Coupled Harmonics: Estimation and Detection

A Thesis

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By

Gene Whipps, B.S.

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The Ohio State University

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Master's Examination Committee:
Dr. Randolph Moses, Adviser
Dr. Lee Potter

Approved by

Adviser
Department of Electrical Engineering
ABSTRACT

This thesis considers the detection, order selection, and parameter estimation of coupled harmonic signal in noise. Coupled harmonic signals are periodic signals that can be written as the sum of a sinusoid at a fundamental frequency and one or more sinusoids at harmonic frequencies. Two main problems are addressed in this work: combined order selection and parameter estimation of coupled harmonic signals in noise, and combined signal detection and parameter estimation of coupled harmonic signals in noise. These problems have importance in a number of applications, including speech processing, vibration analysis, and battlefield seismsics and acoustics. For example, many battlefield vehicles generate frequency coupled harmonic acoustic signals. The features of these signals are useful for tracking and classifying battlefield targets.

First, combined order selection and parameter estimation is considered. A comparative study is performed between nonlinear least-squares (NLS), NLS combined with Minimum Description Length (MDL) order selection, and an approximation to NLS combined with MDL for estimating the parameters of frequency coupled harmonics. Performance results using simulated data are presented in terms of bias and root-mean-squared error (RMSE). The RMSE is compared against the square root of the large-sample Cramér-Rao lower bound (CRLB). In addition, a qualitative analysis is presented for measured acoustic data. We find that the performance of NLS
with MDL and approximated NLS with MDL is well-predicted by the CRLB above a data length-dependent SNR threshold. For the data lengths considered, the approximated NLS with MDL exhibits comparable performance, with fewer computations, to the exact NLS with MDL. Finally, qualitative results are presented for a measured acoustic signal from a military vehicle.

Second, the problem of detecting the presence of a coupled harmonic signal in noise is considered. An introductory analysis is presented for a generalized likelihood ratio test (GLRT) for known number of harmonics and a GLRT combined with MDL for unknown number of harmonics. Theoretical distributions are developed for the detectors in both cases. Performance results using simulated data are presented in terms of test distributions and receiver operating characteristic (ROC) curves. Performance comparisons between simulation and theory are made as a function of the number of harmonics and SNR for two signal amplitude models. We find that the detection statistics are well-modelled by theory in many cases. When the theory fails, simulations show that the detection probability is generally higher than predicted.
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VITA

October 1, 1975 .......................... Born - Columbus, Ohio

2002 ........................................ B.S. Electrical Engineering,
               The Ohio State University,
               Columbus, Ohio

2002-present .............................. Graduate Research Associate,
               The Ohio State University

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CHAPTER 1

INTRODUCTION

In many acoustic and seismic sensing applications, including speech analysis and military applications, coupled harmonics are common signal structures. Coupled harmonic signals are periodic signals that can be written as the sum of a sinusoid at a fundamental frequency (FF) and one or more sinusoids at harmonic frequencies. In many applications, the FF and the number of harmonics (NOH) are not known \textit{a priori}, and it is of interest to estimate them.

This thesis considers two related problems in coupled harmonic signals. First, combined order selection and parameter estimation is addressed. In this problem, it is of interest to estimate the number of harmonic terms present (the order), and to simultaneously estimate the FF along with the amplitudes and phases of each harmonic term. This problem has application in speech processing [30], vibration analysis [8], and battlefield acoustics [34]. The battlefield acoustics application is the primary motivating application of this thesis. For this application, features such as relative amplitudes and phases of harmonic components have been found to be useful for classifying and tracking military vehicles from acoustic signal measurements taken by one or more sensors [16, 34].
The second problem considered is the detection of a coupled harmonic signal in noise. This problem also has application in military sensing applications. One such application involves using a network of sensors. In one scenario, a subset of sensors act as detection devices, operating at low power. If a signal is detected, these sensors alert other sensors in the network to “wake up” and collect more detailed information about the source signal (e.g., to estimate model order and harmonic features). More details on each of these two topics are provided in Sections 1.2 and 1.3 below.

1.1 Outline of Thesis

The thesis is organized as follows. In Chapter 2, the coupled harmonic model is presented. In Chapter 3, nonlinear parameter estimation and model order selection procedures are discussed. Chapter 4 discusses the combined signal detection and parameter estimation problem, and presents an analysis of the detection and false alarm probabilities in this case. This thesis is then concluded in Chapter 5.

1.2 Parameter Estimation

Chapter 3 presents a combined detection-estimation algorithm for determining the parameters of coupled harmonics. Estimating the frequencies in harmonic models is nonlinear in the FF and the number of harmonics (NOH). A search over candidate FF and NOH values is used; for each candidate FF and NOH, the Cartesian amplitude parameters can be computed in closed form from a set of linear equations.

One application of the coupled harmonic estimation problem is classification of battlefield targets from measured acoustic signals. As discussed in [34], harmonic line association (HLA) is a feasible approach to target identification in single or multiple
target scenarios. In addition, the harmonic line estimates may be useful in improving target tracking and counting. For distributed sensor networks, the harmonic line estimates can also be used in conjunction with direction-of-arrival (DOA) estimates to separate targets temporally and spatially [3]. Once the coupled harmonic parameters are estimated and subtracted from the measured signal, the residual allows any broadband energy to be further processed and exploited.

This work investigates the case of a single source generating frequency coupled harmonics in the presence of Gaussian noise. The observations are recorded from a single acoustic sensor. The parameters of frequency coupled harmonics are estimated using nonlinear least-squares (NLS) [27]. Previous works have assumed the NOH is known [6, 9, 14, 15, 27]. In this work, as with [5, 10], the NOH is assumed unknown. The NLS method is combined with order selection methods (such as Rissanen’s Minimum Description Length (MDL) [20]), to estimate the NOH and generate statistically efficient parameter estimates in both white and colored noise.

Previous work by Dommermuth [5] has shown that the FF can be estimated to within an integer multiple or rational fraction of the true FF. Integer fractions and integer factors of the FF will be referred to as sub-harmonics and super-harmonics, respectively. For estimators based on the minimization of the squared error, the difficulty lies in the multimodal shape of the error (or loss) function. For the harmonic model, the loss function will typically have deep troughs at multiples of the true FF. The relative levels of the troughs depend on the NOH in the candidate signal and the amplitudes of the harmonics. For example, for the case in which a noiseless signal contains equal amplitude harmonics related by a FF, if the NOH in the signal is underestimated as one-half the true number, then the troughs corresponding to
\( \omega_0 \) and \( 2\omega_0 \) will be the lowest in the loss function and nearly equal relative to each other. This trend can be seen in Figure 3.1(a). A similar pattern arises for the super-harmonics when the NOH is underestimated. The mis-estimation of a FF as a sub-or super-harmonic results in large estimate variances. The dependence of the loss function on the FF and the NOH is the motivating factor for this work.

In [5], combined order selection and FF estimation is also considered. There, estimators are developed assuming a more restrictive model of frequency coupled harmonics with equal amplitudes. Consequently, the problem of estimating the amplitudes and phases is not considered. We develop an approach in which we also estimate the amplitudes and phases of a more general coupled harmonic model. We also consider several order selection strategies.

Methods are proposed in [10], similar to those provided here, to jointly estimate the coupled harmonics and autoregressive (AR) noise parameters and model orders. In contrast, we assume the AR noise parameters are known and used to whiten the measurements prior to estimating the harmonic signal parameters. Assuming the noise model is known may have validity for battlefield acoustics. In some scenarios, long periods of inactivity allows sensors to estimate the local noise properties to a higher degree of accuracy (i.e., large sample lengths) compared to the shorter sample length estimates in [10]. In addition, the algorithms proposed here take advantage of the shape of the loss function in order to reduce the computational complexity.

Algorithms are proposed in [9, 15] to track the time-varying parameters of coupled harmonics. These algorithms rely on accurate initial FF estimates and assume the
NOH is known. In applications where the parameters are slowly varying, the algorithms proposed here can be used to initialize and periodically update the tracking algorithms.

1.3 Signal Detection

Chapter 4 presents combined estimation-detection procedures for detecting frequency coupled harmonics in noise. As discussed above, some applications include post-processing of signal parameter estimates (e.g., harmonic line association or direction-of-arrival estimation). When power constraints limit processing times, knowledge of the existence or absence of a signal can be used to turn on or off power-consuming post-processing algorithms. In other applications, the presence or absence of a harmonic signal may be the only necessary information.

Detector structures are developed for both the cases of known and unknown NOH. The detection decision for known NOH is generated using a generalized likelihood ratio test (GLRT). In the case of unknown NOH, the GLRT is combined with the MDL. Theoretical receiver operating characteristics (ROCs) are developed for both cases.

Relatively little work has been done in the area of signal detection for frequency coupled harmonics. Previous work by [36], considered a frequency coupled harmonic model under the assumption the FF and the NOH is known. Scharf considered linear subspace signals in noise with [23] and without [22] subspace interference. In each work, the signal subspace was assumed to be known (i.e., for comparison, FF and NOH known) with both known and unknown coordinates. We extend the work in [22]
by considering the effects of estimating the FF and NOH. The work in [22] provides the statistical groundwork for the signal detectors analyzed in this thesis.

Detection of frequency coupled harmonics is also considered in [5] for the case of unknown NOH. However, they consider a more restrictive model of equal energy harmonics. We develop a detection approach that also considers a more general coupled harmonic model.
CHAPTER 2

SIGNAL MODEL

Many signals are well-represented as a sum of harmonic components and broadband energy. Periodicities evident in seismic and acoustic measurements are due to, among other things, rotating mechanical components or resonant vibrations. Broadband energy is partly due to the impulsive nature of events or "noise-like" processes, such as turbine engines. Figure 2.1 is a spectrogram of the measured acoustic signature of a heavy-tracked vehicle. In addition to the broadband signal, a coupled harmonic structure is evident.

In this chapter, the sum-of-harmonics plus broadband noise model is presented. The continuous-time signal for a single source generating frequency coupled harmonics is given by

\[
s(t) = \sum_{k=1}^{q} \alpha_k \cos(2\pi kf_0 t + \phi_k), \quad (2.1)
\]

\[
= \sum_{k=1}^{q} u_k \cos(2\pi kf_0 t) + v_k \sin(2\pi kf_0 t), \quad (2.2)
\]

where \( f_0 \) is the fundamental frequency (FF) in Hertz, \( k \) is the integer corresponding to the \( k \)th harmonic, and \( q \) is the number of harmonics (NOH) present in the signal. As observed in Figure 2.1, the FF may be time-varying. Here, it is assumed the FF is
Figure 2.1: Spectrogram from the acoustic signature of a heavy-tracked vehicle collected at Aberdeen Proving Grounds (APG) for the U.S. Army Research Laboratory’s battlefield acoustic database.

constant over the observation window. An observation window of one second is often used for acoustic measurements of vehicles [17].

Some previous works, for example [6, 12, 13, 14, 15], assume the NOH is known. Assuming a known NOH may be too restrictive. As mentioned above, a frequency search may result in a biased fundamental frequency estimate (FFE), which may be a multiple of the true FF. Estimating an integer multiple of the true FF may be acceptable provided the assumed NOH is less than the true value by an integer fraction. For example, if the FF estimate is twice \( f_0 \), then the estimated harmonics correspond to the even harmonics of the true signal. Then, the assumed NOH should be less than half the true number. Otherwise, some periodic components in the estimated signal will correspond to noise only components from the measured signal. Furthermore, estimating a fraction of the FF will always generate components from
noise only spectral regions. The noisy components adversely affect, for example, DOA estimates. Therefore, in this work the NOH, \( q \), is treated as a parameter to be estimated.

Assuming a perfectly calibrated acoustic sensor, the observed sampled signal is

\[
y(n) = s(n) + \epsilon(n),
\]

\[
= \sum_{k=1}^{q} u_k \cos(k\omega_0 n) + v_k \sin(k\omega_0 n) + \epsilon(n),
\]

(2.3)

where \( \omega_0 = 2\pi f_0 T \) is the fundamental angular frequency satisfying \( \omega_0 \in (0, \pi/q) \), \( \epsilon(n) \) is modelled as a zero-mean, additive Gaussian noise sequence, \( n \) is the sample index, and \( T \) is the sampling period. The term \( \epsilon(n) \) can also represent noise and unmodelled broadband energy. It is of interest to estimate \( \omega_0, q \), the amplitudes \( \{\alpha_k\} \), and phases \( \{\phi_k\} \). The desired amplitudes and phases are calculated from estimates of the Cartesian amplitudes \( \{u_k\} \) and \( \{v_k\} \) by

\[
\alpha_k = \sqrt{u_k^2 + v_k^2},
\]

\[
\phi_k = \arctan(-v_k/u_k).
\]

(2.4)

Although the focus of this document is on estimating the harmonic signal structure, this work is also useful for enhancing the broadband information by subsequent removal of estimated narrowband components.

The observed signal may also be written in matrix form. Let \( \mathbf{y} \) be a vector of sampled sensor data for sample indices \( n = 0, \ldots, N - 1 \). Using Equation (2.3), the harmonic model can be written as

\[
\mathbf{y} = \mathbf{C}(\omega_0)\mathbf{u} + \mathbf{S}(\omega_0)\mathbf{v} + \mathbf{\epsilon}, \quad (N \times 1)
\]

\[
= \mathbf{A}(\omega_0)\mathbf{\theta} + \mathbf{\epsilon},
\]

(2.5)
where

$$
\mathbf{u} = [u_1, \ldots, u_q]^T, \quad (q \times 1) \\
\mathbf{v} = [v_1, \ldots, v_q]^T, \quad (q \times 1)
$$

are the amplitude vectors, and the elements of the \((N \times q)\) matrices \(\mathbf{C(\omega)}\) and \(\mathbf{S(\omega)}\) are given by

$$
[\mathbf{C(\omega)}]_{n,k} = \cos(k\omega n + \varphi_k), \quad 0 \leq n \leq N - 1 \\
[\mathbf{S(\omega)}]_{n,k} = \sin(k\omega n + \varphi_k), \quad 1 \leq k \leq q
$$

(2.7)

where \(n\) and \(k\) are the row and column indices, respectively, and \(\varphi_k = k\omega(N - 1)/2\).

The phase term, \(\varphi_k\), defines the phase at the middle of the observation window and guarantees \(\mathbf{C(\omega)^T S(\omega)} = \mathbf{0}\). From the second line in Equation (2.5), it follows that \(\mathbf{A(\omega)} = [\mathbf{C(\omega)}, \mathbf{S(\omega)}]\) and \(\mathbf{\beta} = [\mathbf{u}^T, \mathbf{v}^T]^T\). Amplitude and phase vectors are defined as

$$
\mathbf{\alpha} = [\alpha_1, \ldots, \alpha_q]^T, \quad (q \times 1) \\
\mathbf{\phi} = [\phi_1, \ldots, \phi_q]^T, \quad (q \times 1)
$$

(2.8)

(2.9)

The amplitude vectors \(\mathbf{u}\) and \(\mathbf{v}\) are viewed as the Cartesian coordinates in the signal subspace \(\langle \mathbf{A(\omega_0)} \rangle\), where the notation \(\langle \mathbf{H} \rangle\) denotes the space spanned by the columns of matrix \(\mathbf{H}\). The Cartesian coordinates are related to their polar counterparts by

$$
\mathbf{u} = \mathbf{\alpha} \odot \cos(\mathbf{\phi}), \\
\mathbf{v} = -\mathbf{\alpha} \odot \sin(\mathbf{\phi}),
$$

(2.10)

(2.11)

where \(\mathbf{x} \odot \mathbf{z}\) is the element-wise product of \((m \times 1)\) vectors \(\mathbf{x}\) and \(\mathbf{z}\), and \(\cos(\mathbf{x}) = [\cos(x_1), \ldots, \cos(x_m)]^T\) with \(\sin(\mathbf{x})\) similarly defined.
The last term in Equation (2.5) is a vector of noise samples distributed as $\epsilon \sim \mathcal{N}(0, \Sigma)$, where $\Sigma$ is the $(N \times N)$ covariance matrix. The harmonic signal and broadband noise are considered statistically independent.
CHAPTER 3

COUPLED HARMONIC PARAMETER ESTIMATION

Maximum likelihood (ML) techniques are a popular class of algorithms used for estimating deterministic parameters of line spectra. When the number of harmonic lines are unknown, the ML algorithms are commonly combined with order selection methods, such as Rissanen’s Minimum Description Length (MDL) [20]. The maximum likelihood methods (MLMs) for parameter estimation in white and colored noise are presented in Section 3.1 under the assumption the number of harmonics (NOH) is known. Although the structure of the noise is assumed to be known in this thesis, techniques are provided to estimate noise parameters. Then, the order selection procedures are introduced in Section 3.2. In addition, the proposed combined detection-estimation algorithms for coupled harmonics are presented. Some practical issues related to the algorithms are discussed. Section 3.3 presents practical numerical examples that demonstrate the statistical properties of the algorithms. Section 3.4 compares parameter estimates with the short-time Fourier transform (STFT) of measurement data. Then, concluding remarks and observations on the combined algorithms are given in Section 3.5.
3.1 Maximum Likelihood Estimation

The parameter vectors in terms of the polar and Cartesian coordinates are, respectively, defined as

\[
\theta^{\text{pol}} = [\omega_0, \alpha_1, \ldots, \alpha_q, \phi_1, \ldots, \phi_q]^T, \quad (2q + 1 \times 1) \tag{3.1}
\]
\[
\theta^{\text{cart}} = [\omega_0, u_1, \ldots, u_q, v_1, \ldots, v_q]^T. \quad (2q + 1 \times 1) \tag{3.2}
\]

It is desired to estimate the parameter vector \( \theta^{\text{pol}} \). Due to a 1:1 transformation, estimating \( \theta^{\text{cart}} \) is equivalent to estimating \( \theta^{\text{pol}} \). The maximum likelihood estimate (MLE) of \( \theta = \theta^{\text{cart}} \) is found by

\[
\hat{\theta} = \arg \max_{\theta \in \Lambda} p(y|\theta), \tag{3.3}
\]

where \( p(y|\theta) \) is the density function of \( y \) conditioned on \( \theta \), and \( \theta \) is assumed to be deterministic and unknown, and \( \Lambda \subseteq \mathbb{R}^{2q+1} \). The conditional density is often referred to as the likelihood function. For the given signal and noise models, Equation (3.3) can be written as

\[
\hat{\theta} = \arg \max_{\theta \in \Lambda} \frac{1}{\sqrt{2\pi} \det \Sigma} \exp \left\{ \frac{1}{2} (y - s(\theta))^T \Sigma^{-1} (y - s(\theta)) \right\}, \tag{3.4}
\]

where \( s(\theta) = A(\omega_0)\beta \). Due to the monotonicity of the natural logarithm, maximizing the likelihood function is equivalent to minimizing the negative log-likelihood function given by

\[
J(\theta) = \frac{N}{2} \ln (2\pi) + \frac{1}{2} \ln (\det \Sigma) + \frac{1}{2} (y - s(\theta))^T \Sigma^{-1} (y - s(\theta)). \tag{3.5}
\]

The number of parameters to be estimated in general sinusoidal summation models with \( q \) components (i.e., \( \{\omega_k, \alpha_k, \phi_k\}_{k=1}^q \)) is \( 3q \). However, in the coupled harmonics
model, the number of parameters is reduced to \(2q + 1\) as all frequency components are described by a single parameter, \(\omega_0\). In this work the parameter vector \(\theta\) and \(q\) are presumed to be constant in \(N\) samples. This chapter describes the procedure used to estimate \(\theta\) from \(N\) finite samples.

### 3.1.1 White Gaussian Noise

With the additional assumption of white noise \((\Sigma = \sigma^2 I)\), the parameter vector \(\theta\) that minimizes \(J(\theta)\) also minimizes

\[
L(\theta) = \|y - s(\theta)\|^2, \\
= \|e\|^2. \tag{3.6}
\]

Equation (3.6) is the squared norm of the difference between the measurement and the signal model. Note that the constant terms in Equation (3.5) are dropped since they do not impact the minimization of \(J(\theta)\). For the chosen model of \(s(n)\), the estimation of the parameter vector \(\theta\) is a highly nonlinear process. However, the minimization of Equation (3.6) can be achieved by the method of linear least-squares (LLS) if \(\omega_0\) and \(q\) are assumed known. In practice, a grid search over candidate \(\omega\) is performed. For a candidate \(\omega\) and a fixed \(q\), the Cartesian amplitude estimates are found by

\[
\hat{\beta}(\omega) = \arg \min_\beta \|y - A(\omega)\beta\|^2, \\
= \arg \min_{[u,v]} \|y - C(\omega)u - S(\omega)v\|^2. \tag{3.7}
\]

The LLS solutions to Equation (3.7) can be decoupled and obtained by

\[
\hat{u} = (C^T C)^{-1} C^T y, \\
\hat{v} = (S^T S)^{-1} S^T y, \tag{3.8}
\]
where the dependance on the frequency has been suppressed to simplify the notation.

Given \( \hat{u} \) and \( \hat{v} \), the estimates of the amplitudes \( \{ \hat{a}_k \} \) and phases \( \{ \hat{\phi}_k \} \) are found using Equation (2.4).

It is straightforward to show that the product \( A^T A \) must be block diagonal to ensure exact decoupling of the amplitude estimates. Diagonality is guaranteed by defining the phase of the candidate signal at the center of the observation window. Otherwise, the above product is only approximately block diagonal for large data lengths. As noted in [14], there is a computational benefit from decoupling the amplitude estimates. Assuming \( N \gg q \), the computational complexity, defined as the number of multiply-and-accumulate operations, of computing the LS amplitude estimates in Equation (3.8) is \( \mathcal{O}(2Nq^2) \). The computational complexity more than doubles without decoupling.

Using the LS amplitude estimates of Equation (3.8), the loss function for FFES is given by

\[
L(\omega) = \| y - C\hat{u} - S\hat{v} \|^2,
\]

\[
= \| y - C(C^T C)^{-1} C^T y - S(S^T S)^{-1} S^T y \|^2,
\]

\[
= \| y - A(A^T A)^{-1} A^T y \|^2,
\]

\[
y^T P_q^*(\omega)y,
\]

(3.9)

where \( P_q^*(\omega) = I - P_q(\omega) = I - A(A^T A)^{-1} A^T \) projects the observation into the left null space of \( A \). Finally, the ML FFE is given by

\[
\hat{z}_0 = \arg \min_{\omega \in \Lambda_0} L(\omega),
\]

\[
= \arg \min_{\omega \in \Lambda_0} y^T P_q^*(\omega)y,
\]

(3.10)
where $\Lambda_\omega = (0,\pi/q)$. This minimization procedure is also known as the nonlinear least-squares (NLS) method. In determining $\hat{\omega}_0$, the estimator attempts to minimize (maximize) the energy in the null space (column space) of $A$. Although there are $2q+1$ free parameters, the NLS method reduces the parameter search to a 1-dimensional (1-D) search. Once $\hat{\omega}_0$ has been computed, the ML Cartesian amplitude estimates are simply given by Equation (3.8).

The rank of $P_\omega(\omega)$ is equal to the dimension of the column space of $A$, denoted $\text{dim} \mathcal{R}(A)$. The rank of $P_\omega^\perp(\omega)$ is equal to the dimension of the left null space of $A$, denoted $\text{dim} \mathcal{N}(A^T)$. It is evident from the definition of $A$ that $\text{dim} \mathcal{R}(A) = 2q$ and, consequently, $\text{dim} \mathcal{N}(A^T) = N - 2q$, provided that $\omega \in (0,\pi)$. Thus, the subscript of $P_\omega(\omega)$ and $P_\omega^\perp(\omega)$ symbolizes the dependence of the projection matrices on the NOH.

Now, an approximation to the NLS method is considered. For large sample lengths (i.e., $N \to \infty$) the approximations $C^T C \approx \frac{N}{2} I$ and $S^T S \approx \frac{N}{2} I$ are made. Therefore, the loss function of Equation (3.9) can be approximated, as similarly done in [11, 14, 19], by

$$L(\omega) \approx \left\| y - \frac{2}{N} CC^T y - \frac{2}{N} SS^T y \right\|^2,$$

$$= \left\| \left(I - \frac{2}{N} AA^T\right)y \right\|^2. \quad (3.11)$$

The FFEs from the minimization of Equation (3.11) are not ML for finite data lengths, but are asymptotically. Furthermore, the approximation to the FF MLE provides computational savings when $q > 2$. The computational complexity of Equation (3.9) is $O(2Nq^2)$, whereas the complexity of Equation (3.11) is $O(4Nq)$. 

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3.1.2 Colored Gaussian Noise

Next, the case of colored noise is examined. The symmetric positive definite matrix $\Sigma^{-1}$ can be written as

$$\Sigma^{-1} = W^TW. \quad (3.12)$$

Note that $W = \sigma^{-1}I$ in the uncorrelated noise case (i.e., when $\Sigma = \sigma^2I$). In either the correlated or uncorrelated Gaussian noise case, minimizing Equation (3.5) is equivalent to minimizing

$$L_W(\theta) = ||W(y - s(\theta))||^2,$$

$$= ||We||^2. \quad (3.13)$$

For known $\omega_0$, the ML amplitude estimates are the weighted linear least-squares (WLLS) solutions given by [28]

$$\hat{u}_W = (C^TW^TWC)^{-1}C^TW^TWy,$$

$$\hat{v}_W = (S^TW^TWS)^{-1}S^TW^TWy. \quad (3.14)$$

Thus, the loss function for FFEs becomes

$$L_W(\omega) = ||W(y - C\hat{u}_W - S\hat{v}_W)||^2,$$

$$= ||P^W_\gamma(\omega)Wy||^2, \quad (3.15)$$

where $P^W_\gamma(\omega) = I - WA(A^TW^TWA)^{-1}A^TW$. Then, the FF MLE is found by minimizing $L_W(\omega)$.

When the statistics of the noise are not known a priori, the weighting matrix $W$ must be estimated for Equations (3.14) and (3.15) to be of practical use. In
some applications, sensors may have multiple functions, such as signal detection (i.e.,
deciding whether a signal is present or not) and parameter estimation (i.e., estimating
the noiseless signal if one is present). If a source is not detected, it would be practical
for the sensing module to operate at lower levels to conserve power. During these
conservative operational modes, the local noise structure can be modelled. In such a
situation, the covariance matrix can be replaced by an estimate such as the standard
unbiased autocorrelation sequence (ACS) estimate, given by [25]

\[ \hat{\Sigma} = \frac{1}{M - N + 1} \sum_{n=N}^{M} \begin{bmatrix} \epsilon(n) \\ \vdots \\ \epsilon(n - N + 1) \end{bmatrix} \begin{bmatrix} \epsilon(n) & \ldots & \epsilon(n - N + 1) \end{bmatrix}, \quad (N \times N) \]  

(3.16)

for \( M \gg 2N - 2 \). Using the relationship in Equation (3.12) and an estimate of the
covariance, Equation (3.14) is approximated by

\[ \hat{u}_w \approx \left( C^T \hat{\Sigma}^{-1} C \right)^{-1} C^T \hat{\Sigma}^{-1} y, \]

\[ \hat{v}_w \approx \left( S^T \hat{\Sigma}^{-1} S \right)^{-1} S^T \hat{\Sigma}^{-1} y. \]  

(3.17)

Then, the FFEs are computed using Equation (3.15) with an estimate, \( \hat{W} \), of \( W \),
which can be obtained as a matrix square root of \( \hat{\Sigma}^{-1} \). Note that matrix square roots
are not unique. Of course, using the covariance estimate assumes the noise is suffi-
ciently time-invariant. Also, it is assumed that \( \hat{\Sigma}^{-1} \) exists. Given these conditions, it
is still necessary to determine \( \hat{W} \) from \( \hat{\Sigma}^{-1} \), or \( W \) if the noise covariance is known,
in order to evaluate Equation (3.15). Because of its structure, the covariance matrix
of Equation (3.5) can be factored using the singular value decomposition given as

\[ \Sigma = U \Lambda U^T, \]  

(3.18)

where \( \Lambda \) is a diagonal matrix containing the singular values of \( \Sigma \) and \( U \) is an orthog-
onal matrix of eigenvectors corresponding to the eigenvalues \( \Sigma^T \Sigma \). Consequently, the
weighting matrix $W$ can be defined by

$$W = \Lambda^{-1/2}U^T.$$  \hspace{1cm} (3.19)

The weighting matrix may also be computed using the Cholesky factorization [28].

When the noise is white, $\hat{\Sigma}$ or $W$ can be replaced with the identity matrix. Then, Equations (3.14) and (3.17) simplify to Equation (3.8). When the noise is correlated and $\Sigma$ is estimated from approximately $N$ samples, the amplitude estimates in Equation (3.17) and corresponding FFE are no longer ML and the estimates may not be efficient in any sense. Furthermore, the use of Equation (3.15) may be restricted by the increased computational complexity for cases of non-white Gaussian noise. Ignoring the computations involved in estimating $\Sigma$ and consequently $W$, the additional computational complexity over the white noise case is $O(2N^2q)$ for each candidate frequency.

An alternative approach is to pre-whiten the samples and then use un-weighted NLS \textit{(i.e.,} Equations (3.8) and (3.10)), thereby enjoying a computational savings. However, this approach requires $W$ be representative of a linear time-invariant (LTI) system. Therefore, $W$ must be constrained to be Toeplitz. This constraint can be met by treating the noise as an autoregressive (AR) signal. Formally, the AR noise is described as

$$\epsilon(n) = \frac{1}{A(z)}\epsilon(n),$$  \hspace{1cm} (3.20)

where $A(z)$ is a stable, rational linear filter, $z^{-1}$ is the unit delay operator ($z^{-p}x(n) = x(n-p)$), and $\epsilon(n)$ is zero-mean white noise with variance $\sigma^2_{AR}$. The filter $A(z)$ has the form

$$A(z) = 1 + a_1z^{-1} + \ldots + a_pz^{-p}.$$  \hspace{1cm} (3.21)
The parameters of $A(z)$ are then given as

$$
\theta_{AR} = [a_1, \ldots, a_p]^T. \quad (p \times 1)
$$  \hspace{1cm} (3.22)

If the AR parameters are not known, then they must be estimated. Accordingly, Equation (3.20) can be re-written as a series of linear equations, given by

$$
e(n) = e(n) + \sum_{i=1}^{p} a_i e(n-i).
$$  \hspace{1cm} (3.23)

A LS solution to determine the AR parameters of Equation (3.23) is given by [25] as

$$
\hat{\theta}_{AR} = -(\Psi^T \Psi)^{-1} \Psi^T \psi,
$$  \hspace{1cm} (3.24)

where $\psi = [e(p+1), \ldots, e(M)]^T$ and

$$
\Psi = \begin{bmatrix}
e(p) & e(p-1) & \cdots & e(1) \\
e(p+1) & e(p) & \cdots & e(2) \\
\vdots & \vdots & \ddots & \vdots \\
e(M-1) & e(M-2) & \cdots & e(M-p)
\end{bmatrix}. \quad (M - p \times p)
$$  \hspace{1cm} (3.25)

It is assumed $M \gg N$ and $N \gg p$. More detailed analysis, remarks, and additional AR parameter estimation techniques are treated in [25].

Assuming the AR parameters are known, the whitened samples are given by filtering the observations with $A(z)$ (i.e., $\hat{y}(n) = A(z)y(n)$). The matrix representation is an alternative view of the filtering. Given the AR parameters, the weighting matrix is defined as

$$
W = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\theta_{AR} & 1 & \ddots & \vdots \\
0 & \theta_{AR} & \ddots & 0 \\
\vdots & 0 & \ddots & 1 \\
0 & \cdots & 0 & \theta_{AR}
\end{bmatrix}. \quad (N + p - 1 \times N)
$$  \hspace{1cm} (3.26)

Notice the Toeplitz structure of Equation (3.26). Accordingly, the whitened observation vector is equivalently given by $\tilde{y} = Wy$. After the pre-whitening procedure, the
harmonic model is given by

\[
\tilde{y} = C(\omega_0)\tilde{u} + S(\omega_0)\tilde{v} + \tilde{\epsilon}, \quad (N \times 1)
\]

\[
= s(\tilde{\theta}) + \tilde{\epsilon}, \quad (3.27)
\]

where \(\tilde{\theta} = \tilde{\theta}^{\text{corr}} = [\omega_0, \tilde{u}^T, \tilde{v}^T]^T\) and \(\tilde{\epsilon}\) is a vector of i.i.d. Gaussian random variables. Then, the FFE is given by Equation (3.10) after replacing the observation \(y\) with \(\tilde{y}\).

The estimates of \(\tilde{u}\) and \(\tilde{v}\) are simply determined by Equation (3.8). Notice, the standard NLS method can be used to estimate the FF after pre-whitening. Nevertheless, it is also of interest to estimate \(\alpha\) and \(\phi\). After computing the estimates of \(\tilde{u}\) and \(\tilde{v}\), the estimates of \(\tilde{\alpha}\) and \(\tilde{\phi}\) are determined by Equation (2.4). Then, \(\alpha\) and \(\phi\) are found by removing the effects of the whitening filter from the estimates of \(\tilde{\alpha}\) and \(\tilde{\phi}\).

This can be achieved by

\[
\hat{\alpha}_k = \tilde{\alpha}_k / |A(e^{jk\omega_0})|,
\]

\[
\hat{\phi}_k = \tilde{\phi}_k - \angle A(e^{jk\omega_0}), \quad (3.28)
\]

where \(A(e^{jk\omega_0})\) is the frequency response of the whitening filter given by Equation (3.42) evaluated at the estimated frequency of the \(k\)th harmonic and \(\angle x\) is the angle in radians of a complex-valued scalar \(x\).

The benefit of using the whitening procedure with the AR noise model, as compared to the ACS method, is twofold. First, there is the obvious computational savings in using the standard NLS method over the weighted NLS method of Equation (3.15). Second, when \(\theta_{AR}\) or \(\Sigma\) are not known, there is also a computational savings in estimating the AR parameters over the ACS estimate. The computational complexity involved in estimating the AR parameters is \(O(Mp^2)\), whereas the complexity of the ACS estimate is \(O(MN^2)\). In addition, when the noise power spectral
density (PSD) is a smooth function of frequency, it is appropriate to model the PSD with orders significantly less than the data length. In this case, the ACS method is seen as over-parameterizing the noise model. In battlefield acoustics, long periods of inactivity are common and provide greater data lengths to estimate the noise parameters as compared to the data lengths used for estimating harmonic source parameters. Thus, the accuracy of the parameters for the assumed stationary noise is significantly greater than that of the source parameters. Consequently, in this work it is assumed the noise is a smooth function of frequency and the AR noise parameters are known.

As demonstrated by [24] for complex sinusoids and [27] for real sinusoids, the NLS method still gives consistent, although no longer ML, parameter estimates in colored Gaussian noise without the pre-whitening step. However, the order selection methods used here, as discussed below, require the noise be uncorrelated. Therefore, pre-whitening is a necessary step in the proposed algorithm.

3.1.3 Loss Function Characteristics

The frequency estimate defined in Equation (3.10) is derived under the assumption $q$ is known. If the true number of harmonic lines is not known, FFs can be highly biased if the number of columns, $r$, in Equation (2.7) is set incorrectly (i.e., $r \neq q$). Here, $r$ is referred to as the system order. The loss function defined by Equation (3.9) is multi-modal. This property of the loss function gives rise to biased frequency estimates when the system order is incorrect. Two examples that follow demonstrate this behavior.

Figure 3.1 shows two plots for the loss function of the frequency estimates of a noiseless signal. The true FF is $\omega_0 = 0.1$ radians/sample and the true NOH is $q = 5$.
The amplitudes of the harmonics are set to unity and the phases are random. The simulated signals are generated from a 1 second window with $N = 512$ samples. The loss function values for candidate frequencies at 0.1 and 0.2 radians/sample are nearly equal when the system order is set to $r = 2$, as shown in Figure 3.1(a). In the presence of noise, the global minimum of the loss function may well be $2\omega_0$ for this case. As a result, Equation (3.9) could give highly biased FFs. If the NOH in the signal is known, then Figure 3.1(b) is representative of the expected loss function for candidate frequencies. Here, the global minimum corresponds to the correct FF.

In the previous examples, the harmonic amplitudes were uniform. A similar situation occurs if the signal amplitudes follow a $1/\sqrt{\omega}$ rule. Figure 3.2 shows the loss functions for a noisy signal using the same parameters as above except for the signal amplitudes. Figure 3.2(a) corresponds to the loss function when the system order is set to $r = 10$, which is higher than its true value. Here, the global minimum corresponds to $\omega_0/2$. With the order set correctly, the global minimum of the loss function shown in Figure 3.2(b) corresponds to the correct FF.

The frequency estimates using Equation (3.9), when biased, are multiples of the true FF for sufficiently high SNR and frequency resolution. Conditions for the frequency resolution are discussed below. For the coupled harmonic model, the initial estimates are given by the NLS solution for a reasonable choice of $r$. Then, an accurate estimate of the true FF can be determined by the use of order estimates.

As determined by [26], harmonics embedded in noise can be estimated very accurately given good initial estimates. This is evident in the deep, narrow troughs in Figures 3.1 and 3.2. This also suggests that a rather fine search grid is required to
Figure 3.1: Loss function for a noiseless signal and the system order set to (a) $r = 2$ (under-set order) and (b) $r = 5$ (correct order) in estimating the FF. The actual NOH is $q = 5$ with $\omega_0 = 0.1$ radians/sample and uniform harmonic signal amplitudes.

Figure 3.2: Loss function for a noisy signal and the system order set to (a) $r = 10$ (over-set order) and (b) $r = 5$ (correct order) in estimating the FF. The actual NOH is $q = 5$ with $\omega_0 = 0.1$ radians/sample and amplitudes inversely proportional to $\sqrt{\omega}$.  

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ensure a candidate frequency lands in a trough of the loss function. So, it is of interest to determine an adequate frequency resolution. The following analysis provides a rudimentary guide to determine a suitable frequency search grid spacing.

To simplify the analysis, it is assumed the coupled harmonics have unit amplitudes and \( q \) is known. Due to sampling with a finite length window, each harmonic has an associated spectrum. For a rectangular window, each harmonic will have a \( \sin(\omega N/2) / \sin(\omega/2) \) spectrum centered at the harmonic frequency. Consequently, each spectrum has a corresponding main beam in which most of the energy is located. The width of the main beam is defined here using the bandwidth between the first nulls (BWBN). The BWBN for the spectrum of a rectangular window is \( 4\pi/N \).

When the candidate fundamental frequency is offset from the actual fundamental frequency by \( \delta \omega \), the spectrum of the \( k^{th} \) candidate harmonic is shifted in frequency by \( k\delta \omega \) from that of the \( k^{th} \) true harmonic. When \( k\delta \omega = 2\pi/N \), the \( k^{th} \) candidate spectrum is orthogonal to the spectra of the true harmonic frequencies (i.e., \( C(\omega_0)^T C(\omega_0 + 2\pi/kN) = 0 \) and \( S(\omega_0)^T S(\omega_0 + 2\pi/kN) = 0 \), where \( C(\omega) \) and \( S(\omega) \) are the \( k^{th} \) columns of \( C(\omega) \) and \( S(\omega) \), respectively). Consequently, no energy from the \( k^{th} \) candidate harmonic contributes in minimizing the NLS loss function. In addition, orthogonality for the \( k^{th} \) harmonic also holds when \( \delta \omega \) is an integer multiple of \( 2\pi/kN \). So for example, when \( q \) is even and \( \delta \omega = 4\pi/qN \), the spectra of the \( q/2^{th} \) and \( q^{th} \) candidate harmonics are orthogonal to each of those in the true signal. Incidentally, when \( \delta \omega = 4\pi/qN \), the spectral main lobes from the last candidate and true harmonic no longer overlap. It follows that the loss function evaluated at candidate frequencies in the vicinity of the true fundamental with increasing offset in the
range $2\pi/qN < \delta \omega < 4\pi/qN$ will take on increasingly large values. As a result, the suggested frequency search grid spacing is $\Delta \omega = BWBN/2q$.

### 3.1.4 Comments

In [5], a loss function is proposed that averages the errors over possibly non-overlapping time windows. However, time averaging may increase estimate variances because of the large sample requirements for the NLS method and the increased potential for model mismatches. A similar but alternate method could be used in sensor array applications. Instead of averaging $L(\omega)$ over data blocks, the loss function can be averaged over sensors. It should be noted that this alternative averaging approach assumes parameter estimation is done prior to beamforming.

### 3.1.5 Cramér-Rao Lower Bounds

The Cramér-Rao lower bound (CRLB) provides a good comparison tool for evaluating estimator performance. The CRLB, which bounds the minimum achievable variance of unbiased estimators, has been well developed for harmonic retrieval problems in white [14, 19, 26, 29], and correlated [7, 14, 24] Gaussian noise. The CRLBs for frequency coupled harmonics are developed in [14] and [29]. The CRLBs of coupled harmonics are also given here. The finite- and large-sample (as $N \to \infty$) CRLBs are developed in Appendix A. The large-sample Cramér-Rao lower bounds for unbiased estimators of the amplitudes, phases, and fundamental frequency are [14]

\[
\begin{align*}
CRB_{\infty}(\hat{a}_k) &= \frac{2\sigma_k^2}{N}, \\
CRB_{\infty}(\hat{\phi}_k) &= \frac{2\sigma_k^2}{N\alpha_k^2}, \\
CRB_{\infty}(\hat{\omega}_0) &= \frac{12}{N^3} \left( \sum_{k=1}^{q} \frac{k^2\alpha_k^2}{2\sigma_k^2} \right)^{-1}.
\end{align*}
\]
In the case of white noise, $\sigma_k^2 = \sigma^2$. For colored noise, the local variance is given by $\sigma_k^2 = |H(e^{j\omega_k})|^2 \sigma^2$, where $H(e^{j\omega}) = |A(e^{j\omega})|^{-1} \angle A(e^{-j\omega})$.

The finite-sample CRLB for an individual parameter is denoted by $CRB_N(\cdot)$. The reason for using $CRB_\infty$ in this work is twofold: $CRB_\infty$ is easier to compute than $CRB_N$, and $CRB_\infty$ approximates $CRB_N$ well when the minimum frequency separation is sufficiently large [26]. The minimum frequency separation, derived in [19], for multi-harmonic models is $\omega_{\min} > 2\pi/N$. For the coupled harmonic model, the minimum resolvable fundamental frequency is also lower bounded by $2\pi/N$.

### 3.2 Model Order Selection

In general, the number of significant harmonics is unknown. Therefore, it is of interest to use the model order information to properly choose the correct fundamental frequency. Several standard order selection techniques are well suited to this task. These methods include or Akaike’s information criterion (AIC) [1], MDL [20], and maximum a posteriori probability (MAP) [4] proposed by Djurić. AIC and MDL are derived from Information Theoretic Criterion (ITC), whereas MAP is derived from asymptotic Bayesian decision theory. Each method has a similar form with a data term and a penalty term. The penalty term accounts for the reduced fit error when the model order is overestimated. For sinusoidal summation models, the order selection criteria have the form [35]

\[
\hat{q}_{\text{AIC}} = \arg\min_r \left\{ N \ln J(\hat{\theta}) + 3r \right\}, \quad (3.32)
\]

\[
\hat{q}_{\text{MDL}} = \arg\min_r \left\{ N \ln J(\hat{\theta}) + \frac{3r}{2} \ln N \right\}, \quad (3.33)
\]

\[
\hat{q}_{\text{MAP}} = \arg\min_r \left\{ N \ln J(\hat{\theta}) + \frac{5r}{2} \ln N \right\}, \quad (3.34)
\]
where \( J(\theta) \) is the negative log-likelihood function evaluated at the ML parameter vector \( \hat{\theta} \) and \( \hat{q} \) is the estimate of the number of sinusoids. Each respective method is denoted by the corresponding subscript. The number of free parameters in this case is \( 3q \). However, the coupled harmonic model has \( 2q + 1 \) free parameters. The MAP criterion penalizes each unknown amplitude and phase parameter by \( \frac{1}{2} \ln N \) and each unknown frequency by \( \frac{3}{2} \ln N \) \[4\]. For the frequency coupled harmonic model, the penalty for the fundamental frequency is \( \frac{3}{2} \ln N + \ln r \). As a result, the order selection criteria for the coupled harmonic model are

\[
\hat{q}_{AIC} = \arg\min_r \left\{ N \ln J(\hat{\theta}) + 2r + 1 \right\},
\]

\[
= \arg\min_r \left\{ N \ln J(\hat{\theta}) + 2r \right\}, \quad (3.35)
\]

\[
\hat{q}_{MDL} = \arg\min_r \left\{ N \ln J(\hat{\theta}) + \frac{2r + 1}{2} \ln N \right\},
\]

\[
= \arg\min_r \left\{ N \ln J(\hat{\theta}) + r \ln N \right\}, \quad (3.36)
\]

\[
\hat{q}_{MAP} \approx \arg\min_r \left\{ N \ln J(\hat{\theta}) + \frac{2r + 3}{2} \ln N \right\},
\]

\[
= \arg\min_r \left\{ N \ln J(\hat{\theta}) + r \ln N \right\}, \quad (3.37)
\]

where \( J(\theta) \) is defined by Equation (3.5). The second equality in the three equations above are obtained by removing terms that do not depend on \( r \). Note that the MDL and MAP criteria are equivalent. They differ from AIC by a factor of \( \frac{1}{2} \ln N \) in the second term. This second term penalizes large model orders, so in general AIC tends to give higher model orders than MDL.

The above decision rules were developed under a white Gaussian noise assumption. Another selection rule, proposed by Wang in \[32\], for the colored noise case has the form

\[
\hat{q}_{COL} = \arg\min_r \left\{ N \ln J(\hat{\theta}) + \frac{cr}{2} \ln N \right\}, \quad (3.38)
\]
where \( c \) is a constant greater than a threshold \( \gamma \), which depends on the characteristics of the noise.

It was noted in [4] that Equation (3.38) can give inconsistent estimates based on the choice of \( c \). In addition, it was determined in [33] that AIC produces inconsistent estimates and tends to overestimate the model order, whereas MDL yields consistent estimates for large sample records. Due to the consistency of MDL, it is the preferred order selection method considered here. The rule proposed by Wang is not examined further, but is a possible extension of this work.

The combined detection-estimation algorithm has the form

\[
\{ \hat{\theta}, \hat{q} \} = \arg \min_{\{\theta, r\}} \{ N \ln J(\theta) + r \ln N \}, \\
= \arg \min_{\{\theta, r\}} \{ N \ln L(\theta) + r \ln N \}, \tag{3.39}
\]

where the MDL criterion represents the detection component and \( L(\theta) \), given by Equation (3.6), represents the estimation component. Since the estimation component can be reduced to a 1-D search it follows that the combined algorithm can be reduced to a 2-D search. Thus, combining Equation (3.9) with Equation (3.39) the FF and order estimates are found by

\[
\{ \hat{\omega}_0, \hat{q} \} = \arg \min_{\{\omega, r\}} N \ln y^T P^{-1}_r (\omega) y + r \ln N, \\
= \arg \min_{\{\omega, r\}} h(\omega, r). \tag{3.40}
\]

Then, the amplitude estimates are generated using Equation (3.8) with the estimates \( \hat{\omega}_0 \) and \( \hat{q} \). When \( \hat{q} = q \), \( \hat{\omega}_0 \) is the MLE. Otherwise, when \( \hat{q} \neq q \), Equation (3.40) can still be used to generate statistically efficient FFES (i.e., \( \text{var}(\hat{\omega}_0) = \sigma^2_{\infty}(\omega_0) \)), as it will be shown through simulations. In the case of colored noise, \( y \) is simply replaced by \( \tilde{y} \) in Equation (3.40).
3.2.1 Proposed Algorithm

It is important to estimate both the parameter set and model order together. Since the order selection methods depend on the parameter estimates, the order estimates may be highly biased when the FF estimates are biased. This frequency-order dependence is evident in Figure 3.3. The simulated signal is composed of \( q = 7 \) harmonics with \( \omega_0 = 0.1 \) radians/sample. The SNR, defined as

\[
\rho = \frac{\alpha^T \alpha}{2\sigma^2},
\]

(3.41)

of the simulated white noise is set to \( \rho = 3 \)dB. Each curve in Figure 3.3 represents the loss function defined by Equation (3.40) evaluated at a fixed frequency (precisely \( \omega_0/3, \omega_0/2, \omega_0, 2\omega_0, 3\omega_0 \)) for a range of \( r \in [2, \min(32, r_{\text{Nyq}})] \), where \( r_{\text{Nyq}} < [\pi/\omega] \) satisfies the Nyquist criterion. As seen in Figure 3.3, the global minimum corresponds to the correct FF and order. Notice that the minimum of the loss function for a frequency other than the correct FF does not correspond to the correct order. Also apparent in Figure 3.3 is that the loss function evaluated at \( \omega_0 \) has a range of \( r \) such that, although not the global minimum, the function is less than the minimum at candidate frequencies at the same order.

An overestimated order results in an estimated signal with harmonics that were not present in the true signal (\( e.g., \) if \( \hat{\omega}_0 = \omega_0 \) with \( \hat{q} = q + 1 \)). The extra harmonics correspond to noise only spectral regions and are, in general, undesirable. In some applications, correctly selecting the FF with an underestimated number of harmonic lines may be tolerable. Some of the harmonics are missed, but the estimates do not correspond to noise regions. From a practical standpoint, it is more desirable to estimate the true FF with the correct or possibly underestimated NOH. On the
Figure 3.3: Loss function for a noisy signal using MDL, evaluated at exact values of \( \omega_0/3, \omega_0/2, \omega_0, 2\omega_0, \text{ and } 3\omega_0 \), with \( \omega_0 = 0.1 \) radians/sample. The true NOH is \( q = 7 \) with uniform amplitudes and \( \rho = 3 \text{ dB} \).

On the other hand, as seen in the CRLB for FFs (Equation (3.31)), the estimate variance is minimized by including at least all the harmonic lines with significant amplitudes.

In practice, the procedure of Equation (3.40) requires a fine grid search over frequency and all possible integer orders. This approach is computationally burdensome. However, it is possible to find the global minimum with a reduced search grid. Recall the general pattern of the loss function versus frequency (no order selection). As seen in Figures 3.1 and 3.2, deep, narrow troughs occur at the FF, sub- and superharmonics. As noted previously, this suggests that these frequencies, although not necessarily the true FF, can be estimated with a high degree of accuracy. With \( r \) properly set, an initial FF estimate, denoted \( \hat{\omega}_f \), will likely correspond to the true FF or multiple thereof. A reduced frequency search set can be defined using the initial frequency estimate (e.g., \( \omega \in \{\hat{\omega}_f/2, \hat{\omega}_f, 2\hat{\omega}_f, 3\hat{\omega}_f\} \)). Then, the loss function can be
1. Pre-whiten the samples if the noise is correlated using the known AR model by \( \tilde{y}(n) = A(z)y(n) \).
2. Obtain an initial estimate, \( \hat{\omega}_1 \), of the FF using Equation (3.9) with a fine frequency search grid from \( \Lambda_\omega \) and \( r \) set to \( r_{\text{max}} \).
3. Compute a refined initial estimate, \( \hat{\omega}_1 \), using an optimization technique (e.g., \( f\text{minbnd} \) in MATLAB), Equation (3.9), and \( r \) set to \( r_{\text{max}} \).
4. Create a new frequency search set from the refined estimate:
   \[ \Lambda_{\omega_i} = \{ \omega = a \hat{\omega}_1 | a \in \{1/b\} \cup \{b\}, b \in \mathbb{Z} \} \subset \Lambda_\omega. \]
5. Minimize Equation (3.40) over \( \Lambda_{\omega_i} \) and candidate orders in \( \Lambda_r = \{r_{\text{min}}, \ldots, r_{\text{max}}\} \) to get \( \hat{\omega}_0 \) and \( \hat{q} \).
6. Finally, use Equations (3.8) and (2.4) with \( \hat{\omega}_0 \) and \( \hat{q} \) to get estimates \( \hat{\alpha} \) and \( \hat{\phi} \).
7. Remove the effects of pre-whitening from \( \hat{\alpha} \) and \( \hat{\phi} \). If the noise is white, skip Steps 1 and 7.

**Table 3.1**: Summary of the NLS-MDL Algorithm.

minimized over the new frequency set and model order. This initialization and the combined order estimation is the basis behind the algorithms proposed in this paper.

The first algorithm is detailed in Table 3.1. The algorithm is referred to as the NLS-MDL method. The algorithm is basically a two stage procedure: first, generate an initial FFE, and, second, generate the order and parameter vector estimates.

It is assumed that the signals are anti-alias filtered and any DC bias is removed. Therefore, each of the harmonics must satisfy \( \omega \in (0, \pi) \). This requirement bounds above the model order corresponding to each FF. However, high orders are possible for lower fundamental frequencies. Therefore, order searches for lower frequencies require more computations than for higher frequencies. In general, the frequency and order search regions would normally be confined by prior knowledge. For example, the frequency search region for the simulations in this work is confined to \( f \in \Lambda_f = [2, 25] \).
Hz, which is a relaxed region based on prior knowledge on battlefield acoustics [34]. If the precise probability density function of \( \theta \), denoted by \( p(\theta) \), is known, then maximum a posteriori (MAP) estimation could be considered [18]. However, \( p(\theta) \) is generally unknown. Assuming \( \theta \) is deterministic can be thought of as a worst-case uniform prior [18]. It has also been determined that a sampling rate on the order of 0.5-1 kHz is sufficient [16, 17, 34] for most acoustic vehicle detection and classification applications. On the other hand, order estimates presented here are upper bounded by the criterion \( r_{\text{max}} = \min(r_{ub}, r_{nyq}) \), where \( r_{ub} \) is chosen as a practical limit. Alternatively, to employ less ad hoc means, methods such as those proposed by [21] and [35] could be implemented to bound the order search region by initial periodogram estimates.

The initial FFEs in Steps 2 and 3 are generated by assuming the true order is \( q = r_{\text{max}} \). If \( r_{\text{max}} \) is set properly and for sufficiently large SNR (as it will be seen through simulation), the initial FFEs will be unbiased with variance nearly equal to the CRLB. Otherwise, the estimates will be close to the true FF or a sub-harmonic. Then, any large FFE bias is removed using order selection, hence the procedure in Step 5.

A second algorithm, referred to as ANLS-MDL, utilizes the approximated NLS method of Equation (3.11). The ANLS-MDL algorithm substitutes Equation (3.11) for (3.9) in Step 3 of the NLS-MDL algorithm. Also, Equation (3.11) is combined with Equation (3.39), which is then substituted for Equation (3.40) in Step 5. It was determined empirically that the estimate variance of ANLS-MDL is improved by repeating Step 3 with Equation (3.11) and \( r = \hat{q} \) after Step 5. Repeating Step 3 after Step 5 for NLS-MDL does not provide any noticeable improvement.
For both algorithms, it is assumed that the frequency search grid for the initial estimates is fixed and the corresponding cosine and sine matrices defined in Equation (2.7) are pre-computed and stored. However, in situations where storage space is limited, an FFT based technique as described in [14] can be substituted to generate initial frequency estimates. The FFT based initialization has similar initialization accuracy as the NLS method and reduces the storage requirements.
3.3 Simulation Studies

The following are numerical examples that demonstrate the statistical properties of the combined detection-estimation algorithm. This study compares the NLS-MDL algorithm with ANLS-MDL and NLS with known or fixed order. First, the algorithms are examined with simulated white Gaussian noise. Then, the colored noise case is investigated. Pre-whitening is used in the colored noise case for all algorithms.

First, the performance of each algorithm is tested against the large-sample Cramér-Rao lower bounds of Equations (3.29)-(3.31) on estimate variances versus SNR. Then, the algorithms are compared against the CRLBs as the data length and as the true NOH vary. Comparisons are made between the estimate RMSEs and the corresponding large-sample root-CRLBs (i.e., $\sqrt{\sigma^2_\omega(\hat{\omega}_0)}$). The root-CRLBs will simply be referred to as the CRLBs.

The simulation parameters common to all simulations are as follows:

- The sampling period $T$ is set to $1/512$ s.
- The FF is $\omega_0 = 0.1$ rads/sample ($\approx 8$ Hz).
- Two amplitude models are examined: $\alpha_k = 1$ and $\alpha_k = 1/\sqrt{k\omega_0}$.
- The phases are set to $\phi_k = k\pi/100$.
- The order search range is set to $\Lambda_r = [2, r_{\text{max}}]$.
- The frequency range is set to $\Lambda_\omega = [\pi/128, 25\pi/256]$ rads/sample ([2,25] Hz).
- The frequency grid resolution is $\Delta \omega = \pi/8N$.
- The simulation results are generated from 500 Monte Carlo simulations.
3.3.1 Algorithm Performance versus SNR in White Noise

In this section, NLS-MDL and ANLS-MDL are compared with NLS with known order. The NLS method with correct order \((i.e., r = q)\) will simply be referred to as NLS or standard NLS. The SNR ranges considered are \(\rho \in [-20, 40]\) dB in 4 dB increments. The noise variance, \(\sigma^2\), is properly adjusted, as defined in Section 3.2.1, for the desired SNR and amplitude models. The signal, given by Equation (2.1), is composed of \(q = 10\) harmonics and a data length of \(N = 256\) samples. The results are first presented for the uniform amplitude rule and then the \(1/\sqrt{\omega}\) rule.

The standard NLS method is implemented as follows. The system order is set to \(r = q\). Then, Equation (3.9) is evaluated at frequencies specified above. Finally, an optimization technique, such as \textit{fminbnd} in MATLAB, is used to refine the frequency estimate.

For the uniform amplitude case, the absolute value of bias and the RMSE of the FF estimates are plotted in Figures 3.4(a) and (b). The RMSEs of the amplitude and phase estimates for the first, fifth, and tenth harmonics \((i.e., \hat{a}_k \text{ and } \hat{\phi}_k \text{ for } k = 1, 5, \text{ and } 10)\) are plotted in Figures 3.4(c), (d) and 3.5, respectively. In each figure of estimate RMSE the results are shown with the corresponding CRLB. In addition, Figure 3.8(a) is a histogram of the FFEs normalized by the true FF at \(\rho = -4\) dB for the uniform amplitude case. The histograms in each figure are normalized by 500. The results for the \(1/\sqrt{\omega}\) amplitude case are similarly plotted in Figures 3.6, 3.7, and 3.9.

As seen from Figures 3.4 and 3.5, the RMSEs from each algorithm correspond well with the CRLB for a large range of SNRs. The estimation accuracy of the NLS-MDL and ANLS-MDL algorithms degrades rapidly below 0 dB SNR. This characteristic
is common for both amplitude models. At $\rho = -4$ dB, a small number of FFEs become largely biased, as seen in Figure 3.8(a). Below -4 dB SNR, the RMSE of frequency estimates begins to escalate faster than the CRLB. The performance of the standard NLS algorithm (Equation (3.10) without model order selection) does not diverge from the CRLB for decreasing SNR until approximately -4 dB. Note that the NOH of NLS is set to the correct number. It follows that the loss in performance of the NLS-MDL and ANLS-MDL between -4 and 0 dB SNR is mainly due to the variance in the order estimates. A histogram of the order estimates at -4 dB SNR is shown in Figure 3.8(b). At this SNR level, most of the order selection estimates equal the true value, but tends to underestimate the NOH. Consequently, outlier FFEs tend to correspond to super-harmonics. At lower SNR levels, the MDL increasingly underestimates the order.

The RMSEs of the frequency and amplitude estimates from each algorithm also diverge from the CRLBs at high SNR. Above approximately 30 dB SNR, bias in the estimates dominates the RMSEs. As noted in [26], the amplitude estimates can be improved by using fewer than $N$ samples since the RMSE of the FFEs only diverges slightly from the CRLB. In Section 3.3.3, it is shown that the biases decrease with increasing data length. It was determined empirically that the biases also decrease with increasing NOH in the signal. Yet, the biases are insignificant compared to the true values, even for a data length of $N = 256$.

The performance trends described above are similarly observed for the $1/\sqrt{\omega}$ amplitude rule. Consequently, the following discussion will concentrate on the noticeable differences from the uniform amplitude rule.
As seen in Figure 3.6, the RMSE for frequency estimates from NLS diverges from the CRLB below 0 dB SNR, as compared to -4 dB in the uniform amplitude case. Three estimates out of 500 (0.6%) are outliers and are nearly equal to $1/2\omega_0$. At higher SNR levels, the difference between the NLS loss function evaluated at $\omega_0$ and $1/2\omega_0$ is relatively small for the $1/\sqrt{\omega}$ rule, as seen in Figure 3.2. Increasing the frequency resolution of the grid search or re-evaluating and comparing the loss function values at refined estimates $\hat{\omega}_0$ and integer multiples thereof should alleviate this issue but at the expense of an increased computational complexity for a slight SNR performance gain.

From Figure 3.6(b), a jump in the RMSE is observed for the FFEs from the NLS-MDL algorithm at 8 dB SNR. This anomaly is due to a single estimate that is close to $\omega_0/2$. The corresponding order estimate is $\hat{q} = 20$, which is twice the true value. At this point, it is not known why this anomaly occurs. However, although not shown, it was determined that increasing the data length eliminates the problem.

It is also observed in the figures that the FFEs from the ANLS-MDL method have an order of magnitude higher bias than those of the NLS-MDL and NLS methods. Again, the biases from each algorithm are insignificant compared to the true values from a practical standpoint. Despite the higher bias, the amplitude and phase estimates from ANLS-MDL have variance close to the CRLBs for a large range of SNR values. Consequently, it seems fair to reason that the ANLS-MDL method exhibits comparable performance to the NLS-MDL and NLS methods for both amplitude models.
Figure 3.4: Parameter estimate results for the uniform amplitude model versus SNR in white noise: FFE (a) bias and (b) RMSE, along with (c) amplitude and (d) phase estimate RMSEs for the 1st harmonic. The estimates are generated using (+) NLS-MDL, (○) ANLS-MDL, and (<kappa>) NLS with r = q. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.5: Parameter estimate RMSEs for the uniform amplitude model versus SNR in white noise: (a) amplitude and (b) phase RMSEs for the 5th harmonic, and (c) amplitude and (d) phase RMSEs for the 10th harmonic. The estimates are generated using (+) NLS-MDL, (o) ANLS-MDL, and (△) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.6: Parameter estimate results for the $1/\sqrt{\omega}$ amplitude model versus SNR in white noise: FFE (a) bias and (b) RMSE, along with (c) amplitude and (d) phase estimate RMSEs for the 1st harmonic. The estimates are generated using (+) NLS-MDL, (o) ANLS-MDL, and (o) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.7: Parameter estimate RMSEs for the $1/\sqrt{\omega}$ amplitude model versus SNR in white noise: (a) amplitude and (b) phase RMSEs for the $5^{th}$ harmonic, and (c) amplitude and (d) phase RMSEs for the seventh harmonic. The estimates are generated using (+) NLS-MDL, (c) ANLS-MDL, and (>) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.8: Normalized histograms of the (a) FFEs normalized by the true value and (b) estimates of the number of harmonics $q$ at $\rho = -4$ dB for the uniform amplitude model in white noise. The FFEs are generated using NLS-MDL, ANLS-MDL, and NLS with $r = q$ in (a). The order estimates in (b) are generated using NLS-MDL and ANLS-MDL.

Figure 3.9: Normalized histograms of the (a) FFEs normalized by the true value and (b) estimates of the number of harmonics $q$ at $\rho = -4$ dB for the $1/\sqrt{\omega}$ amplitude model in white noise. The FFEs are generated using NLS-MDL, ANLS-MDL, and NLS with $r = q$ in (a). The order estimates in (b) are generated using NLS-MDL and ANLS-MDL.
3.3.2 Algorithm Performance versus SNR in Colored Noise

Now, the performance is examined versus SNR with colored noise. The signal parameters are the same as in the simulations with white noise. The CRLBs in this case are calculated using Equations (3.29)-(3.31) with the local noise variance defined as $\sigma_k^2 = |H(e^{j\omega_n})|^2\sigma^2$. The following results are generated using pre-whitening in Step 1 of the NLS-MDL and ANLS-MDL algorithms. These algorithms are compared to the standard NLS as described in Section 3.3.1 with the additional pre-whitening step.

The noise is generated by filtering zero-mean, unit variance Gaussian noise with a fifth order AR coloring filter given by

$$H(z) \approx \frac{1}{1 - 2.0z^{-1} + 1.57z^{-2} - 0.28z^{-3} - 0.36z^{-4} + 0.23z^{-5}},$$  \hspace{1cm} (3.42)

and then scaled by $\sigma = (\alpha^T\alpha/2\rho\sigma_{AR}^2)^{1/2}$ to achieve the desired SNR. The choice of this model is based on measurements collected at Aberdeen Proving Grounds (APG). The lowpass filter represented by Equation (3.42) is specific to the local environment at the time in which the measurements were recorded. However, a model needed to be adopted for these simulations. An extension of this work may include a performance analysis with the use of bandpass and/or highpass coloring filters. The frequency response of the coloring filter of Equation (3.42) is plotted in Figure 3.10.

For the uniform amplitude case, the absolute value of bias and the RMSE of the frequency estimates are plotted in Figures 3.11(a) and (b). The RMSEs of the amplitudes and phases for the first, fifth, and tenth harmonic are plotted in Figures 3.11(c),(d) and 3.12, respectively. In each figure of estimate RMSE the results are shown with the corresponding root-CRLB. Figure 3.15(a) and (b) are histograms.
of the FFEs normalized by the true FF and order estimates at $\rho = 0$ dB for the uniform amplitude case. The results for the $1/\sqrt{\omega}$ amplitude case are similarly plotted in Figures 3.13, 3.14, and 3.16.

As seen in the figures for each amplitude model, the performance characteristics of each estimator are similar to those in the white noise case. For both amplitude models, the amplitude and phase estimates correspond well with the CRLBs above 4 dB SNR for the NLS-MDL and ANLS-MDL methods and above 0 dB SNR for standard NLS. However, the 4 dB threshold, below which the performance rapidly degrades, for the NLS-MDL and ANLS-MDL methods is slightly higher for the correlated noise case as compared to the threshold in the white noise case. On the other hand, fewer than 4% of the FFEs for the uniform amplitude model and fewer than 1% for the $1/\sqrt{\omega}$ model from NLS-MDL and ANLS-MDL constitute largely biased estimates at 0 dB SNR, as evident in Figures 3.15 and 3.16. The biased frequency estimates correspond to outlying order estimates.
Figure 3.10: Magnitude (top) and phase (bottom) response of the coloring filter, $H(e^{j\omega})$.  

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Figure 3.11: Parameter estimate results for the uniform amplitude model versus SNR in correlated noise: FFE (a) bias and (b) RMSE, along with (c) amplitude and (d) phase estimate RMSEs for the 1st harmonic. The estimates are generated using (+) NLS-MDL, (o) ANLS-MDL, and (x) NLS with r = q. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.12: Parameter estimate RMSEs for the uniform amplitude model versus SNR in correlated noise: (a) amplitude and (b) phase RMSEs for the 5th harmonic, and (c) amplitude and (d) phase RMSEs for the 10th harmonic. The estimates are generated using (+) NLS-MDL, (o) ANLS-MDL, and (x) NLS with \( r = q \). The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.13: Parameter estimate results for the $1/\sqrt{\omega}$ amplitude model versus SNR in correlated noise: FFE (a) bias and (b) RMSE, along with (c) amplitude and (d) phase estimate RMSEs for the 1st harmonic. The estimates are generated using (+) NLS-MDL, (o) ANLS-MDL, and (<) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.14: Parameter estimate RMSEs for the $1/\sqrt{\omega}$ amplitude model versus SNR in correlated noise: (a) amplitude and (b) phase RMSEs for the 5th harmonic, and (c) amplitude and (d) phase RMSEs for the seventh harmonic. The estimates are generated using (+) NLS-MDL, (o) ANLS-MDL, and (x) NLS with $r = q$. The RMSEs are plotted against the corresponding (−) root-CRLBs.
Figure 3.15: Normalized histograms of the (a) FFEs normalized by the true value and (b) estimates of the number of harmonics $q$ at $\rho = 0$ dB for the uniform amplitude model in correlated noise. The estimates are generated using (black) NLS-MDL, (gray) ANLS-MDL, and (white) NLS with $r = q$.

Figure 3.16: Normalized histograms of the (a) FFEs normalized by the true value and (b) estimates of the number of harmonics $q$ at $\rho = 0$ dB for the $1/\sqrt{\omega}$ amplitude model in correlated noise. The estimates are generated using (black) NLS-MDL, (gray) ANLS-MDL, and (white) NLS with $r = q$. 

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3.3.3 Algorithm Performance versus Data Length in White Noise

In the following simulation, the performance of the NLS-MDL, ANLS-MDL, and NLS methods is compared against the CRLBs as a function of the data length in white noise. The data lengths considered are powers of 2 with $N \in [64, 1024]$ samples. The noise variance is adjusted for each amplitude model and a desired SNR of $\rho = 0$ dB, as described in Section 3.2.1. The performance trends are similar for both amplitude models. Therefore, the results for both models are reported together.

For the uniform amplitude case, the absolute value of bias and RMSE of the frequency estimates are plotted in Figures 3.17(a) and (b), respectively. The RMSEs of the amplitudes and phases for the first, fifth, and tenth harmonic are plotted in Figures 3.17(c), (d) and 3.18. In each figure of estimate RMSE the results are shown with the corresponding root-CRLB. The results for the $1/\sqrt{\omega}$ amplitude case are similarly plotted in Figures 3.19 and 3.20.

Examining Figures 3.17-3.20 one could suggest a sample length threshold of $N \geq 256$ at $\rho = 0$ dB for the three algorithms. Above $N = 256$, the estimate RMSEs for each algorithm are consistent with the CRLBs. The estimate RMSEs from the standard NLS method approaches the CRLBs at $N = 128$ for the uniform amplitude rule. However, this is not the case for NLS when the true signal has a $1/\sqrt{\omega}$ amplitude model. The variance of the NLS parameter estimates increases and departs from the CRLB at $N = 128$, in contrast to the uniform amplitude case. In practice, the signal amplitude model is unknown. Thus, the suggested data length threshold for NLS is $N \geq 256$. 

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As seen in Figures 3.17(a) and 3.19(a), the bias of the FFEs from each algorithm generally decreases with increasing data length. Consequently, as the data length increases, the CRLBs for unbiased estimators well-characterize the performance of the three algorithms in the presence of independent noise.
Figure 3.17: Parameter estimate results for the uniform amplitude model versus data length in white noise: FFE (a) bias and (b) RMSE, along with (c) amplitude and (d) phase estimate RMSEs for the 1st harmonic. The estimates are generated using the (+) NLS-MDL, (o) ANLS-MDL, and (a) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.18: Parameter estimate RMSEs for the uniform amplitude model versus data length in white noise: (a) amplitude and (b) phase RMSEs for the 5th harmonic, and (c) amplitude and (d) phase RMSEs for the 10th harmonic. The estimates are generated using the (+) NLS-MDL, (o) ANLS-MDL, and (x) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.19: Parameter estimate results for the $1/\sqrt{\omega}$ amplitude model versus data length in white noise: FFE (a) bias and (b) RMSE, along with (c) amplitude and (d) phase estimate RMSEs for the 1st harmonic. The estimates are generated using the (+) NLS-MDL, (o) ANLS-MDL, and (x) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) CRLBs.
Figure 3.20: Parameter estimate RMSEs for the $\frac{1}{\sqrt{\omega}}$ amplitude model versus data length in white noise: (a) amplitude and (b) phase RMSEs for the 5th harmonic, and (c) amplitude and (d) phase RMSEs for the seventh harmonic. The estimates are generated using the (+) NLS-MDL, (○) ANLS-MDL, and (△) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) CRLBs.
3.3.4 Algorithm Performance versus Data Length in Colored Noise

Now, the performance is compared against the CRLBs as a function of data length in the presence of colored noise. The data lengths considered are powers of 2 with \( N \in [64, 1024] \) samples. The noise variance is adjusted as described in Section 3.3.2 for each amplitude model and a desired SNR of \( \rho = 0 \) dB. The trends of the parameter estimates are similar for both amplitude models. Therefore, the results for both models are reported together.

For the uniform amplitude case, the absolute value of bias and RMSE of the frequency estimates are plotted in Figures 3.21(a) and (b), respectively. The RMSEs of the amplitudes and phases for the first, fifth, and tenth harmonic are plotted in Figures 3.21(c), (d) and 3.22. In each figure of estimate RMSE the results are shown with the corresponding root-CRLB for varying data lengths. The results for the \( 1/\sqrt{\omega} \) amplitude case are similarly plotted in Figures 3.23 and 3.24.

Examining Figures 3.21-3.24, it appears the data length threshold increases from that in the white noise case. At \( N = 256 \), a small number of FFEs from NLS-MDL (3.2%, 2.4%) and ANLS-MDL (1.2%, 1.6%) are close to sub- and super-harmonics. The values in the parenthesis are the percentage of estimates in error for the uniform and \( 1/\sqrt{\omega} \) amplitude cases, respectively. As seen in Figures 3.21(a) and 3.23(a), the bias of the FFEs from each algorithm generally decreases with increasing data length. For \( N > 256 \), the estimate RMSEs for each algorithm are consistent with the CRLBs, except for NLS-MDL in the \( 1/\sqrt{\omega} \) amplitude case. For this exception and \( N > 256 \), the phase estimate RMSEs in the higher harmonics are dominated by variance, which exceeds the corresponding CRLBs until \( N = 1024 \).
Figure 3.21: Parameter estimate results for the uniform amplitude model versus data length in correlated noise: FFE (a) bias and (b) RMSE, along with (c) amplitude and (d) phase estimate RMSEs for the 1st harmonic. The estimates are generated using the (+) NLS-MDL, (o) ANLS-MDL, and (x) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.22: Parameter estimate RMSEs for the uniform amplitude model versus data length in correlated noise: (a) amplitude and (b) phase RMSEs for the 5th harmonic, and (c) amplitude and (d) phase RMSEs for the 10th harmonic. The estimates are generated using the (+) NLS-MDL, (o) ANLS-MDL, and (<) NLS with \( r = q \). The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.23: Parameter estimate results for the $1/\sqrt{\omega}$ amplitude model versus data length in correlated noise: FFE (a) bias and (b) RMSE, along with (c) amplitude and (d) phase estimate RMSEs for the 1$st$ harmonic. The estimates are generated using the (+) NLS-ML, (o) ANLS-ML, and (s) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) root-CRLBs.
Figure 3.24: Parameter estimate results for the $1/\sqrt{\omega}$ amplitude model versus data length in correlated noise: FFE (a) bias and (b) RMSE, along with (c) amplitude and (d) phase estimate RMSEs for the 1st harmonic. The estimates are generated using the (+) NLS-MDL, (o) ANLS-MDL, and (s) NLS with $r = q$. The RMSEs are plotted against the corresponding (-) root-CRLBs.
3.3.5 Algorithm Performance versus the Number of Harmonics in White Noise

The following examples examine the performance of NLS-MDL, ANLS-MDL, and NLS with fixed orders as a function of the true NOH in white noise. The NOH considered is \( q \in [2, 30] \) in increments of 2. The orders of the NLS method are fixed at \( r = 10 \) and 16. The data length is set to \( N = 256 \). The noise variance is adjusted as described in Section 3.2.1 for each amplitude model and a desired SNR of \( \rho = 10 \) dB.

As seen in the previous sections, the performance of the amplitude and phase estimates are directly related to the performance of the FFEs. Accordingly, only the FFE results are reported in this section. The bias of the FFEs is plotted in Figure 3.25(a) and (b) for the uniform and \( 1/\sqrt{\omega} \) amplitude models, respectively. The RMSE for the FFEs is similarly plotted in Figures 3.25(c) and (d).

From Figure 3.25, it is observed that RMSEs of the NLS-MDL and ANLS-MDL methods correspond well with the CRLB when \( q \geq 8 \). Below this level, the FFEs for NLS-MDL and ANLS-MDL become inconsistent. For the uniform amplitude case, 40% of the FFEs from ANLS-MDL and 7.8% from NLS-MDL are close to \( \omega_0/2 \) when \( q = 6 \). In the \( 1/\sqrt{\omega} \) case and \( q = 6 \), 51% of the FFEs from ANLS-MDL and 25% from NLS-MDL are close to \( \omega_0/4 \). Nonetheless, the outlier FFEs correspond to inconsistent order estimates.
It is also observed that the NLS method performs well for a large range of \( q \) when \( r = 16 \) in the uniform amplitude case. With \( r = 16 \), NLS produces nearly unbiased estimates with RMSE corresponding to the CRLB for \( 10 \leq q \leq 16 \). When \( q > 16 \), the RMSE for the NLS method is higher than the CRLB for the uniform amplitude signals. With \( r = 10 \), the NLS estimate RMSE corresponds well with the CRLB for \( 6 \leq q \leq 10 \). In addition, the NLS method performs well for both fixed orders when \( q \geq 10 \) in the \( 1/\sqrt{\omega} \) case. In the \( 1/\sqrt{\omega} \) case, the effects of order mismatch are reduced since the amplitudes are decreasing with increasing harmonic number. However, for NLS with \( r = 10 \), the RMSE is offset from the CRLB for \( q > 10 \).
Figure 3.25: FFE results for the uniform (left) and $1/\sqrt{\omega}$ (right) amplitude models versus the true NOH in white noise: FFE bias (top) and RMSE (bottom). The estimates are generated using the (+) NLS-MDL, (o) ANLS-MDL, (a) NLS with $r = 10$, and (x) NLS with $r = 16$. The RMSEs are plotted against the (-) root-CRLB for FFEs.
3.3.6 Algorithm Performance versus the Number of Harmonics in Colored Noise

Now, the performance is compared to the CRLBs as a function of the true NOH in colored noise. The NOH considered is $q \in [2, 30]$ in increments of 2. The orders of the NLS method are fixed at $r = 10$ and $16$. The data length is set to $N = 256$. The noise variance is adjusted as described in Section 3.2.1 for each amplitude model and a desired SNR of $\rho = 10$ dB.

The bias of the FFEs is plotted in Figure 3.26(a) and (b) for the uniform and $1/\sqrt{\omega}$ amplitude models, respectively. The RMSE for the FFEs are similarly plotted in Figures 3.26(c) and (d).

From Figure 3.26, it is observed that RMSEs of the NLS-MDL and ANLS-MDL methods correspond well with the CRLB when $q \geq 8$. For the uniform amplitude case, 11% of the FFEs from ANLS-MDL and 0.2% from NLS-MDL are close to $\omega_0/4$ when $q = 6$. In the $1/\sqrt{\omega}$ case and when $q = 6$, 16.8% of the FFEs from ANLS-MDL and 3.4% from NLS-MDL are close to $\omega_0/4$. Again, the outlier FFEs correspond to over-estimated orders. However, the percentages are improved compared to the white noise case.

In contrast to the white noise case, the NLS methods with fixed orders only perform well over a small range of $q$. The estimate variances coincide with the CRLBs only in the ranges $6 \leq q \leq 10$ for NLS with $r = 10$ and $10 \leq q \leq 16$ for NLS with $r = 16$. Outside these ranges, the estimates become biased toward sub- and super-harmonics. This suggests that the performance of the NLS method is comparable to the CRLB as long as the fixed order is in the range $q \leq r < 2q$. 

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Figure 3.26: FFE results for the uniform (left) and $1/\sqrt{\omega}$ (right) amplitude models versus the true NOH in correlated noise: FFE bias (top) and RMSE (bottom). The estimates are generated using the (+) NLS-MDL, (o) ANLS-MDL, (<) NLS with $r = 10$, and (x) NLS with $r = 16$. The RMSEs are plotted against the (-) root-CRLB for FFEs.
3.4 Field Measurement Data

The following example is a comparison between the STFT and ANLS-MDL parameter and order estimates from measured data. The measurement data consists of noise due to the local environment and a single source generating coupled harmonics and unmodelled broadband energy. The source is a heavy-tracked battlefield vehicle. The data was collected at APG using a seven-sensor, circular microphone array.

The data was recorded at $T = 1/1024$ seconds/sample. The noise is assumed to be stationary and is modelled as a fifth-order ($p = 5$) AR process. The AR parameters are generated using the first 10 seconds ($M = 10240$ samples) of data using Equation (3.24). The data length for the STFT is $N = 1024$, whereas the data length for ANLS-MDL is $N = 512$. Non-overlapping rectangular windows are used for both the STFT and ANLS-MDL. The frequency and order search regions are the same as those itemized in Section 3.3, except that the minimum selectable order is set to $r_{\min} = 0$. The frequency loss function is averaged using data from all seven sensors, as briefly discussed in Section 3.1.4. Consequently, amplitude and phase estimates are generated from each sensor’s data. However, only the results from Sensor 1 are presented.

The STFT of the raw data from Sensor 1 is represented by a spectrogram in Figure 3.27(a). In Figure 3.27(b), the harmonic frequency estimates from each half-second data block are plotted along the vertical axis. The horizontal axis represents the progression of time. The various levels of colors in Figures 3.27(a) and (b) represent the relative amplitudes, scaled in decibels, of the spectral data.
Up to approximately 200 seconds, ANLS-MDL estimates, for the most part, that there is no harmonic signal. Beyond 200 seconds, it appears the ANLS-MDL frequency and amplitude estimates are well-related to the measured harmonic source. The range of the source from the sensor array at 200 seconds is approximately 1 km. The closest point of approach (CPA) of the source occurs at 380 seconds. The estimates from ANLS-MDL also appear to improve up to and beyond the CPA.

The parameters are generated using independent half-second blocks of data. Although the parameter estimates are independent from block to block, there is an obvious continuity in the low to mid-range harmonics over time, as seen in Figure 3.27(b).

The filtered signal estimates are subtracted from the whitened data, resulting in the residual signal. The spectrogram for the residual data is shown in Figure 3.28. As seen in the figure, most of the remaining energy corresponds to unmodelled broadband energy and a pair of possibly coupled harmonic lines.
Figure 3.27: Spectrogram (a) and harmonic line estimates (b) of the acoustic signature from a single heavy-tracked vehicle. The relative amplitudes are in decibels, which is represented by the level of color.
Figure 3.28: Spectrogram of the acoustic signature from Figure 3.27(a) after removing estimated harmonic lines.
3.5 Conclusions

Two algorithms have been introduced which combine parameter estimation and order selection for coupled harmonic signals in Gaussian noise. These methods and the standard NLS method with an assumed order were evaluated in numeric simulations. The NLS method with the hypothesized model orderset to the true model order corresponds to the maximum likelihood estimator. However, it is shown that the NLS with order selection \( i.e., \) NLS-MDL exhibits only slight loss in performance compared to the MLEs, and at the expense of additional computational complexity. The loss in performance is accredited to the uncertainty in the true NOH. However, the performance differences quickly diminish with increasing SNR and data length. Additionally, the ANLS-MDL method offers similar performance to, and in some situations better than, NLS-MDL with fewer computations.

Each algorithm has an associated SNR and sample length thresholds. For sufficiently large SNR \( \rho \approx 0 \text{ dB} \) and data length \( N \approx 256 \), the NLS method provides statistically efficient estimates when the NOH is known. However, when the NOH is not known, but the system order is assumed, the NLS method still provides nearly statistically efficient estimates provided \( q \leq r < 2q \). It is also observed that there is a higher cost \( i.e., \) higher estimate variance in under-estimating the order as opposed to over-estimating the order. Also, when the NOH is not known, the proposed algorithms provide statistically efficient estimates for sufficiently large SNR \( \rho \approx 3 \text{ dB} \), data length \( N \approx 256 \), and number of harmonics \( q \geq 8 \). In battlefield acoustics, the NOH is generally not known and can vary over time, as seen in Figure 3.27(a).

In conclusion, the proposed algorithms can be used to extract features, such as the FF or phase parameters, of single sources generating coupled harmonics. These
features are useful in target classification [34] or in DOA estimation [16]. In addition, these algorithms can also be used to initialize and periodically update algorithms designed to track time-varying parameters, which generally require prior knowledge of the number of parameters to track. In the case of multiple sources, these algorithms can be combined with beamforming to temporally and spatially separate targets [3].
CHAPTER 4

COUPLED HARMONIC DETECTION

In this chapter, signal detection using a generalized likelihood ratio test (GLRT) is examined for the coupled harmonic model. The GLRT algorithm is generally reserved for situations where the statistical properties of signal parameters are unknown. In such situations, maximum likelihood estimates (MLEs) are substituted for the true values.

The GLRT is analyzed for both known and unknown number of harmonics. In Section 4.2, the GLRT is presented for known NOH. Section 4.3 extends the GLRT to the case of unknown NOH. In Sections 4.2 and 4.3, the respective performance properties are derived. Section 4.4 presents practical numerical examples that demonstrate the statistical properties of the detection test, and compares the theoretical and empirical detection performance. Concluding remarks and observations are given in Section 4.5.

4.1 Generalized Likelihood Ratio Test

Based on \( N \) noisy observations,

\[ y = [y(0), \ldots, y(N - 1)]^T, \]

the problem at hand is to decide between two hypothesis; the null hypothesis \( H_0 \), which defines noise only data, and hypothesis \( H_1 \), which defines the sum of a frequency
coupled harmonic signal and noise. More precisely, the hypotheses are

$$H_0 : y = \epsilon \quad \text{versus} \quad H_1 : y = s(\theta) + \epsilon,$$  \hspace{1cm} (4.2)

where the noise vector is distributed as $\epsilon \sim \mathcal{N}(0, \Sigma)$, the signal is $s(\theta) = A(\omega_0)\beta$, the parameter vector is $\theta = \theta^{\text{cart}}$, and where $A(\omega_0)$ and $\beta$ are defined in Chapter 2, and $\theta^{\text{cart}}$ is defined in Section 3.1.

A general linear subspace model given by $s = H\vartheta$, where $H$ was assumed known and $\vartheta$ unknown was considered in [22]. For the general signal model, the hypothesis test is given by Equation (4.2) with $H\vartheta$ substituted for $s(\theta)$. Generalized likelihood ratio tests (GLRTs) were developed for the general signal model in white Gaussian noise with both known and unknown level. In addition, GLRTs were developed in [23] for the hypothesis test given by

$$H_0 : y = G\varphi + \epsilon \quad \text{versus} \quad H_1 : y = H\vartheta + G\varphi + \epsilon,$$  \hspace{1cm} (4.3)

where $G$ is a known ($N \times m$) interference subspace signal with $(m \times 1)$ coordinate vector $\varphi$. Several cases were considered ($e.g.$, known $\varphi$ and unknown $\vartheta$) in [23]. In both [22] and [23], the GLRTs were shown to be uniformly most powerful (UMP). A UMP test is optimal in the class of detectors when prior distributions of the signal (and interference) parameters are unknown and the signal parameter space exists in two disjoint regions, namely $H_0 : \vartheta = 0$ and $H_1 : \vartheta > 0$ [18].

In this work, a partially unknown subspace signal with unknown coordinates $\beta$ and noise of unknown variance is considered. The subspace signal is said to be partially unknown since it is known except for two parameters: the fundamental frequency (FF), denoted $\omega_0$, and the number of harmonics (NOH), denoted $q$. However, the GLRT is derived assuming the NOH is known. Also, the noise covariance is assumed
to be known to within a constant. Note that in Chapter 3, the level of the noise
did not influence the design of the estimators for the unknown signal parameters. In
contrast, the noise variance does impact the signal detector designs presented below.

At this point, the noise samples have not been constrained to be independent.
On the other hand, the structure of the signal detector is simplified in the case of
independent noise. When correlated noise samples are whitened by a filter that can
be represented by a FIR filter (e.g., using the AR noise model in Section 3.1.2), the
two cases, correlated or independent noise, share a common detector structure. In
the case of correlated noise, it is assumed the FIR filter is known and the data is
pre-whitened. Consequently, without loss of generality, it is assumed that $\Sigma = \sigma^2 I$,
where $\sigma^2$ is unknown.

The detection problem in terms of the density functions is

$$H_0 : y \sim \mathcal{N}(0, \sigma_0^2 I) \text{ versus } H_1 : y \sim \mathcal{N}(s(\theta), \sigma_1^2 I).$$

(4.4)

Although $\sigma_0^2 = \sigma_1^2 = \sigma^2$ in the formulation in Equation (4.2), the distinction in the
notation is made to differentiate between the estimators of the noise level for each
hypothesis. Then, the likelihood ratio, conditioned on $\theta$, $\sigma_0^2$, and $\sigma_1^2$, is given by

$$L(y|\theta, \sigma_0^2, \sigma_1^2) = \frac{p(y|\theta, \sigma_1^2, H_1)}{p(y|\sigma_0^2, H_0)},$$

$$= \frac{\frac{1}{(2\pi\sigma_1^2)^{N/2}} \exp \left\{ -\frac{1}{2\sigma_1^2} \| y - s(\theta) \|^2 \right\}}{\frac{1}{(2\pi\sigma_0^2)^{N/2}} \exp \left\{ -\frac{1}{2\sigma_0^2} \| y \|^2 \right\}},$$

$$= \left( \frac{\sigma_0^2}{\sigma_1^2} \right)^{N/2} \exp \left\{ -\frac{1}{2\sigma_1^2} \| y - s(\theta) \|^2 + \frac{1}{2\sigma_0^2} \| y \|^2 \right\},$$

(4.5)

where $p(y|\theta, \sigma_1^2, H_1)$ and $p(y|\sigma_0^2, H_0)$ are the conditional density functions of the ob-
servation vector under each hypothesis, and where $\sigma_0^2$ and $\sigma_1^2$ are the noise variances
under each hypothesis.
Assuming for now that $\theta$ and $\sigma^2$ are random and their joint density, $f(\theta, \sigma^2)$, is known, then the likelihood ratio is determined by

$$L(y) = \int_{\Lambda} \left( \frac{\sigma_0^2}{\sigma_1^2} \right)^{N/2} \exp \left\{ -\frac{1}{2\sigma_1^2} \| y - s(\theta) \|^2 + \frac{1}{2\sigma_0^2} \| y \|^2 \right\} f(\theta, \sigma^2) d\lambda,$$

$$= \int_{\Lambda} \exp \left\{ -\frac{1}{2\sigma_1^2} \| y - s(\theta) \|^2 + \frac{1}{2\sigma_0^2} \| y \|^2 \right\} f(\theta, \sigma^2) d\lambda,$$

$$= \int_{\Lambda} \exp \left\{ \frac{1}{\sigma_2^2} y^T s(\theta) - \frac{1}{2\sigma_2^2} s(\theta)^T s(\theta) \right\} f(\theta, \sigma^2) d\lambda,$$

where $\Lambda$ is the region of integration for $\lambda = [\theta^T, \sigma^2]^T$. Consequently, a standard likelihood ratio test (LRT) could be developed using Neyman-Pearson (NP) or Bayesian techniques. Note that Equation (4.6) involves integration in $(2q + 2)$ dimensions. In general, $f(\theta, \sigma^2)$ and the prior probabilities $P(H_i)$, for $i \in \{0, 1\}$, are not known.

In addition, $f(\theta, \sigma^2)$ may not be a separable function, which further adds to the intractability of Equation (4.6). Therefore, $\lambda$ is assumed to be deterministic. For the provided assumptions and considering the complexity of Equation (4.6), the GLRT is an appropriate alternative decision rule. While it is not optimal compared to a Bayes or NP rule, the GLRT usually yields acceptable performance [31].

The hypothesis test is then a test of deciding between $H_0 : s(\theta) = 0$ and $H_1 : s(\theta) \neq 0$. Each element of the signal vector $s(\theta)$ can take on positive or negative values, hence no UMP exists [18]. Consequently, the GLRT is derived in hopes of acceptable and predictable performance, albeit suboptimal.

The GLRT, which is a NP-type decision rule, has the form

$$\frac{\arg \max_{\{\theta, \sigma_1^2\}} p(y|\theta, \sigma_1^2, H_1)}{\arg \max_{\sigma_0^2} p(y|\sigma_0^2, H_0)} \overset{H_1}{\underset{H_0}{\sim}} \eta,$$

(4.7)

where the detector selects $H_1$ ($H_0$) if $L(y) > (\leq) \eta$. The threshold $\eta$ controls the performance of the detector. The performance of NP-type detectors is usually gauged
by comparing the probability of detection, $P_D$, against the probability of false alarm, $P_{FA}$, and for various values of $\beta$ and $\sigma^2$. The $P_D$ is defined as the probability of correctly selecting hypothesis $H_1$, and the $P_{FA}$ is defined as the probability of selecting $H_1$, given that $H_0$ is the true event. Before analyzing the performance, a simplified expression for Equation (4.7) is developed.

The GLRT for known NOH is developed in Section 4.2. Then, a signal detector combining the GLRT and order selection is developed in Section 4.3.

### 4.2 GLRT for Known Number of Harmonics

As seen from Equation (4.7), the generalized likelihood ratio (GLR) utilizes the MLEs for the unknown parameters (i.e., $L(y) = L(y|\hat{\theta}, \hat{\sigma}_1^2, \hat{\sigma}_0^2)$). The MLEs for the parameters of $\theta$ are given in Section 3.1.1 for known $q$. For each hypothesis, the MLEs for the noise variances are given by

$$\hat{\sigma}_0^2 = \frac{1}{N} \|y\|^2; \tag{4.8}$$

and

$$\hat{\sigma}_1^2 = \frac{1}{N} \|y - s(\hat{\theta})\|^2 = \frac{1}{N} \|y - A(\hat{\omega}_0)\hat{\theta}\|^2. \tag{4.9}$$

Substituting Equations (4.8) and (4.9) and $\hat{\theta}$ into Equation (4.7) results in

$$L(y) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{N/2} \exp \left\{ -\frac{1}{2\hat{\sigma}_1^2} \|y - A(\hat{\omega}_0)\hat{\theta}\|^2 + \frac{1}{2\hat{\sigma}_0^2} \|y\|^2 \right\},$$

$$= \left(\frac{\|y\|^2}{\|y - A(\hat{\omega}_0)\hat{\theta}\|^2}\right)^{N/2},$$

$$= \left(\frac{y^T y}{y^T P_{\hat{\omega}_0}(\hat{\omega}_0) y}\right)^{N/2}. \tag{4.10}$$

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where \( \mathbf{P}_q^\perp(\tilde{\omega}_0) = \mathbf{I} - \mathbf{P}_q(\tilde{\omega}_0) = \mathbf{I} - \hat{\mathbf{A}}(\hat{\mathbf{A}}^T\hat{\mathbf{A}})^{-1}\hat{\mathbf{A}}^T \), \( \hat{\mathbf{A}} = \mathbf{A}(\tilde{\omega}_0) \). It is recognized that \( y^T\mathbf{P}_q^\perp(\omega)\mathbf{y} > 0 \) with probability 1 if \( q < N/2 \). Since the arguments inside the parenthesis on the last line of Equation (4.10) are positive, an equivalent test is given by

\[
T(y) = \frac{y^T y}{y^T \mathbf{P}_q^\perp(\tilde{\omega}_0) y} \frac{H_1}{H_0} \geq \eta^{2/N}.
\]  

(4.11)

To parallel the work by Scharf [22], the test statistic is referenced to unity, which produces

\[
t_q(y) = T(y) - 1,
\]

\[
= \frac{y^T \mathbf{P}_q(\tilde{\omega}_0) y}{y^T \mathbf{P}_q^\perp(\tilde{\omega}_0) y}.
\]

(4.12)

Since the maximum likelihood (ML) fundamental frequency estimate (FFE) is determined by maximizing the energy in the signal subspace (see Section 3.1.1), the test statistic in Equation (4.12) is equivalent to maximizing the ratio of the energy in the signal subspace to that in the noise subspace. Analytically, the test statistic is also given by

\[
t_q(y) = \max_{\omega \in \Lambda_\omega} \frac{y^T \mathbf{P}_q y}{y^T \mathbf{P}_q^\perp y},
\]

\[
= \max_{\omega \in \Lambda_\omega} t_q(y, \omega)
\]

(4.13)

where the dependence of \( \mathbf{P}_q \) and \( \mathbf{P}_q^\perp \) on \( \omega \) is suppressed to simplify the notation. The projection matrices \( \mathbf{P}_q \) and \( \mathbf{P}_q^\perp \) also depend on the NOH. The rank of \( \mathbf{P}_q \) is equal to the dimension of the column space of \( \mathbf{A} \), denoted \( \text{dim} \mathcal{R}(\mathbf{A}) \). The rank of \( \mathbf{P}_q^\perp \) is equal to the dimension of the left null space of \( \mathbf{A} \), denoted \( \text{dim} \mathcal{N}(\mathbf{A}^T) \). It is straightforward to show that \( \text{dim} \mathcal{R}(\mathbf{A}) = 2q \) and \( \text{dim} \mathcal{N}(\mathbf{A}^T) = N - 2q \), provided
that $\omega \in (0, \pi)$. Thus, the subscript of $P_q$ and $P_q^\perp$ symbolizes the dependence of the projection matrices on the NOH. Note, it is assumed that $q \ll N$.

It was shown in [22] that the argument of the max operator in Equation (4.13) is a maximal invariant statistic. That is, the statistic is unaffected by an unknown scaling of the noise variance. Consequently, the decision threshold can be determined without knowledge of the noise level. Finally, the GLRT for the case of known number of harmonic lines is given by

$$t_q(y) \overset{H_1}{\gtrless} \eta' \overset{H_0}{\lesssim} \eta',$$

where $\eta' = \eta^{2/N} - 1$.

**Receiver Operating Characteristic**

Next, the performance of the GLRT is examined. The performance is described in terms of the receiver operating characteristic (ROC).

The GLR in Equation (4.13) is a ratio of quadratic forms. Alternatively, the numerator and denominator can be regarded as the norm-squared of the statistics $P_qy$ and $P_q^\perp y$, respectively. These statistics are distributed as

$$P_qy \sim \begin{cases} 
\mathcal{N}(0, \sigma^2 P_q) & \text{under } H_0 \\
\mathcal{N}(P_q A(\omega_0) \beta, \sigma^2 P_q) & \text{under } H_1
\end{cases}$$

$$P_q^\perp y \sim \begin{cases} 
\mathcal{N}(0, \sigma^2 P_q^\perp) & \text{under } H_0 \\
\mathcal{N}(P_q^\perp A(\omega_0) \beta, \sigma^2 P_q^\perp) & \text{under } H_1
\end{cases}$$

Since, by design, $P_q P_q^\perp = 0$, the vectors $P_qy$ and $P_q^\perp y$ are uncorrelated (and independent due to Gaussianity) [23]. Consequently, the quadratic forms $y^T P_q y / \sigma^2$ and $y^T P_q^\perp y / \sigma^2$ are independent RVs [36]. These quadratics are chi-squared distributed
and given by

\[ y^T \mathbf{P}_q y / \sigma^2 \sim \begin{cases} \chi^2_{2q}(0) & \text{under } H_0 \\ \chi^2_{2q}(\lambda^2) & \text{under } H_1 \end{cases} \]

\[ y^T \mathbf{P}^{-1}_q y / \sigma^2 \sim \begin{cases} \chi^2_{N-2q}(0) & \text{under } H_0 \\ \chi^2_{N-2q}(\lambda^2) & \text{under } H_1, \end{cases} \]  

(4.16)

where \( \chi^2_m(\delta^2) \) represents a noncentral \( \chi^2 \)-distributed RV with \( m \) degrees of freedom and noncentrality parameter \( \delta^2 \). A RV distributed as \( \chi^2_m(\delta^2) \) with \( \delta^2 = 0 \) is simply denoted by \( \chi^2_m \). The number of degrees of freedom is equal to the number of independent, squared Gaussian RVs in the summation. Also, the parameter \( \delta^2 \) is equal to the sum of the squared means of the \( m \) independent Gaussian RVs [22].

To show that Equation (4.16) is valid under the null hypothesis for both quadratics, it is recognized that \( \mathcal{E}\{x\} = m \), where \( x \sim \chi^2_m \) and \( \mathcal{E}\{\cdot\} \) denotes the expectation operator. Observing that \( y \) is zero-mean under \( H_0 \), the number of degrees of freedom for the numerator of \( t_q(y, \omega) \) is

\[
\mathcal{E}\left\{ y^T \mathbf{P}_q y / \sigma^2 \right\} = \mathcal{E}\left\{ \text{tr}\left( y^T \mathbf{P}_q y / \sigma^2 \right) \right\},
\]

\[ = \frac{1}{\sigma^2} \text{tr}\left( \mathbf{P}_q \mathcal{E}\left\{ y y^T \right\} \right), \]

\[ = \text{tr}(\mathbf{P}_q) = \dim \mathcal{R}(\mathbf{A}), \]

\[ = 2q. \]  

(4.17)

Similarly, the number of degrees of freedom for the denominator of \( t_q(y, \omega) \) is equivalent to \( \dim \mathcal{N}(\mathbf{A}^T) = N - 2q \). Additionally, the noncentrality parameter is clearly zero under \( H_0 \). On the other hand, the parameters \( \lambda^2_1 \) and \( \lambda^2_2 \) have yet to be determined for the alternate hypothesis. First, the statistics of the GLR under \( H_0 \) are further developed. Then, the statistics under \( H_1 \), including the parameters \( \lambda^2_1 \) and \( \lambda^2_2 \), are discussed.
Scaling these statistically independent quadratic forms by their respective degrees of freedom and taking their ratio results in the following modified test statistic:

\[
\frac{N - 2q}{2q} t_q(y, \omega) \sim \begin{cases} 
F_{2q, N-2q}(0, 0) & \text{under } H_0 \\
F_{2q, N-2q}(\lambda_1^2, \lambda_2^2) & \text{under } H_1
\end{cases}
\]

(4.18)

where \( F_{m, k}(\delta_1^2, \delta_2^2) \) is the doubly noncentral \( F \)-distribution with \( m \) numerator and \( k \) denominator degrees of freedom and noncentrality parameters \( \delta_1^2 \) and \( \delta_2^2 \) [22]. Notice that, under \( H_0 \), the modified test statistic is the usual centered \( F \)-distribution, which is simply denoted by \( F_{m, k} \). Henceforth, the GLRT will be given by

\[
f_q(y) = \frac{N - 2q}{2q} t_q(y) \quad \text{under } H_1 \quad \overset{H_0}{\underset{\geq \gamma''}{\geq}} \eta'',
\]

(4.19)

where \( \eta'' = \frac{N - 2q}{2q} (\eta''^N - 1) \). The reason for using \( f_q(y) \) instead of \( t_q(y) \) is twofold: \( f_q(y) \) avoids numerical instabilities encountered in MATLAB when evaluating distributions of \( t_q(y) \), and \( f_q(y) \) minimizes the bookkeeping otherwise required when the NOH is not known.

It is of interest to determine the statistics of the GLR given by Equation (4.19). Since the statistic \( f_q(y, \omega) \) is a continuous function of frequency, it may be impractical to determine the precise statistics of the GLR. Therefore, simplifying assumptions are made, which are discussed in what follows.

In general, a fine frequency grid search is performed, followed by optimization techniques, to maximize \( f_q(y, \omega) \). Under \( H_0 \), the GLR is approximated by

\[
f_q(y) \approx \frac{1}{\gamma} \max\{f_q(y, \omega_1), \ldots, f_q(y, \omega_{M_q})\},
\]

(4.20)

where \( M_q \) is the number of frequency points in a search grid and \( \gamma \) is a constant to account for interpolating the test statistic between adjacent samples of \( \omega \). Assuming \( q = 1 \), it is shown in [25] that the RVs \( \{f_q(y, \omega_k)\}_{k=1}^{M_q} \) are asymptotically (in data
length) independent provided that \( \min_{k \neq j} |\omega_k - \omega_j| \geq 2\pi/N \). Consequently, \( M_q \) is given by the number of uniformly spaced frequency points, separated by \( 2\pi/N \), taken from the continuous region

\[
\Lambda_\omega = [\omega_{\min}, \omega_{\max}] \cap (2\pi/N, \pi/q) \tag{4.21}
\]

where \( \omega_{\min} \) and \( \omega_{\max} \) may be set by some prior knowledge, and \( \omega < \pi/q \) satisfies the Nyquist criterion. Here, it is assumed that independence holds for \( q > 1 \). Accordingly, the approximate statistics of \( f_q(y) \) under \( H_0 \) is given by

\[
D_{f_q(y)}(f) \approx \prod_{k=1}^{M_q} D_{f_q(y; \omega_k)}(\gamma f), \\
= F^{M_q}(\gamma f; 2q, N - 2q), \tag{4.22}
\]

where \( f \) is a realization of the RV \( f_q(y) \), \( D_X(x) \) is the cumulative distribution function (CDF) of the RV \( X \), and \( F(x; m, k) \) is the CDF of a RV distributed as \( X \sim F_{m,k} \). The second line in Equation (4.22) follows from the fact that the statistics of the RVs \( \{f_q(y, \omega_k)\}_{k=1}^{M_q} \), given in Equation (4.18), do not individually depend on \( \omega_k \) under \( H_0 \). On the other hand, the correlation between the RVs is directly related to \( \omega_k \) and \( q \). For example, when \( q = 2 \), the test statistics at \( \omega_1 \) and \( \omega_2 = 2\omega_1 \) have a direct correlation through \( 2\omega_1 \). Despite this, the independence assumption in Equation (4.22) is made to make the problem more tractable.

In general, estimators of FF search over a frequency grid with spacings \( \Delta \omega < 2\pi/N \), as discussed in Section 3.1.3. To see how this sub-sampling affects the distribution of the test statistic, the case with a single harmonic is examined. For a small deviation in \( \omega \), the relative change in energy in the noise subspace (i.e., the denominator of \( t_q(y, \omega) \)) would vary little as compared to the relative energy change in the signal subspace (i.e., the numerator), where relative energy is defined as \( E(\omega)/E(\omega_c) \).
Focusing on the numerator of $t_q(y, \omega)$, the expected value of the test statistic evaluated using finely spaced frequency points will be higher than the expected value if the spacing is more coarse.

To see this, imagine samples of the PSD of white noise. For the most part, the power is uniformly distributed with respect to frequency. The PSD is made up of overlapping energy spectrums with uniformly distributed frequency shifts. However, more likely than not, one frequency bin will have more energy than the rest. This peak bin corresponds to sampling of the energy spectrum nearly centered at that frequency bin, with some, but probably negligible, contribution from neighboring energy spectrums. Using a smaller frequency spacing, there is a higher probability of capturing the peak of this energy spectrum in a frequency bin. The normalized energy values are the samples of shifted and squared sinc functions given by

$$|S(e^{i\omega_k})|^2 = \frac{\sin^2(N(\omega_k - \omega_c)/2)}{N^2 \sin^2((\omega_k - \omega_c)/2)},$$  \hspace{1cm} (4.23)

where $S(e^{i\omega_k})$ is the discrete Fourier transform (DFT) of a rectangular window of length $N$ modulated to $\omega_c$ and evaluated at $\omega_k$. The peak energy bin (i.e., the value of $k$ for which $|S(e^{i\omega_k})|^2$ is maximum) will be such that $|\omega_k - \omega_c|$ is minimum, where $\omega_k = 2\pi k/M$ is the center of a frequency bin and $\omega_c$ is the center frequency of the energy spectrum. To make the independence assumption, it is required that $M \leq N$. For $M = N$ and $\omega_c \sim U(\omega_k - \pi/N, \omega_k + \pi/N)$, the expected value of the normalized peak energy is given by

$$\gamma = \frac{N}{2\pi} \int_{-\pi/N}^{\pi/N} \left( \frac{\sin(N\omega/2)}{N\sin(\omega/2)} \right)^2 d\omega,$$

$$\approx 2 \int_0^{1/2} \left( \frac{\sin(\pi x)}{\pi x} \right)^2 dx,$$

$$\approx 0.7737.$$  \hspace{1cm} (4.24)
Figure 4.1: Example uniform sampling of a $\text{sinc}^2(N(\omega - \omega_c)/2)$ energy spectrum. The spectrum is sampled with $\Delta \omega = 2\pi/N$ (o) and $\pi/2N$ (x).

An example energy spectrum is plotted in Figure 4.1 with vertical lines that represent sampling of the spectrum with spacings $\Delta \omega = 2\pi/N$ (o) and $\pi/2N$ (x). The dashed horizontal line labelled 'Expected Value' corresponds to the expected value of the energy relative to the true peak of the spectrum for when the center frequency is uniformly distributed $2\pi/N$ and the bin center. The 'Expected Value' line will typically increase when the frequency spacing decreases.

Now that the statistics have been developed for $H_0$, the attention is directed toward developing the statistics of the GLR under $H_1$. For sufficiently high SNR, the ML FFEs have nearly zero bias. For low SNR levels, the ML FFE tends to be highly biased, with estimates generally close to multiples of $\omega_0$. Despite this, it is speculated that most of the signal energy is captured by the GLR even in cases of low SNR. Therefore, it is assumed that the FFEs are unbiased in approximating the statistics of the GLR for a large range of SNRs.
The distribution of a MLE asymptotically converges to a normal density with covariance matrix equal to the inverse of the Fisher's information matrix (FIM). From Equation (A.18) in Appendix A, the ML FFE errors are $O(1/N^{3/2})$, whereas the errors of the MLE $\hat{\beta}$ are $O(1/N^{1/2})$. For large data lengths, the projections in Equation (4.12) can be approximated by

$$
\text{P}_q(\hat{\omega}_0)y = A(\omega_0)\hat{\beta},
$$

$$
\approx A(\omega_0)\hat{\beta},
$$

$$
\text{P}_q^\perp(\hat{\omega}_0)y = y - A(\omega_0)\hat{\beta},
$$

$$
\approx y - A(\omega_0)\hat{\beta},
$$

(4.25)

which amounts to assuming that $\max_{\omega} t_q(y, \omega) = t_q(y, \omega_0)$. Using Equation (A.18) in Appendix A, $\hat{\beta}$ is asymptotically distributed as $\hat{\beta} \sim \mathcal{N}(\beta, \frac{2\sigma^2}{N}\mathbf{I})$. Using the large-sample approximation $A^T(\omega)A(\omega) \approx \frac{N}{2}\mathbf{I}$ and similar arguments for Equations (4.16)-(4.18), the GLR under $H_1$ is distributed as $f_q(y) \sim F_{2q, N-2q}(\lambda_1^2, \lambda_2^2)$, where $\lambda_1^2 \approx \|A(\omega_0)\beta\|^2$ and $\lambda_2^2 \approx 0$. Thus, the test statistic can be modelled by a singly noncentral $F$-distribution, denoted by $F_{2q, N-2q}(\lambda^2)$ where $\lambda^2 = \lambda_1^2$.

In the presence of a signal (i.e., under $H_1$), the noncentrality parameter is $\lambda^2 \approx \|A(\omega_0)\beta\|^2/\sigma^2 \approx N\alpha^T\alpha/2\sigma^2$. The noncentrality parameter can be physically interpreted as the ratio of the total signal energy to the noise variance. One would expect the performance of the GLRT to improve with increasing $\lambda^2$. Increasing $\lambda^2$ can be viewed as increasing the separation of the distribution 'centers', or, equivalently in this case, increasing the total signal energy to noise power ratio. In passive sensor systems, the designer may only have control over the data length. On the other hand, shorter data lengths may be preferable to minimize signal model mismatching due
to time variations. As a result, there is a trade-off between increasing total signal energy and reducing modelling mismatches.

Putting together the theoretical distributions of $f_q(y)$ under each hypothesis, the false alarm and detection probabilities are

$$P_{FA}(\eta'') = \Pr \{ f_q(y) > \eta'' | H_0 \},$$

$$\approx 1 - D_{f_q(y)}(\eta''),$$

$$P_D(\eta'') = \Pr \{ f_q(y) > \eta'' | H_1 \},$$

$$\approx 1 - F(\eta''; 2q, N - 2q, \lambda^2),$$

(4.26)

(4.27)

where $F(\tau; m, k, \delta^2)$ is the CDF of a singly noncentered $F$-distributed RV, with $m$ numerator and $k$ denominator degrees of freedom and noncentrality parameter $\delta^2$, evaluated at $\tau$. The threshold $\eta''$ is the solution that satisfies $P_{FA} \leq \alpha$, where $\alpha$ is known as the significance level of the decision rule. It is commonplace to set $P_{FA} = \alpha$.

Curves for the theoretical ROC are plotted in Figure 4.2. Figure 4.2(a) shows the $P_D$ versus $P_{FA}$ for $\rho = -11$, -9, and -7 dB and for $q = 10$. Figure 4.2(b) shows the $P_D$ versus SNR for $q = 2$ (a), 10 (+), and 30 (o) for $P_{FA} = 10^{-3}$ and $P_{FA} = 10^{-2}$. In both plots, the signal amplitudes follow a $1/\sqrt{\omega}$ model and the data length is set to $N = 256$. Consequently, $\lambda^2 \approx \text{SNR} + 24$ dB. The noise variance is set according to Equation (3.41). From Figure 4.2(a), it is apparent that the GLRT is predicted to perform well. For example the detection probability is nearly 90% for a SNR as low as $\rho = -7$ dB and 1% false alarm rate. As seen in Figure 4.2(b), the $P_D$ for a given SNR decreases with increasing NOH. Although not shown here, the SNR to achieve the same $P_D$ is reduced by approximately 3 dB for a given $q$ by doubling the data length.

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Figure 4.2: Predicted ROC curves for $P_D$ versus $P_{FA}$ (a) and $P_D$ versus SNR (b). In (a), the NOH is set to $q = 10$ and the ROC curves are plotted for various SNR values, along with the $P_D = P_{FA}$ line. In (b), ROC curves are plotted for $P_{FA} = 10^{-3}$ and $P_{FA} = 10^{-2}$ with $q = 2$ (•), 10 (+), and 30 (◦). The data length is set to $N = 256$.

4.3 Detector for Unknown Number of Harmonics

The event $s(\theta) = 0$ is considered differently for the cases of known and unknown NOH. When the number is assumed known, $H_0$ is considered to occur if $\beta = 0$ for some nonzero $q$. When the number is not known, $H_0$ is considered to occur if $\beta = 0$ and $q \geq 1$. Consequently, this detection test is quite different than order selection. In order selection, $q = 0$ is generally an acceptable NOH. One might then question if allowing $q \geq 0$ might yield a detector with performance superior to one that restricts the order to be $q \geq 1$. Examining Equation (4.12), the GLR evaluated with $q = 0$ is $f_0(y) = 0$, regardless of the observation $y$. Consequently, the theoretical PDF of $f_0(y)$ for $q = 0$ is a Dirac delta function under both hypotheses. The goal in NP-type tests is to maximize $P_D$ subject to some constant false alarm rate (CFAR) threshold.
Although not shown in this work, it was determined empirically that allowing the detector structure to select the NOH including $q = 0$ did not reduce the decision threshold from that in the $q \geq 1$ case for typical values of $P_{FA}$. In turn, there is no gain in the detection probability. Consequently, the NOH is chosen from the set $\Lambda_r = \{1, 2, \ldots, r_{ub}\}$, where $r_{ub}$ is defined in Section 3.2.1.

It is recognized that the event $H_0$ defined as $\beta = 0$ for some nonzero $q$ or $q = 0$ are mathematically equivalent for both the cases of known and unknown NOH. However, the assumed null hypotheses discussed above are just a matter of definition.

In [36], computationally efficient yet ad hoc detector structures are developed assuming the FF and NOH are known. Optimal detectors are developed in [22, 23] for the case of known FF and NOH. When the NOH is not known, optimal finite-sample min-max tests for signal detection, order selection, and classification are developed in [2] for signals than can be selected from a finite set of orthogonal waveforms. The decision tests in [2] are given by weighted GLRTs. However, the weighting functions require solving nonlinear, data-dependent integral equations. It was shown in [2] that the Minimum Description Length (MDL) criterion for order selection is nearly optimal in terms of worst case classification performance as the data length increases. In addition, there is only slight loss in performance when using the optimal classification criterion for signal detection. Therefore, it is reasonable to consider the MDL criterion in a signal detector for its near optimal quality and the simplicity of applying the data-independent penalty. On the other hand, for the problem under study, the signals are chosen from an infinite set (i.e., $\omega_0$ exists in an uncountably infinite set) that may not be made orthogonal. Despite this, the MDL criterion exhibited acceptable performance characteristics when used for combined parameter
estimation and order selection, as seen in Section 3.3. Therefore, a signal detector using the GLR and MDL order selection is pursued as an initial study and to attempt to predict detection performance.

The simultaneous order selection and FF estimation method is given by Equation (3.40). This joint method can be related to the GLR. Equation (3.40) can be rewritten as

\[
\{\hat{\omega}_0, \hat{q}\} = \arg \min_{\{\omega, r\}} N^{-\frac{r}{2}} y^T P_r^\perp y, \\
= \arg \max_{\{\omega, r\}} N^{-\frac{r}{2}} \frac{1}{y^T P_r^\perp y}, \\
= \arg \max_{\{\omega, r\}} \left\{ N^{-\frac{r}{2}} \left( \frac{y^T y}{y^T P_r^\perp y} - 1 \right) + N^{-\frac{r}{2}} \right\}, \\
= \arg \max_r \left\{ N^{-\frac{r}{2}} \left( \max_{\omega} \frac{y^T P_r^\perp y}{y^T P_r^\perp P_r^\perp y} \right) + N^{-\frac{r}{2}} \right\}. \tag{4.28}
\]

The argument in the parenthesis of the last line of Equation (4.28) is a GLR given by Equation (4.13) for \(r\) harmonics. It follows that the arguments that minimize the MDL criterion given by Equation (3.40) also maximize a scaled and shifted GLR.

There is an inherent relationship between the combined order selection and parameter estimation algorithms discussed in Chapter 3 and the combined signal detection and order selection pursued in this chapter. This relationship leads to two approaches: using NLS-MDL or ANLS-MDL to generate the test statistic. The ANLS-MDL algorithm is an approximation to the NLS-MDL and provides computational savings. The detector using the NLS-MDL will be referred to as the GLR-MDL, and the detector using the ANLS-MDL will be referred to as the AGLR-MDL.

For either approach, the detection method is a two-step bootstrapping method. First, the quadratic term \(L_q(\hat{\omega}_0) = y^T P_q^\perp(\hat{\omega}_0)y\), termed the residual error or loss
function, is generated using NLS-MDL or approximated by ANLS-MDL. Then, the GLR given by Equation (4.19) is evaluated and compared to a CFAR threshold. Consequently, computing the test statistic only requires approximately $N$ additional computations over the NLS-MDL or ANLS-MDL methods.

**Receiver Operating Characteristic**

In this section, the performance of the signal detector using the GLR and the MDL order selection criterion is examined. Most of the statistical derivations are covered in Section 4.2.

In general, CFAR signal detectors are designed such that the decision threshold is set independently of the unknown parameters (e.g., $\sigma^2$, $\beta$, $q$, etc.). The previous detector structure only requires knowledge of $q$ in order to set the threshold. Given the statistics of order estimates, the unknown parameter can be averaged out of the $H_0$ and $H_1$ statistics. In other words, the marginal density functions can be found by averaging the density functions conditioned on the NOH using the probability mass function of the order estimates.

The conditional density functions are given by Equations (4.26) and (4.27). Analytically, the new statistic under $H_0$ can then be computed by

$$D_{f(y)}(f) = \sum_{r=1}^{r_{\text{max}}} D_{f_{r}(y)}(f) P(r), \quad (4.29)$$

where $f(y) = f_q(y)$, $r_{\text{max}}$ is the largest selectable NOH, and $P(r)$ is the probability mass function of the order estimates. Then, using the new statistic, a CFAR threshold may be computed independent of any unknown parameter.

Under $H_1$, it is assumed that the test statistic is well-modelled by the statistics developed in the case of known NOH (i.e., $P(r = q) = 1$). Only the statistics under
\( H_0 \) need to be analyzed. Therefore, it is necessary to determine the probability mass function (PMF) for the MDL order estimates under the null hypothesis.

Once more, the MDL order selection criterion can be rewritten as

\[
\hat{q} = \arg \min_r \left\{ N \ln y^T P_r^\perp(\hat{\omega}_{0,r})y + r \ln N \right\},
\]

\[
= \arg \min_r \frac{N^{r/N}}{\sigma^2} y^T P_r^\perp(\hat{\omega}_{0,r})y,
\]

\[
= \arg \min_r \frac{N^{r/N}}{\sigma^2} y^T P_r^\perp(\hat{\omega}_{0,r})y,
\]

(4.30)

where \( \hat{\omega}_{0,r} \) is the FFE for a particular model order \( r \). Now, the random variable \( Z_r \) is defined as

\[
Z_r = \frac{N^{r/N}}{\sigma^2} y^T P_r^\perp(\hat{\omega}_{0,r})y,
\]

\[
= \frac{N^{r/N}}{\sigma^2} \min_{\omega} y^T P_r^\perp(\omega)y,
\]

\[
\approx \min \{ z_{r,1}, \ldots, z_{r,\mathcal{M}_r} \},
\]

(4.31)

where \( z_{r,k} = N^{r/N} y^T P_r^\perp(\omega_k)y/\sigma^2 \) and \( \mathcal{M}_r \) is the number of frequency points. It is of interest to find

\[
P_q(r) = \Pr\{Z_r < Z_i | i \in \Lambda_r^*\},
\]

(4.32)

where \( \Lambda_r^* \) is the set of possible orders excluding order \( r \). Using similar simplifying assumptions made in Section 4.2 for the GLR under \( H_0 \), the PMF is given by

\[
P_q(r) = \int_0^\infty \left( \prod_{i \in \Lambda_r^*} \left( 1 - D_{z_i}(z) \right) \right) p_{z_i}(z)dz,
\]

(4.33)

where \( D_X(x) \) and \( p_X(x) \) are the CDF and PDF of a RV \( X \), respectively. The CDF and PDF of \( Z_k \) are approximated by

\[
D_{z_k}(z) \approx 1 - (1 - D_{z_{k,1}}(N^{-k/N} z))^{\mathcal{M}_k},
\]

\[
p_{z_k}(z) \approx N^{-k/N} \mathcal{M}_k (1 - D_{z_{k,1}}(N^{-k/N} z))^{\mathcal{M}_k - 1} p_{z_{k,1}}(N^{-k/N} z),
\]

(4.34)
where $z_{k,1}$ is distributed as $z_{k,1} \sim \chi^2_{N-2k}$. The false alarm rate and detection probability are then given by

$$P_{FA}(\eta^*) \approx 1 - \sum_{r=1}^{r_{max}} D_{f_t(y)}(\eta^*) P_q(r),$$

$$P_D(\eta^*) \approx 1 - F(\eta^*; 2q, N - 2q, \lambda^2),$$

where $\eta^*$ is the solution to the $\alpha$-level test.

Curves for the theoretical ROC are plotted in Figure 4.3. Figure 4.3(a) shows the $P_D$ versus $P_{FA}$ for $\rho = -8, -6$, and $-4$ dB and for $q = 10$. Figure 4.3(b) displays the $P_D$ versus SNR for $q = 2$ (\diamond), 10 (\oplus), and 30 (\circ) for $P_{FA} = 10^{-3}$ and $P_{FA} = 10^{-2}$. The signal amplitudes follow a $1/\sqrt{\omega}$ model and the data length is set to $N = 256$. The noise level is adjusted as discussed in for each SNR by the relationship in Equation (3.41). As seen in Figure 4.3(a), the detector is predicted to have high detection probabilities with low false alarm rates for SNRs at and above $-4$ dB. Also, there is a significant loss in performance as the NOH increases. As seen in Figure 4.3, the $P_D$ for a given SNR decreases with increasing NOH. However, compared to the GLR for known NOH, there is a greater performance difference for varying NOH, and a significant loss in performance for larger NOH. This loss is related to the statistics under $H_0$, since the statistics under $H_1$ are assumed to be the same for both the known and unknown NOH cases. The PDF of $f(y)$ under $H_0$ is heavier-tailed due to the uncertainty in the NOH. Despite this, the detector appears to have good $P_D$ versus $P_{FA}$ for low SNRs when the signal contains few harmonics.
Figure 4.3: Predicted ROC curves for $P_D$ versus $P_{FA}$ (a) and $P_D$ versus SNR (b). In (a), the NOH is set to $q = 10$ and the ROC curves are plotted for various SNR values, along with the $P_D = P_{FA}$ line. In (b), ROC curves are plotted for $P_{FA} = 10^{-5}$ and $P_{FA} = 10^{-2}$ with $q = 2$ (s), 10 (+), and 30 (o). The data length is set to $N = 256$. 

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4.4 Simulation Studies

The following are numerical examples that demonstrate the statistical properties of the signal detectors. This study first examines the GLR for known NOH, and then the signal detector that combines the GLR and MDL order selection. For each case, the theoretical distributions of the signal detectors are compared against theory. Then, estimated ROC curves are compared against theoretical ones. The ROC curves for $P_D$ versus SNR are plotted with the $P_{FA}$ set to $\alpha = 10^{-2}$.

The simulation parameters common to all simulations are as follows:

- The sampling period $T$ is set to $1/512$ s in a $1/2$ s observation window.
- The FF is $\omega_0 = 0.1$ rads/sample ($\approx 8$ Hz).
- Two amplitude models are examined: $\alpha_k = 1$ and $\alpha_k = 1/\sqrt{k\omega_0}$.
- The phases are set to $\phi_k = k\pi/100$ rads.
- The order search range is set to $\Lambda_r = [1, r_{\text{ub}}]$.
- The frequency range is set to $\Lambda_\omega = [\pi/128, 25\pi/256]$ rads/sample ($[2,25]$ Hz).
- The frequency grid resolution is $\Delta\omega = \pi/2048$ rads/sample (1/8 Hz).
- The results are generated from 10 000 Monte Carlo simulations.

Due to the number of Monte Carlo trials ($10^4$) performed for each simulation parameter, there is confidence in reported event probabilities down to $10^{-3}$.
4.4.1 Performance for Known Number of Harmonics

In this section, the distributions of the GLR test statistic are compared to theory. Comparisons are also made between simulated and theoretical ROC curves. Performance is gauged in terms of the ROC curves.

Figure 4.4 is plot of the $P_{FA}(\eta)$ versus $\eta$. The theoretical $P_{FA}(\eta)$ is given by Equation (4.26). Each curve in Figure 4.4 corresponds to the NOH, which range from $q = 1$ to $q = 30$. The simulation results are represented by the solid blue curves, whereas the theoretical $P_{FA}$ curves are dashed black lines. The $P_{FA}$ values are plotted on a log scale to accentuate the probabilities much less than one.

As seen in the figure, the $P_{FA}$ curves correspond well with the empirical curves for larger values of $q$. For smaller values of $q$, the theory curves tend to be larger than those determined through simulation. Consequently, for a desired false alarm rate, theory would report a higher threshold than what may be necessary. Despite this, the proposed distribution of the test statistic under $H_0$ captures the general curvature as a function of $\eta$. We conjecture that the differences between theory and simulation under $H_0$ are due mainly to a $q$-dependent scaling that is not taken into account, and any differences due to invalid independence assumptions pale in comparison. It may then be sufficient to replace $\gamma$ in Equation (4.20) by some $\gamma(q)$ to improve the agreement between the respective $P_{FA}$ curves. At this point, the function $\gamma = \gamma(q)$ has not been explored.

Figure 4.5 is plot of the $P_D(\eta)$ versus $\eta$. The theoretical $P_D(\eta)$ is given by Equation (4.27). Each curve in Figure 4.5(a) corresponds to a signal with $q = 2, 5, 10$ and 30 harmonics and a SNR of $\rho = -6$ dB. In Figure 4.5(b), each curve corresponds to a signal with $q = 10$ harmonics and $\rho = -4, -6, -8, -10$, and $-12$ dB. In (a) and (b), the
Figure 4.4: False alarm probability versus $\eta$ for varying NOH. The simulated NOH are $q = 1, 2, 3, 5, 10, \text{ and } 30$ with theoretical curves (dashed) plotted against simulation results (solid). The data length is set to $N = 256$.

signal is made of coupled harmonics with equal amplitudes. The simulation results are represented by the solid curves, whereas the theoretical $P_D$ curves are dashed lines. The $P_D$ is similarly plotted in Figures 4.6(a) and (b) for a $1/\sqrt{\omega}$ amplitude model. Notice that the $P_D$ curves are displayed using a linear scale, as opposed to the log scale used in the $P_{FA}$ curves. The $P_D$ plots are scaled as such to accentuate the probabilities closer to one.

As seen in Figure 4.5(a) and Figure 4.6(a), theory and empirical results match very well for a wide range of NOH, even at a SNR as low as $\rho = -6$ dB. It is also evident in Figure 4.5(b) and Figure 4.6(b) that theory and simulation diverge at SNRs below $\rho = -8$ dB. Comparing the uniform and $1/\sqrt{\omega}$ amplitude cases, the detection statistics appear to be better modelled for signals with uniform amplitudes. As seen
Figure 4.5: Detection probability versus $\eta$ for varying NOH (a) and varying SNR levels (b). The simulated NOH are $q = 2, 3, 5, \text{and} 10$ with the SNR set to $\rho = -6$ dB in (a). The simulated SNR levels are $\rho = -12, -10, -8, -6, -4, \text{and} -2$ dB with $q = 10$ in (b). The theoretical curves (dashed) are plotted against simulation results (solid). The data length is set to $N = 256$ and the signal amplitudes follow a uniform model.

In Figure 4.6(b), there is a slight offset, which is not as evident in Figure 4.6(a) due to the horizontal axis scaling, between theory and simulation.

Figures 4.7(a) and (b) are ROC curves for a CFAR threshold set by a $P_{FA} = 10^{-2}$ for the uniform and $1/\sqrt{\omega}$ amplitude models, respectively. The series of curves in each figure correspond to coupled harmonic signals with $q = 2, 5, 10, \text{and} 30$ components. The theoretical curves (dashed) are plotted against simulation results (solid).

As seen in Figures 4.7(a) and (b), the empirical results agree with the theoretical curves for the NOH values up to $q = 10$. There is a better agreement between theory and simulation when the signal amplitudes follow the uniform model, which is expected from the results in Figures 4.5 and 4.6.
Figure 4.6: Detection probability versus $\eta$ for varying NOH (a) and varying SNR levels (b). The simulated NOH are $q = 2, 3, 5, \text{ and } 10$ with the SNR set to $\rho = -6$ dB in (a). The simulated SNR levels are $\rho = -12, -10, -8, -6, -4, \text{ and } -2$ dB with $q = 10$ in (b). The theoretical curves (dashed) are plotted against simulation results (solid). The data length is set to $N = 256$ and the signal amplitudes follow a $1/\sqrt{\omega}$ model.

The ROC curves could have been displayed in the alternate form of $P_D$ versus $P_{FA}$ for a given SNR (e.g., Figure 4.2(a)). However, the latter ROCs from simulation would convey little useful information. For SNR values above $\rho = -9$ dB (below this level the theoretical statistics become questionable), the ROC curves have a strong concavity for values below $P_{FA} = 10^{-3}$. Consequently, in order for simulation results to have any merit requires an exceedingly large, beyond practical, number of Monte Carlo simulations.
Figure 4.7: ROC for varying NOH for a uniform (a) and $1/\sqrt{\omega}$ (b) amplitude model. The NOH are from $q = 2, 5, 10$ and $30$ with the $P_{FA}$ set to $\alpha = 10^{-2}$. The theoretical curves (dashed) are plotted against simulation results (solid). The data length is set to $N = 256$.

4.4.2 Performance for Unknown Number of Harmonics

In this section, the theoretical distributions of the GLR using MDL test statistic are compared to empirical distributions. Comparisons are also made between simulated and theoretical ROC curves.

Figure 4.8 presents PMFs of the MDL order estimates (a) and $P_{FA}$ versus $\eta$ (b) for the GLR using MDL. The PMFs are generated using theory (black), ANLS-MDL (green), and NLS-MDL (blue). In Figure 4.8(b), the empirical $P_{FA}(\eta)$ is generated using ANLS-MDL (solid green) and NLS-MDL (solid blue) and plotted against the theoretical curve (dashed black).

It is observed from Figure 4.8(a) that the theoretical PMF is more heavy tailed compared to the empirical ones. Despite this, the theoretical and empirical $P_{FA}$ curves are in strong agreement for unknown NOH. The curves in Figure 4.8(b) are
Figure 4.8: Probability mass function (a) for the MDL order estimates and false alarm probability versus \( \eta \) (b) for the GLR using MDL. PMFs are generated using theory, GLR-MDL, and AGLR-MDL. The theoretical distribution curve (dashed) in (b) is plotted against the estimates using GLR-MDL (+) and AGLR-MDL (o).

essentially weighted averages of \( P_{FA}(\eta) \) conditioned on a particular NOH. As seen in Figure 4.4, the theoretical conditional \( P_{FA}(\eta) \) is better modelled for increasing NOH. Consequently, it seems reasonable that the theoretical marginal \( P_{FA}(\eta) \) compares well to the empirical curves due to the heavier tailed PMF. On the other hand, the differences in the individual statistics warrants further research.

Figure 4.9 are plots of the \( P_D(\eta) \) versus \( \eta \). The theoretical \( P_D(\eta) \) is given by Equation (4.36). Each curve in Figure 4.9(a) corresponds to a signal with \( q = 2, 5, 10 \) and \( 30 \) harmonics and a SNR of \( \rho = -3 \) dB. In Figure 4.9(b), each curve corresponds to a signal with \( q = 2 \) harmonics and \( \rho = -4, -6, -8, -10, \) and \( -12 \) dB. In (a) and (b), the signal is made up of coupled harmonics with equal amplitudes. The simulation results are represented by the solid curves, whereas the theoretical
Figure 4.9: Detection probability versus $\eta$ for varying NOH (a) and varying SNR levels (b). The simulated NOH are $q = 2, 5, 10, \text{ and } 30$ with the SNR set to $\rho = -3$ dB in (a). The simulated SNR levels are $\rho = -12, -10, -8, -6, \text{ and } -4$ dB with $q = 2$ in (b). The theoretical curves (dashed) are plotted against simulation results using GLR-MDL (+) and AGLR-MDL (o). The data length is set to $N = 256$ and the signal amplitudes follow a uniform model.

$P_D$ curves are dashed lines. The $P_D$ is similarly plotted in Figures 4.10(a) and (b) for a $1/\sqrt{\omega}$ amplitude model.

In the case of unknown NOH, theory well-predicts the detection probability for signals with equal amplitudes, with the exception of low NOH. However, theory and simulation tend to agree more for low NOH as the SNR decreases. Comparing the uniform and $1/\sqrt{\omega}$ amplitude cases, the detection statistics appear to be better modelled for signals with uniform amplitudes.

Figure 4.11 contains ROC curves of $P_D$ versus SNR for the uniform (a) and $1/\sqrt{\omega}$ (b) amplitude models. The series of curves in each figure correspond to coupled harmonic signals with $q = 2, 5, 10, \text{ and } 30$ harmonics. The CFAR threshold $\eta^* \approx 7.76$ corresponds to a $P_{FA} = 10^{-2}$. 

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Figure 4.10: Detection probability distributions for varying NOH (a) and varying SNR levels (b). The simulated NOH are \( q = 2, 5, 10, \) and 30 with the SNR set to \( \rho = -3 \) dB in (a). The simulated SNR levels are \( \rho = -12, -10, -8, -6, \) and \(-4 \) dB with \( q = 2 \) in (b). The theoretical curves (dashed) are plotted against simulation results using GLR-MDL (+) and AGLR-MDL (○). The data length is set to \( N = 256 \) and the signal amplitudes follow a \( 1/\sqrt{\omega} \) model.

As seen in Figures 4.11(a), the theoretical ROCs can be used for predicting the detection performance for a signal with up to 30, and possible more, equal energy harmonics. From Figures 4.11(b), it is clear that the performance is not as well predicted for detection probabilities below 40% for the \( 1/\sqrt{\omega} \) amplitude model. We conjecture that the empirical detection probabilities are higher than predicted due to the increased difficulty in estimating the higher harmonics. However, the detection probabilities below 40% are generally not of interest anyway.
Figure 4.11: ROC for varying NOH for a uniform (a) and $1/\sqrt{\omega}$ (b) amplitude models. The NOH are $q = 2, 5, 10,$ and $30$ with the $P_{FA} = 10^{-2}$. The theoretical curves (dashed) are plotted against simulation results using GLR-MDL (+) and AGLR-MDL ($\circ$). The data length is set to $N = 256$.

4.4.3 Field Measurement Data

The following example illustrates some robustness of the signal detector to deviations in the signal model. The signal detector utilizes the ANLS-MDL estimates. The measurement data consists of noise due to the local environment and a single source generating coupled harmonics and unmodelled broadband energy. The source is a heavy-tracked battlefield vehicle. The data was collected at Aberdeen Proving Grounds (APG) using a seven-sensor, circular microphone array.

The algorithm parameters for estimation are the same as described in Section 3.4, except that the minimum allowable NOH is set to $r_{min} = 1$ (as opposed to $r_{min} = 0$ in Section 3.4). Again, only the results from Sensor 1 are presented.
The STFT of the raw data from Sensor 1 is represented by a spectrogram in Figure 4.12(a). In Figure 4.12(b), the harmonic frequency estimates from each half-second data block are plotted along the vertical axis. The horizontal axis represents the progression of time. The various levels of colors in Figures 4.12(a) and (b) represent the relative amplitudes, scaled in decibels, of the spectral data.

The GLR values, reported in decibels, are plotted in Figure 4.13(a). To achieve a false alarm rate of $P_{FA} = 10^{-2}$, the GLR values are compared against a corresponding threshold $\eta^* \approx 8.9$ dB. Using this decision threshold, the parameter estimates are 'filtered' by passing the parameter estimates for GLR values above $\eta^*$. The resulting filtered estimates are plotted in Figure 4.13(b). There are noticeable similarities in Figures 4.13(b) and 3.27(b). In both figures, the time windows in which no signals are detected are nearly the same. However, the signal detectors presented in this chapter provide variable control over the false alarm rate.
Figure 4.12: Spectrogram (a) and harmonic line estimates (b) of the acoustic signature from a single heavy-tracked vehicle. The relative amplitudes are in decibels, which is represented by the level of color.
Figure 4.13: GLR values (a) subjected to a CFAR threshold corresponding to $P_{FA} = 10^{-2}$ and 'filtered' harmonic line estimates (b).
4.5 Conclusions

Signal detection methods for frequency coupled harmonic signals for both the cases of known and unknown NOH were developed and examined using theoretical ROC curves. The detectors presented here extend the work in [22] to include estimation of the FF and the NOH. A GLRT was analyzed for the case of known NOH, and a GLRT combined with MDL order selection was analyzed for the case of unknown NOH.

As seen in Section 3.4, the MDL may also be used as a signal detector by allowing the minimum selectable order be $r_{\text{min}} = 0$. Since the MDL criterion is independent of a threshold test, there is no control over the false alarm rate. Consequently, the GLRT is combined with order selection in the case of unknown NOH.

The theoretical ROC curves were developed for both the known and unknown NOH cases using a number of approximations to keep the theory analytically tractable. For known NOH, the theoretical ROC curves suggest that the GLRT has desirable $P_D$ versus $P_{FA}$ characteristics for SNRs above $\rho = -9$ dB. Above $\rho = -9$ dB, the detection probability increases rapidly for small and slowly increasing false alarm rates. It is also observed that the detection performance decreases with increasing NOH.

Similar performance characteristics were observed from the theoretical ROC curves for the GLRT combined with MDL. Although, desirable $P_D$ versus $P_{FA}$ characteristics are predicted to occur at higher SNRs ($\rho \geq -4$ dB) as compared to the known NOH case. Similar to the known NOH case, the detection performance decreases with increasing NOH. On the other hand, the loss in detection performance is greater for increasing number of harmonics when the number is not known.
To test the theoretical performance, empirical results were generated for various simulation parameters; including two signal amplitude models, several NOH, and various SNRs. For the case of a uniform amplitude model, the theoretical ROC curves well-predicted the performance of the signal detector for both unknown and known NOH. On the other hand, for a $1/\sqrt{\omega}$ amplitude model, the theoretical performance is only well-modelled for detection probabilities above, say, 40% for the case of unknown NOH.

Although the theoretical and empirical results, for the most part, agree, the statistics of the GLR for the case of known NOH under the null hypothesis need to be reevaluated. In addition, the theoretical PMF for the MDL order estimates needs to be reexamined.
CHAPTER 5

CONCLUSIONS

Algorithms for signal detection and parameter estimation are developed and analyzed for a frequency coupled harmonic model. Due to the uncertainties in the fundamental frequency and the number of harmonics, estimation of the parameters of the harmonics is a difficult problem that requires a computationally costly search over candidate frequencies and model orders. Both the NLS-MDL and ANLS-MDL methods are shown to have comparable performance that is very close to the Cramér-Rao bound for sufficiently large signal-to-noise ratios and data lengths. In addition, the ANLS-MDL algorithm provides a computational savings over the NLS-MDL when the number of harmonics is greater than two. In addition, the proposed methods do not assume prior fundamental frequency estimates are available. Therefore, these estimators can be used as stand-alone algorithms or to initialize more advanced parameter tracking algorithms.

There is an inherent coupling of the parameter estimation algorithms and the signal detectors studied here. Consequently, the proposed estimation algorithms were also used in a detector structure. Despite the intractability of analyzing the exact statistics of the detector, appropriate approximations led to theoretical ROCs that were shown, for the most part, to well-predict the performance of the signal detectors.
APPENDIX A

CRAMÉR-RAO LOWER BOUNDS

The CRLBs for parameters of coupled harmonics in polar coordinates, $\theta^{pol}$, are well-published [14, 29]. However, the CRLBs for the parameters in Cartesian coordinates, $\theta^{ort}$, for both finite and large data lengths do not appear to be available in the literature. For completeness, the finite- and large-sample CRLBs are derived for both the Cartesian and polar cases below. The bounds are derived also assuming the NOH is known.

The time vector is defined as $n = [n_1, n_2, \ldots, n_N]^T = [-(N - 1)/2, -(N - 3)/2, \ldots, (N - 1)/2]^T$. Also, $\sin(\omega n) = [\sin(\omega n_1), \sin(\omega n_2), \ldots, \sin(\omega n_N)]^T$ and similarly for $\cos(\omega n)$. Under a white noise assumption, the log-likelihood function in various forms is

\begin{align*}
\ln p(y|\theta) &= -\frac{N}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \| y - s(\theta) \|^2, \quad (A.1) \\
&= -\frac{N}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \left\| y - \sum_{k=1}^{q} \alpha_k \cos(k\omega n_k - \phi_k) \right\|^2, \quad (A.2) \\
&= -\frac{N}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \left\| y - \sum_{k=1}^{q} u_k \cos(k\omega n_k n) + v_k \sin(k\omega n_k) \right\|^2, \quad (A.3) \\
&= -\frac{N}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \| y - C(\omega_0)u - S(\omega_0)v \|^2, \quad (A.4) \\
&= -\frac{N}{2} \ln(2\pi \sigma^2) - \frac{1}{2\sigma^2} \| y - A(\omega_0)\theta \|^2. \quad (A.5)
\end{align*}

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Since each of the above forms are equivalent, the one that is most convenient will be used. The Fisher’s information matrix (FIM) is given by \( I(\theta) = -\mathcal{E} \left\{ \frac{\partial^2 \ln p(y|\theta)}{\partial \theta \partial \theta^T} \right\} \) [18]. The substitutions \( C = C(\omega_0) \) and \( S = S(\omega_0) \) will be used for notational convenience.

### A.1 CRLB for Cartesian Parameter Vector

First, the FIM for \( \theta = \theta^{cart} \), where \( \theta^{cart} = [\omega_0, u_1, \ldots, u_q, v_1, \ldots, v_q]^T \), is considered. The first partial derivatives of the log–likelihood are

\[
\frac{\partial (\ln p(y|\theta))}{\partial \theta} = \begin{bmatrix}
\frac{\partial (\ln p(y|\theta))}{\partial \omega_0} \\
\frac{\partial (\ln p(y|\theta))}{\partial u_1} \\
\vdots \\
\frac{\partial (\ln p(y|\theta))}{\partial u_q} \\
\frac{\partial (\ln p(y|\theta))}{\partial v_1} \\
\vdots \\
\frac{\partial (\ln p(y|\theta))}{\partial v_q}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-\frac{1}{\sigma^2} \left[ \sum_{k=1}^{q} k \omega_0 \mathbf{n} \odot \sin(k \omega_0 \mathbf{n} + \phi_k) \right]^T (y - s(\theta)) \\
\frac{1}{\sigma^2} (\cos(\omega_0 \mathbf{n}))^T (y - s(\theta)) \\
\vdots \\
\frac{1}{\sigma^2} (\cos(q \omega_0 \mathbf{n}))^T (y - s(\theta)) \\
\frac{1}{\sigma^2} (\sin(\omega_0 \mathbf{n}))^T (y - s(\theta)) \\
\vdots \\
\frac{1}{\sigma^2} (\sin(q \omega_0 \mathbf{n}))^T (y - s(\theta))
\end{bmatrix}, \quad (A.6)
\]

where \( \mathbf{x} \odot \mathbf{z} \) is the element-wise product of vectors \( \mathbf{x} \) and \( \mathbf{z} \). Next, the second partial derivatives, along with the expectation, are developed for the FIM.
The first entry in $I(\theta)$ is

$$
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(\mathbf{y}|\theta))}{\partial \omega_0^2} \right\} = -\mathcal{E} \left\{ -\frac{1}{\sigma^2} \left( \sum_{k=1}^{q} k^2 \alpha_k \mathbf{n} \odot \mathbf{n} \odot \cos(k\omega_0 \mathbf{n} + \phi_k) \right)^T (\mathbf{y} - \mathbf{s}(\theta)) \right. \\
\left. - \frac{1}{\sigma^2} \left\| \sum_{k=1}^{q} k \alpha_k \mathbf{n} \odot \sin(k\omega_0 \mathbf{n} + \phi_k) \right\|^2 \right\},
$$

$$
= \frac{1}{\sigma^2} \left( \sum_{k=1}^{q} k^2 \alpha_k \mathbf{n} \odot \mathbf{n} \odot \cos(k\omega_0 \mathbf{n} + \phi_k) \right)^T \mathcal{E} \{ \mathbf{y} - \mathbf{s}(\theta) \} \\
+ \frac{1}{\sigma^2} \left\| \sum_{k=1}^{q} k \alpha_k \mathbf{n} \odot \sin(k\omega_0 \mathbf{n} + \phi_k) \right\|^2,
$$

$$
= \frac{1}{\sigma^2} \left\| \mathbf{T} \mathbf{S} \mathbf{u} - \mathbf{TC} \mathbf{Q} \mathbf{v} \right\|^2,
$$

$$
= \frac{1}{\sigma^2} \left\| \mathbf{T} \mathbf{S} \mathbf{u} \right\|^2 + \frac{1}{\sigma^2} \left\| \mathbf{TC} \mathbf{Q} \mathbf{v} \right\|^2, \quad (A.7)
$$

where $\mathbf{T} = \text{diag}\{- (N - 1)/2, - (N - 3)/2, \ldots, (N - 3)/2, (N - 1)/2\}$ and $\mathbf{Q} = \text{diag}\{1, 2, \ldots, q\}$. The next $q$ entries in the first column and first row of $I(\theta)$ are

$$
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(\mathbf{y}|\theta))}{\partial \omega_0 \partial u_m} \right\} = -\mathcal{E} \left\{ \frac{\partial^2 (\ln p(\mathbf{y}|\theta))}{\partial u_m \partial \omega_0} \right\},
$$

$$
= -\mathcal{E} \left\{ \frac{-m}{\sigma^2} (\mathbf{n} \odot \sin(m\omega_0 \mathbf{n}))^T (\mathbf{y} - \mathbf{s}(\theta)) \right. \\
\left. + \frac{1}{\sigma^2} (\cos(m\omega_0 \mathbf{n}))^T \left( \sum_{k=1}^{q} k \alpha_k \mathbf{n} \odot \sin(k\omega_0 \mathbf{n} + \phi_k) \right) \right\},
$$

$$
= \frac{m}{\sigma^2} (\mathbf{n} \odot \sin(m\omega_0 \mathbf{n}))^T \mathcal{E} \{ \mathbf{y} - \mathbf{s}(\theta) \} \\
- \frac{1}{\sigma^2} (\cos(m\omega_0 \mathbf{n}))^T \left( \sum_{k=1}^{q} k \alpha_k \mathbf{n} \odot \sin(k\omega_0 \mathbf{n} + \phi_k) \right),
$$

$$
= - \frac{1}{\sigma^2} (\cos(m\omega_0 \mathbf{n}))^T \left( \sum_{k=1}^{q} k \alpha_k \mathbf{n} \odot \sin(k\omega_0 \mathbf{n} + \phi_k) \right),
$$

$$
= - \frac{1}{\sigma^2} \mathbf{C}_m^T \mathbf{T} \mathbf{S} \mathbf{u}, \quad (A.8)
$$
where \( C_m \) and \( S_m \) are the \( m^{th} \) columns of \( C \) and \( S \), respectively, for \( m \in \{1, 2, \ldots, q\} \).

Similarly, the last \( q \) entries of the first row and column are

\[
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial \omega_0 \partial v_m} \right\} = -\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial v_m \partial \omega_0} \right\},
\]
\[
= \frac{1}{\sigma^2} S_m^T T C \Omega v,
\]

for \( m \in \{1, 2, \ldots, q\} \). The \((m, k)\) element of \( I(\theta) \) (i.e., \( [I(\theta)]_{m,k} \)) for \( k, m \in \{2, 3, \ldots, q+1\} \) is given by

\[
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial u_{m-1} \partial u_{k-1}} \right\} = -\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial u_{k-1} \partial u_{m-1}} \right\},
\]
\[
= -\mathcal{E} \left\{ \frac{1}{\sigma^2} C_{m-1}^T C_{k-1} \right\},
\]
\[
= \frac{1}{\sigma^2} C_{m-1}^T C_{k-1}.
\]

Similarly, the entry \([I(\theta)]_{m,k}\) for \( m \in \{q+2, \ldots, 2q+1\} \) and \( k \in \{q+2, \ldots, 2q+1\} \) is given by

\[
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial v_{m-q-1} \partial v_{k-q-1}} \right\} = -\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial v_{k-q-1} \partial v_{m-q-1}} \right\},
\]
\[
= -\mathcal{E} \left\{ \frac{1}{\sigma^2} S_{m-q-1}^T S_{k-q-1} \right\},
\]
\[
= \frac{1}{\sigma^2} S_{m-q-1}^T S_{k-q-1}.
\]

The entries \([I(\theta)]_{m,k}\) and \([I(\theta)]_{k,m}\) for \( m \in \{2, 3, \ldots, q+1\} \) and \( k \in \{q+2, \ldots, 2q+1\} \) is given by

\[
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial u_{m-1} \partial v_{k-q-1}} \right\} = -\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial v_{k-q-1} \partial u_{m-1}} \right\},
\]
\[
= -\mathcal{E} \left\{ \frac{1}{\sigma^2} C_{m-1}^T S_{k-q-1} \right\},
\]
\[
= 0.
\]

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Recall that the phase is referenced to the center of the sampling window to guarantee $C(\omega)^T S(\omega) = 0$. Piecing the above components together, the FIM for $\theta^{\text{cart}}$ is

$$ I(\theta^{\text{cart}}) = \frac{1}{\sigma^2} \begin{bmatrix} \|TSQu\|^2 + \|TCQv\|^2 & -(GQu)^T & (G^T Qv)^T \\ -GQu & C^T C & 0 \\ G^T Qv & 0 & S^T S \end{bmatrix}, \quad (A.13) $$

where $G = C^T TS$. The finite-sample CRLB is then determined by $CRB_N^{\text{cart}} = [I(\theta^{\text{cart}})]^{-1}$. The task of computing $CRB_N^{\text{cart}}$ is well-suited to a computer.

Alternatively, a closed-form expression for the large-sample CRLB exists. As discussed in [26], the large-sample CRLBs are useful approximations to finite-sample CRLBs.

The following results are a consequence of using Lemma A.1 and its corollary from [26].

$$ \lim_{N\to\infty} \frac{1}{N} C(\omega)^T C(\omega) = \frac{1}{2} I_q, $$

$$ = \lim_{N\to\infty} \frac{1}{N} S(\omega)^T S(\omega). \quad (A.14) $$

$$ \lim_{N\to\infty} \frac{1}{N^2} C(\omega)^T TS(\omega)_k = \lim_{N\to\infty} \frac{1}{2N^2} \sum_n n \{ \sin((m - k)\omega n) + \sin((m + k)\omega n) \}, $$

$$ = 0. \quad (A.15) $$

$$ \lim_{N\to\infty} \frac{1}{N^3} C(\omega)^T T^2 C(\omega) = \frac{1}{24} I_q, $$

$$ = \lim_{N\to\infty} \frac{1}{N^3} S(\omega)^T T^2 S(\omega). \quad (A.16) $$

where $I_m$ is the $(m \times m)$ identity matrix. Similarly to [26], defining the matrix

$$ K = \begin{bmatrix} N^{3/2} & 0 \\ 0 & N^{1/2} I_{2q} \end{bmatrix}, \quad (A.17) $$

the large-sample CRLB on the covariance matrix of the error vector $K(\hat{\theta} - \theta)$ is given by $\lim_{N\to\infty} KCRB_N K = CRB_\infty$. 

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Using the above results, the large-sample CRLB for the error vector $\mathbf{K}(\hat{\theta}^{\text{cart}} - \theta^{\text{cart}})$ is given by

\[ CRB_{\infty}^{\text{cart}} = 2\sigma^2 \begin{bmatrix} 12 (\alpha^T Q^2 \alpha)^{-1} & 0 \\ 0 & I_{2q} \end{bmatrix}. \]  

(A.18)

Note that $u^T Q^2 u + v^T Q^2 v = \alpha^T Q^2 \alpha$.

### A.2 CRLB for Polar Parameter Vector

Next, the FIM is derived for $\theta = \theta^{\text{pol}}$, where $\theta^{\text{pol}} = [\omega_0, \alpha_1, \ldots, \alpha_q, \phi_1, \ldots, \phi_q]^T$.

The first partial derivatives of the log-likelihood function are

\[
\frac{\partial \ln p(y|\theta)}{\partial \theta} = \begin{bmatrix}
-\frac{1}{\sigma^2} (\sum_{k=1}^q k\alpha_k \mathbf{n} \odot \sin(k\omega_0 \mathbf{n} + \phi_k))^T (y - s(\theta)) \\
\frac{1}{\sigma^2} (\cos(\omega_0 \mathbf{n} + \phi_1))^T (y - s(\theta)) \\
\vdots \\
\frac{-\alpha_1}{\sigma^2} (\cos(q\omega_0 \mathbf{n} + \phi_q))^T (y - s(\theta)) \\
\frac{-\alpha_2}{\sigma^2} (\sin(\omega_0 \mathbf{n} + \phi_1))^T (y - s(\theta)) \\
\vdots \\
\frac{-\alpha_q}{\sigma^2} (\sin(q\omega_0 \mathbf{n} + \phi_q))^T (y - s(\theta))
\end{bmatrix}. \tag{A.19}

Then, the following are the entries of $I(\theta)$. The first entry in $I(\theta)$ is identical to that given in the Cartesian system. The next $q$ entries in the first column and first row of $I(\theta)$ are

\[
-E \left\{ \frac{\partial^2 \ln p(y|\theta)}{\partial \omega_0 \partial \alpha_m} \right\} = -E \left\{ \frac{\partial^2 \ln p(y|\theta)}{\partial \alpha_m \partial \omega_0} \right\} = -E \left\{ \frac{-m}{\sigma^2} (\mathbf{n} \odot \sin(m\omega_0 \mathbf{n} + \phi_m))^T (y - s(\theta)) + \frac{1}{\sigma^2} (\cos(m\omega_0 \mathbf{n} + \phi_m))^T \left( \sum_{k=1}^q k\alpha_k \mathbf{n} \odot \sin(k\omega_0 \mathbf{n} + \phi_k) \right) \right\}
\]

\[
= -\frac{\cos(\phi_m)}{\sigma^2} \mathbf{C}_m^T \textbf{T} \mathbf{S} \mathbf{Q} \mathbf{u} - \frac{\sin(\phi_m)}{\sigma^2} \mathbf{S}_m^T \mathbf{T} \mathbf{C} \mathbf{Q} \mathbf{v}, \tag{A.20}
\]
for $m \in \{1, 2, \ldots, q\}$. Similarly, the last $q$ entries of the first row and column are

$$
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial \omega_0 \partial \phi_m} \right\} = -\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial \phi_m \partial \omega_0} \right\} = -\mathcal{E} \left\{ \frac{-\alpha m_m}{\sigma^2} (n \odot \cos(m \omega_0 n + \phi_m))^T (y - s(\theta)) \right. \\
\left. - \frac{\alpha m_m}{\sigma^2} (\sin(m \omega_0 n + \phi_m))^T \left( \sum_{k=1}^{q} k \alpha_k n \odot \sin(k \omega_0 n + \phi_k) \right) \right\} = -\frac{v_m}{\sigma^2} C^T_m TSQu - \frac{u_m}{\sigma^2} S^T_m TCQv,
$$

(A.21)

for $m \in \{1, 2, \ldots, q\}$. The element $[I(\theta)]_{m,k}$ for $k, m \in \{2, 3, \ldots, q+1\}$ is given by

$$
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial \alpha_{m-1} \partial \alpha_{k-1}} \right\} = -\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial \alpha_{k-1} \partial \alpha_{m-1}} \right\} = -\mathcal{E} \left\{ \frac{1}{\sigma^2} \left[ \cos((m-1) \omega_0 n + \phi_{m-1}) \right]^T \right. \\
\times \left[ -\cos((k-1) \omega_0 n + \phi_{k-1}) \right] \left\} = \frac{1}{\sigma^2} \cos(\phi_{m-1}) \cos(\phi_{k-1}) C^T_{m-1} C_{k-1} + \sin(\phi_{m-1}) \sin(\phi_{k-1}) S^T_{m-1} S_{k-1} \right\} \right.
$$

(A.22)

The entry $[I(\theta)]_{m,k}$ for $m \in \{q+2, \ldots, 2q+1\}$ and $k \in \{q+2, \ldots, 2q+1\}$ is given by

$$
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial \phi_{m-q-1} \partial \phi_{k-q-1}} \right\} = -\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial \phi_{k-q-1} \partial \phi_{m-q-1}} \right\} = \left\{ \begin{array}{ll}
-\mathcal{E} \left\{ \frac{1}{\sigma^2} [-\alpha_{m-q-1} \cos((m-q-1) \omega_0 n + \phi_{m-q-1})]^T \right. \\
\times [y - s(\theta)] \\
\left. - \frac{1}{\sigma^2} [\alpha_{k-q-1} \sin((k-q-1) \omega_0 n + \phi_{k-q-1})]^T \times [\alpha_{m-q-1} \sin((m-q-1) \omega_0 n + \phi_{m-q-1})] \right\} & \text{if } m = k \\
-\mathcal{E} \left\{ \frac{1}{\sigma^2} [\alpha_{k-q-1} \sin((k-q-1) \omega_0 n + \phi_{k-q-1})]^T \times [\alpha_{m-q-1} \sin((m-q-1) \omega_0 n + \phi_{m-q-1})] \right\} & \text{if } m \neq k \\
\right.
$$

$$
= \frac{1}{\sigma^2} \left\{ u_{m-q-1} u_{k-q-1} S^T_{m-q-1} S_{k-q-1} + v_{m-q-1} v_{k-q-1} C^T_{m-q-1} C_{k-q-1} \right\}
$$

(A.23)
The entries \([I(\theta)]_{m,k}\) and \([I(\theta)]_{k,m}\) for \(m \in \{2, 3, \ldots, q+1\}\) and \(k \in \{q+2, \ldots, 2q+1\}\) are given by

\[
-\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial \phi_{m-1} \partial \phi_{k-q-1}} \right\} = -\mathcal{E} \left\{ \frac{\partial^2 (\ln p(y|\theta))}{\partial \phi_{k-q-1} \partial \alpha_{m-1}} \right\} \\
= \begin{cases} 
-\mathcal{E} \left\{ \left[ -\frac{1}{\sigma} \sin((k - q - 1)\omega_0 n + \phi_{k-q-1}) \right] \right\} T \\
\times \left[ \sin((k - q - 1)\omega_0 n + \phi_{k-q-1}) \right] \right\} & \text{if } m = k \\
-\mathcal{E} \left\{ \left[ \frac{2\alpha_{m-1}}{\sigma^2} \cos((m - 1)\omega_0 n + \phi_{m-1}) \right] \right\} T \\
\times \left[ \sin((k - q - 1)\omega_0 n + \phi_{k-q-1}) \right] \right\} & \text{if } m \neq k 
\end{cases}
\]

\[
= \frac{1}{\sigma^2} \{ u_{k-q-1} \sin(\phi_{m-1})S_{m-1}^T S_{k-q-1}^T + v_{k-q-1} \cos(\phi_{m-1})C_{m-1}^T C_{k-q-1} \} \quad (A.24)
\]

Defining

\[
\Phi_s = \text{diag}\{\sin(\phi_1), \sin(\phi_2), \ldots, \sin(\phi_q)\}, \quad (A.25)
\]

\[
\Phi_c = \text{diag}\{\cos(\phi_1), \cos(\phi_2), \ldots, \cos(\phi_q)\}, \quad (A.26)
\]

\[
\Gamma_u = \text{diag}\{u_1, u_2, \ldots, u_q\}, \quad (A.27)
\]

\[
\Gamma_v = \text{diag}\{v_1, v_2, \ldots, v_q\}, \quad (A.28)
\]

the FIM for \(\theta_{\text{pol}}\) is

\[
I(\theta_{\text{pol}}) = \frac{1}{\sigma^2} \left[ \begin{array}{ccc}
\|TSQU\|^2 + \|TCQV\|^2 & - (\Phi_s GQU + \Phi_c G^T QV) & - (\Gamma_v GQU + \Gamma_u G^T QV) \\
- (\Phi_s GQU + \Phi_c G^T QV) & \Phi_c G^T \Phi_c + \Phi_s S^T S \Phi_s & \Phi_c G^T \Gamma_v + \Phi_s S^T S \Gamma_u \\
- (\Gamma_v GQU + \Gamma_u G^T QV) & \Phi_c G^T \Gamma_v + \Phi_s S^T S \Gamma_u & \Gamma_v G^T \Gamma_v + \Gamma_u S^T S \Gamma_u
\end{array} \right] \quad (A.29)
\]

As with the CRLB for \(\theta_{\text{ort}}\), a closed-form expression exists for the large-sample CRLB of \(\theta_{\text{pol}}\). The large-sample CRLB on the covariance matrix of the error vector
\( K(\theta_{\text{pol}} - \theta_{\text{pol}}) \) is given by

\[
CRB_{\infty}^{\text{pol}} = 2\sigma^2 \begin{bmatrix}
12 \left( \alpha^T Q^2 \alpha \right)^{-1} & 0 & 0 \\
0 & I_q & 0 \\
0 & 0 & \alpha_1^{-2} \\
& & \ddots \\
& & & \alpha_q^{-2}
\end{bmatrix}.
\] (A.30)

Note that \( \Gamma_u \Phi_s = -\Gamma_v \Phi_c, \Phi_c^2 + \Phi_s^2 = I_q \), and \( \Gamma_u^2 + \Gamma_v^2 = \text{diag} \{ \alpha_1^2, \ldots, \alpha_q^2 \} \). As in [14], since the phase is defined at the middle of the observation window, the phase and frequency estimates are asymptotically uncorrelated.

It is recognized that the relationship between the respective FIMs is

\[
I(\theta_{\text{pol}}) = H^T I(\theta_{\text{cart}}) H,
\] (A.31)

where

\[
H = \begin{bmatrix}
1 & 0 & 0 \\
0 & \Phi_c & \Gamma_v \\
0 & -\Phi_s & -\Gamma_u
\end{bmatrix},
\] (A.32)

is the Jacobian of the transformation from \( \theta_{\text{pol}} \) to \( \theta_{\text{cart}} \). The Jacobian relationship, which holds for one-to-one mappings that produce continuous, bounded likelihood functions, is developed in [22].

It was shown in [26] that the estimates of the noise level, denoted \( \hat{\sigma}^2 \), are uncorrelated from the estimates of the signal parameters in general sinusoidal summation models, even for finite data lengths. Accordingly, it is straightforward to show that the CRLB on the covariance matrix of the error vector \( \left[ (\theta_{\text{pol}} - \theta_{\text{pol}})^T, (\hat{\sigma}^2 - \sigma^2) \right]^T \) is given by

\[
CRB_N = \begin{bmatrix}
CRB_{\theta_{\text{pol}}}^{\text{pol}} & 0 \\
0 & 2\sigma^4 / N
\end{bmatrix},
\] (A.33)

and similarly if the polar coordinates are replaced by the Cartesian coordinates.
BIBLIOGRAPHY


