INFORMATION-REVELATION IN INCOMPLETE-INFORMATION GAMES

DISSERTATION

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ABSTRACT

Our thesis concentrates on information-revelation in games with incomplete information. In chapter 2, we examine a one-shot version of a particular formulation of games with incomplete information due to Harsanyi [10]. The chapter examines the extent to which all interior Bayesian equilibria of such games are fully information-revealing as well as the extent to which the number of interior Bayesian equilibria is finite. One of our theorems states that, given a fixed common prior, the two properties hold generically in the space of twice continuously differentiable utility function bundles. Another theorem states that for a generic choice of a sufficiently smooth utility function bundle, the two properties hold generically in the space of common priors.

In chapter 3, we model information in a one period setting using information partitions for the players. Our main result in this chapter states that in the space of twice continuously differentiable utility function bundles, it is a generic property of interior equilibria to yield full information-revelation.

In chapter 4, we consider a two period model with incomplete information. Again, the information of the players is modeled using information partitions. One of the results in this chapter demonstrates that second period beliefs satisfying certain symmetry assumptions are uniquely determined even for information sets that are unreached in equilibrium. Using these beliefs, we formulate a two-period equilibrium.
concept. Another result demonstrates that if the information structure satisfies an informational dispersion assumption, generic existence of two-period equilibria can be demonstrated in the space of twice continuously differentiable first period utility function bundles.
To Einar and Gunborg Nygren.
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CHAPTER 1

INTRODUCTION

In games with incomplete information, players are asymmetrically informed about the true state of nature. If actions of players are observable, this allows for the possibility of inferences about the true state of nature being drawn from the actions undertaken by the players in an equilibrium. In this thesis, we examine the extent to which players tend to reveal their information in equilibria to games with incomplete information.

Before proceeding, we here briefly discuss the distinction between games with complete information and games with incomplete information. The distinction between these two types of games was present already in von Neumann and Morgenstern [24]. The latter differ from the former in the fact that some or all of the players lack full information about the "rules" of the game, or equivalently about its normal form (or about its extensive form). This lack of information may be concerning other players' or even their own utility functions, about the physical facilities and strategies available to other players or even to themselves, about the amount of information the other players have about various aspects of the game situation, etc.
Harsanyi [10] demonstrated that any incomplete information can be reduced to uncertainty about utility functions. He then introduced Bayesian games in order to allow for an easier analysis of games with incomplete information. In a Bayesian game, a player is allowed to have more than one type. Different type-profiles are then characterized by different utility function bundles. The true type profile is drawn at the outset of the game with each player learning his own type before he takes his first action. When he takes his first action, each player hence knows his own type but typically not the true type of the other players. A Bayesian equilibrium then requires the actions of every type of a player to optimize the expected utility of that type given the strategies of the other players.

In chapter 3, we examine one shot Bayesian games where all players take their actions at the same time. We examine the extent to which different types of a player takes different actions in equilibria to such games. Several of our results demonstrate that for smooth one shot Bayesian games of this type, it is a generic property for all interior Bayesian equilibria to be fully-information revealing in the sense that no two types of a player take the same actions in an interior equilibrium. Hence the players types are fully-revealed ex post in any such interior equilibrium.

In many ways, information-revelation is more interesting in a dynamic context. That is, in models with more than one period. In order to be able to discuss information more explicitly in such models we first return to an instance of one-period
incomplete information games in chapter 4. We model the incomplete information by assuming that there are a finite number of states of the world, and that each player has an information partition in the first period. In such an incomplete information game, a player's strategy thus needs to specify a first period action for every element of his first period information partition.

In the one-period incomplete information games, all the players take their actions at the same time after first learning the element of their individual information partitions that have occurred. The one-period incomplete information game we consider can be re-interpreted as a version of Harsanyi's [10] Bayesian game if a player's information partition is re-interpreted as his type space. Given this re-interpretation, our solution concept for the one-period model in fact coincides with Harsanyi's Bayesian equilibrium concept.

Theorem 4.2.1 demonstrates that it is a generic property of smooth one-period incomplete information games to have a finite number of interior equilibria all of which are fully information-revealing. This result parallels one of the results in the one-shot Bayesian game setting. By a fully information-revealing one-period equilibrium, we here mean an equilibrium in which every player's equilibrium strategy is a one-to-one function from his information partition to his pure action space. At first sight, this result may appear a bit surprising. In one-period games, however, all players move
at the same time. Hence no player can use the information revealed by the other players in the first period. Because of this, it is only in exceptional cases where a player takes the same actions in two states which he can distinguish between.

In chapter 5, we then turn to two-period games with incomplete information and observable actions. In such games, belief updating is a significant issue that poses some real difficulties. The main reason for this is that actions undertaken in the first period can signal important information about the true state. Hence a player's beliefs about the true state should be a function of the actions undertaken in the first period. For some action combinations a player can observe in the first period, the most natural way to update beliefs is by using Bayes rule. For some action combinations a player can observe, however, Bayes rule may fail to provide an answer as to how beliefs should be updated. In particular, this is true if a player observes an action combination that was not supposed to occur in any state of the world.

One of the issues we address is how beliefs can be updated even when players observe action combinations that weren't supposed to occur in any of the states. In a different type of model, similar issues have been studied by the equilibrium refinement literature, such as Selten [21] and Kreps-Wilson [13]. Our approach share the characteristic with their model that beliefs are formulated by using the limits of beliefs that would arise if players had some small positive probability of making mistakes. Theorem 5.3.2 demonstrates that the limiting beliefs of a sequence of beliefs
associated with distributions that satisfy certain symmetry assumptions in fact are uniquely determined and are given by a relatively simple formula. Beliefs satisfying the formula are referred to as reasonable beliefs.

The restrictions on beliefs imposed by our formula are stronger than the restrictions on beliefs imposed by the sequential equilibrium concept. There are two requirements that must be satisfied in a sequential equilibrium. One requirement is that beliefs must be the limiting beliefs of a sequence of beliefs associated with completely mixed strategies converging to the equilibrium strategies. The second requirement is that the strategies must be optimal in every information set given the beliefs. In our thesis, the condition on the beliefs is strengthened. Beliefs are required to be the limit of a sequence of beliefs associated with distributions that satisfy certain symmetry assumptions.

The formula for beliefs so derived is used to formulate a two-period equilibrium concept with the properties that the equilibrium strategy of every player is optimal in both the first and the second period. That is, the strategy is optimal in the first period regardless of the element of the players first period information partition in question. It is also optimal in the second period regardless of the element of the product space of the first period action space with the players second period information partition. That is, given the player's reasonable beliefs, he can not find another action that increases his expected utility.
We then address the question of existence of two-period equilibria. Theorem 5.4.1 demonstrates the generic existence of two-period fully information-revealing equilibria under an informational dispersion assumption. The main reason for generic existence when information is dispersed is that no player can hide information if it is revealed by other players.
CHAPTER 2

LITERATURE REVIEW

We here discuss parts of the literature that in one way or another relate closely to the work in the current thesis. Section 1 discusses two-player zero-sum games. A lot of the early important contributions in this area has had parallel developments in the more general theory and hence the motivation for including this discussion. We place a particular emphasis on the solution concepts developed for two-player zero-sum games.

In section 2, we discuss the Nash equilibrium concept which is the most commonly used solution concept for non-cooperative games. We also discuss various parts of the refinement literature. In section 3, we discuss incomplete information games. We pay particular attention to the Bayesian game formulation due to Harsanyi [10] and the associated solution concept of a Bayesian equilibrium. The Bayesian equilibrium is the natural extension of the Nash equilibrium to Bayesian games. Much of the work in our current paper studies generic properties of equilibria in incomplete information games and hence the significant relationship between the current thesis and such games.
In section 4, we discuss dynamic games. The last chapter in this thesis deals with a two-period model and we therefore discuss some papers dealing with related issues. Section 5 deals with generic results in economics, and section 6 with information-revelation in economics.

2.1 TWO-PLAYER ZERO-SUM GAMES

Significant works in game theory began appearing as far back as the early parts of this century. In this early period, most of the work was concentrated on strictly competitive games, more commonly known as two-person zero sum games. Some of the concepts that grew out of the study of these games has played a significant role is more general contexts. These include:

The extensive (or tree) form of a game, consisting of a complete formal description of how the game is played, with a specification of the sequence in which players move, what they know at the times they move, how chance occurrences enter the picture, and the payoff to each player at the end of play. The form was introduced by von Neumann [23] in 1928 and was later generalized by von Neumann and Morgenstern [24] in 1944 using a set theoretic approach and by Kuhn [14] in 1953 using a graph theoretical representation.

The concept of a pure strategy for a player, defined as a complete plan for that player to play the game, as a function of what he observes during the course of play,
about the play of others and about chance occurrences affecting the game. Given
a set of pure strategies for the players, the rules of the game determine a unique
outcome of the game and the associated payoff for each player.

The normal (or matrix) form of a game. The normal form is simply the function
that associates to each pure strategy combination for the players an associated payoff
profile.

The concept of a mixed strategy, which allows player to randomize over their pure
strategies. That is, rather than necessarily specifying a single pure strategy, a player
may play each one of his pure strategies with a specified probability.

The concept of 'individual rationality'. The security level of a player $i$ is the
amount that he guarantees himself, independent of what the other players do. That is,
he can find a strategy that regardless of the strategies of the other players guarantees
him that payoff. Here the security level is the actual payoff if the strategies refer to
pure strategies and an expected payoff if the strategies refer to mixed strategies.
A payoff distribution is said to be individually rational if every player gets a payoff at
least as high as his security level. A game is said to be strictly determined if there is
only one individually rational payoff distribution in the 'pure' sense and determined
if there is only one individually rational payoff distribution in the 'mixed' sense.

The first theorem of game theory due to Zermelo [25] in 1913 asserts that chess
is strictly determined. The proof is easily extended to a much larger class of games.
The precise condition for the proof to work is that the game be a (finite) two-person zero-sum game of perfect information. This means that there are no simultaneous moves, and that everything is open and 'above board': at any given time, all relevant information known to one player is known to all players.

Zermelo's theorem does not hold for general (finite) two-person zero-sum games with imperfect information. That is, in games where some of the information sets in the extensive form contain more than one vertex. However, in 1928 von Neumann [23] proved the minimax theorem which asserts that every two-person zero-sum game with finitely many pure strategies for each player is determined; that is, when mixed strategies are admitted, it has precisely one individually rational payoff vector. This had been previously verified by E. Borel (e.g., [4]) for several special cases, but not in general.

While this result in many ways was elegant, it did not apply to the non-zero-sum theory and not to games with more than 2 players. During the period following this result, significant progress was made in the development of what has become known as cooperative game theory. The single most important contributions in this area were probably contained in the work of von Neumann and Morgenstern [24] published in 1944. Because the present thesis deals with the area of game theory now commonly referred to a non-cooperative game theory, we will restrict our attention to this area for the remainder of this literature review.
2.2 NASH EQUILIBRIA AND REFINEMENTS OF NASH EQUILIBRIA

In the early 50s, Nash [11], [12] introduced to modern game theory the concept of an equilibrium (Nash Equilibrium) for games in normal form. The basic idea behind the solution concept, however, goes back at least to Cournot [5] in 1838. A Nash equilibrium is a (pure or mixed) strategy combination in which each player's strategy maximizes his payoff given that the others are using their strategies. In his paper, Nash proved an equilibrium existence theorem that asserts that any normal form game with a finite number of players and a finite number of pure strategies always has at least one equilibrium in mixed strategies.

For games with imperfect information, there is typically no guarantee that there will always exist a Nash equilibrium in pure strategies. In fact, there are relatively simple counter-examples (such as matching pennies) to the general existence of pure strategy Nash equilibria. In 1953, however, Kuhn [14] proved the existence of pure strategy equilibria in finite n-person games with perfect information. In proving his theorem, Kuhn used the notion of a subgame; this has turned out to be crucial in later developments. If at any time, all the players know everything that has happened in the game up to that time, then what happens from then on constitutes a subgame.

It follows from Kuhn's proof that every equilibrium (not necessarily pure) of a subgame can be extended to the whole game (Zermelo's [25] proof had followed a similar reasoning). This, in turn, implies that every game has equilibria that remain
equilibria when restricted to any subgame. Selten [21] called such equilibria subgame perfect. In games with perfect information, the equilibria that the Kuhn proof yields are all subgame perfect.

Not all Nash equilibria, however, are subgame perfect, even if the game has perfect information. Subgame perfection implies that when making choices, a player looks forward and assumes that the choices that will be made subsequently, will be rational; i.e. in equilibrium. For games with perfect information, this rules out any threats which would be irrational to carry out. This kind of forward looking rationality is in many ways most suited to economic applications. This has lead to the publication of a number of significant works in the area which is now commonly known as the refinement literature.

When working with extensive form games, perfect recall and behavior strategies play a crucial role. An extensive form game satisfies perfect recall if each player remembers what he has known or done, that is, if information is increasing over time. A behavior strategy prescribes local randomization among choices at information sets rather than the randomization over pure strategies that mixed strategies amount to. For games without perfect recall, behavior strategies may be inferior to mixed strategies in the sense that an expected payoff attainable with a mixed strategy may fail to be attainable with some behavior strategy. An important result due to
Kuhn [14], however, asserts that in finite games with perfect recall, any expected payoff that is attainable with a mixed strategy also can be attained with some behavior strategy. Aumann [2] has demonstrated that the same result holds in infinite games.

In a 1965 paper, Selten [20] demonstrated that any finite n-person game with perfect recall, has at least one subgame perfect equilibrium. The set of subgame perfect equilibria is typically much smaller than the set of Nash equilibria and the number of irrational threats in subgame perfect equilibria are typically much smaller than in those equilibria that are not subgame perfect. Still, it turns out that some irrational threats still are allowed in subgame perfect equilibria. The reason for this is that the subgame perfection may fail to impose any restrictions on strategies in information sets that are reached with zero probability in the smallest subgame in which that information set is a part. It hence still allows players to make irrational threats at such information sets.

To address this issue, Selten [21] introduced another concept in 1975. This concept is most commonly known as the perfect (trembling hand) equilibrium. The idea behind this concept is that one first looks at an $\epsilon$-perturbed game. In an $\epsilon$-perturbed game, each choice must be selected with some minimal strictly positive probability $\epsilon$. The minimal probability is best viewed as the probability that an action is mistakenly selected. In such a perturbed game, players are not allowed to use pure strategies in any of the information sets. Their choice of behavior strategies is thus constrained.
In an equilibrium to such an $\epsilon$-perturbed game, every information set is reached and the strategies in every information set is optimal given the constraint. By looking at a sequence of $\epsilon$-perturbed games where the minimal probabilities approach zero and an associated convergent sequence of equilibria, one gets a limiting equilibrium for the original game. A perfect equilibrium is then any equilibrium of the original game for which there exists such a sequence of equilibria that converges to the equilibrium in the original game. The perfect equilibrium concept rules out threats that would always be irrational to carry out.

A closely related concept is the sequential equilibrium of Kreps and Wilson [13]. This concept is based on 'Bayesian' players who construct subjective beliefs about where they are in tree when an information set is reached unexpectedly and who maximize expected payoffs based on such beliefs. The requirement that beliefs be shared by players and that they be consistent with the strategies being played (Bayesian updating) imply that the difference from perfection is only marginal.

The issue of how beliefs should be updated in unreached information sets is an issue that we address in chapter 5 of the present thesis. The approach taken there shares important features with the sequential and perfect equilibrium concepts.

2.3 GAMES WITH INCOMPLETE INFORMATION

In addition to the distinction between games with perfect and imperfect information, the literature also distinguishes between games with complete and incomplete
information. This distinction goes back to von Neumann and Morgenstern [24]. The latter differ from the former in that in the fact that some or all of the players lack full information about the 'rules' of the game, or equivalently about its normal (or extensive form). For example, they may lack full information about other player's or even their own payoff functions, about the physical facilities and strategies available to other players or even about themselves, about the amount of information the other players have about various aspects of the game situation, etc.

Much of the literature tended to focus on games with complete information until the late 60s and the Bayesian game formulation of Harsanyi [10] that appeared in 1967-68. In his paper, Harsanyi argues that any incomplete information game can be replaced by a new game that involves complete but imperfect information, yet which is essentially equivalent to the original game from a game-theoretic view. This is done by replacing the assumption that certain important attributes of the players are determined by some hypothetical random events at the beginning of the game with the assumption that the players themselves are drawn at random from certain hypothetical populations containing a mixture of individuals of different types, characterized by different attributes. After the draw, each player is assumed to learn his own type but, in general, he is allowed to be ignorant about the types of the remaining players.

He argues that for normal form games, no loss of generality results in assuming that any uncertainty about the normal form amounts to uncertainty about the payoff
functions. In his formulation of Bayesian games, has a type space from which his type is drawn. Different type profiles are then characterized by different utility functions over the pure strategy spaces of the players. At the outset of the game, a particular type profile is drawn from the type profile space using an objective joint probability distribution. Each player then learns his own type and uses Bayesian updating to calculate the probability of the other players taking on various types. Play then proceeds as in a standard normal form game with players simultaneously choosing their pure strategies (possibly by mixing). The Bayesian equilibrium concept is one in which every type of each player responds optimally to the strategies of all the types of the other players.

Throughout our thesis, we work with incomplete information as well as Bayesian games.

2.4 DYNAMIC GAMES

Games played in stages, with some kind of stationary structure are called dynamic games. We will here discuss a particular type of a dynamic game, multi-stage games with observable actions. In such games, (1) all players know the actions chosen at all previous stages when choosing their actions in a stage, and (2) all players move simultaneously in each stage. Here the players take their actions at the same time in a stage if each player takes his action in that stage without knowing the actions of the other players. For a more detailed discussion of such games see e.g., Fudenberg
and Tirole [8]. Because preferences in such multi-stage games typically are assumed to be time-separable and all actions in a given stage are taken at the same time, it is possible to allow players actions spaces that have dimension one or higher. In chapter 5, we discuss a two period game with such properties.

A particularly important type of multi-stage games are those known as repeated games. In such games, players play the game a repeated number of times. The future is here usually discounted. For infinitely repeated games, cooperation is an important possibility that is not present in finitely repeated games. An important result that has been known since the late 50s, but whose authorship is obscure is the so called Folk Theorem. This theorem asserts that the equilibrium outcomes in an infinitely repeated game coincide with the feasible and strongly individually rational outcomes in a one-shot games. This is so even if equilibria are required to be perfect. Repeated games here act as a kind of enforcement mechanism; agreements are enforced by punishing deviators in subsequent stages.

The equilibrium of the repeated game that typically is used to prove the Folk theorem rely on threats that would be suboptimal if carried out. However, in 1976 Aumann and Shapley [3] proved that if players evaluate sequences of games with the time average criterion, then the set of subgame perfect equilibrium outcomes also
coincide with the set of feasible and strongly individually rational outcomes in a one shot game. Rubinstein [19] established a similar result for games with an overtaking criterion.

For multi-stage games with observable actions and incomplete information, Fudenberg and Trole [9] introduced a weaker refinement concept than the sequential equilibrium called a Bayesian perfect equilibrium. A perfect Bayesian equilibrium requires (i) that when initial beliefs are that types are independent, posterior beliefs be that types are independent; (ii) that any two players have the same beliefs about the type of a third; and (iii) that if player $i$ deviates and player $j$ does not, then beliefs about player $j$ are updated in accordance with Bayes' rule.

2.5 GENERIC RESULTS IN ECONOMICS

Generic properties have been studied extensively in the economics literature in the past. We here discuss a couple of these. Debreu [6] used Sard's Theorem (see e.g., Milnor [15]) in establishing generic finiteness of price equilibria in a context where a set of differentiable demand functions are given and the initial allocations are allowed to vary. Smale [22] made use of transverse regularity (see e.g., Abraham and Robin [1]) to establish a generic finiteness property of price equilibria when the space of economies is taken to be the set of utility functions and initial allocations. In the context of game theory, Harsanyi [10] established the generic oddness of the number of Nash equilibria in the matrix games. Dubey [7] also made use of transverse regularity
in establishing the generic finiteness and inefficiency of Nash equilibria. This was done by allowing utility functions to vary. The techniques used in deriving several of the results in the current thesis in fact closely mirror the techniques used by Dubey. Kreps and Wilson [13] contains a generic result stating that the number of probability distributions on outcomes associated with sequential equilibria is finite.

2.6 INFORMATION-REVELATION IN ECONOMICS

Several papers in economics have dealt with information-revelation. We here mention a couple of these.

Radner's [17] paper on the rational expectation equilibrium is one of these. In the rational expectations equilibrium setup, a forecast function is a mapping from states of the world into price vectors. In any equilibrium, every price vector in the image of the forecast function must satisfy the property that total sophisticated excess demand is zero for every state in the inverse image of that price vector. That is, when all of the consumers are using their private information and the information conveyed by the price vector in order to decide how much they wish to consume of the various commodities at a price, demand turns out to equal supply in every state of the world that maps into that price vector. If an equilibrium forecast function is one-to-one, then all consumers can infer the realized state of the world from the price vector. In this case, the equilibrium is referred to as a revealing rational expectations equilibrium.
The Rothschild-Stiglitz [18] model of competitive insurance markets has a set of competitive insurance firms simultaneously deciding what insurance contracts to offer. Each insurance firm offers only one contract. This leads to the realization of some particular sets of contracts. Each customer, who may be of several different types, then decides which of the offered contracts to accept. An equilibrium in a competitive insurance market is a set of contracts such that, when customers choose contracts to maximize expected utility, (i) no contract in the equilibrium set makes negative expected profit; and (ii) there is no contract outside the equilibrium set that, if offered, will make a nonnegative profit. Such an equilibrium is said to be separating if each type of consumer chooses a different type of contract.
CHAPTER 3

BAYESIAN GAMES

In this chapter, we examine the extent to which all interior Bayesian equilibria are fully information-revealing as well as the extent to which the number of interior Bayesian equilibria is finite. One of our theorems states that, given a fixed common prior, the two properties hold generically in the space of twice continuously differentiable utility function bundles. Another theorem states that for a generic choice of a sufficiently smooth utility function bundle, the two properties hold generically in the space of common priors.

3.1 INTRODUCTION

At the outset of a Bayesian game, the types of the players are drawn from a known type profile space using an objective joint probability distribution. Each player then learns his own type but is typically left uninformed about the types of the other players. When a player is to make his decision as to what action to take, he thus possesses private information. In a Bayesian game, the strategy of a player can be thought of as a mapping from his type space to his action space. Each type of the player thus gets assigned an action. If, in a Bayesian equilibrium, all types of a
player gets assigned different actions and the actions are observable ex post, then the player's type can be inferred from the actions that he undertakes. When this occurs, we will say that his type is fully revealed or that all of his types are separated.

If all actions are observable ex post and the equilibrium strategies of all the players are one-to-one, then the realized type profile can be inferred from an action combination by looking at its inverse image under the equilibrium strategy combination. If the equilibrium strategy combination is one-to-one, we will say that the Bayesian equilibrium is fully information-revealing or separating. The idea of letting strategy combinations and action combinations convey information goes back to, among others, the rational expectations equilibrium (see e.g., Radner [17]) and the Rothschild-Stiglitz [18] formulation of competitive insurance markets.

In the rational expectations equilibrium setup, a forecast function is a mapping from states of the world into price vectors. In any equilibrium, every price vector in the image of the forecast function must satisfy the property that total sophisticated excess demand is zero for every state in the inverse image of that price vector. That is, when all of the consumers are using their private information and the information conveyed by the price vector in order to decide how much they wish to consume of the various commodities at a price, demand turns out to equal supply in every state of the world that maps into that price vector. If an equilibrium forecast function is
one-to-one, then all consumers can infer the realized state of the world from the price vector. In this case, the equilibrium is referred to as a revealing rational expectations equilibrium.

The Rothschild-Stiglitz [18] model of competitive insurance markets has a set of competitive insurance firms simultaneously deciding what insurance contracts to offer. Each insurance firm offers only one contract. This leads to the realization of some particular sets of contracts. Each customer, who may be of several different types, then decides which of the offered contracts to accept. An equilibrium in a competitive insurance market is a set of contracts such that, when customers choose contracts to maximize expected utility, (i) no contract in the equilibrium set makes negative expected profit; and (ii) there is no contract outside the equilibrium set that, if offered, will make a nonnegative profit. Such an equilibrium is said to be separating if each type of consumer chooses a different type of contract.

This chapter examines the extent to which all interior Bayesian equilibria are fully information-revealing; as well as the extent to which the number of interior Bayesian equilibria is finite. It establishes that both properties hold generically in smooth one-shot Bayesian games. We do so by providing four theorems. Theorem 3.2.2 assumes the following framework: Fix the type profile space, the action spaces, and an objective joint probability distribution over type profiles and identify a game with a type-dependent von Neumann-Morgenstern utility function for each of the players.
The space of games is thus the space of type-dependent von Neumann-Morgenstern utility function bundles. Theorem 3.2.2 states that there is an open-dense set in the space of games so that for any game from that set, the set of interior Bayesian equilibria is finite and all fully information-revealing. The finiteness property that is included in the theorem is not all that surprising since it has been shown by Dubey [7] that finiteness and inefficiency are generic properties of smooth complete information games. The main contribution of Theorem 3.2.2 thus lies in the establishment of the full information-revelation property.

We then consider a framework in which we keep the type profile space and the action spaces fixed but allow the objective joint probability distribution to vary. Theorem 3.2.3 relates to a generic property of the space of type-dependent von Neumann-Morgenstern utility function bundles for a given order of imposed smoothness. Here a von Neumann-Morgenstern utility function for player \( i \) is of order \( r \) if, conditional upon every possible realization of a type profile, his utility function is \( r \) times differentiable in the actions of all the players. A von Neumann-Morgenstern utility function bundle is of order \( r \) if for every player, the von Neumann-Morgenstern utility function is of order \( r \). The Theorem states that there is an open-dense set in the space of type-dependent von Neumann-Morgenstern utility function bundles such that any type-dependent von Neumann-Morgenstern utility function bundle from that set satisfies a particular property. The property is that there is an associated open-dense
set of objective joint probability distributions where the types of players with action spaces of sufficiently large dimensions are fully revealed in any interior Bayesian equilibrium. The Theorem also demonstrates that the required dimension of players' action spaces is decreasing in the order of smoothness and increasing in the dimension of the type profile space.

Theorem 3.2.4 works with the same framework as Theorem 3.2.3 but requires the order of smoothness to be sufficiently large. It states that once the order of smoothness is greater than the dimension of the type space, there is an open-dense set in the space of type-dependent von Neumann-Morgenstern utility function bundles such that any utility function bundle from that set satisfies a particularly nice property. The property is that there is an associated open-dense set of objective joint probability distributions where the set of interior Bayesian equilibria are finite and all fully information-revealing. Thus, once the order of smoothness is sufficiently large, there is not only an open-dense set of type-dependent von Neumann-Morgenstern utility function bundles that yield full information-revelation for a given objective joint probability distribution, but there is also an open-dense set of type-dependent von Neumann-Morgenstern utility function bundles that yield full information-revelation at open-dense sets of objective joint probability distributions.

Theorem 3.2.5 fixes the type profile space and the action spaces and identifies a game with a pair of a type-dependent utility function bundle and an objective joint
probability distribution. The theorem states that there is an open-dense set in the space of such games such that for any game from that open-dense set, the set of interior Bayesian equilibria are finite and all fully information revealing.

Our proofs make use of techniques previously applied to various parts of the economics literature. Debreu [6] used Sard’s Theorem (see e.g., Milnor [15]) in establishing generic finiteness of price equilibria in a context where a set of differentiable demand functions are given and the initial allocations are allowed to vary. Smale [22] and Dubey [7] both made use of transverse regularity (see e.g., Abraham and Robbin [1]). Smale’s paper, like Debreu’s, establishes a generic finiteness property of price equilibria but he allows both utility functions and initial allocations to vary. Dubey’s result establishes generic finiteness and inefficiency of Nash equilibria by allowing utility functions to vary. In Section 2, we present our model and the main results. Section 3 provides a more detailed account of the rational expectations equilibrium as well as an example. This serves the purpose of clarifying how information is processed in our setup and to illustrate the significance of the results. Section 4, finally, contains our proofs.

3.2 THE MODEL AND RESULTS

Let \( N = \{1, \ldots, n\} \), \( n \geq 2 \), be the set of players and \( T^i = \{ t^i_1, \ldots, t^i_{m_i}\} \), \( m_i \geq 1 \), the set of types for player \( i \). We write \( T = \times_{i=1}^n T^i \) for the space of possible type profiles and \( T^{N\setminus i} = \times_{j \neq i} T^j \). Let \( Y^i \subset \mathbb{R}^{l_i} \), \( l_i \geq 1 \), be a compact action space with
non-empty interior. We let \( Y^N \setminus \{i\} = \times_{j \neq i} Y^j \) and write the strategy set for player \( i \) as \( X^i := \{ x^i | x^i : T^i \to Y^i \} \). A strategy for player \( i \) thus specifies an action from \( Y^i \) for each of his types. We also write \( X = \times_{i=1}^n X^i \) and \( X^N \setminus \{i\} = \times_{j \neq i} X^j \).

Now, we write \( m = m_1 m_2 \cdots m_n \) for the number of type profiles and define the set of objective joint probability distributions over the type profiles by \( \Pi = \Delta^{m-1} \) with generic element \( p = (p(t))_{t \in \mathcal{T}} \). We let \( \overset{0}{\Pi} = \text{interior of } \Pi \). For generic element of \( \Pi \), we define the marginal probability of player \( i \)'s type being \( t^i \) by \( p^i(t^i) = \sum_{t' \in \mathcal{T} : t' \sim t^i} p(t') \).

We define the set of all objective joint probability distributions that assign a strictly positive probability to each type of every player by

\[
\overset{1}{\Pi} := \{ p \in \Pi : \forall i \in N : \forall t^i \in T^i : p^i(t^i) > 0 \}
\]

For every \( p \) in \( \overset{1}{\Pi} \), we can then uniquely define the conditional probability of a particular type profile given player \( i \)'s type by

\[
p(t \mid t^i) = \begin{cases} \frac{p(t)}{p^i(t^i)} & \text{if } t^i = t^i \\ 0 & \text{otherwise} \end{cases}
\]

Now, every player \( i \) is assumed to have a type-dependent von Neumann-Morgenstern utility function \( u^i : Y \times T \to \mathbb{R} \). We will restrict our attention to utility functions that satisfy certain differentiability assumptions. In particular, we define

\[
U^i = \{ u^i : Y \times T \to \mathbb{R} | u^i(\cdot, t) \text{ is differentiable } r \text{ times in } Y \text{ given each } t \in T \}
\]

We endow this linear space with the following norm

\[
\| u^i \| = \sup \{ \| u^i(y, t) \|, \| Du^i(y, t) \|, \ldots, \| D^r u^i(y, t) \| : (y, t) \in Y \times T \}
\]

\[
2i
\]
Write $U = \times_{i=1}^{n} U_i$ for the space of type-dependent von Neumann-Morgenstern utility function bundles that satisfy the differentiability assumption. Note that a member of $U$ completely specifies a utility function bundle for the players. It is now natural to define the conditional expected utility function of player $i$, $Eu^i(\cdot \mid \bar{t}) : Y^i \times X^{N\setminus\{i\}} \to \mathbb{R}$ by

$$Eu^i(y^i, x^{N\setminus\{i\}} \mid \bar{t}) = \sum_{t \in T} p(t \mid \bar{t}) u^i(y^i, x^{N\setminus\{i\}}(t^{N\setminus\{i\}}), t)$$

Furthermore, we define a vector of first order derivatives of the conditional expected utility function $D_{y^i} Eu^i(\cdot \mid \bar{t}) : Y^i \times X^{N\setminus\{i\}} \to \mathbb{R}^{l_i}$ by

$$D_{y^i} Eu^i(y^i, x^{N\setminus\{i\}} \mid \bar{t}) = \left( \sum_{t \in T} p(t \mid \bar{t}) \frac{\partial u^i(y^i, x^{N\setminus\{i\}}(t^{N\setminus\{i\}}), t)}{\partial y_j} \right)_{j \in \{1, \ldots, l_i\}}$$

We will now discuss a result on generic finiteness and inefficiency of Nash equilibria due to Dubey [7]. His result applies to games with complete information where there is only one type of each player. In particular, he works with the special case where strategy spaces of players are simplexes of dimension greater than or equal to one. As is noted in one of his remarks, however, the result can be interpreted more broadly. The significance of his result lies in the establishment that almost every smooth complete information game satisfies the properties that (i) every interior equilibrium is inefficient in the Pareto sense; (ii) no interior equilibrium is a strong equilibrium; and that (iii) the set of equilibria are finite. An implication of these properties is that
for any interior equilibrium, there is an alternative set of actions by the players with the property that if the players could agree to those actions, every player would be weakly better off and at least one player better off.

Using our notation, we follow Dubey and make the following definitions. Let $S \subset N, e = \{e^i : i \in S\} \in \times_{i \in S} X^i$, and let $(x | e)$ denote the element of $X$ obtained from $x$ by replacing $x^i$ by $e^i$ for each $i \in S$. We denote the only type profile in the game with complete information by $t$. A point $x \in X$ is then called

(1) $S$-efficient if there does not exist any point $e \in \times_{i \in S} X^i$ such that

$$Eu^i((x | e)(t)) \geq Eu^i(x(t)) \text{ for all } i \in S$$

$$Eu^j((x | e)(t)) > Eu^j(x(t)) \text{ for some } j \in S$$

(2) A Nash equilibrium (N.E) if it is $S$-efficient for all $S$ consisting of one element.

(3) Efficient if it is $S$-efficient for $S = N$.

(4) A strong Nash equilibrium if it is $S$-efficient for all subsets $S \subset N$.

Using this notation, we state Dubey’s theorem.

**Theorem 3.2.1. (DUBEY)** There is an open-dense set $U_0$ of $U$ such that, for $u \in U_0$:

(i) The set of Nash equilibria is finite.

(ii) if $x$ is an efficient Nash equilibrium, then at least one $x^i$ is a vertex.

(iii) if $x$ is a strong Nash equilibrium, then at most one $x^i$ is not a vertex.

**Remark 3.2.1.** (ii) implies that for any $u$ in $U_0$, every interior Nash equilibrium is inefficient.

The notion of a Bayesian game was first introduced by Harsanyi [10]. In his work, he demonstrated that any game of incomplete information can be reinterpreted as a
game where the only incomplete information relates to incomplete information about players utility functions. Thus, there is no need to consider cases where action spaces might be different for different types of a player. He also demonstrated that for any game of incomplete information, there is a Bayes equivalent complete information game. In Bayesian games, there is an initial objective joint probability distribution over the type profile space. After a particular type profile has been selected, play proceeds with each player knowing his own type but not, in general, the types of the other players. We will restrict our attention to Bayesian games as defined below.

**Definition 3.2.1.** A Bayesian game is a list of specified data $\Gamma := \{N, X, T, u, p\}$ where $N$ is the set of players, $X$ the strategy sets, $T$ the type profile space, $u$ a type-dependent von Neumann-Morgenstern utility function bundle, and $p$ an objective joint probability distribution over the type profile space.

Our equilibrium notion for the Bayesian game will be that each player should maximize his unconditional utility expectation given the strategies of the other players. As demonstrated in Harsanyi's Theorem 1, it is a necessary and sufficient condition that if a player is to maximize his unconditional utility expectation, then he must also be maximizing his conditional utility expectation for every type that occurs with positive probability. This leads us to the following definition of a Bayesian equilibrium.

**Definition 3.2.2.** For any given Bayesian game $\Gamma$ a Bayesian equilibrium is a strategy bundle $x^*$ in $X$ that satisfies

$$\forall i \in N : \forall t^i \in T^i : \forall y^i \in Y^i : E u^i(x'^i(t^i), x^{*N \setminus \{i\}} | t^i) \geq E u^i(y^i, x^{*N \setminus \{i\}} | t^i)$$

The results established in the rest of this chapter will refer to generic properties of Bayesian equilibria in the interior of $X$. The restriction to interior equilibria is one
that has been used previously in the literature for technical reasons. For instance, Smale [22] used this restriction to establish generic properties of competitive equilibria and Dubey [7] used it to establish generic properties for complete information games. We thus make the following definition.

**Definition 3.2.3.** An interior Bayesian equilibrium is a Bayesian equilibrium in the interior of $X$.

When utility functions are differentiable, as will be assumed throughout this thesis, a necessary condition for any interior Bayesian equilibrium is that for every type of every player, the vector of first order derivatives of the expected utility function with respect to that player's own actions must vanish. While this is a necessary condition for an interior Bayesian equilibrium, it is of course not a sufficient one. Still, for technical reasons that will become clear, it will be useful to introduce the following solution concept that is slightly more general than the interior Bayesian equilibrium.

**Definition 3.2.4.** For any given Bayesian game $\Gamma$, a strategy bundle $x^*$ in $X$ is an extended Bayesian equilibrium if:

$$\forall i \in N : \forall t^i \in T^i : D_{y^i} E_{y^i} (x^* (t^i), x^{*N \setminus \{i\}} | t^i) = 0$$

Since the underlying framework with which we are working are Bayesian games, we ask to what extent an equilibrium reveals the underlying type profile. In fully revealing rational expectations equilibria, consumers can infer the underlying state of the world from the equilibrium price vectors. This is possible simply because no two states of the world map into the same equilibrium price vector. In Bayesian games,
full information-revelation means that no two types of any player undertakes the same action in equilibrium and hence every player's type can be inferred by observing the undertaken actions. To make it more precise, full information-revelation in a Bayesian game requires that the equilibrium strategies of all players are one-to-one. It is this one-to-one property of the Bayesian equilibrium strategies that makes it possible for the type profile to be inferred from the undertaken actions by looking at the inverse image of that action combination under the equilibrium strategy combination. This leads us to the following definition.

**Definition 3.2.5.** A Bayesian equilibrium \( x^* \) in \( X \) is **fully information-revealing** if for every player \( i \), \( x^{*i} \) is one-to-one.

Our first result establishes a generic property for a given objective joint probability distribution. In particular, it states that if we fix the set of players, the action sets for the players, the type space, and an objective joint probability distribution over type profiles that assigns a positive probability to every type of every player, then almost every smooth type-dependent von Neumann-Morgenstern utility function bundle yields a finite number of interior Bayesian equilibria where every interior Bayesian equilibrium is fully information-revealing. Thus almost every smooth type-dependent von Neumann-Morgenstern utility function bundle will reveal the players' types in every interior Bayesian equilibrium.

**Theorem 3.2.2.** Fix \( N, X, T \) and \( p \) in \( \bar{\Pi} \). Let \( r \geq 2 \). Then there is an open-dense set \( U'' \) in \( U \) such that for every \( u \) in \( U'' \):

(i) The set of interior Bayesian equilibria is a finite set.

(ii) All interior Bayesian equilibria are fully information-revealing.

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Theorem 3.2.2 demonstrates that for a given probability we can find an open-dense set of type-dependent von Neumann-Morgenstern utility function bundles that yield the desired properties of finiteness and full information-revelation. The question we now turn to is if for a given type-dependent von Neumann-Morgenstern utility function bundle, there is an associated open-dense set of objective joint probability distributions where the same properties are true. While the properties do not hold true for every type-dependent von Neumann-Morgenstern utility function bundle, Theorems 3.2.3 and 3.2.4 demonstrate generic properties of type-dependent utility function bundles.

Theorem 3.2.3 demonstrates that for a given order of smoothness of the type-dependent von Neumann-Morgenstern utility function bundles, we can find an open-dense set in the space of such utility function bundles so that for any utility function bundle from that set an information-revelation property in fact does hold true. The information revelation property is that for any utility function bundle from that open-dense set, there is an associated open-dense set of probabilities in Π, such that the equilibrium strategies for all players with sufficiently large dimensional action spaces are one-to-one in any interior Bayesian equilibrium. The result also demonstrates how the set of players for which the property is satisfied increases with the differentiability r of the utility functions while it decreases in the dimension of the type profile space m.

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Theorem 3.2.3. Fix \( N, T, X \). Let \( r \geq 2 \). Then there is an open-dense set \( U^* \) in \( U \) such that for every \( u \) in \( U^* \), there is an associated open-dense set \( \Pi^*_u \) in \( \Pi \) such that for every \( p \) in \( \Pi^*_u \), every interior Bayesian equilibrium \( x^* \) associated with \( (u, p) \), satisfies the property that the equilibrium strategy for every player \( i \) with \( l_i > m - r \) is one-to-one.

An implication of Theorem 3.2.3 is that once the utility functions are sufficiently smooth, the property in fact holds true for all the players. In particular, this holds true once \( r \geq m \). Our next result demonstrates that once \( r > m \), an even stronger result holds true. That result is that there is an open-dense set in \( U \) such that for every \( u \) in that open-dense set, there is an associated open-dense set in \( \Pi \) such that for any \( p \) from that open-dense set, the set of interior Bayesian equilibria is finite and all of them are fully information-revealing. More formally stated:

Theorem 3.2.4. Fix \( N, T, X \). Let \( r > m \). Then there is an open-dense set \( U^{**} \) in \( U \) such that for every \( u \) in \( U^{**} \), there is an associated open-dense set \( \Pi^{**}_u \) in \( \Pi \) such that for any \( p \) in \( \Pi^{**}_u \):

(i) The set of interior Bayesian equilibria associated with \((u, p)\) is a finite set.
(ii) All interior Bayesian equilibria associated with \((u, p)\) are fully information-revealing.

If the space of games is taken to be the set of all pairs \((u, p)\) of a type-dependent utility function bundle and an objective joint probability distribution, then the following Theorem can also be established:

Theorem 3.2.5. Fix \( N, X, T \). Let \( r \geq 2 \). Then there is an open-dense set \( Q \) in \( U \times \Pi \) such that for every \((u, p)\) in \( Q \):

(i) The set of interior Bayesian equilibria is a finite set.
(ii) All interior Bayesian equilibria are fully information-revealing.
This demonstrates that there is thus also an open-dense set in the cross product of these spaces that satisfy the same nice properties of full information-revelation and finiteness.

**Remark 3.2.2.** Assume the $Y^i$'s are all convex and consider the set $U_c$ of $(u^1, u^2, \ldots, u^n)$ for which (a) each $u^i : Y \times T$ is strictly concave in $Y^i$, and (b) for every player $i$, every type $t^i$ in $T^i$, every $x^{N \setminus \{i\}} \in X^{N \setminus \{i\}}$, there exists $y^i \in Y^i$ such that for all $\tilde{y}^i \in \partial Y^i$, $Eu^i(y^i, x^{N \setminus \{i\}} \mid t^i) > Eu^i(\tilde{y}^i, x^{N \setminus \{i\}} \mid t^i)$. Then $U_c$ is open in $U$. It is a well known property that for strictly concave functions, the set of Bayesian equilibria is non-empty, while (b) guarantees that any Bayesian equilibrium is in the interior. This shows that our result is non-vacuous.

**Remark 3.2.3.** In Dubey's [7] paper, action sets are assumed to be simplexes. If this assumption had been made here, we could have shown that there is an open-dense set of utility function bundles for which the set of Bayesian equilibria are finite and where two types take the same action in some equilibrium only if that action is a vertex.

**Remark 3.2.4.** Dubey [7] also dealt with matrix games. It is straightforward to demonstrate using virtually an identical argument to that put forth in the present thesis, that it is a generic property of Bayesian Matrix games that the set of Bayesian equilibria are finite and that any two types of a player use the same mixed action in a Bayesian equilibrium only if that action is a pure action.

### 3.3 EXAMPLES

As mentioned in the introduction, the idea of letting strategy combinations and action combinations convey information goes back to, among others, the rational expectations equilibrium and the Rothschild-Stiglitz formulation of insurance markets.

We will here give a more detailed account of the rational expectations equilibrium and provide an example of how information is conveyed in the present model.

Our account of the rational expectations equilibrium will most closely resemble that of Radner [17]. In his paper, trader $i$ receives exogenous information signal $s_i$. 

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about the payoff relevant environment $e$. The total exogenous information available in the market is denoted by $s = (s_1, \ldots, s_t)$. Each trader $i$ has a subjective joint probability distribution of $s$ and $e$. A forecast function $\phi$ is a mapping that associates with each total exogenous information signal $s$ a price vector $\phi(s)$. For a given forecast function, each trader can infer something about the total information signal from the realized price vector. This means that the (augmented) information available to trader $i$ is given by $[s_i, \phi(s)]$. Each trader is assumed to choose among portfolios according to his conditional expected utility given the available (augmented) information.

For each total signal, this generates an excess demand for each trader as well as a total excess demand, which Radner refers to as the sophisticated excess demand. This sophisticated excess demand thus depends on the forecast function and the particular total signal $s$. A rational expectations equilibrium is then defined as a forecast function such that the corresponding sophisticated excess demand is zero for every total information signal. When the forecast function is one-to-one, a rational expectations equilibrium is said to be revealing.

Our present thesis, in the same manner as the rational expectations equilibrium, makes use of the one-to-one property of a function in order to establish the full information revelation property. In the present setup, however, the self-fulfilling forecast function, known to traders in equilibrium, is replaced by the equilibrium strategies of the players.
3.4 PROOFS

3.4.1 THEOREM 3.2.2

Before proceeding with the proof, we will provide an outline as to how the proof will proceed.

(i) Construct a compact set \( \tilde{X} \) and two open sets \( W, \tilde{W} \) such that \( X \subset \tilde{W} \subset \tilde{X} \subset W \).

(ii) Construct an open set \( \tilde{U} \) such that \( U \subset \tilde{U} \).

(iii) Letting \( q = \sum_{i=1}^{n} l_im_i \), construct a mapping \( \Psi : \tilde{U} \to C^{n-1}(W, \mathbb{R}^q) \) where \( u \to \Psi_u \) and \( \Psi_u : W \to \mathbb{R}^q \) is defined by

\[
\Psi_u(x) := (D_{x^i} E u^i(x^i(t^i), x^{N\setminus\{i\}} | t^i))_{t \in N, t_i \in T^i}
\]

(iv) Define \( \Delta_u \subset \mathbb{R}^q \) to be the zero vector and note that when \( u \in U, x \in X \), and \( \Psi_u(x) \in \Delta_u \), then \( x \) is an extended Bayesian equilibrium.

(v) Letting \( \tilde{U}_{\Delta_u} = \{ u \in \tilde{U} \mid \Psi_u \tilde{p} \in \Delta_u \} \), show using the Transversal Density Theorem of Abraham and Robbin \cite{1} that \( \tilde{U}_{\Delta_u} \) is dense in \( \tilde{U} \).

(vi) Letting \( \tilde{U}_{\tilde{X} \Delta_u} = \{ u \in \tilde{U} \mid \Psi_u \tilde{p} \in \Delta_u, \forall x \in \tilde{X} \} \), show using Abraham and Robbin's \cite{1} Theorem on the Openness of the Transversal Intersection and step (v) that \( \tilde{U}_{\tilde{X} \Delta_u} \) is open-dense in \( \tilde{U} \).

(vii) For every player \( i \) and any two distinct types \( t_j^i, t_k^i \) of player \( i \), construct a mapping \( \Phi'_{t_j^i, t_k^i} : \tilde{U} \to C^{n-1}(W, \mathbb{R}^{q+M_i}) \), where \( u \to \Phi'_{t_j^i, t_k^i}(u) \) and \( \Phi'_{t_j^i, t_k^i}(u) : W \to \mathbb{R}^{q+M_i} \).
\( \mathbb{R}^{q+2n} \) is defined by

\[
\Phi_{(t^i_j, e^i_k), u}(x) = (\Psi_u(x), x^i(t^i_j), x^i(e^i_k))
\]

(viii) Define

\[
\Lambda^i_{(t^i_j, e^i_k)} := \{(0, x^i(t^i_j), x^i(e^i_k)) \in \mathbb{R}^{q+2n} \mid x^i(t^i_j) = x^i(e^i_k)\}
\]

and note that when \( u \in U, x \in X \), and \( \Phi_{(t^i_j, e^i_k), u}(x) \in \Lambda^i_{(t^i_j, e^i_k)} \), then \( x \) is an extended Bayesian equilibrium where types \( t^i_j, e^i_k \) take the same actions.

(ix) Using the Transversal Density theorem as applied to \( W \), show that there is a dense set in \( \tilde{U} \) such that for every \( u \) in that dense set \( \Phi_{(t^i_j, e^i_k), u} \) is transversal to \( \Lambda^i_{(t^i_j, e^i_k)} \) for every \( x \) in \( W \).

(x) Using the Theorem on the openness of the Transversal Intersection together with the result in (vii), show that there is an open-dense set \( \tilde{U}_{\Lambda^i_{(t^i_j, e^i_k)}} \) in \( \tilde{U} \) such that for every \( u \) in \( \tilde{U}_{\Lambda^i_{(t^i_j, e^i_k)}} \), \( \Phi_{(t^i_j, e^i_k), u} \) is transversal to \( \Lambda^i_{(t^i_j, e^i_k)} \) for every \( x \) in \( \tilde{X} \).

(xi) Letting \( \tilde{Q} = \tilde{U}_{\tilde{X}_{\tilde{A}_u}} \cap (\bigcap_{i=1}^n \bigcap_{j=1}^{m_i-1} \bigcap_{k=j+1}^{m_i} \tilde{U}_{\Lambda^i_{(t^i_j, e^i_k)}}) \), show that \( \tilde{Q} \) is open-dense in \( \tilde{U} \).

(xii) Letting \( Q = \{u \in U \mid \exists \tilde{u} \in \tilde{Q} : \tilde{u} \mid x = u\} \), show that \( Q \) is an open-dense set in \( U \).

(xiii) Show that for any \( u \) in \( Q \), every interior Bayesian equilibrium is fully information-revealing.

(xiv) Show that for any \( u \) in \( Q \), the set of interior Bayesian equilibria is finite.
(xv) Conclude from (xii), (xiii), and (xiv) that \( Q \) can be taken as the required open-dense set in Theorem 3.2.2.

We can now proceed with the proof.

**Proof of Theorem 3.2.2.**

We define the sets in (i) using the following procedure. Let \( V^i \) be any open set in \( \mathbb{R}^i \) containing \( Y^i \), \( V = \times_{i=1}^n V^i \), and \( V^{N\setminus(t)} = \times_{j\neq i} V^j \). Let \( \hat{W}^i = \{ w^i \mid w^i : T^i \to V^i \} \). We define \( W = \times_{i=1}^n \hat{W}^i \), and \( W^{N\setminus(t)} = \times_{j\neq i} \hat{W}^j \). Let \( \hat{V}^i \) be an open set and \( \hat{Y}^i \) a compact set such that the following inclusions hold: \( Y^i \subset \hat{V}^i \subset \hat{Y}^i \subset V^i \). Let \( \hat{W}^i = \{ \hat{w}^i \mid \hat{w}^i : T^i \to \hat{V}^i \} \) and \( \hat{X}^i = \{ \hat{x}^i \mid \hat{x}^i : T^i \to \hat{X}^i \} \). We also write \( \hat{W} = \times_{i=1}^n \hat{W}^i \), \( \hat{W}^{N\setminus(t)} = \times_{j\neq i} \hat{W}^j \), \( \hat{X} = \times_{i=1}^n \hat{X}^i \), and \( \hat{X}^{N\setminus(t)} = \times_{j\neq i} \hat{X}^j \). From this, it is clear that \( X \subset \hat{W} \subset \hat{X} \subset W \) and that \( \hat{X} \) is compact, while \( W \) and \( \hat{W} \) are open.

We define the set \( \hat{U} \) in (ii) by letting

\[
\hat{U}^i = \{ \hat{v}^i : V \times T \to \mathbb{R} \mid \hat{v}^i(\cdot, t) \text{ is differentiable } r \text{ times in } V \text{ given each } t \in T \}
\]

We endow this linear space with the following norm

\[
\| \hat{a}^i \| = \sup \{ \| \hat{a}^i(y, t) \|, \| D\hat{a}^i(y, t) \|, \ldots, \| D^r\hat{a}^i(y, t) \| : (y, t) \in V \times T \}
\]

Write \( \hat{U} = \times_{i=1}^n \hat{U}^i \) for the space of utility profiles that satisfy the differentiability assumption.

Steps (iii) and (iv) are clear. To do step (v), note that \( \hat{U}, W, \mathbb{R}^q \) are \( C^{r-1} \) manifolds, \( \Delta_n \) a closed submanifold, \( \text{dim}(W) = \text{codimension } \Delta_n = q \). \( \hat{U} \) and \( W \) are second
countable, \( \hat{X} \) compact and \( \Psi : \hat{U} \rightarrow C^{r-1}(W, \mathbb{R}^q) \) a \( C^{r-1} \) representation. To apply the Transversal Density theorem, the only thing left to show is that the evaluation map \( ev_{\Psi} : \hat{U} \times W \rightarrow \mathbb{R}^q \) is transverse regular to \( \Delta_u \) (i.e., \( ev_{\Psi}^{-1} \Delta_u \)). In demonstrating this, we closely follow Dubey's [7] argument. Choose for any \((u, x)\) in \( \hat{U} \times W \) and \( y = \Psi_u(x) \). If \( y \neq 0 \) we are done. If \( y = 0 \), we need to show that the image of the tangent space to \( \hat{U} \times W \) at \((u, x)\) given by the mapping of equivalence classes of tangent curves in \( \hat{U} \times W \) at \((u, x)\) to equivalence classes of tangent curves in \( \mathbb{R}^q \) at 0 contains a closed complement to \( T_y \Delta_u \) in \( T_y \mathbb{R}^q \) (i.e., \( (T_{(u, x)}ev_{\Psi})(T_{(u, x)}(\hat{U} \times W)) \)) contains a closed complement to \( T_y \Delta_u \) in \( T_y \mathbb{R}^q \). We will show that \( T_{(u, x)}ev_{\Psi} : T_{(u, x)}(\hat{U} \times W) \rightarrow T_y \mathbb{R}^q \) is in fact onto and hence contains the required closed complement to \( T_y \Delta_u \). Choose any \( z \in T_y \mathbb{R}^q \). Write \( z = (z_{it}) \) and \( z_{it}^T \) for the transpose of \( z_{it} \). Define

\[
\hat{u}^{ij}(x^j(t), x^{N \setminus \{i\}}(t^{N \setminus \{i\}}), t) = u^{ij}(x^j(t), x^{N \setminus \{i\}}(t^{N \setminus \{i\}}), t) + rx^j(t)z_{it}^T
\]

Also, define

\[
E\hat{u}^{ij}(x^j(t), x^{N \setminus \{i\}} | t) = \sum_{t \in \bar{t}} p(t | \bar{t})\hat{u}^{ij}(x^j(t), x^{N \setminus \{i\}}(t^{N \setminus \{i\}}), t)
\]

\[
= rx^j(t)z_{it}^T + \sum_{t \in \bar{t}} p(t | \bar{t})u^{ij}(x^j(t), x^{N \setminus \{i\}}(t^{N \setminus \{i\}}), t)
\]

Now, because \( \hat{U} \) is an open set, we can choose a sufficiently small interval \( (r_x^1, r_x^2) \) around 0 in \( \mathbb{R} \) so that \( c_x(r) \mid_{r=r_x^1} \) is a curve at \((u, x)\) in \( \hat{U} \times W \). It follows that \( ev_{\Psi}(c_x(r)) \mid_{r=r_x^1} = ev_{\Psi}(\hat{u}^T, x) \mid_{r=r_x^1} \) is a curve at 0 in \( \mathbb{R}^q \). Since

\[
D_{x^j(t)}E\hat{u}^{ij}(x^j(t), x^{N \setminus \{i\}} | \bar{t}) = rx_{it}^1 + D_{x^j(t)}Ev^{ij}(x^j(t), x^{N \setminus \{i\}} | \bar{t})
\]

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we have \( d/\sqrt{r} ( ev_\Psi (c_x(r))) |_{r=0} = z \) which demonstrates that \( T_{(u,x)} ev_\Psi : T_{(u,x)} \tilde{U} \times W \to T_{\Psi} \mathbb{R}^q \) is onto as required. Because the finite-dimensionality implies that the inverse image \((T_{(u,x)} ev_\Psi)^{-1}(T_{\tilde{u}} \Delta_u)\) splits, it follows that \( ev_\Psi \tilde{u} \Delta_u \). Given this property of the evaluation map and that fact that all the other assumptions are satisfied, we can apply the Transversal Density Theorem and thus \( \tilde{U}_{\Delta_u} = \{ u \in \tilde{U} \mid \Psi_u \tilde{u} \Delta_u \} \) is dense in \( \tilde{U} \).

To do step (vi), note that \( \tilde{U}, W, \mathbb{R}^q \) are \( C^{r-1} \) manifolds, \( \Delta_u \) a closed submanifold, \( \dim(W) = \text{codimension} \Delta_u = q \), \( \mathbb{U} \) and \( W \) are second countable, \( \tilde{X} \) compact and \( \Psi : \tilde{U} \to C^{r-1}(W, \mathbb{R}^q) \) a \( C^{r-1} \) representation. This implies that we can apply the Theorem on the Openness of the Transversal Intersection to get \( \tilde{U}_{\tilde{X} \Delta_u} = \{ u \in \tilde{U} \mid \Psi_u \tilde{u} \Delta_u \} \) \( \forall x \in \tilde{X} \} \) open. Since \( \tilde{U}_{\tilde{X} \Delta_u} \) contains \( \tilde{U}_{\Delta_u} \), we conclude that \( \tilde{U}_{\tilde{X} \Delta_u} \) is open-dense in \( \tilde{U} \).

Now, steps (vii) and (viii) are clear so we turn to step (ix). Step (ix) is carried out very much like step (v). The differences between step (vii) and step (iv) are that \( \dim(W) - \text{codimension}(\Lambda_{(t_l, t_u)}^1) = q - (q + i) = -t_i < 0 \), that we are working with \( \Phi^1_{(t_l, t_u, \tilde{x}, z_1)} \) instead of \( \Psi \) and that we now choose \( z = (z_1, z_2, z_3) \in T_{\Psi} \mathbb{R}^{q+2i} \), where \( y = \Phi^1_{(\tilde{t}_l, \tilde{t}_u, \tilde{z}_1, \tilde{z}_1)} \) and \( z_1 \) is the same as \( z \) in the earlier proof. We define \( \tilde{u}^t(x(t_l), x^{N\setminus i}(t^{N\setminus i}), t) \) and \( \tilde{E}u^{t^i}(x^t(\tilde{F}), x^{N\setminus i} | \tilde{F}) \) as before but now also define \( \tilde{u}^t := (x^1(t_l), x^1(t_2), \ldots, x^1(t_{m_1}), \ldots, x^1(t_l), r z_2, x^1(t_{l+1}), \ldots, x^1(t_k) + rz_3, x^1(t_{k+1}), \ldots, x^N(t_n)) \). Since \( \tilde{U} \) and \( W \) are open sets, we can select a sufficiently small interval around zero, \( (r_1, r_2) \) such that \( c^1_{(\tilde{t}_l, \tilde{t}_u)}(r) \big|_{r=r_1} = (\tilde{u}^t, \tilde{x}^t) \) is a curve
at \((u,x)\) in \(\tilde{U} \times \tilde{W}\). By a similar token to that in step (v), we can show that \(d/dr (ev_{\tilde{U}, x} (c^i_{(t^i_j, t^i_k)} (r))) \Big|_{r=0}= x\). By applying the same argument as in step (v), we then get the required dense set.

Step (x) now follows from step (ix) in the same way as step (vi) followed from step (v). We thus get the required open-dense set \(\tilde{U}_{\Lambda^i_{(t^i_j, t^i_k)}}\).

In step (xi), \(\tilde{Q}\) is open-dense in \(\tilde{U}\) because it is the finite intersection of open-dense sets in a Baire space. Because of a well known property noted by Dubey [7], the openness of \(Q\) in step (xii) follows from the open-denseness of \(\tilde{Q}\) in step (xi). To do step (xiii), first note that if \(\tilde{u}\) is an element of \(\tilde{Q}\), then it follows from the definition of \(\tilde{Q}\) that it is also an element of \(\tilde{U}_{\Lambda^i_{(t^i_j, t^i_k)}}\). From step (x), it then follows that \(\Phi^i_{\tilde{U}_{(t^i_j, t^i_k)}, \tilde{u}}\) is transversal to \(\Lambda^i_{(t^i_j, t^i_k)}\) for every \(x\) in \(\tilde{X}\). In particular, \(\Phi^i_{\tilde{U}_{(t^i_j, t^i_k)}, \tilde{u}}\) is transversal to \(\Lambda^i_{(t^i_j, t^i_k)}\) for every \(x\) in \(\tilde{W}\). Using \(\tilde{W}\), we apply Corollary 17.2 of Abraham and Robbin [1] and get codimension \(\Phi^i_{\tilde{U}_{(t^i_j, t^i_k)}, \tilde{u}} (\Lambda^i_{(t^i_j, t^i_k)}) \cap \tilde{W} = \emptyset\). This implies dimension \(\Phi^i_{\tilde{U}_{(t^i_j, t^i_k)}, \tilde{u}} (\Lambda^i_{(t^i_j, t^i_k)}) \cap \tilde{W} = \emptyset\) which equals \(q + 2l_t - l_i = q + l_t\). This implies dimension \(\Phi^i_{\tilde{U}_{(t^i_j, t^i_k)}, \tilde{u}} (\Lambda^i_{(t^i_j, t^i_k)}) \cap \tilde{W} = \emptyset\) and that \(\Phi^i_{\tilde{U}_{(t^i_j, t^i_k)}, \tilde{u}} (\Lambda^i_{(t^i_j, t^i_k)}) \cap X = \emptyset\). Thus in turn implies that there are no interior Bayesian equilibria of the game associated with \(u\) where \(t^i_j\) and \(t^i_k\) take the same actions and thus as any interior Bayesian equilibrium associated with \(u\) must be fully information-revealing. This concludes step (xiii).

For step (xiv), note that \(\tilde{u}\) in \(\tilde{Q}\) implies that \(\tilde{u}\) is in \(\tilde{U}_{\tilde{X}_{\Delta u}}\). It then follows from step...
(vi) that $\Psi_\varphi$ is transversal to $\Delta_u$ for every $x$ in $\tilde{X}$. In particular, $\Psi_\varphi$ is transversal to $\Delta_u$ for every $x$ in $\tilde{W}$. Applying Corollary 17.2 to $\tilde{W}$, we get that codimension $\Psi_\varphi^{-1}(\Delta_u) \cap \tilde{W}$ = codimension $\Delta_u = q$ which implies that the dimension of $\Psi_\varphi^{-1}(\Delta_u) \cap \tilde{W} = q - q = 0$.

This in turn implies that the dimension of $\Psi_\varphi^{-1}(\Delta_u) \cap X = 0$. It also follows from the Corollary 17.2 that $\Psi_\varphi^{-1}(\Delta_u) \cap X$ has only finitely many connected components. This together with the zero-dimensionality of each component implies that $\Psi_\varphi^{-1}(\Delta_u) \cap X$ consists of only finitely many points which in turn implies that $u$ has only finitely many extended Bayesian equilibria and hence a finite number of interior Bayesian equilibria. This concludes step (xiv).

Given steps (xii), (xiii), and (xiv), we now note in step (xv) that any $u$ in $Q$ satisfies the required properties so $Q$ can be used as the required open-dense set.

Q.E.D.

3.4.2 THEOREM 3.2.3

Before proceeding with the proof of Theorem 3.2.3, we will provide an outline as to how the proof will proceed.

(i) Construct three open sets $\tilde{\tilde{W}}, \tilde{W}, W$ and two compact sets, $\tilde{X}, \tilde{X}$ such that $X \subset \tilde{\tilde{W}} \subset \tilde{X} \subset \tilde{W} \subset \tilde{X} \subset W$.

(ii) Construct an open set $\tilde{U}$ such that $U \subset \tilde{U}$.
(iii) Define a sequence of sets \( \{ (\Pi^{(n)}, \overline{\Pi}^{(n)}) \}_{n=1}^{\infty} \) by

\[
\Pi^{(n)} = \{ p \in \Pi \mid \text{dist}(p, \partial \Pi) > 1/(n + 1) \}
\]

\[
\overline{\Pi}^{(n)} = \{ p \in \Pi \mid \text{dist}(p, \partial \Pi) \geq 1/(n + 1) \}
\]

(iv) Define \( \tilde{D}_p E u^i(\cdot \mid \tilde{t}^i) : \tilde{\Pi} \times Y^i \times X^N \setminus \{i\} \rightarrow \mathbb{R}^i \) by

\[
\tilde{D}_p E u^i(p, y^i, x^N \setminus \{i\} (t^N \setminus \{i\}) \mid \tilde{t}^i) = \left( \sum_{t \in T} p(t \mid \tilde{t}^i) \frac{\partial u^i(y^i, x^N \setminus \{i\} (t^N \setminus \{i\}), i)}{\partial y^i_j} \right)_{j \in \{1, \ldots, i\}}
\]

(v) Letting \( q = \sum_{i=1}^n \lambda_i m_i \), define a mapping \( \beta : \tilde{U} \rightarrow C^{(r-1)}(\tilde{\Pi} \times \tilde{W}, \tilde{\mathbb{R}}^q) \) where

\( \tilde{u} \rightarrow \beta_\tilde{u} \) and \( \beta_\tilde{u} : \tilde{\Pi} \times W \rightarrow \tilde{\mathbb{R}}^q \) is defined by

\[
\beta_\tilde{u}(p, x) := (\tilde{D}_p E u^i(p, x^i(t_j^i), x^N \setminus \{i\} (t^N \setminus \{i\}) \mid \tilde{t}^i))_{i \in N; t \in T}.
\]

(vi) For every player \( i \) such that \( l_i > m - r \), and any two distinct types \( t_j^i, t_k^i \) of player \( i \), construct a mapping \( \alpha_{(t_j^i, t_k^i)}^i : \tilde{U} \rightarrow C^{r-1}(\tilde{\Pi} \times \tilde{W}, \tilde{\mathbb{R}}^{q+2l_i}) \), where \( \tilde{u} \rightarrow \alpha_{(t_j^i, t_k^i)}^i \tilde{u} \) and \( \alpha_{(t_j^i, t_k^i)}^i : \tilde{\Pi} \times \tilde{W} \rightarrow \tilde{\mathbb{R}}^{q+2l_i} \) is defined by

\[
\alpha_{(t_j^i, t_k^i)}^i(p, x) = (\beta_\tilde{u}(p, x), x^i(t_j^i), x^i(t_k^i))
\]

(vii) Define \( \Delta_u \) and \( \Lambda_{(t_j^i, t_k^i)}^i \) as in the proof of Theorem 3.2.2.

(viii) Letting \( \tilde{\Omega}_{(t_j^i, t_k^i)} := \{ \tilde{u} \in \tilde{U} \mid \alpha_{(t_j^i, t_k^i)}^i \tilde{u} \in \tilde{\Pi} \times \tilde{W} \} \), show that \( \tilde{\Omega}_{(t_j^i, t_k^i)} \) is dense in \( \tilde{U} \).
Remark 3.4.1. It is in this step of the proof that the assumption that \( l_i > m - r \) is crucial. If this assumption is not made, the inequality \( r - 1 > \max \{0, |q + (m - 1) - (q + l_i)|\} = \max \{0, m + l_i - 1\} \) fails. That this inequality holds is one of the assumptions required in order for the transversal density theorem to apply and without this, we can not show the denseness property.

(ix) Letting \( \Omega_i^{(n)} \) := \( \{ \tilde{u} \in \tilde{U} \mid \alpha_{(j_i, k_i)}(\tilde{\nu}, p, x) = \lambda_i^{(n)}(j_i, k_i) \forall (p, x) \in \Pi^{(n)} \times \tilde{X} \} \), show that \( \Omega_i^{(n)} \) is open-dense in \( \tilde{U} \).

(x) Letting \( \tilde{U}^{(n)} = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{n_i-1} \bigcap_{k=1}^{n_j-1} \Omega_i^{(n)} \), show that \( \tilde{U}^{(n)} \) is open-dense in \( \tilde{U} \).

(xi) Letting \( \tilde{U}^* = \bigcap_{p=1}^{\infty} \tilde{U}^{(n)} \), show that \( \tilde{U}^* \) is open-dense in \( \tilde{U} \).

(xii) Letting \( U^* = \{ u \in U \mid \exists \tilde{u} \in \tilde{U}^* : \tilde{u} |_X = u \} \), show that \( U^* \) is open-dense in \( U \).

(xiii) For every \( \tilde{u} \in \tilde{U}^* \), every \( n \), every player \( i \) such that \( l_i > m - r \), and any two distinct types \( t_j^i, t_k^i \) of player \( i \), construct a mapping \( \rho_i^{(n)} \) : \( \Pi^{(n)} \to C^{r-1}(\tilde{W}, \tilde{R}^{q+2l_i}) \) where \( p \to \rho_i^{(n)}(t_j^i, t_k^i, \tilde{x}, \tilde{p}) \) and \( \rho_i^{(n)}(t_j^i, t_k^i, \tilde{x}, \tilde{p}) = (\beta_{(j_i, k_i)}(p, x), x(t_j^i), x(t_k^i)) \).

(xiv) Letting \( \Pi_{\lambda_i^{(n)}(j_i, k_i)} \) := \( \{ p \in \Pi^{(n)} \mid \rho_i^{(n)}(t_j^i, t_k^i, \tilde{x}, \tilde{p}) = \lambda_i^{(n)}(j_i, k_i) \forall (\tilde{x}, \tilde{p}) \in \tilde{X} \} \), show that \( \Pi_{\lambda_i^{(n)}(j_i, k_i)} \) is dense in \( \Pi^{(n)} \).

(xv) Letting \( \Pi_{\tilde{X}, \lambda_i^{(n)}(j_i, k_i)} = \{ p \in \Pi^{(n)} \mid \rho_i^{(n)}(t_j^i, t_k^i, \tilde{x}, \tilde{p}) = \lambda_i^{(n)}(j_i, k_i) \forall (\tilde{x}, \tilde{p}) \in \tilde{X} \} \), show that \( \Pi_{\tilde{X}, \lambda_i^{(n)}(j_i, k_i)} \) is open-dense in \( \Pi^{(n)} \) and open in \( \Pi \).

(xvi) Letting \( \Pi_u^{(n)} = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{n_i-1} \bigcap_{k=1}^{n_j-1} \Pi_{\lambda_i^{(n)}(j_i, k_i)} \), show that \( \Pi_u^{(n)} \) is open-dense in \( \Pi^{(n)} \) and open in \( \Pi \).

(xvii) Letting \( \Pi_0^* = \cup_{r=1}^{\infty} \Pi_u^{(n)} \), show that \( \Pi_0^* \) is open-dense in \( \Pi \).
(xviii) Show that $U^\ast$ can be taken to be the open-dense set in Theorem 3.2.3.

We now proceed with the proof.

**Proof of Theorem 3.2.3**

We define the sets in (i) using the following procedure. Let $\tilde{V}^i$, $\tilde{X}_i$, and $V^i$ be open sets while $\tilde{Y}^i$, and $\tilde{Y}^i$ are compact sets such that the following inclusion holds: $Y^i \subset \tilde{V}^i \subset \tilde{Y}^i \subset \tilde{V}^i \subset V^i$. Write $V = \times_{i=1}^n V^i$ and $V^{N\setminus\{i\}} = \times_{j \neq i} V^j$. Let $\tilde{W}^i = \{\tilde{w}^i : T^i \to \tilde{V}^i\}$, $\tilde{W}^i = \{w^i : T^i \to \tilde{V}^i\}$, $W^i = \{w^i : T^i \to V^i\}$, $\tilde{X}^i = \{\tilde{x}^i : T^i \to \tilde{Y}^i\}$, and $\tilde{X}^i = \{x^i : T^i \to \tilde{Y}^i\}$. We also write $\tilde{W} = \times_{i=1}^n \tilde{W}^i$, $\tilde{W} = \times_{i=1}^n \tilde{W}^i$, $W = \times_{i=1}^n W^i$, $\tilde{X} = \times_{i=1}^n \tilde{X}^i$, $\tilde{X} = \times_{i=1}^n \tilde{X}^i$, $\tilde{W}^{N\setminus\{i\}} = \times_{j \neq i} \tilde{W}^j$, $W^{N\setminus\{i\}} = \times_{j \neq i} W^j$, $\tilde{X}^{N\setminus\{i\}} = \times_{j \neq i} \tilde{X}^j$, and $\tilde{X}^{N\setminus\{i\}} = \times_{j \neq i} \tilde{X}^j$. From these definitions, it is clear that the required properties are satisfied for $\tilde{W}$, $\tilde{W}$, $W$, $\tilde{X}$, and $\tilde{X}$.

Define the set $\tilde{U}$ in (ii) in the same manner in which it was defined in the proof of Theorem 3.2.2 and endow it with the same norm. Steps (iii), (iv), (v), (vi), and (vii) are clear. In step (viii), first demonstrate the transverse regularity property of the evaluation map using the same procedure as in the proof of Theorem 3.2.2. Then after observing that all the other assumptions of the Transversal density theorem are satisfied, simply make use of that Theorem to get the required dense set. By applying the Theorem on the openness of the transversal intersection and combining this with step (viii), we can get the required open-dense set in step (ix). In step (x),
\( \tilde{U}^{(n)} \) is open-dense as the finite intersection of open-dense sets. \( \tilde{U}^* \) in step (xi) is open-dense as the countable intersection of open-dense sets in a Baire space. By the same argument as in the proof of Theorem 3.2.2, \( U^* \) in step (xii) can be shown to be open-dense.

Step (xiii) is clear. In step (xiv), first note that \( \tilde{W} \) has finite dimension \( q \), \( \Lambda_{i_j, t_k}^{(q)} \) has finite codimension \( q - l_i, \pi^{(n)} \), and \( \tilde{W} \) is second countable and \( \rho < \max(0, q - (q + l_i)) \). Also note that since \( \tilde{u} \in \tilde{U}^* \), it follows that \( \alpha_{\rho, x}^{(q)}(\tilde{u}) = \Lambda_{i_j, t_k}^{(q)} \forall (\rho, x) \in \pi^{(n)} \times \tilde{W} \). But \( \alpha_{\rho, x}^{(q)}(\tilde{u}) \) is simply the evaluation map for \( \rho_{(i_j, t_k, \tilde{u})} \). Therefore, all the assumptions of the Transversal Density Theorem (Abraham and Robbin [1]), are satisfied implying the denseness property.

In step (xv), the openness property follows directly from Theorem 18.2 in Abraham and Robbin while the denseness property is an immediate consequence of step (xiv). The set in step (xvi) is open because it is the finite intersection of open sets and it is dense in \( \pi^{(n)} \) as the finite intersection of open-dense sets. In step (xvii), the set is open as the union of open sets and dense because it is dense in \( \pi^{(n)} \) for every \( n \).

In step (xviii), choose any \( u \in U^* \). Then there exists \( \tilde{U} \in \tilde{U}^* \) such that \( \tilde{u} |_X = u \). Choose any such \( \tilde{u} \). We will show that \( \Pi^{(n)}_u \) can be taken as the open-dense set associated with \( u \). Choose any \( p \in \Pi^{(n)}_u \), then \( p \in \Pi^{(n)}_u \) for some \( n \). Consider any player \( i \) such that \( l_i > m - r \) and any two distinct types of player \( i, t_{j_i}, t_{k_i} \). Then \( p \in \Pi^{(n)}_X \). From this it is clear that \( \rho_{(i_j, t_k, \tilde{u})}^{(q)}(\tilde{u}) = \Lambda_{i_j, t_k}^{(q)} \forall x \in \tilde{W} \). It then follows
from Corollary 17.2 in Abraham and Robbin that $\Lambda^i_{q, l_i}$ and $\rho_{(q, l_i), p}^{-1}(\Lambda^i_{q, l_i}) \cap \tilde{W}$ have the same codimension. This codimension equals $q + l_i$. From this, it follows that the dimension of $\rho_{(q, l_i), p}^{-1}(\Lambda^i_{q, l_i}) \cap \tilde{W}$ is $q - (q + l_i) = -l_i < 0$. Hence $\rho_{(q, l_i), p}^{-1}(\Lambda^i_{q, l_i}) \cap \tilde{W} = \rho_{(q, l_i), p}^{-1}(\Lambda^i_{q, l_i}) \cap X = \emptyset$. From this, it is clear that in every extended Bayesian equilibrium and hence in every interior Bayesian equilibrium, types $t^i_j$ and $t^i_k$ take different actions. From this, we can conclude that for every player $i$ with $l_i > m - r$, all types take different actions in any interior Bayesian equilibrium. Hence that player’s type is fully revealed in every interior Bayesian equilibrium associated with $u$ and $p$. Thus $U^*$ can be taken to be the open-dense set in $U$ while for $u \in U^*$, $\Pi^*_u$ can be taken as the associated open-dense set in $\Pi$.

3.4.3 THEOREM 3.2.4

Before proceeding with the proof of Theorem 3.2.4, we will provide an outline as to how the proof will proceed.

(i) Construct sets $\tilde{W}$, $\hat{W}$, $W$, $\tilde{X}$, $\hat{X}$ and $\tilde{U}$ as in the proof of Theorem 3.2.3.

(ii) Define a sequence of sets $\{(\Pi^{(n)}, \Pi^{(n)})\}_{n=1}^{\infty}$ as well as mappings $\beta$ and $\alpha_{(q, l_i)}$ as in the proof of Theorem 3.2.3.

Remark 3.4.2. $i, > m - r$ for every player $i$ under the assumptions of Theorem 3.2.4.

(iii) Define $\Delta_u$ and $\Lambda^i_{(q, l_i)}$ as in the proof of Theorem 3.2.3.

(iv) In the same manner as in the proof of Theorem 3.2.3, construct and demonstrate that $\tilde{U}^{(n)}$ is open-dense in $\tilde{U}$. 

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(v) Letting $\tilde{Z} := \{ \tilde{u} \in \tilde{U} \mid \beta_{\tilde{u}} \tilde{\Pi}((p, x)) \Delta_u \forall (p, x) \in \tilde{\Pi} \times W \}$, show that $\tilde{Z}$ is dense in $\tilde{U}$.

(vi) Letting $Z^{(n)} := \{ u \in U \mid \beta_{u} \tilde{\Pi}(x) \Delta_u \forall (p, x) \in \tilde{\Pi}^{(n)} \times \tilde{X} \}$, show that $Z^{(n)}$ is open-dense in $\tilde{U}$.

(vii) Letting $\tilde{U}^{(n)} = \tilde{U}^{(n)} \cap Z^{(n)}$, show that $\tilde{U}^{(n)}$ is open-dense in $\tilde{U}$.

(viii) Letting $\tilde{U}^{* * } = \cap_{n=1}^{\infty} \tilde{U}^{(n)}$, show that $\tilde{U}^{* * }$ is open-dense in $\tilde{U}$.

(ix) Letting $U^{* * } = \{ u \in U \mid \exists \tilde{u} \in \tilde{U}^{* * } : \tilde{u} |_{X} = u \}$, show that $U^{* * }$ is open-dense in $U$.

(x) For every $\tilde{u} \in \tilde{U}^{* * }$, define $\Pi^{(n)}_{\tilde{u}}$ as in the proof of Theorem 3.2.3. Show that for any $\tilde{u} \in \tilde{U}^{* * }$, $\Pi^{(n)}_{\tilde{u}}$ is open-dense in $\Pi^{(n)}$ and open in $\Pi$.

(xi) For every $\tilde{u} \in \tilde{U}^{* * }$, and every $n$, construct a mapping $\delta^{(n)}_{\tilde{u}} : \Pi^{(n)} \to C^{\infty} (\hat{W}, R^{\nu})$, where $\nu \to \delta^{(n)}_{\tilde{u}}(p)$ and $\delta^{(n)}_{\tilde{u}}(x) := \beta_{\tilde{u}}(p, x)$.

(xii) Letting $\Pi^{(n)}_{(\tilde{\Delta}, \tilde{u})} = \{ p \in \Pi^{(n)} \mid \delta^{(n)}_{\tilde{u}} \tilde{\Pi} \Delta_u \forall \tilde{x} \in \hat{W} \}$, show that $\Pi^{(n)}_{(\tilde{\Delta}, \tilde{u})}$ is dense in $\Pi^{(n)}$.

(xiii) Letting $\Pi^{(n)}_{(\tilde{\Delta}, \tilde{u})} = \{ p \in \Pi^{(n)} \mid \delta^{(n)}_{\tilde{u}} \tilde{\Pi} \Delta_u \forall \tilde{x} \in \hat{X} \}$, show that $\Pi^{(n)}_{(\tilde{\Delta}, \tilde{u})}$ is open-dense in $\Pi^{(n)}$ and open in $\Pi$.

(xiv) Letting $\hat{\Pi}^{(n)}_{\tilde{u}} = \Pi^{(n)}_{\tilde{X}, \tilde{u}} \cap \Pi^{n}_{\tilde{u}}$, show that $\hat{\Pi}^{(n)}_{\tilde{u}}$ is open-dense in $\Pi^{(n)}$ and open in $\Pi$.

(xv) Letting $\Pi^{* * }_{\tilde{u}} = \cup_{n=1}^{\infty} \hat{\Pi}^{(n)}_{\tilde{u}}$, show that $\Pi^{* * }_{\tilde{u}}$ is open-dense in $\Pi$.

(xvi) Show that $U^{* * }$ can be taken as the open-dense set in Theorem 3.2.4.
We are now ready to proceed with the proof.

**Proof of Theorem 3.2.4.**

Steps (i),(ii),(iii), and (iv) are all clear. Step (v) follows from the Transversal Density Theorem (note that the evaluation map is transverse regular to $\Delta_u$). Step (vi) follows from the Theorem on the Openness of the Transversal Intersection and step (v). Step (vii) follows because the intersection of two open-dense sets is itself open-dense. Step (viii) follows because the countable intersection of open-dense sets in a Baire space is itself open-dense.

Step (ix) is straightforward. In step (x), note that if $\tilde{u}$ is in $\tilde{U}^{**}$, then $\tilde{u}$ must also be in $\tilde{U}^{(n)}$. Since we showed in the proof of Theorem 3.2.3 that for any $\tilde{u} \in \tilde{U}^{(a)}$, $\Pi_u^{(n)}$ is open-dense in $\Pi^{(n)}$ and open in $\Pi$ we conclude that the same must apply here. Step (xi) is straightforward.

In step (xii), first note that $\bar{W}$ has finite dimension $q$, $\Delta_u$ has finite codimension $q$, $\Pi^{(n)}$, and $\bar{W}$ are second countable and $(r-1) \geq \max(0, q - q)$. Also note that since $\tilde{u} \in \tilde{U}^{**}$, it follows that $\beta_{\tilde{u}}^{\Pi_u(p, x)} \Delta_u \forall (p, x) \in \Pi^{(n)} \times \bar{W}$. But $\beta_{\tilde{u}}$ is simply the evaluation map for $\delta_{\tilde{u}}$. Therefore, all the assumptions of the Transversal Density Theorem (Abraham and Robbin [1]), are satisfied implying the denseness property.

In step (xiii), the openness follows from the Theorem on the Openness of the Transversal Intersection and the denseness follows from step (xii). In step (xiv), $\Pi_u^{(n)}$ is open-dense as the intersection of two open-dense sets. In step (xv), $\Pi_u^{**}$ is open as
the union of open sets and dense because it is dense in every $\Pi^{(n)}$.

In step (xvi), choose any $u \in U^{**}$. Then there exists $\tilde{U} \in U^{**}$ such that $\tilde{u} |X = u$. Choose any such $\tilde{u}$. We will show that $\Pi^{**}_{\tilde{u}}$ can be taken as the open-dense set associated with $u$. Choose any $p \in \Pi^{**}_{\tilde{u}}$, then $p \in \Pi^{(n)}_{\tilde{u}}$ for some $n$. Hence it is in both $\Pi^{(n)}_{\tilde{u}}$ and $\Pi^{(n)}_{(\tilde{X}, \Delta), \tilde{a}}$. Because of the assumption on $r$, it can be shown using the same procedure as in the proof of Theorem 3.2.3 that it being in $\Pi^{(n)}_{\tilde{u}}$ implies that in any interior Bayesian equilibrium, every players equilibrium strategy is one-to-one. Now, it being in $\Pi^{(n)}_{(\tilde{X}, \Delta), \tilde{a}}$ implies that $\delta_{(\tilde{u}, p), \tilde{a}} \Delta_u \forall x \in \tilde{W}$. It then follows from Corollary 17.2 in Abraham and Robbin that $\Delta_u$ and $\delta_{(\tilde{u}, p)^{-1}(\Delta_u)} \cap \tilde{W}$ have the same codimension. This codimension equals $q$. From this, it follows that the dimension of $\delta_{(\tilde{u}, p)^{-1}(\Delta_u)} \cap \tilde{W}$ is $q - q = 0$. An implication of this is that $\delta_{(\tilde{u}, p)^{-1}(\Delta_u)} \cap X = \text{is a finite set.}$ From this, it is clear that the set of extended Bayesian equilibria and hence the set of interior Bayesian equilibria associated with $(u, p)$ are both finite sets. This shows that $U^{**}$ can be taken to be the open-dense set in $U$ while for $u \in U^{**}$, $\Pi^{**}_{u}$ can be taken as the associated open-dense set in $\Pi$.

3.4.4 THEOREM 3.2.5

Before proceeding with the proof, we will provide an outline as to how the proof will proceed.

(i) Define $\tilde{X}, W, \tilde{W}, \tilde{G}, \Delta_{\tilde{u}}$, and $\Lambda^{(i, \alpha)}_{(\tilde{u}, \alpha)}$ as in the proof of Theorem 3.2.5.
(ii) Letting \( q = \sum_{i=1}^{n_i} l_i m_i \), construct a mapping \( \Psi : \tilde{U} \times \tilde{\Pi} \rightarrow C^{q-1}(W, \mathbb{R}^q) \) where 
\((u, p) \rightarrow \Psi_{(u, p)}\) and \( \Psi_{(u, p)} : W \rightarrow \mathbb{R}^q \) is defined by 
\[\Psi_{(u, p)}(x) := \{D_y E u^t x^N(t) x^N(t') \mid t, t' \in T\} \in \mathbb{R}^q\] 

(iii) Letting \( S_{\Delta_u} = \{(u, p) \in \tilde{U} \times \tilde{\Pi} \mid \Psi_{(u, p)} \Delta_u \} \), show using the Transversal Density Theorem of Abraham and Robbin [1] that \( S_{\Delta_u} \) is dense in \( \tilde{U} \times \tilde{\Pi} \).

(iv) Letting \( S_{\tilde{X}_{\Delta_u}} = \{(u, p) \in \tilde{U} \times \tilde{\Pi} \mid \Psi_{(u, p)} \tilde{\Delta}_u \forall x \in \tilde{X} \} \), show using Abraham and Robbin's [1] Theorem on the Openness of the Transversal Intersection and step (iii) that \( S_{\tilde{X}_{\Delta_u}} \) is open-dense in \( \tilde{U} \times \tilde{\Pi} \).

(v) For every player \( i \) and any two distinct types \( t^i_j, t^i_k \) of player \( i \), construct a mapping \( \Phi^{t^i_j, t^i_k}_i : \tilde{U} \times \tilde{\Pi} \rightarrow C^{q-1}(W, \mathbb{R}^{q+2l_i}) \), where 
\((u, p) \rightarrow \Phi^{t^i_j, t^i_k}_i (u, p)\) and \( \Phi^{t^i_j, t^i_k}_i : W \rightarrow \mathbb{R}^{q+2l_i} \) is defined by 
\[\Phi^{t^i_j, t^i_k}_i (x) = (\Psi_{(u, p)}(x), x^i(t^i_j), x^i(t^i_k))\] 

(vi) Letting \( S_{\Lambda_{t^i_j, t^i_k}} = \{(u, p) \in \tilde{U} \times \tilde{\Pi} \mid \Phi^{t^i_j, t^i_k}_i (u, p) \Lambda_{t^i_j, t^i_k} \} \), show using the Transversal Density Theorem that \( S_{\Lambda_{t^i_j, t^i_k}} \) is dense in \( \tilde{U} \times \tilde{\Pi} \).

(vii) Letting \( S_{\tilde{X}_{\Lambda_{t^i_j, t^i_k}}} = \{(u, p) \in \tilde{U} \times \tilde{\Pi} \mid \Phi^{t^i_j, t^i_k}_i (u, p) \Lambda_{t^i_j, t^i_k} \forall x \in \tilde{X} \} \), show that \( S_{\tilde{X}_{\Lambda_{t^i_j, t^i_k}}} \) is open-dense in \( \tilde{U} \times \tilde{\Pi} \).

(viii) Letting \( \tilde{Q} = S_{\Delta_u} \cap (\cap_{i=1}^{l_{m_i}} \cap_{p=j_{i-1}+1}^{m_{i-1}+1} S_{\tilde{X}_{\Lambda_{t^i_j, t^i_k}}}) \), show that \( \tilde{Q} \) is open-dense in \( \tilde{U} \times \tilde{\Pi} \) and therefore also open-dense in \( U \times \Pi \).

(ix) Letting \( Q = \{(u, p) \in U \times \Pi \mid \exists (\tilde{u}, \tilde{p}) \in \tilde{Q} : (u, p) \in \tilde{Q} \land (u, p) \} \), show that \( Q \) is an open-dense set in \( U \times \Pi \).
(x) Show that for any \((u, p)\) in \(Q\), every interior Bayesian equilibrium is fully information-revealing.

(xi) Show that for any \((u, p)\) in \(Q\), the set of interior Bayesian equilibria is finite.

(xii) Conclude from (ix), (x), and (xi) that \(Q\) can be taken as the required open-dense set in Theorem 3.2.5.

We are now ready to proceed with the proof.

**Proof of Theorem 3.2.5**

Steps (i) and (ii) are clear. Step (iii) is possible because it can be shown, just like in the proof of Theorem 3.2.2, that the evaluation map is transverse regular to \(\Delta_u\) and hence the Transversal density theorem can be applied. Step (iv) is an immediate consequence of the Theorem on the Openness of the Transversal intersection and step (iii). Step (v) is clear. Steps (vi) and (vii) are again direct applications of the Transversal Density Theorem and the Theorem on the Openness of the Transversal intersection.

Step (viii) is true because the finite intersection of open-dense sets is also open-dense. Steps (x) and (xi) are carried out in the same fashion as steps (xiii) and (xiv) were carried out in the proof of Theorem 3.2.5. Step (xii) then concludes the proof.
CHAPTER 4

ONE-PERIOD INCOMPLETE INFORMATION GAMES

We examine information-revelation in a one-period incomplete information model. The main result states that in the space of twice continuously differentiable utility function bundles, it is a generic property of interior equilibria to be finite and to yield full information-revelation.

4.1 INTRODUCTION

In this chapter, we examine informational issues in one-period incomplete information games. The one key property that separates games of this kind from games with complete information is that players are asymmetrically informed about the true state of the world. We model this incomplete information by assuming that there are a finite number of states of the world, and that each player has an information partition in the first period. In such an incomplete information game, a player's strategy thus needs to specify a first period action for every element of his first period information partition.

In a one-period incomplete information game, all the players take their actions at the same time after first learning the element of their individual information partitions that have occurred. The one-period incomplete information game we consider
can be re-interpreted as a version of Harsanyi's [10] Bayesian game if a player's information partition is re-interpreted as his type space. Given this re-interpretation, our solution concept for the one-period model in fact coincides with Harsanyi's Bayesian equilibrium concept.

Our main object of study is the extent to which players reveal their information. Theorem 4.2.1 demonstrates that it is a generic property of smooth one-period incomplete information games to have a finite number of interior equilibria all of which are fully information-revealing. By a fully information-revealing one-period equilibrium, we mean an equilibrium in which every player's equilibrium strategy is a one-to-one function from his information partition to his pure action space. At first sight, this result may appear a bit surprising. In one-period games, however, all players move at the same time. Hence no player can use the information revealed by the other players in the first period. Because of this, it is only in exceptional cases where a player takes the same actions in two states which he can distinguish between. The idea of letting strategy combinations and action combinations convey information goes back to, among others, the rational expectations equilibrium (see e.g., Radner [17] and the Rothschild-Stiglitz [18] formulation of competitive insurance markets.

The proof of theorem 4.2.1 makes use of techniques previously applied to various parts of the economics literature. Debreu [6] used Sard's Theorem (see e.g., Milnor [15]) in establishing generic finiteness of price equilibria in a context where a set of
differentiable demand functions are given and the initial allocations are allowed to vary. Smale [22], Dubey [7], and Nygren [16] all made use of transverse regularity (see e.g., Abraham and Robbin [1]). Smale's paper, like Debreu's, establishes a generic finiteness property of price equilibria but he allows both utility functions and initial allocations to vary. Dubey's result establishes generic finiteness and inefficiency of Nash equilibria by allowing utility functions to vary. In chapter 2, we established generic full information-revelation and finiteness of equilibria in Bayesian games. The proof of our first result adopts the method of proof used there to the more explicit model of information in the current chapter.

The rest of this chapter is organized as follows. In section two, we introduce one-period incomplete information games and state Theorem 4.2.1. Section three contains the proof of Theorem 4.2.1.

4.2 THE MODEL AND RESULTS

In this chapter, Incomplete Information Games will refer to games with pure strategies as opposed to behavior strategies. An example of a one-period incomplete information game of this type would be a game where firms facing uncertainty about cost and demand functions simultaneous choose their prices. Clearly in such a setting, strategies are pure in the sense that they specify a pure action (in our example a price) as opposed to a probability distribution over the prices.
We now more formally introduce our notation for a one-period incomplete information game. The first ingredient is a finite set of players $N$. As the second ingredient, we add a finite state space $S$. Nature at the outset selects the state which is to occur. This is done using our next ingredient, an objective joint probability distribution over the state space denoted $p$. Throughout, it will be assumed that each state occurs with a positive probability and the probability with which state $s$ occurs will be denoted $p(s)$.

The fourth ingredient is a set of fields $F_i := \{F_i^j\}_{i \in N}$ over the state space $S$. An event $E$ is in the field $F_i^j$ if and only if player $i$ can distinguish the occurrence of event $E$ from the occurrence of the complementary event $S \setminus E$. The set of minimal non-empty events in $F_i^j$ form a partition of $S$ and will be denoted $\mathcal{PF}_i^j$. The collection of all such partitions $\mathcal{PF}_i := \{\mathcal{PF}_i^j\}_{i \in N}$ will be taken as our fifth ingredient. Throughout, the notation $\mathcal{PF}_i^j(s)$ will refer to the event in $\mathcal{PF}_i^j$ that contains state $s$.

The next ingredient is a pure action space $Y_i := \times_{i \in N} Y_i^j$, where $Y_i^j$ is player $i$'s action space. Our seventh ingredient is the strategy space $X_i := \times_{i \in N} X_i^j$ where $X_i^j := \{x_i^j : S \to Y_i^j | x_i^j \text{ is } F_i^j\text{-measurable}\}$. Player $i$'s strategy specifies an action for him to take in every state. Any feasible strategy also has to be $F_i^j$-measurable. That is, if player $i$ can not distinguish between two states, he has to take the same action in both of them.
Our final ingredient will be a von Neumann-Morgenstern expected utility function bundle $u_i := \{u_i^j\}_{j \in N}$ where $u_i^j : S \times Y_i \to \mathbb{R}$ is player $i$'s von Neumann-Morgenstern expected utility function. We now have all the ingredients required to define a one-period incomplete information game.

**Definition 4.2.1.** A one period Incomplete Information Game is a list of specified data $\Gamma_i := \{S, p, F_i, PF_i, Y_i, X_i, u_i\}$ of a state space $S$, a probability distribution $p$, a collection of fields $F_i$, a collection of partitions $PF_i$, a pure action space $Y_i$, a strategy space $X_i$, and a von Neumann-Morgenstern utility function bundle $u_i$.

The solution concept we will use for this one-period incomplete information game will be a strategy bundle $x_i^* \in X_i$ satisfying the property that no player in any of the events in his information partition can find an alternative strategy that increases his expected utility conditional upon realization of that state. More formally, the definition is the following.

**Definition 4.2.2.** A one period equilibrium of a one-period incomplete information game $\Gamma_i := \{S, p, F_i, PF_i, Y_i, X_i, u_i\}$ is a strategy bundle $x_i^* \in X_i$:

$$-\exists i \in N : \exists E \in PF_i : \exists \tilde{x}_i^j \in X_i^j :$$

$$\sum_{s \in E} p(s|E)u_i^j(s, \tilde{x}_i^j(s), x_i^{N\setminus\{i\}}(s)) > \sum_{s \in E} p(s|E)u_i^j(s, x_i^*(s))$$

By simply treating a player's first period information partition as that player's type space, one can easily re-interpret the one-period incomplete information game and the one-period equilibrium as a Bayesian game and a Bayesian equilibrium. The existence of Bayesian equilibria is well known under standard assumptions.

In many economic models, it may be of value to know not only if equilibria exists but also the properties of the equilibria. For pure exchange economies there are
several known results concerning the generic finiteness of the set of competitive equilibria. For complete information games, there are also several generic finiteness of equilibria results. Dubey (1986) also established the generic inefficiency of interior Nash equilibria and in chapter 2, we established generic full information-revelation in Bayesian games.

We now turn to examining some generic properties of one-period incomplete information games. There are two such properties that we will be particularly interested in. The first one concerns the extent to which equilibria of one-period incomplete information games tend to be finite. Given the finiteness properties of complete information games, this would probably be expected. The second generic property we will examine is the extent to which one-period equilibria tends to reveal all the information of the players. That is, do different "types" of a player take different actions in interior equilibria? We formally state what we mean by full information-revelation in the following definition.

**Definition 4.2.3.** An equilibrium strategy bundle \( x_i^* \) of a one-period incomplete information game \( \Gamma_i := \{S, p, F_i, P F_i, Y_i, X_i, u_i\} \) is said to be fully information-revealing if for every player \( i \), it is false that there exists distinct events \( E, E' \) in \( PF_i \) such that \( x_i^i(E) = x_i^i(E') \).

In order to talk about generic properties of incomplete information games, we first have to define what we mean by the space of games. We hence introduce the following definitions.

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For each player $i$, define

$$U^i := \{ u^i : S \times Y_1 \rightarrow \mathbb{R} | \forall s \in S : u^i(s, \cdot) \text{ is } C^2 \}$$

That is, $U^i$ are the set of all twice continuously differentiable utility functions for player $i$. We endow this linear space with the following norm

$$\|u^i\| = \sup\{\|u^i(s, y_1)\|, \|Du^i(s, y_1)\|, \|D^2u^i(s, y_1)\| : (s, y_1) \in S \times Y_1\}$$

Define $U_1 := \times_{i \in N} U^i$. In the theorem that follows, $U_1$ will be taken as the space of games. Hence our Theorem will state properties that hold for almost any choice of such a utility function bundle. A proof of this Theorem is available in the appendix.

**Theorem 4.2.1.** Fix $\{S, p, F_1, PF_1, Y_1, X_1\}$. Assume that for every player $i$, $Y^i_1$ is a compact, convex, nonempty subset of a Euclidean space of dimension at least one and has nonempty interior. Then there is an open-dense subset $U^*_1$ of $U_1$ such that for any $u_1$ in $U^*_1$, the one-period incomplete information game $\{S, p, F_1, PF_1, Y_1, X_1, u_1\}$ satisfies the following two properties.

(i) the set of interior one-period equilibria is a finite set; and
(ii) every interior one-period equilibrium strategy bundle is fully information-revealing.

Some comments are in order concerning this result. Our first comment is that the theorem does not address the existence of fully information-revealing equilibria. In fact, the conclusions of theorem 4.2.1 would hold even without the convexity assumption on the action spaces. Our second comment is that the set of utility function bundles in $U_1$ satisfying the property that every player's utility function is strictly concave in the action space forms an open subset of $U_1$. Given the rest of the assumptions in the theorem, the existence of at least one equilibrium is well known for such
strictly concave utility functions. For the open subset of such strictly concave utility functions, we can thus conclude that it is a generic property of games to either have a fully information-revealing equilibrium or to have all equilibria, of which there is at least one, in the boundary.

Our third comment concerns the underlying intuitive reasons for this result. In simultaneous move one-period incomplete information games of this kind, no player has to consider the impact his actions today may have on other players' actions in future periods. The actions of the other players remain the same regardless of the action a player takes in the first period. Hence, taking the actions of the other players as given in each of the states, player $i$ simply chooses his optimal response to those actions without worrying about any responses in the future. This in turn is what causes the optimal actions for two different elements of player $i$'s information partition to be different from each other in all but exceptional cases where the utility functions are "similar" enough.

4.3 PROOF OF THEOREM 4.2.1

Before proceeding with the proof of Theorem 4.2.1, we will provide an outline as to how the proof will proceed.

(i) Construct a compact set $\tilde{X}_1$ and two open sets $W_1, \tilde{W}_1$ such that $X_1 \subset \tilde{W}_1 \subset \tilde{X}_1 \subset W_1$.

(ii) Construct an open set $\tilde{U}_1$ such that $U_1 \subset \tilde{U}_1$.
(iii) Letting $q = \sum_{i=1}^{n_i} i \cdot m_i$, construct a mapping $\Psi : \tilde{U}_1 \to C^1(W_1, \mathbb{R}^q)$ where $u_1 \to \Psi_{u_1}$ and $\Psi_{u_1} : W_1 \to \mathbb{R}^q$ is defined by

$$\Psi_{u_1}(x_1) := (D_{x_1} E u_1^i(x_1^i(E), x_1^{N-i}(E), | E))_{i \in N, E \in PF_i}$$

(iv) Define $\Delta_{u_1} \subset \mathbb{R}^q$ to be the set consisting of the zero vector and note that when $u_1 \in U_1$, $x_1 \in X_1$, and $\Psi_{u_1}(x_1) \in \Delta_{u_1}$, then $x_1$ is an extended one-period equilibrium.

(v) Letting $\tilde{U}_{1, \Delta_{u_1}} = \{u_1 \in \tilde{U}_1 \mid \Psi_{u_1}(\Delta_{u_1})\}$, show using the Transversal Density Theorem of Abraham and Robbin [1] that $\tilde{U}_{1, \Delta_{u_1}}$ is dense in $\tilde{U}_1$.

(vi) Letting $\tilde{U}_{1, \tilde{X}_{1, \Delta_{u_1}}} = \{u_1 \in \tilde{U}_1 \mid \Psi_{u_1}(\tilde{X}_{1, \Delta_{u_1}}) \forall x_1 \in \tilde{X}_1\}$, show using Abraham and Robbin's [1] Theorem on the Openness of the Transversal Intersection and step (v) that $\tilde{U}_{1, \tilde{X}_{1, \Delta_{u_1}}}$ is open-dense in $\tilde{U}_1$.

(vii) For every player $i$ and any two distinct events $E_j^i, E_k^i$ in $PF_i$, construct a mapping $\Phi_{(E_j^i, E_k^i), u_1} : \tilde{U}_1 \to C^1(W_1, \mathbb{R}^{q+2l_i})$, where $u_1 \to \Phi_{(E_j^i, E_k^i), u_1}$ and $\Phi_{(E_j^i, E_k^i), u_1} : W_1 \to \mathbb{R}^{q+2l_i}$ is defined by

$$\Phi_{(E_j^i, E_k^i), u_1}(x_1) = (\Psi_{u_1}(x_1), x_1^i(E_j^i), x_1^i(E_k^i))$$

(viii) Define

$$\Lambda_{(E_j^i, E_k^i)} := \{(0, x_1^i(E_j^i), x_1^i(E_k^i)) \in \mathbb{R}^{q+2l_i} \mid x_1^i(E_j^i) = x_1^i(E_k^i)\}$$

and note that when $u_1 \in U_1$, $x_1 \in X_1$, and $\Phi_{(E_j^i, E_k^i), u_1}(x_1) \in \Lambda_{(E_j^i, E_k^i)}$, then $x_1$ is an extended equilibrium where the events $E_j^i, E_k^i$ map into the same actions.

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(ix) Using the Transversal Density theorem as applied to $W_1$, show that there is a dense set in $\tilde{U}_1$ such that for every $u_1$ in that dense set $\Phi^{i}_{(x_1), u_1}$ is transversal to $\Lambda^{i}_{(x_1), u_1}$ for every $x_1$ in $W_1$.

(x) Using the Theorem on the openness of the Transversal Intersection together with the result in (vii), show that there is an open-dense set $\tilde{U}_{1, \Lambda^{i}_{(x_1), u_1}}$ in $\tilde{U}_1$ such that for every $u_1$ in $\tilde{U}_{1, \Lambda^{i}_{(x_1), u_1}}$, $\Phi^{i}_{(x_1), u_1}$ is transversal to $\Lambda^{i}_{(x_1), u_1}$ for every $x_1$ in $\tilde{X}_1$.

(xi) Letting $\tilde{Q}_1 = \tilde{U}_{1, \Delta_{u_1}} \cap \cap_{i=1}^{n_2} \cap_{j=1}^{n_4} \cap_{k=1}^{n_6} \cap_{l=1}^{n_8} \tilde{U}_{1, \Lambda^{i}_{(x_1), u_1}}$, show that $\tilde{Q}_1$ is open-dense in $\tilde{U}_1$.

(xii) Letting $Q_1 = \{ u_1 \in U_1 \mid \exists \tilde{u}_1 \in \tilde{Q}_1 : \tilde{u}_1 \approx x_1 = u_1 \}$, show that $Q_1$ is an open-dense set in $U_1$.

(xiii) Show that for any $u_1$ in $Q_1$, every interior one period equilibrium is fully information-revealing.

(xiv) Show that for any $u_1$ in $Q_1$, the set of interior one period equilibria is finite.

(xv) Conclude from (xii), (xiii), and (xiv) that $Q_1$ can be taken as the required open-dense set in Theorem 4.2.1.

We are now ready to proceed with the proof.

**Proof of Theorem 4.2.1.**

We define the sets in (i) using the following procedure. Let $V_1^{0}$ be any open set in $\mathbb{R}^{k}$ containing $Y_1^{0}$, $V_1 = \times_{i=1}^{n_1} V_1^{0}$, and $V_1^{N(t)} = \times_{j \neq i} V_1^{0}$. Let $W_1^{0} = \{ w_1 : S \rightarrow 1 \cup u^{0} \text{ is } F_t^{1}\text{-measurable} \}$. We define $W_1 = \times_{i=1}^{n_1} W_1^{0}$, and $W_1^{N(t)} = \times_{j \neq i} W_1^{0}$. Let
\( \bar{V}_i \) be an open set and \( \bar{Y}_i \) a compact set such that the following inclusions hold:
\( Y_i \subset \bar{V}_i \subset \bar{Y}_i \subset V_i \) Let \( \bar{\tilde{W}}_i := \{ \tilde{w}_i : S \to \bar{V}_i | \tilde{w}_i \text{ is } F_i \text{-measurable} \} \) and \( \bar{X}_i := \{ \tilde{x}_i : S \to \bar{Y}_i | \tilde{x}_i \text{ is } F_i \text{-measurable} \} \) We also write \( \bar{W}_i = \times_{i=1}^n \bar{W}_1, \bar{X}_i^{N(i)} = \times_{j \neq i} \bar{X}_j \). From this, it is clear that \( X_1 \subset \bar{W}_1 \subset \bar{X}_1 \subset W_1 \) and that \( \bar{X}_1 \) is compact, while \( W_1 \) and \( \bar{W}_1 \) are open.

We define the set \( \bar{U}_1 \) in (ii) by letting
\[
\bar{U}_1 = \{ \tilde{u}_i : S \times V \to \mathbb{R}_+ \mid \tilde{u}_i(s,.) \text{ is differentiable } r \text{ times in } V \text{ given each } s \in S \}
\]
We endow this linear space with the following norm
\[
\| \tilde{u}_i \| = \sup \{ \| \tilde{u}_i(s, y_1) \|, \| D\tilde{u}_i(s, y_1) \|, \| D^2\tilde{u}_i(s, y_1) \| : (s, y_1) \in S \times V \}
\]
Write \( \bar{U}_1 = \times_{i=1}^n \bar{U}_1 \) for the space of utility profiles that satisfy the differentiability assumption.

Steps (iii) and (iv) are clear. To do step (v), note that \( \bar{U}_1, W_1, \mathbb{R}^q \) are \( C^r \) manifolds, \( \Delta_{u_1} \) a closed submanifold, \( \dim (W_1) = \text{codimension } \Delta_{u_1} = q \), \( \bar{U}_1 \) and \( W_1 \) are second countable, \( \bar{X}_1 \) compact and \( \Psi : \bar{U}_1 \to C^1(W_1, \mathbb{R}^q) \) a \( C^1 \) representation. To apply the Transversal Density theorem, the only thing left to show is that the evaluation map \( ev_\psi : \bar{U}_1 \times W_1 \to \mathbb{R}^q \) is transverse regular to \( \Delta_{u_1} \) (i.e., \( ev_\psi(x, \overline{\Delta_{u_1}}) \)). In demonstrating this, we closely follow Dubey's [7] argument. Choose for any \((u_1, x_1)\) in \( \bar{U}_1 \times W_1 \) and \( y_1 = \Psi_u(x_1) \). If \( y_1 \neq 0 \) we are done. If \( y_1 = 0 \), we need to show that the image of the tangent space to \( \bar{U}_1 \times W_1 \) at \((u_1, x_1)\) given by the mapping of equivalence classes of

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tangent curves in \( \tilde{U}_1 \times W_1 \) at \((u_1, x_1)\) to equivalence classes of tangent curves in \( \mathbb{R}^q \) at \(0\) contains a closed complement to \( T_{y_1} \Delta_{u_1} \) in \( T_{y_1} \mathbb{R}^q \) (i.e., \((T_{(u_1, x_1)}e_{v_\Psi})(T_{(u_1, x_1)}\tilde{U}_1 \times W_1)\)) contains a closed complement to \( T_{y_1} \Delta_{u_1} \) in \( T_{y_1} \mathbb{R}^q \). We will show that \( T_{(u_1, x_1)}e_{v_\Psi} : T_{(u_1, x_1)}(\tilde{U}_1 \times W_1) \to T_{y_1} \mathbb{R}^q \) is in fact onto and hence contains the required closed complement to \( T_{y_1} \Delta_{u_1} \). Choose any \( z \in T_{y_1} \mathbb{R}^q \). Write \( z = (z_E^i)_{i \in N : E \in P_{\mathcal{F}^1}} \) and \( z_E^i \) for the transpose of \( z_E^i \). Define

\[
\tilde{u}_1^{i^*}(s, x_1^i(E), x_1^{N\setminus\{i\}}(s)) = u_1^i(s, x_1^i(E), x_1^{N\setminus\{i\}}(s)) + r x_1^i(E) z_E^i
\]

Also, define

\[
E \tilde{u}_1^{i^*}(x_1^i(E), x_1^{N\setminus\{i\}}(s)) = \sum_{s \in E} p(s \mid E) \tilde{u}_1^{i^*}(s, x_1^i(E), x_1^{N\setminus\{i\}}(s))
\]

\[
= r x_1^i(E) z_E^i + \sum_{s \in E} p(s \mid E) u_1^i(s, x_1^i(E), x_1^{N\setminus\{i\}}(s))
\]

Now, because \( \tilde{U}_1 \) is an open set, we can choose a sufficiently small interval \((r_2^1, r_2^2)\) around \(0\) in \( \mathbb{R} \) so that \( c_z(\tau^2) \mid_{\tau=r_2^1} : = (\tilde{u}_1^i, x_1) \) is a curve at \((u_1, x_1)\) in \( \tilde{U}_1 \times W_1 \). It follows that \( e_{v_\Psi}(c_z(\tau)) \mid_{\tau=r_2^1} = e_{v_\Psi}(\tilde{u}_1^i, x_1) \mid_{\tau=r_2^1} \) is a curve at \(0\) in \( \mathbb{R}^q \). Since

\[
D_{x_1^i(E)}E \tilde{u}_1^i(s, x_1^i(E), x_1^{N\setminus\{i\}}(s)) = r x_1^i(E) + D_{x_1^i(E)}E u_1^i(x_1^i(E), x_1^{N\setminus\{i\}}(s))
\]

we have \( d/d\tau (e_{v_\Psi}(c_z(\tau))) \mid_{\tau=0} = z \) which demonstrates that \( T_{(u_1, x_1)}e_{v_\Psi} : T_{(u_1, x_1)}\tilde{U}_1 \times W_1 \to T_{y_1} \mathbb{R}^q \) is onto as required. Because the finite-dimensionality implies that the inverse image \((T_{(u_1, x_1)}e_{v_\Psi})^{-1}(T_{y_1} \Delta_{u_1})\) splits, it follows that \( e_{v_\Psi} \Delta \Delta_{u_1} \). Given this property of the evaluation map and that fact that all the other assumptions are
satisfied, we can apply the Transversal Density Theorem and thus $\tilde{U}_{1, \Delta_{u_1}} = \{ u_1 \in \tilde{U}_1 \mid \Psi_{u_1, \tilde{x}_1 \Delta_{u_1}} \}$ is dense in $\tilde{U}_1$.

To do step (vi), note that $\tilde{U}_1, W_1, \mathbb{R}^q$ are $C^1$ manifolds, $\Delta_{u_1}$ a closed submanifold, $\dim (W_1) = \text{codimension } \Delta_{u_1} = q$, $\tilde{U}_1$ and $W_1$ are second countable, $\tilde{x}_1$ compact and $\Psi : \tilde{U}_1 \rightarrow C^1(W_1, \mathbb{R}^q)$ a $C^1$ representation. This implies that we can apply the Theorem on the Openness of the Transversal Intersection to get $\tilde{U}_{1, \tilde{x}_1 \Delta_{u_1}} = \{ u_1 \in \tilde{U}_1 \mid \Psi_{u_1, \tilde{x}_1 \Delta_{u_1}} \forall x_1 \in \tilde{x}_1 \}$ open. Since $\tilde{U}_{1, \tilde{x}_1 \Delta_{u_1}}$ contains $\tilde{U}_{1, \Delta_{u_1}}$, we conclude that $\tilde{U}_{1, \tilde{x}_1 \Delta_{u_1}}$ is open-dense in $\tilde{U}_1$.

Now, steps (vii) and (viii) are clear so we turn to step (ix). Step (ix) is carried out very much like step (v). The differences between step (vii) and step (iv) are that $\dim (W) - \text{codimension } (\Lambda^i_{(E_i, E_j)}) = q - (q + l_i) = -l_i < 0$, that we are working with $\Phi^i_{(E_i, E_j)}$ instead of $\Psi$ and that we now choose $z = (z_1, z_2, z_3) \in T_y \mathbb{R}^{q + 2l_i}$, where $y = \Phi^i_{(E_i, E_j), \nu_1}(x_1)$ and $z_1$ is the same as $z_1$ in the earlier proof. We define $\tilde{u}_1^{ir}(s, x_1^i(E), x_1^{N_i}(s))$ and $E \tilde{u}_1^{ir}(x_1^i(E), x_1^{N_i}(s) \mid E)$ as before but now also define $\tilde{x}_1^i := (x_1^i(E_1^1), x_1^i(E_2^1), \ldots, x_1^i(E_{n_1}^1), \ldots, x_1^i(E_1^3), x_1^i(E_4^1), \ldots, x_1^i(E_{n_2}^1), x_1^i(E_1^3), \ldots, x_1^i(E_4^1), \ldots, x_1^i(E_{n_3}^1))$. Since $\tilde{U}_1$ and $W_1$ are open sets, we can select a sufficiently small interval around zero, $(r_1^2, r_2^2)$ such that $c^i_{(E_j, E_j)}(r) \mid_{r = r_1^2}^{r_2^2} := (\tilde{u}_1^i, \tilde{x}_1^i)$ is a curve at $(u_1, x_1)$ in $\tilde{U}_1 \times W_1$. By a similar token to that in step (v), we can show that $d/dr (ev_{\Phi^{ir}_{(E_j, E_j)}}(c^i_{(E_j, E_j)}(r))) \mid_{r = 0} = z$. By applying the same argument as in step (v), we then get the required dense set.
Step (x) now follows from step (ix) in the same way as step (vi) followed from step (v). We thus get the required open-dense set $\tilde{U}_{1, \Lambda_{i}^i_{(E_j^i), a_1}}$.

In step (xii), $\tilde{Q}_1$ is open-dense in $\tilde{U}_1$ because it is the finite intersection of open-dense sets in a Baire space. Because of a well-known property noted by Dubey [7], the open-denseness property of $Q_1$ in step (xii) follows from the open-denseness of $\tilde{Q}_1$ in step (xi). To do step (xiii), first note that if $\tilde{u}_1$ is an element of $\tilde{Q}_1$, then it follows from the definition of $\tilde{Q}_1$ that it is also an element of $\tilde{U}_{1, \Lambda_{i}^i_{(E_j^i), a_1}}$. From step (x), it then follows that $\Phi^i_{(E_j^i), a_1}^{-1}(\Lambda_{i}^i_{(E_j^i), a_1})$ is transversal to $\Lambda_{(E_j^i), a_1}$ for every $x_1$ in $\tilde{X}_1$. In particular, $\Phi^i_{(E_j^i), a_1}$ is transversal to $\Lambda_{(E_j^i), a_1}$ for every $x_1$ in $\tilde{W}_1$. Using $\tilde{W}_1$, we apply Corollary 17.2 of Abraham and Robbin [1] and get codimension $\Phi^i_{(E_j^i), a_1}^{-1}(\Lambda_{(E_j^i), a_1}) \cap \tilde{W}_1$ equal to the codimension of $\Lambda_{(E_j^i), a_1}$ which equals $q + 2l_i^i - l_i^i = q + l_i^i$. This implies dimension $\Phi^i_{(E_j^i), a_1}^{-1}(\Lambda_{(E_j^i), a_1}) \cap \tilde{W}_1 = q - (q + l_i^i) = -l_i^i < 0$ which in turn implies that $\Phi^i_{(E_j^i), a_1}^{-1}(\Lambda_{(E_j^i), a_1}) \cap \tilde{W}_1 = \emptyset$ and that $\Phi^i_{(E_j^i), a_1}^{-1}(\Lambda_{(E_j^i), a_1}) \cap X_1 = \emptyset$. This in turn implies that there are no interior Bayesian equilibria of the game associated with $u_i$ where $E_j^i$ and $E_j^i$ take the same actions and thus that any one period interior equilibrium associated with $u_i$ must be fully information-revealing. This concludes step (xiii).

For step (xiv), note that $\tilde{u}_1$ in $\tilde{Q}_1$ implies that $\tilde{u}_1$ is in $\tilde{U}_{1, X_1, A_{u_1}}$. It then follows from step (vi) that $\Psi_{\tilde{u}_1}$ is transversal to $A_{u_1}$ for every $x_1$ in $\tilde{X}_1$. In particular, $\Psi_{\tilde{u}_1}$ is transversal to $A_{u_1}$ for every $x_1$ in $\tilde{W}_1$. Applying Corollary 17.2 to $\tilde{W}_1$, we get...
that codimension $\Psi_{\mathfrak{u}_1}^{-1}(\Delta_{u_1}) \cap \bar{W}_1 = \text{codimension } \Delta_{u_1} = q$ which implies that the dimension of $\Psi_{\mathfrak{u}_1}^{-1}(\Delta_{u_1}) \cap \bar{W} = q - q = 0$. This in turn implies that the dimension of $\Psi_{\mathfrak{u}_1}^{-1}(\Delta_{u_1}) \cap X_1 = 0$. It also follows from the Corollary 17.2 that $\Psi_{\mathfrak{u}_1}^{-1}(\Delta_{u_1}) \cap X_1$ has only finitely many connected components. This together with the zero-dimensionality of each component implies that $\Psi_{\mathfrak{u}_1}^{-1}(\Delta_{u_1}) \cap X_1$ consists of only finitely many points which in turn implies that $u_1$ has only finitely many extended equilibria and hence a finite number of interior one period equilibria. This concludes step (xiv).

Given steps (xii), (xiii), and (xiv), we now note in step (xv) that any $u_1$ in $Q_1$ satisfies the required properties so $Q_1$ can be used as the required open-dense set.

Q.E.D.
CHAPTER 5

TWO-PERIOD INCOMPLETE INFORMATION GAMES

We examine the updating of beliefs and information-revelation in two-period incomplete information models. Two main results are derived. The first result concerns information in a two-period model. It demonstrates that in a two-period model, second period beliefs satisfying certain symmetry assumptions are uniquely determined even for information sets that are unreached in equilibrium. Using these beliefs, we formulate a two-period equilibrium concept. Our second result demonstrates that if the information structure satisfies an informational dispersion assumption, generic existence of two-period equilibria can be demonstrated in the space of twice continuously differentiable first period utility function bundles.

5.1 INTRODUCTION

In this chapter, we examine informational issues in two-period incomplete information games with observable actions. We model this incomplete information by assuming that there are a finite number of states of the world, and that each player has an information partition in the first period and another finer information partition in the second period. In such an incomplete information game, a player's strategy thus needs to specify a first period action for every element of his first period
information partition. It also needs to specify a second period action for every element of the product of the first period action space with his second period information partition.

In two-period games with incomplete information and observable actions, belief updating is a significant issue that poses some real difficulties. The main reason for this is that actions undertaken in the first period can signal important information about the true state. Hence a player’s beliefs about the true state should be a function of the actions undertaken in the first period. For some action combinations a player can observe in the first period, the most natural way to update beliefs is by using Bayes rule. For some action combinations a player can observe, however, Bayes rule may fail to provide an answer as to how beliefs should be updated. In particular, this is true if a player observes an action combination that was not supposed to occur in any state of the world.

One of the issues we address is how beliefs can be updated even when players observe action combinations that weren’t supposed to occur in any of the states. In a different type of model, similar issues have been studied by the equilibrium refinement literature, such as Selten [21] and Kreps-Wilson [13]. Our approach share the characteristic with their model that beliefs are formulated by using the limits of limits of beliefs that would arise if players had some small positive probability of making mistakes. Theorem 5.3.2 demonstrates that the limiting beliefs of a sequence
of beliefs associated with distributions that satisfy certain symmetry assumptions in fact are uniquely determined and are given by a relatively simple formula. Beliefs satisfying the formula are referred to as reasonable beliefs.

The restrictions on beliefs imposed by our formula are stronger than the restrictions on beliefs imposed by the sequential equilibrium concept. There are two requirements that must be satisfied in a sequential equilibrium. One requirement is that beliefs must be the limiting beliefs of a sequence of beliefs associated with completely mixed strategies converging to the equilibrium strategies. The second requirement is that the strategies must be optimal in every information set given the beliefs. In the present thesis, the condition on the beliefs is strengthened. Beliefs are required to be the limit of a sequence of beliefs associated with distributions that satisfy certain symmetry assumptions.

The formula for beliefs so derived is used to formulate a two-period equilibrium concept with the properties that the equilibrium strategy of every player is optimal in both the first and the second period. That is, the strategy is optimal in the first period regardless of the element of the players first period information partition in question. It is also optimal in the second period regardless of the element of the product space of the first period action space with the players second period information partition. That is, given the player's reasonable beliefs, he can not find another action that increases his expected utility.
We then address the question of existence of two-period equilibria. Theorem 5.4.1 demonstrates the generic existence of two-period fully information-revealing equilibria under an informational dispersion assumption. The main reason for generic existence when information is dispersed is that no player can hide information if it is revealed by other players.

The rest of this chapter is organized as follows. In section 2, we formulate two-period incomplete information games, in section 3, we discuss belief-updating in the two-period model, section four finally contains proofs of two of our key Theorems.

5.2 FORMULATION OF TWO-PERIOD INCOMPLETE INFORMATION GAMES

We will here consider a class of two-period incomplete information games with observable actions. The type of interaction we have in mind is the following. At the start of the first period, nature selects the true state. After learning the event in their information partition that has occurred, all the players then simultaneously take their first period actions. At the end of the first period, the actions taken by all of the players are observed and the players learn the element of their second period information partition that has occurred. The players then simultaneously take their second period actions.

In order to be able to state the definition of our two-period incomplete information game, we now introduce some notation in addition to that already introduced in the
previous chapter. The first ingredient is a collection of second period information fields \( F_2^i := \{F_2^i\}_{i \in N} \). Here \( F_2^i \) denotes player \( i \)'s information field over the states in the second period. The second ingredient is a collection of information partitions \( PF_2^i := \{PF_2^i\}_{i \in N} \) where the information partition \( PF_2^i \) consists of the set of minimal non-empty events in the field \( F_2^i \). Throughout, the notation \( PF_2^i(s) \) will be used to denote the unique element of \( PF_2^i \) that contains state \( s \).

The third ingredient is an action space \( Y_2^i := \times_{i \in N} Y_2^i \). Here \( Y_2^i \) denotes player \( i \)'s action space in the second period. The fourth ingredient is a strategy space \( X_2 := \times_{i \in N} X_2^i \) where \( X_2^i := \{x_2^i : S \times Y_1 \to Y_2^i | \forall y_1 \in Y_1 : x_2^i(\cdot, y_1) \text{ is } F_2^i \text{-measurable}\} \).

Note that player \( i \)'s second period strategy needs to specify an action for every state and every first period action combination. The final ingredient is a utility function bundle \( u_2 := \{u_2^i\}_{i \in N} \) where player \( i \)'s utility function is of the form \( u_2^i : S \times Y_2 \to \mathbb{R} \).

We are now ready to define a two period incomplete information game.

**Definition 5.2.1.** A two-period Incomplete Information Game is a list of specified data \( \Gamma_1 := \{S, p, \{F_t, PF_t, Y_t, X_t, u_t\}_{t=1}^2\} \) of a state space \( S \), a probability distribution \( p \), a collection of fields \( F_t \), a collection of partitions \( PF_t \), a pure action space \( Y_t \), a strategy space \( X_t \), and a von Neumann-Morgenstern utility function bundle \( u_t \) for both time periods.

Note that we are using time separable utility functions here. This is the assumption that most frequently is made in complete information multi-period games as well. The reason for this is largely that equilibrium existence results are difficult to get without time separability. In complete information games with time-separable
utility functions, existence of Nash equilibria is relatively easy to establish. The method most commonly used is to first find an equilibrium of the second period game. If second period actions are made constant as functions of the first period actions and are made to be the equilibrium actions in the second period, one can find an equilibrium of the overall game by simply letting the first period actions be equilibrium actions in a myopic game. In some instances, this may be the only way to find a subgame perfect equilibrium. In complete information games with infinitely many periods, a hole host of other equilibria may arise. In the particular case of repeated games, the well known folk theorem holds true.

We wish now to formulate an appropriate solution concept for two-period incomplete information games. It is clear that one would want to impose the standard type of restrictions that strategies should be optimal responses. In the first period, the natural condition to impose is that any equilibrium strategy \((x_1^*, x_2^*) \in X_1 \times X_2\) should satisfy the following:

\[
\exists i \in N : \exists E \in PF^i : \exists \bar{x}^i \in X_1^i \times X_2^i :
\]

\[
\sum_{s \in E} p(s|E)[u_1^i(s, \bar{x}^i(s), x_1^* \setminus \{i\}(s)) + u_2^i(s, \bar{x}^i(s), x_2^* \setminus \{i\}(s), x_1^* \setminus \{i\}(s), x_2^* \setminus \{i\}(s))]
\]

\[
> \sum_{s \in E} p(s|E)[u_1^i(s, x_1^i(s)) + u_2^i(s, x_2^i(s), x_1^i(s))]
\]

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This is the standard condition that regardless of the information a player may have in the first period, he cannot find an alternative strategy that would increase his expected utility when he takes his first period action. This condition, one may argue, is insufficient since it imposes no restrictions for second period strategies of the equilibrium path. That is, without some additional condition being added, a strategy may satisfy the condition even though in some circumstances it requires a player to take actions that he under no circumstances would be willing to take once the second period is reached. In order to formulate additional restrictions on second period strategies, we first have to specify how probabilities over the state space should be updated in the second period. This is what we therefore devote the next section to.

5.3 BELIEF-UPDATING IN THE TWO-PERIOD MODEL

We now explore the issue of how players should update their beliefs in the second period. Recall that player i’s second period strategy needs to specify an action for every element of the product space \( PF_2 \times Y_1 \). Consider any such element \((E, y_1)\). In order to be able to judge the relative attractiveness of the various alternatives available, player i has to have well defined beliefs over the state space. Without such well defined beliefs, there is no sense in talking about expected utility. Denote by \( b^i(s|E, y_1) \) the conditional probability that player i assigns to the true state being \( s \) if he observes \((E, y_1) \in PF_2 \times Y_1\). Since we presume that player i knows that the true
state is in event $E$ whenever $(E, y_1) \in PF_i^2 \times Y_1$ is observed, we shall restrict player $i$'s beliefs to those that assign a zero probability to any state in $S \setminus E$. This means that the set of all possible second period beliefs can be described by the following set of functions.

**Definition 5.3.1.** The set of all possible second period belief functions for player $i$ is given by:

$$B^i := \left\{ b^i : S \times PF_i^2 \times Y_1 \to [0, 1] \mid \begin{array}{l}
\forall (E, y_1) \in PF_i^2 \times Y_1 : \\
(i) \sum_{s \in E} b^i(s|E, y_1) = 1 \\
(ii) \forall s \in S \setminus E : b^i(s|E, y_1) = 0
\end{array} \right\}$$

Define $B := \times_{i \in N} B^i$. Let us suppose for now that player $i$ knew that the strategy bundle used by the players was given by $(x_1, x_2)$ and ask how the player should update his beliefs in such circumstances. The players' strategy bundle being given by $(x_1, x_2)$ means that if the true state is $s$, the players take the first period action combination $x_1(s)$ with probability 1. The ex ante probability of $(E, y_1)$ occurring would then be given by $p((E, y_1)) = \sum_{s \in E : x_1(s) = y_1} p(s)$.

If the ex ante probability of $(E, y_1)$ occurring was positive, one could naturally define player $i$'s beliefs using the conditional probability of the true state being $s$:

$$b^i(s|E, y_1) = \begin{cases} 
\frac{p(s)}{\sum_{s' \in E : x_1(s') = y_1} p(s')} & \text{if } x_1(s) = y_1 \\
0 & \text{otherwise}
\end{cases} \quad (5.3.1)$$

If the ex ante probability of $(E, y_1)$ was zero, however, such a procedure no longer works. The question then becomes how player $i$ should update his beliefs given that he observes something that had a zero ex ante probability of occurrence.
To address this question, we propose a hypothesis of how player \( i \) updates his belief. It explains the updated belief for every element of \( PF_2^i \times Y_1 \) regardless of whether that element occurs with a positive or zero ex ante probability. Furthermore, for any element which has a positive ex ante probability, formula (5.3.1) applies.

First, we consider how player \( i \) would update his beliefs if players did not always take their actions according to \( x_1 \). In particular, we consider the possibility that every player \( j \), with some strictly positive probability, makes a mistake causing his action to be selected using a uniform distribution over his action space. The mistakes here are presumed to satisfy two key symmetry assumptions. The first is that the likelihood of a player making a mistake is the same in every state, and the second is that the likelihood of a mistake is directly proportional to the Lebesgue measures of the action spaces. That is, a player that has an action space with double the Lebesgue measure of another player makes a mistake twice as often as the other player. These two assumptions together imply the existence of a strictly positive number \( \epsilon \) which for every player \( j \) gives the ratio of the probability of that player making a mistake to the measure of the player’s action space. For each such strictly positive number \( \epsilon \), every action combination is taken with a strictly positive probability density in every state \( s \). This in turn allows us to update beliefs using Bayes rule for each element \((E, y_1) \in PF_2^i \times Y_1\).
Second, we consider a sequence of such positive numbers $\epsilon$ as they approach zero and consider the limiting beliefs. Given the assumptions, the limiting beliefs are the same regardless of the sequence selected. The limiting beliefs can actually be given by a relatively simple formula that provides some important intuitive insights. Clearly, the limiting beliefs are also fully consistent with formula (5.3.1).

We now enter into a more formal treatment of how beliefs would be formed for a given set of probabilities associated with mistakes. For each state $s$, denote by $q^j(s)$ the probability that player $j$ makes a mistake in state $s$. The set of all functions assigning a strictly positive probability to a mistake in every state for player $j$ is then given by:

$$Q^j := \{q^j : S \to [0,1] | \forall s \in S : q^j(s) > 0\}$$

Now, if player $j$ makes a mistake in state $s$, his action is selected from his action space using some probability density function. Denote by $r^j(\cdot | s)$ the probability density function with which his action is selected if he makes a mistake in state $s$. The set of all functions that assign such a probability density function to each state $s$ is then given by:

$$R^j := \{r^j : Y^j_1 \times S \to [0,1] | \forall s \in S : \int_{y^j_1 \in Y^j_1} r^j(y^j_1 | s) dy^j_1 = 1\}$$
Now, define \( Q := \times_{j \in N} Q^j \), and \( R := \times_{j \in N} R^j \). Assuming that each \( Y^j_1 \) is Lebesgue-measurable, we also denote by \( \mu(Y^j_1) \) the Lebesgue-measure of player \( j \)'s action space \( Y^j_1 \).

We now introduce the following assumptions, all of which involve symmetry of one form or another.

\[ \mathbf{A1} \quad \forall j \in N : [s, s' \in S] \Rightarrow [q^j(s) = q^j(s')] \]

\[ \mathbf{A2} \quad \forall (j, s) \in N \times S : \forall y^j_1 \in Y^j_1 : r^j(y^j_1 | s) = \frac{1}{\mu(Y^j_1)} \]

\[ \mathbf{A3} \quad \forall s \in S : [j, k \in N] \Rightarrow \left[ q^j(s) \mu(Y^j_1) = q^k(s) \mu(Y^k_1) \right] \]

The first assumption states that a player \( j \) has the same probability of making a mistake in every state. The second assumption states that once player \( j \) makes a mistake in state \( s \), the probability density function is uniform on his action space. Finally, the third assumption states that the ratio of the likelihood of making a mistake to the measure of the player's action space coincides for all the players.

Together, the three assumptions above turn out to have an implication of the following sort: Let \((q, r)\) be any pair in \( Q \times R \) satisfying assumptions A1-A3, then there exists a unique positive number \( \epsilon \) that uniquely identifies \((q, r)\). The following claim clarifies this.

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Claim 5.3.1. Let \( \langle q, r \rangle \) be any pair in \( Q \times R \) satisfying assumptions A1-A3. Then there exists a unique positive real number \( \epsilon \) such that for all \( \langle j, s \rangle \) in \( N \times S \) and for all \( y_i^j \in Y_i^j \),

\[
r^i(y_i^j|s)q^i(s) = \epsilon
\]

Proof. Choose any \( \langle k, s' \rangle \in N \times S \). Set \( \epsilon = \frac{q^i(s')}{\mu(Y_i^j)} \). Choose any \( \langle j, s \rangle \in N \times S \). Then from assumption A3 it follows that \( \frac{q^i(s)}{\mu(Y_i^j)} = \epsilon \). Assumption A1 then in turn implies that \( \frac{q^i(s)}{\mu(Y_i^j)} = \epsilon \). Using assumption A2, it is then straightforward to see that \( r^i(y_i^j|s)q^i(s) = \epsilon \) as required. \( \square \)

Let us now suppose that a strategy bundle \( \langle x_1, x_2 \rangle \in X_1 \times X_2 \) has been specified. If we considered a particular pair \( \langle q, r \rangle \in Q \times R \) satisfying assumptions A1-A3, the ex ante probability of the true state being \( s \) and \( y_1 \) being selected would given by:

\[
p(s, y_1) = p(s)[ \prod_{j \in N: x_i^j(s) \neq y_i^j} r^i(y_i^j|s)q^i(s) ] [ \prod_{j \in N: x_i^j(s) = y_i^j} (1 - \int_{Y_i^j \setminus \{y_i^j\}} r^i(y_i^j|s)q^i(s) \, dy_i^j) ]
\]

(5.3.2)

Using the positive number \( \epsilon \) given in our claim above, we can easily see that this in turn can be rewritten as

\[
p_{q, \epsilon}(s, y_1) = p(s)[ \prod_{j \in N: x_i^j(s) \neq y_i^j} \epsilon ] [ \prod_{j \in N: x_i^j(s) = y_i^j} (1 - \int_{Y_i^j \setminus \{y_i^j\}} \epsilon \, dy_i^j) ]
\]

(5.3.3)

which is easily seen to be a positive number.

Now, given that this in fact is a positive number, the conditional belief function for player \( i \) is well defined by
\[ \theta_{q,r}^i(s|E, y_1) = \left\{ \begin{array}{ll} \frac{p(s,y_1)}{\sum_{r' \in R} p(r', y_1)} & \text{if } s \in E \\ 0 & \text{otherwise} \end{array} \right. \] (5.3.4)

Now consider a sequence of such pairs \((q, r)\) in \(Q \times R\) where each pair satisfies assumptions A1-A3. As stated in our claim, each such pair is associated with a number \(\epsilon\). Suppose that the sequence of \(\epsilon\)'s associated with the sequence of pairs converges to zero. In this instance, we may ask what the limiting belief function of player \(i\) would look like. It turns out that the limiting belief function exists and that it is given by a relatively simple formula. That is, if beliefs are derived as limits of belief functions associated with distributions satisfying our three symmetry assumptions, the limiting belief function is uniquely determined. The following Theorem clarifies this.

**Theorem 5.3.2.** Let \(\{q_k, r_k\}_{k=1}^{\infty}\) be a sequence of pairs in \(Q \times R\) for which each pair satisfies assumptions A1-A3 and for which the associated sequence of positive numbers given in the claim converges to zero. Then the sequence of associated belief functions converges and for all \((E, y_1) \in PF_2^i \times Y_1\), and all \(s \in E\), the limit is explicitly given by the following formula:

\[ \theta^i(s|E, y_1) = \left\{ \begin{array}{ll} \frac{p(s)}{\sum_{s' \in E \setminus \{y_1\}} p(s')} & \text{if } \# \{j \in N \setminus \{i\} | x^i_j(s) \neq y^i_j \} = n^*(E, y_1) \\ 0 & \text{otherwise} \end{array} \right. \] (5.3.5)

where \(n^*(E, y_1) := \min_{s' \in E} \# \{j \in N \setminus \{i\} | x^i_j(s') \neq y^i_j \}\).

**Proof.** Letting \(\epsilon_k\) denote the unique positive number associated with the \(k\)th member of the sequence, it is straightforward to verify that for each \((E, y_1) \in PF_2^i \times Y_1\), and for each \(s \in E\), the following equalities holds for the belief function associated with the \(k\)th member of the sequence:
\[ b^t_{q_k, r_k}(s, E, y_1) = \frac{p(s)\prod_{j \in N: x_j^t(s) \neq y_j^t} \epsilon_k \prod_{j \in N: x_j^t(s) = y_j^t} (1 - \epsilon_k \int_{Y_j^{(i)}} d\hat{y}_j)}{\sum_{s' \in E} p(s')\prod_{j \in N: x_j^{s'}(s') \neq y_j^t} \epsilon_k \prod_{j \in N: x_j^{s'}(s') = y_j^t} (1 - \epsilon_k \int_{Y_j^{(i)}} d\hat{y}_j) \prod_{j \in N: x_j^{s'}(s') = y_j^t} (1 - \epsilon_k \int_{Y_j^{(i)}} d\hat{y}_j)} \]

\[ = \frac{p(s)\prod_{j \in N \setminus \{i\}: x_j^t(s) \neq y_j^t} \prod_{j \in N \setminus \{i\}: x_j^t(s) = y_j^t} (1 - \epsilon_k \int_{Y_j^{(i)}} d\hat{y}_j)}{\sum_{s' \in E} p(s')\prod_{j \in N \setminus \{i\}: x_j^{s'}(s') \neq y_j^t} \prod_{j \in N \setminus \{i\}: x_j^{s'}(s') = y_j^t} (1 - \epsilon_k \int_{Y_j^{(i)}} d\hat{y}_j)} \]

\[ = \frac{p(s)\prod_{j \in N \setminus \{i\}: x_j^t(s) \neq y_j^t} \prod_{j \in N \setminus \{i\}: x_j^t(s) = y_j^t} \prod_{j \in N \setminus \{i\}: x_j^t(s') \neq y_j^t} \prod_{j \in N \setminus \{i\}: x_j^t(s') = y_j^t} n^t(E, y_1)}{\sum_{s' \in E} p(s')\prod_{j \in N \setminus \{i\}: x_j^{s'}(s') \neq y_j^t} \prod_{j \in N \setminus \{i\}: x_j^{s'}(s') = y_j^t} n^t(E, y_1)} \]

The first of the above equalities simply uses the definition of the beliefs. The second equality should be obvious for the reader. The third is an immediate implication of the fact that \( x_j^t(s') = x_j^t(s) \) for all \( s' \in E \). The fourth equality results by dividing both the numerator and the denominator by \( \prod_{j \in N \setminus \{i\}} n^t(E, y_1) \).

Now, from this expression it is obvious to see that

\[ \lim_{\epsilon_k \to 0} b^t_{q_k, r_k}(s, E, y_1) = \begin{cases} \frac{p(y)}{\sum_{s' \in E: \#(j \in N \setminus \{i\}) x_j^{s'}(s') \neq y_j^t} \prod_{j \in N \setminus \{i\}} p(s')} & \text{if} \ \#(j \in N \setminus \{i\}) x_j^t(s) \neq y_j^t = n^t(E, y_1) \\
0 & \text{otherwise} \end{cases} \]

as required. \( \square \)

Henceforth, we will refer to belief functions satisfying the above formula as *reasonable* belief functions. Given the formula, some comments are here in order. Suppose that there exists a state \( s \in E \) satisfying the property that \( x_j^t(s) = y_1 \). Then \( n^t(E, y_1) = 0 \) and the formula in fact reduces to being the same as formula (1). More generally,
suppose that \( n^*(E; y_i) = k \). Then there is no state \( s' \in E \) satisfying the property that fewer than \( k \) players in \( N \setminus \{i\} \) have taken actions other than those they should have taken if the true state was \( s' \). However, there is at least one state \( s' \in E \) satisfying the property that the actions of exactly \( k \) of the players in \( N \setminus \{i\} \) are different from the actions they should have taken if the true state in fact was \( s' \). If player \( i \)'s belief function is reasonable, it updates using the same procedure that would be used if player \( i \) actually knew that the true state fell into the subset of states in \( E \) satisfying the property that for each state in that subset, exactly \( k \) players in \( N \setminus \{i\} \) have taken actions other than those they should have taken had it actually been the true state. That is, any state \( s' \in E \) where more than \( k \) players in \( N \setminus \{i\} \) are assigned actions other than those actually observed is assigned a probability of zero. The other states in \( E \) gets assigned the conditional probability given the information that one of them has occurred.

The formation of reasonable beliefs is thus guided by the notion that the conditional probability of strictly more than \( k \) players having made mistakes given only the information that \( k \) or more players have made a mistake should be equal to zero.

Remark 5.3.1. The problem addressed in this section is related to the refinement literature for finite games. A couple of papers deal with refinements in games where some information sets are reached with zero probability in a Nash equilibrium. There too, players are faced with the difficulty of updating beliefs upon observing something.
that had a zero ex ante probability of occurrence. Two papers in particular have addressed the problem that arises in such games and we find it appropriate here to discuss the approaches taken there.

Selten [21] addressed the problem by introducing a new equilibrium concept which he labeled a perfect equilibrium. A perfect equilibrium is the limit of a sequence of equilibria associated with $\epsilon$-perturbed games. In an $\epsilon$-perturbed game, players in information sets are forced to use local strategies that assign a minimum $\epsilon$-probability to every available alternative in that information set. For every possible strategy in such an $\epsilon$-perturbed game, every information set is always reached with a positive probability. This means that beliefs always can be updated using Bayes rule. Selten then defines an equilibrium of an $\epsilon$-perturbed game to be a set of behavior strategies satisfying the property that the local strategies in every information set is optimal given the restriction that each alternative has to be selected with some minimal probability. A perfect equilibrium is then the limit of a sequence of such equilibria for $\epsilon$-perturbed games as the minimum probability assigned to the actions in the information sets goes to zero.

Kreps and Wilson [13] formulated a related concept (sequential equilibria) that includes a formal specification of beliefs. They first consider completely mixed strategies that uniquely determine beliefs in every information set. They then restrict beliefs to those that are limits of beliefs associated with some sequence of completely mixed
strategies. Such beliefs, they refer to as consistent beliefs. A sequential equilibrium is then a pair of such consistent beliefs combined with behavior strategies that are optimal responses at every information set given the consistent beliefs.

The current model shares the approach of using limiting beliefs with the two papers discussed above. In their papers, beliefs are first defined for completely mixed strategies and then, in the limit, for general behavior strategies. Beliefs in our thesis are generated in a similar manner. The sequence of beliefs used can be viewed as resulting from a sequence of completely mixed strategies. The restriction on the sequence is stronger than that of the sequential equilibrium, since any valid sequence must satisfy the above discussed symmetry assumption. The symmetry assumptions in fact uniquely determines the limiting beliefs. Hence every two-period equilibrium can be interpreted as a sequential equilibrium, while the converse is not generally true. No comparison can be easily made between the perfect equilibrium and our solution concept. In fact, there may be perfect equilibria that fails to be supported by reasonable beliefs. Conversely, there may be two-period equilibrium strategies supported by reasonable beliefs that fails to be perfect.

5.4 SOLUTION CONCEPT AND EXISTENCE OF TWO-PERIOD EQUILIBRIA

We now turn to the formulation of an appropriate two-period equilibrium concept. The concept we formulate relies on the notion that an equilibrium concept should be optimal both in the first period and in the second period. That is, for every element
of $PF_i$, player $i$'s strategy should maximize his combined two period expected utility and for every element of $PF_i \times Y_i$, his strategy should maximize his second period expected utility. We hence propose that a two-period equilibrium be defined according to the following.

**Definition 5.4.1.** A two period equilibrium is a pair $(x^*, y^*) \in X \times B$ such that

(i) $\exists i \in N : \exists E \in PF_i^1 : \exists \tilde{x}^i \in X^i$:

$$\sum_{s \in E} p(s|E)[u^i_1(s, \tilde{x}^i_1(s), x^{N\setminus\{i\}}_1(s)) + u^i_2(s, \tilde{x}^i_2(s), x^{N\setminus\{i\}}_2(s), x^{N\setminus\{i\}}_1(s))]$$

$$> \sum_{s \in E} p(s|E)[u^i_1(s, x^i_1(s)) + u^i_2(s, x^i_2(s), x^i_1(s))]$$

(ii) $\exists i \in N : \exists (E, y_1) \in PF_i^2 \times Y_i : \exists y^i_2 \in Y^i_2$:

$$\sum_{s \in E} b^i(s|E, y_1)u^i_2(s, y^i_2, x^{N\setminus\{i\}}_2(s, y_1)) > \sum_{s \in E} b^i(s|E, y_1)u^i_2(s, x^i_2(s), y_1)$$
(iii) For every player $i$, for every $(E, y_1) \in PF_2 \times Y_1$,

$$u'(s|E, y_1) =$$

$$\begin{cases} \frac{p(s)}{\sum_{s' \in E} \# \{j \in N \setminus \{i\}| x_j(s') \neq y_j'\} = n^*(E, y_1) p(s')} & \text{if } \# \{j \in N \setminus \{i\}| x_j(s) \neq y_j'\} = n^*(E, y_1) \\ 0 & \text{otherwise} \end{cases}$$

where $n^*(E, y_1) := \min_{s' \in E} \# \{j \in N \setminus \{i\}| x_j(s') \neq y_j'\}$.

The first condition imposed says that for every element of $PF_2^i$, player $i$’s two-period expected utility should be maximized. The second condition says that for every element of $PF_2^i \times Y_1$, player $i$’s second period expected utility should be maximized given his beliefs. The third condition finally requires his beliefs to be reasonable.

In the two period model we consider here, players can learn important information from the actions undertaken in the first period. If all the players have first period strategies that are one-to-one as functions from the information partitions to the first period actions spaces, the players can infer all the first period information available from the action undertaken. That is, by looking at the inverse image of the equilibrium strategy combination, they can infer the information that the players must have had in the first period. We will therefore refer to equilibria that satisfy this property as fully information-revealing two period equilibria according to the following definition:

**Definition 5.4.2.** A fully information-revealing two period equilibrium is a two period equilibrium where it is false that there exists a player $i$ and two distinct events $E, E' \in PF_i^i$ such that $x_i(E) = x_i(E')$.
It turns out that as long as information is dispersed, it is possible to provide a generic existence result for two period fully information-revealing equilibria. The following theorem illustrates this.

**Theorem 5.4.1.** Fix \( \{S, p, F_1, PF_1, Y_1, X_1, F_2, PF_2, Y_2, X_2, u_2\} \). Assume the following:

(i) For every \((i, s) \in N \times S\), \( PF_1^i(s) \subset PF_1^i(s) \);

(ii) For every \((i, s) \in N \times S, [s' \in PF_2^i(s), s' \notin PF_2^i(s) \cap \bigcap_{k \in N} PF_k^i(s)] \Rightarrow \left| \left\{ j \in N \mid PF_1^j(s') \neq PF_1^j(s) \right\} \right| \geq 3 \};

(iii) \( p(s) > 0 \) for all \( s \) in \( S \);

(iv) For every player \( i \), \( Y_1^i \) is a compact, convex, nonempty subset of a Euclidean space of dimension at least one and has a nonempty interior;

(v) For every player \( i \), \( Y_2^i \) is a compact, convex, nonempty subset of a Euclidean space;

(vi) For every player \( i \), for every \( s \in S \), \( u_2^i(s, .) \) is a continuous function in \( Y_2 \); and

(vii) For every player \( i \), for every \( s \in S \), for every \( y_2^{N \setminus \{i\}} \in Y_2^{N \setminus \{i\}} \), \( u_2^i(s, ., y_2^{N \setminus \{i\}}) \) is a concave function in \( Y_2^i \).

Then there is an open-dense subset \( \tilde{U}_1 \) of \( U_1 \) such that for every \( u_1 \in \tilde{U}_1 \), either

(a) the one period incomplete information game \( \{S, p, F_1, PF_1, Y_1, X_1, u_1\} \) has no interior equilibrium; or

(b) the two period game \( \{S, p, \{F_1, PF_1, Y_1, X_1, u_1\}_1, \{F_2, PF_2, Y_2, X_2, u_2\}_2\} \) has a fully information-revealing two period equilibrium.
The first assumption states that player $i$'s second period information partition is finer than his second period information partition. This implies that whenever player $i$ can distinguish state $s$ from state $s'$ in the first period, he can also distinguish state $s$ from state $s'$ in the second period. That is, he does not forget information.

The second assumption is our informational dispersion assumption. It concerns any two states $s$ and $s'$ that player $i$ is unable to distinguish between using his own second period information. If pooling his second period information with the first period information of all the players would allow him to distinguish between the two states, the assumption requires that at least three of the other players could distinguish between the two states. That is, his ability to distinguish between the two states does not depend on two or fewer players.

The third assumption states that the probability of every state is strictly positive. The fourth assumption concerns the first period action spaces. Every player's action space needs to be compact, convex, non-empty valued, and of dimension at least one. The fifth assumption requires every player's second period action space to be compact, convex, and non-empty valued. Assumptions six and seven requires that for every player $i$, the second period utility function is continuous in $Y_2$, and concave in $Y_2^i$.

These assumptions together then implies that there is an open-dense subset of $U_1$.
satisfying the property that for every first period utility function bundle in that open-dense set, either the myopic one-period game has no interior equilibrium equilibrium, or the two period incomplete information game has a two period fully information-revealing equilibrium. A comment is in order concerning this result. Note that the set of first period utility function bundles satisfying the property that every players utility function is strictly concave in \( Y_1 \) is an open subset of \( U_1 \). For such functions the myopic game is known to have an equilibrium given the present assumptions. We can thus conclude that for an open-dense subset of all such strictly concave functions, either all one period equilibria, of which at least one exists, are on the boundary, or the two-period game has an equilibrium.

Remark 5.4.1. One may ask if fully information-revealing equilibria exist for two period games even when the informational dispersion assumption fails to be satisfied. Demonstrating existence under more general informational assumptions turns out to be quite difficult and we here enter into a discussion as to why the current proof fails for such games in general.

Before we consider this issue in general, we first consider the special case of a complete information game with only one state. In such games, demonstrating existence of a two period equilibrium turns out to be relatively simple. The best approach is perhaps the following.

First, find an action combination for the first period that would be an equilibrium
in a one period game. That is, a first period action combination satisfying the property that no player can improve his first period utility by changing his first period action. Such an action combination exists under standard assumptions. Now, pick any such action combination and define each player's first period strategy to be the strategy that assigns him the first period action that we found for him.

Second, find an action combination that would form an equilibrium in the second period. That is, a second period action combination satisfying the property that no player could improve his second period utility by changing his second period action as long as the other players sticks to their second period actions. Again, under standard assumptions, such an action combination in fact exists. Now, pick any such action combination and define the second period strategy of every player to be a constant function that takes on the second period action we found for that player.

Third, demonstrate that this strategy combination in fact forms a two period equilibrium. Because there is only one state, all possible beliefs are in fact reasonable. Because of how the second period strategies were defined, it is also obvious that every players second period expected utility is maximized. To see that the two period expected utility also is maximized, we note that the second period strategies are constant functions of the first period actions. This has two important implications. The first is that no player can influence the second period actions of the other players by changing his first period action. The second implication, which is a consequence
of the first, is that regardless of the choice of first period action a player may select, the second period contribution to his overall utility is maximized by the same second period action. Hence it will always be optimal for him to specify his second period strategy as we did above.

Now, from this it is easily seen that the maximum second period contribution to his overall utility the player can attain in fact is the same regardless of his first period action. Hence his first period action only needs to maximize his first period utility in order to be optimal. But this is in fact done by the first period action specified by the first period strategy we formulated for him above. Hence the strategy combination specified above in fact is an equilibrium for the complete information game.

Given this discussion, we are now ready to return to the general two-period incomplete information game. Suppose that we attempted to follow the same procedure that could successfully be used for the two-period complete information game. The first step could still be successfully employed. As demonstrated by Harsanyi [10], the one period incomplete information in fact has an equilibrium under standard assumptions. We could hence specify the first period strategy to simply be an equilibrium for this one period game.

When we get to the second step, however, we run into some difficulties. Once we specify the first period strategy as in step one, the second period beliefs are uniquely determined. For any action combination \( y_1 \in Y_1 \), we could then successfully find
an equilibrium for an associated one-period incomplete information game. However, there is no guarantee that there would exist a one-period strategy that would be an equilibrium for every first period action combination specified. This is because reasonable beliefs are functions of the first period actions and that the second period utility functions of the players therefore change as the first period action combination changes.

An equilibrium for one set of second period utility functions clearly needs not be an equilibrium for a different set of second period utility functions. Hence we cannot, in general, specify a second period strategy that is a constant function of the first period actions and at the same time is an equilibrium for the one-period incomplete information game associated with each $y_1 \in Y_1$.

If we think about step three above, however, what we really need is something weaker than that the second period strategies be constant functions of the first period action combination. What we really need is that the second period strategies be constant as functions of any one player's first period strategy given the strategies of the other players. If so, the two important implications we mentioned above would still hold. That is, no player could affect the second period actions selected by the other players by changing his first period strategy. This in turn would imply that the maximum attainable second period contribution to the player's two-period expected utility function would be the same regardless of his choice of first-period strategy.
So, under what conditions would it be possible for us to have this satisfied? Let us for a minute examine the special case where the first period strategies of the players were fully information-revealing. From Theorem 1, we know that it is a generic property of one-period incomplete information games to have all interior equilibria be fully information revealing. Denote a particular such first period strategy bundle by $x_1$. Consider player $i$. If player $i$ observes $E \in PF_1^i$, he clearly only cares about the actions the players end up taking in states that are in $E$.

Choose any state $s$ in $E$ and denote by $PF_2^k(s)$ the element of player $k$'s second period information partition that contains state $s$. If player $i$ used first period strategy $x_1^i$, then the resulting beliefs for player $k$ in state $s$ would be given by $b_k(x_1^i(s), x_1(s))$. If player $i$ instead used first period strategy $\tilde{x}_1^i$, then the resulting beliefs for player $k$ would be given by $b_k(x_1^i(s), x_1^N(s), \tilde{x}_1^i(s))$. The problem we face clearly arises if there is some player $k$ in $N$ for which these beliefs are not constant as functions of player $i$'s first period strategy. Now, given the formula for reasonable beliefs, we know that the beliefs are given as follows if player $i$ uses strategy $x_1^i$:

$$b_k(s' | PF_2^k(s), x_1(s)) =$$

$$\left\{ \begin{array}{ll}
\frac{p(s')}{\sum_{s'' \in PF_2^k(s), \#\{j \in N \setminus \{i\} \mid x_1^j(s'') \neq x_1^i(s)\} = 0} & \text{if } \#\{j \in N \setminus \{k\} \mid x_1^j(s'') \neq x_1^i(s)\} = 0 \\
0 & \text{otherwise} \end{array} \right.$$
If player $i$ uses strategy $\tilde{x}_1^i$, define $y_1 := (x_1^{N\setminus\{i\}}(s), \tilde{x}_1^i(s))$. Then player $k$'s beliefs are given by:

$$b^k(s''|PF^k_{2}(s), x_1^{N\setminus\{i\}}(s), \tilde{x}_1^i(s)) =$$

$$\begin{cases} \frac{\mu(c')}{{\sum_{s'' \in E: \#\{j \in N \setminus \{k\}: x_1^j(s') \neq y_1^j\} = n^*(PF^k(s), y_1)}} & \text{if } \#\{j \in N \setminus \{k\}: x_1^j(s) \neq y_1^j\} = n^*(PF^k(s), y_1) \\ 0 & \text{otherwise} \end{cases}$$

Note that for any strategy $\tilde{x}_1^i$ for which $\tilde{x}_1^i(s) \neq x_1^i(s)$, $n^*(PF^k(s), y_1)$ can be either one or zero. $n^*(PF^k(s), y_1) = 0$ can happen only if there exists $s' \in PF^k(s) \setminus \cap_{m \in N} PF^m_1(s)$ such that $x_1(s') = y_1$. That is, player $i$ is the only player that takes different actions in states $s$ and $s'$. Given the full information-revelation property we assumed about $x_1$, the only way this can happen is if player $i$ is the only player that can distinguish between the two states in the first period. Hence we would face some serious difficulties if player $i$ was the only player that could distinguish between the two states.

Suppose for a minute that there did not exist a state $s' \in PF^k(s) \setminus \cap_{m \in N} PF^m_1(s)$ which only player $i$ could distinguish from state $s$ in period one. Would player $k$'s beliefs then coincide with those he would have if player $i$ used strategy $x_1^i$ regardless of the strategy player $i$ chooses? Not necessarily. Note that player $k$'s beliefs assigns a probability of zero to any state $s' \in PF^k_2(s) \setminus \cap_{m \in N} PF^m_1(s)$ when player $i$ uses strategy $x_1^i$. Suppose exactly one player in addition to player $i$ could distinguish between states
$s$ and $s'$. In this instance, that player would also take different actions in the two states. If player $i$ used a strategy that sets $\tilde{x}_1^i(s) = x_1^i(s')$, $\nu^*(PF^k(s), y_1) = 1$ and state $s'$ would get assigned a positive probability by player $k$. Hence we would still face difficulties if there was some state $s'$ in $PF^k_2(s) \setminus E$ which only one player in addition to player $i$ could distinguish from state $s$ in the first period.

It turns out that the problem would disappear if for any state $s'$ in $PF^k_2(s) \setminus_{m \in N} PF^m_2(s)$, there are at least two players other than player $i$ that can distinguish state $s'$ from state $s$. To see this, suppose that player $i$ uses some alternative strategy $\tilde{x}_1^i$ for which $\tilde{x}_1^i(s) \neq x_1^i(s)$. Then given the full information-revelation property we know that $\nu^*(PF^k(s), y_1) = 1$. Now, consider any state $s' \in PF^k_2(s) \setminus \cap_{m \in N} PF^m_2(s)$. Then it clear from above that $\# \{ j \in N \setminus \{ k \} | x_1^j(s') \neq y_1^j \} \geq 2$. This is so since at least two players other than player $i$ takes different actions in state $s'$ than they do in state $s$. Hence player $k$ updates the same way regardless of how player $i$ chooses his first period strategy.

So it thus turns out that this kind of informational assumption allows us to prove the generic existence result for two-period incomplete information games. Note that the assumption in essence represents that information is sufficiently widely dispersed.
5.5 PROOF OF THEOREM 5.4.1

By Theorem 4.2.1, there exists an open-dense subset $\tilde{U}_1$ of $U_1$ such that for every $u_1 \in \tilde{U}_1$, all interior equilibria of the one-period game $\{S, p, F_1, PF_1, Y_1, X_1, u_1\}$ are fully information-revealing. We will show that this open-dense set can be used as the open-dense set in the present Theorem as well.

To see this, choose any $u_1 \in \tilde{U}_1$. If $\{S, p, F_1, PF_1, Y_1, X_1, u_1\}$ has no interior equilibrium, we are done. If it has an interior equilibrium, pick any such equilibrium and denote it by $\tilde{x}_1^*$. Clearly, $\tilde{x}_1^*$ is fully information-revealing.

We now carefully define a particular set of first period strategies, beliefs, and second period strategies and demonstrate that when defined in this matter, the resulting pair $(x^*, b^*)$ in fact is a two period equilibrium.

**Defining First Period Strategies:** Simply set $x_i^* = \tilde{x}_i^*$. Clearly, $x_i^*$ is a valid first period strategy bundle which furthermore, as required, is fully information-revealing.

**Defining Beliefs:** Simply let the beliefs be the reasonable beliefs associated with $x_i^*$. That is, let $b^*$ be defined by the following:
\[ b^i(s|E, y_i) = \]
\[
\begin{cases} 
\frac{P(s)}{\sum_{s' \in E : \#\{j \in N \setminus \{i\}|x^j_1(s') \neq y^j_1\}} P(s')} & \text{if } \#\{j \in N \setminus \{i\}|x^j_1(s) \neq y^j_1\} = n^*(E, y_i) \\
\emptyset & \text{otherwise}
\end{cases}
\]

**Defining the Second Period Strategies:**

First construct a fictitious one-period incomplete information game. Denote by \(I^1_2\) and \(PI^1_2\) respectively the information field and information partition respectively for which \(PI^1_2(s) = PF^1_2(s) \cap_{k \in N} PF^k_1(s)\). Let \(I_2 := \{I^1_2\}_{i \in N}\), and \(PI_2 := \{PI^1_2\}_{i \in N}\). Define \(G^i_2 := \{g^i_2 : S \rightarrow Y^i_2 | g^i_2 \text{ is } I^1_2\text{-measurable} \} \) and \(G_2 := \times_{i \in N} G^i_2\). Now, consider the fictitious game given by \(\{S, p, I_2, PI_2, Y_2, G_2, u_2\}\). Under the current assumptions, this game has a one-period equilibrium. Let \(g^*_2\) be any such equilibrium.

Given \(x^*_1, b^*, \) and \(g^*_2\) we use the following procedure in order to specify \(x^*_2\). Define

\[ \tilde{X}^i_2 := \{\tilde{e}^i_2 : S \rightarrow Y^i_2 | \tilde{e}^i_2 \text{ is } F^i_2\text{-measurable} \} \]

Set \(\tilde{X}_2 = \times_{i \in N} \tilde{X}^i_2\). We claim that for every \(y_1 \in Y_1\), there exists \(\tilde{x}^*_{2,y_1} \in \tilde{X}_2\) satisfying the following for every player \(i\):

(a) \(\forall E \in PF^i_2 : \exists s' \in E : \#\{j \in N|y^j_1 \neq x^j_1(s')\} \leq 1\), every \(s \in E\) satisfies the property that \(\tilde{x}^*_{2,y_1}(s) = g^*_2(s')\) for every \(s' \in E\) for which \(\#\{j \in N|y^j_1 \neq x^j_1(s')\} \leq 1\).
(b) $\forall E \in PF_2 : \exists s' \in E : \#\{j \in N | y_j^1 \neq x_j^1(s')\} \leq 1$, it is false that there exists $y_1^1 \in Y_2^1$ such that

$$\sum_{s' \in E} b(s'|E, y_1) u_2(s', y_2, x_2^{N\setminus\{i\}, y_1}(s')) >$$

$$\sum_{s' \in E} b(s'|E, y_1) u_2(s', x_2^{N\setminus\{i\}, y_1}(s'), x_2^{N\setminus\{i\}, y_1}(s'))$$
To see this, define, for each player $i$, a correspondence $\Phi^i_2 : \tilde{X}^i_2 \to \tilde{X}^i_2$ by:

$$\Phi^i_2(x_2) = \{ \tilde{x}^i_2 \in \tilde{X}^i_2 \}$$

(i) $\forall E \in PF^i_2 : \exists s' \in E : \#\{ j \in N | y^j_1 \neq x^j_1(s') \} \leq 1$, every $s \in E$ satisfies the property that $\tilde{x}^i_2(s) = g^*_2(s^*)$ for every $s^* \in E$ for which $\#\{ j \in N | y^j_1 \neq x^j_1(s^*) \} \leq 1$.

(ii) $\forall E \in PF^i_2 : \neg \exists s' \in E : \#\{ j \in N | y^j_1 \neq x^j_1(s') \} \leq 1$, it is false that there exists $y^i_1 \in Y^i_2$ such that

$$\sum_{s' \in E} b^i(s'|E, y_1) u^i_2(s', y^i_2, x^{N\setminus\{i\}}_2(s')) >$$

$$\sum_{s' \in E} b^i(s'|E, y_1) u^i_2(s', \tilde{x}^i_2(s'), x^{N\setminus\{i\}}_2(s'))$$

We claim this correspondence is non-empty valued. To see this, simply note that for any $E \in PF^i_2 \times Y^i_1 : \exists s' \in E : \#\{ j \in N | y^j_1 \neq x^j_1(s') \} \leq 1$, $g^*_2(s') \in Y^i_2$. Furthermore, $g^*_2(s'') = g^*_2(s')$ for any $s'' \in PI^i_2(s')$ and for any $s'' \in PF^i_2(s) \setminus PI^i_2(s')$, it follows from assumption (ii) and the full information-revelation property of $x^*_1$ that $\#\{ j \in N | y^j_1 \neq x^j_1(s'', s') \} \geq 2$. Hence, there is always at least one function satisfying the first requirement. Furthermore, since $Y^i_2$ is a non-empty compact set and $u^i_2$ is a
continuous function for every player $i$, there is always at least one function satisfying
the second requirement and the correspondence is hence non-empty valued.

It is also straightforward to verify that the correspondence is upper semi-continuous,
closed-valued, and convex valued.

Now, define a correspondence $\Phi_2 : \tilde{X}_2 \to \tilde{X}_2$ by $\Phi_2(x_2) = \{\Phi_2(x_2)\}_{i \in N}$.

Since $\tilde{X}_2$ can be viewed as a nonempty, convex, compact subset of a Euclidean
space and $\Phi_2$ easily can be shown to be an upper semi-continuous, non-empty-valued,
closed-valued, and convex-valued correspondence, we know from Kakutani’s fixed
point Theorem that $\Phi_2$ has a fixed point.

Given $y_i \in Y_i$, let $\tilde{x}^{*\{y\}}_2$ be any such fixed point and set $x^*_2(s, y_i) = \tilde{x}^{*\{y\}}_2$. This
procedure completely specifies $x^*_2$.

$(x^*, b^*)$ satisfies property (i).

To see this, choose any player $i$, and any $E \in PP_i^i$. We claim that the following
equalities hold:

$$
\sum_{s \in E} p(s | E) [u_1^i(s, x^*_i(s), x_{1}^{*\{i\}}(s)) + \\
(\sum_{s \in E} p(s | E) [u_1^i(s, x^*_i(s), x_{1}^{*\{i\}}(s)) + u_2^i(s, x^*_2(s, x^*_i(s), x_{1}^{*\{i\}}(s))))]
= \sum_{s \in E} p(s | E) [u_1^i(s, x^*_i(s), x_{1}^{*\{i\}}(s)) + u_2^i(s, x^*_2(s, x^*_i(s), x_{1}^{*\{i\}}(s)), g_{2}^{*\{i\}}(s))]$$

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\[
= \sum_{s \in E} p(s|E)[u_1^*(s, x_1^*(s), x_1^{*N\{i\}}(s))] \\
+ \sum_{E' \in P_{I_2}^i; E' \subseteq E} p(E'|E) \sum_{s \in E'} p(s|E')u_2^*(s, x_2^*(s, x_1^*(s), x_1^{*N\{i\}}(s)), g_2^{*N\{i\}}(s)) \]

To see that the first equality holds, set \( y_1 = (x_1^i(s), x_1^{*N\{i\}}(s)) \). Then \( \# \{ j \in N | y_1^j \neq x_1^{*j}(s) \} \leq 1 \). It then follows from the definition of \( x_1^{*j} \) that \( x_2^{*j}(s, x_1^i(s), x_1^{*N\{i\}}(s)) = g_2^{*j}(s) \) for all \( j \in N \setminus \{ i \} \) as required.

The second equality is gotten by making use of the fact that for any \( E' \in P_{I_2}^i \) satisfying the property that \( E' \subseteq E \), it is true for every \( s \in E' \) that \( p(s|E) = p(s|E') * p(E'|E) \).

We now show that the last expression in fact is maximized by strategy \( x^{*i} \). To see this, pick any first period strategy \( x_1^i \) for player \( i \). Set \( y_1 = (x_1^i(s), x_1^{*N\{i\}}(s)) \).

We claim that given any such strategy \( x_1^i \), the remaining terms are maximized by setting \( x_2^*(s, x_1^i(s), x_1^{*N\{i\}}(s)) = g_2^*(s') \) for all \( s' \in P_{I_2}^i(s) \) for which \( \# \{ j \in N | y_1^j \neq x_1^{*j}(s) \} \leq 1 \). To demonstrate this, we show that for any first period strategy choice \( x_1^i \) for player \( i \), player \( i \) is free to set \( x_2^*(s', x_1^i(s'), x_1^{*N\{i\}}(s')) \neq x_2^*(s, x_1^i(s), x_1^{*N\{i\}}(s)) \) if and only if \( s' \notin P_{I_2}^i(s) \). To see this, suppose \( s' \notin P_{I_2}^i(s) \). Then \( s' \in P_{I_2}^i(s) \), and \( (x_1^i(s'), x_1^{*N\{i\}}(s')) = (x_1^i(s), x_1^{*N\{i\}}(s)) \). In order for our measurability requirement to be satisfied, it must be that for any acceptable second period strategy,

\[
= x_2^*(s, x_1^i(s), x_1^{*N\{i\}}(s)) \]

If \( s' \notin P_{I_2}^i(s) \), then either \( s' \notin P_{I_2}^i(s) \).
or there exists a player $j \in N \setminus \{i\}$ for which $x_1^{i*}(s') \neq x_1^{ij}(s)$. In either of these two cases, player $i$ is free to set $x_1^j(s', x_1^i(s'), x_1^{N \setminus \{i\}}(s')) \neq x_2^j(s, x_1^i(s), x_1^{N \setminus \{i\}}(s))$.

From this, it is clear that for a given strategy $x_1^i$, our above expression is maximized if and only if the term for each $E' \in PI_2 : E' \subset E$ is individually maximized. Clearly this is achieved by setting $x_1^j(s, x_1^i(s), x_1^{N \setminus \{i\}}(s)) = g_2^i(s)$. Given this, it is straightforward to see that the maximum attainable utility from the second period is the same regardless of what first period strategy the player selects. Hence, in order to maximize his overall utility, player $i$ should choose his first period strategy so as to maximize his first period utility. This is clearly done by setting $x_1^i(s) = x_1^{i*}(s)$ for all $s \in S$. Player $i$ hence solves his maximization problem by using a strategy that sets $x_1^i(s) = x_1^{i*}(s)$ for all $s$ in $S$ and $x_1^j(s', x_1^{i*}(s), x_1^{N \setminus \{i\}}(s)) = g_2^i(s)$ for all $s' \in PF_2(s)$ as required. Since the equilibrium strategy does so, it is in fact a solution to player $i$’s two-period problem and it hence satisfies condition (i).

$(x^*, b^*)$ satisfies condition (ii). To see this, choose any $i \in N$, and any $(E, y_1) \in PF_2 \times Y_1$. If it is false that there exist $s' \in E$ for which $\# \{j \in N | y_1^j \neq x_1^{ij}(s')\} \leq 1$, we are done because of how the correspondence $\Phi_2^i$ was defined. If there exists $s' \in E$ such that $\# \{j \in N | y_1^j \neq x_1^{ij}(s')\} \leq 1$, let $s^*$ be any such state. We claim that the following equalities hold.
\[
\sum_{s \in E} b^*(s|E, y_1) u^i_2(s, y_2, x_2^{N\{i\}}(s, y_1)) = \sum_{s \in \text{PI}_2^*(s^*)} b^*(s, E, y_1) u^i_2(s, y_2, x_2^{N\{i\}}(s, y_1))
\]

\[
= \sum_{s \in \text{PI}_2^*(s^*)} p(s|\text{PI}_2^*(s^*)) u^i_2(s, y_2, x_2^{N\{i\}}(s, y_1))
\]

\[
= \sum_{s \in \text{PI}_2^*(s^*)} p(s|\text{PI}_2^*(s^*)) u^i_2(s, y_2, y_2^{N\{i\}}(s))
\]

The first equality follows because \( b^* \) satisfies the property that \( b^*(s|E, y_1) = 0 \) for all \( s \in E \setminus \text{PI}_2^*(s^*) \). To see this, note that \( n^*(E, y_1) \leq 1 \) under our current assumptions.

Now, \( b^*(s|E, y_1) > 0 \) for a state \( s \in E \) if and only if \( \#\{j \in N \setminus \{i\}| x_1^j(s) = y_1^j \} = n^*(E, y_1) \). Suppose there exist such an \( s \) in \( E \setminus \text{PI}_2^*(s^*) \). Then since \( x_1^i(s) = x_1^i(s^*) \), it must be that \( \#\{j \in N \setminus \{i\}| x_1^j(s) = x_1^j(s^*) \} \leq 2 \). It then follows from assumption (ii) that there exist a player \( k \in N \setminus \{i\} \) for which \( x_1^k(s) = x_1^k(s^*) \) and \( PF_1^k(s) \neq PF_1^k(s^*) \).

But this contradict the full information-revelation property of \( x_1^* \). Hence \( b^*(s|E, y_1) \) must be equal to zero for all \( s \in E \setminus \text{PI}_2^*(s^*) \).

The second equality follows from the definition of \( b^* \) and the fact that \( x_1^i(s) = x_1^i(s^*) \) for all \( s \in \text{PI}_2^*(s^*) \). This is so since \( s \in \text{PI}_2^*(s^*) \) requires \( s \in PF_2^k(s^*) \) for all \( k \in N \). Hence the measurability requirement on the first period strategies leads to \( x_1^k(s) = x_1^k(s^*) \) for all \( s \in \text{PI}_2^*(s^*) \). To see that the third equality holds, note that
for all $s \in PL^{i}_{2}(s^{*})$, $\#\{j \in N | y_{i}^{j} \neq x_{1}^{*j}(s)\} \leq 1$. Then for any such $s$, it follows from the definition of $x^{*}$ that for all $j \in N \setminus \{i\}$, $x_{2}^{*j}(s, y_{1}) = g_{2}^{*j}(s)$.

Clearly, due to the definition of $g_{2}^{*}$, this last expression is maximized for player $i$ by setting $x_{2}^{*i}(s, y_{1}) = g_{2}^{*i}(s)$ for all $s \in PL^{i}_{2}(s^{*})$ as required.

$(x^{*}, b^{*})$ satisfies condition (iii).

Trivial since we picked $b^{*}$ to be the reasonable beliefs associated with $x_{1}^{*}$. This concludes our proof. □
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