \ldots \rightarrow^{P_{r-2}} \pi_{r-1} \). In general a chain sequence of length \( r \) will be denoted as \( \pi \rightarrow^r \varphi \) and a chain sequence will be noted by \( \pi \rightarrow \varphi \). The following result has been long known about (Kempe) chains:

**Lemma 2.1.1.** For any graph \( H \) and integer \( k \geq 1 \), “\( \rightarrow \)” is an equivalence relation in \( C_k(H) \).

In this case, the equivalence class of a coloring \( \pi \in C_k(H) \) is denoted by \([\pi]\). Clearly, any coloring on a chain sequence starting with \( \pi \) is in \([\pi]\).

### 2.2 Vizing Fans

Suppose \( G \) is an edge critical graph with \( \chi'(G) = k + 1 \) for an integer \( k \geq 1 \). Let \( e \in E(G), x \in V(e) \) and \( \pi \in C_k(G \setminus e) \).

By a multi-fan at \( x \) with respect to \( e \) and \( \pi \), we mean a sequence \( F = (e_1, y_1, \ldots, e_n, y_n) \) consisting of edges \( e_1, \ldots, e_n \) and vertices \( y_1, \ldots, y_n \) satisfying the following two conditions.

(F1) The edges \( e_1, \ldots, e_n \) are distinct, \( e_1 = e \) and \( e_i \in E_G(x, y_i) \) for \( i = 1, \ldots, n \).

(F2) For every edge \( e_i \) with \( 2 \leq i \leq n \) there is a vertex \( y_j \) with \( 1 \leq j < i \) such that \( \pi(e_i) \in \pi(y_j) \).

Please, see Figure 2.1 for an example of a fan at \( x_0 \). Let \( F = (e_1, y_1, \ldots, e_n, y_n) \) be a multi-fan at \( x \). Since the vertices of \( F \) need not be distinct, the set \( V(F) = \{y_1, \ldots, y_n\} \) may have cardinality smaller than \( n \). For \( z \in V(F) \), let \( \mu_F(x, z) = |E_G(x, z) \cap \{e_1, \ldots, e_n\}| \).

**Theorem 2.2.1** (Vizing [44], 1965). Let \( G \) be a graph with \( \chi'(G) = k + 1 \) for an integer \( k \geq \Delta(G) \) and, for a pair of vertices \( x, y \), let \( e \in E_G(x, y) \) be a critical edge
Figure 2.1: A fan at $x_0$ with respect to some coloring. ($\alpha$□) represents a color missing at that vertex.

of $G$. Furthermore, let $F = (e_1, y_1, \ldots, e_n, y_n)$ be a multi-fan at $x$ with respect to $e$ and $\pi \in C_k(G \setminus e)$. Then the following statements hold:

(a) $\bar{\pi}(x) \cap \bar{\pi}(y_i) = \emptyset$ for $i = 1, \ldots, n$.

(b) If $\alpha \in \bar{\pi}(x)$ and $\beta \in \bar{\pi}(y_i)$ for $1 \leq i \leq n$, then there is an $(\alpha, \beta)$-chain with respect to $\pi$ having endvertices $x$ and $y_i$.

(c) If $y_i \neq y_j$ for $1 \leq i, j \leq n$, then $\bar{\pi}(y_i) \cap \bar{\pi}(y_j) = \emptyset$.

(d) If $F$ is a maximal multi-fan at $x$ with respect to $e$ and $\pi$, then $|V(F)| \geq 2$ and $\sum_{z \in V(F)} (d_G(z) + \mu_F(x, z) - k) = 2$.

**Theorem 2.2.2** (Favrholdt, Stiebitz, Toft [11], 2006). Let $G$ be a graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G)$ and let $e \in E_G(x, y)$ be a critical edge of $G$. Then
there are \( m \geq 2 \) distinct neighbours \( z_1, \ldots, z_m \) of \( x \) in \( G \) satisfying \( y \in \{z_1, \ldots, z_m\} \) and, moreover,

\[
\begin{align*}
&a \sum_{i=1}^{m} (d_G(z_i) + \mu_G(x, z_i) - k) \geq 2, \\
&b \quad d_G(z_1) + \mu_G(x, z_1) \geq k + 1, \\
&c \quad d_G(z_1) + \mu_G(x, z_1) + d_G(z_2) + \mu_G(x, z_2) \geq 2(k + 1), \text{ and} \\
&d \quad d_G(x) + d_G(z_1) + d_G(z_2) \geq 2(k + 1).
\end{align*}
\]

We will refer to the equation in statement (d) of Theorem 2.2.1 as the fan equation. If \( G \) is a graph and \( k \geq \Delta(G) + \mu(G) \), then the fan equation fails for \( G \) as well as for any subgraph of \( G \). Consequently, \( G \) does not contain a \((k + 1)\)-critical subgraph, which implies \( \chi'(G) \leq k \). Thus, Vizing’s theorem follows from Theorem 2.2.1.

**Theorem 2.2.3** (Vizing [43], 1964 and Gupta [19], 1967). For a graph \( G \), we have \( \chi'(G) \leq \Delta(G) + \mu(G) \).

The definition of a fan at the beginning of this section is due to Favrholdt et. al. [11] and it differs slightly from the classical definition going back to Vizing [43, 44, 45]. It allows multiple edges and only requires the color of an edge of the fan to be missing at some previous vertex of the fan (instead of missing exactly at the previous vertex). This change makes proofs easier and is essential for obtaining the fan equation in Theorem 2.2.1 (d) and the fan inequality in Theorem 2.2.2 (a).

The fan equation and inequality are unifying results from which all the classical results on edge colorings seem easily derivable. The fan inequality appears in several earlier papers as part of proofs, rather than as a separate result of interest in its own right; versions of it can be found in the papers by Andersen [1], Goldberg [16], Hilton and Jackson [20] and Chodoum and Kayathri [6].
There have been three independent papers that have explicitly mentioned the fan equation/inequality as an important result and tool, namely, in chronological order, the M.Sc. thesis of Favrholdt [10], the paper by Reed and Seymour [33] and the Ph.D. thesis by Cariolaro [2]. Cariolaro goes even further in his analysis of fans in graphs by associating a directed walk in a certain directed graph to a fan; see [3] for details. Favrholdt, Stiebitz and Toft [11] also improved Theorem 2.2.3 in their recent preprint.

Suppose $H$ is a graph and $\pi \in C_k(H)$. Let $X$ be a subgraph of $H$ or be a subset of $V(H) \cup E(H)$. We denote the set of colors missing at $X$ with respect to $\pi$ by $\bar{\pi}(X) = \bigcup_{x \in V(X)} \bar{\pi}(x)$. We call $X$ elementary with respect to $\pi$ if, for all distinct $u, v \in X$, $\bar{\pi}(u) \cap \bar{\pi}(v) = \emptyset$.

We note that, for a multi-fan $F$ at $x$ with respect to $e$ and $\pi$, $V(F) \cup \{x\}$ is elementary with respect to $\pi$.

The concept of elementariness defined in this section is not the same as the elementariness of a graph defined in Chapter 1 even though it is one of the main tools used in proving Theorems 1.6.1- 1.6.6 as well as our main results.

We conclude this section with some basic facts about elementary sets that are useful for our further investigations. The following result due to Favrholdt et. al. [11] is essentially at the core of the results of Gupta, Goldberg, Andersen and many others and it is even used to formulate Conjectures 1.3.4 and 1.3.5 by Gupta in 1967:

**Proposition 2.2.4** (Favrholdt, Stiebitz, Toft: 2006). Let $G$ be a graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta = \Delta(G)$ and let $e \in E_G(x, y)$ be a critical edge of $G$. If $X \subseteq V(G)$ is an elementary set with respect to a coloring $\pi \in C_k(G \setminus e)$ such that both endvertices of $e$ are contained in $X$, then the following statements hold:

\[(a) \ |X| \leq \frac{|\bar{\pi}(X)| - 2}{k - \Delta} \leq \frac{k - 2}{k - \Delta} \text{ provided that } k \geq \Delta + 1,\]

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(b) $\sum_{v \in X} d_G(v) \geq k(|X| - 1) + 2$, and

(c) if $\chi'(G) > \frac{m}{m-1} \Delta + \frac{m-3}{m-1}$ for an integer $m \geq 3$, then $|X| \leq m - 1$ and, moreover, $|\bar{\pi}(X)| \geq \Delta + 1$ provided that $|X| = m - 1$.

We note that Proposition 2.2.4(c) implies that the number of vertices in an elementary set determines to the bound in Theorems 1.6.1-1.6.6 as well as our main results.

2.3 Kierstead Paths

Another method for obtaining an elementary set is due to Kierstead [23]. While Vizing’s fan can be seen as the first step of a Breadth-First Search algorithm, Kierstead’s path can be seen as the first step of a Depth-First Search algorithm.

Let $K = (y_0, e_1, y_1, \ldots, e_n, y_n)$ be a sequence consisting of edges $e_1, \ldots, e_n$ and vertices $y_0, \ldots, y_n$ of $G$. We call $K$ a Kierstead path with respect to an edge $e \in E(G)$ and a coloring $\pi \in \mathcal{C}_k(G \setminus e)$ if the following two conditions are satisfied.

(K1) The vertices $y_0, \ldots, y_n$ are distinct, $e_1 = e$, and $e_i \in E_G(y_i, y_{i-1})$ for $1 \leq i \leq n$.

(K2) For every edge $e_i$ with $2 \leq i \leq n$, there is a vertex $v_j$ with $0 \leq j < i$ such that $\pi(e_i) \in \bar{\pi}(y_j)$.

Please, see Figure 2.2 for an example of a Kierstead path. If $K$ is a Kierstead path, then (K1) implies that the corresponding graph $(V(K), E(K))$ is indeed a path in $G$ with endvertices $y_0$ and $y_n$.

Theorem 2.3.1 (Kierstead [23], 1984). Let $G$ be a graph with $\chi'(G) = k + 1$ for an integer $k > \Delta(G) + 1$, let $e \in E(G)$ be a critical edge of $G$ and $\pi \in \mathcal{C}_k(G \setminus e)$. 18
If \( K = (y_0, e_1, y_1, \ldots, y_{n-1}, e_n, y_n) \) is a Kierstead path with respect to \( e \) and \( \pi \), then \( V(K) \) is elementary with respect to \( \pi \).

Kierstead [23] used Theorem 2.3.1 to give an alternative proof for Vizing’s bound in Theorem 2.2.3 and found some other results in [23].

### 2.4 Tashkinov Trees

Let \( G \) be a graph and let \( e \in E(G) \) be an edge such that \( \chi'(G \setminus e) = k \). By a Tashkinov tree with respect to \( e \) and \( \pi \in C_k(G \setminus e) \) we mean a sequence \( T = (y_0, e_1, y_1, \ldots, e_n, y_n) \) consisting of edges \( e_1, \ldots, e_n \) and vertices \( y_0, \ldots, y_n \) satisfying the following two conditions.

1. (T1) The vertices \( y_0, \ldots, y_n \) are distinct, \( e_1 = e \), and, for \( 1 \leq i \leq n \), there exists \( 0 \leq j < i \) such that \( e_i \in E_G(y_i, y_j) \).

2. (T2) For every edge \( e_i \) with \( 2 \leq i \leq n \), there is a vertex \( y_j \) with \( 0 \leq j < i \) such that \( \pi(e_i) \in \bar{\pi}(y_j) \).

Please, see Figure 2.3 for an example of a Tashkinov tree. If \( T \) is a Tashkinov tree, then (T1) implies that the graph \((V(T), E(T))\) is indeed a tree. If \( F = (e_1, y_1, \ldots, e_n, y_n) \) is a multi-fan at \( x \) with respect to \( e \) and \( \pi \in C_k(G \setminus e) \), then

![Figure 2.2: A Kierstead path at e with respect to some coloring. (α□) represents a color missing at that vertex.](image)
Figure 2.3: A Tashkinov at e with respect to some coloring. (α□) represents a color missing at that vertex.

\[ T = (x, e_1, y_1, \ldots, e_n, y_n) \] is a Tashkinov tree with respect to e and \( \pi \) provided that the vertices \( x, y_1, \ldots, y_n \) are distinct. Furthermore, every Kierstead path with respect to e and \( \pi \in C_k(G \setminus e) \) is a Tashkinov tree with respect to e and \( \pi \).

Let \( T = (y_0, e_1, y_1, \ldots, e_n, y_n) \) be a Tashkinov tree with respect to e and \( \pi \in C_k(G \setminus e) \). We define, for \( 0 \leq i \leq j \leq n \), the section of T between \( y_i \) and \( y_j \) to be \( y_iTy_j = (y_i, e_{i+1}, \ldots, e_j, y_j) \). If \( i = 0 \), then we denote \( Ty_j = y_0Ty_j \) and, if \( j = n \), then we denote \( y_iT = y_iTy_n \). Clearly, \( Ty_i \) is a Tashkinov tree with respect to e and
\( \pi \). Furthermore, \( y_n T = (y_n) \) is a path of \( G \) of length 0. Hence, there is a smallest integer \( i \in \{0, \ldots, n\} \) such that the sequence \( y_i T = (y_i, e_{i+1}, \ldots, e_n, y_n) \) corresponds to a path of \( G \), that is \( e_j \in E_G(y_{j-1}, y_j) \) for \( j = i+1, \ldots, n \). We refer to this number \( i \) as the path number of \( T \) and write \( p(T) = i \). Clearly, if \( p(T) = 0 \), then \( T \) is a Kierstead path with respect to \( e \) and \( \pi \).

We say that a color \( \alpha \) is used on \( T \) with respect to \( \pi \) if \( \pi(e) = \alpha \) for some edge \( e \in E(T) \). Otherwise, we say that \( \alpha \) is unused on \( T \) with respect to \( \pi \). Furthermore, we say that \( (u, v) \) is a \((\gamma, \delta)\)-pair with respect \( \pi \) if \( u, v \in V(G), \gamma \in \pi(u) \) and \( \delta \in \pi(v) \).

The following result is due to Tashkinov [41]. In case of \( \chi'(G) \geq \Delta(G) + 2 \) this result generalizes both Theorem 2.2.1 and Theorem 2.3.1. The proof of this result is similar to the proof of Theorem 2.3.1, but much more sophisticated.

**Theorem 2.4.1** (Tashkinov [41], 2000). *Let \( G \) be a graph with \( \chi'(G) = k + 1 \) for an integer \( k \geq \Delta(G) + 1 \) and let \( e \in E(G) \) be a critical edge of \( G \). If \( T \) is a Tashkinov tree with respect to \( e \) and a coloring \( \pi \in C_k(G \setminus e) \), then \( V(T) \) is elementary with respect to \( \pi \).*

Up to now, the Tashkinov tree seems to be the most general method for obtaining an elementary set. All other known results about elementary sets in a given graph \( G \) are corollaries of Theorem 2.4.1 or small improvements based on Theorem 2.4.1 such that \( \chi'(G) > \Delta(G) + 1 \).

Let \( G \) be a graph and let \( e \in E_G(x, y) \) be an edge such that \( \chi'(G \setminus e) = k \). Moreover, let \( X \subseteq V(G) \) and \( \pi \in C_k(G \setminus e) \). The set \( X \) is called closed with respect to \( \pi \) if for every edge \( f \in E_G(X, V(G) \setminus X) \) the color \( \pi(f) \) is present at every vertex of \( X \), i.e. \( \pi(f) \in \pi(v) \) for every \( v \in X \). Furthermore, the set \( X \) is called strongly closed with respect to \( \pi \) if \( X \) is closed and if \( \pi(f) \neq \pi(f') \) for every two distinct edges \( f, f' \in E_G(X, V(G) \setminus X) \).
In fact, the following is equivalent to Conjecture 1.3.4

**Conjecture 2.4.2.** Let $G$ be a critical graph with $\chi'(G) = k + 1$ where $k > \Delta(G)$. Then there exists $e \in E(G)$, $\pi \in C_k(G \setminus e)$ and $X \subseteq V(G)$ with $V(e) \subseteq X$ such that $X$ is strongly closed and elementary.

To observe that Conjecture 1.3.4 implies Conjecture 2.4.2, it is enough to assume that $G$ is critical with $\chi'(G) > k + 1$ for an integer $k \geq \Delta(G) + 1$. If we assume Conjecture 1.3.4, then $\chi' = \lceil \chi'' \rceil = \left\lceil \frac{2|E(S)|}{|S| - 1} \right\rceil$ for some odd vertex set $S$ with $|S| \geq 3$. We must have $|E(S)| \geq \left( \frac{|S| - 1}{2} \right) (\chi'' - 1) + 1$. If equality does not hold, then, for $e \in E(S)$, deletion of $e$ does not affect the inequality, which implies that $G$ is not critical. Hence, $|E(S)| = \left( \frac{|S| - 1}{2} \right) (\chi'' - 1) + 1$ must hold. Let $e \in E(G)$ and let $\pi \in C_k(G \setminus e)$. The restriction of each color class of $\pi$ to $E(S) \setminus e$ must be a matching of size $\frac{|S| - 1}{2}$, which implies that $S$ is strongly closed and elementary with respect to $e$ and $\pi$. Hence, Conjecture 1.3.4 implies Conjecture 2.4.2. Conversely, if $X$ is strongly closed and elementary with respect to some $e$ and $\pi \in C_k(G \setminus e)$, then $|E(X)| = \left( \frac{|X| - 1}{2} \right) (\chi'' - 1) + 1$ since each color class in $E(X) \setminus e$ is a matching of size $\frac{|X| - 1}{2}$ and $e$ is not counted. But this implies that $\chi''(G) > \chi'(G) - 1$ as in Conjecture 1.3.4.

Let $G$ be a graph and let $H$ be its underlying simple graph, that is $V(H) = V(G)$ and, for $x, y \in V(H)$, $\mu_H(x, y) = 1$ iff $\mu_G(x, y) \geq 1$. The girth of $G$ is the smallest number of edges in a cycle of $H$ provided $H$ has a cycle. Otherwise, the girth of $G$ is infinite. The odd girth of $G$ is defined similarly; where cycle is replaced by odd cycle, that is a cycle having an odd number of edges. The number of edges of a cycle is also called its length.

**Theorem 2.4.3** (Tashkinov [41], 2000). Let $G$ be a graph with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$, let $e \in E(G)$ be a critical edge of $G$, and let $\pi \in C_k(G \setminus e)$ be a
coloring. Moreover, let $T$ be a maximal Tashkinov tree with respect to $e$ and $\pi$ and let $T' = (y_0, e_1, y_1, \ldots, e_n, y_n)$ be an arbitrary Tashkinov tree with respect to $e$ and $\pi$. Then the following statements hold:

(a) $V(T)$ is elementary and closed with respect to $\pi$.

(b) $|V(T)| \equiv 1 \mod 2$ and $\delta(G[V(T)]) \geq (|V(T)| - 1)(\chi'(G) - \Delta(G) - 1) + 2$.

(c) $V(T') \subseteq V(T)$.

(d) There is a Tashkinov tree $\tilde{T}$ with respect to $e$ and $\pi$ satisfying $V(\tilde{T}) = V(T)$ and $\tilde{T}y_n = T'$.

(e) Suppose that $(y_i, y_j)$ is a $(\gamma, \delta)$-pair with respect to $\pi$ where $1 \leq i < j \leq n$. Then $\gamma \neq \delta$ and there is a $(\gamma, \delta)$-chain $P$ with respect to $\pi$ satisfying the following conditions:

1. $P$ is a path with endvertices $y_i$ and $y_j$.
2. $|E(P)|$ is even.
3. $V(P) \subseteq V(T)$.
4. If $\gamma$ is unused on $T'y_j$, then $T'$ is a Tashkinov tree with respect to the edge $e$ and the coloring $\pi' \in C_k(G \setminus e)$ obtained from $\pi$ by recoloring the $(\gamma, \delta)$-chain $P$.

(f) $G[V(T)]$ contains an odd cycle.

One advantage of Tashkinov’s method is that the vertex set of a maximal Tashkinov tree is not only elementary, but also closed. However, as stated in Conjecture 2.4.2, in order to prove Conjecture 1.3.4, we need an elementary set that is not only closed, but also strongly closed. But even for the vertex set $X$ of a maximum Tashkinov tree, we are not able to prove that $X$ is always strongly closed. Hence, it is
natural to search for a way to extend $X$ to a larger elementary set provided that $X$

is not strongly closed. One such way will be discussed in Chapters 3 and 4.

For the rest of this chapter, let $G$ be a critical graph such that $\chi'(G) = k + 1$

for a given integer $k \geq \Delta(G) + 1$. Since $G$ is critical, for every edge $e \in E(G)$ and

every coloring $\pi \in C_k(G \setminus e)$, there is a Tashkinov tree $T$ with respect to $e$ and $\pi$.

Hence there is a largest number $n$ such that $n = |V(T)|$ for such a Tashkinov tree $T$.

We call $n$ the Tashkinov order of $G$ and write $t(G) = n$. Furthermore, we denote

by $T(G)$ the set of all triples $(T, e, \pi)$ such that $e \in E(G)$, $\pi \in C_k(G \setminus e)$, and $T$ is a

Tashkinov tree on $n$ vertices with respect to $e$ and $\pi$. Evidently, $T(G) \neq \emptyset$.

For a triple $(T, e, \pi) \in T(G)$ we introduce the following notation. For a color

$\alpha \in \{1, \ldots, k\}$, let $E_\alpha(e, \pi) = \{e' \in E(G) - \{e\} \mid \pi(e') = \alpha\}$; further, let

$$E_\alpha(T, e, \pi) = E_\alpha(e, \pi) \cap E_G(V(T), V(G) \setminus V(T))$$

and, for non-empty sets $X, Y \subseteq V(G)$, let

$$E_\alpha(X, Y) = E_G(X, Y) \cap E_\alpha(e, \pi).$$

The color $\alpha$ is said to be defective with respect to $(T, e, \pi)$ if $|E_\alpha(T, e, \pi)| \geq 2$. The set

of all defective colors with respect to $(T, e, \pi)$ is denoted by $\Gamma^d(T, e, \pi)$. Furthermore,

the color $\alpha$ is said to be free with respect to $(T, e, \pi)$ if $\alpha \in \bar{\pi}(V(T))$ and $\alpha$ is unused

on $T$ with respect to $\pi$. The set of all free colors with respect to $(T, e, \pi)$ is denoted

by $\Gamma^f(T, e, \pi)$.

**Proposition 2.4.4** (Favrholdt, Stiebitz, Toft [11], 2006). Let $G$ be a critical graph

with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$ and let $(T, e, \pi) \in T(G)$. Then the following

statements hold:

(a) $|V(T)| = t(G) \geq 3$ and is odd.

(b) $V(T)$ is elementary and closed with respect to $\pi$. 

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(c) $V(T)$ is strongly closed with respect to $\pi$ iff $\Gamma^d(T,e,\pi) = \emptyset$.

(d) If $\alpha \in \bar{\pi}(V(T))$, then $E_\alpha(T,e,\pi) = \emptyset$.

(e) If $\alpha \in \Gamma^d(T,e,\pi)$, then $|E_\alpha(T,e,\pi)| \geq 3$ and is odd.

(f) For a vertex $x \in V(T)$, we have $|\bar{\pi}(x)| = k - d_G(x) + 1 \geq 2$ if $e \in E_G(x)$ and $|\bar{\pi}(x)| = k - d_G(x) \geq 1$ otherwise.

(g) Every color in $\Gamma^d(T,e,\pi) \cup \Gamma^f(T,e,\pi)$ is unused on $T$ with respect to $\pi$. Moreover, $|\Gamma^f(T,e,\pi)| \geq 4$.

The following results are also due to the research done by Favrholdt, Stiebitz and Toft:

**Lemma 2.4.5** (Favrholdt, Stiebitz, Toft [11], 2006). Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T,e,\varphi) \in \mathcal{T}(G)$. If $\alpha \in \bar{\varphi}(x) \cap \Gamma^f(T,e,\varphi)$ for some $x \in V(T)$ and $\delta \notin \bar{\varphi}(T)$ then $E_\delta(T,e,\varphi) \subseteq P_x(\alpha,\delta,\varphi)$.

**Proposition 2.4.6** (Favrholdt, Stiebitz, Toft [11], 2006). Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T,e,\varphi) \in \mathcal{T}(G)$. If $\bar{\varphi}(x) \cap \Gamma^f(T,e,\varphi) \neq \emptyset$ for all $x \in V(T)$ then $G$ is elementary.

### 2.5 Absorbing Sets

In [11], Favrholdt et. al. defines absorbing vertices. The reader should note that, while more general, the idea of the absorbing vertices in this section are similar to theirs.

For $\alpha \in \bar{\pi}(T)$, a vertex $x \in V(G) \setminus V(T)$ is called $\alpha$-absorbing if, for all $\beta \in \bar{\pi}(x)$, $P_x(\alpha,\beta,\pi)$ ends in $T$. If $C \subseteq \bar{\pi}(T)$, then $x$ is called $C$-absorbing if, for all $\alpha \in C$, $x$ is $\alpha$-absorbing and absorbing if $C = \bar{\pi}(T)$. The set of all $C$-absorbing vertices with
respect to $\pi$ will be denoted by $A_C(T, e, \pi)$. We define $A(T, e, \pi) = A_{\pi}(T, e, \pi)$ and $A_f(T, e, \pi) = A_{\Gamma_f(T, e, \pi)}$.

The following lemmas are useful both in this section and later.

**Lemma 2.5.1.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T, e, \pi) \in \mathcal{T}(G)$. For any $\alpha \in \bar{\pi}(T)$ and $\beta \notin \bar{\pi}(T)$, there is a unique acyclic $(\alpha, \beta)$-chain $P^\alpha(\alpha, \beta, \pi)$ intersecting $T$.

*Proof.* Note that, if $(T, e, \pi)$ is maximum and $\beta \notin \Gamma^d(T, e, \pi)$, then the lemma trivially follows. Hence, we may assume that $\beta \in \Gamma^d(T, e, \pi)$. Moreover, if $\alpha \in \Gamma_f(T, e, \pi)$, then $E_\beta(T, e, \pi) \subseteq E(P^\alpha)$ by Lemma 2.4.5 and hence the proof follows. We next assume that $(T, e, \pi) \in \mathcal{T}(G)$ and $\alpha \notin \Gamma_f(T, e, \pi)$. But then, for some $\varepsilon \in \Gamma_f(T, e, \pi)$, we may re-color all $(\alpha, \varepsilon)$-chains within $T$ and get a coloring $\pi'$ such that $(T, e, \pi') \in \mathcal{T}(G)$ and $\alpha \in \Gamma_f(T, e, \pi')$. As was shown before, there is a unique $(\alpha, \beta)$-path that starts in $T$ but does not end in $T$ with respect to $\pi'$. This still holds with respect to $\pi$ since re-coloring all $(\alpha, \varepsilon)$-chains within $V(T)$ effect only the edges in $G[V(T)]$. This implies that there exists a unique acyclic $(\alpha, \beta)$-chain intersecting $T$ with respect to $\pi$ since the $(\alpha, \beta)$-path leaving $T$ can not return to $T$. \qed

Let $(T, e, \pi)$ be a maximal Tashkinov triple. It is called *pseudo-maximum* if for any $\alpha \in \bar{\pi}(T)$, $\beta \notin \bar{\pi}(T)$ and $(\alpha, \beta)$-chain $P$, $V(T)$ is not a proper subset of $V(T')$ where $(T', e, \pi')$ is the maximal Tashkinov triple obtained by re-coloring $P$.

**Lemma 2.5.2.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T, e, \pi)$ be a pseudo-maximum Tashkinov triple. Then, for any $\alpha \in \bar{\pi}(T)$ and $\beta \notin \bar{\pi}(T)$, there is a unique acyclic $(\alpha, \beta)$-chain $P^\alpha(\alpha, \beta, \pi)$ intersecting $T$.

*Proof.* We may assume $\beta \in \Gamma^d(T, e, \pi)$. We first let $\alpha \in \Gamma_f(T, e, \pi)$, say $\alpha \in \bar{\pi}(x)$ for some $x \in V(T)$. If $E_\beta(T, e, \pi) \setminus P_x(\alpha, \beta, \pi) \neq \emptyset$, by re-coloring $P_x$, we get a Tashkinov
triple \((T', e, \pi')\) such that \(V(T) \subset V(T')\) contrary to the fact that \((T, e, \pi)\) is pseudo-maximum. Hence, there is a unique acyclic \((\alpha, \beta)\)-chain intersecting \(T\) with respect to \(\pi\).

We next assume \(\alpha \notin \Gamma^f(T, e, \pi)\). Then we pick \(\varepsilon \in \Gamma^f(T, e, \pi)\) and re-color all \((\alpha, \varepsilon)\)-chains within \(T\). If we call the new coloring \(\pi'\), then \((T, e, \pi')\) is a pseudo-maximum Tashkinov triple and \(\alpha \in \Gamma^f(T, e, \pi)\), implying that there is a unique acyclic \((\alpha, \beta)\)-chain \(P\) intersecting \(T\) with respect to \(\pi'\). But then there must be a unique acyclic \((\alpha, \beta)\)-chain intersecting \(T\) with respect to \(\pi\). This completes the proof. \(\square\)

The following is the most general information about the nature of chains intersecting maximal Tashkinov trees.

**Lemma 2.5.3.** Let \(G\) be critical with \(\chi'(G) > \Delta(G) + 1\) and \((T, e, \pi)\) be a maximal Tashkinov triple. Then, for any \(\alpha \in \bar{\pi}(T)\) and \(\beta \notin \bar{\pi}(T)\), there is a unique acyclic \((\alpha, \beta)\)-chain \(P^a(\alpha, \beta, \pi)\) intersecting \(T\).

**Proof.** We may assume \(\beta \in \Gamma^d(T, e, \pi)\). We first let \(\alpha \in \Gamma^f(T, e, \pi)\); say \(\alpha \in \bar{\pi}(x)\) for some \(x \in V(T)\). If \(E_\beta(T, e, \pi) \subseteq P_x(\alpha, \beta, \pi)\), then there exists a unique acyclic \((\alpha, \beta)\)-chain intersecting \(T\). Hence, we assume \(E_\beta \setminus P_x \neq \emptyset\). Re-coloring \(P_x\), we get a coloring \(\pi_1\) such that \((T_1, e, \pi_1)\) is a maximal Tashkinov triple with \(V(T) \subset V(T')\). If \(|E_\alpha(T_1, e, \pi_1)| = 1\), then \(V(P') \subseteq V(T')\) for any other \((\alpha, \beta)\)-chain \(P'\) intersecting \(T\) with respect to \(\pi\). But this implies that \(P'\) is a cyclic chain since \(T_1\) is elementary with respect to \(\pi_1\). Hence, we may assume \(\alpha \in \Gamma^d(T_1, e, \pi_1)\). Let \(\varepsilon \in \Gamma^f(T_1, e, \pi_1)\). Re-coloring all \((\varepsilon, \beta)\)-chains within \(T_1\), we get a coloring \(\pi'_1\) such that \(\beta \in \Gamma^f(T_1, e, \pi'_1)\); say \(\beta \in \bar{\pi}'_1(x_1)\) for some \(x_1 \in V(T_1)\). Repeating the same argument, either we get a maximal Tashkinov triple \((T_2, e, \pi_2)\) such that \(V(T_1) \subset V(T_2)\) and \(\beta \in \Gamma^d(T_2, e, \pi_2)\) or there exists a unique acyclic \((\alpha, \beta)\)-chain intersecting \(T_1\) with
respect to $\pi'_1$, and, hence, $\pi_1$. This second case also implies that there is a unique acyclic $(\alpha, \beta)$-chain intersecting $T$ since only $(\alpha, \beta)$-chains outside $T_1$ are exchanged during the re-colorings. Repeating this process, we either get a pseudo-maximum Tashkinov triple, in which case, the proof follows by Lemma 2.5.2, or complete the proof.

Next, we assume $\alpha \notin \Gamma^f(T, e, \pi)$. As before, we pick $\gamma \in \Gamma^f(T, e, \pi)$ and re-color all $(\alpha, \gamma)$-chains within $V(T)$ to get the previous case. □

**Lemma 2.5.4.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T, e, \pi)$ be a maximal Tashkinov triple. Then $A_C(T, e, \pi) \cup V(T)$ is elementary with respect to $\pi$ for all non-empty $C \subseteq \overline{\pi}(T)$.

**Proof.** We assume, on the contrary that $X = A_C(T, e, \pi) \cup V(T)$ is not elementary. Let $x, y \in X$ be distinct and $\alpha \in \overline{\pi}(x) \cap \overline{\pi}(y) \neq \emptyset$. If $x, y \in A_C(T, e, \pi)$ then for all $\beta \in C$, $P_x(\alpha, \beta, \pi)$ and $P_y(\alpha, \beta, \pi)$ both end in $T$, contrary to Lemma 2.5.3. Hence, we may assume that $x \in A_C(T, e, \pi)$ and $y \in V(T)$. If $\alpha \in C$ then $P_y(\alpha, \alpha, \pi) = (y)$ does not end in $T$, contrary to the definition of $A_C$. If, on the other hand, $\alpha \notin C$, then for all $\beta \in C$, $P_y(\alpha, \beta, \pi)$ does not visit $T$ since $V(T)$ is closed with respect to $\pi$. This is a contradiction and the proof follows.

□

The following lemma of Favrholdt et. al. directly follows from Lemma 2.5.4:

**Lemma 2.5.5** (Favrholdt, Stiebitz, Toft [11], 2006). Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$, $(T, e, \pi) \in \mathcal{T}(G)$. Then $A^f(T, e, \pi) \cup V(T)$ is elementary.

The following lemmas are simple observations about absorbing sets and will be stated without any proof:

**Lemma 2.5.6.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T, e, \pi)$ be a maximal Tashkinov triple. If $C_1 \subseteq C_2 \subseteq \overline{\pi}(T)$ then $A_{C_1}(T, e, \pi) \supseteq A_{C_2}(T, e, \pi)$. 28
Lemma 2.5.7. Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T, e, \pi)$ be a maximal Tashkinov triple. If $C_1, C_2 \subseteq \pi(T)$ with $C_1 \cap C_2 \neq \emptyset$ then $A_{C_1}(T, e, \pi) \cup A_{C_2}(T, e, \pi) \cup V(T)$ is elementary with respect to $\pi$.

Let $P$ be a path and let $u, v$ be two vertices of $P$. Then there is a unique subpath $P'$ of $P$ having $u$ and $v$ as endvertices. We denote this subpath by $uPv$ or $vPu$. If we fix an endvertex of $P$, say $u$, then we obtain a linear order $\preceq_{(u,P)}$ of the vertex set of $P$ in a natural way, where $x \preceq_{(u,P)} y$ if the vertex $x$ belongs to the subpath $uPy$.

The next two propositions we learned from Tashkinov [41] and Favrholdt et. al. [11], respectively.

Proposition 2.5.8 (Tashkinov [41], 2000). Let $G$ be a critical graph with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$ and let $(T, e, \pi) \in T(G)$. Let $\alpha \in \Gamma^d(T, e, \pi)$ be a defective color and let $u$ be a vertex of $T$ such that $\bar{\pi}(u)$ contains a free color $\gamma \in \Gamma^f(T, e, \pi)$. Then, for the $(\alpha, \gamma)$-chain $P = P_u(\alpha, \gamma, \pi)$, the following statements hold:

(a) $P$ is a path where one endvertex is $u$ and the other endvertex is some vertex $v \in V(G) \setminus V(T)$.

(b) $E_{\alpha}(T, e, \pi) = E(P) \cap E_G(V(T), V(G) \setminus V(T))$.

(c) In the linear order $\preceq_{(u,P)}$ there is a first vertex $v^1$ that belongs to $V(G) \setminus V(T)$ and there is a last vertex $v^2$ that belongs to $V(T)$ where $v^1 \preceq_{(u,P)} v^2$.

(d) $\bar{\pi}(v^2) \cap \Gamma^f(T, e, \pi) = \emptyset$.

(e) $V(T) \cup \{v^1\}$ is elementary with respect to $\pi$.

Proposition 2.5.9 (Favrholdt, Stiebitz, Toft [11], 2006). Let $G$ be a critical graph with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$ and let $(T, e, \pi) \in T(G)$. Let $u$ be a vertex of $T$ such that $\bar{\pi}(u)$ contains a free color $\gamma \in \Gamma^f(T, e, \pi)$ and let $\delta$ be an arbitrary color that is distinct from $\gamma$. Then the following statements hold:

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(a) $E_\gamma(T,e,\pi) = \emptyset$.

(b) $P_u(\gamma, \delta)$ contains all edges from $E_\delta(T,e,\pi)$

(c) If $\delta \in \bar{\pi}(v)$ for some vertex $v \in V(T) \cup A^f(T,e,\pi)$, then $v$ belongs to $P_u(\gamma, \delta)$.

(d) If $P$ is a $(\gamma, \delta)$-chain with respect to $\pi$ that is distinct from $P_u(\gamma, \delta)$, then every endvertex of $P$ belongs to $V(G) \setminus V(T) \setminus A^f(T,e,\pi)$.

(e) If $P$ is a $(\gamma, \delta)$-chain with respect to $\pi$ such that $P$ is a path whose endvertices belong to $V(G) \setminus V(T)$, then $P$ and $T$ are vertex disjoint.

We call $v \in V(G)$ a defective vertex with respect to $(T,e,\pi) \in \mathcal{T}(G)$ if there are two distinct colors $\alpha$ and $\gamma$ such that $\alpha \in \Gamma^d(T,e,\pi)$ is a defective color, $\gamma \in \Gamma^f(T,e,\pi)$ is a free color, and $v$ is the first vertex in the linear order $\preceq_{(u,P)}$ that belongs to $V(G) \setminus V(T)$, where $u$ is the unique vertex in $T$ with $\gamma \in \bar{\pi}(u)$ and $P = P_u(\alpha, \gamma)$. The set of all defective vertices with respect to $(T,e,\pi)$ is denoted by $D(T,e,\pi)$. Favrholdt et. al. proved the following result:

**Proposition 2.5.10** (Favrholdt, Stiebitz, Toft [11], 2006). Let $G$ be a critical graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$ and let $(T,e,\pi)$. Then $D(T,e,\pi) \subseteq A^f(T,e,\pi)$.

We have the following improvement to Proposition 2.5.10. In Chapter 3, we prove (Corollary 3.1.7) the existence of a more general elementary set than $D(T,e,\pi)$ on the boundary of $T$:

**Proposition 2.5.11.** Let $G$ be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$ and let $(T,e,\pi) \in \mathcal{T}(G)$. Then $D(T,e,\pi) \subseteq A(T,e,\pi)$.

**Proof.** By Proposition 2.5.10, $V(T) \cup D(T,e,\pi)$ is elementary with respect to $\pi$ and $D(T,e,\pi) \subseteq A^f(T,e,\pi)$. By the definition of $D$, there exists $z \in V(T)$ with $\gamma \in$
\( \bar{\pi}(z) \cap \Gamma^f(T, e, \pi) \) and \( \delta \in \Gamma^d(T, e, \pi) \) such that \( x \) is the first vertex outside \( T \) that is visited by \( P = P_z(\gamma, \delta, \pi) \). For any \( \beta \in \bar{\pi}(T) \) \( \setminus \) \( \Gamma^f(T, e, \pi) \), re-coloring the union of all \( (\gamma, \beta) \)-chains in \( G[V(T)] \) we get a coloring \( \pi_\beta \) such that \( (T, e, \pi_\beta) \in T(G) \), \( \beta \in \Gamma^f(T, e, \pi_\beta) \) and \( x \in D(T, e, \pi_\beta) \) since \( \beta \in \bar{\pi}_\beta(z) \) and \( P_z(\beta, \delta, \pi_\beta)x = P_z(\gamma, \delta, \pi)x \). Since \( D(T, e, \pi) = D(T, e, \pi_\beta) \subseteq A^f(T, e, \pi_\beta) \), we have \( D(T, e, \pi_\beta) = D(T, e, \pi) \) is a set of \( \Gamma^f(T, e, \pi) \cup \Gamma^f(T, e, \pi_\beta) \)-absorbing vertices. Since \( \beta \) was arbitrarily chosen, this implies \( D(T, e, \pi) \) is a set of absorbing vertices, implying \( D(T, e, \pi) \subseteq A(T, e, \pi) \). \( \square \)

The next few results are due to Favrholdt et. al.

**Proposition 2.5.12** (Favrholdt, Stiebitz, Toft [11], 2006). Let \( G \) be a critical graph with \( \chi'(G) = k + 1 \) where \( k \geq \Delta(G) + 1 \) and let \( (T, e, \pi) \in T(G) \). Let \( P \) be a \((\gamma, \delta)\)-chain with respect to \( \pi \) where \( \gamma \in \Gamma^f(T, e, \pi) \) is a free color and \( \delta \) is an arbitrary color distinct from \( \gamma \). Suppose that \( P \) is a path and every endvertex of \( P \) belongs to \( V(G) \setminus V(T) \). Then, for the coloring \( \pi_1 = \pi/P \), the following statements hold:

(a) \((T, e, \pi_1) \in T(G)\).

(b) \( \bar{\pi}_1(V(T)) = \bar{\pi}(V(T)) \) and \( \Gamma^f(T, e, \pi_1) = \Gamma^f(T, e, \pi) \).

(c) \( \Gamma^d(T, e, \pi_1) = \Gamma^d(T, e, \pi) \) and \( E_\alpha(T, e, \pi_1) = E_\alpha(T, e, \pi) \) for every defective color \( \alpha \in \Gamma^d(T, e, \pi) \).

(d) \( D(T, e, \pi_1) = D(T, e, \pi) \).

**Proposition 2.5.13** (Favrholdt, Stiebitz, Toft [11], 2006). Let \( G \) be a critical graph with \( \chi'(G) = k + 1 \) where \( k \geq \Delta(G) + 1 \) and let \( (T, e, \pi) \in T(G) \). Furthermore, let \( Y \subseteq D(T, e, \pi) \) and \( Z = V(T) \cup Y \). Suppose that \( e' \in E_G(z, v) \) is an edge such that \( z \in Z \), \( v \in V(G) \setminus Z \), and, \( \pi(e') \in \bar{\pi}(Z) \). Then the vertex set \( Z \cup \{v\} \) is elementary with respect to \( \pi \).
Consider a triple \((T, e, \pi) \in T(G)\) and a set \(Z \subseteq V(G)\) that contains all vertices of \(T\). Furthermore, let \(F = (e_1, u_1, \ldots, e_p, u_p)\) be a sequence consisting of distinct edges \(e_1, \ldots, e_p \in E(G)\) and distinct vertices \(u_1, \ldots, u_p \in V(G)\). The sequence \(F\) is called a \textit{fan} at \(Z\) with respect to \(\pi\) if for every \(i \in \{1, \ldots, p\}\) there are two vertices \(z, z'\) satisfying \(z \in Z, z' \in Z \cup \{u_1, \ldots, u_{i-1}\}, e_i \in E_G(z, u_i),\) and \(\pi(e_i) \in \bar{\pi}(z')\).

**Theorem 2.5.14** (Favrholdt, Stiebitz, Toft [11], 2006). Let \(G\) be a critical graph with \(\chi'(G) = k + 1\) where \(k \geq \Delta(G) + 1\) and let \((T, e, \pi) \in T(G)\). Furthermore, let \(Y \subseteq D(T, e, \pi)\) and \(Z = V(T) \cup Y\). If \(F\) is a fan at \(Z\) with respect to \(\pi\), then \(Z \cup V(F)\) is elementary with respect to \(\pi\).

We note that Theorem 2.5.14 implies the elementariness part of Proposition 3.2.1. We will give our own proof of this result in Chapter 3.

### 2.6 Balanced Tashkinov Trees

Let \(G\) be a critical graph with \(\chi'(G) = k + 1\) and \(k \geq \Delta + 1\) and \(\pi \in C_k(G \setminus e)\). There are more than one maximal Tashkinov trees with respect to \(e, \pi\). However, for any such trees \(T, T'\), we have \(V(T) = V(T')\). While it is best to find a way to use all possible trees, it is also important to understand the size and nature of Tashkinov trees through the use of some graph invariants related to Tashkinov trees such as the Tashkinov order, \(t(G)\).

Consider an arbitrary triple \((T, e, \pi) \in T(G)\). Let \(\gamma \in \bar{\pi}(u)\) for a vertex \(u \in V(T)\) and let \(\delta \in \Gamma^d(T, e, \pi)\). Clearly, the \((\gamma, \delta)\)-chain \(P = P_u(\gamma, \delta, \pi)\) is a path where \(u\) is one endvertex of \(P\) and, moreover, exactly one of the two colors \(\gamma\) or \(\delta\) is missing at the second endvertex of \(P\) with respect to \(\pi\). Since \(V(T)\) is elementary and \(\delta\) is present at every vertex in \(V(T)\) with respect to \(\pi\), the second endvertex of \(P\) belongs to \(V(G) \setminus V(T)\). Hence, in the linear order \(\preceq_{(u, P)}\), there is a last vertex \(v\) that belongs
to $V(T)$. This vertex is said to be an exit vertex with respect to $(T, e, \pi)$. The set of all exit vertices with respect to $(T, e, \pi)$ is denoted by $F(T, e, \pi)$. The following lemmas are due to Scheide and Stiebitz:

**Lemma 2.6.1** (Scheide, Stiebitz [35], 2009). Let $G$ be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \pi) \in \mathcal{T}(G)$. Then $\bar{\pi}(F(T, e, \pi)) \cap \Gamma^f(T, e, \pi) = \emptyset$.

Let $(T, e, \pi) \in \mathcal{T}(G)$. Then $T$ has the form

$$T = (y_0, e_1, y_1, \ldots, e_{n-1}, y_{n-1})$$

where $n = t(G)$. Then $T$ is called a normal Tashkinov tree with respect to $e$ and $\pi$ if and only if there are two colors $\alpha \in \bar{\pi}(y_0)$ and $\beta \in \bar{\pi}(y_1)$, an integer $2 \leq p \leq n - 1$ and an edge $f \in E_G(y_0, y_{p-1})$ such that $P(y_1, e_2, y_2, \ldots, y_{p-1}, f, y_0)$ is an $(\alpha, \beta)$-chain with respect to $\pi$. In this case, $Ty_{p-1}$ is called the $(\alpha, \beta)$-trunk of $T$ and the number $p$ is called the height of $T$, denoted by $h(T) = p$. Furthermore, let $\mathcal{T}^N(G)$ denote the set of all triples $(T, e, \pi) \in \mathcal{T}(G)$ for which $T$ is a normal Tashkinov tree, and $h(G)$ denote the greatest number $p$ such that there is a triple $(T, e, \pi) \in \mathcal{T}^N(G)$ with $h(T) = p$. The following lemma shows that normal Tashkinov trees can be generated from arbitrary ones, which implies that $\mathcal{T}^N(G) \neq \emptyset$.

**Lemma 2.6.2** (Scheide, Stiebitz [35], 2009). Let $G$ be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, and let $(T, e, \pi) \in \mathcal{T}(G)$ with $e \in E_G(x, y)$. Then there are two colors $\alpha \in \bar{\pi}(x)$ and $\beta \in \bar{\pi}(y)$, and there is a Tashkinov tree $T'$ with respect to $e$ and $\pi$ satisfying $V(T') = V(T)$, $(T', e, \pi) \in \mathcal{T}^N(G)$ and $h(T') = |V(P)|$ where $P = P_e(\alpha, \beta, \pi)$.

Next, consider an arbitrary triple $(T, e, \pi) \in \mathcal{T}^N(G)$ where $T$ has the form $T = (y_0, e_1, y_1, \ldots, e_{n-1}, y_{n-1})$. Then $(T, e, \pi)$ is called a balanced triple with respect to $e$. 33
and π if and only if $h(T) = h(G)$ and $\pi(e_{2j}) = \pi(e_{2j-1})$ for $p < 2j < n$. Let $T^B(G)$ denote the set of all balanced triples $(T, e, \pi) \in T(G)$. The following lemma shows that $T^B(G) \neq \emptyset$.

**Lemma 2.6.3** (Scheide, Stiebitz [35], 2009). Let $G$ be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. Then the following statements hold:

1. $h(G) \geq 3$ is odd.

2. If $(T, e, \pi) \in T^N(G)$ with $h(T) = h(G)$, then there is a Tashkinov tree $T'$ with respect to $e$ and $\pi$ satisfying $V(T') = V(T)$ and $(T', e, \pi) \in T^B(G)$. Moreover, all colors used on $T'$ are also used on $T$.

3. $T^B(G) \neq \emptyset$.

The following lemma, while proven by Scheide and Stiebitz too, is ingrained in the proofs of Shannon [39] and Gupta [19].

**Lemma 2.6.4.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$. If $h(G) = 3$ then $G$ is elementary.

We note that we have proven the following lemma as a result of a structure theorem. The proof will not be included in this thesis and will be published later in [24].

**Lemma 2.6.5.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$. If $h(G) = 5$ then $G$ is elementary.

We also note that, when $h(G) > 5$, the structure of a Tashkinov tree gets more complicated. This suggests that a further improvement to this bound will be rather tricky.
Consider a graph $G$ and a balanced triple $(T, e, \pi) \in \mathcal{T}^B(G)$. Then $T$ has the form
\[ T = (y_0, e_1, y_1, \ldots, e_{n-1}, y_{n-1}) \]
and $T_{y_{p-1}}$ is the $(\alpha, \beta)$-trunk of $T$, where $p = h(G)$, $\alpha \in \bar{\pi}(y_0)$ and $\beta \in \bar{\pi}(y_1)$. Moreover, there is an edge $f_p \in E_G(y_0, y_{p-1})$ with $\pi(f_p) = \beta$. For $i = 1, \ldots, p - 1$, let $f_i = e_i$. Clearly, the edges $f_1, \ldots, f_p$ form a cycle in $G$. Furthermore, the edge $f_1 = e$ is uncolored and the edges $f_2, \ldots, f_p$ are colored alternately with $\alpha, \beta$ with respect to $\pi$. Now, choose a $j \in \{1, \ldots, p - 1\}$. Since $(y_0, y_1)$ is an $(\alpha, \beta)$-pair with respect to $\pi$, there is a coloring $\pi' \in \mathcal{C}_k(G - f_{j+1})$ such that $\pi'(e') = \pi(e')$ for all $e' \in E(G) \setminus \{f_1, \ldots, f_p\}$ and the edges $f_{j+2}, \ldots, f_p, f_1, \ldots, f_j$ are colored alternately with $\alpha$ and $\beta$ with respect to $\pi'$. Then
\[ T' = (y_j T y_{p-1}, f_p, y_0 T y_{j-1}, e_p, y_p, \ldots, e_{n-1}, y_{n-1}) \]
is a normal Tashkinov tree where $T'_{y_{j-1}}$ is the $(\alpha, \beta)$-trunk of $T'$ and, moreover, the triple $(T', f_{j+1}, \pi')$ is balanced. We finish this section by stating the following important result by Scheide [35].

**Theorem 2.6.6** (Scheide, Stiebitz [35], 2009). Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$. If $t(G) < 11$ then $G$ is elementary.

In [24], we make the following improvement to Theorem 2.6.6. Its proof will not be included in this thesis since it is rather long and we think that it is not essential to proving Conjecture 1.3.4. To give the idea behind the proof, we note that we use some general methods together with some structural lemmas while making use of the structural lemma proven by Scheide and Stiebitz in [35]. Some of these tools will be introduced in Chapter 5.

**Theorem 2.6.7.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$. If $t(G) < 13$ then $G$ is elementary.
We note that as Theorem 2.6.7 will not be used in the proof of Theorem 1.6.9, it is not proven here.
Throughout this chapter, we let $G$ be a critical graph with $\chi'(G) = k + 1$ and $k \geq \Delta(G) + 1$. It is shown that $V(T)$ is closed and elementary with respect to $\pi$. While we do not know whether $V(T)$ is strongly elementary or strongly closed, it is closest known result to both. Hence, one can treat it like the starting point of a fan, path, tree or some other structures. Throughout this chapter, we will give various formulations of Conjecture 1.3.4 and give results that are very important, if not essential, in proving it.

We also have some understanding of the chains in $G \setminus e$ under $\pi$. If $\alpha, \beta \in \pi(T)$, then any $(\alpha, \beta)$-chain is either in $G[V(T)]$ or outside since $V(T)$ is closed with respect to $\pi$. Similarly, if $\alpha \in \pi(T)$ and $\beta \notin \pi(T)$ then there exists a unique acyclic $(\alpha, \beta)$-chain that intersects $T$ by Lemma 2.5.3. These facts will be used repeatedly throughout this chapter without any reference. On the other hand, we do not have any knowledge of the behavior of the $(\alpha, \beta)$-chains if $\alpha, \beta \notin \pi(T)$ and at least one of them is in $\Gamma^d(T, e, \pi)$. Hence, we will try to abstain from using any color interchange that involves such a chain unless we are certain that it does not effect the structures defined in this chapter.

Absorbing sets defined in Section 2.5 will also be employed in giving and simplifying proofs thanks to Propositions 2.5.4, 2.5.6 and 2.5.7.
3.1 Handles of a Tashkinov Tree

In the well-known Ear or Handle Lemma for 2-connected graphs, one can generate a 2-connected graph by recursively adding handles to a cycle. The following proposition suggests that one can try to add handles to a cycle one of whose edges is the uncolored edge recursively to generate a strongly closed, elementary graph; namely, to prove Conjecture 1.3.4.

Proposition 3.1.1. Let $G$ be critical with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. If $G$ is elementary then there exists a triple $(X, e, \pi)$ such that $e \in E(G)$, $\pi \in C_k(G\backslash e)$ and $V(e) \subset X \subseteq V(G)$ satisfying the following:

(1) $X$ is strongly closed and elementary with respect to $\pi$.

(2) $X$ is minimal under set inclusion.

(3) There exists an integer $r \geq 0$ and, for $0 \leq i \leq r$, there exist colors $\alpha_i, \beta_i$, vertices $x_i, y_i$ and $(\alpha_i, \beta_i)$-alternating paths $P_i = P_i(x_i, y_i)$ such that

(i) $e \in E_G(x_0, y_0)$, $\alpha_0 \in \pi(x_0)$, $\beta_0 \in \pi(y_0)$ and $P_0 = P_{x_0}(\alpha_0, \beta_0, \pi)$,

(ii) for $1 \leq i \leq r$, if we denote $X_{i-1} = \bigcup_{j<i} P_j$, then $P_i \cap X_{i-1} = \{x_i, y_i\}$,

(iii) for $1 \leq i \leq r$, $\{\alpha_i, \beta_i\} \cap \pi(X_{i-1}) \neq \emptyset$, and

(iv) $X = X_r = \bigcup_{j \leq r} V(P_j)$.

Proof. (1) and 2 directly follows from Conjecture 2.4.2. Hence, we only need to show (3).

$P_0$ is well-defined since for any $\alpha_0 \in \pi(x_0)$ and $\beta_0 \in \pi(y_0)$, $P_x(\alpha_0, \beta_0, \pi) = P_y(\alpha_0, \beta_0, \pi)$. Assume, by induction, that $X_{i-1} = \bigcup_{j<i} P_j$ is well-defined for some $i \geq 1$. We need to show that either $V(X_{i-1}) = X$ or there exists an $(\alpha_i, \beta_i)$-alternating path $P_i = P_i(x_i, y_i)$ with $P_i \cap X_{i-1} = \{x_i, y_i\}$ and $\{\alpha_i, \beta_i\} \cap \pi(X_{i-1}) \neq \emptyset$. 

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Assume that \( V(X_{i-1}) \neq X \). Note that \( X \supset V(X_{i-1}) \) and hence, \( V(X_{i-1}) \) is elementary but not strongly closed with respect to \( \pi \) by minimality of \( X \).

If \( X_{i-1} \) is not closed, then there exists \( \alpha_i \in \bar{\pi}(X_{i-1}) \) such that \( E_{\alpha_i}(X_{i-1}, V(G) \setminus V(X_{i-1})) \neq \emptyset \). Let \( \beta_i \in \bar{\pi}(X_{i-1}) \setminus \alpha_i \) and let \( P_i \) be an \((\alpha_i, \beta_i)\)-path starting with one of the \( \alpha_i \)-edges in \( E_{\alpha_i}(X_{i-1}, V(G) \setminus V(X_{i-1})) \) at a vertex \( x_i \in X_{i-1} \) and leaving \( X_{i-1} \) such that either \( P_i \) is a path turning back to \( X_{i-1} \) at a vertex \( y_i \) and ending at \( y_i \) or it never comes back to \( X_{i-1} \). In the first case, \( P_i \) satisfies the desired properties and, by induction, the proof follows. In the latter case, \( X_{i-1} \cup V(P_i) \subseteq X \) is not elementary since \( \alpha_i, \beta_i \in \bar{\pi}(X_{i-1}) \) and \( \{\alpha_i, \beta_i\} \cap \bar{\pi}(P_i \setminus X_{i-1}) \neq \emptyset \), contrary to the fact that \( X \) is elementary.

If \( X_{i-1} \) is closed, then \( \Gamma^d(X_{i-1}, e, \pi) \neq \emptyset \). Let \( \alpha_i \) be a deficient color with respect to \( \pi \) and, for some \( z \in X_{i-1} \), let \( \beta_i \in \bar{\pi}(z) \) be an unused color in \( X_{i-1} \) with respect to \( \pi \). We claim that there exists a unique acyclic \((\alpha_i, \beta_i)\)-chain intersecting \( X_{i-1} \). If not, there exists at least two acyclic \((\alpha_i, \beta_i)\)-chains intersecting \( X \) contrary to the definition of elementariness and strong closedness. But this implies that there exist an \((\alpha_i, \beta_i)\)-alternating path \( P_i = P_i(x_i, y_i) \) such that \( P_i \cap X_{i-1} = \{x_i, y_i\} \) since there exist at least three \( \alpha_i \)-edges leaving \( X_{i-1} \). But this completes the proof by induction since we were able to add a handle to \( X_{i-1} \) with the desired properties.

\( \square \)

We note that Proposition 3.1.1(3) suggests a handle-type recursive algorithm to find a strongly closed, elementary set. We call \( X \subseteq V(G) \) a \textit{k-handle-set or a handle-set} with respect to \( e \in E(G) \) and \( \pi \in C_k(G \setminus e) \) if \( X \) can be found by adding \( k \) handles to \( e \) as described in Proposition 3.1.1(3). In this case, \((X, e, \pi)\) denotes a \textit{handle-set triple}. Moreover, an \((\alpha, \beta)\)-\textit{handle} at \( X \) is an \((\alpha, \beta)\)-alternating path \( P = (y_1, f_2, y_2, \ldots, f_m, y_m) \) such that \( y_i \notin X \) for \( 1 \leq i \leq m \) and there exists \( x, y \in X \) and edges \( f_1 \in E_G(x, y_1) \) and \( f_2 \in E_G(y, y_m) \) such that \((x, f_1, P, f_1', y)\) is also an
(\alpha, \beta)$-alternating path. In this case, we denote $P = P(x, y)$ if the knowledge of $x, y$ is necessary. Please, see Figure 3.1 for an example of a handle of a Tashkinov tree.

**Conjecture 3.1.2.** Let $G$ be critical with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$. Then any handle-set triple $(X, e, \pi)$ is elementary with respect to $e$ and $\pi$.

Note that Conjecture 3.1.2 implies that a maximal handle triple $(X, e, \pi)$ is elementary. While it will not be shown here, a maximal handle triple is also strongly closed and this implies Conjecture 2.4.2. Hence, Conjecture 3.1.2 is a restatement of Goldberg conjecture.

In this section, we will show that handles can be added to the vertex set of a maximal Tashkinov triple $(T, e, \pi)$. Note that $(V(T), e, \pi)$ is also a handle-set triple.

**Lemma 3.1.3.** Let $G$ be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$, $(T, e, \pi)$ be a maximal Tashkinov triple with $T = (x_0, e_1, \ldots, e_n, x_n)$, $\alpha \in \Gamma^d(T, e, \pi)$ and $\beta \in \overline{\pi}(T)$. If, for some $0 \leq i_1 < i_2 \leq n$, $Q = Q(x_{i_1}, x_{i_2})$ is an $(\alpha, \beta)$-alternating
path with $Q \cap T = \{x_{i_1}, x_{i_2}\}$ and $P = P(x_{i_1}, x_{i_2}) = Q - \{x_{i_1}, x_{i_2}\}$ is an $(\alpha, \beta)$-handle at $T$, then $V(P) \subseteq A_{C_{\beta}}(T, e, \pi)$ where $C_{\beta} = \bar{\pi}(T) \setminus \{\beta\}$.

**Proof.** Let $P = (y_1, f_2, y_2, \ldots, y_m)$ be such that $E_\alpha(x_{i_2}, y_1) \neq \emptyset$ and $E_\alpha(x_{i_1}, y_m) \neq \emptyset$.

(a) $V(T) \cup \{y_1\}$ is elementary with respect to $\pi$

**Proof of (a):** Assume, on the contrary, that there exists $\varepsilon \in \bar{\pi}(T) \cap \bar{\pi}(y_1)$. By definition of $P$, $\varepsilon \neq \alpha, \beta$. If $\beta \notin \bar{\pi}(Tx_{i_2-1})$, then, for some $\delta \in \bar{\pi}(Tx_{i_2-1})$, we can re-color all $(\delta, \beta)$-chains and get a coloring $\varphi$ such that $\beta \in \varphi(Tx_{i_2-1})$. Hence, we may assume $\beta \in \bar{\pi}(Tx_{i_2-1})$. Let $\gamma \in \bar{\pi}(x_{i_2})$. Re-coloring all $(\gamma, \varepsilon)$-chains within $T$, we get a coloring $\pi'$ such that $(T, e, \pi')$ is a maximal Tashkinov triple and $\varepsilon \in \bar{\pi}(x_{i_2}) \cap \bar{\pi}(y_1)$.

Now, re-coloring $f_1$ by $\varepsilon$, we get a maximal Tashkinov triple $(T'', e, \pi'')$ such that $Tx_{i_2} = T''x_{i_2}$. Since $\alpha \in \bar{\pi}''(x_{i_2})$ and $\beta \in \bar{\pi}(Ty_{i_2})$, $V(P) \subseteq V(T'')$. But this implies $y_1, x_{i_2} \in V(T'')$. This gives a contradiction since $\alpha \in \bar{\pi}''(x_{i_2}) \cap \bar{\pi}''(y_1)$. This completes the proof.

(b) $V(T) \cup \{y_1, y_2, \ldots, y_m\}$ is elementary with respect to $\pi$.

**Proof of (b):** Assume that $r > 1$ is minimal such that $V(T) \cup \{y_1, \ldots, y_{r-1}\}$ is elementary but $V(T) \cup \{y_1, y_2, \ldots, y_r\}$ is not elementary. Namely, there exists $\gamma \in \bar{\pi}(y_r) \cap \bar{\pi}(u)$ for some $u \in V(T)$ or $u = y_s$ with $s < r$. By definition of $P$, $\gamma \neq \alpha, \beta$.

If $u = y_{r-1}$, then re-coloring $f_r$ by $\gamma$, we get a coloring $\pi'$ such that $(T, e, \pi')$ is a maximal Tashkinov triple and there exist at least two acyclic $(\alpha, \beta)$-chains intersecting $T$ contrary to Lemma 2.5.3. Hence, we may assume $u \in V(T) \cup \{y_1, \ldots, y_{r-2}\}$.

Let $\varepsilon \in \bar{\pi}(y_{r-1})$. We note that $\varepsilon \neq \gamma, \alpha, \beta$ by definition of $P$ and the assumption that $V(T) \cup \{y_1, \ldots, y_{r-1}\}$ is elementary with respect to $\pi$. We also note that if $Q_1 = P_{y_{r-1}}(\gamma, \varepsilon, \pi)$ does not end in $u$, then $Q \cap T = \emptyset$ and re-coloring $Q_1$, we get a coloring $\pi_1$ such that $(T, e, \pi_1)$ is a maximal Tashkinov triple, $P$ is still an $(\alpha, \beta)$-handle and $V(T) \cup \{y_1, \ldots, y_{r-1}\}$ is not elementary contrary to the minimality
of \( r \). Hence, \( Q_1 \) ends in \( u \), implying that \( Q_2 = P_{y_r}(\gamma, \varepsilon, \pi) \) does not visit \( T \) and ends outside \( V(T) \cup \{y_1, \ldots, y_{r-1}\} \). Re-coloring \( Q_2 \), we get a coloring \( \pi_2 \) for which \( \bar{\pi}_2(y_{r-1}) \cap \bar{\pi}_2(y_{r}) \neq \emptyset \), \((T, e, \pi_2)\) is a maximal Tashkinov triple and \( P \) is again an \((\alpha, \beta)\)-handle. But it was proven that this is not possible previously. Hence, the proof of \((b)\) is completed by induction.

\[(c) V(P) \subseteq AC_{\beta}(T, e, \pi)\]

**Proof of (c):** We note that for any \( \varepsilon \in \bar{\pi}(T) \setminus \{\beta\} \) and \( \gamma \in \bar{\pi}(y_s) \) for \( 1 \leq s \leq r \), if \( Q = P_{y_s}(\varepsilon, \gamma, \pi) \) does not end in \( T \), re-coloring \( Q \), we get a coloring \( \varphi \) for which \((T, e, \varphi)\) is a maximal Tashkinov triple, \( P \) is an \((\alpha, \beta)\)-handle and \( V(T \cup P) \) is not elementary. Hence, \( Q \) must end in \( T \) implying that \( y_s \in AC_{\beta}(T, e, \pi) \) and completing the proof of \((c)\) and the lemma.

For \( X \subset V(G) \) and \( \alpha, \beta \cap \bar{\pi}(X) \neq \emptyset \), we denote \( L_{(\alpha, \beta)}(X, e, \pi) \) as the vertex set of the union of all \((\alpha, \beta)\)-handles at \( X \) with respect to \( \pi \) and \( L_1(X, e, \pi) \) as the union of all \( L_{(\alpha, \beta)}(X, e, \pi) \) with \( \{\alpha, \beta\} \cap \bar{\pi}(T) \neq \emptyset \). The following are directly implied by Lemma 3.1.3 and Lemma 2.5.4:

**Corollary 3.1.4.** Let \( G \) be a critical graph with \( \chi'(G) = k + 1 \) for an integer \( k \geq \Delta(G) + 1 \), \((T, e, \pi)\) be a maximal Tashkinov triple, \( \alpha \in \Gamma^d(T, e, \pi) \) and \( \beta \in \bar{\pi}(T) \). Then \( L_{(\alpha, \beta)}(T, e, \pi) \subseteq AC_{\beta}(T, e, \pi) \) where \( C_{\beta} = \bar{\pi}(T) \setminus \{\beta\} \).

**Proof.** Since any \((\alpha, \beta)\)-handle \( P \) at \( T \) with respect to \( \pi \) satisfies \( V(P) \subseteq AC_{\beta} \) by Lemma 3.1.3, we have \( L_{(\alpha, \beta)}(T, e, \pi) = \bigcup \{V(P) : \; P \; is \; an \; (\alpha, \beta) - \; handle\} \subseteq AC_{\beta} \).

**Corollary 3.1.5.** Let \( G \) be a critical graph with \( \chi'(G) = k + 1 \) for an integer \( k \geq \Delta(G) + 1 \), \((T, e, \pi)\) be a maximal Tashkinov triple and \( \alpha \in \Gamma^d(T, e, \pi) \). Then \( X = L_1(T, e, \pi) \cup V(T) \) is elementary with respect to \( \pi \).
Proof. Let \( v_1, v_2 \in V(\mathcal{L}_1) \). Then, for \( i = 1, 2 \), there exists \( \alpha_i \in \Gamma^d(T, e, \pi) \), \( \beta_i \in \bar{\pi}(T) \) and an \((\alpha_i, \beta_i)\)-handle \( P_i \) such that \( v_i \in V(P_i) \). This implies that, for \( i = 1, 2 \), \( v_i \in A_{C_{\alpha_i}}(T, e, \pi) \) by Corollary 3.1.4. We set \( C = C_{\beta_1} \cap C_{\beta_2} = \bar{\pi}(T) \setminus \{\beta_1, \beta_2\} \) which is not empty. So, by Lemma 2.5.6, we have \( v_1, v_2 \in A_{C}(T, e, \pi) \) implying that \( V(T) \cup \{v_1, v_2\} \) is elementary with respect to \( \pi \) by Lemma 2.5.4. Since any two element of \( \mathcal{L}_1 \) can be represented the same way with \( v_1, v_2, X \) is elementary with respect to \( \pi \).

We define \( I(T, e, \pi) \) as set of vertices which are the end points of at least two handles at \((T, e, \pi)\) using different colors from \( \bar{\pi}(T) \).

Corollary 3.1.6. Let \( G \) be a critical graph with \( \chi'(G) = k + 1 \) for an integer \( k \geq \Delta(G) + 1 \), \((T, e, \pi)\) be a maximal Tashkinov triple. Then \( I(T, e, \pi) \subseteq A(T, e, \pi) \).

Proof. Let \( x \in I(T, e, \pi) \). Then, for \( i = 1, 2 \), there exist \( \alpha_i \in \Gamma^d(T, e, \pi) \), distinct \( \beta_i \in \bar{\pi}(T) \) and \((\alpha_i, \beta_i)\)-handle \( P_i \) such that \( \beta_1 \neq \beta_2 \) and \( x \) is an end point of both \( P_1 \) and \( P_2 \). This implies \( x \in A_{C_i}(T, e, \pi) \) where \( C_i = \bar{\pi}(T) \setminus \{\beta_i\} \) for \( i = 1, 2 \). But then we have \( x \in A(T, e, \pi) \) since \( C_1 \cup C_2 = \bar{\pi}(T) \).

Corollary 3.1.7. Let \( G \) be a critical graph with \( \chi'(G) = k + 1 \) for an integer \( k \geq \Delta(G) + 1 \), \((T, e, \pi)\) be a maximal Tashkinov triple. Then \( D(T, e, \pi) \subset I(T, e, \pi) \).

For \( \alpha \in \Gamma^d(T, e, \pi) \), we denote \( V_\alpha(T, e, \pi) = V(E_\alpha(T, e, \pi)) \setminus V(T) \) as the end points of \( \alpha \)-edges which are not in \( V(T) \). Note that since \( |V(T)| + 2 \leq |\bar{\pi}(T)| \), at least \( |E_\alpha(T, e, \pi)| - 1 = |V_\alpha| - 1 \) of the vertices of \( V_\alpha \) are also in \( I(T, e, \pi) \). We denote this set as \( I_\alpha(T, e, \pi) \).
3.2 Fans at a Tashkinov Tree

In 1964, Vizing [43] defined fans for the first time and used them in his well-known proof of Theorem 2.2.3. One can say that one needs to generalize fans first if one wishes to generalize the concepts of Kierstead path and Tashkinov trees.

Let \((T, e, \pi)\) be a maximal Tashkinov triple. By a \((T, e, \pi)\)-fan, we mean a sequence \(F = (e_1, y_1, e_2, y_2, \ldots, e_n, y_n)\) such that:

1. \(y_1 \in I(T, e, \pi)\).
2. For all \(i = 1, \ldots, n\), \(e_i \in E_G(V(T), y_i)\).
3. For all \(i = 2, \ldots, n\), \(y_i \in I(T, e, \pi)\) or there exists \(y \in V(T) \cup \{y_1, \ldots, y_{i-1}\}\) with \(\pi(e_i) \in \bar{\pi}(y)\).

**Proposition 3.2.1.** Let \(G\) be critical with \(\chi'(G) = k + 1\) where \(k \geq \Delta(G) + 1\) and \((T, e, \pi)\) be a maximal Tashkinov tree. For any \((T, e, \pi)\)-fan \(F = (e_1, y_1, e_2, y_2, \ldots, e_n, y_n)\), \(V(F) \subseteq A(T, e, \pi)\).

**Proof.** Let \(F = (e_1, y_1, \ldots, y_n)\) be a minimal counterexample. Since \(y_1 \in I(T, e, \pi)\), we may assume \(n > 1\). By minimality of \(n\), we have \(\{y_1, \ldots, y_{n-1}\} \subseteq A(T, e, \pi)\) and \(\pi(e_n) \in \bar{\pi}(y_{n-1})\) for, otherwise, \(F - y_{n-1}\) is a smaller fan containing \(x_n\) and, hence, \(y_n \in A(T, e, \pi)\). We may also assume \(y_i \notin I(T, e, \pi)\) if \(i > 1\), by minimality of \(n\).

(a) \(V(T) \cup \{y_n\}\) is elementary

**Proof of (a):** Assume, on the contrary, that \(\alpha \in \bar{\pi}(T) \cap \bar{\pi}(y_n)\). Let \(\pi(e_n) = \beta \in \bar{\pi}(y_{n-1})\). So, \(P_{y_n}(\alpha, \beta, \pi) \cap T \neq \emptyset\) implying that \(y_{n-1} \notin A(T, e, \pi)\) contrary to the assumption. Hence, the proof of (a) follows.

(b) \(\{y_1, \ldots, y_n\}\) is elementary

**Proof of (b):** Assume, on the contrary, that \(\alpha \in \bar{\pi}(y_r) \cap \bar{\pi}(y_n)\) for some \(r < n\). Since \(y_r \in A(T, e, \pi)\), \(P_{y_n}(\alpha, \gamma, \pi) \cap T = \emptyset\) for \(\gamma \in \bar{\pi}(T)\). Re-coloring \(P_{y_n}\), we get a
coloring \( \pi' \) for which \( F \) is still a fan and \( \{y_n\} \cap V(T) \) is not elementary contrary to (a). Hence, (b) holds.

(a) and (b) together implies that \( V(T) \cup V(F) \) is elementary with respect to \( \pi \). Moreover, for any \( \alpha \in \bar{\pi}(T) \) and \( \beta \in \bar{\pi}(y_n) \), \( P_{y_n}(\alpha, \beta, \pi) \) ends in \( T \) for otherwise, re-coloring \( P_{y_n} \), we get the same contradiction. Hence, \( y_n \in A(T, e, \pi) \).

For any \( X \subseteq V(G) \), we denote, by \( J(X, e, \pi) \), the set of vertices which, for some \( \alpha \in \bar{\pi}(X) \), can be added to \( X \) through an \( \alpha \)-edge and, if \( X \) is closed, \( J(X, e, \pi) = I(X, e, \pi) \). Given a maximal Tashkinov triple \( (T, e, \pi) \), we denote \( Y_{-1} = V(T) \), \( J_0 = I(T, e, \pi) \) and \( F_0 \) as the maximal \( (Y_{-1}, e, \pi) \)-fan. For \( i \geq 0 \), a \( (Y_i, e, \pi) \)-fan \( F = (e_1, y_1, \ldots, y_n) \) is defined as follows:

1. \( Y_i = Y_{i-1} \cup F_i \) where \( F_i \) is the maximal \( (Y_{i-1}, e, \pi) \)-fan.

2. \( J_i = J(Y_i, e, \pi) \)

3. \( e_1 \in E_G(Y_i, J_i), y_1 \in J_i \).

4. For all \( s = 1, \ldots, n \), \( e_s \in E_G(Y_i, s) \).

5. For all \( s = 2, \ldots, n \), \( y_s \in J_i \) or there exists \( y \in Y_i \cup \{y_1, \ldots, y_{i-1}\} \) with \( \pi(e_i) \in \bar{\pi}(y) \).

The following results will be used in showing that a \( (Y_0, e, \pi) \)-fan is elementary.

**Proposition 3.2.2.** Let \( G \) be critical with \( \chi'(G) > \Delta(G) + 1 \), \( (T, e, \pi) \) be a maximal Tashkinov triple and \( F_0 \) be the unique maximal \( (T, e, \pi) \)-fan. Then for any \( x \in J_1 \), \( Y_0 \cup \{x\} \) is elementary with respect to \( \pi \). Moreover, \( x \in A_C(T, e, \pi) \) where \( C = \bar{\pi}(T) \setminus \{\alpha\} \) if \( \pi(E_G(Y_0, x)) \cap \bar{\pi}(Y_0) = \{\beta\} \) for some \( \beta \in \bar{\pi}(T) \) and \( C = \bar{\pi}(T) \), otherwise.
Proof. Let \( f \in E_G(y, x) \) for some \( y \in Y_0 \) and \( \pi(f) = \beta \) for some \( \beta \in \pi(Y_0) \). Then \( y \in V(F_0) \) for, otherwise, \( F_0 \) is not maximal. Moreover, assume on the contrary, that there exists \( \alpha \in \pi(x) \cap \pi(z) \) for some \( z \in Y_0 \). Throughout this proof \( \square(\star) \) will be used to describe a set “\( \square \)” under a coloring “\( \star \)”. There are four cases:

**Case 1:** \( \beta \in \pi(T) \) and \( z \in V(T) \).

Proof of (1): Let \( \delta \in \pi(y) \). Then \( P_\pi(\alpha, \beta, \pi) \cap T = \emptyset \) by Lemma 2.5.3 and \( P_\pi(\alpha, \beta, \pi) \cap T \neq \emptyset \). Re-coloring \( P_\pi \), we get a coloring \( \varphi \) for which \( T(\varphi) = T(\pi) = T \), \( F_0 = F_0(\pi) = F_0(\varphi) \) and \( P_\pi(\delta, \beta, \varphi) \cap T = \emptyset \) implying that \( y \notin A(T, e, \varphi) \) contrary to Proposition 3.2.1. Hence, the proof follows.

**Case 2:** \( \beta \in \pi(T) \) and \( z \in V(F_0) \).

Proof of (2): Let \( \gamma \in \pi(T) \setminus \{\beta\} \). Since \( z \in V(F_0) \subseteq A(T, e, \pi) \), \( P_\pi(\gamma, \alpha, \pi) \cap T = \emptyset \). Re-coloring \( P_\pi \), we get Case 1.

**Case 3:** \( \beta \in \pi(V(F_0)) \) and \( z \in V(T) \).

Proof of (3): Since \( \beta \in \pi(v) \) for some \( v \in V(F_0) \subseteq A(T, e, \pi) \), we have \( P_\pi(\alpha, \beta, \pi) \cap T = \emptyset \). Re-coloring \( P_\pi \), we get Case 2.

**Case 4:** \( \beta \in \pi(V(F_0)) \) and \( z \in V(F_0) \).

Proof of (4): Let \( \varepsilon \in \pi(T) \). Since \( z \in V(F_0) \subseteq A(T, e, \pi) \), we have \( P_\pi(\alpha, \varepsilon, \pi) \cap T = \emptyset \). Re-coloring \( P_\pi \), we get Case 3.

This completes the proof that \( Y_0 \cup \{x\} \) is elementary.

Let \( \gamma \in \pi(T) \setminus \{\alpha\} \). Then for any \( \varepsilon \in \pi(x) \), if \( P_\pi(\varepsilon, \gamma, \pi) \cap T = \emptyset \), then re-coloring \( P_\pi \) we get a coloring \( \pi' \) for which \( Y_0 \cup \{x\} \) is not elementary with respect to \( \pi' \) contrary to the first part of the proof. Hence, \( x \in A_{\pi(T) \setminus \{\alpha\}}(T, e, \pi) \). If \( \{\alpha\} = \pi(E_G(Y_0, x)) \cap \pi(Y_0) \subseteq \pi(T) \), then \( \pi(T) \setminus \{\alpha\} = C \). If \( \{\alpha\} = \pi(E_G(Y_0, x)) \cap \pi(Y_0) \) with \( \alpha \notin \pi(T) \), then \( C = \pi(T) \). If there exist distinct \( \alpha, \alpha' \in \pi(E_G(Y_0, x)) \cap \pi(Y_0) \),
then $C = (\bar{\pi}(T) \setminus \{\alpha\}) \cup (\bar{\pi}(T) \setminus \{\alpha'\}) = \bar{\pi}(T)$. This completes the second part of the proof.

The rest of the discussion on fans will continue on Chapter 5 since it is not needed in Chapter 4, where we prove our main results. We refer the interested reader to Section 5.3.

### 3.3 Paths at a Tashkinov Tree

Prior to Tashkinov trees, the only way to reach a vertex that is not in a fan or a handle was to use Kierstad’s path argument. In the previous section, we showed that Vizing’s fan argument can be fully generalized if we replace the initial vertex with a maximal Tashkinov tree and the other end vertex of the uncolored edge with vertices of $I(T, e, \pi)$. In this section, we will show that generalizing Kierstad’s path argument is possible under some conditions.

Throughout this section, $G$ will be a critical graph with $\chi'(G) = k + 1$ for $k \geq \Delta(G) + 1$, and all colorings will be in $\mathcal{C}_k(G \setminus e)$ for some $e \in E(G)$. Let $(T, e, \pi)$ be a maximal Tashkinov triple and $F_0$ be the maximal $(T, e, \pi)$-fan. As before, we set $Y_0 = V(T) \cup V(F_0)$.

$K = (e_1, y_1, e_2, y_2, \ldots, e_n, y_n)$ is called a $(Y_0, e, \pi)$-path if it satisfies the followings:

1. $y_1 \notin Y_0$ and $e_1 \in E_{\bar{\pi}(Y_0)}(Y_0, y_1)$.
2. For all $i = 2, \ldots, n$, $e_i \in E_G(y_{i-1}, y_i)$.
3. For all $i = 2, \ldots, n$, there exists $y \in Y_0 \cup \{y_1, \ldots, y_{i-1}\}$ with $\pi(e_i) \in \bar{\pi}(y)$.

The following is the main result of this section.
Proposition 3.3.1. Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$, $(T,e,\pi)$ be a maximal Tashkinov triple and $F_0$ be the maximal $(T,e,\pi)$-fan. Let also $K = (e_1, y_1, e_2, y_2, \ldots, e_n, y_n)$ be a $(Y_0, e, \pi)$-path with $n < |\bar{\pi}(T)|$. Then $Y_0 \cup V(K)$ is elementary with respect to $\pi$. Moreover, if $B(y_j, \pi) = \bar{\pi}(T) \setminus \pi(\{e_1, e_2, \ldots, e_j\})$ then $y_j \in A_{B(y_j,\pi)}(T,e,\pi) = B_j(\pi)$ for $1 \leq j \leq n$.

Proof. We first note that $n < |\bar{\pi}(T)|$ implies $B(y_j, \pi) \neq \emptyset$ and $B(y_j, \pi) \subseteq B(y_{j-1}, \pi)$ for $1 \leq j \leq n$. By Lemma 2.5.6, $V(T) \cup B_n(\pi)$ is elementary. Assume $K$ is a counterexample with the smallest $n$. Then $n \geq 2$ by Lemma 3.2.2. By minimality of $n$, $y_j \in B_j(\pi)$ for $0 \leq j < n$. The proof has two parts:

(A) $Y_0 \cup V(K)$ is elementary with respect to $\pi$.

Proof of (A): It is enough to show $\bar{\pi}(Y_0 \cup K \setminus \{y_n\}) \cap \bar{\pi}(y_n) = \emptyset$ for minimality implies that $Y_0 \cup Ky_{n-1}$ is elementary with respect to $\pi$. There are two cases:

Case 1: $\bar{\pi}(y_n) \cap \bar{\pi}(T) \neq \emptyset$.

Proof of Case (1): We define $u_\alpha(\pi) = \min\{j : \pi(e_j) = \alpha\}$. If $\{j : \pi(e_j) = \alpha\} = \emptyset$, then $u_\alpha(\pi) = \infty$. Assume $Y_0, e, \pi, K, \alpha \in \bar{\pi}(T) \cap \bar{\pi}(y_n)$ are chosen such that $u_\alpha(\pi)$ is maximum.

If $u_\alpha(\pi) = \infty$, then we pick $\beta \in \bar{\pi}(y_{n-1})$. Since $\alpha \in B(y_{n-1}, \pi)$ and $y_{n-1} \in B_{n-1}(\pi)$, $P_{y_{n-1}}(\beta, \alpha, \pi)$ ends in $T$. Hence, by Lemma 2.5.3, $P = P_{y_n}(\alpha, \beta, \pi)$ does not visit $T$. Recoloring $P$, we get a coloring $\varphi$ such that $Y_0(\varphi) = Y_0$, $K$ is a $(Y_0, e, \varphi)$-path, $\varphi(e_n) = \pi(e_n) = \varepsilon \in \varphi(Y_0 \cup Ky_{n-2})$ and $\beta \in \varphi(y_n) \cap \varphi(y_{n-1})$. Recoloring $e_n$ by $\beta$, we get a coloring $\psi$ such that $Y_0(\psi) = Y_0$, $Ky_{n-1}$ is a $(Y_0, e, \psi)$-path, $\psi(y_j) = \varphi(y_j) = \bar{\pi}(y_j)$ for $0 \leq j < n-1$ and $\varepsilon \in \bar{\psi}(y_{n-1})$. But this means, $\varepsilon \in \bar{\psi}(y_{n-1}) \cap \bar{\psi}(Y_0 \cup Ky_{n-2})$ contrary to minimality of $n$.

If $u_\alpha(\pi) = j_0$ for some $1 \leq j_0 \leq n-1$ then $\alpha \in B(y_{j_0-1}, \pi)$. Let $\gamma \in \bar{\pi}(y_{j_0-1})$. Then $P_{y_{j_0-1}}(\gamma, \alpha, \pi)$ ends in $T$. Hence, by Lemma 2.5.3, $P = P_{y_n}(\alpha, \gamma, \pi)$ does not visit $T$. 48
Recoloring $P$, we get a coloring $\pi'$ such that $Y_0(\pi') = Y_0$, $K$ is a $(Y_0, e, \pi')$-path and $\gamma \in \pi'(y_{j_0-1}) \cap \pi'(y_n)$. Let $\delta \in B(y_{j_0}, \pi')$. Clearly, $\pi'(e_r) \neq \delta$ for $0 < r < j_0 + 1$. Moreover, $P_{y_{j_0}}(\gamma, \delta, \pi')$ ends in $T$. Hence, by Lemma 2.5.3, $Q = P_{y_n}(\gamma, \delta, \pi')$ does not visit $T$. Re-coloring $Q$, we get a coloring $\varphi$ such that $Y_0(\varphi) = Y_0$, $K$ is a $(Y_0, e, \varphi)$-path and $\delta \in \varphi(T) \cap \varphi(y_n)$. Since $\delta \in B(y_{j_0}, \varphi)$, $u_\delta(\varphi) \geq j_0 + 1$ contrary to the assumption that $j_0$ is maximal. This completes the proof.

**Case 2:** $\pi(y_n) \cap \pi(T) = \emptyset$ and $\pi(y_n) \cap \pi(y_j) \neq \emptyset$ for some $0 < j < n$ provided that, if $j = 0$ then $y_0 \in V(F_0)$.

**Proof of Case (2):** Let $\alpha \in \pi(y_j) \cap \pi(y_n)$ and $\beta \in B(y_j, \pi)$. Then $P_{y_j}(\alpha, \beta, \pi)$ ends in $T$. Hence, by Lemma 2.5.3, $R = P_{y_n}(\alpha, \beta, \pi)$ does not visit $T$. Re-coloring $R$, we get a coloring $\varphi$ such that $Y_0(\varphi) = Y_0$, $K$ is a $(Y_0, e, \varphi)$-path and $\beta \in \varphi(T) \cap \varphi(y_n)$. Hence, this case is equivalent to Case 1 and the proof of (A) is completed.

**(B)** $y_n \in B_n$

**Proof of (B):** Let $\alpha \in \pi(y_n)$ and $\beta \in B(y_n, \pi)$. If $P = P_{y_n}(\alpha, \beta, \pi)$ does not visit $T$, then re-coloring $P$, we get a coloring $\varphi$ such that $Y_0(\varphi) = Y_0$, $K$ is a $(Y_0, e, \varphi)$-path and $\beta \in \varphi(T) \cap \varphi(y_n)$ contradicting (A). Hence, $P$ must visit $T$ implying $y_n \in B_n$.

If one can drop the condition “$n < |\pi(T)|$” in the proof of Proposition 3.3.1, then its highly possible to prove a generalization of Tashkinov’s Theorem.

### 3.4 Trees at a Tashkinov Tree

In Section 3.2, we have shown that fans at a Tashkinov tree indeed behave exactly like the Vizing fans at the uncolored edge, while we have shown in Section 3.3 that paths at a Tashkinov tree behave like a Kierstead path under special circumstances.
In this section, we will use even a more special condition to show that trees at a Tashkinov tree are elementary.

Throughout this section, $G$ will be a critical graph with $\chi'(G) = k + 1$ for $k \geq \Delta(G) + 1$, and all colorings will be in $C_k(G \setminus e)$ for some $e \in E(G)$. Let $(T, e, \pi)$ be a maximal Tashkinov triple and $F_0$ be the maximal $(T, e, \pi)$-fan.

$T^1 = (e_1, y_1, e_2, y_2, \ldots, e_n, y_n)$ is called a $(Y_0, e, \pi)$-tree if it satisfies the followings:

1. $y_1, y_2, \ldots, y_n \notin Y_0$ are distinct.
2. For all $i = 1, \ldots, n$, $e_i \in E_G(x, y_i)$ for some $x \in Y_0 \cup \{y_1, \ldots, y_{i-1}\}$.
3. For all $i = 1, 2, \ldots, n$, there exists $y \in Y_0 \cup \{y_1, \ldots, y_{i-1}\}$ with $\pi(e_i) \in \bar{\pi}(y)$.

The following result is inspired by the proof of Tashkinov. As in Tashkinov’s proof, if $T'$ is a $(Y_0, e, \pi)$-tree, then $p(T')$ is the path number of $T'$; that is $p(T') = p$ is the smallest number such that $y_pT'$ is a path. Note that if $p = 0$ then $T'$ is a $(Y_0, e, \pi)$-path.

**Proposition 3.4.1.** Let $G$ be critical with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$, $(T^0, e, \pi)$ be a maximal Tashkinov triple, $F_0$ be the maximal $(T^0, e, \pi)$-fan. Let also $T^1$ be a closed $(Y_0, e, \pi)$-tree. If $|V(T^1)| < |\bar{\pi}(T^0)|$ then $V(T^1) \cup Y_0$ is elementary with respect to $\pi$. Moreover, $V(T^1) \subseteq A(T^0, e, \pi)$.

**Proof.** Let $T = (e_1, y_1, \ldots, e_n, y_n)$ be a $(Y_0, e, \pi)$-tree such that

1. $T$ is included in a closed $(Y_0, e, \pi)$-tree $S$ with $V(S) = V(T^1)$.
2. $V(T) \cup Y_0$ is not elementary with respect to $\pi$.
3. $p(T)$ is smallest with respect to (1) and (2).
4. \(|V(T)|\) is smallest with respect to (1), (2) and (3).

We first note that any color interchange used in this proof uses colors from \(\pi(Y_0 \cup T^1)\) and is within \(V(T^1)\). Hence, the closed \((Y_0, e, \pi)\)-tree obtained by these interchanges is a subset of \(V(T^1)\) and, therefore, the condition of the proposition holds throughout the proof. Moreover, this implies that no chain used throughout the proof can have more than \(|\pi(T^0)|\) vertices unless it intersects \(Y_0\).

Given a color \(\alpha \in \pi(y_n) \cap \pi(T^0)\), we define \(u_\alpha(\pi)\) as the smallest \(j\) such that \(\pi(e_j) = \alpha\) as before. If \(\alpha \notin \pi(T)\) then \(u_\alpha(\pi) = \infty\). We assume \(\alpha \in \pi(x) \cap \pi(y_n)\) for some \(x \in Y_0 \cup \{y_1, y_2, \ldots, y_{n-1}\}\). We have the following cases:

**Case 1:** \(x \in V(T^0)\) and \(u_\alpha(\pi) = \infty\).

*Proof of (1):* We pick \(\beta \in \pi(y_{n-1})\).

If \(p(T) < n\) then re-coloring \(P = P_{y_0}(\alpha, \beta, \pi)\), we get a coloring \(\pi_1\) such that \(Y_0(\pi_1) = Y_0\) and \(Ty_{n-1}\) is a \((Y_0, e, \pi_1)\)-tree since \(P_{y_{n-1}}(\alpha, \beta, \pi) \cap T \neq \emptyset\) implies \(P \cap T = \emptyset\). Assume \(\gamma = \pi_1(e_n) \in \bar{\pi}_1(z)\) for some \(z \in Y_0 \cup V(Ty_{n-2})\). Re-coloring \(P_{y_n}(\beta, \gamma, \pi_1) = (y_n, e_n, y_{n-1})\), we get a coloring \(\pi_2\) such that \(Y_0(\pi_2) = Y_0, Ty_{n-1}\) is a \((Y_0, e, \pi_2)\)-tree and \(\gamma \in \bar{\pi}_2(y_{n-1}) \cap \bar{\pi}_2(z)\). This gives a contradiction since \(Ty_{n-1}\) is a proper subtree of \(T\).

If, on the other hand, \(p(T) = n\), then we may assume \(e_n \in E_G(y_n, Y_0 \cup V(Ty_{n-2}))\) and \(\pi(e_n) = \beta \in \bar{\pi}(y_{n-1})\) for otherwise, \(T - y_{n-1}\) is a smaller tree which is not elementary. Re-coloring \(P_{y_n}(\alpha, \beta, \pi) = P\), we get a coloring \(\pi_1\) such that \(Y_0(\pi_1) = Y_0, T' = Ty_{n-1}\) and \(T'' = T - y_{n-1}\) are both \((Y_0, e, \pi_1)\)-trees with \(n - 1\) vertices where \(n < |\pi(T^0)| = |\bar{\pi}_1(T^0)|\). Hence, \(C = \bar{\pi}_1(T^0) \setminus (\pi_1(Ty_{n-1}) \cup \{\alpha\}) \neq \emptyset\). Hence, we have \(y_n, y_{n-1} \in A_G(T^0, e, \bar{\pi}_1)\) contrary to the fact that \(\beta\) is missing at both vertices. This completes the proof of Case 1.

**Case 2:** \(x \in V(T^0)\) and \(u_\alpha(\pi) = u\) for some \(u\) with \(1 \leq u < n\).
Proposition 3.4.2. Let \( \pi \) be a maximal Tashkinov triple, \( T_0, e, \pi \) be a critical \((T_0, e, \pi)\)-fan and \( T^1 \) be a closed \((Y_0, e, \pi)\)-tree. If \( |V(T^1)| \geq |\pi(T^0)| \) then there exists a \((Y_0, e, \pi)\)-tree \( T \) within \( V(T^1) \) such that \( |V(T)| = |\pi(T^0)| - 1 \) and \( V(S) \cup Y_0 \) is elementary with respect to \( \pi \).

Proof. Let \( T = (e_1, y_1, \ldots, e_n, y_n) \) be a \((Y_0, e, \pi)\)-tree such that \( T \) is a \((Y_0, e, \pi)\)-tree itself, \( V(T^1) \) is elementary with respect to \( \pi \) and, since \( X \) is closed, \( V(T^1) \subseteq A(T^0, e, \pi) \) follows. □

Proposition 3.4.2. Let \( G \) be critical with \( k' = k + 1 \) where \( k \geq \Delta(G) + 1 \), \((T_0, e, \pi)\) be a maximal Tashkinov triple, \( F_0 \) be the maximal \((T_0, e, \pi)\)-fan and \( T^1 \) be a closed \((Y_0, e, \pi)\)-tree. If \( |V(T^1)| \geq |\pi(T^0)| \) then there exists a \((Y_0, e, \pi)\)-tree \( S \) within \( V(T^1) \) such that \( |V(S)| = |\pi(T^0)| - 1 \) and \( V(S) \cup Y_0 \) is elementary with respect to \( \pi \).

Proof. Let \( T = (e_1, y_1, \ldots, e_n, y_n) \) be a \((Y_0, e, \pi)\)-tree such that
1. \( V(T) \cup Y_0 \) is not elementary with respect to \( \pi \).

2. \( p(T) \) is smallest with respect to (1).

3. \( |V(T)| \) is smallest with respect to (1) and (2).

If \( |V(T)| \ge |\bar{\pi}(T^0)| \), then \( Ty_{n-1} \) is elementary. Hence, we may assume \( |V(T)| \le |\bar{\pi}(T^0)| - 1 \).

Given a coloring \( \alpha \in \bar{\pi}(y_0) \), we define \( u_\alpha(\pi) \) as the smallest \( j \) such that \( \pi(e_j) = \alpha \). If \( \alpha \notin \pi(T) \) then \( u_\alpha(\pi) = \infty \). Assume \( \alpha \in \bar{\pi}(x) \cap \bar{\pi}(y_0) \). We have the following cases:

**Case 1**: \( x \in V(T^0) \) and \( u_\alpha(\pi) = \infty \).

*Proof of (1)*: The proof is exactly the same with Case 1 of Propositions 3.4.1.

**Case 2**: \( x \in V(T^0) \) and \( u_\alpha(\pi) = u \) for some \( u \) with \( 1 \le u < n \).

*Proof of (2)*: We choose \((T^0,e,\pi), T \) and \( a \in \bar{\pi}(T^0) \cap \bar{\pi}(y_0) \) such that, not only the conditions (1)-(3) above holds but also \( u = u_\alpha(\pi) \) is as large as possible.

If \( u > p \), then the exact arguments in the proof of Proposition 3.3.1 repeats.

If \( u \le p \), then we let \( \beta \in \bar{\pi}(T^0) \setminus \pi(T) \). If \( e_u \notin P_{y_0}(\alpha, \beta, \pi) = P \), we can re-color \( P \), to get a coloring \( \pi' \) such that \( \beta \in \bar{\pi}'(T^0) \cap \bar{\pi}(y_0) \) and \( u_\beta(\pi') > u \) contrary to the choice of \( u \). Hence we may assume \( e_u \in P_{y_0}(\alpha, \beta, \pi) = P \). Let \( P' \) be the smallest section of \( P \) with \( y_n \in V(P') \) and \( V(P') \cap Ty_u \neq \emptyset \). In any case, we have \( P' \cap Ty_u = \{z\} \) for some vertex in \( T \) by the minimal choice of \( P' \). If \( z \in Y_0 \cup \{y_1, y_2, \ldots, y_u-2\} \), then \( T' = Ty_{u-2} + P' \) is a \((Y_0,e,\pi)-tree\) with \( p(T') \le u - 1 < p \). If \( z = y_{u-1} \), then \( T'' = Ty_{u-1} + P' \) is a \((Y_0,e,\pi)-tree\) with \( p(T'') \le u - 1 \). If \( z = y_u \), then \( T''' = Ty_u + P - y_{u-1} \) is a \((Y_0,e,\pi)-tree\) with \( p(T''') \le u - 1 \). In any case, either

\[ |V(X)| < |\pi(T^0)| \]

and by (2), \( V(X) \cup V(T^0) \) is elementary with respect to \( \pi \) for some \( X \in \{V(T'), V(T''), V(T''')\} \) contrary to the fact that \( \alpha \in \bar{\pi}(y_n) \cap \bar{\pi}(T^0) \) or \( |V(X)| > |\pi(T^0)| - 1 \) and \( P' \) has a section \( P'' \) starting from \( z \) and ending at some \( v \).
such that $S = Xv$ is elementary with $|V(S)| = |\bar{\pi}(T^0)| - 1$. This completes the proof of Case 2.

**Case 3:** $x = y_r$ for some $1 \leq r < n$.

Proof of Case 3: The proof is exactly the same with Case 3 of Propositions 3.4.1.

These three cases complete the proof that there exists $S$ with $|V(S)| = |\bar{\pi}(T^0)| - 1$ and $Y_0 \cup V(S)$ is elementary with respect to $\pi$. \qed

It was Gupta [19], who defined the trees as the first time. In fact, his recursion kept the vertex set of a tree odd by adding two edges of the same color each time in his recursion. Hence, his trees were also balanced. One can generalize the above results to Gupta trees. By a *Gupta tree* with respect to $\pi$, we mean a $(Y_0, e, \pi)$-tree $T_1$ such that, for all $i \geq 1$, $\pi(e_{2i}) = \pi(e_{2i-1})$. We note the following proposition without proving:

**Proposition 3.4.3.** Let $G$ be critical with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$, $(T^0, e, \pi)$ be a maximal Tashkinov triple, $F_0$ be the maximal $(T^0, e, \pi)$-fan. Let also $T^1$ be a closed $(Y_0, e, \pi)$-tree. If $|V(T^1)| < 2|\bar{\pi}(T^0)|$ then $V(T^1) \cup Y_0$ is elementary with respect to $\pi$. Moreover, $V(T^1) \subseteq A(T^0, e, \pi)$.

**Proposition 3.4.4.** Let $G$ be critical with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$, $(T^0, e, \pi)$ be a maximal Tashkinov triple, $F_0$ be the maximal $(T^0, e, \pi)$-fan and $T^1$ be a closed $(Y_0, e, \pi)$-tree. If $|V(T^1)| \geq 2|\bar{\pi}(T^0)|$ then there exists a Gupta tree $S$ within $V(T^1)$ such that $|V(S)| = 2|\bar{\pi}(T^0)| - 2$ and $V(S) \cup Y_0$ is elementary with respect to $\pi$. 54
3.5 Notes

Most of the results given in this chapter are not only used in Chapter 4 to prove our main results but also stepping stones in an attempt to prove Conjecture 1.3.4. In fact, proving that any \((Y_0, e, \pi)\)-tree is elementary is almost equivalent to proving Conjecture 1.3.4.

Secondly, while using absorbing sets is a very nice way to get results, one needs to capture the elements of \(A(T, e, \pi)\) under some recursion \(\mathcal{R}\) such as the tree algorithm, fan algorithm or handle algorithm.

Thirdly, it is even more important to find the behavior of \((\gamma, \varepsilon)\)-chains on the boundary of \(T\) whenever \(\gamma, \varepsilon \not\in \bar{\pi}(T)\). If we assume Conjecture 1.3.4, there is at most one acyclic \((\gamma, \varepsilon)\)-chain intersecting \(T\) whenever \(\gamma, \varepsilon \not\in \bar{\pi}(T)\). So, naturally, one needs to prove this result to have any chance of proving this conjecture. Interestingly, it is very hard to prove even that it can be done for the colors missing at the vertices of the maximal \((T, e, \pi)\)-fans.

While there are other approaches that can be taken in proving Conjecture 1.3.4, these two seem to be essential as long as one uses criticality and Kempe chains as the main tools of their proof.
CHAPTER 4
ASYMPTOTIC AND LINEAR BOUNDS ON THE CHROMATIC INDEX

In this chapter, we aim to prove linear and asymptotic results. The following are the main results of this thesis:

**Theorem 1.6.8.** Let $G$ be a graph with $\chi'(G) > \Delta(G) + 1$. If $\chi'(G) > \Delta(G) + \sqrt[3]{\frac{\Delta(G)}{2}}$, then $G$ is elementary.

**Theorem 1.6.9.** Let $G$ be a graph with $\chi'(G) > \Delta(G) + 1$. If $\chi'(G) > \frac{25}{24}\Delta(G) + \frac{22}{24}$, then $G$ is elementary.

We note that it is possible to improve Theorem 1.6.8 to $\frac{3\sqrt{\Delta}}{2}$ and Theorem 1.6.9 to $\frac{43}{42}\Delta + \frac{40}{42}$ by using Propositions 3.4.3, 3.4.4 and Theorem 2.6.7. However, such a result will not be included here since we have not given the proofs of these results are not included in this thesis. In fact, Chen, Yu and Zang [5] claims a proof of this fact.

### 4.1 VKT*-trees

In [4], Chen, Yu and Zang introduced a more general tree definition, called VKT*-trees, than that of Tashkinov. In this section, we first modify their definition of a VKT-tree in the light of Chapter 3.
Let $G$ be a critical graph with $\chi'(G) = \Delta(G) + k + 1$ for $k \geq 1$. Let $e \in E(G)$ and $\pi$ be a $(\Delta + k)$-coloring of $G \setminus e$.

A VKT-tree $T = (y_0, e_1, y_1, e_2, \ldots, e_n)$ is a sequence satisfying the following conditions:

(V1) $e_1 = e \in E_G(y_0, y_1)$ and for all $i \geq 0$, $Ty_i = T_i$ is a tree.

(V2) If $\pi(e_i) \notin \bar{\pi}(T_{i-1})$ (for each $2 \leq i \leq n$), then

(a) $T_{i-1}$ is closed with respect to $\pi$.

(b) $|E_{\pi(e_i)}(T_{i-1}, e, \pi)| > 1$.

(c) If $S_0(T, e, \pi)$ denotes the maximal sequence $T_k$ such that $\pi(e_j) \in \bar{\pi}(T_{j-1})$ whenever $2 \leq j \leq k$, then, for some $0 \leq i \leq k$ and $\alpha_i \in \bar{\pi}(x_i) \setminus \pi(T_i)$, $e_i \in P_x(\alpha_i, \pi(e_i)) = P$ and $V(Px_i) \setminus V(T) = \{x_i\}$.

The following improvement to VKT-tree definition is thanks to Corollary 3.1.7.

A $\text{VKT}^*$-tree $T = (X_0, E_1, X_1, E_2, \ldots, E_n, X_n)$ is a sequence satisfying the following conditions:

(V0*) $\bigcup_{i=0}^{n} X_i = \sum_{i=1}^{n} |X_i|$; that is all vertices are distinct.

(V1*) $E_1 = \{e\}$ where $X_0 = \{x_0\}$, $X_1 = \{x_1\}$ and $e \in E_G(x_0, x_1)$.

(V2*) For all $i \geq 1$, $E_i \subset E_G(TX_{i-1}, X_i)$

(V3*) For all $i \geq 2$, if $T_i = TX_i$ is not closed, then $E_i = \{e_i\}$, $X_i = \{x_i\}$ and $\pi(e_i) \in \bar{\pi}(T_{i-1})$.

(V4*) For all $i \geq 2$, if $T_{i-1}$ is closed then there exists $\delta_i \in \Gamma_d(T_{i-1}, e, \pi)$ such that $\pi(E_i) = \delta_i$ and $X_i = I_{\delta_i}(T_{i-1}, e, \pi) = V(E_i) \setminus V(T_{i-1})$.  

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Any edge set $E_i$ with $\pi(E_i) \notin \bar{\pi}(T_{i-1})$ is said to be connecting in $(T,e,\pi)$. We define $S_0(T,e,\pi)$ as the maximal sequence $T_k$ such that $|E_j| = 1$ and $|X_j| = 1$ whenever $2 \leq j \leq k$. Note that whenever $\pi(E_i) \notin \bar{\pi}(T_{i-1})$, $V_0(T,e,\pi) = V(S_0(T,e,\pi)) \subseteq V(T_{i-1})$. A pair of distinct vertex sets, say $\{X,Y\}$, in $(T,e,\pi)$ is said to be divided if there is a connecting edge set $E_i$ such that $X,Y$ belong to different components of $T - E_i$; and is undivided otherwise. For notational simplicity, we also write $T_0 = \emptyset$ and $\pi(E_1) \in \bar{\pi}(T_0)$. We denote, by $c(T)$, the number of connecting edge sets in $T$. Moreover, if $(T,e,\pi)$ is a VKT-triple with $c(T) = c$, then $S_0, S_1, \ldots, S_c$ denotes its sections where a section $S_i = (E_{r_i}, X_{r_i}, E_{r_i+1}, E_{r_i+1}, \ldots, X_{r_{i+1}-1})$ is a part of $T$ such that $E_{r_i}$ and $E_{r_i+1}$ are connecting edge sets of $T$ while $E_s$ is not a connecting edge set for $r_i < s < r_{i+1}$. The union of the first $i + 1$ sections of $T$ will be denoted by $Y_i(T) = \bigcup_{j=0}^i S_j$ or $Y_i$ if no confusion arises. We denote the connection order by $\omega(T) = (|V(S_0)|, |V(S_1)|, \ldots, |V(S_{c-1})|)$ if $T$ is not closed with respect to $\pi$ and $\omega(T) = (|V(S_0)|, |V(S_1)|, \ldots, |V(S_c)|)$, otherwise.

Note that the only difference in our definition is that we add a connecting edge set instead of a connecting edge. Hence, the proofs of Chen, Yu and Zang for VKT-trees can be generalized to VKT*-trees with the help of Corollaries 3.1.6 and 3.1.7.

**Lemma 4.1.1** (Chen, Yu and Zang [4], 2008). Let $G$ be critical with $\chi'(G) = \Delta(G) + k + 1$ for an integer $k \geq 1$ and $(T,e,\pi)$ be a VKT*-tree. If $c(T) = c < k$ then $G$ is elementary.

We will improve the proof of the first part of the following lemma while proving our main results. We include its proof due to Chen et. al. to help the reader see the similarity in our counting argument and theirs. We note that we state the following lemma in the terminology of this thesis to prevent confusion:

**Lemma 4.1.2** (Chen, Yu and Zang [4], 2008). Let $G$ be critical with $\chi'(G) = \Delta(G) +$
Let \((T, e, \pi)\) be a VKT*-tree with \(c(T) = c \geq 0\) connecting edge sets such that \(T = (X_0, E_1, x_1, \ldots, X_m)\). Let also \(T_{m-1}\) be elementary with respect to \(\pi\). If \(k \geq \sqrt{\Delta(G)/2}\) then \(c(T) < k\).

**Proof.** We assume on the contrary that \(c \geq k \geq 1\). \(|V(S_0)| \geq 3\) and \(|V(S_i)| \geq 2\) for \(1 \leq i < c\). Since \(T_{m-1}\) is elementary with respect to \(\pi\), \(Y_{c-1}\) is elementary with respect to \(\pi\). Counting the number of missing colors in \(Y_{c-1}\), we get

\[
\Delta + k \geq |\bar{\pi}(Y_{c-1})| \geq (3 + 2(c - 1))k + 2 \\
\geq 2k^2 + k + 2 \\
> 2k^2 + k
\]

This implies \(\Delta > 2k^2\) or \(k < \sqrt{\Delta/2}\) contrary to the assumption. This completes the proof.

4.2 An Asymptotic Bound for \(\chi'\)

Let \((T, e, \pi)\) be a VKT*-tree with \(c(T) = c\). If \(c = 0\), then \(T\) is a Tashkinov tree, which is elementary by Theorem 2.4.1. On the other hand, the initial section of \(T\), \(S_0\) is a maximal Tashkinov tree always. In this section, we will find a lower bound on the number of vertices of a Tashkinov tree. Together with the idea of the proof of Lemma 4.1.2, we will give a proof of the following theorem:

**Theorem 4.2.1.** Let \(G\) be a graph with \(\chi'(G) > \Delta(G) + 1\). If \(\chi'(G) > \Delta(G) + \sqrt{\Delta(G)}/2\), then \(G\) is elementary.

was initially the same as that of Favrholdt et. al., we did not know about their result at the time we proved our result. We later used the techniques in [4], to get the final form of Theorem 4.2.1.

While we had our own version of the proof of the following lemmas, we will use the notation of balanced Tashkinov trees introduced by Scheide and Stiebitz [35].

Throughout this section, let $G$ be critical with $\chi'(G) = \Delta(G) + k + 1$ for an integer $k \geq 1$, $(T, e, \pi) \in T^B(G)$ with $t(G) = t$, $h(G) = h \geq 5$ and $|F(T, e, \pi)| = d$.

**Lemma 4.2.2.** If $t = \Delta + k$ then $G$ is elementary.

**Proof.** If $t = \Delta + k$ then $V(G) = V(T)$ for, otherwise, $G$ is disconnected contrary to its criticality. \hfill \Box

The following simple observation is due to Favrholdt et. al. [11].

**Lemma 4.2.3.** If $\bar{\pi}(x) \setminus \bar{\pi}(T) \neq \emptyset$ for all $x \in V(T)$, then $G$ is elementary.

Lemma 4.2.3 suggests that, for all $x \in F(T, e, \pi)$, $\bar{\pi}(x) \subseteq \pi(T)$.

**Lemma 4.2.4.** $|\pi(T)| \leq \frac{t - h}{2} + 2$.

**Proof.** For the first $h - 1$ edges of $T$, only 2 colors are used and there are $t - h$ edges of $T$ left. Since $T$ is balanced, every color used on the rest of $T$ are used even number of times. Hence, the rest of $T$ uses at most $\frac{t - h}{2}$ edges. Hence, the proof follows. \hfill \Box

**Lemma 4.2.5.** $t(G) \geq 2dk + h$.

**Proof.** If $V(e) \cap F(T, e, \pi) = \emptyset$ then at least $dk$ colors are used on $T$ by Lemma 4.2.3. In this case, the edges that are used on the trunk of $T$ are not counted since they are colored by the colors missing at $\bar{\pi}(V(e))$. Hence, 2 more colors must be used on $T$. On the other hand, for the addition of each of the end points of $e$ to $F(T, e, \pi)$,
we add one color to the used colors list while subtracting one from the colors used on the trunk. Hence, $|\pi(T)| \geq dk + 2$.

Combining this result with Lemma 4.2.4, we get $dk + 2 \leq \frac{t - h}{2} + 2$. This implies $t(G) \geq 2dk + h$. \hfill \Box

**Corollary 4.2.6.** $t(G) \geq 2k + 5$.

**Proof.** If $d = 0$ then $G$ is elementary by Lemma 4.2.3. Hence, we may assume $d \geq 1$. We also have $h(G) \geq 5$ by Lemma 2.6.4. Hence, the proof follows. \hfill \Box

**Proposition 4.2.7.** Let $G$ be a critical graph with $\chi'(G) = \Delta(G) + k + 1$ for an integer $k \geq 1$ and $(T, e, \pi)$ be a VKT*-triple with $T = (X_0, E_1, X_1, \ldots, X_m)$ and $c(T) = c$. Let $S_0, S_1, \ldots, S_c$ be its sections with $|S_0| = t(G)$ and $T_{m-1}$ be elementary with respect to $\pi$. If $k \geq \sqrt{\frac{\Delta}{4}}$ then $c(T) < k$.

**Proof.** We repeat the idea in the proof of Lemma 4.1.2. We assume, on the contrary, $c \geq k \geq 1$. By Corollary 4.2.6, $|V(S_0)| \geq 2k + 5$ and $|V(S_i)| \geq 2$ for $1 \leq i < s$. Since $T_{m-1}$ is elementary, we can calculate the number of its missing colors as follows:

$$
\Delta + k \geq |\bar{\pi}(T_{m-1})| \\
\geq \sum_{i=0}^{c-1} |\bar{\pi}(S_i)| \\
\geq k \sum_{i=0}^{c-1} |V(S_i)| + 2 \\
\geq k((2k + 5) + 2(c - 1)) + 2 \\
> 4k^2 + k
$$

Hence, $\Delta > 4k^2$ which implies $k < \sqrt{\frac{\Delta}{4}}$ contrary to the assumption that $k \geq \sqrt{\frac{\Delta}{4}}$. This completes the proof. \hfill \Box
Together with Lemma 4.1.1, Proposition 4.2.7 implies Theorem 4.2.1 as a corollary.

4.3 A Better Asymptotic Bound

Let $G$ be a critical graph with $\chi'(G) \geq \Delta(G) + k + 1$ where $k \geq 1$ and $(T, e, \pi)$ be a VKT*-triple with $T = (X_0, E_1, \ldots, X_n)$, $c(T) = c$ and sections $S_0, S_1, \ldots, S_c$. Let, for $0 < i_1 < i_2 < \ldots < i_c \leq n$, $E_{i_1}, E_{i_2}, \ldots, E_{i_c}$ be the connecting edge sets of $(T, e, \pi)$. Suppose, for $1 \leq j \leq c$, $\pi(E_{i_j}) = \delta_j$.

The smallest closed VKT*-triple having $(T, e, \pi)$ as a subtree is called the closure of $(T, e, \pi)$ and denoted by $(\bar{T}, e, \pi)$. We note that $\bar{T} = T$ if $T$ is closed with respect to $\pi$. We say that $(T, e, \pi)$ is bounded if $c(T) = 0$ or, for $0 \leq j < c$, $|V(S_j) \setminus I_{\delta_j}(Y_{j-1}, e, \pi)| < |\bar{\pi}(Y_{j-1})|$ and $|V(S_c(\bar{T})) \setminus I_{\delta_c}(T, e, \pi)| < \bar{\pi}(Y_{c-1})$. $G$ is called $\chi'$-bounded if any VKT*-triple $(T, e, \pi)$ is bounded.

**Lemma 4.3.1.** Let $G$ be critical with $\chi'(G) = \Delta(G) + k + 1$ for an integer $k \geq 1$ and $(T, e, \pi)$ be a closed VKT*-triple with sections $S_0, S_1, \ldots, S_c$. Then, for $0 \leq i \leq c$ and $\alpha, \beta \in \bar{\pi}(Y_i)$, the family of $(\alpha, \beta)$-chains can be partitioned into chains with whose vertex set is a subset of one of $V(Y_i)$, $V(S_{i+1}), \ldots, V(S_c)$ and $V(G) \setminus V(S)$.

**Proof.** It is enough to observe that $Y_0, Y_1, \ldots, Y_c = T$ are all closed under $\pi$. Hence, if $\alpha, \beta \in \bar{\pi}(Y_i)$ then, by the definition of a VKT*-tree, an $(\alpha, \beta)$-chain $P$ is either within or outside $Y_i$ since $Y_i$ is closed. Similarly, if $P$ is outside $Y_i$ and intersects $Y_{i+1}$, then $P$ must be within $Y_{i+1}$ and outside $Y_i$. This means that $P$ is within $V(S_{i+1})$. But this idea generalizes to all $Y_j$ with $j \geq i$.

The following small observation allows us to treat the VKT*-trees the same with Tashkinov trees.
Lemma 4.3.2. Let $G$ be critical with $\chi'(G) = \Delta(G) + k + 1$ for an integer $k \geq 1$ and $(T, e, \pi)$ be a closed and bounded VKT*-triple with sections $S_0, S_1, \ldots, S_c$. Then there exists a VKT*-triple $(T, e, \pi')$ such that $\pi'(S_0) \cap \Gamma^f(T, e, \pi') \neq \emptyset$.

Proof. Let $\alpha \in \Gamma^f(T, e, \pi)$. We may assume $\alpha \notin \bar{\pi}(S_0)$. We pick $\beta \in \Gamma^f(S_0, e, \pi)$. By Lemma 4.3.1, the set of $(\alpha, \beta)$-chains within $T$ are either within $V(S_0)$ or within $V(T) \setminus V(S_0)$. Re-coloring all $(\alpha, \beta)$-chains within $V(T) \setminus V(S_0)$, we get a coloring $\pi'$ for which $(T, e, \pi')$ is a VKT*-triple and $\beta \in \Gamma^f(T, e, \pi') \cap \bar{\pi}'(S_0)$. This completes the proof.

Proof of Theorem 1.6.8. We assume on the contrary that $k > \frac{3 \sqrt{\Delta(G)}}{2}$. We would like to show that any VKT*-triple $(T, e, \pi)$ with $T = (X_0, E_1, X_1, \ldots, X_m)$ such that $c(T) = c > 0$ and $|V(S_0)| = t(G)$ is elementary. We will call such trees as special VKT*-trees.

Assume on the contrary that there exists a special VKT*-triple $(T, e, \pi)$ such that

1. $T$ is not elementary.
2. $c(T) = c$ is smallest with respect to (1).
3. The path number $p(T)$ is smallest with respect to (1), (2).
4. $|V(S_c)|$ is smallest with respect to (1) – (3)

Let, for $0 < i_1 < i_2 < \ldots < i_c \leq n$, $E_{i_1}, E_{i_2}, \ldots, E_{i_c}$ be the connecting edge sets of $(T, e, \pi)$. Suppose, for $1 \leq j \leq c$, $\pi(E_{i_j}) = \delta_j$. For $1 \leq i \leq c$, we denote $I_{\delta_j} = I_j$.

If $T$ is closed, then $T_{m-1}$ is elementary and $\bar{\pi}(T_{m-1}) \cap \bar{\pi}(X_m) \neq \emptyset$. This implies that either $T_{m-1}$ is also closed or $X_m = \{x_m\}$ and $\bar{\pi}(T_{m-1}) \subseteq \bar{\pi}(x_m)$ for, otherwise, we need another vertex by a simple parity argument. Hence, we may assume that either $T$ and $T_{m-1}$ are both closed, or $T$ is not closed.
(a) $Y_{c-1}$ is bounded.

Proof of (a): By minimality of $c$, $Y_{c-1}$ is elementary. We assume on the contrary that it is not bounded; say, $|V(S_j) \setminus I_j| \geq |ar{\pi}(Y_{j-1})|$ for some $1 \leq j < c$. Then we have

$$\Delta + k \geq |ar{\pi}(Y_{c-1})|$$
$$= \sum_{i=0}^{c-1} |\bar{\pi}(S_i)|$$
$$> k(2k + 5) + k(2(j - 1)) + k(k(2k + 5 + 2(j - 1)))$$
$$> 2k^3 + k.$$

This implies that $k < \sqrt[3]{\Delta}/2$ contrary to the initial assumption on $k$. Hence, the proof of (a) is completed.

(b) For any $\alpha \in \bar{\pi}(Y_{c-1})$ and $\beta \notin \bar{\pi}(Y_{c-1})$, there is a unique acyclic $(\alpha, \beta)$-chain intersecting $Y_{c-1}$.

Proof of (b): We may assume $\Gamma^I(Y_{c-1}, e, \pi) \cap \bar{\pi}(S_0) \neq \emptyset$ by Lemma 4.3.2. But then the proof of (b) is exactly same with the proof of Lemma 2.5.3.

(c) $V(Y_{c-1}) \cup A_C(Y_{c-1}, e, \pi)$ is elementary with respect to $\pi$

Proof of (c): This is exactly same proof with Lemma 2.5.4. We only need to use (b) instead of Lemma 2.5.3.

(d) $I_c \subset I(Y_{c-1}, e, \pi) \subseteq A(Y_{c-1}, e, \pi)$

Proof of (d): This is exactly same proof with Lemma 2.5.11 and Corollary 3.1.6. We only need to use (b), (c) instead of Lemmas 2.5.3, 2.5.4.

We note that this implies $T$ is not closed since, if $T$ and $T_{m-1}$ are both closed then by (d), $T$ is elementary.

(e) Let $F$ be an $(Y_{c-1}, e, \pi)$-fan. Then $V(F) \subseteq A(Y_{c-1}, e, \pi)$.
Proof of (e): This is exactly same proof with Proposition 3.2.1. We only need to use (b) – (d) instead of Lemmas 2.5.3, 2.5.4, 2.5.11 and Corollary 3.1.6.

(f) Let \( K \) be an \( (Y_{c-1} \cup F_c, e, \pi) \)-path where \( F_c \) the maximal \( (Y_{c-1}, e, \pi) \)-fan and \( |V(K)| < |\overline{\pi}(Y_{c-1})| \), then \( K \) is elementary.

Proof of (f): This is exactly the same proof with Proposition 3.3.1. We only need to use (b) – (e) instead of Lemmas 2.5.3, 2.5.4, 2.5.11 and Corollary 3.1.6.

We do not need (f) in full capacity since, by the definition of a VKT*-triple, only the knowledge of \( (Y_{c-1} \cup I_c, e, \pi) \)-paths is required in this proof. This facts follows since \( I_c \subset V(F_c) \).

Let \( |V(S_c(\overline{T})) \setminus I_c| = \overline{a}_c \).

(g) If \( \overline{a}_c < |\overline{\pi}(Y_{c-1})| \), then \( T \) is elementary.

Proof of (g): This is exactly same proof with Proposition 3.4.1. We only need to use (b) – (f) instead of the respective results used in the original proof.

Since \( T \) is not elementary, (g) implies that \( s_c \geq |\overline{\pi}(Y_{c-1})| \).

(h) There exists an elementary special VKT-triple \( (R, e, \pi) \) such that \( c(R) = c \), \( R \) is a subtree of \( \overline{T} \) with \( S_i(R) = S_i(T) \) for \( 0 \leq i < c \) and \( |V(R_c)| = |\overline{\pi}(Y_{c-1})| - 1 \).

Proof of (h): This is exactly same proof with Proposition 3.4.2. We only need to use (b) – (g) instead of the respective results used in the original proof.

By (g), we have

\[
\Delta + k \geq |\overline{\pi}(R)| \geq \sum_{i=0}^{c-1} |\overline{\pi}(S_i)| + |\overline{\pi}(R_c)| \geq [k((2k + 5 + 2(c - 1)) + 2] + k(2k + 5 + 2(c - 1) + 2 - 1) > 2k^3 + k.
\]
This implies that \( k < \sqrt[3]{\Delta/2} \) contrary to our initial assumption on \( k \). Hence, any special VKT*-triple is elementary. This completes the proof.

As one can easily observe, boundedness allows us generalize the results from Chapters 2 and 3.

### 4.4 A Linear Bound

In this section, we will prove Theorem 1.6.9.

**Proof of Theorem 1.6.9:** Let \( \chi' > \frac{25}{24} \Delta + \frac{22}{24} \). Let \( (T, e, \pi) \) with \( T = (X_0, E_1, \ldots, X_m) \) be a smallest special VKT*-triple which is not elementary. Then \( c(T) \geq 1 \) and \( T_{m-1} \) is elementary. It is enough to show that \( Y_{c-1} \) is bounded since the rest follows from (b) – (g) of Theorem 1.6.8.

If \( c(T) = c = 1 \), then \( Y_0 = S_0 \) is bounded by the definition of boundedness. Hence, we may assume \( c \geq 2 \). By Theorem 2.6.6, \( |V(S_0)| \geq 11 \) and by Corollary 3.1.6, \( |I_1| \geq 2 \). On the other hand, by Lemma 2.2.4, any elementary set \( X \subset V(G) \) with \( V(e) \subset X \) must satisfy \( |X| \leq 24 \). Hence, \( |V(Y_{c-1}) \setminus (V(S_0) \cup I_1)| \leq 11 < |\bar{\pi}(S_0)|. \) But this implies \( Y_{c-1} \) is bounded.

Since \( T \) is not elementary with respect to \( \pi \), this implies that there exists an elementary special VKT*-triple \( (R, e, \pi) \) with \( Y_{c-1}(R) = Y_{c-1}(T) \) and \( |V(R_{c-1})| = |\bar{\pi}(Y_{c-1})| - 1 \). But then \( |V(R)| \geq |V(S_0)| + |I_1| + (|\bar{\pi}(S_0)| - 1) \geq 11 + 2 + (13 - 1) = 25 \) contrary to Lemma 2.2.4. This completes the proof.

**4.5 Notes**

We note that improving Theorems 1.6.8 and 1.6.9 is possible by using a generalization of the Gupta trees to VKT*-trees instead of regular VKT*-trees and employing...
Theorem 2.6.7 instead of Theorem 2.6.6. Chen et al. [5] in fact has already claimed this result.

The following theorem whose proof is ingrained in the proof of Theorem 1.6.8 allows us generalize our results in Chapters 2 and 3 to special VKT*-trees. In fact, Theorems 1.6.8 and my3 can be seen as its corollaries.

**Theorem 4.5.1.** Let $G$ be critical with $\chi'(G) = \Delta(G) + k + 1$ for an integer $k \geq 1$. If any special VKT*-triple $(T, e, \pi)$ in $G$ is $\chi'$-bounded, then $G$ is elementary.

This result suggests that anyone who wishes to solve Conjecture 1.3.4 through Kempe chains must prove “Any VKT*-triple $(T, e, \pi)$ with $c(T) = 1$ is elementary.”

On the other hand, it would be an enormous contribution if some reasonable lower bound in terms of $\Delta$ to $t(G)$, or any such constraint for that matter, is found. For now, the research on that end gives lower bounds in terms of $\chi' - \Delta - 1$ which can only be replaced by small numbers in reality since $\chi' - \Delta - 1 \geq 1$.

Lastly, There are two separate streams of research attacking Conjecture 1.3.4: Coloring Theory and Combinatorial Optimization. It seems that it would lead to significant results if one were able to incorporate the products of these two streams into one. We suggest the interested researcher to find ways to use both sides of the research on the subject.
CHAPTER 5
NEW CONCEPTS AND LEMMAS

In this chapter, we will present some new concepts and results that are either used in [24] or introduced in various attempts to prove Conjecture 1.3.4. We note that some of these tools, though only definitions, are quite useful in the proof of Theorem 2.6.7.

5.1 Strongly Elementary Sets

One of the intermediate problems in finding a proof of Conjectures 1.3.4 (and 1.3.5 under some careful adjustments) is the following:

**Problem 5.1.1.** Let $G$ be a critical graph with $\chi'(G) > \Delta(G) + 1$, $k = \chi'(G) - 1$ and $\pi \in C_k(G \setminus e)$ for some $e \in E(G)$. Does there exist a set $X \subseteq V(G)$ such that $V(e) \subseteq X$ and $X$ behaves “like” $V(e)$?

The term “like” needs to be mathematically stated in this problem. The following observations will be used for this reason:

Let $V(e) = \{x, y\}$. We easily observe that $V(e)$ is elementary with respect to $\pi$. We are indeed able to find larger elementary sets containing $V(e)$. We also observe that $\bar{\varphi}(x) \cap \bar{\varphi}(y) = \emptyset$ for all $\varphi \in [\pi]$. This is a lot stronger than elementariness.

$X \subseteq V(G)$ is called **strongly elementary** if $\bar{\varphi}(v) \cap \bar{\varphi}(w) = \emptyset$ for all $\varphi \in [\pi]$ and distinct $v, w \in X$. If $X$ is not strongly elementary with respect to $\pi$, then it is called **fragile**.
It is important to observe how fragile a set is. \( X \) is called \( s \)-fragile with respect to \( \pi \) if there exists a chain sequence of length \( s \), \( \pi \rightarrow^s \pi' \) such that \( X \) is not elementary with respect to \( \pi' \), while for any \( r < s \) and any chain sequence of length \( r \), \( \pi \rightarrow^r \pi'' \), \( X \) is elementary with respect to \( \pi'' \). \( X \) is called \( \infty \)-fragile if it is strongly elementary with respect to \( \pi \).

The following is a reformulation Problem 5.1.1:

**Problem 5.1.2.** Let \( G \) be a critical graph with \( \chi'(G) > \Delta(G) + 1 \), \( k = \chi'(G) - 1 \) and \( \pi \in \mathcal{C}_k(G \setminus e) \) for some \( e \in E(G) \). Does there exist a set \( X \subseteq V(G) \) such that \( V(e) \subseteq X \) and \( X \) is strongly elementary?

This problem is believed to be really hard and Mcclain [30] even claims that if one can find a strongly elementary set \( X \) with \( V(e) \subseteq X \) and \( |X| = 3 \), then one can prove Conjecture 1.3.4. We note that a strongly closed, elementary set is indeed strongly elementary.

A “Yes” answer to Problem 5.1.2 means that we can then treat the set \( X \) like the initial uncolored edge and repeat many of the results that are given recursively for \( e \) such as the fan, path and tree arguments without encountering the challenges that we had in Chapter 3. More importantly, constructive arguments that are previously used to find a strongly elementary set can be omitted ultimately giving a set that behaves exactly like the end points of an uncolored edge.

The following conjecture is equivalent to Conjecture 2.4.2. Hence, answering Problem 5.1.2 positively is rather important:

**Conjecture 5.1.3.** Let \( G \) be critical with \( \chi'(G) > \Delta(G) + 1 \) and \( k = \chi' - 1 \). Then there exists \( e \in E(G) \) and \( \pi \in \mathcal{C}_k(G \setminus e) \) such that the maximal strongly elementary set \( X \subseteq V(G) \) with \( V(e) \subseteq X \) with respect to \( \pi \) is also strongly closed.
5.2 Chain Sequences and Flowers of Trees

In this section, we will describe a way to find a strongly elementary set starting with a maximal Tashkinov tree. Our reason for starting with a maximal Tashkinov tree is simple: It is the smallest closed set containing the end points of an uncolored edge.

The following conjecture is very important and useful for anyone who attempts to prove Conjecture 1.3.4 or, equivalently, Conjectures 2.4.2 or 5.1.3:

**Conjecture 5.2.1.** Let \( G \) be critical with \( \chi'(G) = k + 1 \) where \( k \geq \Delta(G) + 1 \) and \( (T, e, \pi) \) be a maximal Tashkinov triple. Then, for any \( r \geq 0 \) and \( (T, e, \pi) \rightarrow^r (T', e, \pi') \), \( V(T) \cup V(T') \) is elementary with respect to \( \pi \).

We call \( Fl(T, e, \pi) = \bigcup \{ V(T') : (T', e, \varphi) \text{ is maximal, } \varphi \in [\pi] \} \) as the flower at \( (T, e, \pi) \) or \( \pi \). We will use \( Fl(\pi) \) instead when no confusion arises. Note that almost all of the results such as fans, paths and trees can be generalized by simply starting with \( Fl(\pi) \) instead of the uncolored edge \( e \) or its end points. Hence, this small conjecture is rather important.

We next define a \( k \)-flower at \( (T, e, \pi) \) as the union of vertices of all maximal Tashkinov trees which can be reached from \( \pi \) by a sequence of length \( k \) or less:

\[
Fl_k(T, e, \pi) = \bigcup \{ V(T') : (T', e, \varphi) \text{ is maximal, } \varphi \rightarrow^p \pi \text{ for some } p \leq k \}
\]

. Note that \( Fl(\pi) = \bigcup_{k=0}^{\infty} Fl_k(\pi) \). Figure 5.1 represents \( Fl_1(T, e, \pi) \).

In what follows, we give a series of results proving that \( Fl_1(\pi) \) is elementary with respect to \( \pi \).

**Lemma 5.2.2.** Let \( G \) be critical with \( \chi'(G) = k + 1 \) for an integer \( k \geq \Delta(G) + 1 \) and let \( (T, e, \pi) \) and \( (T', e, \pi') \) be maximal Tashkinov triples such that \( (T, e, \pi) \rightarrow^P (T', e, \pi') \) where \( P \) is a \((\alpha, \beta)\)-chain.

1. If \( \alpha, \beta \in \bar{\pi}(T) \) then \( V(T') \subseteq V(T) \).
Figure 5.1: $Fl_1(T, e, \pi)$ for some maximal Tashkinov triple $(T, e, \pi)$.

(2) If $\alpha, \beta \in \bar{\pi}'(T')$ then $V(T) \subseteq V(T')$.

(3) If $\alpha, \beta \not\in \bar{\pi}(T)$ then $T = T'$.

Proof. We note that it is enough to prove (1) since (2) follows by symmetry with $(T', e, \pi') \rightarrow P'' (T, e, \pi)$ where $P' = P$ under $\pi'$ and (3) clearly holds. Since $V(T)$ is closed with respect to $\pi$ and $\alpha, \beta \in \bar{\pi}(T)$, $P$ is either in $G[V(T)]$ or outside. In any case, $V(T)$ is still closed after recoloring $P$ and hence, $V(T')$, being the smallest closed set containing $V(e)$, must be a subset of $V(T)$. This completes the proof.

Lemma 5.2.3. Let $G$ be critical with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$ and let $(T, e, \pi)$ and $(T', e, \pi')$ be maximal Tashkinov triples such that $(T, e, \pi) \rightarrow P' (T', e, \pi')$ where $P$ is a $(\alpha, \beta)$-chain. Then the following holds:

1. $V(T) \cup V(T')$ is elementary with respect to both $\pi$ and $\pi'$
2. If $\alpha, \beta \in \bar{\pi}(T)$ or $\alpha, \beta \in \bar{\pi}'(T)$ or $\alpha, \beta \not\in \bar{\pi}(T)$ then $V(T') \setminus V(T) \subseteq A(T, e, \pi)$. 

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(3) If $\alpha \in \tilde{\pi}(T)$ and $\beta \notin \tilde{\pi}(T)$ then $V(T') \setminus V(T) \subseteq A_C(T, e, \pi)$ where $C = \tilde{\pi}(T) \setminus \{\alpha\}$.

Proof. If $\alpha, \beta \in \tilde{\pi}(T)$ or $\alpha, \beta \in \tilde{\pi}'(T)$ or $\alpha, \beta \notin \tilde{\pi}(T)$ then we have either $V(T') \subseteq V(T) = \emptyset$ or $V(T) \subseteq V(T')$ in which case $V(T')$ is closed under $\pi$ and $\pi'$ implies that any two colors inside are chained to each other. This implies (2). In this very case, (1) also follows directly. To complete the proof of (1), it is enough to show that $V(T) \cup V(T')$ is elementary with respect to $\pi$ when $\alpha \in \tilde{\pi}(T)$ and $\beta \notin \tilde{\pi}(T \cup T')$ since the other cases follow directly from Lemma 5.2.2.

We assume, on the contrary, that, for some distinct $x, y \in V(T) \cup V(T')$, there exists $\gamma \in \tilde{\pi}(x) \cap \tilde{\pi}(y)$. Clearly, we may assume $x \in V(T)$ and $y \in V(T') \setminus V(T)$ for if $x, y \in V(T)$ then $\tilde{\pi}(x) \cap \tilde{\pi}(y) = \emptyset$ and if $x, y \in V(T') \setminus V(T)$ then $\gamma \neq \alpha, \beta$ implies $\tilde{\pi}'(x) \cap \tilde{\pi}'(y) = \emptyset$. Since $y \notin V(T)$, we have $\gamma \neq \alpha, \beta$.

Let $C_1 = \tilde{\pi}(V(T) \cap V(T')) \setminus \{\alpha\}$. Note that $Tx = T'x$ and $|V(T) \cap V(T)|$ is odd. Hence, we have $|V(T) \cap V(T')| \geq 3$ implying that $C_1 \neq \emptyset$. We also note that $C_1 \subseteq \tilde{\pi}'(T)$ and for all $\varepsilon \in C_1$ there exists $z \in V(T) \cap V(T')$ such that $\varepsilon \in \tilde{\pi}(z) \cap \tilde{\pi}'(z)$. This implies $P_z(\varepsilon, \gamma, \pi) = P_z(\varepsilon, \gamma, \pi') = P_y(\gamma, \varepsilon, \pi') = P_y(\gamma, \varepsilon, \pi)$ since $y, z \in V(T')$ and $\varepsilon, \gamma$ are different from $\alpha, \beta$. On the other hand, $\gamma, \varepsilon \in \tilde{\pi}(T)$ and $V(T)$ is closed with respect to $\pi$. So, $P_z(\varepsilon, \gamma, \pi) \cap P_y(\varepsilon, \gamma, \pi) = \emptyset$ since $y \notin V(T)$ contrary to what was shown before. This completes the proof of (1).

Moreover, for all $v \in V(T') \setminus V(T)$, $\gamma \in \tilde{\pi}(v)$ and $\varepsilon \in C_1$, $P_x(\varepsilon, \gamma, \pi) = P_x(\varepsilon, \gamma, \pi')$ must end in $T'$ since $\varepsilon, \gamma \in \tilde{\pi}(T') \setminus \{\alpha\} \subseteq \tilde{\pi}'(T')$. Hence, $V(T) \setminus V(T') \subseteq A_{C_1}(T, e, \pi)$.

Secondly, let $\varepsilon \in \tilde{\pi}(u)$ for some $u \in V(T) \setminus V(T')$ and $\gamma \in \tilde{\pi}(z)$ for some $z \in 72
$V(T') \setminus V(T)$. Assume on the contrary that $P_u = P_u(\varepsilon, \gamma, \pi) \neq P_z(\varepsilon, \gamma, \pi) = P_z$ which implies that $P_z \cap T = \emptyset$ by Lemma 2.5.3. Using $P_z$ we get

$$
\begin{array}{ccc}
(T, \pi) & \xrightarrow{P} & (T', \pi') \\
\downarrow_{P_z} & & \downarrow_{P'_z} \\
(T_1, \pi_1) & \xrightarrow{P'_z} & (T'_1, \pi'_1)
\end{array}
$$

where $P = P'$ and $P_z = P'_z$. We note that $T_1 = T$ and $T'_z = T'_1z$ and $\varepsilon \in \bar{\pi}_1(z) \cap \bar{\pi}_1(u)$ contrary to (1). Hence, $V(T') \setminus V(T) \subseteq A_{C_2}(T, e, \pi)$ where $C_2 = \bar{\pi}(V(T) \setminus V(T'))$. Therefore, we have $V(T') \setminus V(T) \subseteq A_C(T, e, \pi)$ where $C = C_1 \cup C_2 = \bar{\pi}(T) \setminus \{\alpha\}$. This completes the proof of (3).

\[\Box\]

![Figure 5.2](image_url)

Figure 5.2: A representation of Lemma 5.2.3
Proposition 5.2.4. Let $G$ be critical with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$ and let $(T, e, \pi)$ be a maximal Tashkinov triple. Then $Fl_1(T, e, \pi)$ is elementary with respect to $\pi$.

Proof. We assume that $(T_1, \pi_1) \xrightarrow{(\alpha_1, \beta_1)} (T, \pi) \xrightarrow{(\alpha_2, \beta_2)} (T_2, \pi_2)$. We may also assume $\alpha_1, \alpha_2 \in \bar{\pi}(T)$ and $\beta_1, \beta_2 \notin \bar{\pi}(T)$ without loss of generality for, otherwise, $V(T_i) \subseteq V(T)$ for some $i = 1, 2$ and the proof follows by Lemma 5.2.3. Again by Lemma 5.2.3, $V(T_i) \setminus V(T) \subseteq A_{C_i}(T, e, \pi)$ where $C_i \supseteq \bar{\pi}(T) \setminus \{\alpha_i\}$ for $i = 1, 2$. Set $C = C_1 \cap C_2$. Clearly, $C \neq \emptyset$ implying that $V(T_1) \cup V(T_2) \setminus V(T) \subseteq A_C(T, e, \pi)$ by Lemma 2.5.6. But then this implies that $X = V(T) \cup V(T_1) \cup V(T_2)$ is elementary with respect to $\pi$ by Lemma 2.5.4 and completes the proof since for any $x, y \in Fl_1(T, e, \pi)$, there exists a diagram as the one used above such that $x, y \in V(T) \cup V(T_1) \cup V(T_2)$.

Proposition 5.2.4 is used in the proof of Theorem 2.6.7 and, to some extent, simplifies the proof of the structural lemma in [35].

If one wishes to repeat the same results for $Fl_2(T, e, \pi)$, one has to either find a more delicate approach or show that $V(T') \setminus V(T) \subseteq A(T, e, \pi)$ in Lemma 5.2.2(3).

5.3 Fans over Fans

The following result directly follows from Lemmas 3.2.2 and 2.5.4

Corollary 5.3.1. Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$, $(T, e, \pi)$ be a maximal Tashkinov triple and $F_0$ be the unique maximal $(T, e, \pi)$-fan. Then $Y_0 \cup J_1$ is elementary with respect to $\pi$.

Vizing [43] defined fans strictly in a way that the color of the next edge in the fan must miss in the previous vertex added to the fan. Though the fans defined in this section are a lot more general, such fans will be called Vizing-fans. We note that a
minimal fan is usually a Vizing-fan. We also note that the maximal \((Y_i, e, \pi)-\text{fan}\) is the union of all \((Y_i, e, \pi)-\text{Vizing-fans}\).

**Lemma 5.3.2.** Let \(G\) be critical with \(\chi'(G) > \Delta(G) + 1\), \((T, e, \pi)\) be a maximal Tashkinov triple and \(F_0\) be the unique maximal \((T, e, \pi)-\text{fan}\). For any \((Y_0, e, \pi)-\text{Vizing-fan}\) \(F_1 = (e_1, y_1, \ldots, e_n, y_n)\), \(V(F_1) \subseteq A_C(T, e, \pi)\) where \(C = \bar{\pi}(T)\) unless \(\bar{\pi}(Y_0) \cap E_G(Y_0, y_1) = \{\bar{\pi}(e_1)\} \subset \bar{\pi}(T)\) and \(C = \bar{\pi}(T) \setminus \bar{\pi}(e_1)\) otherwise.

**Proof.** Let \(F_1 = (e_1, y_1, \ldots, e_k, y_n)\) be a smallest \((Y_0, e, \pi)-\text{Vizing-fan}\) satisfying the following conditions:

1. \(V(F_1 y_{n-1}) \subseteq A_C(T, e, \pi)\).
2. There exists \(y_q \in V(T) \cup V(F_0) \cup V(F_1 y_{n-1})\) such that \(\bar{\pi}(y_q) \cap \bar{\pi}(y_n) \neq \emptyset\) with the assumption that \(y_0 \in V(T) \cup V(F_0)\) if \(q = 0\).

By Lemma 3.2.2, we may assume \(n > 1\). Let \(\alpha \in \bar{\pi}(y_n) \cap \bar{\pi}(y_q)\). We have the following cases:

**Case 1:** \(q = 0\), \(\alpha \in \bar{\pi}(T)\) and \(\alpha \neq \pi(e_1)\)

**Proof of (1):** Let \(\beta = \pi(e_n) \in \bar{\pi}(y_{n-1})\). Since \(y_{n-1} \in A_C\), \(P_{y_n}(\alpha, \beta, \pi) \cap T = \emptyset\).

Re-coloring \(P_{y_n}\), we get a coloring \(\pi'\) such that \(Y_0(\pi') = Y_0(\pi) = Y_0\), \(F_1 y_{n-1}\) is a Vizing-fan and \(y_n \in J_1(\pi')\). By Lemma 3.2.2, \(y_n \in A_{C'}(T, e, \pi')\) where \(C' = \pi'(T) \setminus \{\alpha\}\).

Moreover, \(y_{n-1} \in A_{C''}(T, e, \pi')\) where \(C'' = C \cap C'' \neq \emptyset\) contrary to the fact that \(\beta \in \bar{\pi}'(y_n) \cap \bar{\pi}'(y_{n-1})\).

Hence, the proof follows.

**Case 2:** \(q = 0\), \(\pi(e_1) = \alpha \in \bar{\pi}(T)\)

**Proof of (2):** Let \(e_1 \in E_G(x, y_1)\). Then \(x \in V(F_0)\) by the maximality of \(F_0\). Let \(\varepsilon \in \bar{\pi}(x)\). Then \(e_1 \in P_x(\alpha, \varepsilon, \pi)\) which ends in \(T\). Re-coloring \(P_{y_n}(\alpha, \varepsilon, \pi)\), we get 75
a coloring $\pi'$ for which $Y_0$, $F_0$, $F_1$ are all well-defined and $\varepsilon \in \pi'(y_n) \cap \pi'(x)$ and $\pi'(e_1) = \alpha$. Next, let $\gamma \in \pi(T) \setminus \{\alpha\}$. Now, re-coloring $P_{y_n}(\varepsilon, \gamma, \pi')$, we get Case 1.

Case 3: $q = 0$, $\alpha \in \pi(F_0)$

Proof of 3: We assume $\alpha \in \pi(z)$ for some $z \in V(F_0)$. Let $\gamma \in \pi(T) \setminus \{\pi(e_1)\}$. Since $z \in A(T, e, \pi)$, we have $P_{y_n}(\alpha, \gamma, \pi) \cap T = \emptyset$. Re-coloring $P_{y_n}$, we get one of Cases 1 or 2.

Case 4: $q > 0$

Proof of 4: Let $\varepsilon \in \pi(T) \setminus \{\pi(e_1)\}$. Then $P_{y_q}(\alpha, \varepsilon, \pi) \cap T \neq \emptyset$ by minimality of $n$. If $e_{q+1} \in P_{y_q}$, then re-coloring $P_{y_q}(\alpha, \varepsilon, \pi)$, we get a coloring $\pi'$ for which Case 1 holds. Otherwise, re-coloring $P_{y_q}$, we get a coloring $\pi''$ for which $Y_0$ is well-defined and $e_{q+1}F_1$ is a smaller $(Y_0, e, \pi'')$-Vizing-fan for which $\varepsilon \in \pi''(T) \cap \pi(y_n)$ contrary to the minimality of $|V(F_1)|$. Hence, the proof follows.

This completes the proof that $Y_0 \cup V(F_1)$ is elementary with respect to $\pi$. Moreover, $y_n \in A_{C_1}(T, e, \pi)$ follows directly from elementariness. \qed

The following is a direct result of Lemma 5.3.2 for any vertex of a maximal fan can be seen as a vertex of a Vizing-fan. We note that this is possible since, unlike the definition of a fan at $(T, e, \pi)$, there is only one edge colored by the previous colors in a $(Y_0, e, \pi)$-fan.

Corollary 5.3.3. A maximal $(Y_0, e, \pi)$-fan $V(F_1) \cup Y_0$ is elementary with respect to $\pi$. Moreover, $V(F_1) \subseteq A_{C_1}(T, e, \pi)$ where $\pi(F_1) \cap \pi(T) = \{\alpha\}$ and $C_\alpha = \pi(T) \setminus \{\alpha\}$.

We denote $F_1(T, e, \pi)$ as the union of all $(Y_0, e, \pi)$-fans. Then the following is a direct result Lemma 5.3.2 and Corollary 5.3.3 because of the properties of absorbing sets.

Corollary 5.3.4. $F_1(T, e, \pi) \cup V(F_0) \cup V(T)$ is elementary with respect to $\pi$. 76
We let $Y_1 = Y_0 \cup V(F_1)$ where $F_1$ is the maximal $(Y_0, e, \pi)$-fan. The following results are important since they allow us work with $(Y_1, e, \pi)$-fans and show that the proof gets harder as we start to move away from the original Tashkinov tree.

**Lemma 5.3.5.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$, $(T, e, \pi)$ be a maximal Tashkinov triple and $F_0$ be the unique maximal $(T, e, \pi)$-fan and $F_1$ be a maximal $(Y_0, e, \pi)$-fan. Let $x \in V(G) \setminus Y_1$ with $f \in E_G(x, y)$ for some $y \in V(F_1)$ and $\pi(f) \in \bar{\pi}(Y_1)$ where $Y_1 = V(T) \cup V(F_0) \cup V(F_1)$. Then $Y_1 \cup \{x\}$ is elementary with respect to $\pi$. Moreover, $x \in A_C(T, e, \pi)$ where $C = \bar{\pi}(T) \setminus \pi(F_1 \cup \{f\})$.

**Proof.** Let $\alpha \in \bar{\pi}(x) \cap \bar{\pi}(v), \gamma \in \bar{\pi}(y)$ and $\pi(f) = \varepsilon \in \bar{\pi}(z)$ for some $z, v \in Y_1$. By $Y_1 y$ we mean $V(T) \cup V(F_0) \cup V(F_1 y)$. We have the following cases:

**Case 1:** $v \in V(T), \alpha \in C$ and $z \in Y_1 y$

**Proof of 1:** We have $P_y(\gamma, \alpha, \pi) \cap T \neq \emptyset$. Re-coloring $P_x(\alpha, \gamma, \pi)$, we get a coloring $\pi'$ for which $T, F_0, F'_1$ with $F'_1 y = F_1 y$ and $Y'_1 = V(T \cup F_0 \cup F'_1)$ are well-defined. Moreover, $\gamma \in \bar{\pi}'(y) \cap \bar{\pi}'(x)$. Re-coloring $f$ by $\gamma$, we get a coloring $\pi''$ for which $Y'_1$ is well-defined. However, we have $\varepsilon \in \bar{\pi}''(z) \cap \bar{\pi}''(y)$ contrary to Lemma 5.3.3. Hence, the proof follows.

**Case 2:** $v \in V(T), \alpha \in C$ and $z \notin Y_1 y$

**Proof of 2:** We have $P_x(\alpha, \varepsilon, \pi) \cap T = \emptyset$. Re-coloring $P_x(\alpha, \varepsilon, \pi)$, we get a coloring $\varphi$ such that $Y'''_1 = V(T) \cup V(F_0) \cup V(F''_1)$ with $F''_1 z = F_1 z$ is well-defined. Moreover, we have $\varepsilon \in \varphi(x) \cap \varphi(z)$ and $\varphi(f) = \alpha \in \varphi(T)$. Let $\delta \in \varphi(T) \setminus \varphi(F_1 \cup \{\alpha\})$. We have $P_x(\varepsilon, \delta, \varphi) \cap T \neq \emptyset$. Re-coloring $P_x(\varepsilon, \delta, \pi)$, we get a coloring $\varphi'$ such that $Y'''_1 = V(T \cup F_0 \cup F''_1)$ with $F''_1 z = F_1 z$ is well-defined. But then Case 2 is reduced to Case 1.

**Case 3:** $v \in V(T), \alpha \notin C$. 

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Proof of 3: Let \( e \in F_1 \) be such that \( e \in E_G(u, w) \) with \( u \in V(F_0) \) and \( \pi(e) = \alpha \). Let also \( \lambda \in \bar{\pi}(u) \). Since \( u \in V(F_0) \subseteq A(T, e, \pi) \), \( P_u(\lambda, \beta, \pi) \cap T = \emptyset \) for all \( \beta \in \bar{\pi}(T) \). Hence, re-coloring \( P_x(\alpha, \lambda, \pi) \), we get a coloring \( \pi' \) for which \( Y_1 \) is well-defined and \( \lambda \in \bar{\pi}'(u) \cap \bar{\pi}'(x) \). Next, we pick \( \beta \in \bar{\pi}(T) \setminus \{\alpha, \varepsilon\} \) and re-coloring \( P_x(\beta, \lambda, \pi') \), we get a coloring \( \pi'' \) for which \( Y_1 \) is well-defined and \( \beta \in \bar{\pi}''(x) \cap \bar{\pi}''(T) \). Since \( \beta \in C' \) where \( C' = \bar{\pi}''(T) \setminus \pi(F_1 \cup \{\pi''(f)\}) \), we reduced this case to one of Cases 1 or 2.

Case 4: \( v \in V(F_0) \)

Proof of Case 4: Let \( \beta \in \bar{\pi}(T) \). Since \( v \in V(F_0) \subseteq A(T, e, \pi) \), \( P_v(\alpha, \beta, \pi) \cap T = \emptyset \) implying that \( P_x(\alpha, \beta, \pi) = P \) does not visit \( T \). Re-coloring \( P \), we get a coloring \( \pi' \) and reduce this case to one of the previous ones.

Case 5: \( v \in V(F_1) \).

Proof of 5: The following diagram will be used to prove this case. Indeed, our purpose is to show either that one of the previous cases hold or that there is a contradiction to Lemma 5.3.4.

We first explain Figure 5.3. Each chain in Figure 5.3 starts at \( x \) and every chain except the ones with “\( \ast \)” are between a color missing in the original tree and one from outside. In fact, they are all outside \( T \) and hence, \( T, F_0 \) are preserved under each \( \pi_i \). That is the reason that our notation above only involves the \( (Y_0, e, \cdot) \)-fans containing \( y, z \) represented by a subscript of these vertices. The end of each directed path (there are three of them) represents colorings which can either be reduced to a previous case or Lemma 5.3.4 or combination of both.

Let \( \beta \in \bar{\pi}(T) \setminus \pi(F \cup \{f\}) \). Then \( P_v(\alpha, \beta, \pi) \cap T \neq \emptyset \). This implies that \( P_x(\alpha, \beta, \pi) = P \) does not visit \( T \). Re-coloring \( P \), we get either (1) which is a reduction to previous cases or \( [F^1_x, F^1_y, \pi_1] \). Our purpose is to find a coloring \( \varphi \) for
which the union the respective fans $F_z, F_y$ is not elementary with respect to $\varphi$. Let $g_i$ denote the first edge of $F_i^1$ and $u_i$ denote the end of $g_i$ in $F_0$ for $i = z, y$.

If $g_y$ is the first edge in $F_1$ then $\pi_1(g_y) = \pi(g_y) \neq \beta$ and $g_z$ is an edge of $F_1$ with $\pi(g_z) = \alpha$ but $\pi_1(g_z) = \beta$. Let $\delta \in \bar{\pi}_1(u_z)$. Since $u_z \in F_0 \subseteq A(T, e, \pi_1)$, $P_{u_z}(\delta, \beta, \pi_1) \cap T = \emptyset$ implying $P_1 = P_x(\beta, \delta, \pi)$ does not visit $T$. Re-coloring $P_1$, we get $[F_{y}^{2}, F_{z}^{2}, \pi_2]$ where $F_i^2 = F_i^1$ for $i = y, z$. Let $\lambda \in \bar{\pi}_2(T) \setminus \{\beta, \pi_2(f), \pi_2(g_y)\}$. Then, for the same reason, $P_{u_z}(\lambda, \delta, \pi_2) \cap T \neq \emptyset$ implying $P_2 = P_x(\lambda, \delta, \pi_2)$ does not visit $T$. Re-coloring $P_2$, we get $[F_{y}^{3}, F_{z}^{3}, \pi_3]$ where again $F_i^3 = F_i^2$ for $i = y, z$. Note that $p_3(g_z) = \lambda$ is possible.

If $\pi_3(g_y) \neq \lambda$, then $P_y(\lambda, \gamma, \pi_3) \cap T \neq \emptyset$ implying that $P_3 = P_x(\lambda, \gamma, \pi)$ does not visit $T$. Re-coloring $P_3$, we get $[F_{y}^{4}, F_{z}^{4}, \pi_4]$ such that $F_i^4 = F_i^3$ for $i = y, z$. Moreover, we have $\gamma \in \bar{\pi}_4(y) \cap \bar{\pi}_4(x)$. Re-coloring $P_4 = P_x(\gamma, \varepsilon, \pi_4) = [x, f, y]$, we get $[F_{y}^{5}, F_{z}^{5}, \pi_5]$ (or (2)) for which $F_i^5 = F_i^4$ for $i = y, z$. But then $\varepsilon \in \bar{\pi}_5(y) \cap \bar{\pi}_5(z)$ contrary to Lemma 5.3.4.
If \( \pi_3(g_y) = \lambda \), let \( \delta' \in \bar{\pi}_3(u_y) \). Since \( u_y \in V(F_0) \subseteq A(T, e, \pi_3) \), \( P_{u_y}(\delta', \lambda, \pi_3) \cap T \neq \emptyset \) implying that \( P_6 = P_x(\lambda, \delta', \pi) \) does not visit \( T \). Re-coloring \( P_6 \), we get \([F^7_y, F^7_z, \pi_7]\) where \( F_i^7 = F_i^3 \) for \( i = y, z \). Let \( \lambda' \in \bar{\pi}_7(T) \setminus \{ \beta, \pi_7(f), \pi_7(g_y) = \lambda \} \). Then, for the same reason, \( P_{u_y}(\delta', \lambda', \pi_7) \cap T \neq \emptyset \) implying that \( P_7 = P_x(\lambda', \delta', \pi_7) \) does not visit \( T \). Re-coloring \( P_7 \), we get \([F_8^y, F_8^z, \pi_8]\) with \( F_i^8 = F_i^7 \) for \( i = y, z \). The rest of the proof follows as in the downward interchanges of (2) giving a contradiction to Lemma 5.3.4.

If \( g_z \) is the first edge in \( F_1 \) then the proof follows similarly since, this time, \( \pi_1(g_y) = \beta \in \bar{\pi}_1(x) \) must be assumed. It is enough to change the proof above by simply replacing the places \( y, z \) in the proof.

If none of \( g_y, g_z \) is the first edge of \( F_1 \), then either there exists \( F_1' \) containing both of \( y \) and \( z \) as in (1) or \( \pi_1(g_y) = \pi_1(g_z) = \beta \) in which case (2) or (3) holds.

Therefore, we completed the proof that \( Y_1 \cup F_1 \cup \{ x \} \) is elementary. But then setting \( C = \pi(T) \setminus \pi(F_1 \cup \{ f \}) \), \( x \in A_C(T, e, \pi) \).

We let \( J_2 = J(Y_1, e, \pi) \) where \( Y_1 = V(T) \cup V(F_0) \cup V(F_1) \). We have the following result directly implied by Lemma 5.3.5:

**Corollary 5.3.6.** Let \( G \) be a critical graph with \( \chi'(G) > \Delta(G) + 1 \), \( (T, e, \pi) \) be a maximal Tashkinov triple, \( F_0 \) be the maximal \((T, e, \pi)\)-fan and \( F_1 \) be a maximal \((Y_0, e, \pi)\)-fan where \( Y_0 = V(T) \cup V(F_0) \). Then \( Y_1 \cup J_2 \) is elementary where \( Y_1 = Y_0 \cup V(F_1) \).

The next result natural result would be showing that any \((Y_i, e, \pi)\)-fan is elementary. However, it will not be proven here since it is too lengthy. Naturally, we would like to prove that, for \( i \geq 0 \), any \((Y_i, e, \pi)\)-fan is elementary. This proves that any \((Y_0, e, \pi)\)-tree is elementary which is probably the next best thing after Tashkinov’s result.
5.4 Closed Colors

In Chapter 2, closed sets were defined. While they are very useful, the biggest drawback in the use of closed sets is that they are either closed or not and, unfortunately, one cannot say much about the sets in between. Hence, there is a need for concepts that are useful even while a set is not closed.

Let $H$ be a graph and $\pi \in C_k(H)$. For $X \subseteq V(G)$ and $\alpha \in \bar{\pi}(X)$, we say $\alpha$ is a closed color in $X$ with respect to $\pi$ or $X$ is $\alpha$-closed with respect to $\pi$ if $E_\alpha(X, V(G) \setminus X) = \emptyset$. Similarly, if $C \subseteq \bar{\pi}(X)$, then $X$ is $C$-closed if, for all $\alpha \in C$, $X$ is $\alpha$-closed. Clearly, $X$ is closed with respect to $\pi$ if $X$ is $\bar{\pi}(X)$-closed with respect to $\pi$.

This new improvement to “closed-ness” provides a better understanding of the sets that are not closed. However, it will not be used in our thesis at all and will only be used to simplify the language in [24] even though it suggests a certain degree of freedom while working on proofs. One should note that we need to define an order to understand “how closed a set is”. Lack of such an order might be the reason for us not to be able to find a direct use for this new concept. Hence, we suggest the interested reader to describe such an order and give an alternative proof to Theorem 2.4.1 using this new concept.

5.5 Independence of Colors

In Chapter 2, the missing colors at a Tashkinov tree were grouped as either used or unused (free) colors, which is due to Tashkinov [41]. In this section, we will introduce a concept that is more useful in elementary and closed sets such as the vertex set of a maximal Tashkinov tree.

The fact that at least 4 free colors are present in every Tashkinov tree $T$ with
$|V(T)| > 1$ was at the heart of the proof of Tashkinov’s theorem which provides the most promising methodological improvement to Conjecture 1.3.4 to this date. While these definitions are very much useful in improving the lower bounds through finding a large elementary sets, they are not as strong as one wishes, perhaps due to the general setting they were defined.

For a maximal Tashkinov triple $(T, e, \varphi)$, $(T', e, \varphi)$ is called a realization of $(T, e, \varphi)$ if $V(T') = V(T)$. The set of realizations of $(T, e, \varphi)$ will be denoted by $\mathcal{R}(T, e, \varphi)$.

A color $\alpha \in \bar{\varphi}(T)$ is called independent with respect to $\varphi$ if, for some $T' \in \mathcal{R}(T, e, \varphi)$, $\alpha \in \Gamma_f(T', e, \pi)$. Otherwise, it is called dependent. The set of independent missing colors in $(T, e, \varphi)$ is denoted by $\Gamma^i(T, e, \varphi)$ while the set of dependent missing colors in $(T, e, \varphi)$ is denoted by $\Gamma^D(T, e, \varphi)$.

While all free colors are independent, not all independent colors are free and similarly, while the dependent colors are used, not all used colors are dependent. Hence, we have $\Gamma^f(T, e, \varphi) \subseteq \Gamma^i(T, e, \varphi)$ and $\bar{\varphi}(T) \setminus \Gamma^f(T, e, \varphi) \supseteq \Gamma^D(T, e, \varphi)$. We can understand dependent colors as the set of colors that must be used while constructing

![Diagram](image)

**Figure 5.4:** $\alpha$ is a dependent color in the above Tashkinov tree. It separated the tree in two parts: before the $\alpha$-edge ($X$) and after the $\alpha$-edge ($Y$).
a maximal Tashkinov tree with respect to a fixed $e$ and $\varphi$. In what follows, a series of results that are due to Favrholdt et. al. and Scheide will be repeated for independent and dependent colors.

The following lemma is a generalization of Lemma 2.4.5:

**Lemma 5.5.1.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T, e, \varphi) \in \mathcal{T}(G)$. If, for some $x \in V(T)$, $\alpha \in \bar{\varphi}(x) \cap \Gamma^i(T, e, \varphi)$ and $\delta \notin \bar{\varphi}(T)$ then $E_\delta(T, e, \varphi) \subseteq P_x(\alpha, \delta, \varphi)$.

**Proof.** If $\alpha \in \Gamma^i(T, e, \varphi)$ then there exists $(T', e, \varphi) \in \mathcal{R}(T, e, \varphi)$ such that $\alpha \in \Gamma^i(T', e, \varphi)$. By Lemma 2.4.5, $E_\delta(T, e, \varphi) = E_\delta(T', e, \varphi) \subseteq P_x(\alpha, \delta, \varphi)$. This completes the proof. \qed

The next lemma is generalization of Proposition 2.4.6:

**Proposition 5.5.2.** Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T, e, \varphi) \in \mathcal{T}(G)$. If, for all $x \in V(T)$, $\bar{\varphi}(x) \cap \Gamma^i(T, e, \varphi) \neq \emptyset$ then $G$ is elementary.

**Proof.** Assume, on the contrary, that $G$ is not elementary. This implies that $\Gamma^d(T, e, \varphi) \neq \emptyset$. So, we may assume that there is some color $\beta \in \Gamma^d(T, e, \pi)$. Let $\varepsilon \in \bar{\varphi}(x)$ be an independent color for some $x \in V(T)$. This implies $E_\varepsilon(T, e, \varphi) \subseteq P_x(\varepsilon, \alpha, \varphi)$ by Lemma 5.5.1. Assume $y$ is the last vertex of $T$ visited by $P_x$ and $\gamma \in \bar{\varphi}(y)$ is independent. Re-coloring $G_T(\alpha, \gamma)$ -the graph induced by the colors $\alpha, \gamma$ in $G[V(T)]$-, we get a coloring $\psi$ such that $(T, e, \psi)$ is a Tashkinov triple and $\alpha \in \bar{\psi}(y)$ is independent. However, this implies that $V(P_y(\alpha, \beta, \psi)) \cap T = \{y\}$ and re-coloring $P_y$ gives a coloring $\pi$ such that $T$ is a Tashkinov tree which is not closed with respect to $\pi$. Hence, there exists a maximal Tashkinov tree $T'$ such that $|V(T')| > |V(T)|$ contradicting that $(T, e, \varphi)$ is a maximum triple. This completes the proof. \qed

Note that the proof suggests the following more general result which is again the independence version of Proposition 2.6.1.
Proposition 5.5.3. Let $G$ be critical with $\chi'(G) > \Delta(G) + 1$ and $(T, e, \varphi) \in \mathcal{T}(G)$. If there exists $x \in \Gamma_f(T, e, \pi)$ such that $\overline{\varphi}(x) \cap \Gamma_i(T, e, \varphi) \neq \emptyset$ then $G$ is elementary.

Proposition 5.5.3 suggests that there exists at least one vertex all of whose missing colors are also dependent.

Lastly, we note that the structural lemma in [35] can be simplified even more if Proposition 5.5.3 is used in its proof. Moreover, we use dependence and independence heavily in [24] while proving Theorem 2.6.7.

5.6 Notes

Many of the results in this chapter are employed in the proof of Theorem 2.6.7. Its proof is quite long and this suggests that focuses on the Tashkinov tree itself is not likely. We also note that, to our knowledge, no graph containing a maximum Tashkinov tree which is not strongly closed is presented to this day.
CHAPTER 6

A GENERAL METHOD USING COLORATIONS

6.1 Introduction

In 1967, Gupta [19] proved a series of results regarding Conjectures 1.3.4 and 1.3.5. He proved the following results around the same time with Vizing using a different approach. His approach was not only proving results in Edge Coloring Theory, it was also proving results in Edge-Cover Coloring Theory.

While Vizing and many others focused on critical graphs and their chromatic index, Gupta’s focus was on the “criticality” of the $k$-coloration of an arbitrary graph for an almost arbitrary integer $k > 1$.

His method was more complicated than that of Kempe chains since chains were a lot more complicated when one has a coloration. If $\pi$ is a $k$-coloring of a graph $G$, then an $(\alpha, \beta)$-chain is not only the edge set of a maximal connected graph but it is also minimal. While working with colorations, an $(\alpha, \beta)$-chain is assumed to be minimal and not necessarily maximal. Hence, it is essential to understand the nature of such chains when one has a coloration. The first part of this chapter aims to describe the chains under colorations; that is whether we can describe chains as alternating sequences.

Gupta is also, while unknown, the first person who attempted to prove a tree argument 33 years before Tashkinov. While not successful, his method gave rise to
Conjectures 1.3.4 and 1.3.5 long before Goldberg [15] introduced Conjecture 1.3.4. Moreover, he was able to prove many of the results, as mentioned in Chapter 1, before 1980 using his method. In the second part of this chapter, we will generalize the concepts of fan and path to colorations and then prove that they indeed are “elementary” in this new setting. We note that our goal in the short run is to show that Tashkinov trees are in fact elementary in this setting and, hopefully, repeat our main results of Chapter 4 or more for Conjecture 1.3.5.

Let $G$ be a graph and, for an integer $k > 1$, $\pi$ be a $k$-coloration of $G$. Let also $x$ be a vertex and $S$ be a non-empty subset of $V(G)$. A color $\beta$ is said to appear at a vertex $x$ under $\pi$ if $\pi(e) = \beta$ for some $\beta \in E_G(x, V - x)$. For an integer $i \geq 1$, we denote $\pi^i(x)$ as the set of colors that appear at $x$ at least $i$ times and denote $\pi^i(S) = \bigcup_{x \in S} \pi^i(x)$ where $i > 0$. We denote $\pi^0(x)$ as the set of colors that does not appear at $x$ under $\pi$ or more generally the set of colors missing at $x$. We denote $\bar{\pi}(x) = \pi^2(x) \cup \pi^0(x)$ as the set of colors that does not appear exactly once at $x$ and, similarly, $\bar{\pi}(S) = \bigcup_{x \in S} \bar{\pi}(x)$. We denote by $\pi(x) = \pi^1(x)$ as the set of colors that appear at $x$ and $|\pi(x)|$ is clearly the number of colors that appear at $x$. Clearly, $|\pi(x)| \leq \min\{d_G(x), k\}$. If $|\pi(x)| = d_G(x)$ for all $x \in V(G)$, then $\pi$ is a $k$-coloring of $G$ and $G$ is $k$-colorable. Similarly, if $|\pi(x)| = k$ for all $x \in V(G)$, then $\pi$ is a $k$-cover coloring of $G$ and $G$ is $k$-cover-colorable.

Let $\pi, \tau$ be two $k$-colorations of $G$. We say $\tau \geq \pi$ if $|\tau(x)| \geq |p(x)|$ for all $x \in V(G)$ and $\tau > \pi$ if $\tau \geq \pi$ and $|\tau(x_0)| > |p(x_0)|$ for some $x_0 \in V(G)$. In this case, we say $\tau$ improves $\pi$ (at $x_0$) and $\pi$ is said to be improvable at $x_0$ by $\tau$. Otherwise, $x_0$ is called $\pi$-deficient. A coloration that is not improvable is called optimal. A coloration $\pi$ for which $\sum_{x \in X} |\pi(x)|$ is maximum is called optimum.

For a $k$-coloration $\pi$ of $G$, we call $X \subseteq V(G)$ is elementary with respect to $\pi$ if

(i) For all $x \in X$, $\pi^3(x) = \emptyset$.  

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(ii) For all distinct pairs \( x, y \in X \), \( \bar{\pi}(x) \cap \bar{\pi}(y) = \emptyset \).

We note that the definition of elementariness is slightly different than that of Chapter 2. However, the closed and strongly closed sets are defined as before. We can re-state Conjectures 1.3.4 and 1.3.5 as follows:

**Conjecture 6.1.1.** Let \( G \) be a graph with \( \chi'(G) > \Delta(G) + 1 \) and set \( k = \chi' - 1 \). Then there exists an optimum \( k \)-coloration, \( \pi \), of \( G \) and a strongly closed, elementary set \( X \) with respect to \( \pi \) such that for some \( x \in X \), \( x \) is \( \pi \)-deficient.

**Conjecture 6.1.2.** Let \( G \) be a graph with \( \chi'_c(G) < \delta(G) - 1 \) and set \( k = \chi'_c + 1 \). Then there exists an optimum \( k \)-coloration, \( \pi \), of \( G \) and a strongly closed, elementary set \( X \) with respect to \( \pi \) such that, for some \( x \in X \), \( x \) is \( \pi \)-deficient.

The equivalence of Conjectures 6.1.1 and 6.1.2 to Conjectures 1.3.4 and 1.3.5, respectively, will not be proven here. We refer the interested reader to the wonderful yet fairly unknown thesis work of Gupta. In [19], he does not only show the equivalence of these results but also give the proofs of the initial steps of a unified proof. In fact, he introduces the concept of Tashkinov trees in 1967 even though he could not prove their elementariness.

While not exactly dual of each other, Conjectures 1.3.5 and 1.3.4 are similar to Min-Max Problems in nature. Conjecture 6.1.1 suggests that one should find a vertex set \( S \) so dense that every color class in \( G[S] \) is an almost perfect matching or more. On the other hand, Conjecture 6.1.2 suggests that one should find a vertex set \( R \) such that every color class in \( G : E'(R) \) is a smallest edge cover or less. The interesting part is that \( E(R) \) must be very dense to minimize the number of edges in \( E'(R) = E(R) \cup E(R, V \setminus R) \). This is one of the main reasons why the use of optimal colorations leads to a more general method in solving Conjectures 1.3.4 and 1.3.5 at the same time.
On the other hand, in [37], Schrijver gives proofs to some theorems on edge-cover colorings by using the analogous results in edge colorings such as those of Vizing [43]. However, his methods are not enough to show the equivalence of Conjectures 1.3.4 and 1.3.5 and, hence, whether they are equivalent is unknown.

Lastly, we note that the nature of chains in this new setting is a lot more complicated than that of edge colorings. However, they deserve the attention for two reasons: Firstly, the criticality in edge-cover colorings is rather complicated when compared to that in edge colorings. Secondly, the results pertaining to Conjecture 1.3.4 has a chance to work in this new setting even though their proof is a lot harder.

6.2 Changers, trails and chains

In solving edge coloring problems, using re-coloring techniques is essential. Let \( \pi \) be a \( k \)-coloration of \( G \) where \( k > 1 \). Given two distinct colors \( \alpha, \beta \), an \((\alpha, \beta)\)-Kempe chain is the edge set of a component of the graph induced by \( \pi^{-1}(\{\alpha, \beta\}) \). They are very useful in solving coloring problems because they are minimal sets. However, they are maximal connected sets if defined for a coloration. Hence, there is a need to understand the structure of minimal \((\alpha, \beta)\)-colored edge sets which allows the use of re-coloring techniques.

Given a \( k \)-coloration \( \pi \) and colors \( \alpha, \beta \), an \((\alpha, \beta)\)-changer is a subset \( R \) of \( \pi^{-1}(\{\alpha, \beta\}) \) on which interchanging colors gives a coloration \( \varphi \) which satisfies \( |\varphi(x)| \geq |\pi(x)| \) for all vertices \( x \) of \( G \). In this case, we write \( \pi \rightarrow^R \varphi \) and we say \( \varphi \) is obtained from \( \pi \) through an interchange on \( R \). If such a coloration is not improvable through any interchange on any changer then we call it a maximal coloration. A connected component of \( \pi^{-1}(\{\alpha, \beta\}) \) containing a vertex \( x \) is called a complete changer and is denoted by \( H_x(\alpha, \beta) \). If a complete changer \( R \) is a cycle then it will be called a
cycle changer and if a complete changer satisfies $1 \leq \sum_{x \in R} |\bar{\pi}(x) \cap \{\alpha, \beta\}| \leq 2$ then it is called a perfect changer. The set of all $(\alpha, \beta)$-changers is denoted as $Ch_{G, \pi}(\alpha, \beta)$. Note that $\emptyset, \pi^{-1}(\{\alpha, \beta\}) \in Ch_{G, \pi}(\alpha, \beta)$. A minimal non-empty $(\alpha, \beta)$-changer is called an $(\alpha, \beta)$-chain.

It is easy to show that $Ch_{G, \pi}(\alpha, \beta)$ is a partial order under subset relation and we are interested in the minimal non-empty elements of this family.

Let $\pi$ be a $k$-coloration for an integer $k > 1$ and $\alpha, \beta$ be any two colors. A trail

$$P(x, y) = [x = x_0, e_1, x_1, e_2, x_2, \ldots, x_{r-1}, e_r, x_r = y], \ r \geq 1,$$

where $e_1, e_2, \ldots, e_r$ are distinct edges of $G$ with $e_i \in E_G(x_{i-1}, x_i)$ for $i = 1, \ldots, r$ is called an $(\alpha, \beta)$-alternating trail if $|\pi(\{e_{2k} : 0 < k \leq \lfloor r/2 \rfloor\}| \leq 1, |\pi(\{e_{2k-1} : 0 < k \leq \lfloor (r + 1)/2 \rfloor\}| \leq 1, \pi(E(P)) \subseteq \{\alpha, \beta\}$, and if $r > 1$, the equality holds for all three inequalities. For $0 \leq i < j \leq r$, $x_i P x_j = [x_i, e_{i+1}, \ldots, x_j]$ is called a section of $P(x, y)$ and $\bar{P} = [y = x_r, e_r, x_{r-1}, \ldots, x_1, e_1, x_0 = x]$. We denote $x_i P = x_i P x_r$ and $P x_j = x_0 P x_j$.

An $(\alpha, \beta)$-alternating trail is called suitable if it is an $(\alpha, \beta)$-changer and a suitable $(\alpha, \beta)$-alternating trail is called minimal if it is not empty and $P(x, x_t) = [x = x_0, e_1, x_1, e_2, x_2, \ldots, x_{t-1}, e_t, x_t]$ is not suitable for any $t, 1 \leq t < r$. A minimal suitable $(\alpha, \beta)$-alternating trail is briefly referred as an $(\alpha, \beta)$-trail. The following lemma gives the structure of $(\alpha, \beta)$-trails.

**Lemma 6.2.1.** Let $G$ be a graph, $\pi$ be a $k$-coloration of $G$ for an integer $k > 1$ and $\alpha, \beta$ be any two colors. Let also $P = P(x, y) = [x = x_0, e_1, x_1, \ldots, e_n, x_n = y]$ be an $(\alpha, \beta)$-alternating trail. Then $P$ is an $(\alpha, \beta)$-trail if and only if one of the following holds for $P$ or $\bar{P}$:

1. $P$ is a minimal path starting from $x$ and satisfying at least one of $\pi(e_1) \in \pi^2(x)$ and $\pi(e_2) \in \pi^0(x)$; and at least one of $\pi(e_n) \in \pi^2(y)$ or $\pi(e_{n-1}) \in \pi^0(y)$.
2. $P$ is an even cycle with $\bar{\pi}(V(P)) \cap \{\alpha, \beta\} = \emptyset$

3. $P$ is an even cycle with $\bar{\pi}(V(P) \setminus \{x\}) \cap \{\alpha, \beta\} = \emptyset$ and $\bar{\pi}(x) \cap \{\alpha, \beta\} \neq \emptyset$

4. For some $i \geq 2$ and $0 < r_1 < r_2 < \ldots < r_i < n$, $P$ is an even cycle with $\bar{\pi}(V(P) \setminus \{x, x_{r_1}, x_{r_2}, \ldots, x_{r_i}\}) \cap \{\alpha, \beta\} = \emptyset$ and $|\bar{\pi}(x) \cap \{\alpha, \beta\}| = |\bar{\pi}(x_{r_j}) \cap \{\alpha, \beta\}| = 1$ for $j = 1, \ldots, i$. Moreover, both $\pi(e_1) \in \bar{\pi}(x)$ and $\pi(e_{r_j}) \notin \bar{\pi}(x_{r_j})$ for $j = 1, \ldots, i$ must be satisfied.

5. $P$ is an odd cycle with $\bar{\pi}(V(P) \setminus \{x\}) \cap \{\alpha, \beta\} = \emptyset$ such that $\pi(e_1) \in \bar{\pi}^2(x)$ and $\{\alpha, \beta\} - \{\pi(e_1)\} \subseteq \pi^0(x)$ or $\pi(e_1) \in \pi^3(x)$.

6. There exists $r, j$ with $0 < r < j < n$ such that $Px_r$ is an odd cycle with $\bar{\pi}(V(Px_r) \setminus \{x\}) \cap \{\alpha, \beta\} = \emptyset$, $x_rPx_j$ is a path with $\pi(e_t) \notin \bar{\pi}(x_{t+1})$ for $r < t < j$ and $x_jP$ is an odd cycle with $\bar{\pi}(V(x_jP) \setminus \{y\}) \cap \{\alpha, \beta\} = \emptyset$.

7. There exists $r$ with $0 < r < n$ such that $Px_r$ is an odd cycle with $\bar{\pi}(V(Px_r) \setminus \{x\}) \cap \{\alpha, \beta\} = \emptyset$ and $x_rP$ is a path with $\pi(e_t) \notin \bar{\pi}(x_{t+1})$ for $r < t < n - 1$ and at least one of $\pi(e_n) \in \bar{\pi}(y)$ or $\pi(e_{n-1}) \in \pi^0(y)$ happens.

8. There exists $r$ with $0 < r < n$ such that $Px_r$ is an odd cycle with $\bar{\pi}(V(Px_r) \setminus \{x\}) \cap \{\alpha, \beta\} = \emptyset$ and $x_rP$ is an odd cycle with $\bar{\pi}(V(x_rP) \setminus \{y\}) \cap \{\alpha, \beta\} = \emptyset$. Moreover, $x = y$ and $\alpha, \beta \in \pi^2(x) \setminus \pi^3(x)$.

Proof. To prove that if one of (1) – (8) is satisfied than $P$ is an $(\alpha, \beta)$-trail is straightforward.

We will only show that if $P$ is an $(\alpha, \beta)$-trail then it must satisfy one of (1) – (8). We assume that $P$ is an $(\alpha, \beta)$-trail. It is clear that any even cycle which is an alternating chain is a changer and hence suitable. Showing such a changer is minimal if and only if (2)-(4) happens is quite straightforward. Hence, we may assume that
\[ \tilde{\pi}(x) \cap \{\alpha, \beta\} \neq \emptyset \] and it does not contain any even cycle as a part of the chain. Without loss of generality, we assume \( \alpha \in \tilde{\pi}(x) \).

If \( P \) does not contain an odd cycle then clearly (1) holds. Hence, we may also assume without loss of generality that, for some \( r \) with \( 0 < r \leq n \), \( x = x_r \) and \( P \) starts with an odd cycle \( Px_r \). This implies that \( \alpha = \pi(e_1) = \pi(e_r) \in \pi^2(x_r) \). If \( \beta \in \pi^0(x) \) or \( \alpha \in \pi^3(x) \) then \( r = n \) must hold and this implies (5). If \( r < n \), then \( \alpha \notin \pi^3(x) \) and \( \beta \notin \pi^0(x) \). Moreover, \( \pi(e_{r+1}) = \beta \).

If \( x_rP \) is a path then \( \pi(e_n) \in \pi^2(y) \) or \( \pi(e_{n-1}) \in \pi^0(y) \). This implies (7). If there exist \( j \) with \( r < j < n \) such that \( x_jP \) is an odd cycle then (6) is satisfied. If \( x_rP \) is an odd cycle then (8) is satisfied. This implies that \( P \) or \( \bar{P} \) must satisfy one of (1) – (8) and completes the proof.

Trails are very useful structures. Indeed, the next lemma shows that the set of \((\alpha, \beta)\)-trails of a graph \( G \) contains the set of \((\alpha, \beta)\)-chains of a graph.

**Lemma 6.2.2.** Let \( G \) be a graph, \( \pi \) be a \( k \)-coloration of \( G \) for an integer \( k > 1 \) and \( \alpha, \beta \) be any two colors. Then every \((\alpha, \beta)\)-chain is an \((\alpha, \beta)\)-trail.

**Proof.** Let \( A \) be an \((\alpha, \beta)\)-chain. A chain must be connected and it is clear that a chain is an even cycle of type (2) – (4) or it does not contain an even cycle. If it does not contain any cycle, it is again clearly is of type (1). Hence, we may assume that \( A \) is not of type (1)-(4). This also implies that \( S = \{x \in V(A) : \pi(x) \cap \{\alpha, \beta\} \neq \emptyset \} \neq \emptyset \). We choose \( x_0 \in S \) such that \( T = (x_0, e_1, \ldots, x_n) \) is a smallest \((\alpha, \beta)\)-trail starting in \( V(A) \). If \( G : E(T) \) is a subgraph of \( G : A \) then \( E(T) = A \) must happen. We choose a smallest \( r \) with \( 0 < r < n \) such that, for \( 0 \leq i \leq r \), \( x_i \in V(T) \cap V(A) \) and \( x_{r+1} \in V(T) \setminus V(A) \).

Assume \( \pi(e_r) = \alpha \). Then \( \alpha \notin \pi(x_r) \) and \( \beta \notin \pi^0(x_r) \) for, otherwise, \( r = n \) must hold. We have two cases:
Case 1: $\beta \notin \pi^2(x_r)$.

Proof of (1): Re-coloring $A$, we get a coloration $\varphi$ such that $\beta \in \varphi^2(x_r)$ and $\alpha \in \varphi^0(x_r)$. Hence, $|\varphi(x_r)| < |\pi(x_r)|$ contrary to the definition of an $(\alpha, \beta)$-chain.

Case 2: $\beta \in \pi^2(x)$.

Proof of 2: If $E_{\alpha}(x,V(G) \setminus \{x\}) \cap A = \emptyset$, then, re-coloring $A$, we get a coloration $\varphi$ such that $\beta \in \varphi^3(x_r)$ and $\alpha \in \varphi^0(x_r)$. Hence, $|\varphi(x_r)| < |\pi(x_r)|$ contrary to the definition of an $(\alpha, \beta)$-chain. Therefore, we may assume that there exists $y_{r+1} \in V(G)$ and $f \in E_{G}(x_r, y_{r+1}) \setminus \{e_{r+1}\}$ such that $\pi(f) = \beta$ and $f \in A$.

If $x_n = x_r$ and $\beta \in \pi^3(x_r)$, or $x_n \neq x_r$, then $x_r \in S$ and $x_r T$ is a smaller $(\alpha, \beta)$-trail starting in $V(A)$ contrary to the choice of $T$. Hence, we may assume $x_n = x_r$ and $\beta \in \pi^2(x_r) \setminus \pi^3(x_r)$. By Lemma 6.2.1, $\bar{\pi}(x_j) \cap \{\alpha, \beta\} = \emptyset$ for $r < j < n$. We denote $T' = T_{x_r} + x_r T$, that is we reverse the odd cycle at the end. Since $f \in A$, $f = e_n$ and there exists $j$ with $r < j < n$ such that $E(T'x_j) \subset A$ but $e_j \notin A$. But then $\bar{\pi}(x_j) \cap \{\alpha, \beta\} = \emptyset$ reduces this case to (1).

Therefore, $E(T) \subseteq A$ and by minimality of $A$, equality holds.

The next lemma shows which trails are chains:

Lemma 6.2.3. Let $G$ be a graph and $\pi$ be a $k$-coloration of $G$ for an integer $k > 1$. An $(\alpha, \beta)$-chain $P(x, y) = [x = w_0, e_1, ..., e_n, w_n = y]$ is an $(\alpha, \beta)$-trail satisfying the following:

(a) For any $z \in V(P(x, y)) \setminus \{x, y\}$, $\bar{\pi}(z) \cap \{\alpha, \beta\} = \emptyset$ unless $P$ is type (4) or (4).

(b) If $\pi^3(\{x, y\}) \cap \{\alpha, \beta\} = \emptyset$ and $P(x, y)$ is any of types (2), (5), (6), (8) then it is also a perfect chain.

(c) If $P$ is of type (1), $\pi(e_1) \notin \pi^2(x)$ and $\pi(e_n) \notin \pi^2(y)$ then $P$ is a perfect chain.
(d) If \( P \) is of type (7) and \( \pi(e_n) \notin \pi^2(y) \) then \( P \) is a perfect chain.

Proof. Let \( P \) be an \((\alpha, \beta)\)-chain with \( P = [x = w_0, e_1, ..., e_n, w_n = y] \) and \( S = \{v \in V(P) : \pi(v) \cap \{\alpha, \beta\} \neq \emptyset\} \). We have the following cases to understand the structure of \( P \):

Case 1: \( S = \emptyset \)

Proof of (1): This implies that \( P \) is of type-(2) in Lemma 6.2.1 and it is perfect.

Case 2: \( S = \{x\} \)

Proof of (2): This implies that \( x = y \) and \( P \) is either type-(3) or -(5). Moreover, in each case, if \( \pi^3(x) \cap \{\alpha, \beta\} = \emptyset \) then \( P \) is perfect.

Case 3: \(|S| = 2\)

Proof of (3): We either have \( S = \{x, y\} \) or \( S = \{x, z\} \) for some \( z \neq y \). In the first case, \( P \) is either of type (1), (3), (6) or (7) and it is perfect. In the latter case, \( x = y \) and \( P \) is of type-(4) since, otherwise, there is \( r \) with \( 0 < r < n \) such that \( x_rP \) or \( Px_r \) is smaller chain contrary to the definition of a chain.

Case (4): \(|S| \geq 3\)

Proof of (4): This is only possible if \( P \) is an even cycle which is type-(4).

Based on the above observations, (a) follows directly. Moreover, if \( P \) is an even cycle it is perfect if and only if (2) holds and it is not perfect, otherwise. If \( P \) is a path then clearly (c) holds. Hence, without loss of generality, we may assume that \( P \) is not an even cycle and it starts with an odd cycle \( Pw_r \). Assume \( \pi(e_1) = \pi(e_r) = \alpha \). If \( \alpha \in \pi^3(x) \) then \( r = n \) and \( P \) is not perfect. Hence, for \( P \) to be perfect, \( \alpha \in \pi^2(x) \setminus \pi^3(x) \).

If \( \beta \in \pi^0(x) \) then \( r = n \). Hence, if \( P \) is of type (5) and \( \alpha \notin \pi^3(x) \) then \( P \) is perfect. Similarly, if \( P \) is of type (8) and \( \alpha, \beta \notin \pi^3(x) \) then \( P \) is perfect. If \( P \) is of type (6) and \( \beta \in \pi^2(x) \), then clearly, \( x_rP \) is a smaller chain contained in \( P \). Hence, \( \beta \notin \pi^2(x) \)
and $P$ is complete. For the same reason, if $P$ is of type (7) and $\pi(e_n) \notin \bar{\pi}(y)$ then $\pi(e_{n-1}) \in \pi^0(y)$ and $P$ is perfect. Hence, $(b) - (d)$ holds.

We note that not all $(\alpha, \beta)$-trails are $(\alpha, \beta)$-chains. Let $P = (x_0, e_1, \ldots, x_n)$ be of type-(7) and $\alpha, \beta \in \pi^2(x)$. Let also $r$ be such that $0 < r < n$ and $Px_r$ is an odd cycle with $\pi(e_1) = \pi(e_r) = \alpha$ and $\pi(e_{r+1}) = \beta$. Since $\beta \in \pi^2(x)$, $P$ is an $(\alpha, \beta)$-trail but $x_rP$ is also an $(\alpha, \beta)$-trail.

In fact, we have the following simple observation:

**Corollary 6.2.4.**

Perfect changers are essential in the study of Goldberg-Gupta conjecture. Hence, the following lemma is useful.

**Lemma 6.2.5.** Let $G$ be a graph and $\pi$ be a $k$-coloration of $G$ for an integer $k > 1$.

1. An $(\alpha, \beta)$-trail $P$ is an $(\alpha, \beta)$-chain if and only if $\bar{P}$ is also an $(\alpha, \beta)$-trail.

2. Let $\pi$ be a maximal $k$-coloration, $\alpha \in \bar{\pi}(x)$, $\beta \in \bar{\pi}(y)$ be distinct colors. If any $(\alpha, \beta)$-trail starting at $x$ ends at $x$ or $y$ and any $(\alpha, \beta)$-trail starting at $y$ ends at $x$ or $y$, then $H_x(\alpha, \beta)$ is perfect.

**Proof.** To prove (1), assume $z \in V(P) \setminus \{x, y\}$ satisfies $\{\alpha, \beta\} \cap \bar{\pi}(z) \neq \emptyset$. But then $\bar{P}(y, z)$ is an $(\alpha, \beta)$-trail contradicting to the minimality of the trail $\bar{P}$. Therefore, $P$ is an $(\alpha, \beta)$-chain.

The fact that the only nodes of $H_x(\alpha, \beta)$ are $x, y$ follows from (1). But this means that it is perfect by the maximality of $\pi$. Hence, the proof of (2) follows. \qed
6.3 Fans of colorations

In this section, we will prove that Vizing’s fan definition [43] can be used for colorations. However, we will use the analog of the more general fan definition given by Carliori [3].

Let $G$ be a graph, $\pi$ be a $k$-coloration of $G$ for an integer $k > 1$ and $x_0 \in V(G)$. We define an $(x_0, \pi)$-fan $F = (x_0, e_1, x_1, ..., e_n, x_n)$ as follows:

1. $x_0, x_1, ..., x_n$ are all distinct,
2. For all $i$ with $0 < i \leq n$, $e_i \in E_G(x_0, x_i)$,
3. For all $i$ with $0 < i \leq n$, there exists $j$ with $0 \leq j < i$ such that $\pi(e_i) \in \bar{\pi}(x_j)$.

We define, for $1 \leq i \leq n$, $Fx_i = (x_0, e_1, x_1, ..., e_i, x_i)$ as an $(x_0, \pi)$-subfan of $F$.

In the original fan definition of Vizing [43], there is a need for a critical edge. Hence, a useful fan definition requires us to work on a special coloration as well as a special vertex $x_0$.

Throughout the rest of this section, let $k > 1$ be an integer with $\min\{|k - d_G(v)| : v \in V(G)\} > 0$ and $\pi$ be an optimum $k$-coloration of $G$ and $x_0$ be a $\pi$-deficient vertex of $G$; namely, $|\pi(x_0)| < \min\{k, d_G(x_0)\}$. It is easy to show that $\pi^0(x_0) \neq \emptyset$ and $\pi^2(x_0) \neq \emptyset$.

We define a seedling at $x_0$ with respect to $\pi$ as $\Lambda(x_0, \pi) = \{x \in V(G) : \exists e \in E_G(x_0, x) \ni \pi(e) \in \bar{\pi}(x_0)\}$. This definition is inspired by the following result of Gupta:

**Lemma 6.3.1** (Gupta [19], 1967). For all $\alpha \in C_2(x_0, \pi)$, there exists distinct vertices $x_1, x_2 \in V(G)$ such that $\alpha \in \pi(E_G(x_0, x_1)) \cap \pi(E_G(x_0, x_2))$. Moreover, $\{x_0, x_1, x_2\}$ is elementary with respect to $\pi$.

The following result on seedlings are inspired by this lemma:
Lemma 6.3.3. 1. \( S = \{x_0\} \cup \Lambda(x_0, \pi) \) is elementary with respect to \( \pi \).

2. For any \( x, y \in S \), \( \alpha \in \bar{\pi}(x), \beta \in \bar{\pi}(y) \), \( H_x(\alpha, \beta) \) is a perfect changer with nodes \( x, y \).

Proof. By Lemma 6.3.1, \( \bar{\pi}(x) \cap \bar{\pi}(\Lambda(x_0, \pi)) = \emptyset \), \( \pi^3(x) = \emptyset \) for all \( x \in S \) and if, for some distinct \( x, y \in \Lambda(x_0, \pi) \), \( \bar{\pi}(x) \cap \bar{\pi}(y) \neq \emptyset \) then \( \pi(\Gamma(x_0, x)) \cap \pi(\Gamma(x_0, y)) = \emptyset \). Let \( \alpha \in \bar{\pi}(x) \cap \bar{\pi}(y) \) and \( \beta \in \pi^0(x_0) \). Let \( P(x_0, z) \) be a \((\beta, \alpha)\)-trail. Clearly, \( z \neq x_0 \).

Moreover, \( z \neq x \) or \( z \neq y \). Letting \( \pi \rightarrow P', \pi' \), we get \( \Lambda(x_0, \pi') = \Lambda(x_0, \pi) \) and at least one of \( \alpha \in \bar{\pi}'(x_0) \cap \bar{\pi}(x) \) and \( \alpha \in \bar{\pi}'(x_0) \cap \bar{\pi}(y) \) both of which contradicts Lemma 6.3.1. Hence, for all distinct \( x, y \in S \), \( \bar{\pi}(x) \cap \bar{\pi}(y) = \emptyset \) and \( S \) is elementary. This completes the proof of (1).

If \( x = x_0 \), then (2) follows directly from Lemma 6.3.1. Assume that \( x, y \neq x_0 \). Let \( P_1(x, z) \) be an \((\alpha, \beta)\)-trail. If \( z \neq x, y \) then \( \pi \rightarrow P_1 \varphi \) with \( \Lambda(x_0, \varphi) = \Lambda(x_0, \pi) \) and \( \beta \in \bar{\varphi}(x) \cap \bar{\varphi}(y) \) which contradicts to (1). Similarly, if \( P_2(y, w) \) is an \((\alpha, \beta)\)-trail and \( w \neq x, y \) then \( \pi \rightarrow P_2 \varphi' \) with \( \Lambda(x_0, \varphi') = \Lambda(x_0, \pi) \) and \( \alpha \in \bar{\varphi'}(x) \cap \bar{\varphi'}(y) \) which contradicts (1). But then by Lemma 6.2.5, \( H_x(\alpha, \beta) \) is a perfect changer with nodes \( x, y \). Hence, the proof of (2) is completed. \( \square \)

Lemma 6.3.3. Let \( F = (x_0, e_1, x_1, \ldots, e_n, x_n) \) be an \((x_0, \pi)\)-fan. Then the smallest subfan \( F' \) ending at \( x_n \) of \( F \) contains a unique element of \( \Lambda(x_0, \pi) \).

Proof. If \(|V(F)| = 2 \) then the result is trivial by fan definition. So, we may assume that for any fan with less than \( k \) vertices where \( 0 < k \leq n \), the statement holds. Assume that \( F \) is a fan on \( n + 1 \) vertices. \( \pi(e_n) \in \bar{\pi}(x_{i_1}) \) for some \( i_1 \) with \( 0 \leq i_1 < n \). If \( i_1 = 0 \) then the proof is completed. Otherwise, there exists \( F'x_{i_1} = (x_0, e_{i_1}, x_{i_1}, e_{i_{r-1}}, x_{i_{r-1}}, \ldots, x_{i_1}) \), a subfan of \( Fx_{i_1} \) such that \( F'x_{i_1} \) contains a unique element of \( \Lambda \). But then \( F' = (x_0, e_{i_r}, x_{i_r}, e_{i_{r-1}}, x_{i_{r-1}}, \ldots, x_{i_1}, e_n, x_n)\pi(e_{i_1}) \in \bar{\pi}(x_{i_2}) \) is a minimal subfan of \( F \) containing a unique element of \( \Lambda \). This completes the proof. \( \square \)
Lemma 6.3.4. An \((x_0, \pi)\)-fan \(F = [x_0, e_1, x_1, ..., e_n, x_n]\) is elementary. Namely, it satisfies the following:

(i) For all \(i\) with \(1 \leq i \leq n\), \(\bar{\pi}(x_0) \cap pb(x_i) = \emptyset\).

(ii) For all \(i\) with \(1 \leq i \leq n\), \(p^3(x_i) = \emptyset\).

(iii) For all \(i, j\) with \(1 \leq j < i \leq n\), \(\bar{\pi}(x_j) \cap \bar{\pi}(x_i) = \emptyset\).

Proof. Assume that \((F, x_0, \pi)\) is a triple giving a minimal counterexample. Hence, \(\bar{\pi}(x_i) \cap \bar{\pi}(x_j) = \emptyset\) for all \(i\) with \(0 \leq i < j < n\) and \(\pi^3(x_i) = \emptyset\) for all \(i\) with \(0 \leq i < n\). Hence, at least one of the following must hold for \(F\):

(1) \(\bar{\pi}(x_0) \cap \bar{\pi}(x_n) \neq \emptyset\),

(2) For some \(j\) with \(0 \leq j < n\), \(\bar{\pi}(x_j) \cap \bar{\pi}(x_n) \neq \emptyset\),

(3) \(\pi^3(x_n) \neq \emptyset\).

We will first show that (6.3) does not hold. Assume \(\alpha \in \bar{\pi}(x_0) \cap \bar{\pi}(x_n)\) and \(\beta = \pi(e_n)\). \(\beta \in \bar{\pi}(x_i)\) for some \(i\) with \(0 \leq i < n\). Clearly, \(\beta \neq \alpha\) for otherwise, \(Fx_i\) is a smaller fan which is not elementary. Moreover, if \(\alpha \in \pi^2(x_0)\) and \(\gamma \in \pi^0(x_0)\), then \(H = H_{x_0}(\alpha, \gamma)\) must be an odd cycle forming a perfect changer. Hence, we have \(\pi \rightarrow^H \varphi\) with \(\alpha \in \varphi^0(x_0)\), \(F\) an \((x_0, \varphi)\)-fan and \(\alpha \in \varphi(x_0) \cap \varphi(x_n)\). Therefore, we may assume that \(\alpha \in \bar{\pi}(x_0)\) from the beginning.

Let \(P_1(x_i, z_1)\) be an \((\alpha, \beta)\)-trail with \(z_1 \neq x_i\). Note that \(z_1 \neq x_0\) for, otherwise, \(P_1\) must end with the section \((x_n, e_n, x_0)\) which is not possible since \(\alpha \in \bar{\pi}(x_n)\). Hence, we have \(\pi \rightarrow^{P_1} \pi''\) for which \(Fx_i\) is an \((x_0, \pi'')\)-fan with \(\alpha \in \bar{\pi''}(x_0) \cap \bar{\pi''}(x_i)\). This contradicts to the minimality of \(F\) and completes the proof of (6.3.4).

Next, we will show that (6.3) does not hold. Assume \(\alpha \in \bar{\pi}(x_i) \cap \bar{\pi}(x_n)\) for some \(i\) with \(0 < i < n\) and \(\varepsilon \in \pi^0(x_0)\). Let \(P(x_0, z)\) be the unique \((\alpha, \varepsilon)\)-trail at \(x_0\).
Re-coloring $P(x_0, z)$, we get a coloration $\tau$. If $z \neq x_i$ then $\alpha \in \tilde{\tau}(x_0) \cap \tilde{\tau}(x_i)$ and $Fx_i$ is an $(x_0, \tau)$-fan which is not elementary, contrary to the minimality of $F$. Hence, $z = y$. Otherwise, $\pi(e_n) \neq \alpha$ and hence, $F$ is an $(x_0, \tau)$-fan with $\alpha \in \tilde{\tau}(x_0) \cap \tilde{\tau}(x_n)$. But this contradicts (6.3.4). Hence, we completed the proof of (6.3.4).

Lastly, we will show that (6.3) cannot hold. If $\gamma \in \pi^3(x_n)$ then, choosing $\varepsilon \in \pi^0(x_0)$, let $P = P(x_n, z)$ be an $(\gamma, \varepsilon)$-chain. Clearly, $P$ is not an even cycle. Re-coloring $P$, we get a coloration $\tau$ for which $F$ is an $(x_0, \tau)$-fan. If $z \neq x_0$, then $\varepsilon \in \tilde{\tau}(x_0) \cap \tilde{\tau}(x_n)$ and, otherwise, $\gamma \in \tilde{\tau}(x_0) \cap \tilde{\tau}(x_n)$. In both cases, we get a contradiction to (6.3.4). This shows (6.3) does not hold, completing the proof of (6.3.4) and the theorem.

\section*{6.4 Paths of Colorations}

Kierstead\[23\] has shown that it is possible to find an elementary set of vertices which are, unlike Vizing’s fans, not necessarily adjacent to the initial vertex. We will first define a generalization of Kierstead paths to colorations of graphs:

Let $G$ be a graph, $\pi$ be a $k$-coloration of $G$ for an integer $k > 1$ and $x_0 \in V(G)$. We define an $(x_0, \pi)$-path $K = (x_0, e_1, x_1, ..., e_n, x_n)$ as follows:

1. $x_0, x_1, ..., x_n$ are all distinct,

2. For all $i$ with $0 < i \leq n$, $e_i \in E_G(x_{i-1}, x_i)$,

3. For all $i$ with $0 < i \leq n$, there exists $j$ with $0 \leq j < i$ such that $\pi(e_i) \in \tilde{\pi}(x_j)$.

We define $Kx_i = (x_0, e_1, x_1, ..., e_i, x_i)$ as an $(x_0, \pi)$-subpath of $K$.

In the original path definition of Kierstead [23], there is a need for a critical edge. Hence, a useful path definition requires us to work on a special coloration as well as a special vertex $x_0$. 98
Throughout the rest of this section, let $k > 1$ be an integer satisfying $\min\{|k = d_G(v)| : v \in V(G)\} > 1$, $\pi$ be an optimal $k$-coloration and $x_0$ be a $\pi$-deficient vertex. It is easy to show that $\pi(e_1) \in \pi^2(x_0)$.

The following result is the coloration version of Kierstead’s result:

**Proposition 6.4.1.** An $(x_0, \pi)$-path $K = (x_0, e_1, x_1, ..., e_n, x_n)$ is elementary with respect to $\pi$. Namely, it satisfies the following:

(i) For all $i, j$ with $0 \leq i < j \leq n$, $\bar{\pi}(x_i) \cap \bar{\pi}(x_j) = \emptyset$.

(ii) For all $i$ with $0 \leq i \leq n$, $\pi^3(x_i) = \emptyset$.

**Proof.** Let $K$ be a minimal counterexample. We may assume that $n \geq 2$ by Lemma 6.3.1. Moreover, $Kx_{n-1}$ is elementary by the minimality of $K$ and $|\bar{\pi}(v)| > 1$ for all $v \in V(G)$ by $\min\{|k = d_G(v)| : v \in V(G)\} > 1$.

We will first show the following results for $K$:

(a) For $0 \leq s < n - 1$, $\varepsilon \in \bar{\pi}(x_s)$ and $\gamma \in \bar{\pi}(x_{s+1})$, there exists an $(\varepsilon, \gamma)$-changer $C$ with $\pi \rightarrow^C \pi_1$ satisfying:

(i) $K$ is an $(x_0, \pi_1)$-path.

(ii) $\bar{\pi}_1(x_s) = \bar{\pi}(x_s) \cup \{\gamma\} - \{\varepsilon\}$, $\bar{\pi}_1(x_{s+1}) = \bar{\pi}(x_{s+1}) \cup \{\varepsilon\} - \{\gamma\}$ and $\bar{\pi}_1(x) = \bar{\pi}(x)$ otherwise.

(iii) $C$ is either an $(\varepsilon, \gamma)$-trail $P = P(x_s, x_{s+1})$ or union of $P$ with an even cycle.

(iv) $H_{x_s}(\varepsilon, \gamma)$ is a perfect changer with nodes $x_s, x_{s+1}$.

(b) For $0 \leq s < n$, $\varepsilon, \gamma \in \bar{\pi}(x_s)$ with $\varepsilon \neq \gamma$, $H_{x_s}(\varepsilon, \gamma)$ is a perfect changer with $x_s$ as its only node.

**Proof of (a):** We first note that $\pi(e_{s+1}) \neq \gamma$ by path definition.
We will first assume \( \pi(e_{s+1}) \neq \varepsilon \). Let \( P_1(x_{s+1}, z_1) \) be an \((\varepsilon, \gamma)\)-trail with \( z_1 \neq x_{s+1} \). Then \( \pi \to^{P_1} \pi_1 \) satisfying that \( K \) is an \((x_0, \pi_1)\)-path. If \( z_1 \neq x_s \), then \( \varepsilon \in \bar{\pi}_1(x_s) \cap \bar{\pi}_1(x_{s+1}) \). This means that \( Kx_{s+1} \) is not elementary contrary to the minimality of \( K \). Therefore, \( z_1 = x_s, \bar{\pi}_1(x_s) = \bar{\pi}(x_s) \cup \{ \gamma \} - \{ \varepsilon \}, \bar{\pi}_1(x_{s+1}) = \bar{\pi}(x_{s+1}) \cup \{ \varepsilon \} - \{ \gamma \} \) and \( \bar{\pi}_1(x) = \bar{\pi}(x) \) otherwise.

We next assume \( \pi(e_{s+1}) = \varepsilon \). Then \( \varepsilon \in \bar{\pi}^2(x_s) \). Let \( P_2(x_s, z_2) \) be an \((\varepsilon, \gamma)\)-trail starting with \( e_{s+1} \). If \( z_2 \neq x_s \), then \( \pi \to^{P_2} \pi_1 \) satisfying that \( K \) is an \((x_0, \pi_1)\)-path. If we also assume \( z_2 \neq x_{s+1} \) then \( \gamma \in \bar{\pi}_1(x_s) \cap \bar{\pi}_1(x_{s+2}) \) which gives \( Kx_{s+1} \) is not elementary contrary to the minimality of \( K \). Hence, we may assume \( z_2 = x_{s+1} \) which gives \( \bar{\pi}_1(x_s) = \bar{\pi}(x_s) \cup \{ \gamma \} - \{ \varepsilon \}, \bar{\pi}_1(x_{s+1}) = \bar{\pi}(x_{s+1}) \cup \{ \varepsilon \} - \{ \gamma \} \) and \( \bar{\pi}_1(x) = \bar{\pi}(x) \) otherwise. Therefore, we may assume that \( z_2 = x_s \). Since \( \gamma \notin \bar{\pi}(x_s) \) and \( \varepsilon \notin \bar{\pi}^2(x_s) \), \( P_2 \) is an even cycle. Then, by choosing \( P_3(x_s, z_3) \) as the \((\varepsilon, \gamma)\)-trail starting with the other \( \varepsilon \)-edge at \( x_s \), we have \( \pi \to P_2 \cup P_3 \pi_2 \) for which \( K \) is an \((x_0, \pi_2)\)-path. Clearly, \( z_3 \neq x_s \). If \( z_3 \neq x_{s+1} \) then \( \gamma \in \bar{\pi}_2(x_s) \cap \bar{\pi}_2(x_{s+1}) \) which implies that \( Kx_{s+1} \) is a not elementary contrary to the minimality of \( K \). Therefore, \( z_3 = x_{s+1} \) and \( \pi_2 \) satisfies \( \bar{\pi}_2(x_s) = \bar{\pi}(x_s) \cup \{ \gamma \} - \{ \varepsilon \}, \bar{\pi}_2(x_{s+1}) = \bar{\pi}(x_{s+1}) \cup \{ \varepsilon \} - \{ \gamma \} \) and \( \bar{\pi}_2(x) = \bar{\pi}(x) \) otherwise. Throughout the proof of (a), we used either a trail or the union of a trail and an even cycle. Therefore, this completes the proof of (a). We have shown that every \((\varepsilon, \gamma)\)-trail starting at \( x_s \) ends at \( x_s \) or \( x_{s+1} \). If we also show that every \((\varepsilon, \gamma)\)-trail starting at \( x_{s+1} \) ends at \( x_s \) or \( x_{s+1} \), then \( H_{x_s}(\gamma, \varepsilon) \) is a perfect changer with nodes \( x_s, x_{s+1} \). Let \( Q_1(x_{s+1}, v_1) \) be an \((\varepsilon, \gamma)\)-trail with \( v_1 \neq x_s, x_{s+1} \). Note that if \( e_{s+1} \in E(Q_1) \) then \( x_s Q_1 \) is also an \((\varepsilon, \gamma)\)-trail and it must end at \( x_s \) or \( x_{s+1} \). Hence, we may assume that \( e_{s+1} \notin E(Q_1) \). But then we get \( \pi \to^{Q_1} \pi_1 \) for which \( Kx_{s+1} \) is an \((x_0, \pi_1)\)-path and \( \varepsilon \notin \bar{\pi}_1(x_s) \cap \bar{\pi}_1(x_{s+1}) \). Hence, \( Kx_{s+1} \) is not elementary with respect to \( \pi_1 \), contrary to the minimality of \( K \). Therefore, we must have \( v_1 = x_s \) or \( v_1 = x_{s+1} \) and \( H_{x_s}(\varepsilon, \gamma) \) is perfect with nodes \( x_s, x_{s+1} \).
Proof of (b) is fairly easy and hence, is not included here.

We next set $I(\pi) = \{ j : \pi(x_j) \cap \pi(x_n) \neq \emptyset, \ 0 \leq j < n \}$. We will first show that (6.4.1) holds. Assume to the contrary that $I(\pi) \neq \emptyset$.

(c) If $I(\pi) \neq \emptyset$ then $n - 1 \notin I(\pi)$

Proof of (c): Assume on the contrary that there exists a color $\alpha \in \pi(x_n) \cap \pi(x_{n-1})$. Let $\pi(e_n) = \beta \in \pi(x_j)$ for some $j$ with $0 \leq j \leq n - 1$. We choose $\pi$ such $j$ is largest. We would like to show that $j \geq n - 2$. We can assume $n > 2$ for otherwise, the result is trivial. Assume $j < n - 2$. By using (a) multiple times, we get a coloration $\tau$ such that $K$ is a $(\tau, x_0)$-path, $\tau(e_n) = \beta \in \tau(x_{j+1})$, $\alpha \in \tau(x_j) \cap \tau(x_n)$. Let $\lambda \in \tau(x_{j+1}) \setminus \{\beta\}$. By (a), re-coloring $H_{x_j}(\lambda, \alpha)$, we get a coloration $\phi$ satisfying $\phi(e_n) = \beta \in \phi(x_{j+1})$, $\alpha \in \phi(x_{j+1}) \cap \phi(x_n)$. But then by using (a) repeatedly, we get a coloration $\psi$ satisfying $\psi(e_n) = \beta \in \psi(x_{j+1})$, $\alpha \in \psi(x_{n-1}) \cap \psi(x_n)$. Moreover, $K$ is an $(x_0, \psi)$-path by (a), $n - 1 \in I(\psi)$ and $\psi(e_n) \in \psi(x_{j+1})$ contrary to the maximality of $j$. Therefore, we may assume that $\pi(e_n) = \beta \in \pi(x_j)$ with $j \geq n - 2$.

If $j = n - 2$ then, by (a), $H_{x_{n-2}}(\varepsilon, \gamma)$ is perfect with nodes $x_{n-1}, x_{n-2}$. But $x_n \in V(H_{x_{n-2}}(\varepsilon, \gamma))$ contrary to the perfectness of $H_{x_{n-2}}(\varepsilon, \gamma)$. Hence, we may assume $j = n - 1$. This implies, by (b), that $H_{x_{n-1}}(\varepsilon, \gamma)$ is perfect with a unique node $x_{n-1}$. But $x_n \in V(H_{x_{n-1}}(\varepsilon, \gamma))$ contrary to the perfectness of $H_{x_{n-1}}(\varepsilon, \gamma)$. Therefore, $n - 1 \notin I(\pi)$.

(d): If $I(\pi) \neq \emptyset$, then there exists a coloration $\varphi$ obtained from $\pi$ by a series of interchanges such that $K$ is an $(x_0, \varphi)$-path and $n - 1 \in I(\varphi)$.

Proof of (d): Let $\varphi$ be a coloration obtained from $\pi$ such that

1. $K$ is an $(x_0, \varphi)$-path and
2. $m = \max I(\varphi)$ is maximal with respect to (1).

We claim that $m = n - 1$. Assume $m < n - 1$. Let $\alpha \in \varphi(x_m) \cap \varphi(x_n)$ and
\(\beta \in \varphi(x_{m+1}).\) By (a), \(H_1 = H_{x_m}(\alpha, \beta)\) is perfect and re-coloring \(H_1\), we get a coloration \(\varphi'\) for which \(K\) is an \((x_0, \varphi')\)-path, \(\varphi'(x_m) = \varphi(x_m) \cup \{\beta\} - \{\alpha\}, \ \varphi'(x_{m+1}) = \varphi(x_{m+1}) \cup \{\alpha\} - \{\beta\}\) and \(\varphi'(x) = \varphi(x)\) otherwise. But then \(\alpha \in \varphi'(x_{m+1}) \cap \varphi'(x_n)\) and hence, \(\max I(\varphi') > \max I(\varphi)\) contrary to the assumption that \(m\) is maximal. Hence, \(n - 1 \in I(\varphi)\). This completes the proof of (d).

However, (c) and (d) contradict each other and this completes the proof of (6.4.1).

Hence, we may assume that \(I(\pi) = \emptyset\) for any choice of \(K, \pi, x_0\) but \(\alpha \in \pi^3(x_n)\) for some color \(\alpha\). Assume \(\beta \in \bar{\pi}(x_{n-1})\) and \(P(x_n, v)\) be an \((\alpha, \beta)\)-trail which is not an even cycle. Then we have \(\pi \to \pi'\). Note that if \(\pi(e_n) = \beta\) then \(e_n \notin E(P)\) for otherwise, \(P\) is an even cycle. Hence, if \(v \neq x_{n-1}\) then \(K\) is an \((x_0, \pi')\)-path and \(\beta \in \bar{\pi}'(x_n) \cap \pi'(x_{n-1})\) contrary to the assumption that \(I(\pi) = \emptyset\) for any choice of \(K, \pi, x_0\). Hence, we may assume \(v = x_{n-1}\). Then we have \(\alpha \in \pi'(x_{n-1}) \cap \pi'(x_n)\).

If \(\pi(e_n) \neq \beta\) then \(K\) is an \((x_0, \pi')\)-path. This contradicts to the assumption that \(I(\pi) = \emptyset\) for any choice of \(K, \pi, x_0\). Hence, we may assume \(\pi(e_n) = \beta\). But then we may choose \(P'(x_n, v')\) as another \((\alpha, \beta)\)-trail. But then \(v' \neq x_{n-1}\) and hence by using the same arguments as above we get \(\pi \to P'' \pi''\) for which \(K\) is an \((x_0, \pi'')\)-path and \(\beta \in D(x_n, \pi'') \cap D(x_{n-1}, \pi'')\). But then \(I(\pi'') \neq \emptyset\) contrary to the assumption that \(I(\pi) = \emptyset\) for any choice of \(K, \pi, x_0\). This completes the proof of (6.4.1).

It is worth to note that \(\min\{|k - d_G(x)| : x \in V(G)\} > 0\) is enough while proving the upper bound for the chromatic index. However, since a chain is not necessarily complete, it is rather complicated to show even the existence of an \((\alpha, \beta)\)-chain between two vertices \(x, y\) such that \(e \in E(x, y), \alpha \in \bar{\pi}(x) \cap \bar{\pi}(y)\) and \(\pi(e) = \beta\).

If one thinks that this is a rather trivial chain when working with edge colorings, the challenge that this new setting brings will be understood better. On the other hand, this method promises a unified approach since it does not focus on the criticality of graphs but on the improvability of colorations. While it is introduced by Gupta in
1967 [19], it is not used much due to its complicated nature. We believe that this approach might also be used in $[f, g]$-colorings of graphs which are widely studied by Liu [25].
CHAPTER 7
ON THE COVER CLUSTERS OF A GRAPH

In [17], Goldberg studies the nature of the dense sets in a graph. In this chapter, we give the same results for the “co-dense” sets in a graph.

7.1 Introduction

Let $G$ be a graph. We define $Density$, $\Gamma(G)$, of $G$ and $Co-Density$, $\Gamma_c(G)$, of $G$, respectively, as follows:

$$
\Gamma(G) = \max_{H \subseteq G} \left\lceil \frac{e(H)}{\lfloor v(H)/2 \rfloor} \right\rceil,
$$

$$
\Gamma_c(G) = \min_{H \subseteq G} \left\lfloor \frac{e'(H)}{\lceil v(H)/2 \rceil} \right\rfloor,
$$

where $e(H) = |E(H)|$, $v(H) = |V(H)|$ and $e'(H) = |E'(H)|$ and $v(H) \geq 3$.

We note that $\Gamma = \lceil \chi^{*} \rceil$ and $\Gamma_c = \lfloor \chi^{*}_c \rfloor$ provided that $\Gamma > \Delta - 1$ and $\Gamma_c < \delta + 1$, respectively.

Let $F \subseteq E(G)$. We denote the size of the largest matching in $F$ by $m(G)$. We define the exact density of $G$ as follows:

$$
\Omega(G) = \max_{F \subseteq E(G)} \left\lceil \frac{|F|}{m(F)} \right\rceil.
$$

Let $S \subseteq V(G)$. We denote the size of the smallest edge cover in $E'(S)$ by $c(G)$. We also define the exact co-density of $G$ as follows:

$$
\Omega_c(G) = \min_{S \subseteq V(G)} \left\lfloor \frac{e'(S)}{c(S)} \right\rfloor.
$$
It is fairly easy to see that

\[ \Gamma(G) \leq \Omega(G) \leq \chi'(G), \]

\[ \Gamma_e(G) \geq \Omega_e(G) \geq \chi'_e(G). \]

## 7.2 Upper Bounds of \( \Omega_e \)

In [17], Goldberg recently proved the following result:

**Theorem 7.2.1** (Goldberg [17], 2007). Let \( G \) be a graph. Then \( \Omega(G) = \max \{ \Delta(G), \Gamma(G) \} \).

In what follows, we will repeat the same result for the exact co-density of a graph. Although for some graphs, \( \Omega_e \) is a stronger upper bound for \( \chi'_e \) than \( \Gamma_e \), it turns out that it is not stronger than \( \delta \) and \( \Gamma_e \) combined.

**Lemma 7.2.2.** For any graph \( G \),

\[ \min \{ \Gamma_e(G), \delta(G) \} \geq \Omega_e(G). \]

**Proof.** For any \( S \subseteq V(G), c(S) \geq \lceil |S|/2 \rceil \) and hence, setting \( H = G : E'(S) \), we get

\[ \chi'_e(H) \leq \left\lfloor \frac{e'(S)}{c(S)} \right\rfloor \leq \left\lfloor \frac{e'(S)}{\lceil |S|/2 \rceil} \right\rfloor. \]

To complete the proof, set \( S = x \) where \( d_G(x) = \delta \). Clearly, \( e'(S) = \delta \) and \( c(S) = 1 \) and, hence, \( \delta \geq \Omega_e(G). \)

**Lemma 7.2.3.** Let \( G \) be a graph. If \( S \subseteq V(G) \) is a minimal set satisfying \( \Omega_e(G) = \left\lfloor \frac{e'(S)}{c(S)} \right\rfloor \), then \( G[S] \) is connected.

**Proof.** Assume \( S \subseteq V(G) \) is a minimal set satisfying \( \Omega_e(G) = \left\lfloor \frac{e'(S)}{c(S)} \right\rfloor \), but \( G[S] \) is not connected to get a contradiction. We then have \( S = S_1 \cup S_2, \ S_1 \cap S_2 = \emptyset \) and \( G[S] = G[S_1] \cup G[S_2] \). But this means \( c(S) = c(S_1) + c(S_2) \). By minimality of \( S \), we
have \((\Omega + 1)c(S_i) \leq e'(S_i)\) for \(i = 1, 2\). By disconnectedness of \(S_1\) and \(S_2\), we have 
\(e'(S) = e'(S_1) + e'(S_2) \geq (\Omega + 1)(c(S_1) + c(S_2)) = (\Omega + 1)c(S)\). But this means 
\[\frac{e'(S)}{c(S)} \geq \Omega_c(G) + 1\] contrary to the choice of \(S\). Hence, \(S\) must be connected. \(\Box\)

**Theorem 7.2.4.** For every graph \(G\), \(\min\{\Gamma_c(G), \delta(G)\} = \Omega_c(G)\).

**Proof.** By Lemma 7.2.2, we only need to prove \(\min\{\Gamma_c(G), \delta(G)\} \leq \Omega_c(G)\). Assume

\[\min\{\Gamma_c(G), \delta(G)\} > \Omega_c(G) \tag{7.2.1}\]

Let \(S \subseteq V(G)\) be a minimal set satisfying \(\Omega_c(G) = \left\lfloor \frac{e'(S)}{c(S)} \right\rfloor\). If \(c(S) = \left\lceil \frac{S}{2} \right\rceil\), then the proof follows immediately.

By Lemma 7.2.3, \(H = G[S]\) is connected and, hence, it contains a edge cover having size \(c(H) = c(S)\).

By Tutte’s theorem [7], there is a subset \(K \subseteq S\) such that the number \(q\) of odd connected components of \(H - K\) is \(q = 2c(S) + k - |S|\) where \(|K| = k\). We may also assume that there is no even components. For \(1 \leq i \leq q\), let \(V_i\) denote the vertex set of a component and \(|V_i| = 2v_i - 1\). By the minimality of \(S\), we have

\[(\Omega_c + 1) \left\lfloor \frac{|A|}{2} \right\rfloor \leq e'(A) \tag{7.2.2}\]

for any non-empty proper subset \(A\) of \(S\). Using the basic fact that \(e'(K) - e(K, S - K) \geq 0\) where \(e(K, S - K)\) is the number of edges between \(K\) and \(S - K\), (7.2.1) and (7.2.2) together with basic set calculations, we get

\[e'(S) = \sum_{i=1}^{q} e'(V_i) + e'(K) - e(K, S - K) \geq (\Omega_c + 1) \left( \sum_{i=1}^{q} v_i \right) \tag{7.2.3}\].

Since

\[|S| = k + \sum_{i=1}^{q} (2v_i - 1) = k - q + 2 \sum_{i=1}^{q} v_i\]

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, we have
\[ c(S) = \sum_{i=1}^{q} v_i. \]

Using this expression for \( c(S) \) and inequality (7.2.3),
\[ \Omega_C = \left\lfloor \frac{e'(S)}{c(S)} \right\rfloor \geq (\Omega_c + 1). \]

This contradiction disproves the assumption (7.2.1) and completes the proof. \( \square \)

7.3 Notes

This chapter suggests that one does not need to look for the behavior of the matchings while finding the chromatic index. It is enough to focus on the fractional chromatic index \( \chi'^* \) or fractional cover index \( \chi'_c \) of a graph to understand \( \chi' \) or \( \chi'_c \).
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