Bäcklund Transformation and Homoclinic Solutions to the Coupled Nonlinear Schrödinger System

Dissertation

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dedicated to my wife and my daughter
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CHAPTER I

Introduction

The envelope of a quasi-monochromatic electromagnetic wave in a medium with weakly nonlinear index of refraction is governed by a nonlinear Schrödinger (NLS) equation. The theory of such phenomena in a two-dimensional geometry has been developed by Zakharov and Shabat for waves having the same polarization which leads to the ubiquitous single NLS equation[30]. For applications in optical communications, the solitons are lossless and balance nonlinearity and dispersion, so that each soliton is envisioned to represent one bit of information. In order to pack many bits into an optical fiber, however, one must consider trains of solitons, which requires the study of oscillatory NLS wavetrains. Rather than infinite-line boundary conditions appropriate for solitons, one must consider periodic and quasiperiodic solutions. Kotlyarov and Its[20] developed exact oscillatory wavetrains in explicit form using theta function representations. The linearized stability of NLS periodic solutions and modulation equations for quasiperiodic solutions was developed by Forest and Lee[13]. A more rigorous, function-theoretic analysis for the periodic NLS equation has recently appeared due to Li and McLaughlin[21]. A generalized model which represents the birefringent effects with two polarizations and the governing equations are a coupled nonlinear Schrödinger (CNLS) system. When the two polarizations interact in the
medium a much more unstable propagation will occur, which is actually used to advantage in that the instabilities saturate to a more oscillatory state. Control over the instability is necessary, as governed to leading order by the integrable CNLS system. Such analytical control of these instabilities is the goal of this study. In fact, the simple plane waves in both focusing and defocusing channels may be unstable. These unstable plane waves in an integrable system lead to the existence of homoclinic solutions. This fact has been developed previously in the periodic sine-Gordon equation by Ercolani, Forest, and McLaughlin[7]. The goal of the present paper is to construct the spatially periodic, homoclinic solutions associated to the plane waves of the CNLS system, laying the foundation for general constructions similar to [7]. We adapt the method developed by Chen[4] in constructing the Bäcklund transformations for KdV and sine-Gordon equations, and the Bäcklund-gauge transformation theory of Sattinger and Zirkowski[24] for the $3 \times 3$ AKNS system to obtain the Bäcklund transformation of the CNLS system. The specific results presented include a restriction of a general Bäcklund transformation to construct the spatially periodic, homoclinic solutions arising from the unstable plane waves. These formulas provide an explicit parameterization of one-dimensional fibers in the unstable manifolds of the plane waves.

In this paper the Bäcklund transformations for the $2 \times 2$ AKNS system is generalized to the $3 \times 3$ case. The Bäcklund transformation is constructed by taking quotients of the components of the eigenfunction of the AKNS system and by generalizing the Riccati equation from scalar to vector equations. The Bäcklund transformation is
obtained by choosing a specific automorphism in the hierarchy of commuting flows of the AKNS system. This automorphism is chosen to fix the potential of the AKNS system. In fact, there are different Lie algebraic structures for the focusing and the defocusing systems and the specific automorphisms are completely determined by their Lie algebraic structures. When the Bäcklund transformation is cast in terms of Riccati equations, the automorphism can be identified as an element of the Weyl group of the Lie algebra of the system. This is also a widely observed phenomenon in the $2 \times 2$ AKNS systems[24]. The transformation of eigenfunctions associated with the Bäcklund transformation is given by a linear gauge transformation. This gauge transformation corresponds to a change of local sections of a principal fiber bundle associated with the AKNS system, and it induces transformations on the potentials as well as the eigenfunctions. For the specific Bäcklund transformation obtained from Riccati equations, a linear gauge transformation can be found by closely examining the algebraic structure generated by the potential. This linear gauge transformation provides an iterated Bäcklund transformation.

After identifying the unstable simple plane waves and constructing the Bäcklund transformation we construct the homoclinic solutions via the Bäcklund transformation. The spectral parameters associated with the homoclinic solutions can also be identified by the Bäcklund transformation. In comparison with the scalar nonlinear Schrödinger equation or the sine-Gordon equation, the spectral parameters associated with the homoclinic solutions of the CNLS system correspond to special multiple or branch points of the Floquet multiplier curve and not necessarily to the periodic ele-
ments of the spectrum. In the $2 \times 2$ AKNS system associated to NLS or sine-Gordon equations, all multiple points are periodic points. Finally, we will show that the homoclinic solutions constructed by the Bäcklund transformation are truly homoclinic to the plane waves: they approach the plane wave solutions in different orientations as $t \longrightarrow \pm \infty$. 
CHAPTER II

Plane Waves and Linearized Stability

The following system of coupled nonlinear Schrödinger (CNLS) equations was analyzed by Manakov[22] and shown to have the formal integrable structure shared by more familiar nonlinear wave equations (KdV, single NLS, sine-Gordon, etc.):

\[
\begin{align*}
    u_t - iu_{xx} + 2i\sigma(|u|^2 + |v|^2)u &= 0 \\
    v_t - iv_{xx} + 2i\sigma(|u|^2 + |v|^2)v &= 0,
\end{align*}
\]

(2.1)

where, \(\sigma = -1\) or 1, corresponding to the focusing or the defocusing cases, respectively. This coupled system reduces to the single NLS equation if \(u = 0\) or \(v = 0\), or if \(u = v\).

The CNLS system has the simple plane wave solution

\[
\begin{pmatrix}
    u \\
    v
\end{pmatrix} = \begin{pmatrix}
    u_0 e^{iax + i\Omega t} \\
    v_0 e^{iax + idt}
\end{pmatrix},
\]

(2.2)

with the constants \(u_0, v_0, a, b, c, d\) satisfying the dispersion relation
\[ b = -a^2 - 2\sigma(u_0^2 + u_0^2), \]
\[ d = -c^2 - 2\sigma(u_0^2 + v_0^2) \] (2.3)

Under x-periodic boundary conditions \((a = \frac{2\pi\kappa_e}{L}, c = \frac{2\pi\kappa_p}{L}, \kappa_e, \kappa_p \in \mathbb{Z}, L > 0)\), the linearized stability of the plane wave solutions can be examined by classical Fourier analysis[3, 28]. Expand \(u, v\) in a neighborhood of the solution (2.2):

\[ \tilde{u} = (u_0 + \epsilon w_1(x, t))e^{iux + ibt} \]
\[ \tilde{v} = (v_0 + \epsilon w_2(x, t))e^{icx + idt} \] (2.4)

Inserting (2.4) into (2.1), retaining terms of \(O(\epsilon)\), we find

\[ (i\partial_t + iA\partial_x + \partial_{xx} - 2\sigma M)W = 2\sigma M\tilde{W}, \] (2.5)

where \(W = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}\), \(A = \begin{pmatrix} 2a & 0 \\ 0 & 2c \end{pmatrix}\), \(M = \begin{pmatrix} u_0^2 & u_0v_0 \\ u_0v_0 & v_0^2 \end{pmatrix}\), and \(\tilde{W}\) is the complex conjugate of \(W\).

Using the conjugate equation for \(W\), one easily find a single equation for \(W\):

\[ (\partial_t + 2A\partial_{xt} + (A^2 - 4\sigma M)\partial_{xx} + 2i\sigma(AM - MA) + \partial_{xxx})W = 0. \] (2.6)

We now assume that \(W\) has a formal Fourier expansion
\[ W = \sum_{n \in \mathbb{Z}} W_n(t)e^{i\kappa_n x}, \kappa_n = \frac{2\pi n}{L}. \] (2.7)

From (2.6), the Fourier coefficient \( W_n \) satisfies

\[
(\partial_{tt} + 4ia\kappa_n \partial_t - 4\kappa_n^2(a^2 - \sigma u_0^2) + \kappa_n^4)W_{n1} = -4\sigma \kappa_n u_0 v_0 (\kappa_n - (a - c))W_{n2},
\]

\[
(\partial_{tt} + 4ic\kappa_n \partial_t - 4\kappa_n^2(c^2 - \sigma v_0^2) + \kappa_n^4)W_{n2} = -4\sigma \kappa_n u_0 v_0 (\kappa_n - (a - c))W_{n1},
\] (2.8)

where \( W_n = \begin{pmatrix} W_{n1} \\ W_{n2} \end{pmatrix} \).

Hence, we have

\[
(\partial_{tt} + 4ic\kappa_n \partial_t - 4\kappa_n^2(c^2 - \sigma v_0^2) + \kappa_n^4)(\partial_{tt} + 4ia\kappa_n \partial_t - 4\kappa_n^2(a^2 - \sigma u_0^2) + \kappa_n^4)W_{n1}
\]

\[ = 16\kappa_n^4 u_0^2 v_0^2 W_{n1} - 16\kappa_n^2 u_0^2 v_0^2 (a - c)^2 W_{n1}, \] (2.9)

and the analogous equation for \( W_{n2} \).

Now, let \( W_{nj} = a_j e^{i\omega_n t}, j = 1, 2 \).

From (2.9), \( \omega_n \) satisfies the following equation

\[
(\omega_n^2 + 4a\kappa_n \omega_n + 4\kappa_n^2(a^2 - \sigma u_0^2) - \kappa_n^4)(\omega_n^2 + 4ic\kappa_n \omega_n + 4\kappa_n^2(c^2 - \sigma v_0^2) - \kappa_n^4)
\]

\[-16\kappa_n^4 u_0^2 v_0^2 + 16\kappa_n^2(a - c)^2 u_0^2 v_0^2 = 0. \] (2.10)

The linear instability to modes of wavenumber \( \kappa_n \) is deduced if \( \omega_n \) has non-real
solutions. We can solve the equation (2.10) and summarize the results as follows.

Case 1:

When $a = c$, we have

$$
(\omega_n^2 + 4a\kappa_n\omega_n + 4\kappa_n^2(\kappa_n^2 + \sigma\mu^2) - \kappa_n^4)(\omega_n^2 + 4a\kappa_n\omega_n + 4\kappa_n^2a^2 - \kappa_n^4) = 0. \tag{2.11}
$$

Therefore, $\omega_n = 2a\kappa_n \pm \kappa_n^2, 2a\kappa_n \pm \kappa_n\sqrt{\kappa_n^2 + 4\sigma\mu^2}$, where $\mu^2 = u_0^2 + v_0^2$.

Hence, the Fourier mode $W_{nj} = a_j e^{i\omega nt}$ is stable if $\kappa_n^2 + 4\sigma(u_0^2 + v_0^2) \geq 0$.

From above, for $a = c$ the plane wave solution of defocusing CNLS system ($\sigma = 1$) is linearly stable. And, the plane wave solution of focusing CNLS system is linearly unstable when $\kappa_n^2 < 4(u_0^2 + v_0^2)$.

Case 2:

When $a \neq c$, the equation (2.10) has non-real roots for $\omega_n$ if the discriminant $\Delta < 0$. This linearized instability condition can be found in both focusing and defocusing CNLS systems. We summarize the results in the following:
(1): If $u_0 = v_0$, then all four roots are real in the defocusing case.

(2): For fixed $a \neq c$ and fixed $u_0 \neq v_0$, the stability of wavenumber $\kappa_n$ can be determined by analyzing the discriminant of (2.10). For both focusing and defocusing CNLS systems there are unstable plane wave solutions. The discriminant is given by $\Delta = 256(I^3 - 27J^2)$, with

$$I = c - 4bd + 3c^2 \text{ and } J = \det \begin{pmatrix} 1 & b & c \\ b & c & d \\ c & d & e \end{pmatrix},$$

where

$$b = \kappa_n(a + c)$$

$$c = \frac{1}{3}[(2a^2 + 2c^2 + 8ac - 2\sigma \mu^2)\kappa_n^2 - \kappa_n^4]$$

$$d = 4\kappa_n^3(ca^2 + ac^2 - \sigma(cu_0^2 + av_0^2)) - \kappa_n^5(a + c)$$

$$e = [4a^2\kappa_n^2 - \kappa_n^4 - 4\sigma\kappa_n^2u_0^2][4c^2\kappa_n^2 - \kappa_n^4 - 4\sigma\kappa_n^2v_0^2].$$

The above linearized results imply that there exist solutions homoclinic to the plane wave solutions (2.2) of the focusing CNLS system for any $a = \frac{2\pi k_a}{L}$ and $c = \frac{2\pi k_c}{L}$ and of the defocusing CNLS system for $a \neq c$. The homoclinic solutions are defined in the sense that as $t \rightarrow \pm \infty$ the solutions will approach the simple plane wave solutions. In the next sections, we will construct the Bäcklund transformation of the CNLS system and use the transformation to construct the homoclinic solutions which arise from the unstable plane wave solutions.
CHAPTER III

The Lax Representation of the CNLS System

To find the Bäcklund transformation of the CNLS system we need to find a Lax representation. This representation is a $3 \times 3$ AKNS system which was first found by Manakov in 1973[22]. In this section we examine the representation systematically and in the next section we will explore the Lie algebraic structure of the system and construct the Bäcklund transformation of the CNLS system.

Usually, a nonlinear evolution equation can be obtained from the compatibility condition, $[D_x, D_t] = 0$, of a pair of Dirac operators of the following form:

\[ D_x(z, Q) = \partial/\partial x - A(x, t, z) \]
\[ D_t(z, Q) = \partial/\partial t - B(x, t, z), \]

where $A$ and $B$ are analytic group-valued functions.

The Lax representation of the evolution can be obtained with restriction of $A$ and $B$ in the following form:
\[ A(x,t,z) = izP + Q(x,t) \]  
\[ B(x,t,z) = \sum_{j=0}^{n} z^{n-j} B_j(x,t), \]  

with \( P \) a constant diagonal matrix, \( z \in \mathbb{C} \) the spectral parameter, and the potential \( Q(x,t) \) an off-diagonal matrix function.

The system of differential equations associated with (3.1) is given by the spectral equation

\[ \Psi_x = A\Psi, \]  
and the auxiliary spectral equation

\[ \Psi_t = B\Psi. \]  

The system is integrable if the matrices \( A \) and \( B \) satisfy the well-known Zakharov-Shabat equation

\[ A_t - B_x + [A, B] = 0. \]  

Some interesting nonlinear partial differential equations can be obtained by specifying the potential \( Q \) in a certain semi-simple Lie algebra and seeking the matrix \( B \) of the form \( B = azA + B_1 \), where \( a \) is a constant to be determined and \( B_1 \) is a matrix function depending on \( x \) and \( t \). Under this assumption, the compatibility condition \([D_x, D_t] = 0\) is satisfied iff \( A \) and \( B \) satisfy the Zakharov-Shabat equation (3.5):
\[ A_t - B_x + [A, B] = 0 \]

\[ \iff Q_t - B_{1x} - azQ_x + [A, azA + B_1] = 0 \quad (3.6) \]

\[ \iff (Q_t - B_{1x} + [Q, B_1]) + z(i[P, B_1] - aQ_x) = 0. \quad (3.7) \]

The matrix function \( B_1 \) then satisfies

\[ i[P, B_1] - aQ_x = 0, \]

\[ Q_t - B_{1x} + [Q, B_1] = 0. \quad (3.8) \]

From the above equations (3.8), we will be able to solve for the constant \( a \), the diagonal matrix \( P \), and the matrix \( B_1 \). The representation of the evolution equation for the matrix \( B \) then can be completely determined.

Example 1:

Choose the potential \( Q = \begin{pmatrix} 0 & -\bar{q} \\ \bar{q} & 0 \end{pmatrix} \) in the Lie algebra \( su(2) \), and assume that the constant diagonal matrix \( P = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \).

Let \( B_1 = i \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \). Then, from the first term of (3.8) we have

\[ -\begin{pmatrix} 0 & (a_{11} - a_{22})b_{12} \\ (a_{22} - a_{11})b_{21} & 0 \end{pmatrix} = a \begin{pmatrix} 0 & -\bar{q}_x \\ q_x & 0 \end{pmatrix}. \]
Therefore, the constant \( a = (a_{11} - a_{22}) \), and the matrix \( B_1 = i \begin{pmatrix} b_{11} & \bar{q}_x \\ q_x & b_{22} \end{pmatrix} \).

From the second term of (3.8),

\[
\begin{pmatrix} 0 & -\bar{q}_t \\ q_t & 0 \end{pmatrix} - i \begin{pmatrix} b_{11x} & \bar{q}_{xx} \\ q_{xx} & b_{22x} \end{pmatrix} + i \begin{pmatrix} -|q|_x^2 \\ q(b_{11} - b_{22}) \end{pmatrix} = 0.
\]

Therefore, \( b_{11} = -|q|^2, b_{22} = |q|^2 \), and \( q_t - iq_{xx} - 2i|q|^2q = 0 \), which is the scalar focusing nonlinear Schrödinger equation.

Example 2:

To obtain the Lax representation of the CNLS system we assume that the matrices \( P \) and \( Q \) have the following forms:

\[
P = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix},
\]

and

\[
Q = \begin{pmatrix} 0 & \sigma \bar{u} & \sigma \bar{v} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix}, \text{ in the Lie algebra } gl(3).
\]
Assume that $B_1 = i \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$. From the first term of (3.8),

$$i[P, B_1] - aQ_x = 0$$

iff

$$\begin{pmatrix} 0 & (a_{11} - a_{22})b_{12} & (a_{11} - a_{33})b_{13} \\ (a_{22} - a_{11})b_{21} & 0 & (a_{22} - a_{33})b_{23} \\ (a_{33} - a_{11})b_{31} & (a_{33} - a_{22})b_{32} & 0 \end{pmatrix} = -a \begin{pmatrix} 0 & \sigma \tilde{u}_x & \sigma \tilde{v}_x \\ u_x & 0 & 0 \\ v_x & 0 & 0 \end{pmatrix}.$$

Comparing the entries of the above matrices, we have

$$a_{22} = a_{33},$$

$$a = (a_{11} - a_{22}),$$

and $B_1 = i \begin{pmatrix} b_{11} & -\sigma \tilde{u}_x & -\sigma \tilde{v}_x \\ u_x & b_{22} & b_{23} \\ v_x & b_{32} & b_{33} \end{pmatrix}.$$

From the second term of (3.8),

$$Q_t - B_{1x} + [Q, B_1] = 0$$

iff

$$\begin{pmatrix} 0 & -\tilde{u}_t & -\tilde{v}_t \\ u_t & 0 & 0 \\ v_t & 0 & 0 \end{pmatrix} - i \begin{pmatrix} b_{11x} & -\sigma \tilde{u}_{xx} & -\sigma \tilde{v}_{xx} \\ u_{xx} & b_{22x} & b_{23x} \\ v_{xx} & b_{32x} & b_{33x} \end{pmatrix}$$
\[ +i \begin{pmatrix} \sigma(|u|^2 + |v|^2)_x & \sigma(u_{b_{11} - b_{11}}) + \sigma \bar{v}b_{32} & \sigma \bar{u}b_{23} + \sigma \bar{v}(b_{33} - b_{11}) \\ u(b_{11} - b_{22}) - v_{b_{23}} & -\sigma(|u|^2)_x & -\sigma(u\bar{u})_x \\ -ub_{32} + v(b_{11} - b_{33}) & -\sigma(\bar{u}v)_x & -\sigma(|v|^2)_x \end{pmatrix} = 0. \]

Therefore, \( b_{11} = \sigma(|u|^2 + |v|^2), b_{22} = -\sigma|u|^2, b_{33} = -\sigma|v|^2, b_{23} = -\sigma u\bar{v}, b_{32} = -\sigma \bar{u}v, \)
and
\[
\begin{align*}
    u_t - iu_{xx} + i[u(b_{11} - b_{22}) - v_{b_{23}}] &= 0 \\
    v_t - iv_{xx} + i[-ub_{32} + v(b_{11} - b_{33})] &= 0 \\
    -\bar{u}_t - i\bar{u}_{xx} + i[(b_{11} - b_{22})\bar{u} - b_{32}\bar{v}] &= 0 \\
    -\bar{v}_t - i\bar{v}_{xx} + i[-b_{23}\bar{u} + (b_{11} - b_{33})\bar{v}] &= 0
\end{align*}
\]
or
\[
\begin{align*}
    u_t - iu_{xx} + 2i\sigma(|u|^2 + |v|^2)u &= 0 \\
    v_t - iv_{xx} + 2i\sigma(|u|^2 + |v|^2)v &= 0 \\
    \bar{u}_t + i\bar{u}_{xx} - 2i\sigma(|u|^2 + |v|^2)\bar{u} &= 0 \\
    \bar{v}_t + i\bar{v}_{xx} - 2i\sigma(|u|^2 + |v|^2)\bar{v} &= 0,
\end{align*}
\]
which is the CNLS system (2.1) together with its conjugate.

From this formalism we have therefore elicited a Lax representation of the CNLS system:
Property 1:

Let

\[ D_x(z, Q) = \partial / \partial x - zi \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} - \begin{pmatrix} 0 & \sigma \bar{u} & \sigma \bar{v} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix}, \]  

(3.9)

\[ D_t(z, Q) = \partial / \partial t - (a - b)z^2i \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \]

(3.10)

\[ - \begin{pmatrix} \sigma i(|u|^2 + |v|^2) & -\sigma i\bar{u}_x + \sigma (a - b)z\bar{u} & -\sigma i\bar{v}_x + \sigma (a - b)z\bar{v} \\ iu_x + (a - b)zu & -\sigma i|u|^2 & -\sigma i\bar{u}\bar{v} \\ iv_x + (a - b)zv & -\sigma i\bar{u}v & -\sigma i|v|^2 \end{pmatrix}, \]

with \( a \neq b \) and \( a \neq 0, b \neq 0 \). Then

\[ [D_x, D_t] = 0 \text{ iff } u \text{ and } v \text{ satisfy the CNLS equations} \]

\[ u_t - iu_{xx} + 2i\sigma(|u|^2 + |v|^2)u = 0 \]

\[ v_t - iv_{xx} + 2i\sigma(|u|^2 + |v|^2)v = 0. \]

We note that the Lax representation of NLS or CNLS equations is not unique. Different choices of \( P \) will determine different Lie groups of eigenfunctions. If we choose \( P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \), then the group of eigenfunctions lies in the Lie group
\( SL(3, \mathbb{C}) \). We will use this specific group structure to deduce a linear gauge-Bäcklund transformation and to analyze the Floquet spectrum of the CNLS system.
CHAPTER IV

Bäcklund Transformation of the CNLS System

The Bäcklund transformation for $2 \times 2$ AKNS systems has been studied by many authors by using different approaches\cite{4, 7, 10, 17, 19, 24}. Those systems include the KdV equation, nonlinear Schrödinger equation, sine-Gordon equation, etc. The above methods include inverse scattering techniques, the Hirota formalism, canonical transformations, etc. In differential geometric language, the Bäcklund transformation can be viewed as a condition which keeps the flat connection determined by the Dirac equation (3.1) and the explicit form can be obtained by deriving a Ricatti equation for the eigenfunctions. More generally, one can view the Bäcklund transformation as a special case of a gauge transformation\cite{24, 2}. In some $2 \times 2$ AKNS equations, like the sine-Gordon equation and nonlinear Schrödinger equation, the classical Bäcklund transformations correspond to the linear gauge transformation of the system\cite{24}. Detailed discussions of the definition of a Bäcklund transformation as a flat connection will be given in section 4.2 and the definition of gauge transformation will be given in chapter 5. The relations between Bäcklund transformations and gauge transformations will also be discussed in chapter 5. We also refer the reader to references\cite{24, 2}.

In this chapter a Bäcklund transformation of the CNLS system is constructed from a formalism based on Riccati equations for eigenfunctions. There are two aspects in
viewing the Bäcklund transformation from the Riccati equations. From one point of view, the Riccati equations are derived from the linear equations of the eigenfunctions, and the Bäcklund transformation is obtained from a suitable automorphism of the Riccati functions (quotients of eigenfunction components). This approach has been developed by Chen[4], P. Winternitz[27], and others. From another point of view, the Bäcklund transformation can be viewed as integrability conditions for a flat connection and it can be derived by solving the commuting flow of the Dirac operators. This method is illustrated as follows: First, one can express a given solution \((u, v)\) of the evolution equation in terms of the Riccati functions with a given spectral value \(z_0\). The new solution under the Bäcklund transformation can also be expressed in terms of the same Riccati functions with spectral value \(\tau(z_0)\), where \(\tau\) is an automorphism that fixes the potential in the Lie algebra generated by the potential. In fact, this automorphism \(\tau\) of the Lie algebra can be identified from the discrete automorphism of the Riccati functions as an element of the Weyl group of the Lie algebra in the \(2 \times 2\) AKNS system[24]. In this section we will apply this ansatz to the \(3 \times 3\) AKNS system. We will show that for the focusing and the defocusing CNLS systems there are different Lie algebraic structures of the systems and the Bäcklund transformations can be derived by identifying the automorphism of the Lie algebras generated by the potentials.

We can illustrate these two aspects of Bäcklund transformations with the sine-Gordon equation \(u_{xt} = \sin u\). A Lax representation of the sine-Gordon equation is given as follows:
\[ D_x = \frac{\partial}{\partial x} - iz_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & \frac{u_x}{2} \\ -\frac{u_x}{2} & 0 \end{pmatrix}, \]  
\[ D_t = \frac{\partial}{\partial t} - \frac{i}{4z_0} \begin{pmatrix} \cos u & \sin u \\ \sin u & -\cos u \end{pmatrix}, \]  
with the potential \( Q \) in the Lie algebra \( so(2) \).

Let \( \Psi = (\psi_1, \psi_2) \) be the common eigenfunction of the Dirac operators and let \( m = \frac{\psi_2}{\psi_1} \) be the candidate Riccati function. We can derive the Riccati equation \( m_x = -2iz_0m + u(1 + m^2) \) directly from the system \( D_x \Psi = 0 \). This equation then yields

\[ u = F(z_0, m) = \frac{m_x + 2iz_0m}{1 + m^2}. \]  

Since the automorphism of \( so(2) \) is \( \sigma : A \rightarrow -A^t \) and the action of the automorphism on the Dirac operators leaves \( Q \) invariant and changes the spectral value \( z_0 \) to \( -z_0 \), a new solution of the commuting flow \( D_x^\sigma \Psi = 0 \) is

\[ \tilde{u} = F(-z_0, m) = \frac{m_x - 2iz_0m}{1 + m^2}. \]  

The Bäcklund transformation is then given by comparison of these two expressions:

\[ \tilde{u} = u - \frac{4iz_0m}{1 + m^2}. \]
On the other hand, we observe that

\[ F(-z_0, m) = F(z_0, -\frac{1}{m}). \]  \hfill (4.6)

Thus the Bäcklund transformation \( u \rightarrow \hat{u} \) is also achieved by the mapping \( m \rightarrow -\frac{1}{m} \). In passing from the real vector \((\psi_1, \psi_2)\) to \( m = \frac{\psi_1}{\psi_2} \) we are passing from the vector space \( \mathbb{R}^2 \) to the projective space \( P_1(\mathbb{R}) \). Since the pair of coordinates \( m \) and \(-\frac{1}{m}\) give a covering of \( P_1(\mathbb{R}) \) the Bäcklund transformation was the result of the automorphism \( m \rightarrow -\frac{1}{m} \) on the projective space \( P_1(\mathbb{R}) \). And, we can identify this automorphism as an element of the Weyl group \( S_2 \) of the Lie algebra \( so(2) \). This phenomenon is observed not only in the sine-Gordon equation but also in KdV and scalar NLS equations. The clear geometric interpretations are still unknown, as far as we know.

Before constructing the Bäcklund transformation of the CNLS system we examine some of the fundamental properties of the Lie algebra of the Dirac equations (3.9) and (3.10). With these Lie algebraic structures in mind we will define the Bäcklund transformation as a condition that keeps the connection flat and in the next chapter we will define the gauge transformation of an AKNS system. We refer the reader to reference[16] for the details.

4.1: The Lie algebra of the CNLS system
Property 2:[16]

If $A$ and $B$ satisfy the Zakharov-Shabat equation, then for each fixed $z_0$ and for any $(x_0, t_0) \in \mathbb{R}^2$, there exists a matrix-valued function $\Psi(x, t; z_0) : \mathbb{R}^2 \rightarrow GL(n, \mathbb{C})$ such that

$$\Psi_x = A\Psi$$
$$\Psi_t = B\Psi$$
$$\Psi(x_0, t_0; z_0) = I,$$

the identity matrix,

and the general solution $\Phi$ of (3.4) is given by the following formula

$$\Phi(x, t; z_0) = \Psi(x, t; z_0)\Phi(x_0, t_0; z_0).$$

Property 3:[16]

The smallest Lie subgroup $G$ of $GL(n, \mathbb{C})$, to which the fundamental matrix $\Psi$ belongs, is the connected Lie subgroup of $GL(n, \mathbb{C})$ generated by the Lie algebra $\mathcal{G}$, where the Lie algebra $\mathcal{G}$ is the subalgebra of the $n \times n$ complex matrices spanned by the matrices $\{A(x, t) : (x, t) \in \mathbb{R}^2\}$.

The Lie algebra generated by the matrix functions $A(x, t; z)$ depends on the choice of the matrix $P$. The most important are the zero trace matrices $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for
the NLS equation, and $P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ for the CNLS equation. The associated Lie algebra is $sl(n, C)$ and the Lie group formed by the eigenfunctions is $SL(n, C)$. Later, we will use these Lie algebraic structures to solve the eigenfunctions and to deduce the gauge transformation of the CNLS system.

With Lie algebraic structure in mind we can view the Bäcklund transformation as a Lie transformation in geometry. We will define the Bäcklund transformation as a condition that keeps the connection flat. The transformation itself can be constructed from a concrete point of view.

4.2: The Bäcklund transformation as the condition of a flat connection

We can associate with (3.1) a principal fiber bundle over $R^2 \times C$ with fiber in some Lie subgroup $G$ of $GL(n, C)$ as follows:

$$E = \{(x, t, z; \Phi) : x, t \in R, z \in C\}, \text{ and } \Phi \text{ the solution matrix}$$

$$\pi$$

$$B = R^2 \times C,$$
where \( \pi(x,t,z; \Phi) = (x,t,z) \) and the Lie group \( G \) is formed by the solution matrices.

The equations of (3.1) then define a linear connection for \( E \), the Ehresman-Cartan connection, with the connection form

\[
\omega = d\Phi - A\Phi dx - B\Phi dt. \tag{4.7}
\]

Here, the connection form is defined as a group-valued connection. See Appendix A.

We regard this construction as

\[(A, B) \longrightarrow \omega, \tag{4.8}\]

which assigns a connection to a pair of matrix-valued functions.

The tangent vectors to \( E \) that annihilate the Pfaffian form \( \omega \) are the horizontal distribution of the connection. We can easily show that

**Property 4:**

The linear connection \( \omega \) defined by the form (4.7) is flat, i.e., the asso-
ciated curvature form $\Omega = 0$, if and only if $A$ and $B$ satisfy the Zakharov-Shabat equation $A_t - B_x + [A, B] = 0$.

Proof:

The flatness condition of the connection form means that the Pfaffian system $\omega = 0$ is completely integrable in Frobenius sense.

$$d\omega = d(d\Phi) - dA\Phi \wedge dx - Ad\Phi \wedge dx - dB\Phi \wedge dt - Bd\Phi \wedge dt$$

$$= -A_t dt \Phi \wedge dx - B_x dx \Phi \wedge dt - AB\Phi dt \wedge dx - BA\Phi dx \wedge dt$$

$$= (A_t - B_x) \Phi dx \wedge dt + [A, B] \Phi dx \wedge dt.$$ 

Therefore, the Pfaffian system $\omega = 0$ is completely integrable if and only if $A_t - B_x + [A, B] = 0$, which establishes the results.

We can summarize the results as follows:

Let $D_x(z, Q)$ and $D_t(z, Q)$ be defined by (3.9) and (3.10). Then

$u, v$ satisfy the CNLS equations

iff $A, B$ satisfy the Zakharov-Shabat equation

iff $B_1$ satisfies (3.8)
iff the connection form $\omega$ given by (4.7) is flat.

We can now regard the Bäcklund transformation

$$B : (Q, z_1) \rightarrow (\bar{Q}, z_2)$$

(4.9)

as a transformation of potentials in (3.9) and (3.10) such that the connection form associated with the new potential is flat.

4.3: Construction of a Bäcklund transformation for the CNLS system

We now turn to construct the Bäcklund transformation of the CNLS system. First, we closely examine the Lie algebras generated by the potential $Q$ in the focusing or defocusing CNLS systems.

**Focusing Case:**

The real Lie algebra $\mathcal{G}$ is generated by the potential $Q = \begin{pmatrix} 0 & -\bar{u} & -\bar{v} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix}$. The basis contains at least four generators.
\[ E_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ E_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \]

They generate \[ E_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \]

\[ E_7 = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_8 = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}. \]

The Lie structure can be easily checked. See Appendix B. The smallest Lie algebra generated by the potential is \( su(3) \), the skew Hermitian matrices with trace zero. The map \( \tau : A \to -A^* \), the skew hermitian operator is an automorphism of \( su(3) \) that fixes the potentials of the focusing CNLS system.

**Defocusing Case:**

The real Lie algebra \( \mathcal{G} \) is generated by the potential \( Q = \begin{pmatrix} 0 & \bar{u} & \bar{v} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix} \). The basis contains at least four generators.
\[ E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \]

\[ E_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \]

They generate \( E_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad E_6 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ E_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad E_8 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix}. \]

The smallest Lie algebra generated by the potential is

\[ \mathcal{G} = \{ M \in \mathfrak{gl}(3) | M^\sigma g + gM = 0 \}, \] where \( g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \) and \( \sigma \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \)

\[ \begin{pmatrix} -\bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ \bar{a}_{12} & -\bar{a}_{22} & -\bar{a}_{32} \\ \bar{a}_{13} & -\bar{a}_{23} & -\bar{a}_{33} \end{pmatrix}. \]

\( \sigma \) is an automorphism of the Lie algebra \( \mathcal{G} \) and it fixes the potentials of the defocusing CNLS system.

**Property 5:**
Let $Q = \begin{pmatrix} 0 & \sigma \bar{u} & \sigma v \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix}$ be a matrix function in the Lie algebra $\mathcal{G}$ such that the associated connection form $\omega$ associated with $D_x(z_0, Q)$ and $D_t(z_0, Q)$ is flat. Then the new connection form $\tilde{\omega}$ associated with $D_x(z_0, Q)\tau$ and $D_t(z_0, Q)\tau$ is also flat, where $\tau$ is an automorphism that fixes the potential $Q$.

Proof:

We know that

$\omega$ is flat

iff $-i[P, B_1] - aQ_x = 0$, and

$Q_t - B_{1x} + [Q, B_1] = 0$.

The automorphism $\tau$ acts on $D_x(z_0, Q)$ and $D_t(z_0, Q)$ leaving $P, Q, B_1$ invariant and changing $z_0 \rightarrow \tilde{z}_0$. The second statement of the above is independent of the spectral parameter $z_0$. Therefore, the new connection form $\tilde{\omega}$ is also flat. This completes the proof.

We now assume that $B : Q \rightarrow \tilde{Q}$ is a Bäcklund transformation, which means that the connection form $\omega$ associated with $D_x(z_0, Q)$ and $D_t(z_0, Q)$ is flat as is the connection form associated with $D_x(z_0, \tilde{Q})\tau$ and $D_t(z_0, \tilde{Q})\tau$. From the classical method given by Chen[4] in constructing the Bäcklund transformations of the KdV and sine-
Gordon equations, or the Bäcklund-gauge transformation method given by Sattinger and Zarkowski[24] we note that the criterion to construct the Bäcklund transformation is that the Riccati functions of $D_{x}(z_{0}, Q)$ and $D_{x}(z_{0}, \hat{Q})$ are the same. If we express a given solution $(u, v)$ in terms of Riccati functions with spectral parameter $z$, $u = F(z, m, n)$ and $v = G(z, m, n)$, then the new solution can be expressed as $\hat{u} = F(\hat{z}, m, n)$ and $\hat{v} = G(\hat{z}, m, n)$, where $m = \frac{\psi_{2}}{\psi_{1}}$ and $n = \frac{\psi_{3}}{\psi_{1}}$. The typical Bäcklund transformation $B : Q \rightarrow \hat{Q}$ for the CNLS system now can be obtained by stipulating the equations

$$D_{x}(z_{0}, \hat{Q})g\Phi = 0 = D_{x}(z_{0}, Q)\Phi,$$  \hspace{1cm} (4.10)

where $g$ is a scalar function and $\Phi$ is a solution vector.

From (4.10), we can deduce a pair of Riccati equations. Use the Riccati equations we obtain the Bäcklund transformations of the CNLS system. The result and the proof are given in the following theorem.

**Theorem 1 :**

The Bäcklund transformation $B : Q \rightarrow \hat{Q}$ for the focusing CNLS system is given by
\[ \tilde{u} = u + \frac{\alpha \text{im}(\bar{z}_0 - z_0)}{1 + |m|^2 + |n|^2} = u + \frac{\alpha \text{im}(\bar{z}_0 - z_0)\bar{\psi}_1\psi_2}{|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2} \]

(4.11)

\[ \tilde{v} = v + \frac{\alpha \text{im}(\bar{z}_0 - z_0)}{1 + |m|^2 + |n|^2} = v + \frac{\alpha \text{im}(\bar{z}_0 - z_0)\bar{\psi}_1\psi_3}{|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2} \]

where \( \alpha = a - b \), \( m = \frac{\psi_2}{\psi_1} \), \( n = \frac{\psi_3}{\psi_1} \), and \( \Psi = (\psi_1, \psi_2, \psi_3)^t \) is the wave function satisfying \( D_x(z_0, Q)\Psi = 0 \) and \( D_t(z_0, Q)\Psi = 0 \). Here, we assume that the Riccati functions \( m \) and \( n \) are analytic in \( x \) and \( t \).

Proof:

Let \( \Psi = (\psi_1, \psi_2, \psi_3)^t \) be the wave function of equation (3.9) associated with the spectral parameter \( z_0 \in C \). Let \( \tilde{\Psi} = g\Psi \). From the identity

\[ D_x(z_0, Q)\Psi = 0 = D_x(z_0, \tilde{Q})\tilde{\Psi}, \]

we have

\[
\begin{align*}
\psi_{1x} &= a_0 \bar{\psi}_1 - \bar{u}\psi_2 - \bar{v}\psi_3 \\
\psi_{2x} &= u\psi_1 + b_0 \bar{i}\psi_2 \\
\psi_{3x} &= v\psi_1 + b_0 \bar{i}\psi_3
\end{align*}
\]

and

\[
\begin{align*}
\tilde{\psi}_{1x} &= a_0 \bar{\tilde{\psi}}_1 - \bar{\tilde{u}}\tilde{\psi}_2 - \bar{\tilde{v}}\tilde{\psi}_3, \\
\tilde{\psi}_{2x} &= \tilde{u}\tilde{\psi}_1 + b_0 \bar{i}\tilde{\psi}_2, \\
\tilde{\psi}_{3x} &= \tilde{v}\tilde{\psi}_1 + b_0 \bar{i}\tilde{\psi}_3
\end{align*}
\]
Let \[ m = \frac{\psi_2}{\psi_1} = \frac{\bar{\psi}_2}{\bar{\psi}_1}, \quad \text{and} \quad n = \frac{\psi_3}{\psi_1} = \frac{\bar{\psi}_3}{\bar{\psi}_1}. \]

Then we have
\[
m_x = \frac{\psi_2 \psi_1 - \psi_1 \psi_2}{\psi_1^2} = \frac{(u \psi_1 + k \bar{z}_0 i \psi_2) \psi_1 + (u \bar{\psi}_2 + \bar{v} \psi_3 - a \bar{z}_0 i \psi_1) \psi_2}{\psi_1^2} = u - \alpha z_0 i m + \bar{\alpha} m^2 + \bar{\alpha} n.
\]

and
\[
n_x = \frac{\psi_3 \psi_1 - \psi_1 \psi_3}{\psi_1^2} = \frac{(u \psi_1 + k \bar{z}_0 i \psi_3) \psi_1 + (u \bar{\psi}_2 + \bar{v} \psi_3 - a \bar{z}_0 i \psi_1) \psi_3}{\psi_1^2} = v - \alpha z_0 i n + \bar{\alpha} n^2 + \bar{\alpha} mn,
\]

where \( \alpha = a - b. \)

Also
\[
m_x = \frac{\psi_2 \psi_1 - \psi_1 \psi_2}{\psi_1^2} = \bar{u} - \alpha \bar{z}_0 i m + \bar{\alpha} m^2 + \bar{\alpha} mn
\]

\[
n_x = \frac{\psi_3 \psi_1 - \psi_1 \psi_3}{\psi_1^2} = \bar{v} - \alpha \bar{z}_0 i n + \bar{\alpha} n^2 + \bar{\alpha} mn
\]

We get the following Riccati equations:
\[
m_x = u - \alpha z_0 i m + \bar{\alpha} m^2 + \bar{\alpha} mn
\]
\[
m_x = \bar{u} - \alpha \bar{z}_0 i m + \bar{\alpha} m^2 + \bar{\alpha} mn
\]
\[
n_x = v - \alpha z_0 i n + \bar{\alpha} n^2 + \bar{\alpha} mn
\]
\[
n_x = \bar{v} - \alpha \bar{z}_0 i n + \bar{\alpha} n^2 + \bar{\alpha} mn.
\]
We now use the Riccati equations to derive the Bäcklund transformation.

From (4.12),

\[ u = m_x + \alpha_z \bar{m} - \bar{u} m^2 - \bar{v} mn \]
\[ = m_x + \alpha_z \bar{m} - m^2 (\bar{m}_x - \alpha \bar{z} \bar{m} - u \bar{m}^2 - v \bar{m} \bar{n}) \]
\[ - mn (\bar{n}_x - \alpha \bar{z} \bar{n} - v \bar{n}^2 - u \bar{n} \bar{m}) . \]

\[ v = n_x + \alpha_z \bar{n} - \bar{v} n^2 - \bar{u} mn \]
\[ = n_x + \alpha_z \bar{n} - n^2 (\bar{n}_x - \alpha \bar{z} \bar{n} - v \bar{n}^2 - u \bar{n} \bar{m}) \]
\[ - mn (\bar{m}_x - \alpha \bar{z} \bar{m} - u \bar{m}^2 - v \bar{m} \bar{n}) . \]

We have the following equations

\[ u (1 - |m|^4 - |m|^2 |n|^2) - v (m \bar{m} (|m|^2 + |n|^2)) \]
\[ = m_x + \alpha_z \bar{m} - m^2 \bar{m}_x + \alpha \bar{z} \bar{m} |m|^2 - mn \bar{m}_x + \alpha \bar{z} \bar{m} |n|^2 \]

and

\[ - u (m \bar{m} (|m|^2 + |n|^2)) + v (1 - |n|^4 - |m|^2 |n|^2) \]
\[ = n_x + \alpha_z \bar{n} - n^2 \bar{n}_x + \alpha \bar{z} \bar{n} |n|^2 - mn \bar{n}_x + \alpha \bar{z} \bar{n} |m|^2 . \]

Eliminate \( v \) from the above equations. After some computations, we have

\[ u = \frac{(|m|^2 + |n|^2) (m \bar{n} n_x - |n|^2 m_x + \alpha \bar{z} \bar{m}) + m_x + \alpha z \bar{m} - m^2 \bar{m}_x - mn \bar{m}_x}{1 - (|m|^2 + |n|^2)^2} . \]

(4.13)
Eliminating $u$, we get

$$v = \frac{(|m|^2 + |n|^2)(mn \bar{m}_x - |m|^2 n_x + \alpha \bar{z}_0 in) + n_x + \alpha z_0 in - n^2 \bar{n}_x - mn \bar{n}_x}{1 - (|m|^2 + |n|^2)^2}. \quad (4.14)$$

From similar computations for $\tilde{u}, \tilde{v}$,

$$\tilde{u} = \frac{(|m|^2 + |n|^2)(mn \bar{m}_x - |n|^2 m_x + \alpha z_0 im) + m_x + \alpha \bar{z}_0 im - m^2 \bar{m}_x - mn \bar{m}_x}{1 - (|m|^2 + |n|^2)^2} \quad (4.15)$$

$$\tilde{v} = \frac{(|m|^2 + |n|^2)(mn \bar{m}_x - |m|^2 n_x + \alpha z_0 in) + n_x + \alpha \bar{z}_0 in - n^2 \bar{n}_x - mn \bar{n}_x}{1 - (|m|^2 + |n|^2)^2}. \quad (4.16)$$

Therefore,

$$\tilde{u} - u = \frac{\alpha im(|m|^2 + |n|^2)(z_0 - \bar{z}_0) + \alpha im(\bar{z}_0 - z_0)}{1 - (|m|^2 + |n|^2)^2} = \frac{\alpha im(\bar{z}_0 - z_0)}{1 + (|m|^2 + |n|^2)}. \quad (4.17)$$

$$\tilde{v} - v = \frac{\alpha in(|m|^2 + |n|^2)(z_0 - \bar{z}_0) + \alpha in(\bar{z}_0 - z_0)}{1 - (|m|^2 + |n|^2)^2} = \frac{\alpha in(\bar{z}_0 - z_0)}{1 + (|m|^2 + |n|^2)}. \quad (4.18)$$

Finally, we get the Bäcklund transformation $B : (u, v) \rightarrow (\tilde{u}, \tilde{v})$, with
\[ \tilde{u} = u + \frac{\alpha m(z_0 - z_0)}{1 + |m|^2 + |n|^2} \tag{4.19} \]

\[ \tilde{v} = v + \frac{\alpha n(z_0 - z_0)}{1 + |m|^2 + |n|^2} \tag{4.20} \]

which is the desired result.

Using the same method we derive the Bäcklund transformation of the defocusing CNLS system. See Appendix C.

**Theorem 2:**

The Bäcklund transformation for the defocusing CNLS system is given by

\[ \tilde{u} = u + \frac{\alpha m(z_0 - z_0)}{1 - |m|^2 - |n|^2} = u + \frac{\alpha i(z_0 - z_0)\bar{\psi}_1\psi_2}{|\psi_1|^2 - |\psi_2|^2 - |\psi_3|^2} \tag{4.21} \]

\[ \tilde{v} = v + \frac{\alpha n(z_0 - z_0)}{1 - |m|^2 - |n|^2} = v + \frac{\alpha i(z_0 - z_0)\bar{\psi}_1\psi_2}{|\psi_1|^2 - |\psi_2|^2 - |\psi_3|^2} \]

Here, we assume that \( m \) and \( n \) are analytic in \( x \) and \( t \) and \( |m|^2 + |n|^2 \neq 1 \).

We close this section by making some observations.

From (4.13) and (4.15), we note that the Bäcklund transformations given in (4.11) or (4.21) can be defined in the other form.
\[ F(z_0, m, n) \rightarrow F(\bar{z}_0, m, n) \]
simply changing the spectral parameter \( z_0 \rightarrow \bar{z}_0 \), with generating function
\[ F(z_0, m, n) = (\mathcal{U}, \mathcal{V}), \]
where \( \mathcal{U}, \mathcal{V} \) are given in (4.13) and (4.14) respectively.

Also, we note that this Bäcklund transformation is invariant under the transformation \((1, m, n) \rightarrow (1, \frac{m}{|m|^2 + |n|^2}, \frac{n}{|m|^2 + |n|^2})\). In passing from the vector \((\psi_1, \psi_2, \psi_3)\) to the Riccati functions \( m = \frac{\psi_2}{\psi_1} \) and \( n = \frac{\psi_3}{\psi_1} \) we are passing from the vector space \( C^3 \) to the projective space \( \mathbb{P}_2(C) \). This invariant transformation can be viewed as an element in the Weyl group \( S_3 \) acting on the complex projective space \( \mathbb{P}_2(C) \). This property is also observed in the sine-Gordon equation and nonlinear Schrödinger equation which is the key ingredient to construct the gauge transformation of those equations[24]. Using this property one can construct a simple gauge transformation that generates the single soliton solutions of the CNLS system. We use those known formulas to confirm the results of this section. The detail discussions of gauge transformation are in the next chapter.

4.4: Example: single soliton solutions of the CNLS system

Choose \( a = 2, b = -1 \) in the Lax representation (3.9). Start from the trivial solution \((u, v) = (0, 0)\) of the CNLS system. The eigenfunction \( \Psi = (\psi_1, \psi_2, \psi_3) \) satisfies the following ode’s:
\[ \psi_{1x} = 2z_0 i \psi_1 \]
\[ \psi_{2x} = -z_0 i \psi_2 \]
\[ \psi_{3x} = -z_0 i \psi_3, \]
and
\[ \psi_{1t} = 6z_0^2 i \psi_1 \]
\[ \psi_{2t} = -3z_0^2 i \psi_2 \]
\[ \psi_{3t} = -3z_0^2 i \psi_3. \]

Therefore,
\[ m = \psi_2 / \psi_1 = e^{-9z_0^2 it - 3z_0 ix + \theta_1} \tag{4.22} \]
\[ n = \psi_3 / \psi_1 = e^{-9z_0^2 it - 3z_0 ix + \theta_2}, \tag{4.23} \]

where \( \theta_1 \) and \( \theta_2 \) are phase constants.

Let \( z_0 = a + bi, \theta_1 = c_1 + c_2 i, \theta_2 = c_3 + c_4 i. \) Then

\[ \tilde{u} = \frac{3im(z_0 - z_0)}{1 + |m|^2 + |n|^2} = 6b \frac{e^{i(-9(a^2 - b^2)t - 3ax + c_2)}}{2 \cosh(18abt + 3bx + c_1) + e^{18abt + 3bx + 2c_1 - c_1}}, \tag{4.24} \]

\[ \tilde{v} = \frac{3im(z_0 - z_0)}{1 + |m|^2 + |n|^2} = 6b \frac{e^{i(-9(a^2 - b^2)t - 3ax + c_4)}}{2 \cosh(18abt + 3bx + c_3) + e^{18abt + 3bx + 2c_1 - c_3}}, \tag{4.25} \]

which is the single soliton solution[22].
CHAPTER V

Gauge Transformation and Iterated Bäcklund Transformation

A gauge transformation corresponds to a change of local sections of the principal fiber bundle $E$ over the base $B$ with fibers in the Lie group of the fiber bundle. We can regard it as a change of basis on the fibers of the fiber bundles. I.e. if $\Psi$ is a section of the fiber bundle $E$, then $\tilde{\Psi} = G\Psi$ is a section in the new basis. With the principal fiber bundle defined in section 4.2, the gauge transformation $G$ induces a transformation on the connection potentials $Q \rightarrow \tilde{Q}$ or on the connection matrix functions $\begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix}$ with the property that

$$(\tilde{D}\Psi) = \tilde{D}\tilde{\Psi},$$  \hspace{1cm} (5.1)

where $\tilde{D}$ is associated with the new linear connection $\tilde{\omega} = d\tilde{\Psi} - \tilde{A}\tilde{\Psi}dx - \tilde{B}\tilde{\Psi}dt$.

If we write (5.1) in matrix form, we can regard a gauge transformation $G$ from potential $Q \rightarrow \tilde{Q}$ as a matrix function $G(x, t; z)$, which is bounded in $x$ and analytic in $z$ such that

$$GD_x(z, Q) = D_x(z, \tilde{Q})G,$$

$$GD_t(z, Q) = D_t(z, \tilde{Q})G,$$  \hspace{1cm} (5.2)

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It is clear that if $D_x(z, Q)\Psi = 0$, then $D_x(z, \tilde{Q})\tilde{\Psi} = 0$ for $\tilde{\Psi} = G\Psi$. The gauge transformation $G$ transfers the wave function $\Psi$ of potential $Q$ to the wave function $\tilde{\Psi}$ of the new potential $\tilde{Q}$. To obtain a Bäcklund transformation $B : Q \rightarrow \tilde{Q}$ from a gauge transformation $G$ there are two questions which must be answered.

(1) How does one find the gauge transformation $G$ without knowing the potential $\tilde{Q}$? Does there exist such a transformation?

(2) If the gauge transformation $G$ is known, how does one derive the Bäcklund transformation $B : Q \rightarrow \tilde{Q}$ from $G$?

Some of the answers are given by M. Boiti, and G. Tu[2]; D. Sattinger and V. Zurkowski[24], and others based on the assumption that the gauge transformation $G$ is a polynomial function in $z$, i.e. $G(x, t; z) = G_0(x, t) + \sum_{j=1}^{n} z^j G_j(x, t)$. Under this assumption one considers the gauge transformation $G$ on the wave functions $\tilde{\Psi} = G\Psi$. The wave functions $\Psi$ and $\tilde{\Psi}$ satisfy the following equations:

$$\Psi_x = (izP + Q(x, t))\Psi, \quad (5.3)$$

$$\tilde{\Psi}_x = (izP + \tilde{Q}(x, t))\tilde{\Psi}. \quad (5.4)$$
Since $\hat{\Psi} = G\Psi$, from (5.2)

$$G_x = (izP + \hat{Q})G - G(izP + Q)$$

$$= iz[P, G] + \hat{Q}G - GQ$$

$$= iz[P, G] + [Q, G] + RG$$

(5.5)

where $\hat{Q} = Q + R$.

The equation (5.5) determines the gauge transformation $G$ and is called the gauge equation. Usually, a general gauge transformation satisfying (5.5) is hard to obtain. If we assume that $G(x, t; z) = G_0(x, t) + \sum_{j=1}^{n} z^j G_j(x, t)$, then from (5.5) we have

$$G_{0,x} = \hat{Q}G_0 - G_0Q$$

$$G_{j,x} = i[P, G_{j-1}] + (\hat{Q}G_j - G_jQ), j = 1, \ldots, n$$

$$0 = [P, G_n].$$

(5.6)

The matrix functions $G_j(x, t)$ might be solved recursively if we specify the potentials $Q$ and $\hat{Q}$. Therefore, for a given Bäcklund transformation $B : Q \longrightarrow \hat{Q}$ one can find a corresponding gauge transformation.

Specifically, if $G$ is linear in $z$, then the gauge transformation has a simple form which is completely determined by a set of linearly independent wave functions. This is the main result obtained by D. Sattinger and V. Zurkowski[24]. The theorem is stated as follows:

**If there exist a gauge transformation** $G(x, t; z) = G_0(x, t) + z G_1(x, t)$ **which**
solves (5.5), then \( G \) takes the form

\[
G = \Phi(z - C)\Phi^{-1}, \Phi = [\Psi_1, \ldots, \Psi_n],
\]

(5.7)

where \( C = \text{diag}(z_1, \ldots, z_n) \) and \( \Psi_1, \ldots, \Psi_n \) are linear independent wave functions satisfying the equations \( D_x(z_j, Q)\Psi_j = 0 \), and \( D_t(z_j, Q)\Psi_j = 0 \).

Here, we note that the existence of such matrix function \( G \) is a necessary condition of proving the existence of a linear gauge transformation. The sufficient condition of the existence of a linear gauge transformation is still an open problem. However, the theorem suggests that one can seek a linear gauge transformation of the form \( G(x, t; Z) = zI + G_0(x, t) \) and try to verify that \( G(x, t; z) \) satisfies the gauge equation (5.5).

5.1: Bäcklund transformation as a linear gauge transformation

We assume that there exists a linear gauge transformation that solves (5.5). From (5.6), we can deduce a Bäcklund transformation from a gauge transformation. Seeking a linear gauge transformation of the form \( G = G_0 + zI \) and substituting into (5.6), we have

\[
G_{0x} = [Q, G_0] + RG_0,
\]

(5.8)

\[
i[P, G_0] + R = 0,
\]

(5.9)

where \( \tilde{Q} = Q + R \).
If we can determine $G_0$ from the above equations, then the Backlund transformation is given by

$$\tilde{Q} = Q - i[P, G_0]. \quad (5.10)$$

We illustrate the method by the following example.

Example 1: The Bäcklund transformation of focusing NLS equation

We choose $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Assume that $G_0 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ and $\tilde{Q} = Q + R = \begin{pmatrix} 0 & -\bar{q} \\ q & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\bar{r} \\ r & 0 \end{pmatrix}$.

From (5.9),

$$i \begin{pmatrix} 0 & 2g_{12} \\ -2g_{21} & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\bar{r} \\ r & 0 \end{pmatrix} = 0$$

Therefore, $r = 2ig_{21}$, and the Backlund transformation $B: q \rightarrow \tilde{q}$ is given by $\tilde{q} = q + 2ig_{21}$.

Now, let $\Psi_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ be the wave function associated with the spectral parameter $z_0$. Then $\Psi_2 = \begin{pmatrix} -\overline{\psi}_2 \\ \overline{\psi}_1 \end{pmatrix}$ is a solution associated with the spectral parameter $\overline{z}_0$. 
We now seek a linear gauge transformation of the form

\[ G = zI + G_0, \text{ with } G_0 = \Phi \begin{pmatrix} -z_0 & 0 \\ 0 & -\bar{z}_0 \end{pmatrix} \Phi^{-1}, \]

where

\[ \Phi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix}. \]

We get \( g_{21} = \frac{(z_0 - \bar{z}_0)\bar{\psi}_1 \psi_2}{|\psi_1|^2 + |\psi_2|^2} \), and the new solution \( \tilde{q} \) is given by

\[ \tilde{q} = q + 2i \frac{(z_0 - \bar{z}_0)\bar{\psi}_1 \psi_2}{|\psi_1|^2 + |\psi_2|^2}. \tag{5.11} \]

This is the Bäcklund transformation of the focusing NLS equation.

Unfortunately, this method can not be applied to construct a Bäcklund transformation of the CNLS system. The difficulty involves the different components \( u \) and \( v \) in the potential and it is hard to deduce a set of three linearly independent wave functions. However, a simple gauge transformation can be constructed to generate the single soliton solution.

Example 2:

Now, let us assume that \( G_0 = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \), and
\[ \tilde{Q} = Q + R = \begin{pmatrix} 0 & -\bar{u} & -\bar{v} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\bar{s} & -\bar{t} \\ s & 0 & 0 \\ t & 0 & 0 \end{pmatrix}. \]

Then, from (5.9),

\[ i \begin{pmatrix} 0 & \alpha g_{12} & \alpha g_{13} \\ -\alpha g_{21} & 0 & 0 \\ -\alpha g_{31} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\bar{s} & -\bar{t} \\ s & 0 & 0 \\ t & 0 & 0 \end{pmatrix} = 0. \]

Therefore, \( s = \alpha i g_{21} \) and \( t = \alpha i g_{31} \). The Bäcklund transformation \( B : (u, v) \rightarrow (\bar{u}, \bar{v}) \) is given by \( \bar{u} = u + s = u + \alpha i g_{21}, \) and \( \bar{v} = v + t = v + \alpha i g_{31} \).

Let us choose \( P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \) and \((u, v) = (0, 0)\) the trivial solution of CNLS equation.

Let \( \Psi_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \) be the wave function associated with the spectral parameter \( z_0 \). It is clear that \( (-\bar{\psi}_2, \bar{\psi}_1, 0) \) and \( (-\bar{\psi}_3, 0, \bar{\psi}_1) \) are two linearly independent solutions associated with the spectral parameter \( \bar{z}_0 \).

The linear gauge transformation is then given by

\[ G = zI + G_0, \text{ with } G_0 = \Phi \begin{pmatrix} -z_0 & 0 & 0 \\ 0 & -\bar{z}_0 & 0 \\ 0 & 0 & -\bar{z}_0 \end{pmatrix} \Phi^{-1}, \text{ where} \]
\[ \Phi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & -\bar{\psi}_3 \\ \psi_2 & \bar{\psi}_1 & 0 \\ \psi_3 & 0 & \bar{\psi}_1 \end{pmatrix}. \]

After some computations, we get

\[ g_{21} = \frac{(z_0 - z_0)\bar{\psi}_1 \psi_2}{|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2} \] and \[ g_{31} = \frac{(z_0 - z_0)\bar{\psi}_1 \psi_3}{|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2}. \]

The new solution \((\tilde{u}, \tilde{v})\) is then given by

\[ \tilde{u} = 2i \frac{(z_0 - z_0)\bar{\psi}_1 \psi_2}{|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2}, \quad \text{(5.12)} \]
\[ \tilde{v} = 2i \frac{(z_0 - z_0)\bar{\psi}_1 \psi_3}{|\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2}. \quad \text{(5.13)} \]

This is the single soliton solutions of CNLS system which we derived in previous chapter.

5.2: An iterated scheme of Bäcklund transformation

In this section we will derive an iterated scheme of the Bäcklund transformations with the help of gauge transformations. In previous chapter we only construct the Bäcklund transformation of the potentials. To perform the Bäcklund transformations iterately we need to find the wave function at each stage of the transformation. An iterated scheme relies on finding a gauge transformation that transfers the wave
functions. From (5.6), if we assume that the gauge transformation is polynomial in $z$ and the potentials $Q$ and $\tilde{Q}$ are specified, we may solve the gauge matrix $G$ recursively. Now, let $\tilde{Q} = Q + R$ be the Bäcklund transformation given in (4.11), where

$$R = \begin{pmatrix} 0 & -\tilde{s} & -\tilde{t} \\ s & 0 & 0 \\ t & 0 & 0 \end{pmatrix}, \text{ with } s = \frac{3i(z_0 - z_0) m}{1 + |m|^2 + |n|^2} \text{ and } t = \frac{3i(z_0 - z_0) n}{1 + |m|^2 + |n|^2}.$$  

Seek a linear gauge transformation of the form $G = G_0 + z_1 I$, where $G_0$ is a matrix function in $x$ and $t$, and is in the Lie algebra $su(3)$. From (5.6), $G_0$ satisfies the following equations

$$G_{0,x} = [Q, G_0] + RG_0,$$

$$i[P, G_0] + R = 0. \tag{5.14}$$

From the above equations one can solve for the matrix function $G_0$.

Let $G_0 = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$.

From $R = -i[P, G_0]$, we have

$$g_{12} = -\frac{i}{3}\tilde{s}, \quad g_{13} = -\frac{i}{3}\tilde{t},$$

$$g_{21} = -\frac{i}{3}s, \quad g_{31} = -\frac{i}{3}t,$$

and $G_0$ takes the form

$$G_0 = \begin{pmatrix} g_{11} & -\frac{i}{3}\tilde{s} & -\frac{i}{3}\tilde{t} \\ -\frac{i}{3}s & g_{22} & g_{23} \\ -\frac{i}{3}t & g_{32} & g_{33} \end{pmatrix}, \text{ with } g_{ii} \in iR \text{ and } g_{11} + g_{22} + g_{33} = 0.$$
From $G_{0,x} = [Q, G_0] + RG_0$, we can solve for $G_0$. Comparing the entries of the matrices, we have

\[ -\frac{i}{3} s_x = u(g_{11} - g_{22}) - v g_{23} + s g_{11} \quad (1) \]

\[ -\frac{i}{3} \bar{s}_x = \bar{u}(g_{11} - g_{22}) - \bar{v} g_{32} - s g_{22} - i g_{32} \quad (2) \]

\[ -\frac{i}{3} t_x = -u g_{32} + v(g_{11} - g_{33}) + t g_{11} \quad (3) \]

\[ -\frac{i}{3} \bar{t}_x = -\bar{u} g_{23} + \bar{v}(g_{11} - g_{33}) - \bar{s} g_{23} - i \bar{g}_{33} \quad (4) \]

\[ g_{32x} = -\frac{i}{3}(\bar{u} t + \bar{s} v + \bar{s} t) \quad (5) \]

\[ g_{22x} = -\frac{2i}{3} \text{Re}(u \bar{s}) - \frac{i}{3}|s|^2 \quad (6) \]

\[ g_{33x} = -\frac{2i}{3} \text{Re}(v \bar{t}) - \frac{i}{3}|t|^2 \quad (7). \]

From (1) and (2), $g_{32} = \frac{\bar{s}}{i} g_{33}$.

From (3) and (4), $g_{32} = \frac{t}{i} g_{22}$.

Therefore, $g_{22} = \frac{|t|^2}{|u|^2} g_{33}$,

We summarize the results in the following property.

**Theorem 3:**

Let $\tilde{Q} = Q + R$ be the Bäcklund transformation given in (4.11). The linear gauge transformation associated with the Bäcklund transformation is given by $G(x, t; z) = G_0(x, t) + z I$ with
\[ G_0 = -\frac{i}{3} \begin{pmatrix} -\left(1 + \frac{|p|^2}{|t|^2}\right)g & s & \bar{t} \\ s & \frac{|p|^2}{|t|^2}g & \bar{s}g \\ \bar{t} & \bar{s}g & g \end{pmatrix}, \]

where \( g = -\frac{i}{3} \int_{0}^{\pi} 2 \text{Re}(u\bar{t}) + |t|^2 \, dx, \) \( s = \frac{3i(z_0 - \bar{z}_0)m}{1 + |m|^2 + |n|^2}, \) and \( t = \frac{3i(z_0 - \bar{z}_0)n}{1 + |m|^2 + |n|^2}. \)

The iterated scheme of the Bäcklund transformation can be described as follows: starting from a given potential \( Q_0 \) (\( (0,0) \) or plane waves), one solves the wave function \( \Psi_0 \) of the potential and applies the Bäcklund transformation to obtain the new potential \( Q_1 \) and the gauge matrix \( G_0 \). The wave function associated with the potential \( Q_1 \) is then given by \( \Psi_1 = G_0 \Psi_0 \) and one can obtain the next potential \( Q_2 \), and so on. This iterated scheme requires one to solve the wave function of the system once and the rest of the wave functions can be obtained from the gauge matrices.

We note that an analogous results of the gauge transformation of the defocusing CNLS system can be obtained from the same method. See appendix D. In next chapter, we will apply the Bäcklund transformation to construct the homoclinic solutions.
CHAPTER VI

Homoclinic Solutions of the CNLS System

In this section we construct the solutions homoclinic to the plane waves for the CNLS system by using the Bäcklund transformation given in (4.11) and (4.21). The scheme for obtaining such solutions is the following:

(a) Choose the unstable plane wave solution $u = u_0 e^{i(ax+bt)}$,
   
   $v = v_0 e^{i(cx+dt)}, \ a = \frac{2\pi \alpha}{L}, \ c = \frac{2\pi \gamma}{L}, \ L > 0.$

(b) Solve for the eigenfunction $\Psi(x, t; z_0)$ of the Dirac operators (3.9) and (3.10).

(c) Use the Bäcklund transformations given in (4.11) and (4.21) to pick $z_0$ so that the new solutions $(\tilde{u}, \tilde{v})$ are x-periodic.

(d) Show that the solutions $(\tilde{u}, \tilde{v})$ are homoclinic to the plane wave solutions.

Since the Bäcklund transformations given in (4.11) and (4.21) are expressed in
Riccati functions the spectral value $z_0$ which gives the periodicity in $x$-space can be determined by the Riccati functions. Later, we will show that this spectral point $z_0$ is associated with a “multiple point” of the Floquet multiplier curve of the system (3.9) and it will determine the homoclinic solution. The construction of the homoclinic solutions relies on finding the eigenfunctions of the Dirac operators in (3.9) and (3.10). For the simple plane waves, the eigenfunctions of (3.9) and (3.10) can be solved analytically from a decoupled system of ode’s and then the Riccati functions can be found.

6.1: The homoclinic solution connected to the $x$-independent plane wave solution $u = u_0 e^{2i(u_0^2 + v_0^2)t}$, $v = v_0 e^{2i(u_0^2 + v_0^2)t}$ of the focusing CNLS system.

Let $(u, v)$ be an $x$-independent plane wave solution of the focusing CNLS system. Assume that $\Psi = (\psi_1, \psi_2, \psi_3)^t$ is an eigenfunction associated with the plane wave solution $(u, v)$ and spectral parameter $z_0$. Here, we leave the spectral parameter $z_0$ arbitrary for now, and later we will pick $z_0$ so that the solutions generated by the Bäcklund transformation is $x$-periodic. The eigenfunction $\Psi$ satisfies the following differential equations.

$$
\Psi_x = \begin{pmatrix}
2z_0i & -\bar{u} & -\bar{v} \\
 u & -z_0i & 0 \\
v & 0 & -z_0i
\end{pmatrix} \Psi, 
\Psi_t = \begin{pmatrix}
6z_0^2i - \mu^2i & -3z_0\bar{u} & -3z_0\bar{v} \\
3z_0u & -3z_0^2i + u_0^2i & iuv \\
3z_0v & iuv & -3z_0^2i + v_0^2i
\end{pmatrix} \Psi,
$$

(6.1) (6.2)
where \( \mu^2 = u_0^2 + v_0^2 \)

The characteristic polynomial of (6.1) is

\[
(\lambda + z_0 i)(\lambda^2 - z_0 i \lambda + \mu^2 + 2z_0^2) = 0.
\]

(6.3)

Therefore, the system (6.1) has eigenvalues

\[
\lambda_1 = -z_0 i, \quad \lambda_{2,3} = \frac{z_0 i \pm \xi}{2},
\]

(6.4)

where \( \xi = \sqrt{-9z_0^2 - 4\mu^2} \).

The normalized eigenvectors associated with \( \lambda_i \)'s are

\[
Y_1(t) = \frac{1}{\sqrt{q^2 + q^2}} \begin{pmatrix} 0 \\ \bar{v} \\ -\bar{u} \end{pmatrix},
\]

(6.5)

\[
Y_{2,3} = \frac{1}{\sqrt{4(u^2 + v^2) + (3z_0 \pm \xi)^2}} \begin{pmatrix} 3z_0 i \pm \xi \\ 2u \\ 2v \end{pmatrix}.
\]

Let \( P = (Y_1, Y_2, Y_3) \). Then

\[
PAP^{-1} = D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \text{ a diagonal matrix.}
\]

And, \( \Psi_x = A\Psi \) iff \( \Phi_x = D\Phi \), where \( \Phi = P\Psi = \begin{pmatrix} e^{\lambda_1 x} & 0 & 0 \\ 0 & e^{\lambda_2 x} & 0 \\ 0 & 0 & e^{\lambda_3 x} \end{pmatrix} \).
The Floquet transfer matrix is

$$
\Phi(L) = \begin{pmatrix}
\rho_1 & 0 & 0 \\
0 & \rho_2 & 0 \\
0 & 0 & \rho_3
\end{pmatrix} = \begin{pmatrix}
e^{\lambda_1 L} & 0 & 0 \\
0 & e^{\lambda_2 L} & 0 \\
0 & 0 & e^{\lambda_3 L}
\end{pmatrix}.
$$ (6.6)

From (6.2), one can easily check that $\rho_1 \rho_2 \rho_3 = e^{(\lambda_1 + \lambda_2 + \lambda_3)L} = 1$

We now show that the Bäcklund transformation generates the homoclinic solutions only if the multipliers are degenerate: $\rho_i = \rho_j, i \neq j$. First, we solve the eigenfunction of the systems (6.1) and (6.2), and find the Riccati functions. By the formula of Bäcklund transformations given in (4.11) we can determine the spectral value $z_0$ so that $(\tilde{u}, \tilde{v})$ are x-periodic. We can show that there are discrete values of such $z_0$ and the Floquet multipliers are degenerate.

(a): The eigenfunction $\Psi(x, t; z_0)$

The general eigenfunction solution is of the following form

$$
\Psi(x, t) = X_1(t)e^{\lambda_1 x} + X_2(t)e^{\lambda_2 x} + X_3(t)e^{\lambda_3 x},
$$ (6.7)

where $X_1(t) = c_1(t) \begin{pmatrix} 0 \\ \tilde{v} \\ -\tilde{u} \end{pmatrix}$, $X_2(t) = c_2(t) \begin{pmatrix} 3z_0 i + \xi \\ 2u \\ 2v \end{pmatrix}$, and $X_3(t) = c_3(t) \begin{pmatrix} 3z_0 i - \xi \\ 2u \\ 2v \end{pmatrix}$.

Insert (6.7) into (6.2), then $c_1, c_2,$ and $c_3$ satisfy the following equations:
\[ c_{1,t} = (-3z_0^2 i + 2i \mu^2) c_1 \]
\[ c_{2,t} = (6z_0^2 - \mu^2)i + \frac{6z_0 \mu^2}{3z_0 + i \xi} c_2 \]
\[ c_{3,t} = (6z_0^2 - \mu^2)i + \frac{6z_0 \mu^2}{3z_0 - i \xi} c_3 . \]

Therefore,
\[ c_1(t) = Ae^{(-3z_0^2 i + 2i \mu^2)t} \]
\[ c_2(t) = Be^{(6z_0^2 - \mu^2) i + \frac{6z_0 \mu^2}{3z_0 + i \xi}t} \]
\[ c_3(t) = Ce^{(6z_0^2 - \mu^2) i + \frac{6z_0 \mu^2}{3z_0 - i \xi}t} , \]

where \( A, B, \) and \( C \) are some constants.

We then get the eigenfunction \( \Psi = (\psi_1, \psi_2, \psi_3) \) associated with the \( x \)-independent plane wave solution \((u, v)\):

\[ \psi_1(x, t) = (3z_0i + \xi)c_2(t)e^{\lambda_2x} + (3z_0i - \xi)c_3(t)e^{\lambda_3x} , \]
\[ \psi_2(x, t) = \bar{u}c_1(t)e^{\lambda_1x} + 2u(c_2(t)e^{\lambda_2x} + c_3(t)e^{\lambda_3x}) , \]
\[ \psi_3(x, t) = -\bar{u}c_1(t)e^{\lambda_1x} + 2v(c_2(t)e^{\lambda_2x} + c_3(t)e^{\lambda_3x}) , \]

where \( \lambda_i \) \((i = 1, 2, 3)\) are given in (6.4) and \( c_i(t) \) \((i = 1, 2, 3)\) are given in (6.9).

(b): The spectral parameter \( z_0 \)

We now turn to determine the spectral parameter which generates x-periodic
solutions.

Since \( \tilde{u} = u + \frac{\sin(z_0 - z_0)}{1 + |m|^2 + |n|^2} \), and \( \tilde{v} = v + \frac{\sin(z_0 - z_0)}{1 + |m|^2 + |n|^2} \), the periodicity of \( \tilde{u} \) and \( \tilde{v} \) is completely determined by \( m \) and \( n \). Also, in order to get a nontrivial transformation, we assume that the spectral parameter \( z_0 \) is not a real number. The spectral values associated with the x-periodic solutions can be determined by the following corollary.

**Corollary 1:**

\[
\left( \frac{\tilde{u}}{\tilde{v}} \right) \text{ is periodic in } x \text{ if and only if } c_1(t) = 0 \text{ in (6.9) and } \xi = \frac{2k\pi}{L} i.
\]

From the corollary we have the following results immediately.

1. Since \( \xi = \sqrt{-9z_0^2 - 4\mu^2} \) and from the corollary \( \xi = \frac{2k\pi}{L} i \), we find the spectral value \( z_0 \) must take the form

\[
z_0 = \frac{2i}{3} \sqrt{\mu^2 - k^2\pi^2/L^2}, \text{ when } \xi = \frac{2k\pi}{L} i, \text{ for } k = 0, -1, -2, \ldots \quad (6.11)
\]

\[
z_0 = -\frac{2i}{3} \sqrt{\mu^2 - k^2\pi^2/L^2}, \text{ when } \xi = \frac{2k\pi}{L} i, \text{ for } k = 0, 1, 2, \ldots
\]

2. From (6.6) and corollary, if \((\tilde{u}, \tilde{v})\) is x-periodic, then \( \lambda_2 - \lambda_3 = \xi = \frac{2k\pi}{L} i \).

Thus, the Floquet multipliers \( \rho_2 = e^{\lambda_2 L} = e^{\lambda_3 L} = \rho_3 \). The Floquet multipliers are degenerate.

The Proof of corollary 1 is given in appendix E.
(c): The homoclinic solutions

We now construct the homoclinic solutions.

**Theorem 3:**

Let \((u_v) = (u_0 \ v_0) e^{2i(u_0^2 + v_0^2)t}\) be a plane wave solution of the focusing CNLS system. The new solution \((\tilde{u}_\tilde{v}) = B(u_v)\) generated by the Bäcklund transformation (4.11) with the spectral value \(z_0\) given in (6.11) and eigenfunction given in (6.10) is a homoclinic solution to the focusing CNLS system in the sense that as \(t \rightarrow \pm \infty\), the solution \((\tilde{u}_\tilde{v})\) converges to a pure phase translation of \((u_v)\).

The homoclinic solution with \(u_0 = 1, v_0 = 2\) is shown in figure 1.

proof:

We note that

\[
\frac{c_3(t)}{c_2(t)} = Ce^{\sqrt{\mu^2 - k^2 \pi^2 / L^2} \frac{kr}{L} t},
\]

where we choose \(\xi = 2k\pi / L, k \geq 0\), and \(z_0 = -\frac{2}{\mu} \sqrt{\mu^2 - k^2 \pi^2 / L^2 i}\).

Hence, \(\frac{c_3(t)}{c_2(t)} \rightarrow 0\) as \(t \rightarrow -\infty\), and \(\frac{c_2(t)}{c_3(t)} \rightarrow 0\) as \(t \rightarrow \infty\).

Now, as \(t \rightarrow -\infty\),

\[
\tilde{u} = u + \frac{\alpha i(z_0 - z_0)}{1 + |m|^2 + |n|^2} = u + \frac{\alpha i(z_0 - z_0)}{\frac{1}{m} + \frac{z_0}{n}}
\]

\[
\rightarrow u \left(\frac{\mu^2 - 2k^2 \pi^2}{\mu^2} - i - \frac{2k\pi}{\mu^2} \sqrt{\mu^2 - k^2 \pi^2 / L^2 i}\right)
\]
\[ \equiv (s + ti)u, \text{ where } s = \frac{\mu^2 - \frac{2k^2\pi^2}{L^2}}{\mu^2} \text{ and } t = -\frac{2k\pi}{L\mu^2} \sqrt{\mu^2 - \frac{K^2\pi^2}{L^2}}. \]

The same computation for \( \tilde{v} \) yields, as \( t \to -\infty \),

\[ \tilde{v} = v + \frac{\alpha i(\xi_0 - z_0)}{\frac{1}{m + \bar{m} + \frac{1}{a\bar{a}}}} \]

\[ \to v + (4\sqrt{\mu^2 - \frac{k^2\pi^2}{L^2}})(2\sqrt{\mu^2 - \frac{k^2\pi^2}{L^2}} - \frac{2k\pi}{L}) \frac{v}{4\mu^2} \]

\[ = (s + ti)v. \]

As \( t \to -\infty \),

\[ \tilde{u} = u + \frac{\alpha i(\xi_0 - z_0)}{\frac{1}{m + \bar{m} + \frac{1}{a\bar{a}}}} \]

\[ \to u + (4\sqrt{\mu^2 - \frac{k^2\pi^2}{L^2}})(2\sqrt{\mu^2 - \frac{k^2\pi^2}{L^2}} + \frac{2k\pi}{L}) \frac{u}{4\mu^2} \]

\[ = (s - ti)u. \]

The same computation for \( \tilde{v} \) yields, as \( t \to \infty \),

\[ \tilde{v} \to (s - ti)v. \]

And, we note that
\[ s^2 + t^2 = 1 - \frac{4k^2 \pi^2}{\mu^2 L^2} + \frac{4k^4 \pi^4}{\mu^4 L^4} + \frac{4k^2 \pi^2}{\mu^2 L^2} (\mu^2 - \frac{k^2 \pi^2}{L^2}) = 1 \]

Therefore, the solution \( \left( \hat{\vec{u}}, \hat{\vec{v}} \right) \) converges to the plane wave solution \( \left( \frac{\vec{u}}{\vec{v}} \right) \) with different orientations. This completes the proof.

We note that the condition (6.11), \( z_0 = \pm \frac{2}{\sqrt{\mu^2 - \frac{k^2 \pi^2}{L^2}}} i \) with \( \mu^2 - \frac{k^2 \pi^2}{L^2} > 0 \), determines the spectral parameter so that the Bäcklund transformation will generate the homoclinic orbit. The number of such parameter values is the greatest integer \( k \) with \( k \leq \sqrt{u_0^2 + v_0^2 \frac{L}{\pi}} \). This is identical with the calculation in section 2 of the dimension of the unstable manifold. Each single transformation yields a one-parameter orbit on the unstable manifold.

6.2: The homoclinic solutions connected to the plane wave solution \( u = u_0 e^{ix + ibt}, v = v_0 e^{icx + idt}, a = \frac{2\sigma_2 \pi}{L}, c = \frac{2\sigma_0 \pi}{L} \).

We consider the Dirac equations \( \Psi_x = A\Psi, \Psi_t = B\Psi \) in (3.9) and (3.10) with \( u = u_0 e^{iax + ibt}, v = v_0 e^{icx + idt} \), where \( a = \frac{2\sigma_2 \pi}{L} \) and \( c = \frac{2\sigma_0 \pi}{L} \).

Consider the transformation \( \phi_1 = \psi_1, \phi_2 = e^{-ix} \psi_2 \), and \( \phi_3 = e^{-icx} \psi_3 \).

We have the decoupled systems for \( (\phi_1, \phi_2, \phi_3) \):

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}_x =
\begin{pmatrix}
2z_0 i & \sigma u_0 e^{-ibt} & \sigma v_0 e^{-idt} \\
u_0 e^{ibt} & -(z_0 + a)i & 0 \\
v_0 e^{idt} & 0 & -(z_0 + c)i
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix} = \tilde{A}\Phi,
\]

(6.12)
\[
\begin{pmatrix}
  \phi_1 \\
  \phi_2 \\
  \phi_3
\end{pmatrix}_t = \begin{pmatrix}
  6z_0i + \sigma \mu^2i & \sigma(3z_0 - a)u_0e^{-ibt} & \sigma(3z_0 - c)v_0e^{-idt} \\
  (3z_0 - a)u_0e^{ibt} & -\sigma iu_0^2 - 3z_0^2i & -\sigma iu_0v_0e^{id(\beta - \delta)t} \\
  (3z_0 - c)v_0e^{idt} & -\sigma iv_0u_0e^{-i(\beta - \delta)t} & -\sigma iv_0^2 - 3z_0^2i
\end{pmatrix} \begin{pmatrix}
  \phi_1 \\
  \phi_2 \\
  \phi_3
\end{pmatrix} \equiv \tilde{B}\Phi.
\]

(6.13)

The characteristic polynomial of $\tilde{A}$ is the following cubic algebraic equation,

\[
\lambda^3 + (a + c)i\lambda^2 + [3z_0^2 + (a + c)z_0 - ac - \sigma \mu^2]\lambda + [2z_0^3 + 2(a + c)z_0^2 + (2ac - \sigma \mu^2)z_0 - \sigma(cu_0^2 + av_0^2)]i = 0.
\]

(6.14)

Assume that $\lambda_1, \lambda_2$ and $\lambda_3$ are three distinct eigenvalues. The associated eigenvectors are given by

\[
v_k = \begin{pmatrix}
  (\lambda_k + (z_0 + a)i)(\lambda_k + (z_0 + c)i) \\
  u_0e^{ibt}(\lambda_k + (z_0 + c)i) \\
  v_0e^{idt}(\lambda_k + (z_0 + a)i)
\end{pmatrix}, k = 1, 2, 3.
\]

(6.14)

The Floquet transfer matrix is given by

\[
\Phi(L) = \begin{pmatrix}
  \rho_1 & 0 & 0 \\
  0 & \rho_2 & 0 \\
  0 & 0 & \rho_3
\end{pmatrix} = \begin{pmatrix}
  e^{\lambda_1L} & 0 & 0 \\
  0 & e^{\lambda_2L} & 0 \\
  0 & 0 & e^{\lambda_3L}
\end{pmatrix}.
\]

(6.15)

The general solution of system (6.12) is given by

\[
\Phi(x, t) = \sum_{k=1}^{3} c_k(t)v_ke^{\lambda_kx}
\]

(6.16)
Insert into (6.13). Comparing the coefficients of $e^{\lambda_k x}$, we have

$$\frac{c_{kt}}{c_k} = \Lambda_k = \left(6z_0^2 i + \sigma \mu^2 i + \frac{\sigma(3z_0 - a)\psi_0^2}{\lambda_k + (z_0 + a)i} + \frac{\sigma(3z_0 - c)\psi_0^2}{\lambda_k + (z_0 + c)i}\right), \quad k = 1, 2, 3. \tag{6.17}$$

Therefore, $c_k(t) = A_k e^{\Lambda_k t}$, where $A_k$ is a constant.

The Riccati functions $m = \frac{\psi_2}{\psi_1}$ and $n = \frac{\psi_3}{\psi_1}$ are then given by

$$m = \frac{u_0[r_1 e^{ib t + ia x} + r_2 e^{(\lambda_2 - \lambda_1 + ib)t + (\lambda_2 - \lambda_1 + ia)x} + r_3 e^{(\lambda_3 - \lambda_1 + ib)t + (\lambda_3 - \lambda_1 + ia)x}]}{s_1 r_1 + s_2 r_2 e^{(\lambda_2 - \lambda_1 + ib)t + (\lambda_2 - \lambda_1)x} + s_3 r_3 e^{(\lambda_3 - \lambda_1 + ib)t + (\lambda_3 - \lambda_1)x}}. \tag{6.18}$$

$$n = \frac{v_0[s_1 e^{id t + ic x} + s_2 e^{(\lambda_2 - \lambda_1 + id)t + (\lambda_2 - \lambda_1 + ic)x} + s_3 e^{(\lambda_3 - \lambda_1 + id)t + (\lambda_3 - \lambda_1 + ic)x}]}{s_1 r_1 + s_2 r_2 e^{(\lambda_2 - \lambda_1 + id)t + (\lambda_2 - \lambda_1)x} + s_3 r_3 e^{(\lambda_3 - \lambda_1 + id)t + (\lambda_3 - \lambda_1)x}}. \tag{6.19}$$

where, $r_1 = (\lambda_1 + (z_0 + c)i), r_2 = (\lambda_2 + (z_0 + c)i), r_3 = (\lambda_3 + (z_0 + c)i)$, and $s_1 = (\lambda_1 + (z_0 + a)i), s_2 = (\lambda_2 + (z_0 + a)i), s_3 = (\lambda_3 + (z_0 + a)i)$.

If $m$ and $n$ are $x$-periodic, then the solutions generated by Bäcklund transformation given in (4.11) or (4.21) are $x$-periodic as well. It can be seen that two possibilities arise:

either (1): if $\lambda_2 - \lambda_1 = \frac{2k_1 \pi}{L} i$ and $\lambda_3 - \lambda_1 = \frac{2k_2 \pi}{L} i$, (i.e., the Floquet multipliers $\rho_1 = \rho_2 = \rho_3$), then $m$ and $n$ are $x$-periodic with period $L$.

or (2): Without loss of generality, if $A_3 = 0$ and $\lambda_2 - \lambda_1 = \frac{2k_3 \pi}{L} i$, (i.e., the
Floquet multipliers \( \rho_1 = \rho_2 \), then \( m \) and \( n \) are \( x \)-periodic with period \( L \).

From the above two cases and (6.14), we can determine the spectral parameter \( z_0 \) which gives the \( x \)-periodicity of the new solution generated by the Bäcklund transformation. Comparing the coefficients of

\[
(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 + (a + c)i\lambda^2 + (3z_0^2 + (a + c)z_0 - ac - \sigma \mu^2)\lambda + (2z_0^3 + 2(a + c)z_0^2 + (2ac - \sigma \mu^2)z_0 - \sigma cu_0^2 - \sigma av_0^2)i,
\]

in case (1) : \( \lambda_1 = -\frac{i}{3}[(a+c)+2(k_1+k_2)\frac{c}{L}]i, \lambda_2 = \lambda_1 + 2k_1\frac{c}{L}i, \) and \( \lambda_3 = \lambda_1 + 2k_2\frac{c}{L}i, \)

\[
3z_0^2 + (a + c)z_0 - (ac + \sigma \mu^2) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3,
\]

\[
(2z_0^3 + 2(a + c)z_0^2 + (2ac - \sigma \mu^2)z_0 - \sigma cu_0^2 - \sigma av_0^2)i = -\lambda_1 \lambda_2 \lambda_3,
\]

in case (2) : \( \lambda_2 = \lambda_1 + \frac{2k\pi}{L}i, \lambda_3 = -2\lambda_1 - (a + c)i - \frac{2k\pi}{L}i, \)

\[
3z_0^2 + (a + c)z_0 - ac - \sigma \mu^2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3,
\]

\[
(2z_0^3 + 2(a + c)z_0^2 + (2ac - \sigma \mu^2)z_0 - \sigma cu_0^2 - \sigma av_0^2)i = -\lambda_1 \lambda_2 \lambda_3.
\]

For case (1), the eigenvalues \( \lambda_k \)'s are in \( iR \). Therefore, \( \Lambda_k \)'s are in \( iR \) and the solution generated by the Bäcklund transformation is simply a plane wave solution, related to the original solution by a phase shift.
For case (2), if the constants $u_0, v_0, a, b, c, d$ are chosen so that (2.10) has non-real solutions for $\omega_n$, then the solutions generated by the Bäcklund transformation are homoclinic solutions. The homoclinic solutions with $a = 3, c = 2, u_0 = 1, v_0 = 1.5$ are shown in figure 2 and figure 3 for the focusing and defocusing CNLS system, respectively. It can be easily checked that:

As $t \rightarrow \infty$, $\hat{u} \rightarrow [u_0 + 3i(\bar{z}_0 - z_0)/R_1]e^{iax + ibt}$, where $R_1 = \frac{1}{u_0} [\lambda_2 + (z_0 - a)i + \frac{u_0^2}{\lambda_2 - (z_0 + a)i} + v_0^2 \frac{\lambda_2 + (z_0 + a)i}{|\lambda_2 + (z_0 + a)i|^2}]$.

As $t \rightarrow -\infty$, $\hat{u} \rightarrow [u_0 + 3i(\bar{z}_0 - z_0)/R_2]e^{iax + ibt}$, where $R_2 = \frac{1}{u_0} [\lambda_1 + (z_0 - a)i + \frac{u_0^2}{\lambda_1 - (z_0 + a)i} + v_0^2 \frac{\lambda_1 + (z_0 + a)i}{|\lambda_1 + (z_0 + a)i|^2}]$, and

$|u_0 + 3i(\bar{z}_0 - z_0)/R_1| = |u_0 + 3i(\bar{z}_0 - z_0)/R_2| = u_0$. 
CHAPTER VII

Conclusions and Discussions

In this paper we derive a Bäcklund transformation of the CNLS system and use it to generate the homoclinic solutions arising from unstable plane wave solutions. The Bäcklund transformations in either the focusing or defocusing CNLS systems are obtained by identifying the automorphisms of the Lie algebras generated by the potentials. In the representation of the Riccati functions the Bäcklund transformation can also be obtained from the transformation $(1, m, n) \rightarrow (1, \frac{m}{|m|^2 + |n|^2}, \frac{n}{|m|^2 + |n|^2})$. This property is also widely observed in $2 \times 2$ AKNS systems, like the KdV equation, sine-Gordon equation, and scalar NLS equations. The geometric or algebraic interpretations of these properties are still unknown in general. However, the relations between the automorphism of a Lie algebra and the automorphism of its associated Weyl group may give a possible Lie-algebraic interpretation of the Bäcklund transformation. The conjecture[24] is that the Bäcklund transformation of $n \times n$ AKNS systems can be obtained from the action of the automorphism $\sigma$ of the Weyl group on $P_{n-1}(C)$, where $\sigma$ is associated with the automorphism of the Lie algebra generated by the potential in the AKNS system. Also, we note that to obtain an iterated Bäcklund transformation one must find the gauge transformation of the system. As we mentioned in chapter 5, without knowing the transformation $Q \rightarrow \tilde{Q}$ a gauge
transformation of the $3 \times 3$ AKNS system is very hard to find. For the CNLS system, the gauge matrix associated with the Bäcklund transformations (4.11) and (4.21) are in the Lie algebras generated by the potentials. This gauge transformation of the AKNS system not only transfers the potentials of the system but also control the eigenfunctions of the transformed potentials. This allows us to perform the iterated Bäcklund transformation and to produce the exact solutions for the full homoclinic manifold. In our work, we construct the solutions homoclinic to the unstable plane waves. The higher dimensional orbits of homoclinic solutions can be obtained by performing iterated Bäcklund transformations. This will be pursued in our further study.

Also, we note that existence of such homoclinic solutions impacts on the dynamics of the CNLS system with small perturbations. An important class of perturbed systems come from birefringent optical fibers in which the physical system retains the Hamiltonian structure while losing integrability[3]. This system has attracted significant research due to its application in tele-communication. Our further study will also be on such systems.
Appendix A

The Group-Valued Connection Form

Let $X$ be an $n$-manifold and $T(X)$ be the tangent space of $X$. Let $C^\infty(X)$ be the vector space of $C^\infty$-functions on $X$. Let $\mathcal{G}$ be a real Lie algebra.

An $p$-th degree $\mathcal{G}$-valued differential form is a mapping

$$\omega : T(X) \times T(X) \times \ldots \times T(X) \rightarrow \mathcal{F}(\mathcal{X}, \mathcal{G})$$

which is $C^\infty$-multilinear and skew-symmetric, where $\mathcal{F}(\mathcal{X}, \mathcal{G})$ denotes the space of mappings $X \rightarrow \mathcal{G}$.

The exterior product of $\mathcal{G}$-valued differential form is not defined as the usual $C^\infty$-valued form, instead it is defined by the Lie algebra structure on $\mathcal{G}$.

Let $\omega_1$ be a $p$-form and $\omega_2$ be a $q$-form. The exterior product is denoted by $(\omega_1, \omega_2) \rightarrow [\omega_1, \omega_2]$, where $[\omega_1, \omega_2]$ is a $p + q$-form with
\[ [\omega_1, \omega_2](v_1, ..., v_{p+q}) = \Sigma_{\sigma \in S_{p+q}}(\text{sign}\sigma) [\omega_1(v_{\sigma(1)}, ..., v_{\sigma(p)}), \omega_2(v_{\sigma(p+1)}, ..., v_{\sigma(p+q)})] \]

the sum on the right hand side is over all permutations of \((1, 2, ..., p + q)\), and \([ , ]\) denotes the Lie bracket on \(\mathcal{G}\).

A \(\mathcal{G}\)-valued differential form can be described in terms of scalar-valued differential form. Suppose \(\text{dim } \mathcal{G} = m\) and \((g_a), a = 1, ..., m\) is a basis for \(\mathcal{G}\). Let \((C_{ab}^c)\) be the structure constants of \(\mathcal{G}\) with respect to the basis \((g_a)\). A \(\mathcal{G}\)-valued form on \(X\) can be written as \(\omega = \omega^a g_a\), where \((\omega^a), a = 1, ..., m\) are the scalar-valued forms.

Let \(\omega\) be a Lie algebra valued one-form on the manifold \(X\). The two-form

\[ \Omega = d\omega + \frac{1}{2}[\omega, \omega] \]

is called the curvature form of \(\omega\).

Let \(G\) be a Lie group whose Lie algebra is \(\mathcal{G}\). Let \((g_a)\) be a basis of \(\mathcal{G}\) and \((\theta^a)\) is the dual basis for the left-invariant one-form, i.e.

\[ \theta^a g_b = \delta^a_b. \]
The $G$-valued one-form defined on $G$ given by

$$\theta = \theta^a g_b$$

is called the Maurer-Cartan form.
Appendix B

The Structure of the Lie Algebra Generated by the Potential

We note that the Lie algebras generated by the potential in focusing or defocusing CNLS system are subalgebras of $gl(3)$. These subalgebras can be determined by finding the generators and structure constants.

Property: $S$ is a subalgebra of the Lie algebra $\mathcal{G}$ if $S$ is closed under commutation, i.e. $[s_1, s_2] \in S$ whenever $s_1$ and $s_2 \in S$.

The real Lie algebra $g$ generated by the focusing potential $Q = \begin{pmatrix} 0 & -\bar{u} & -\bar{v} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix}$ is apparently generated by

\[
e_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}.
\]

They generate a basis $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$ with
\[ e_5 = \begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \]

\[ e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad e_8 = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}. \]

The structure constants are given as follows:

\[
\begin{array}{cccccccc}
e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
e_1 & 0 & 2e_5 & e_6 & e_7 & -2e_2 & -e_3 & -e_4 & -e_2 \\
e_2 & -2e_5 & 0 & -e_7 & e_6 & 2e_1 & -e_4 & e_3 & e_1 \\
e_3 & -e_6 & e_7 & 0 & 2e_8 & -e_5 & e_1 & -e_2 & -2e_4 \\
e_4 & -e_7 & -e_6 & -2e_8 & 0 & e_3 & e_2 & e_1 & 2e_3 \\
e_5 & 2e_5 & -2e_1 & e_5 & -e_3 & 0 & -e_7 & e_6 & 0 \\
e_6 & e_3 & e_4 & -e_1 & -e_2 & e_7 & 0 & e_8 & -e_5 & -e_7 \\
e_7 & e_4 & -e_3 & e_2 & -e_1 & -e_6 & e_5 & -e_8 & 0 & e_6 \\
e_8 & e_2 & -e_1 & 2e_4 & -2e_3 & 0 & e_7 & -e_6 & 0 & \end{array}
\]

The real Lie algebra \( g \) generated by the defocusing potential \( Q = \begin{pmatrix} 0 & \bar{u} & \bar{v} \\ u & 0 & 0 \\ v & 0 & 0 \end{pmatrix} \)

is apparently generated by

\[ e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]
\[ e_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ e_4 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \]

They generate a basis \( e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \) with

\[ e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \ e_6 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \ e_8 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}. \]

The structure constants are given as follows:

\[
\begin{array}{cccccccc}
  e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
  e_1 & 0 & e_5 & 2e_6 & -e_7 & e_2 & 2e_3 & -e_4 & e_3 \\
  e_2 & -e_5 & 0 & -e_7 & 2e_8 & -e_1 & -e_3 & 2e_4 & 0 \\
  e_3 & -2e_6 & e_7 & 0 & e_5 & e_4 & -2e_1 & e_2 & -e_1 \\
  e_4 & e_7 & -2e_8 & -e_5 & 0 & -e_3 & -e_2 & e_1 & -2e_2 \\
  e_5 & -e_2 & e_1 & -e_4 & e_3 & 0 & e_7 & 2e_8 & -2e_6 & -e_7 \\
  e_6 & -2e_3 & -e_4 & 2e_1 & e_2 & -e_7 & 0 & e_5 & 0 \\
  e_7 & e_4 & e_3 & -e_2 & -e_1 & 2e_6 & -2e_8 & -e_5 & 0 & e_5 \\
  e_8 & -e_3 & -2e_4 & e_1 & 2e_2 & e_7 & 0 & -e_5 & 0 & 0
\end{array}
\]
Appendix C

Bäcklund Transformation of the Defocusing CNLS System

We consider the automorphism $\sigma$ on the Lie algebra generated by the defocusing potential. The automorphism acts on $D_x(z_0, Q)$ and $D_t(z_0, Q)$ leaving $Q$ invariant and changing $z_0$ to $\bar{z}_0$. Therefore, the action keeps the flat connection determined by $D_x(z_0, Q)$ and $D_t(z_0, Q)$.

Stipulating the equation $D_x(z_0, \tilde{Q})^\sigma g\Phi = D_x(z_0, Q)\Phi$ with $g$ a scalar function and $\Phi$ a solution vector, we have

\begin{align*}
(C1) \quad \Phi_x &= ( \begin{array}{cc} a z_0 i & \bar{u} & \bar{v} \\ u & b z_0 i & 0 \\ v & 0 & b \bar{z}_0 i \end{array} ) \Phi, \text{ and} \\
&C2 \quad \tilde{\Phi}_x = ( \begin{array}{cc} a \bar{z}_0 i & \bar{\bar{u}} & \bar{\bar{v}} \\ \bar{\bar{u}} & b \bar{z}_0 i & 0 \\ \bar{\bar{v}} & 0 & b \bar{z}_0 i \end{array} ) \tilde{\Phi}, \text{ where } \tilde{\Phi} = g\Phi.
\end{align*}

Assume that $\Phi = (\begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \end{array})$ and $\tilde{\Phi} = (\begin{array}{c} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{array})$, we have the following system of ode's

\begin{align*}
\phi_{1x} &= (a z_0 i) \phi_1 + \bar{u} \phi_2 + \bar{v} \phi_3 
\end{align*}
\[ \phi_{2x} = u \phi_1 + (bz_0i) \phi_2 \]
\[ \phi_{3x} = v \phi_1 + (bz_0i) \phi_3 \]

and

\[ \tilde{\phi}_{1x} = (a \tilde{z}_0i) \tilde{\phi}_1 + \tilde{u} \tilde{\phi}_2 + \tilde{v} \tilde{\phi}_3 \]
\[ \tilde{\phi}_{2x} = u \tilde{\phi}_1 + (bz_0i) \tilde{\phi}_2 \]
\[ \tilde{\phi}_{3x} = v \tilde{\phi}_1 + (bz_0i) \tilde{\phi}_3. \]

Let \( m = \frac{\phi}{\phi_1} = \frac{\tilde{\phi}}{\tilde{\phi}_1} \) and \( n = \frac{\phi}{\phi_1} = \frac{\tilde{\phi}}{\tilde{\phi}_1} \)

\[ m_x = \frac{(u \phi_1 + bz_0i \phi_2) \phi_1 - (u \phi_2 + v \phi_3 + sz_0i \phi_1) \phi_2}{\phi_1^2} \]
\[ = u - \alpha z_0 i m - \tilde{u} m^2 - \tilde{v} mn, \text{ where } \alpha = a - b. \]

\[ n_x = \frac{(v \phi_1 + bz_0i \phi_3) \phi_1 - (v \phi_2 + u \phi_3 + sz_0i \phi_1) \phi_3}{\phi_1^2} \]
\[ = v - \alpha z_0 i n - \tilde{v} n^2 - \tilde{u} mn. \]

Also,

\[ m_x = \frac{(u \phi_1 + bz_0i \phi_2) \phi_1 - (u \phi_2 + v \phi_3 + sz_0i \phi_1) \phi_2}{\phi_1^2} \]
\[ = \tilde{u} - \alpha \tilde{z}_0 i m - \tilde{u} m^2 - \tilde{v} mn \]

\[ n_x = \frac{(v \phi_1 + bz_0i \phi_3) \phi_1 - (v \phi_2 + u \phi_3 + sz_0i \phi_1) \phi_3}{\phi_1^2} \]
\[ = \dot{v} - \alpha z_0 \dot{in} - \ddot{\bar{m}} n^2 - \ddot{\bar{u}} m n. \]

We get the Riccati equations:

\begin{align*}
(C3.1) \quad & m_x = u - \alpha z_0 \dot{m} - \ddot{\bar{m}} m^2 - \dot{\bar{v}} m n \\
(C3.2) \quad & \bar{m}_x = \bar{u} - \alpha \bar{z}_0 \dot{\bar{m}} - \ddot{\bar{m}} \bar{m}^2 - \dot{\bar{v}} \bar{m} n \\
(C3.3) \quad & n_x = v - \alpha z_0 \dot{n} - \ddot{\bar{v}} n^2 - \dot{\bar{u}} m n \\
(C3.4) \quad & \bar{n}_x = \bar{v} - \alpha \bar{z}_0 \dot{\bar{n}} - \ddot{\bar{v}} \bar{n}^2 - \dot{\bar{u}} m n.
\end{align*}

From the Riccati equations,

\[ u = m_x + \alpha z_0 \dot{m} + m^2 (\ddot{\bar{m}} m_x - \alpha \bar{z}_0 \dot{\bar{m}} + u m^2 + v m \bar{n}) + m n (\ddot{\bar{m}} m_x - \alpha \bar{z}_0 \dot{\bar{m}} + v n^2 + u m \bar{n}), \]

and

\[ v = n_x + \alpha z_0 \dot{n} + n^2 (\ddot{\bar{v}} n_x - \alpha \bar{z}_0 \dot{\bar{n}} + v n^2 + u m \bar{n}) + m n (\ddot{\bar{v}} n_x - \alpha \bar{z}_0 \dot{\bar{n}} + m n \bar{n} + v m \bar{n}). \]

This implies

\[ u (1 - |m|^4 - |m|^2 |n|^2) - v (m \bar{n} (|m|^2 + |n|^2)) \]

\[ = m_x + \alpha z_0 \dot{m} + m^2 \ddot{m}_x - \alpha \bar{z}_0 \dot{\bar{m}} m^2 + m n \ddot{m}_x - \alpha \bar{z}_0 \dot{\bar{m}} |n|^2, \text{ and} \]

\[ -u (m \ddot{\bar{m}} (|m|^2 + |n|^2)) + v (1 - |n|^4 - |m|^2 |n|^2) \]

\[ = n_x + \alpha z_0 \dot{n} + n^2 \ddot{n}_x - \alpha \bar{z}_0 \dot{\bar{n}} n^2 + m n \ddot{n}_x - \alpha \bar{z}_0 \dot{\bar{n}} |m|^2. \]

Eliminate \( v \):
\[(C4.1)\quad u = \frac{(|m|^2+|n|^2)(m \bar{n} m_x - |n|^2 m_x - \alpha z_0 \bar{m} + \alpha z_0 i m + m^2 \bar{n} m_x + m m \bar{n} x)}{1-((|m|^2+|n|^2)^2}.
\]

Eliminate \(u\):

\[(C4.2)\quad v = \frac{(|m|^2+|n|^2)(m m m_x - |m|^2 n_x - \alpha z_0 \bar{m} + n_x + \alpha z_0 i m + n^2 n_x + m m m_x)}{1-((|m|^2+|n|^2)^2}.
\]

Similarly, we have

\[\hat{u}(1 - |m|^4 - |m|^2|n|^2) - \hat{v}(m \bar{n}(|m|^2 + |n|^2))
= m_x + \alpha z_0 i m + m^2 \bar{n} m_x - \alpha z_0 i m |m|^2 + m m \bar{n} m_x - \alpha z_0 i m |n|^2,\]
and

\[-\hat{u}(m \bar{n}(|m|^2 + |n|^2)) + \hat{v}(1 - |n|^4 - |m|^2|n|^2)
= n_x + \alpha z_0 i n + n^2 \bar{n}_x - \alpha z_0 i n |n|^2 + m m \bar{n}_x - \alpha z_0 i n |m|^2.
\]

Eliminate \(\hat{v}\):

\[(C4.3)\quad \tilde{u} = \frac{(|m|^2+|n|^2)(m \bar{n} n_x - |n|^2 m_x - \alpha z_0 \bar{m} + \alpha z_0 i m + m^2 \bar{n} m_x + m m \bar{n} x)}{1-((|m|^2+|n|^2)^2}.
\]

Eliminate \(\tilde{u}\):

\[(C4.4)\quad \tilde{v} = \frac{(|m|^2+|n|^2)(m m m_x - |m|^2 n_x - \alpha z_0 \bar{m} + n_x + \alpha z_0 i m + n^2 n_x + m m m_x)}{1-((|m|^2+|n|^2)^2}.
\]
Therefore, \( \hat{u} - u = \frac{\alpha \text{in}(z_0 - z_0)}{1 - |m|^2 - |n|^2} \), and \( \hat{v} - v = \frac{\alpha \text{in}(z_0 - z_0)}{1 - |m|^2 - |n|^2} \).

Finally, we get the Bäcklund transformation of the defocusing CNLS system

\[(C5.1) \quad \hat{u} = u + \frac{\alpha \text{in}(z_0 - z_0)}{1 - |m|^2 - |n|^2} \]

\[(C5.2) \quad \hat{v} = v + \frac{\alpha \text{in}(z_0 - z_0)}{1 - |m|^2 - |n|^2} \].
Appendix D

Gauge Transformation Associated with the Bäcklund Transformation of the CNLS System

Let $\tilde{Q} = Q + R$ be the Bäcklund transformation of the defocusing CNLS system given in (4.9), where

$$R = \begin{pmatrix} 0 & \tilde{s} & \tilde{t} \\ s & 0 & 0 \\ t & 0 & 0 \end{pmatrix}, \text{ with } s = \frac{3i(z_0 - z_0)m}{1-|m|^2-|n|^2} \text{ and } t = \frac{3i(z_0 - z_0)n}{1-|m|^2-|n|^2}.$$

Seek a linear gauge transformation of the form $G = G_0 + z_1 I$, where $G_0$ is a matrix function in $x$ and $t$. From (5.3), $G_0$ satisfies the following equations

$$G_{0,x} = [Q, G_0] + RG_0,$$
$$i[P, G_0] + R = 0.$$

From the above equations one can solve for the matrix function $G_0$.

Let $G_0 = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$ be a matrix in the Lie algebra generated by the potential.

From $R = -i[P, G_0]$, we have

75
\[
\begin{pmatrix}
0 & \bar{s} & i \\
\bar{s} & 0 & 0 \\
i & 0 & 0
\end{pmatrix}
= -3i
\begin{pmatrix}
0 & g_{12} & g_{13} \\
g_{21} & 0 & 0 \\
g_{31} & 0 & 0
\end{pmatrix}.
\]

Therefore,

\[g_{12} = \frac{i}{3}\bar{s}, \quad g_{13} = \frac{i}{3}i,
\]

\[g_{21} = -\frac{i}{3}s, \quad g_{31} = -\frac{i}{3}t,
\]

and \(G_0\) takes the form

\[
G_0 = \begin{pmatrix}
g_{11} & \frac{i}{3}\bar{s} & \frac{i}{3}i \\
-\frac{i}{3}s & g_{22} & g_{23} \\
-\frac{i}{3}i & g_{32} & g_{33}
\end{pmatrix}, \text{ with } g_{ii} \in iR \text{ and } g_{11} + g_{22} + g_{33} = 0.
\]

From \(G_{0,x} = [Q, G_0] + RG_0\), we have

\[
\begin{pmatrix}
g_{11x} & \frac{i}{3}s_x & \frac{i}{3}t_x \\
-\frac{i}{3}s_x & g_{22x} & g_{23x} \\
-\frac{i}{3}t_x & g_{32x} & g_{33x}
\end{pmatrix}
= \begin{pmatrix}
\frac{-2i}{3}[Re(u\bar{s}) + Re(v\bar{t})] & \bar{u}(-g_{11} + g_{22}) + \bar{v}g_{33} & \bar{u}g_{23} + \bar{v}(-g_{11} + g_{33}) \\
u(g_{11} - g_{22}) - vg_{23} & \frac{2i}{3}Re(u\bar{s}) & \frac{i}{3}(\bar{u}t + \bar{v}s) \\
-ug_{32} + v(g_{11} - g_{33}) & \frac{1}{3}(\bar{u}t + \bar{v}s) & \frac{2i}{3}Re(v\bar{t})
\end{pmatrix}
\]

\[+
\begin{pmatrix}
\frac{-i}{3}(|s|^2 + |t|^2) & \bar{s}g_{22} + \bar{t}g_{32} & \bar{s}g_{23} + \bar{t}g_{33} \\
s g_{11} & \frac{i}{3}|s|^2 & \frac{i}{3}s \bar{t} \\
t g_{11} & \frac{i}{3}s \bar{t} & \frac{i}{3}|t|^2
\end{pmatrix}.
\]

Comparing the entries of the matrices, we have

\[-\frac{i}{3}s_x = u(g_{11} - g_{22}) - vg_{23} + sg_{11}——(1)\]
\[
\begin{align*}
\bar{s}_x &= \bar{u}(-g_{11} + g_{22}) + \bar{v}g_{32} + \bar{s}g_{22} + i\bar{t}g_{32} \quad (2) \\
-\frac{i}{3}t_x &= -ug_{32} + v(g_{11} - g_{33}) + tg_{11} \quad (3) \\
\frac{i}{3}t_x &= \bar{u}g_{23} + \bar{v}(-g_{11} + g_{33}) + \bar{s}g_{23} + \bar{t}g_{33} \quad (4) \\
g_{32x} &= \frac{i}{3}(\bar{u}t + v\bar{s} + \bar{s}t) \quad (5) \\
g_{22x} &= \frac{2i}{3}Re(\bar{u}s) + \frac{i}{3}|s|^2 \quad (6) \\
g_{33x} &= \frac{2i}{3}Re(v\bar{t}) + \frac{i}{3}|t|^2 \quad (7).
\end{align*}
\]

From (1) and (2), \( g_{32} = \frac{2}{i}g_{33} \).

From (3) and (4), \( g_{32} = \frac{i}{x}g_{22} \).

Therefore, \( g_{22} = \frac{|s|^2}{|t|^2}g_{33} \).

We summarize the results in the following property.

**Property:**

Let \( Q = Q + R \) be the Bäcklund transformation given in (4.9). The linear gauge transformation associated with the Bäcklund transformation is given by \( G(x, t; z) = G_0(x, t) + zI \) with

\[
G_0 = -\frac{i}{3} \begin{pmatrix} -(1 + \frac{|s|^2}{|t|^2})g & -\bar{s} & -\bar{t} \\ s & \frac{|s|^2}{|t|^2}g & \bar{s}g \\ t & (\bar{s})g & g \end{pmatrix},
\]

where \( g = \frac{i}{3} \int_0^x 2Re(v\bar{t}) + |t|^2d\bar{x} \).
\[ s = \frac{3i(z_0 - z_0)m}{1 - |m|^2 - |n|^2}, \quad \text{and} \quad t = \frac{3i(z_0 - z_0)n}{1 - |m|^2 - |n|^2}. \]

The iterated scheme of the Bäcklund transformation can be described as follows: starting from a given potential \( Q_0 \) (\((0,0)\) or plane waves), one solves the wave function \( \Psi_0 \) of the potential and applies the Bäcklund transformation to obtain the new potential \( Q_1 \) and the gauge matrix \( G_0 \). The wave function associated with the potential \( Q_1 \) is then given by \( \Psi_1 = G_0 \Psi_0 \) and one can obtain the next potential \( Q_2 \), and so on. This iterated scheme requires one to solve the wave function from the system of ode only once and the rest of the wave functions can be obtained from the gauge matrices.
Appendix E

Proof of Corollary 1

We want to show that the Bäcklund transformation generates the homoclinic solutions only if the Floquet multipliers are degenerate: $\rho_i = \rho_j, i \neq j$. First, we solve the eigenfunction of the systems (6.1) and (6.2), and find the Riccati functions. By the formula of Bäcklund transformations given in (4.8) we can determine the spectral value $z_0$ so that $(\tilde{u}, \tilde{v})$ are x-periodic. We can show that there are discrete values of such $z_0$ and the Floquet multipliers are degenerate. First, we note that the eigenfunction $\Psi$ is given as follows:

$$
\psi_1(x, t) = (3z_0i + \xi)c_2(t)e^{\lambda_2x} + (3z_0i - \xi)c_3(t)e^{\lambda_3x},
$$

$$
\psi_2(x, t) = \tilde{v}c_1(t)e^{\lambda_1x} + 2u(c_2(t)e^{\lambda_2x} + c_3(t)e^{\lambda_3x}),
$$

$$
\psi_3(x, t) = -\tilde{u}c_1(t)e^{\lambda_1x} + 2v(c_2(t)e^{\lambda_2x} + c_3(t)e^{\lambda_3x}),
$$

where $\lambda_i \ (i = 1, 2, 3)$ are given in (6.4) and $c_i(t) \ (i = 1, 2, 3)$ are given in (6.9).

Since $\tilde{u} = u + \frac{\alpha \sin(z_0 - \gamma)}{1 + |x|^2 + |\eta|^2}$, and $\tilde{v} = v + \frac{\alpha \sin(z_0 - \gamma)}{1 + |x|^2 + |\eta|^2}$, the periodicity of $\tilde{u}$ and $\tilde{v}$ is completely determined by $m$ and $n$. Also, in order to get a nontrivial transforma-
tion, we assume that the spectral parameter $z_0$ is not a real number. The spectral values associated with the x-periodic solutions can be determined by the following corollary.

**Corollary 1:**

\[
\begin{pmatrix}
\begin{align*}
\tilde{u} \\
\tilde{v}
\end{align*}
\end{pmatrix}
\] is periodic in x if and only if $c_1(t) = 0$ in (6.5) and $\xi = \frac{2k\pi}{L} i$.

Proof:

If $c_1(t) = 0$ and $\xi = \frac{2k\pi}{L} i$, then

\[
m(x, t) = \frac{2\nu(c_2(t) + c_3(t)e^{-\xi t})}{(\alpha z_0 i + \xi) c_2(t) + (\alpha z_0 i - \xi) c_3(t) e^{-\xi t}}, \text{ and}
\]

\[
n(x, t) = \frac{2\nu(c_2(t) + c_3(t)e^{-\xi t})}{(\alpha z_0 i + \xi) c_2(t) + (\alpha z_0 i - \xi) c_3(t) e^{-\xi t}}
\]

are clearly x-periodic. Therefore, $(\tilde{u}, \tilde{v})$ are x-periodic.

On the other hand, we show that if $(\tilde{u}, \tilde{v})$ are x-periodic, then $\frac{\tilde{m}}{m}$ is x-periodic.

Let $\dot{m} = m(x + L, t)$, $\dot{n} = n(x + L, t)$ and $m = m(x, t)$, $n = n(x, t)$.

\[
\tilde{u}(x + L, t) = \tilde{u}(x, t) \quad \text{and} \quad \tilde{v}(x + L, t) = \tilde{v}(x, t)
\]

iff

\[
\frac{\dot{m}}{1 + |m|^2 + |n|^2} = \frac{\dot{\tilde{m}}}{1 + |\tilde{m}|^2 + |\tilde{n}|^2}, \quad \text{and} \quad \frac{\dot{n}}{1 + |m|^2 + |n|^2} = \frac{\dot{\tilde{n}}}{1 + |\tilde{m}|^2 + |\tilde{n}|^2}
\]
 iff \[ \frac{m}{\bar{n}} = \frac{1 + |m|^2 + |n|^2}{1 + |m|^2 + |\bar{n}|^2} = \frac{n}{\bar{n}}. \]

 iff \[ \frac{n}{m} = \frac{\bar{n}}{\bar{m}} = \frac{\bar{n}(1 + |m|^2 + |n|^2)}{m(1 + |\bar{m}|^2 + |\bar{n}|^2)} \]

Therefore, periodicity of \(\frac{n}{m}\) is necessary condition for the periodicity of \(\bar{u}\) and \(\bar{v}\).

It can then be seen that \(\frac{n}{m} = \frac{\psi_3}{\psi_2} = \frac{ae^{\omega_1 x} + be^{\omega_2 x} + k_1}{ce^{\omega_1 x} + de^{\omega_2 x} + k_2}\)

where \(\omega_1 = \frac{-3\mu i - \xi}{2}\), \(\omega_2 = -\xi\), \(a = -\bar{u} c_1\), \(b = 2 uc_3\), \(c = \bar{v} c_1\), \(d = 2 uc_3\), \(k_1 = 2 uc_2\), and \(k_2 = 2 uc_2\).

Now, we can show that \(\frac{n}{m}\) is periodic of period \(L\) iff \(c_1 = 0\) and \(\xi = \frac{2 \pi i}{L}\). First, we note that if \(\omega_1 = \omega_2\), then \(-3z_0 i = \xi\).

This implies that \(-9z_0^2 = -9z_0^2 - 4\mu^2\). However, \(\mu^2 = |u|^2 + |v|^2 \neq 0\) since \(u\) and \(v\) can not be zero at the same time. Therefore, \(\omega_1 \neq \omega_2\).

Now, \(\frac{n}{m}\) is periodic

iff \(\frac{ae^{\omega_1 x} + be^{\omega_2 x} + k_1}{ce^{\omega_1 x} + de^{\omega_2 x} + k_2} = \frac{ae^{\omega_1(x+L)} + be^{\omega_2(x+L)} + k_1}{ce^{\omega_1(x+L)} + de^{\omega_2(x+L)} + k_2}\).

iff \((ae^{\omega_1 x} + be^{\omega_2 x} + k_1)(ce^{\omega_1(x+L)} + de^{\omega_2(x+L)} + k_2)\)
\[= (ae^{\omega_1(x+L)} + be^{\omega_2(x+L)} + k_1)(ce^{\omega_1 x} + de^{\omega_2 x} + k_2).\]

iff \(ad c^{(\omega_1 + \omega_2) x}(e^{\omega_2 L} - e^{\omega_1 L}) + bcc^{(\omega_1 + \omega_2) x}(e^{\omega_1 L} - e^{\omega_2 L})\)
\[+ ak_2 e^{\omega_1 x}(1 - e^{\omega_1 L}) + ck_1 e^{\omega_1 x}(e^{\omega_1 L} - 1)\]
\[ +bk_2 e^{\omega_2 x} (1 - e^{\omega_2 L}) + dk_1 e^{\omega_2 x} (e^{\omega_2 L} - 1) \]
\[ = 0. \]

iff \[ (ad - bc)e^{(\omega_1 + \omega_2)x} (e^{\omega_2 L} - e^{\omega_1 L}) + (ak_2 - ck_1)e^{\omega_1 x}(1 - e^{\omega_1 L}) \]
\[ +(bk_2 - dk_1)e^{\omega_2 x}(1 - e^{\omega_2 L}) \]
\[ = 0. \]

iff \[ ad - bc = 0, ak_2 - ck_1 = 0, \text{ and } bk_2 - dk_1 = 0. \]

We note that \[ bk_2 - dk_1 = (2vc_3)(2uc_2) - (2uc_3)(2vc_2) = 0. \] This is identically zero.

\[ ak_2 - ck_1 = -2(|u|^2 + |v|^2) c_1 c_2 = 0 \text{ iff } c_1 = 0 \text{ or } c_2 = 0. \]

\[ bc - ad = 2(|u|^2 + |v|^2) c_1 c_3 = 0 \text{ iff } c_1 = 0 \text{ or } c_3 = 0. \]

We also note that \( c_2 \) and \( c_3 \) can not be zero at the same time, otherwise \( m \) and \( n \) are not defined.

Therefore, if \( \frac{\pi}{m_0} \) is periodic of period \( L \) then \( c_1 = 0 \) and \( \xi = \frac{2k_2 i}{L} \). This complete the proof.
Appendix F

Near-Integrable Construction and Stability of the Spatially-Coherent, Time-Periodic Solutions of the Damped and Driven Sine-Gordon Equation

Abstract: Previously we have used near-integrable perturbation theory to construct branches of "breather-type" solutions to the damped and periodically driven sine-Gordon pde. Here we extend this construction to higher order which is necessary to capture stability information of these solutions, and in particular, to locate bifurcation values of the perturbation parameter. These constructions and stability properties are compared with direct numerical results provided by C. Xiong.

section 1: Introduction

We consider the weakly perturbed, damped and driven sine-Gordon equation on a fixed, finite spatial interval with even periodic boundary conditions:

\[(F1.1) \quad q_{tt} - q_{xx} + \sin q = \varepsilon [-\alpha_0 q_t + \Gamma_0 \cos(\omega_d t)],\]

with

\[q(x + L, t) = q(x, t)\text{ for all } t \text{ and for all } x \in [0, L],\]

\[q_t(x + L, t) = q_t(x, t)\text{ for all } t \text{ and for all } x \in [0, L],\]
\[ q_\varepsilon(0, t) = q_\varepsilon(L, t) \text{ for all } t, \]

where \( 0 < \varepsilon \ll 1 \), the driven frequency \( \omega_d = 0.87 \), \( \varepsilon \alpha_0 = 0.04 \) and \( L = 12 \). The remaining free parameter is the driven amplitude, \( \varepsilon \Gamma_0 \). We are interested in branches of solutions of the system (F1.1) which are parameterized by \( \varepsilon \Gamma_0 \).

Direct numerical experiments of the pde reveal a widely observed phenomenon in nonlinear evolution equations: a temporal quasi-periodic route to chaos with corresponding low-dimensional coherent spatial structure [23, 1]. The large time \( (t \gg 1) \) attractors of this system as a function of the single bifurcation parameter, \( \varepsilon \Gamma_0 \), can be specified by their spatial structure and temporal behavior. When \( L = 12, \omega = 0.87 \), and \( \varepsilon \alpha_0 = 0.04 \), the spatial structure and temporal behavior are as follows: spatially, the attractor changes with increasing stress from flat, to one localized excitation per period, to two localized excitations per period; temporally, the asymptotic state is periodic, then quasi-periodic, and finally chaotic. For the flat state solutions, the governing equation is the perturbed pendulum equation \( q_{tt} + \sin q = \varepsilon [-\alpha_0 q_t + \Gamma_0 \cos(\omega_d t)] \).

The time periodic solutions, commensurate with the driven frequency \( \omega_d \), can be found by constructing the bifurcation diagram of the perturbed pendulum equation. These periodic solutions serve as the "flat" or uniform state attractors of the system. For the x- and t- periodic solutions which subsequently bifurcate to quasi-periodic and chaotic states, we will construct approximate solutions which at leading order are given by two-phase solutions of the unperturbed pde. We derive the phase locking conditions for the two phase solutions and show that the Melnikov function of the
two phase theta function solution predicts two branches of perturbed solutions. Our previous study[12] shows that for small $\Gamma_0$ two-phase solutions of the unperturbed sine-Gordon equation approximate the perturbed solutions with $O(\varepsilon)$ accuracy. We then carry the approximation to higher order in order to capture bifurcation points inaccessible from a leading order approximation. Finally, we compare our result with the bifurcation diagram of (F1.1) as generated by C. Xiong with different numerical methods. These studies are similar to the nonlinear Schrödinger studies of Terrones et.al[26], with the difference that our problem is non-autonomous so that fixed points of [26] are replaced with elliptic functions.

section 2: The Flat State Solutions

First, we study the flat state (x-independent) solutions of the sine-Gordon equation. The results of this section are standard “textbook” calculations and are included for self-containment to illustrate the simplest examples before we examine the non-trivial space- and time-dependent solutions in the next section. Uniform solutions of (F1.1) are governed by the perturbed pendulum equation

\begin{equation}
(F2.1) \quad \ddot{q} + \sin q = \varepsilon[-\alpha_0 \dot{q} + \Gamma_0 \cos(\omega_0 t)].
\end{equation}

The classical solution of the unperturbed pendulum equation is given by

\begin{equation}
(F2.3) \quad q^0(t, H; t_0) = 2 \arcsin(\sqrt{H/2} \sin(t - t_0; \sqrt{H/2}),
\end{equation}

where $sn$ is the Jacobi elliptic function, $H$ is the conserved energy,
(F2.4) \[ H = \frac{1}{2} q_t^2 + 1 - \cos q, \]

and \( t_0 \) is an arbitrary phase constant, and the relation between the frequency \( \omega \) and the energy \( H \) is given by \( \omega = \pi / (2K(\sqrt{H/2}) \)

where \( K(\sqrt{H/2}) \) is the complete elliptic integral of the first kind.

We fix the parameters \( \alpha_0, \omega_d \) and seek the periodic solutions for varying \( \Gamma_0 \). First, for the fixed driven frequency \( \omega_d \) we can find the Hamiltonian \( H \) which determines the periodicity of \( q^0 \) with period \( 2\pi / \omega_d \).

2.1: The phase locking condition

In this section we want to find the phase locking condition in which discrete values \( t_0 \) of the unperturbed arbitrary phase shifts are selected. This phase locking condition will determine the existence of the time periodic solutions of the perturbed system. In a perturbed 2-dimensional system \( \dot{x} = f(x) + \varepsilon g(x,t) \), the subharmonic Melnikov function of period \( T \) is given by [15]

\[ M(t_0) = \int_0^T f(q^0(t - t_0)) \wedge g(q^0(t - t_0)) dt, \]

which also takes the form

\[ M(t_0) = \int_0^T \frac{dH}{dt}(q^0(t - t_0)) dt, \]

where \( q^0 \) is the solution of the unperturbed system.

Property: The phase locking condition of the perturbed pendulum equa-
tion

The subharmonic Melnikov function of period $2\pi/\omega_d$ of the perturbed pendulum equation is given by

\[(F2.5) \quad M(t_0) = -2\sqrt{H/2}[2\alpha_0\sqrt{H/2}\frac{\pi}{\omega_d} - \Gamma_0 \cos(\omega_d t_0) \int_{0}^{2\pi/\omega_d} cn(t) \cos(\omega_d t) dt].\]

\[(F2.6) \quad M(t_0) = 0 \iff t_0^\pm = \pm \frac{1}{\omega_d} \arccos\left(\frac{2\alpha_0}{\Gamma_0^2}\right),\]

where $I = \int_{0}^{2\pi/\omega_d} cn(t) \cos(\omega_d t_0) dt$.

The perturbed pendulum equation $(F2.1)$ has a subharmonic orbit with period $2\pi/\omega_d$ if $M(t_0)$ has simple zeros. First, we note that for fixed damping coefficient $\alpha_0$, there is a minimum critical driving amplitude, $\Gamma_c$, for existence of a phase-locked constant $t_0$. The critical driving amplitude $\Gamma_c$ is plotted against the damping coefficient $\alpha$ in figure 4. In our example, we fix the parameters $\varepsilon \alpha_0 = 0.04$, $\omega_d = 0.87$, and $L = 12$. The Melnikov function predicts that for $\varepsilon \Gamma_0 > 0.050938$ there exist periodic solutions of period $2\pi/\omega_d$ of the perturbed pendulum equation. Secondly, for each $\varepsilon \Gamma_0 > 0.050938$ there are two values for the phase constant $t_0$. These two phase constants $t_0^\pm$ determine two branches of periodic solutions which are the $O(\varepsilon)$ approximation of the large amplitude periodic solutions of the perturbed pendulum equation. The two branches of phase constants are shown in figure 5. The periodic solutions to the perturbed pendulum equation can be solved numerically. This numerical method will be discussed in section 4, and the bifurcation diagram of the periodic solutions will be constructed and shown in figure 6.
2.2: The linear stability analysis

The linear stability of the periodic solutions to the perturbed pendulum equation can be determined analytically[15] or numerically. Analytically, we study the linear stability by perturbation theory based on action-angle coordinates. Numerically, we can compute the Floquet multipliers of the linearized system about the periodic solutions and use those Floquet multipliers to determine the stability. The two methods are illustrated as follows.

First, we consider the action-angle transformation of the pendulum equation,

\[(q, p) \rightarrow (I, \Theta),\]

with \(I = \frac{\Pi(H)}{2\pi} = \frac{4}{\pi}[(H - 2)K(\sqrt{\frac{H}{2}}) + 2E(\sqrt{\frac{H}{2}})]\) and \(\Theta = \frac{\partial H}{\partial I} sn^{-1}(\sqrt{\frac{1 - \cos^2}{H}}; \sqrt{\frac{H}{2}})\), where \(q(t, t_0) = 2 \sin^{-1}(\sqrt{\frac{H}{2}} sn(t - t_0; \sqrt{\frac{H}{2}}))\), \(p = \dot{q}\), and \(\Pi(H)\) is the area of the closed region formed by the periodic solution with energy \(0 < H < 2\).

Under the transformation \((q, p) \rightarrow (I, \Theta)\), the perturbed pendulum equation can be expressed in the action-angle coordinates[15]

\[
\dot{I} = \varepsilon[\frac{\partial I}{\partial p}(\alpha_0 p + \Gamma_0 \cos(\omega_d t))]
\]

(F2.6)

\[
\dot{\Theta} = \frac{\partial H}{\partial I} + \varepsilon[\frac{\partial \Theta}{\partial p}(\alpha_0 p + \Gamma_0 \cos(\omega_d t))],
\]

where \(\frac{\partial H}{\partial I} = \omega\) is the angular frequency.

Choosing the resonant orbit with period \(T = \frac{2\pi}{\omega_d} = 4K(\sqrt{\frac{H}{2}})\) and perturb the
system (F2.6) about the orbit by the following transformation \( I = I^0 + \sqrt{\varepsilon} h, \Theta = \omega(t - t_0(t)) \), where \( I^0 \) is the fixed action of the chosen orbit. The averaged system for \( h \) and \( t_0 \) to \( O(\sqrt{\varepsilon}) \) is

\[
\dot{h} = \frac{\sqrt{\varepsilon}}{2\pi} M(t_0)
\]

\[(F2.7)\]

\[
\dot{t}_0 = -\frac{\sqrt{\varepsilon}}{\omega} \frac{\partial \omega}{\partial I} h.
\]

The linear stability of the periodic solutions to the perturbed pendulum equation is completely determined by (F2.7). The averaged system (F2.7) has fixed points when \( h = 0 \) and \( M(t_0) = 0 \). These fixed points are saddles if \( \frac{\partial M}{\partial t_0} > 0 \) and centers if \( \frac{\partial M}{\partial t_0} < 0 \). Since \( \frac{\partial M}{\partial t_0} = -2\sqrt{H/2} \omega \Gamma_0 \sin(\omega dt_0) \int_0^T \cos(\omega dt) dt \), the fixed point is a center when \( t_0 > 0 \) and a saddle when \( t_0 < 0 \).

The numerical method to study the linear stability is the following: First, we find the periodic solutions of the perturbed pendulum equation. The method is to find the initial condition so that the Poincare map \( P \) determined by (F2.1) fixes the initial condition after period \( 2\pi/\omega_d \). It is equivalent to find the zeros of the map \( F = P - I \), where \( I \) is the identity map. The initial approximations of the fixed point are given by the unperturbed solution and the phase locking condition.

Numerically, we study the linear stability of periodic solutions by computing the Floquet multipliers. First, we linearize the system about the periodic solutions and choose the initial condition of the linearized system to be \( Y(t = 0) = I \), the identity
matrix. We then integrate the ode to time $2\pi/\omega_d$. The linearized Poincare map after $2\pi/\omega_d$ period is then given by $DP = Y(\frac{2\pi}{\omega_d})$ and the Floquet multipliers are the eigenvalues of $DP$. The periodic solutions are unstable if there exist eigenvalues $\lambda$ with module $|\lambda| > 1$. The result shows that the periodic solutions are unstable if $t_0 < 0$ and they are stable if $t_0 > 0$. This is consistent with the analytical study.

**section 3: Construction of the Breather Solutions**

In this section we turn to construct an approximate time-periodic breather solution to the perturbed sine-Gordon equation (F1.1). The approximate solutions can be constructed by solving the truncated systems of ode derived from the pde.[29] Our construction of the solution is based on the study of the two phase theta functions of the unperturbed sine-Gordon equation. In [12], we have shown that the two phase theta function solution of the unperturbed sine-Gordon equation approximates the breather solution of the perturbed equation to $O(\varepsilon)$ accuracy for small $\Gamma_0$. To construct an approximate solution with higher order accuracy we assume that the perturbed sine-Gordon equation has a solution of the following form:

\begin{equation}
(F3.1) \quad q^\varepsilon(x,t) = q^0(x,t) + \varepsilon q^1(x,t) + \varepsilon^2 q^2(x,t) + \ldots,
\end{equation}

where $q^0(x,t)$ is the two phase solution to the unperturbed sine-Gordon equation $q_{tt} - q_{xx} + \sin q = 0$.

Our method is to compute the corrections $q^1(x,t), q^2(x,t), \ldots$ through Fourier
expansions in $x$ of $q^i(x,t)$. We will compare the results from the theta function approximation and the truncated ode approximation in the next section. Before doing that we note some facts about two phase theta function solutions of the unperturbed sine-Gordon equation. We refer the reader to [11] and [12] for details.

### 3.1: The time-periodic two phase solutions to the unperturbed s-G equation

The two phase, x- and t-periodic theta function solutions of the unperturbed sine-Gordon equation $q_{tt} - q_{xx} + \sin q = 0$ can be found by the inverse spectral transform (IST)[11]. The two phase theta function solution can be written as

\[ (F3.2) \quad q_2(x,t; \sum_{N=2}^{(s)}; t_0, x_0) = i \ln \left[ \frac{\Theta(\tilde{\ell}(x,t) + \tilde{I}(B),)}{\Theta(\tilde{I}; B)} \right] = 4 \tan^{-1} \left( \frac{Im(\Theta(\tilde{I}; B))}{Re(\Theta(\tilde{I}; B))} \right), \]

with the prescribed spectrum $\Sigma_N^{(s)} = \{ E_1, E_2, E_3, E_4 | E_j = \frac{1}{16} e^{i\psi_j}, \psi_{2j} = -\psi_{2j-1}, 0 < \psi_1 < \psi_2 < \pi \}$, where

1. $\Theta(\tilde{I}, B)$ is the Riemann theta function of genus 2, $B$ is the $2 \times 2$ normalized “period matrix” defined on the Riemann surface $R$ of genus 2 with the prescribed spectrum $\Sigma_N^{(s)}$, where $R = \{(E, R(E)) | R^3(E) = E \Pi_1^4(E - E_j)\}$.

2. $\tilde{I} = (\frac{1}{2}, \frac{1}{2})^t$ and $\tilde{l}(x,t) = (l_1(x,t), l_2(x,t))^t$ is the vector of complex phases.

With the prescribed spectrum, the period matrix $B$ and $\tilde{l}$ have the following special symmetries:
\[
B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} + i \begin{pmatrix} \alpha & \alpha \\ \alpha & \beta \end{pmatrix}
\]
\[
\tilde{l}(x, t) = i \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ -b \end{pmatrix} (t - t_0),
\]
where \(a, b, \alpha, \beta\) are constants uniquely specified by \(\psi_1\) and \(\psi_2\) in the spectrum.

With these special symmetries, the Riemann theta function can be factored into the form:
\[
\Theta(\tilde{l}; B) = f_1(x)g_1(t) + if_2(x)g_2(t).
\]

It can be represented by Jacobian elliptic functions explicitly. This result will greatly facilitate the analysis of the phase-locking conditions of the two phase wave train.

Property 2:

The unperturbed sine-Gordon equation has the time-periodic two phase solution
\[
(F3.3) \quad q^0(x, t-t_0) = 4 \tan^{-1} \left[ \sqrt{\frac{k_1}{k_2}} \frac{dn(4\pi K(k_1) (t - t_0); k_1^2)}{cn(2\pi K(k_2) (t - t_0); k_2^2)} \right],
\]
where the constants \(k_1\) and \(k_2\) are determined by \(L = \frac{K'(k_1)}{2\pi K(k_1)}, \omega = \frac{2K'(k_2)}{\delta K(k_2)}\),
and \(t_0\) is the phase constant.

proof:

First, we note that the Riemann theta function
\[
\Theta(\tilde{l}, B) = \Sigma_{\vec{n} \in \mathbb{Z}^2} \exp[i\pi < B\vec{n}, \vec{n}> + 2i\pi < \tilde{l}, \vec{n}>]
\]
\[
= \Sigma_{(n_1, n_2) \in \mathbb{Z}^2} \exp[i\pi (n_1^2 + n_2^2) - \pi < \vec{n}, A\vec{n}> - 2\pi < \tilde{l}, \vec{n}>],
\]
where \(\vec{n} = (n_1, n_2)\), and \(A = \begin{pmatrix} \alpha & \alpha \\ \alpha & \beta \end{pmatrix}\).
Let \( m_1 = n_1 + n_2 \) and \( m_2 = n_2 \). Then
\[
< \vec{n}, A\vec{n} > = \alpha m_1^2 + (\beta - \alpha)m_2^2, \\
< \vec{l}, \vec{n} > = am_1 x - bm_2 t
\]
and \( n_1^2 + n_2^2 = \alpha^2 - 2m_1 m_2 + 2m_2^2 \).

We break the summation into four parts
\[
\Sigma_{m_2 \text{even}, m_1 \text{odd}} + \Sigma_{m_2 \text{even}, m_1 \text{even}} + \Sigma_{m_2 \text{odd}, m_1 \text{odd}} + \Sigma_{m_2 \text{odd}, m_1 \text{even}}.
\]
We find
\[
\Theta(x, t - t_0) = \theta_3(2\pi ia x, \tau_1) \theta_4(-ib(t - t_0), \tau_2) \\
+ i\theta_2(2\pi ia x, \tau_1) \theta_3(ib(t - t_0), \tau_2),
\]
where \( \tau_1 = 4i\alpha \) and \( \tau_2 = i(\beta - \alpha) \).

The two phase solution given in (F3.2) then can be represented by a Jacobian elliptic function
\[
q^0(x, t - t_0) = 4 \tan^{-1} \left[ \frac{k_1}{\sqrt{k_2}} \frac{1}{\frac{dn(4aK(k_1)x;k_1^2)}{cn(2bK(k_2)(t-t_0);k_2^2)}} \right],
\]
where the constants \( k_1 \) and \( k_2 \) are determined by \( L = \frac{K''(k_1)}{2aK(k_1)}, \frac{2\pi}{\omega} = \frac{2K'(k_2)}{bK(k_2)} \).

3.2: The phase locking condition

The analogous result of the phase locking condition can be formally derived for the two phase breather train. The IST variables which are equivalent to the conserved sine-Gordon energy are the elements of the invariant simple periodic spectrum, \( \Sigma_N^{(a)} \), which is invariant under the symmetry \( E \rightarrow \frac{1}{16\pi E} \). The perturbed modulation equations for these even two phase breather trains reduce to a single ode [10]. Let
\[
Z_j = \frac{1}{2}(E_j + \frac{1}{16\pi E_j}),
\]
then the perturbed breather modulation equations are

\[(Z_j)_t = \frac{(Z_j + (-1)^m/16)\tau_1}{(Z_j - \chi)}, \quad j = 1, 2,\]

where

\[\tau_1 = \frac{1}{32} \int_0^L \left(q_t + q_x^2 - 2 \cos q dx,\right)\]

\(m\) is a discrete parameter which may take the values 0 or 1, and \(\chi\) is completely determined from \(\Sigma_{N=2}^{(s)}\) by Riemann surface period information as follows. Define the meromorphic differential

\[\Omega(x) = \left(-\frac{1}{2}E^3 + (-1)^{m+1}16\alpha_1 E^2 + \alpha_1 E + (-1)^m \frac{\sqrt{144E_k}}{32}\right) \frac{dE}{E^2(E)},\]

\[R^2(E) = E\Pi_1(E - E_j),\]

\[\alpha_1 = -\frac{1}{32} \left[\frac{1}{16} + (-1)^m \chi\right],\]

and \(\alpha_1\) is fixed by the condition

\[\mu_j \epsilon\text{cycle } \Omega(x) = 0, \quad j = 1, 2,\]

where \(\mu_j = a_j - 2b_j\) and the cycles \(a_1, a_2, b_1, b_2\) are given in figure 7.

The phase locking condition for the two phase breather train of frequency \(\omega_d\) is then given by

\[M(t_0) = \int_0^{2\pi/\omega_d} (Z_j)_t dt = 0.\]

**Property 3: The phase locking condition of two phase breather train**

The Melnikov function of the two phase breather train is

\[(F3.4) \quad M(t_0) = -\alpha C_1 - \Gamma \sin(\omega_d t_0) C_2,\]

where \(C_1 = \int_0^{2\pi/\omega_d} \int_0^L q_t(q_0^0 + q_x^0) dx dt\), and \(C_2 = \int_0^{2\pi/\omega_d} \int_0^L q_t^0 \sin(\omega_d t) dx dt\).
The phase locking condition is

\[
\begin{align*}
( F3.5 ) & \quad \begin{cases} 
t_0^+ = \frac{1}{\omega_d} \arcsin\left( \frac{-a_c G_1}{\Gamma C_2} \right), \\
t_0^- = \frac{\pi}{\omega_d} - t_0^+ 
\end{cases}
\end{align*}
\]

With parameters \( \omega_d \) and \( L \) fixed one can search the two parameters \( \psi_1, \psi_2 \) in the spectrum \( \sum_{N=2}^{(e)} \) numerically. For \( \omega_d = 0.87 \) and \( L = 12 \), \( \psi_1 = 42.255^o \) and \( \psi_2 = 71.923^o \). The related constants are \( a = 0.074315, b = 0.10787, \alpha = 0.44589, \beta = 0.83540 \). Again, for fixed damping coefficient and fixed driving frequency, there is a minimum critical driving amplitude, \( \Gamma^* \), for existence of a phase-locked even breather train. The critical driving amplitude \( \Gamma_c \) is plotted against the damping coefficient in figure F8. The Melnikov function predicts that for \( \Gamma > 0.04892 \), there are two branches of breather solutions to the perturbed sine-Gordon equation and the two phase solution given in \( (F3.3) \) approximates the breather solution to \( O(\epsilon) \) accuracy.

The two branches of phase constants are shown in figure F9.

3.3: Construction of the breather solutions

From the even boundary conditions in \( (F1.1) \) we assume that the perturbed breather solution \( q^e(x,t) \) has a formal Fourier expansion

\[
( F3.6 ) \quad q^e(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t) \cos(nkx), \text{ where } k = \frac{2\pi}{L}.
\]

Inserting \( (F3.6) \) into \( (F1.1) \), the temporal coefficients \( a_n \) satisfy the following system of ode's:
\[ (F3.7) \quad \begin{cases} \ddot{a}_0 + \alpha \dot{a}_0 = -F_0(a_0, a_1, ...) + \Gamma \cos(\omega_d t), \\ \ddot{a}_m + \alpha \dot{a}_m + k^2 m^2 a_m = -2F_m(a_0, a_1, ...) \end{cases}, \text{ for } m = 1, 2, 3, \ldots, \]

where

\[ F_m = \left[ \frac{1}{L} \int_0^L \left( \sum_{n=1}^{\infty} a_n \cos(nkx) \right) \cos(mkx) \, dx \right] \sin a_0 \]

\[ + \left[ \frac{1}{L} \int_0^L \left( \sum_{n=1}^{\infty} a_n \cos(nkx) \right) \cos(mkx) \, dx \right] \cos a_0, \text{ for } m = 0, 1, 2, \ldots \]

The two-mode truncated system of the above system has the following simple form:

\[ (F3.8) \quad \begin{cases} \ddot{a}_0 + \alpha \dot{a}_0 = -J_0(a_1) \sin a_0 + \Gamma \cos(\omega_d t), \\ \ddot{a}_1 + \alpha \dot{a}_1 + k^2 a_1 = -2J_1(a_1) \cos a_0. \end{cases} \]

where \( J_1 \) and \( J_2 \) are Bessel functions.

Directly, one can solve the truncated system of \((F3.7)\) and get an approximation of \( q^e(x, t) \). However, this will need to solve a large system to get a good approximation. Since we know from extensive numerical studies that the two phase solution to the unperturbed pde is a good approximation to the perturbed breather solution we can use it as the initial approximation and then find the corrections.

We assume that the system \((F3.7)\) has a solution of the form

\[ (F3.9) \quad a_m(t) = a_m^0(t) + \varepsilon a_m^1(t) + \varepsilon^2 a_m^2(t) + \ldots, m = 0, 1, 2, \ldots, \]

from \((F3.1)\) and \((F3.6)\) we have

\[ (F3.10) \quad a_m^0(t) + \varepsilon a_m^1(t) + \varepsilon^2 a_m^2(t) + \ldots = <q^0, \cos mkx> + \varepsilon <q^1, \cos mkx> + \varepsilon^2 <q^2, \cos mkx> + \ldots. \]
We can then approximate the corrections \( q^i(x,t), q^j(x,t), \ldots \) by a finite Fourier expansion

\[
(F3.11) \quad q^i(x,t) = a^i_0(t) + \sum_{m=1}^{n} a^i_m \cos mkx, \ i = 1, 2, 3, \ldots
\]

This perturbative construction of the breather solution \( q^c(x,t) \) is then related to find the corrections \( a^i_m \) in the truncated system of \((F3.7)\).

Inserting \((F3.9)\) into the truncated system of \((F3.7)\) and expanding the system into Taylor series, the first and the second order corrections \( a^1_m(t), a^2_m(t) \) can be solved numerically via the following nonautonomous ode's

\[
(F3.12) \quad \begin{cases} 
\ddot{a}^1_0 &= -\sum_{i=1}^{n} \left[ \frac{\partial F_0}{\partial a_i} (a_0^0, a_0^1, \ldots) a_i^1 \right] - \alpha \dot{a}^0_0 + \Gamma \cos(\omega_d t), \\
\ddot{a}^1_m &= -2 \sum_{i=1}^{n} \left[ \frac{\partial F_m}{\partial a_i} (a_0^0, a_0^1, \ldots) a_i^1 \right] - m^2 k^2 a^1_m - \alpha \dot{a}^0_m,
\end{cases}
\]

\[
(F3.13) \quad \begin{cases} 
\ddot{a}^2_0 &= -\sum_{i=1}^{n} \left[ \frac{\partial F_0}{\partial a_i} (a_0^0, a_0^1, \ldots) a_i^2 \right] - \frac{1}{2} \sum_{i,j=1}^{n} \left[ \frac{\partial^2 F_0}{\partial a_i \partial a_j} (a_0^0, a_0^1, \ldots) a_i^1 a_j^1 \right] - \alpha \dot{a}^1_0, \\
\ddot{a}^2_m &= -2 \sum_{i=1}^{n} \left[ \frac{\partial F_m}{\partial a_i} (a_0^0, a_0^1, \ldots) a_i^2 \right] - \sum_{i,j=1}^{n} \left[ \frac{\partial^2 F_m}{\partial a_i \partial a_j} (a_0^0, a_0^1, \ldots) a_i^1 a_j^1 \right] - \alpha \dot{a}^1_m.
\end{cases}
\]

Using the phase-locked conditions, we can find the periodic solutions of period \( 2\pi/\omega_d \) of the first order correction \( a^1_m(t) \) and the second order correction \( a^2_m(t) \) numerically. The technique is the following: (1) For fixed \( \Gamma_0 \), we use the phase-locked condition to pick the phase constant \( t_0 \) and let \( a^0_m(t) = < q^0, \cos mkx > \). (2) Use the equations \((F3.12)\) and \((F3.13)\) to find the initial conditions so that the Poincare map of period \( 2\pi/\omega_d \) has fixed points. The corrections \( a^1_m \) and \( a^2_m \) can be obtained by integrating the equations over a \( 2\pi/\omega_d \) period. We compare the results \( a^0_m = a^0_m + \varepsilon a^1_m \) with the solutions solved directly from the 3-mode truncated system of \((F3.7)\) for
\( \Gamma = 0.055 \). See figure 10. The numerical comparison shows that the theta function approximation with one correction term in a two-mode projection achieves the accuracy of the approximation by three-modes in the truncated system \((F3.7)\). The advantage of the theta function approximation is that we can avoid integrating a very large system of ode's, but the front-end expense of theta function codes far outweighs this numerical advantage. The real issue here is that the integrable solutions yield the order one approximation of the near-integrable pde, with higher order corrections available by a systematic procedure.

### 3.4: The Linear Stability

In this section we will study the stability of the temporal periodic solution \( \Theta(t) = (a_0(t), a_1(t), a_2(t), ...) \) derived in our perturbative construction. Since we know that \( \Theta(t) \) is a good approximation to the solution of the truncated system of \((F3.7)\) we can linearize the truncated system of \((F3.7)\) about the periodic solution \( \Theta(t) \) and compute the Floquet multipliers to investigate the linear stability of the periodic solution. The linearized flow of \((F3.7)\) about the periodic solution \( \Theta(t) \) is given by

\[
\begin{align*}
\dot{\delta}_{01} &= \delta_{02} \\
\dot{\delta}_{02} &= -\alpha \delta_{02} - \sum_{n=0}^{\infty} \left( \frac{\partial F_0}{\partial \Theta} \right)^{\delta_{n1}} \\
\dot{\delta}_{m1} &= \delta_{m2} \\
\dot{\delta}_{m2} &= -\alpha \delta_{m2} - m^2 k^2 \delta_{m1} - \sum_{n=0}^{\infty} \left( \frac{\partial F_m}{\partial \Theta} \right)^{\delta_{n1}}, \text{ for } m = 1, 2, 3, ...
\end{align*}
\]

The numerical computation of Floquet multipliers is as follows: we integrate
the truncated system of (F3.14) with initial condition $Y(t = 0) = I$, where I is the identity matrix. The linearized Poincare map is then given by $DP = Y(\frac{2\pi}{\omega_d})$, and the Floquet multipliers are the eigenvalues of $DP$. The results show that for the branch with phase condition $t_0^+$ the periodic solutions are one dimensionally unstable. For the branch with phase condition $t_0^-$ the periodic solutions are stable for $\Gamma < 0.059$, two dimensionally unstable for $\Gamma > 0.059$, and the point $\Gamma = 0.059$ corresponds to a Hopf bifurcation. This numerical result will also be confirmed by solving the truncated system directly. We will compare the results obtained by both methods in the next section.

section 4: Numerical Experiments of Truncated ODE's

The results in the previous section suggest that the low mode truncated system will be a good model to study the chaotic behavior of the perturbed sine-Gordon equation. In this section we study the truncated system of (F3.7) directly. We construct the bifurcation diagram of the system and compare the results with previous sections. These results are reproduced with the help of C. Xiong, whose thesis[29] contains a detailed study far beyond the issues of concern here.

Fix the parameters $\varepsilon \alpha = 0.04$, $L = 12$, and $\omega_d = 0.87$. We try to find the periodic solutions of period $2\pi/\omega_d$ for changing $\Gamma_0$. Note that when $\Gamma_0 = 0, a_i = 0$ is a trivial solution at $t = 0$. Choosing the time section $T = 0$, the periodic solutions of period $2\pi/\omega_d$ can be determined by the fixed points of the Poincare map $P$. To find the fixed point of the Poincare map is equivalent to find the zero of the map $F = P - I$,
where \( I \) is the identity map. The bifurcation diagrams of two-mode and three-mode truncated ode's are shown in figure 11 and 12. The flat state solutions determined by the perturbed pendulum equation are indicated as the branch 0-T1-A-T2, and the breather solutions are indicated as the branch A-T3-B-, where T1, T2, T3 are turning points and B is a Hopf bifurcation point. Quantitively, we compare 2-mode truncation, 3-mode truncation, and the perturbation construction. See the table. In fact, for 3-mode and 4-mode ode truncations there is no significant difference in the periodic solutions.

<table>
<thead>
<tr>
<th></th>
<th>Two Modes</th>
<th>Three Modes</th>
<th>Perturbation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_1 )</td>
<td>0.13518</td>
<td>0.13518</td>
<td></td>
</tr>
<tr>
<td>( T_2 )</td>
<td>0.05094</td>
<td>0.05094</td>
<td>0.05094</td>
</tr>
<tr>
<td>( T_3 )</td>
<td>0.04942</td>
<td>0.04899</td>
<td>0.04892</td>
</tr>
<tr>
<td>( B )</td>
<td>0.0714</td>
<td>0.0592</td>
<td>0.0585</td>
</tr>
</tbody>
</table>

We now summarize the results of the bifurcation diagram of the two-mode and three-mode truncated systems as follows:

(1): For the flat state solutions (0-T1-A-T2-), the branch 0-T1 is stable, the branch T1-A is one dimensional unstable, the branch A-T2 is two dimensionally unstable, and the branch T2- is one dimensionally unstable.

(2): For the breather solutions (A-T3-B-), the branch A-T3 is one dimensionally
unstable, the branch T3-B is stable, the branch B-is two dimensionally unstable.

(3): As \( \Gamma \) crosses B, there emerges a pair of complex Floquet multipliers with moduli equal 1 and their imaginary parts are not 0. There exists a Hopf bifurcation at B and quasi-periodic solutions emerge.

(5): For the two-mode truncated system, the Hopf bifurcation around B is subcritical and the quasi-periodic solutions are unstable. For the 3-mode truncated system, the Hopf bifurcation around B is supercritical and the quasi-periodic solutions are stable. This is the essential flaw in the two-mode truncated system: the Hopf bifurcation is identified, but the resolution of the third mode is necessary both to stabilize the quasi-periodic branch and to assess that it is supercritical.

(6): The results from the direct computation of the truncated system and from the perturbative construction coincide. The turning points \( T2 \) and \( T3 \) are predicted precisely by the phase locking conditions and the Hopf bifurcation is found in both computations.

section 5: Discussions

The purpose of the above study is to exhibit a near-integrable perturbation construction of special families of solutions to the damped, driven sine-Gordon pde. In particular, we focus primarily on spatially coherent states of breather-type which frequency lock to the time-periodic driving term. Our previous results\[12\] yield leading order approximations for the solution branches, provided by theta function solutions of the unperturbed sine-Gordon pde. In order to capture stability and bifurcation in-
formation, we extend the previous near-integrable “averaging” results to higher order in small parameter series expansion of the solution. Our near-integrable construction and higher order stability are confirmed by alternative direct numerical studies with the assistance of C. Xiong.

section 6: Figure Captions

Figure 4: The critical driving amplitude $\Gamma_c$ versus the damping coefficient $\alpha_0$ of the perturbed pendulum equation. For $\Gamma < \Gamma_c$, there does not exist a phase-locked constant $t_0$, and for $\Gamma > \Gamma_c$ there are two branches of solutions, $t_0^\pm$.

Figure 5: Two branches of phase locking constants, $t_0^\pm$, of the perturbed pendulum equation.

Figure 6: The bifurcation diagram of periodic solutions of the perturbed pendulum equation.

Figure 7: A basis for homology cycle on the genus 2 Riemann surface associated with two-phase, breather-type, sine-Gordon wavetrains.

Figure 8: The critical driving amplitude $\Gamma^*$ versus the damping coefficient $\alpha_0$ of the two phase breather train. For $\Gamma < \Gamma^*$, there does not exist a phase-locked constant $t_0$, and for $\Gamma > \Gamma^*$ there are two branches of solutions, $t_0^\pm$.

Figure 9: Two branches of phase locking constants, $t_0^\pm$, of two phase solutions.

Figure 10: Comparison of periodic solutions from two mode, three mode, and perturbation constructions; perturbation solid line, two mode dotted line, three mode dashed line.
Figure 11: The bifurcation diagram of periodic solutions of two-mode ode.

Figure 12: The bifurcation diagram of periodic solutions of three-mode ode.
BIBLIOGRAPHY


[28] O. Wright, private discussions.


Figure 1: A solution homoclinic to the x-independent plane wave.
Figure 2: A solution homoclinic to the focusing plane wave.
Figure 3: A solution homoclinic to the defocusing plane wave.
Figure 4: The critical driving amplitude versus the damping coefficient of the perturbed pendulum equation.
Figure 5: Two branches of phase locking constants of the perturbed pendulum equation.
Figure 6: The bifurcation diagram of periodic solutions of the perturbed pendulum equation.
Figure 7: A basis for homology cycles on the genus 2 Riemann surface associated with two-phase, breather-type, sine-Gordon wavetrains.
Figure 8: The critical driving amplitude versus the damping coefficient of the two phase breather train.
Figure 9: Two branches of phase locking constants of two phase solutions.
Figure 10: Comparison of periodic solutions of two-mode, three-mode, and perturbative constructions.
Figure 11: The bifurcation diagram of periodic solutions of 2-mode ode.
Figure 12: The bifurcation diagram of periodic solutions of 3-mode ode.